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Black Hole Surface Gravity in Doubly Special Relativity Geometries

José Javier Relancio [1,2,3,*] and Stefano Liberati [4,5,6]

1 Dipartimento di Fisica "Ettore Pancini", Università di Napoli Federico II, 80125 Naples, Italy
2 INFN, Sezione di Napoli, 80125 Naples, Italy
3 Departamento de Física Teórica and Centro de Astropartículas y Física de Altas Energías (CAPA), Universidad de Zaragoza, 50009 Zaragoza, Spain
4 SISSA, International School for Advanced Studies, Via Bonomea 265, 34136 Trieste, Italy; liberati@sissa.it
5 IFPU—Institute for Fundamental Physics of the Universe, Via Beirut 2, 34014 Trieste, Italy
6 INFN Sezione di Trieste, Via Valerio 2, 34127 Trieste, Italy
* Correspondence: relancio@unizar.es

Abstract: In a quantum gravity theory, spacetime at mesoscopic scales can acquire a novel structure very different from the classical concept of general relativity. A way to effectively characterize the quantum nature of spacetime is through a momentum dependent space-time metric. There is a vast literature showing that this geometry is related to relativistic deformed kinematics, which is precisely a way to capture residual effects of a quantum gravity theory. In this work, we study the notion of surface gravity in a momentum dependent Schwarzschild black hole geometry. We show that using the two main notions of surface gravity in general relativity we obtain a momentum independent result. However, there are several definitions of surface gravity, all of them equivalent in general relativity when there is a Killing horizon. We show that in our scheme, despite the persistence of a Killing horizon, these alternative notions only agree in a very particular momentum basis, obtained in a previous work, so further supporting its physical relevance.

Keywords: quantum gravity phenomenology; relativistic deformed kinematics; Hamilton geometry; surface gravity; black holes

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1. Introduction

It is common lore that the most daunting challenge of theoretical physics is nowadays the unification of General Relativity (GR) and Quantum Field Theory (QFT), or equivalently, the formulation of a Quantum Gravity Theory (QGT). Indeed, while GR and QFT had a stunning success in describing the observed natural phenomena, they also showed fundamental incompatibilities, mostly stemming from the role that spacetime plays in them (a dynamical variable in GR, a static frame in QFT).

Such apparent incompatibility is paradigmatically illustrated by the so-called information loss problem associated to quantum black hole evaporation [1]. On the one hand, accepting the standard scenario dictated by GR seems in fact to imply a non-unitary evolution of quantum states, so violating a basic tenet of QFT. On the other hand, avoiding this loss of unitarity seems to require a radical departure from the equivalence principle that would predict no radical departure from standard physics at the event horizon of the black hole (see e.g., the so-called firewall paradigm proposed in [2–7] for quantum entropy and information evaporation from black holes).

In order to solve these inconsistencies between general relativity and quantum field theory, several QGT proposals have been advanced and developed in the past decades. Examples of these attempts are string theory [8–10], loop quantum gravity [11,12], causal dynamical triangulations [13] or causal set theory [14–16]. In most of these theories, a minimum length appears [17–19], which is normally associated with the Planck length $\ell_P \sim 1.6 \times 10^{-33}$ cm. It is believed that this minimum length could mark somehow the transition to a "quantum" spacetime which replaces our concept of "classical" spacetime. Unfortunately, the aforementioned theories are not yet fully satisfactory in the sense that they do not have yet well defined testable predictions which might serve us as a guidance in building a definitive theory of quantum gravity.

However, a complementary approach towards the realization of a quantum gravity theory can be represented by a bottom up strategy where the possible scenarios of a low-energy (sub-Planck scale) limit of quantum gravity are considered and put to the observational/experimental tests. In particular, it is expected that the transition from a full quantum and discrete spacetime to the standard classical continuum one, will not happen abruptly but will give rise to a mesoscopic regime where a continuum spacetime is endowed with different local symmetries due to remnant structure inherited from the super-Planckian regime. Deviation from standard Lorentz invariance is the most investigated scenario (but not the only one, see e.g., [20]). In this respect there are two main scenarios: one can consider that for high energies a Lorentz invariance violation (LIV) [21,22] can arise, or that the symmetry is deformed, leading to the theories known as deformed special relativity (DSR) [23].

LIV scenarios modify the kinematics of special relativity (SR) with the introduction of a preferred frame associated to some extra geometrical structure such as a fixed norm vector field. Such framework allows then to write for elementary particles a modified, no more Lorentz invariant, dispersion relation. Generally, new terms proportional to the inverse of a high-energy scale (normally considered to be the Planck scale) are added to the usual quadratic expression of SR.

In DSR theories the relativity principle is instead preserved, albeit at the cost of introducing a non-linear realization of the Lorentz group which allows for an invariant (observer independent) energy scale, leading to a relativistic deformed kinematics. Also in this case, this quantum gravity scale can be associated to a deformed dispersion relation, although in this framework this is not fully capturing the new physics. Indeed, one can even choose a special basis in momentum space, the so-called "classical basis" of κ-Poincaré [24], where the usual dispersion relation of SR is recovered. However, even in this case one gets that the deformed symmetry requires a deformed composition law for the momenta. This implies that the total momentum of a system of two (or more) particles is not derivable as the trivial sum of the initial momenta as in SR (it involves instead additional terms depending on both momenta and on the high-energy scale).

Within the above scenario a natural question concerns the possibility for then to admit a geometrical description, and it was soon understood that a momentum dependence of spacetime would naturally arise. Indeed, in [25] it was rigorously shown that all the ingredients of a relativistic deformed kinematics can be obtained from a maximally symmetric momentum space. In particular, κ-Poincaré kinematics [26–29], as well as other very known examples, such as Snyder [30] and hybrids models [31], can be obtained identifying the isometries (translations and Lorentz isometries) and the squared distance of the metric with the deformed composition law, deformed Lorentz transformations and deformed dispersion relation, respectively (the last two facts were previously contemplated in Refs. [32,33]). In [34] the proposal of [25] was generalized so allowing the metric to describe a curved spacetime, leading to a metric in the cotangent bundle depending on all the phase-space variables. This is a generalization of previous works in the literature, the so-called generalized Hamilton spaces, in which a metric that depends on the velocities (Finsler geometries) [35–37] and momenta (Hamilton geometries) [38–40] were regarded[1].

As discussed in [23], DSR theories have a long standing history in the literature since their emergence from quantum deformations of the Poincaré group [26–29] whose physical interpretation is nicely summarized in [45]. In [46] it was also recognized that such a framework could be rigorously derived from a top down approach in $2+1$ quantum gravity and soon after it was suggested that DSR could be the outcome of an energy (rainbow) spacetime [47], showing therefore a clear connection between a momentum dependent spacetime and quantum gravity. Moreover, a quantum spacetime it is often described by noncommutative geometry, and a clear connection between the above discussed geometrical setup and such space-time noncommutativity can be rigorously established [48–50]. Finally, a momentum dependent metric naturally arise also in other approaches to quantum gravity, such as asymptotically safe gravity [51]. In conclusion, while a definitive quantum gravity theory able to make observational predictions it is not yet available, it is still fair to say that a momentum dependent geometry appears to be a plausible effective description of spacetime at least in some mesoscopic regime between the full quantum gravitational one and the classical/continuum one of GR.

As we already mentioned, black holes are the natural test bench for exploring such transition region between classical and quantum regimes. The study of black holes in LIV frameworks was considered in [52–55]. In these works it was shown that particles with different energies see different horizons. Moreover, the fact that generically there is not a common Killing horizon in LIV theories is problematic as it seems that black holes could violate the second law of thermodynamics already at the classical level as well as via particle dependent Hawking radiation (see e.g., [52]), albeit the dynamical realization of such violations might be precluded (see e.g., [56]). The UV completion of LIV theories could remedy this problem by introducing further geometrical structure to the black holes (the so-called Universal Horizon [54]) which might fix a universal temperature and restore black hole thermodynamics [57].

In the context of rainbow geometries (not directly linked to DSR), the study of the Hawking radiation of Schwarzschild black holes in the presence of momentum corrections of the black hole metric, and then on the dispersion relation, was studied in [58,59]. In these papers it was considered that the energy of the modified dispersion relation was the mass of the black hole. In another vein, in [60] the momentum dependency arises from the energy of the particle emitted by the Hawking radiation. Moreover, the energy scale of the modification of Hawking radiation was taken to be the inverse of the Schwarzschild radius in [61–69]. In all these scenarios, it was shown that the new physics could lead to a remnant mass, i.e., prevent the black hole from fully evaporating. Noticeably, in [70] it was also studied the Unruh effect, obtaining a similar modification to that found in the previous papers.

In previous works these authors have advanced a proposal for a geometrical description of DSR in curved spacetimes via cotangent bundle geometries [34,71,72]. Black holes were specifically considered in [34] where a common horizon for all particles, independently of their energy, was shown to exist, hence providing a strong hint that, with the chosen framework, the universal character and salient features of these gravitational objects might be preserved. Among such crucial features, a particular importance is associated to the black hole thermodynamics and its associated implications.

In this sense, a very relevant quantity appears to be the horizon surface gravity which describes the forces near the horizon in the reference frame of a distant observer and in GR sets the scale of the black hole temperature measured at infinity via a QFT in curved spacetimes derivation [73,74]. In DSR such a link is not as straightforward, given that so far no rigorous extension of QFT to the DSR framework has been carried on. Such a dynamical theory, going beyond the usual relativistic QFT, would have to take into account a relativistic deformed kinematics, so that while the usual QFT has at its base the Poincaré symmetry, a DSR QFT would have to rest on a deformed symmetry group characterizing a relativistic deformed kinematics. While there are some works on this topic [75–79], such a theory is far from being realized. Consequently, a direct connection between surface gravity and black hole temperature within DSR scenarios is so far at best conjectural. Nonetheless, proving that a universal, momentum independent, notion of surface gravity can be still provided in the context of cotangent bundle geometries implementing the DSR scenario, would be an important milestone on the way to prove such a conjecture.

For this reason, we shall focus here on the notion of surface gravity in the geometrical interpretation of DSR advanced in Refs. [34,71,72,80]. While, from both a geometrical and algebraical point of view, different choices of the kinematics of κ-Poincaré (different choices of coordinates in a de Sitter momentum space or different bases in Hopf algebras [81]) represent the same relativistic deformed kinematics (with the same properties, such as the associativity of the composition law and the relativity principle), there is an ambiguity about what are the momentum variables associated to physical measurements. The fact that different bases could represent different physics was deeply considered in the literature [82]. In [80] we showed that only Lorentz covariant metrics are allowed in our geometrical scheme of lifting the deformed symmetries to a curved spacetime, and in [71] we proposed a way to select a "physical" basis by imposing the conservation of the Einstein tensor. Here, we will see that only for this particular basis all the definitions of surface gravity, which are equivalent in GR for Killing horizons, are also coinciding in the proposed framework for the geometrical description of DSR.

The structure of the paper is as follows. We start by summarizing the concepts of the cotangent bundle geometry we use in the following in Section 2, where we also briefly discuss our main results of previous works. All the notions of surface gravity coincide in GR if Killing equation is satisfied. However, we find that only one particular momentum basis is able to do so in Section 3. In Section 4 we compute the two main notions of surface gravity of GR, the peeling off and inaffinity of null geodesics, showing that, in any basis allowed in our scheme, are always momentum independent, obtaining then the same result of GR. In Section 5 we discuss that, despite having a Killing horizon, different notions of surface gravity considered in GR lead to different (momentum dependent) results. The only way in which this can be avoided is by considering a particular momentum basis obtained in Section 3. We check that for this preferred basis several notions agree in Section 6. Therefore, we are able to propose the Hawking temperature to be linked to the surface gravity through the same GR expression, avoiding the necessity of a DSR QFT. Finally, we end with the conclusions in Section 7.

2. Cotangent Bundle in a Nutshell

In this section we review the main geometrical ingredients in the cotangent bundle approach that we shall use in the following. Also we shall recall the main results from our previous papers about how to consider a relativistic deformed kinematics in a curved space-time background.

2.1. Main Properties of the Geometry in the Cotangent Bundle

In [83] a line element in the cotangent bundle is defined as

$$\mathcal{G} \,=\, g_{\mu\nu}(x,k)dx^\mu dx^\nu + g^{\mu\nu}(x,k)\delta k_\mu \delta k_\nu \,, \tag{1}$$

where

$$\delta k_\mu \,=\, dk_\mu - N_{\nu\mu}(x,k)\, dx^\nu \,, \tag{2}$$

being $N_{\nu\mu}(x,k)$ the so-called nonlinear connection coefficients.

In [83] it is shown that a horizontal path in the cotangent bundle is determined by the geodesic motion in spacetime

$$\frac{d^2x^\mu}{d\tau^2} + H^\mu{}_{\nu\sigma}(x,k)\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} \,=\, 0\,, \tag{3}$$

and by the change of momentum obtained from

$$\frac{\delta k_\lambda}{d\tau} \,=\, \frac{dk_\lambda}{d\tau} - N_{\sigma\lambda}(x,k)\frac{dx^\sigma}{d\tau} \,=\, 0\,, \tag{4}$$

where

$$H^{\rho}{}_{\mu\nu}(x,k) = \frac{1}{2}g^{\rho\sigma}(x,k)\left(\frac{\delta g_{\sigma\nu}(x,k)}{\delta x^{\mu}} + \frac{\delta g_{\sigma\mu}(x,k)}{\delta x^{\nu}} - \frac{\delta g_{\mu\nu}(x,k)}{\delta x^{\sigma}}\right), \tag{5}$$

is the affine connection of spacetime, and

$$\frac{\delta}{\delta x^{\mu}} \doteq \frac{\partial}{\partial x^{\mu}} + N_{\nu\mu}(x,k)\frac{\partial}{\partial k_{\nu}}. \tag{6}$$

Here, τ plays the role of the proper time or the affine parameter depending if one is considering a massive or a massless particle respectively.

The choice of the nonlinear connection coefficients is not unique but, as it is shown in [83], there is one and only one choice of nonlinear connection coefficients that leads to a space-time affine connection which is metric compatible and torsion free. In GR, the coefficients of the nonlinear connection are given by

$$N_{\mu\nu}(x,k) = k_{\rho}\Gamma^{\rho}{}_{\mu\nu}(x), \tag{7}$$

where $\Gamma^{\rho}_{\mu\nu}(x)$ is the affine connection. Then, when the metric does not depend on the space-time coordinates, these coefficients vanish.

In [83] it was defined the covariant derivatives in space-time

$$\begin{aligned} T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s;\mu}(x,k) = \frac{\delta T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}(x,k)}{\delta x^{\mu}} + T^{\lambda\alpha_2\ldots\alpha_r}_{\beta_1\ldots\beta_s}(x,k)H^{\alpha_1}{}_{\lambda\mu}(x,k) + \cdots + T^{\alpha_1\ldots\lambda}_{\beta_1\ldots\beta_s}(x,k)H^{\alpha_r}{}_{\lambda\mu}(x,k) \\ - T^{\alpha_1\ldots\alpha_r}_{\lambda\beta_2\ldots\beta_s}(x,k)H^{\lambda}{}_{\beta_1\mu}(x,k) - \cdots - T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\lambda}(x,k)H^{\lambda}{}_{\beta_s\mu}(x,k). \end{aligned} \tag{8}$$

Also, it is shown that given a metric, there is always a symmetric non-linear connection leading to the affine connections in spacetime making that the covariant derivative of the metric vanishes:

$$g_{\mu\nu;\rho}(x,k) = 0. \tag{9}$$

In order to study the properties of the horizon, we need to know how the Lie derivative is deformed in this context. In [34,38] the modified Killing equation for a metric in the cotangent bundle was derived

$$\frac{\partial g_{\mu\nu}(x,k)}{\partial x^{\alpha}}\chi^{\alpha} - \frac{\partial g_{\mu\nu}(x,k)}{\partial k_{\alpha}}\frac{\partial \chi^{\gamma}}{\partial x^{\alpha}}k_{\gamma} + g_{\alpha\nu}(x,k)\frac{\partial \chi^{\alpha}}{\partial x^{\mu}} + g_{\alpha\mu}(x,k)\frac{\partial \chi^{\alpha}}{\partial x^{\nu}} = 0, \tag{10}$$

where $\chi^{\alpha} = \chi^{\alpha}(x)$ is momentum independent. In GR, where the metric does not depend on the momentum, the previous condition reduces as expected to the usual Killing equation

$$\chi_{\mu\,;\nu} + \chi_{\nu\,;\mu} = 0. \tag{11}$$

We can wonder about the generality of this assumption, i.e., the possibility that the Killing vector could depend on the momentum, as considered in [36]. In [71] we showed that within the chosen framework the line element (and hence also the metric) is invariant only under space-time diffeomorphisms, not under a change of coordinates in momentum space. This means that all tensors and properties of our geometrical construction are not equivalent when regarding such kind of transformation. Indeed, this is the reason why one is lead to seek for a privileged basis in momentum space. Allowing for a possible momentum dependence of the Killing vector, in an attempt to characterize isometries in the full cotangent bundle, would end up in results which would be not invariant under momentum basis changes. We shall instead study isometries in spacetimes and their associated properties such as Killing horizons in order to identify a privileged momentum basis.

2.2. Relativistic Deformed Kinematics in Curves Spacetimes

We summarize here our previous results about supplementing a relativistic deformed kinematics within a curved space-time.

The relativistic deformed kinematics of DSR are usually obtained from Hopf algebras [84], being the κ-Poincaré kinematics [85] the most studied example of this kind of construction. This kinematics has been understood from a geometrical point of view in [25]. Given a de Sitter momentum metric $\bar{g}$, translations can be used to define the associative deformed composition law, the Lorentz isometries lead to the Lorentz transformations, and the (squared of the) distance in momentum space is identified with the deformed Casimir.

In [34,80], we extended [25] in order to consider in the same framework a relativistic deformed kinematics and a curved spacetime. For that aim, it is mandatory to consider the cotangent bundle geometry above discussed. The metric tensor $g_{\mu\nu}(x,k)$ in the cotangent bundle depending on space-time coordinates was constructed with the tetrad of spacetime and the original metric in momentum space, $\bar{g}$, explicitly

$$g_{\mu\nu}(x,k) = e^{\alpha}_{\mu}(x)\bar{g}_{\alpha\beta}(\bar{k})e^{\beta}_{\nu}(x), \tag{12}$$

where $\bar{k}_\alpha = \bar{e}^{\nu}_{\alpha}(x)k_\nu$, and $\bar{e}$ denotes the inverse of the tetrad of spacetime.

In [71] it was also proved that this Hamiltonian can be identified with the square of the minimal geometric distance of a momentum k from the origin of momentum space, measured by the momentum space length measure induced by the metric, relating the Casimir and the metric in the following way [34]

$$\mathcal{C}(x,k) = \frac{1}{4}\frac{\partial\mathcal{C}(x,k)}{\partial k_\mu}g_{\mu\nu}(x,k)\frac{\partial\mathcal{C}(x,k)}{\partial k_\nu}. \tag{13}$$

A very important relation that the Casimir satisfies is that its delta derivative (6) is zero, i.e.,

$$\frac{\delta\mathcal{C}(x,k)}{\delta x^\mu} = 0. \tag{14}$$

This is a necessary condition derived from the fact the Hamilton equations of motions are horizontal curves [71].

In [80] we showed that the most general form of the metric, in which the construction of a relativistic deformed kinematics in a curved space-time background is allowed, is a momentum basis whose Lorentz isometries are linear transformations in momenta, i.e., a metric of the form

$$\bar{g}_{\mu\nu}(k) = \eta_{\mu\nu}f_1(k^2) + \frac{k_\mu k_\nu}{\Lambda^2}f_2(k^2), \tag{15}$$

where Λ is the high-energy scale parametrizing the momentum deformation of the metric and kinematics. From Equation (12) one obtains the following metric in the cotangent bundle when a curvature in spacetime is present

$$g_{\alpha\beta}(x,k) = a_{\alpha\beta}(x)f_1(\bar{k}^2) + \frac{k_\alpha k_\beta}{\Lambda^2}f_2(\bar{k}^2), \tag{16}$$

where $a_{\mu\nu}(x) = e^{\alpha}_{\mu}(x)\eta_{\alpha\beta}e^{\beta}_{\nu}(x)$ is the GR metric. Therefore, one can use the definition of the space-time affine connection (5) to show that it is momentum independent, ending up being the same affine connection $\Gamma^{\rho}_{\mu\nu}(x)$ of GR [80].

Since we want the momentum metric (15) to be a maximally symmetric momentum space, allowing us to define a relativistic deformed kinematics, a relationship between the functions f_1 and f_2 must hold. In particular, we are interested in the special case of de Sitter space, where κ-Poincaré kinematics can be described. In [71], the conservation of the Einstein tensor defined in [83] it was shown to lead to a conformally flat metric. Then, taking $f_2 = 0$ and imposing a momentum de Sitter space, one obtains from Equation (16)

$$g_{\mu\nu}(x,k) \;=\; a_{\mu\nu}(x)\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)^2 . \tag{17}$$

3. Killing Equation in the Cotangent Bundle

We shall now see that the above result concerning the cotangent bundle metric can be derived also on the base of simple requirements concerning the Killing Equation (10).

3.1. Killing Equation Revisited

In this subsection we consider a Schwarzschild black hole metric in some static (not time dependent) coordinates [74]. The Killing vector is $\partial/\partial t$, as can be easily derived from Equation (10) when considering that the metric is momentum independent [34], since the GR metric is independent of time[2]. Then, as the metric in the cotangent bundle is constructed from the GR metric, this will be also independent of time, and therefore, Equation (10) is automatically satisfied for any metric of the form of (12), and in particular, for (16).

In the GR case, Equation (11) can be written as

$$\chi^{\rho}{}_{;\nu}a_{\rho\mu}(x)+\chi^{\rho}{}_{;\mu}a_{\rho\nu}(x) \;=\; 0 . \tag{18}$$

Let us now require instead that Equation (11) holds also for a generic cotangent bundle metric (of the form allowing for the connection with the consistent lift of a relativistic deformed kinematics to curved spacetimes of Equation (16)), i.e., that

$$\chi^{\rho}{}_{;\nu}g_{\rho\mu}(x,k)+\chi^{\rho}{}_{;\mu}g_{\rho\nu}(x,k) \;=\; 0 , \tag{19}$$

is satisfied. Following our previous discussion, both Equations (18) and (19) hold simultaneously. This will assure that all the surface gravity notions considered in the literature agree [86].

Let us now now make use of the explicit form of the metric (16) to write Equation (18) as

$$\begin{aligned}\chi^{\rho}{}_{;\nu}\left(f_1(\bar{k}^2)a_{\rho\mu}(x)+\frac{k_\rho k_\mu}{\Lambda^2}f_2(\bar{k}^2)\right)+\chi^{\rho}{}_{;\mu}\left(f_1(\bar{k}^2)a_{\rho\nu}(x)+\frac{k_\rho k_\nu}{\Lambda^2}f_2(\bar{k}^2)\right) \;=\\ \chi^{\rho}{}_{;\nu}\frac{k_\rho k_\mu}{\Lambda^2}f_2(\bar{k}^2)+\chi^{\rho}{}_{;\mu}\frac{k_\rho k_\nu}{\Lambda^2}f_2(\bar{k}^2) \;=0 ,\end{aligned} \tag{20}$$

where we have used Equation (18) in order to cancel the first two terms, since the function f_1 is a common factor multiplying the GR Killing equation. There are two possible ways in which the previous equation can be satisfied: either $f_2 = 0$ or

$$\chi^{\rho}{}_{;\nu}k_\rho k_\mu+\chi^{\rho}{}_{;\mu}k_\rho k_\nu \;=\; 0 . \tag{21}$$

Taking into account that the Killing vector χ^μ is momentum independent, then its covariant derivative will also be (since as discussed above, the affine connection is the same one of GR). Therefore, we can derive Equation (21) with respect to the momentum two times, obtaining

$$\chi^{\alpha}{}_{;\nu}\delta^{\beta}_{\mu}+\chi^{\beta}{}_{;\nu}\delta^{\alpha}_{\mu}+\chi^{\alpha}{}_{;\mu}\delta^{\beta}_{\nu}+\chi^{\beta}{}_{;\mu}\delta^{\alpha}_{\nu} \;=\; 0 . \tag{22}$$

For the considered Killing vector $\partial/\partial t$ this equation does not hold, which implies that the only way in which Equation (19) will be satisfied is by considering $f_2 = 0$. Imposing this condition on the metric (16) and asking it to be a de Sitter space (so we are able to define a relativistic deformed kinematics, as explained in the introduction) we fix the function f_1, which is precisely the one reproducing Equation (17).

3.2. Killing Equation in a Conformally Flat Metric

We shall now prove that in the momentum coordinates for which the metric takes the form of Equation (17), Equation (19) will be satisfied for every Killing vector. Let us assume that for the metric (17), Equation (11) holds, which is tantamount to saying

$$\frac{\partial a_{\mu\nu}(x)}{\partial x^\alpha}\chi^\alpha + a_{\alpha\nu}(x)\frac{\partial \chi^\alpha}{\partial x^\mu} + a_{\alpha\mu}(x)\frac{\partial \chi^\alpha}{\partial x^\nu} = 0\,. \tag{23}$$

In order to prove our ansatz, we start by noticing that for the metric (17) there is a simple relation between the δ derivative and the ∂ one. As it was shown in [71], the δ derivative of a function of $\bar{k}^2$ is zero if the affine connection is the one of GR, i.e.,

$$\frac{\delta f(\bar{k}^2)}{\delta x^\mu} = 0\,, \qquad \text{if} \qquad H^\gamma{}_{\rho\alpha}(x,k) = \Gamma^\gamma{}_{\rho\alpha}(x)\,. \tag{24}$$

This is due to the fact that the δ derivative of the Casimir vanishes, as stated in Equation (14), which also implies that the δ derivative of any function of the Casimir, that is, of any function of $\bar{k}^2$, is zero. This implies that

$$\frac{\delta g_{\mu\nu}(x,k)}{\delta x^\rho} = \left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)^2 \frac{\partial a_{\mu\nu}(x)}{\partial x^\rho}\,. \tag{25}$$

Therefore, the first term of Equation (10) can be written as the sum of two terms

$$\frac{\partial g_{\mu\nu}(x,k)}{\partial x^\alpha}\chi^\alpha = \left(\frac{\delta g_{\mu\nu}(x,k)}{\delta x^\alpha} - \frac{\partial g_{\mu\nu}(x,k)}{\partial k_\rho}N_{\rho\alpha}(x,k)\right)\chi^\alpha = \left(\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)^2\frac{\partial a_{\mu\nu}(x)}{\partial x^\alpha} - \frac{\partial g_{\mu\nu}(x,k)}{\partial k_\rho}N_{\rho\alpha}(x,k)\right)\chi^\alpha\,, \tag{26}$$

where in the last step we have used Equation (25).

We can now expand the second term of the previous equation, obtaining

$$\begin{aligned}
-\frac{\partial g_{\mu\nu}}{\partial k_\rho}N_{\rho\alpha}(x,k)\chi^\alpha &= -2a_{\mu\nu}(x)\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)\left(-\frac{1}{4\Lambda^2}\right)k_\sigma a^{\rho\sigma}(x)k_\gamma \Gamma^\gamma{}_{\rho\alpha}(x)\chi^\alpha \\
&= 2a_{\mu\nu}(x)\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)\left(\frac{1}{4\Lambda^2}\right)k_\sigma a^{\rho\sigma}(x)k_\gamma\frac{1}{2}a^{\gamma\delta}(x)\left(\frac{\partial a_{\delta\rho}(x)}{\partial x^\alpha}+\frac{\partial a_{\delta\alpha}(x)}{\partial x^\rho}-\frac{\partial a_{\alpha\rho}(x)}{\partial x^\delta}\right)\chi^\alpha \\
&= a_{\mu\nu}(x)\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)\left(\frac{1}{4\Lambda^2}\right)k_\sigma a^{\rho\sigma}(x)k_\gamma\frac{\partial a_{\delta\rho}(x)}{\partial x^\alpha}\chi^\alpha \\
&= -2a_{\mu\nu}(x)\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)\left(\frac{1}{4\Lambda^2}\right)k_\sigma a^{\rho\sigma}(x)\frac{\partial \chi^\gamma}{\partial x^\alpha}k_\gamma = \frac{\partial g_{\mu\nu}(x,k)}{\partial k_\alpha}\frac{\partial \chi^\gamma}{\partial x^\alpha}k_\gamma\,,
\end{aligned} \tag{27}$$

where in the first step we have used Equation (7), in the second one the definition of the affine connection in GR,

$$\Gamma^\rho{}_{\mu\nu}(x) = \frac{1}{2}a^{\rho\sigma}(x)\left(\frac{\partial a_{\sigma\nu}(x)}{\partial x^\mu}+\frac{\partial a_{\sigma\mu}(x)}{\partial x^\nu}-\frac{\partial a_{\mu\nu}(x)}{\partial x^\sigma}\right)\,, \tag{28}$$

in the third one the symmetry under the exchange of indexes $\delta \leftrightarrow \rho$, and in the forth one, Equation (23) and the same symmetry. This expression can be written as a momentum derivative of the cotangent bundle metric, as we did in the first step. Therefore, since this term is the same one of the second of Equation (10) with opposite sign, we can write Equation (10) as

$$\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)^2\left(\frac{\partial a_{\mu\nu}(x)}{\partial x^\alpha}\chi^\alpha + a_{\alpha\nu}(x)\frac{\partial \chi^\alpha}{\partial x^\mu} + a_{\alpha\mu}(x)\frac{\partial \chi^\alpha}{\partial x^\nu}\right) = 0\,, \tag{29}$$

which is automatically true if Equation (23) holds. Hence, with the particular choice of the momentum metric, Equation (17), the standard Killing equation Equation (11) is still satisfied.

4. Main Notions of Surface Gravity

In GR, there are different definitions for the surface gravity [86]. In this section, we compute the two main ones related respectively to the peeling off properties near the horizon and the inaffinity of null geodesics on the horizon.

4.1. Peeling off Properties of Null Geodesics

Due to the form of the metric (16), the Casimir defined as the squared of the distance in momentum space as in Equation (13) is a function of $\bar{k}^2$. Then, for massless particles the same relationship between energy and momentum of GR holds in this deformed scenario. This also means that photons in this scenario follow the same trajectories of GR, implying an existence of an universal horizon at $2M$, independently of the energy of the particle.[3]

As it was shown in [86], the surface gravity can be defined as the peeling off of null geodesics

$$\frac{d|r_1(t)-r_2(t)|}{dt} \approx 2\kappa_{\text{peeling}}(t)|r_1(t)-r_2(t)|\,, \tag{30}$$

where $r_1(t)$ and $r_2(t)$ are two null geodesics on the same side of the horizon and the normalization of κ_{peeling} is chosen so to coincide with $\kappa_{\text{inaffinity}}$ in the GR limit.

In order to compute it, we need to use some coordinates for which the final result does not diverge at the horizon. While there is no problem in GR, due to the momentum dependence of the metric (16) some coordinates are not well behaved at the horizon. This means that, for example, Schwarzschild coordinates used in [86] cannot be employed here. We will consider the Eddington-Finkelstein coordinates [87] in the following

$$a_{vv} = -\left(1-\frac{2M}{r}\right), \qquad a_{vr} = 1\,, \qquad a_{rr} = 0\,, \qquad a_{\theta\theta} = r^2\,, \qquad a_{\varphi\varphi} = r^2\sin^2\theta\,. \tag{31}$$

This makes that radial component of the momentum is related to the zero component as

$$k_\mu a^{\mu\nu}(x)k_\nu = 0 \qquad \Longrightarrow \qquad k_r = -\frac{2k_v}{1-2M/r}\,. \tag{32}$$

Therefore, from Equation (16) one finds

$$\frac{dr}{dt} = \frac{1}{2}\left(1-\frac{2M}{r}\right), \tag{33}$$

which is independent of the energy. This result is an obvious outcome from the fact that the Casimir is underformed for massless particles. Hence, the same result of GR is obtained from Equation (30)

$$\kappa_{\text{peeling}} = \frac{1}{4M}\,. \tag{34}$$

This differs from the result obtained in [34] for the momentum metric corresponding to the bicrossproduct basis of κ-Poincaré [25,88]. However, it is important to note that this basis cannot be consistently lifted to curved spacetimes from our proposal, as it was shown in [80].

4.2. Inaffinity of Null Geodesics

Also, in [86] it was shown that for a Killing vector χ^μ one finds

$$\chi^\nu \chi^\mu{}_{;\nu} = \kappa_{\text{inaffinity}}\chi^\mu\,. \tag{35}$$

Again, we consider the Eddington-Finkelstein coordinates. As commented before, it is easy to see from Equation (10) that, if the Killing vector of a metric in GR does not depend on the coordinates, it will be also constant in this framework, being in this case $\partial/\partial t$. Due to the fact that the space-time affine connection is the same one of GR, it is easy to obtain

$$\kappa_{\text{inaffinity}} = \frac{1}{4M}\,. \tag{36}$$

5. Killing Equation and Selection of Momentum Basis

In [86], several alternative definitions of surface gravity were discussed, showing that all of these coincide for Killing horizons. As explained previously, in our framework the same Killing vector of GR is present. However, the same framework implies a modification of the standard GR Killing equation Equation (11): the momentum dependent Equation (10). We shall show below that this reflects in an inequivalence of the other common definitions of surface gravity, which in general lead to a momentum dependent results different from what was found in the previous section. Nevertheless, Equation (11) indeed holds for the momentum basis of Equation (17) (which we stress was derived by completely different arguments in [71]). Therefore, different definitions of surface gravity will again coincide in these particular momentum coordinates. This disambiguation of the definition of surface gravity lends then further support to the physical relevance of this particular momentum basis.

6. Different Notions of Surface Gravity

We are going to check that other definitions of surface gravity coincides with the particular choice of the metric (17).

6.1. Null Normal Derivative

Another definition with respect to the previously discussed peeling and inaffinity ones is the null normal derivative evaluated in the horizon (see for example [86])

$$\left(\chi^{\nu}\chi_{\nu}\right)_{;\mu} = -2\kappa_{\text{normal}}\chi_{\mu}\,, \tag{37}$$

which is equivalent to

$$\chi^{\nu}\chi_{\nu;\mu} = -\kappa_{\text{normal}}\chi_{\mu}\,. \tag{38}$$

Using Equation (11) (as it holds for our metric (17)), one can rewrite the previous expression as

$$\chi^{\nu}\chi_{\mu;\nu} = \kappa_{\text{normal}}\chi_{\mu} \qquad \Longrightarrow \qquad \chi^{\nu}\chi^{\mu}{}_{;\nu} = \kappa_{\text{normal}}\chi^{\mu}\,, \tag{39}$$

from which we see that $\kappa_{\text{normal}} = \kappa_{\text{inaffinity}}$.

6.2. Generator

Another possible definition of the black hole surface gravity is the so-called κ-generator defined as [86]

$$\kappa^2_{\text{generator}} = -\frac{1}{2}\chi^{\mu}{}_{;\sigma}\chi^{\nu}{}_{;\lambda}g_{\mu\nu}(x,k)g^{\sigma\lambda}(x,k)\,. \tag{40}$$

As we have seen previously, the same Killing vector of GR is a Killing vector in this scheme. Then, the only momentum dependency arises from the metric. Due to the conformal form of the metric (17), from the previous equation one obtains

$$\kappa^2_{\text{generator}} = -\frac{1}{2}\chi^{\mu}{}_{;\sigma}\chi^{\nu}{}_{;\lambda}\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)^2 a_{\mu\nu}(x)\left(1-\frac{\bar{k}^2}{4\Lambda^2}\right)^{-2} a^{\sigma\lambda}(x) = -\frac{1}{2}\chi^{\mu}{}_{;\sigma}\chi^{\nu}{}_{;\lambda}a_{\mu\nu}(x)a^{\sigma\lambda}(x)\,. \tag{41}$$

Hence, since this definition is equivalent to the others in GR [74,87], it leads to the same value also in our scheme.

6.3. Wick Rotation

A different way to obtain the surface gravity in GR is by the Wick rotation (explained in Ch.6 of [89]). We firstly resume the case of GR using the Schwarzschild coordinates

$$a_{tt}(x) = -\left(1 - \frac{2M}{r}\right), \qquad a_{rr}(x) = \left(1 - \frac{2M}{r}\right)^{-1}, \quad a_{\theta\theta}(x) = r^2, \qquad a_{\phi\phi}(x) = r^2 \sin^2(\theta). \tag{42}$$

We consider the t-r line element

$$d\gamma^2 = -dt^2\left(1 - \frac{2M}{r}\right) + dr^2\left(1 - \frac{2M}{r}\right)^{-1}. \tag{43}$$

We start by making a Wick rotation obtaining

$$d\gamma^2 = dt_E^2\left(1 - \frac{2M}{r}\right) + dr^2\left(1 - \frac{2M}{r}\right)^{-1}. \tag{44}$$

We can define in the vicinity of the horizon the proper length distance from the horizon [89]

$$\rho = \int_{2M}^{r} \frac{dr}{\sqrt{1 - 2M/r}} \implies 1 - \frac{2M}{r} = 16M^2\rho^2. \tag{45}$$

Then, the Euclidean line element takes the following form

$$d\gamma^2 = \left(16M^2\rho^2 dt_E^2 + d\rho^2\right). \tag{46}$$

This line element will not be singular if the temporal coordinate behaves as an "angular coordinate" of the plane t-r. This implies that time must be periodical:

$$d\gamma^2 = \rho^2 \frac{dt_E}{\kappa^2}^2 + d\rho^2, \tag{47}$$

being $\kappa = 1/4M$.

It is obvious that the only way in which this procedure can be followed in our scheme is by using the conformally flat metric (17), obtaining then the same result of GR.

7. Conclusions

In this work we have studied different notions of surface gravity of a Schwarzschild black hole in a rainbow geometry in the DSR scenario. This study differs from previous works in the literature because here we have taken into account that, in order describe the relativistic deformed kinematics of DSR, the momentum metric must be a maximally symmetric momentum space. In this way, the relativistic deformed kinematics are encoded in the geometrical ingredients of the cotangent bundle metric.

As we have seen, both main notions of surface gravity lead to the same result of GR. However, the only way in which different definitions lead to the same momentum independent result is by selecting a conformally flat momentum metric, selecting a particular momentum basis of κ-Poincaré. This basis is the same one found in a previous work by imposing the conservation of the Einstein tensor. Therefore, all the notions of surface gravity discussed in [86] are equivalent for this metric, due to the fact that the same Killing equation of GR is also valid here.

The fact that all the notions are equivalent in this scheme, indicating that there is a clear way to define the surface gravity in these cotangent bundle geometries, could seem to imply that the Hawking temperature is universal, independent of the energy of the emitted particle, when computed from the GR formula $T = \kappa/2\pi$ (see however [90] for an example where κ and T are not trivially related). This result differs from the one obtained in LIV scenarios, where a different temperature for different particles is obtained. Therefore, while in LIV theories it could be possible a violation of the second law of thermodynamics, already at the classical level, we see that in the DSR geometrical framework proposed in this work this possible issue is avoided. This points out to the consistency of our model and DSR kinematics in curved spacetimes. Nonetheless, as discussed in the introduction, the formal derivation of the Hawking radiation requires a QFT in DSR, which at present is unknown.

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Notes

1 It is worth mentioning that a Finsler/momentum dependent spacetime can be introduced also as an alternative description of Lorentz breaking physics [41–44].

2 Similar argument can be done for different space-time coordinates, since Equation (10) is invariant under diffeomorphisms, as can be seen for its construction [34]

3 As we mentioned above, due to the form of the metric (16), the Casimir is a function of $\bar{k}^2$. This means that, on the one hand there is not any modification of the dispersion relation for massless particles, and on the other, the modification of massive particles is of the order of m^2/Λ^2, being m the mass of the particle, which is completely negligible. Therefore, in the ultraviolet regime in which particles escape from the horizon of the black hole, and then masses can be neglected, massive particles see the same horizon. This is a very important check of consistency since, as commented in the introduction, otherwise the black hole would be a perpetuum mobile [52].

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