

Exploring replica-Potts CFTs in two dimensions

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ABSTRACT: We initiate a numerical conformal bootstrap study of CFTs with $S_n \ltimes (S_Q)^n$ global symmetry. These include CFTs that can be obtained as coupled replicas of two-dimensional critical Potts models. Particular attention is paid to the special case $S_3 \ltimes (S_3)^3$, which governs the critical behaviour of three coupled critical 3-state Potts models, a multi-scalar realisation of a (potentially) non-integrable CFT in two dimensions. The model has been studied in earlier works using perturbation theory, transfer matrices, and Monte Carlo simulations. This work represents an independent non-perturbative analysis. Our results are in agreement with previous determinations: we obtain an allowed peninsula within parameter space for the scaling dimensions of the three lowest-lying operators in the theory, which contains the earlier predictions for these scaling dimensions. Additionally, we derive numerous bounds on admissible scaling dimensions in the theory, which are compatible with earlier results. Our work sets the necessary groundwork for a future precision study of these theories in the conformal bootstrap.

KEYWORDS: Conformal and W Symmetry, Scale and Conformal Symmetries, Renormalization Group

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1 Introduction

Following the resurgence of the conformal bootstrap program with [1], a considerable amount of effort has been devoted to studying higher-dimensional CFTs ($d > 2$). For a thorough account, see [2], and for a more recent account, see [3]. The flagship results of this program include particularly precise critical exponents for the Ising [4], $O(2)$ [5], $O(3)$ [6], GNY [7]

and super-Ising [8] universality classes. These theories correspond to three-dimensional CFTs which are not exactly solvable/solved, but are accessible via perturbative techniques such as the ε -expansion and non-perturbative techniques such as lattice simulations (see [9] for a review) and, more recently, also the fuzzy sphere regularisation technique [10]. While the numerical conformal bootstrap method can be applied in $d = 2$ using global conformal blocks, and the technology required to do this is indeed available, there are no analogues of the above-mentioned studies for two-dimensional theories. This is despite older literature [11], as well as more recent literature [12], suggesting the existence of families of interesting non-integrable CFTs. For a somewhat recent overview of $2d$ CFT see [13], for an alternative approach to the 4-pt function bootstrap see [14].

In this work, we take the first steps to remedy this by initiating a study of multi-scalar theories with $S_3 \ltimes (S_3)^3$ global symmetry, and more generally $S_n \ltimes (S_Q)^n$.¹ The universality class of these CFTs is supposed to be realized by coupled Potts models [11]. It is believed to be among those $2d$ CFTs with the simplest spectrum, whose exact solution is not known. More pedantically, within the classification of $2d$ CFTs with central charge $c > 1$, this model is expected to be among the few examples of unitary, compact, irrational,² non-supersymmetric, and non-multicritical CFTs. The universality class can be understood as the IR fixed point of an “energy-energy” deformation around copies of critical Potts models

$$S = \sum_{a=1}^n S_{\text{Potts}}^a + g \int d^d x \sum_{a \neq b} \epsilon^a \epsilon^b, \quad (1.1)$$

where S_{Potts}^a is the action of the a -th replica of a Q -state Potts model, and the operators ϵ^a are the energy operators, i.e. the operators that would multiply the mass term.³ Finally, g is a coupling which is set to zero at the decoupled fixed point. This deformation possesses controlled perturbative fixed points in the $Q = 2 + \varepsilon$ limit [17–20], where ε is taken to be infinitesimal, and also in the large- n limit (large number of replicas/copies). In the case of the large- n limit, the computations have not been carried out explicitly for $S_n \ltimes (S_Q)^n$ in $d = 2$, however, they have been carried out to leading order for $S_n \ltimes (\mathbb{Z}_2)^n$ in $d = 3$ [21]. Notice also that this theory cannot be reached by a naive $d = d_c - \varepsilon$ dimensional expansion, such as the $d = 4 - \varepsilon$ expansion for ϕ^4 theories, and the $d = 6 - \varepsilon$ expansion for ϕ^3 theories. This is due to the theory, morally speaking, being a $\phi^3 + \phi^4$ theory, the ϕ^3 coming from each replica and the ϕ^4 coming from the term that couples the replicas. This is in contrast, for example, with the critical 3-state Potts model, which can in principle be treated with the $d = 6 - \varepsilon$ expansion, albeit with a very long flow in terms of ε , needed to reach $d = 2$. The theory may however be amenable to a fixed dimension expansion in the spirit of [22].

Using a lattice Hamiltonian of the schematic type [11]

$$H = \sum_{\text{n.n.}} [a(\delta_{\sigma,\sigma'} + \delta_{\tau,\tau'} + \delta_{\eta,\eta'}) + b(\delta_{\sigma,\sigma'}\delta_{\tau,\tau'} + \delta_{\tau,\tau'}\delta_{\eta,\eta'} + \delta_{\eta,\eta'}\delta_{\sigma,\sigma'})], \quad (1.2)$$

¹For the study of theories with S_3 global symmetry in $d = 2$ and above, which corresponds to a single 3-state Potts model, versus replicas which we will study in this work, see [15, 16].

²That is, with an infinite number of Virasoro primaries and without an extended algebra organizing them into a finite number of representations.

³This would for example be the operator ϕ^2 in a perturbative treatment around the free theory. More generally, it’s the lowest-dimensional non-trivial singlet of the single critical Q -state Potts model.

the theory can also be studied non-perturbatively. The sum runs over lattice sites and nearest neighbours (denoted with a prime) while σ , τ and η represent spin configurations in each of the $n = 3$ Potts replicas. For $Q = 3$, the spins in each replica can be thought of vectors pointing to any of the three vertices of an equilateral triangle. The term multiplying the coupling a is just the sum of three decoupled Potts Hamiltonians, whereas b multiplies the “energy-energy” interaction. The non-perturbative methods used in [11], such as transfer matrices and Monte Carlo simulations, also seem to indicate the existence of a fixed point. We will compare to these results in much of the present work, hence we postpone their discussion.

It is currently believed that unitary non-trivial fixed points exist for $2 < Q \leq 4$ and $n > 2$. The pedestrian understanding for the constraint $2 < Q \leq 4$ is that it guarantees the decoupled theories one perturbs off are unitary. Given that the β function is proportional to $\varepsilon g + A g^2 + O(g^3)$ with A a number, we do not get a controlled fixed point when $\varepsilon = 0$ or equivalently when $Q = 2$. As far as the limitation $n > 2$ is concerned, we refer to the discussion in [11].

In this work, we will make use of the numerical conformal bootstrap using $SL(2, \mathbb{R})$ (global) conformal blocks, i.e. the same bootstrap formalism which has led to the precision results in $d = 3$. See [23] for lecture notes. That is, we will be working in terms of *quasi-primary* operators and only considering the $sl(2, \mathbb{R})$ sub-algebra of the Virasoro algebra spanned by L_1 , L_0 and L_{-1} . In practice, we impose self-consistency, i.e. crossing symmetry, in various 4-point correlators involving three of the lowest spin-0 operators in the spectrum, in terms of scaling dimension. These in our notation will be called ϕ , X and Z_3 and have dimensions respectively $\Delta_\phi = 2/15$, $\Delta_X = 4/5$, and $\Delta_{Z_3} = 6/15$ for $n = 3$ and $Q = 3$ at the decoupled fixed point, i.e. for $g = 0$ in eq. (1.1). The second lowest operator in the theory is also constrained by our analysis, where it enters as an exchanged operator. We call this operator Z and it has dimension $\Delta_Z = 4/15$ at the decoupled fixed point. For all four operators, we compare to available perturbative and non-perturbative results from the literature, which include data from Monte Carlo simulations [11] as well as transfer matrices [11, 24]. In particular, we obtain an allowed peninsula in the parameter space spanned by $(\Delta_\phi, \Delta_Z, \Delta_{Z_3})$ which nicely includes the earlier predictions of [11]. This indicates that the conformal bootstrap is indeed strong enough to constrain the low-lying spectrum of these theories. Notice that because the singlet in the theory has a dimension $\Delta_S \simeq 1.2$,⁴ this differs from the usual situation regarding most theories studied in $d = 3$ where the first scalar singlet is typically one of the lowest-lying operators in the theory. In practice, this means that in order to access the low-lying part of the spectrum we will need to study a much more numerically costly set of sum rules. This is because the correlators of the non-singlet operators have more tensor structures and thus the crossing symmetry constraint produces more sum rules.

With respect to conventions throughout this work, a four-point correlator denoted as $\langle O_1 O_2 O_3 O_3 \rangle$ will always imply that the OPE is to be taken between the first two and last two operators respectively. Also, from this point forth, the phrase “global symmetry”, will refer to

⁴This value can also be understood qualitatively from the large- n expansion [21]. In the large- n limit the lowest lying scalar, i.e. the auxiliary field, has the shadow dimension of the lowest lying singlet in the decoupled theory, i.e. $\Delta_S \rightarrow d - \Delta_S^{\text{dec}} = 2 - 4/5 = 1.2$. See later table 2 for perturbative and non-perturbative estimates from [11]. That being said, we will see operators, such as Z_3 , which acquire particularly large anomalous dimensions. Hence large- n intuition should be used in moderation.

the global “flavour” symmetry i.e. $S_n \ltimes (S_Q)^n$, and *not* the global part of the Virasoro algebra. The latter of which is included in what we will call the “spacetime symmetry”. Lastly, when discussing the lowest-lying operators in the theory, we do not refer to the identity operator, so that e.g. by “lowest-lying singlet” we refer to a non-trivial field.

The paper is organised as follows. In section 2 we outline all the necessary group theory for building operators in a given representation and deriving the sum rules. This is done in a way appropriate for generic n and Q . In section 3 we review previous results from the literature which we will make contact to in the rest of the work. Then, in section 4 we present the numerical constraints we have derived, and conclude with section 5, where we discuss various directions we expect to prove fruitful in future studies. In order to aid readability, we have relegated further technical details to appendices A, B, C, D and E. These can be skipped on a first read-through.

2 Elements of group and field theory

In this work, we will follow the naming conventions of [25], which dealt with the symmetry group $S_n \ltimes (\mathbb{Z}_2)^n$ describing theories with hypercubic symmetry. The group we will cover here is a special case of a wreath product $G \wr S_n = S_n \ltimes G^n$, where $G = S_3$ or more generally $G = S_Q$. This class of theories, their group theory, as well as their tensor structures as appropriate to the bootstrap have been studied in [26], from where we will also borrow. Note that while $S_3 \ltimes (S_3)^3$ is itself a finite group and its character table can be obtained with *GAP* [27]⁵ (see table 8), the way we will treat the group theory in this work will allow us to treat the groups $S_n \ltimes (S_Q)^n$ for arbitrary values of Q and n , including non-integer, as well as large parameter limits such as large- n which would be infeasible with the software currently available.⁶ It will also enable us to study complicated mixed correlators in theories, such as the $S_n \ltimes O(m)^n$ symmetric multi-scalar theories of the $d = 4 - \varepsilon$ dimension expansion, in future work with only small modifications. See [9, 26, 29, 30] for more information on these theories.

2.1 The defining representation

One intuitive way to start discussing the group theory, which will also make direct contact with the field content of the CFT, is to consider a field transforming in the “defining”, representation of $S_n \ltimes (S_Q)^n$ dubbed

$$\phi_i^a. \quad (2.1)$$

Here the upper index is acted on by S_n , running from $a = 1$ to n , and it labels the copy/replica to which the field belongs. Whereas the lower index i runs from 1 to Q , and transforms in the defining representation of S_Q . This corresponds to the field with which the Lagrangian is built in the perturbative limit. The dimension of the irrep carried by ϕ_i^a under $S_n \ltimes (S_Q)^n$ is $\dim(\phi) = n(Q - 1)$, which is n times the dimension of the defining representation of S_Q , since we have n replicas of it. We will label this representation as V and ϕ interchangeably.

⁵<https://www.gap-system.org/>.

⁶Let us also remark that while *autoboot* [28] can in principle treat all finite groups, in practice the order of the group $S_3 \ltimes (S_3)^3$ is so large that its usage is impractical if not infeasible with the current implementation.

For a generic wreath product $G \wr S_n$, we can build “simple” representations in a similar way. We consider n copies of a field, denoted collectively as O_i^a , which transforms in each copy under an irrep ρ of G , while the copy index can be permuted under S_n :

$$O_i^a \xrightarrow{(\vec{g}; \pi)} \sum_{b=1}^n \sum_{i=1}^{\dim(\rho)} \rho(g_a)_{ij} P^{ab} O_j^b. \quad (2.2)$$

Here $(\vec{g}; \pi) = ((g_1, \dots, g_n); \pi) \in G \wr S_n$ (namely $g_a \in G$ and $\pi \in S_n$) and P is the permutation matrix associated to π . This representation is irreducible and has dimension $\dim(O) = n \dim(\rho)$ unless ρ is the trivial representation of G . In the latter case the field O^a is effectively a vector of dimension n which transforms under S_n only. This representation is reducible and can be split into a singlet and a vector representation of S_n :

$$S = \sum_{a=1}^n O^a, \quad (2.3)$$

$$X^a = O^a - \frac{1}{n} \sum_{b=1}^n O^b, \quad (2.4)$$

of dimension $\dim(S) = 1$ and $\dim(X) = n - 1$ respectively. For a mathematically precise treatment of representations of wreath products see [31].

2.2 Decomposing the product of two defining representations

The rest of the representations in the group can now be built by taking products of ϕ ’s and decomposing them into irreducible components. See table 1 for the complete list of irreps when $n = 3$, $Q = 3$. Let us work out the decomposition of the product between two ϕ ’s to give an explicit example in the $Q = 3$ case. This has also been outlined for $S_n \ltimes O(m)^n$ in [26], of which the groups we are considering are subgroups. We have

$$\phi_i^a \otimes \phi_j^b \sim \delta^{ab} \delta_{ij} S + \delta_{ij} X^{ab} + A_{ij}^{ab} + C_{ijk}^{abc} \phi_k^c + Z_{ij}^{ab} + B_{ij}^{ab}, \quad (2.5)$$

where the decomposition rules are as follows:

- if $a = b$ we decompose the lower indices onto irreps of S_3 (or more generally S_Q);
- if $a \neq b$ then we symmetrize and antisymmetrize the ϕ fields, one of which carries a prime so that the antisymmetric combination doesn’t vanish identically.

The tensors C_{ijk}^{abc} are defined as

$$C_{ijk}^{abc} = \delta^{abc} c_{ijk}, \quad (2.6)$$

where c_{ijk} is proportional to the tensor structure of a three-point function of fields transforming in the defining representation of S_3 , $\langle O_i O_j O_k \rangle$.

Symbol	dimension	Symbol	dimension
S	1	XV	6
$\overline{AAA}$	1	VAA	6
$\overline{XX}$	1	$\phi = V$	6
AAA	1	$B_3 = \overline{VVV}$	8
$\overline{AAA}$	2	$Z_3 = VVV$	8
X	2	VA	12
XA	3	$AB = \overline{AVV}$	12
$\overline{AA}$	3	$B = \overline{VV}$	12
A	3	$AZ = AVV$	12
AA	3	$Z = VV$	12
$V\overline{AA}$	6	$VB = \overline{VVV}$	16

Table 1. Some low-lying irreps of $S_n \ltimes (S_3)^n$. The list is complete for $n = 3$. The dimension of each representation is reported for $n = 3$, but each irrep in this table also exists for $n > 3$.

The first four irreps on the right hand side of eq. (2.5), $(S, X^{ab}, A_{ij}^{ab}, \phi_k^c)$, only appear when $a = b$, whereas Z_{ij}^{ab} and B_{ij}^{ab} only appear when $a \neq b$. These can be expressed as

$$S = \delta^{ab} \delta_{ij} \phi_i^a \phi_j^b, \quad (2.7)$$

$$X^{ab} = \left(\delta^{abcd} - \frac{1}{n} \delta^{ab} \delta^{cd} \right) \delta_{ij} \phi_i^c \phi_j^d, \quad (2.8)$$

$$A_{ij}^{ab} = \delta^{abcd} P_{ijkl}^A \phi_i^c \phi_j^d, \quad (2.9)$$

$$Z_{ij}^{ab} = \left((\delta^{ac} \delta^{bd} - \delta^{abcd}) \delta_{ik} \delta_{jl} + (\delta^{ad} \delta^{bc} - \delta^{abcd}) \delta_{il} \delta_{jk} \right) \phi_i^c \phi_j^d, \quad (2.10)$$

$$B_{ij}^{ab} = \left((\delta^{ac} \delta^{bd} - \delta^{abcd}) \delta_{ik} \delta_{jl} - (\delta^{ad} \delta^{bc} - \delta^{abcd}) \delta_{il} \delta_{jk} \right) \phi_i^c \phi_j^d, \quad (2.11)$$

where P^A is the tensor that projects a product of two fields onto the rank-two antisymmetric representation of S_3 (or S_Q), see appendix A.1. The projectors typically contain additional factors of ω_i , which satisfy $\sum_i \omega_i = Q$, and each component of the vector ω_i is equal to one. Hence in the above relations, they can be omitted, since $\sum_i \omega_i \phi_i^a = \sum_i \phi_i^a = 0$, where the last equality follows from the definition of the $Q - 1$ dimensional defining irrep of S_Q .

For a generic wreath product $G \wr S_n$, we can decompose the tensor product of two “simple” irreps $O_i^a \otimes \tilde{O}_j^b$ (transforming in the ρ and $\tilde{\rho}$ irreps under G) with a similar logic.

- If $a = b$ we decompose the lower indices into irreps of G : $\rho \otimes \tilde{\rho} = r_1 \oplus \dots \oplus r_m$. Each of them is irreducible, unless r_k is the trivial representation; in the latter case, as previously discussed, the representation is reducible and can be decomposed in $S \oplus X$. This is what happens in eq. (2.5): the tensor product of two defining representations of S_3 decomposes into singlet (S^a), defining (ϕ_i^a) and antisymmetric (A_{ij}^a) representations; the “replicated” singlet is then further decomposed into S and X components.
- If $a \neq b$ and $\rho \neq \tilde{\rho}$ the representation is irreducible.

- If $a \neq b$ and $\rho = \tilde{\rho}$ the representation is reducible under the action of S_n and can be decomposed into a symmetric and antisymmetric part. In eq. (2.5) these are the Z_{ij}^{ab} and B_{ij}^{ab} representations.

2.3 Higher representations

While carrying out the above procedure of decomposing an arbitrary product of ϕ 's would prove to be highly impractical very quickly, it is easy to infer (and check with the help of the character table), the entire representation content of the theory. We restrict to $S_3 \ltimes (S_3)^3$, and $S_3 \ltimes (S_Q)^3$, for practical reasons.

As an example, let us work out all the representations that can be built using the three replicas of the antisymmetric representation, (A^1, A^2, A^3) , of each of the three S_Q factors included in $S_3 \ltimes (S_Q)^3$. We can take the totally symmetric product of three of these $A^{(1}A^2A^3)$, or of two $A^{(1}A^2)$ or of just one of them A^1 . These are precisely the irreps AAA , AA and A in table 1. Similarly, one can also antisymmetrize the replicas of A . One has $A^{[1}A^2A^3]$ and $A^{[1}A^2]$, which are denoted as $\overline{AAA}$, and $\overline{AA}$ in table 1. Notice that whenever factors of a representation are antisymmetrized we place a bar over them, this is the notation used in [25]. Beyond totally symmetric and totally antisymmetric combinations of A 's, we can form combinations that form other Young tableaux of S_3 , which produces $\overline{AA}A$. Lastly, in cases where all three replicas have not been exhausted, which in this example are AA , A and $\overline{AA}$, we can also adjoin to them another representation to create a new irrep. For example AA can be adjoined with V to produce $V^1A^{(2}A^3)$, which is VAA in table 1.

Together with the representation X from earlier in the product of two fields, these considerations explain all representations in table 1. In the following Z , B , Z_3 , B_3 and VB are simply shorthand for VV , $\overline{VV}$, VVV , $\overline{VVV}$, $V\overline{VV}$. This understanding of the group theory in terms of “composite” representations, will provide important intuition later when we consider gap assumptions to be imposed on admissible spectra. In the case of $S_n \ltimes (S_3)^n$, for example, the scaling dimensions of all operators are given in terms of the ones of the single critical 3-state Potts model (see appendix C) in the decoupled or large- n limits.

3 Previous results

We point the reader to [11] and references therein for a number of useful resources. To the best of our knowledge, the theories under consideration in this paper have been studied using perturbation theory, transfer matrices and Monte Carlo simulations.

With respect to perturbation theory, while one may in principle perform an expansion continuing in flavours, an expansion at large number of copies [21], or a fixed dimension expansion [22], only the first has been carried out explicitly [17–20].⁷ In this case, one considers an expansion of $S_n \ltimes (S_{2+\varepsilon})^n$ around ε infinitesimal, where the flow triggered by the “energy-energy” interaction of eq. (1.1) becomes controlled. In other words, one perturbs around n replicas of the Ising model ($Q = 2$). A caveat in this case is that for $Q = 2$ a lot of representations do not appear: all irreps in table 1 that include a factor of A cease to exist, and thus data for them cannot be extracted.

⁷Additional references include [32–37].

Operator	Expression	$\varepsilon = 2/15$	Tr. Matrix	M. Carlo
S	$\Delta_\epsilon(\varepsilon) + 6\varepsilon - 9\varepsilon^2$	1.44	1.27190	1.27 ± 0.13
S'	$2\Delta_\epsilon(\varepsilon) + 3\varepsilon + \frac{9}{2}\varepsilon^2$	2.08	~ 2.1	
S''	$3\Delta_\epsilon(\varepsilon) - \frac{27}{4}\varepsilon^2$	2.28	~ 2.4	
X	$\Delta_\epsilon(\varepsilon) - 3\varepsilon + \frac{9}{2}\varepsilon^2$	0.48	0.63 ± 0.03	0.63 ± 0.04
X'	$2\Delta_\epsilon(\varepsilon) - \frac{3}{2}\varepsilon - 9\varepsilon^2$	1.24		
Z	$2\Delta_\sigma(\varepsilon) + \frac{3\varepsilon}{4(n-2)}\left(1 - \frac{3\varepsilon}{n-2}\left((n-2)\log 2 + \frac{11}{12}\right)\right)$	0.30227	0.30960	
Z_3	$3\Delta_\sigma(\varepsilon) + \frac{9\varepsilon}{4(n-2)}\left(1 - \frac{3\varepsilon}{n-2}\left((n-2)\log 2 + \frac{11}{12} + \frac{\alpha}{24}\right)\right)$	0.60482	0.59222	
$\phi = V$	$\Delta_\sigma(\varepsilon) - \frac{27}{32}\frac{n-1}{(n-2)^2}\mathcal{F}\varepsilon^3$	0.12805	0.13162	0.13 ± 0.03

Table 2. Dimensions of operators from [11] (and [24] in the case of Transfer Matrices for X). For the original computations in each case see [11] and references therein. The results are reported in conventions where $\varepsilon = 2/15$ corresponds to replicas of 3-state Potts models and $\varepsilon = 1/3$ to 4-state Potts models, see below equation (3.2) in [11]. Some transfer matrix results are not reported with error bars in the original work, hence we also omit them. Thus the choice of digits does not imply any error bar unless explicitly stated. Primes denote subleading operators and α and $\mathcal{F}$ are defined as $\alpha = 33 - 29\sqrt{3}\pi/3$ and $\mathcal{F} = 2\Gamma(-\frac{2}{3})^2\Gamma(\frac{1}{6})^2 / \left(\Gamma(-\frac{1}{3})^2\Gamma(-\frac{1}{6})^2\right)$ respectively. The perturbative results for the S and X representations are reported at $n = 3$. The factors $\Delta_\sigma(\varepsilon)$ and $\Delta_\epsilon(\varepsilon)$ are the decoupled theory dimensions and are equal to $2/15$ and $4/5$ respectively at $Q = 3$.

The perturbative results are complemented by numerous transfer matrix computations. In this approach, one computes an approximation to the partition function of the lattice model on the cylinder, from which it is possible to extract the eigen-energies. These in turn are proportional to the scaling dimensions of the theory in the continuum limit. To this end, in [11] the authors calculated the central charge and the scaling dimensions of what we call ϕ , Z , Z_3 and S . More concretely, they calculated these for the leading spin-0 operator in each representation. Furthermore, in the singlet sector S they computed the first 8 eigen-energies, which correspond to both $SO(2)$ scalars and spinning operators (e.g. the stress tensor), see table 17 in that work. Consequently, the scaling dimension of the leading spin-0 X operator was computed in [24].

Lastly, in [11], the authors also performed a Monte Carlo study in order to corroborate the existence of a critical fixed point. They computed the decay of two-point functions with distance and found a behaviour in agreement with criticality. This led to the extraction of the scaling dimension for operators in the ϕ , X and S representations. These were also found to be in good agreement with the transfer matrix results.

4 Results

We will study a number of different four-point correlator systems on which we will impose self-consistency (i.e. crossing symmetry). We remind the reader of the convention where $\langle O_1 O_2 O_3 O_4 \rangle$ implies that the given four-point correlator is computed by taking the OPE

of $O_1 \times O_2$ and $O_3 \times O_4$. Hence, an equation of the form

$$\langle O_1 O_2 O_3 O_4 \rangle = \langle O_3 O_2 O_1 O_4 \rangle$$

will simply imply that the OPEs are taken in a different order, and is shorthand for

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) O_4(x_4) \rangle = \langle O_3(x_3) O_2(x_2) O_1(x_1) O_4(x_4) \rangle.$$

With this in mind, let us specify the correlator systems we will study.

These are:

- the system involving only ϕ_i^a , which will be referred to as the single correlator ϕ bootstrap:

$$\langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle = \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle; \quad (4.1)$$

- the mixed system involving both ϕ_i^a and X^{ab} , which will be referred to as the mixed correlator $\phi - X$ bootstrap:

$$\begin{aligned} \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle, \\ \langle X^{ab} X^{cd} X^{ef} X^{gh} \rangle &= \langle X^{ef} X^{cd} X^{ab} X^{gh} \rangle, \\ \langle \phi_i^a X^{cd} \phi_j^b X^{ef} \rangle &= \langle \phi_j^b X^{cd} \phi_i^a X^{ef} \rangle, \\ \langle \phi_i^a \phi_j^b X^{cd} X^{ef} \rangle &= \langle X^{cd} \phi_j^b \phi_i^a X^{ef} \rangle; \end{aligned} \quad (4.2)$$

- the system involving only Z_3^{abc} , which will be referred to as the single correlator Z_3 bootstrap:

$$\langle Z_3^{a_1 b_1 c_1} Z_3^{a_2 b_2 c_2} Z_3^{a_3 b_3 c_3} Z_3^{a_4 b_4 c_4} \rangle = \langle Z_3^{a_3 b_3 c_3} Z_3^{a_2 b_2 c_2} Z_3^{a_1 b_1 c_1} Z_3^{a_4 b_4 c_4} \rangle; \quad (4.3)$$

- the mixed system involving both ϕ_i^a and Z_3^{abc} , which will be referred to as the mixed correlator $\phi - Z_3$ bootstrap:

$$\begin{aligned} \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle, \\ \langle \phi_i^a Z_3^{bcd} \phi_m^e Z_3^{fgh} \rangle &= \langle \phi_m^e Z_3^{bcd} \phi_i^a Z_3^{fgh} \rangle, \\ \langle \phi_i^a \phi_m^e Z_3^{bcd} Z_3^{fgh} \rangle &= \langle Z_3^{bcd} \phi_m^e \phi_i^a Z_3^{fgh} \rangle, \\ \langle Z_3^{a_1 b_1 c_1} Z_3^{a_2 b_2 c_2} Z_3^{a_3 b_3 c_3} Z_3^{a_4 b_4 c_4} \rangle &= \langle Z_3^{a_3 b_3 c_3} Z_3^{a_2 b_2 c_2} Z_3^{a_1 b_1 c_1} Z_3^{a_4 b_4 c_4} \rangle. \end{aligned} \quad (4.4)$$

As a heavy-handed indicator of numerical cost, we mention that for $n = 3$, $Q = 3$ the above systems give 6, 15, 10 and 28 sum rules respectively.

Each of these systems offers various advantages and disadvantages. The ϕ single correlator bootstrap has a small number of sum rules, hence it is numerically less costly and also has the lightest operator in the theory as an external. It is however less constraining than the mixed correlator systems we will study. The $\phi - X$ system contains fewer sum rules than the $\phi - Z_3$ system but does not allow us (as we will see) to easily segregate the coupled and the decoupled fixed points in parameter space. On the other hand, while a potential $\phi - Z$ mixed correlator system would involve the two lightest operators in the theory, and would

presumably be the most constraining, it is considerably more costly numerically than the $\phi - Z_3$ system. In addition to this, the first Z operator at spin-0 has a similar dimension at both the coupled and decoupled fixed points, making it hard to distinguish the two fixed points in the numerics.⁸ Hence, we omit an analysis of the $\phi - Z$ system in the present work.

Note that in addition to the correlator systems discussed above, we also performed some preliminary tests with the mixed correlator $\phi - \phi'$ bootstrap

$$\begin{aligned}
 \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle, \\
 \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle, \\
 \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle, \\
 \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle.
 \end{aligned} \tag{4.5}$$

The rationale behind this system is that it exchanges the S and V representations at spin-1, and the A representation at spin-0. At least at the decoupled fixed point, whose scaling dimensions are determined through the 3-state Potts model, these operators possess large gaps (see appendix C). However, we found this system less constraining than the ones we present in this work. While there may be many reasons for this, one obvious guess is that since $\Delta_{\phi'} = 4/3$ at the decoupled fixed point (and presumably somewhat close in the coupled one), it is significantly heavier than all the other operators we study as externals. In particular, we observed the “no mixing” problem,⁹ which is not observed in the other systems we study, at least in the vicinity of the fixed point.

Let us finally mention that we also experimented with a mixed correlator $\phi - S$ bootstrap

$$\begin{aligned}
 \langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle &= \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle, \\
 \langle SSSS \rangle &= \langle SSSS \rangle, \\
 \langle \phi_i^a S \phi_j^b S \rangle &= \langle \phi_j^b S \phi_i^a S \rangle, \\
 \langle \phi_i^a \phi_j^b SS \rangle &= \langle S \phi_j^b \phi_i^a S \rangle,
 \end{aligned} \tag{4.6}$$

since this is the “cheapest” mixed correlator system one might consider. We found this system to also be less constraining than the ones we present in this work. As for the $\phi - \phi'$ system, the most probable reason is the fact that the singlet is expected to have $\Delta_S \simeq 1.2$ and is thus a relatively heavy operator.

The sum rules we derive have the schematic form

$$\mathcal{V}_{\mathbb{1}}^{\Delta_{\text{ext}}} + \vec{\theta} \cdot \mathcal{V}_{\theta}^{\Delta_{\text{ext}}} \cdot \vec{\theta} + \sum_{O=(R,\Delta,\ell)} \vec{\lambda}_O \cdot \mathcal{V}_O^{\Delta_{\text{ext}}} \cdot \vec{\lambda}_O = 0 \tag{4.7}$$

where $\mathcal{V}$ are vectors made out of convolved conformal blocks (see appendix A for more details). $\mathcal{V}_O^{\Delta_{\text{ext}}}$ are generically vectors of matrices in mixed correlator systems. In the sum, R labels

⁸Whereas the first spin-0 Z_3 operator obtains a rather large anomalous dimension at the coupled fixed point. See table 2.

⁹Described in the following lectures <https://perimeterinstitute.ca/events/mini-course-numerical-conformal-bootstrap>. Essentially, the bootstrap functional reduces to that of the single correlator system by putting all components involving the mixed parts to what is practically zero. See the slides at http://scgp.stonybrook.edu/video_portal/video.php?id=4327, slides 11 and 12 in particular.

the irrep under $S_n \times (S_Q)^n$. The vector $\vec{\lambda}_O$ is a collection OPE coefficients between a pair of externals and the exchanged operator O . The identity is always exchanged in our sum rules and we separate it explicitly $\mathcal{V}_{\mathbb{1}}^{\Delta_{\text{ext}}} = \vec{\lambda}_{\mathbb{1}} \cdot \mathcal{V}_{\mathbb{1}}^{\Delta_{\text{ext}}} \cdot \vec{\lambda}_{\mathbb{1}}$. Moreover, since in the present work the external operators also appear as exchanged (see table 3), we explicitly separate them in the term $\mathcal{V}_{\theta}^{\Delta_{\text{ext}}} = \mathcal{V}_{O=\text{ext}}^{\Delta_{\text{ext}}}$ and $\vec{\theta} = \vec{\lambda}_{O=\text{ext}}$.

The numerical bootstrap approach rules out putative CFT spectra by looking for a linear functional α satisfying $\alpha(\mathcal{V}_{\mathbb{1}}) = 1$ and certain positivity conditions on $\alpha(\mathcal{V}_O)$ which depend on the gap assumptions on the spectrum. If the functional exists the assumed gaps are inconsistent with crossing symmetry. When looking for the optimal bound Δ^* on the dimension of the lightest operator in the R^* representation and of spin ℓ^* , we look for functionals satisfying schematically the following conditions

$$\begin{aligned}
 \alpha(\mathcal{V}_{\mathbb{1}}^{\Delta_{\text{ext}}}) &= 1, \\
 \alpha(\mathcal{V}_{\theta}^{\Delta_{\text{ext}}}) &\geq 0, \\
 \alpha(\mathcal{V}_{R_{\text{ext}}, \Delta, \ell}^{\Delta_{\text{ext}}}) &\geq 0 \quad \forall \ell; \Delta \geq \Delta_{\text{gap}}, \\
 \alpha(\mathcal{V}_{R^*, \Delta^*, \ell^*}^{\Delta_{\text{ext}}}) &\geq 0 \quad \forall \Delta \geq \Delta^*, \\
 \alpha(\mathcal{V}_{R, \Delta, \ell}^{\Delta_{\text{ext}}}) &\geq 0 \quad \forall \ell \neq \ell^*; \forall R \neq R^*, R_{\text{ext}}; \forall \Delta \geq \Delta_{\text{gap}}.
 \end{aligned} \tag{4.8}$$

Here $\Delta_{\text{gap}} = \Delta_{\text{gap}}(R, \ell)$ is a set of fixed gap assumptions (see table 4 and subsequent comments). As already mentioned the externals are always exchanged so we explicitly look for a functional $\alpha(\mathcal{V}_{\theta}^{\Delta_{\text{ext}}}) \geq 0$ and impose a fixed gap for the next operator in that sector. The problem is parametric in Δ_{ext} and Δ^* , which are thus scanned over to produce the plots reported in this work.

In some cases, we look for the allowed region for the dimension of the lowest-lying operator $O_{R^*, \Delta^*, \ell^*}$ by putting it as isolated in the spectrum and imposing a fixed gap above it (in a similar way to what we do for external operators). In this case the positivity condition reads

$$\begin{aligned}
 \alpha(\mathcal{V}_{\mathbb{1}}^{\Delta_{\text{ext}}}) &= 1, \\
 \alpha(\mathcal{V}_{\theta}^{\Delta_{\text{ext}}}) &\geq 0, \\
 \alpha(\mathcal{V}_{R_{\text{ext}}, \Delta, \ell}^{\Delta_{\text{ext}}}) &\geq 0 \quad \forall \ell; \forall \Delta \geq \Delta_{\text{gap}}, \\
 \alpha(\mathcal{V}_{R^*, \Delta^*, \ell^*}^{\Delta_{\text{ext}}}) &\geq 0, \\
 \alpha(\mathcal{V}_{R^*, \Delta, \ell^*}^{\Delta_{\text{ext}}}) &\geq 0 \quad \forall \Delta \geq \Delta_{\text{gap}}, \\
 \alpha(\mathcal{V}_{R, \Delta, \ell}^{\Delta_{\text{ext}}}) &\geq 0 \quad \forall \ell \neq \ell^*; \forall R \neq R^*, R_{\text{ext}}; \forall \Delta \geq \Delta_{\text{gap}}.
 \end{aligned} \tag{4.9}$$

The search for the functional is done again parametrically in $(\Delta_{\text{ext}}, \Delta^*)$.

In favour of readability we have relegated the resulting sum rules from the above crossing equations, (4.1) to (4.4), and their derivation to a number of appendices, namely A.1 through A.9. Tables 3 and 4 collect all OPEs and gap assumptions relevant to the figures that will follow. If a gap assumption is mentioned in the caption of a figure, which concerns an irrep already covered in table 4, *then the assumption in the caption replaces the one in table 4*. Also, when the dimension of an operator is being bounded, we don't impose the corresponding gap for that operator found in the table. This choice was made to shorten the presentation and make the present manuscript more readable.

OPE	exchanged representations with spin parity
$\phi \times \phi$	$S_+ + X_+ + \phi_+ + A_- + Z_+ + B_-$
$X \times X$	$S_+ + X_+ + (\overline{X}X)_-$
$Z_3 \times Z_3$	$S_+ + (AA)_+ + (VAA)_+ + (AZ)_- + A_- + \phi_+ + (VA)_- + Z_+ + (Z_3)_+ + (AAA)_-$
$\phi \times X$	$\phi_\pm + (XV)_\pm$
$\phi \times Z_3$	$Z_\pm + (AZ)_\pm + (VB)_\pm + (Z_3)_\pm$

Table 3. OPEs relevant to the present work for $n = 3$, $Q = 3$. While we use $\phi \times \phi$ at $n = 100$ later on, its form remains as reported in this table (same exchanged irreps). A plus (resp. minus) subscript on a representation signifies it is exchanged with even (resp. odd) spins. OPEs with non-identical external irreps exchange both spins.

Operator	Gap Assumption	Operator	Gap Assumption
S_0	≥ 1	A_1	≥ 1.1
$T_{\mu\nu} = S_2$	$= 2$	$(VA)_1$	≥ 1.2
S'_2	≥ 2.5	$(AZ)_1$	≥ 1.2
$S_{\ell=1 \text{ or } \geq 3}$	$\geq \ell$	$(AA)_0$	≥ 2.2
X_0	≥ 0.5	$(VAA)_0$	≥ 2.3
$\phi'_0 = V'_0$	≥ 0.7	$(AAA)_0$	≥ 3.3
Z_0	≥ 0.2	$(Z'_3)_0$	≥ 1

Table 4. Gap assumptions imposed throughout most of the present work. Operators not mentioned below satisfy the unitarity bound ($d - 2 + \ell$ for spin ℓ primaries) plus a small twist gap $\tau = 10^{-6}$, i.e. $\Delta_O \geq \ell + \tau$. For singlets, we impose just unitarity since quasi-primaries that are Virasoro descendants of the identity have zero twist. Primes on operators denote subleading operators. Subscripts denote the spin. Motivation for these gaps is given in parts of the main text. *When an assumption for an operator is given both in this table and the caption of a figure, then the assumption in the figure replaces the one of the table.*

All figures that will follow concern $n = 3$, $Q = 3$, which the exception of a sub-figure in figure 10 computed at $n = 100$ and $Q = 3$. In all plots, shaded areas present the admissible parameter space, surrounded by the excluded region. The blue dot denotes the coupled replica fixed point (transfer matrix values in table 2), whereas the magenta dot the fixed point of decoupled replicas.

4.1 ϕ single correlator bootstrap

In figures 1 through 11 we show bounds obtained on some of the first few operators in the theory. This is done using the crossing equation in eq. (4.1). We use the notation $O_{R,\ell}$ where R is the global symmetry irrep and ℓ the $SO(2)$ spin-label. The operators bounded are $O_{S,0}$, $O_{Z,0}$, $O_{A,1}$, $O_{B,1}$, $O_{A,3}$, $O_{S',2}$, $O_{X,0}$, $O_{X,2}$ and $O_{V',0}$, whose scaling dimensions are constrained as a function of $\Delta_\phi = \Delta_{V,0}$.

An important comment is that in all plots we have enforced several assumptions on operators, beyond the ones being plotted. Typically, one would like to start by obtaining bounds with no assumptions whatsoever (except for the operator dimension being bounded), and then progressively impose more assumptions in order to single out a theory of interest. We have found this not possible in our case because, in the absence of any assumptions, the bounds are trivial, i.e. there is no boundary between allowed and disallowed points, only allowed points. Hence the situation in $2d$ is different from the one typically encountered in $3d$. This phenomenon has already been observed in $2d$ in earlier work, for instance in the lower bound on the central charge [38].

The assumptions we use are outlined in table 4 and in the captions of the figures. See the caption of each figure for the precise assumptions in each case. The gaps on $\Delta_{S,0}$, $\Delta_{X,0}$ and $\Delta_{Z,0}$ found in table 4 are motivated by transfer matrix results reported in table 2. The rest of the assumptions are motivated by perturbative intuition, i.e. perturbation theory around decoupled theories or the large- n limit, and are somewhat conservative. For example, the first two sub-leading spin-0 V operators are built with $\sigma\phi_i^a$ and $\phi_i'^a$, which both have a dimension of $4/3$ at zeroth order in the large- n limit. The first is just the field ϕ_i^a multiplied by the auxiliary “Hubbard-Stratonovich” field, whereas the second is the subleading vector representation quasi-primary of the (single/decoupled) critical 3-state Potts model (see appendix C). We generically expect the scaling eigenoperators to be some combination of these two building blocks, which will obtain an anomalous dimension.

A few comments are in hand. First, we observe that the bound in figure 1 on the dimension of $O_{S,0}$ is rather unconstraining. We attribute this, at least in part, to the fact that this operator is relatively heavy. Secondly, and perhaps unsurprisingly, we observe that the most constraining bound is the one on the $O_{Z,0}$ operator as a function of $O_{V,0}$, figure 2 and figure 3. This is because these two operators are the two lightest in the theory, and the bootstrap is known to be the most constraining as close as possible to the unitarity bound ($(d-2)/2 = 0$ for scalars in $d = 2$). In addition to the interacting theory, the bound can also be seen to include the decoupled theory, which we remind the reader has the same symmetry. Thus, unfortunately, while we seem to obtain a peninsula including our theory of interest we are unable to segregate it from other theories in the same parameter space, e.g. the decoupled theory. This is despite the gap imposed on the first singlet $\Delta_{S,0} \geq 1$ which would in principle exclude the decoupled theory, having the lightest singlet operator at $\Delta_{S,0} = 0.8$. We will improve on this later on in the present text by mixing in additional external operators.

The rest of the bounds do not seem to present any striking features close to the expected fixed point. A sample of them includes the $O_{V,0}$ – $O_{A,1}$ bound figure 4 which includes the decoupled theory (and presumably the coupled one) in the bulk of the allowed region. The $O_{V,0}$ – $O_{B,1}$ bound figure 5. The $O_{V,0}$ – $O_{A,3}$ bound figure 6, which is of interest since at the decoupled fixed point the $O_{A,3}$ operator becomes the spin-3 conserved current of the critical 3-state Potts model. The $O_{V,0}$ – $O_{S',2}$ bound figure 7, which bounds the sub-leading operator after the stress tensor. While this plot has a significant feature around the value of $O_{V,0}$ corresponding to our CFT of interest, we do not attribute it to our theory of interest given the values of $\Delta_{S',2}$ which it is located at. Consequently, we observe that while the $O_{V,0}$ – $O_{X,0}$ bound figure 8 has a feature, it does not seem to be close to our theory of interest. Indeed, we will see it disappears later when using $\phi - X$ mixed correlator system eq. (4.2) (figures 14 and 15).

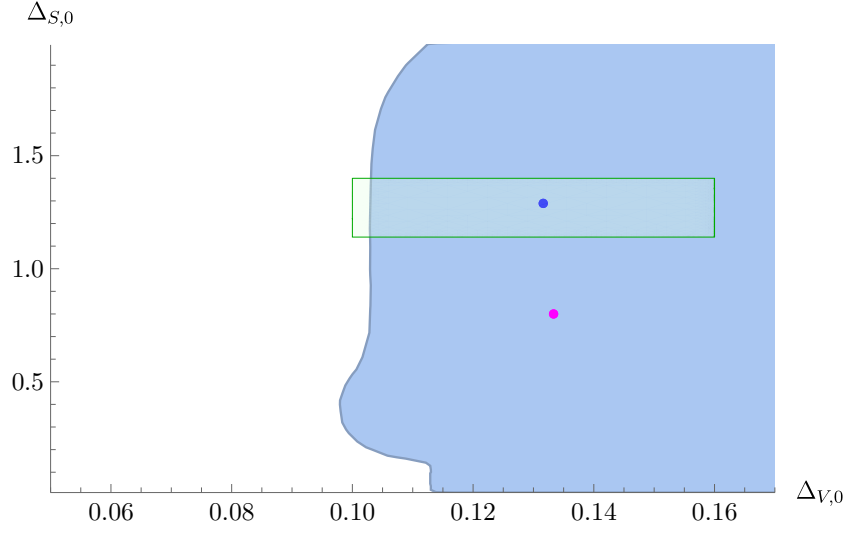


Figure 1. Single correlator (ϕ external) bound on the scaling dimension of the $O_{S,0}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{S',0} \geq 2$. For more details see the discussion around and above table 4.

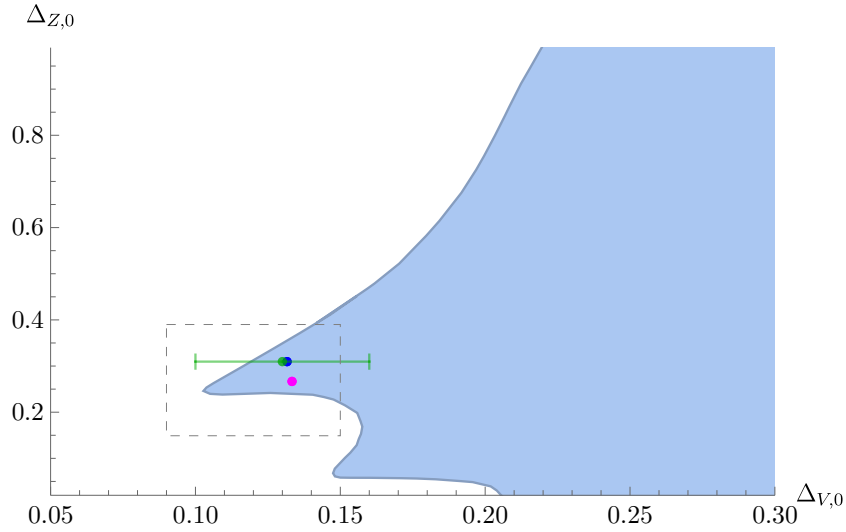


Figure 2. Single correlator (ϕ external) allowed region for the scaling dimension of the $O_{Z,0}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{Z',0} \geq 1$. For more details see the discussion around and above table 4.

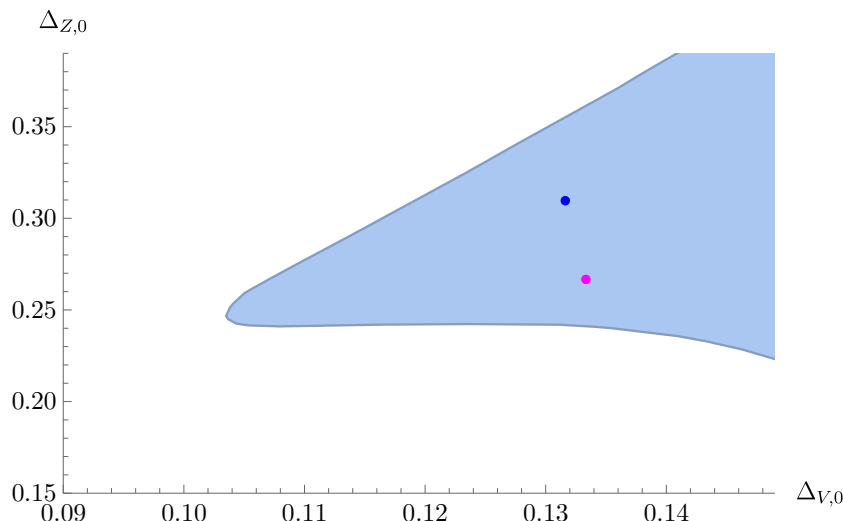


Figure 3. Zoomed in version of the peninsula in figure 2.

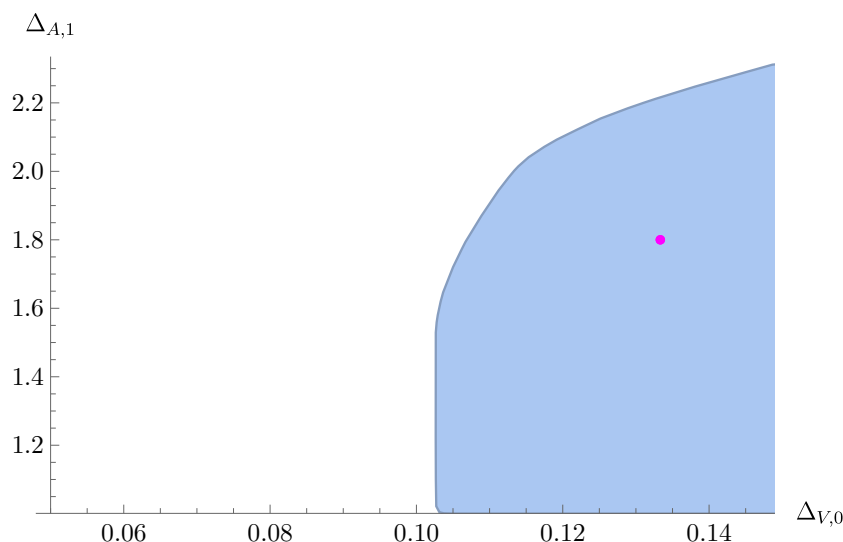


Figure 4. Single correlator (ϕ external) bound on the scaling dimension of the leading $O_{A,1}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

We also show the $O_{V,0}$ – $O_{X,2}$ bound figure 9, where $O_{X,2}$ becomes conserved at the decoupled fixed point,¹⁰ but is expected to obtain an anomalous dimension at the interacting one.

In figure 10 we overlay the bounds on $\Delta_{Z,0}$ obtained at $n = 3$ and at $n = 100$ (all previous and subsequent plots concern $n = 3$). As $n \rightarrow \infty$, $\Delta_{Z,0}$ should approach $2\Delta_\phi$. Indeed, we observe that the peninsula shrinks to a very thin “dagger”, which we relate to the fact that the coupled and decoupled theories sit on top of each other in this limit, and in this particular slice of parameter space. The slope of the dagger is $\Delta_{Z,0} = 2\Delta_\phi$, as expected. This

¹⁰Since it is a linear combination of stress tensors in the individual Potts replicas.

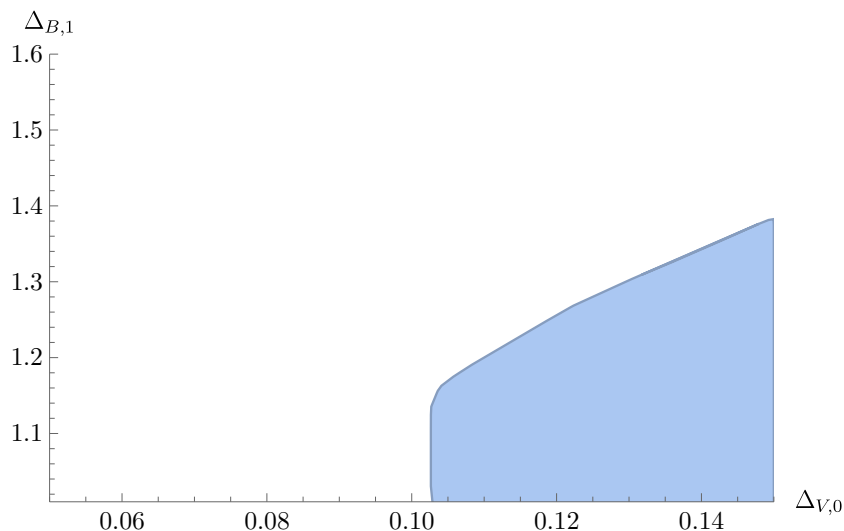


Figure 5. Single correlator (ϕ external) bound on the scaling dimension of the leading $O_{B,1}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

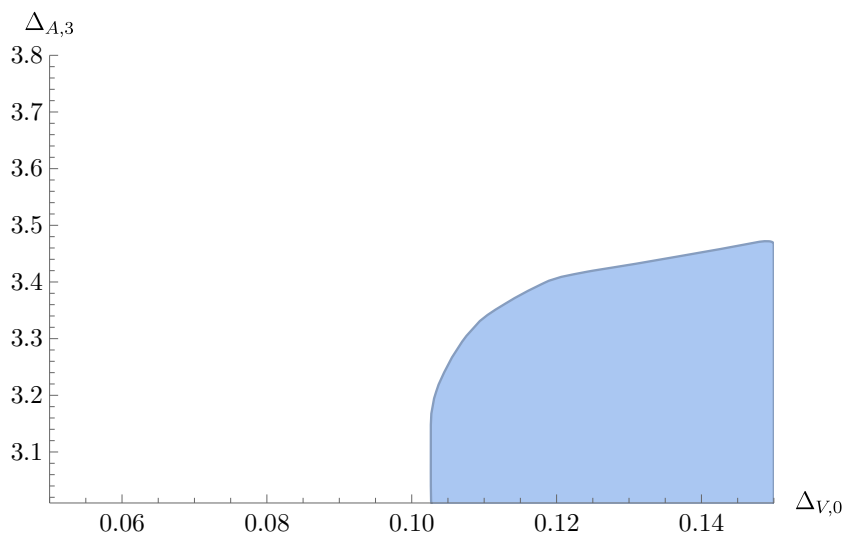


Figure 6. Single correlator (ϕ external) bound on the scaling dimension of the leading $O_{A,3}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. We remind that the spin-3 operator $O_{A,3}$ is conserved at the decoupled fixed point. For more details see the discussion around and above table 4.

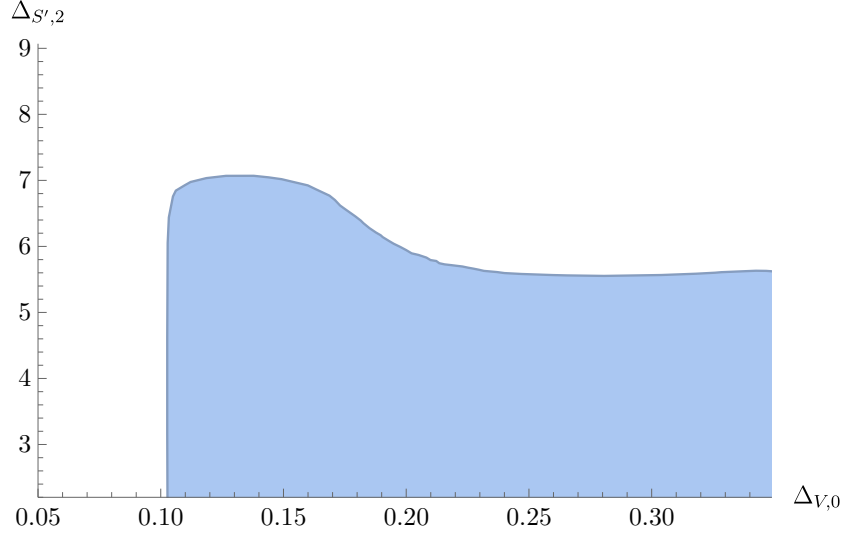


Figure 7. Single correlator (ϕ external) bound on the scaling dimension of the $O_{S',2}$ operator, i.e. the subleading spin-2 operator after the stress-tensor, as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

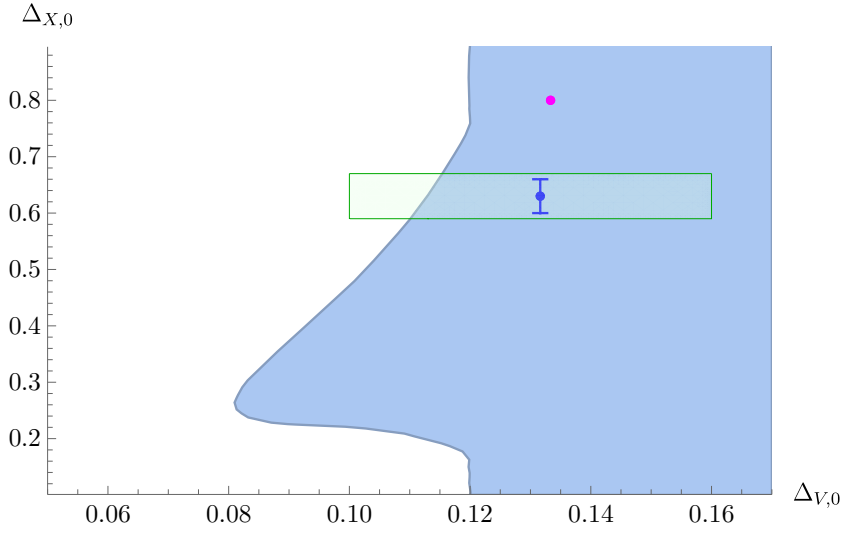


Figure 8. Single correlator (ϕ external) allowed region for the scaling dimension of the $O_{X,0}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{X',0} \geq 0.75$. For more details see the discussion around and above table 4.

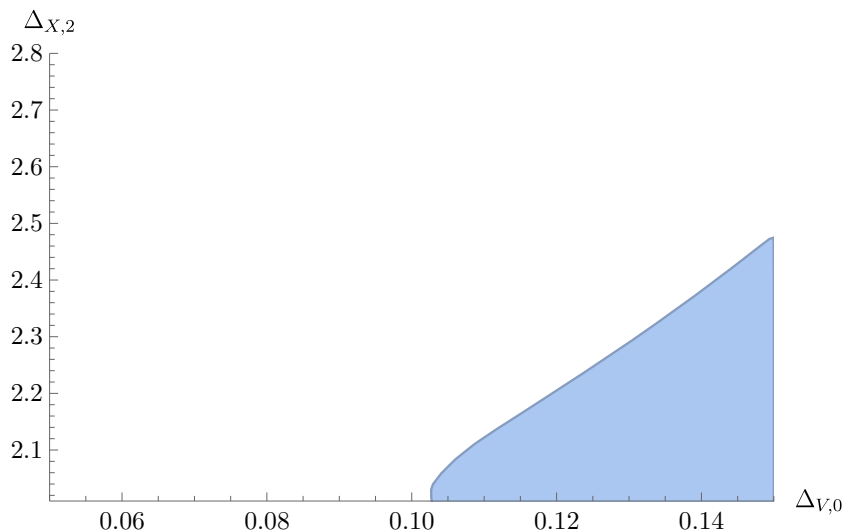


Figure 9. Single correlator (ϕ external) bound on the scaling dimension of the $O_{X,2}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. In the decoupled theory $O_{X,2}$ is a linear combination of the stress-tensors of the Potts models and is thus conserved. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

reinforces the large- n intuition for the theory. However, as also discussed earlier, we have no knowledge of the $1/n$ corrections, and as indicated by transfer matrices and our plots later on, certain operators do acquire rather large anomalous dimensions. For this reason, we opt to use assumptions that are somewhat conservative, such that our plots present somewhat “universal” bounds that are not too heavily gap-dependent. Lastly, in figure 11 we present the $O_{V,0}$ – $O_{V',0}$ exclusion bound. This is saturated by the decoupled theory, similarly to the corresponding plot in [15] studying CFTs with S_3 symmetry.

A note on error bars in plots. In some of the plots, where available, we have indicatively included error bars on the predictions obtained in the literature, as summarised in table 2. Green error bars, and error boxes in cases where estimates exist for both axes, refer to Monte Carlo determinations. Whereas, blue error bars refer to transfer matrices. We have opted to display the error bars only on some of the plots, in order to avoid visual clutter. Another reason for which we opted to selectively show error bars, is that the majority of them are from Monte Carlo, which in [11] was used mainly as a means to confirm criticality, and not necessarily for precision calculations of CFT data. Thus, we suspect that the Transfer Matrix results, which present the bulk of results from [11], are more precise, even though error bars for these results are not reported.

4.2 Z_3 single correlator bootstrap

We briefly showcase in figure 12 the bound on $\Delta_{V,0}$ obtained using the Z_3 single correlator system in eq. (4.3). This will serve to demonstrate the improvement achieved when mixing both ϕ and Z_3 later on using the mixed correlator system in eq. (4.4). Similarly we can also

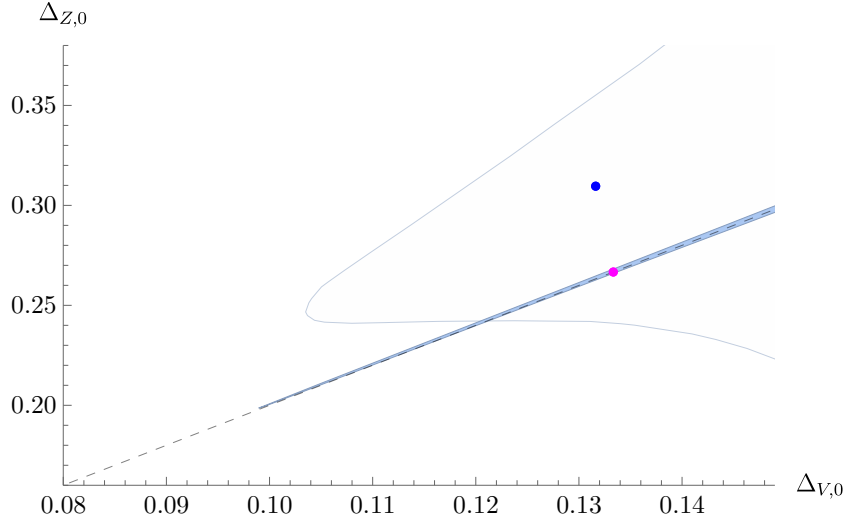


Figure 10. Single correlator (ϕ external) allowed region for the scaling dimension of the leading $O_{Z,0}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$, both for $n = 3$ and $n = 100$, the second of which presenting the narrow daggerlike bound. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{Z',0} \geq 1$. For more details see the discussion around and above table 4.

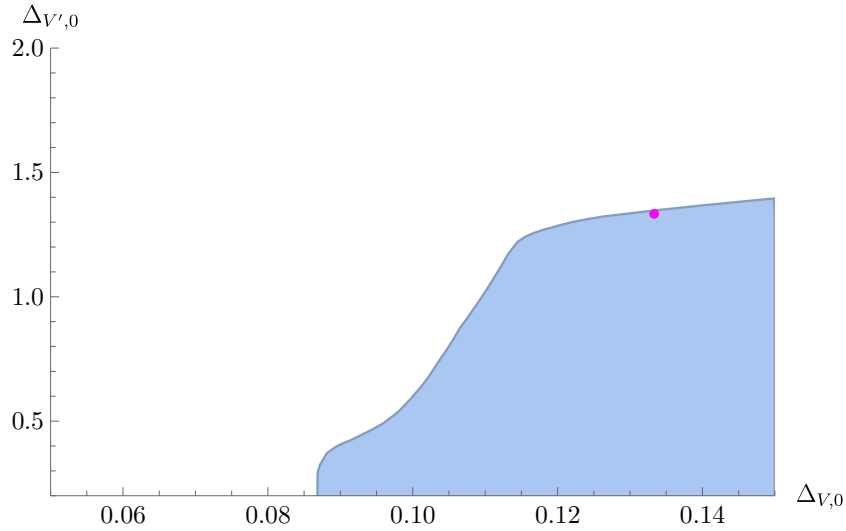


Figure 11. Single correlator (ϕ external) bound on the scaling dimension of the $O_{V',0}$ operator, i.e. the subleading operator after $O_{V,0}$ in the V spin-0 irrep, as a function of the scaling dimension of $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

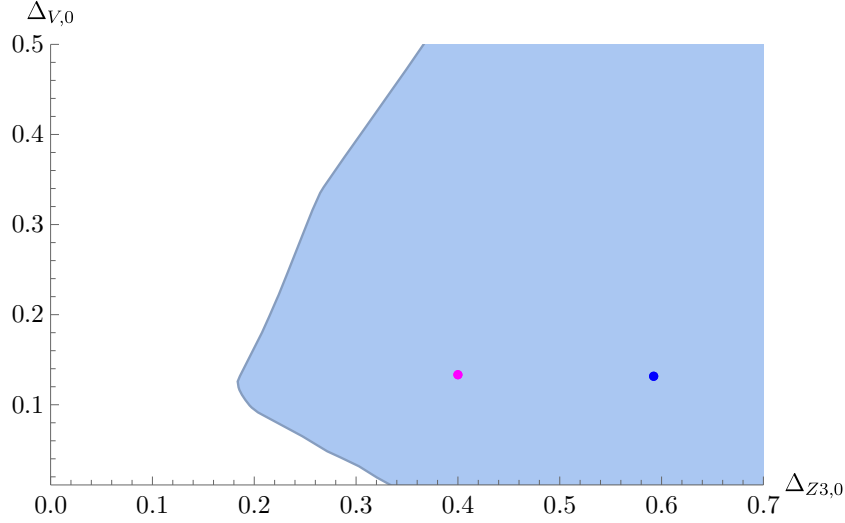


Figure 12. Single correlator (Z_3 external) allowed region for the scaling dimension of $O_{V,0}$ operator as a function of the scaling dimension of $Z_3 = O_{Z_3,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

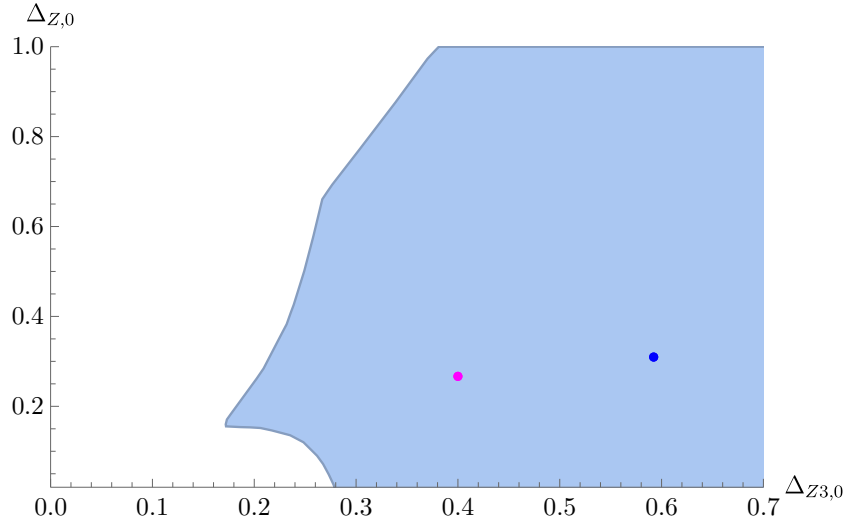


Figure 13. Single correlator (Z_3 external) allowed region for the scaling dimension of $O_{Z,0}$ operator as a function of the scaling dimension of $Z_3 = O_{Z_3,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4.

bound $\Delta_{Z,0}$ as a function of $\Delta_{Z_3,0}$ using the same correlator system, which we showcase in figure 13.

In addition to assumptions outlined in subsection 4.1, we have imposed some new ones due to the new irreps that can appear in the OPE involving Z_3 . These are reported again in table 4. An example is $\Delta_{AA,0} \geq 3.3$. To motivate these assumptions it helps to think of the “composite” irreps discussed in subsection 2.3. At the decoupled fixed point (or large- n),

the operator $O_{AAA,0}$ must be built by taking an A operator from each of the three replicas and symmetrizing them (possibly with insertions of derivatives). The A operators of each replica can be of any spin as long as the combination reduces to spin-0. Thus the lowest lying $O_{AAA,0}$ operator at the decoupled (or large- n) fixed point cannot be lighter than the combination of the three lightest A operators, i.e. 3×1.8 ,¹¹ which is much bigger than the gap $\Delta_{AAA,0} \geq 3.3$ we impose.

4.3 $\phi - X$ mixed correlator bootstrap

Firstly, in figure 14 we present the mixed correlator $O_{V,0}-O_{X,0}$ exclusion plot. This is obtained by mixing in $O_{X,0}$ as an external, which corresponds to crossing eq. (4.2). We superimpose the corresponding single correlator plot, figure 8, to illustrate that the apparent feature has gone away. In figure 15, we present the same plot but at two different values of Λ , $\Lambda = 11$ and 19. The apparent feature is again observed to disappear. We now proceed by taking a closer look at the $O_{V,0}-O_{Z,0}$ allowed region in figure 16, which is the projection of the allowed region in $\Delta_{V,0}-\Delta_{X,0}-\Delta_{Z,0}$ space onto the $\Delta_{V,0}-\Delta_{Z,0}$ plane. The projection is done for the slice $0.55 < \Delta_{X,0} < 0.75$ which corresponds to the area around the coupled replica CFT, see table 2. The two bounds in figure 16 differ in the gap assumption on $O_{X',0}$, which is the gap to which we found the plots to be the most sensitive. In particular, we find that the blue dot, corresponding to the interacting fixed point, seems to be excluded both for $\Delta_{X',0} \geq 0.75$ (slightly) and for $\Delta_{X',0} \geq 1$. We recall that at the decoupled fixed $\Delta_{X',0} = 1.6$. It would thus appear that $\Delta_{X',0}$ acquires a large negative anomalous dimension. The operator does acquire a negative anomalous dimension in perturbation theory, see table 2. However, since the results are known only to two orders in ε it is hard to draw a definitive conclusion. Notice also that the coefficient of the ε^2 term has a large negative value, which may also point in this direction.

4.4 $\phi - Z_3$ mixed correlator bootstrap

Finally, we proceed to the constraints obtained using the crossing system in eq. (4.4). As we will see eq. (4.4) carries the advantage of being able to differentiate between the coupled and decoupled fixed points. In other words, we can obtain an allowed region that contains the coupled fixed point while the decoupled one being disallowed. This is the main advantage of this system over eq. (4.2). The improvement is perhaps in part due to the operator $O_{Z_3,0}$ gaining a relatively large anomalous dimension. Notice that this anomalous dimension is especially large compared to the one $O_{Z,0}$ obtains, see table 2.¹²

In figure 17 we show the $\Delta_{V,0}-\Delta_{Z_3,0}$ allowed region. On the other hand, in figures 18 and 19, we present the projections of the $\Delta_{V,0}-\Delta_{Z,0}-\Delta_{Z_3,0}$ allowed region onto two different planes. In figure 17 ($\Delta_{V,0}-\Delta_{Z_3,0}$ plane) and in figure 18 ($\Delta_{Z_3,0}-\Delta_{Z,0}$ plane) we explicitly observe that the decoupled theory (magenta dot) is excluded, while the coupled theory (blue dot) is included in the allowed region. Thus, while in the projection of figure 19 ($\Delta_{V,0}-\Delta_{Z,0}$ plane) it would naively appear that the decoupled point is allowed, we find out that it is not. We have thus found a sub-region of parameter space where the interacting theory is present and the decoupled is not.

¹¹For the quasi-primaries see appendix C.

¹²For the reader's convenience we remind that these operators have dimensions $2\Delta_\phi$ and $3\Delta_\phi$ at the decoupled fixed point.

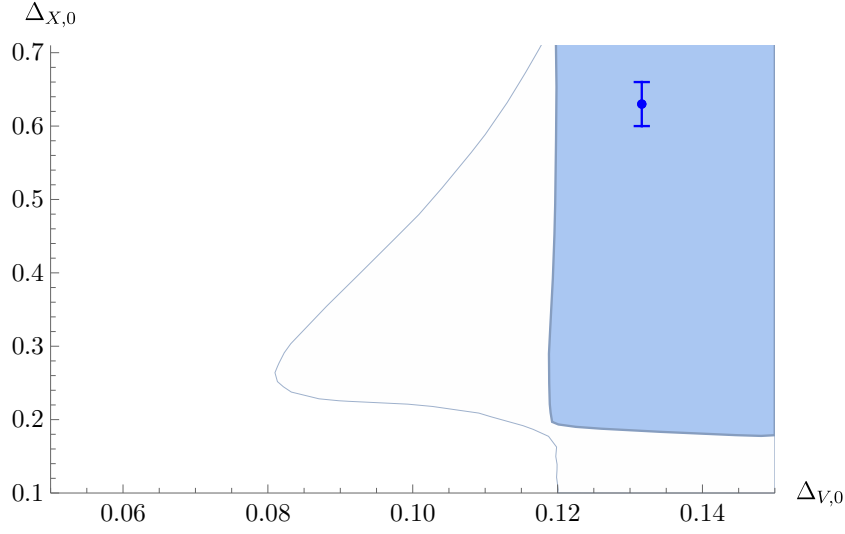


Figure 14. Single versus mixed correlator ($\phi - X$ externals) allowed region for scaling dimensions of $X = O_{X,0}$ and $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{X',0} \geq 0.75$. For more details see the discussion around and above table 4. We also fix the ratio of OPE coefficients $\lambda_{\phi\phi T_{\mu\nu}}/\lambda_{XX T_{\mu\nu}} = \Delta_\phi/\Delta_X$.

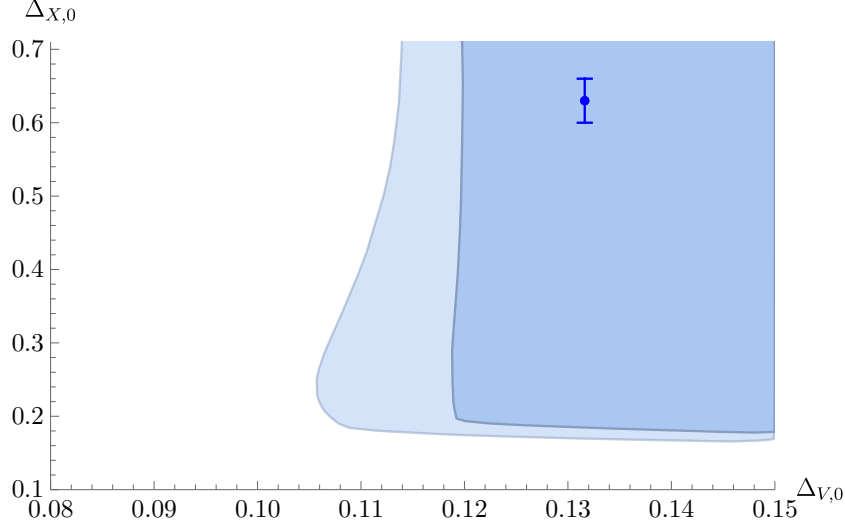


Figure 15. Mixed correlator ($\phi - X$ externals) allowed region for scaling dimensions of $X = O_{X,0}$ and $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B, in particular, we obtain the plot at two different values of Λ ($\Lambda = 11$ and 19) to illustrate convergence. Shaded areas present the admissible parameter space, surrounded by the excluded region. The assumptions imposed are those of table 4 in addition to $\Delta_{X',0} \geq 0.75$. For more details see the discussion around and above table 4. We also fix the ratio of OPE coefficients $\lambda_{\phi\phi T_{\mu\nu}}/\lambda_{XX T_{\mu\nu}} = \Delta_\phi/\Delta_X$.

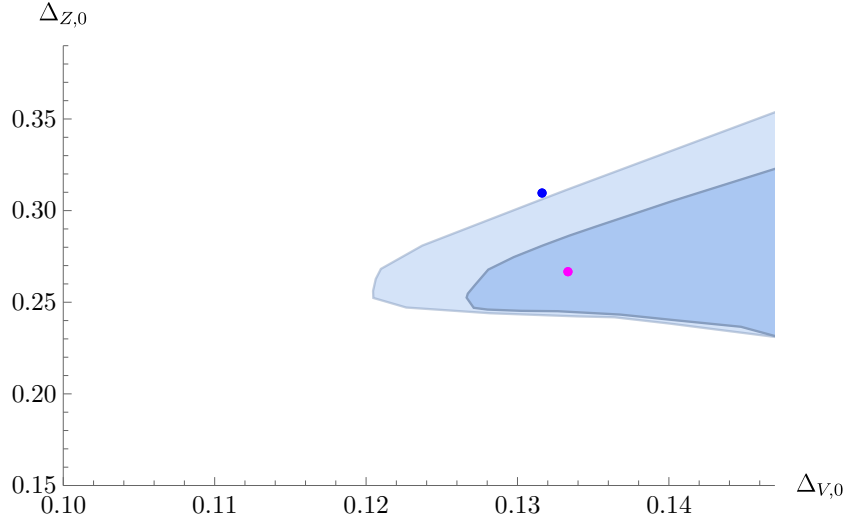


Figure 16. Mixed correlator ($\phi - X$ externals) allowed region for the scaling dimension of the leading $O_{Z,0}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. In order to obtain the plot we projected the $\Delta_{V,0}-\Delta_{X,0}-\Delta_{Z,0}$ allowed space onto the $\Delta_{V,0}-\Delta_{Z,0}$ subspace. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{Z',0} \geq 1$ and $\Delta_{X',0} \geq 0.75$ or $\Delta_{X',0} \geq 1$. For more details see the discussion around and above table 4. We also fix the ratio of OPE coefficients $\lambda_{\phi\phi T_{\mu\nu}}/\lambda_{XX T_{\mu\nu}} = \Delta_{\phi}/\Delta_X$.

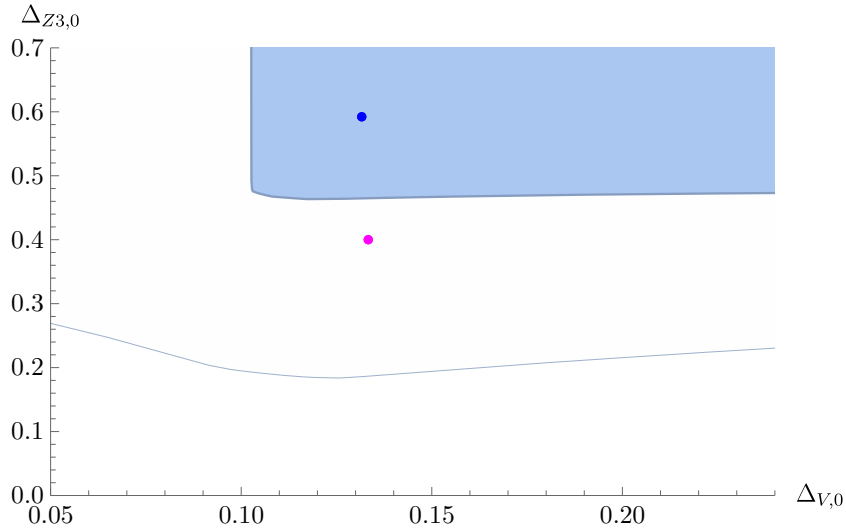


Figure 17. Mixed correlator ($\phi - Z_3$ externals) allowed region for scaling dimensions of $Z_3 = O_{Z_3,0}$ and $\phi = O_{V,0}$. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4. For more details see the discussion around and above table 4. We also fix the ratio of OPE coefficients $\lambda_{\phi\phi T_{\mu\nu}}/\lambda_{Z_3 Z_3 T_{\mu\nu}} = \Delta_{\phi}/\Delta_{Z_3}$.

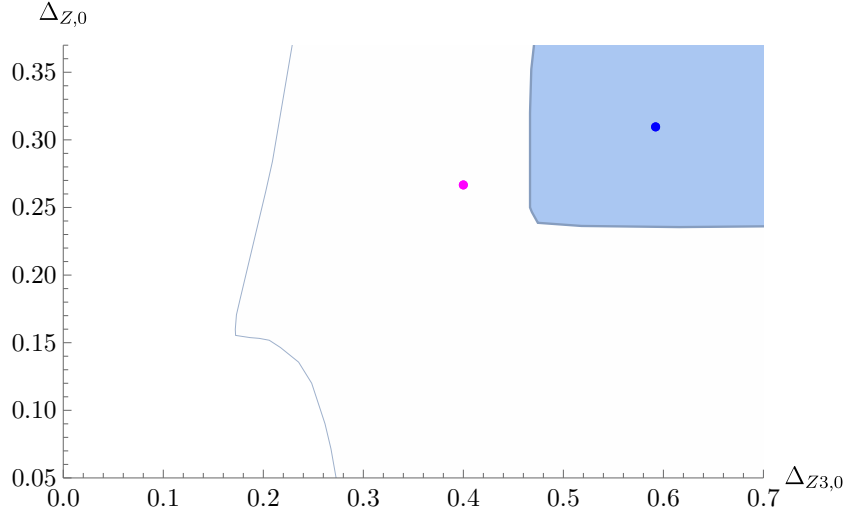


Figure 18. Mixed correlator ($\phi - Z_3$ externals) allowed region for $O_{Z,0}$ operator as a function of the scaling dimension of $Z_3 = O_{Z_3,0}$. To obtain the plot we projected the $\Delta_{V,0}-\Delta_{Z,0}-\Delta_{Z_3,0}$ allowed space onto the $\Delta_{Z_3,0}-\Delta_{Z,0}$ subspace. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{Z',0} \geq 1$. For more details see the discussion around and above table 4. We also fix the ratio of OPE coefficients $\lambda_{\phi\phi T_{\mu\nu}}/\lambda_{Z_3 Z_3 T_{\mu\nu}} = \Delta_\phi/\Delta_{Z_3}$.

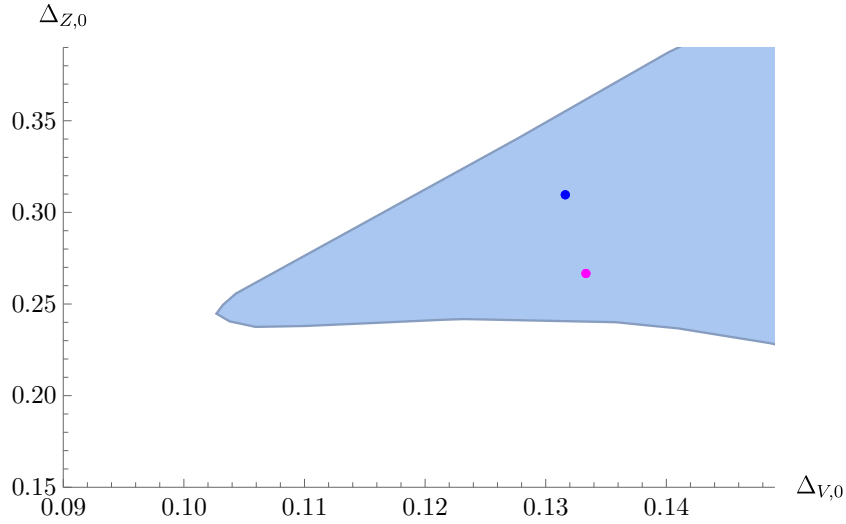


Figure 19. Mixed correlator ($\phi - Z_3$ externals) allowed region for $O_{Z,0}$ operator as a function of the scaling dimension of $\phi = O_{V,0}$. In order to obtain the plot we projected the $\Delta_{V,0}-\Delta_{Z,0}-\Delta_{Z_3,0}$ allowed space onto the $\Delta_{V,0}-\Delta_{Z,0}$ subspace. The numerical parameters used to obtain this plot are given in appendix B. The assumptions imposed are those of table 4 in addition to $\Delta_{Z',0} \geq 1$. For more details see the discussion around and above table 4. We also fix the ratio of OPE coefficients $\lambda_{\phi\phi T_{\mu\nu}}/\lambda_{Z_3 Z_3 T_{\mu\nu}} = \Delta_\phi/\Delta_{Z_3}$.

While the $\Delta_{V,0}-\Delta_{Z,0}$ plot obtained via the projection of $\Delta_{V,0}-\Delta_{Z,0}-\Delta_{Z_3,0}$ does not show any particular improvement versus the corresponding plot obtained from the single correlator system eq. (4.1), the $\Delta_{Z_3,0}-\Delta_{Z,0}$ projection compared to the plot from the single correlator system eq. (4.3) does. Additionally, the $\Delta_{V,0}-\Delta_{Z_3,0}$ exclusion bound can either be obtained directly by placing a gap $\Delta_{Z,0} \geq 0.2$, or by adding $O_{Z,0}$ as an isolated exchanged operator and imposing $\Delta_{Z',0} \geq 1$ (and then projecting). We have found no noticeable difference between the two methods. We emphasise that figure 17 is the first of the two options, i.e. we just impose $\Delta_{Z,0} \geq 0.2$.

A note on the navigator [39] and skydiving [40] methods. While we did experiment with using the navigator and skydiving methods, specifically in an analogue of figure 19, where we also scanned over the OPE vectors $(\lambda_{VZ_3Z_3}, \lambda_{VVV}, \lambda_{Z_3Z_3Z_3})$ and $(\lambda_{VVZ}, \lambda_{VZ_3Z}, \lambda_{Z_3Z_3Z})$, we did not find any significant improvement both in terms of numerical efficiency as well as constraining power. In fact, the computations became considerably more expensive. Thus, we omit any discussion of these methods as applied to our problem. Let us stress, however, that we do not exclude the possibility that these methods could become more practical at higher values of Λ (which controls the amount of components in the bootstrap functional).

5 Discussion and outlook

In the present work, we initiated a study of the space of admissible two-dimensional CFTs with $S_n \times (S_Q)^n$ global symmetry, with a particularly strong emphasis on the $n = 3, Q = 3$ case. Our main results concern bounds on the space of allowed dimensions for the four lightest (in the UV)¹³ operators in the theory. These results were derived by studying two mixed correlator systems, namely the $\phi - X$ and $\phi - Z_3$ systems. The first of which contains the 1st and 4th lightest operators in the theory and the second the 1st and 3rd lightest. From the $\phi - X$ systems, we seem to have (re)discovered that the second operator in the spin-0 X representation, X' , has a rather large negative anomalous dimension. This appears to be hinted at from perturbative results, see table 2. However, we are not currently aware of any non-perturbative predictions for comparison. Next, using the $\phi - Z_3$ system we were able to constrain the theory of interest within an allowed peninsula in parameter space, see figure 19. Most importantly, we were able to exclude the decoupled $S_3 \times (S_3)^3$ fixed point (figures 17 and 18), which corresponds to three decoupled replicas of the critical 3-state Potts model. This peninsula, however, did not show any signs of becoming an island when increasing the number of derivatives or using stronger assumptions.¹⁴

In practice, we found that our study was impeded by the lack of detailed knowledge on the spectrum of anomalous dimensions in the theory. With the exception of some earlier results (table 2) from perturbation theory and non-perturbative techniques, such as transfer matrices and Monte Carlo, not much is known about the precise spectrum. To a large extent, this is understandable, since for example operators in representations such as AAA in table 1

¹³These operators may as well prove to be the four lightest also in the IR, however without precise knowledge of the anomalous dimensions it is not certain.

¹⁴While not included in this manuscript, we did try imposing some large gap assumptions inspired by the dimensions of quasi-primaries at the decoupled fixed point. See more generally the discussion in appendix C.

are probably of no practical use outside the conformal bootstrap. However, in the numerical conformal bootstrap, precise knowledge of such operators can inform detailed gap assumptions on the spectrum which can lead to much more constraining results. We saw that in cases where we did have knowledge of the spectrum, specifically for the leading spin-0 ϕ , X , Z and Z_3 operators, we were able to make more progress. It goes without saying that further perturbative and non-perturbative results in the spirit of [11] would be invaluable.

Another promising direction will be to study these theories through a quantum critical spin-chain in the spirit of [41]. In this approach, one may realise the Virasoro generators in terms of lattice degrees of freedom allowing one to single out both Virasoro primary and quasi-primary operators.

Given that we are dealing with two-dimensional conformal field theories, another obvious direction is take advantage of the full conformal symmetry. That is, proceeding with a generalisation of the usual numerical conformal bootstrap technique using Virasoro conformal blocks, from which we expect more constraining results. This is because, under Virasoro, the spectrum of primaries in the theories is expected to be much more sparse. Moreover, since Virasoro blocks depend explicitly on the central charge c , this is a parameter that can then be scanned over. It would thus be interesting to be able to compare with the estimate of $c \simeq 2.36$ obtained in [11] for 3 coupled 3-state Potts models. We hope that Virasoro blocks, together with more constraining assumptions inspired by results from independent methods, will provide the edge needed to obtain an island in parameter space. We aim to report on these issues in the near future.

Two more directions to consider would be to study the theories in question through a “truncation method”, for example along the lines of [42], and also to study more complicated mixed correlators (such as mixing in as an external the leading spin-0 Z operator). In the former case, one truncates the sum rule assuming a fixed number of exchanged operators and then tries to best determine their dimensions so that the sum rule is satisfied as best as possible, an idea going back to [43]. This would provide data for anomalous dimensions needed when imposing gaps in the “vanilla” conformal bootstrap, either Global or Virasoro. On the point of more mixed correlators, e.g. including Z , these could prove useful since they can both provide more constraining power and also because different mixed correlators can exchange different representations from table 1. This is something to consider because some representations can have (much) larger gaps than others, which would aid in obtaining an island in parameter space. Of course, the benefits of mixing in additional operators should be weighed against the increased numerical cost.

Lastly, let us note that our analysis has been carried out in a way that will allow us to recycle a lot of technology in the future when studying other theories. One such example are theories with $S_3 \ltimes (O(2))^3$ global symmetry, which are interesting due to applications in critical phenomena [44].

Acknowledgments

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A Calculation of tensor structures and sum rules

Let us note that in all the following we will use a particular choice of notation to simplify the presentation as much as possible. For example a sum rule of the form

$$0 = \sum_O \left(\lambda_{ijO} \lambda_{klO} F_{\mp, \Delta, l}^{ij, kl} \pm \lambda_{kjO} \lambda_{ilO} F_{\mp, \Delta, l}^{kj, il} \right), \quad (\text{A.1})$$

found in [45], which in the case of a single correlator Ising bootstrap would reduce to

$$0 = \sum_O \left(\lambda_{\phi\phi O}^2 F_{-, \Delta, l}^{\phi\phi, \phi\phi} \right), \quad (\text{A.2})$$

will be in this work shortened to

$$0 = F_S^-. \quad (\text{A.3})$$

This will render larger expressions significantly more readable. In cases where there might be ambiguities in deducing what OPE coefficients should be multiplying the convolved conformal blocks F , we will state it explicitly. This can happen in mixed correlator settings, such as the crossing equation $\langle OOO'O' \rangle = \langle O'O O O' \rangle$, where if both OPEs $O \times O$ and $O \times O'$ exchange an operator in common, then there would be an ambiguity to assigning OPE coefficients in our notation. We hope that this notation will not be confusing to the reader.

A.1 Tensor structures for $\langle \phi\phi\phi\phi \rangle$

The tensor structures for a correlator of four fields in the defining representation

$$\langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle \quad (\text{A.4})$$

can be easily worked out applying the rules of [26]. In order to obtain the structures for $S_n \times (S_Q)^n$, we start with those of S_Q . The OPE of two fundamentals ϕ_i, ϕ_j of S_Q decomposes as

$$\phi_i \times \phi_j \sim \left(\delta_{ij} - \frac{1}{Q} \omega_i \omega_j \right) S + C_{ijk} \phi_k + T_{ij} + A_{ij} \quad (\text{A.5})$$

¹⁵<https://perimeterinstitute.ca/events/mini-course-numerical-conformal-bootstrap>.

for $Q \geq 4$, and

$$\phi_i \times \phi_j \sim \left(\delta_{ij} - \frac{1}{Q} \omega_i \omega_j \right) S + c_{ijk} \phi_k + A_{ij} \quad (\text{A.6})$$

for $Q = 3$. For $Q = 2$ the OPE reduces to the one of an Ising-like CFT. Here ω_i is a column vector of units, $\sum_{i=1}^Q \omega_i = Q$. The irreps on the r.h.s. include the singlet S , the defining irrep ϕ_k which is both external and exchanged, a two-index symmetric irrep T_{ij} and a two-index antisymmetric irrep A_{ij} . OPEs of this type have been studied in [15, 46, 47]. That being said, we will use a slightly different notation, including the vector ω_i , to avoid a mismatch of indices. The tensor c_{ijk} is proportional to $\langle \phi_i \phi_j \phi_k \rangle$ and satisfies $c_{ijr} c_{klr} = P_{ijkl}^V$ (to be defined below). The tensor structures, and projectors, for these irreps are

$$\begin{aligned} P_{ijkl}^S &= \frac{1}{Q-1} P_{ij} P_{kl}, \\ P_{ijkl}^V &= \frac{Q}{Q-2} d_{ijm} d_{klme}, \\ P_{ijkl}^T &= \frac{P_{ik} P_{jl} + P_{il} P_{jk}}{2} - \frac{Q}{Q-2} d_{ijm} d_{klm} - \frac{1}{Q-1} P_{ij} P_{kl}, \\ P_{ijkl}^A &= \frac{P_{ik} P_{jl} - P_{il} P_{jk}}{2}, \end{aligned} \quad (\text{A.7})$$

where $d_{ijk} = \delta_{ijk} - \frac{1}{Q}(\delta_{ij}\omega_k + \delta_{ik}\omega_j + \delta_{kl}\omega_i) + \frac{2}{Q^2}\omega_i\omega_j\omega_k$ and $P_{ij} = \delta_{ij} - \frac{1}{Q}\omega_i\omega_j$. Consequently, the projectors for the full group $S_n \ltimes (S_Q)^n$ are

$$\begin{aligned} P^{Sabcd}_{ijkl} &= (Q-1) \delta^{ab} \delta^{cd} P_{ijkl}^S, \\ P^{Xabcd}_{ijkl} &= (Q-1) \left(\delta^{abcd} - \frac{1}{n} \delta^{ab} \delta^{cd} \right) P_{ijkl}^S, \\ P^{Zabcd}_{ijkl} &= \left(\delta^{ac} \delta^{bd} - \delta^{abcd} \right) P_{ik} P_{jl} + \left(\delta^{ad} \delta^{bc} - \delta^{abcd} \right) P_{il} P_{jk}, \\ P^{Babcd}_{ijkl} &= \frac{1}{2} \left(\delta^{ac} \delta^{bd} - \delta^{abcd} \right) P_{ik} P_{jl} - \frac{1}{2} \left(\delta^{ad} \delta^{bc} - \delta^{abcd} \right) P_{il} P_{jk}, \\ P^{\phi abcd}_{ijkl} &= 2 \delta^{abcd} P_{ijkl}^V, \\ P^{Aabcd}_{ijkl} &= 2 \delta^{abcd} P_{ijkl}^A, \\ P^{Tabcd}_{ijkl} &= \delta^{abcd} P_{ijkl}^T. \end{aligned} \quad (\text{A.8})$$

Notice that some of the above tensors are not unit normalised, this is for future convenience when considering mixed correlator systems. In order to obtain the sum rules from these tensor structures it is convenient to contract the correlator $\langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle$ with vectors v_i satisfying $\sum_i v_i = 0$. This eliminates all factors of ω without modifying the sum rules. Thus one can obtain the sum rules simply by reading off the coefficients of the Kronecker deltas. In parts of this work, we use conventions that omit the ω 's from the beginning for practical purposes.

A.2 ϕ single correlator sum rules

We obtain the sum rules for the crossing equation $\langle \phi_i^a \phi_j^b \phi_k^c \phi_l^d \rangle = \langle \phi_k^c \phi_j^b \phi_i^a \phi_l^d \rangle$ by reading the coefficients of eq. (A.1), with respect to $\delta^{ac} \delta^{bd} \delta_{ik} \delta_{jl}$, $\delta^{ad} \delta^{bc} \delta_{il} \delta_{jk}$, $\delta^{ab} \delta^{cd} \delta_{ij} \delta_{kl}$, $\delta^{abcd} \delta_{ij} \delta_{kl}$,

$\delta^{abcd}\delta_{ik}\delta_{jl}$, $\delta^{abcd}\delta_{il}\delta_{jk}$, and $\delta^{abcd}\delta_{ijkl}$. Notice that in the special case of $Q = 3$ the four index Kronecker delta δ_{ijkl} can be written as a function of $\delta_{ijk}\omega_l$, $\delta_{ij}\delta_{kl}$, $\delta_{ij}\omega_i\omega_j$ and $\omega_i\omega_j\omega_k\omega_l$ and their index permutations. Concretely, $\delta_{ijkl} = \frac{1}{3}\omega_{(i}\delta_{jkl)} + \frac{1}{6}\delta_{(ij}\delta_{kl)} - \frac{1}{6}\omega_{(i}\omega_j\delta_{kl)} + \frac{1}{6}\omega_i\omega_j\omega_k\omega_l$. For $n \geq 2$ and $Q = 3$ the sum rules read

$$\begin{aligned}
 0 &= n(Q-1)F_S^- - 2F_A^- - 2(n-1)F_B^-, \\
 0 &= (Q-1)F_X^- - 2F_A^- + 2F_B^-, \\
 0 &= 2F_V^- + 2F_A^-, \\
 0 &= 2F_Z^- + F_B^-, \\
 0 &= n(Q-1)F_S^+ - 4F_V^+ + 2F_A^+ - 2(n-1)F_Z^+ + (n-1)F_B^+, \\
 0 &= (Q-1)F_X^+ - 4F_V^+ + 2F_A^+ + 2F_Z^+ - F_B^+,
 \end{aligned} \tag{A.9}$$

whereas for $n \geq 2$ and $Q > 3$ they read

$$\begin{aligned}
 0 &= n(Q-1)F_S^- - 2(Q-2)F_A^- - (Q-1)(n-1)F_B^-, \\
 0 &= (Q-1)F_X^- - 2(Q-2)F_A^- + (Q-1)F_B^-, \\
 0 &= 2F_V^- + 2F_A^-, \\
 0 &= F_T^- + 2F_A^-, \\
 0 &= 2F_Z^- + F_B^-, \\
 0 &= n(Q-1)F_S^+ - \frac{2(Q-1)}{Q-2}F_V^+ - \frac{Q(Q-3)}{2(Q-2)}F_T^+ + (Q-1)F_A^+ - (Q-1)(n-1)F_Z^+ + \frac{(Q-1)(n-1)}{2}F_B^+, \\
 0 &= (Q-1)F_X^+ - \frac{2(Q-1)}{Q-2}F_V^+ - \frac{Q(Q-3)}{2(Q-2)}F_T^+ + (Q-1)F_A^+ + (Q-1)F_Z^+ - \frac{Q-1}{2}F_B^+.
 \end{aligned} \tag{A.10}$$

Notice that the former ones coincide with those for $S_n \ltimes (O(2))^n$, up to adding/subtracting rows and redefining OPE coefficients, originally derived in [29]. This is expected due to the group-subgroup relationship between $O(2)$ and S_3 . Indeed, after removing factors of ω ,¹⁶ the tensor structures for both groups, with respect to this specific correlator, coincide.

A.3 X single correlator sum rules

To start off, we remind the reader that the $X \times X$ OPE decomposes as

$$X \times X \sim S + X + \overline{X\overline{X}} \tag{A.11}$$

for $n = 3$ and as

$$X \times X \sim S + X + XX + \overline{X\overline{X}} \tag{A.12}$$

for $n \geq 4$. In order to eventually define sum rules mixing ϕ_i^a and X_{ij}^{ab} as externals it is convenient to define some OPE conventions. Tensors in OPEs can be defined modulo numerical coefficients, which can be absorbed into the OPE coefficients. Different choices

¹⁶Which as discussed above, does not affect the sum rules.

correspond to different conventions, which are of course all equivalent. In particular, in order to obtain the projectors in eq. (A.8) we have already implicitly defined

$$\phi_i^a \times \phi_j^b \sim \lambda_{VVS} \delta^{ab} \left(\delta_{ij} - \frac{1}{Q} \omega_i \omega_j \right) S + \lambda_{V VX} X_{ij}^{ab} + \dots, \quad (\text{A.13})$$

where the dots refer to other irreps whose exact convention is not currently important. In our convention, $\langle X_{ij}^{ab} X_{kl}^{cd} \rangle \sim P_{ijkl}^{Xabcd}$, with P^X as defined in eq. (A.8). Consequently, associativity of the OPE forces

$$\phi_i^a \times X_{jk}^{bc} \sim \lambda_{VXV} P_{iljk}^{Xadbc} \phi_l^d + \dots \quad (\text{A.14})$$

Lastly, we define

$$X_{ij}^{ab} \times X_{kl}^{cd} \sim \lambda_{XXS} P_{ijkl}^{Xabcd} S + \lambda_{XXX} \frac{1}{4} P_{ijd_1 d_2}^{Xabu_1 u_2} P_{kld_1 d_3}^{Xcd u_1 u_3} P_{mnd_2 d_3}^{Xef u_2 u_3} X_{mn}^{ef} + \dots \quad (\text{A.15})$$

The above considerations fix the sum rules $\langle X_{ij}^{ab} X_{kl}^{cd} X_{mn}^{ef} X_{op}^{gh} \rangle = \langle X_{mn}^{ef} X_{kl}^{cd} X_{ij}^{ab} X_{op}^{gh} \rangle$ for $n = 3$ to be

$$\begin{aligned} 0 &= \frac{1}{6} F_X^- + \frac{1}{2} F_{\overline{XX}}^-, \\ 0 &= F_S^- - \frac{1}{2} F_{\overline{XX}}^-, \\ 0 &= F_S^+ - \frac{2}{6} F_X^+ + \frac{1}{2} F_{\overline{XX}}^+, \end{aligned} \quad (\text{A.16})$$

which have been derived using

$$\begin{aligned} P_{ijklmnop}^{Sabcdefg h} &= P_{ijkl}^{Xabcd} P_{mnop}^{Xefg h}, \\ P_{ijklmnop}^{Xabcdefg h} &= \frac{1}{6} \left(P_{ijmn}^{Xabef} P_{klop}^{Xcdgh} + P_{ijop}^{Xabgh} P_{mnkl}^{Xefcd} - P_{ijkl}^{Xabcd} P_{mnop}^{Xefg h} \right), \\ P_{ijklmnop}^{\overline{XX}abcdefg h} &= \frac{1}{2} P_{ijmn}^{Xabef} P_{klop}^{Xcdgh} - \frac{1}{2} P_{ijop}^{Xabgh} P_{mnkl}^{Xefcd}, \end{aligned} \quad (\text{A.17})$$

where we have also used the simplifying identity for $\delta_{abcdefg h}$ when $n = 3$, which expresses it in terms of Kronecker deltas with less indices, see eq. (2.22) in [48]. On the other hand, for $n \geq 4$ we obtain

$$\begin{aligned} 0 &= (n-1) F_S^- - (n-2) F_{\overline{XX}}^-, \\ 0 &= \frac{(n-1)(n-2)}{n} F_X^- + F_{\overline{XX}}^-, \\ 0 &= F_{XX}^- + F_{\overline{XX}}^-, \\ 0 &= (n-1) F_S^+ - \frac{(n-1)^2}{n} F_X^+ - \frac{n(n-3)}{2(n-2)} F_{XX}^+ + \frac{n-1}{2} F_{\overline{XX}}^+, \end{aligned} \quad (\text{A.18})$$

by using

$$\begin{aligned}
 P^{S\,abcdefgh}_{ijklmnop} &= P^{X\,abcd}_{ijkl} P^{X\,efgh}_{mnop}, \\
 P^{X\,abcdefgh}_{ijklmnop} &= d^{abcdrs} d^{efghrs} P_{ij} P_{kl} P_{mn} P_{op}, \\
 P^{XX\,abcdefgh}_{ijklmnop} &= \frac{1}{2} \left(P^{X\,abef}_{ijmn} P^{X\,cdgh}_{klop} + P^{X\,abgh}_{ijop} P^{X\,efcd}_{mnkl} \right) - \frac{n}{n-2} P^{X\,abcdefgh}_{ijklmnop} - \frac{1}{n-1} P^{S\,abcdefgh}_{ijklmnop}, \\
 P^{\overline{XX}\,abcdefgh}_{ijklmnop} &= \frac{1}{2} \left(P^{X\,abef}_{ijmn} P^{X\,cdgh}_{klop} - P^{X\,abgh}_{ijop} P^{X\,efcd}_{mnkl} \right),
 \end{aligned} \tag{A.19}$$

where $d^{abcdef} = \delta^{abcdef} - \frac{1}{n} \left(\delta^{abcd} \delta^{ef} + \delta^{abef} \delta^{cd} + \delta^{cdef} \delta^{ab} \right) + \frac{2}{n^2} \delta^{ab} \delta^{cd} \delta^{ef}$. Notice that the above projectors, mirror those of S_Q written earlier, by swapping Q for n . This is expected since the operator X_{ij}^{ab} transforms under $S_n \times (S_Q)^n$ in the same way that the defining representation of S_n transforms under S_n . Indeed these sum rules coincide (up to normalization conventions) with the ones derived in [15]. In our treatment, the tensor structures carry “redundant indices” ($X_{ij}^{ab} \propto \delta^{abc} P_{ij} X^c$) which prove however useful when considering mixed correlator systems.

A.4 $\phi - X$ mixed correlator sum rules

The sum rules resulting from the $\langle \phi_i^a \phi_j^b X_{kl}^{cd} X_{mn}^{ef} \rangle = \langle X_{kl}^{cd} \phi_j^b \phi_i^a X_{mn}^{ef} \rangle$ and $\langle \phi_i^a X_{kl}^{cd} \phi_j^b X_{mn}^{ef} \rangle = \langle \phi_j^b X_{kl}^{cd} \phi_i^a X_{mn}^{ef} \rangle$ crossing equations can be obtained using the following tensors

$$\begin{aligned}
 P^{S\,abcdef}_{ijklmn} &= P_{ij} \delta^{ab} P^{X\,cdef}_{klmn}, \\
 P^{X\,abcdef}_{ijklmn} &= P_{ij} P_{kl} P_{mn} d^{abcdef},
 \end{aligned} \tag{A.20}$$

which appear in the $\langle \phi_i^a \phi_j^b X_{kl}^{cd} X_{mn}^{ef} \rangle$ correlator, and

$$\begin{aligned}
 P^V\,abcdef_{ijklmn} &= P_{ij} P_{kl} P_{mn} \left(\delta^{abcdef} - \frac{1}{n} \left(\delta^{abcd} \delta^{ef} + \delta^{abef} \delta^{cd} \right) + \frac{1}{n^2} \delta^{ab} \delta^{cd} \delta^{ef} \right), \\
 P^{VX}\,abcdef_{ijklmn} &= P_{ij} P_{kl} P_{mn} \left(-\delta^{abcdef} + \frac{1}{n} \left(\delta^{abcd} \delta^{ef} + \delta^{abef} \delta^{cd} \right) + \frac{n-1}{n} \delta^{ab} \delta^{cd} \delta^{ef} + \frac{1}{n} \delta^{ab} \delta^{cd} \delta^{ef} \right),
 \end{aligned} \tag{A.21}$$

which appear in the $\langle \phi_i^a X_{kl}^{cd} \phi_j^b X_{mn}^{ef} \rangle$ correlator. These can be checked to be compatible with the OPE definitions of the previous section. We obtain

$$0 = F_V^- \tag{A.22}$$

and

$$0 = F_{XV}^- \tag{A.23}$$

resulting from $\langle \phi_i^a X_{kl}^{cd} \phi_j^b X_{mn}^{ef} \rangle = \langle \phi_j^b X_{kl}^{cd} \phi_i^a X_{mn}^{ef} \rangle$. As well as

$$0 = F_S^\mp \pm (-1)^\ell \frac{1}{n} F_V^\mp \pm (-1)^\ell \frac{1}{n} F_{XV}^\mp \tag{A.24}$$

and

$$0 = F_X^\mp \pm (-1)^\ell F_V^\mp \pm (-1)^\ell F_{XV}^\mp \tag{A.25}$$

resulting from $\langle \phi_i^a \phi_j^b X_{kl}^{cd} X_{mn}^{ef} \rangle = \langle X_{kl}^{cd} \phi_j^b \phi_i^a X_{mn}^{ef} \rangle$.

A.5 Tensor structures for $\langle Z_3 Z_3 Z_3 Z_3 \rangle$

In order to make contact with previous work in the literature, i.e. [11], we will work specifically in the case $n = 3$, which is also the smallest value of n for which there is a non-trivial fixed point. This will considerably simplify the derivation of the sum rules. A field in the Z_3 irrep can be represented with three upper (replica) and three lower indices, $Z_3^{abc}_{ijk}$. Since by the definition of this representation, the replica indices must be different, for $n = 3$ we can *wlog* take $a = 1$, $b = 2$ and $c = 3$. We will thus consider the following correlator $\langle Z_3^{123}_{ijk} Z_3^{123}_{lmn} Z_3^{123}_{opr} Z_3^{123}_{stu} \rangle$. The $Z_3 \times Z_3$ OPE decomposes as follows

$$Z_3 \times Z_3 \sim AA + VAA + AZ + S + A + V + VA + Z + Z_3 + AAA. \quad (\text{A.26})$$

Remembering that the lower indices can only “talk” to each other if they are in the same copy we obtain the following tensor structures

$$\begin{aligned} P^{AA}_{ijklmnoprstu} &= P^A_{ilos} P^A_{jmpt} P^S_{knru} + P^A_{ilos} P^S_{jmpt} P^A_{knru} + P^S_{ilos} P^A_{jmpt} P^A_{knru}, \\ P^{AAV}_{ijklmnoprstu} &= P^A_{ilos} P^A_{jmpt} P^\phi_{knru} + P^A_{ilos} P^\phi_{jmpt} P^A_{knru} + P^\phi_{ilos} P^A_{jmpt} P^A_{knru}, \\ P^{AZ}_{ijklmnoprstu} &= P^A_{ilos} P^\phi_{jmpt} P^\phi_{knru} + P^\phi_{ilos} P^\phi_{jmpt} P^A_{knru} + P^\phi_{ilos} P^A_{jmpt} P^\phi_{knru}, \\ P^S_{ijklmnoprstu} &= P^S_{ilos} P^S_{jmpt} P^S_{knru}, \\ P^A_{ijklmnoprstu} &= P^A_{ilos} P^S_{jmpt} P^S_{knru} + P^S_{ilos} P^A_{jmpt} P^S_{knru} + P^S_{ilos} P^S_{jmpt} P^A_{knru}, \\ P^V_{ijklmnoprstu} &= P^\phi_{ilos} P^S_{jmpt} P^S_{knru} + P^S_{ilos} P^\phi_{jmpt} P^S_{knru} + P^S_{ilos} P^S_{jmpt} P^\phi_{knru}, \\ P^{VA}_{ijklmnoprstu} &= P^S_{ilos} P^\phi_{jmpt} P^A_{knru} + P^S_{ilos} P^A_{jmpt} P^\phi_{knru} + P^A_{ilos} P^S_{jmpt} P^\phi_{knru}, \\ &\quad + P^A_{ilos} P^\phi_{jmpt} P^S_{knru} + P^\phi_{ilos} P^A_{jmpt} P^S_{knru} + P^\phi_{ilos} P^S_{jmpt} P^A_{knru}, \\ P^Z_{ijklmnoprstu} &= P^\phi_{ilos} P^S_{jmpt} P^S_{knru} + P^S_{ilos} P^\phi_{jmpt} P^S_{knru} + P^S_{ilos} P^S_{jmpt} P^\phi_{knru}, \\ P^{Z_3}_{ijklmnoprstu} &= P^\phi_{ilos} P^\phi_{jmpt} P^\phi_{knru}, \\ P^{AAA}_{ijklmnoprstu} &= P^A_{ilos} P^A_{jmpt} P^A_{knru}, \end{aligned} \quad (\text{A.27})$$

where the P^S , P^A and P^ϕ on the right hand side of eq. (A.27) are those defined earlier in eq. (A.7), but multiplied by a factor of 2.

A.6 Z_3 single correlator sum rules

To obtain the sum rules from the $\langle Z_3^{123}_{ijk} Z_3^{123}_{lmn} Z_3^{123}_{opr} Z_3^{123}_{stu} \rangle = \langle Z_3^{123}_{opr} Z_3^{123}_{lmn} Z_3^{123}_{ijk} Z_3^{123}_{stu} \rangle$ crossing equation we first make the following convenient substitutions

$$\begin{aligned} P^A_{ilos} &= P^0_{ilos} - \frac{P^+_{ilos} - P^-_{ilos}}{2}, \\ P^V_{ilos} &= P^0_{ilos} + \frac{P^+_{ilos} - P^-_{ilos}}{2} - \frac{P^+_{ilos} + P^-_{ilos}}{2}, \\ P^S_{ilos} &= \frac{P^+_{ilos} + P^-_{ilos}}{2}, \end{aligned} \quad (\text{A.28})$$

and similarly for P^A_{jmpt} , P^A_{knru} and the rest. We have defined P^0 and P^+ to be even under crossing, and P^- to be odd. Reading the coefficients of each possible combination of P^+ ,

P^- and P^0 we obtain the following sum rules

$$\begin{aligned}
 0 &= 2F_{AAA}^- + 6F_{AAV}^- + 6F_{AZ}^- + 2F_{Z_3}^-, \\
 0 &= F_{AA}^- - F_{AAA}^- - 2F_{AAV}^- - F_{AZ}^- + 2F_{VA}^- + F_Z^-, \\
 0 &= \frac{1}{2}F_A^- - F_{AA}^- + \frac{1}{2}F_{AAA}^- + \frac{1}{2}F_{AAV}^- + \frac{1}{2}F_V^- - F_{VA}^-, \\
 0 &= \frac{1}{2}F_A^- + F_{AA}^- + \frac{1}{2}F_{AAA}^- - \frac{3}{2}F_{AAV}^- + \frac{1}{2}F_V^- - F_{VA}^- - 2F_Z^- + 2F_{Z_3}^-, \\
 0 &= -\frac{3}{4}F_A^- + \frac{3}{4}F_{AA}^- - \frac{1}{4}F_{AAA}^- + \frac{1}{4}F_S^-, \\
 0 &= \frac{1}{4}F_A^- - \frac{1}{4}F_{AA}^- - \frac{1}{4}F_{AAA}^- + F_{AAV}^- - F_{AZ}^- + \frac{1}{4}F_S^-, -F_V^- + F_Z^-,
 \end{aligned} \tag{A.29}$$

and

$$\begin{aligned}
 0 &= F_{AA}^+ + F_{AAA}^+ - 3F_{AZ}^+ + 2F_{VA}^+ + F_Z^+ - 2F_{Z_3}^+, \\
 0 &= \frac{1}{2}F_A^+ - \frac{1}{2}F_{AAA}^+ + \frac{1}{2}F_{AAV}^+ + F_{AZ}^+ + \frac{1}{2}F_V^+ - F_{VA}^+ - F_Z^+, \\
 0 &= -\frac{1}{4}F_A^+ - \frac{1}{4}F_{AA}^+ + \frac{1}{4}F_{AAA}^+ - \frac{1}{2}F_{AAV}^+ + \frac{1}{4}F_S^+ - \frac{1}{2}F_V^+ + F_{VA}^+, \\
 0 &= \frac{3}{4}F_A^+ + \frac{3}{4}F_{AA}^+ + \frac{1}{4}F_{AAA}^+ - \frac{3}{2}F_{AAV}^+ + 3F_{AZ}^+ + \frac{1}{4}F_S^+ - \frac{3}{2}F_V^+ - 3F_{VA}^+ + 3F_Z^+ - 2F_{Z_3}^+.
 \end{aligned} \tag{A.30}$$

A.7 Tensor structures for $\langle \phi Z_3 \phi Z_3 \rangle$

To study the tensor structures of the $\langle \phi Z_3 \phi Z_3 \rangle$ correlators, as in the preceding section we will plug in explicit index values for the upper indices. Without loss of generality, the two independent cases to consider are $\langle \phi_i^1 Z_3^{123}_{jkl} \phi_m^1 Z_3^{123}_{nop} \rangle$ and $\langle \phi_i^1 Z_3^{123}_{jkl} \phi_m^2 Z_3^{123}_{nop} \rangle$. The $\phi \times Z_3$ OPE decomposes as

$$\phi \times Z_3 \sim Z + AZ + VB + Z_3. \tag{A.31}$$

In order to understand what tensors will appear, let us look more carefully at how the VB and Z_3 representations come about¹⁷

$$\begin{aligned}
 \phi_i^1 \times Z_3^{123}_{jkl} &\sim C_{ijs} V_s^1 Z_{kl}^{23} \\
 &\sim C_{ijs} (V_s^1 Z_{kl}^{23} + V_k^2 Z_{sl}^{13} + V_l^3 Z_{sk}^{12}) \\
 &\quad + C_{ijs} (V_s^1 Z_{kl}^{23} - V_k^2 Z_{sl}^{13}) + C_{ijs} (V_s^1 Z_{kl}^{23} - V_l^3 Z_{sk}^{12}) \\
 &\sim C_{ijs} (Z_3^{123}_{skl} + V_l^3 B_{sk}^{12} + V_k^2 B_{sl}^{13}).
 \end{aligned} \tag{A.32}$$

Plugging this equation into the four-point correlator one can find the tensor structures for each irrep. For the first of the two correlators, namely $\langle \phi_i^1 Z_3^{123}_{jkl} \phi_m^1 Z_3^{123}_{nop} \rangle$, we have

$$\begin{aligned}
 P_{ijklmnop}^Z &= \delta_{ij} \delta_{mn} \delta_{ko} \delta_{lp}, \\
 P_{ijklmnop}^{AZ} &= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \delta_{ko} \delta_{lp},
 \end{aligned}$$

¹⁷We temporarily omit other representations from the product $\phi \times Z_3$.

$$\begin{aligned}
 P^{VB}_{ijklmnop} &= 2 \left(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} - \frac{2}{m} \delta_{ij} \delta_{mn} \right) \delta_{ko} \delta_{lp}, \\
 P^{Z_3}_{ijklmnop} &= \left(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} - \frac{2}{m} \delta_{ij} \delta_{mn} \right) \delta_{ko} \delta_{lp}.
 \end{aligned}
 \tag{A.33}$$

Notice that while the VB and Z_3 irreps above have the same tensor structure up to a numerical prefactor, this prefactor can and will be different in other correlators. Thus, the overall tensor structure will be different, as it should. Moving on to the second correlator of interest, $\langle \phi_i^1 Z_3^{123} \phi_m^2 Z_3^{123} \rangle$, we have

$$\begin{aligned}
 P^{Z_3}_{ijklmnop} &= C_{ijr} C_{mos} \delta_{sk} \delta_{rn} \delta_{lp} = C_{ijn} C_{mok} \delta_{lp}, \\
 P^{VB}_{ijklmnop} &= -C_{ijr} C_{mos} \delta_{sk} \delta_{rn} \delta_{lp} = -C_{ijn} C_{mok} \delta_{lp},
 \end{aligned}
 \tag{A.34}$$

where now the tensors corresponding to the two irreps differ by a minus sign. The AZ and Z irreps do not appear in this correlator. As an aside, note that these two tensor structures will give rise to a sum rule of the form $F_{Z_3}^- - F_{VB}^- = 0$. This sum rule is characteristic of “symmetry breaking”: it appears because the irrep VZ of $S_3 \times (O(2))^3$ becomes reducible when breaking the symmetry down to $S_3 \times (S_3)^3$, due to the existence of the invariant tensor C_{ijk} .¹⁸

A.8 Tensor structures for $\langle \phi \phi Z_3 Z_3 \rangle$

Using similar reasoning to the preceding subsection, we only need to consider the correlators $\langle \phi_m^1 \phi_i^1 Z_3^{123} Z_3^{123} \rangle$ and $\langle \phi_m^2 \phi_i^1 Z_3^{123} Z_3^{123} \rangle$. The irreps that contribute to these correlators are the ones that are exchanged in both the $\phi \times \phi$ and the $Z_3 \times Z_3$ OPE. These are S , A , V and Z . The tensor structures obtained for $\langle \phi_m^1 \phi_i^1 Z_3^{123} Z_3^{123} \rangle$ are

$$\begin{aligned}
 P^S_{mijklnop} &= \delta_{im} \delta_{jn} \delta_{ko} \delta_{lp}, \\
 P^A_{mijklnop} &= (\delta_{in} \delta_{mj} - \delta_{ij} \delta_{mn}) \delta_{ko} \delta_{lp}, \\
 P^V_{mijklnop} &= (\delta_{in} \delta_{mj} + \delta_{ij} \delta_{mn} - \delta_{im} \delta_{jn}) \delta_{ko} \delta_{lp},
 \end{aligned}
 \tag{A.35}$$

whereas for $\langle \phi_m^2 \phi_i^1 Z_3^{123} Z_3^{123} \rangle$ we have

$$P^Z_{mijklnop} = C_{ijn} C_{mok} \delta_{lp},
 \tag{A.36}$$

which is found using $C_{ijk} \sim \langle \phi_i \phi_j \phi_k \rangle$.

A.9 $\phi - Z_3$ mixed correlator sum rules

Having worked out the tensor structures in the two previous subsections it is straightforward to obtain the sum rules. We start from the $\langle \phi_i^1 Z_3^{123} \phi_m^1 Z_3^{123} \rangle = \langle \phi_m^1 Z_3^{123} \phi_i^1 Z_3^{123} \rangle$ crossing equation from which we obtain

$$\begin{aligned}
 0 &= 2F_{AZ}^- + 4F_{VB}^- + 2F_{Z_3}^-, \\
 0 &= -F_{AZ}^- + 2F_{VB}^- + F_Z^- + F_{Z_3}^- - 2F_{VB}^- - F_{Z_3}^-,
 \end{aligned}
 \tag{A.37}$$

¹⁸Which in turn is invariant due to the breaking of the Z_2 contained in $O(2)$, when breaking $O(2)$ to S_3 .

and

$$0 = F_{AZ}^+ + F_Z^+ - 4F_{VB}^+ - 2F_{Z_3}^+. \quad (\text{A.38})$$

Consequently, from the crossing equation $\langle \phi_i^1 Z_3^{123} \phi_m^2 Z_3^{123} \rangle = \langle \phi_m^2 Z_3^{123} \phi_i^1 Z_3^{123} \rangle$ one obtains

$$0 = F_{Z_3}^- - F_{VB}^-. \quad (\text{A.39})$$

These exhaust the number of sum rules one can obtain from crossing equations of the schematic type $\langle \phi Z_3 \phi Z_3 \rangle = \langle \phi Z_3 \phi Z_3 \rangle$. We thus move on to crossing equations of the schematic type $\langle \phi \phi Z_3 Z_3 \rangle = \langle Z_3 \phi \phi Z_3 \rangle$. These are $\langle \phi_m^1 \phi_i^1 Z_3^{123} Z_3^{123} \rangle = \langle Z_3^{123} \phi_i^1 \phi_m^1 Z_3^{123} \rangle$ and $\langle \phi_m^2 \phi_i^1 Z_3^{123} Z_3^{123} \rangle = \langle Z_3^{123} \phi_i^1 \phi_m^2 Z_3^{123} \rangle$. Notice that while we will omit the OPE coefficients with which exchanged operators appear, we have taken into account the minus signs that occur due to $\lambda_{\phi Z_3 O} = (-1)^\ell \lambda_{Z_3 \phi O}$, where ℓ is the spin of the exchanged operator O . From the first of the two aforementioned crossing equations we obtain

$$\begin{aligned} 0 &= (-1)^{1+\ell} F_{AZ}^- + F_S^- - F_A^- + (-1)^\ell F_Z^-, \\ 0 &= (-1)^{1+\ell} F_{AZ}^- + F_S^- + F_A^- - 2F_V^- + 4(-1)^\ell F_{VB}^- + 2(-1)^\ell F_{Z_3}^- + (-1)^{1+\ell} F_Z^-, \\ 0 &= (-1)^\ell F_{AZ}^- + F_A^- + F_V^- + 2(-1)^\ell F_{VB}^- + (-1)^\ell F_{Z_3}^-, \end{aligned} \quad (\text{A.40})$$

and

$$\begin{aligned} 0 &= (-1)^\ell F_{AZ}^+ + F_S^+ - F_A^+ + (-1)^{1+\ell} F_Z^+, \\ 0 &= (-1)^\ell F_{AZ}^+ + F_S^+ + F_A^+ - 2F_V^+ - 4(-1)^\ell F_{VB}^+ - 2(-1)^\ell F_{Z_3}^+ + (-1)^\ell F_Z^+, \\ 0 &= (-1)^{1+\ell} F_{AZ}^+ + F_A^+ + F_V^+ - 2(-1)^\ell F_{VB}^+ + (-1)^{1+\ell} F_{Z_3}^+, \end{aligned} \quad (\text{A.41})$$

where operators in the Z , AZ , Z_3 and VB representations appear with $\lambda_{\phi Z_3 O}^2$ and S , A and V and V_B appear with $\lambda_{\phi \phi O} \lambda_{Z_3 Z_3 O}$. From the second of the two crossing equations we obtain

$$0 = (-1)^{1+\ell} F_{VB}^- + F_Z^- + (-1)^\ell F_{Z_3}^- \quad (\text{A.42})$$

and

$$0 = (-1)^\ell F_{VB}^+ + F_Z^+ + (-1)^{1+\ell} F_{Z_3}^+, \quad (\text{A.43})$$

where operators in the Z_3 and VB representations appear with $\lambda_{\phi Z_3 O}^2$ and Z now appears with $\lambda_{\phi \phi O} \lambda_{Z_3 Z_3 O}$. We emphasise that Z appears with a different combination of OPE coefficients in this crossing equation compared to the previous one.

B Numerical implementation

All the results we presented in this text have been computed with the use of *simpleboot*.¹⁹ The parameters we used in this software are $d = 2$, $\Lambda = 19$, $\kappa = 14$, $r_N = 56$ and $\ell_{\text{set}} = \{0, \dots, 26, 49, 50\}$, the last of which corresponds to the spins of exchanged operators. In the case of figure 15, and only there, we also present an allowed region obtained with the parameters $d = 2$, $\Lambda = 11$, $\kappa = 12$, $r_N = 48$ and $\ell_{\text{set}} = \{0, \dots, 20, 49, 52\}$. While calling *SDPB* [49, 50], we used the parameters in table 5. During the early stages of this work, we also performed tests using *PyCFTBoot* [51] and *qboot* [52].

¹⁹<https://gitlab.com/bootstrapcollaboration/simpleboot>. See [here](#) for lectures on this software.

parameter	value
precision	1024
maxIterations	1000
maxComplementarity	10^{70}
dualityGapThreshold	10^{-40}
primalErrorThreshold	10^{-60}
dualErrorThreshold	10^{-60}
initialMatrixScalePrimal	10^{20}
initialMatrixScaleDual	10^{20}
detectPrimalFeasibleJump	True
detectDualFeasibleJump	True

Table 5. SDPB parameters.

C Quasi primaries of the critical 3-state Potts model

The quasi-primary spectrum of the $Q = 3$ (bi-)critical Potts model can be obtained from the partition function by standard character techniques. The CFT has a finite number of Virasoro primaries reported in table 6 along with the S_3 representation where they belong to. The modular invariant partition function is a particular bilinear combination of “left” and “right” characters of singular Verma modules [53]:

$$\mathcal{Z}(q, \bar{q}) = \sum_{r=1,2} \left(|\chi_{r,1}(q) + \chi_{r,5}(q)|^2 + 2|\chi_{r,3}(q)|^2 \right), \quad (\text{C.1})$$

where

$$\begin{aligned} \chi_{r,s}(q) &= K_{\lambda_{r,s}}(q) - K_{\lambda_{r,s}}(\bar{q}), \\ K_{\lambda}(q) &= \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{\frac{(Nn+\lambda)^2}{2N}}, \end{aligned} \quad (\text{C.2})$$

and $\lambda_{r,s} = pr - p's$, $N = 2pp'$, $\eta(q)$ is the Dedekind η -function. For the 3-state Potts model $p = 6$, $p' = 5$.

The above partition function can be split in sectors of S_3 representation as follows

$$\begin{aligned} \mathcal{Z}_{\text{sing}}(q, \bar{q}) &= |\chi_{1,1}(q)|^2 + |\chi_{2,1}(q)|^2 + |\chi_{2,5}(q)|^2 + |\chi_{1,2}(q)|^2, \\ \mathcal{Z}_{\text{fund}}(q, \bar{q}) &= |\chi_{2,3}(q)|^2 + |\chi_{1,3}(q)|^2, \\ \mathcal{Z}_{\text{asym}}(q, \bar{q}) &= \chi_{2,1}(q)\chi_{2,5}(\bar{q}) + \chi_{2,5}(q)\chi_{2,1}(\bar{q}) + \chi_{1,1}(q)\chi_{1,5}(\bar{q}) + \chi_{1,5}(q)\chi_{1,1}(\bar{q}). \end{aligned} \quad (\text{C.3})$$

Each sector can be then decomposed in characters of the global conformal group to read off the quasi-primary spectrum

$$\mathcal{Z}_R(s, x) = \mathcal{Z}_R(q = sx, \bar{q} = s/x) = s^{-\frac{c}{12}} \sum_{(\Delta, \ell)} p_{R, \Delta, \ell} \chi_{\Delta, \ell}(s, x), \quad (\text{C.4})$$

operator	Δ	ℓ	$((r, s); (\bar{r}, \bar{s}))$	S_3 irrep
$\mathbb{1}$	0	0	$((1, 1); (1, 1))$	singlet
ϵ	4/5	0	$((2, 1); (2, 1))$	singlet
σ	2/15	0	$((2, 3); (2, 3))$	fundamental
σ'	4/3	0	$((1, 3); (1, 3))$	fundamental
γ	9/5	1	$((2, 1); (2, 5)) \oplus ((2, 5); (2, 1))$	antisymmetric
W	3	3	$((1, 1); (1, 5)) \oplus ((1, 5); (1, 1))$	antisymmetric
ϵ'	14/5	0	$((2, 5); (2, 5))$	singlet
ϵ''	6	0	$((1, 5); (1, 5))$	singlet

Table 6. Virasoro primaries of the modular invariant critical 3-state Potts model.

where $R = \text{sing, fund, asym}$, $c = 4/5$ is the central charge, p denotes the degeneracy of the quasi-primary and the characters are given by

$$\begin{aligned}
 \chi_{\Delta,0}(s, x) &= \frac{s^\Delta}{(1-sx)(1-s/x)} & (\Delta > 0), \\
 \chi_{\Delta,\ell}(s, x) &= \frac{s^\Delta}{(1-sx)(1-s/x)} \left(x^\ell + \frac{1}{x^\ell} \right) & (\Delta > \ell, \ell \geq 1), \\
 \chi_{0,0}(s, x) &= 1, \\
 \chi_{\ell,\ell}(s, x) &= \frac{1}{(1-sx)(1-s/x)} \left[s^\ell \left(x^\ell + \frac{1}{x^\ell} \right) - s^{\ell+1} \left(x^{\ell-1} + \frac{1}{x^{\ell-1}} \right) \right].
 \end{aligned} \tag{C.5}$$

In table 7 we report the scaling dimensions and spins of the first 19 operators for each S_3 sector.

These are of use in our work since both at the decoupled limit, as well as the large- n limit, the dimensions of operators in the replica theories are completely determined from those of a single copy. Hence, they provide a minimal intuition for what gap assumptions one may impose on the theory.

We observe that the operators $O_{S,1}$, $O_{V,1}$ and $O_{A,0}$ have particularly large gaps. While performing our numerical bootstrap bounds we experimented with imposing large gaps in these sectors, but this unfortunately did not produce an island, or alternatively, a peninsula which appears to be converging towards an island. In particular, we experimented with using $O_{V,0}$ and $O_{V',0}$ as externals in a single Potts model,²⁰ since this exchanges all three operators in question, and imposed $O_{S,1} \geq 7.5$, $O_{V,1} \geq 3.5$, $O_{A,0} \geq 6.5$ and $O_{V'',0} \geq 4.1$ (while the exact value is $\Delta_{V'',0} \simeq 4.13$) at $\Lambda = 11$.

For that reason, as also mentioned in the main text, throughout our work we aimed to instead impose rather conservative gap assumptions so that our bounds present somewhat “universal” (i.e. not too heavily gap-dependent) constraints on our theories of interest. This is especially important since we do not know anything about the anomalous dimensions that the aforementioned operators obtain at the coupled replica fixed point.

²⁰Which totals to 21 sum rules that can be obtained via *autoboot* [28].

ℓ	Δ	p	ℓ	Δ	p	ℓ	Δ	p
0	0	1	0	0.13	1	0	6.8	2
	0.8	1		1.33	1	1	1.8	1
	2.8	1		4.13	1		5	1
	4	1		5.33	1		7	1
	6	1		6.13	1		7.8	1
	6.8	2	1	5.13	1	2	4.8	1
1	7.8	1		7.13	2	3	3	1
2	2	1	2	2.13	1		3.8	1
	4.8	1		3.33	1		5.8	1
	6	1		6.13	2		7	1
3	3.8	1	3	3.13	1	4	6.8	1
4	4	1		7.13	2	5	5	1
	4.8	1	4	4.13	2		5.8	2
	6.8	2		5.33	2		7.8	2
5	5.8	1	5	5.13	2	6	6	1
	7.8	1		6.33	1		6.8	1
6	6	2	6	6.13	3	7	7	1
	6.8	2		7.33	3		7.8	3
7	7.8	2		7.13	4	8	8	1

(a) Singlet. (b) Fundamental. (c) Antisymmetric.

Table 7. Quasi-primary spectrum of critical 3-state Potts model. Coloured rows are the Virasoro primaries. The dimensions are all integers or rational ($x.13$ and $y.33$ should be read as $x + 2/15$ and $y + 1/3$).

D Character table

c.c.s.	1	6	12	8	27	54	72	144	18	36	36	72	162	27	9	36	36	216	54	108	54	108
S	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\overline{AAA}$	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
$\overline{XX}$	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1	-1	-1	-1	-1
AAA	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\overline{AAA}$	2	2	2	2	2	2	-1	-1	0	0	0	0	0	-2	-2	-2	-2	1	0	0	0	0
X	2	2	2	2	2	2	-1	-1	0	0	0	0	0	2	2	2	2	-1	0	0	0	0
XA	3	3	3	3	-1	-1	0	0	-1	-1	-1	-1	1	-3	1	1	1	0	1	1	-1	-1
$\overline{AA}$	3	3	3	3	-1	-1	0	0	-1	-1	-1	-1	1	3	-1	-1	-1	0	-1	-1	1	1
A	3	3	3	3	-1	-1	0	0	1	1	1	1	-1	-3	1	1	1	0	-1	-1	1	1
AA	3	3	3	3	-1	-1	0	0	1	1	1	1	-1	3	-1	-1	-1	0	1	1	-1	-1
$V\overline{AA}$	6	3	0	-3	2	-1	0	0	-2	-2	1	1	0	0	-4	-1	2	0	0	0	2	-1
XV	6	3	0	-3	2	-1	0	0	-2	-2	1	1	0	0	4	1	-2	0	0	0	-2	1

c.c.s.	1	6	12	8	27	54	72	144	18	36	36	72	162	27	9	36	36	216	54	108	54	108
VAA	6	3	0	-3	2	-1	0	0	2	2	-1	-1	0	0	-4	-1	2	0	0	0	-2	1
V	6	3	0	-3	2	-1	0	0	2	2	-1	-1	0	0	4	1	-2	0	0	0	2	-1
B_3	8	-4	2	-1	0	0	2	-1	-4	2	2	-1	0	0	0	0	0	0	0	0	0	0
Z_3	8	-4	2	-1	0	0	2	-1	4	-2	-2	1	0	0	0	0	0	0	0	0	0	0
VA	12	6	0	-6	-4	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
AB	12	0	-3	3	0	0	0	0	-2	1	-2	1	0	0	-4	2	-1	0	2	-1	0	0
B	12	0	-3	3	0	0	0	0	-2	1	-2	1	0	0	4	-2	1	0	-2	1	0	0
AZ	12	0	-3	3	0	0	0	0	2	-1	2	-1	0	0	-4	2	-1	0	-2	1	0	0
Z	12	0	-3	3	0	0	0	0	2	-1	2	-1	0	0	4	-2	1	0	2	-1	0	0
VB	16	-8	4	-2	0	0	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 8. Character table of $S_3 \ltimes (S_3)^3$, namely the value of the character $\chi_\rho([g])$ for each representation ρ and for all the conjugacy classes $[g]$. The first row gives the sizes of the conjugacy classes.

E Sum rules from finite group characters

We describe an alternative method to derive the sum rules for specific correlators. This allows us to obtain the sum rules directly from the character table of a given discrete group (accessible e.g. with [GAP](#) [27]). This method has provided an additional check for part of the sum rules derived using the projectors/“tensor” methods.

Let G be a finite group, V a representation and R an irreducible representation. The projection map $V \rightarrow R$ is known to be given as an average of the representation V over the group weight by the character of the representation R

$$(P_R^V)_{ij} = \frac{d_R}{|G|} \sum_g \chi_R(g)^* V(g)_{ij}, \quad (\text{E.1})$$

where d_R signifies the dimension of the representation. We henceforth assume that all matrix representations are put in a unitary form, but the result will be basis-independent.

Thus, the complex conjugate of the projection matrix is

$$(P_R^V)_{ij}^* = (P_R^V)_{ji}. \quad (\text{E.2})$$

Hence, the projectors satisfy the following orthogonality relation (sum over repeated indices)

$$(P_R^V)_{ji}^* (P_S^V)_{jk} = (P_R^V)_{ij} (P_S^V)_{jk} = \delta_{RS} (P_R^V)_{jk}, \quad (\text{E.3})$$

and are normalised to

$$\text{tr}[P_R^V] = \frac{d_R}{|G|} \sum_g \chi_R(g)^* \chi_V(g) = d_R \langle \chi_R, \chi_V \rangle = d_R n_{R,V}, \quad (\text{E.4})$$

where $n_{R,V}$ is multiplicity of the R representation in the decomposition of V into irreps and $\langle \chi, \chi' \rangle$ is the usual scalar product of characters.

Moreover, if V is irreducible, then the projector is the identity if $R = V$ and zero otherwise by the usual orthogonality relation of irreducible representations

$$(P_R^V)_{ij} = \frac{d_R}{|G|} \sum_g \chi_R(g)^* V(g)_{ij} = \frac{d_R}{|G|} \sum_g R(g)_{kk}^* V(g)_{ij} = \delta_{RV} \delta_{ij}. \quad (\text{E.5})$$

If $V = V_1 \otimes V_2$, then the projection to $R \in V_1 \otimes V_2$ is

$$(P_R^{V_1 \otimes V_2})_{ijkl} = \frac{d_R}{|G|} \sum_g \chi_R(g)^* V_1(g)_{ij} V_2(g)_{kl}. \quad (\text{E.6})$$

This corresponds to the contraction of two Clebsh-Gordans (CG) for the decomposition $V_1 \otimes V_2 \rightarrow R$. Indeed if CG are defined by

$$(V_1 \otimes V_2)(g)_{ij,kl} = V_1(g)_{ij} V_2(g)_{kl} = \sum_{R \in V_1 \otimes V_2} \sum_{\ell=1}^{n_R} \left\{ \begin{matrix} [cc|c] V_1 & V_2 & R \\ i & k & n \end{matrix} \right\}_\ell \left\{ \begin{matrix} [cc|c] V_1 & V_2 & R \\ j & l & m \end{matrix} \right\}_\ell^* R(g)_{nm} \quad (\text{E.7})$$

where n_R is the degeneracy of the irrep R in the $V_1 \otimes V_2$ decomposition. Then

$$\begin{aligned} (P_R^{V_1 \otimes V_2})_{ijkl} &= \frac{d_R}{|G|} \sum_g \chi_R(g)^* \sum_{S \in V_1 \otimes V_2} \sum_{\ell=1}^{n_R} \left\{ \begin{matrix} [cc|c] V_1 & V_2 & S \\ i & k & n \end{matrix} \right\}_\ell \left\{ \begin{matrix} [cc|c] V_1 & V_2 & S \\ j & l & m \end{matrix} \right\}_\ell^* S(g)_{nm} \\ &= \sum_{S \in V_1 \otimes V_2} \sum_{\ell=1}^{n_R} \left\{ \begin{matrix} [cc|c] V_1 & V_2 & S \\ i & k & n \end{matrix} \right\}_\ell \left\{ \begin{matrix} [cc|c] V_1 & V_2 & S \\ j & l & m \end{matrix} \right\}_\ell^* \left(\frac{d_R}{|G|} \sum_g R(g)_{pp}^* S(g)_{nm} \right) \\ &= \sum_{S \in V_1 \otimes V_2} \sum_{\ell=1}^{n_R} \left\{ \begin{matrix} [cc|c] V_1 & V_2 & S \\ i & k & n \end{matrix} \right\}_\ell \left\{ \begin{matrix} [cc|c] V_1 & V_2 & S \\ j & l & m \end{matrix} \right\}_\ell^* \delta_{RS} \delta_{np} \delta_{mp}. \end{aligned}$$

Thus

$$(P_R^{V_1 \otimes V_2})_{ijkl} = \sum_{\ell=1}^{n_R} \left\{ \begin{matrix} [cc|c] V_1 & V_2 & R \\ i & k & n \end{matrix} \right\}_\ell \left\{ \begin{matrix} [cc|c] V_1 & V_2 & R \\ j & l & m \end{matrix} \right\}_\ell^*. \quad (\text{E.8})$$

Following the method described in [28], one can write down the sum rules for correlators of $\text{SO}(d)$ scalars in the presence of some global symmetry in terms of contractions of CG. We shall then make use of the above formula to write the sum rules in terms of group characters.

The method applies to 4-point functions of type

$$\langle v_i(x_1) w_j(x_2) v_k(x_3)^* w_l(x_4)^* \rangle, \quad (\text{E.9})$$

where v, w are scalar primaries in the V and W irreducible representations of the global symmetry group G . They admit the following OPE decomposition in the (12)(34) channel

$$\langle v_i(x_1) w_j(x_2) v_k(x_3)^* w_l(x_4)^* \rangle = \mathbf{K}_{12,34} \sum_{R \in V \otimes W} \left\{ \begin{matrix} [cc|c] V & W & R \\ i & j & n \end{matrix} \right\} \left\{ \begin{matrix} [cc|c] V^* & W^* & R^* \\ k & l & n \end{matrix} \right\} g_R^{vw,vw}(z, \bar{z}), \quad (\text{E.10})$$

where

$$\mathbf{K}_{ij,kl} = \frac{1}{|x_{12}|^{\Delta_i + \Delta_j} |x_{34}|^{\Delta_k + \Delta_l}} \left(\frac{|x_{24}|}{|x_{14}|} \right)^{\Delta_i - \Delta_j} \left(\frac{|x_{14}|}{|x_{13}|} \right)^{\Delta_k - \Delta_l} \quad (\text{E.11})$$

and $u = z\bar{z} = x_{12}^2 x_{34}^2 / (x_{13}^2 x_{24}^2)$, $v = (1-z)(1-\bar{z}) = x_{23}^2 x_{14}^2 / (x_{13}^2 x_{24}^2)$ are the usual cross ratios ($x_{ij} = x_i - x_j$). We have further assumed that the irrep decomposition of $V \otimes W$ is degeneracy-free.

Observing that

$$\left\{ \begin{matrix} [cc|c]V^* & W^* & R^* \\ & k & l & n \end{matrix} \right\} = \left\{ \begin{matrix} [cc|c]V & W & R \\ & k & l & n \end{matrix} \right\}^*, \quad (\text{E.12})$$

we recognise the projector $V \otimes W \rightarrow R$

$$\langle v_i(x_1) w_j(x_2) v_k(x_3)^* w_l(x_4)^* \rangle = \mathbf{K}_{12,34} \sum_{R \in V \otimes W} (P_R^{V \otimes W})_{ikjl} g_R^{vw, vw}(z, \bar{z}). \quad (\text{E.13})$$

If we perform the OPE in the (14)(32) channel instead we have

$$\langle v_i(x_1) w_j(x_2) v_k(x_3)^* w_l(x_4)^* \rangle = \mathbf{K}_{14,32} \sum_{S \in V \otimes W^*} \left\{ \begin{matrix} [cc|c]V & W^* & S \\ & i & l & n \end{matrix} \right\} \left\{ \begin{matrix} [cc|c]V^* & W & S^* \\ & k & j & n \end{matrix} \right\} g_S^{vw, vw}(1-z, 1-\bar{z}), \quad (\text{E.14})$$

and again using eq. (E.12) we reconstruct the projector from $V \otimes W^* \rightarrow S$

$$\langle v_i(x_1) w_j(x_2) v_k(x_3)^* w_l(x_4)^* \rangle = \mathbf{K}_{14,32} \sum_{S \in V \otimes W^*} (P_S^{V \otimes W^*})_{iklj} g_S^{vw, vw}(1-z, 1-\bar{z}). \quad (\text{E.15})$$

The crossing equation is then

$$\mathbf{K}_{14,32} \sum_{S \in V \otimes W^*} (P_S^{V \otimes W^*})_{iklj} g_S^{vw, vw}(1-z, 1-\bar{z}) = \mathbf{K}_{12,34} \sum_{R \in V \otimes W} (P_R^{V \otimes W})_{ikjl} g_R^{vw, vw}(z, \bar{z}). \quad (\text{E.16})$$

We contract the above equation with $(P_R^{V \otimes W})_{ijkl}$, then on the r.h.s.

$$(P_R^{V \otimes W})_{ikjl}^* (P_{R'}^{V \otimes W})_{ikjl} = \delta_{RR'} \text{tr}[P_R^{V \otimes W}] = \delta_{RR'} d_R, \quad (\text{E.17})$$

while on the l.h.s. we get

$$N_{RS} = (P_R^{V \otimes W})_{ikjl}^* (P_S^{V \otimes W^*})_{iklj}. \quad (\text{E.18})$$

We observe that even though we used multiple times that the representations are unitary, N_{RS} has a basis-independent expression. Firstly,

$$N_{RS} = (P_R^{V \otimes W})_{ikjl}^* (P_S^{V \otimes W^*})_{iljk} = (P_R^{V \otimes W})_{jlik}^* (P_S^{V \otimes W^*})_{iljk} = \text{tr}[P_R^{V \otimes W} \text{CP}_S^{V \otimes W}], \quad (\text{E.19})$$

where CP stands for “crossed projector”, i.e.

$$(\text{CP}_S^{V \otimes W})_{ikjl} = (P_S^{V \otimes W^*})_{iljk} = \frac{d_S}{|G|} \sum_g \chi_S(g)^* V(g)_{ij} W(g)_{lk}^*. \quad (\text{E.20})$$

Now recall that for unitary representations $W(g)_{lk}^* = W(g^{-1})_{kl} = W^{-1}(g)_{kl}$, thus

$$\text{CP}_S^{V \otimes W} = P_S^{V \otimes W^{-1}}. \quad (\text{E.21})$$

Therefore

$$N_{RS} = \text{tr} \left[P_R^{V \otimes W} P_S^{V \otimes W^{-1}} \right]. \quad (\text{E.22})$$

N_{RS} can then be written in terms of characters as follows

$$N_{RS} = \frac{d_R d_S}{|G|^2} \sum_{g_1, g_2 \in G} \chi_R(g_1)^* \chi_S(g_2)^* \chi_V(g_1 g_2) \chi_W(g_1 g_2^{-1}). \quad (\text{E.23})$$

The crossing equation can thus be written as

$$\mathbf{K}_{14,32} \sum_{S \in V \otimes W^*} M_{RS} g_S^{vw, vw} (1 - z, 1 - \bar{z}) - \mathbf{K}_{12,34} g_R^{vw, vw} (z, \bar{z}) = 0, \quad \forall R \in V \otimes W, \quad (\text{E.24})$$

where

$$M_{RS} = \frac{N_{RS}}{d_R} = \frac{d_S}{|G|^2} \sum_{g_1, g_2 \in G} \chi_R(g_1)^* \chi_S(g_2)^* \chi_V(g_1 g_2) \chi_W(g_1 g_2^{-1}). \quad (\text{E.25})$$

We can take advantage of the fact that characters are constant over conjugacy classes to make the above formula more computationally approachable with GAP:

$$M_{RS} = \frac{d_S}{|G|^2} \sum_{[i_1], [i_2] \in \text{c.c.}} \chi_R([i_1])^* \chi_S([i_2])^* \left(\sum_{g_1 \in i_1, g_2 \in i_2} \chi_V(g_1 g_2) \chi_W(g_1 g_2^{-1}) \right), \quad (\text{E.26})$$

where i_1, i_2 runs over the conjugacy classes of the group.

Computing the matrix M_{RS} in this way, upon changing to the basis of convolved conformal blocks, we have checked the sum rules of eq. (E.24) for the correlators $\langle \phi \phi \phi \phi \rangle$, $\langle XXXX \rangle$, $\langle Z_3 Z_3 Z_3 Z_3 \rangle$, $\langle \phi X \phi X \rangle$, $\langle \phi Z_3 \phi Z_3 \rangle$ match the ones previously obtained by writing the explicit tensor structures.

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