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Applications of Grothendieck-Riemann-Roch Theorem for Stacks and Stringy Chow ring of Weighted Blow-ups

Candidate:
Qiangru Kuang

Advisor:
Nicolò Sibilla
Co-advisors:
Mattia Talpo

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Abstract

In this thesis we prove three results. Firstly we apply the Grothendieck-Riemann-Roch theorem for stacks to root stacks to rederive the formula for parabolic bundles. Next we apply the same theorem to a quotient stack to derive a formula for equivariant Euler characteristic. When the quotient is obtained by an action on a smooth projective curve, we explicitly compute the Euler characteristic in terms of ramification data. This agrees with many previous results with different levels of generalities, thereby providing a unified way to prove the result in these settings. Lastly, we study the stringy Chow ring structure of weighted blow-ups with regular centres. We completely determine the ring structure and answering several questions regarding its finite-generation.

Contents

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Chapter 0

Introduction

This thesis is a compilation of three projects of the author. The central ideas of these projects are Grothendieck-Riemann-Roch theorem for stacks and root stacks.

0.1 Themes

Riemann-Roch theorem and its many siblings are some of the most fundamental and important results in algebraic geometry. Proven by Riemann and his student Roch in the 19th century, the theorem in its original form concerns the dimension of the space of meromorphic functions with prescribed zeros and poles on a compact Riemann surface. With the advent of modern algebraic geometry based on commutative algebra, it was soon proven for projective algebraic curves. In the mid 20th century Hirzebruch generalised the theorem greatly to vector bundles on smooth complex algebraic varieties of any dimension. It is also in this formulation, named the Hirzebruch-Riemann-Roch theorem, that the Todd class made its first appearance:

$$\chi(X, E) = \int_X \operatorname{ch} E \operatorname{Td}(X).$$

Another milestone was reached soon after, when Grothendieck revolutionised the field of algebraic geometry by, among other things, introducing the relative point of view. The Grothendieck-Riemann-Roch theorem concerns a proper morphism $f : X \to Y$ between smooth quasi-projective schemes instead of a single variety, and relates the pushforward in K-theory and pushforward of cycle classes:

$$\operatorname{ch}(Rf_* E) \operatorname{Td}(Y) = f_*(\operatorname{ch} E \operatorname{Td}(X)).$$

With the introduction of stacks in algebraic geometry, it is natural to expect that there is a Grothendieck-Riemann-Roch style theorem in this setting. However there are serious challenges and solving the problem requires fundamentally new ideas, even for Deligne-Mumford stacks. Vistoli [Vis89] developed an intersection theory for stacks and in particular a Chow group with rational coefficients for Deligne-Mumford stacks. In the same paper it is shown that the coarse moduli space maps induces an isomorphism of Chow groups.

However, this immediately poses a problem for GRR style theorem. To wit, take the simplest example: the classifying stack BG of a finite abelian group G. Then its K-theory is the representation ring of G, while its Chow group is the same as that of a point. Thus for dimension reason $\mathrm{ch}\colon K(BG)_{\mathbb{Q}} \to \mathrm{A}(BG)_{\mathbb{Q}}$ cannot be an isomorphism, in contrast with the case of schemes where it follows from Grothendieck-Riemann-Roch theorem. This does not directly rule out the *existence* of a Riemann-Roch morphism $\tau\colon K(-) \to \mathrm{A}(-)$ that is covariant with respect to proper pushforward. Unfortunately we will see with a bit more analysis that this is impossible either. In any case, one could deduce from this simple example that the K-theory of a Deligne-Mumford stack records automorphism data while the Chow group only knows the coarse moduli space and is hence "too small" to be the target of a Riemann-Roch style theorem.

The breakthrough was achieved by Toën in [Toë99], who showed that one needs to consider the *inertia stack*. Roughly speaking, the inertia stack of a stack is a stack that records the automorphisms of every object. For a scheme (or more generally, algebraic space) the inertia stack is itself so indeed we can recover Grothendieck-Riemann-Roch theorem. Built on previous work by Vistoli ([Vis92], [Vis91]) on higher equivariant K-theory, Toën proved that there is a covariant Riemann-Roch morphism $\tau : K(-)_{\mathbb{C}} \to \mathrm{A}(I(-))_{\mathbb{C}}$. Its construction is far from obvious and makes heavy use of K-theory. Toën's theory of Riemann-Roch, although comprehensive and aesthetically pleasing, is very abstract. There are very few examples illustrating its use, except a few Hirzebruch-Riemann-Roch style computations (i.e. applications of the theorem to the a morphism of the form $\mathcal{X} \to \operatorname{Spec} k$). Our work in Chapter 2 is a novel application that exploits the full power of the theorem and computes the *Chern character* instead of just the Euler characteristic.

To motivate the main result of Chapter 2 we introduce the second topic in this thesis — root stacks and parabolic sheaves. Root stacks appeared to be first defined in the work of Cadman [Cad07] and Abramovich, Graber and Vistoli [AGV08], although the construction was mentioned in [AGV02] and was almost certainly known to experts before. Given a line bundle L and a section σ on a stack X, the nth root stack $\sqrt[n]{(X, L, \sigma)}$ is a stack with morphism to X that parameterises the nth root of (L, σ). In particular one can take an effective Cartier divisor D and consider the root stack associated to its ideal sheaf $\mathcal{O}_X(-D)$. Note that the pair (X, D) defines a logarithmic structure. This observation leads Borne and Vistoli [BV12] to give a very general definition of root stacks using the language of Deligne-Faltings structures, allowing one to take roots along a more general logarithmic structure and replacing the denominator n by sheaves of monoids.

In the same paper [BV12] Borne and Vistoli established a dictionary between sheaves on root stacks and parabolic sheaves. Parabolic vector bundle was first defined by Mehta and Seshadri [MS80] to study the correspondence between unitary representation of the fundamental group of a Riemann surface and semi-stable vector bundles. It was given in terms of a vector bundle on a compact Riemann surface together with filtrations of the fibres at some given points. It was later generalised by [MY92] to higher dimensions: given an effective Cartier divisor D on a smooth projective variety X, a parabolic vector bundle is a vector bundle E together with a filtration

$$E(-D) = F_{n+1} \subseteq F_n \subseteq \cdots \subseteq F_1 = E$$

for some rational numbers $0 < a_1 < \cdots < a_n < 1$ called weights. When D is not smooth, in particular if it has multiple components, however, it is better to consider different components separately, which was done by Mochizuki [Moc06]. This is essentially the definition of parabolic vector bundle that we will use in this thesis, and we note that we adopt the definition that is closest in literature to that of [Bor09]. We think of a parabolic bundle as a diagram $E : \frac{1}{n}\mathbb{Z} \to \mathrm{Vect}(X)$ satisfying the pseudoperiodicity condition $E_i \otimes \mathcal{O}_X(D) \cong E_{i+1}$ and local freeness of quotients $\mathrm{coker}(E_i \to E_j)$ for $i \leq j < i + 1$. Clearly E is determined by its values at $[0, 1)$ by the pseudoperiodicity condition.

In the aforementioned work [BV12], Borne and Vistoli generalised parabolic sheaves to general denominator systems of monoids, and showed that there is an equivalence of category between the abelian categories of quasicoherent sheaves on a root stack and the category of quasicoherent parabolic sheaves on the logarithmic scheme. This generalises previous work of Borne ([Bor07], [Bor09]).

Combining the equivalence and the isomorphism between the rational Chow group of a root stack and its coarse moduli space, which is given by the canonical map $\sqrt[n]{X} \to X$, it is natural to ask what the Chern character of a parabolic bundle as an element of $\mathrm{A}^*(X)_{\mathbb{Q}}$ is. We will show that it is closely related to the contribution of the component of the inertia stack $I\sqrt[n]{X}$ corresponding to trivial stabiliser. Thus we could extract it by applying the Toën-Riemann-Roch theorem to the map $\sqrt[n]{X} \to X$ and then subtracting the contributions from other components.

In Chapter 3, we use a related but different Riemann-Roch theorem to derive an equivariant Riemann-Roch theorem. When restricted to a curve, the problem is a classical one: given a curve X equipped with an action of a group G and a G-equivariant vector bundle E, find the Euler characteristic $\sum H^i(X, E)$ as a G-representation. We propose a new solution by applying a K-theory valued Riemann-Roch theorem to the structure map $[X/G] \to BG$.

The third topic in this thesis is stringy Chow ring. Motivated by string theory, Chen and Ruan in [CR04], [CR02] introduced a Gromov-Witten invariants for symplectic orbifolds. The algebraic counterpart was developed by Abramovich, Graber and Vistoli in [AGV02] and [AGV08]. The key observation was that in order to generalise the Gromov-Witten theory to a target $\mathcal{X}$ with stacky structure, it is essential to consider maps into $\mathcal{X}$ from families of marked stable curves which acquire stacky structures themselves. This gives rise to the moduli space $\mathcal{K}_{g,n}(\mathcal{X})$ of twisted stable maps. In addition, the evaluation maps should not land in $\mathcal{X}$, rather the inertia stack $I\mathcal{X}$ (this is technically only true for $g = 0, n = 3$, as in general we will need to consider the rigidified inertia stack). In summary, orbifold Gromov-Witten theory gives associative products to $\mathrm{A}_*(I\mathcal{X})$ using the moduli space $\mathcal{K}_{g,n}(\mathcal{X})$. Unlike the case of varieties, even the $g = 0, n = 3$ case gives a non-trivial product on $I\mathcal{X}$, giving rise to the *stringy Chow ring*, denoted $\mathrm{A}^*_{\mathrm{st}}(\mathcal{X})$. We note that in literature it is also known as Chen-Ruan Chow ring, orbifold Chow ring as well as their variants using cohomology.

In the last part of the thesis we focus on the stringy Chow ring of *weighted blow-ups* with regular centres, in particular determining completely the ring structure and answering several questions regarding finite-generation of stringy Chow ring. In terms of literature, the closest previous work in this direction is by Borisov, Chen and Smith [BCS05] and Jiang and Tseng [JT10] to compute the rational and integral stringy Chow ring of toric stacks. However there are two major differences. Firstly although there is a non-empty intersection between weighted blow-ups and toric stacks, neither is contained in another and our result extends to the new setting. Secondly, while [BCS05] (resp. [JT10]) concerns the generation of stringy Chow ring as $\mathbb{Q}$-modules (resp. $\mathbb{Z}$-modules), describing the generators and relations using the combinatorial data of stacky toric fans, we are more interested in the finite generation of $\mathrm{A}^*_{\mathrm{st}}(\mathscr{B}\mathrm{l}_Y\, X)$ as a $\mathrm{A}^*(X)$-algebra. We give a necessary and sufficient condition in terms of the Chow rings of the X and Y at the *scheme* level, which is not obvious at all a priori. Even without knowing finite generation, we demonstrate that there is a subring of *ambient classes* which is always finitely generated, and we give explicit generators and relations.

0.2 Summary of work and main theorems

In the first project we derive a formula for vector bundles on a root stack. This is a novel application of Toën-Riemann-Roch theorem for Deligne-Mumford stacks, as previous literature focuses either on developing the abstract theory of Riemann-Roch theorem, or applications of Hirzebruch-Riemann-Roch style theorem, computing only the degree of vector bundles. We use the full power of Toën-Riemann-Roch theorem to compute the Chern character of parabolic vector bundles. The main theorem is

Theorem (Theorem 2.1)**.** *Let X be a smooth projective variety over an algebraically closed field of characteristic 0 and let D be a smooth effective Cartier divisor on X. Let $E_\bullet$ be a parabolic vector bundle on (X, D) of weight n. Then*

$$\operatorname{ch}^{\mathrm{par}} E_\bullet = \frac{1 - e^{-D/n}}{1 - e^{-D}} \sum_{i=0}^{n-1} \operatorname{ch} E_i \cdot e^{-iD/n}$$

in $\mathrm{A}^*(X)_{\mathbb{Q}}$.

The formula was first derived by Iyer and Simpson in [IS08] by a different method: they showed that the parabolic Chern character depends only on the (ordinary) Chern character of the component bundles and extracted the $\operatorname{ch}^{\mathrm{par}} E_\bullet$ by averaging over the shifts $E_\bullet, E_\bullet[1], \ldots, E_\bullet[n-1]$.

In the second project, joint with Francesco Sala, we compute the Euler characteristic of an equivariant vector bundle on a smooth variety equipped with a finite group action. In particular when the variety is a curve, we derive the formula for the virtual representation in terms of ordinary Euler characteristic and ramification data. The result itself is classical and has been studied by many, such as [EL80], [Kan86], [Nak86], [Köc05], [FWK09] but as far as we know, our approach is unique in the sense that the passage to inertia stack allows us to treat the global space and the local ramification data on equal footing. The main results are

Theorem (Theorem 3.1)**.** *Given a G-equivariant vector bundle $\mathcal{E}$ on X, the Euler characteristic of $\mathcal{E}$ is*

$$\chi_G(X, \mathcal{E}) = \bigoplus_\sigma \bigoplus_i \frac{\chi(A^{\sigma,i})}{\varphi(|\sigma|)} \frac{|\sigma|}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^G \iota(x^i) \in K(BG)$$

where $A^{\sigma,i}$ is a $C(\sigma)$-equivariant vector bundle on X^σ that will be given explicitly.

Theorem (Theorem 3.2)**.** *Let X be a curve and $Y = X/G$ be the quotient. For each point $x \in X$, we denote by G_x the stabiliser of x. Let e_x (resp. e_x^t) be the ramification index (resp. the tame ramification index). Let $N_x^\vee$ be the cotangent space at x. Then in $K(BG)$ there is an equality*

$$\chi_G(X, \mathcal{E}) = \left(\chi(X, \mathcal{E}) + \frac{\operatorname{rk} \mathcal{E}}{2} \sum_x (e_x^t - 1)\right) \frac{kG}{n} + \sum_{x \in X} \frac{e_x}{n} \operatorname{Ind}_{G_x}^G \frac{\mathcal{E}_x}{1 - N_x^\vee}.$$

When the G-action is tame we may write

$$\chi_G(X, \mathcal{E}) = ((1 - g_Y) \operatorname{rk} \mathcal{E} + \frac{1}{n} \deg \mathcal{E}) kG - \frac{1}{n} \sum_{x \in X} \sum_{d=0}^{e_x - 1} d \cdot \operatorname{Ind}_{G_x}^G (\mathcal{E}_x \otimes N_x^{-d}).$$

Finally, in the fourth project, joint with Yeqin Liu, Rachel Webb and Weihong Xu, we study the stringy Chow ring of smooth Deligne-Mumford stacks which are global quotients by abelian groups. We are particularly interested in the case of weighted blow-ups, where additionally we give necessary and sufficient conditions under which the stringy Chow ring is finitely generated. A key formula in the derivation uses the main theorem from Chapter 3. The main result is

Theorem (Theorem 4.1)**.** *Let X be a smooth variety and let $\mathcal{I}_\bullet := (I_1, a_1) + \cdots + (i_m, a_m)$ be a Rees algebra such that each $V(I_k) \hookrightarrow X$ is a regular immersion and $\mathcal{I}_\bullet$ defines a quasi-regular weighted closed immersion. Let $Y \subseteq X$ be the closed subvariety defined by $\mathcal{I}_1$. Let $\mathcal{X} = \mathscr{B}\mathrm{l}_{\mathcal{I}_\bullet} X$ be the weighted blow-up of X along $\mathcal{I}_\bullet$ and let $\mathcal{Y}$ be the exceptional divisor. Then the restriction $A^*(X) \to A^*(Y)$ is surjective if and only if the ring $\mathrm{A}^*_{st}(\mathcal{X})$ is generated as an algebra over $\mathrm{A}^*(\mathrm{I}(1))$ by the elements $1e_\zeta$. In this case, $\mathrm{A}^*_{st}(\mathcal{X})$ is a finitely generated algebra over $\mathrm{A}^*(\mathcal{X})$ modulo explicit relations that will be given in the main text.*

0.3 Organisation of thesis

In Chapter 1 we introduce the preliminary notions, including root stacks, parabolic sheaves, inertia stack, Grothendieck-Riemann-Roch theorem for stacks and weighted blow-ups. To avoid cluttering the preliminary and improve the continuity of exposition, we only introduce what we judged to be common background to all projects, and leave background materials that are more specific to each project to be mentioned later only when they are needed.

Each of the remaining chapters focuses on a project.

In Chapter 2, after motivating the usage of Toën-Riemann-Roch theorem, we first compute the inertia stack of a root stack and describe pullback of parabolic sheaves to it. Next we describe the intersection theory on the root stack, leveraging its geometric description as a weighted blow-up. The final crucial ingredient is the description of the relative tangent of the structure map of a root stack, which as in the case with ordinary Riemann-Roch mediates the interaction between proper pushforward and Chern characters. The main theorem is then proven after a lengthy computation. In the final part we derive the parabolic Chern character for divisors with more than one component, using the iterated root construction. This part is also of independent interest as it expounds on the necessity for a Toën style Riemann-Roch theorem and relative inertia stack.

In Chapter 3, we first recall the Lefschetz-Riemann-Roch map valued in K-theory developed by Sala. This is quite a complicated construction and involves several steps. We do not attempt to recall the setup in full generality, instead making simplifications that are adapted to our need. Along the way we point out the similarities with Toën's Riemann-Roch morphism. One difference is that the work of [Sal24] uses the cyclotomic inertia, a variation of the inertia stack which is better behaved on tame but not necessarily Deligne-Mumford stacks. We then embark on proving the main theorem in Section 3.3. It involves understanding the K-theory of the inertia stack of BG, the classifying stack of a finite abstract group, which are given by the sum of representation rings of centralisers of dual cyclic subgroups. The computation then boils down to understand natural maps of representations among these groups. Finally in Section 3.4 we specialise to curves, where the key idea is to reindex the summation over conjugacy classes to a summation over fixed points.

In Chapter 4, we first recall the definition of stringy Chow ring $\mathrm{A}^*_{\mathrm{st}}(\mathcal{X})$ of a Deligne-Mumford stack $\mathcal{X}$, which shares the same additive structure as $\mathrm{A}^*(I\mathcal{X})$ but has a different product. Along the way we introduce the moduli stack of twisted stable curves, the obstruction sheaf on it and the age grading. Among these the obstruction sheaf is the most essential one. It is defined in terms of a perfect obstruction theory on the moduli stack of twisted stable curves. We compute it explicitly for global quotients by reductive abelian groups, using the main theorem of Chapter 3 in the proof of the essential Proposition 4.6. In the second half we specialise to weighted blow-ups. By describing its twisted sectors as weighted projective bundles and combining with results in the first half we completely determine their stringy Chow rings. Finally we give a characterisation of when the stringy Chow ring is finitely generated as an algebra over the (ordinary) Chow ring.

Chapter 1

Preliminaries

1.1 Root stack

Root stacks were initially introduced by Cadman [Cad07] and Abramovich, Graber and Vistoli [AGV08] to study enumerative problems on stacks. There are many ways to understand root stacks, and in this section we recall its definition and basic properties via its functor of points. We will mainly follow [Ols16, Section 10.3]. Another one via weighted blow-up will be useful for understanding its geometry, in particular its intersection theory, and we will treat it later in Section 1.4.

A *generalised effective Cartier divisor* on a scheme X is a pair (L, σ) of an invertible sheaf L and a cosection $\sigma : L \to \mathcal{O}_X$. Clearly an effective Cartier divisor D gives rise to a generalised Cartier divisor by the canonical section $\mathcal{O}_X(-D) \to \mathcal{O}_X$. An isomorphism between (L, σ) and (L', σ') is an isomorphism of line bundles $\lambda : L' \to L$ such that the diagram

$$\begin{array}{ccc} L' & \xrightarrow{\lambda} & L \\ & {\scriptstyle \sigma'}\searrow \quad \swarrow{\scriptstyle \sigma} & \\ & \mathcal{O}_X & \end{array}$$

commutes.

There is an obvious notion of tensor product of generalised effective Carter divisors.

Definition 1.1. Let X be a scheme and (L, σ) be a generalised effective Cartier divisor. Fix a positive integer n. The *nth root stack* associated to the (L, σ) is the fibred category over the category of schemes whose objects are triples $(f : T \to X, (M, \lambda), \rho)$ where (M, λ) is a generalised effective Cartier divisor on T and $\rho : (M^{\otimes n}, \lambda^{\otimes n}) \to (f^*L, f^*\sigma)$ is an isomorphism of generalised effective Cartier divisors on T. A morphism

$$(f' : T' \to X, (M', \lambda'), \sigma') \to (f : T \to X, (M, \lambda), \sigma)$$

is a pair (g, h) where $g : T' \to T$ is a morphism over X and $h : (M', \lambda') \to (g^*M, g^*\lambda)$ is an isomorphism of generalised effective Cartier divisors on T' that makes the diagram

$$\begin{array}{ccc} M'^{\otimes n} & \xrightarrow{\quad h^{\otimes n} \quad} & g^*M^{\otimes n} \\ & {\scriptstyle \lambda'}\searrow \qquad \swarrow{\scriptstyle g^*\lambda^{\otimes n}} & \\ & f'^*L \cong g^*f^*L & \end{array}$$

commute. We denote the fibred category by $\sqrt[n]{(X, L, \sigma)}$. When the generalised effective Cartier divisor comes from a Cartier divisor D we also denote it by $\sqrt[n]{(X, D)}$.

The following basic properties of root stacks are well-known:

Proposition 1.1.

1. *If n is invertible on X then $\sqrt[n]{(X, L, \sigma)}$ is a Deligne-Mumford stack.*
2. *Let $p : \sqrt[n]{(X, L, \sigma)} \to X$ be the morphism sending $(f : T \to X, (M, \lambda), \sigma)$ to f. Then p is an isomorphism over the complement of the zero scheme of σ in X.*
3. *p is the coarse moduli space of $\sqrt[n]{(X, L, \sigma)}$.*

It is shown in [Ols16, Theorem 10.3.10] that if $L = \mathcal{O}_X$ and σ is given by an element $f \in \Gamma(X, \mathcal{O}_X)$ then $\sqrt[n]{(X, L, \sigma)}$ is isomorphic to the quotient stack

$$[\operatorname{Spec}_X \mathcal{O}_X[T]/(T^n - f)/\mu_n]$$

where the action is given by $(\zeta, T) \mapsto \zeta T$. From the local description it follows that

Proposition 1.2. *If X is smooth and D is a smooth Cartier divisor then $\sqrt[n]{(X, D)}$ is smooth.*

The root stack $\sqrt[n]{(X, L, \sigma)}$ is equipped with a universal nth root of the generalised effective Cartier divisor $(p^*L, p^*\sigma)$. When (L, σ) is associated with a Cartier divisor D, this defines a Cartier divisor on the root stack, which we denote by $\frac{1}{n}D$ if there is no risk of confusion with other notations. The corresponding ideal sheaf will be denoted $\mathcal{O}_{\sqrt[n]{X}}(-\frac{1}{n}D)$. We will see in Section 1.1.1 that the divisor $\frac{1}{n}D$, as a stack, can be defined also using a variant of root stack.

The functor of points description gives us another useful characterisation of root stacks. Consider the Artin stack $[\mathbb{A}^1/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication. It classifies $\mathbb{G}_m$-torsors with an equivariant map to $\mathbb{A}^1$. By sending a $\mathbb{G}_m$-torsor to the sheaf of sections of the associated line bundle, it is clear that $[\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to the category whose objects are triples (X, L, σ) of generalised effective Cartier divisors. The morphisms

$$\begin{aligned} \mathbb{A}^1 &\to \mathbb{A}^1, & \qquad \mathbb{G}_m &\to \mathbb{G}_m \\ x &\mapsto x^n, & \qquad x &\mapsto x^n \end{aligned}$$

defines a morphism of stacks $p : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$. For a morphism $X \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponding to (L, σ), we can form the fibre product $\mathcal{X}$.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow p \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

It follows from the description of functors of points that $\mathcal{X}$ is isomorphic to the root stack $\sqrt[n]{(X, L, \sigma)}$. For this reason we call the map $p : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ the *universal nth root stack*.

In fact we use this construction to generalise the root stack construction to any stack and multiple generalised effective Cartier divisors simultaneously:

Definition 1.2. Let X be an algebraic stack. Let $\{(L_i, \sigma_i) : 1 \leq i \leq k\}$ be a collection of generalised effective Cartier divisors on X, corresponding to a morphism $X \to [\mathbb{A}^k/\mathbb{G}_m^k]$. Let $\mathbf{n} = (n_i)$ be a collection of positive integers. The $\mathbf{n}$*th root stack* of X along $\{(L_i, \sigma_i)\}$ is defined to be the fibre product

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & [\mathbb{A}^k/\mathbb{G}_m^k] \\ \downarrow & & \downarrow p \\ X & \longrightarrow & [\mathbb{A}^k/\mathbb{G}_m^k] \end{array}$$

where p is the map induced by raising to the n_ith power on the ith coordinate of both $\mathbb{A}^1$ and $\mathbb{G}_m$. We denote it by $\sqrt[\mathbf{n}]{(X, (L_i, \sigma_i))}$.

Lemma 1.3. *Let $\{(L_i, \sigma_i) : 1 \leq i \leq k\}$ be a collection of generalised effective Cartier divisors on X and $\mathbf{n} = (n_i)$. Then there is a natural isomorphism*

$$\sqrt[\mathbf{n}]{(X, (L_i, \sigma_i))} \cong \sqrt[n_1]{(X, L_1, \sigma_1)} \times_X \cdots \times_X \sqrt[n_k]{(X, L_k, \sigma_k)}.$$

Furthermore for each $1 \leq i \leq k$, there is a morphism of stacks $p_i : \mathcal{X}_i \to \mathcal{X}_{i-1}$ such that

- $\mathcal{X}_0 = X$,
- $\mathcal{X}_k \cong \sqrt[\mathbf{n}]{(X, (L_i, \sigma_i))}$,
- *each $p_i : \mathcal{X}_i \to \mathcal{X}_{i-1}$ is the n_ith root stack along the generalised effective divisor*

$$(L_i, \sigma_i)|_{\mathcal{X}_{i-1}} = (p_{i-1}^* \cdots p_1^*(L_i), p_{i-1}^* \cdots p_1^*(\sigma_i)).$$

Informally, the second statement is saying that $\sqrt[\mathbf{n}]{(X, (L_i, \sigma_i))}$ is isomorphic to the iterated root stack $\sqrt[n_k]{\cdots \sqrt[n_1]{X}}$.

Proof. The first claim follows from the isomorphism $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \prod_{i=1}^k [\mathbb{A}^1/\mathbb{G}_m]$. Note also that the formation of root stack is compatible with base change, so we have a Cartesian squares

$$\begin{array}{ccccc}
\sqrt[n_2]{\sqrt[n_1]{(X, L_1, \sigma_1)}, p_1^* L_2, p_1^* \sigma_2} & \longrightarrow & \sqrt[n_2]{(X, L_2, \sigma_2)} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle (-)^{n_2}} \\
\sqrt[n_1]{(X, L_1, \sigma_1)} & \xrightarrow{p_1} & X & \xrightarrow{f_2} & [\mathbb{A}^1/\mathbb{G}_m]
\end{array}$$

where f_2 corresponds to the generalised effective Cartier divisor (L_2, σ_2). The second claim then follows from induction on k. □

1.1.1 Root stack along line bundle

There is a similar construction which parameterises roots of line bundles without section, which is closely related to the fibre of a root stack $\sqrt[n]{(X, D)}$ above D.

Definition 1.3. Let X be a scheme, L a line bundle on X and n a positive integer. The *nth generic root stack of X along the line bundle* is the fibred category over the category of schemes whose objects are triples $(f : T \to X, M, \rho)$ where M is a line bundle on T and $\rho : M^{\otimes n} \to L$ is an isomorphism. The morphisms are defined in the obvious way. We denote it by $X_{(L,n)}$.

More generally,

Definition 1.4. Let X be an algebraic stack. Let L be a line bundle on X, corresponding to a morphism $X \to B\mathbb{G}_m$. Let n be a positive integer. The *nth generic root stack of* X *along the line bundle* is the stack defined by the fibre product

$$\begin{array}{ccc} X_{(L,n)} & \longrightarrow & B\mathbb{G}_m \\ \downarrow & & \downarrow \\ X & \longrightarrow & B\mathbb{G}_m \end{array}$$

Clearly this recovers the previous definition when X is a scheme.

Locally the line bundle L is trivial so the morphism $X \to B\mathbb{G}_m$ factors through the constant family $\operatorname{Spec} k \to B\mathbb{G}_m$ and hence $X \cong B\mu_r \times X$ since μ_r is the kernel of the homomorphism $(-)^n : \mathbb{G}_m \to \mathbb{G}_m$. Globally however such a trivialisation does not exist in general so we have

Proposition 1.4. *The canonical map* $p : X_{(L,n)} \to X$ *is a* μ_n*-gerbe. In addition it is the coarse moduli space of* $X_{(L,n)}$.

Note that there is a morphism of stacks $X_{(L,n)} \to \sqrt[n]{(X, L, 0)}$ induced by $(T, M, \sigma) \mapsto (T, (M, 0), \sigma)$. Locally this is given by

$$X \times B\mu_n \to [(\operatorname{Spec}_X \mathcal{O}_X[T]/T^n)/\mu_n]$$

so $\sqrt[n]{(X, L, 0)}$ is an infinitesimal thickening of $X_{(L,n)}$. If X is smooth then $X_{(L,n)}$ is the reduction of $\sqrt[n]{(X, L, 0)}$.

Let $\sqrt[n]{X}$ be the nth root stack of (X, D). The pullback of the (generalised) Cartier divisor D along $D \to X$ is $(\mathcal{N}^\vee_{D/X}, 0)$ where $\mathcal{N}^\vee_{D/X}$ is the conormal bundle, so the fibre product $D \times_X \sqrt[n]{X}$ is a non-reduced closed substack of $\sqrt[n]{X}$. By the discussion above its reduction is the generic root stack. It is the Cartier divisor on the root stack $\sqrt[n]{X}$ corresponding to the universal nth root $\mathcal{O}(\frac{1}{n}D)$.

$$\begin{array}{ccccc} D_{(\mathcal{N}^\vee_{D/X}, n)} & \longrightarrow & D \times_X \sqrt[n]{X} & \longrightarrow & \sqrt[n]{X} \\ & \searrow & \downarrow & & \downarrow \\ & & D & \longrightarrow & X \end{array}$$

1.2 Parabolic sheaves and parabolic vector bundles

Parabolic sheaves were initially introduced by Mehta and Seshadri in [MS80] to study moduli problem of vector bundles on algebraic curves. The definition has been abstracted and generalised ([MY92], [IS08], [Bor09]). Here we use a definition that is similar to that used by Borne in [Bor09] (except a difference in the convention of the indexing set).

Let X be a smooth projective variety over an algebraically closed field of characteristic 0. Let D be a smooth irreducible divisor on X. We denote by $\mathrm{Qcoh}(X)$ the abelian category of quasicoherent sheaves. Given a functor $\mathcal{E}_\bullet : \mathbb{Z} \to \mathrm{Qcoh}(X)$ and an integer i, the shift of $\mathcal{E}_\bullet$ is defined on objects by $\mathcal{E}_\bullet[i]_j = \mathcal{E}_{i+j}$. There is a natural transformation $\mathcal{E}_\bullet[i] \to \mathcal{E}_\bullet[j]$ for each $i \leq j$.

Definition 1.5. A *parabolic sheaf* on (X, D) of weight n is a pair $(\mathcal{E}_\bullet, j)$ consisting of a functor $\mathcal{E}_\bullet : \mathbb{Z} \to \mathrm{Qcoh}(X)$ to the category of quasicoherent sheaves on X and a natural isomorphism $j : \mathcal{E}_\bullet \otimes \mathcal{O}_X(D) \to \mathcal{E}_\bullet[n]$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}_\bullet & \longrightarrow & \mathcal{E}_\bullet[n] \\ & \searrow \quad \nearrow_{j} & \\ & \mathcal{E}_\bullet \otimes \mathcal{O}(D) & \end{array}$$

A morphism $(\mathcal{E}, j) \to (\mathcal{E}', j')$ is a natural transformation $\alpha : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{O}(D) & \xrightarrow{j} & \mathcal{E}[n] \\ \downarrow{\scriptstyle \alpha\otimes 1} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{E}' \otimes \mathcal{O}(D) & \xrightarrow{j'} & \mathcal{E}'[n] \end{array}$$

We denote the category of parabolic sheaves on (X, D) of weight n by $\mathrm{Qcoh}^n(X, D)$.

More generally, suppose $D = D_1 + \cdots + D_k$ is a simple normal crossing divisor. Let $\mathbf{n} = (n_i)$ be a collection of positive integers. A *parabolic sheaf* on (X, D) of weight $\mathbf{n}$ is a pair $(\mathcal{E}, j)$ consisting of a functor $\mathcal{E} : \mathbb{Z}^k \to \mathrm{Qcoh}(X)$ and a natural isomorphism $j_i : \mathcal{E} \otimes \mathcal{O}_X(D_i) \to \mathcal{E}[n_i \mathbf{e}_i]$ for each $1 \leq i \leq k$, where $\mathbf{e}_i$ is the tuple with 1 at ith place and 0 elsewhere, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}[n_i \mathbf{e}_i] \\ & \searrow \quad \nearrow_{j_i} & \\ & \mathcal{E} \otimes \mathcal{O}(D_i) & \end{array}$$

Let $p : \sqrt[\mathbf{n}]{X} \to X$ be the canonical map. Given a vector bundle E on $\sqrt[\mathbf{n}]{X}$, we obtain a functor $\mathcal{E} : \mathbb{Z}^k \to \mathrm{Qcoh}(X)$ by the assignment

$$\mathcal{E}_{m_1,\ldots,m_k} = p_*\left(E \otimes \mathcal{O}_{\sqrt[\mathbf{n}]{X}}\left(\frac{m_1}{n_1}D_1 + \cdots + \frac{m_k}{n_k}D_k\right)\right).$$

Also note that for any $1 \leq i \leq k$, we have

$$\begin{aligned}
\mathcal{E}[n_i\mathbf{e}_i]_{m_1,\ldots,m_k} &= \mathcal{E}_{m_1,\ldots,m_i+n_i,\ldots,m_k} \\
&= p_*\left(E \otimes \mathcal{O}_{\sqrt[\mathbf{n}]{X}}\left(\frac{m_1}{n_1}D_1 + \cdots + (\frac{m_i}{n_i}+1)D_i \cdots + \frac{m_k}{n_k}D_k\right)\right) \\
&= p_*\left(E \otimes \mathcal{O}_{\sqrt[\mathbf{n}]{X}}\left(\sum \frac{m_\ell}{n_\ell}D_\ell\right) \otimes \mathcal{O}_{\sqrt[\mathbf{n}]{X}}(D_i)\right) \\
&= p_*\left(E \otimes \mathcal{O}_{\sqrt[\mathbf{n}]{X}}\left(\sum \frac{m_\ell}{n_\ell}D_\ell\right) \otimes p^*\mathcal{O}_X(D_i)\right) \\
&= p_*\left(E \otimes \mathcal{O}_{\sqrt[\mathbf{n}]{X}}\left(\sum \frac{m_\ell}{n_\ell}D_\ell\right) \otimes\right) \otimes \mathcal{O}_X(D_i) \\
&= \mathcal{E}_{m_1,\ldots,m_k} \otimes \mathcal{O}_X(D_i)
\end{aligned}$$

by projection formula, so $\mathcal{E}$ defines a parabolic sheaf. This defines a functor Φ from the category of quasicoherent sheaves on the root stack and the category of parabolic sheaves on X. The main result of [BV12] is

Theorem 1.5 ([BV12, Theorem 6.1])**.** *Let X be a smooth scheme and $D = D_1 + \cdots + D_k$ be a simple normal crossing divisor. Fix weights $\mathbf{n}$. Then Φ is an equivalence of tensor abelian categories between $\mathrm{Qcoh}(\sqrt[\mathbf{n}]{(X,D)})$ and $\mathrm{Qcoh}^{\mathbf{n}}(X,D)$.*

In Chapter 2, we will be interested in vector bundles on root stacks. In this case there is a refinement of the result:

Proposition 1.6 ([Bor09, Théorème 2])**.** *Φ induces an equivalence of tensor categories between vector bundles on $\sqrt[\mathbf{n}]{(X,D)}$ and parabolic bundles of weight $\mathbf{n}$ on (X,D).*

1.3 Grothendieck-Riemann-Roch for stacks and inertia stack

For a quasi-projective variety X, we write $K(X) := K^0(X)$ for the K-theory of vector bundles on X. As X is non-singular the map $K^0(X) \to K_0(X)$ sending a vector bundle to its class in the K-theory of coherent sheaves is an isomorphism so we may also identify $K(X)$ with $K_0(X)$. The Chern character of X is a ring homomorphism $\mathrm{ch} : K(X) \to \mathrm{A}^*(X)_{\mathbb{Q}}$ from the K-theory to the rational Chow ring of X. It is natural with respect to pullbacks. Its failure to commute with pushforward is quantified precisely by the Grothendieck-Riemann-Roch theorem: suppose $f : X \to Y$ is a proper morphism between non-singular varieties. Define the pushforward in K-theory to be

$$\begin{aligned} Rf_* : K(X) &\to K(Y) \\ [E] &\mapsto \sum (-1)^i [Rf_*^i E]. \end{aligned}$$

Then there is commutative diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}(-)\,\mathrm{Td}\,X} & \mathrm{A}^*(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle f_*} \\ K(Y) & \xrightarrow{\mathrm{ch}(-)\,\mathrm{Td}\,Y} & \mathrm{A}^*(Y)_{\mathbb{Q}} \end{array}$$

Given a smooth Deligne-Mumford stack $\mathcal{X}$, an intersection theory can be constructed ([Vis89]). Moreover it is shown in the same paper that the coarse moduli space map $p : \mathcal{X} \to |\mathcal{X}|$ induces an isomorphism of the rational Chow groups. It is natural to wonder if such a theorem holds for morphisms of stacks.

Unfortunately the word-by-word transcription of Grothendieck-Riemann-Roch theorem cannot work for stacks, due to the existence of coarse moduli spaces. Consider the following example, adapted from [Toë99]. Let $B_{\mathbb{C}}H$ be the classifying stack of $H = \mu_n$, the dual of the cyclic group of order n. Its K-theory can be identified with its representation ring. Its moduli space $p : B_{\mathbb{C}}H \to \operatorname{Spec} k$ is a point, and pushforward along which induces a map $V \mapsto \dim V^H$ taking a representation to the dimension of its invariant part on K-theories. Suppose we had a Grothendieck-Riemann-Roch theorem for p, so there were a commutative diagram

$$\begin{array}{ccc} K(BH)_{\mathbb{C}} & \xrightarrow{\mathrm{ch}} & \mathrm{A}^*(BH)_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

Note that since the tangent bundle of both the source and the target is trivial, the Todd class vanishes and in particular the top horizontal arrow is multiplicative. Let V be a non-trivial character. Then

$$1 = \mathrm{ch}\, 1 = \mathrm{ch}\, V^{\otimes n} = (\mathrm{ch}\, V)^{\otimes n} = 0,$$

absurd.

The key insight of [Toë99] is that if one systematically records the automorphisms then it is possible to establish a version of Grothendieck-Riemann-Roch valued in *inertia stacks*.

1.3.1 Inertia stack

Definition 1.6. Let $\mathcal{X}$ be a Deligne-Mumford stack. Its *inertia stack* is defined to be $I\mathcal{X} = \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$.

As a fibred category, its objects are pairs (x, α) where $x \in \mathcal{X}(T)$ is an object of $\mathcal{X}$ above T and $\alpha \in \mathrm{Aut}_T(x)$ is an automorphism of x. A morphism $f : (x, \alpha) \to (y, \beta)$ from $(x, \alpha) \in I\mathcal{X}(T)$ to $(y, \beta) \in I\mathcal{X}(S)$ consists of an arrow $f : x \to y$ such that the induced morphism $\mathrm{Aut}_T(x) \to \phi^* \mathrm{Aut}_S(y)$ maps α to $\phi^*(\beta)$, where $\phi : T \to S$ is the image of f in the category of schemes.

We can make the description more explicit if $\mathcal{X}$ is a global quotient stack. Suppose $\mathcal{X} = [X/G]$ is the quotient of a scheme X by an abelian group G. Then an easy computation (for example [Sta18, 0373]) shows that there is a decomposition

$$I\mathcal{X} = \coprod_{g\in G} [X^g/G]$$

where X^g is the fixed subscheme of g. More generally if G is not necessarily abelian then

$$I\mathcal{X} = \coprod_{g\in\mathcal{C}(G)} [X^g/C(g)]$$

where $\mathcal{C}(G)$ is the set of conjugacy classes of G and $C(g)$ is the centraliser of g.

If X is smooth then X^g is a smooth closed subscheme. Since $[X^g/G] \to \mathcal{X}$ is a regular embedding, it has a normal bundle $\mathcal{N}_{[X^g/G]/\mathcal{X}}$. They assemble into a vector bundle (of possible different rank) on $I\mathcal{X}$, which we denote by $\mathcal{N}^\vee_{I\mathcal{X}/\mathcal{X}}$.

1.3.2 Toën-Riemann-Roch theorem

The main theorem of [Toë99] is

Theorem 1.7 ([Toë99, Théorème 4.11])**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack which has a quasi-projective coarse moduli space and satisfies the resolution property. Then there is a Riemann-Roch transformation $\tau_{\mathcal{X}} : K^0(\mathcal{X}) \to \mathrm{A}^*(\mathcal{X})_{\mathbb{C}}$ that is covariant with* *respect to proper morphisms. In other words, let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of Deligne-Mumford stacks. Then there is a commutative diagram*

$$\begin{array}{ccc} K^0(\mathcal{X})_{\mathbb{C}} & \xrightarrow{\tau_{\mathcal{X}}} & \mathrm{A}^*(I\mathcal{X})_{\mathbb{C}} \\ \downarrow f_* & & \downarrow f_* \\ K^0(\mathcal{Y})_{\mathbb{C}} & \xrightarrow{\tau_{\mathcal{Y}}} & \mathrm{A}^*(I\mathcal{Y})_{\mathbb{C}} \end{array}$$

Remark 1.1. The resolution property says that every coherent sheaf is the quotient of some locally free sheaf. It is shown in [Tot04] that this implies that the stack $\mathcal{X}$ is a quotient stack. In addition for X a smooth separated scheme and D a smooth Cartier divisor, the root stack $\sqrt[n]{(X,D)}$ has the resolution property by [Tot04, Theorem 1.2]. Thus we may write $K(\sqrt[n]{(X,D)}) = K^0(\sqrt[n]{(X,D)}) = K_0(\sqrt[n]{(X,D)})$ unambiguously.

When $\mathcal{X}$ is smooth, the Riemann-Roch morphism is defined to be the composition

$$K^0(\mathcal{X})_{\mathbb{C}} \xrightarrow{\pi^*} K^0(I\mathcal{X})_{\mathbb{C}} \xrightarrow{\cdot\lambda_{-1}(\mathcal{N}^\vee_{I\mathcal{X}/\mathcal{X}})^{-1}} K^0(I\mathcal{X})_{\mathbb{C}} \xrightarrow{\rho} K^0(I\mathcal{X})_{\mathbb{C}} \xrightarrow{\mathrm{ch}(-)\cdot\mathrm{Td}(I\mathcal{X})} \mathrm{A}^*(I\mathcal{X})_{\mathbb{C}}.$$

We now explain the notations.

Let $\pi : I\mathcal{X} \to \mathcal{X}$ be the inertia. The first map is the pullback in K-theory induced by π.

The λ_{-1} appearing in the second map is defined in terms of λ-operations. For simplicity we recall here the relevant definitions that suffice for our applications, and interested readers are encouraged to consult the textbook [FL13] for the most general definitions. Recall that the K-theory of a scheme or a stack is a λ-*ring*, meaning that it is equipped with operations λ^i for each integer $i \geq 0$ that satisfy certain axioms. For a vector bundle E they are defined by

$$\lambda^i(E) = [\wedge^i E].$$

These operations can be assembled into a power series

$$\lambda_t(x) = \sum_i \lambda^i(x) t^i.$$

Thus for a vector bundle E,

$$\lambda_{-1}(E) = \sum_i (-1)^i [\wedge^i E],$$

which is a finite sum since E has finite sum. By assumption the conormal bundle $\mathcal{N}^\vee_{I\mathcal{X}/\mathcal{X}}$ is a vector bundle on the inertia stack so we have completely described the second map.

The third map ρ is called the *twist map*. Informally it twists each eigenbundle under the action of the automorphism group by its eigenvalue. Recall that an object of $I\mathcal{X}$ is a pair (x, α) where $x \in \mathcal{X}(T)$ and α is an automorphism of x. Given a vector bundle V on $\mathcal{X}$, x^*V is a vector bundle on T equipped with a $\langle\alpha\rangle$-action. Since the ground field is the complex numbers, the action is diagonalisable and hence we may decompose

$$x^*V = \bigoplus_\zeta (x^*V)_\zeta$$

according to characters ζ of $\langle\alpha\rangle$. It can be shown that the subbundle $(x^*V)_\zeta$ is compatible with morphisms in $I\mathcal{X}$ and thus defines a vector bundle on $I\mathcal{X}$. Thus $\bigoplus_\zeta \zeta(x^*V)_\zeta$ is a well-defined element of $K^0(I\mathcal{X})_\mathbb{C}$, which we call $\rho(V)$. The above procedure defines the linear map $\rho : K^0(I\mathcal{X})_\mathbb{C} \to K^0(I\mathcal{X})_\mathbb{C}$.

The last map in the composition is the same as in Grothendieck-Riemann-Roch for varieties. It takes the Chern character of a K-theory class and multiplies it by the Todd class of the tangent bundle $T_{I\mathcal{X}}$, which we abbreviate to $\mathrm{Td}(I\mathcal{X})$.

1.4 Weighted blow-ups

Blow-up is an operation in classical algebraic geometry that principalises an ideal sheaf. Suppose I is an ideal sheaf on a scheme X, defining a closed subscheme Y. The blow-up $\mathrm{Bl}_I X$ is defined to be $\mathrm{Proj}_X \mathcal{O}_X[I]$, where the Rees algebra $\mathcal{O}_X[I]$ is a graded algebra. The inverse image of Y is the exceptional divisor, which is the projectivised conormal cone of Y. Weighted blow-up generalises classical blow-up by allowing an ideal sheaf $I_\bullet$ to have "weights", thus producing a Rees algebra which is not necessarily generated by degree 1 elements. Subsequently the Proj construction is replaced by the stacky version $\mathscr{P}\mathrm{roj}$, producing a stack $\mathscr{B}\mathrm{l}_{I_\bullet} X$. In this subsection we introduce weighted blow-ups following [QR] and interpret root stacks using this perspective.

Definition 1.7. A *Rees algebra* on X is a quasicoherent, finitely generated, graded $\mathcal{O}_X$-subalgebra $R = \bigoplus_{n\geq 0} I_n t^n$ of $\mathcal{O}_X[t]$ such that $I_0 = \mathcal{O}_X$ and $I_n \supseteq I_{n+1}$ for all n.

Example 1.1.

1. An ordinary Rees algebra $\mathcal{O}_X[I]$ where I is an ideal sheaf is a Rees algebra generated in degree 1. Conversely if $I_\bullet$ is a Rees algebra generated by I_1 then $I_\bullet = \mathcal{O}_X[I_1]$. We also denote this Rees algebra by $(I_1, 1)$.

2. Given an ideal I and $n \geq 1$, let (I, d) denote the smallest Rees algebra containing It^n. Explicitly $(I, n)_k = I^{\lceil k/n \rceil}$, i.e.

$$(I, n)_k = \mathcal{O}_X \oplus It \oplus It^2 \oplus \cdots \oplus It^{n-1} \oplus I^2t^{d+1} \oplus I^2t^{d+2} \oplus \cdots .$$

3. Given a finite collection of Rees algebras $I_{i,\bullet}$, we let $\sum_i I_{i,\bullet}$ denote the smallest Rees algebra containing all the $I_{k,\bullet}$. Explicitly

$$(I_{1,\bullet} + \cdots + I_{r,\bullet})_k = \sum_{k=k_1+\cdots+k_r} I_{1,k_1} \cdots I_{r,k_r}.$$

Definition 1.8. Suppose $R = \bigoplus_{n\geq 0} R_n$ be a quasicoherent graded $\mathcal{O}_X$-algebra. Let $R_+ = \bigoplus_{n>0} R_n$ be the irrelevant ideal. The *stack-theoretic Proj* of R is the quotient stack

$$\mathscr{P}\mathrm{roj}_X(R) = \left[\frac{\mathrm{Spec}_X(R) \setminus V(R_+)}{\mathbb{G}_m}\right]$$

where the $\mathbb{G}_m$-action is induced by the grading on R.

Example 1.2.

1. Given vector bundles $E_1, \ldots, E_r$ on X and positive integers $n_1, \ldots, n_r$, the *weighted vector bundle* $E = \bigoplus E_i(-n_i)$ gives the smooth stack

$$\mathscr{P}(E) = \mathscr{P}\mathrm{roj}_X(\bigotimes_{i=1}^{r} \mathrm{Sym}_{\mathcal{O}_X}(E_i(-n_i))).$$

2. An important special case is when L is a line bundle on X. Then $\mathscr{P}(L(-n))$ parameterises nth roots of the line bundle E. To see this, we note the stacky Proj construction satisfies a similar universal property as ordinary Proj: given a morphism $f : T \to X$ and a weighted vector bundle $E = \bigoplus E_i(-n_i)$ on X, a lift to $\mathscr{P}(E)$ corresponds to the data of a line bundle M on T and morphisms $\varphi_i : f^*E_i \to M^{\otimes d_i}$ such that locally on T at least one of the φ_i is surjective (see [QR] Proposition 1.5.1 and Example 2.1.1). Thus for L a line bundle, $\mathscr{P}(L(-n))$ parameterises a line bundle M together with an isomorphism $f^*L \to M^{\otimes n}$. Thus there is an isomorphism $\mathscr{P}(L(-n)) \cong X_{(L,n)}$, the nth generic root stack of X along L.

Definition 1.9. Let $I_\bullet$ be a Rees algebra on X. The *weighted blow-up* of X along $I_\bullet$ is defined as the morphism

$$\pi : \mathscr{B}\mathrm{l}_{I_\bullet} X = \mathscr{P}\mathrm{roj}_X I_\bullet \to X.$$

Let $X' = \mathscr{B}\mathrm{l}_{I_\bullet} X$. The natural inclusion $I_{\bullet+1} \hookrightarrow I_\bullet$ corresponds to the inclusion $\mathcal{O}_{X'}(1) \hookrightarrow \mathcal{O}_{X'}$ of invertible sheaves, and defines an effective Cartier divisor E on X' such that $\mathcal{O}_{X'}(1) = \mathcal{O}_{X'}(-E)$. E is called the *exceptional divisor* of $\mathscr{B}\mathrm{l}_{I_\bullet} X$.

For the rest of this section we consider weighted blow-up along a *regular centre*: let $I_1, \cdots I_k$ be ideal sheaves on X and let $a_1, \ldots, a_k$ be positive integers. Let

$$\mathcal{I}_\bullet = (I_1, a_1) + \cdots + (I_k, a_k).$$

We assume

- each $V(I_k) \hookrightarrow X$ is a regular embedding,
- $\mathcal{I}_\bullet$ defines a quasi-regular weighted closed immersion ([QR, Definition 5.1.3]).

Let $Y = V(\sum I_i)$ be the blow-up centre. Denote the blow-up and the exceptional divisor by $\widetilde{X}$ and $\widetilde{Y}$ respectively. Then there is a commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{j} & \widetilde{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array}$$

which is *not* Cartesian.

Example 1.3. The most important example for us is root stack. Given a generalised effective Cartier divisor (L, σ) on X and a positive integer n, we can define a graded $\mathcal{O}_X$-algebra

$$R = \bigoplus_{i \geq 0} L^{\lceil i/n \rceil}$$

where the multiplication is defined by $\sigma : L \to \mathcal{O}_X$. It is shown in [QR, Example 2.2.2] that it has exactly the same universal property as the root stack, namely if $f : T \to X$ is a morphism then a lift to $\mathscr{P}\mathrm{roj}_X R$ is equivalent to giving a generalised effective Cartier divisor (M, λ) with an isomorphism $r : f^*(L, \sigma) \to (M, \lambda)^{\otimes n}$. Thus $\mathscr{P}\mathrm{roj}_X R \cong \sqrt[n]{(X, L, \sigma)}$. In particular when (L, σ) is derived from a Cartier divisor D, we see $R = (I, n)$ using the notation of Example 1.1 so $\sqrt[n]{(X, D)} \cong \mathscr{P}\mathrm{roj}_X(I, n) = \mathscr{B}\mathrm{l}_{(I,n)} X$.

Chapter 2

Parabolic Chern Character

Let X be a smooth scheme over $\mathbb{C}$ and D a smooth Cartier divisor. The root stack $\sqrt[n]{(X,D)}$ is a smooth Deligne-Mumford stack which admits X as a coarse moduli space. We have seen that vector bundles on $\sqrt[n]{X,D}$ are equivalent to parabolic bundles on (X,D), which are diagrams of vector bundles on X subject to some compatibility conditions. Since it is possible to identify the rational Chow groups of $\sqrt[n]{X,D}$ and X, given a vector bundle on $\sqrt[n]{X,D}$, its Chern character can be expressed as an element of the rational Chow group $\mathrm{A}(X)_{\mathbb{Q}}$. It is then natural to ask how it is related to Chern characters of constituent bundles of the corresponding parabolic bundles. In this paper we gave a complete, self-contained solution to this question using Toën's Grothendieck-Riemann-Roch theorem for stacks:

Theorem 2.1. *Let X be a smooth projective variety over an algebraically closed field of characteristic 0 and let D be a smooth effective Cartier divisor on X. Let $E_\bullet$ be a parabolic bundle on (X,D) of weight n. Then*

$$\mathrm{ch}^{\mathrm{par}} E_\bullet = \frac{1-e^{-D/n}}{1-e^{-D}} \sum_{i=0}^{n-1} \mathrm{ch}\, E_i \cdot e^{-iD/n}$$

in $\mathrm{A}^*(X)_{\mathbb{Q}}$.

This agrees with previous computations by Iyer and Simpson in [IS08], which uses a more elementary approach.

2.1 Introduction

In this chapter we will not distinguish a vector bundle E on the root stack and the parabolic bundle $\mathcal{E}_\bullet = \Phi(E)$ induced by the functor in Section 1.2. We also write

$D_n := \frac{1}{n}D$ for the universal Cartier divisor.

Definition 2.1. Let X be a smooth projective variety and D a smooth Cartier divisor. Given a parabolic vector bundle $E_\bullet$ of weight n on (X, D), the *parabolic Chern character* of $E_\bullet$ is the element

$$p_* \operatorname{ch}(E) \in \mathrm{A}^*(X)_{\mathbb{Q}}$$

in the rational Chow group of X where $p : \sqrt[n]{(X, D)} \to X$ is the canonical map. We denote it by $\operatorname{ch}^{\mathrm{par}} E_\bullet$.

As the parabolic vector bundle $E_\bullet$ is presented using data of vector bundles on X, whose Chern characters is an element of the same group $\mathrm{A}^*(X)_{\mathbb{Q}}$, it is natural to ask what is the relationship between the two. As the comparison involves the $p_* \operatorname{ch} E_\bullet$, the pushforward of the Chern class along a proper morphism, heuristics suggests that it should be related to $\operatorname{ch}(p_* E_\bullet)$ by a Grothendieck-Riemann-Roch-like theorem.

The Toën-Riemann-Roch theorem says that there is a "correction term" that should be accounted for.

2.2 Geometry of root stack

2.2.1 Inertia of root stack

We begin by describing the inertia stack of a root stack.

Proposition 2.2. *The inertia stack of* $\sqrt[n]{(X, D)}$ *is*

$$I \sqrt[n]{(X, D)} = \sqrt[n]{(X, D)} \amalg \coprod_{\zeta \in \mu_n \setminus \{1\}} D_n.$$

Proof. Let $\mathcal{X} = \sqrt[n]{(X, D)}$. Given an object $(f : T \to X, (M, \lambda), \rho)$ of $\mathcal{X}$, the canonical isomorphism $h_M : \mathcal{H}om(M, M) \cong \mathcal{O}_X$ induces a homomorphism

$$\begin{aligned} \operatorname{Aut}_T(f, M, \lambda, \rho) &\to \boldsymbol{\mu}_{n,T} \\ \alpha &\mapsto (f^* h_M)(\alpha). \end{aligned}$$

As $\boldsymbol{\mu}_r$ is constant over $\mathbb{C}$, write $\mu_r = \boldsymbol{\mu}_r(\mathbb{C})$ and the above map induces a decomposition $I\mathcal{X} := \coprod_{\zeta \in \mu_n} I(\zeta)\mathcal{X}$. Similarly $ID_n := \coprod_{\zeta \in \mu_n} I(\zeta) D_n$ and the morphism of inertia stacks $ID_n \to I\mathcal{X}$ induced by the closed immersion $D_n \to \mathcal{X}$ respects the decomposition. Clearly $I(1)\mathcal{X}$ is isomorphic to the image of the diagonal section of

$I\mathcal{X} \to \mathcal{X}$ so it remains to prove that for $\zeta \neq 1$, $I(\zeta)D_n \to I(\zeta)\mathcal{X}$ is an isomorphism. Recall from the description of the functor of points of the root stack and the generic root stack that the morphism is given over T by

$$(f, M, \rho, \alpha) \mapsto (i \circ f, (M, 0), \rho, \alpha)$$

where $i : D \to X$ is the inclusion of the Cartier divisor. Clearly it is fully faithful. To show it is essentially surjective take an object $(f : T \to X, (M, \lambda), \rho, \alpha)$ of $I(\zeta)\mathcal{X}$ over T. Since $\zeta \neq 1$, $1_M - \alpha : M \to M$ is invertible. On the other hand, both 1_M and α preserve λ so

$$(1_M - \alpha)(\lambda) = 0$$

which forces $\lambda = 0$. It then follows that $f^*(\sigma_D) = \lambda^{\otimes n} = 0$, where $\sigma_D : I_D \to \mathcal{O}_X$ is the inclusion of the ideal sheaf. Thus f factors through D as required. □

Remark 2.1. See also Section 4.2.2 for a general treatment of the decomposition of the inertia stack of global quotients by reductive abelian groups.

Proposition 2.3. *Let $E_\bullet$ be a vector bundle on $\sqrt[n]{(X, D)}$. Then*

$$E_\bullet|_{D_n} = \bigoplus_{i=0}^{n-1} p^* \operatorname{Gr}_{-i} E_\bullet \otimes N^i_{D_n}.$$

More generally, such a decomposition exists for any indexing set which forms a coset representative of $\mathbb{Z}/n\mathbb{Z}$.

Proof. This is [IS08] Lemma 4.1 and Lemma 4.4. □

2.2.2 Chow group of root stack

Proposition 2.4. *The rational Chow group of D_n can be identified with that of D in such a way that p^* induces a ring isomorphism. Moreover under this isomorphism the pushforward $p_* : \mathrm{A}^*(D_n)_\mathbb{Q} \to \mathrm{A}^*(D)_\mathbb{Q}$ is multiplication by $\frac{1}{n}$.*

Proof. We will use results from [AOA23] to describe the Chow ring of a weighted projective bundle. By Example 1.2 $D_n = \mathscr{P}\mathrm{roj}(E)$ where E is the weighted line bundle $N^\vee_{D/X}(-n)$. By viewing X as equipped with a trivial $\mathbb{G}_m$-action, E has an equivariant Euler class

$$e^{\mathbb{G}_m}(E) = nt + e(N^\vee_D) \in \mathrm{A}^*_{\mathbb{G}_m}(X) = \mathrm{A}^*(X)[t]$$

so rationally

$$\mathrm{A}^*(D_n)_{\mathbb{Q}} = \mathrm{A}^*(D)_{\mathbb{Q}}[t]/(nt + e(N_D^\vee)) = \mathrm{A}^*(D)_{\mathbb{Q}}.$$

To prove the second statement note that since $D_n \to D$ is a μ_n-gerbe, the stabiliser of a general geometric point is μ_n so p has degree $\frac{1}{n}$. Given a cycle $[V]$ on D, denote by $V_n = V \times_D D_n$ its pullback to D_n. Since the image of V_n under p is also V, we have

$$p_* p^*[V] = p_*[V'] = \deg(p)[V] = \frac{1}{n}[V].$$

Therefore the identity

$$p_* p^*(\alpha) = \frac{\alpha}{n}$$

holds for any cycle class $\alpha \in A^*(D)$. □

2.2.3 Relative tangent and Todd class

Lemma 2.5. *Given a root stack $p : \sqrt[n]{X} \to X$, the cotangent sequence*

$$0 \to p^*\Omega_X \to \Omega_{\sqrt[n]{X}} \to \Omega_{\sqrt[n]{X}/X} \to 0 \tag{2.1}$$

is exact. Taking its dual, there is another short exact sequence

$$0 \to T_{\sqrt[n]{X}} \to p^*T_X \to \mathcal{E}xt^1(p^*\Omega_X, \mathcal{O}_{\sqrt[n]{X}}) \to 0 \tag{2.2}$$

where $T_{\sqrt[n]{X}} = \Omega^\vee_{\sqrt[n]{X}}$ and $T_X = \Omega^\vee_X$.

Proof. Firstly note that Ω_X and $\Omega_{\sqrt[n]{X}}$ are both locally free due to smoothness. The coarse moduli space $\sqrt[n]{X} \to X$ is a generic isomorphism so the kernel and cokernel of $p^*\Omega_X \to \Omega_{\sqrt[n]{X}}$ are torsion. Thus the kernel, being a torsion subsheaf of a locally free sheaf, must vanish and sequence (2.1) is exact on the left. Similarly $\mathcal{H}om(\Omega_{\sqrt[n]{X}/X}, \mathcal{O}_{\sqrt[n]{X}}) = 0$ so sequence (2.2) is exact on the left. Exactness on the right follows from the local freeness of $\Omega_{\sqrt[n]{X}}$. □

Lemma 2.6. *The classifying map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ induced by a smooth Cartier divisor is flat.*

Proof. Consider the pullback of the universal family

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where P is the $\mathbb{G}_m$-bundle corresponding to the line bundle (in other words the line bundle minus the zero section). Since both P and $\mathbb{A}^1$ are regular we can apply miracle flatness and it suffices to show the fibres of f have constant dimension, which is clear. □

Proposition 2.7. *The relative tangent of a root stack* $\sqrt[n]{X} \to X$ *is given by the class* $[\mathcal{O}(D_n)] - \mathcal{O}(nD_n) \in K(\sqrt[n]{X})$.

Proof. Recall that the relative tangent of a morphism $f : X \to Y$ is the K-theory class $[T_X] - [T_Y|_X]$. By Lemma 2.5 the relative tangent is represented by $\mathcal{E}xt^1(p^*\Omega_X, \mathcal{O}_{\sqrt[n]{X}})$. Consider the universal root stack

$$\begin{array}{ccc} \sqrt[n]{X} & \xrightarrow{g} & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

We first prove the result for the universal root stack $[\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$. For clarity we write the $\mathbb{A}^1$ appearing in the domain and codomain as $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k[y]$ respectively. The morphism between the quotient stacks is then induced by the nth power map $(-)^n : X \to Y$ (and the nth power map $\mathbb{G}_m \to \mathbb{G}_m$). Take the smooth cover $Y \to [Y/\mathbb{G}_m]$ and form the fibre product which is $[X/\mu_n]$. Note that X is the cover of both $[X/\mu_n]$ and $[X/\mathbb{G}_m]$ and a sheaf on $[X/\mathbb{G}_m]$ is the same as a $\mathbb{G}_m$-equivariant sheaf on X, which is the same as a $\mathbb{Z}$-graded $k[x]$-module.

$$\begin{array}{ccccc} X & \longrightarrow & [X/\mu_n] & \longrightarrow & Y \\ & & \downarrow & & \downarrow \\ & & [X/\mathbb{G}_m] & \longrightarrow & [Y/\mathbb{G}_m] \end{array}$$

The relative cotangent sequence reads

$$\Omega_Y|_X \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \to 0$$

which we spell out as

$$k[x]dy \xrightarrow{nx^{n-1}} k[x]dx \longrightarrow k[x]/nx^{n-1}dx \longrightarrow 0$$

which happens to be exact on the left as well. Here the generator dx has degree -1 and dy has degree $-n$ so we may write it explicitly as

$$k[x][n] \longrightarrow k[x][1] \longrightarrow k[x]/nx^{n-1}[1] \longrightarrow 0$$

Note that as $X \to Y$ is generically étale, $\Omega_{X/Y}$ is torsion. Dualising the above sequence, we get

$$0 \longrightarrow k[x][-1] \longrightarrow k[x][-n] \longrightarrow k[x]/nx^{n-1}[-n] \longrightarrow 0$$

The $\mathbb{Z}$-graded module $k[x][-1]$ corresponds to the universal root $\mathcal{O}(D_n)$, so it follows that the relative tangent is $\mathcal{O}(D_n) - \mathcal{O}(nD_n)$.

For a general root stack $\sqrt[n]{X}$, as the classifying map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ is flat by Lemma 2.6, the claim follows by pulling back $\mathcal{E}xt^1(p^*\Omega_{[\mathbb{A}^1/\mathbb{G}_m]}, \mathcal{O}_{[\mathbb{A}^1/\mathbb{G}_m]})$, which represents the relative tangent in K-theory by Lemma 2.5. □

2.3 Proof of main theorem

In this section we will prove:

Theorem 2.8. *Let $E_\bullet$ be a parabolic vector bundle on (X, D). Then*

$$\mathrm{ch}^{\mathrm{par}} E_\bullet = \frac{1 - e^{-D/n}}{1 - e^{-D}} \sum_{i=0}^{n-1} \mathrm{ch}\, E_i \cdot e^{-iD/n}$$

in $\mathrm{A}^*(X)_{\mathbb{Q}}$.

It suffices to prove the equality in $\mathrm{A}^*(X)_{\mathbb{C}}$. Apply Toën-Riemann-Roch to the coarse moduli space $p : \sqrt[n]{(X, D)} \to X$, we get the following commutative diagram

$$\begin{array}{ccccccc}
K(\sqrt[n]{X}) & \xrightarrow{\frac{\pi^*(-)}{\lambda_{-1}(\mathcal{N}^\vee)}} & K(I\sqrt[n]{X}) & \xrightarrow{\rho} & K(I\sqrt[n]{X}) & \xrightarrow{c} & \mathrm{A}^*(I\sqrt[n]{X}) = \mathrm{A}^*(\sqrt[n]{X}) \oplus \mathrm{A}^*(D_n)^{\oplus \mu_n - 1} \\
\Big\downarrow p_* & & & & & & \Big\downarrow p_* \\
 & & & & & & \mathrm{A}^*(X) \oplus \mathrm{A}^*(D)^{\oplus \mu_n - 1} \\
 & & & & & & \Big\downarrow (\mathrm{id}, i_*) \\
K(X) & & & \xrightarrow{\mathrm{ch}^X(-)\,\mathrm{Td}\,X} & & & \mathrm{A}^*(X)
\end{array}$$

where the superscript on each Chern character indicates the space on which it operates. In the diagram c is the map

$$\mathrm{ch}^{I\sqrt[n]{X}}(-)\,\mathrm{Td}(I\sqrt[n]{X}).$$

It is helpful to recall that we have a (non-Cartesian) commutative square

$$\begin{array}{ccc} D_n & \longrightarrow & \sqrt[n]{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ D & \xrightarrow{i} & X \end{array}$$

Consider a vector bundle $E_\bullet$ on $\sqrt[n]{X}$. We are going to compute its image in the bottom right corner using the two paths in the commutative diagrams and equate them.

The left-and-bottom composition is straightforward: p_* takes the 0th component in the parabolic diagram so $E_\bullet$ is sent to

$$\operatorname{ch}^X E_0 \cdot \operatorname{Td} X.$$

For the upper-and-right composition, it is convenient to split the contributions from the identity and non-identity sectors. On the identity sector the inclusion $I\sqrt[n]{X}_1 \to \sqrt[n]{X}$ is identity so π restricts to identity and the conormal bundle vanishes. In addition the twisting is trivial so we get an element

$$\operatorname{ch}^{\sqrt[n]{X}}(E_\bullet) \cdot \operatorname{Td} \sqrt[n]{X} \in \mathrm{A}^*(\sqrt[n]{X}).$$

On the non-identity sector labelled by $\xi \in \mu_n - 1$, the upper horizontal composition gives

$$\begin{aligned} \operatorname{ch}^{D_n} \rho(\frac{E_\bullet|_{D_n}}{1 - N_{D_n}^\vee}) \cdot \operatorname{Td} D_n &= \operatorname{ch}^{D_n} \rho\left(\sum_{i=1}^{n} \frac{p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1 - N_{D_n}^\vee}\right) \cdot \operatorname{Td} D_n \\ &= \sum_{i=1}^{n} \operatorname{ch}^{D_n} \frac{\zeta^{-i} p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1 - \zeta^{-1} N_{D_n}^\vee} \cdot \operatorname{Td} D_n \in \mathrm{A}^*(D_n), \end{aligned}$$

to be followed by the right vertical composition:

$$\sum_{i=1}^{n} i_* p_* \left(\operatorname{ch}^{D_n} \frac{\zeta^{-i} p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1 - \zeta^{-1} N_{D_n}^\vee} \cdot \operatorname{Td} D_n\right).$$

Summing up the contributions from all sectors and equating it with the other composition gives the main equation

$$\operatorname{ch} E_0 \cdot \operatorname{Td} X = p_*(\operatorname{ch}^{\sqrt[n]{X}}(E_\bullet) \cdot \operatorname{Td} \sqrt[n]{X}) + \sum_{\zeta \in \mu_n \setminus 1} \sum_{i=1}^{n} i_* p_* \left(\operatorname{ch}^{D_n} \frac{\zeta^{-i} p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1 - \zeta^{-1} N_{D_n}^\vee} \cdot \operatorname{Td} D_n\right).$$

Recall that by definition the parabolic Chern character is $p_* \operatorname{ch}^{\sqrt[n]{X}}(E_\bullet)$. To extract it from the equation it is natural to divide both sides by $\operatorname{Td} X$, thereby expressing the difference between the Todd class of $\sqrt[n]{X}$ and X in terms of the Todd class of the relative tangent.

In (2.3) we use the projection formula to combine the tangent bundles into the relative tangent, just as one would do for Grothendieck-Riemann-Roch theorem. Then in (2.4) we apply Proposition 2.7 to obtain its Todd class. Recall that for a line bundle L with Chern root is given by

$$\operatorname{Td} L = \frac{x}{1-e^{-x}}$$

and it is multiplicative in the sense that $\operatorname{Td}(L_1 \oplus L_2) = \operatorname{Td} L_1 \operatorname{Td} L_1$. Denoting the Todd class of the relative tangent of a morphism $\mathcal{X} \to \mathcal{Y}$ by $\operatorname{Td}(\mathcal{X}/\mathcal{Y})$, we have

$$\operatorname{Td}(\sqrt[n]{X}/X) = \frac{\operatorname{Td} \mathcal{O}(D_n)}{\operatorname{Td} \mathcal{O}(nD_n)} = \frac{D/n}{1-e^{-D/n}} \cdot \frac{1-e^{-D}}{D} = \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}}.$$

At this stage we have isolated the sought-after parabolic Chern character, so it remains to simplify the summands due to the non-identity sectors.

As $D_n \to D$ is étale, we can identify the their tangent bundles and subsequently $\operatorname{Td}(D_n/X) = p^* \operatorname{Td}(D/X)$. Then in (2.6) we invoke Proposition 2.4 to pushforward the classes on the gerbe D_n to D. Finally in (2.7), combining the relative tangent of $D \to X$ with the Chern characters of $\operatorname{Gr}_i E_\bullet$, we get an expression in terms of the Chern character on X of

$$i_* \operatorname{Gr}_i E_\bullet = E_i/E_{i-1}.$$

$$\begin{aligned}\operatorname{ch} E_0 = {} & p_*(\operatorname{ch}^{\sqrt[n]{X}} E_\bullet \cdot \operatorname{Td}(\sqrt[n]{X}/X)) \\ & + \sum_{\zeta\in\mu_n\setminus 1} \sum_{i=1}^{n} i_* p_* \left(\operatorname{ch}^{D_n} \frac{\zeta^{-i} p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1-\zeta^{-1} N_{D_n}^\vee} \operatorname{Td}(D_n/X) \right) \end{aligned} \tag{2.3}$$

`projection formula`

$$\begin{aligned} = {} & p_* \operatorname{ch}^{\sqrt[n]{X}} E_\bullet \cdot \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}} \\ & + \sum_{\zeta\in\mu_n\setminus 1} \sum_{i=1}^{n} i_* p_* \left(\operatorname{ch}^{D_n} \frac{\zeta^{-i} p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1-\zeta^{-1} N_{D_n}^\vee} \operatorname{Td}(D_n/X) \right) \end{aligned} \tag{2.4}$$

$$\texttt{expression for } T_{\sqrt[n]{X}/X} \texttt{ and projection formula for } p$$

$$= p_* \operatorname{ch}^{\sqrt[n]{X}} E_\bullet \cdot \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}} + \sum_{\zeta\in\mu_n\setminus 1}\sum_{i=1}^{n} i_* p_* \left(\operatorname{ch}^{D_n} \frac{\zeta^{-i} p^* \operatorname{Gr}_i E_\bullet \otimes N_{D_n}^{-i}}{1-\zeta^{-1}N_{D_n}^\vee} \cdot p^* \operatorname{Td}(D/X)\right) \tag{2.5}$$

$$D_n \to D \texttt{ is étale}$$

$$= \operatorname{ch}^{\mathrm{par}} E_\bullet \cdot \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}} + \frac{1}{n}\sum_{i=1}^{n}\sum_{\zeta\in\mu_n\setminus 1} i_* \left(\operatorname{ch}^{D} \frac{\zeta^{-i} \operatorname{Gr}_i E_\bullet \otimes N_{D}^{-i/n}}{1-\zeta^{-1}N_{D}^\vee} \operatorname{Td}(D/X)\right) \tag{2.6}$$

$$\texttt{projection formula and } p_* : \mathrm{A}^*(D_n) \to \mathrm{A}^*(D)$$

$$= \operatorname{ch}^{\mathrm{par}} E_\bullet \cdot \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}} + \frac{1}{n}\sum_{i=1}^{n}\sum_{\zeta\in\mu_n\setminus 1} \frac{\zeta^{-i} \operatorname{ch}^X E_i/E_{i-1} \otimes e^{-iD/n}}{1-\zeta^{-1}e^{-D/n}} \tag{2.7}$$

$$\texttt{GRR for } i : D \to X$$

$$= \operatorname{ch}^{\mathrm{par}} E_\bullet \cdot \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}} + \sum_{i=1}^{n} \operatorname{ch} E_i/E_{i-1} \cdot P(i,n). \tag{2.8}$$

On the last line (2.8), we define for $1 \le i \le n$

$$P(i,n) = \frac{1}{n}\sum_{\zeta\in\mu_n\setminus 1} \frac{\zeta^{-i}e^{-iD/n}}{1-\zeta^{-1}e^{-D/n}}.$$

Lemma 2.9. *For $1 \le i \le n$, we have*

$$P(i,n) = \frac{e^{-D}}{1-e^{-D}} - \frac{e^{-iD/n}}{n(1-e^{-D/n})}.$$

Proof. Let $\zeta_n = \exp(2\pi i/n)$. Then for $1 \le i \le n$ we have

$$\sum_{j=1}^{n-1} \frac{\zeta_n^{-ij}}{1-\zeta_n^{-j}x} = \frac{nx^{n-i}}{1-x^n} - \frac{1}{1-x}.$$

□

Lemma 2.10. *Let $f(x)$ be a rational function in x. Suppose it has Laurent series expansion $f(x) = \sum a_i x^i$. Then there is an equality of Laurent series*

$$\sum_{\zeta \in \mu_n} f(\zeta x) = n \sum_k a_{nk} x^{nk}.$$

Proof. Observe that for any integer i, the sum over ith powers of roots of unity satisfies

$$\sum_{\zeta \in \mu_n} \zeta^i = \begin{cases} n & \text{if } n \text{ divides } i \\ 0 & \text{otherwise} \end{cases}$$

which can be easily checked. The result then follows by extracting terms in the Laurent series whose degree is divisible by n. $\square$

Corollary 2.11. *For $1 \leq i \leq n$, there is an equality of rational functions*

$$\sum_{\zeta \in \mu_n} \frac{\zeta^{-i}}{1 - \zeta^{-1} x} = \frac{n x^{n-i}}{1 - x^n}.$$

Proof. It suffices to show that they have the same Laurent series expansions. Apply Lemma 2.10 to $f(x) = \frac{x^i}{1-x} = x^i + x^{i+1} + x^{i+2} + \cdots$ to obtain

$$\sum_{\zeta \in \mu_n} \frac{\zeta^i x^i}{1 - \zeta x} = n(x^n + x^{2n} + \cdots) = \frac{n x^n}{1 - x^n}.$$

The result then follows. $\square$

Remark 2.2. Expressions of the form

$$\sum_{\zeta \in \mu_n - 1} \frac{\zeta^i}{1 - \zeta}$$

are examples of *generalised Dedekind sum* due to their connection with the Dedekind η-function and are well studied by number theorists. More intriguing to us is their frequent appearances in characteristic class, more precisely contributions from inertia (resp. fixed point sets) on a stack (resp. manifolds with group actions). For example as mentioned in [Zag73] the sums appear in the work by Atiyah and Singer on equivariant signature ([AS68]). They also appeared in the work by Buckley, Reid and Zhou on orbifold Euler characteristic ([BRZ13]). It would be an interesting question to investigate if there are other contexts in which these sums appear.

As a consequence of Corollary 2.11, for $1 \leq i \leq n$ we can express $P(i,n)$ as

$$P(i,n) = \frac{e^{-D}}{1-e^{-D}} - \frac{e^{-iD/n}}{n(1-e^{-D/n})}.$$

For $1 \leq i < n$, the difference telescopes:

$$P(i+1,n) - P(i,n) = \frac{e^{-iD/n}}{n}.$$

Rearranging the equation, we proceed to complete the final steps of manipulation and conclude the proof:

$$\begin{aligned}
\operatorname{ch}^{\mathrm{par}} E_\bullet \cdot \frac{1}{n}\frac{1-e^{-D}}{1-e^{-D/n}} &= \operatorname{ch} E_0 - \sum_{i=1}^{n} \operatorname{ch} E_i/E_{i-1} \cdot P(i,n) \\
&= \operatorname{ch} E_0 + \sum_{i=1}^{n-1} (P(i+1,n) - P(i,n)) \operatorname{ch} E_i \\
&\qquad + P(1,n) \operatorname{ch} E_0 - P(n,n) \operatorname{ch} E_n \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{ch} E_i \cdot e^{-iD/n}.
\end{aligned}$$

Chapter 3

Equivariant Euler Characteristic

In this joint work with Francesco Sala, we prove an equivariant Riemann-Roch theorem using the theory of Lefschetz-Riemann-Roch morphism developed in [Sal24]. We first present a general formula that computes the equivariant Euler characteristic (regarded as a virtual representation) of an equivariant sheaf on a smooth projective scheme equipped with a faithful action by a finite group. Next we specialise to the case of a curve where the individual terms in the formula can be made explicit. We thus derive a new version of the equivariant Riemann-Roch theorem, which was a classical problem studied in different context and using different approaches, such as [EL80], [Kan86], [Nak86], [Köc05], [FWK09] etc.

Our work not only unifies the previous results and generalises the equivariant Riemann-Roch theorem to a new setting, but also provides an intuitive approach that explains why the formula assumes such a form. In the formula there are two types of contributions: a global term which is determined by the non-equivariant Euler characteristic, and local terms which are determined by ramification data. One could either view the latter as a correction to the naïve, non-equivariant Euler characteristic, or use the inertia stack to treat the local and global contributions on an equal footing.

3.1 Introduction

Fix a base field k which is algebraically closed. Let X be a smooth projective variety equipped with an action of a finite group $G \subseteq \mathrm{Aut}(X)$ of order n. Denote the stack quotient by $\mathcal{X} = [X/G]$. Given an element $g \in G$, let σ be the cyclic subgroup generated by g. We denote by X^σ the fixed loci by the subgroup, which inherits an action by the centraliser $C(\sigma)$.

Theorem 3.1. *Given a G-equivariant vector bundle $\mathcal{E}$ on X, the Euler characteristic of $\mathcal{E}$ is*

$$\chi_G(X, \mathcal{E}) = \bigoplus_\sigma \bigoplus_i \frac{\chi(A^{\sigma,i})}{\varphi(|\sigma|)} \frac{|\sigma|}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^G \iota(x^i) \in K(BG)$$

where $A^{\sigma,i}$ is a $C(\sigma)$-equivariant vector bundle on X^σ that will be given explicitly.

Suppose furthermore X is a curve. Then the quotient $Y = X/G$ exists and is again a smooth curve. We can give a formula for $A^{\sigma,i}$ in terms of the cotangent space of the ramification points.

Theorem 3.2. *Let X be a curve and $Y = X/G$ be the quotient. For each point $x \in X$, we denote by G_x the stabiliser of x. Let e_x (resp. e_x^t) be the ramification index (resp. the tame ramification index). Let $N_x^\vee$ be the cotangent space at x. Then in $K(BG)$ there is an equality*

$$\chi_G(X, \mathcal{E}) = \left(\chi(X, \mathcal{E}) + \frac{\operatorname{rk} \mathcal{E}}{2} \sum_x (e_x^t - 1) \right) \frac{kG}{n} + \sum_{x \in X} \frac{e_x}{n} \operatorname{Ind}_{G_x}^G \frac{\mathcal{E}_x}{1 - N_x^\vee}.$$

When the G-action is tame we may write

$$\chi_G(X, \mathcal{E}) = ((1 - g_Y) \operatorname{rk} \mathcal{E} + \frac{1}{n} \deg \mathcal{E}) kG - \frac{1}{n} \sum_{x \in X} \sum_{d=0}^{e_x - 1} d \cdot \operatorname{Ind}_{G_x}^G (\mathcal{E}_x \otimes N_x^{-d}).$$

Before we give an outline of the proof, let us briefly recall the history. The problem of equivariant Riemann-Roch theorem for curves is a classical one, dating at least to Chevalley and Weil [CW34], who determined the G-equivariant structure on the space of global holomorphic differentials on a compact Riemann surface. Ellingsrud and Lønsted [EL80] found the Euler characteristic of a G-equivariant sheaf on a curve over an algebraically closed field of characteristic zero. It is generalised by Kani [Kan86] and Nakajima [Nak86] to tamely ramified G-cover over any algebraically closed field. A new approach by applying character theory to the virtual G-representation $\chi_G(X, \mathcal{E})$ is worked out by Köck [Köc05] under the assumption of tameness and algebraically closed gound field. It is further generalised by Fischbacher-Weitz and Köck [FWK09] to weakly ramified cover.

In our work, we compute the equivariant Euler characteristic as an element in $K(BG)$, the K-theory of locally free sheaves on BG. When the order of G is invertible in k, this completely determines the isotypical decomposition of a G-representation. However when the order is not invertible $K(BG)$ only determines the indecomposable factors. However although the conclusion is weaker than the main result of [FWK09], we *do not* place any restrictions on ramification of the map $X \to X/G$.

The result is obtained by applying the Lefschetz-Riemann-Roch map to the morphism $\mathcal{X} = [X/G] \to BG$:

$$\begin{array}{ccc} K(\mathcal{X}) & \xrightarrow{\mathcal{L}_{\mathcal{X}}} & \bigoplus_{r,\sigma\in\overline{C}_r} K(I_\sigma\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \\ \big\downarrow{\scriptstyle \pi_*} & & \big\downarrow \\ K(BG) & \xrightarrow{\mathcal{L}_{BG}} & \bigoplus_{r,\sigma\in\overline{C}_r} K(I_\sigma BG)_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \end{array}$$

Since $\mathcal{X}$ is smooth, the upper row can be identified with the composition

$$K(\mathcal{X}) \xrightarrow{\lambda_{-1}(\mathcal{N}^\vee)^{-1}\cdot\rho^*} K(I\mathcal{X}) \xrightarrow{m^*} K(\widetilde{I_{\boldsymbol{\mu}}\mathcal{X}}) \xrightarrow{\bigoplus \frac{r}{\phi(r)}} K(\widetilde{I_{\boldsymbol{\mu}}\mathcal{X}}),$$

which by construction lands in the tautological part of the K-theory of the inertia stack. On the other hand, $\mathcal{L}_{BG}^{-1}$ is the composition

$$\bigoplus_\sigma K(BC(\sigma))_{\mathbf{g}} \otimes \tilde{R}\sigma \to \bigoplus_\sigma K(BC(\sigma)) \otimes R\sigma \xrightarrow{m_*} \bigoplus_\sigma K(BC(\sigma)) \xrightarrow{\mathrm{Ind}_{C(\sigma)}^G} K(BG).$$

We now explain the notations by recalling the work of [Sal24].

3.2 Lefschetz-Riemann-Roch morphism

In [Sal24], Sala developed a theory of covariant Riemann-Roch morphism for tame stacks. Very briefly, given a tame stack $\mathcal{X}$, there is a morphism of $\mathbb{Q}$-modules called *Lefschetz-Riemann-Roch morphism*

$$\mathcal{L}_{\mathcal{X}} : K'_*(\mathcal{X}) \to \bigoplus_{r\geq 1} K'_*(I_{\mu_r}\mathcal{X}) \otimes \mathbb{Q}(\zeta_r)$$

which is an isomorphism onto its image (which can be very precisely characterised) and is covariant with respect to proper morphisms. Here K'_* denotes the higher K-groups of coherent sheaves and $I_{\boldsymbol{\mu}_r}\mathcal{X}$ is the rth cyclotomic inertia stack, which is closely related to the inertia. One might wonder why $\mathcal{L}$ deserves its name despite being valued in K-theory. It is because after post-composing it with Chern character and multiplication by Todd class we can recover Toën-Riemann-Roch theorem for stacks. More importantly, the target of $\mathcal{L}$ also appeared in [Toë99] under the name *étale K-theory* (denoted as $K_{*,ét}$ in loc. cit.) and is used as an important intermediate step in establishing the Toën-Riemann-Roch theorem ([Toë99, Lemme 4.12]).

In this section we summarise the main results in [Sal24], making suitable simplifications adapting to our application. We fix a base field k. Assume $\mathcal{X} = [X/G]$ is a tame or Deligne-Mumford quotient stack where X is a separated scheme over k and G is an affine group scheme of finite type over k. With these assumption there is an identification $K'_*(\mathcal{X}) = K_*(\mathcal{X})$ between the higher K-theory of coherent and locally free sheaves on $\mathcal{X}$. Since only need the 0th K-group, we abbreviate it to $K(\mathcal{X})$. Lastly whenever we work with K-groups or representation rings we mean their tensor product with $\mathbb{Q}$.

The fundamental decomposition We call a subgroup scheme $\sigma \subseteq G$ *dual cyclic* when it is isomorphic to $\boldsymbol{\mu}_{r,k}$ for some $r \geq 1$. For each $r \geq 1$, denote by $\overline{\mathcal{C}}_r(G)$ the set of conjugacy classes of monomorphisms $\boldsymbol{\mu}_r \to G$, and by $\overline{\mathcal{C}}(G)$ the union over all $r \geq 1$. Given an element $\sigma \in \overline{\mathcal{C}}_r(G)$ we choose a monomorphism $\boldsymbol{\mu}_r \to G$ representing it and also denote it by $\sigma : \boldsymbol{\mu}_r \to G$. The fixed point subscheme for the induced action of $\boldsymbol{\mu}_r$ on X is denoted X^σ. We also set $\mathcal{C}_r(G)$ to be the set of conjugacy classes of dual cyclic groups of order r. The union over all r is denoted $\mathcal{C}(G)$. We note that there is an action of $\mathrm{Aut}(\boldsymbol{\mu}_r)$ on $\overline{\mathcal{C}}_r(G)$.

Given a group H we denote its representation ring by RH. In particular if $H = \boldsymbol{\mu}_r$ then $RH = \mathbb{Q}[x]/(x^r - 1)$. We denote by $\tilde{R}H$ the projection to $\mathbb{Q}[x]/\Phi_r(x) \cong \mathbb{Q}(\zeta_r)$ where $\Phi_r(x)$ is the rth cyclotomic polynomial in x.

Given an RG-module M and a dual cyclic group $\sigma \subseteq G$, the *σ-localisation* M_σ is the localisation $M_{\mathfrak{m}_\sigma}$ where $\mathfrak{m}_\sigma$ is the kernel of the composition $RG \to R\sigma \to \tilde{R}\sigma$, which depends only on the conjugacy class of σ.

For each $r \geq 1$, define the multiplicative system $\Sigma_r = \Sigma_r^{\mathcal{X}} \subseteq K(\mathcal{X})$ as follows. An element $\alpha \in K(\mathcal{X})$ is in Σ_r if for all representable morphisms $\phi : B_K\boldsymbol{\mu}_r \to \mathcal{X}$ where K is an extension of k, the projection of $\phi^*\alpha \in R\boldsymbol{\mu}_r$ in $\tilde{R}\boldsymbol{\mu}_r$ is non-zero. The *$\boldsymbol{\mu}_r$-localisation* $K(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ of $K(\mathcal{X})$ is the $K(\mathcal{X})$-module $\Sigma_r^{-1}K(\mathcal{X})$.

Theorem 3.3 ([Sal24, Theorem 4.2])**.** *The projections $K(\mathcal{X}) \to K(\mathcal{X})_{(\boldsymbol{\mu}_r)}$ induce an isomorphism*

$$K(\mathcal{X}) \cong \prod_{r \geq 1} K(\mathcal{X})_{(\boldsymbol{\mu}_r)}.$$

In the decomposition, we call the factor corresponding to $r = 1$ the geometric part *of the K-theory of $\mathcal{X}$ and denote it by $K(\mathcal{X})_{\mathbf{g}}$. The product of the rest is called the* algebraic part *and is denoted $K(\mathcal{X})_{\mathbf{a}}$. Thus we have the* fundamental decomposition

$$K(\mathcal{X}) = K(\mathcal{X})_{\mathbf{g}} \oplus K(\mathcal{X})_{\mathbf{a}}.$$

Proposition 3.4 ([Sal24, Proposition 4.5, Proposition 4.7]). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of tame stacks.*

1. *If f is representable then the pushforward $f_* : K(\mathcal{X}) \to K(\mathcal{Y})$ preserves the fundamental decomposition.*

2. *There exists a homomorphism $f_* : K(\mathcal{X})_{\mathbf{g}} \to K(\mathcal{Y})_{\mathbf{g}}$ such that the diagram*

$$\begin{array}{ccc} K(\mathcal{X})_{\mathbf{g}} & \hookrightarrow & K(\mathcal{X}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ K(\mathcal{Y})_{\mathbf{g}} & \hookrightarrow & K(\mathcal{X}) \end{array}$$

   commutes. The horizontal arrows are the inclusions coming from the fundamental decomposition.

There is a similar statement for functoriality with respect to (representable) pullbacks, but we will not need it here.

Remark 3.1. The name *geometric part* is justified by the following theorem:

Theorem 3.5 ([Sal24, Theorem 4.13]). *Let $\pi : \mathcal{X} \to M$ be the coarse moduli space. Then the pushforward $\pi_* : K(\mathcal{X})_{\mathbf{g}} \to K(M)_{\mathbf{g}} \to K(M)$ is an isomorphism.*

In particular for a finite group G, the projection to the geometric part $K(BG) \to K(BG)_{\mathbf{g}} = \mathbb{Q}$ is the augmentation map, and it can be shown that the inclusion of the geometric part is the section $1 \mapsto \frac{1}{|G|} \operatorname{Ind}_1^G 1 = \frac{kG}{|G|}$.

Cyclotomic inertia stack and twist For a positive integer r, the *rth cyclotomic inertia stack* $I_{\boldsymbol{\mu}_r}\mathcal{X}$ is the fibred category which parameterises order r automorphisms of objects of $\mathcal{X}$. Its objects are pairs (x, α) where $x \in \mathcal{X}(T)$ is an object of $\mathcal{X}$ above T and $\alpha : \boldsymbol{\mu}_{r,S} \to \operatorname{Aut}_S(x)$ is a monomorphism of group schemes. A morphism $f : (x, \alpha) \to (y, \beta)$ from $(x, \alpha) \in I_{\boldsymbol{\mu}_r}\mathcal{X}(T)$ to $(y, \beta) \in I_{\boldsymbol{\mu}_r}\mathcal{X}(S)$ consists of an arrow $f : x \to y$ in $\mathcal{X}$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{\mu}_{r,T} & = & \phi^* \boldsymbol{\mu}_{r,S} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi^* y} \\ \operatorname{Aut}_T(x) & \longrightarrow & \phi^* \operatorname{Aut}_S(y) \end{array}$$

commutes. Here $\phi : T \to S$ is the image of f in the category of schemes and the bottom row is induced by f.

There is a morphism $I_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$ sending (x, α) to x. The *cyclotomic inertia stack* $I_{\boldsymbol{\mu}}\mathcal{X}$ is the disjoint union of the $I_{\boldsymbol{\mu}_r}\mathcal{X}$ for all $r \geq 1$. We denote by $\rho = \rho_{\mathcal{X}} : I_{\boldsymbol{\mu}_r}\mathcal{X} \to \mathcal{X}$.

Remark 3.2. Over $\mathbb{C}$ the canonical isomorphisms $\boldsymbol{\mu}_r \cong \mathbb{Z}/r\mathbb{Z}$ induce natural isomorphisms $I_{\boldsymbol{\mu}}\mathcal{X} \cong I\mathcal{X}$.

Similar to the inertia stack, we can describe the cyclotomic inertia stack of the global quotient $\mathcal{X} = [X/G]$ in equivariant terms as

$$I_{\boldsymbol{\mu}_r}\mathcal{X} = \coprod_{\sigma \in \overline{\mathcal{C}}_r(G)} [X^\sigma / C_G(\sigma)]$$

where $C_G(\sigma)$ is the centraliser of $\sigma : \boldsymbol{\mu}_r \to G$ in G. For this reason, for a $\sigma \in \overline{\mathcal{C}}_r(G)$ we also write $I_\sigma\mathcal{X}$ for the component $[X^\sigma / C_G(\sigma)]$.

For a fixed σ, there is a multiplication map

$$m : C_G(\sigma) \times \sigma \to C_G(\sigma)$$

which is a homomorphism of groups. It induces a map

$$\alpha_{\mathcal{X}} : I_{\boldsymbol{\mu}_r}\mathcal{X} \times B\boldsymbol{\mu}_r \cong [X^\sigma / C_G(\sigma) \times \sigma] \to [X^\sigma / C_G(\sigma)] \cong I_{\boldsymbol{\mu}_r}\mathcal{X}.$$

We call it the *twist map*. It assembles into a morphism

$$\alpha_{\mathcal{X}} : \widetilde{I_{\boldsymbol{\mu}_r}\mathcal{X}} := \coprod_{r \geq 1} I_{\boldsymbol{\mu}_r}\mathcal{X} \times \boldsymbol{\mu}_r \to I_{\boldsymbol{\mu}}\mathcal{X}.$$

Remark 3.3. The twist map induces a pullback on K-theory, which we will call the *twisting operation* due to its close relation with the twist map ρ in appeared in Toën-Riemann-Roch theorem (see Section 1.3.2). It also induces a pushforward which we will call *anti-twisting* that will be investigated later.

Tautological part of K-theory The *tautological part* of $K(\widetilde{I_{\boldsymbol{\mu}_r}\mathcal{X}})$ is defined to be its localisation with respect to some multiplicative system. The actual definition is quite complicated, but the projection to the tautological part has a natural splitting. We describe the image of the splitting, which suffices for our purpose: for each r it is the image of the inclusion

$$K(I_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \hookrightarrow K(I_{\boldsymbol{\mu}_r}\mathcal{X}) \otimes R\boldsymbol{\mu}_r = K(I_{\boldsymbol{\mu}_r}\mathcal{X} \times B\boldsymbol{\mu}_r) = K(\widetilde{I_{\boldsymbol{\mu}_r}\mathcal{X}})$$

where the first map is the splitting of the projections to the geometric part.

Finally composing the inclusion with the pushforward induced by the twist map $\alpha_* : K(\widetilde{I_{\boldsymbol{\mu}}\mathcal{X}}) \to K(I_{\boldsymbol{\mu}}\mathcal{X})$ we get a map

$$\beta_{\mathcal{X}} : \bigoplus_{r \geq 1} K(I_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \to K(I_{\boldsymbol{\mu}}\mathcal{X}).$$

Lefschetz-Riemann-Roch morphism We are ready to state the main theorem of [Sal24]. Recall that $\rho : I_{\boldsymbol{\mu}}\mathcal{X} \to \mathcal{X}$ is the cyclotomic inertia stack.

Theorem 3.6 ([Sal24, Theorem 7.2])**.** *Let $\mathcal{X}$ be a tame quotient stack. Let $r_* : \left(\bigoplus_r K(I_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\right)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty} \to K(\mathcal{X})$ be the composition $\rho_* \circ \beta_{\mathcal{X}}$. Then it is an isomorphism. Furthermore the map $\mathcal{L} = r_*^{-1}$ gives a* Lefschetz-Riemann-Roch isomorphism *which is covariant with respect to proper morphisms of stacks*

$$\mathcal{L} : K(\mathcal{X}) \to \left(\bigoplus_r K(I_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\right)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}.$$

While the morphism $\mathcal{L}$ is valid for any stack, it is not particularly easy to calculate in general. When $\mathcal{X}$ is regular, however, it is possible to express it in a different way which is reminiscent of the Toën-Riemann-Roch morphism. Consider the map r^* given by the composition

$$K(\mathcal{X}) \xrightarrow{\lambda_{-1}(\mathcal{N}_\rho^\vee)^{-1}\cdot\rho^*} K(I\mathcal{X}) \xrightarrow{\alpha^*} K(\widetilde{I_{\boldsymbol{\mu}}\mathcal{X}}) \to \left(\bigoplus_r K(I_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)\right)^{\mathrm{Aut}\,\boldsymbol{\mu}_\infty}.$$

Theorem 3.7 ([Sal24, Theorem 7.3])**.** *The composition $r^* \circ r_*$ is equal to the endomorphism that is multiplication by the rational number $\frac{\phi(r)}{r}$ on the component $K(I_{\boldsymbol{\mu}_r}\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r)$. In particular the Lefschetz-Riemann-Roch morphism, in the regular case, is given by*

$$\mathcal{L} = \bigoplus_d \frac{d}{\phi(d)} \cdot r^*.$$

3.3 Proof of Theorem 3.1

The goal of this section is to prove

Theorem 3.1. *Given a G-equivariant vector bundle $\mathcal{E}$ on X, the Euler characteristic of $\mathcal{E}$ is*

$$\chi_G(X,\mathcal{E}) = \bigoplus_\sigma \bigoplus_i \frac{\chi(A^{\sigma,i})}{\varphi(|\sigma|)} \frac{|\sigma|}{|C(\sigma)|} \cdot \mathrm{Ind}_\sigma^G\, \iota(x^i) \in K(BG)$$

where $A^{\sigma,i}$ is a $C(\sigma)$-equivariant vector bundle on X^σ that will be given explicitly.

We will do so by computing the Euler characteristics using the composition $\mathcal{L}_{BG} \circ I\pi_* \circ \mathcal{L}_{\mathcal{X}}^{-1}$ where $I\pi_*$ is the map on K-theory induced by $\pi : \mathcal{X} \to BG$.

3.3.1 Lefschetz-Riemann-Roch for BG

In this section we compute $\mathcal{L}_{BG}$, or more precisely its inverse. It is a technical computation in representation theory. Index the components of the inertia stack of BG by $\overline{\mathcal{C}} = \coprod \overline{\mathcal{C}}_r$, where $\overline{C}_r$ is the conjugacy classes of monomorphisms $\mu_r \to G$:

$$IBG = \coprod_{\sigma \in \overline{C}_r} BC(\sigma).$$

We remind the readers that $\mathcal{L}_{BG}^{-1} = \rho_* \circ \beta_{BG}$ is defined on the component corresponding to σ as

$$K(BC(\sigma))_{\mathbf{g}} \otimes \tilde{R}\sigma \hookrightarrow K(BC(\sigma)) \otimes R(\sigma) \xhookrightarrow{\alpha_*} K(BC(\sigma)) \xhookrightarrow{\rho_*} K(BG).$$

Inclusion of geometric part of BH Given any group H, the inclusion $K(BH)_{\mathbf{g}} \to K(BH)$ sends the unit to $\frac{kH}{|H|}$.

Inclusion $\tilde{R}\sigma \to R\sigma$ Suppose σ is dual cyclic of order r. Recall that by Chinese remainder theorem there is a splitting

$$R\sigma = \mathbb{Q}[x]/x^r - 1 \cong \prod_{d|r} \mathbb{Q}[x]/\Phi_d(x).$$

We denote by $\iota : \mathbb{Q}[x]/\Phi_r(x) \to \mathbb{Q}[x]/x^r - 1$ the section of the projection to the factor $\mathbb{Q}[x]/\Phi_r(x) \cong \tilde{R}\sigma$.

Anti-twist Let σ be a dual cyclic subgroup of G of order r and let $H = C(\sigma)$ be its centraliser. The twisting operation α^* on the rth cyclotomic inertia stack is given by the pullback induced by multiplication $m : H \times \sigma \to H$. The anti-twist α_* is given by m_*, whose effect is taking invariants relative to the subgroup $\ker(m) = \{(x^{-1}, x) : x \in \sigma\}$. In formula, for an H-representation V and a character χ of σ,

$$\begin{aligned} m_* : RH \otimes R\sigma &\to RH \\ V \otimes \chi &\mapsto \chi\text{-isotypical part of } \operatorname{Res}_\sigma^H V, \end{aligned}$$

which inherits an H-action.

Since H is the centraliser of σ, we have $\operatorname{Res}_\sigma^H \operatorname{Ind}_\sigma^H \chi = \frac{|H|}{|\sigma|}\chi$ for any σ-character χ and

$$\operatorname{Res}_\sigma^H k[H] = \operatorname{Res}_\sigma^H \operatorname{Ind}_\sigma^H k[\sigma] = k[\sigma]^{\oplus|H/\sigma|} = \frac{|H|}{|\sigma|}\bigoplus_{\chi\in\widehat{\sigma}}\chi = \bigoplus_{\chi\in\widehat{\sigma}}\operatorname{Res}_\sigma^H \operatorname{Ind}_\sigma^H \chi.$$

As a consequence we note that the χ-isotypic part of $\operatorname{Res}_\sigma^H k[H]$ is exactly $\operatorname{Ind}_\sigma^H \chi$, whence

$$m_*(\frac{k[H]}{|H|}\otimes\chi) = \frac{1}{|H|}\operatorname{Ind}_\sigma^H \chi.$$

In particular, the composition

$$\tilde{R}\sigma \simeq K(BH)_{\mathbf{g}}\otimes\tilde{R}\sigma \hookrightarrow K(BC(\sigma))\otimes R\sigma \xrightarrow{m_*} K(BC(\sigma))$$

is the same as

$$\tilde{R}\sigma \simeq K(B\sigma)_{\mathbf{g}}\otimes\tilde{R}\sigma \hookrightarrow K(B\sigma)\otimes R\sigma \xrightarrow{m_*} K(B\sigma) \xrightarrow{\operatorname{Ind}_\sigma^{C(\sigma)}} K(BC(\sigma)).$$

up to a correcting factor of $\frac{r}{|H|} = \frac{|\sigma|}{|C(\sigma)|}$.

Finally, by [Sal24, proof of Theorem 7.3], the map

$$\tilde{R}\sigma \simeq K(B\sigma)_{\mathbf{g}}\otimes\tilde{R}\sigma \hookrightarrow K(B\sigma)\otimes R\sigma \xrightarrow{m_*} K(B\sigma)$$

is equal to $\frac{1}{r}\cdot\iota\colon \mathbb{Q}(\zeta_r)\to\mathbb{Q}[x]/x^r-1$.

Pushforward from inertia stack The pushforward $\rho : K(BC(\sigma))\to K(BG)$ is simply induction from the subgroup $C(\sigma)$.

The composition Now we are ready to state the result. On the summand labelled by $\sigma\in\overline{\mathcal{C}}_r$, $\mathcal{L}_{BG}^{-1}$ is given by

$$\begin{array}{ccccccc}
K(BC(\sigma))_{\mathbf{g}}\otimes\tilde{R}\sigma & \hookrightarrow & K(BC(\sigma))\otimes R\sigma & \xrightarrow{m_*} & K(BC(\sigma)) & \xrightarrow{\operatorname{Ind}_{C(\sigma)}^G} & K(BG) \\
\| & & \| & & & & \\
\mathbb{Q}\otimes\mathbb{Q}[x]/(\Phi_r(x)) & & K(BC(\sigma))\otimes\mathbb{Q}[x]/x^r-1 & & & &
\end{array}$$

$$1\otimes x \longmapsto \frac{kC(\sigma)}{|C(\sigma)|}\otimes\iota(x) \longmapsto \frac{r}{|C(\sigma)|}\cdot\frac{1}{r}\operatorname{Ind}_\sigma^{C(\sigma)}\iota(x) \longmapsto \frac{1}{|C(\sigma)|}\operatorname{Ind}_\sigma^G\iota(x)$$

3.3.2 Lefschetz-Riemann-Roch for $\mathcal{X}$

The upper composition is considerably easier, thanks to 3.7 and the fact that the twisting map α^* is much easier to describe (see Section 1.3 for the computation). Write

$$I_{\boldsymbol{\mu}}\mathcal{X} = \bigoplus_{r,\sigma\in\overline{\mathcal{C}}_r} [X^\sigma/C(\sigma)].$$

Given a G-equivariant vector bundle $\mathcal{E}$ on X, let $(\mathcal{E}_\sigma)_\sigma = \rho^*\mathcal{E}$ be its pullback along $\rho : I\mathcal{X} \to \mathcal{X}$ and let N_σ be the normal bundle of ρ on the component labelled by σ. For each σ, we split the virtual bundle

$$\frac{\mathcal{E}_\sigma}{\lambda_{-1}(N_\sigma^\vee)} := A^\sigma = \bigoplus_{i\in\hat{\sigma}} A^{\sigma,i}$$

into isotypical components as σ-representations, obtained by restriction from $C(\sigma)$. Here $\hat{\sigma}$ is the character group of σ, which can be identified with $\mathbb{Z}/r\mathbb{Z}$.

The Lefschetz-Riemann-Roch map for $\mathcal{X}$ is thus given by

$$\mathcal{E} \mapsto \bigoplus_{r,\sigma\in\overline{C}_r, i\in\hat{\sigma}} \frac{r\cdot\zeta_r^i}{\phi(r)}\cdot A_{\mathbf{g}}^{\sigma,i}$$

where $A_{\mathbf{g}}^{\sigma,i}$ denotes the projection to the geometric part.

3.3.3 Total composition

We have the vertical map

$$\mathcal{I}\pi_* : \bigoplus_{r,\sigma\in\overline{\mathcal{C}}_r} K(I_\sigma\mathcal{X})_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \longrightarrow \bigoplus_{r,\sigma\in\overline{\mathcal{C}}_r} K(I_\sigma BG)_{\mathbf{g}} \otimes \mathbb{Q}(\zeta_r) \simeq \bigoplus_{r,\sigma\in\overline{\mathcal{C}}_r} \mathbb{Q}(\zeta_r)$$

which, since π is representable, is the identity on the $\mathbb{Q}(\zeta_r)$ components and sends $A_{\mathbf{g}}^{\sigma,i}$ to $\chi(A^{\sigma,i})$, the *non-equivariant Euler characteristic* of the virtual bundle $A^{\sigma,i}$ on X^σ. Indeed, by Proposition 3.4, we have a commutative diagram

$$\begin{array}{ccc} K(I_\sigma\mathcal{X}) & \longrightarrow & K(I_\sigma\mathcal{X})_{\mathbf{g}} \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle I\pi_*} \\ K(I_\sigma BG) & \longrightarrow & K(I_\sigma BG)_{\mathbf{g}} \end{array}$$

the horizontal maps being projections to the geometric part. In particular the lower horizontal map is the augmentation morphism which forgets the group action. The bottom-left composition is then easily seen to be exactly the Euler characteristic map.

Combining this with the previous calculation of the lower composition, we immediately get

$$\chi_G(X,\mathcal{E}) = \bigoplus_{r,\sigma\in\overline{C}_r}\bigoplus_{i\in\hat{\sigma}} \frac{\chi(A^{\sigma,i})}{\phi(r)}\frac{r}{|C(\sigma)|}\cdot \mathrm{Ind}_\sigma^G\,\iota(\zeta_r^i).$$

which completes the proof of Theorem 3.1.

3.4 Proof of Theorem 3.2

In this section we specialise to smooth projective curves and prove

Theorem 3.2. *Let X be a curve and $Y = X/G$ be the quotient. For each point $x \in X$, we denote by G_x the stabiliser of x. Let e_x (resp. e_x^t) be the ramification index (resp. the tame ramification index). Let $N_x^\vee$ be the cotangent space at x. Then in $K(BG)$ there is an equality*

$$\chi_G(X,\mathcal{E}) = \left(\chi(X,\mathcal{E}) + \frac{\mathrm{rk}\,\mathcal{E}}{2}\sum_x (e_x^t - 1)\right)\frac{kG}{n} + \sum_{x\in X}\frac{e_x}{n}\mathrm{Ind}_{G_x}^G\frac{\mathcal{E}_x}{1-N_x^\vee}.$$

When the G-action is tame we may write

$$\chi_G(X,\mathcal{E}) = ((1-g_Y)\,\mathrm{rk}\,\mathcal{E} + \frac{1}{n}\deg\mathcal{E})kG - \frac{1}{n}\sum_{x\in X}\sum_{d=0}^{e_x-1} d\cdot \mathrm{Ind}_{G_x}^G(\mathcal{E}_x\otimes N_x^{-d}).$$

We need to give a concrete description of $\chi(A^{\sigma,i})$'s. Over the identity sector corresponding to $\sigma = 1$ it is the Euler characteristic of $\mathcal{E}$. For $\sigma \neq 1$, as the coarse moduli space of $[X^\sigma/C(\sigma)]$ is zero-dimensional, $\chi(A^{\sigma,i})$ is nothing but the dimension of the virtual representation $A^{\sigma,i}$. Thus

$$\chi_G(X,\mathcal{E}) = \chi(X,\mathcal{E})\frac{kG}{n} + \bigoplus_{r>1,\sigma\in\overline{C}_r}\bigoplus_{i\in\hat{\sigma}}\frac{\dim A^{\sigma,i}}{\phi(r)}\frac{r}{|C(\sigma)|}\cdot\mathrm{Ind}_\sigma^G\,\iota(\zeta_r^i).$$

Let us reindex the summation. We fix a representative for each $\tilde{\sigma}\in\overline{C}_r$, which we also call σ. Choose also a set of representatives $\{\tilde{x}\}$ for each G-orbit with non-trivial stabilisers. The action being effective, for each $1 \neq \tilde{\sigma}$ the fixed locus X^σ is zero-dimensional, so we can regroup uniquely the $\{\tilde{\sigma}\}$'s to the sets $\{\tilde{\sigma}\subseteq G_{\tilde{x}}\}_{\tilde{x}}$. For each

$x \in X$, denote by e_x (resp. e_x^t) the ramification index (resp. the *tame* ramification index). We have:

$$\chi_G(X,\mathcal{E}) = \chi(X,\mathcal{E})\frac{kG}{n} + \sum_{\tilde{x}} \bigoplus_{1\neq\tilde{\sigma}\subseteq G_{\tilde{x}}} \bigoplus_{i\in\hat{\sigma}} \frac{|X^\sigma|\cdot\dim A^{\sigma,i}}{\phi(r)}\frac{r}{|C(\sigma)|}\cdot \operatorname{Ind}_\sigma^G \iota(\zeta_r^i)$$
$$= \chi(X,\mathcal{E})\frac{kG}{n} + \sum_{x\in X} \frac{e_x}{n}\operatorname{Ind}_{G_x}^G \bigoplus_{1\neq\tilde{\sigma}\subseteq G_x} \bigoplus_{i\in\hat{\sigma}} \frac{|X^\sigma|\cdot\dim A^{\sigma,i}}{\phi(r)}\frac{r}{|C(\sigma)|}\cdot \operatorname{Ind}_\sigma^{G_x} \iota(\zeta_r^i).$$

At this point we observe that making use of [Sal24, Theorem 7.3], we can rewrite the expression

$$\bigoplus_{1\neq\tilde{\sigma}\subseteq G_x} \bigoplus_{i\in\hat{\sigma}} \frac{\dim A^{\sigma,i}}{\phi(r)}\frac{r}{|C(\sigma)|}\cdot \operatorname{Ind}_\sigma^{G_x} \iota(\zeta_r^i)$$

Indeed, let $A^\sigma = \operatorname{Res}_{C(\sigma)}^{G_x} V^\sigma$, where V^σ is the G_x-virtual sheaf $\frac{\mathcal{E}_\sigma}{\lambda_{-1}(N_\sigma^\vee)}$.

Lemma 3.8. *We have*

$$\bigoplus_{1\neq\tilde{\sigma}\subseteq G_x} \bigoplus_{i\in\hat{\sigma}} \frac{|X^\sigma|\cdot\dim A^{\sigma,i}}{\phi(r)}\frac{r}{|C(\sigma)|}\cdot \operatorname{Ind}_\sigma^{G_x} \iota(\zeta_r^i) = A_x - \iota_{\sigma=1}\dim A_x$$

Proof. This is an application of Theorem 3.7 to BG_x. The composition of maps

$$R(G_x) \xrightarrow{\quad\text{Res}\quad} \bigoplus_{\substack{r,\sigma\\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma)) \xrightarrow{m^*} \bigoplus_{\substack{r,\sigma\\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma))_{\mathbf{g}}\otimes\mathbb{Q}(\zeta_r)$$

$$\bigoplus_{\substack{r,\sigma\\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma))_{\mathbf{g}}\otimes\mathbb{Q}(\zeta_r) \xrightarrow{m_*} \bigoplus_{\substack{r,\sigma\\ \tilde{\sigma}\subseteq G_x}} R(C_{G_x}(\sigma)) \xrightarrow{\quad\text{Ind}\quad} R(G_x)$$

given explicitly by

$$V^\sigma \longrightarrow \sum_{\substack{r,\sigma\\ \tilde{\sigma}\subseteq G_x}} \operatorname{Res}_{C_{G_x}(\sigma)}^{G_x}(V^\sigma) \longrightarrow \sum_{\substack{r,\sigma\\ i\in\hat{\sigma}}} \dim(V^{\sigma,i})\cdot\zeta_r^i\cdot\tfrac{r}{\phi(r)}$$

$$\sum_{\substack{r,\sigma\\ i\in\hat{\sigma}}} \dim(V^{\sigma,i})\cdot\zeta_r^i\cdot\tfrac{r}{\phi(r)} \longrightarrow \sum_{\substack{r,\sigma\\ i\in\hat{\sigma}}} \dim(V^{\sigma,i})\cdot\tfrac{\iota(\zeta_r^i)}{|C_{G_x}(\sigma)|}\cdot\tfrac{r}{\phi(r)} \longrightarrow \sum_{\substack{r,\sigma\\ i\in\hat{\sigma}}} \tfrac{\dim(V^{\sigma,i})}{\phi(r)}\tfrac{r}{|C_{G_x}(\sigma)|}\cdot\operatorname{Ind}_\sigma^{G_x}(\iota(\zeta_r^i))$$

is the identity. Note that we computed the map m_* exactly as we did at the end of Section 3.3.1.

We must have $\frac{|X^\sigma|}{|C(\sigma)|} = \frac{1}{|C_{G_x}(\sigma)|}$, since the connected components of the inertia stack corresponding to $1 \neq \tilde{\sigma} \subseteq G_x$ are equal to those of $\mathcal{I}BG_x$. This allows us to write

$$\bigoplus_{1\neq\tilde{\sigma}\subseteq G_x} \bigoplus_{i\in\hat{\sigma}} \frac{|X^\sigma| \cdot \dim A^{\sigma,i}}{\phi(r)} \frac{r}{|C(\sigma)|} \cdot \operatorname{Ind}_\sigma^{G_x} \iota(\zeta_r^i) = A_x - \iota_{\sigma=1} \dim A_x$$

since the missing term corresponding to $\sigma = 1$ corresponds exactly to the geometric part of A_x. □

What remains to do is to expand $A_x = \frac{\mathcal{E}_x}{1-N_x^\vee}$ in terms of the cotangent space. We invoke the following lemma:

Lemma 3.9. *Let H be a cyclic group of order m and χ a non-trivial character. Then*

$$\frac{1}{1-\chi} = -\frac{1}{m}\sum_{d=1}^{m-1} \chi^{-d}.$$

Proof. Same as [Köc05, Lemma 1.2]. □

Noting that the action of G_x on $N_x^\vee$ factors through the tame part, we use Lemma 3.9 to write

$$\begin{aligned}\dim A_x = \dim \frac{\mathcal{E}_x}{1-N_x^\vee} &= \frac{1}{e_x^t} \dim \sum_{d=1}^{e_x^t-1} (-d)\mathcal{E}_x \otimes N_x^{-d} \\ &= -\frac{1}{e_x^t} \cdot \operatorname{rk} \mathcal{E}_x \frac{e_x^t(e_x^t-1)}{2} = -\operatorname{rk}\mathcal{E}\frac{e_x^t-1}{2}.\end{aligned}$$

and

$$A_x - \iota_{\sigma=1} \dim A_x = A_x - \dim A_x \frac{kG_x}{e_x} = A_x + \operatorname{rk}\mathcal{E}\frac{e_x^t-1}{2}\frac{kG_x}{e_x}.$$

Putting all ingredients together,

$$\begin{aligned}\chi_G(X,\mathcal{E}) &= \chi(X,\mathcal{E})\frac{kG}{n} + \sum_{x\in X} \frac{e_x}{n} \operatorname{Ind}_{G_x}^G (A_x + \operatorname{rk}\mathcal{E} \cdot \frac{e_x^t-1}{2}\frac{kG_x}{e_x}) \\ &= \left(\chi(X,\mathcal{E}) + \frac{\operatorname{rk}\mathcal{E}}{2}\sum_x (e_x^t-1)\right)\frac{kG}{n} + \sum_{x\in X}\frac{e_x}{n}\operatorname{Ind}_{G_x}^G \frac{\mathcal{E}_x}{1-N_x^\vee}.\end{aligned}$$

Finally when the G-action is tame, we can use Hirzebruch-Riemann-Roch and Riemann-Hurwitz to express $\chi(X, \mathcal{E})$ in terms of rank and degree of $\mathcal{E}$ and genus of Y:

$$\chi(X, \mathcal{E}) = \deg \mathcal{E} + \operatorname{rk} \mathcal{E}(1 - g_X) = \deg \mathcal{E} + \operatorname{rk} \mathcal{E} \left(n(1 - g_Y) - \frac{1}{2} \sum_{x \in X} (e_x - 1) \right)$$

and expand $\frac{\mathcal{E}}{1 - N_x^\vee}$ using Lemma 3.9.

Chapter 4

Stringy Chow Rings and Weighted Blow-ups

We compute the stringy chow ring of weighted blow up of a smooth variety along a smooth center. We explore finite generation properties of this ring and of the usual Chow ring of the weighted blowup. We also compute the obstruction classes arising in the string chow product for general Deligne-Mumford stacks of the form $[X/\mathbb{G}_m^k]$ for a smooth variety X.

4.1 Introduction

In [AGV02], Abramovich-Graber-Vistoli define the *stringy chow ring* $A^*_{st}(\mathcal{X})$ for any smooth tame Deligne-Mumford stack $\mathcal{X}$ over a field k. In this article we study $A^*_{st}(\mathcal{X})$ when $\mathcal{X} = [\widetilde{X}/G]$ for some smooth variety $\widetilde{X}$ and reductive abelian group G. This context generalizes that of [BCS05] and [JT10], which compute $A^*_{st}(\mathcal{X})_{\mathbb{Q}}$ and $A^*_{st}(\mathcal{X})$, respectively, when $\mathcal{X} = [(\mathbb{A}^n \setminus Z)/\mathbb{G}_m^k]$ for certain choices of closed subset $Z \subset \mathbb{A}^n$. We are particularly interested in the example of *weighted blowups*: if X is variety and $Y \subset X$ is a subvariety, a weighted blowup $\mathscr{B}\mathrm{l}_Y X$ of X along Y is a certain Deligne-Mumford stack with coarse space equal to the usual blowup $Bl_Y X$ (see 4.3 for a more detailed review).

4.1.1 Results

The results in this article are in two directions. The first is the problem of computing $A^*_{st}(\mathcal{X})$ when $\mathcal{X} = [\widetilde{X}/G]$ for some smooth variety $\widetilde{X}$ and reductive abelian group G. In Proposition 4.5 we give a formula for the obstruction class arising in the definition of the product on $A^*_{st}(\mathcal{X})$. Our formula is reminiscent of the formula in [BCS05, Proposition 6.3] for toric Deligne-Mumford stacks. We apply this formula in Section 4.4.4 to completely describe $A^*_{st}(\mathcal{X})$ when $\mathcal{X} = \mathscr{B}\mathrm{l}_Y X$ is a weighted blowup.

The second direction is the problem of determining when $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)$ is finitely generated over a more familiar ring. As a group, $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)$ decomposes as a direct sum of *sectors*

$$A^*_{st}(\mathscr{B}\mathrm{l}_Y X) = \bigoplus_{\zeta \in \mathbb{G}_m} A^*(\mathrm{I}(\zeta))e_\zeta$$

where $\mathrm{I}(\zeta)$ is a certain (often empty) substack of $\mathscr{B}\mathrm{l}_Y X$. A special example is $\mathrm{I}(1) = \mathscr{B}\mathrm{l}_Y X$. In fact $A^*(\mathrm{I}(1))e_1$ is a subring of $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)$, and a natural question is whether $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)$ is generated as an algebra over $A^*(\mathrm{I}(1))$ by the elements $1e_\zeta$. It turns out that this happens exactly when $A^*(X) \to A^*(Y)$ is surjective:

Theorem 4.1 (Theorem 4.13)**.** *The restriction $A^*(X) \to A^*(Y)$ is surjective if and only if the ring $\mathrm{A}^*_{st}(\mathcal{X})$ is generated as an algebra over $A^*(\mathrm{I}(1))$ by the elements $1e_\zeta$. In this case, $A^*_{st}(\mathcal{X})$ is a finitely generated algebra over $A^*(\mathcal{X})$ modulo explicit relations* (4.19)*.*

In the course of proving Theorem 4.1, we show in Lemma 4.11 and Corollary 4.12 that $A^*(Y)$ is equal to the restriction image of $A^*(X)$ (resp. finite as a module or finitely generated over the image) if and only if the chow group of the exceptional divisor is equal to the restriction image of $A^*(\mathscr{B}\mathrm{l}_Y X)$ (resp. finite or finitely generated over the image). As far as we know, this observation is new even for ordinary blowups.

Even when $A^*(X) \to A^*(Y)$ is not surjective, we define the subgroup $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)^{\mathrm{amb}}$ of $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)$ to be the set of all elements of the form $\alpha e_1 \star e_\zeta$. We prove the following:

Proposition 4.2 (Proposition 4.10)**.** *$A^*_{st}(\mathscr{B}\mathrm{l}_Y X)^{\mathrm{amb}}$ is a subring of $A^*_{st}(\mathscr{B}\mathrm{l}_Y X)$, equal to the $A^*(\mathscr{B}\mathrm{l}_Y X)$-subalgebra generated by the e_ζ.*

4.1.2 Further questions

In Proposition 4.5 we compute the obstruction classes for $\mathcal{X} = [\widetilde{X}/G]$ where G is a reductive abelian group. This is a significant step towards computing the ring $A^*_{st}(\mathcal{X})$ and it leads to several questions, of which a sampling follows.

Let $\mathcal{R}_\bullet$ be a sheaf of finitely generated $\mathbb{Z}^k$-graded algebras on a smooth scheme B, so $\mathrm{Spec}_B(\mathcal{R}_\bullet)$ has an action by $\mathbb{G}_m^k$. Let $\widetilde{X} \subset \mathrm{Spec}_B(\mathcal{R}_\bullet)$ be an open subscheme such that $\mathcal{X} := [\widetilde{X}/\mathbb{G}_m^k]$ is a smooth Deligne-Mumford stack.

Problem 4.1. Describe $A^*_{st}(\mathcal{X})$ as a graded algebra: give formulas for the twisted sectors, age grading, and product in terms of $\mathcal{R}_\bullet$.

One could investigate Problem 1 in general, for example in terms of the degrees of generators of $\mathcal{R}_\bullet$ (these degrees are the rays of the stacky fan when $\mathcal{X}$ is a toric Deligne-Mumford stack). On the other hand, there are families of examples where Problem 1 may have a particularly nice solution, for example weighted Grassmannians or weighted blowups of arbitrary $\mathbb{G}_m^k$ quotients. In both these general and specific contexts, one can also investigate finite generation properties of $A^*_{st}(\mathcal{X})$.

Problem 4.2. Define $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ in analogy with the case when $\mathcal{X} = \mathscr{B}\mathrm{l}_Y X$. Determine when $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ is subring of $A^*_{st}(\mathcal{X})$ and when it is equal to all of $A^*_{st}(\mathcal{X})$.

4.1.3 Conventions and notation

If X is a variety, and $\mathcal{I}_\bullet$ is a sheaf of graded $\mathcal{O}_X$-algebras, then as explained in [Sta18, 0EKJ] the scheme $\mathrm{Spec}_X(\mathcal{I}_\bullet)$ naturally has a $\mathbb{G}_m$ action and we write

$$\mathscr{P}\mathrm{roj}_X(\mathcal{I}_\bullet) := \left[\left(\mathrm{Spec}_X\left(\mathcal{I}_\bullet\right) \setminus V(\mathcal{I}_+)\right)/\mathbb{G}_m\right],$$

where $\mathcal{I}_+$ is the ideal of elements of positive degree. We note that the degree of an element $x \in \mathcal{I}_\bullet$ is the negative of its weight as a function on the $\mathbb{G}_m$-scheme $\mathrm{Spec}_X(\mathcal{I}_\bullet)$.

4.2 Stringy chow rings

If $\mathcal{X}$ is a smooth Deligne-Mumford stack, there is a moduli space $\mathcal{K}(\mathcal{X})$ of degree-zero stable maps to $\mathcal{X}$ from weighted $\mathbb{P}^1$ with three stacky points. If $\pi : \mathcal{C} \to \mathcal{K}(\mathcal{X})$ denotes the universal curve and $f : \mathcal{C} \to \mathcal{X}$ the universal stable map, then $R\pi_* f^* T\mathcal{X}$ is the virtual tangent complex of $\mathcal{K}(\mathcal{X})$. There are moreover evaluation maps $ev_1, ev_2, ev_3 : \mathcal{K}(\mathcal{X}) \to \mathrm{I}(\mathcal{X})$, where $\mathrm{I}(\mathcal{X})$ is the inertia stack of $\mathcal{X}$.[1]

Defined by Abramovich, Graber, and Vistoli in [AGV02], the stringy chow ring of $\mathcal{X}$ is a graded ring structure on $A^*(\mathrm{I}(\mathcal{X}))$ defined using $\mathcal{K}(\mathcal{X})$, its virtual tangent complex, and evaluation maps. We now recall the definition (Section 4.2.1) and give fairly explicit descriptions when $\mathcal{X} = [\widetilde{X}/G]$ is a global quotient of a smooth k-variety $\widetilde{X}$ by a reductive abelian k-group scheme G (Sections 4.2.2 and 4.2.4). These descriptions are similar to those in [BCS05] when $\mathcal{X}$ is a toric Deligne-Mumford stack and we use many of the arguments in that paper; however, our computation of the obstruction bundle in 4.2.4 is new at the level of generality presented here.

[1]The evaluation maps from $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ normally land in the rigidified inertia stack, but since the universal curve on $\mathcal{K}(\mathcal{X})$ is locally trivial (this is special to the genus zero, $n = 3$ situation), one gets the maps to $\mathrm{I}(\mathcal{X})$. See [AGV02, Lemma 6.2.1].

4.2.1 Stringy chow ring

The *age* of $\mathrm{I}(\mathcal{X})$ at a point $x \in \mathrm{I}(\zeta)$ is defined as follows. The tangent space $T_{\mathcal{X},\tilde{x}}$ is a d-dimensional representation of the cyclic subgroup $\langle\zeta\rangle$ of G generated by ζ. Let a be the order of this group, so $\langle\zeta\rangle \simeq \mu_a$, and let $\mathbb{Z}[x]/(x^a - 1)$ be the representation ring of μ_a where x corresponds to the representation of weight 1. If we write $T_{\mathcal{X},\tilde{x}} = \sum_{i=1}^{d} b_i x^i$ in the representation ring of μ_a, then the age at $x \in \mathrm{I}(\zeta)$ is

$$\mathrm{age}(x,\zeta) := \frac{1}{a}\sum_{i=1}^{d} b_i.$$

Age is a locally constant function on $\mathrm{I}(\zeta)$, so we may speak of the age of a connected component. The *stringy chow ring* has the graded $\mathbb{Z}$-module

$$A^*_{st}(\mathcal{X}) = \bigoplus_i A^{*-\mathrm{age}(\mathcal{I}(\mathcal{X})_i)}(\mathcal{I}(\mathcal{X})_i)$$

where the summation is over all connected components of $\mathrm{I}(\mathcal{X})$. The ring structure on $A^*_{st}(\mathcal{X})$ is defined by

$$\gamma_1 \star \gamma_2 = \overline{ev}_{3,*}(ev_1^*\gamma_1 \cdot ev_2^*\gamma_2 \cdot c_{top}(R^1\pi_* f^* T\mathcal{X})) \tag{4.1}$$

for $\gamma_1, \gamma_2 \in A^*(\mathrm{I}(\mathcal{X}))$, where $\overline{ev}_{3,*} : \mathcal{K}(\mathcal{X}) \to \mathrm{I}(\mathcal{X})$ is the composition of ev_3 with the involution on $\mathrm{I}(\mathcal{X})$ sending (x,ζ) to (x,ζ^{-1}) (here $x \in \mathcal{X}$ is an object and ζ is an automorphism of x). The sheaf $R^1\pi_* f^* T\mathcal{X}$ is called the *obstruction sheaf*.

4.2.2 Computation of the group $A^*_{st}(\mathcal{X})$

Assume $\mathcal{X} = [\widetilde{X}/G]$ is a global quotient of a smooth k-variety $\widetilde{X}$ by a reductive abelian k-group scheme G. To compute the inertia stack of $\mathcal{X}$, note that a point of $\mathrm{I}(\mathcal{X})$ can be written (x,ζ) where $x \in \widetilde{X}$ and $\zeta \in G$ fixes x. In fact, we have a decomposition

$$\mathrm{I}(\mathcal{X}) = \coprod_{\zeta \in G} \mathrm{I}(\zeta), \qquad \mathrm{I}(\zeta) := [\widetilde{X}^\zeta/G], \tag{4.2}$$

where $\widetilde{X}^\zeta$ is the fixed locus of ζ in $\widetilde{X}$ (this follows from [Sta18, 06PB]]). In particular the natural map $\mathrm{I}(\zeta) \to \mathcal{X}$ is a closed embedding. We call the open and closed substacks $\mathrm{I}(\zeta)$ and $\mathrm{II}(\zeta,\eta)$ *sectors*. The sector $\mathrm{I}(1)$ corresponding to the identity element is called *untwisted sector*, while the other sectors of $\mathrm{I}(\mathcal{X})$ are called *twisted sectors*. Note that $\mathrm{I}(1) \cong \mathcal{X}$.

To compute the age on $\mathrm{I}([\widetilde{X}/G])$, we use the following exact sequence, dual to the exact sequence of cotangent sheaves associated to $\widetilde{X} \to \mathcal{X}$:

$$0 \to \mathcal{O}^{\oplus r} \to T_{\widetilde{X}} \to T_{\mathcal{X}}|_{\widetilde{X}} \to 0.$$

Here r is the rank of G. Since G acts trivially on $\mathcal{O}^{\oplus r}$ the age can be computed using $T_{\widetilde{X}}$.

4.2.3 Computation of the product on $A^*_{st}(\mathcal{X})$

To compute the product we describe the moduli space and universal curve $\mathcal{C} \to \mathcal{K}(\mathcal{X})$ and evaluation maps $ev_i : \mathcal{K}(\mathcal{X}) \to \mathrm{I}(\mathcal{X})$ explicitly.

Let $\mathrm{II}(\mathcal{X})$ be the fiber product $\mathrm{I}(\mathcal{X}) \times_{\mathcal{X}} \mathrm{I}(\mathcal{X})$. It follows from (4.2) that we have a decomposition

$$\mathrm{II}(\mathcal{X}) = \coprod_{(\zeta,\eta)\in G\times G} \mathrm{II}(\zeta,\eta), \qquad \mathrm{II}(\zeta,\eta) := [\widetilde{X}^{\zeta} \cap \widetilde{X}^{\eta}/G], \tag{4.3}$$

and we call the $\mathrm{II}(\zeta,\eta)$ the *sectors* of $\mathrm{II}(\mathcal{X})$. Recall from [FG03] the following result.

Lemma 4.3. *Given a finite group H and elements $\zeta, \eta \in H$, there is a unique ramified H-cover $C_{\zeta,\eta} \to \mathbb{P}^1$ ramified over $0, 1,$ and ∞, such that for any $p \in C_{\zeta,\eta}$ in the fiber of 0 (resp. 1, ∞) we have $H_p = \langle\zeta\rangle$ (resp. $\langle\eta\rangle$, $\langle\eta^{-1}\zeta^{-1}\rangle$).*

Lemma 4.4. *The moduli space $\mathcal{K}(\mathcal{X})$ is isomorphic to $\mathrm{II}(\mathcal{X})$; in particular, these are smooth stacks. On a sector $\mathrm{II}(\zeta,\eta)$, the universal curve and morphism to $\mathcal{X}$ are given explicitly by*

$$\begin{array}{ccc} [\widetilde{X}^H/G] \times [C_{\zeta,\eta}/H] & \xrightarrow{f} & [\widetilde{X}/G] \\ \downarrow{\scriptstyle\pi} & & \\ [\widetilde{X}^H/G] & & \end{array}$$

where $H \subset G$ is the subgroup $\langle\zeta,\eta\rangle$ and $\widetilde{X}^H$ is the subvariety fixed by H. The map π is projection to the first factor while the map f is induced by the projection and inclusion $\widetilde{X}^H \times C_{\zeta,\eta} \to \widetilde{X}^H \to \widetilde{X}$ and the product homomorphism $G \times H \to G$. In particular, the (twisted) evaluation maps $ev_1, ev_2, \overline{ev}_3 : \mathcal{K}(\mathcal{X}) \to \mathcal{X}$ send $((x,\zeta),(x,\eta)) \in \mathrm{II}(\mathcal{X})$ to (x,ζ), (x,η), and $(x,\zeta\eta)$, respectively.

Lemma 4.4 enables explicit computation of most parts of the product (4.1). What remains is to compute the obstruction class $c_{top}(R^1\pi_* f^* T\mathcal{X})$; we do this in the next section.

4.2.4 Computation of the obstruction class

Assume $\mathcal{X} = [\widetilde{X}/G]$ is a global quotient of a smooth k-variety $\widetilde{X}$ by a reductive abelian k-group scheme G. If $z \in \mathbb{G}_m$, define $\arg z$ to be the unique value of the argument of z in $[0, 2\pi)$. The following generalizes [BCS05, Proposition 6.3].

Proposition 4.5. *The obstruction sheaf $R^1\pi_* f^* T_{\mathcal{X}}$ is a vector bundle on $\mathrm{II}(\mathcal{X})$. The top chern class of its restriction to the sector $\mathrm{II}(\zeta, \eta)$ is given by*

$$c_{top}(R^1\pi_* f^* T_{\mathcal{X}}) = \prod_{\arg Ai(\zeta) + \arg Ai(\eta) > 2\pi} c^G_{top}((N_{\widetilde{X}^H/\widetilde{X}})_{A^i})$$

where $H \subset G$ is the subgroup generated by ζ and η and $(N_{\widetilde{X}^H/\widetilde{X}})_{A^i}$ is the summand of the normal bundle $N_{\widetilde{X}^H/\widetilde{X}}$ of weight Ai.

Remark 4.1. Since $\mathcal{X}$ is Deligne-Mumford ζ and η have finite order and since G is abelian the subgroup they generate is also finite.

Proof. The first part of this proof follows the proof of [BCS05, Proposition 6.3]. The universal diagram in Lemma 4.4 can be extended to a diagram

$$\begin{array}{ccccccc}
\widetilde{X}^H \times C & \longrightarrow & \widetilde{X}^H \times [C/H] & \longrightarrow & [\widetilde{X}^H/G] \times [C/H] & \xrightarrow{f} & [\widetilde{X}/G] \\
& \searrow^{p} & \downarrow & & \downarrow{\scriptstyle \pi} & & \\
& & \widetilde{X}^H & \longrightarrow & [\widetilde{X}^H/G] & &
\end{array}$$

where the square is fibered and we write $C := C_{\zeta,\eta}$. Recall that the map f sends $(x, z) \in \widetilde{X}^H \times C$ to x and it sends $(\zeta, h) \in G \times H$ to ζh. It follows that if $\mathcal{F}$ is a bundle on $[\widetilde{X}^H/G] \times [C/H]$ corresponding to a $G \times H$-equivariant bundle F on $\widetilde{X}^H \times C$, then $R^i\pi_*\mathcal{F}$ corresponds to the G-equivariant bundle on $\widetilde{X}^H$ equal to $(R^i p_* F)^H$, where the superscript H means to take H-invariants.[2]

Now suppose $\mathcal{F} = f^*\mathcal{E}$, where $\mathcal{E}$ corresponds to a G-equivariant bundle E on $\widetilde{X}$. Then the induced bundle F on $\widetilde{X}^H \times C$ is $E|_{\widetilde{X}^H \times C}$ with $G \times H$ action induced by the product map $G \times H \to G$. From the projection formula (for the map p_*) and the above description of the functor $R^i\pi_*$, we have that

$$R^i\pi_* f^*\mathcal{E} \text{ corresponds to the } G\text{-bundle } (H^i(C, \mathcal{O}_C) \otimes E|_{\widetilde{X}^H})^H \text{ on } \widetilde{X}^H, \qquad (4.4)$$

[2] We are using here that H is reductive hence taking H-invariants is exact, so the derived functor of p_*-then-take-invariants is equal to the derived functor of p_*, followed by taking invariants.

where $E|_{\widetilde{X}^H}$ is the $G \times H$-equivariant bundle on $\widetilde{X}^H$ induced by the product map $G \times H \to G$.

Using this we can compute $R^1\pi_* f^* T_{\mathcal{X}}$ as follows. From the closed embedding of smooth stacks $\mathrm{II}(\zeta, \eta) \to \mathcal{X}$ we have the short exact sequence of vector bundles

$$0 \to T_{[\widetilde{X}^H/G]} \to T_{\mathcal{X}}|_{[\widetilde{X}^H/G]} \to N_{[\widetilde{X}^H/G]/\mathcal{X}} \to 0.$$

Recall from the explicit description of f that f factors through $[\widetilde{X}^H/G]$, where this sequence is supported, so it makes sense to apply f^*. Applying $R^\bullet \pi_* f^*$ we get a long exact sequence

$$\ldots \to R^1\pi_* f^* T_{[\widetilde{X}^H/G]} \to R^1\pi_* f^* T_{\mathcal{X}}|_{[\widetilde{X}^H/G]} \to R^1\pi_* f^* N_{[\widetilde{X}^H/G]/\mathcal{X}} \to R^2\pi_* f^* T_{[\widetilde{X}^H/G]} \to \ldots.$$

Using (4.4) one sees that $R^i\pi_* f^* T_{[\widetilde{X}^H/G]} = (H^i(C, \mathcal{O}_C) \otimes T_{\widetilde{X}^H})^H$, but $T_{\widetilde{X}^H}$ is H-invariant and the H-invariant part of $H^i(C, \mathcal{O}_C)$ vanishes for $i > 0$ by Proposition 4.6 below, so these groups vanish. Using (4.4) again it follows that

$$R^1\pi_* f^* T_{\mathcal{X}} = R^1\pi_* f^* N_{[\widetilde{X}^H/G]/\mathcal{X}} = (H^1(C, \mathcal{O}_C) \otimes N_{\widetilde{X}^H/\widetilde{X}})^H \tag{4.5}$$

is a formula for the G-equivariant bundle on $\widetilde{X}^H$ corresponding to the obstruction sheaf on $\mathrm{II}(\zeta, \eta) = [\widetilde{X}^H/G]$.

It remains to compute $H^1(C, \mathcal{O}_C)$. Let $M(H)$ denote the character group of H and $\mathbb{C}_A i$ be the one-dimensional representation corresponding to the character Ai.

Proposition 4.6. *In the representation ring of H, we have*

$$H^1(C, \mathcal{O}_C) = \sum_{Ai \in M(H) \setminus \{0\}} f(\zeta, \eta, -Ai)\mathbb{C}_A i,$$

where

$$f(\zeta, \eta, Ai) := (2\pi)^{-1}\big(\arg Ai(\zeta) + \arg Ai(\eta) + \arg Ai(\zeta\eta)^{-1}\big) - 1.$$

Proof. The key ingredient is Theorem 3.2, the equivariant Riemann-Roch formula valued in the K-theory of H-representations that we developed in chapter 3. We have

$$Ai(C, \mathcal{O}_C, H) = \mathbb{C}[H] - \frac{1}{|H|} \sum_{p \in C} \sum_{d=0}^{e_p - 1} d \operatorname{Ind}_{H_p}^H (T_{C,p}^\vee)^d \tag{4.6}$$

where $Ai(C, \mathcal{O}_C, H)$ is $H^0(C, \mathcal{O}_C) - H^1(C, \mathcal{O}_C)$ as a virtual H-representation, $\mathbb{C}[H] = \sum_{Ai \in M(H)} \mathbb{C}_A i$ is the regular representation, H_p is the isotropy group at $p \in C$. Since $H^0(C, \mathcal{O}_C)$ is the trivial module, we have

$$H^1(C, \mathcal{O}_C) = \frac{1}{|H|} \sum_{p \in C} \sum_{d=0}^{e_p - 1} d \operatorname{Ind}_{H_p}^{H} (T_{C,p}^{\vee})^d - \sum_{Ai \in M(H) \setminus \{0\}} \mathbb{C}_A i. \tag{4.7}$$

The first sum is in fact finite since the only nonzero contributions come from the fibers over the three branch points of $H \to \mathbb{P}^1$. These branch points correspond to the elements ζ, η, and $(\zeta\eta)^{-1}$ of H, respectively: for example, ζ defines the homomorphism to H from the stabilizer $\mu_{|\zeta|}$ at the corresponding branch point in $[C/H] = w\mathbb{P}^1$.

We compute the contribution to (4.7) from the branch point corresponding to ζ (the contributions of the other two branch points are analogous). The group H acts transitively on the fiber of the branch point, with stabilizer $H_p = \langle \zeta \rangle$, so the cardinality of the fiber is $|H|/|\zeta|$. Hence the contribution of this branch point to (4.7) is

$$\frac{1}{|\zeta|} \sum_{d=0}^{|\zeta|-1} d \operatorname{Ind}_{\langle g \rangle}^{H} (T_{C,p}^{\vee})^d. \tag{4.8}$$

Now to compute the coefficient of $\mathbb{C}_A i$ in (4.8), note that $\operatorname{Ind}_{\langle g \rangle}^{H} (T_{C,p}^{\vee})^d$ is equal to $\mathbb{C}_A i$ for some j. We will show that given Ai, there is at most one $d \in \{0, \ldots, |\zeta| - 1\}$ corresponding to this representation, and we will compute $d/|\zeta|$ which will be the coefficient of $\mathbb{C}_A i$ in (4.8).

It remains to compute the coefficient of $\mathbb{C}_A i$ in (4.8). Since C is constructed from local monodromy data associated to ζ, η and $(\zeta\eta)^{-1}$, the action of ζ on $T_{C,p}$ is via multiplication by $\exp(2\pi i/|\zeta|)$. By Frobenius reciprocity, the multiplicity of $\mathbb{C}_A i$ in $\operatorname{Ind}_{\langle g \rangle}^{H} (T_{C,p}^{\vee})^d$ is the same as that of $\operatorname{Res}_{\langle g \rangle}^{H} Ai$ in $(T_{C,p}^{\vee})^d$, which is 1 if

$$Ai(\zeta) = e^{-2\pi i(d/|\zeta|)}$$

and 0 otherwise. There is a unique d, characterised by $2\pi d/|\zeta| = \arg(Ai(\zeta^{-1}))$, satisfying this condition. Thus

$$\frac{1}{|\zeta|} \sum_{d=0}^{|\zeta|-1} d \operatorname{Ind}_{\langle g \rangle}^{H} (T_{C,p}^{\vee})^d = \sum_{A} i \frac{1}{2\pi} \arg(Ai(\zeta^{-1})) \mathbb{C}_A i. \tag{4.9}$$

In particular the coefficient of the trivial representation in this contribution is zero.

Identical computations for the branch points corresponding to η and $(\zeta\eta)^{-1}$ finish the proof of the lemma. □

To finish the proof of Proposition 4.5, note that from (4.5) and Proposition 4.6 we get a formula

$$c_{top}(R^1\pi_*f^*T_{\mathcal{X}}) = \prod_{Ai\in M(H),\ Ai\neq 0} c^G_{top}((N_{\widetilde{X}^H/\widetilde{X}})_A i)^{f(\zeta,\eta,Ai)}.$$

But when Ai is not zero the value of $f(\zeta,\eta,Ai)$ is either 0 or 1, and it is 1 precisely when $\arg Ai(\zeta)+\arg Ai(\eta)>2\pi$. $\square$

Example 4.1. If $G=\mathbb{G}_m$, then the group H considered above will be equal to $\mu_N\subset G$ for some positive integer N. In this case the characters $M(H)$ are given by $Ai_j(x)=x^j$ for $j=0,\dots,N-1$, and we have

$$f(\zeta,\eta,Ai_j) = (1/2\pi)\Big(\arg(\zeta^{-j})+\arg(\eta^{-j})+\arg((\zeta\eta)^j)\Big)-1.$$

4.3 Weighted blow-ups

In this section we recall material about weighted blow-ups and their (classical) chow rings. Everything in this section is either stated or implicit in [QR] and [AOA23]

Let X be a smooth variety over a field of characteristic zero, let $I_1,\dots,I_m$ be ideal sheaves on X, and let $a_1,\dots,a_m$ be positive integers. Let $\mathcal{I}_\bullet$ be the smallest Rees algebra containing I_i in degree a_i (see [QR, Definition 3.1.5]). In the notation of loc. cit. we have

$$\mathcal{I}_\bullet := (I_1,a_1)+\cdots+(I_m,a_m).$$

In particular, $\mathcal{I}_\bullet$ is a sheaf of graded $\mathcal{O}_X$-algebras, so its relative spectrum over X has a $\mathbb{G}_m$-action. We define the stack $\mathcal{X}$ to be the weighted blowup of X along $\mathcal{I}_\bullet$:

$$\mathcal{X} := \mathscr{B}\mathrm{l}_{\mathcal{I}_\bullet}(X) = \mathscr{P}\mathrm{roj}_X(\mathcal{I}_\bullet).$$

Let $Y\subset X$ be the closed subvariety defined by $\mathcal{I}_1$, and let $\mathcal{Y}\subset\mathcal{X}$ be the divisor defined by the ideal sheaf $\mathcal{I}_{\bullet+1}\hookrightarrow\mathcal{I}_\bullet$, so $\mathcal{Y}=\mathscr{P}\mathrm{roj}_X(\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1})$. Then $\mathcal{Y}$ is the exceptional divisor of the blowup and we have a commuting diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{j} & \mathcal{X} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X. \end{array} \tag{4.10}$$

We moreover make the following regularity assumption on $\mathcal{I}_\bullet$:

Assumption 4.1. Each $V(I_k) \to X$ is a regular immersion, and $\mathcal{I}_\bullet$ defines a quasi-regular weighted closed immersion (see [QR, Definition 5.1.3]).

The key consequence of Assumption 4.1 is the following (both parts of the statement are used in the proof).

Lemma 4.7. *The sum $\oplus_{k=1}^m I_k/I_k^2$ is a graded locally free sheaf on Y where I_k/I_k^2 has degree a_k. The exceptional divisor $\mathcal{Y}$ is isomorphic to a weighted projective bundle on Y:*

$$\mathcal{Y} = \mathscr{P}\mathrm{roj}_Y \left(\mathrm{Sym}^\bullet_{\mathcal{O}_Y} \bigoplus_{k=1}^m I_k/I_k^2 \right).$$

Proof. Since $V(I_k) \to X$ is regular I_k/I_k^2 is locally free on $V(I_k)$, hence on Y. Since $\mathcal{Y} = \mathscr{P}\mathrm{roj}_X(\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1})$, to finish the proof it is enough to demonstrate an isomorphism of $\mathcal{O}_Y$-algebras

$$\mathrm{Sym}^\bullet_{\mathcal{O}_Y} \bigoplus_{k=1}^m I_k/I_k^2 \to \oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}. \tag{4.11}$$

To define (4.11) it is enough to define a $\mathbb{G}_m$-graded $\mathcal{O}_Y$-module homomorphism

$$\bigoplus_{k=1}^m I_k/I_k^2 \to \bigoplus_{n\geq 0} \mathcal{I}_n/\mathcal{I}_{n+1}.$$

But $\mathcal{I}_{a_k}$ contains I_k and $\mathcal{I}_{a_k+1}$ contains I_k^2 by definition of $\mathcal{I}$, so we can map I_k to its natural image in $\mathcal{I}_{a_k}$. To show that (4.11) is an isomorphism it is enough to show that this holds locally, where (4.11) reduces to the morphism α in [QR, Section 5.1]. This is an isomorphism by Assumption 4.1 and [QR, Definition 5.1.1]. □

Since $\mathcal{Y}$ is a weighted projective bundle, from [AOA23, Theorem 3.12] we get a formula for the chow ring of $\mathcal{Y}$:

$$A^*(\mathcal{Y}) = A^*(Y)[t]/P(t) \tag{4.12}$$

where $A^*(Y)[t]$ is identified with $A^*_{\mathbb{G}_m}(Y)$ and t is the first chern class of the topologically trivial line bundle with $\mathbb{G}_m$-weight 1 (see [AOA23, Example 2.2]). Moreover $P(t)$ is the $\mathbb{G}_m$-equivariant top chern class of the locally free sheaf $\oplus(I_k/I_k^2)^\vee$, where $\mathbb{G}_m$ acts on Y trivially and on $(I_k/I_k^2)^\vee$ with *weight* a_k.[3] It follows (see e.g. [QR, Remark 3.2.4]) that

$$t \text{ is the first chern class of } N^\vee_{\mathcal{Y}/\mathcal{X}}. \tag{4.13}$$

[3]The graded sheaf $\oplus I_k/I_k^2$ contains I_k/I_k^2 in *degree* a_k, hence as a graded sheaf $\oplus(I_k/I_k^2)^\vee$ contains $(I_k/I_k^2)^\vee$ in *degree* $-a_k$. Since the degrees of a graded module are dual to the weights of the associated $\mathbb{G}_m$ representation, we see that the weight of $(I_k/I_k^2)^\vee$ is a_k.

From [AOA23, Theorem 6.4] we get the following formula for the chow ring of $\mathcal{X}$:

$$A^*(\mathcal{X}) = A^*(Y)[t] \cdot t \oplus A^*(X)/\langle ((P(t) - P(0))\alpha, -i_*\alpha)\rangle_{\alpha \in A^*(Y)}. \tag{4.14}$$

Here, the ring structure on $A^*(Y)[t] \cdot t \oplus A^*(X)$ is given by the rule

$$(q_1(t), \beta_1) \cdot (q_2(t), \beta_2) = (q_1(t)q_2(t) + q_1(t)i^*\beta_2 + q_2(t)i^*\beta_1,\ \beta_1\beta_2)$$

and $-t$ is equal to the fundamental class of $[\mathcal{Y}]$. The identification in (4.14) is that $(q(t), \beta)$ maps to $-j_*[q_1(t)] + f^*\beta$, where $tq_1(t) = q(t)$ and $[q_1(t)]$ is the element of $A^*(\mathcal{Y})$ defined by the polynomial $q_1(t)$ via the isomorphism (4.12).

Lemma 4.8. *The restriction map $j^* : A^*(\mathcal{X}) \to A^*(\mathcal{Y})$ sends $(q(t), \beta)$ to the class $[q(t) + i^*\beta]$ in the quotient ring* (4.12)*, where $q(t) + i^*\beta$ is viewed as a polynomial in t with coefficients in $A^*(Y)$.*

Proof. This follows from commutativity of (4.10) and the fact that by the self intersection formula and (4.13),

$$-j^*j_*[q_1(t)] = -[q_1(t)] \cdot c_1(N_{\mathcal{Y}/\mathcal{X}}) = [tq_1(t)].$$

Let $\widetilde{Y} \subset \widetilde{X}$ be the preimage of $\mathcal{Y} \subset \mathcal{X}$. We point out that this computation may equivalently be done in the Chow ring $A^*_{\mathbb{G}_m}(\widetilde{Y})$, after replacing j by its lift $\widetilde{j} : \widetilde{Y} \hookrightarrow \widetilde{X}$. □

4.4 Stringy chow ring of a weighted blow-up

We compute each of the ingredients in Section 4.2 for the weighted blowup $\mathcal{X}$.

4.4.1 Sectors

For $\zeta \in \mathbb{G}_m$, let $\mathrm{I}(\zeta)$ (resp. $\mathrm{II}(\zeta, \eta)$) denote the corresponding sector of $\mathrm{I}(\mathcal{X})$ (resp. $\mathrm{II}(\mathcal{X})$). If ζ and η are the identity then $\mathrm{I}(\zeta) = \mathrm{II}(\zeta, \eta) = \mathcal{X}$. If $\zeta \neq 1$ then $\mathrm{I}(\zeta)$ and $\mathrm{II}(\zeta, \eta)$ are isomorphic to closed substacks of $\mathcal{Y}$, and it follows from Lemma 4.7 that these sectors are also weighted projective bundles over Y:

$$\mathrm{I}(\zeta) = \mathscr{P}\mathrm{roj}_Y \left(\mathrm{Sym}^\bullet_{\mathcal{O}_Y} \bigoplus_{\zeta \in \mu_{a_k}} I_k/I_k^2 \right), \qquad \mathrm{II}(\zeta, \eta) = \mathscr{P}\mathrm{roj}_Y \left(\mathrm{Sym}^\bullet_{\mathcal{O}_Y} \bigoplus_{\zeta, \eta \in \mu_k} I_k/I_k^2 \right).$$

Here the sums are taken over $k \in \{1, \ldots, m\}$ satisfying the displayed conditions; e.g., the second sum is over all k such that μ_k contains both ζ_1 and ζ_2. Notice that $\mathrm{I}(\zeta)$ is nonempty if and only if $\zeta^{a_k} = 1$ for one of the weights a_k of the Rees algebra $\mathcal{I}_\bullet$. When $\zeta \neq 1$ we have the following formulas for the normal bundles of the closed embeddings $\mathrm{I}(\zeta) \to \mathcal{Y}$ and $\mathrm{II}(\zeta, \eta) \to \mathcal{Y}$:

$$N_{\mathrm{I}(\zeta)/\mathcal{Y}} = \bigoplus_{\zeta \notin \mu_{a_k}} (I_k/I_k^2)^\vee, \qquad N_{\mathrm{II}(\zeta,\eta)/\mathcal{Y}} = \bigoplus_{\zeta \text{ or } \eta \notin \mu_k} (I_k/I_k^2)^\vee. \tag{4.15}$$

These are in fact weight decompositions of equivariant sheaves as $(I_k/I_k^2)^\vee$ has pure $\mathbb{G}_m$-weight a_k. We note that $(I_k/I_k^2)^\vee$ is most naturally a bundle on $V(I_k) \subset X$, but by restriction can be viewed as a bundle on Y and even on $\mathcal{Y}$ by pullback along $\mathcal{Y} \to Y$.

We define $\mathbf{e}_k$ to be the equivariant euler class in $A^*_{\mathbb{G}_m}(Y) \simeq A^*(Y)[t]$ of the bundle $(I_k/I_k^2)^\vee$ of pure weight a_k and rank r_k; i.e.,

$$\mathbf{e}_k := c_{r_k}\left(\left(I_k/I_k^2\right)^\vee\right) + a_k t c_{r_k - 1}\left(\left(I_k/I_k^2\right)^\vee\right) + \cdots + a_k^{r_k} t^{r_k}$$

where $t \in A^*_{\mathbb{G}_m}(Y)$ is as in [AOA23, Example 2.2]. It follows from [AOA23, Theorem 3.12] that for $\zeta \neq 1$ we have ring isomorphisms

$$A^*(\mathrm{I}(\zeta)) = \frac{A^*(Y)[t]}{\prod_{\zeta \in \mu_{a_k}} \mathbf{e}_k}; \qquad A^*(\mathrm{II}(\zeta)) = \frac{A^*(Y)[t]}{\prod_{\zeta, \eta \in \mu_{a_k}} \mathbf{e}_k}.$$

4.4.2 Age

The age at any point of $\mathrm{I}(1)$ is zero. To compute the age at a point of $\mathrm{I}(\zeta)$ for some $\zeta \neq 1$, use the short exact sequence

$$0 \longrightarrow T_{\mathrm{I}(\zeta)} \longrightarrow T_{\mathcal{X}}|_{\mathrm{I}(\zeta)} \longrightarrow N_{\mathrm{I}(\zeta)/\mathcal{X}} \longrightarrow 0.$$

Since the action of ζ on a fiber of $T_{\mathrm{I}(\zeta)}$ is trivial, it follows that it suffices to consider the action of ζ on a fiber of $N_{\mathrm{I}(\zeta)/\mathcal{X}}$. From the inclusions $\mathrm{I}(\zeta) \subset \mathcal{Y} \subset \mathcal{X}$ we have the short exact sequence

$$0 \longrightarrow N_{\mathrm{I}(\zeta)/\mathcal{Y}} \longrightarrow N_{\mathrm{I}(\zeta)/\mathcal{X}} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}|_{\mathrm{I}(\zeta)} \longrightarrow 0.$$

By (4.13) the bundle $N_{\mathcal{Y}/\mathcal{X}}|_{I(\zeta)}$ has $\mathbb{G}_m$-weight -1. From this and the weight decomposition (4.15) of $N_{\mathrm{I}(\zeta)/\mathcal{Y}}$ it follows that the age at any point of $I(\zeta)$ is

$$\mathrm{age}(\zeta) = \sum_{\zeta \notin \mu_{a_k}} \frac{\arg \zeta^{a_k}}{2\pi} r_k + \frac{\arg \zeta^{-1}}{2\pi}, \tag{4.16}$$

where for $z \in \mathbb{C}^\times$, $\arg z$ is the unique value of the argument of z in the interval $[0, 2\pi)$.

4.4.3 Product

Let $a_0 = -1$, let μ_{a_0} be the trivial group, and let $\mathbf{e}_0 = -t$. Then we may express the product on $A^*_{st}(\mathcal{X})$ as follows.

Proposition 4.9. *Let $\alpha \in A^*(\mathrm{I}(\zeta))$ and let $\beta \in A^*(\mathrm{I}(\eta))$. Then*

$$\alpha e_\zeta \star \beta e_\eta = \alpha\beta C_{\zeta\eta}\, e_{\zeta\eta} \qquad \textit{where} \quad C_{\zeta\eta} = \left(\prod_{\substack{\zeta \textit{ or } \eta \notin \mu_{a_k} \\ \arg \zeta^{a_k} + \arg \eta^{a_k} > 2\pi}} \mathbf{e}_k \right) \left(\prod_{\substack{\zeta \textit{ or } \eta \notin \mu_{a_k} \\ \zeta\eta \in \mu_{a_k}}} \mathbf{e}_k \right)$$

and $e_{\zeta\eta}$ is defined to be zero if $\mathrm{I}(\zeta\eta)$ is empty (i.e., if $(\zeta\eta)^{a_k} \neq 1$ for all $k = 1, \ldots, m$). In the definition of $\star$ we have used an implicit restriction of α and β to $\mathrm{I}(\zeta\eta)$.

Proof. The first factor in $C_{\zeta\eta}$ is the obstruction class $c_{top}(R^1\pi_* f^* T_{\mathcal{X}})$, and the second is c_{top} of the normal bundle to the inclusion $\overline{ev}_3 : \mathrm{II}(\zeta, \eta) \to \mathrm{I}(\zeta\eta)$. The proposition holds if $\zeta = \eta = 1$ because e_1 is the identity for A^*_{st}, so from now on we assume $\zeta \neq 1$.

To compute the obstruction class we use Proposition 4.5. Let $\widetilde{Y} \subset \widetilde{X}$ be the preimage of $\mathcal{Y} \subset \mathcal{X}$. The inclusions $\mathrm{II}(\zeta, \eta) \to \mathcal{Y} \to \mathcal{X}$ lift to inclusions $\widetilde{X}^H \to \widetilde{Y} \to \widetilde{X}$ which yield a short exact sequence

$$0 \to N_{\widetilde{X}^H/\widetilde{Y}} \to N_{\widetilde{X}^H/\widetilde{X}} \to N_{\widetilde{Y}/\widetilde{X}}|_{\widetilde{X}^H} \to 0. \tag{4.17}$$

It follows that for a character $Ai(x) = x^j$ of H, we have

$$c_{top}((N_{\widetilde{X}^H/\widetilde{X}})_A i) = c_{top}((N_{\widetilde{X}^H/\widetilde{Y}})_{Ai}) c_{top}((N_{\widetilde{Y}/\widetilde{X}})_A i).$$

Now the formula for the obstruction class follows from the weight decomposition (4.15) for $N_{\mathrm{II}(\zeta,\eta)/\mathcal{X}} = N_{\widetilde{X}^H/\widetilde{X}}$ and from the fact that $N_{\widetilde{Y}/\widetilde{X}} = N_{\mathcal{Y}/\mathcal{X}}$ has pure weight 1.

To compute c_{top} of the normal bundle to $\mathrm{II}(\zeta, \eta) \to \mathrm{I}(\zeta\eta)$ there are two cases. If $\zeta\eta \neq 1$ then $\mathrm{I}(\zeta\eta) \subset \mathcal{Y}$ and we have

$$N_{\mathrm{II}(\zeta,\eta)/\mathrm{I}(\zeta\eta)} = \bigoplus_{\zeta \text{ or } \eta \notin \mu_{a_k},\, \zeta\eta \in \mu_k} (I_k/I_k^2)^\vee.$$

Otherwise, $\mathrm{I}(\zeta\eta) = \mathcal{X}$ and we use the short exact sequence (4.17). $\square$

4.4.4 Summary

We have an isomorphism of $\mathbb{Z}$-modules

$$A^*_{st}(\mathcal{X}) = A^*(\mathcal{X})e_1 \oplus \left(\bigoplus_{\zeta \neq 1,\, \zeta^{a_k}=1} \frac{A^*(Y)[t]}{\prod_{\zeta \in \mu_{a_k}} \mathbf{e}_k} e_\zeta \right)$$

where the second sum is over all nontrivial ζ such that $\zeta^{a_k} = 1$ for some $k = 1, \ldots, m$ and the degree of αe_ζ is $\deg(\alpha) + \operatorname{age}(\zeta)$ (note $\deg(t) = 1$, and $\operatorname{age}(\zeta)$ is given in (4.16)). The product $\star$ is given by the rule in Proposition 4.9.

4.5 Finite generation

The ring $A^*_{st}(\mathcal{X})$ is naturally an algebra over $A^*(\mathcal{X})$ via the action of the untwisted sector, and $A^*(\mathcal{X})$ is in turn an algebra over $A^*(X)$ via the pullback f^*. In this section we ask the question, when is $A^*_{st}(\mathcal{X})$ finitely generated over $A^*(\mathcal{X})$ or $A^*(X)$?

We first find a subring of $A^*_{st}(\mathcal{X})$ that is always finitely generated over $A^*(\mathcal{X})$.

Definition 4.1. Let $A^*(\mathrm{I}(\zeta))^{\mathrm{amb}}$ denote the image of the restriction map $A^*(\mathcal{X}) \to A^*(\mathrm{I}(\zeta))$. The *ambient* classes of $A^*_{st}(\mathcal{X})$ are the elements of the subgroup

$$A^*_{st}(\mathcal{X})^{\mathrm{amb}} := \oplus_{\zeta \in \mathbb{G}_m} A^*(\mathrm{I}(\zeta))^{\mathrm{amb}}.$$

Proposition 4.10. *The group $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ is a subring of $A^*_{st}(\mathcal{X})$ and a finitely generated algebra over $A^*(\mathcal{X})$. Explicitly, we have*

$$A^*_{st}(\mathcal{X})^{\mathrm{amb}} = A^*(\mathcal{X})[e_\zeta]_{\zeta \in Z}/I$$

where $Z = \{\zeta \in \mathbb{G}_m \mid \zeta^{a_k} = 1$ for some $k = 1, \ldots, m\}$ and I is the ideal generated by elements of the forms:

1. *$e_\zeta e_\eta - C_{\zeta\eta} e_{\zeta\eta}$ for $\zeta, \eta \in Z$;*
2. *αe_ζ for $\zeta \in Z$ and $\alpha \in \ker(A^*(\mathcal{X}) \to A^*(\mathrm{I}(\zeta))$.*

Proof. To show that $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ is closed under multiplication, by Proposition 4.9 we only need to show that the elements $C_{\zeta\eta} \in \mathrm{I}(\zeta\eta)$ are restrictions of classes in $A^*(\mathcal{X})$. The factors of $C_{\zeta\eta}$ are equal to $\mathbf{e}_0 = -t$ and to $\mathbf{e}_k$ where

$$\mathbf{e}_k = c_{r_k}\left(\left(I_k/I_k^2\right)^\vee \right) + a_k t c_{r_k - 1}\left(\left(I_k/I_k^2\right)^\vee \right) + \cdots + a_k^{r_k} t^{r_k} = c_{r_k}(N_{V(I_k)/X}|_Y) + tq(t)$$

for some $q(t) \in A^*(Y)[t]$, where $V(I_k)$ is the subvariety of X defined by the ideal I_k. By self-intersection formula,

$$c_{r_k}(N_{V(I_k)/X}|_Y) = c_{r_k}(N_{V(I_k)/X})|_Y = i^*[V(I_k)]$$

is in the image of i. It follows that $\mathbf{e}_k$ is ambient. The class $\mathbf{e}_0 = -t$ is clearly ambient.

Next, the ring map

$$A^*(\mathcal{X})[e_\zeta]_{\zeta \in Z} \to A^*_{st}(\mathcal{X})^{\mathrm{amb}}$$

sending e_ζ to the fundamental class of $I(\zeta)$ is clearly surjective and contains I in its kernel. We show that the kernel is contained in I. Let α be an element of the kernel. Write $\alpha = \sum_{J \in \mathbb{N}^Z} \alpha_J m^J$ where $\alpha_J \in A^*(\mathcal{X})$ and $m^J = \prod_{\zeta \in Z} e_\zeta^{J_\zeta}$. Modulo I we may write α as $\sum_{\zeta \in Z} \alpha'_\zeta e_\zeta$ for some $\alpha'_\zeta \in A^*(\mathcal{X})$—this uses that $C_{\zeta\eta}$ is ambient. But since $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ is a direct sum and α is zero in here, we must have that α'_ζ is in the kernel of $A^*(\mathcal{X}) \to A^*(\mathrm{I}(\zeta))$. So α is zero modulo I. □

The next question is when $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ is equal to $A^*_{st}(\mathcal{X})$. To answer this question we will need a lemma, which may be of independent interest. Let $Y' \subset X$ be a smooth closed subvariety containing Y, so the normal bundle $N' := N_{Y'/X}$ is a subbundle of $N_{Y/X}$. We have a natural closed immersion $\mathbb{P}_Y(N') \to \mathcal{Y}$. Consider the composition of restriction maps

$$A^*(\mathcal{X}) \to A^*(\mathcal{Y}) \to A^*(\mathbb{P}_Y(N')) \tag{4.18}$$

Lemma 4.11. *The composition* (4.18) *is surjective if and only if $i^* : A^*(X) \to A^*(Y)$ is surjective. In particular, taking $N' = N$, we have that $j^* : A^*(\mathcal{X}) \to A^*(\mathcal{Y})$ is surjective if and only if i^* is surjective.*

Proof. The chow rings and restriction maps in (4.18) may be written explicitly as

$$\begin{array}{ccccc} \frac{A^*(Y)[t]\cdot t \oplus A^*(X)}{J} & \longrightarrow & \frac{A^*(Y)[t]}{c_{top}^{\mathbb{G}_m}(N_{Y/X})} & \longrightarrow & \frac{A^*(Y)[t]}{c_{top}^{\mathbb{G}_m}(N')} \\ (q(t), \beta) & \longmapsto & [q(t) + i^*\beta] & \longmapsto & [q(t) + i^*\beta] \end{array}$$

where J is the ideal of relations described in (4.14). From this it is clear that if i^* is surjective, so is (4.18). Conversely suppose (4.18) is surjective and let $\alpha \in A^*(Y)$. Let $g' : \mathbb{P}_Y(N') \to Y$ be the natural map. Then we can find $\beta \in A^*(X)$ and $q(t) \in A^*(Y)[t] \cdot t$ such that $g'^*(\alpha)$ is (4.18) applied to $(q(t), \beta)$; i.e.,

$$g'^*(\alpha) = [q(t) + i^*\beta] \in A^*(Y)[t]/c_{top}^{\mathbb{G}_m}(N').$$

If we write $Q(t) = c_{top}^{\mathbb{G}_m}(N')$, then we may write

$$\alpha - i^*\beta - q(t) = s(t)Q(t)$$

for some $s(t) \in A^*(Y)[t]$. Equating constant terms, we get

$$\alpha = i^*\beta + c_{top}(N')s(0) \in A^*(Y).$$

We have shown that for arbitrary $\alpha \in A^*(Y)$ we can write $\alpha = i^*\beta + c_{top}(N')\alpha'$ for some $\beta \in A^*(X)$ and $\alpha' \in A^*(Y)$. Note that $c_{top}(N') = i^*[Y']$ is in the image of i^*. Hence, by recursively applying this decomposition to α', we can show for that for any $k > 0$ we can write

$$\alpha = \gamma + c_{top}(N')^k\alpha'$$

where γ is in the image of i^*. Since $c_{top}(N')$ is nilpotent we have that α is in the image of i^*. □

Corollary 4.12. *$i^* : A^*(X) \to A^*(Y)$ is finite (resp. of finite type) if and only if $j^* : A^*(\mathcal{X}) \to A^*(\mathcal{Y})$ is finite (resp. of finite type).*

Proof. Let $p_1(t), \ldots, p_n(t) \in A^*(Y)[t]$ be finitely many polynomials such that the $[p_i(t)] \in A^*(\mathcal{Y})$ generate $A^*(\mathcal{Y})$ as a module (resp. algebra) over $A^*(\mathcal{X})$. Then the natural extension of the proof of Lemma 4.11 shows that the constant terms $p_1(0), \ldots, p_n(0)$ generate $A^*(Y)$ as a module (resp. algebra) over $A^*(X)$. □

Theorem 4.13. *Assume at least one of the weights a_k is not 1. Then the following are equivalent:*

1. *$i^* : A^*(X) \to A^*(Y)$ is surjective*
2. *$A^*_{st}(\mathcal{X})^{\mathrm{amb}}$ is equal to $A^*_{st}(\mathcal{X})$*
3. *$A^*_{st}(\mathcal{X})$ is generated as an algebra over $A^*(\mathcal{X})$ by $\{e_\zeta\}_{\zeta\in Z}$.*

In this case, $A^*_{st}(\mathcal{X})$ is generated as an algebra over $A^*(X)$ by $\{e_\zeta\}_{\zeta\in Z}$ and t, and we can write

$$A^*_{st}(\mathcal{X}) \cong \frac{A^*(X)[t, e_\zeta]_{\zeta\in Z}}{\left(t\cdot\ker i^*, \prod_{k=1}^m \mathbf{e}_k - \prod_{k=1}^m c_{r_k}\left((I_k/I_k^2)^\vee\right) + [Y], e_\zeta\cdot\ker i^*, e_\zeta\cdot\prod_{\zeta\in\mu_{a_k}}\mathbf{e}_k, e_\zeta e_\eta - C_{\zeta\eta}e_{\zeta\eta}\right)}. \tag{4.19}$$

Proof. Let a_k be a degree different from 1 and let $\zeta = \exp(2\pi i/a_k)$. Then from Section 4.4.1 we have that $\mathrm{I}(\zeta) \subset \mathcal{Y}$ is nonempty. It follows from Lemma 4.11 that $A^*(X) \to A^*(Y)$ is surjective if and only if $A^*(\mathcal{X}) \to A^*(\mathrm{I}(\zeta))$ is surjective, and it follows that (1) and (2) are equivalent.

Statements (2) and (3) are equivalent by the presentation in Proposition 4.10 for $A^*_{st}(\mathcal{X})^{\mathrm{amb}}$.

For the explicit presentation, we first apply [AOA23, Corollary 6.5] to write

$$A^*(\mathcal{X}) = \frac{A^*(X)[t]}{(t \cdot \ker i^*, \prod_{k=1}^m \mathbf{e}_k - \prod_{k=1}^m c_{r_k}\left((I_k/I_k^2)^\vee\right) + [Y])},$$

where every $\alpha \in A^*(Y)$ is identified with any $\beta \in A^*(X)$ such that $i^*\beta = \alpha$. With this identification, the kernel of the composition

$$A^*(\mathcal{X}) \to A^*(\mathcal{Y}) \to A^*(I(\zeta)) \tag{4.20}$$

is generated by $e_\zeta \cdot \ker i^*$ and $e_\zeta \cdot \prod_{\zeta \in \mu_{a_k}} \mathbf{e}_k$. The result follows by applying Proposition 4.10. □

4.6 Examples

When the blow-up centre has a single weight, the stringy ring is particularly simple owing to the simple geometry of the inertia stack. Let the weighted ideal be $(\mathcal{I}, d)$. Then a non-identity element $\zeta \in \mu_\infty$ either fixes the exceptional divisor or has no fixed point. Thus the inertia stack is

$$\mathcal{I}\widetilde{X} = \widetilde{X} \amalg \coprod_{\mu_d - 1} \widetilde{Y}.$$

Similarly all twisted sectors of the second inertia is isomorphic to $\widetilde{Y}$.

The stringy Chow ring is

$$A^*_{st}(\widetilde{X}) = A^*(\widetilde{X})e_1 \oplus \bigoplus_{\zeta \in \mu_d - 1} A^*(\widetilde{Y})e_\zeta$$

with non-trivial multiplications given by

$$e_{\zeta_1} \star e_{\zeta_2} = \begin{cases} [\tilde{Y}]e_{\zeta_1\zeta_2} = (-t)e_{\zeta_1\zeta_2} & \text{if } \arg(\zeta_1) + \arg(\zeta_2) \geq 2\pi \\ e_{\zeta_1\zeta_2} & \text{otherwise} \end{cases}$$

Root stack construction along a Cartier divisor falls into this category. Ordinary blow-up also constitutes a rather degenerate case. In fact blowing-up along a centre with a single weight is a combination of these two: by [QR] weighted blow-up along $(\mathcal{I}, d)$ is the same as performing a classical blow-up along I first, followed by taking the root stack along the exceptional divisor. For example one may perform weighted blow-up along the twisted cubic in $\mathbb{P}^3$.

Chapter 5

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