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Quantum Spectral Problems and Isomonodromic Deformations

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Abstract: We develop a self-consistent approach to study the spectral properties of a class of quantum mechanical operators by using the knowledge about monodromies of 2×2 linear systems (Riemann–Hilbert correspondence). Our technique applies to a variety of problems, though in this paper we only analyse in detail two examples. First we review the case of the (modified) Mathieu operator, which corresponds to a certain linear system on the sphere and makes contact with the Painlevé III_3 equation. Then we extend the analysis to the 2-particle elliptic Calogero–Moser operator, which corresponds to a linear system on the torus. By using the Kyiv formula for the isomonodromic tau functions, we obtain the spectrum of such operators in terms of self-dual Nekrasov functions ($\epsilon_1 + \epsilon_2 = 0$). Through blowup relations, we also find Nekrasov–Shatashvili type of quantizations ($\epsilon_2 = 0$). In the case of the torus with one regular singularity we obtain certain results which are interesting by themselves. Namely, we derive blowup equations (filling some gaps in the literature) and we relate them to the bilinear form of the isomonodromic deformation equations. In addition, we extract the $\epsilon_2 \to 0$ limit of the blowup relations from the regularized action functional and CFT arguments.

Contents

1. Introduction

1.1 The results of this paper are based on the interplay between different branches of mathematical physics. The key objects are 1d quantum mechanical operators, Painlevé equations, monodromies of 2×2 linear systems [1–6], 4d Nekrasov partition functions [7,8], blowup relations [9,10] and conformal blocks [11–13]. At the center of this circle of connections lie *linear systems*. We usually denote such system as

$$\frac{\mathrm{d}}{\mathrm{d}z} Y(z) = A(z) Y(z), \tag{1.1}$$

where $A(z)$ is a 2×2 matrix, $Y(z) = (Y_1(z), Y_2(z))^t$.

One can also rewrite the linear system (1.1) as a second order differential equation in $Y_1(z)$. To remove the first derivative term from this equation we also need to rescale $Y_1(z)$ by switching to $\widetilde{Y}_1(z) = (A_{12}(z))^{-1/2}\, Y_1(z)$. This function satisfies a Schrödinger–type equation of the form

$$(-\partial_z^2 + W(z))\widetilde{Y}_1(z) = 0. \tag{1.2}$$

However, $W(z)$ is not yet a good quantum mechanical potential because, first of all, it has extra singularities at the zeros of $A_{12}(z)$ (so-called apparent singularities). To work with an actual quantum mechanical problem we demand that apparent singularities are hidden inside the actual singularities of $A(z)$. In this case the potential simplifies. We call the simplified version $U(z)$. Then we get the actual Schrödinger equation we are interested in:

$$(-\partial_z^2 + U(z))\widetilde{Y}_1(z) = 0. \tag{1.3}$$

Under some special conditions on the monodromies of the linear system (1.1) the function $\widetilde{Y}_1(z)$ becomes square integrable on some one-dimensional domain of the complex plane. Hence we get the eigenfunctions for a certain *1d quantum mechanical operator* (and the corresponding formulas for its discrete spectrum) in terms of the solution to the *isomonodromic deformation equations*. Indeed, the monodromy data of $A(z)$ are encoded in these equations.

In the simplest case these isomonodromic deformation equations are *Painlevé equations*, see [14] for a review and a list of references. Among the corresponding quantum mechanical operators we recover the cubic, quartic and hyperbolic cosine potentials as well as (confluent) Heun's equation whose appearance in the context of Painlevé equations was also discussed in [15–31].

Another case, which is also considered in this paper, is the linear system on the torus with one regular singular point of $A(z)$. The corresponding operator is the 2-particle elliptic Calogero–Moser operator (4.33).

1.2 In the seminal paper [32] Gamayun, Iorgov and Lisovyy suggested a formula for the tau function of Painlevé VI as a sum of $c = 1$ *Virasoro conformal blocks*.[1] In the last ten years this relation was proven and generalized to many other isomonodromic deformation problems, see for example [41–48]. In particular, the generalization to the isomonodromic problem on the torus was recently worked out in [49–51].

Due to the AGT correspondence [52] conformal blocks essentially coincide with *Nekrasov partition functions*. Hence, the aforementioned result can be stated as a formula expressing the solution to the isomonodromic deformation problem in terms of the self-dual (i.e. $\epsilon_1 + \epsilon_2 = 0$) Nekrasov partition functions.[2] To be more precise, the isomonodromic tau function is equal to the Nekrasov–Okounkov dual partition function [8]. This correspondence is sometimes referred to as Isomonodromy/CFT/gauge theory correspondence. The formula for the tau function is usually called "Kyiv formula", named after [32].

Using this correspondence, and the discussion of paragraph 1.1, we get the exact formulas for the quantization conditions of the operators (1.3) in terms of self-dual Nekrasov functions. More precisely the spectrum of (1.3) will be obtained by imposing (among other things) the vanishing of a suitable combination of isomonodromic tau functions.[3]

This also connects with the observation [15, 16, 18–21] that movable poles in Painlevé III_3, II and I are closely connected to the spectrum of a class of quantum mechanical operators. Within our framework this observation can be straightforwardly generalised to other isomonodromic deformation problems, the corresponding quantum operator is simply obtained from (1.3).

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[1] Connection between isomonodromic deformations and two-dimensional quantum field theory was noticed before in [33–40].

[2] The relation between Painlevé equations and supersymmetric gauge theories (their Seiberg–Witten curves) was noticed before in [53, 54].

[3] It is interesting to note that the existence of such expressions agrees with predictions from (the limit of) Topological String/Spectral Theory duality [55].

1.3 There is another remarkable way to write down the discrete spectrum of these operators due to Nekrasov and Shatashvili (NS). In this approach the main ingredient is the NS limit (i.e. $\epsilon_2 \to 0$) [56–58] of Nekrasov functions. Compatibility between these two approaches follows from a special limit of *Nakajima–Yoshioka blowup relations* [9,10]. From this perspective one can view our results as an independent derivation of the NS formulas [57,58].

Let us note however that the NS approach to spectral theory has some restrictions, for example when it comes to study the edges of the bands in periodic potentials, see for instance [59], or the spectrum of relativistic integrable systems, see for instance [55]. On the contrary thinking in terms of vanishing of isomonodromic tau functions provides a unifying framework which naturally extends also to these situations. We discuss this briefly at the end of Sect. 8.

The connection between the self-dual and the NS limits of Nekrasov functions (or $c = 1$ and $c = \infty$ conformal blocks) has been discussed in various contexts over the past few years. For example, by using quantization conditions as motivation, a five dimensional version of such relations was first proposed in [60]. The idea to use the $\epsilon_2 \to 0$ limit of blowup relations for such problem can be found in [61].[4] More recently, in [64–66], a four dimensional version of these relations has been applied in the context of Painlevé equations and spectral theory. We will discuss this further in the main text. Finally, in [67–69] blowup equations for Nekrasov function with defects have been used to provide a direct link between the work of [32] and the work of [21] which relates the $c \to \infty$ limit of the $N_f = 4$ BPZ equation to the Hamilton–Jacobi equation of Painlevé VI, see also [31]. This has provided an alternative derivation for [32] as well as a gauge theoretical meaning of the monodromy parameter η appearing in the Kyiv formula.

1.4 This paper is structured as follows.

In Sect. 2 we accurately formulate the relationship between 2×2 linear systems and quantum mechanical operators. In order to get such operators (and the corresponding spectrum) we have to fulfil three constraints: the singularities matching condition, the reality condition and the square integrability of the solution.

In Sect. 3 we apply this procedure to the example of Painlevé III_3 whose associated 2×2 linear system is (3.2). The corresponding operator is the (modified) Mathieu

$$-\partial_x^2 + \sqrt{t}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right). \tag{1.4}$$

We find that the operator spectrum is given by

$$E_n(t) = -t\partial_t \log \mathcal{T}_0(\sigma_n, \eta_n, t), \tag{1.5}$$

where $\mathcal{T}_0(\sigma, \eta, t)$ is the Painlevé III_3 tau function, and (σ_n, η_n) are solutions of[5]

$$\mathcal{T}_0(\sigma + \frac{1}{2}, \eta, t) = 0, \tag{1.6}$$

$$\sin\frac{\eta}{2} = 0. \tag{1.7}$$

The variables (σ, η) are the monodromy data of the associated 2×2 linear system given in (3.2) and they specify the initial conditions for the Painlevé III_3 equation. In this

[4] This was later used in [62] and in several follow-up papers leading to the formulation of blowup equations for new class of geometries, see for instance [63].

[5] We can actually fix without loss of generality $\eta_n = 0\ \forall n$.

language equation (1.6) corresponds to the singularity matching condition while (1.7) is the normalizability condition. See Sect. 3 for the details. Thanks to the Kyiv formula (3.10), $\mathcal{T}_0$ is computed explicitly by using $c = 1$ conformal blocks. Hence (1.5)–(1.7) completely determine the spectrum of (1.4) in terms of the self-dual ($\epsilon_1 + \epsilon_2 = 0$) Nekrasov function. The results of this section also overlap with [64,65].

In Sect. 4 we extend the analysis to the case of isomonodromic deformation on the one punctured torus. The associated 2×2 linear system is given in (4.1) and the isomonodromy equation corresponds to an elliptic form of Painlevé VI (the non-autonomous classical elliptic Calogero–Moser system (4.4)). In this example our procedure leads naturally to two quantum operators

$$\mathrm{O}_{\mp} = -\partial_z^2 + m(m\mp 1)\wp(z|\tau), \tag{1.8}$$

which correspond to the 2-particle quantum elliptic Calogero–Moser operator. As in the previous case, the operator spectrum is obtained by asking some particular constraints on monodromy data of linear system (4.1). However, unlike in the previous case, here we have two charts parametrising the monodromy data of the linear system. We denote the corresponding coordinates as (σ, η) and $(\sigma, \tilde{\eta})$. In addition, to satisfy the condition of reality here we have several different options which require independent considerations, see Table 1.

Schematically, the spectrum of (1.8) is given by[6]

$$\begin{aligned} E_n^{\mp} &\sim H_\star^{\mp}(\sigma, \eta, \tau)\Big|_{(\sigma,\eta)=(\sigma_n^{\mp},\eta_n^{\mp})} \\ &= \left(2\pi \mathrm{i}\partial_\tau \log Z_0^D(\sigma, m, \eta, \tau) + 2\pi \mathrm{i}\partial_\tau \log \frac{\eta(\tau)}{\theta_3(0|2\tau)} \mp 2m\frac{\theta_3''(0|2\tau)}{\theta_3(0|2\tau)}\right)\Bigg|_{(\sigma,\eta)=(\sigma_n^{\mp},\eta_n^{\mp})} \end{aligned} \tag{1.9}$$

where Z_0 is essentially the isomonodromic tau function on the torus as in (4.8). We also denoted by $(\sigma_n^{\mp}, \eta_n^{\mp})$ the set of values which satisfies the singularity matching condition (4.19)

$$\theta_2(0|2\tau)Z_0^D(\sigma, m, \eta, \tau) - \theta_3(0|2\tau)Z_{1/2}^D(\sigma, m, \eta, \tau) = 0, \tag{1.10}$$

as well as the normalizability condition listed in Table 2. Note that the solutions to such equations contain simultaneously the spectrum of O_+ and O_-. Hence one still has to disentangle such solutions and map them either to O_+ or to O_-. The Kyiv formula (4.13) give us $Z_{1/2}$ and Z_0 in terms of $c = 1$ conformal blocks. Hence the spectrum of (1.8) is completely determined in terms of self-dual ($\epsilon_1 + \epsilon_2 = 0$) Nekrasov function.

In Sect. 5 we discuss the compatibility between our results and the Nekrasov–Shatashvili (NS) exact quantization. For the Painlevé III_3 example we essentially follow [64,65] with slight improvement. For the example of the torus these results are new. Even blowup relations for the four-dimensional $\mathcal{N} = 2^*$ theory were not written explicitly in the literature (see [63] for the 5d version of some of these equations). For example, by using blowup equations we show that the solutions $\eta_\star^{\pm}$ to the singularity matching condition (1.10) can be expressed as

$$\eta_\star^{\pm} = -\mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma, m\mp\frac{1}{2}, \tau), \tag{1.11}$$

[6] Here $\sim$ means up to overall factors and shifts by $\eta_1(\tau)$ which depend on the reality condition we chose. The precise expression for each case is shown in the last column of Table 1.

where $F^{\rm NS}$ is the $c\to\infty$ conformal blocks on the torus.

In Sect. 6 we derive some new results for the isomonodromic problem associated to the linear system on the torus with one regular singular point. More precisely, we show that the isomonodromic equation for the corresponding tau function takes the form of a very simple bilinear relation which is written in equation (4.12) and reads ($\mathfrak{q}={\rm e}^{2\pi {\rm i}\tau}$)

$$\begin{aligned}
&(\tilde Z_0^D)^2\partial^2_{\log\mathfrak{q}}\log\tilde Z_0^D+(\tilde Z_{1/2}^D)^2\partial^2_{\log\mathfrak{q}}\log\tilde Z_{1/2}^D\\
&=2\left(\frac{\partial_{\log\mathfrak{q}}\theta_3(0|\tau))}{\theta_3(0|\tau))}\left(\partial_{\log\mathfrak{q}}-\frac{\partial_{\log\mathfrak{q}}\theta_3(0|\tau))}{\theta_3(0|\tau))}\right)\right.\\
&\left.-m^2\partial^2_{\log\mathfrak{q}}\log(\theta_3(0|\tau))\right)\left(\tilde Z_0^D\tilde Z_0^D+\tilde Z_{1/2}^D\tilde Z_{1/2}^D\right),
\end{aligned}\tag{1.12}$$

where $\tilde Z_\epsilon^D(\sigma,m,\eta,\tau)=\eta(\tau)Z_\epsilon^D(\sigma,m,\eta,\tau)$. Such relation generalise to the torus setup the well known Hirota-like equations characterising the Painlevé III_3 tau functions. We use them, as well as the $\mathcal{N}=2^*$ blowup relations, to provide an alternative proof for the result of [49].

In Sect. 7 we deduce the NS limit of the blowup relations from the regularized action functional and CFT arguments. This is done by following the method developed in [65], which was also inspired by the works of [18,21].

Finally, in Sect. 8 we conclude by discussing some other examples and generalisations.

There are five appendices which contain definitions ("Appendices A and D"), additional tests ("Appendix B"), technical details ("Appendix C"), and some proofs ("Appendix E").

2. General Idea

The main idea can be outlined as follows. We start from a 2×2 linear system

$$\frac{\rm d}{{\rm d}z}\begin{pmatrix}Y_1(z)\\ Y_2(z)\end{pmatrix}=\begin{pmatrix}A_{11}(z) & A_{12}(z)\\ A_{21}(z) & -A_{11}(z)\end{pmatrix}\begin{pmatrix}Y_1(z)\\ Y_2(z)\end{pmatrix}.\tag{2.1}$$

For a given matrix

$$A(z)=\begin{pmatrix}A_{11}(z) & A_{12}(z)\\ A_{21}(z) & -A_{11}(z)\end{pmatrix}\tag{2.2}$$

the global monodromy of the solution

$$Y(z)=\begin{pmatrix}Y_1(z)\\ Y_2(z)\end{pmatrix}\tag{2.3}$$

is fixed. However, the opposite is generically not true. Given a solution $Y(z)$ with a corresponding monodromy, we can find a parametric family of matrices $A(z,t)$ realising such solution. We refer to $A(z,t)$ as the set of isomonodromic deformations of $A(z)$. One can then deduce that $Y(z,t)$ satisfies the following system (see for instance [14, Ch. 4])

$$\begin{aligned}
&\frac{\rm d}{{\rm d}z}Y(z,t)=A(z,t)Y(z,t),\\
&\frac{\rm d}{{\rm d}t}Y(z,t)=B(z,t)Y(z,t),
\end{aligned}\tag{2.4}$$

where $B(z,t)$ can be obtained with a suitable procedure once $A(z,t)$ is known. The system (2.4) comes together with a compatibility condition

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}t}Y(z,t)=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}z}Y(z,t), \tag{2.5}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}A(z,t)=\frac{\mathrm{d}}{\mathrm{d}z}B(z,t)+[B(z,t),A(z,t)]\,. \tag{2.6}$$

If z is a coordinate on the 4-punctured $\mathbb{CP}^1$ and $A(z,t)\in sl(2,\mathbb{C})$ is suitably chosen, then (2.6) takes the form of a Painlevé equation, see for instance [14, Ch. 4]. The matrices $A(z,t)$ and $B(z,t)$ are also known as Lax pairs. We claim that for a given isomonodromic problem, characterised by $A(z,t)$, we can associate a corresponding quantum mechanical operator

$$-\partial_z^2+U(z,t) \tag{2.7}$$

whose exact spectrum is computed using the tau function of the original isomonodromic problem.

We proceed as follows. We wish to rewrite the linear system in the form of a 2nd order linear equation for $Y_1(z,t)$. From the first equation in (2.1) we have:

$$Y_2(z,t)=\frac{1}{A_{12}(z,t)}\left(Y_1'(z,t)-A_{11}(z,t)Y_1(z,t)\right), \tag{2.8}$$

where by $'$ we denote the derivative w.r.t. z. By plugging $Y_2(z,t)$ back into (2.1) we get

$$\begin{aligned}&Y_1''(z,t)+Y_1'(z,t)\left(-\frac{A_{12}'(z,t)}{A_{12}(z,t)}-\operatorname{tr}A(z,t)\right)\\&+Y_1(z,t)\left(\det A(z,t)+\frac{A_{12}'(z,t)}{A_{12}(z,t)}A_{11}(z,t)-A_{11}'(z,t)\right)=0.\end{aligned} \tag{2.9}$$

To remove the first derivative part we define:

$$Y_1(z,t)=\sqrt{A_{12}(z,t)}\widetilde{Y}_1(z,t). \tag{2.10}$$

The resulting equation is

$$\left(-\partial_z^2+W(z,t)\right)\widetilde{Y}_1(z,t)=0, \tag{2.11}$$

where

$$W(z,t)=\left(-\det A+A_{11}'-\frac{A_{11}A_{12}'}{A_{12}}-\frac{2A_{12}A_{12}''-3(A_{12}')^2}{4A_{12}^2}\right). \tag{2.12}$$

Furthermore, we want $\widetilde{Y}_1(z,t)$ to give the eigenfunction for some quantum mechanical problem. To achieve this we need to fulfil several requirements:

1. Let us denote the zero of $A_{12}(z,t)$ by z_0. Due to the change of variable (2.10), the equation (2.11) has apparent singularity (singularity with trivial monodromy (-1)) at $z=z_0$. We require that such apparent singularities match with the existing singularities of $A(z,t)$. We refer to this constraint as *singularities matching condition*. This requirement gives some restrictions on the matrix elements of $A(z,t)$ and has two consequences.

(a) On one hand such restriction leads to a further simplification of the potential

$$W(z,t) \xrightarrow[\text{condition}]{\text{singularities matching}} U(z,t). \tag{2.13}$$

(b) On the other hand $A(z,t)$ are dynamical variables in the isomonodromic problem. Hence the aforementioned condition can be written as a vanishing condition involving some particular combination of isomonodromic tau functions.

2. We want the operator $\mathrm{e}^{\mathrm{i}\alpha}\left(\partial_z^2 - U(z,t)\right)$ to be self-adjoint on some one-dimensional domain $\mathcal{C}$ in the variable z and for some values of α. This requirement gives some reality conditions for the parameters of the potential and for the domain of z.
3. We also demand that $\widetilde{Y}_1(z,t)$ is normalizable. For periodic potentials we don't need this condition. For confining potentials this condition, together with point (b) above, gives an equation for the spectrum. More precisely, it gives some constraints on the monodromy data: the transport matrix between two singular points should map regular solutions to regular solutions. This is a very standard idea from quantum mechanics, but in contrast to the usual quantum mechanical problems, here the monodromies of $\widetilde{Y}_1(z,t)$ are known by construction.

In this way we get a self-consistent approach which allows us to study the spectrum of some quantum mechanical operators by using the knowledge about isomonodromic deformations. In Sects. 3 and 4 we illustrate this procedure in details for the example of Painlevé III_3 and for the isomonodromic deformation on the torus.

3. Modified Mathieu Equation and Painlevé III_3

In this section we will apply the strategy presented in Sect. 2 to the isomonodromic problem leading to the Painlevé III_3 equation. As explained below, the relevant quantum operator in this context is the modified Mathieu, or 2-particle quantum Toda Hamiltonian. Connection between the spectrum of modified Mathieu and the poles of Painlevé III_3 have been observed for instance in [16,18,70] at the level of asymptotic expansions as well as numerically. This interplay was recently revisited in [64,65] from the optic of the Ω background and blowup equations.

We follow [45] (some formulas for Painlevé are taken from [71]). The linear system associated to Painlevé III_3 has the form

$$\frac{\mathrm{d}}{\mathrm{d}z}Y(z,t) = A(z,t)Y(z,t), \tag{3.1}$$

where $z \in \mathbb{CP}^1$ and

$$A(z,t) = z^{-2}\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} + z^{-1}\begin{pmatrix} -p/w & t/w \\ -1 & p/w \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.2}$$

where $w = w(t)$ and $p = p(t)$. The compatibility condition of this isomonodromy problem is given by (see e.g [45, eq. (2.10)])

$$\begin{cases} t\frac{\mathrm{d}w}{\mathrm{d}t} = 2p + w, \\ t\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{2p^2}{w} + p + w^2 - t, \end{cases} \tag{3.3}$$

which can be written as the known Painlevé III_3 equation:

$$\frac{d^2 w}{d^2 t} = \frac{1}{w}\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{2w^2}{t^2} - \frac{2}{t}. \tag{3.4}$$

By using

$$w(t) = -(r/8)^2 e^{iu(r)}, \quad t = (r/8)^4 \tag{3.5}$$

we can write (3.4) as the radial sin-Gordon equation

$$\frac{d^2 u}{d^2 r} + r^{-1}\frac{du}{dr} + \sin u = 0. \tag{3.6}$$

We also introduce the Hamiltonians H_i as

$$H_0 = H_1 + \frac{p}{w} + \frac{1}{4}, \quad H_1 = \frac{p^2}{w^2} - w - \frac{t}{w}. \tag{3.7}$$

Note that here p and w are not canonical coordinates, we have $\{w, p\} = w^2$. Transformation to canonical coordinates from [71] is $p \mapsto pw^2 - w/2$, $w \mapsto w$. There is a Bäcklund transformation of the Painlevé III_3 equation which permutes H_0, $H_1(t)$, it maps $w \mapsto t/w$.

The tau functions are defined as

$$H_i(t) = t\frac{d\log \mathcal{T}_i(t)}{dt}. \tag{3.8}$$

The Painlevé transcendent w can be expressed as

$$w = \frac{-1}{\partial^2_{\log t}\mathcal{T}_0} = t^{1/2}\frac{\mathcal{T}_0^2}{\mathcal{T}_1^2}. \tag{3.9}$$

The tau functions $\mathcal{T}_0(t)$, $\mathcal{T}_1(t)$ are holomorphic on the universal covering of $\mathbb{C}\setminus\{0\}$. Let $\{t_n(\sigma,\eta)\}_{n\geq 0}$ denote the zeros of $\mathcal{T}_1(t)$, they correspond to movable poles of the Painlevé transcendent w.

Remarkably, the tau function of Painlevé III_3 has been computed in [32,41] for generic initial conditions. They found

$$\begin{aligned}
\mathcal{T}_0(\sigma,\eta,t) &= \sum_{n\in\mathbb{Z}} e^{in\eta}\frac{t^{(\sigma+n)^2}}{G(2\sigma+2n+1)\,G(1-2\sigma-2n)}Z(\sigma+n,t),\\
\mathcal{T}_1(\sigma,\eta,t) &= \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{in\eta}\frac{t^{(\sigma+n)^2}}{G(2\sigma+2n+1)\,G(1-2\sigma-2n)}Z(\sigma+n,t),
\end{aligned} \tag{3.10}$$

where $G(z)$ denotes the Barnes G function and $Z(\sigma,t)$ is the irregular $c=1$ Virasoro conformal block[7] whose precise definition can be found for instance in [72, eqs. (3.4)–(3.6)]. The first few terms read

$$Z(\sigma,t) = 1 + \frac{t}{2\sigma^2} + \frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)^2}t^2 + \mathcal{O}(t^3). \tag{3.11}$$

[7] Via AGT [52] this is equivalent to Nekrasov partition function of pure $\mathcal{N}=2$, $SU(2)$ SYM in the four-dimensional self-dual phase ($\epsilon_1+\epsilon_2=0$) of the Ω background [7].

Higher order terms can be computed systematically by using combinatorics and Young diagrams, we refer to [72] for the details. The parameters (σ, η) are related to the monodromies of the linear system (3.1) around $z = 0, \infty$ and parametrise the space of initial conditions (see for instance [45, Sec. 2] or [72, Sec. 2]). It was proven in [72] that, as long as $2\sigma \notin \mathbb{Z}$, the series (3.10) converges uniformly and absolutely on every bounded subset of the universal cover of $\mathbb{C} \setminus \{0\}$.

The expression (3.10) is also known as Kyiv formula for Painlevé III_3.

3.1. Singularities matching condition. It is convenient to introduce $x = \log z$, as well as $\tilde{Y}_1(z) = \mathrm{e}^{x/2}\Psi(x)$. Then (2.11) reads

$$\left(\partial_x^2 - V(x,t)\right)\Psi(x) = 0, \tag{3.12}$$

where

$$V(x,t) = \mathrm{e}^{2x} W(\mathrm{e}^x, t) + \frac{1}{4}. \tag{3.13}$$

By using the explicit expression (3.2) we get

$$V(x,t) = \frac{p^2 + pw - w\left(t + w^2\right)}{w^2} + \frac{t(p+w)}{w(w\mathrm{e}^x - t)} + \frac{3t^2}{4(t - w\mathrm{e}^x)^2} + \frac{t}{\mathrm{e}^x} + \mathrm{e}^x + \frac{1}{4}. \tag{3.14}$$

The linear system (3.1) has singularities at $x = \pm\infty$. However, since

$$A_{12}(x) = -(\mathrm{e}^x - t/w), \tag{3.15}$$

we have an auxiliary pole in the Eq. (3.12) at the point $x = \log(t/w)$. We do not have this pole if $w = \infty$ or $w = 0$. Hence, we need to be at such points (singularities matching condition). Let us analyse these two cases in more detail.

Case $w = \infty$ at some time $t_\star$. Using (3.3) we have

$$w \sim \frac{t_\star^2}{(t - t_\star)^2} + \mathcal{O}(1), \quad p \sim \frac{-t_\star^3}{(t - t_\star)^3} - \frac{t_\star^2}{2(t - t_\star)^2} + \mathcal{O}(1). \tag{3.16}$$

It follows from these expressions, or from (3.9), that

$$\mathcal{T}_1(t_\star) = 0, \tag{3.17}$$

as well as $\mathcal{T}_0(t_\star) \neq 0$, and H_0 is finite.

Case $w = 0$ at some time $t_\star$. We have

$$w \sim t_\star^{-1}(t - t_\star)^2 + \mathcal{O}((t - t_\star)^3), \quad p \sim (t - t_\star) + \mathcal{O}((t - t_\star)^2).$$

It follows from these expressions, or from (3.9), that

$$\mathcal{T}_0(t_\star) = 0, \tag{3.18}$$

as well as $\mathcal{T}_1 \neq 0$, H_1 is finite.

The two cases $w = 0$ and $w = \infty$ are actually related by Bäcklund transformation and are equivalent. In the rest of the work we will focus without loss of generality on the constraints coming from imposing $w = \infty$.

3.2. Quantum mechanical operator. It is easy to see that if we expand the potential (3.14) around $w = \infty$, we obtain

$$V(x,t) \quad \to \quad U(x,t) = \mathrm{e}^{x} + t\mathrm{e}^{-x} + H_0. \tag{3.19}$$

Here and below we use t instead of $t_\star$ for simplicity. The corresponding spectral problem is[8]

$$\left(\partial_x^2 - \left(\sqrt{t}\mathrm{e}^{x} + \sqrt{t}\mathrm{e}^{-x} - E\right)\right)\Psi(x) = 0, \tag{3.20}$$

which is the well known (modified) Mathieu operator. Moreover, from (3.8) and (3.19) we have

$$E = -H_0 = -t\frac{\mathrm{d}\log \mathcal{T}_0(\sigma,\eta,t)}{\mathrm{d}t}. \tag{3.21}$$

If $\sqrt{t} > 0$, $x \in \mathbb{R}$ this operator is self-adjoint with a positive discrete spectrum on $L^2(\mathbb{R})$.

3.3. Quantization conditions and spectrum. According to our general approach illustrated in Sect. 2, the exact quantization condition for the operator (3.20) is obtained by asking simultaneously the singularities matching condition as well as the normalizability of the associated linear problem.

The singularities matching condition is given in Eq. (3.17):

$$\mathcal{T}_1(\sigma,\eta,t) = 0. \tag{3.22}$$

The condition that Y_1 is normalisable can be expressed in terms of the connection matrix $\mathcal{E}$ for the Painlevé III_3 equation. We follow [45, Sec.2] and use a gauge transformation together with the twofold covering $z = \zeta^2$ to write (3.1) in the form

$$\partial_\zeta \widehat{Y}(\zeta,t) = \widehat{A}(\zeta,t)\widehat{Y}(\zeta,t) \tag{3.23}$$

with

$$\widehat{Y}(\zeta,t) = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}\sqrt{\zeta}} & \frac{\mathrm{i}\sqrt{\zeta}}{\sqrt{2}} \\ \frac{\mathrm{i}}{\sqrt{2}\sqrt{\zeta}} & -\frac{\mathrm{i}\sqrt{\zeta}}{\sqrt{2}} \end{pmatrix} Y(\zeta^2,t) \tag{3.24}$$

and

$$\widehat{A}(\zeta,t) = \frac{1}{\zeta^2}\left(\left(w + \frac{t}{w}\right)\sigma_3 + \left(w - \frac{t}{w}\right)\mathrm{i}\sigma_2\right) - \frac{1}{\zeta}\left(\frac{2p}{w} + \frac{1}{2}\right)\sigma_1 - 2\sigma_3, \tag{3.25}$$

where σ_i are the Pauli matrices. The reason for such rewriting is that the matrices multiplying ζ^{-2} and ζ^0 in (3.23) are diagonalizable, hence we can easily write formal solutions around $\zeta = 0$ and $\zeta = \infty$. Around $\zeta = 0$ we have

$$\widehat{Y}^{(0)}_{\mathrm{form}}(\zeta,t) = \left(-\frac{w}{\sqrt{t}}\right)^{-\sigma_1/2}\left(\mathbb{1} + \mathcal{O}(\zeta)\right)\mathrm{e}^{2\sqrt{t}\sigma_3/\zeta}. \tag{3.26}$$

Likewise around $\zeta = \infty$ we have

$$\widehat{Y}^{(\infty)}_{\mathrm{form}}(\zeta,t) = \left(\mathbb{1} + \mathcal{O}(\zeta^{-1})\right)\mathrm{e}^{-2\sigma_3\zeta}. \tag{3.27}$$

[8] We shifted here $x \mapsto x + \frac{1}{2}\log t$.

The connection matrix $\mathcal{E}$ relates solutions around 0 to solutions around ∞ as $\widehat{Y}^{(0)}\mathcal{E} = \widehat{Y}^{(\infty)}$. We have (see [45, eq. (2.7)]):

$$\mathcal{E} = \frac{1}{\sin(2\pi\sigma)}\begin{pmatrix} \sin(\eta/2) & -\mathrm{i}\sin(2\pi\sigma+\eta/2) \\ \mathrm{i}\sin(2\pi\sigma-\eta/2) & \sin(\eta/2) \end{pmatrix}. \tag{3.28}$$

Normalizability of Y requires that we map decaying solutions around $\zeta = 0$ to decaying solutions around $\zeta = \infty$. Hence the diagonal elements of $\mathcal{E}$ have to vanish:

$$\sin\left(\frac{\eta}{2}\right) = 0. \tag{3.29}$$

By combining (3.22), (3.29), and the Bäcklund transformation $\mathcal{T}_1(\sigma, \eta, t) \sim \mathcal{T}_0(\sigma + \frac{1}{2}, \eta, t)$ from [42] we get the quantization condition for modified Mathieu:

$$\mathcal{T}_0(\sigma + \frac{1}{2}, 0, t) = 0. \tag{3.30}$$

From the point of view of spectral theory we think of (3.30) as a quantization condition for σ. If we denote the solutions to such quantization condition by

$$\{\sigma_n\}_{n\geq 1}, \tag{3.31}$$

then the spectrum $\{E_n(t)\}_{n\geq 1}$ of modified Mathieu is obtained from (3.21) and reads

$$E_n(t) = -t\partial_t \log \mathcal{T}_0(\sigma_n, 0, t). \tag{3.32}$$

We also cross-checked against explicit (numerical) computations that (3.32) and (3.30) indeed reproduce the correct spectrum of modified Mathieu. Hence, from that point of view, the exact quantization condition of modified Mathieu follows from the Kyiv formula for the tau function of Painlevé III_3 (3.10) and can be expressed entirely by using $c = 1$ Virasoro conformal blocks.

Some comments.

- In the work [15], which was later made more precise in [18,70], the Author considers the semi-classical Bohr–Sommerfeld quantization for the Mathieu operator as a quantization for the variable t in (3.20). Then he connects such solutions $\{t_n(E)\}$ (in the limit $n \to \infty$) to the poles in the time variable t of the function u satisfying (3.6). Roughly speaking one has $u \sim \log(\sqrt{t} - \sqrt{t_n})$. These poles are the zeros of the tau function.
  Here instead we are using the inverse analysis. We do not start from the quantization for the Mathieu operator: we derive it from tau function of Painlevé III_3 as computed in [41].
- Note that (3.30) is precisely the condition found in [64, Sec. 6] even though the derivation of [64] is different from the approach presented in this section. Moreover in [64] one still needs to relay on Matone relation (hence $c = \infty$ conformal blocks). Instead in our perspective we have (3.32). We will see in Sect. 5.1 that (3.32) and Matone relation are connected via blowup equations.
- Some of the results presented in this section overlap with [65].

4. Weierstrass Potential and Isomonodromic Deformations on the Torus

In this section we will apply the strategy presented in Sect. 2 to the isomonodromic problem on the one-punctured torus which was studied recently in [49], and then in [51]. As explained below, the relevant quantum operator in this context is the 2-particle quantum elliptic Calogero–Moser Hamiltonian.

We follow [49]. We start from the following linear system:

$$\frac{\mathrm{d}}{\mathrm{d}z}Y(z,\tau) = A(z,\tau)Y(z,\tau), \tag{4.1}$$

with

$$A(z,\tau) = \begin{pmatrix} p & mx(2Q,z) \\ mx(-2Q,z) & -p \end{pmatrix}, \tag{4.2}$$

where

$$x(u,z) = \frac{\theta_1(z-u|\tau)\theta_1'(0|\tau)}{\theta_1(z|\tau)\theta_1(u|\tau)}$$

is the Lamé function, θ_1 is the Jacobi theta function, and $\theta_1'(0|\tau) = \partial_z\theta_1(z|\tau)\Big|_{z=0}$. See Appendix A for the conventions. The coordinate z is on the torus $\mathbb{T}^2$ with modular parameter τ. Note that (4.2) has a simple pole at $z=0$. It was shown in [73–76], see also [49] and reference therein, that the compatibility condition of the linear system (4.1) leads to

$$p = 2\pi\mathrm{i}\partial_\tau Q, \tag{4.3}$$

$$(2\pi\mathrm{i})^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Q = m^2\wp(2Q|\tau)', \tag{4.4}$$

where $'$ refers to the derivative w.r.t. the first argument, and $\wp$ is the Weierstrass function defined in (A.8). We see that the potential of this system is the Weierstrass $\wp$-function, where the time τ is identified with the modular parameter: this is the classical non-autonomous 2-particle elliptic Calogero–Moser system.

Notice that (4.4) has to be supplied by two integration constants (σ,η). These are such that at $m=0$ we have

$$Q\Big|_{m=0} = \tau\sigma + \frac{\eta}{4\pi}. \tag{4.5}$$

The variables (σ,η) are also related to the monodromies of $Y(z,t)$ around the A and B cycles of the torus, whereas m determines the monodromy around the singularity $z=0$. This is explained in [49, Sec. 4.2]. We report some of these results in "Appendix C".

One also defines the Hamiltonian associated to (4.4), (4.3) as [49, eq. (3.9)]

$$H = p^2 - m^2\left(\wp(2Q|\tau) + 2\eta_1(\tau)\right), \tag{4.6}$$

where $\eta_1(\tau)$ is defined in (A.9). The tau function $\mathcal{T}$ corresponding to the linear system (4.1) is then defined following [5,6,77,78] as

$$H = 2\pi\mathrm{i}\partial_\tau\log\mathcal{T}. \tag{4.7}$$

It is very convenient to introduce the functions $Z_0^D, Z_{1/2}^D$ as in [49] by the formula

$$\mathcal{T}(\sigma, m, \eta, \tau) = \frac{\eta(\tau) Z_{1/2}^D(\sigma, m, \eta, \tau)}{\theta_2(2Q|2\tau)} = \frac{\eta(\tau) Z_0^D(\sigma, m, \eta, \tau)}{\theta_3(2Q|2\tau)}, \tag{4.8}$$

where $\eta(\tau)$ is the Dedekind's η function defined in (A.4), and $Q = Q(\sigma, m, \eta, \tau)$ is a solution of (4.4). These formulas express indirectly both the tau function $\mathcal{T}$ and the transcendent Q in terms of some functions Z_0^D and $Z_{1/2}^D$. Note that the function Q is determined by Z_0^D and $Z_{1/2}^D$ (up to a sign and shifts by $\mathbb{Z} + \mathbb{Z}\tau$) via the equation

$$\frac{\theta_2\left(2Q(\sigma, m, \eta, \tau)|2\tau\right)}{\theta_3\left(2Q(\sigma, m, \eta, \tau)\,|2\tau\right)} = \frac{Z_{1/2}^D\,(\sigma, m, \eta, \tau)}{Z_0^D\,(\sigma, m, \eta, \tau)}. \tag{4.9}$$

Indeed, suppose that $\tilde{Q}$ solves the same equation, then using the relation

$$\frac{\theta_2(2\tilde{Q}|\tau)}{\theta_3(2\tilde{Q}|\tau)} - \frac{\theta_2(2Q|\tau)}{\theta_3(2Q|\tau)} = \frac{\theta_1(Q - \tilde{Q}|\tau)\theta_1(Q + \tilde{Q}|\tau)}{\theta_3(2\tilde{Q}|2\tau)\theta_3(2Q|2\tau)} \tag{4.10}$$

we get $\tilde{Q} = \pm Q + n\tau + \ell$, with $n, \ell \in \mathbb{Z}$.

In order to write the isomonodromic deformation equations in terms of $Z_0^D, Z_{1/2}^D$ it is convenient to introduce

$$\tilde{Z}_\epsilon^D(\sigma, m, \eta, \tau) = \eta(\tau) Z_\epsilon^D(\sigma, m, \eta, \tau). \tag{4.11}$$

We will show in Sect. 6 that if Q defined by (4.8) satisfies the isomonodromic deformation equations (4.4), then $\tilde{Z}_0^D, \tilde{Z}_{1/2}^D$ satisfy

$$\begin{aligned}
&(\tilde{Z}_0^D)^2 \partial_{\log \mathfrak{q}}^2 \log \tilde{Z}_0^D + (\tilde{Z}_{1/2}^D)^2 \partial_{\log \mathfrak{q}}^2 \log \tilde{Z}_{1/2}^D \\
&\quad = 2\left(\frac{\partial_{\log \mathfrak{q}} \theta_3(0|\tau)}{\theta_3(0|\tau)}\left(\partial_{\log \mathfrak{q}} - \frac{\partial_{\log \mathfrak{q}} \theta_3(0|\tau)}{\theta_3(0|\tau)}\right) - m^2 \partial_{\log \mathfrak{q}}^2 \log \theta_3(0|\tau)\right) \\
&\quad\quad \times\left(\tilde{Z}_0^D \tilde{Z}_0^D + \tilde{Z}_{1/2}^D \tilde{Z}_{1/2}^D\right),
\end{aligned} \tag{4.12}$$

where $\mathfrak{q} = \mathrm{e}^{2\pi i \tau}$.

The main proposal of [49] is the explicit expression of $Z_0^D, Z_{1/2}^D$ as dual Nekrasov partition functions. They found that

$$Z_\epsilon^D(\sigma, m, \eta, \tau) = \sum_{n \in \mathbb{Z}+\epsilon} \mathrm{e}^{in\eta} \frac{\prod_{\epsilon'=\pm} G(1 - m + 2\epsilon'(\sigma+n))}{\prod_{\epsilon'=\pm} G(1 + 2\epsilon'(\sigma+n))} \mathfrak{q}^{(\sigma+n)^2 - 1/24} Z(\sigma + n, m, \mathfrak{q}), \tag{4.13}$$

where $Z(\sigma, m, \mathfrak{q})$ denotes the $c = 1$ conformal block on the torus, i.e. Nekrasov partition function for the $SU(2), \mathcal{N} = 2^*$ theory in the four-dimensional self-dual phase of the Ω background, see eq. (D.13) for the definition. This proposal was proved recently in a more rigorous and mathematical way in [51] using the techniques of Fredholm determinants. Notice that if $m = 0$ we have a very simple expression

$$\mathcal{T}(\sigma, 0, \eta, \tau) = \mathrm{e}^{2i\pi\sigma^2\tau}. \tag{4.14}$$

This formula can be easily deduced from Eqs. (4.5) and (4.7), or from the formula (4.13) using $Z(\sigma, 0, \mathfrak{q}) = \mathfrak{q}^{1/24}\eta(\tau)^{-1}$. In Sect. 6 we will demonstrate that (4.13) indeed satisfy

(4.12), providing in this way another proof of the isomonodromy-CFT correspondence for the 1-punctured torus, alternative to [49,51].

Remark. One might wonder how could it happen that (4.13) is naively non-symmetric under $m \mapsto -m$, while (4.4) is symmetric. The answer is that one needs to accompany this transformation by the transformation of the Barnes functions

$$\frac{G(1-\nu+n)}{G(1-\nu)} = (-1)^{\frac{n(n-1)}{2}} \frac{G(1+\nu-n)}{G(1+\nu)} \left(\frac{\pi}{\sin \pi\nu}\right)^n, \tag{4.15}$$

which leads to the transformation $(m, \eta) \mapsto (-m, \tilde{\eta})$, where $\tilde{\eta}$ is defined by[9]

$$e^{i\frac{\tilde{\eta}}{2}} = e^{i\frac{\eta}{2}} \frac{\sin \pi(2\sigma + m)}{\sin \pi(2\sigma - m)}. \tag{4.16}$$

The latter transformation is not well defined if $\sigma = \frac{1}{2}(\pm m + k)$, $k \in \mathbb{Z}$. In this case we have to chose either η or $\tilde{\eta}$ to be finite. Different choices will correspond to different charts on the monodromy manifold, see also “Appendix C.3”.

4.1. Singularities matching condition. In the current example we have

$$A_{12}(z) = m \frac{\theta_1(z - 2Q|\tau)\theta_1'(0|\tau)}{\theta_1(z|\tau)\theta_1(2Q|\tau)}. \tag{4.17}$$

Hence $A_{12}(z)$ admits several zeroes unless[10]

$$Q = Q(\sigma, m, \eta, \tau) = 0. \tag{4.18}$$

This is our condition of singularities matching. By using (4.9) we see that (4.18) is equivalent to

$$\theta_2(0|2\tau) Z_0^D(\sigma, m, \eta, \tau) - \theta_3(0|2\tau) Z_{1/2}^D(\sigma, m, \eta, \tau) = 0. \tag{4.19}$$

4.2. Quantum mechanical operator. Let us denote by $\tau_\star$ the solution to (4.18)

$$Q(\sigma, m, \eta, \tau_\star) = 0. \tag{4.20}$$

By solving (4.4) around $Q = 0$ we get

$$Q \approx \frac{\exp\left(\mp\frac{i\pi}{4}\right)\left(\sqrt{m}\sqrt{\tau - \tau_\star}\right)}{\sqrt{2\pi}}. \tag{4.21}$$

One is actually free to choose any of the two signs. We keep them both in order to see possible symmetries. We will also denote the quantities corresponding to the two

[9] The transformation $\eta \to \tilde{\eta}$ or $m \to -m$ also appears for other isomonodromic systems in the context of transition between different topologies of the spectral networks [79,80], or as a very close companion of the complex conjugation, used to construct “physical” correlation function [81].

[10] To be precise, one should require $Q = \frac{n}{2} + \frac{\tau k}{2}$. However, due to [49, Appendix C], such a shift can be achieved by a simple Bäcklund transformation $(\sigma, \eta) \mapsto (\sigma + k/2, \eta + 2\pi n)$, so it is sufficient to consider only $Q = 0$.

different solutions by $\mp$: the upper sign always corresponds to the upper sign in (4.21), and vice versa.

By using (4.3) it is easy to see that

$$p = 2\pi i \partial_\tau Q = \pm \frac{m}{2Q} + \mathcal{O}(Q^0). \tag{4.22}$$

Hence the Hamiltonian (4.6) is finite at the point $\tau_\star$.[11] Likewise we can think of (4.18) as an equation for σ or η. In this case we will denote the corresponding solution by $\sigma_\star$ or $\eta_\star$. The corresponding Hamiltonian will always be finite and we will denote its values by $H_\star^\mp$.

Let us now look at the quantum mechanical operator. After some algebra, we find that the potential (2.12) associated to the linear system (4.1) can be written as

$$\begin{aligned} W(z,\tau) = H &+ m^2 (\wp(z) + 2\eta_1(\tau)) - p\,(\zeta(z-2Q|\tau) - \zeta(z|\tau) + 4Q\eta_1(\tau)) \\ &+ \frac{1}{2}(\wp(z-2Q|\tau) - \wp(z|\tau)) + \frac{1}{4}(\zeta(z-2Q|\tau) - \zeta(z|\tau) + 4Q\eta_1(\tau))^2, \end{aligned} \tag{4.23}$$

where the elliptic functions $\wp$, ζ, η_1 are defined in "Appendix A". In deriving (4.23) we also used several identities for the Lamé function $x(u,z)$ which can be found in [49, Appendix A].

The potential (4.23) is quite complicated, especially because it depends on $Q = Q(\sigma, m, \eta, \tau)$. However, when we impose the singularities matching condition (4.18) the second line of (4.23) vanishes. In addition, by using (4.22) we can rewrite the first line in (4.23) as

$$-p\big(\zeta(z-2Q|\tau) - \zeta(z|\tau) + 4Q\eta_1(\tau)\big) = \mp m\big(\wp(z|\tau) + 2\eta_1(\tau)\big) + \mathcal{O}(Q). \tag{4.24}$$

It follows that the relevant potential at $Q=0$ is

$$U(z,\tau) = (m^2 \mp m)\wp(z|\tau) + \Big(H_\star^\mp + 2(m^2 \mp m)\eta_1(\tau)\Big). \tag{4.25}$$

Hence the quantum operator arising from isomonodromic deformations on the torus is the 2-particle *quantum* elliptic Calogero–Moser system with potential $(m^2 \mp m)\wp(z|\tau)$.

To have a physically well-defined spectral problem, in this paper we will restrict without loss of generality to $|m| > 1$. Note that $H_\star^\mp$ can be computed explicitly from (4.7), (4.8) with the help of (4.22). It reads:

$$H_\star^\mp = \left(2\pi \mathrm{i}\partial_\tau \log Z_0^D(\sigma, m, \eta, \tau) + 2\pi \mathrm{i}\partial_\tau \log \frac{\eta(\tau)}{\theta_3(0|2\tau)} \mp 2m \frac{\theta_3''(0|2\tau)}{\theta_3(0|2\tau)} \right)\Bigg|_{Q=0}. \tag{4.26}$$

Here we denote by $'$ the derivative w.r.t. the first argument of the θ function.

Remark. By inverting (4.21) we get $\tau - \tau_\star \approx \pm \frac{2\pi \mathrm{i}}{m} Q^2$. By substituting this into (4.4), (4.6) one can compute further terms

$$\tau - \tau_\star = \pm \frac{2\pi \mathrm{i}}{m} \left(Q^2 - \frac{H_\star^\mp + 2m^2\eta_1(\tau_\star)}{m^2} Q^4 \right) + \mathcal{O}(Q)^6. \tag{4.27}$$

[11] Another way to get this is to notice that $\frac{\mathrm{d}}{\mathrm{d}\tau} H = -m^2 \partial_\tau (\wp(2Q|\tau) + 2\eta_1(\tau))$ is regular when $Q = 0$ (note that the partial τ-derivative is computed at fixed Q). Then one gets the behavior (4.22) from the finiteness of $H_\star$.

To derive this formula it is sufficient to use the approximation $\wp(x|\tau) \sim \frac{1}{x^2}$, since higher order terms in such expansion start to contribute from $\mathcal{O}(Q)^6$. The upper and lower signs in (4.27) agree with the ones in (4.21). We will use (4.27) in Sect. 7.

Remark. By using (4.27) and (4.26) we can compute the first few terms in the $\mathfrak{q}$-expansion of $H_\star^\mp$. We get

$$H_\star^\mp = 4\pi^2 \left(-\sigma^2 + \frac{2m^2(m\mp 1)^2}{1-4\sigma^2}\mathfrak{q}_\star + \mathcal{O}(\mathfrak{q}_\star^2) \right). \tag{4.28}$$

4.3. Reality condition. Now we wish to fulfil another requirement: the reality of the potential in the Schrödinger equation

$$\left(-\partial_z^2 + (m^2 \mp m)\wp(z|\tau) + H_\star^\mp + 2(m^2\mp m)\eta_1(\tau) \right) \widetilde{Y}_1(z) = 0. \tag{4.29}$$

There are several ways to do this. First we study the conjugation of the Weierstrass function: $\overline{\wp(z|\tau)} = \wp(\overline{z}|-\overline{\tau})$. This transformation reflects the fundamental domain of the modular group with respect to the vertical line. There are two (actually intersecting) branches which are invariant under such conjugation: $\tau \in \mathrm{i}\mathbb{R}_{>0}$ and $\tau \in \frac{1}{2} + \mathrm{i}\mathbb{R}_{>0}$.[12] The two special points with additional symmetry of the elliptic curve also lie on these branches: $\tau = \mathrm{i}$ and $\tau = \mathrm{e}^{\mathrm{i}\pi/3}$.

If we want the potential to be real, then z should lie on some suitable domain $\mathcal{C}$. In Table 1 we give a list of all the possible options. In this Table the complex variables τ and z are parametrised by the two real variables

$$\mathfrak{t} \in \mathbb{R}_{>0}, \quad x \in (0,1). \tag{4.30}$$

The corresponding lines in the z-plane are shown in Fig. 1. During the computations we also used the modular transformation for the Weierstrass function:

$$\wp(z|\tau) = \tau^{-2}\wp(\frac{z}{\tau}| - \frac{1}{\tau}). \tag{4.31}$$

After parametrising z and τ via (4.30) we write the Schrödinger equation (4.29) as

$$\left(-\partial_x^2 + u_\mp(x,\mathfrak{t}) + E(\mathfrak{t}) \right) \psi(x,\mathfrak{t}) = 0\,, \tag{4.32}$$

where $E(\mathfrak{t})$ and $u_\mp(x,\mathfrak{t})$ are reported in the last and the second to last column of Table 1. Later we will also use the notation

$$\mathrm{O}_\mp = -\partial_x^2 + u_\mp(x,\mathfrak{t}). \tag{4.33}$$

[12] The latter branch maps to $|\tau| = 1$ by the transformation $\tau' = \frac{\tau}{1-\tau}$, in the Table 1 we map it to the line $-\frac{1}{2} + \mathrm{i}\mathbb{R}_{>0}$.

Table 1. The possible spectral problems associated to the linear system (4.1).

#	τ	z	$[\mathcal{C}]$	Notation	Potential $u_{\mp}(x,\mathfrak{t})$	Energy $E(\mathfrak{t})$
1	$\mathrm{i}\mathfrak{t}$	x	A		$(m^2 \mp m)\wp(x\vert\mathrm{i}\mathfrak{t})$	$-H_\star^{\mp} - 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t})$
2	$\mathrm{i}\mathfrak{t}$	$\mathrm{i}\mathfrak{t}x$	B		$(m^2 \mp m)\wp(x\vert\frac{\mathrm{i}}{\mathfrak{t}})$	$\mathfrak{t}^2\left(H_\star^{\mp} + 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t})\right)$
3	$\mathrm{i}\mathfrak{t}$	$x + \frac{\mathrm{i}\mathfrak{t}}{2}$	A		$(m^2 \mp m)\wp(x + \frac{\mathrm{i}\mathfrak{t}}{2}\vert\mathrm{i}\mathfrak{t})$	$-H_\star^{\mp} - 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t})$
4	$\mathrm{i}\mathfrak{t}$	$\mathrm{i}\mathfrak{t}x + \frac{1}{2}$	B		$(m^2 \mp m)\wp(x + \frac{\mathrm{i}}{2\mathfrak{t}}\vert\frac{\mathrm{i}}{\mathfrak{t}})$	$\mathfrak{t}^2\left(H_\star^{\mp} + 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t})\right)$
5	$\mathrm{i}\mathfrak{t} - \frac{1}{2}$	x	A		$(m^2 \mp m)\wp(x\vert\mathrm{i}\mathfrak{t} - \frac{1}{2})$	$-H_\star^{\mp} - 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t} - \frac{1}{2})$
6	$\mathrm{i}\mathfrak{t} - \frac{1}{2}$	$2\mathrm{i}\mathfrak{t}x$	$C = 2B + A$		$(m^2 \mp m)\wp(x\vert\frac{\mathrm{i}}{4\mathfrak{t}} - \frac{1}{2})$	$4\mathfrak{t}^2\left(H_\star^{\mp} + 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t} - \frac{1}{2})\right)$

The Hamiltonian $H_\star^{\mp}$ is given in (4.26). The cases 3 and 4 correspond to the Bloch waves and will not be discussed here

Table 2. Normalizability conditions for different spectral problems. The Cases # are as in the first column of Table 1

Cases #	$\mathcal{C}$	m	η, σ
1, 5	A	$m>0$	$\sigma \in \pm\frac{m}{2}+\frac{1}{2}\mathbb{Z}, \quad \eta$ — finite
1, 5	A	$m<0$	$\sigma \in \pm\frac{m}{2}+\frac{1}{2}\mathbb{Z}, \quad \tilde{\eta}$ — finite
2	B	$m>0$	$\tilde{\eta} \in 2\pi\mathbb{Z}$
2	B	$m<0$	$\eta \in 2\pi\mathbb{Z}$
6	C	$m>0$	$e^{i\tilde{\eta}} = e^{2\pi i\sigma}\frac{\cos\pi(\sigma+\frac{m}{2})}{\cos\pi(\sigma-\frac{m}{2})}, \quad e^{i\tilde{\eta}} = -e^{2\pi i\sigma}\frac{\sin\pi(\sigma+\frac{m}{2})}{\sin\pi(\sigma-\frac{m}{2})}$
6	C	$m<0$	$e^{i\eta} = e^{2\pi i\sigma}\frac{\cos\pi(\sigma-\frac{m}{2})}{\cos\pi(\sigma+\frac{m}{2})}, \quad e^{i\eta} = -e^{2\pi i\sigma}\frac{\sin\pi(\sigma-\frac{m}{2})}{\sin\pi(\sigma+\frac{m}{2})}$

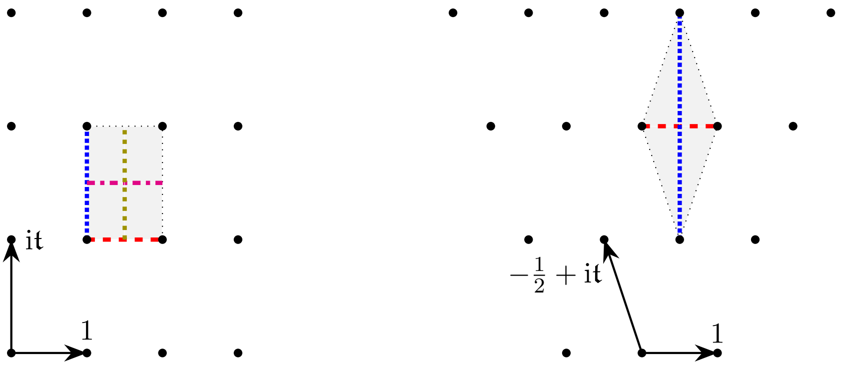

Fig. 1. Lines corresponding to a real potential in (4.29). Left figure: cases 1–4. Right figure: cases 5, 6

4.4. Normalizability conditions. As we saw above, all the potentials with a discrete spectrum that appear in this problem live on a line segment $\mathcal{C}$ bounded by the point 0 and its image, which we denote by $a = 0.\gamma_{\mathcal{C}}$. Normalizability of the linear problem means that we map normalizable solutions at one end to normalizable solutions at the other end. As in Eq. (3.29), we expect normalizability to give us some constraints on the monodromies (σ, η) of the linear system.

Let us first study the solution of the linear system (4.1) around $z=0$. For the purposes of this paper we can focus on the limit $\tau \to \tau_\star$. In order to do this we use (4.22) and further expand (4.2) at $Q \to 0$:

$$A(z,\tau) \approx \begin{pmatrix} \pm\frac{m}{2Q} & \frac{m}{2Q}-\frac{m}{z} \\ -\frac{m}{2Q}-\frac{m}{z} & \mp\frac{m}{2Q}. \end{pmatrix}. \tag{4.34}$$

By substituting (4.34) into (4.1) we get

$$Y(z,\tau) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\mathbb{I} + \frac{mz}{2Q(1\mp 2m)}\begin{pmatrix} 0 & -1\pm 1 \\ 1\pm 1 & 0 \end{pmatrix} + \mathcal{O}(z)^2\right)\begin{pmatrix} z^{-m} & 0 \\ 0 & z^m \end{pmatrix}\mathsf{C}, \tag{4.35}$$

for some z-independent matrix C. We see that this expression has a singularity at $Q \to 0$, so it needs to be renormalized by choosing an appropriate diagonal matrix C. This is done in different ways for the upper and for the lower sign:

- Upper sign, $\mathsf{C} = \mathrm{diag}\,((1-2m)Q/m, 1)$:

$$Y(z,\tau) \approx \begin{pmatrix} \frac{(1-2m)Qz^{-m}}{m} + z^{1-m} + \mathcal{O}(z^{2-m}) & z^m + \mathcal{O}(z^{2+m}) \\ \frac{(1-2m)Qz^{-m}}{m} - z^{1-m} + \mathcal{O}(z^{2-m}) & -z^m + \mathcal{O}(z^{2+m}) \end{pmatrix} + \mathcal{O}(Q^2). \quad (4.36)$$

Hence the leading asymptotics in the limit[13] $Q \to 0$ are

$$(z^{1-m}, z^m). \quad (4.37)$$

- Lower sign, $\mathsf{C} = \mathrm{diag}\,(1, (-1-2m)Q/m)$:

$$Y(z,\tau) \approx \begin{pmatrix} z^{-m} + \mathcal{O}(z^{2-m}) & \frac{z^m(mz-(2m+1)Q)}{m} + \mathcal{O}(z^{2+m}) \\ z^{-m} + \mathcal{O}(z^{2-m}) & \frac{z^m(2mQ+mz+Q)}{m} + \mathcal{O}(z^{2+m}) \end{pmatrix} + \mathcal{O}(Q^2). \quad (4.38)$$

Hence the leading asymptotics in the limit $Q \to 0$ are

$$(z^{-m}, z^{m+1}). \quad (4.39)$$

Notice that in the limit $Q \to 0$ and around $z = 0$, the function $\widetilde{Y}_1(z,\tau)$ differs from $Y_1(z,\tau)$ only by a normalization factor, i.e. by $\sqrt{2Q/m}$. In particular they both have the same asymptotics.

A similar analysis can be repeated for the point $a = 0.\gamma_{\mathcal{C}}$. Hence for $m > 0$ a normalizable solution $Y_{norm}(z)$ should have positive asymptotics near both boundaries. For the upper sign we find

$$Y(z) \sim z^m, \quad Y(z) \sim (z-a)^m, \quad (4.40)$$

while for the lower sign we get

$$Y(z) \sim z^{1+m}, \quad Y(z) \sim (z-a)^{1+m}. \quad (4.41)$$

This means that normalizable solutions should have monodromies $\mathrm{e}^{2\pi i m}$ around both points (for both upper and lower signs):

$$Y_{norm}(z.\gamma_0) = \mathrm{e}^{2\pi \mathrm{i} m} Y_{norm}(z), \quad Y_{norm}(z.\gamma_a) = \mathrm{e}^{2\pi \mathrm{i} m} Y_{norm}(z), \quad (4.42)$$

where γ_0 and γ_a are the contours encircling 0 and a. To fulfil the first requirement it is sufficient to project onto the column of $Y(z)$ with appropriate asymptotics[14]:

$$Y_{norm}(z) = Y(z)(M_0 - \mathrm{e}^{-2\pi \mathrm{i} m}\mathbb{I}). \quad (4.43)$$

We now look at the second requirement in (4.42). Let $M_{\mathcal{C}}$ be the monodromy along the cycle $\mathcal{C}$ as defined in Table 1. We want to map normalisable solutions around 0 to normalisable solution around $a = 0.\gamma_{\mathcal{C}}$ (and the other way around). Hence we ask

$$Y(z) \cdot M_{\mathcal{C}}^{-1}(M_0 - \mathrm{e}^{2\pi \mathrm{i} m}\mathbb{I})M_{\mathcal{C}} \cdot (M_0 - \mathrm{e}^{-2\pi \mathrm{i} m}\mathbb{I}) = 0. \quad (4.44)$$

13 We keep z fixed and send $Q \to 0$, or in other words consider $2Q \ll z$.

14 We notice that one can capture the two asymptotics by their monodromies only for $m \notin \frac{1}{2}\mathbb{Z}$. However, there are no singularities in m in the solution of the spectral problem at $m \in \frac{1}{2}\mathbb{Z}$, so everything can be continued analytically to these points.

Using that $M_{\mathcal{C}}^{-1}$ and $Y(z)$ are non-degenerate we have

$$(M_0 - \mathrm{e}^{2\pi \mathrm{i} m}\mathbb{I})M_{\mathcal{C}}(M_0 - \mathrm{e}^{-2\pi \mathrm{i} m}\mathbb{I}) = 0. \tag{4.45}$$

This is the normalizability equation.

It is, of course, more convenient to rewrite this condition in the basis where M_0 is diagonal:

$$M_0^{(I,II)} = \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} m} & 0 \\ 0 & \mathrm{e}^{-2\pi \mathrm{i} m} \end{pmatrix}. \tag{4.46}$$

The superscripts ${}^{(I)}$, ${}^{(II)}$ denote different charts on the monodromy manifold with coordinates (σ, η) and $(\sigma, \tilde{\eta})$, see Appendix C for more details. The diagonalization of M_0 is performed by a matrix that depends on the chart, see Appendix C.4 for more details. Hence (4.45) means that for $m > 0$ it maps the column vector $(*, 0)$ to itself. For $m < 0$ instead it should map $(0, *)$ to itself. This means that $M_{\mathcal{C}}$ is upper- or lower-triangular:

$$M_{\mathcal{C}}^{(I,II)} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \text{ for } m > 0, \qquad M_{\mathcal{C}}^{(I,II)} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \text{ for } m < 0. \tag{4.47}$$

Hence if $m > 0$ the normalizability condition reads

$$\left(M_{\mathcal{C}}^{(I,II)}\right)_{21} = 0. \tag{4.48}$$

Likewise the condition for $m < 0$ is

$$\left(M_{\mathcal{C}}^{(I,II)}\right)_{12} = 0. \tag{4.49}$$

To simplify our analysis it is convenient to notice that if $2\sigma \in \pm m + \mathbb{Z}$, the matrix elements of $M_{\dots}^{(I,II)}$ are regular in one of the two charts with η or $\tilde{\eta}$ finite,[15] see also (4.16). In addition, the element 12 or 21 does not vanish simultaneously for any pair of matrices of our interest, M_A, M_B, and M_C. So we consider the ratios of the corresponding matrix elements:

$$f_{21}^{\mathcal{C}/A} = \frac{\left(M_{\mathcal{C}}^{(I)}\right)_{21}}{\left(M_A^{(I)}\right)_{21}} = \frac{\left(M_{\mathcal{C}}^{(II)}\right)_{21}}{\left(M_A^{(II)}\right)_{21}}, \quad f_{12}^{\mathcal{C}/A} = \frac{\left(M_{\mathcal{C}}^{(I)}\right)_{12}}{\left(M_A^{(I)}\right)_{12}} = \frac{\left(M_{\mathcal{C}}^{(II)}\right)_{12}}{\left(M_A^{(II)}\right)_{12}}. \tag{4.50}$$

These expressions are better because they are independent from the remaining diagonal conjugation. Their explicit values are (see "Appendix C.4")

$$f_{21}^{B/A} = \frac{\mathrm{e}^{\mathrm{i}\frac{\eta}{2}} \frac{\sin\pi(2\sigma+m)}{\sin\pi(2\sigma-m)} - \mathrm{e}^{-\mathrm{i}\frac{\eta}{2}} \frac{\sin\pi(2\sigma-m)}{\sin\pi(2\sigma+m)}}{-2\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi m}\sin 2\pi\sigma} = -\mathrm{e}^{\pi \mathrm{i} m}\frac{\sin\frac{\tilde{\eta}}{2}}{\sin 2\pi\sigma}, \tag{4.51}$$

$$f_{12}^{B/A} = -\mathrm{e}^{-\pi \mathrm{i} m}\frac{\sin\frac{\eta}{2}}{\sin 2\pi\sigma} = \frac{\mathrm{e}^{\mathrm{i}\frac{\tilde{\eta}}{2}} \frac{\sin\pi(2\sigma-m)}{\sin\pi(2\sigma+m)} - \mathrm{e}^{-\frac{\mathrm{i}\tilde{\eta}}{2}} \frac{\sin\pi(2\sigma+m)}{\sin\pi(2\sigma-m)}}{-2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi m}\sin 2\pi\sigma}, \tag{4.52}$$

[15] We are not talking here about the singularities at $\sigma \in \frac{1}{2}\mathbb{Z}$. These cannot be removed by switching to another chart and they are completely forbidden in this framework. The singularities at $m \in \frac{1}{2} + \mathbb{Z}$ are not visible in the spectral problem and will not appear in the normalizability conditions.

$$f_{21}^{C/A} = \frac{\mathrm{i}e^{\mathrm{i}\pi m - \mathrm{i}\tilde{\eta} + 2\pi\mathrm{i}\sigma}}{\sin^2 2\pi\sigma}\left(e^{\mathrm{i}\tilde{\eta} - 2\pi\mathrm{i}\sigma}\cos\pi(\sigma - \frac{m}{2}) - \cos\pi(\sigma + \frac{m}{2})\right)$$
$$\times\left(e^{\mathrm{i}\tilde{\eta} - 2\pi\mathrm{i}\sigma}\sin\pi(\sigma - \frac{m}{2}) + \sin\pi(\sigma + \frac{m}{2})\right), \tag{4.53}$$

$$f_{12}^{C/A} = \frac{\mathrm{i}e^{-\mathrm{i}\pi m - \mathrm{i}\eta + 2\pi\mathrm{i}\sigma}}{\sin^2 2\pi\sigma}\left(e^{\mathrm{i}\eta - 2\pi\mathrm{i}\sigma}\cos\pi(\sigma + \frac{m}{2}) - \cos\pi(\sigma - \frac{m}{2})\right)$$
$$\times\left(e^{\mathrm{i}\eta - 2\pi\mathrm{i}\sigma}\sin\pi(\sigma + \frac{m}{2}) + \sin\pi(\sigma - \frac{m}{2})\right). \tag{4.54}$$

We conclude that the normalizability condition for $\mathcal{C} = A$ is

$$\begin{aligned} \left(f_{21}^{B/A}\right)^{-1} &= 0, \quad \text{if } m > 0, \\ \left(f_{12}^{B/A}\right)^{-1} &= 0, \quad \text{if } m < 0. \end{aligned} \tag{4.55}$$

If $\mathcal{C} = B$ or $\mathcal{C} = C$ instead we have

$$\begin{aligned} f_{21}^{\mathcal{C}/A} &= 0, \quad \text{if } m > 0\,, \\ f_{12}^{\mathcal{C}/A} &= 0, \quad \text{if } m < 0\,. \end{aligned} \tag{4.56}$$

We write all these conditions explicitly in Table 2.

4.5. Quantization conditions and spectrum. Following the general approach presented in Sect. 2, we want to test that the singularities matching condition (4.19), combined with the normalizability conditions of Table 2, reproduces the correct spectrum of the operators in Table 1. We work out in details the cases # 1 and # 2 of Table 1. The other examples work analogously.

4.5.1. Case # 2 Let us first focus on case # 2 of Table 1. The operator we consider is

$$\mathrm{O}_{\mp} = -\partial_x^2 + (m^2 \mp m)\wp(x|\frac{\mathrm{i}}{\mathfrak{t}}), \quad x \in [0, 1], \quad \mathfrak{t} \in \mathbb{R}_+ \tag{4.57}$$

on $L^2[0, 1]$.

Let us first consider the case $m < 0$. The relevant conditions can be written as ($\tau = \mathrm{i}\mathfrak{t}$)

$$\theta_2(0|2\tau) Z_0^D(\sigma, m, \eta, \tau) - \theta_3(0|2\tau) Z_{1/2}^D(\sigma, m, \eta, \tau) = 0 \quad \text{with} \quad \eta \in 2\pi\mathbb{Z}. \tag{4.58}$$

Notice that there are two inequivalent values of η in (4.58)

$$\eta = \begin{cases} 0 & \mod 4\pi, \\ 2\pi & \mod 4\pi. \end{cases} \tag{4.59}$$

These correspond to even and odd eigenvalues of (4.57). Moreover, the solutions to (4.58) reproduce the spectrum of both O_+ and O_-. Hence it is useful to introduce the notation

$$\eta_n^+ = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2\pi & \text{if } n \text{ is odd} \end{cases}, \quad \eta_n^- = 2\pi - \eta_n. \tag{4.60}$$

Table 3. The ground state energy of $\mathrm{O}_{\mp}$ in (4.57) as computed from (4.58) and (4.64) for $t = 1$, $m = -\sqrt{6}$. We denote by Nb the order q^{Nb} at which we truncate the instanton partition function $Z(\sigma, m, q)$ in (4.13). We underline the stable digits. The numerical result reported in the last line is performed as "Appendix B.1", see also [82,83]

Nb	E_0^+	E_0^-
1	48.68163544578	92.4799329375161
3	48.43513749440	91.8587660448900
5	48.43513819947	91.8587662451199
Num	48.43513819950	91.8587662451245

We think of (4.58) as a quantization condition for σ. More precisely, we can organise the zeroes of (4.58) in ascending order

$$(\sigma, \eta) \in \{(\sigma_0, 0), (\sigma_1, 2\pi), (\sigma_2, 2\pi), (\sigma_3, 0), \dots\}. \tag{4.61}$$

This sequence contains both the spectrum of O_+ and O_-. Experimentally, we find that a pattern to disentangle them is the following. If we wish to study the operator O_- in (4.57), then we have to consider the subset of (4.61) given by

$$(\sigma, \eta) \in \{(\sigma_{2n+1}, \eta_n^-)\}_{n\geq 0} = \{(\sigma_n^-, \eta_n^-)\}_{n\geq 0}. \tag{4.62}$$

Instead, if we wish to study the operator O_+ in (4.57), then we have to consider the subset of (4.61) given by

$$(\sigma, \eta) \in \{(\sigma_{2n}, \eta_n^+)\}_{n\geq 0} = \{(\sigma_n^+, \eta_n^+)\}_{n\geq 0}. \tag{4.63}$$

To obtain the exact spectrum of (4.57) one also needs the relation $E(\tau)$ reported in the last column of Table 1. More precisely, the energy levels $E_n^{\mp}$ of $\mathrm{O}_{\mp}$ are obtained from $(\sigma_n^{\pm}, \eta_n^{\pm})$ as

$$E_n^{\mp}(m, \mathfrak{t}) = \mathfrak{t}^2 \left((H_\star^{\mp})^{(n)} + 2(m^2 \mp m)\eta_1(\mathrm{i}\mathfrak{t}) \right), \tag{4.64}$$

with

$$\begin{aligned} (H_\star^{\mp})^{(n)} = &\left(2\pi \mathrm{i} \partial_\tau \log Z_0^D(\sigma_n^{\mp}, m, \eta_n^{\mp}, \tau) \right) |_{\tau=\mathrm{i}t} \\ &+ 2\pi \mathrm{i} \partial_\tau \left(\log \frac{\eta(\tau)}{\theta_3(0|2\tau)} \right) \Big|_{\tau=\mathrm{i}\mathfrak{t}} \mp 2m \frac{\theta_3''(0|2\mathrm{i}\mathfrak{t})}{\theta_3(0|2\mathrm{i}\mathfrak{t})}, \end{aligned} \tag{4.65}$$

where $'$ denotes the derivative w.r.t. the first argument of the θ function. Some independent tests are provided in Table 3.

If $m > 0$, instead of (4.58) we have

$$\begin{aligned} &\theta_2(0|2\tau) Z_0^D(\sigma, m, \eta, \tau) - \theta_3(0|2\tau) Z_{1/2}^D(\sigma, m, \eta, \tau) = 0, \\ &\mathrm{e}^{\mathrm{i}\frac{\eta}{2}} = \mathrm{e}^{\mathrm{i}\frac{\tilde{\eta}}{2}} \frac{\sin \pi(2\sigma - m)}{\sin \pi(2\sigma + m)}, \quad \tilde{\eta} \in 2\pi\mathbb{Z}. \end{aligned} \tag{4.66}$$

One can perform the same analysis as before. In particular, the zeros of (4.66) are mapped to the spectrum of $\mathrm{O}_{\mp}$ by using (4.64). The only subtlety is that, if $m > 0$, we impose $\tilde{\eta} \in 2\pi\mathbb{Z}$ instead of $\eta \in 2\pi\mathbb{Z}$ as summarised in Table 2.

4.5.2. Case # 1 We study

$$\mathrm{O}_{\mp} = -\partial_x^2 + (m^2 \mp m)\wp(x|i\mathfrak{t}), \quad x \in [0,1], \quad \mathfrak{t} \in \mathbb{R}_+. \tag{4.67}$$

We focus on $m > 1$ without loss of generality. The normalizability condition in Table 2 gives

1. η finite
2. $\sigma = \sigma_{1,2}$, where

$$2\sigma_1 = m + k_1 \quad \text{or} \quad 2\sigma_2 = -m + k_2 \tag{4.68}$$

   with $k_\ell \in \mathbb{Z}$, $\ell = 1, 2$.

Hence we look at the singularities matching condition (4.19) as an equation for η. More precisely, we should find a solution of (4.19) in a form

$$\eta = \eta_\star(\sigma, m, \tau). \tag{4.69}$$

To do this we substitute an Ansatz

$$\mathrm{e}^{\mathrm{i}\eta_\star/2} = \mathfrak{q}^{-\sigma}\mathrm{e}^{\mathrm{i}\eta_0/2}\mathrm{e}^{\sum_{i=1}^{\infty} c_i \mathfrak{q}^i}. \tag{4.70}$$

At the first non-trivial level such substitution gives the quadratic equation:

$$\begin{aligned} &2\frac{G(1-m-2\sigma)G(1-m+2\sigma)}{G(1-2\sigma)G(1+2\sigma)} - \mathrm{e}^{\mathrm{i}\eta_0/2}\frac{G(1-m-2\sigma-1)G(1-m+2\sigma+1)}{G(1-2\sigma-1)G(1-2\sigma+1)} \\ &-\mathrm{e}^{-\mathrm{i}\eta_0/2}\frac{G(1-m-2\sigma+1)G(1-m+2\sigma-1)}{G(1-2\sigma+1)G(1-2\sigma-1)} = 0. \end{aligned} \tag{4.71}$$

Its two solutions are

$$\mathrm{e}^{\mathrm{i}\eta_0/2} = \frac{(2\sigma+m)\Gamma(-m-2\sigma)\Gamma(2\sigma)}{(2\sigma-m)\Gamma(-m+2\sigma)\Gamma(-2\sigma)}, \quad \mathrm{e}^{\mathrm{i}\eta_0/2} = \frac{\Gamma(-m-2\sigma)\Gamma(2\sigma)}{\Gamma(-m+2\sigma)\Gamma(-2\sigma)}. \tag{4.72}$$

By solving (4.19) iteratively as a power series in $\mathfrak{q}$, we get the full answers in both these cases:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\eta_\star^-/2} &= -\mathfrak{q}^{-\sigma}\frac{\Gamma(1-m-2\sigma)\Gamma(2\sigma)}{\Gamma(1-m+2\sigma)\Gamma(-2\sigma)}\exp\left(\frac{8m^2(m-1)^2}{(1-4\sigma^2)^2}\mathfrak{q} + \mathcal{O}(\mathfrak{q}^2)\right) \\ &= \exp\left(\partial_\sigma F^{\mathrm{NS}}(\sigma, m - \frac{1}{2}, \mathfrak{q})/2\right), \end{aligned} \tag{4.73}$$

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\eta_\star^+/2} &= \mathfrak{q}^{-\sigma}\frac{\Gamma(-m-2\sigma)\Gamma(2\sigma)}{\Gamma(-m+2\sigma)\Gamma(-2\sigma)}\exp\left(\frac{8m^2(m+1)^2}{(1-4\sigma^2)^2}\mathfrak{q} + \mathcal{O}(\mathfrak{q}^2)\right) \\ &= \exp\left(\partial_\sigma F^{\mathrm{NS}}(\sigma, m + \frac{1}{2}, \mathfrak{q})/2\right), \end{aligned} \tag{4.74}$$

where we used the definition of F^{NS} in (D.15).[16] The appearance of this quantity will be clarified in Sect. 5. The two solutions $\mathrm{e}^{\mathrm{i}\eta_\star^\pm/2}$ correspond to the two operators $\mathrm{O}_\mp$ in (4.67). We can focus without loss of generality on O_-.

[16] In principle, formulas (4.74), (4.72) work for both cases # 1 and # 2 as we will also see in Sect. 5. However, in # 2 it is more natural to think of the normalizability equation as an equation for σ instead of an equation for η. But the two approaches are equivalent.

If now we impose the normalizability condition for σ (4.68) on this solution, we get

$$e^{i\eta_\star^-/2}\,|_{\sigma=\sigma_\ell} = -\,\mathfrak{q}^{(-1)^\ell m/2-k_\ell/2}\frac{\Gamma(1-(-1)^\ell m-k_\ell)\Gamma((-1)^{\ell+1}m+k_\ell)}{\Gamma(1-(-1)^{\ell+1}m+k_\ell)\Gamma((-1)^\ell m-k_\ell)}(1+\mathcal{O}(\mathfrak{q})). \tag{4.75}$$

Now we consider the condition that η is finite. This means that the gamma functions should not have poles nor zeroes. For σ_1 (i.e. $\ell = 1$ in (4.75)) this means that

$$k_1 \geq 1. \tag{4.76}$$

For σ_2 instead we have

$$k_2 \leq -1. \tag{4.77}$$

In other words, this means

$$2\sigma_\ell = (-1)^{\ell+1}(m+k), \quad k \geq 1, \quad \ell = 1, 2. \tag{4.78}$$

To compute the energy we use Table 2 as well as (4.26)

$$E = -H_\star^-(\sigma, m, \eta_\star^-, \tau) - 2m(m-1)\eta_1(\tau)\Big|_{\sigma=\pm(m-k)/2}. \tag{4.79}$$

The first terms of the expansion are

$$E = -\frac{\pi^2}{3}m(m-1) + \pi^2(m+k)^2 + 8\pi^2 m(m-1)\mathfrak{q}\left(1+\frac{m(m-1)}{(m+k)^2-1}\right) + \mathcal{O}(\mathfrak{q})^2, \tag{4.80}$$

which coincides with perturbative calculation (B.5) (see also [83, Sec.2]).

5. Nekrasov–Shatashvili Quantization from Kyiv Formula

The operators discussed above have an interpretation as (four-dimensional) quantum Seiberg–Witten curves. In particular they also appear in the work of Nekrasov and Shatashvili (NS) in the context of the Bethe/gauge correspondence for (non-relativistic) quantum integrable models [57,58]. In this section we show that the exact quantization condition proposed by [57,58] can in fact be derived from the approach based on the tau function of isomonodromic problems presented above. The key ingredients in this analysis are the Kyiv formulas for tau functions [32,41,49], as well as Nakajima–Yoshioka blowup equations [9,10].

The relation between Painlevé and blowup equations has appeared before in the literature. For example in [42,84] blowup equations on $\mathbb{C}^2/\mathbb{Z}^2$ were used to prove the Kyiv formula [41] and its q-deformation [85]. More recently an alternative proof for the Painlevé VI example was presented in [68,69] based on blowup equation with defects. The interplay between Painlevé and blowup equations appearing in this section is similar to the one of [61,64,65] and does not require any defect.

In order to write differential blowup relations we will use the Hirota differential operators $\mathrm{D}^k_{\epsilon_1,\epsilon_2}$ with respect to $\log\mathfrak{q}$ which are defined by the formula

$$F(\mathfrak{q}e^{\epsilon_1\hbar})G(\mathfrak{q}e^{\epsilon_2\hbar}) = \sum \frac{\hbar^k}{k!}\mathrm{D}^k_{\epsilon_1,\epsilon_2}(F,G). \tag{5.1}$$

For example $\mathrm{D}^1_{\epsilon_1,\epsilon_2}(F,G) = \epsilon_1 G\ \partial_{\log\mathfrak{q}} F + \epsilon_2 F\ \partial_{\log\mathfrak{q}} G$.

5.1. Modified Mathieu. In this section we prove that (3.32) and (3.30) lead to the quantization condition obtained in [57,86] and proven in [87].

The starting point are Nakajima–Yoshioka blowup equations [9] for Nekrasov partition function of pure $\mathcal{N}=2$, $SU(2)$ Seiberg–Witten theory in the four-dimensional Ω background. Such partition function is denoted by $\mathcal{Z}(a,\epsilon_1,\epsilon_2,t)$, see for instance [64, Sec 4.1] for a complete definition and more references. In this paper we are interested in two special limits of this function

The first one is the self-dual limit where $\epsilon_2 \to -\epsilon_1$. In this case we have

$$\mathcal{Z}(a,\epsilon_1,\epsilon_2,t) \xrightarrow{\epsilon_2=-\epsilon_1=-1, a=\sigma} \sim \mathfrak{q}^{\sigma^2}\frac{1}{\prod_{\epsilon'=\pm} G(1+2\epsilon'\sigma)} Z(\sigma,t), \tag{5.2}$$

where $Z(\sigma,t)$ is the $c=1$ Virasoro conformal block appearing in (3.11), and $\sim$ stands for the constant factor.

The second limit is the Nekrasov–Shatashvili limit $\epsilon_2 \to 0$. In this case we have

$$\epsilon_2 \log\left(\mathcal{Z}(\sigma,1,\epsilon_2,t)\right) \xrightarrow{\epsilon_2\to 0} -F_{\mathrm{NS}}(\sigma,t), \tag{5.3}$$

where F_{NS} is the NS free energy for the pure $\mathcal{N}=2$, $SU(2)$ four-dimensional theory. More precisely we have

$$F_{\mathrm{NS}}(\sigma,t) = -\psi^{(-2)}(1+2\sigma) - \psi^{(-2)}(1-2\sigma) + \sigma^2\log(t) + F^{\mathrm{NS}}_{\mathrm{inst}}(\sigma,t), \tag{5.4}$$

where ψ is the polygamma function, and $F^{\mathrm{NS}}_{\mathrm{inst}}$ is the instanton part of the NS free energy (or logarithm of $c\to\infty$ Virasoro conformal blocks). The first few terms read

$$F^{\mathrm{NS}}_{\mathrm{inst}}(\sigma,t) = -\frac{2t}{-4\sigma^2+1} + \frac{t^2\left(7+20\sigma^2\right)}{\left(-4\sigma^2+1\right)^3\left(-4\sigma^2+4\right)} + \mathcal{O}(t^3). \tag{5.5}$$

Higher order terms can be computed by using combinatorics and Young diagrams, we refer to [64, Sec. 4.1] for the details of the definition and a list of references.

It was shown in [61] that the two limits introduced above are in fact closely related by the Nakajima–Yoshioka blowup equations [9] on $\mathbb{C}^2$.

We have several blowup relations on $\mathbb{C}^2$. One relation is [10, Eq. (5.3)]

$$\sum_{n\in\mathbb{Z}+1/2} \mathcal{Z}(a+n\epsilon_1,\epsilon_1,-\epsilon_1+\epsilon_2;z)\mathcal{Z}(a+n\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2;z) = 0. \tag{5.6}$$

As in [61], we take the limit $\epsilon_2 \to 0$ of (5.6) and we get

$$\mathcal{T}_1(\sigma,\eta,t) = 0, \quad \text{for } \eta = \mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma,t). \tag{5.7}$$

This means that if we look at the singularities matching condition (3.22) as an equation for η, the solution is

$$\eta = \mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma, t). \tag{5.8}$$

Moreover, if in addition we impose (3.29) we get

$$\partial_\sigma F^{\mathrm{NS}}(\sigma, t) = 2\mathrm{i}\pi n, \quad n = 1, 2, \dots . \tag{5.9}$$

This is the quantization condition proposed in [57], where it was found that the solutions $\{\sigma_n\}_{n>0}$ of (5.9) are related to the spectrum $\{E_n\}_{n>0}$ of (3.20) via the Matone relation [88–91]

$$E_n(t) = -t\partial_t F^{\mathrm{NS}}(\sigma_n, t). \tag{5.10}$$

To reproduce Matone relation from the point of view of isomonodromic deformations we need another blowup equation which takes the form of a differential bilinear relation and reads [9, eq. (6.14)]

$$\sum_{n\in\mathbb{Z}} \mathrm{D}^1_{\epsilon_1,\epsilon_2}\Big(\mathcal{Z}(a+n\epsilon_1, \epsilon_1, -\epsilon_1+\epsilon_2; t), \mathcal{Z}(a+n\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2; t)\Big) = 0, \tag{5.11}$$

where $\mathrm{D}^1_{\epsilon_1,\epsilon_2}$ was defined in (5.1) . In the limit $\epsilon_2 \to 0$ this equation becomes

$$(t\partial_t \log \mathcal{T}_0(\sigma, \eta, t))\Big|_{\eta=\mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma,t)} = (t\partial_t F^{\mathrm{NS}}(\sigma, t)). \tag{5.12}$$

Note that the t derivative on the l.h.s. does not act on η. Hence we have an equivalence between Matone relation (5.10) and the Hamiltonian (3.32) of Painlevé III_3. This concludes the derivation of the NS quantization from the Kyiv formula.

5.2. Weierstrass Potential. The starting point are Nakajima–Yoshioka blowup equations for Nekrasov partition function of $\mathcal{N} = 2^*$ $SU(2)$ Seiberg–Witten theory in the four-dimensional Ω background. Such partition function is denoted by

$$\mathcal{Z}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}). \tag{5.13}$$

One can find the definition in (D.1). For the purpose of this paper we are interested only in two limits of (5.13).[17] In the self-dual limit we have

$$\mathcal{Z}(a, \alpha, \epsilon_1, \epsilon_2|\mathfrak{q}) \xrightarrow{\epsilon_2=-\epsilon_1=1, a=\sigma, \alpha=m} \mathfrak{q}^{\sigma^2}(2\pi)^{-m}\frac{\prod_{\epsilon'=\pm} G(1-m+2\epsilon'\sigma)}{\prod_{\epsilon'=\pm} G(1+2\epsilon'\sigma)} Z(\sigma, m, \mathfrak{q}), \tag{5.14}$$

where $Z(\sigma, m, \mathfrak{q})$ is the $c = 1$ conformal block on the torus as in (4.13). In the NS limit we have

$$\epsilon_2 \log(\mathcal{Z}(\sigma, m, 1, \epsilon_2|\mathfrak{q})) \xrightarrow{\epsilon_2\to 0} F^{\mathrm{NS}}(\sigma, m - \frac{1}{2}, \mathfrak{q}), \tag{5.15}$$

where F^{NS} is defined in (D.15).

[17] In such limits we can absorb ϵ_1 into a redefinition of parameters. So we set it to $\epsilon_1 = 1$.

5.2.1. Blowup relations We first note that Nakajima–Yoshioka blowup relations for the four dimensional $\mathcal{N} = 2^*$ theory were not written explicitly in the literature. In this section we list the relevant relations that are used in the paper. Some of them have been worked out by one of us (MB) together with A. Litvinov and A. Shchechkin some time ago. The five-dimensional version of some of these equations was recently obtained in [63, Sec. 3.1].

We start with two algebraic blowup relations on $\mathbb{C}^2$

$$\begin{aligned}
&\frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})}\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&\quad=\sum_{n\in\mathbb{Z}}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q}),\\
&\frac{\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&\quad=\sum_{n\in\mathbb{Z}+\frac12}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q}).
\end{aligned}\tag{5.16}$$

Using the symmetry (D.9) we can obtain two more equations:

$$\begin{aligned}
&\frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})}\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&\quad=\sum_{n\in\mathbb{Z}}\mathcal{Z}(a+n\epsilon_1,\alpha-\epsilon_1;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha-\epsilon_2;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q}),\\
&\frac{\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&\quad=\sum_{n\in\mathbb{Z}+\frac12}\mathcal{Z}(a+n\epsilon_1,\alpha-\epsilon_1;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha-\epsilon_2;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q}).
\end{aligned}\tag{5.17}$$

There are also differential relations which are written in terms of the Hirota differential operators defined in (5.1). The first order relations are

$$\begin{aligned}
&\Big((\epsilon_1+\epsilon_2)\beta_0^{1,1}(\mathfrak{q})+\alpha\beta_0^{1,2}(\mathfrak{q})\Big)\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&\quad=\sum_{n\in\mathbb{Z}}\mathrm{D}^1_{\epsilon_1,\epsilon_2}\big(\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q}),\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})\big),
\end{aligned}\tag{5.18}$$

$$\begin{aligned}
&\Big((\epsilon_1+\epsilon_2)\beta_1^{1,1}(\mathfrak{q})+\alpha\beta_1^{1,2}(\mathfrak{q})\Big)\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&\quad=\sum_{n\in\mathbb{Z}+\frac12}\mathrm{D}^1_{\epsilon_1,\epsilon_2}\big(\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q}),\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})\big),
\end{aligned}\tag{5.19}$$

where

$$\beta_0^{1,1}(\mathfrak{q})=\partial_{\log\mathfrak{q}}\Big(\frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})}\Big),\quad \beta_0^{1,2}(\mathfrak{q})=2\frac{\partial_{\log\mathfrak{q}}\theta_3(0|2\tau)}{\varphi(\mathfrak{q})},\tag{5.20}$$

$$\beta_1^{1,1}(\mathfrak{q})=\partial_{\log\mathfrak{q}}\Big(\frac{\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}\Big),\quad \beta_1^{1,2}(\mathfrak{q})=2\frac{\partial_{\log\mathfrak{q}}\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}.\tag{5.21}$$

There are also second order differential relations which look rather cumbersome

$$\Big((\epsilon_1{+}\epsilon_2)^2\beta_0^{2,1}(\mathfrak{q})+\alpha(\epsilon_1{+}\epsilon_2)\beta_0^{2,2}(\mathfrak{q})+\epsilon_1\epsilon_2\beta_0^{2,3}(\mathfrak{q})+\epsilon_1\epsilon_2\beta_0^{2,4}(\mathfrak{q})\partial_{\log\mathfrak{q}}\Big)\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})$$
$$=\sum_{n\in\mathbb{Z}}\mathrm{D}^2_{\epsilon_1,\epsilon_2}\big(\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})\big), \tag{5.22}$$

$$\Big((\epsilon_1{+}\epsilon_2)^2\beta_1^{2,1}(\mathfrak{q})+\alpha(\epsilon_1{+}\epsilon_2)\beta_1^{2,2}(\mathfrak{q})+\epsilon_1\epsilon_2\beta_1^{2,3}(\mathfrak{q})+\epsilon_1\epsilon_2\beta_1^{2,4}(\mathfrak{q})\partial_{\log\mathfrak{q}}\Big)\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})$$
$$=\sum_{n\in\mathbb{Z}+\frac{1}{2}}\mathrm{D}^2_{\epsilon_1,\epsilon_2}\big(\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})\big), \tag{5.23}$$

where

$$\beta_0^{2,1}(\mathfrak{q})=\partial^2_{\log\mathfrak{q}}\frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})},\ \ \beta_0^{2,2}(\mathfrak{q})=4\partial_{\log\mathfrak{q}}\frac{\partial_{\log\mathfrak{q}}\theta_3(0|2\tau)}{\varphi(\mathfrak{q})},\ \ \beta_0^{2,4}(\mathfrak{q})=-4\frac{\partial_{\log\mathfrak{q}}\theta_3(0|2\tau)}{\varphi(\mathfrak{q})} \tag{5.24}$$

$$\beta_0^{2,3}(\mathfrak{q})=\left(-\frac{1}{3}\frac{\partial_{\log\mathfrak{q}}\theta_3(0|2\tau)}{\varphi(\mathfrak{q})}-4\frac{\partial_{\log\mathfrak{q}}\theta_3(0,2\tau)\partial_{\log\mathfrak{q}}\varphi(\mathfrak{q})}{\varphi(\mathfrak{q})^2}+\frac{4}{3}\frac{\partial^2_{\log\mathfrak{q}}\theta_3(0|2\tau)}{\varphi(\mathfrak{q})}\right), \tag{5.25}$$

$$\beta_1^{2,1}(\mathfrak{q})=\partial^2_{\log\mathfrak{q}}\frac{\theta_2(0|2\tau)}{\varphi(\mathfrak{q})},\ \ \beta_1^{2,2}(\mathfrak{q})=4\partial_{\log\mathfrak{q}}\frac{\partial_{\log\mathfrak{q}}\theta_2(0|2\tau)}{\varphi(\mathfrak{q})},\ \ \beta_1^{2,4}(\mathfrak{q})=-4\frac{\partial_{\log\mathfrak{q}}\theta_2(0|2\tau)}{\varphi(\mathfrak{q})} \tag{5.26}$$

$$\beta_1^{2,3}(\mathfrak{q})=\left(-\frac{1}{3}\frac{\partial_{\log\mathfrak{q}}\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}-4\frac{\partial_{\log\mathfrak{q}}\theta_2(0,2\tau)\partial_{\log\mathfrak{q}}\varphi(\mathfrak{q})}{\varphi(\mathfrak{q})^2}+\frac{4}{3}\frac{\partial^2_{\log\mathfrak{q}}\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}\right). \tag{5.27}$$

We will not use the relations (5.22), (5.23) to study the Nekrasov–Shatashvili quantization conditions, but they will be used later in Sect. 6.

We do not claim that this is the full list of blowup relations, this is just the list needed in this paper. Note also that, as far as we know, there is no rigorous proof of these relations. However we believe that this can be done either by using the geometric methods of [9,92] or by using the representation theory methods of [93].

5.2.2. From blowup to NS quantization conditions and spectrum By combining the relations (5.16) we obtain

$$\theta_2(0|2\tau)\sum_{n\in\mathbb{Z}}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})$$
$$-\theta_3(0|2\tau)\sum_{n\in\mathbb{Z}+\frac{1}{2}}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a$$
$$+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})=0. \tag{5.28}$$

Similarly from (5.17) we obtain

$$\theta_2(0|2\tau)\sum_{n\in\mathbb{Z}}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha{+}\epsilon_1-\epsilon_2;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})$$
$$-\theta_3(0|2\tau)\sum_{n\in\mathbb{Z}+\frac{1}{2}}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+n\epsilon_2,\alpha{+}\epsilon_1-\epsilon_2;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})=0. \tag{5.29}$$

Taking the NS limit $a = \sigma, \alpha = m, \epsilon_1 = 1, \epsilon_2 \to 0$ we are left with[18]

$$\begin{aligned} &\theta_2(0|2\tau) Z_0^D(\sigma, m, \eta_\star^-, \tau) - \theta_3(0|2\tau) Z_{1/2}^D(\sigma, m, \eta_\star^-, \tau) = 0, \\ &\quad \eta_\star^- = -\mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma, m - \frac{1}{2}, \mathfrak{q}), \end{aligned} \tag{5.30}$$

$$\begin{aligned} &\theta_2(0|2\tau) Z_0^D(\sigma, m, \eta_\star^+, \tau) - \theta_3(0|2\tau) Z_{1/2}^D(\sigma, m, \eta_\star^+, \tau) = 0, \\ &\quad \eta_\star^+ = -\mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma, m + \frac{1}{2}, \mathfrak{q}). \end{aligned} \tag{5.31}$$

In turn, this means that if we consider (4.19) as an equation for η, then we have two solutions: $\eta_\star^-$ and $\eta_\star^+$. The solution $\eta_\star^-$ makes contact with the operator O_-, while the solution $\eta_\star^+$ makes contact with the operator O_+. This is in perfect agreement with what we discussed around (4.61) and (4.73).

Let us now consider the operator corresponding to the case #2 in Table 1 (the other cases work analogously). From the above discussion it follows that the singularities matching condition and the normalizability of the linear problem are equivalent to

$$\begin{aligned} &\partial_\sigma F^{\mathrm{NS}}(\sigma, m \mp \frac{1}{2}, \mathfrak{q}) = 2\pi\mathrm{i}(n+1), \quad n = 0, 1, 2, \ldots \quad \text{if} \quad m < -1, \\ &\partial_\sigma F^{\mathrm{NS}}(\sigma, -m \pm \frac{1}{2}, \mathfrak{q}) = 2\pi\mathrm{i}(n+1), \quad n = 0, 1, 2, \ldots \quad \text{if} \quad m > 1. \end{aligned} \tag{5.32}$$

This is precisely the quantization condition proposed in [57,58] where it was found that the solutions $\{\sigma_n\}_{n\geq 0}$ of (5.32) are related to the spectrum $\{E_n\}_{n\geq 0}$ of $\mathrm{O}_\mp$ via Matone relation

$$E_n(t) = (\log \mathfrak{q})^2 \left(\mathfrak{q}\partial_{\mathfrak{q}} F^{\mathrm{NS}}(\sigma, m \mp \frac{1}{2}, \mathfrak{q}) + \frac{1}{4\pi^2} 2m(m \mp 1)\eta_1(\tau) \right). \tag{5.33}$$

As in Sect. 5.1, to reproduce Matone relation in the context of isomonodromic deformations, we need another set of blowup equations. These take the form of differential bilinear relations. It follows from (5.18) and (5.16) that

$$\begin{aligned} &\left((\epsilon_1 + \epsilon_2)\partial_{\log \mathfrak{q}} \big(\log \frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})} \big) + 2\alpha \frac{\partial_{\log \mathfrak{q}} \theta_3(0|2\tau)}{\theta_3(0|2\tau)} \right) \\ &\quad \times \sum_{n\in\mathbb{Z}} \mathcal{Z}(a + n\epsilon_1, \alpha; \epsilon_1, \epsilon_2 - \epsilon_1|\mathfrak{q}) \mathcal{Z}(a + n\epsilon_2, \alpha; \epsilon_1 - \epsilon_2, \epsilon_2|\mathfrak{q}) \\ &= \sum_{n\in\mathbb{Z}} \mathrm{D}^1_{\epsilon_1,\epsilon_2} \big(\mathcal{Z}(a + n\epsilon_1, \alpha; \epsilon_1, \epsilon_2 - \epsilon_1|\mathfrak{q}), \mathcal{Z}(a + n\epsilon_2, \alpha; \epsilon_1 - \epsilon_2, \epsilon_2|\mathfrak{q}) \big). \end{aligned} \tag{5.34}$$

Taking the NS limit $a = \sigma, \alpha = m, \epsilon_1 = 1, \epsilon_2 \to 0$ we get

$$\begin{aligned} &\left(\partial_{\log \mathfrak{q}} \big(\log \frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})} \big) + 2m \frac{\partial_{\log \mathfrak{q}} \theta_3(0|2\tau)}{\theta_3(0|2\tau)} \right) \mathfrak{q}^{1/24} Z_0^D(\sigma, m, \eta_\star^-, \tau) \\ &\quad = \partial_{\log \mathfrak{q}} \left(\mathfrak{q}^{1/24} Z_0^D(\sigma, m, \eta_\star^-, \tau) \right) + \mathfrak{q}^{1/24} Z_0^D(\sigma, m, \eta_\star^-, \tau) \partial_{\log \mathfrak{q}} F^{\mathrm{NS}}(\sigma, m - \frac{1}{2}, \mathfrak{q}), \end{aligned} \tag{5.35}$$

[18] The possibility of a relation between (4.9) and blowup equations has also been hypothesised by G. Bonelli, F. Del Monte, A. Tanzini in [94].

where we use

$$\eta_\star^- = -\mathrm{i}\partial_\sigma F^{\mathrm{NS}}(\sigma, m - \frac{1}{2}, \mathfrak{q}), \tag{5.36}$$

as in (5.30). Using (4.26) we obtain

$$\partial_{\log \mathfrak{q}} \log Z_0^D(\sigma, m, \eta_\star^-, \tau) = -\frac{1}{4\pi^2} H_\star^- - \partial_{\log \mathfrak{q}} \log \frac{\varphi(\mathfrak{q})}{\theta_3(0|2\tau)} - \frac{1}{24} + 2m \frac{\partial_{\log \mathfrak{q}} \theta_3(0|2\tau)}{\theta_3(0|2\tau)} \tag{5.37}$$

and get

$$\frac{1}{4\pi^2} H_\star^- = \mathfrak{q}\partial_\mathfrak{q} F^{\mathrm{NS}}(\sigma, m - \frac{1}{2}, \mathfrak{q}). \tag{5.38}$$

The relation for $\eta_\star^+$ can be obtained similarly by using the symmetry (D.9). We have

$$\frac{1}{4\pi^2} H_\star^+ = \mathfrak{q}\partial_\mathfrak{q} F^{\mathrm{NS}}(\sigma, m + \frac{1}{2}, \mathfrak{q}). \tag{5.39}$$

Hence we have a complete equivalence between the Hamiltonian (4.64) and Matone relation (5.33). This concludes the derivation of the NS quantization from the Kyiv formula on the torus.

As a final remark we note that, unlike in the example of modified Mathieu, to our knowledge there is no proof of the NS quantization for the example of the quantum elliptic Calogero–Moser system.[19] Our derivation here is based on monodromy arguments, which are rigorous, and blowup relations, which we believe can be proven.

6. Bilinear Relations on the Torus

In this section we show that the isomonodromic equation (4.4) in Q is equivalent to the bilinear relation (4.12) for the $\mathcal{T}$ function, more precisely for Z_ϵ^D. Moreover, by using blowup equations, we demonstrate that (4.13) indeed satisfies such bilinear relation. This provides an alternative proof for the work of [49]

6.1. From blowup relations. Let us first note that by substituting (4.13) into (4.12) one gets a bilinear relation for the function Z. As already noted in a related context [42], this type of relations cannot be a specialization of the $\mathbb{C}^2$ blowup equations. Usually such relations come from the so-called (-2) or $\mathbb{C}^2/\mathbb{Z}_2$ blowup equations. In this terminology the relations from Sect. 5.2.1 are called (-1) blowup equations. The (-2) blowup relations can be obtained using representation theory (see e.g. [42]) or algebraic geometry (see e.g. [98,99]) arguments. There is a transparent algebraic method to deduce them from the standard (-1) blowup relations.[20] This method goes back to the papers on Donaldson invariants [100,101]. Recently this method was applied to the case of the pure theory in [102]. Here we apply it to the $\mathcal{N} = 2^*$ case. As a result we get a first order differential relation

[19] There are however several independent tests which have been performed (see for instance [83, Sec.2], as well as [59,95–97], or also "Appendix B").

[20] We a grateful to H. Nakajima for explaining this idea and help with references.

$$\sum_{2n\in\mathbb{Z}} \mathrm{D}^1_{2\epsilon_1,2\epsilon_2}\big(\mathcal{Z}(a+2n\epsilon_1,\alpha;2\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+2n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,2\epsilon_2|\mathfrak{q})\big)$$
$$=(\epsilon_1+\epsilon_2)\gamma_0(\mathfrak{q})\sum_{2n\in\mathbb{Z}}\mathcal{Z}(a+2n\epsilon_1,\alpha;2\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+2n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,2\epsilon_2|\mathfrak{q}) \tag{6.1}$$

and a second order differential relation

$$\sum_{2n\in\mathbb{Z}} \mathrm{D}^2_{2\epsilon_1,2\epsilon_2}\big(\mathcal{Z}(a+2n\epsilon_1,\alpha;2\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+2n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,2\epsilon_2|\mathfrak{q})\big)$$
$$=\Big(\epsilon_1\epsilon_2\gamma_1(\mathfrak{q})+\alpha^2\gamma_2(\mathfrak{q})+(\epsilon_1+\epsilon_2)^2\gamma_3(\mathfrak{q})+\alpha(\epsilon_1+\epsilon_2)\gamma_4(\mathfrak{q})+\epsilon_1\epsilon_2\gamma_5(\mathfrak{q})\partial_{\log\mathfrak{q}}\Big)$$
$$\sum_{2n\in\mathbb{Z}}\mathcal{Z}(a+2n\epsilon_1,\alpha;2\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\mathcal{Z}(a+2n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,2\epsilon_2|\mathfrak{q}), \tag{6.2}$$

where the Hirota differential operators were defined in (5.1) and we use

$$\gamma_0(\mathfrak{q})=\log'\left(\frac{\theta_3^2+\theta_2^2}{\varphi^2}\right)=2\partial_{\log\mathfrak{q}}\log\left(\frac{\theta_3(0|\tau)}{\varphi(\mathfrak{q})}\right),$$
$$\gamma_2(\mathfrak{q})=-\gamma_4(\mathfrak{q})=-8\frac{(\theta_3')^2+(\theta_2')^2}{\theta_3^2+\theta_2^2}=-4\big(\partial^2_{\log\mathfrak{q}}\log(\theta_3(0|\tau))\big),$$
$$\gamma_5(\mathfrak{q})=-8\log'(\theta_3^2+\theta_2^2)=-16\big(\partial_{\log\mathfrak{q}}\log(\theta_3(0|\tau))\big),$$
$$\gamma_1(\mathfrak{q})=-8\frac{\theta_3^2\log''(\theta_3/\varphi)+\theta_2^2\log''(\theta_2/\varphi)}{\theta_2^2+\theta_3^2}+8\log'\left(\frac{\theta_3^2+\theta_2^2}{\varphi^2}\right)\log'(\theta_3^2+\theta_2^2)$$
$$=8\big(\partial^2_{\log\mathfrak{q}}\log\varphi(\mathfrak{q})\big)-32\big(\partial_{\log\mathfrak{q}}\log\varphi(\mathfrak{q})\big)\big(\partial_{\log\mathfrak{q}}\log(\theta_3(0|\tau))\big)$$
$$+16\big(\partial_{\log\mathfrak{q}}\log\theta_3(0|\tau)\big)^2,$$
$$\gamma_3(\mathfrak{q})=\frac{2(3\varphi'\varphi'-\varphi\varphi'')}{\varphi^2}-\frac{16\varphi'(\theta_3\theta_3'+\theta_2\theta_2')}{\varphi(\theta_3^2+\theta_2^2)}$$
$$-\frac{18(\theta_3'\theta_3'+\theta_2'\theta_2')+10(\theta_3\theta_3''+\theta_2\theta_2'')-(\theta_3\theta_3'+\theta_2\theta_2')}{3(\theta_3^2+\theta_2^2)}.$$

with the understanding that $\theta_i=\theta_i(0|2\tau)$, $\varphi=\varphi(\mathfrak{q})$, and $'$ means that the derivatives are taken with respect to $\log\mathfrak{q}$. We give a detailed proof of the relation (6.1) in "Appendix E.1". The proof of the relation (6.2) is based on the same ideas, but involves more cumbersome calculations.

After the substitution $\epsilon_2=-\epsilon_1=\frac{1}{2}$, $a=\sigma$, $\alpha=m$, Eq. (6.2) leads to the following bilinear relation for the self-dual Nekrasov function $\mathcal{Z}(\sigma,m,-1,1|\mathfrak{q})$

$$\sum_{2n\in\mathbb{Z}}\mathrm{D}^2_{-1,1}\big(\mathcal{Z}(\sigma-n,m,-1,1|\mathfrak{q})\mathcal{Z}(\sigma+n,m,-1,1|\mathfrak{q})\big)$$
$$=\Big(-\frac{1}{4}\gamma_1(\mathfrak{q})+m^2\gamma_2(\mathfrak{q})-\frac{1}{4}\gamma_5(\mathfrak{q})\partial_{\log\mathfrak{q}}\Big)\sum_{2n\in\mathbb{Z}}\big(\mathcal{Z}(\sigma-n,m,-1,1|\mathfrak{q})\mathcal{Z}(\sigma+n,m,-1,1|\mathfrak{q})\big). \tag{6.3}$$

This is equivalent to

$$\begin{aligned}&\mathrm{D}^2_{-1,1}(Z_0^D, Z_0^D) + \mathrm{D}^2_{-1,1}(Z_{1/2}^D, Z_{1/2}^D)\\&= \Big(-\frac{1}{4}(\gamma_1(\mathfrak{q}) - \frac{1}{12}\gamma_5(\mathfrak{q})) + m^2\gamma_2(\mathfrak{q}) - \frac{1}{4}\gamma_5(\mathfrak{q})\partial_{\log\mathfrak{q}}\Big)\big(Z_0^D Z_0^D + Z_{1/2}^D Z_{1/2}^D\big).\end{aligned} \tag{6.4}$$

Indeed, by comparing terms with given η exponents (say $\mathrm{e}^{\mathrm{i}k\eta}$ in (6.4)) we obtain (6.3). This is the same argument as in [42, Sec. 4.2]. Using the notation (4.11) and the relation $\mathrm{D}^2_{-1,1}(F, F) = 2F^2\big(\partial^2_{\log\mathfrak{q}} \log F\big)$ we get

$$\begin{aligned}&(\tilde{Z}_0^D)^2\partial^2_{\log\mathfrak{q}} \log \tilde{Z}_0^D + (\tilde{Z}_{1/2}^D)^2\partial^2_{\log\mathfrak{q}} \log \tilde{Z}_{1/2}^D\\&= \Big(-2\big(\partial_{\log\mathfrak{q}} \log(\theta_3(0|\tau))\big)^2 - 2m^2\big(\partial^2_{\log\mathfrak{q}} \log(\theta_3(0|\tau))\big) + 2\big(\partial_{\log\mathfrak{q}} \log(\theta_3(0|\tau))\big)\partial_{\log\mathfrak{q}}\Big)\\&\quad\big(\tilde{Z}_0^D\tilde{Z}_0^D + \tilde{Z}_{1/2}^D\tilde{Z}_{1/2}^D\big),\end{aligned} \tag{6.5}$$

which is precisely equation (4.12). To have a better intuition on the meaning of (6.5) let us consider two particular cases.

Example 1. Let $m = 0$. Then we have $Z(\sigma, m, \mathfrak{q}) = 1/\varphi(\mathfrak{q})$ and get (cf. formulas (4.5) and (4.14))

$$\tilde{Z}_0^D = \mathfrak{q}^{\sigma^2}\theta_3(\frac{\eta}{2\pi} + 2\sigma\tau|2\tau), \quad \tilde{Z}_{1/2}^D = \mathfrak{q}^{\sigma^2}\theta_2(\frac{\eta}{2\pi} + 2\sigma\tau|2\tau). \tag{6.6}$$

It is convenient to reduce everything to theta functions with modular parameter τ by using

$$\theta_3(z|2\tau)^2 + \theta_2(z|2\tau)^2 = \theta_3(z|\tau)\theta_3(0|\tau). \tag{6.7}$$

We have

$$\begin{aligned}&\theta_3(z|2\tau)^2\left(\partial^2_{\log\mathfrak{q}} \log\theta_3(z|2\tau)\right) + \theta_2(z|2\tau)^2\left(\partial^2_{\log\mathfrak{q}} \log\theta_2(z|2\tau)\right)\\&= 2\big(\partial_{\log\mathfrak{q}}\theta_3(z|\tau)\big)\left(\partial_{\log\mathfrak{q}}\theta_3(0|\tau)\right),\end{aligned} \tag{6.8}$$

$$\theta_3(z|2\tau)^2\left(\partial^2_{2\pi\mathrm{i}z} \log\theta_3(z|2\tau)\right) + \theta_2(z|2\tau)^2\left(\partial^2_{2\pi\mathrm{i}z} \log\theta_2(z|2\tau)\right) = \theta_3(z|\tau)\left(\partial_{\log\mathfrak{q}}\theta_3(0|\tau)\right), \tag{6.9}$$

$$\begin{aligned}&\theta_3(z|2\tau)^2\left(\partial_{\log\mathfrak{q}}\partial_{2\pi\mathrm{i}z} \log\theta_3(z|2\tau)\right) + \theta_2(z|2\tau)^2\left(\partial_{\log\mathfrak{q}}\partial_{2\pi\mathrm{i}z} \log\theta_2(z|2\tau)\right)\\&= \left(\partial_{2\pi\mathrm{i}z}\theta_3(z|\tau)\right)\left(\partial_{\log\mathfrak{q}}\theta_3(0|\tau)\right),\end{aligned} \tag{6.10}$$

$$\theta_3(z|2\tau)^2\left(\partial_{2\pi\mathrm{i}z} \log\theta_3(z|2\tau)\right) + \theta_2(z|2\tau)^2\left(\partial_{2\pi\mathrm{i}z} \log\theta_2(z|2\tau)\right) = \left(\partial_{2\pi\mathrm{i}z}\theta_3(z|\tau)\right)\theta_3(0|\tau). \tag{6.11}$$

These relations can be easily proven by using the definition of theta function as power series in $\mathfrak{q}$. We present the details of the calculation for (6.8) and (6.10) in "Appendix E.2". The other relations are similar. By using (6.7), we can write the relation (6.5) at $m = 0$ as

$$\begin{aligned}&\mathfrak{q}^{2\sigma^2}\Big(\theta_3(\tfrac{\eta}{2\pi}{+}2\sigma\tau|2\tau)^2\,\partial^2_{\log\mathfrak{q}}\theta_3(\tfrac{\eta}{2\pi}{+}2\sigma\tau|2\tau) + \theta_2(\tfrac{\eta}{2\pi}{+}2\sigma\tau|2\tau)^2\,\partial^2_{\log\mathfrak{q}}\theta_2(\tfrac{\eta}{2\pi}{+}2\sigma\tau|2\tau)\Big)\\&= 2\frac{\partial_{\log\mathfrak{q}}\theta_3(0|\tau)}{\theta_3(0|\tau)}\left(\partial_{\log\mathfrak{q}} - \frac{\partial_{\log\mathfrak{q}}\theta_3(0|\tau)}{\theta_3(0|\tau)}\right)\left(\mathfrak{q}^{2\sigma^2}\theta_3(\tfrac{\eta}{2\pi}{+}2\sigma\tau|\tau)\theta_3(0|\tau)\right).\end{aligned} \tag{6.12}$$

This equality can be easily proven from (6.8), (6.9), (6.10).

Example 2. Consider the limit $m \to \infty$, $\mathfrak{q}m^4 \to t$. In this limit $Z_\epsilon^D \to \mathcal{T}_{2\epsilon}$, where $\mathcal{T}_{2\epsilon}$ is the Painlevé III_3 tau function given in (3.10). The Eq. (6.4) becomes the Toda equation in the form [71, Prop 2.3]

$$\mathrm{D}^2_{-1,1}(\mathcal{T}_0, \mathcal{T}_0) + \mathrm{D}^2_{-1,1}(\mathcal{T}_1, \mathcal{T}_1) = 2t^{1/2}(\mathcal{T}_0^2 + \mathcal{T}_1^2). \tag{6.13}$$

6.2. From isomonodromic deformations. In this part we deduce the bilinear equation (6.5) from the isomonodromic equations (4.3),(4.4) and the formula for the tau function (4.8). The logic of the calculation is the same as in the example $m = 0$ above but instead of the simple formula for Q given in (4.5), we have to use the formulas (4.3),(4.4). It is convenient to rewrite them as

$$\partial_{\log \mathfrak{q}}(2\pi \mathrm{i} Q) = \frac{p}{2\pi \mathrm{i}}, \quad \partial_{\log \mathfrak{q}} \frac{p}{2\pi \mathrm{i}} = -m^2 \partial_1^3 \log \theta_1(2Q|\tau). \tag{6.14}$$

In this section we use the following short notation for the derivatives of theta functions:

$$\partial_1 \theta_j(z, \tau) = \partial_{2\pi \mathrm{i} z} \theta_j(z, \tau), \quad \partial_2 \theta_j(z, \tau) = \partial_{2\pi \mathrm{i} \tau} \theta_j(z, \tau). \tag{6.15}$$

After these preparations we compute the left and the right sides of Eq. (6.5). We have

$$\begin{aligned}
\mathrm{LHS} &= \tilde{Z}_0^D \tilde{Z}_0^D \partial^2_{\log \mathfrak{q}} \log \tilde{Z}_0^D + \tilde{Z}_{1/2}^D \tilde{Z}_{1/2}^D \partial_{\log \mathfrak{q}} \log \tilde{Z}_{1/2}^D \\
&= \mathcal{T}^2 \big(\theta_3(2Q|2\tau)^2 + \theta_2(2Q|2\tau)^2\big) \partial^2_{\log \mathfrak{q}} \log \mathcal{T} \\
&\quad + \mathcal{T}^2 \Big(\theta_3(2Q|2\tau)^2\, \partial^2_{\log \mathfrak{q}} \log(\theta_3(2Q|2\tau) + \theta_2(2Q|2\tau)^2\, \partial^2_{\log \mathfrak{q}} \log(\theta_2(2Q|2\tau)\Big) \\
&= \mathcal{T}^2 \Big(m^2 \theta_3(2Q|\tau)\, \theta_3(0|\tau)\, \partial_2 \partial_1^2 \log \theta_1(2Q|\tau) - m^2 \partial_1 \theta_3(2Q|\tau)\, \theta_3(0|\tau)\, \partial_1^3 \log \theta_1(2Q|\tau) \\
&\quad + 2\partial_2 \theta_3(2Q|\tau)\, \theta_3(0|\tau) - \frac{p^2}{\pi^2} \theta_3(2Q, \tau)\, \partial_2 \theta_3(0|\tau) + \frac{2p}{\pi \mathrm{i}} \partial_1 \theta_3(2Q|\tau)\, \partial_2 \theta_3(0|\tau)\Big),
\end{aligned} \tag{6.16}$$

where we used the relations (4.7),(4.6) for the tau functions and the relations (6.8)–(6.11) for the theta functions. On the other side we have

$$\begin{aligned}
\mathrm{RHS} &= 2\Big(\partial_2 \log(\theta_3(0|\tau))\, \Big(\partial_{\log \mathfrak{q}} - \partial_2 \log(\theta_3(0|\tau))\Big) \\
&\qquad - m^2 \partial_2^2 \log(\theta_3(0|\tau))\Big) \big(\tilde{Z}_0^D \tilde{Z}_0^D + \tilde{Z}_{1/2}^D \tilde{Z}_{1/2}^D\big) \\
&= 2\Big(\partial_2 \log(\theta_3(0|\tau))\, \Big(\partial_{\log \mathfrak{q}} - \partial_2 \log(\theta_3(0|\tau))\Big) \\
&\quad - m^2 \partial_2^2 \log(\theta_3(0|\tau))\Big) \big(\mathcal{T}^2 \theta_3(2Q|\tau) \theta_3(0|\tau)\big) \\
&= 2\theta_3(0|\tau) \Big(\partial_2 \log(\theta_3(0|\tau)\, \partial_{\log \mathfrak{q}} - m^2 \partial_2^2 \log(\theta_3(0|\tau)))\Big) \big(\mathcal{T}^2 \theta_3(2Q|\tau)\big) \\
&= \mathcal{T}^2 \Big(m^2 \theta_3(2Q|\tau) \Big(-2\theta_3(0|\tau)\, \partial_2^2 \log \theta_3(0|\tau) + 4\partial_1^2 \log \theta_1(2Q|\tau)\, \partial_2 \theta_3(0|\tau)\Big) \\
&\quad + 2\partial_2 \theta_3(0|\tau)\, \partial_2 \theta_3(2Q|\tau) - \frac{p^2}{\pi^2} \theta_3(2Q|\tau)\, \partial_2 \theta_3(0|\tau) + \frac{2p}{\pi \mathrm{i}} \partial_2 \theta_3(0|\tau)\, \partial_1 \theta_3(2Q|\tau)\Big),
\end{aligned} \tag{6.17}$$

where we used (4.7) (4.6) for the tau function and (6.7) for the theta functions. The last three terms in (6.16) agree with the last three terms in (6.17). The equality of the two other terms is equivalent to a theta function identity

$$\theta_3(2Q|\tau) \Big(-2\theta_3(0|\tau)\, \partial_2^2 \log \theta_3(0|\tau) + 4\partial_1^2 \log \theta_1(2Q|\tau)\, \partial_2 \theta_3(0|\tau)\Big)$$

$$= \theta_3(0|\tau)\Big(\theta_3(2Q|\tau)\,\partial_2\partial_1^2\log\theta_1(2Q|\tau) - \partial_1\theta_3(2Q|\tau)\,\partial_1^3\log\theta_1(2Q|\tau)\Big) \quad (6.18)$$

which we are going to prove now.

After some simple algebra (6.18) reduces to a relation $F(z,\tau)=0$, where (here all derivatives are taken with respect to $2\pi\mathrm{i}z$)

$$\begin{aligned}F(z,\tau) = {}&\theta_1(z|\tau)^3\theta_3(z|\tau)\Big(\theta_3^{(4)}(0|\tau)\theta_3(0|\tau) - \theta_3^{(2)}(0|\tau)^2\Big)\\ &-4\theta_1(z|\tau)\Big(\theta_1^{(2)}(z|\tau)\theta_1(z|\tau) - \theta_1^{(1)}(z|\tau)^2\Big)\theta_3(z|\tau)\theta_3^{(2)}(0|\tau)\theta_3(0|\tau)\\ &-2\Big(\theta_1^{(3)}(z|\tau)\theta_1(z|\tau)^2 - 3\theta_1^{(2)}(z|\tau)\theta_1^{(1)}(z|\tau)\theta_1(z|\tau)\\ &+2\theta_1^{(1)}(z|\tau)^3\Big)\theta_3^{(1)}(z|\tau)\theta_3(0|\tau)^2\\ &+\Big(\theta_1^{(4)}(z|\tau)\theta_1(z|\tau)^2 + \theta_1^{(2)}(z|\tau)^2\theta_1(z|\tau) - 2\theta_1^{(3)}(z|\tau)\theta_1^{(1)}(z|\tau)\theta_1(z|\tau)\\ &+2\theta_1^{(2)}(z|\tau)\theta_1^{(1)}(z|\tau)^2\Big)\theta_3(z|\tau)\theta_3(0|\tau)^2. \end{aligned} \quad (6.19)$$

Using the power series expression for $F(z,\tau)$ one can deduce the following modular properties

$$F(z+\tau,\tau) = -\mathrm{e}^{-4\pi\mathrm{i}\tau}\mathrm{e}^{-8\pi\mathrm{i}z}F(z,\tau), \quad F(z+1,\tau) = -F(z,\tau). \quad (6.20)$$

The function $F(z,\tau)$ does not have poles. Therefore it should have exactly 4 zeroes in the fundamental domain otherwise $F=0$ (as we will see). Clearly F is an odd function $F(z,\tau) = -F(-z,\tau)$, and it is easy to see that $F^{(1)}(0,\tau)=0$. Hence F has a zero of order at least 3 at $z=0$. Moreover one can check that $F(\frac{\tau+1}{2},\tau)=0$, $F(\frac{1}{2},\tau)=0$. Hence we must have $F=0$. Actually the only nontrivial check is $F(\frac{1}{2},\tau)=0$. It reduces to the identity

$$\begin{aligned}&\Big(\theta_2^{(4)}(0|\tau)\theta_2(0|\tau) - \theta_2^{(2)}(0|\tau)^2\Big)\theta_3(0|\tau)^2 + \theta_2(0|\tau)^2\Big(\theta_3^{(4)}(0|\tau)\theta_3(0|\tau) - \theta_3^{(2)}(0|\tau)^2\Big)\\ &-4\theta_2(0|\tau)\theta_2^{(2)}(0|\tau)\theta_3(0|\tau)\theta_3^{(2)}(0|\tau) = 0. \end{aligned} \quad (6.21)$$

This identity can be proven directly using the $\mathfrak{q}$ series expansion for the θ functions.

Remark. Equation (6.5) can be viewed as a system of two second order differential equations. Hence its general solution depends on four constants of integration. But there is a simple two-parametric set of transformations of the form

$$\tilde{Z}_\epsilon \mapsto C_1 \exp^{C_2\int\theta_3(0|\tau)^4 d\tau}\,\tilde{Z}_\epsilon, \quad \epsilon = 0, 1/2, \quad (6.22)$$

which preserves Eq. (6.5). These transformations preserve the ratio $Z_0/Z_{1/2}$, hence the equation for Q (which follows (4.8))

$$\frac{\theta_3(2Q|2\tau)}{\theta_2(2Q|2\tau)} = \frac{\tilde{Z}_0^D(\sigma,m,\eta,\tau)}{\tilde{Z}_{1/2}^D(\sigma,m,\eta,\tau)} \quad (6.23)$$

will depend only on two constants of integration. In terms of the Ansatz (4.13) these constants are σ,η. As we explained at the beginning of Sect. 4, the Eq. (6.23) determines Q essentially uniquely. In addition we proved that for any solution of the isomonodromic deformation equation (4.4), the corresponding functions $\tilde{Z}_0, \tilde{Z}_{1/2}$ satisfy (6.5). Since a generic solution of (4.4) depends on two parameters, we get a correspondence. Therefore any generic solution $\tilde{Z}_0, \tilde{Z}_{1/2}$ of the Eq. (6.5) determines Q which solves the isomonodromic deformation equation (4.4).

7. Blowup Equations from Regularized Action Functional

In this section we derive the $\epsilon_2 \to 0$ limit of blowup equations from the regularized action functional. This was first understood in the example of the pure $\mathcal{N} = 2$, $SU(2)$ SW theory in [65], which was also inspired by the works of [18,21].

7.1. Definition and derivatives of the action functional. The Lagrangian of the non-autonomous classical Calogero–Moser system is given by

$$\mathcal{L} = (2\pi \mathrm{i}\partial_\tau Q)^2 + m^2\wp(2Q|\tau) + 2m^2\eta_1(\tau). \tag{7.1}$$

Now we study its asymptotics in two limits: $\tau \to \mathrm{i}\infty$ and $\tau \to \tau_\star$. We know that in the second limit the Hamiltonian is finite, so the leading singularity is given by $2m^2\wp(2Q|\tau) \approx \frac{m^2}{2Q^2}$. By using Eq. (4.27) we get

$$\mathcal{L} = \pm\frac{\pi \mathrm{i} m}{\tau - \tau_\star} + \mathcal{O}(1). \tag{7.2}$$

In the other limit $\tau \to \mathrm{i}\infty$ the leading asymptotics is given by the derivative term, the constant term of $\wp(2Q|\tau)$ from (A.13) cancels with the asymptotics of $\eta_1(\tau)$ (A.16):

$$\mathcal{L} = (2\pi \mathrm{i}\sigma)^2 + \mathcal{O}(e^{2\pi \mathrm{i}\tau}). \tag{7.3}$$

We need to subtract both these asymptotics and define the regularized Lagrangian[21]:

$$\tilde{\mathcal{L}}^{\mp} = \mathcal{L}\mp\frac{\pi \mathrm{i} m}{\tau - \tau_\star}\pm\frac{\pi \mathrm{i} m}{\tau} - (2\pi \mathrm{i}\sigma)^2. \tag{7.4}$$

Now we define the regularized action functional:

$$\begin{aligned}\tilde{S}^{\mp}(\sigma, m, \tau_\star) &= \int_{\mathrm{i}\infty}^{\tau_\star} \tilde{\mathcal{L}}^{\mp}\frac{\mathrm{d}\tau}{2\pi \mathrm{i}}\\ &= \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{i}\infty}^{\tau_\star}\Big(\left((2\pi \mathrm{i}\partial_\tau Q)^2 + m^2\left(\wp(2Q|\tau) + 2\eta_1(\tau)\right)\right)\mathrm{d}\tau\mp\pi \mathrm{i} m\,\mathrm{d}\log\frac{\tau-\tau_\star}{\tau}\\ &\quad -(2\pi \mathrm{i}\sigma)^2\mathrm{d}\tau\Big).\end{aligned} \tag{7.5}$$

The $\tau_\star$ derivative is:

$$\partial_{\tau_\star}\tilde{S}^{\mp}(\sigma, m, \tau_\star) = \frac{\tilde{\mathcal{L}}^{\mp}(\tau_\star)}{2\pi \mathrm{i}} + \int_{\mathrm{i}\infty}^{\tau_\star}\mathrm{d}\left(4\pi \mathrm{i}\partial_\tau Q\partial_{\tau_\star}Q\pm\frac{m/2}{\tau-\tau_\star}\right), \tag{7.6}$$

[21] We also subtract a finite part. This is needed to get the (7.12) and (7.14) in the current form.

where we used the equation of motion (4.4). For this computation we need the expansions of Q in both limits. The expansion around $\tau_\star$ is given by (4.27):

$$Q \approx \sqrt{\pm\frac{m}{2\pi \mathrm{i}}(\tau-\tau_\star)}\left(1\pm\frac{H_\star^\mp+2m^2\eta_1(\tau_\star)}{4\pi \mathrm{i} m}(\tau-\tau_\star)\right). \tag{7.7}$$

The expansion around $\mathrm{i}\infty$ is given by (C.5): $Q\approx\sigma\tau+\beta$, where

$$\beta=\frac{\eta}{4\pi}+\frac{1}{2\pi\mathrm{i}}\log\frac{\Gamma(-m+2\sigma)\Gamma(1-2\sigma)}{\Gamma(1-m-2\sigma)\Gamma(2\sigma)}=:\frac{\eta}{4\pi}+\frac{\phi(\sigma,m)}{2\pi\mathrm{i}}. \tag{7.8}$$

We now compute the value of the regularized Lagrangian at the upper limit:

$$\tilde{\mathcal{L}}^\mp(\tau_\star)\approx\frac{1}{2}\left(H_\star^\mp+2m^2\eta_1(\tau)\right)+2m^2\eta_1(\tau)\pm\frac{\pi\mathrm{i}m}{\tau_\star}-(2\pi\mathrm{i}\sigma)^2. \tag{7.9}$$

Other useful expressions are the expansions around $\tau_\star$

$$4\pi\mathrm{i}\partial_\tau Q\partial_\alpha Q\approx\mp\frac{\frac{m}{2}\partial_\alpha\tau_\star}{\tau-\tau_\star}-3\frac{\partial_\alpha\tau_\star(H_\star^\mp+2m^2\eta_1(\tau))}{4\pi\mathrm{i}}, \tag{7.10}$$

and around $\mathrm{i}\infty$:

$$4\pi\mathrm{i}\partial_\tau Q\partial_\alpha Q\approx 4\pi\mathrm{i}\sigma(\tau\partial_\alpha\sigma+\partial_\alpha\beta), \tag{7.11}$$

where α is either σ or $\tau_\star$. Using these relations we finally find

$$2\pi\mathrm{i}\partial_{\tau_\star}\tilde{S}^\mp(\sigma,m,\tau_\star)=-H_\star^\mp\pm\frac{\pi\mathrm{i}m}{\tau_\star}-(2\pi\mathrm{i}\sigma)^2-2\pi\mathrm{i}\partial_{\tau_\star}(4\pi\mathrm{i}\sigma\beta). \tag{7.12}$$

In the same way we also compute the σ-derivative:

$$\begin{aligned}\partial_\sigma\tilde{S}^\mp(\sigma,m,\tau_\star)&=\int_{\mathrm{i}\infty}^{\tau_\star}\mathrm{d}\,(4\pi\mathrm{i}\partial_\tau Q\partial_\sigma Q-4\pi\mathrm{i}\tau\sigma)=-4\pi\mathrm{i}\sigma\partial_\sigma\beta-4\pi\mathrm{i}\sigma\tau_\star\\&=\partial_\sigma\left(-4\pi\mathrm{i}\sigma\beta-2\pi\mathrm{i}\sigma^2\tau_\star\right)+4\pi\mathrm{i}\beta.\end{aligned} \tag{7.13}$$

Now we consider the following equality:

$$\begin{aligned}&\tilde{S}^\mp-m\partial_m\tilde{S}^\mp\\&=\frac{1}{2\pi i}\int_{\mathrm{i}\infty}^{\tau_\star}\left((2\pi\mathrm{i}\partial_\tau Q)^2-m^2\left(\wp(2Q|\tau)+2\eta_1(\tau)\right)-(2\pi\mathrm{i}\sigma)^2\right)\mathrm{d}\tau-\mathrm{d}\left(8(\pi\mathrm{i})^2\partial_\tau Q\partial_m Q\right)\\&=\int_{\mathrm{i}\infty}^{\tau_\star}\mathrm{d}\left(\log\mathcal{T}-2\pi\mathrm{i}\sigma^2\tau-4\pi\mathrm{i}\partial_\tau Q\,m\partial_m Q\right)\end{aligned} \tag{7.14}$$

To complete this computation we need to know the asymptotics of $\mathcal{T}$ at $+\mathrm{i}\infty$. This can be found from (4.8) and (4.13):

$$\mathcal{T}\approx\mathrm{e}^{2\pi\mathrm{i}\tau\sigma^2}\prod_{\epsilon=\pm1}\frac{G(1-m+2\epsilon\sigma)}{G(1+2\epsilon\sigma)}. \tag{7.15}$$

Therefore

$$\exp\left(\tilde{S}^{\mp} - m\partial_m \tilde{S}^{\mp} - m\partial_m\left(4\pi \mathrm{i}\sigma\beta\right)\right) = \prod_{\epsilon=\pm 1} \frac{G(1+2\epsilon\sigma)}{G(1-m+2\epsilon\sigma)} \mathrm{e}^{-2\pi \mathrm{i}\tau_\star \sigma^2} \mathcal{T}(\sigma, m, \eta^{\mp}, \tau_\star). \tag{7.16}$$

We introduce the new function

$$\mathcal{S}^{\mp} = \tilde{S}^{\mp} \mp m/2 \log\tau_\star + 2\pi \mathrm{i}\sigma^2\tau_\star + 4\pi \mathrm{i}\sigma\beta + \varphi(\sigma, m) \tag{7.17}$$

with some function $\varphi(\sigma, m)$, which will be chosen later. By using the $\mathcal{S}$ identities (7.12), (7.13), and (7.16) we get

$$\begin{aligned}
&2\pi \mathrm{i}\partial_{\tau_\star}\mathcal{S}^{\mp} = -H_\star^{\mp},\\
&\partial_\sigma \mathcal{S}^{\mp} = \mathrm{i}\eta + 2\phi(\sigma, m) + \partial_\sigma\varphi(\sigma, m),\\
&\exp(\mathcal{S}^{\mp} - m\partial_m\mathcal{S}^{\mp} - \mathrm{i}\sigma\eta - 2\sigma\phi(\sigma, m) - \varphi(\sigma, m) + m\partial_m\varphi(\sigma, m))\\
&\quad = \prod_{\epsilon=\pm 1} \frac{G(1+2\epsilon\sigma)}{G(1-m+2\epsilon\sigma)} \mathcal{T}(\sigma, m, \eta^{\mp}, \tau_\star).
\end{aligned} \tag{7.18}$$

We would like to cancel some unwanted terms, namely, to find $\varphi(\sigma, m)$ such that

$$\begin{aligned}
&\partial_\sigma\varphi(\sigma, m) = -2\phi(\sigma, m),\\
&\varphi(\sigma, m) - m\partial_m\varphi(\sigma, m) = -2\sigma\phi(\sigma, m) + \ldots,
\end{aligned} \tag{7.19}$$

where "…" stands for the logarithm of the Barnes functions and is almost completely defined by the first equation (σ-derivative). We can solve the first equation by integration:

$$\begin{aligned}
\varphi(\sigma, m) &= \varphi_0(m) - \int^{\sigma} \mathrm{d}(2\sigma) \log\frac{\Gamma(-m+2\sigma)\Gamma(1-2\sigma)}{\Gamma(1-m-2\sigma)\Gamma(2\sigma)}\\
&= \log\frac{G(-m+2\sigma)G(1-m-2\sigma)}{G(1-2\sigma)G(2\sigma)}\\
&\quad -2\sigma\log\Gamma(1-2\sigma) + (2\sigma-1)\log\Gamma(2\sigma)\\
&\quad -(2\sigma-m-1)\log\Gamma(-m+2\sigma) + (2\sigma+m)\log\Gamma(1-m-2\sigma)\\
&\quad +m^2 + m\log 2\pi + \varphi_0(m).
\end{aligned} \tag{7.20}$$

Here we used the following identity:

$$\int^x \mathrm{d}x \log\Gamma(x) = \frac{x(1-x)}{2} + \frac{x}{2}\log 2\pi + (x-1)\log\Gamma(x) - \log G(x). \tag{7.21}$$

We now take $\varphi_0(m) = -m^2 + am$, where a is an arbitrary constant, which can be fixed after identification of $\mathcal{S}$ with the properly normalized conformal block. In this way we get

$$\varphi(\sigma, m) - m\partial_m\varphi(\sigma, m) + 2\sigma\phi(\sigma, m) = \log\frac{G(1-m+2\sigma)G(1-m-2\sigma)}{G(1-2\sigma)G(1+2\sigma)}. \tag{7.22}$$

After choosing $\varphi(\sigma, m)$ in such a way we have

$$\begin{aligned}
2\pi \mathrm{i}\partial_{\tau_\star}\mathcal{S}^{\mp}(\sigma, m, \tau_\star) &= -H_\star^{\mp}(\sigma, m, \tau_\star),\\
\partial_\sigma\mathcal{S}^{\mp}(\sigma, m, \tau_\star) &= \mathrm{i}\eta^{\mp}(\sigma, m, \tau_\star),\\
\exp(\mathcal{S}^{\mp} - m\partial_m\mathcal{S}^{\mp} - \sigma\partial_\sigma\mathcal{S}^{\mp}) &= \mathcal{T}(\sigma, m, \eta^{\mp}, \tau_\star).
\end{aligned} \tag{7.23}$$

7.2. Relation to classical conformal blocks. **Classical conformal blocks and BPZ equations.** Following [21,65] $\mathcal{S}$ should be identifies with the $c = \infty$ conformal blocks (or classical conformal blocks, or NS free energy), and therefore (7.18) reproduces the $\epsilon_2 \to 0$ limit of blowup equations used in Sect. 5.2. Following [103–107], we start from consideration of the correlators with heavy degenerate field $\phi_{(1,2)}(w)$, light degenerate field $\phi_{(2,1)}(w)$, and energy-momentum tensor $T(z)$. We define

$$
\begin{aligned}
G(\sigma, m, \tau) &= \mathrm{tr}_{\Delta(\sigma)}\, \mathcal{R}\left(\mathfrak{q}^{L_0-\frac{c}{24}} V_{\Delta(m)}(0)\right),\\
G_T(\sigma, m, z, \tau) &= \mathrm{tr}_{\Delta(\sigma)}\, \mathcal{R}\left(\mathfrak{q}^{L_0-\frac{c}{24}} V_{\Delta(m)}(0) T(z)\right),\\
G_h(\sigma, m, w, \tau) &= \mathrm{tr}_{\Delta(\sigma)}\, \mathcal{R}\left(\mathfrak{q}^{L_0-\frac{c}{24}} V_{\Delta(m)}(0) \phi_{(1,2)}(w)\right),\\
G_l(\sigma, m, w, \tau) &= \mathrm{tr}_{\Delta(\sigma)}\, \mathcal{R}\left(\mathfrak{q}^{L_0-\frac{c}{24}} V_{\Delta(m)}(0) \phi_{(2,1)}(w)\right),\\
G_{T,h}(\sigma, m, z, w, \tau) &= \mathrm{tr}_{\Delta(\sigma)}\, \mathcal{R}\left(\mathfrak{q}^{L_0-\frac{c}{24}} V_{\Delta(m)}(0) T(z) \phi_{(1,2)}(w)\right),
\end{aligned}
\tag{7.24}
$$

where $\mathcal{R}$ denotes the cyclic ordering on a cylinder (analog of the radial ordering), and we used the following parameterization of Δ:

$$
\Delta(m) = \frac{1}{4}(b + b^{-1})^2 - m^2/b^2 = b^{-2}(1/4 - m^2) + \mathcal{O}(1). \tag{7.25}
$$

Now we use the OPE with degenerate field:

$$
T(z)\phi_{(1,2)}(w) = \frac{\Delta_{(1,2)}\phi_{(1,2)}(w)}{(z-w)^2} + \frac{\partial_w \phi_{(1,2)}(w)}{z-w} + (\mathcal{L}_{-2}\phi_{(1,2)})(w) + \ldots \tag{7.26}
$$

Using the explicit form of the null-vector and the formula for $\Delta_{(1,2)}$ we rewrite it as

$$
T(z)\phi_{(1,2)}(w) = \frac{-\frac{2b^2+3}{4b^2}\phi_{(1,2)}(w)}{(z-w)^2} + \frac{\partial_w \phi_{(1,2)}(w)}{z-w} - b^2 \partial_w^2 \phi_{(1,2)}(w) + \ldots \tag{7.27}
$$

Another OPE is

$$
T(z)V_{\Delta(m)}(0) = \frac{\Delta(m) V_{\Delta(m)}(0)}{z^2} + \frac{\partial V_{\Delta(m)}(0)}{z} + \ldots. \tag{7.28}
$$

Combining these OPE's together, and using the fact that correlator depends only on difference of coordinates, we finally write

$$
\begin{aligned}
G_{T,h}(\sigma, m, z, w, \tau) = I(\sigma, m, w, \tau) &+ \frac{2b^2+3}{4b^2}\left(\log\theta_1(z-w)\right)'' G_h(\sigma, m, w, \tau)\\
&+ \left(\log\frac{\theta_1(z-w)}{\theta_1(z)}\right)' \partial_w G_h(\sigma, m, w, \tau)\\
&- \Delta(m)\left(\log\theta_1(z)\right)'' G_h(\sigma, m, w, \tau).
\end{aligned}
\tag{7.29}
$$

To derive this formula we first wrote explicitly the globally defined functions $\Big(\log\theta_1(z-w)\Big)''$, $(\log\theta_1(z))''$, $\left(\log\frac{\theta_1(z-w)}{\theta_1(z)}\right)'$, which are fixed up to constants by their singular behavior. Coefficients in front of these functions are dictated by the OPE's with $T(z)$

(conformal Ward identities). These terms have vanishing A-cycle integral in the variable z. The constant term $I(\sigma, m, w, \tau)$ is not fixed by the singular parts of the OPE's. We can find it in two different ways, which will give a non-trivial equation (7.33) on $G_h(\sigma, m, w, \tau)$. This is the analog of the BPZ equation [11] on the torus, see [108].

On one side,

$$\oint_A T(z)\mathrm{d}z = (2\pi \mathrm{i})^2 \left(L_0 - \frac{c}{24}\right), \tag{7.30}$$

therefore

$$I(\sigma, m, w, \tau) = 2\pi \mathrm{i}\partial_\tau G_h(\sigma, m, w, \tau). \tag{7.31}$$

On the other side, we have not used the regular part of (7.27) yet. To do this first rewrite (7.29) in a more suitable form using (A.8) and (A.10):

$$\begin{aligned} G_{T,h}(\sigma, m, z, w, \tau) = &-\frac{2b^2+3}{4b^2}\left(\wp(z-w|\tau) + 2\eta_1(\tau)\right) G_h(\sigma, m, w, \tau) \\ &+ \left(\zeta(z-w|\tau) - \zeta(z|\tau) + 2\eta_1(\tau)w\right) \partial_w G_h(\sigma, m, w, \tau) \\ &+\Delta(m)\left(\wp(z|\tau) + 2\eta_1(\tau)\right) G_h(\sigma, m, w, \tau) + 2\pi\mathrm{i}\partial_\tau G_h(\sigma, m, w, \tau). \end{aligned} \tag{7.32}$$

The regular part at $z = w$ is:

$$\begin{aligned} -b^2\partial_w^2 G_h(\sigma, m, w, \tau) = &-\frac{2b^2+3}{2b^2}\eta_1(\tau) G_h(\sigma, m, w, \tau) \\ &+ \left(2\eta_1(\tau)w - \zeta(w|\tau)\right) \partial_w G_h(\sigma, m, w, \tau) \\ &+\Delta(m)\left(\wp(w|\tau) + 2\eta_1(\tau)\right) G_h(\sigma, m, w, \tau) \\ &+2\pi\mathrm{i}\partial_\tau G_h(\sigma, m, w, \tau). \end{aligned} \tag{7.33}$$

To get rid of the first derivative we redefine [103,104]

$$G_h(\sigma, m, w, \tau) = \theta_1(w|\tau)^{\frac{1}{2b^2}} \eta(\tau)^{-1-\frac{3}{2b^2}} \tilde{G}_h(\sigma, m, w, \tau) \tag{7.34}$$

and then get

$$\begin{aligned} &2\pi\mathrm{i}\partial_\tau \tilde{G}_h(\sigma, m, w, \tau) + b^2\partial_w^2 \tilde{G}_h(\sigma, m, w, \tau) \\ &+\left(\frac{1}{2}\frac{\partial_w^2\theta_1(w|\tau)}{\theta_1(w|\tau)} + \left(\frac{1}{4b^2} + (m^2 - 1/4)b^{-2} + \mathcal{O}(1)\right)\partial_w^2 \log\theta_1(w|\tau)\right) \\ &\times\tilde{G}_h(\sigma, m, w, \tau) = 0, \end{aligned} \tag{7.35}$$

where we substituted the $b \to 0$ expansion of $\Delta(m)$ (7.25). It is convenient to make the following Ansatz

$$\begin{aligned} G_h(\sigma, m, w, \tau) &= \mathrm{e}^{b^{-2} f_h(\sigma,m,w,\tau)+\mathcal{O}(1)}, \qquad \tilde{G}_h(\sigma, m, w, \tau) = \mathrm{e}^{b^{-2}\tilde{f}_h(\sigma,m,w,\tau)+\mathcal{O}(1)}, \\ G(\sigma, m, \tau) &= \mathrm{e}^{b^{-2} f(\sigma,m,\tau)+\mathcal{O}(1)}, \end{aligned} \tag{7.36}$$

where f's are called classical conformal blocks. Using the leading order of (7.36) one gets:

$$2\pi\mathrm{i}\partial_\tau \tilde{f}_h(\sigma, m, w, \tau) + (\partial_w \tilde{f}_h(\sigma, m, w, \tau))^2 - m^2\left(\wp(w|\tau) + 2\eta_1(\tau)\right) = 0. \tag{7.37}$$

Identification with the action functional Equation (7.37) is nothing but the Hamilton–Jacobi equation for a system with Hamiltonian given by

$$H(p, w, \tau) = p^2 - m^2(\wp(w|\tau) + 2\eta_1(\tau)). \tag{7.38}$$

The equations of motion are $2\pi i\partial_\tau w = 2p$, $2\pi i\partial_\tau p = m^2\wp'(w|\tau)$. If we define

$$w = 2Q \tag{7.39}$$

we recover (4.3) and (4.4). Moreover if we evaluate Q on the solution of the equations of motions, we have

$$\begin{aligned}&\tilde{f}_h(\sigma, m, 2Q(\tau_2), \tau_2) - \tilde{f}_h(\sigma, m, 2Q(\tau_1), \tau_1)\\ &= \int_{\tau_1}^{\tau_2} \frac{d\tau}{2\pi i}\left((2\pi i\partial_\tau Q)^2 + m^2(\wp(2Q|\tau) + 2\eta_1(\tau))\right).\end{aligned} \tag{7.40}$$

To get a non-trivial statement we consider the limit $\tau_1 \to i\infty$ and $\tau_2 \to \tau_\star$. First we look at the $\tau_1 \to i\infty$ limit. In this region $Q \approx \sigma\tau_1 + \beta$ (C.5). Hence

$$\begin{aligned}&G_h(\sigma, m, 2Q(\tau_1), \tau_1)\\ &\approx e^{-\pi i\tau_1\frac{c}{12}} \operatorname{tr}_{\Delta(\sigma)}\left(V_{\Delta(m)}(0)e^{2\pi i(2\sigma\tau_1+2\beta)L_0}\phi_{(1,2)}(0)e^{2\pi i(\tau_1-2\sigma\tau_1-2\beta)L_0}\right).\end{aligned} \tag{7.41}$$

Taking the logarithm we get in the leading order

$$\begin{aligned}f_h(\sigma, m, 2Q(\tau_1), \tau_1) \approx{}& 2\pi i(2\sigma\tau_1 + 2\beta)\left(\frac{1}{4} - (\sigma - \frac{1}{2})^2\right)\\ &+2\pi i(\tau_1 - 2\sigma\tau_1 - 2\beta)(\frac{1}{4} - \sigma^2)\\ &-\frac{\pi i\tau_1}{2} + c(\sigma, m, \sigma - \frac{1}{2}) + c(\sigma - \frac{1}{2}, 1, \sigma)\\ ={}& 2\pi i\tau_1\sigma^2 + 4\pi i\sigma\beta - \pi i(\sigma\tau_1 + \beta)\\ &+c(\sigma, m, \sigma - \frac{1}{2}) + c(\sigma - \frac{1}{2}, 1, \sigma),\end{aligned} \tag{7.42}$$

where c's are NS limits of the 3-point functions: $\log C(\alpha_1, \alpha_2, \alpha_3) = b^{-2}c(\alpha_1, \alpha_2, \alpha_3) + \mathcal{O}(1)$.

Now we switch to $\tilde{f}_h(\sigma, m, 2Q(\tau_1), \tau_1) = f_h(\sigma, m, 2Q(\tau_1), \tau_1) - \frac{1}{2}\log\theta_1(2Q(\tau_1)|\tau_1) + \frac{3}{2}\log\eta(\tau_1)$. By using the asymptotic

$$\theta_1(2Q(\tau_1)|\tau_1) \approx ie^{\pi i\tau_1/4}e^{-2\pi i(\sigma\tau_1+\beta)} \tag{7.43}$$

we get:

$$\tilde{f}_h(\sigma, m, 2Q(\tau_1), \tau_1) \approx -\pi i/4 + 2\pi i\tau_1\sigma^2 + 4\pi\sigma\beta + c(\sigma, m, \sigma - \frac{1}{2}) + c(\sigma - \frac{1}{2}, 1, \sigma). \tag{7.44}$$

Let us now focus on the $\tau_2 \to \tau_\star$ limit. Here we use the OPE

$$V_{\Delta(m)}(0)\phi_{(1,2)}(2Q) = C(m+\frac{1}{2},1,m)(-2Q)^{\Delta(m+\frac{1}{2})-\Delta(1)-\Delta(m)}V_{\Delta(m+\frac{1}{2})}(0) \\ +C(m-\frac{1}{2},1,m)(-2Q)^{\Delta(m-\frac{1}{2})-\Delta(1)-\Delta(m)}V_{\Delta(m-\frac{1}{2})}(0)+\dots \tag{7.45}$$

In the $b \to 0$ limit we can neglect the sub-leading term. However for different signs of m, different terms will dominate. We will again describe these two possibilities by using $\pm$ sign:

$$V_{\Delta(m)}(0)\phi_{(1,2)}(2Q) = C(m\mp\frac{1}{2},1,m)(-2Q)^{b^{-2}(\pm m+\frac{1}{2})}V_{\Delta(m\mp\frac{1}{2})}(0)+\dots \tag{7.46}$$

Using this OPE we write the asymptotics of $\tilde{f}_h$:

$$\begin{aligned}\tilde{f}_h(\sigma,m,2Q(\tau_2),\tau_2) &\approx \left(\frac{1}{2}\pm m\right)\mathrm{i}\pi\pm m\log 2Q-\frac{1}{2}\log\theta_1'(0|\tau_\star) \\ &\quad+\frac{3}{2}\log\eta(\tau_\star) \\ &\quad+c(m\mp\frac{1}{2},1,m)+f(\sigma,m-\frac{1}{2},\tau_\star) \\ &= \left(\frac{1}{2}\pm m\right)\mathrm{i}\pi\pm m\log\left(\sqrt{\frac{2m}{\pi}}\mathrm{e}^{\mp\frac{\mathrm{i}\pi}{4}}\sqrt{\tau_2-\tau_\star}\right)-\frac{1}{2}\log\theta_1'(0|\tau_\star) \\ &\quad+\frac{3}{2}\log\eta(\tau_\star) \\ &\quad+c(m\mp\frac{1}{2},1,m)+f(\sigma,m-\frac{1}{2},\tau_\star) \\ &= \left(\frac{1}{2}\pm m\right)\mathrm{i}\pi\pm\frac{m}{2}\log\frac{2m}{\pi} \\ &\quad-\frac{\mathrm{i}\pi m}{4}\pm\frac{m}{2}\log(\tau_2-\tau_\star)+c(m\mp\frac{1}{2},1,m)+f(\sigma,m\mp\frac{1}{2},\tau_\star),\end{aligned} \tag{7.47}$$

where we used

$$G(\sigma,m,\tau) = \mathrm{e}^{b^{-2}f(\sigma,m,\tau)+\mathcal{O}(1)}. \tag{7.48}$$

Combining together (7.40), (7.44), (7.47) we get the following equality:

$$\begin{aligned}f(\sigma,m\mp\frac{1}{2},\tau_\star) &= \lim_{\substack{\tau_1\to\mathrm{i}\infty\\ \tau_2\to\tau_\star}}\int_{\tau_1}^{\tau_2}\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}\left((2\pi\mathrm{i}\partial_\tau Q)^2+m^2(\wp(2Q|\tau)+2\eta_1(\tau))\right) \\ &\quad+2\pi\mathrm{i}\tau_1\sigma^2+4\pi\sigma\beta+c(\sigma,m,\sigma-\frac{1}{2})+c(\sigma-\frac{1}{2},1,\sigma)-\pi\mathrm{i}/4 \\ &\quad-\left(\frac{1}{2}\pm m\right)\mathrm{i}\pi\mp\frac{m}{2}\log\frac{2m}{\pi}+\frac{\mathrm{i}\pi m}{4}\mp\frac{m}{2}\log(\tau_2-\tau_\star)-c(m\mp\frac{1}{2},1,m).\end{aligned} \tag{7.49}$$

Now we would like to compare (7.49) with the regularized and redefined action $\mathcal{S}$. To do that we combine (7.5) and (7.17) and rewrite $\mathcal{S}$ as:

$$\begin{aligned}\mathcal{S}^{\mp}(\sigma, m, \tau_\star) =& \lim_{\substack{\tau_1\to i\infty\\ \tau_2\to\tau_\star}} \int_{\tau_1}^{\tau_2} \frac{d\tau}{2\pi i}\Big((2\pi i\partial_\tau Q)^2 + m^2(\wp(2Q|\tau) + 2\eta_1(\tau))\Big)\\ &+2\pi i\sigma^2(\tau_1 - \tau_\star)\mp\frac{m}{2}\log(\tau_2 - \tau_\star)\pm\frac{m}{2}\log\tau_\star\\ &+2\pi i\sigma^2\tau_\star + 4\pi i\sigma\beta\mp\frac{m}{2}\log\tau_\star + \varphi(\sigma, m)\\ =& \lim_{\substack{\tau_1\to i\infty\\ \tau_2\to\tau_\star}} \int_{\tau_1}^{\tau_2} \frac{d\tau}{2\pi i}\Big((2\pi i\partial_\tau Q)^2 + m^2(\wp(2Q|\tau) + 2\eta_1(\tau))\Big)\\ &+2\pi i\sigma^2\tau_1 + 4\pi i\sigma\beta\mp\frac{m}{2}\log(\tau_2 - \tau_\star) + \varphi(\sigma, m).\end{aligned} \tag{7.50}$$

By comparing the above expressions we get the following identification:

$$\begin{aligned}f(\sigma, m\mp\frac{1}{2}, \tau_\star) =& \mathcal{S}^{\mp}(\sigma, m, \tau_\star) + c(\sigma, m, \sigma - \frac{1}{2}) + c(\sigma - \frac{1}{2}, 1, \sigma) - c(m\mp\frac{1}{2}, 1, m)\\ &-\varphi(\sigma, m) + \frac{i\pi(m-1)}{4} - \left(\frac{1}{2}\pm m\right) i\pi\mp\frac{m}{2}\log\frac{2m}{\pi}.\end{aligned} \tag{7.51}$$

This proves that the regularized action is equals to the classical conformal block up to some possible τ-independent constant. Together with (7.23) this gives an additional proof of the classical/$c = 1$ blowup relations.

Energy from classical conformal blocks. Though we already know that the energy for the spectral problem can be described by $-H_\star^{\mp} = 2\pi i\partial_\tau f(\sigma, m\mp\frac{1}{2}, \tau)$, we can see how this fact follows directly from CFT. To do this we consider the equivalent of Eq. (7.33) for the light degenerate field. More precisely we have $\phi_{(2,1)}(w)$:

$$\begin{aligned}-b^{-2}\partial_w^2 G_l(\sigma, m, w, \tau) =& -\frac{2+3b^2}{2}\eta_1(\tau)G_l(\sigma, m, w, \tau)\\ &+(2\eta_1(\tau)w - \zeta(w|\tau))\,\partial_w G_l(\sigma, m, w, \tau)\\ &+\Delta(m)\,(\wp(w|\tau) + 2\eta_1(\tau))\,G_l(\sigma, m, w, \tau)\\ &+2\pi i\partial_\tau G_l(\sigma, m, w, \tau).\end{aligned} \tag{7.52}$$

As before we use the Ansatz

$$G_l(\sigma, m, w, z) = \psi(\sigma, m, w, \tau)e^{b^{-2}f(\sigma,m,\tau)+\mathcal{O}(1)}, \tag{7.53}$$

where $\mathcal{O}(1)$ is a function of σ, m, τ only. In the $b \to 0$ limit we get

$$\Big(\partial_w^2 - (m^2 - 1/4)\,(\wp(w|\tau) + 2\eta_1(\tau)) + 2\pi i\partial_\tau f(\sigma, m, \tau)\Big)\,\psi(\sigma, m, w, \tau) = 0. \tag{7.54}$$

By shifting m we end up with

$$\begin{aligned}&\Big(-\partial_w^2 + m(m\mp 1)\wp(w|\tau)\Big)\,\psi(\sigma, m\mp\frac{1}{2}, w, \tau)\\ &= \left(2\pi i\partial_\tau f(\sigma, m\mp\frac{1}{2}, \tau) - 2m(m\mp 1)\eta_1(\tau)\right)\psi(\sigma, m\mp\frac{1}{2}, w, \tau),\end{aligned} \tag{7.55}$$

in agreement with [103,104]. One can also add that to get the solution of the general 2×2 system, or equivalently the solution of the equation with apparent singularity (2.11), (4.23), one should also insert heavy degenerate field $\phi_{(1,2)}(2Q)$ into G_l. It has (-1) monodromy with $\phi_{(2,1)}$, so its insertion will give precisely apparent singularity.

Now using the AGT relation between conformal blocks and Nekrasov partition functions [52] we can identify classical conformal block with the NS limit of Nekrasov partition function:

$$f(\sigma, m\mp\frac{1}{2}, \tau) = F^{\mathrm{NS}}(\sigma, m\mp\frac{1}{2}, \mathrm{e}^{2\pi\mathrm{i}\tau}). \tag{7.56}$$

8. Other Examples and Generalizations

In Sects. 3 and 4 we illustrated in detail the example of isomonodromic deformations on the torus and the one of Painlevé III_3. In this section we briefly comment on other examples, even though we do not spell out all the details. We limit ourselves to list the quantum operators corresponding to the other Painlevé equations. These operators coincide with the quantum Seiberg–Witten (SW) curves of the gauge theories underlying Painlevé equations, in agreement with several existing results in the literature that we will discuss below. Some of these operators have been studied recently in [109]. According to the procedure spelled out in Sect. 2, the exact spectrum of such operators should be obtained by imposing vanishing of some combination of tau functions with suitable normalizability conditions on the monodromy parameters of the associated linear system.

Painlevé I. The associated linear problem is defined by the following Lax matrix (see for instance [110, eq. (2.2)])

$$A(z,t) = \begin{pmatrix} -p & q^2 + qz + \frac{t}{2} + z^2 \\ 4z - 4q & p \end{pmatrix}. \tag{8.1}$$

The compatibility conditions are

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t} &= p, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= 6q^2 + t, \end{aligned} \tag{8.2}$$

leading to the Painlevé I equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}^2 t} = 6q^2 + t. \tag{8.3}$$

The Hamiltonian is

$$H_0 = \frac{p^2}{2} - 2q^3 - qt. \tag{8.4}$$

It is also useful to define

$$H_1 = 2H_0 + \frac{p}{q}. \tag{8.5}$$

Moreover, since

$$A_{12} = q^2 + qz + \frac{t}{2} + z^2, \tag{8.6}$$

we have auxiliary poles unless $q = \infty$. This is our requirement for singularities matching and it gives the following operator

$$-\partial_z^2 + 2tz + 4z^3 + H_1 \tag{8.7}$$

in agreement with expectations from [15,19,20]. It is also straightforward to see that the singularities matching condition imposes vanishing of the Painlevé I tau function. The condition of normalizability instead is more subtle and will not be addressed in this work. Notice that, as expected, the operator (8.7) is the one arising in the quantization of the Seiberg–Witten curve corresponding to the $\mathcal{H}_0$ Argyres–Douglas theory. It is also well known that the quantization condition for this potential involves the $\mathcal{H}_0$ NS free energy (see for instance [111,112]). It should be possible to relate such NS type quantization to the vanishing of the Painlevé I tau function. For that we would need some kind of blowup equations for Argyres–Douglas theories which, at present, are not known. Alternatively one can try to develop a functional approach similar to [65], see also Sect. 7. This is under investigation and will appear in [113].

Painlevé II. The linear problem is obtained from the following Lax matrix (see for instance [46, eq. (3.14)])

$$A(z,t) = \begin{pmatrix} p + \frac{t}{2} + z^2 & u(z-q) \\ -\frac{2(\theta+pq+pz)}{u} & -p - \frac{t}{2} - z^2 \end{pmatrix}. \tag{8.8}$$

The associated compatibility condition leads to the Painlevé II equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}^2 t} = 2q^3 + qt + \frac{1}{2} - \theta. \tag{8.9}$$

The Hamiltonian is

$$H_0 = \frac{p^2}{2} + pq^2 + \frac{pt}{2} + \theta q. \tag{8.10}$$

We also define

$$H_1 = 2H_0 - 2q. \tag{8.11}$$

Since

$$A_{12} = u(z-q), \tag{8.12}$$

we have singularities matching if $u = \infty$ (or $q = \infty$). It is easy to show that at this special point the relevant operator reads

$$-\partial_z^2 + tz^2 + z^4 - 2\theta z + \frac{t^2}{4} + H_1. \tag{8.13}$$

As expected, this is the operator arising in the quantization of the Seiberg–Witten curve to the $\mathcal{H}_1$ Argyres–Douglas theory. It is also straightforward to see that condition of singularities matching imposes the vanishing of the Painlevé II tau function as a quantization condition for the potential (8.13). As before, the condition of normalizability instead is more subtle and will not be addressed in this work. Likewise we do not know blowup equations that would link the vanishing of tau function to the NS quantization. Hence it would be nice to develop a functional approach to this problem as done in [65] for Painlevé III_3 and in Sect. 7 for the torus.

Painlevé IV. The linear problem is obtained from the following Lax matrix (see for instance [46, eq. (3.36)])

$$A(z,t) = \begin{pmatrix} \frac{\theta_0 - pq}{z} + t + z & u\left(1 - \frac{q}{2z}\right) \\ \frac{2(-\theta_0-\theta_\infty+pq)}{u} + \frac{2p(pq-2\theta_0)}{uz} & -\frac{\theta_0-pq}{z} - t - z \end{pmatrix}. \tag{8.14}$$

The compatibility condition leads to the Painlevé IV equation:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}^2 t} = \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 \frac{1}{2q} + \frac{3q^3}{2} + 4tq^2 + 2q(t^2 - 2\theta_\infty + 1) - \frac{8\theta_0^2}{q}. \tag{8.15}$$

The Painlevé IV Hamiltonian is defined as

$$H_0 = 2p^2 q - p\left(4\theta_0 + q^2 + 2qt\right) + q(\theta_0 + \theta_\infty). \tag{8.16}$$

We also introduce

$$H_1 = H_0 - 2p + t + 2t\theta_0. \tag{8.17}$$

After imposing the singularities matching condition we obtain the following operator in agreement with [114]

$$-\partial_z^2 - 2\theta_\infty + t^2 + 2tz + \frac{H_1}{z} + \frac{\theta_0^2 - \frac{1}{4}}{z^2} + z^2. \tag{8.18}$$

This is the operator appearing in quantization of SW curve to $\mathcal{H}_2$ Argyres–Douglas theory.

Painlevé III_2. We follow [115]. The relevant Lax matrix is

$$A(z,t) = \begin{pmatrix} -\frac{2p+q^2(-t)-\theta_\infty q}{2z^2} - \frac{t}{2} - \frac{\theta_\infty}{2z} & -\frac{4p^2-4pq^2t+q^4t^2-\theta_\infty^2 q^2-4q}{4q^2 z} - \frac{\left(-2p+q^2t+\theta_\infty q\right)^2}{4qz^2} \\ \frac{q}{z^2} - \frac{1}{z} & \frac{2p+q^2(-t)-\theta_\infty q}{2z^2} + \frac{t}{2} + \frac{\theta_\infty}{2z}. \end{pmatrix}, \tag{8.19}$$

leading to the Painlevé III_2 equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}^2 t} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{\mathrm{d}q}{\mathrm{d}t}\frac{1}{t} + \frac{q^2(1+\theta_\infty)}{t} + q^3 - \frac{2}{t^2}. \tag{8.20}$$

The Hamiltonian is

$$H_0 = -\frac{p^2}{q^2 t} + \frac{q^2 t}{4} + \frac{1}{2}(\theta_\infty + 1)q + \frac{1}{qt}. \tag{8.21}$$

It is useful to define

$$H_1 = \frac{1}{2}(qt - \theta_\infty) - H_0 t. \tag{8.22}$$

Singularities matching leads to

$$-\partial_z^2 + \frac{t^2}{4} + t\,\frac{\frac{\theta_\infty}{2} - 1}{z} + \frac{1}{z^3} + \frac{H_1}{z^2}. \tag{8.23}$$

This is precisely the operator corresponding to the quantization of the $N_f = 1$ SW curve, see for instance [116, 117].

Painlevé III_1. The relevant Lax matrix is (see for instance [46, eq. (A.23)]

$$A(z,t)=\begin{pmatrix} \frac{(2p-1)\sqrt{t}}{2z^2}+\frac{\sqrt{t}}{2}-\frac{\theta_*}{z} & -\frac{pqu}{z}-\frac{p\sqrt{t}u}{z^2} \\ \frac{2(\theta_*+\theta_\star)-2p^2q-4\theta_*p+2pq}{2puz}+\frac{(p-1)\sqrt{t}}{uz^2} & -\frac{(2p-1)\sqrt{t}}{2z^2}-\frac{\sqrt{t}}{2}+\frac{\theta_*}{z} \end{pmatrix}, \tag{8.24}$$

leading to the Painlevé III_1 equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}^2t}=\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2\frac{1}{q}-\frac{\mathrm{d}q}{\mathrm{d}t}\frac{1}{t}+\frac{q^3}{t^2}+\frac{2q^2\theta_\star}{t^2}+\frac{1-2\theta_*}{t}-\frac{1}{q}. \tag{8.25}$$

The Hamiltonian is

$$H_0=\theta_*^2+p^2q^2-pq^2+2\theta_*pq+pt-q(\theta_*+\theta_\star)-\frac{t}{2}, \tag{8.26}$$

and we define

$$H_1=H_0+q-pq-\theta_*. \tag{8.27}$$

We have singularities matching if $p=\infty$, $q=0$, such that $pq=\infty$ and u finite. After some algebra this leads to the following operator

$$-\partial_z^2+\frac{H_1}{z^2}+\frac{t}{4z^4}-\frac{\theta_\star\sqrt{t}}{z^3}-\frac{\theta_*\sqrt{t}}{z}+\frac{t}{4}, \tag{8.28}$$

which is the quantum SW curve with $N_f=2$, see for instance [116,117].

Painlevé V. The relevant Lax matrix is (see for instance [115, Sec. 4.3])

$$A(z,t)=\begin{pmatrix} \frac{2p+(q-z)(\theta_\infty+t(q+z-1))}{2(z-1)z} & \frac{(q-1)\left(\left(p+\frac{1}{2}q(\theta_\infty+(q-1)t)\right)^2-\frac{\theta_0^2}{4}\right)}{qz}+\frac{\frac{\theta_1^2}{4}-\left(p+\frac{1}{2}(q-1)(\theta_\infty+qt)\right)^2}{z-1} \\ \frac{q}{z-qz}+\frac{1}{z-1} & -\frac{2p+(q-z)(\theta_\infty+t(q+z-1))}{2(z-1)z} \end{pmatrix}. \tag{8.29}$$

The compatibility condition leads to the Painlevé V equation:

$$\frac{\mathrm{d}^2q}{\mathrm{d}^2t}=\frac{2q-1}{2(q-1)q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2-\frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{(q-1)q(2qt-t+2\theta_\infty-2)}{2t}+\frac{\theta_0^2}{2qt^2}+\frac{\theta_1^2}{2(q-1)t^2}. \tag{8.30}$$

The Hamiltonian is

$$H_0=-\frac{p^2}{(q-1)(qt)}-\frac{\theta_0^2}{(4q)t}+\frac{\theta_1^2}{(q-1)(4t)}+\frac{1}{4}q(2\theta_\infty+qt-t-2), \tag{8.31}$$

we also define

$$H_1=tH_0+\frac{\theta_0^2}{4}+\frac{\theta_1^2}{4}+\frac{\theta_\infty}{2}+\frac{qt}{2}-\frac{\theta_\infty t}{2}+\frac{t}{2}-\frac{1}{2}. \tag{8.32}$$

Singularities matching leads to confluent Heun equation

$$\begin{aligned} &-\partial_z^2+\frac{1}{4z^2(z-1)^2}\left(\theta_0^2+t^2z^4-1+z\left(-2\theta_0^2+4H_1+2(\theta_\infty-2)t+2\right)\right.\\ &\left.+z^2\left(\theta_0^2+\theta_1^2-4H_1+t^2-4(\theta_\infty-2)t-2\right)+z^3\left(2(\theta_\infty-2)t-2t^2\right)\right), \end{aligned} \tag{8.33}$$

which is the operator appearing in quantization of $N_f = 3$ SW curve, see for instance [116,117]. We note that this curve also plays a role in the study of black hole quasinormal modes. By adapting the procedure illustrated in Sect. 2 to potentials with resonance eigenstates, one should be able to reproduce [26,27]. In addition by using Sect. 5 one should be able to provide a more direct link between [26,27] and [116,118]. It would also be interesting to further investigate the connection with the Rabi model [119] by using the NS approach.

Painlevé VI. The relevant Lax matrix in this case is quite complicated. We write it as

$$A(z,t) = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & -A_{11}(z) \end{pmatrix}, \tag{8.34}$$

where

$$A_{12}(z) = -\frac{(t-1)t(z-q)}{(z-1)z(t-z)}. \tag{8.35}$$

The expressions for A_{11} and A_{21} take the following forms

$$A_{21} = \frac{1}{64\theta_\infty^2(t-1)^2t^2(t-q)^2(q-1)^2q^2}\left(\sum_{i=0}^{7} u_i(t,z,p)q^i\right), \tag{8.36}$$

$$A_{11} = \frac{1}{8\theta_\infty(z-1)z(t-q)(q-1)q(t-z)}\left(\sum_{i=0}^{7} s_i(t,z,p)q^i\right). \tag{8.37}$$

The expressions for $u_i(t,z,p)$ and $s_i(t,z,p)$ are quite cumbersome, hence we do not write them explicitly. Compatibly condition for this system is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}q &= -\frac{2}{(t-1)t}\left(-pq^3 + pq^2t + pq^2 + pq(-t)\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}p &= \frac{1}{4t}\left(\frac{4\theta_1^2}{(q-1)^2} + \frac{(2\theta_\infty+1)^2}{t-1}\right) + \frac{1-4\theta_t^2}{4(t-q)^2} \\ &\quad -\frac{4\left(P(t)^2q^2\left(3q^2-2(t+1)q+t\right)+\theta_0^2 t\right)}{4(t-1)tq^2}, \end{aligned} \tag{8.38}$$

leading to Painlevé VI

$$\begin{aligned} \frac{\mathrm{d}^2q}{\mathrm{d}^2t} =& \frac{1}{2}\left(\frac{1}{q-t}+\frac{1}{q}+\frac{1}{q-1}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \left(\frac{1}{q-t}+\frac{1}{t}+\frac{1}{t-1}\right)\frac{\mathrm{d}q}{\mathrm{d}t} \\ &+ \frac{(q-1)q(q-t)}{2(t-1)^2t^2}\left((2\theta_\infty+1)^2 - \frac{4\theta_0^2 t}{q^2} + \frac{4\theta_1^2(t-1)}{(q-1)^2} - \frac{\left(4\theta_t^2-1\right)(t-1)t}{(t-q)^2}\right). \end{aligned} \tag{8.39}$$

From (8.35) is easy to see that the singularities matching condition leads to the Heun operator appearing in the quantization of the $N_f = 4$ SW curve. This example was studied in [21,23–25,29–31,68,69,120]. In particular the self-dual approach to the Heun equation was studied in great details in [23–25,30].

Some further comments and generalisations:

- In this work we took the approach of studying the spectral properties of quantum mechanical operators by using the knowledge about isomonodromic deformations. However, one can read our result by taking the inverse logic and, in line with [16], use the spectral properties of quantum operators to study the distribution of movable poles in solutions of second order nonlinear ODEs arising as compatibility conditions of isomonodromic deformations.
- All the examples listed above correspond to isomonodromic problems associated to 2×2 linear system. Nevertheless, a similar story is expected to hold also for the higher rank situation. In this case some related properties for the corresponding tau function(s) and generalisation of the Kyiv formula can be found in [50, 121–126].
- We also note that recently a new class of nonlinear eigenvalue problems has been related to a set of generalized Painlevé equations [127]. It would be interesting to study these problems and their stability/instability notion within our gauge theoretic framework.
- In this work we studied in detail examples of operators with confining potential. However our formalism can also be applied straightforwardly to study the band structure of periodic potentials, including the band edges and the corresponding energy splitting. These results will appear somewhere else. It would also be interesting to extend our analysis to the study of potentials which admit a spectrum of resonance modes.
- Another set of generalised problems which it would be interesting to investigate are these connected to q-deformed Painlevé equations and five-dimensional gauge theories [85, 123, 128–133]. In this case the relevant quantum spectral problems are the ones associated to relativistic quantum integrable systems.

  For example, we know that the NS quantization condition does not extend directly to the five-dimensional/relativistic setup. In particular, to compute the exact spectrum of relativistic integrable systems one needs to supply the naive NS quantization [57, 134] with additional non-perturbative corrections [55]. Nevertheless, if we think of the four dimensional quantization as the vanishing of Painlevé tau functions, then this fact extends directly to the five dimensional/relativistic integrable system setup. Indeed it was found in [128] that the zeroes of the tau functions for q-Painlevé compute the exact spectrum of relativistic integrable systems. Hence thinking of the quantization condition as vanishing of (q-) Painlevé tau functions provides a unifying framework for both relativistic and non-relativistic quantum systems. From that perspective it would be interesting to understand how the quantum mirror map is realised on the q-Painlevé side.

  Recent interesting related work in this direction is also [135].
- In the Painlevé III$_3$ example we have an intriguing bridge between the following two operators. On one side we have the modified Mathieu

$$\partial_x^2 - \sqrt{t}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right), \tag{8.40}$$

  and on the other side we have a "dual" Fermi gas operator, which reads [136, eq (1.3)]

$$\mathrm{e}^{4t^{1/4}\cosh(x)}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\partial_x} + \mathrm{e}^{-\frac{\mathrm{i}}{2}\partial_x}\right)\mathrm{e}^{4t^{1/4}\cosh(x)}. \tag{8.41}$$

  In particular, the spectral properties of both operators are encoded in the isomonodromic deformation equations of the linear system (3.1). For example, the quantization condition of both operators can be expressed as vanishing of the Painlevé III$_3$

tau function. This provided a concrete link between the results of [38, 137] and [41], which was also generalised to the q-deformed/five-dimensional framework, see [136] and [128] for more details.[22] It would be very interesting to find such "dual" operator for other Painlevé equations by using the geometrical guideline coming from the TS/ST correspondence [55]. This could provide some concrete realisation of ideas presented in [139].[23]

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A. Conventions for Elliptic Functions

For the elliptic functions we use the same conventions and definitions as in Appendix A of [49]. We always use

$$\mathsf{q} = \mathrm{e}^{2\pi\mathrm{i}\tau}. \tag{A.1}$$

The conventions for Jacobi theta functions are

$$\begin{aligned}
\theta_1(z|\tau) &= -\mathrm{i}\sum_{n\in\mathbb{Z}}(-1)^n \mathrm{e}^{\mathrm{i}\pi\tau(n+\frac{1}{2})^2}\mathrm{e}^{2\pi\mathrm{i}z(n+\frac{1}{2})},\\
\theta_2(z|\tau) &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi\tau(n+\frac{1}{2})^2}\mathrm{e}^{2\pi\mathrm{i}z(n+\frac{1}{2})},\\
\theta_3(z|\tau) &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi\tau n^2}\mathrm{e}^{2\pi\mathrm{i}zn},\\
\theta_4(z|\tau) &= \sum_{n\in\mathbb{Z}}(-1)^n \mathrm{e}^{\mathrm{i}\pi\tau n^2}\mathrm{e}^{2\pi\mathrm{i}zn}.
\end{aligned} \tag{A.2}$$

There is also a useful infinite product representation for θ_1:

$$\theta_1(z|\tau) = 2\mathsf{q}^{\frac{1}{4}}\sin\pi z\prod_{k=1}^{\infty}(1-\mathsf{q}^k)(1-\mathsf{q}^k e^{2\pi\mathrm{i}z})(1-\mathsf{q}^k e^{-2\pi\mathrm{i}z}). \tag{A.3}$$

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[22] Similarly, relations between Kyiv formulas and tt* equations are also discussed in [138] and forthcoming publications by the same Authors.

[23] We thank N. Nekrasov for discussions on this point and bringing our attention to this reference.

The Dedekind η is defined as

$$\eta(\tau) = \mathfrak{q}^{1/24} \prod_{n \geq 1} (1 - \mathfrak{q}^n), \tag{A.4}$$

and satisfies

$$\eta(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau). \tag{A.5}$$

We also define

$$\varphi(\mathfrak{q}) = \prod_{k=1}^{\infty} (1 - \mathfrak{q}^k) = \mathfrak{q}^{-1/24} \eta(\tau). \tag{A.6}$$

One also has a relation

$$\partial_z \theta_1(z|\tau)|_{z=0} = 2\pi \eta(\tau)^3. \tag{A.7}$$

The Weierstrass $\wp$ function is

$$\wp(z|\tau) = -\partial_z^2 \log \theta_1(z|\tau) - 2\eta_1(\tau), \tag{A.8}$$

where

$$\eta_1(\tau) = -2\pi \mathrm{i} \partial_\tau \log \eta(\tau) = -\frac{1}{6} \left(\frac{\partial_z^3 \theta_1(z|\tau)}{\partial_z \theta_1(z|\tau)} \right) \Big|_{z=0}. \tag{A.9}$$

We also use

$$\zeta(z|\tau) = \partial_z \log \theta_1(z|\tau) + 2z\eta_1(\tau). \tag{A.10}$$

We now list some useful representations of the Weierstrass function. The first one is

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{(n,k) \neq (0,0)} \left(\frac{1}{(z - k\tau - n)^2} - \frac{1}{(k\tau + n)^2} \right). \tag{A.11}$$

This representation makes the modular transformation obvious:

$$\wp(z|\tau) = \tau^{-2} \wp(z/\tau | -1/\tau). \tag{A.12}$$

If we take (A.11) and we perform the sum in the n-direction we get another representation:

$$\wp(z|\tau) = \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3} + \sum_{k \neq 0} \left(\frac{\pi^2}{\sin^2 \pi(z - k\tau)} - \frac{\pi^2}{\sin^2(\pi k\tau)} \right). \tag{A.13}$$

From (A.13) one can easily get the expansion in the limit $\tau \to \mathrm{i}\infty$:

$$\wp(z|\tau) = \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3} + 16\pi^2 \mathrm{e}^{2\pi i\tau} \sin^2 \pi z + \mathcal{O}(\mathrm{e}^{4\pi i\tau}). \tag{A.14}$$

Another option is to compute the sum in (A.11) along the k-direction:

$$\wp(z|\tau) = \frac{\pi^2/\tau^2}{\sin^2 \frac{\pi z}{\tau}} - \frac{\pi^2}{3\tau^2} + \sum_{n \neq 0} \left(\frac{\pi^2/\tau^2}{\sin^2 \frac{\pi(z-n)}{\tau}} - \frac{\pi^2/\tau^2}{\sin^2 \frac{\pi n}{\tau}} \right). \tag{A.15}$$

There are also some useful expansions of $\eta_1(\tau)$. For example

$$\eta_1(\tau) = \frac{\pi^2}{6}\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\mathrm{e}^{2\pi i n\tau}\right), \tag{A.16}$$

where $\sigma_1(n)$ is sum of all divisors of n:

$$\sigma_1(n) = \sum_{d|n} d. \tag{A.17}$$

Another useful expansion can be obtained by using the modular transformation of the Dedekind function:

$$\eta_1(\tau) = \frac{\pi \mathrm{i}}{\tau} + \frac{1}{\tau^2}\eta_1(-1/\tau) = \frac{\pi^2}{6\tau^2} + \frac{\pi \mathrm{i}}{\tau} - \frac{4\pi^2}{\tau^2}\sum_{n=1}^{\infty}\sigma_1(n)\mathrm{e}^{-2\pi \mathrm{i}/\tau}. \tag{A.18}$$

B. Perturbative Study of Quantum Mechanical Potentials

The results presented in this Appendix are not new and can be found in various textbook, as well as in [83, Sec. 2] where they also discuss them in relation to gauge theory. We added this Appendix in order to perform another verification of the computations done in the main part of the paper, and also to study several limiting cases in more detail.

B.1. Perturbed Pöschl–Teller potential. Here we study the quantum mechanical problem of the Weierstrass potential by considering its expansion (A.14) in the limit $\tau \to \mathrm{i}\infty$. The corresponding quantum mechanical Hamiltonian is

$$\begin{aligned} \mathrm{O}_- &= -\partial_x^2 + m(m-1)\wp(x|\tau) \\ &= -\partial_x^2 + \frac{\pi^2 m(m-1)}{\sin^2 \pi x} - \frac{\pi^2}{3}m(m-1) \\ &\quad + 16\pi^2 m(m-1)\mathrm{e}^{2\pi \mathrm{i}\tau}\sin^2\pi x + \mathcal{O}(\mathrm{e}^{4\pi \mathrm{i}\tau}). \end{aligned} \tag{B.1}$$

The first term is the well-known trigonometric Pöschl–Teller potential. Its eigenfunctions can be constructed explicitly, for example, using supersymmetric quantum mechanics (we take $m > 1$ so that the eigenfunctions are well defined for $x \in [0, 1]$):

$$|\Psi_k\rangle = (\partial_x + \pi m \cot \pi x)(\partial_x + \pi(m+1)\cot\pi x)\cdots(\partial_x + \pi(m+k-1)\cot\pi x)\,(\sin\pi x)^{m+k}. \tag{B.2}$$

The eigenvalue equation has the form

$$\left(-\partial_x^2 + \frac{m(m-1)\pi^2}{\sin^2\pi x}\right)|\Psi_k\rangle = \pi^2(m+k)^2|\Psi_k\rangle. \tag{B.3}$$

Now we compute the first order correction to the energy (we computed it for the first few levels and then guessed the general form):

$$\frac{\langle\Psi_k|\sin^2\pi x|\Psi_k\rangle}{\langle\Psi_k|\Psi_k\rangle} = \frac{1}{2} + \frac{m(m-1)}{2((m+k)^2-1)}. \tag{B.4}$$

Hence

$$\begin{aligned} E_k &= \frac{\langle \Psi_k | \hat{H} | \Psi_k \rangle}{\langle \Psi_k | \Psi_k \rangle} + \mathcal{O}(\mathrm{e}^{4\pi \mathrm{i}\tau}) \\ &= -\frac{\pi^2}{3} m(m-1) + \pi^2 (m+k)^2 + 8\pi^2 \mathrm{e}^{2\pi \mathrm{i}\tau} \left(1 + \frac{m(m-1)}{(m+k)^2 - 1} \right) + \mathcal{O}(\mathrm{e}^{4\pi \mathrm{i}\tau}), \end{aligned} \tag{B.5}$$

see also [83, formula (2.36)]. This is in perfect agreement with the gauge theory computation, see (4.80).

We notice that this formula is applicable also for $\tau \to -\frac{1}{2} + \mathrm{i}\infty$. In this case the potential is still real, and all the formulas for the energy can be applied directly. The only difference is that now $\mathfrak{q} = \mathrm{e}^{2\pi \mathrm{i}\tau}$ will be a small negative real number.

B.2. Distant potential walls approximation. We now switch to another approximation. We consider the operator

$$\mathrm{O}_- = -\partial_x^2 + m(m-1)\wp(x, \widehat{\tau}), \tag{B.6}$$

where

$$x \in [0, 1], \quad \widehat{\tau} \in \mathrm{i}\mathbb{R}_+, \quad m > 1. \tag{B.7}$$

We introduce here $\widehat{\tau}$ to distinguish it from the modular parameter in the conformal blocks, which will be $\tau = -1/\widehat{\tau}$. We want to study the limit

$$-\mathrm{i}\widehat{\tau} \to 0. \tag{B.8}$$

In this regime the potential takes the form of an infinite collection of walls

$$\frac{\pi^2 m(m-1)/\widehat{\tau}^2}{\sin^2 \frac{\pi(x-n)}{\widehat{\tau}}} \tag{B.9}$$

located at integer points $n \in \mathbb{Z}$, see (A.15). Each wall decays exponentially at a distance of the order $-\mathrm{i}\widehat{\tau}$. We could naively rescale the x coordinate as

$$y = \pi x / (-\mathrm{i}\widehat{\tau}) \tag{B.10}$$

and say that the limiting potential is $\frac{m(m-1)}{\sinh^2 y}$. However, such procedure produces some inconvenient artifacts like a continuous spectrum (it was discrete before the limit). In fact the correct way to solve this problem is to keep also the second neighboring wall before taking the limit (B.8).[24] This procedure will give us a result which is valid up to exponentially small corrections of the form $\mathcal{O}(\mathrm{e}^{-2\pi \mathrm{i}/\widehat{\tau}})$.

We proceed as follows. We first consider the scattering on the single potential wall and then glue the two scattered wave functions corresponding to the two neighboring walls. The equation for the $n = 0$ wall is (hyperbolic Pöschl–Teller)

$$-\partial_x^2 \Psi + \frac{\pi^2 m(m-1)/\widehat{\tau}^2}{\sin^2 \frac{\pi x}{\widehat{\tau}}} \Psi = \frac{\pi^2 \kappa^2}{\widehat{\tau}^2} \Psi, \tag{B.11}$$

[24] In this limit the second wall goes to infinity in the y-coordinate.

where κ parameterizes the energy. Two independent solutions of this equation are

$$\Psi^{(1)}(x) = \left(2\mathrm{i}\sin\frac{\pi x}{\widehat{\tau}}\right)^m {}_2F_1(m+\kappa, m-\kappa, m+\frac{1}{2}, \sin^2\frac{\pi x}{2\widehat{\tau}}), \tag{B.12}$$

$$\Psi^{(2)}(x) = \left(\sin\frac{\pi x}{\widehat{\tau}}\right)^m \left(\sin\frac{\pi x}{2\widehat{\tau}}\right)^{1-2m} {}_2F_1(\frac{1}{2}+\kappa, \frac{1}{2}-\kappa, \frac{1}{2}-m, \sin^2\frac{\pi x}{2\widehat{\tau}}). \tag{B.13}$$

We see that if $m > 1$ only the first one is regular as $x \to 0$. So in order to study the scattering on the potential wall it is sufficient to consider $\Psi^{(1)}(x)$. We also need to re-expand $\Psi^{(1)}(x)$ in the limit (B.8). This can be done by using the formula (C.15). We have

$$\begin{aligned}\Psi^{(1)}(x) \approx\ & \frac{\Gamma(-2\kappa)\Gamma(m+1/2)}{\Gamma(m-\kappa)\Gamma(1/2-\kappa)}\left(-\sin^2\frac{\pi x}{2\widehat{\tau}}\right)^{-m-\kappa}\left(2\mathrm{i}\sin\frac{\pi x}{\widehat{\tau}}\right)^m \\ & +\frac{\Gamma(2\kappa)\Gamma(m+1/2)}{\Gamma(m+\kappa)\Gamma(1/2+\kappa)}\left(-\sin^2\frac{\pi x}{2\widehat{\tau}}\right)^{-m+\kappa}\left(2\mathrm{i}\sin\frac{\pi x}{\widehat{\tau}}\right)^m \approx \\ \approx\ & \frac{\Gamma(-2\kappa)\Gamma(m+1/2)}{\Gamma(m-\kappa)\Gamma(1/2-\kappa)}2^{2m+2\kappa}\mathrm{e}^{-\kappa\frac{\mathrm{i}\pi x}{\widehat{\tau}}} \\ & +\frac{\Gamma(2\kappa)\Gamma(m+1/2)}{\Gamma(m+\kappa)\Gamma(1/2+\kappa)}2^{2m-2\kappa}\mathrm{e}^{\kappa\frac{\mathrm{i}\pi x}{\widehat{\tau}}}+\mathcal{O}(e^{-\pi\mathrm{i}x/\widehat{\tau}}).\end{aligned} \tag{B.14}$$

Likewise the wave function coming from the potential wall at $x = 1$ should have the form

$$\begin{aligned}\Psi^{(\tilde{1})} = \Psi^{(1)}(1-x) \approx\ & \frac{\Gamma(-2\kappa)\Gamma(m+1/2)}{\Gamma(m-\kappa)\Gamma(1/2-\kappa)}2^{2m+2\kappa}\mathrm{e}^{-\frac{\kappa\mathrm{i}\pi}{\widehat{\tau}}}\mathrm{e}^{\kappa\frac{\mathrm{i}\pi x}{\widehat{\tau}}} \\ & +\frac{\Gamma(2\kappa)\Gamma(m+1/2)}{\Gamma(m+\kappa)\Gamma(1/2+\kappa)}2^{2m-2\kappa}\mathrm{e}^{\frac{\kappa\mathrm{i}\pi}{\widehat{\tau}}}\mathrm{e}^{-\kappa\frac{\mathrm{i}\pi x}{\widehat{\tau}}}+\mathcal{O}(e^{-\pi\mathrm{i}(1-x)/\widehat{\tau}}).\end{aligned} \tag{B.15}$$

One should have $\Psi^{(1)}(x) \approx \pm\Psi^{(\tilde{1})}(x)$, because the potential is symmetric. This gives us the following relation:

$$\mathrm{e}^{\pi\mathrm{i}\kappa/\widehat{\tau}}\frac{\Gamma(2\kappa)\Gamma(1/2-\kappa)\Gamma(m-\kappa)}{\Gamma(-2\kappa)\Gamma(1/2+\kappa)\Gamma(m+\kappa)}2^{-4\kappa} \approx \pm 1. \tag{B.16}$$

This can be simplified using Legendre duplication formula

$$\Gamma(2\kappa) = \frac{2^{2\kappa-1}}{\sqrt{\pi}}\Gamma(\kappa)\Gamma(1/2+\kappa). \tag{B.17}$$

We get

$$\mathrm{e}^{\pi\mathrm{i}\kappa/\widehat{\tau}}\frac{\Gamma(1+\kappa)\Gamma(m-\kappa)}{\Gamma(1-\kappa)\Gamma(m+\kappa)} \approx \mp 1. \tag{B.18}$$

We can also rewrite it in the logarithmic form:

$$\pi\mathrm{i}k \approx \frac{\pi\mathrm{i}\kappa}{\widehat{\tau}} + \log\frac{\Gamma(1+\kappa)\Gamma(m-\kappa)}{\Gamma(1-\kappa)\Gamma(m+\kappa)}, \quad k \in \mathbb{Z}. \tag{B.19}$$

This is in perfect agreement with the gauge theory computation. Indeed this example is case # 2 in Table 1. Therefore the quantization condition coming from the gauge theory (4.59), (4.73) give us

$$e^{i\pi(k+1)} = -\mathfrak{q}^{-\sigma}\frac{\Gamma(m-2\sigma)\Gamma(2\sigma)}{\Gamma(m+2\sigma)\Gamma(-2\sigma)}e^{\partial_\sigma F^{\mathrm{NS}}_{\mathrm{inst}}(\sigma,m-1/2,\mathfrak{q})/2}. \tag{B.20}$$

Taking the logarithm of this equation we get

$$i\pi k = -2i\pi\tau\sigma + \log\frac{\Gamma(m-2\sigma)\Gamma(2\sigma)}{\Gamma(m+2\sigma)\Gamma(-2\sigma)} + \frac{1}{2}\partial_\sigma F^{\mathrm{NS}}_{\mathrm{inst}}(\sigma, m-1/2, \mathfrak{q}). \tag{B.21}$$

We see that this expression coincides with the approximate quantum mechanical quantization condition (B.19) after identification

$$2\sigma = \kappa, \quad \widehat{\tau} = -\frac{1}{\tau}. \tag{B.22}$$

The difference between these two expressions is given by the derivative of the conformal block, which is exponentially small i.e. $\mathcal{O}(e^{-2\pi i/\hat{\tau}})$. On the quantum mechanical side this corresponds to the "interaction" between the potential walls.

We can also try to analyze the spectrum in the limit $\widehat{\tau} \to 0$. To do this we expand the gamma functions around $\sigma = 0$ with $\pi > \arg\sigma \geq 0$:

$$\begin{aligned}\log\frac{\Gamma(m-2\sigma)\Gamma(2\sigma)}{\Gamma(m+2\sigma)\Gamma(-2\sigma)} = -i\pi - 4\sigma(\psi^{(0)}(m)+\gamma_{\mathrm{Euler}}) + \frac{8}{3}\sigma^3(\psi^{(2)}(1)\\ -\psi^{(2)}(m)) + O\left(\sigma^5\right),\end{aligned} \tag{B.23}$$

where ψ is the polygamma function. Now we can find σ in a form of a double series expansion

$$\begin{aligned}\sigma = \frac{-k-1}{2\tau} - \frac{i(k+1)(\psi^{(0)}(m)+\gamma_{\mathrm{Euler}})}{\pi\tau^2} + \frac{2(k+1)(\psi^{(0)}(m)+\gamma_{\mathrm{Euler}})^2}{\pi^2\tau^3}\\ +\mathcal{O}(\tau^{-4}) + \mathcal{O}(\mathfrak{q}).\end{aligned} \tag{B.24}$$

By substituting this expression into the formula for the energy of the one-wall problem we get[25]

$$E \approx -\frac{\pi^3}{3}m(m-1) + \pi^2\tau^2\kappa^2. \tag{B.25}$$

Hence

$$E = -\frac{\pi^3}{3}m(m-1) + \pi^2(k+1)^2 + \frac{4i\pi(k+1)^2}{\tau}(\gamma_{\mathrm{Euler}} + \psi^{(0)}(m)) + \ldots. \tag{B.26}$$

At the leading order (B.26) coincides with the spectrum of two infinite walls potentials. Corrections of order τ^{-1} and higher come from the fact that the potential walls have a non-zero width. Other exponentially small corrections of order $\mathcal{O}(\mathfrak{q})$ come from all the other terms in the expansion (A.15) and also from the fact that we used the one-wall wave functions for the two-wall problem.

We also notice that a similar computation can be done for the case #6 in Table 1. In this case the limiting quantum-mechanical potential consists of potential walls $\frac{m(m-1)}{\sinh^2 y}$ with potential wells $-\frac{m(m-1)}{\cosh^2 y}$ between them.

[25] The first term comes from the $\pi^2/3$ in (A.15).

C. Computation of Monodromies

To make the exposition self-consistent we report in this Appendix the explicit computations of monodromies in terms of η and σ. These results were obtained in [49]. We assume everywhere that $2\sigma \notin \mathbb{Z}$ and the same for m.

C.1. B-cycle monodromy M_B. The fundamental solution of the linear system (4.1) has the following properties

$$Y(z+1) = \mathrm{T}_A Y(z) M_A, \qquad Y(z+\tau) = \mathrm{T}_B Y(z) M_B. \tag{C.1}$$

Here the matrices the $\mathrm{T}_A, \mathrm{T}_B$ are called twists, they encode nontrivial shifts of the matrix $A(z)$. Geometrically they encodes the fact that we have connection in nontrivial holomorphic bundle on $\mathbb{T}^2$. It follows from the formula (4.2) that

$$\mathrm{T}_A = 1, \quad \mathrm{T}_B = \mathrm{e}^{2\pi \mathrm{i} Q \sigma_3}, \tag{C.2}$$

where σ_3 denotes the Pauli matrix.

The matrices M_A, M_B are called monodromy matrices. In this section we find their explicit expressions for an appropriate choice of $Y(z)$ (see [49, eq. (D.32)]), the reader in hurry can skip the derivation and go to the answers (C.20), (C.22), (C.24). We will assume through this derivation that σ, m are generic, then expect possible singularities in these parameters.

Since the monodromies do not depend on τ and z, we can consider the system (2.1) in the limit

$$\tau \to \mathrm{i}\infty, \quad \mathrm{Re}(\tau) = 0. \tag{C.3}$$

Moreover, we take z in the vicinities of four points, $\frac{1}{2} - \frac{\tau}{2}, \frac{1}{2} + \frac{\tau}{2}, -\frac{1}{2} + \frac{\tau}{2}, -\frac{1}{2} - \frac{\tau}{2}$. As a first step we find the asymptotics of Q in this regime. By using (4.9) we get the following expansion:

$$\begin{aligned} &\mathrm{e}^{\pi \mathrm{i}\tau/2}(\mathrm{e}^{2\pi\mathrm{i}Q} + \mathrm{e}^{-2\pi\mathrm{i}Q}) \frac{\prod_{\epsilon=\pm} G(1-m+2\epsilon\sigma)}{\prod_{\epsilon=\pm} G(1+2\epsilon\sigma)} \mathrm{e}^{2\pi\mathrm{i}\sigma^2} \\ &\quad\approx \frac{\prod_{\epsilon=\pm} G(1-m+\epsilon(2\sigma+1))}{\prod_{\epsilon=\pm} G(1+\epsilon(2\sigma+1))} \mathrm{e}^{\mathrm{i}\eta/2} \mathrm{e}^{2\pi\mathrm{i}\tau(\sigma+1/2)^2} + \\ &\quad + \frac{\prod_{\epsilon=\pm} G(1-m+\epsilon(2\sigma-1))}{\prod_{\epsilon=\pm} G(1+\epsilon(2\sigma-1))} \mathrm{e}^{-\mathrm{i}\eta/2} \mathrm{e}^{2\pi\mathrm{i}\tau(\sigma-1/2)^2}. \end{aligned} \tag{C.4}$$

The leading term has the following form:

$$\mathrm{e}^{2\pi\mathrm{i}Q} \approx \mathrm{e}^{2\pi\mathrm{i}\tau\sigma + \mathrm{i}\eta/2} \frac{\Gamma(-m+2\sigma)\Gamma(1-2\sigma)}{\Gamma(1-m-2\sigma)\Gamma(2\sigma)} =: \mathrm{e}^{2\pi\mathrm{i}(\sigma\tau+\beta)}. \tag{C.5}$$

Hence we have $Q \approx \sigma\tau + \beta$. The corresponding momentum is $p \approx 2\pi\mathrm{i}\sigma$. Here we are focusing on the upper sign in (4.22), but this does not matter for the computation of the monodromies.

To compute the limit of the Lax matrix we first analyze the element 12 of (4.17). Using (A.3) we have

$$A_{12}(z) = m \frac{(\mathrm{e}^{\pi\mathrm{i}(z-2Q)} - \mathrm{e}^{-\pi\mathrm{i}(z-2Q)}) \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2\pi\mathrm{i}n\tau} \mathrm{e}^{2\pi\mathrm{i}(z-2Q)})(1 - \mathrm{e}^{2\pi\mathrm{i}n\tau} \mathrm{e}^{-2\pi\mathrm{i}(z-2Q)})}{(\mathrm{e}^{\pi\mathrm{i}z} - \mathrm{e}^{-\pi\mathrm{i}z}) \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2\pi\mathrm{i}n\tau} \mathrm{e}^{2\pi\mathrm{i}z})(1 - \mathrm{e}^{2\pi\mathrm{i}n\tau} \mathrm{e}^{-2\pi\mathrm{i}z})}$$

$$\times \frac{\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})^2}{(e^{2\pi i Q}-e^{-2\pi i Q})\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}e^{4\pi i Q})(1-e^{2\pi i n\tau}e^{-4\pi i Q})}\times 2\pi i. \tag{C.6}$$

It is convenient to consider

$$\mathrm{Re}\sigma \in (0, 1/4), \tag{C.7}$$

and then perform analytic continuation to all values of σ. We also consider z in the following region containing the fundamental domain:

$$\mathrm{Im}z \in (-\frac{1}{2}\mathrm{Im}\tau, \frac{1}{2}\mathrm{Im}\tau). \tag{C.8}$$

Under such assumptions we can just drop the infinite products in (C.6) and write

$$\begin{aligned} A_{12}(z) &\approx \frac{2\pi i m(e^{\pi i(z-2Q)}-e^{-\pi i(z-2Q)})}{(e^{\pi i z}-e^{-\pi i z})(e^{2\pi i Q}-e^{-2\pi i Q})} \\ &= \frac{-2\pi i m}{e^{2\pi i z}-1}+\frac{2\pi i m}{e^{4\pi i Q}-1}\approx \frac{-2\pi i m e^{2\pi i z}}{e^{2\pi i z}-1}. \end{aligned} \tag{C.9}$$

The same procedure can be done for $A_{21}(z)$:

$$A_{21}(z) \approx \frac{-2\pi i m}{e^{2\pi i z}-1}+\frac{2\pi i m}{e^{-4\pi i Q}-1}\approx \frac{-2\pi i m}{e^{2\pi i z}-1}. \tag{C.10}$$

Therefore in our approximation the connection matrix has the form

$$A(z) \approx 2\pi i \begin{pmatrix} \sigma & -\frac{m e^{2\pi i z}}{e^{2\pi i z}-1} \\ -\frac{m}{e^{2\pi i z}-1} & -\sigma \end{pmatrix}. \tag{C.11}$$

The solution of the linear system (4.1) becomes

$$\begin{aligned} Y(z) &\approx (1-e^{2\pi i z})^m \\ &\times \begin{pmatrix} {}_2F_1(m, m+2\sigma, 2\sigma, e^{2\pi i z}) & \frac{-m e^{2\pi i z}}{2\sigma-1}\,{}_2F_1(1+m, 1+m-2\sigma, 2-2\sigma, e^{2\pi i z}) \\ \frac{m}{2\sigma}\,{}_2F_1(1+m, m+2\sigma, 1+2\sigma, e^{2\pi i z}) & {}_2F_1(m, 1+m-2\sigma, 1-2\sigma, e^{2\pi i z}) \end{pmatrix} \\ &\times \mathrm{diag}\left((-e^{2\pi i z})^{\sigma}, (-e^{2\pi i z})^{-\sigma}\right). \end{aligned} \tag{C.12}$$

We now compare this solution in the vicinities of the two following points. The first one is

$$z=\frac{1}{2}+\frac{\tau}{2} \tag{C.13}$$

which corresponds to $e^{2\pi i z} \to -0$, so in this case (C.12) can be used. The second point is

$$z=\frac{1}{2}-\frac{\tau}{2}. \tag{C.14}$$

In this case $e^{2\pi i z} \to -\infty$, so one has to perform analytic continuation of $Y(z)$ along the straight line from $\frac{1}{2}+\frac{\tau}{2}$ to $\frac{1}{2}-\frac{\tau}{2}$. This can be done by using the standard formula for hypergeometric function:

$$\begin{aligned} {}_2F_1(a, b, c, x) = &\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}\,{}_2F_1(a, a-c+1, a-b+1, x^{-1}) \\ &+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}\,{}_2F_1(b-c+1, b, b-a+1, x^{-1}). \end{aligned} \tag{C.15}$$

The result of this analytic continuation $Y_{cont}(z)$ is given by the following explicit formula

$$\begin{aligned} Y_{cont}(z) &\approx (1-\mathrm{e}^{-2\pi\mathrm{i}z})^m \\ &\times \begin{pmatrix} {}_2F_1(h, 1+m-2\sigma, 1-2\sigma, \mathrm{e}^{-2\pi\mathrm{i}z}) & \frac{m}{2\sigma}\,{}_2F_1(1+m, m+2\sigma, 1+2\sigma, \mathrm{e}^{-2\pi\mathrm{i}z}) \\ \frac{-m\mathrm{e}^{-2\pi\mathrm{i}z}}{2\sigma-1}\,{}_2F_1(1+m, 1+m-2\sigma, 2-2\sigma, \mathrm{e}^{-2\pi\mathrm{i}z}) & {}_2F_1(m, m+2\sigma, 2\sigma, \mathrm{e}^{-2\pi\mathrm{i}z}) \end{pmatrix} \\ &\times \operatorname{diag}\left((-\mathrm{e}^{2\pi\mathrm{i}z})^{\sigma}, (-\mathrm{e}^{2\pi\mathrm{i}z})^{-\sigma}\right) \begin{pmatrix} \frac{\Gamma(2\sigma)^2}{\Gamma(2\sigma-m)\Gamma(2\sigma+m)} & -\frac{\sin\pi m}{\sin 2\pi\sigma} \\ \frac{\sin\pi m}{\sin 2\pi\sigma} & \frac{\Gamma(1-2\sigma)^2}{\Gamma(1-m-2\sigma)\Gamma(1+m-2\sigma)} \end{pmatrix}. \end{aligned} \tag{C.16}$$

Now we rewrite the B-cycle monodromy equation (C.1) as

$$M_B^{(0)} \approx Y_{cont}(z)^{-1} \operatorname{diag}\left(\mathrm{e}^{-2\pi\mathrm{i}\sigma\tau}\mathrm{e}^{-2\pi\mathrm{i}\beta}, \mathrm{e}^{2\pi\mathrm{i}\sigma\tau}\mathrm{e}^{2\pi\mathrm{i}\beta}\right) Y(z+\tau). \tag{C.17}$$

We now use this formula for z near $\frac{1}{2}-\frac{\tau}{2}$. In this region $\mathrm{e}^{2\pi\mathrm{i}z}$ is large and $\mathrm{e}^{2\pi\mathrm{i}(z+\tau)}$ is small. Hence we get

$$\begin{aligned} M_B^{(0)} &\approx \begin{pmatrix} \frac{\Gamma(1-2\sigma)^2}{\Gamma(1-m-2\sigma)\Gamma(1+m-2\sigma)} & \frac{\sin\pi m}{\sin 2\pi\sigma} \\ -\frac{\sin\pi m}{\sin 2\pi\sigma} & \frac{\Gamma(2\sigma)^2}{\Gamma(-m+2\sigma)\Gamma(m+2\sigma)} \end{pmatrix} \begin{pmatrix} 1 & \frac{m}{2\sigma}(-\mathrm{e}^{2\pi\mathrm{i}z})^{-2\sigma} \\ 0 & 1 \end{pmatrix} \\ &\quad \times \operatorname{diag}\left(\mathrm{e}^{-2\pi\mathrm{i}\beta}, \mathrm{e}^{2\pi\mathrm{i}\beta}\right) \begin{pmatrix} 1 & 0 \\ (-\mathrm{e}^{2\pi\mathrm{i}(z+\tau)})^{2\sigma} & 1 \end{pmatrix}. \end{aligned} \tag{C.18}$$

In this computation we already neglected the terms of order $\mathrm{e}^{-2\pi\mathrm{i}z}$ and $\mathrm{e}^{2\pi\mathrm{i}(z+\tau)}$. Therefore, given (C.7), we can also neglect the terms of order $(\mathrm{e}^{2\pi\mathrm{i}z})^{-2\sigma}$. Finally, by using the definition of β from (C.5) we obtain

$$M_B^{(0)} = \begin{pmatrix} \frac{\sin\pi(2\sigma-m)}{\sin 2\pi\sigma}\mathrm{e}^{-\mathrm{i}\eta/2} & \frac{\sin\pi m}{\sin 2\pi\sigma}\frac{\Gamma(1-2\sigma)\Gamma(2\sigma-m)}{\Gamma(1-m-2\sigma)\Gamma(2\sigma)}\mathrm{e}^{-\mathrm{i}\eta/2} \\ -\frac{\sin\pi m}{\sin 2\pi\sigma}\frac{\Gamma(1-m-2\sigma)\Gamma(2\sigma)}{\Gamma(1-2\sigma)\Gamma(2\sigma-m)}\mathrm{e}^{\mathrm{i}\eta/2} & \frac{\sin\pi(2\sigma+m)}{\sin 2\pi\sigma}\mathrm{e}^{\mathrm{i}\eta/2} \end{pmatrix}. \tag{C.19}$$

To get rid of the factors with gamma functions and the two $\mathrm{e}^{\pm\mathrm{i}\eta/2}$ in the out of diagonal elements, we can perform conjugation by a diagonal matrix. We get

$$M_B = \begin{pmatrix} \frac{\sin\pi(2\sigma-m)}{\sin 2\pi\sigma}\mathrm{e}^{-\mathrm{i}\eta/2} & \frac{\sin\pi m}{\sin 2\pi\sigma} \\ -\frac{\sin\pi m}{\sin 2\pi\sigma} & \frac{\sin\pi(2\sigma+m)}{\sin 2\pi\sigma}\mathrm{e}^{\mathrm{i}\eta/2} \end{pmatrix}. \tag{C.20}$$

This matrix can also be written in terms of $\tilde{\eta}$ defined by

$$\mathrm{e}^{\mathrm{i}\tilde{\eta}/2} = \frac{\sin\pi(2\sigma+m)}{\sin\pi(2\sigma-m)}\mathrm{e}^{\mathrm{i}\eta/2}. \tag{C.21}$$

We have

$$M_B = \begin{pmatrix} \frac{\sin\pi(2\sigma+m)}{\sin 2\pi\sigma}\mathrm{e}^{-\mathrm{i}\tilde{\eta}/2} & \frac{\sin\pi m}{\sin 2\pi\sigma} \\ -\frac{\sin\pi m}{\sin 2\pi\sigma} & \frac{\sin\pi(2\sigma-m)}{\sin 2\pi\sigma}\mathrm{e}^{\mathrm{i}\tilde{\eta}/2} \end{pmatrix}. \tag{C.22}$$

These two representations give different monodromies in the dangerous cases when

$$\sin\pi(2\sigma\pm m) = 0, \tag{C.23}$$

which will be explained in "Appendix C.3".

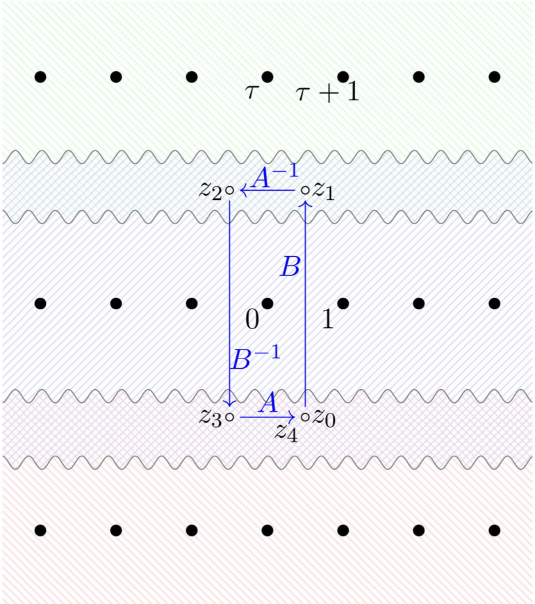

Fig. 2. Degenerate limit of the linear system

C.2. Other monodromies: M_A, M_0. Now we compute the remaining monodromies M_A, M_0. We take the starting point to be z_0 in the vicinity of $\frac{1}{2}-\frac{\tau}{2}$. Using expression (C.12) for Y(z) we get

$$M_A = \begin{pmatrix} e^{2\pi i\sigma} & 0 \\ 0 & e^{-2\pi i\sigma} \end{pmatrix}. \tag{C.24}$$

Now we compute the analytic continuation around the singular point as in Fig. 2:

$$\begin{aligned} Y(z_1) &= Y(z_0+\tau) = \mathrm{T}_B Y(z_0) M_B, \\ Y(z_2) &= Y(z_1-1) = \mathrm{T}_B Y(z_0-1) M_B = T_B Y(z_0) M_A^{-1} M_B, \\ Y(z_3) &= Y(z_2-\tau) = \mathrm{T}_B Y(z_0-\tau) M_A^{-1} M_B = Y(z_0) M_B^{-1} M_A^{-1} M_B, \\ Y(z_4) &= Y(z_3+1) = Y(z_0+1) M_B^{-1} M_A^{-1} M_B = Y(z_0) M_A M_B^{-1} M_A^{-1} M_B = Y(z_0) M_0, \end{aligned} \tag{C.25}$$

where $T_B = e^{2\pi i Q\sigma_3}$. Thus

$$M_0 = M_A M_B^{-1} M_A^{-1} M_B. \tag{C.26}$$

We see from this relation that the monodromies corresponding to consecutive pieces of the path should be written from the right to the left.

Another monodromy which we also need is the one over the straight line C connecting the point 0 with $2\tau+1$, see Fig. 3. It corresponds to the self-adjoint operator living on C when $\tau = -\frac{1}{2} + \mathrm{i}t$. Looking at Fig. 3 we conclude that

$$M_C = M_A M_B^2. \tag{C.27}$$

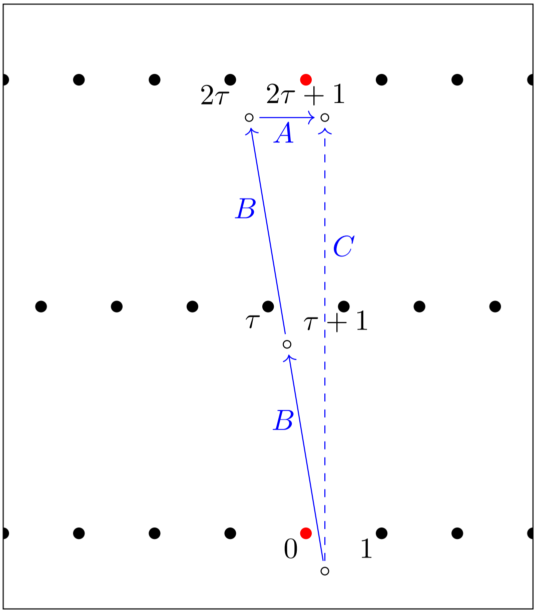

Fig. 3. Monodromy M_C

C.3. Trace coordinates. Using the formulas (C.20), (C.22), (C.24) we will consider (σ, η) or $(\sigma, \tilde{\eta})$ as coordinates on the moduli space of monodromy data on torus.[26] This monodromy manifold can be described in terms of traces of some products of matrices

$$p_{AB} = \operatorname{tr} M_A M_B, \quad p_A = \operatorname{tr} M_A, \quad p_B = \operatorname{tr} M_B, \quad p_0 = \operatorname{tr} M_0. \tag{C.28}$$

Here $p_0 = 2\cos 2\pi m$ is considered as a fixed parameter. By using the explicit expressions above we easily get the following formulas:

$$\begin{aligned} p_{AB} - \mathrm{e}^{2\pi i\sigma} p_B &= -2\mathrm{i}\sin\pi(2\sigma + m)\mathrm{e}^{\mathrm{i}\frac{\eta}{2}} = -2\mathrm{i}\sin\pi(2\sigma - m)\mathrm{e}^{\mathrm{i}\frac{\tilde{\eta}}{2}}, \\ p_{AB} - \mathrm{e}^{-2\pi i\sigma} p_B &= 2\mathrm{i}\sin\pi(2\sigma - m)\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}} = 2\mathrm{i}\sin\pi(2\sigma + m)\mathrm{e}^{-\mathrm{i}\frac{\tilde{\eta}}{2}}, \end{aligned} \tag{C.29}$$

By multiplying the above expressions we get the following relation

$$\left(p_{AB} - \mathrm{e}^{2\pi \mathrm{i}\sigma} p_B\right)\left(p_{AB} - \mathrm{e}^{-2\pi \mathrm{i}\sigma} p_B\right) = p_0 - p_A^2 + 2, \tag{C.30}$$

or after simplification

$$p_{AB}^2 + p_A^2 + p_B^2 - p_A\, p_B\, p_{AB} = p_0 + 2. \tag{C.31}$$

This equation defines a surface in $\mathbb{C}^3$ with coordinates p_A, p_B, p_{AB}. This surface is the monodromy manifold in our case.

[26] To be precise, η and σ are not actual coordinates. They are defined up to simultaneous sign inversion $(\eta, \sigma) \sim (-\eta, -\sigma)$, and also up to integer shifts.

Now we can look to the dangerous points (C.23). In terms of trace coordinates this equations reads $p_A^2 = p_0 + 2$. Hence it defines two points on the monodromy manifold

$$p_{AB} = \mathrm{e}^{\pm 2\pi \mathrm{i}\sigma} p_B. \tag{C.32}$$

It follows from equations (C.29) that these two points belong to two different charts, one for finite η and another for finite $\tilde{\eta}$. For example for $\sin(2\sigma + m) = 0$ we have finite η for " + " sign in (C.32) and finite η for " $-$ " sign in (C.32).

The consequence of these considerations is that for generic m the corresponding monodromy manifold (without the very bad points $2\sigma \in \mathbb{Z}$) can be covered by two charts where either η or $\tilde{\eta}$ is finite.

C.4. Diagonalization of M_0. To study normalizability of the solution to the linear system (4.1) we will need to study its asymptotics around 0. To do this it is very convenient to work in a basis where M_0 is diagonal. There are two possible diagonalizations corresponding to two different charts, where either η or $\tilde{\eta}$ are kept finite. These diagonalizations will be denoted by $^{(I)}$ and $^{(II)}$, respectively:

$$M_\nu^{(I,II)} = \left(T^{(I,II)}\right)^{-1} M_\nu T^{(I,II)}. \tag{C.33}$$

The first diagonalization of M_0 is done by the matrix

$$T^{(I)} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\eta}{2}+2\pi\mathrm{i}\sigma} \frac{\sin\pi(2\sigma+m)}{\sin 2\pi\sigma\cos\pi m} & 1 \\ -\frac{\sin\pi(2\sigma-m)}{\sin 2\pi\sigma\cos\pi m} & \mathrm{e}^{-\mathrm{i}\frac{\eta}{2}-2\pi\mathrm{i}\sigma} \end{pmatrix}. \tag{C.34}$$

It is always non-degenerate, since $\det T^{(I)} = 2$. The corresponding conjugated monodromy matrices are

$$M_0^{(I)} = M_0^{(II)} = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}m} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}m} \end{pmatrix}, \tag{C.35}$$

$$M_A^{(I)} = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}m}\frac{\cos 2\pi\sigma}{\cos\pi m} & \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}-2\pi\mathrm{i}\sigma}\sin 2\pi\sigma \\ \frac{\mathrm{i}\sin\pi(2\sigma+m)\sin\pi(2\sigma-m)}{\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}-2\pi\mathrm{i}\sigma}\sin 2\pi\sigma\cos^2\pi m} & \mathrm{e}^{-\pi\mathrm{i}m}\frac{\cos 2\pi\sigma}{\cos\pi m} \end{pmatrix}, \tag{C.36}$$

$$M_B^{(I)} = \begin{pmatrix} \frac{\mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\sin\pi(2\sigma+m)+\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\sin\pi(2\sigma-m)}{2\mathrm{e}^{-\pi\mathrm{i}m}\cos\pi m\sin 2\pi\sigma} & \frac{\mathrm{e}^{-\mathrm{i}\eta}-1}{2\mathrm{e}^{\pi\mathrm{i}(2\sigma+m)}} \\ \frac{\sin^2\pi(2\sigma-m)-\mathrm{e}^{\mathrm{i}\eta}\sin^2\pi(2\sigma+m)}{2\mathrm{e}^{-\pi\mathrm{i}(2\sigma+m)}\cos^2\pi m\sin^2 2\pi\sigma} & \frac{\mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\sin\pi(2\sigma+m)+\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\sin\pi(2\sigma-m)}{2\mathrm{e}^{\mathrm{i}\pi m}\cos\pi m\sin 2\pi\sigma} \end{pmatrix}. \tag{C.37}$$

The matrix M_C has a cumbersome expression, so we present here only its off-diagonal entries:

$$\left(M_C^{(I)}\right)_{12} = \frac{\left(\cos\pi(\sigma-\frac{m}{2}) - \mathrm{e}^{\mathrm{i}\eta-2\pi\mathrm{i}\sigma}\cos\pi(\sigma+\frac{m}{2})\right)\left(\mathrm{e}^{\mathrm{i}\eta-2\pi\mathrm{i}\sigma}\sin\pi(\sigma+\frac{m}{2}) + \sin\pi(\sigma-\frac{m}{2})\right)}{\mathrm{e}^{3\mathrm{i}\frac{\eta}{2}+\pi\mathrm{i}m}\sin 2\pi\sigma}, \tag{C.38}$$

$$\left(M_C^{(I)}\right)_{21} = \frac{\left(\mathrm{e}^{\mathrm{i}\eta-2\pi\mathrm{i}\sigma}\sin\pi(2\sigma+m)\cos\pi(\sigma+\frac{m}{2}) + \sin\pi(2\sigma-m)\cos\pi(\sigma-\frac{m}{2})\right)}{\mathrm{e}^{\mathrm{i}\frac{\eta}{2}-\pi\mathrm{i}m-4\pi\mathrm{i}\sigma}\sin^3 2\pi\sigma\cos^2\pi m} \times\left(\sin\pi(2\sigma-m)\sin\pi(\sigma-\frac{m}{2}) - \mathrm{e}^{\mathrm{i}\eta-2\pi\mathrm{i}\sigma}\sin\pi(2\sigma+m)\sin\pi(\sigma+\frac{m}{2})\right) \tag{C.39}$$

The second diagonalization of M_0 is done by the matrix

$$T^{(II)} = \begin{pmatrix} 1 & e^{i\frac{\tilde{\eta}}{2}+2\pi i\sigma}\frac{\sin\pi(2\sigma-m)}{\sin 2\pi\sigma\cos\pi m} \\ -e^{-i\frac{\tilde{\eta}}{2}-2\pi i\sigma} & \frac{\sin(2\sigma+m)}{\sin 2\pi\sigma\cos\pi m} \end{pmatrix}. \tag{C.40}$$

The conjugated monodromy matrices are

$$M_A^{(II)} = \begin{pmatrix} e^{i\pi m}\frac{\cos 2\pi\sigma}{\cos\pi m} & \frac{i\sin\pi(2\sigma+m)\sin\pi(2\sigma-m)}{e^{-i\frac{\tilde{\eta}}{2}-2\pi i\sigma}\sin 2\pi\sigma\cos^2\pi m} \\ ie^{-i\frac{\tilde{\eta}}{2}-2\pi i\sigma}\sin 2\pi\sigma & e^{-\pi i m}\frac{\cos 2\pi\sigma}{\cos\pi m} \end{pmatrix}, \tag{C.41}$$

$$M_B^{(II)} = \begin{pmatrix} \frac{e^{i\frac{\tilde{\eta}}{2}}\sin\pi(2\sigma-m)+e^{-i\frac{\tilde{\eta}}{2}}\sin\pi(2\sigma+m)}{2e^{-\pi i m}\sin 2\pi\sigma\cos\pi m} & \frac{\sin^2\pi(2\sigma+m)-e^{i\tilde{\eta}}\sin^2\pi(2\sigma-m)}{2e^{\pi i(m-2\sigma)}\sin^2 2\pi\sigma\cos^2\pi m} \\ \frac{e^{-i\tilde{\eta}}-1}{2e^{\pi i(2\sigma-m)}} & \frac{e^{i\frac{\tilde{\eta}}{2}}\sin\pi(2\sigma-m)+e^{-i\frac{\tilde{\eta}}{2}}\sin\pi(2\sigma+m)}{2e^{\pi i m}\sin 2\pi\sigma\cos\pi m} \end{pmatrix}, \tag{C.42}$$

$$\begin{aligned}\left(M_C^{(II)}\right)_{12} = {} & \frac{\left(e^{i\tilde{\eta}-2\pi i\sigma}\sin\pi(2\sigma-m)\cos\pi(\sigma-\frac{m}{2})+\sin\pi(2\sigma+m)\cos\pi(\sigma+\frac{m}{2})\right)}{e^{i\frac{\tilde{\eta}}{2}+\pi i m-4\pi i\sigma}\sin^3 2\pi\sigma\cos^2\pi m}\times \\ & \times\left(\sin\pi(2\sigma+m)\sin\pi(\sigma+\frac{m}{2})-e^{i\tilde{\eta}-2\pi i\sigma}\sin\pi(2\sigma-m)\sin\pi(\sigma-\frac{m}{2})\right),\end{aligned} \tag{C.43}$$

$$\left(M_C^{(II)}\right)_{21} = \frac{\left(\cos\pi(\sigma+\frac{m}{2})-e^{i\tilde{\eta}-2\pi i\sigma}\cos\pi(\sigma-\frac{m}{2})\right)\left(e^{i\tilde{\eta}-2\pi i\sigma}\sin\pi(\sigma-\frac{m}{2})+\sin\pi(\sigma+\frac{m}{2})\right)}{e^{3i\frac{\tilde{\eta}}{2}-\pi i m}\sin 2\pi\sigma}. \tag{C.44}$$

D. Conformal Blocks on the Torus: Conventions

The torus conformal blocks almost coincide with the Nekrasov partition functions for $\mathcal{N} = 2^*$ four-dimensional theory. Hence here we write the formulas for Nekrasov functions. We follow the notations of [84], or rather adopt the notations from loc. cit. since there is no $\mathcal{N} = 2^*$ there.

The Nekrasov function is a product

$$\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q}) = \mathcal{Z}_{\mathrm{cl}}\mathcal{Z}_{1-\mathrm{loop}}\cdot\varphi(\mathfrak{q})^{1-2\frac{\alpha(\epsilon_1+\epsilon_2-\alpha)}{\epsilon_1\epsilon_2}}\mathcal{Z}_{\mathrm{inst}}^{U(2)}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q}), \tag{D.1}$$

where

$$\mathcal{Z}_{\mathrm{cl}}(a;\epsilon_1,\epsilon_2|\mathfrak{q}) = \mathfrak{q}^{-a^2/\epsilon_1\epsilon_2}, \tag{D.2}$$

$$\begin{aligned}\mathcal{Z}_{1-\mathrm{loop}}(a,\alpha;\epsilon_1,\epsilon_2) = \exp\big(&\gamma_{\epsilon_1,\epsilon_2}(2a-\alpha;1)+\gamma_{\epsilon_1,\epsilon_2}(-2a-\alpha;1) \\ &-\gamma_{\epsilon_1,\epsilon_2}(2a;1)-\gamma_{\epsilon_1,\epsilon_2}(-2a;1)\big),\end{aligned} \tag{D.3}$$

with [10, App. E]

$$\gamma_{\epsilon_1,\epsilon_2}(x,\Lambda) = \frac{d}{ds}\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty\frac{dt}{t}t^s\frac{e^{-tx}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_2 t}-1)}\Big|_{s=0}, \quad \mathrm{Re}(x)>0. \tag{D.4}$$

The function $\mathcal{Z}_{\text{inst}}^{U(2)}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q})$ is defined as a sum over partitions:

$$\mathcal{Z}_{inst}(a, \alpha; \epsilon_1, \epsilon_2|\Lambda) = \sum_{\lambda^{(1)},\lambda^{(2)}} \frac{\prod_{i,j=1}^{2} \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(\alpha + a_i - a_j; \epsilon_1, \epsilon_2)}{\prod_{i,j=1}^{2} \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(a_i - a_j; \epsilon_1, \epsilon_2)} \mathfrak{q}^{|\lambda^{(1)}|+|\lambda^{(2)}|},$$

$$\mathsf{N}_{\lambda,\mu}(a; \epsilon_1, \epsilon_2) = \prod_{s\in\lambda}(a - \epsilon_2(a_\mu(s) + 1) + \epsilon_1 l_\lambda(s))$$
$$\prod_{s\in\mu}(a + \epsilon_2 a_\lambda(s) - \epsilon_1(l_\mu(s) + 1)), \tag{D.5}$$

where $\lambda^{(1)}, \lambda^{(2)}$ are partitions and $|\lambda| = \sum \lambda_j$ denotes the number of boxes. We also use $a_\lambda(s), l_\lambda(s)$ to denote the lengths of arms and legs for the box s in the Young diagram corresponding to the partition λ. The parameters a_1 and a_2 satisfy $a_1 = a, a_2 = -a$. The first terms of these functions are

$$\mathcal{Z}_{\text{inst}}^{U(2)}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q})$$
$$= 1 + \mathfrak{q}\frac{2(\alpha - \epsilon_1)(\alpha - \epsilon_2)\left(-4a^2 + \alpha^2 - \alpha\epsilon_1 - \alpha\epsilon_2 + \epsilon_1^2 + 2\epsilon_1\epsilon_2 + \epsilon_2^2\right)}{\epsilon_1\epsilon_2(-2a + \epsilon_1 + \epsilon_2)(2a + \epsilon_1 + \epsilon_2)} + \mathcal{O}(\mathfrak{q}^2).$$

It is sometimes convenient to factor out the $U(1)$ part of the partition function by defining

$$\mathcal{Z}_{\text{inst}}^{SU(2)}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}) = \varphi(\mathfrak{q})^{1-2\frac{\alpha(\epsilon_1+\epsilon_2-\alpha)}{\epsilon_1\epsilon_2}} \mathcal{Z}_{\text{inst}}^{U(2)}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}). \tag{D.6}$$

On the CFT side this is the transition from the sum of Virasoro and Heisenberg to Virasoro algebra. The first few terms of this function are

$$\mathcal{Z}_{\text{inst}}^{SU(2)}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}) = 1 + \mathfrak{q}\left(1 - \frac{2\alpha(\epsilon_1 - \alpha)(\epsilon_2 - \alpha)(-\alpha + \epsilon_1 + \epsilon_2)}{\epsilon_1\epsilon_2(-2a + \epsilon_1 + \epsilon_2)(2a + \epsilon_1 + \epsilon_2)}\right) + \mathcal{O}(\mathfrak{q}^2).$$

The function $\mathcal{Z}_{\text{inst}}^{U(2)}$ has the following reflection symmetry

$$\mathcal{Z}_{\text{inst}}^{U(2)}(a, \epsilon_1 + \epsilon_2 - \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}) = \mathcal{Z}_{\text{inst}}^{U(2)}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q})\ . \tag{D.7}$$

This follows from the relation

$$\mathsf{N}_{\lambda,\mu}(2a + \alpha; \epsilon_1, \epsilon_2) = (-1)^{|\lambda|+|\mu|}\mathsf{N}_{\mu,\lambda}(\epsilon_1 + \epsilon_2 - 2a - \alpha; \epsilon_1, \epsilon_2) \tag{D.8}$$

The classical part $\mathcal{Z}_{\text{cl}}(a; \epsilon_1, \epsilon_2|\mathfrak{q})$ also posses such symmetry, but not $\mathcal{Z}_{1-\text{loop}}(a; \epsilon_1, \epsilon_2|\mathfrak{q})$. Hence we have

$$\mathcal{Z}(a, \epsilon_1 + \epsilon_2 - \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}) = C^{\text{refl}}_{\epsilon_1,\epsilon_2}(a, \alpha)\mathcal{Z}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q}) \tag{D.9}$$

where

$$C^{\text{refl}}_{\epsilon_1,\epsilon_2}(a, \alpha) = \exp\Big(\gamma_{\epsilon_1,\epsilon_2}(2a+\alpha-\epsilon_1-\epsilon_2; 1) + \gamma_{\epsilon_1,\epsilon_2}(-2a+\alpha-\epsilon_1-\epsilon_2; 1)$$
$$-\gamma_{\epsilon_1,\epsilon_2}(2a-\alpha; 1) - \gamma_{\epsilon_1,\epsilon_2}(-2a-\alpha; 1)\Big). \tag{D.10}$$

In the main text we mainly need two special cases of $\mathcal{Z}(a, \alpha; \epsilon_1, \epsilon_2|\mathfrak{q})$, namely the self-dual case ($\epsilon_1 + \epsilon_2 = 0$) and the Nekrasov–Shatashvili case ($\epsilon_2 \to 0$).

Self-dual case In this case we impose the condition $\epsilon_1 + \epsilon_2 = 0$. It is also convenient to fix some rescaling freedom and impose $\epsilon_1 = -1$, $\epsilon_2 = 1$. As a consequence we change the notation and use $a = \sigma$, $\alpha = m$. We have

$$\begin{aligned}&\mathcal{Z}(a, \alpha; 1, -1|\mathfrak{q})\\&= \mathfrak{q}^{\sigma^2}(2\pi)^m \frac{G(1+2\sigma-m)G(1-2\sigma-m)}{G(1+2\sigma)G(1-2\sigma)}\varphi(\mathfrak{q})^{1-2m^2}\mathcal{Z}^{U(2)}_{\text{inst}}(\sigma, m, -1, 1|\mathfrak{q}).\end{aligned} \tag{D.11}$$

Here for the $\gamma_{\epsilon_1,\epsilon_2}$ we used the transformation (see e.g. [84, App. A,B])

$$\begin{aligned}\exp(\gamma_{1,-1}(x;1)) &= \exp(-\gamma_{-1,-1}(x+1)) = \Gamma_2(x+1;1;1)^{-1}\\&= G(x)\mathrm{e}^{-\zeta'(-1)}(2\pi)^{1-x/2}.\end{aligned} \tag{D.12}$$

In the main text we use the self-dual limit of the function $\mathcal{Z}^{SU(2)}_{\text{inst}}$ and we denote it by Z:

$$Z(\sigma, m, \mathfrak{q}) = \varphi(\mathfrak{q})^{1-2m^2}\mathcal{Z}^{U(2)}_{\text{inst}}(\sigma, m|\mathfrak{q}) = \mathcal{Z}^{SU(2)}_{\text{inst}}(\sigma, m|\mathfrak{q}). \tag{D.13}$$

The first few orders read

$$Z(\sigma, m, \mathfrak{q}) = 1 + \left(1 + \frac{\left(m^2-1\right)m^2}{2\sigma^2}\right)\mathfrak{q} + \mathcal{O}(\mathfrak{q}^2). \tag{D.14}$$

Nekrasov–Shatashvili limit In the limit $\epsilon_2 \to 0$, we have

$$\begin{aligned}\lim_{\epsilon_2\to 0}\epsilon_2 \log \mathcal{Z}(\sigma, \mu+\frac{1}{2}; 1, \epsilon_2|\mathfrak{q}) &= F^{\text{NS}}(\sigma, \mu, \mathfrak{q})\\&= -\sigma^2\log\mathfrak{q} + F^{\text{NS}}_{1-\text{loop}}(\sigma, \mu, \mathfrak{q}) + F^{\text{NS}}_{\text{inst}}(\sigma, \mu, \mathfrak{q})\\&\quad + \left(2\mu^2 - \frac{1}{2}\right)\log(\varphi(\mathfrak{q})),\end{aligned} \tag{D.15}$$

where

$$\begin{aligned}F^{\text{NS}}_{1-\text{loop}}(\sigma, \mu, \mathfrak{q}) &= -\psi^{(-2)}(\frac{1}{2}+2\sigma-\mu) - \psi^{(-2)}(\frac{1}{2}-2\sigma-\mu)\\&\quad +\psi^{(-2)}(1+2\sigma) + \psi^{(-2)}(1-2\sigma) + (\mu+\frac{1}{2})\log(2\pi),\end{aligned} \tag{D.16}$$

and

$$\begin{aligned}F^{\text{NS}}_{\text{inst}}(\sigma, \mu, \mathfrak{q}) &= \lim_{\epsilon_2\to 0}\mathcal{Z}^{U(2)}_{\text{inst}}(\sigma, \mu+\frac{1}{2}; 1, \epsilon_2|\mathfrak{q}) = \frac{(4\mu^2-1)(3+4\mu^2-16\sigma^2)}{8(1-4\sigma^2)}\mathfrak{q}\\&\quad + \left(\frac{3}{4}(4\mu^2-1) + \frac{(4\mu^2-1)^2}{64}\left(\frac{3\left(1-4\mu^2\right)^2}{4\left(1-4\sigma^2\right)^2} - \frac{(1-4\mu^2)^2}{(1-4\sigma^2)^3}\right.\right.\\&\quad \left.\left.+\frac{16\mu^4-72\mu^2-15}{4(4\sigma^2-1)} + \frac{(9-4\mu^2)^2}{16(1-\sigma^2)}\right)\right)\mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3).\end{aligned} \tag{D.17}$$

The formula for $F^{\mathrm{NS}}_{1-\mathrm{loop}}$ is somehow ambiguous, it depends on the branch of the function $\psi^{(-2)}$. Fortunately F^{NS} appears in the main text only through $\mathfrak{q}$ and σ derivatives. The term $F^{\mathrm{NS}}_{1-\mathrm{loop}}$ do not depend on $\mathfrak{q}$. For the σ derivative we have

$$\begin{aligned}\partial_\sigma F^{\mathrm{NS}}_{1-\mathrm{loop}}(\sigma,\mu,\mathfrak{q}) =& -2\log\Gamma\left(-\mu+2\sigma+\frac{1}{2}\right)+2\log\Gamma\left(-\mu-2\sigma+\frac{1}{2}\right)\\ &-2\log\Gamma(1-2\sigma)+2\log\Gamma(1+2\sigma).\end{aligned} \tag{D.18}$$

Formally speaking, this function also has monodromy, but now its exponent $\exp(\partial_\sigma F^{\mathrm{NS}}_{1-\mathrm{loop}})$ is well defined, so the eqs. (4.73) and (4.74) make sense. The formula (D.18) is actually used in the main text.

In order to obtain the formula (D.16) it is useful to consider $\gamma^{\mathrm{NS}}(x)=\lim_{\epsilon_2\to 0}\epsilon_2\gamma_{1,\epsilon_2}(x;1)$. By using the expansion of the exponent in terms of Bernoulli numbers, we get the asymptotic series (for $x>0$)

$$\begin{aligned}\gamma^{\mathrm{NS}}(x) &= \left(\frac{3}{4}-\frac{1}{2}\log x\right)x^2+\frac{1}{2}(1-\log x)x-\frac{1}{12}\log x+\sum_{m=3}\frac{B_m}{m(m-1)(m-2)}x^{2-m}\\ &= -\psi^{(-2)}(x+1)+\frac{1}{2}(x+1)\log(2\pi)+\log(A),\end{aligned} \tag{D.19}$$

where A is Glaisher–Kinkelin constant.

We used the polygamma function $\psi^{(-2)}(x)$ in the formula (D.16), but it can be also written by using Barnes G functions (as in self-dual case) thanks to the formula (for $x>0$)

$$G(x+1)=\exp\left(-\psi^{(-2)}(x)+x\log\Gamma(x)-\frac{1}{2}x^2+\frac{1+\log(2\pi)}{2}x\right). \tag{D.20}$$

E. Some Proofs

E.1. Proof of the relation (6.1). The proof is similar to the one in [102]. We will use the notation β_0^0 and β_1^0 for the functions which appear in the algebraic blowup relations (5.16)

$$\beta_0^0(\mathfrak{q})=\frac{\theta_3(0|2\tau)}{\varphi(\mathfrak{q})},\quad \beta_1^0(\mathfrak{q})=\frac{\theta_2(0|2\tau)}{\varphi(\mathfrak{q})}. \tag{E.1}$$

It follows from the blowup relations (5.16) and (5.18), (5.19) that for $j=0,1$

$$\begin{aligned}&\sum_{n\in\mathbb{Z}+\frac{j}{2}}\left(\partial_{\log\mathfrak{q}}\mathcal{Z}(a+n\epsilon_1,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q})\right)\mathcal{Z}(a+n\epsilon_2,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q})\\ &=\frac{1}{\epsilon_1-\epsilon_2}\left(\epsilon_1\beta_j^{1,1}(\mathfrak{q})+\alpha\beta_j^{1,2}(\mathfrak{q})-\epsilon_2\beta_j^0(\mathfrak{q})\partial_{\log\mathfrak{q}}\right)\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q}).\end{aligned} \tag{E.2}$$

We will use Nekrasov functions depending on different ϵ parameters. Let us denote

$$\begin{aligned}&\mathcal{Z}^{(1)}(a)=\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2-\epsilon_1|\mathfrak{q}), && \mathcal{Z}^{(2)}(a)=\mathcal{Z}(a,\alpha;\epsilon_1-\epsilon_2,2\epsilon_2-\epsilon_1|\mathfrak{q}),\\ &\mathcal{Z}^{(3)}(a)=\mathcal{Z}(a,\alpha;\epsilon_1-2\epsilon_2,\epsilon_2|\mathfrak{q}), && \mathcal{Z}^{(4)}(a)=\mathcal{Z}(a,\alpha;\epsilon_1-\epsilon_2,\epsilon_2|\mathfrak{q}).\end{aligned} \tag{E.3}$$

By $\partial^{(l)}$ we denote the operator $\partial_{\log \mathfrak{q}}$ acting on the argument of $\mathcal{Z}^{(l)}$, for $l = 1, 2, 3, 4$. We also denote

$$\widehat{\mathcal{Z}}(a) = \sum_{2r\in\mathbb{Z}} \Big(\epsilon_1\partial^{(1)} + (2\epsilon_2-\epsilon_1)\partial^{(2)} - \epsilon_2\gamma_0(\mathfrak{q})\Big)\, \mathcal{Z}^{(1)}(a+r\epsilon_1)\mathcal{Z}^{(2)}(a+r(2\epsilon_2-\epsilon_1)). \quad \text{(E.4)}$$

Then Eq. (6.1) is equivalent to $\widehat{\mathcal{Z}} = 0$. We have

$$\begin{aligned}
&\sum_{2s\in\mathbb{Z}} \widehat{\mathcal{Z}}(a+s\epsilon_1)\mathcal{Z}^{(3)}(a+2s\epsilon_2)\\
&= \sum_{2r,2s\in\mathbb{Z}} \Big(\epsilon_1\partial^{(1)} + (2\epsilon_2-\epsilon_1)\partial^{(2)} - \epsilon_2\gamma_0\Big)\\
&\quad \mathcal{Z}^{(1)}(a+(r+s)\epsilon_1)\mathcal{Z}^{(2)}(a+r(2\epsilon_2-\epsilon_1)+s\epsilon_1)\mathcal{Z}^{(3)}(a+2s\epsilon_2)\\
&= \sum_{2n\in 2\mathbb{Z}+j,\, j=0,1} \Big(\epsilon_1\beta_j^0\partial^{(1)} - \Big((\epsilon_1-\epsilon_2)\beta_j^{1,1} + \alpha\beta_j^{1,2} - \epsilon_2\beta_j^0\partial^{(4)}\Big) - \epsilon_2\gamma_0\beta_j^0\Big)\\
&\quad \mathcal{Z}^{(1)}(a+n\epsilon_1)\mathcal{Z}^{(4)}(a+n\epsilon_2)\\
&= \sum_{j=0,1} \Big(\beta_j^0\big((\epsilon_1+\epsilon_2)\beta_j^{1,1} + \alpha\beta_j^{1,2}\big) - \big((\epsilon_1-\epsilon_2)\beta_j^{1,1} + \alpha\beta_j^{1,2}\big)\beta_j^0 - \epsilon_2\gamma_0(\beta_j^0)^2\Big)\\
&\quad \mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q})\\
&= \epsilon_2\Big(\partial_{\mathfrak{q}}\big((\beta_0^0)^2 + (\beta_1^0)^2\big) - \gamma_0\big((\beta_0^0)^2 + (\beta_1^0)^2\big)\Big)\mathcal{Z}(a,\alpha;\epsilon_1,\epsilon_2|\mathfrak{q}) = 0. \qquad \text{(E.5)}
\end{aligned}$$

Here for shortness we omit the $\mathfrak{q}$ dependence in the β and γ functions. In the first transformation we change variables to $n = r + s$ and use (E.2) for $\partial^{(2)}$. In the second transformation we used the blowup relations (5.16),(5.18),(5.19). Then we used the relation $\beta_j^{1,1}(\mathfrak{q}) = \partial_{\log \mathfrak{q}}\beta_j^0(\mathfrak{q})$ and the definition of $\gamma_0(\mathfrak{q})$, $\beta_j^0(\mathfrak{q})$.

If we decompose

$$\begin{aligned}
\widehat{\mathcal{Z}}(a) &= \mathfrak{q}^{2a^2/(\epsilon_1-2\epsilon_2)\epsilon_1} \sum_{2N\in\mathbb{Z}_{\geq 0}} \widehat{\mathcal{Z}}_N(a)\mathfrak{q}^N,\\
\mathcal{Z}^{(3)}(a) &= \mathfrak{q}^{a^2/(2\epsilon_2-\epsilon_1)\epsilon_2} \sum_{N\in\mathbb{Z}_{\geq 0}} \mathcal{Z}_N^{(3)}(a)\mathfrak{q}^N,
\end{aligned} \qquad \text{(E.6)}$$

then using $\mathcal{Z}_0^{(3)}(a) \neq 0$ and (E.5) we get by induction that $\widehat{\mathcal{Z}}_N(a) = 0$ for any N.

E.2. Proofs of the relations (6.8) *and* (6.10)*. Proof of the relation* (6.8). The proof is just a computation based on the power series expansion of theta function. Let $y = \mathrm{e}^{2\pi \mathrm{i} z}$

$$\begin{aligned}
&\theta_3(z|2\tau)^2\left(\partial^2_{\log \mathfrak{q}} \log\theta_3(z|2\tau)\right) + \theta_2(z|2\tau)^2\left(\partial^2_{\log \mathfrak{q}} \log\theta_2(z|2\tau)\right)\\
&= \Big(\theta_3(z|2\tau)\,\partial^2_{\log \mathfrak{q}}\theta_3(z|2\tau) - \big(\partial_{\log \mathfrak{q}}\theta_3(z|2\tau)\big)^2\\
&\quad + \theta_2(z|2\tau)\,\partial^2_{\log \mathfrak{q}}\theta_2(z|2\tau) - \big(\partial_{\log \mathfrak{q}}\theta_2(z|2\tau)\big)^2\Big)\\
&= \frac{1}{2}\sum_{2m,2n\in\mathbb{Z}+j,\, j=0,1} \mathfrak{q}^{m^2+n^2} y^{n+m}\Big(n^4 - 2n^2m^2 + m^4\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{a,b\in\mathbb{Z}} \mathfrak{q}^{(a^2+b^2)/2} y^a a^2 b^2 \\
&= \frac{1}{2}\Big(\sum_{a\in\mathbb{Z}} a^2 \mathfrak{q}^{a^2/2} y^a\Big)\Big(\sum_{b\in\mathbb{Z}} b^2 \mathfrak{q}^{b^2/2}\Big) \\
&= 2\big(\partial_{\log\mathfrak{q}}\theta_3(z|\tau)\big)\big(\partial_{\log\mathfrak{q}}\theta_3(0|\tau)\big).
\end{aligned} \tag{E.7}
$$

Here we changed variables in the sum by $a = n + m$ and $b = n - m$. □
Proof of the relation (6.10). The notations are as before,

$$
\begin{aligned}
&\theta_3(z|2\tau)^2 \left(\partial_{\log\mathfrak{q}}\partial_{2\pi i z} \log\theta_3(z|2\tau)\right) + \theta_2(z|2\tau)^2 \left(\partial_{\log\mathfrak{q}}\partial_{2\pi i z} \log\theta_2(z|2\tau)\right) \\
&= \Big(\theta_3(z|2\tau)\, \partial_{2\pi i z}\partial_{\log\mathfrak{q}}\theta_3(z|2\tau) \\
&\quad - \big(\partial_{2\pi i z}\theta_3(z|2\tau)\big)\big(\partial_{\log\mathfrak{q}}\theta_3(z|2\tau)\big) + [\theta_3 \leftrightarrow \theta_2]\Big) \\
&= \frac{1}{2} \sum_{2m,2n\in\mathbb{Z}+j,\, j=0,1} \mathfrak{q}^{m^2+n^2} y^{n+m}\left(n^3 - n^2 m - n m^2 + m^3\right) \\
&= \frac{1}{2} \sum_{a,b\in\mathbb{Z}} \mathfrak{q}^{(a^2+b^2)/2} y^a a b^2 \\
&= \frac{1}{2}\Big(\sum_{a\in\mathbb{Z}} a \mathfrak{q}^{a^2/2} y^a\Big)\Big(\sum_{b\in\mathbb{Z}} b^2 \mathfrak{q}^{b^2/2}\Big) \\
&= \big(\partial_{2\pi i z}\theta_3(z|\tau)\big)\big(\partial_{\log\mathfrak{q}}\theta_3(0|\tau)\big).
\end{aligned} \tag{E.8}
$$

□

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