Physics Area - Ph.D. course in Astroparticle Physics

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Renormalization and running couplings in higher derivative theories

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Abstract

The renormalization group has a crucial role in modern physics, however some of its features have not been completely understood yet. While its perturbative realization in 2-derivative theories in $d = 4$ spacetime has been widely studied, other classes of theories can still hide some subtleties.

Higher derivative theories, and in particular quadratic gravity, could furnish a UV completion to general relativity within the framework of quantum field theory. For this reason, a detailed study of the renormalization group of this class of theories is of great importance and is the main object of this thesis.

Higher derivative theories suffer from the Ostrogradskij instability at the classical level, which translates into ghost particles in the spectrum at the quantum level, with related problems with unitarity and negative norm states. In recent years many solutions to this pathology have been suggested in order to obtain a well-defined quantum theory and we review some of them.

Then, we study the nonperturbative renormalization group of a higher derivative shift-invariant scalar model. In the theory space, we find an interesting region where the renormalization group trajectories flow between the two free Gaussian fixed points corresponding respectively to the 2- and 4-derivative kinetic term.

From the perturbative point of view, the fourth power of transferred momentum in the propagator reduces the degree of UV divergence of Feynman diagrams, but at the same time it introduces new off-shell IR divergences. We notice that not all renormalization prescriptions are sensible to this type of infrared effects, potentially leading to running couplings that do not resum all the large logarithms of momenta in scattering amplitudes. We define a "physical" prescription using a momentum subtraction renormalization scheme and we apply it to various higher derivative theories.

In particular, we focus on the higher derivative scalar toy model already studied non-perturbatively and on some quantum field theories in curved spacetime. We find that shift invariance seems to protect the universality of one-loop beta functions from IR effects and that quadratic gravity, according to its physical running, has an asymptotically free sector without tachyonic particles, in contradiction with older results which predict asymptotic freedom only in the presence of a scalar tachyon.

Finally, we observe that the same type of IR effects can also emerge in 2-derivative theories in $d = 2$ spacetime. For this reason, we study the renormalization group of the $CP(1)$ non-linear sigma model (NLSM). We observe that in this case the symmetries of the theory seem to protect the running of couplings from IR effects, preserving one-loop universality.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following published works:

- D. Buccio and R. Percacci, "Renormalization group flows between Gaussian fixed points," JHEP 10 (2022), 113 [arXiv:2207.10596 [hep-th]]
- D. Buccio, J. F. Donoghue and R. Percacci, "Amplitudes and Renormalization Group Techniques: A Case Study," Phys.Rev.D 109 (2024) 4, 045008 [arXiv:2307.00055 [hep-th]]
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- D. Buccio, L. Parente and O. Zanusso, "Physical Running in Conformal Gravity and Higher Derivative Scalars," [2410.21475[hep-th]]

Contents

Chapter 1

Introduction

It is well known that the same physical system can be described in different ways at different scales. An ensemble of molecules can be treated as independent particles interacting with each others or, at a larger distance, as a continuous medium obeying laws of hydrodynamics. Quarks can be described as single particles whose interactions become weaker and weaker with the increase of energy, but below the Λ_{QCD} scale they become strongly interacting and form bound states as baryons and mesons. In particle physics there are many examples of light particles interacting with heavy ones with an effective description in terms of light particles only below the heavy mass threshold. This effective field theory contains all possible nonrenormalizable interactions.

The fundamental interactions between microscopic states manifest themselves via new effective interactions between low-energy degrees of freedom. Above and below the threshold, the scale dependence of the theory can be described by a continuous modification of its parameters (coupling constants, masses, wave function renormalization constants,...). Such a scale dependence is described by the renormalization group. This idea has proven to be one of the most revolutionary, powerful and effective of modern physics, reaching great successes in many areas, from quantum to classical physics, from solid state to high-energy physics, from fundamental particles to atoms, passing trough phase transitions. This flow between different scales is well defined going from the microscopic scale to the macroscopic one, but the same is not true in the inverse process. It is generally not possible to reconstruct the information about the microscopical system lost during the coarse graining process.

In high-energy physics one is often interested in guessing the UV completion of a known theory. As in the examples described above, this UV completion typically contains new degrees of freedom, that are unobservable below the energy scale of their masses. This new particle content is usually added by introducing new fields. An alternative, that we extensively discuss in this thesis, consists in writing higher derivative kinetic terms for the existing ones. This kind of UV completion gained some popularity in gravitational theories, since Einstein general relativity is nonrenormalizable as a quantum field theory and including operators quadratic in curvatures in the action makes gravity renormalizable by introducing a higher derivative kinetic term for metric fluctuations.

On the other hand, higher derivative theories suffer from problems related to the fact that the new degrees of freedom are actually ghost particles. Moreover, they have peculiar features from the point of view of the renormalization group, in particular in how the running of couplings (namely their dependence on the energy scale) emerges from quantum loop corrections. This particular behaviour will be the main object of this thesis.

Going more into details, in Chapter 2 we review how the renormalization group has been defined in different ways in the context of high energy physics. We compare momentum subtraction and minimal subtraction schemes, as defined in perturbation theory, with the cutoff running of the Wilsonian renormalization group and of the functional renormalization group. We consider couplings that run either as logarithms or powers of the energy scale and we describe how infrared divergences are treated in theories with 2-derivative kinetic terms in $d = 4$ spacetime dimensions. We observe that the same arguments cannot be applied to higher derivative theories or standard 2-derivative theories in $d = 2$.

In Chapter 3, we briefly describe the intrinsic pathologies that affect higher derivative theories. At the classical level, if the Lagrangian depends on the second or higher rank time derivative of its argument, the associated Hamiltonian turns out to be unbounded from below. This effect is known as the Ostrograskij instability. At the quantum level, it produces either a quantum Hamiltonian with infinite negative energy states, or states with negative norms, implying problems with unitarity and the notion of transition probability. In the second part of the chapter we review some of the most popular approaches to solve these problems without losing the good features of higher derivative theories with respect to renormalization.

In Chapter 4, we start investigating the features of the renormalization group flow of higher derivative theories. From the nonperturbative point of view, the presence of more than one free Gaussian fixed points opens the possibility of a nontrivial flow between two different Gaussian theories, giving rise to a theory free both in the IR and in the UV. As a toy model, we study a higher derivative shift invariant scalar model. The theory shows an interesting flow in the coupling's space, linking the two Gaussian fixed points and a third interacting one.

In Chapter 5, we study the perturbative renormalization group of the same scalar theory via its two-point function and the $2 \to 2$ scattering amplitude. We match the higher derivative theory with the low-energy effective field theory where the quartic kinetic term is suppressed. We compute the beta functions and the anomalous dimension of the field with different renormalization prescriptions and compare them with the one-loop approximation of the exact renormalization group equation. We observe that, since the tadpole diagram is logarithmically divergent, some unexpected discrepancies between the momentum subtraction scheme and other prescriptions emerge. We observe something similar happening in the higher derivative ϕ^4 theory. This motivates us to study the problem in a more systematic way. It turns out that off-shell IR divergences, absent in two derivative theories in $d = 4$, can contribute to the running of couplings in scattering amplitudes, but not all renormalization prescriptions are sensitive to them.

In Chapter 6, we study the same phenomenon in higher derivative theories in curved spacetime. In this case, the beta functions are extracted from the form factors in the one-loop quantum effective action. We consider a scalar field theory non-minimally coupled to the background metric, quadratic gravity and Weyl conformal gravity. In the scalar field, we observe a discrepancy between different prescriptions that disappears when the theory is shift invariant. Quadratic gravity surprisingly shows asymptotic freedom also without tachyonic particles, in contradiction with old beta functions computed using only UV contributions. In conformal gravity the coefficient of the new beta function is slightly different from the one already present in literature, however no qualitative changes are produced in this case.

2-derivative theories in $d = 2$ spacetime dimensions share some features with 4-derivative theories in $d = 4$. In particular, off-shell infrared logarithmic divergences could be generated. In Chapter 7, we consider the well-known $CP(1)$ NLSM. We explicitly compute some of its scattering amplitudes and extract from them the beta functions of the theory using the same procedure introduced for higher derivative theories. We observe that the running of coupling is universal in this case, even if it is generated by different diagrams in different schemes. This happens because, unlike in higher derivative gravity, all IR effects cancel out in the quantum corrections, removing the discrepancy between various renormalization prescriptions.

Finally, in Chapter 8, we resume the main results and we discuss their implications and possible further developments.

Chapter 2

Many faces of the renormalization group

In theoretical physics, there exist different definitions of Renormlaization Group (RG) and running couplings. The Renormalization Group appeared for the first time in the context of high-energy physics, where the problem of UV divergences was preventing physicists from obtaining experimental predictions from Quantum Field Theory computations. Another possible approach is the formalism introduced by Wilson [1] in the 1970s in the context of statistical physics. Here an artificial cutoff is introduced in the energy spectrum of the theory as a way to reproduce the finite resolution of an experiment testing physics at a given energy scale. We will see that these two conceptions are strictly related, however, when we move to spacetime dimensionalities different from 4 or consider higher derivative theories, the connection becomes less trivial, and the two definitions can even give different results and predictions.

2.1 The renormalization group in scattering amplitudes

In Quantum Field Theory, the standard procedure of regularization and renormalization permits us to hold under control infinities coming out of perturbative computations if the theory taken in account is renormalizable. By means of a redefinition of a finite set of parameters of the bare Lagrangian in such a way as to reproduce the experimental results of some measurement processes, one can make predictions on the expectation value of all n-point functions of the theory. Unfortunately, this process introduces an arbitrariness in the choice of the experimental tests used to fix the parameters of the theory. Moreover, logarithms of the ratio between the energy scale of the correlation function that is being computed and the energy of the experiments that were used to fix the renormalized couplings start to plague the results. These logarithms in the Green functions can grow arbitrarily large with energy up to the failure of perturbativity of the loop expansion, even if the renormalized couplings are small. In this section, we will review how the arbitrariness intrinsic in the process of renormalization can be used to tame these large logs. We will mainly follow the discussions in [2, 3].

A typical example used to explain the properties of the renormalization group is the scalar $\lambda\phi^4$ theory

$$\mathcal{L} = -\frac{1}{2} Z_B \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_B^2 \phi - \frac{1}{4!} \lambda_B \phi^4 \ , \tag{2.1.1}$$

where the subscript B stands for bare quantities. At one loop, the quantum correction to the two point function computed with a cutoff regularization is

$$\Gamma^{(2)} = Z_B p^2 + m_B^2 + \frac{\lambda}{32\pi^2}\left[\Lambda^2 + m_B^2 \log\left(\frac{m_B^2}{m_B^2+\Lambda^2}\right)\right] \tag{2.1.2}$$

The common "physical" renormalization prescription consists in fixing the field normalization in such a way that the pole in the propagator is at the measured rest mass of the particle m and the related residue has value one. These two conditions,

$$\Gamma^{(2)}|_{p^2=-m^2} = 0, \qquad \partial_p^2 \Gamma^{(2)}|_{p^2=-m^2} = 1 \ , \tag{2.1.3}$$

can be easily fulfilled by introducing the counterterm $\delta m^2 = m_B^2 - m^2$ and setting simultaneously

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 + m_B^2 \log\left(\frac{m_B^2}{m_B^2+\Lambda^2}\right)\right] \tag{2.1.4}$$

and $Z_B = 1$. With such a choice, the one-loop contribution is completely reabsorbed in the bare mass and canceled by the mass counterterm δm. This is possible because the related Feynman integral is independent of the particle momentum p^μ and therefore the one-loop correction is just a constant shift.

However, things are trickier when considering the four-point amplitude for the $2 \to 2$ scattering process. In this case, the one-loop corrected amplitude explicitly depends on the momenta of ingoing and outgoing particles p_1^μ, p_2^μ, p_3^μ and p_4^μ via the Mandel'stam variables

$$s = -(p_1+p_2)^2 \ , \tag{2.1.5}$$
$$t = -(p_1+p_3)^2 \ , \tag{2.1.6}$$
$$u = -(p_1+p_4)^2 \ . \tag{2.1.7}$$

Indeed we have

$$\mathcal{M} = \lambda_B - \frac{\lambda_B^2}{32\pi^2}\int_0^1 dx \left\{\log\left[\frac{m^2 - sx(1-x) + \Lambda^2}{m^2 - sx(1-x)}\right] - \frac{\Lambda^2}{m^2 - sx(1-x)+\Lambda^2}\right\} + (s\to t) + (s \to u) + O(\lambda^3) \ , \tag{2.1.8}$$

that can be approximated with

$$\mathcal{M} = \lambda_B - \frac{\lambda_B^2}{32\pi^2}\int_0^1 dx \left\{\log\left[\frac{\Lambda^2}{m^2 - sx(1-x)}\right] + \log\left[\frac{\Lambda^2}{m^2 - tx(1-x)}\right] + \log\left[\frac{\Lambda^2}{m^2 - ux(1-x)}\right] - 3\right\} + O(\lambda^3) \ , \tag{2.1.9}$$

since we chose the cutoff Λ much bigger than the mass m and momenta s, t and u. If one tries to impose a renormalization prescription, for example requiring

$$\mathcal{M} = \lambda|_{p_i=0} \tag{2.1.10}$$

to match the low energy limit of the amplitude with the static classical interaction λ, one easily finds $\lambda_B = \lambda + \delta\lambda$ with

$$\delta\lambda = \frac{3\lambda_B^2}{32\pi^2}\left[\log\left(\frac{\Lambda^2}{m^2}\right) - 1\right] \ , \tag{2.1.11}$$

while the renormalized amplitude becomes

$$\mathcal{M} = \lambda - \frac{\lambda^2}{32\pi^2}\int_0^1 dx \left\{ \log\left[\frac{m^2}{m^2 - sx(1-x)}\right] + \log\left[\frac{m^2}{m^2 - tx(1-x)}\right] + \log\left[\frac{m^2}{m^2 - ux(1-x)}\right]\right\} + O(\lambda^3) \ . \quad (2.1.12)$$

At this point, the renormalized amplitude is finite, because the Λ dependence has been canceled by the counterterm, but it does not behave well in the high energy limit: if s, t or u grow larger than m^2, the amplitude reduces to

$$\mathcal{M} \sim \lambda + \frac{\lambda^2}{32\pi^2}\left[\log\left(\frac{-s}{m^2}\right) + \log\left(\frac{-t}{m^2}\right) + \log\left(\frac{-u}{m^2}\right) - 6\right] \quad (2.1.13)$$

and $\lambda \log\left(\frac{-s,t,u}{m^2}\right)$ can grow bigger than one and break the weak coupling perturbative expansion. Such large logs emerge every time the diverging loop integrals depend on external momenta and cannot be avoided by any smart choice of renormalization scheme.

A way around was suggested in the context of QED by Gell-Mann and Low [4]: the energy scale used to define the renormalized coupling does not have to be equal to the on-shell masses of the particles in the theory or other physical observables; indeed, the renormalization point can be taken as close as desired to the energy scale of the experiment, avoiding the occurrence of large logs. The relation between the effective high-energy coupling and the results of classical low-energy measurements will be established by means of the so-called renormalization group.

Let us see it explicitly in the $\lambda\phi^4$ case: one can take as renormalization condition

$$\mathcal{M} = \lambda(\mu_R)|_{s,t,u=\mu_R} \quad (2.1.14)$$

defining as $\lambda(\mu_R)$ the coupling at the unphysical symmetric point $s, t, u = -\mu_R \gg m^2$. With this choice the amplitude is

$$\mathcal{M} = \lambda(\mu_R) - \frac{\lambda(\mu_R)^2}{32\pi^2}\int_0^1 dx \left\{ \log\left[\frac{m^2 + \mu_R^2 x(1-x)}{m^2 - sx(1-x)}\right] + \log\left[\frac{m^2 + \mu_R^2 x(1-x)}{m^2 - tx(1-x)}\right] + \log\left[\frac{m^2 + \mu_R^2 x(1-x)}{m^2 - ux(1-x)}\right]\right\} + O(\lambda^3) \ . \quad (2.1.15)$$

and, if s, t and u are of the same order of magnitude of μ_R, the logs remain of order unity and no breakdown of the perturbative expansion occurs, as far as $\lambda(\mu_R) \ll 1$. However, a careful reader will immediately notice that requiring $\lambda(\mu_R)$ to be small at $\mu_R \gg m$ is a very strong constraint. By comparing the last expression with the amplitude in terms of the classical coupling λ written a few lines above (2.1.13), one can see that the relation between the two is

$$\lambda(\mu_R) \sim \lambda + \frac{3\lambda^2}{32\pi^2}\log\left(\frac{\mu_R^2}{m^2}\right) \ , \quad (2.1.16)$$

So $\lambda(\mu_R)$ gets large with growing μ_R. Up to now, we are just moving the large logarithm from one place to another. Instead of making one big leap and defining $\lambda(\mu_R)$ from λ, an alternative method consists of considering many successive redefinitions of the coupling $\lambda(\mu_R)$ at nearby scales μ_R and μ'_R, such that $\log\left(\frac{\mu_R}{\mu'_R}\right) \sim 1$. In this way, both $\lambda(\mu_R)$ and $\lambda(\mu'_R)$ are kept small, preserving perturbativity, and a resummation of the effects of small change of scale can be done at a later stage. Physical amplitudes must be independent of the chosen renormalization scale, so we can exploit this property to find how the couplings must change with the scale μ_R. If the shift from $\lambda(\mu_R)$ to $\lambda(\mu'_R)$ has to leave the amplitude (2.1.15) unchanged, $\lambda(\mu_R)$ and $\lambda(\mu'_R)$ must respect the relation

$$\lambda(\mu'_R) = \lambda(\mu_R) - \frac{3\lambda(\mu_R)^2}{32\pi^2}\int_0^1 dx \log\left[\frac{m^2+\mu_R^2 x(1-x)}{m^2+\mu_R'^2 x(1-x)}\right] \tag{2.1.17}$$

and one can define the beta function of λ as the logarithmic derivative of the coupling with respect to the scale μ_R

$$\beta_\lambda := \mu_R \frac{d}{d\mu_R}\lambda(\mu_R) \ . \tag{2.1.18}$$

In this case we have

$$\beta_\lambda = \frac{3\lambda(\mu_R)^2}{16\pi^2}\int_0^1 dx \frac{\mu_R^2 x(1-x)}{m^2+\mu_R^2 x(1-x)} \ . \tag{2.1.19}$$

Now the beta function can be integrated back to obtain the effect of a finite shift in μ_R: in the approximation $\mu_R > m$, the beta function simplifies to

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2} \ , \tag{2.1.20}$$

that gives

$$\lambda(\mu_R) \sim \frac{\lambda}{1-\frac{3\lambda}{16\pi^2}\log\left(\frac{\mu_R}{m}\right)} \ . \tag{2.1.21}$$

Notice that $\lambda(\mu_R)$ can remain small even when $\lambda \log\left(\frac{\mu_R}{m}\right)$ is of order one, hence the renormalization group and the definition of running couplings via beta functions permit to extend the perturbative regime with respect to the naive approach of (2.1.16).

The prescription we used here, consisting in fixing the value of renormalized couplings using the expressions of scattering amplitudes at particular values of external momenta, is part of a larger family of renormalization schemes called Momentum Subtraction Schemes (MOM). We will often call the running couplings defined using these schemes "physical", because they are defined using explicitly the dependence of scattering amplitudes on external momenta, which are the physical observables of scattering processes. In the following, we will see other schemes where the running of coupling constants is defined using the dependence of correlation functions or their generating functionals on unphysical parameters introduced by hand in the theory.

2.1.1 The Callan-Symanzik equation

The statement that physical observables, such as Green functions or scattering amplitudes, must be invariant with respect to changes of the renormalization point, is formalized in the Callan-Symanzik equation [5, 6], which allows one to compute beta functions and anomalous dimensions of fields and couplings that appear in a given observable. Let's consider a connected n-point Green function

$$G^n_B(x_1,...,x_n,\Lambda,\lambda_B) = \langle\Omega| T\phi_B(x_1)...\phi_B(x_n)|\Omega\rangle \ . \tag{2.1.22}$$

We can define the renormalized Green function in terms of the renormalized fields and couplings as

$$G^n(x_1,...,x_n,\mu_R,\lambda(\mu_R)) = \langle\Omega| T\phi(x_1)...\phi(x_n)|\Omega\rangle = Z^{-n/2}(\mu_R)G^n_B(x_1,...,x_n,\Lambda,\lambda_B) \ . \tag{2.1.23}$$

The renormalized Green function depends both explicitly and implicitly (via the running coupling $\lambda(\mu_R)$) on the renormalization scale μ_R and also the field normalization $Z(\mu_R)$ depends on the choice of the renormalization conditions. However, the bare Green function G_B^n is independent of μ_R, so

$$\mu_R \frac{d}{d\mu_R} G^n = \frac{n}{2}\eta \ , \tag{2.1.24}$$

where

$$\eta = -\mu_R \frac{d}{d\mu_R} \log Z \tag{2.1.25}$$

is the anomalous dimension of the field, and we can conclude that

$$\left[\mu_R \frac{\partial}{\partial \mu_R} + \beta_\lambda \frac{\partial}{\partial \lambda} - \frac{n}{2}\eta \right] G^n(x_1, ..., x_n, \mu_R, \lambda(\mu_R)) = 0 \ . \tag{2.1.26}$$

This is called the Callan-Symanzik equation and can be easily extended to theories with many different fields ϕ_j and couplings g_i just by defining their respective beta functions β_{g_i} and anomalous dimensions η_j and adding to the equation as many terms of the type $\beta_{g_i} \frac{\partial}{\partial g_i}$ and $\frac{n_j}{2}\eta_j$ as needed, where n_j is the number of fields ϕ_j in the Green's function $G^{n_1,...,n_j}$. Hence, by substituting in it a good set of n-particles correlators computed in perturbation theory, one can obtain the RG flow of couplings and field normalizations of a given theory.

2.2 The Wilsonian renormalization group

The Wilsonian method [1] comes from statistical mechanics and the theory of critical phenomena. As well known in fluid dynamics and thermodynamics, it is not always necessary to keep track of all the microscopic degrees of freedom of a system, in order to give a good description of it. A reduced number of effective variables could be good enough to describe the observed physics. Suppose to have a statistical system composed by a lattice of spins with spacing L, such that the system shows the same macroscopic behaviour until its total extension is bigger than $\xi \gg L$, while below that size the collective behaviour is lost and a microscopic description of the degrees of freedom is needed. The length scale ξ is called correlation length. The partition function can be computed from the local microscopic Hamiltonian H_1, however one can also introduce a new reduced lattice with spacing $2L$ and Hamiltonian H_2 describing the same collective behaviour via local interactions between the new spin variables. This process, as suggested by Kadanoff [7], can be repeated over and over until arriving to a a macroscopic description in term of extended degrees of freedom of size $2^n L \sim \xi$ interacting via the Hamiltonian H_n. In this context, the renormalization group describes the relations between the various Hamiltonians H_i, all leading to the same partition function via differently coarse grained variables. Now we would like to translate this idea to theories with an infinite number of continuous degrees of freedom, like field theories. The partition function of a statistical field theory is

$$Z = \int D\phi e^{-S_E[\phi]/T} \ , \tag{2.2.1}$$

where the Hamiltonian is now called S_E. This peculiar naming is due to the fact that we are actually interested in quantum field theories and the associated path integral assumes precisely the from of a partition function of a statistical field theory after Wick rotating to the imaginary axis the time coordinate. After this process the Euclidean action takes the role of the Hamiltonian of the statistical system and many notions, as the renormalization group itself, can be transferred quite easily. In a continuous theory there is nothing like a lattice scale that we can change to reduce the number of effective degrees of freedom, however one can observe that on a lattice the Fourier space is periodic with $0 < p < 2\pi/L$. Hence, increasing the lattice spacing by a factor b corresponds to multiplying by the inverse $1/b$ the domain of periodicity of the Fourier space. We can reproduce the step of neglecting microscopic degrees of freedom by introducing a cutoff over momenta: the collective degrees of freedom of size L will be described only by Fourier modes with $|p| < 2\pi/L$. If we are interested in physics at energy scales below a given scale Λ, the modes with $|p| > \Lambda$ are not strictly necessary to describe it, in the same way as not all the spins in the lattice were necessary, if the correlation length was larger than the spacing L. That means we can just adopt an effective theory that uses only the modes with $|p| < \Lambda$ and automatically embodies all the effects of high energy modes. To define it, we can decompose $\phi(p)$ in heavy and light modes

$$\phi_L(p) = \theta(\Lambda - |p|)\phi(p), \qquad \phi_H(p) = \theta(|p| - \Lambda)\phi(p) \ ,$$

and rewrite the partition function as

$$Z = \int D\phi_H D\phi_L e^{-S_E[\phi_H+\phi_L]/T} \ . \tag{2.2.2}$$

Since in the end we are interested only in correlators between light fields, we can do the functional integral over ϕ_H, obtaining

$$Z = \int D\phi_L e^{-S_E^{EFF}[\phi_L,\Lambda]/T} \ . \tag{2.2.3}$$

Here $S_E^{EFF}[\phi_L, \Lambda]$ is the new effective action that permits to describe the physics at energies below Λ just in terms of light fields. The Wilsonian renormalization group describes how the effective action changes if the cutoff is lowered to $\Lambda' < \Lambda$. All we have to do is integrating over all modes with $\Lambda' < |p| < \Lambda$ and find $S_E^{EFF}[\phi_L, \Lambda']$. Obviously the effective action will change in such a way that the correlators of fields at energies below Λ' are exactly the same as if they were computed using the old cutoff Λ and effective action $S_E^{EFF}[\phi_L, \Lambda]$. If S_E^{EFF} is a local action, it can be expanded on a complete basis of operators which are spatial integrals of monomials of light fields and their derivatives at coincident points

$$S_E^{EFF}[\phi_L, \Lambda] = \sum_i g_i(\Lambda)\mathcal{O}_i(\phi_L) \tag{2.2.4}$$

with

$$\mathcal{O}_i(\phi) = \int d^dx \prod_n (\partial^n \phi(x))^{m_n} \tag{2.2.5}$$

with both n and $m > 0$, or equivalently, in momentum space,

$$\mathcal{O}_i(\phi) = \prod_n \left(\int d^dp_1 ... \int d^dp_{m_n} \right) \delta^d(\sum_n \sum_{k=1}^{m_n} p_k) \prod_n \prod_{j_n=1}^{m_n} p_{j_n}^n \phi(p_{j_n}) \ . \tag{2.2.6}$$

We want to establish a relation between $g_i(\Lambda)$ and $g_i(\Lambda')$, however to do it we have to bring $S_E^{EFF}[\phi_L, \Lambda]$ and $S_E^{EFF}[\phi_L, \Lambda']$ in a comparable form. First of all, we rescale the integrated momenta in $S_E^{EFF}[\phi_L, \Lambda']$ by Λ'/Λ in such a way as to have integrals ranging from zero to Λ in both actions. After that, there is still a certain degree of arbitrariness in the definition of the coefficients g_i. For example, one has to fix the normalization of the field. Wilson, in its work, reabsorbs the Λ dependence in the operator

$$\frac{1}{2}Z(\Lambda)\int d^dx\,(\partial_\mu\phi)^2 \tag{2.2.7}$$

in the field via the redefinition $\phi \to Z^{-\frac{1}{2}}\phi$, in such a way that the kinetic term in the action has always the coefficient $\frac{1}{2}$ in front. The new field depends on Λ, so its transformation under the RG is not determined only by its classical mass dimension, but it acquires also a non-zero anomalous dimension. There are also other couplings that can be fixed by a redefinition of the field; they are called redundant or inessential and their behaviour does not affect the physics of the system. Now that the two effective actions have the same structure, the RG flow of the theory can be calculated via infinitesimal changes of Λ and described in terms of the beta functions of the dimensionless couplings $\gamma_i = \Lambda^{-\Delta_i}Z^{-\frac{1}{2}(n_i+m_i)}g_i$

$$\beta_{\gamma_i} = \Lambda\frac{d}{d\Lambda}\gamma_i(\Lambda)\ , \tag{2.2.8}$$

where Δ_i is the mass dimension of g_i. In the Wilsonian picture, one does not actually have to start from a bare action S, as in standard perturbation theory, and subsequently find S_E^{EFF} by integrating out heavy modes. In fact, all measures and observations actually give some information which is meaningful at a particular energy scale and permits only to estimate the couplings of the effective theory that describe the system at that scale. Hence, what the physicist has to do to understand the behaviour of the same system at a different scale is to integrate out shells of momenta and see how the effective couplings change, without caring of the UV divergences of the bare theory and its regularization. Obviously, this procedure of integrating over momentum shells can be very tricky: it can be done via a diagrammatic computation as in [1], however, it can become very complicated from a technical point of view and it works only in the perturbative regime. Moreover, the use of a hard cutoff in momentum space usually leads to a breakdown of covariance and is not gauge invariant in theories with local symmetries. Despite these limitations, the Wilsonian conception of the renormalization group inspired many successive attempts to find a simpler way to describe how the effective description of a system changes at different scales [8, 9]. One of these is the Functional Renormalization Group (FRG) [10, 11], a nonperturbative technique that permits to compute directly the dependence of a new type of effective action on a cutoff over the spectrum of the kinetic operator. We will briefly review this tool in the next section, following the discussion of [12].

2.2.1 The Functional Renormalization Group

The main object of the functional renormalization group is the Effective Average Action (EAA). Let's consider the partition function of a quantum field theory describing a scalar field ϕ associated with an external source J

$$Z[J] = \int D\phi e^{-S_E[\phi]+\int d^Dx\,\phi J}\ . \tag{2.2.9}$$

In order to reproduce the Wilsonian idea of integrating over heavy modes, i.e. those with momenta above a given cutoff, we add to the action an infrared regulator $\Delta S_k[\phi]$ built as

$$\Delta S_k[\phi] = \frac{1}{2}\int d^dx\,\phi R_k(\Delta)\phi\ , \tag{2.2.10}$$

where k is an energy scale and Δ is the Laplace operator. If R_k is chosen in such a way that the weight in the functional integration of modes with momentum below k becomes exponentially suppressed, the new partition function $Z_k[J]$ defined using $S + \Delta S_k$ as bare action will look very similar to the effective action $S_E^{EFF}[\phi_L, \Lambda]$ defined before with $\Lambda = k$. The Laplace operator has a complete basis of eigenfunctions λ_i, with associated eigenvalues z_n, so we can define R_k as acting on this basis as a scalar function $R_k(z)$. The implementation of the cutoff over the spectrum of the Laplace operator and the use of the Heat Kernel, that we will introduce in section 6.1.2, will permit us to preserve spacetime covariance and gauge symmetries. In the particular case of Euclidean flat space, the eigenvalues of Δ are simply $z = p^2$. To obtain the suppression discussed above and have some good properties that will be useful in the followings, we require the cutoff function R_k to:

- be monotonically decreasing with z at fixed k;
- be monotonically increasing with k at fixed z;
- go to 0 $\forall$ z in the limit $k \to 0$;
- go to zero at least exponentially when $z > k^2$;
- satisfy $R_k(0) = k^2$.

By the introduction of the cutoff, no new interactions are added to the theory, only the bare propagator is modified. In fact the new propagator is

$$G(z) = \frac{1}{z + R_k(z)} \ . \tag{2.2.11}$$

In this way the eigenmodes of the Laplacian with eigenvalue above k^2 remain unchanged in their propagation, while modes with $z \ll k^2$ acquire a fictitious mass of order k which suppresses their propagation length and their contribution in loops. From the k-dependent partition function, one can define all the other generating functionals used in standard QFT. So we proceed as usual by introducing the free energy $W_k[J]$ as

$$W_k[J] = \log Z_k[J] \ . \tag{2.2.12}$$

This functional is the generator of connected n-points functions and can be used to compute the expectation value of the field with a given external source J via the relation $\varphi = \frac{\delta W_k[J]}{\delta J}$. At this point we can define a quantum effective action $\tilde{\Gamma}_k[\varphi]$ via a Legendre transform

$$\tilde{\Gamma}_k[\varphi] = -W_k[J_\varphi] + \int d^D x \varphi J_\varphi \ , \tag{2.2.13}$$

where J_φ is obtained by inverting the relation used before to write φ as a function of J. At the end, we arrive at the Effective Average Action after subtracting from the k-dependent effective action the regulator, giving

$$\Gamma_k[\varphi] = \tilde{\Gamma}_k[\varphi] - \Delta S_k[\varphi] \ . \tag{2.2.14}$$

Thanks to the particular features of the regulator, the EAA has the nice property of interpolating between the bare action S and the full quantum effective action Γ, which is the generating functional of 1-point irreducible (1PI) correlation functions. Indeed it is trivial to see that, in the limit $k \to 0$, the regulator can be neglected and we have the full effective action, while, in the limit $k \to \infty$, all modes are exponentially suppressed by an infinite factor, hence there is no integration at all and the EAA is equal to the bare action.

The Effective Average Action obeys a simple exact renormalization group equation: the Wetterich-Morris equation. The one-loop approximation of the quantum effective action can be written as

$$\Gamma^1 = S + \frac{1}{2}\mathrm{Tr}\log\frac{\delta^2 S}{\delta\phi\delta\phi} \tag{2.2.15}$$

and this expression can easily be used to find a one-loop approximation of the EAA: $\tilde{\Gamma}_k$ is just the effective action given by the bare action $S + \Delta S_k$, so its one-loop expansion is the same as Γ, while to obtain Γ_k we just have to subtract the regulator. Hence,

$$\Gamma^1_k = S + \Delta S_k + \frac{1}{2}\mathrm{Tr}\log\frac{\delta^2(S+\Delta S_k)}{\delta\phi\delta\phi} - \Delta S_k = S + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right) . \tag{2.2.16}$$

What we are actually interested in is the dependence of the EAA on k, so we act with a logarithmic derivative $\mathrm{d}_t = k\frac{d}{dk}$ on both sides of the last expression, obtaining

$$d_t\Gamma^1_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right)^{-1} d_t R_k \ , \tag{2.2.17}$$

since the k-dependence in the right hand side comes only from R_k. Up to this point, we have an equation that contains both the bare action and the one-loop EAA, however it is possible to make it a closed exact equation of only Γ_k. Let us return to the free energy W_k, its derivative turns out to be

$$\begin{aligned} d_t W_k &= -\frac{\int D\phi\, e^{-S_E[\phi]-\Delta S_k[\phi]+\int d^D x\,\phi J} d_t\Delta S_k[\phi]}{Z_k} \\ &= -d_t\langle\Delta S_k\rangle = -\frac{1}{2}\mathrm{Tr}\langle\phi\phi\rangle d_t R_k \ , \end{aligned} \tag{2.2.18}$$

where we have introduced the expectation value with respect to the partition function Z_k defined as

$$\langle\mathcal{O}\rangle = \frac{\int D\phi\, e^{-S_E[\phi]-\Delta S_k[\phi]+\int d^D x\,\phi J}\mathcal{O}}{Z_k} . \tag{2.2.19}$$

Hence, we immediately have $d_t\tilde{\Gamma}_k = -d_t W_k$, since the two functionals are related by the Legendre transform (2.2.13). The derivative of the full EAA is

$$d_t\Gamma_k[\varphi] = -d_t W_k - d_t\Delta S_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left(\langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle\right) d_t R_k \ , \tag{2.2.20}$$

because $\varphi = \langle\phi\rangle$ by definition. Notice that $\langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle$ is the connected two-point function and W_k is the generator functional of this type of correlators, then we can write

$$d_t\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\frac{\delta^2 W_k}{\delta J\delta J} d_t R_k \ . \tag{2.2.21}$$

Due to the properties of the Legendre transform (2.2.13), the external source J written as a function of φ is equal to $\frac{\delta\tilde{\Gamma}_k}{\delta\varphi}$, so we can write, respectively,

$$\frac{\delta^2\tilde{\Gamma}_k}{\delta\varphi\delta\varphi} = \frac{\delta J}{\delta\varphi} \qquad \frac{\delta^2 W_k}{\delta\varphi\delta\varphi} = \frac{\delta\varphi}{\delta J} \ , \tag{2.2.22}$$

that implies

$$\frac{\delta^2 W_k}{\delta\varphi\delta\varphi} = \left(\frac{\delta^2\tilde{\Gamma}_k}{\delta\varphi\delta\varphi}\right)^{-1} \quad . \tag{2.2.23}$$

Using the last relation, one obtains the Wetterich-Morris equation or Exact Renormalization Group Equation (ERGE) [10, 11]

$$d_t\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1} d_t R_k \ . \tag{2.2.24}$$

What we have done is basically an "RG improvement" of the approximate eq (2.2.17), since we have substituted both the one-loop EAA $\Gamma_k^{(1)}$ and the bare action S with the full EAA. That means that the one-loop structure of the ERGE is preserved, and we can diagrammatically represent the equation as in figure 2.1.

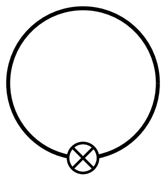
$d_t\Gamma_k[\varphi] = \frac{1}{2}$ 

Figure 2.1: Diagrammatic representation of the ERGE: the line represents the full propagator $\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1}$, while the crossed insertion represents the derivative of the regulator $d_t R_k$.

As in the case of the Wilsonian RG, arrived at this point we can forget of the existence of a bare action and start the computations from an ansatz for the EAA at a given scale k. Moreover, despite R_k being introduced as an infrared regulator, the Wetterich-Morris equation is free of UV divergences. This is ensured by the fact that the regulator decreases at least exponentially when $z > k$, thus the contribution of UV modes in the functional trace is suppressed by $d_t R_k \to 0$. From a qualitative point of view, the derivative with respect to t of Γ_k can be seen as the difference between two EAAs that differ only by a small change in k. Modes with $z \gg k^2$ are almost unaffected by the regulators in both terms of the subtraction, and hence their contributions cancel each other. At the same time, modes with eigenvalues well below k are suppressed by the cutoff itself, thus only modes with $z \sim k^2$ really contribute to the scale derivative of the average effective action. Thanks to this feature, the ERGE really resembles the integration over momentum shells of the Wilsonian RG.

At this point, there are two ways to use the ERGE: one corresponds to exploiting the property of the Average Effective Action to interpolate between the bare action S and the full $1PI$ generating functional Γ. Choosing a "quasi-bare" action at a very high k, preferably close to a UV attractive fixed point of the RG flow of the theory, and integrating the Wetterich-Morris equation down to $k = 0$, one can obtain a nonperturbative expression for Γ. From this point of view, the FRG turns out to be a tool to compute the quantum effective action of a theory without stumbling in all the UV divergences that usually emerge in perturbation theory. Although feasible in some simple cases [13], this approach turns out to be extremely hard from a technical point of view in theories like Gravity and only in recent years some steps have been made in this direction [14, 15]. The alternative path consists of considering k as a good proxy for the energy dependence of the theory, as happens with Λ in the Wilsonian case, and using the k running to predict the changes in physical observables at different energy scales. One takes an ansatz for Γ_k at a fixed scale, usually expands it in terms of local operators of fields and derivatives of the field (2.2.5) and finds the beta functions $\beta_{\gamma_i} = k\frac{d}{dk}\gamma_i(k)$, where $\gamma_i = k^{-\Delta_i} g_i$ are the dimensionless couplings and Δ_i represents the mass dimension of g_i. In the FRG, the coupling Z is usually treated at the same level as any other coupling g_i, leaving the fields independent of k. The only difference with respect to other couplings is that the anomalous dimension $\eta = -d_t \log Z$ is usually used in spite of β_Z to describe its flow. So we have obtained a very promising non-perturbative equation that describes the dependence of the average effective action on the cutoff scale k and permits to extract the beta functions of the theory; however, this powerful tool also comes with a price. The basis of local operators has infinite cardinality and at every new step of the FRG flow all these operators could potentially have a nonzero beta function. This makes the ERGE practically impossible to handle. What is usually done is to reduce the theory space, the space of local operators taken in to account, to a finite-dimensional space by means of a truncation and compute the flow of the theory restricted to this subspace. Hence, whether the predictions of experimental observable obtained from FRG calculations match or not with measurements of real physical systems depends equally on the choice of the starting EAA and of the truncation of the theory space.

The Wilsonian RG and other techniques inspired by it, such as the FRG, are very powerful tools capable of giving insights on non-perturbative effects in QFT and describing the RG flow of a theory even far away from the perturbative (small coupling) regime, where the standard approach in terms of Feynman diagrams cannot be applied. On the other hand, these realizations of the renormalization group are based on the dependence of the effective theory on an artificially introduced cutoff. While in condensed matter physics this cutoff can be easily related to a characteristic scale of the molecular structure of the system, as the spacing of a crystal lattice, below which the effective description in terms of excitations is not faithful, in high-energy physics, due to Poincarè invariance, there are no such physical cutoffs. In these cases, usually physicists try to relate the cutoff scale with the energy scale of the experiment, which was used to define the RG in Section 2.1, and then compare it with the masses of the particles in the spectrum of the theory. However, this identification sometimes turns out to be not unique. In the next Section we will discuss the relation between different definitions of the renormalization group in the particular case of QFTs with 2 derivatives in the kinetic term in four spacetime dimensions.

2.3 Universality in the renormalization group

Differences and analogies between various definitions of the Renormalization Group have been widely discussed in the literature in the standard case of 2 derivative theories in 4 spacetime dimensions. In this discussion, we will distinguish two cases: the logarithmic running and the power-law running. We will see that, while the logarithmic part, under certain conditions, is a universal feature of the RG, that means it produces the same beta functions at one-loop for all definitions of the RG or renormalization schemes, the power-law running of coupling has been object of fierce discussions along the years and there seems not to exist a universal notion of it.

2.3.1 The logarithmic running

A first hint of the universality of logarithmic running can already be found in the example of section 2.1. To define the physical running, we used the dependence of the amplitude on the kinematical variables to calculate the beta functions, however, in a Wilsonian approach, we can consider the bare parameters appearing there as the Λ dependent parameters of the effective theory at scale Λ. Hence, to extract the beta functions, we have to impose the independence of $\mathcal{M}$ on Λ and apply the same machinery introduced with the Callan-Symanzik equation (2.1.26) to the cutoff instead of the renormalization scale μ_R. The result is

$$\beta_{\lambda_B} = -\Lambda \frac{\partial}{\partial \Lambda} \mathcal{M} = \frac{3\lambda_B^2}{16\pi^2} \tag{2.3.1}$$

One can immediately see that this expression is identical to the high-energy $(s, t, u \gg m^2)$ limit of the physical beta function (2.1.20). This should not be surprising, after looking at the one-loop amplitude (2.1.9) in the same limit, that is simply

$$\mathcal{M} \sim \lambda_B + \frac{\lambda_B^2}{32\pi^2} \left[\log\left(\frac{-s}{\Lambda^2}\right) + \log\left(\frac{-t}{\Lambda^2}\right) + \log\left(\frac{-u}{\Lambda^2}\right) \right] . \tag{2.3.2}$$

Notice that the logarithms of the kinematical variables s, t and u are all coupled with logarithms of the cutoff Λ, hence they have the same coefficients in front. This is not a coincidence, but a general property of one-loop Feynman diagrams of 2-derivative theories in 4 spacetime dimensions, where large logarithms in momentum variables always come with the regulator of UV divergences in the high-energy limit. A good intuition of why this is true can be obtained by looking at a generic one-loop Feynman diagram as the one depicted in fig. 2.2.

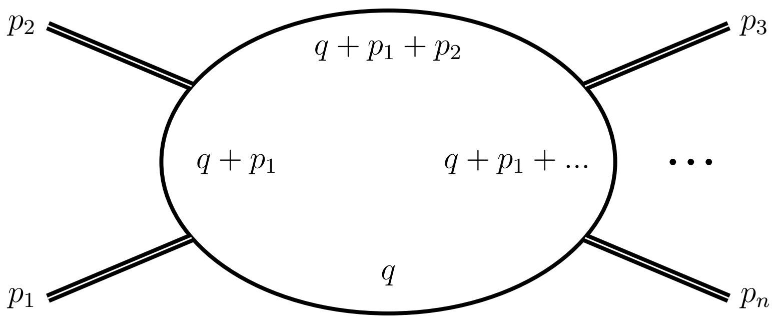


Figure 2.2: a generic 1-loop diagram. Doubled external lines mean that they do not necessarily correspond to single particles, in fact they can stand also for 2 or more fields interacting at the same point with total momentum p_i.

Let us start considering a theory where there are no intrinsic scales as masses or dimensionful couplings. In this case, the integral related to the loop looks like

$$\int_k^\Lambda d^4q \, \frac{N(q, p_i)}{q^2(q + p_1)^2 \times \cdots \times (q + p_1 + \cdots + p_{n-1})^2} , \tag{2.3.3}$$

where N is a generic polynomial in q and p_i and the integral is regulated in the UV by the cutoff Λ and in the infrared by k. The external momenta p_i and the two cutoffs are the only dimensionful quantities, so if we take $\Lambda \gg p_i \gg k$, the logarithms that a priori can be produced have the structures $\log \frac{f(p_i)}{\Lambda}$, $\log \frac{f(p_i)}{k}$, $\log \frac{f(p_i)}{g(p_i)}$ and $\log \frac{\Lambda}{k}$, where $f(p_i)$ and $g(p_i)$ are generic functions of mass dimension 1 of the external momenta. Logs containing Λ come from UV divergences, while those with k are the results of the regularization of IR divergences. Notice that, in the limit $q \ll p_i \; \forall \, i$, the integrand reduces to

$$\frac{N(0, p_i)}{q^2(p_1)^2 \times \cdots \times (p_1 + \cdots + p_{n-1})^2} , \tag{2.3.4}$$

if none of the sums of external momenta is equal to zero, i.e. virtual particles in loops are not on-shell. Hence, the integral is IR finite in 4 dimensions and k does not appear in any log in the result, because the limit $k \to 0$ must be regular. We will discuss the on-shell IR divergences later on, for the moment, it is enough to know that the RG is not needed to handle them. If we choose a renormalization point where all the relevant kinematical variables f and g in the amplitude have similar magnitudes, $\log \frac{f(p_i)}{g(p_i)}$ will be of order 1, hence the only remaining source of large logs dangerous for the perturbativity of the series expansion will be terms like $\log \frac{f(p_i)}{\Lambda}$, exactly as we have seen in our example. The direct consequence is that we can be sure that, in massless theories with an ordinary 2 derivative kinetic term in 4 spacetime dimensions, large logs of large momenta in one-loop amplitudes always have the same coefficient as logs of the UV regulator, hence the physical running is equivalent to the logarithmic Wilsonian running. Concerning the k-running in the FRG, since it substantially realizes the Wilsonian idea of integration over momentum shells, we expect it to produce results compatible with the Wilsonian RG. Moreover, when one computes an n-point function from the EAA, it turns out that the sums of external momenta in the denominator of (2.3.3) act as effective masses for the propagators in a way similar to the cutoff R_k, so it is natural to obtain amplitudes that depend in the same way on the cutoff scale k and the typical energy scale of external momenta E [16, 17].

Dimensional regularization

The relation between large logs and UV divergences holds independently of the regularization method used to avoid infinities, then other techniques are often preferred to cutoff regularization. A very popular one, due to its property of preserving gauge symmetries, is dimensional regularization (dim reg) [18], where the momentum integral is analytically extended to $d \in \mathbb{C}$ spacetime dimensions. In this way all loop integrals are reduced to euclidean integrals of the form

$$\mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{q^{2l}}{(q^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d(d+2)\dots(d+2(l-1))}{2^l} \frac{\Gamma\left(n-\frac{d}{2}-l\right)}{\Gamma(n)} \Delta^{l+\frac{d}{2}-n} \ , \tag{2.3.5}$$

where μ is a mass parameter introduced to preserve the overall dimensionality of the integral. The singular behaviour of the integral is now totally enclosed in the Euler gamma function in the numerator of the right hand side, which has poles in the complex plane in correspondence to 0 and negative integer numbers. Near zero we have

$$\Gamma(x) = \frac{1}{x} - \gamma_E + O(x) \ , \tag{2.3.6}$$

where γ_E is the Eulero-Mascheroni constant. The Laurent expansion near the other poles can be calculated from the latter using the well-known property of the gamma function

$$\Gamma(x-1) = \frac{\Gamma(x)}{x-1} \ , \tag{2.3.7}$$

thus

$$\Gamma(x) = \frac{(-1)^n}{n!}\left(\frac{1}{x+n} - \gamma_E + 1 + \dots + \frac{1}{n} + O(x+n)\right) \tag{2.3.8}$$

for $x \sim -n$. If the integral was divergent before the analytical continuation, $d = 4$ will correspond to a pole of the Γ function in the numerator and, taking $d = 4 - 2\varepsilon$ with small ϵ, we will obtain

$$\mu^{2\epsilon} \int \frac{d^{4-2\varepsilon} q}{(2\pi)^{4-2\varepsilon}} \frac{q^{2l}}{(q^2+\Delta)^n} = a_0 \Delta^{l-n-2}\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log\left(\frac{\mu^2}{\Delta}\right)\right) + O(\varepsilon) \ , \tag{2.3.9}$$

where a_0 is a constant number. In the particular case of the $\lambda\phi^4$ theory we have considered up to now, the one-loop amplitude regulated with dim reg is

$$\mathcal{M} \sim \lambda_B + \frac{\lambda_B^2}{32\pi^2}\left[-\frac{1}{\varepsilon} + \gamma_E - \log 4\pi + \log\left(\frac{-s}{\mu^2}\right) + \log\left(\frac{-t}{\mu^2}\right) + \log\left(\frac{-u}{\mu^2}\right)\right] \tag{2.3.10}$$

at high energies. Using the so-called $\overline{MS}$ renormalization scheme, the divergent part $\frac{1}{\varepsilon}$ and the finite terms γ_E and $\log 4\pi$ are reabsorbed in the definition of the renormalized coupling $\lambda(\mu)$. So, the renormalized amplitude is

$$\mathcal{M} \sim \lambda(\mu) + \frac{\lambda(\mu)^2}{32\pi^2}\left[\log\left(\frac{-s}{\mu^2}\right) + \log\left(\frac{-t}{\mu^2}\right) + \log\left(\frac{-u}{\mu^2}\right)\right] \ , \tag{2.3.11}$$

that is free of large logs that break the perturbative expansion, as far as we take $\mu^2 \sim s, t, u$. Even in this case, the physical amplitude has to be independent of the arbitrary parameter μ, then the coupling $\lambda(\mu)$ must cancel the explicit dependence on it. That means we can find its beta function with a Callan-Symanzik-like equation, exactly the same way we defined the cutoff beta function in eq. (2.3.1):

$$\beta_\lambda^\mu = -\mu\frac{\partial}{\partial\mu}\mathcal{M} = \frac{3\lambda(\mu)^2}{16\pi^2} \ . \tag{2.3.12}$$

Again, we recover the same beta function, since different regularization procedures are just different way to identify the same logarithmic divergences. During the rest of this thesis, we will often call the running of coupling induced by $\overline{MS}$ or similar renormalization schemes μ-running, from the standard name given to the unphysical energy scale introduced by dimensional regularization.

The general integral of (2.3.5), regulated with an UV cutoff, would be

$$\int^\Lambda \frac{d^4q}{(2\pi)^4}\frac{q^{2l}}{(q^2+\Delta)^n} = a_i\Lambda^{2l-2n+4} + a_{i-1}\Delta\Lambda^{2l-2n+4-2} + \dots a_0\Delta^{l-n-2}\log\left(\frac{\Lambda^2}{\Delta}\right) + O(1) \ . \tag{2.3.13}$$

Notice that only the logarithmic divergence was already there in the result of dim reg, while the coefficients in front of power-law divergences are automatically set to zero by the analytical extension procedure. This discrepancy will have a crucial role in the discussion of power-law divergences later on, however, if we limit ourselves to logarithmic divergences, the two regularizations give the same coefficient a_0 and the same beta function.

One-loop universality of beta functions

In general, it can be shown that all alternative definitions of the running coupling g that grant perturbativity of the scattering amplitudes at small renormalized coupling end up giving the same beta function at leading and subleading order in g itself [3]. Let us consider two renormalized couplings $g(\mu)$ and $\tilde{g}(\mu)$, defined with different prescriptions. If they are both parametrically small and allow us to reabsorb all the large logs, one can be written in terms of the other in the following way

$$\tilde{g}(g) = g + ag^2 + O\left(g^3\right) \ , \tag{2.3.14}$$

or equivalently

$$g(\tilde{g}) = \tilde{g} - a\tilde{g}^2 + O\left(\tilde{g}^3\right) \ , \tag{2.3.15}$$

On the other hand, also the beta functions can be perturbatively expanded in g:

$$\begin{aligned}\beta_g &= bg^2 + cg^3 + O\left(g^4\right) \\ &= b\tilde{g}^2 + (c - 2ab)\tilde{g}^3 + O\left(\tilde{g}^4\right) \ ,\end{aligned} \tag{2.3.16}$$

while, from the chain rule

$$\beta_{\tilde{g}} = \mu \frac{d}{d\mu}\tilde{g} = \frac{d\tilde{g}}{dg}\beta_g \ . \tag{2.3.17}$$

So, putting all together,

$$\begin{aligned}\beta_{\tilde{g}} &= \left[1 + 2a\tilde{g} + O\left(\tilde{g}^2\right)\right]\left[b\tilde{g}^2 + (c - 2ab)\tilde{g}^3 + O\left(\tilde{g}^4\right)\right] \\ &= b\tilde{g}^2 + c\tilde{g}^3 + O\left(\tilde{g}^4\right) \ ,\end{aligned} \tag{2.3.18}$$

implying that the coefficients b (given by the one-loop corrections) and c are independent of the renormalization scheme. However, the same cannot be said of subsequent terms in the small g expansion.

massive theories

In the case of theories containing massive particles, the picture is more complicated. The presence of another energy scale m different from the external momenta permits us to construct new logarithms such as $\log \frac{\Lambda}{m}$ and $\log \frac{f(p_i)}{m}$. Moreover, the so-called threshold effects emerge when the energy scale of the scattering process falls below m.

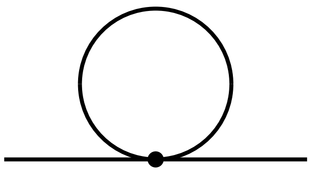


Figure 2.3: The tadpole diagram contributing to the self energy in scalar $\lambda\phi^4$ theory

When a virtual massive particle propagates in loops, the tadpole diagram (fig. 2.3), associated to the integral

$$\int d^4q \, \frac{1}{q^2 + m^2} \ , \tag{2.3.19}$$

acquires a logarithmic UV divergent term, as in the case of the one-loop two-point function (2.1.4). As observed in that case, such an infinite one-loop correction to the bare propagator does not require introducing a physical running for the mass, since there is no dependence on the particle's four-momentum in the logarithm and the divergence can be reabsorbed once for all energy scales in the renormalized mass. However, if we would have used instead a different definition of the RG, as the Wilsonian one or dimensional regularization with $\overline{MS}$ prescription, we would have found a running mass. Following the Wilsonian recipe of section 2.2, we require the independence of the two point function (2.1.4) from the cutoff, obtaining

$$\Lambda \frac{dm_B^2}{d\Lambda} = -\frac{\lambda}{16\pi^2}\frac{\Lambda^4}{\Lambda^2 + m_B^2} \ , \tag{2.3.20}$$

then we introduce the dimensionless coupling $\tilde{m}^2(\Lambda) = \Lambda^{-2}m_B^2$ and calculate the beta function of $\tilde{m}^2$

$$\beta_{\tilde{m}^2} = -2\tilde{m}^2 - \frac{\lambda}{16\pi^2}\frac{1}{1 + \tilde{m}^2} \ . \tag{2.3.21}$$

On the other hand, the FRG and the Wetterich equation with the optimized or Litim cutoff $R_k(z) = (z - k^2)\theta(k^2 - z)$ [19] gives [12]

$$\beta_{\tilde{m}^2} = -2\tilde{m}^2 - \frac{\lambda}{32\pi^2}\frac{1}{(1+\tilde{m}^2)^2} \ . \tag{2.3.22}$$

In dim reg the one-loop two-point function is

$$\Gamma^{(2)} = p^2 + m_B^2 - \frac{\lambda m_B^2}{32\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log\left(\frac{\mu^2}{m_B^2}\right)\right] \ , \tag{2.3.23}$$

so the dimensionless renormalized coupling $\tilde{m}^2(\mu) = \mu^{-2}(m_B^2 - \delta m^2)$ is defined in $\overline{MS}$ by setting

$$\delta m^2 = \frac{\lambda m_B^2}{32\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi\right] \ . \tag{2.3.24}$$

Again the beta function is defined via the Callan-Symanzik equation for μ, so we get

$$\beta^{\mu}_{\tilde{m}^2} = -2\tilde{m}^2(\mu) + \frac{\lambda \tilde{m}^2(\mu)}{16\pi^2} \ . \tag{2.3.25}$$

In this case, we had a power-law divergence together with the logarithmic one. We have already anticipated that the running produced by powers of the cutoff is not universal, still the logarithmic part of the running, namely the term proportional to $\lambda\tilde{m}^2$, is a common feature of all these schemes. In fact, it is present at subleading order in the small mass expansion of (2.3.21) and (2.3.22). This term comes from logarithms of the type $\log\frac{\Lambda}{m}$ or $\log\frac{\mu}{m}$, hence it has no equivalent in the physical running, because no external momenta can enter in the tadpole integral and then no large logarithms of kinematical variables can be generated. This kind of running is unphysical in the sense that it cannot be observed by comparing the outcome of scattering experiments at different energy scales. That could seem a threat to the universality of logarithmic running, but it is important to notice that, in the small coupling limit, the dominant term in both beta functions is the classical contribution $-2\tilde{m}^2$ given by the classical dimension of the mass. Since it is negative, in the UV the renormalized dimensionless mass becomes small as far as we remain in the perturbative regime. So, the quantum corrections are negligible and we can consider the massless case discussed before as a good high-energy approximation in all renormalization schemes. In general, all logarithmic divergences coming from tadpoles are multiplied by positive powers of m, since they are produced by subleading terms in the $q^2 \gg m^2$ expansion of (2.3.19); thus, the logs are actually suppressed in the high-energy limit equivalent to $m \to 0$. This result is strictly related to the property of 2-derivative theories in 4 dimensions of not having off-shell large logarithms containing the IR cutoff: sending the mass to zero in a logarithm not suppressed by powers of the mass would produce exactly that type of IR divergences we have excluded before.

The other typical phenomenon of massive theories is the presence of mass thresholds. When the energy of the scattering process is much above the mass scales of the theory, the RG flow inherits trivially the universality observed in the massless case: the large logarithms and UV divergences are all generated in the large q part of the loop integral and in this region all the propagators can be rewritten as their massless versions up to $O(\frac{m}{q})$ corrections. Hence, in the high-energy limit the beta functions are the same as those of massless theories and universal. On the other hand, if we look at the physical beta function of $\lambda\phi^4$ theory 2.1.19 in the low energy limit, i.e. $\mu_R^2 < m^2$, we notice that it simply goes to zero. This is a physical feature related to the decoupling of massive particles. When the energy scale of the process is below the mass of a particle, the external momenta p_i in loop propagators of this kind of particle can be neglected, since

$$\frac{1}{(q+p)^2+m^2} \sim \begin{cases} \frac{1}{q^2} + O\left(\frac{p}{q}\right), \; if \; q^2 \geq m^2 \; ; \\ \frac{1}{m^2} + O\left(\frac{p^2}{m^2}\right), \; if \; q^2 < m^2 \; ; \end{cases} \tag{2.3.26}$$

so the loop integral becomes independent of external momenta. This means the energy of the process is too low to efficiently generate virtual excitations of this kind of particles, which stop to contribute significantly to quantum corrections. Thus, we expect the physical beta functions to go to zero in the limit $p_i^2 \ll m^2$, as we observed above in our toy model. The freezing out of heavy modes below the energy scale of their mass is reproduced also in the Wilsonian RG. If we naively take the expression (2.1.9) for the amplitude and we extract from it the dependence on the cutoff, we find the constant result (2.3.1) and we miss the decoupling process, but the correct implementation of the Wilsonian RG requires to integrate shells of momenta until we arrive to energy scales close to the experiment. Hence, we have to take $\Lambda < m$, a region where the amplitude (2.1.9) is not correct. If we start instead from eq (2.1.8), the dependence of the regularized amplitude on the cutoff disappears in the limit $m \gg \Lambda > s, t, u$. The cancellation of beta functions below mass thresholds is even more clear when the Wilsonian RG is implemented via the FRG: in this case the beta function of λ is [12]

$$\beta_\lambda = \frac{3\lambda}{16\pi^2}\frac{1}{(1+\tilde{m}^2)^3} \; , \tag{2.3.27}$$

that goes to zero as $\tilde{m}^2 = m^2/k^2 \to \infty$. From a more technical point of view, when the cutoff scale k goes below the mass of the particle, the action of R_k in propagators is already outperformed by the much bigger physical mass. In this way the cutoff becomes ineffective and the effective action stops to run. On the other hand, $\overline{MS}$ and other mass-independent schemes usually used together with dimensional regularization completely miss the mass threshold: the beta function (2.3.12) remains constant independently of masses and other scales of the theory. A more physical example can be found in QED, where the vacuum polarization correction involving a top quark loop yields the correction

$$\Pi(p^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m_t^2}{\mu^2} + \frac{p^2}{5m_t^2} + ...\right] \tag{2.3.28}$$

at low energy. However, despite the dependence on $\log \mu$, this does not imply that the top quark loop contributes to the running of the electric charge at low energy. The top quark makes a contribution to the running of α only at energies above m_t, where the logarithmic factor involves $\log p^2$ instead of $\log m_t^2$. On the other hand, the high energy version of the same process is

$$\Pi(p^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m_t^2}{\mu^2} - \log\frac{-p^2}{m_t^2} + ...\right] \; . \tag{2.3.29}$$

Asymptotically the mass cancels out, and we could have performed the renormalization using a mass-independent scheme. However, since we previously chose to renormalize the electric charge at low energy, absorbing the $\log m_t^2/\mu^2$ factor into the coupling, the latter logarithm is potentially a large logarithm and should be resummed in a running coupling constant. The top quark contribution to the electric charge is constant at low energy and runs at high energy.

The only way to take in account the process of decoupling of heavy modes using these types of prescriptions consists in defining an effective action below the mass scale and match the coefficients of the two theories at $\mu = m$ [20]. We well see an example of this procedure in higher derivative theories later on.

IR divergences

We have shown how UV divergences are strictly related to large logarithms emerging in off-shell momenta configurations. However, we have not yet discussed the particular kinematical configurations in which at least one of the sums of external momenta in the numerator of (2.3.3) goes to zero. In these cases, at least one more $\frac{1}{q^2}$ factor remains in the $q \ll p_i$ sector of the integral, causing an IR divergence regulated by k and the appearance of $\log \frac{f(p_i)}{k}$ or $\log \frac{\Lambda}{k}$ terms in the amplitude. These IR divergences are even more frequent in Lorentzian signature, where on-shell external massless particles have 0 squared momentum. In massive theories, the propagators in (2.3.3) acquire a mass term that avoids IR divergences, but it becomes singular in the small mass limit, that is actually equivalent to the high-energy regime. The crucial idea used to treat this type of divergences is that, in a theory containing massless modes, states with an arbitrary number of massless particles with very low-energy (soft) or moving all in the same direction (collinear) cannot be distinguished by real detectors with a finite energy sensitivity and angular resolution. If the total energy of soft particles is too low or the angular separation of collinear ones is too small, they will not be detected. That means the transition probabilities of all these indistinguishable states must be summed to describe the probability of physically meaningful processes, where single-particle in and out states are replaced by jets of high-energy particles surrounded by an arbitrary number of soft and collinear massless particles. Surprisingly, it turns out that the amplitudes describing the emission of these massless particles are IR divergent too, but in such a way that, when summed with IR divergent loop diagrams, all divergences cancel out order by order in perturbation theory [3]. Thus, transition probabilities between inclusive states, which are the only meaningful states from an operational point of view, are IR safe. A general theorem that shows this cancellation was formulated by Kinoshita [21] and independently by Lee and Nauenberg [22]. Consider a generic quantum mechanical system with a Hamiltonian $H = (H_0 + gH_1)$ such that it is diagonalized via the action respectively on the left and on the right of two unitary matrices U_- and U_+ in the following way

$$U_-^\dagger (H_0 + gH_1) U_+ = E \ , \tag{2.3.30}$$

with E and H_0 diagonal matrices. In this theory the scattering matrix is given by

$$S = U_-^\dagger U_+ \tag{2.3.31}$$

and the transition probability from an asymptotic state a to another asymptotic state b is given by

$$\sum_{i,j} \left[(U_-)^*_{ib} (U_-)_{jb}\right] \left[(U_+)_{ia} (U_+)^*_{ja}\right] \ . \tag{2.3.32}$$

For small g, we can apply perturbation theory to the time evolution operators $U_\pm$ and write

$$(U_\pm)_{ij} = \delta_{ij} + g \frac{(1-\delta_{ij})(H_1)_{ij}}{E_j - E_i \pm i\epsilon} + O(g^2) \ , \tag{2.3.33}$$

where ϵ is a small positive quantity. Infrared divergences emerge when there are degenerate states in the spectrum, in such a way that the denominator becomes singular. Thus, also

(2.3.32) suffers of IR divergences when the states i and j in the sum are degenerate with a or b. In order to have a better grasp on the problem, let's insert in the Hamiltonian a parameter μ that breaks the degeneracy, like a small artificial mass. We call $D(E_i)$ the set of states that have energy E_i in the $\mu \to 0$ limit. The theorem shows that the quantity

$$\sum_{D(E_a)} U_{ia} U^*_{ja} \tag{2.3.34}$$

is finite in the limit $\mu \to 0$ order by order in g both for U_+ and U_-. So, if instead of the transition between single states a and b we consider the inclusive transition probability between the sums over all in and out states respectively in $D(E_a)$ and $D(E_b)$

$$\sum_{i,j} \left[\sum_{D(E_b)} (U_-)^*_{ib} (U_-)_{jb} \right] \left[\sum_{D(E_a)} (U_+)_{ia} (U_+)^*_{ja} \right] , \tag{2.3.35}$$

the result is well-behaved in the limit $\mu \to 0$, hence IR finite.

2.3.2 The power-law running

In the high-energy limit, when the mass scales of the theory are negligible, logarithmic running of couplings has a universal behaviour. Universality means that the related beta function is, up to next to leading order in perturbative expansion, equal in all renormalization schemes and that the same renormalized coupling fits to all processes involving its interaction vertex. When divergences are of the power-law type, both these universalities fail. We already observed that dimensional regularization automatically removes power-law divergences. A simple proof of this can be derived from the prototypical dimensionally regularized integral (2.3.5). A pure power-law UV divergent integral has a structure like

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^{2n}} \tag{2.3.36}$$

with $d/2 - n > 0$. This integral corresponds to the limit $\Delta \to 0$ of the left hand side of (2.3.5) with $l = 0$, so it is quite simple to see that the same limit of the right hand side is power-law suppressed when $d/2 - n > 0$. We already observed how this blindness of dim reg with respect to power-law divergences manifests itself in the calculation of the mass beta function in $\lambda\phi^4$ theory. Moreover, even the Wilsonian RG showed a dependence of the part of the beta function containing powers of the cutoff on the way this cutoff is imposed, as can be seen, for example, by comparing (2.3.21) and (2.3.22).

There is also an intrinsic problem in the definition of a power-law running coupling [23]. The peculiar properties of logarithms allow us to define a universal running coupling capable of reabsorbing loop corrections of all processes where this coupling appears, up to finite terms. On the other hand, one can try to reabsorb in a renormalized coupling the powers of kinematical variables appearing in one-loop corrections to a given scattering amplitude in an analogous way to the definition of the physical running, but this new running coupling will describe in an efficient way only one particular process. Other amplitudes, for example those corresponding to a crossing of external states, can depend polynomially on different kinematical variables or have quantum corrections with the opposite sign. Thus, such a definition would be practically useless and the power-law quantum corrections must be interpreted as the generation of some new higher dimensional operator containing more derivatives of the fields [24, 25].

A power-law running coupling is meaningful only in the Wilsonian sense [26], where a cutoff is inserted and all couplings end up depending on it. Beta functions containing powers of the cutoff cannot be trivially related to scattering amplitudes, however they are necessary to describe the non-perturbative RG flow in systems close to criticality. The non-universality of power-law divergences has as an effect the non-universality of the position of fixed points in theories' space; anyway, the RG flow close to fixed points permits one to extract the universal observables that are used to describe phase transitions, as critical exponents.

2.3.3 Higher derivative theories and different dimensions

In the discussion of universality of one-loop beta functions in the high-energy limit, the observation that there are no IR divergences in off-shell loop integrals had a crucial role. However, this statement is not true anymore if we consider different spacetime dimensionalities or theories with a higher number of derivatives in the kinetic term. In higher derivative theories with $\frac{1}{q^4}$ propagators, the integral related to the diagram 2.2 is

$$\int_k^\Lambda d^4q \, \frac{N(q,p_i)}{q^4(q+p_1)^4 \times \cdots \times (q+p_1+\cdots+p_{n-1})^4} \ , \tag{2.3.37}$$

that, in the integration region near $q = 0$, reduces to

$$\int_k^{\lambda' \ll p_i} d^4q \, \frac{N(0,p_i)}{q^4(p_1)^4 \times \cdots \times (p_1+\cdots+p_{n-1})^4} \ . \tag{2.3.38}$$

This integral is at least logarithmically divergent independently of the particular kinematical configuration, being it on-shell or off-shell. The result is the appearance of terms containing $\log(k)$ in scattering amplitudes, potentially capable of disrupting the correspondence between UV divergences and large logs of external momenta. A similar phenomenon occurs in theories with ordinary 2-derivative kinetic terms in $2d$ spacetimes, where the integrand (2.3.4) turns out to be logarithmically divergent in the IR with generic external momenta. In the rest of this thesis we will discuss how different definitions of the RG flow behave in these particular cases and how to extract the actual high-energy behaviour of scattering amplitudes. While in $d = 2$ the unitarity of well-behaved QFTs allows us to apply the Kinoshita-Lee-Nauberg (KLN) theorem and find IR safe scattering amplitudes, in the higher derivative case unitarity is not ensured for the degenerate massless theories, resulting in some cases in a discrepancy between the physical running and other implementations of the renormalization group.

Chapter 3

Higher derivative theories and the ghost

Since the formulation of the second principle of Newtonian dynamics

$$F = m\frac{d^2x}{dt^2} \ , \tag{3.0.1}$$

the laws of nature have been written in terms of equations of motion which are second order differential equations in time. However, one could be interested in seeing what happens when considering a Lagrangian depending on time derivatives of order larger than one of the coordinates. It was shown in 1850 by Michail Ostrogradskij [27] that any classical theory described by a nondegenerate higher derivative Lagrangian is plagued by the presence of negative energy modes, so it is linearly unstable. Any attempt to canonically quantize them leads to the presence of a pathological class of particles called ghosts in the spectrum of the Hamiltonian, which can be seen equivalently as negative norm or negative energy states [28].

On the other hand, higher derivative quantum field theories are power counting much less divergent in the UV compared to two derivative theories and this feature comes to be very attractive when dealing with renormalization. The most important case is quantum gravity: when one tries to quantize Einstein general relativity (GR) as a standard quantum field theory, it turns out to be non-renormalizable, at two loops in the case of pure gravity [29], already at one loop when coupled with matter [30]. Loops in quantum GR generate new terms in the action with more and more powers of curvature tensors; however, if we consider a bare action containing also terms quadratic in curvatures, the theory becomes higher derivative and, more important, it is renormalizable [31]. In fact, operators with more than two curvatures are no longer generated by quantum corrections, so this modification of Einstein theory, usually known as quadratic gravity (QG) or higher derivative gravity (HDG), is a possible UV completion to general relativity and a potential solution to the problem of quantum gravity.

All these good features of quadratic gravity would be useless without fixing the Ostrogradskij instability. There is the possibility that the quantization process could remove the issue. Often classical instabilities disappear when looking in the quantum, the most famous example of it being the infinite well in the classical potential of the hydrogen atom compared to the spectrum bounded from below of the quantum Hamiltonian. Moreover, history of physics teaches us that the same quantization prescription does not fit to all particles, as one can see from the quantization of spin $\frac{1}{2}$ particles: the classical Hamiltonian has an infinite sea of negative energy states and canonical quantization with commutation relations gives an Hamiltonian unbounded from below, only quantization conditions written in terms of anticommutation rules return a well behaved theory. In this chapter, we will see how the Ostrogradskij instability emerges in classical theories and how different approaches tried to solve the problems related to ghost particles in the quantum world. Then, in the last section, we will expand the discussion of the effects of infrared divergences on the beta functions of higher derivative theories already introduced in the final part of chapter 2 and discuss other features of the renormalization group of this class of models.

3.1 Classical and quantum Ostrogradskij instability

As we anticipated, the naive treatment of higher derivative theories leads to problems both at classical and quantum level. We will see that the classical instability discovered more than 150 years ago produces in the canonically quantized theory an unbounded Hamiltonian or, equivalently, a nonunitary time evolution. We will see how these features manifest themselves in a simple quantum mechanical toy model and in quantum field theories using both canonical and path integral quantizations.

3.1.1 Classical instability

In the discussion of the classical instability of nondegenerate higher derivative theories we will follow [28]. In a classical mechanical system describing a point particle moving in a one-dimensional space, the standard Lagrangian depends only on the coordinate of the particle and its first derivative, namely $L = L(x, \dot{x})$. The equation of motion is given by

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{3.1.1}$$

and, if the Lagrangian is not degenerate, that means $\frac{\partial^2 L}{\partial \dot{x}^2} \neq 0$, it takes the form of a second order differential equation corresponding to the second Newton law (3.0.1). Like all second order differential equations, a solution $x(t)$ is determined by two boundary conditions, which are usually chosen to be be the initial values of $x(t = 0)$ and $\dot{x}(x = 0)$. Thus, the phase space is actually two-dimensional and the canonical choice of coordinates is

$$X = x \qquad \text{and} \qquad P = \frac{\partial L}{\partial \dot{x}} \ . \tag{3.1.2}$$

In a nondegenerate theory, one can write the velocity $\dot{x}$ in terms X and P by inverting the definition of P and introduce the Hamiltonian via a Legendre transform with respect to $\dot{x}$

$$H(X, P) = P\dot{x}(X, P) - L(X, P) \ . \tag{3.1.3}$$

The Hamilton equations governing the motion in phase space are

$$\dot{X} = \frac{\partial H}{\partial P} \ , \tag{3.1.4}$$

$$\dot{P} = -\frac{\partial H}{\partial X} \tag{3.1.5}$$

and they are equivalent to the Euler-Lagrange equation of motion (3.1.1). The Hamiltonian generates time evolution and, if it has no explicit dependence on time, it is a conserved quantity, hence it is usually identified with the energy of the system. If the starting Lagrangian has the usual nondegenerate structure

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \ , \tag{3.1.6}$$

where $V(x)$ is a potential bounded from below, the corresponding Hamiltonian is

$$H = \frac{P^2}{2m} + V(X) \tag{3.1.7}$$

with $P = m\dot{x}$ and is bounded from below by the minimum of the potential.

On the other hand, if we take a higher derivative Lagrangian $L(x, \dot{x}, \ddot{x})$ that is nondegenerate in $\ddot{x}$, i.e. $\frac{\partial^2 L}{\partial \ddot{x}^2} \neq 0$, the Euler-Lagrange equation reads

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0 \ . \tag{3.1.8}$$

In this case the equation of motion can be rewritten as

$$\ddddot{x} = F(x, \dot{x}, \ddot{x}, \dddot{x}) \ , \tag{3.1.9}$$

a very different structure with respect to the Newton equation (3.0.1). Being a fourth order differential equation, four initial conditions are necessary to identify a particular solution. To these initial data correspond four canonical coordinates, and the set chosen by Ostrogradskij is

$$X_1 = x \ , \qquad P_1 = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} \ , \tag{3.1.10}$$

$$X_2 = \dot{x} \ , \qquad P_2 = \frac{\partial L}{\partial \ddot{x}} \ . \tag{3.1.11}$$

Again, thanks to nondegeneracy, the acceleration $\ddot{x}$ can be considered as a function of X_1 X_2 and P_2 by inverting the definition of P_2 itself, and one can define the higher derivative Hamiltonian by a double Legendre transformation

$$H(X_1, X_2, P_1, P_2) = P_1 \dot{x} + P_2 \ddot{x}(X_1, X_2, P_2) - L(X_1, X_2, P_2) \ . \tag{3.1.12}$$

Even in this case the Hamilton equations

$$\dot{X}_i = \frac{\partial H}{\partial P_i} \ , \tag{3.1.13}$$

$$\dot{P}_i = -\frac{\partial H}{\partial X_i} \tag{3.1.14}$$

contain the time evolution of the system and H is a conserved quantity if it does not depend explicitly on time, so it provides a good notion of energy. However, since P_1 appears only linearly in the Hamiltonian, the energy can reach arbitrary negative values, hence the Hamiltonian is not bounded from below and the theory is unstable. Notice that the only request needed to obtain this result was the Lagrangian to be nondegenerate. The expected consequence of this instability is the presence of runaway trajectories in the configuration space and the impossibility of stable localized trajectories.

3.1.2 Quantization of the higher derivative harmonic oscillator

The harmonic oscillator is one of the first models to be quantized in all quantum mechanics courses, so we will start our discussion of higher derivative quantum theories from the higher derivative Pais-Uhlenbeck oscillator [32]. Its simplest version is the higher derivative harmonic oscillator with Lagrangian

$$\begin{aligned} L &= -\frac{1}{2}q\left(\frac{d^2}{dt^2}+\omega_1^2\right)\left(\frac{d^2}{dt^2}+\omega_2^2\right)q \\ &= -\frac{1}{2}\ddot{q}^2+\frac{\omega_1^2+\omega_2^2}{2}\dot{q}^2-\frac{\omega_1^2\omega_2^2}{2}q^2 \ , \end{aligned} \tag{3.1.15}$$

where ω_1 and ω_2 are real parameters. The associated Euler-Lagrange equation is

$$\ddddot{q}+(\omega_1^2+\omega_2^2)\ddot{q}+\omega_1^2\omega_2^2\dot{q} \ , \tag{3.1.16}$$

while the phase space coordinates in Ostrogradskij analysis are defined as

$$x_1 = q \ , \qquad p_1 = (\omega_1^2+\omega_2^2)\dot{q}+\dddot{q} \ , \tag{3.1.17}$$

$$x_2 = \dot{q} \ , \qquad p_2 = -\ddot{q} \ . \tag{3.1.18}$$

The corresponding Hamiltonian reads

$$H = p_1x_2-\frac{1}{2}p_2^2-\frac{\omega_1^2+\omega_2^2}{2}x_2^2+\frac{\omega_1^2\omega_2^2}{2}x_1^2 \tag{3.1.19}$$

and, as expected, it is linear in p_1, thus it is not bounded from below. Assuming $\omega_1 > \omega_2$, we can define a smart change of coordinates [33, 34]

$$X_1 = \frac{1}{\omega_1}\frac{p_1-\omega_1^2x_2}{\sqrt{\omega_1^2-\omega_2^2}} \ , \qquad P_1 = \omega_1\frac{-p_2+\omega_2^2x_1}{\sqrt{\omega_1^2-\omega_2^2}} \ , \tag{3.1.20}$$

$$X_2 = \frac{-p_2+\omega_1^2x_1}{\sqrt{\omega_1^2-\omega_2^2}} \ , \qquad P_2 = \frac{p_1-\omega_2^2x_2}{\sqrt{\omega_1^2-\omega_2^2}} \ , \tag{3.1.21}$$

such that the Hamiltonian reduces to the more familiar form

$$H = -\frac{P_1^2+\omega_1^2X_1^2}{2}+\frac{P_2^2+\omega_2^2X_2^2}{2} \ . \tag{3.1.22}$$

It corresponds to two harmonic oscillators with opposite signs in front. Notice that the canonical transformation (3.1.21) becomes singular when $\omega_1 = \omega_2$, while it is well defined if $\omega_1 < \omega_2$, since the initial Pais-Uhlenbeck oscillator was symmetric in ω_1 and ω_2 and we can just exchange the roles of the two frequencies. In the degenerate case, another canonical transformation allows us to write the quantum Hamiltonian as the energy of two free point particles plus the difference between their angular momenta [32, 35].

Returning to the nondegenerate case, the two oscillators are completely decoupled from each other, so the sign in front of their Hamiltonian is irrelevant: it does not enter in the equations of motions at classical level, while in the quantum theory there is no dynamics, hence the sign of the energy has no relevance. A proof of the fact that the free Pais-Uhlenbeck oscillator is actually well defined can be seen in the fact that there exists another coordinate transformation that gives a positive definite Hamiltonian where the two harmonic oscillators are summed together and not subtracted [36]

$$H = \frac{P_1^2+\omega_1^2X_1^2}{2}+\frac{P_2^2+\omega_2^2X_2^2}{2} \ . \tag{3.1.23}$$

However, if we add an interaction $V(q)$ to the Lagrangian (3.1.15) and to the two Hamiltonians (3.1.22) and (3.1.23), only the former gives equations of motion equivalent to the Lagrangian ones. In fact, the presence of an interaction between the two harmonic oscillators allows positive energy states to decay in negative energy states and makes manifest the difference between a positive definite Hamiltonian and an indefinite one.

Canonical quantization

Starting from the Hamiltonian (3.1.22), canonical quantization can be easily implemented separately in the two harmonic oscillators. We promote P_i to $\hat{P}_i = -i\partial_{X_i}$ and impose the canonical commutation relations $[X_i, P_j] = \delta_{ij} i$, $[X_i, X_j] = 0$ and $[P_i, P_j] = 0$. It is convenient to introduce, as usual, creation and annihilation operators, respectively $a_i^\dagger$ and a_i, which commute as $[a_i, a_j^\dagger] = \delta_{ij}$ and act on the vacuum state $|0\rangle$ as $a_i |0\rangle = 0$. The quantum Hamiltonian, written in terms of these operators, takes the form

$$H = -\omega_1(a_1^\dagger a_1) + \omega_2(a_2^\dagger a_2) + \frac{1}{2}(\omega_2 - \omega_1) \ . \tag{3.1.24}$$

Since the Hilbert space is the direct sum of the Hilbert spaces of two harmonic oscillators, the eigenfunctions of the Hamiltonian will be a product of eigenfunctions of the two harmonic oscillators with labels n_1 and n_2. So, the energy eigenvalues associated to the eigenstates

$$|n_1, n_2\rangle = \frac{\left(a_1^\dagger\right)^{n_1}}{\sqrt{n_1!}} \frac{\left(a_2^\dagger\right)^{n_2}}{\sqrt{n_2!}} |0\rangle \tag{3.1.25}$$

are

$$E_{n_1,n_2} = \omega_2 \left(n_2 + \frac{1}{2}\right) - \omega_1 \left(n_1 + \frac{1}{2}\right) \ . \tag{3.1.26}$$

Hence, the spectrum of H is discrete and not bounded from below, due to the minus sign in front of the oscillator with frequency ω_1. On the other hand, the degenerate case $\omega_1 = \omega_2$ has a continuous spectrum summed with discrete steps ranging from $-\infty$ to $+\infty$ due to the quantized eigenvalues of angular momenta. The result is an infinitely degenerate spectrum with non-normalizable eigenfunctions [37].

To avoid an energy spectrum unbounded from below, a different quantization prescription can be used in the nondegenerate case, leading to a positive definite Hamiltonian at the price of a breakdown of unitarity [31]. By defining a new ground state $|\bar{0}\rangle$ such that it obeys the relations

$$a_2 |\bar{0}\rangle = 0 = a_1^\dagger |\bar{0}\rangle \ , \tag{3.1.27}$$

one can invert the roles of creation and annihilation operators in the harmonic oscillator with a minus sign in front. This vacuum state is non-normalizable [28]: its wave function can be calculated and turns out to be

$$\psi(x_1, x_2) = \exp\left[\frac{-x_1^2\omega_1\omega_2 + x_2}{2}(\omega_1 + \omega_2) - i x_1 x_2 \omega_1 \omega_2\right] \tag{3.1.28}$$

and the integral over x_2 of $|\psi|^2$ is divergent. Anyway, it is still possible to formally introduce the new set of eigenstates

$$|\overline{n_1, n_2}\rangle = \frac{\left(a_2^\dagger\right)^{n_2}}{\sqrt{n_2!}} \frac{a_1^{n_1}}{\sqrt{n_1!}} |\bar{0}\rangle \tag{3.1.29}$$

with the associated positive spectrum

$$E_{\overline{n_1,n_2}} = \omega_1 \left(n_1 + \frac{1}{2} \right) + \omega_2 \left(n_2 + \frac{1}{2} \right) \ . \tag{3.1.30}$$

Unfortunately, this comes with a price: the commutation relations have not been changed, so the norm of states (3.1.30) is

$$\frac{\langle \overline{n_1,n_2} | \overline{n_1,n_2} \rangle}{\langle \bar{0} | \bar{0} \rangle} = \frac{\langle \bar{0} | \, [a_2, a_2^\dagger]^{n_2} [a_1^\dagger, a_1]^{n_1} \, | \bar{0} \rangle}{\langle \bar{0} | \bar{0} \rangle} = (-1)^{n_1} \ , \tag{3.1.31}$$

that is negative if n_1 is odd. These negative norm states are usually called ghosts in the literature.

The Copenhagen interpretation of quantum mechanics as a probabilistic theory requires physical states to have a positive norm, in order to define a consistent notion of probability. In particular, we are going to show that the usual notion of probability in the measurement process in quantum mechanics and the requirement of total probabilities summing up to one are in contradiction in presence of negative norm states. Consider a theory with a complete orthogonal basis that can be divided in a subset U of states $|u_i\rangle$ normalized to 1 and another subset V of states $|v_j\rangle$ normalized to -1. The indefinite inner product allows us to introduce the completeness relation

$$1 = \sum_U |u_i\rangle \langle u_i| - \sum_V |v_j\rangle \langle v_j| \ . \tag{3.1.32}$$

It is quite trivial to check that it works well with the elements of the basis. Imagine now that it is possible to construct an experimental apparatus that measures an observable A having this basis as eigenstates. As usual, we would like to define the probability to obtain from a measurement on a system prepared in the state $|\phi\rangle$ the result a_k associated with the eigenstate $|a_k\rangle$ belonging to U or V. In standard quantum mechanics this probability is given by the Born rule

$$P_{\phi,a_k} = \frac{|\langle \phi | a_k \rangle|^2}{\langle \phi | \phi \rangle \langle a_k | a_k \rangle} \ . \tag{3.1.33}$$

Obviously, if one of ϕ or a_k is a negative norm state, we will obtained negative probability, which is inadmissible. An alternative way could be defining the probability as

$$P_{\phi,a_k} = \frac{|\langle \phi | a_k \rangle|^2}{|\langle \phi | \phi \rangle \langle a_k | a_k \rangle|} \ , \tag{3.1.34}$$

however, if we take as ϕ a positive norm state normalized to 1 and we introduce a resolution of the identity in its norm, we find

$$1 = \langle \phi | \phi \rangle = \sum_U \langle \phi | u_i \rangle \langle u_i | \phi \rangle - \sum_V \langle \phi | v_j \rangle \langle v_j | \phi \rangle = \sum_U P_{\phi,u_i} - \sum_V P_{\phi,v_j} \ . \tag{3.1.35}$$

This implies that, if the overlap between $|\phi\rangle$ and at least one negative norm state v_i is non zero, the total probability

$$\sum_U P_{\phi,u_i} + \sum_V P_{\phi,v_j} \tag{3.1.36}$$

is bigger than one. Thus, in order to have a sensible definition of probability, we have to exclude negative norm states from the physical spectrum. Nevertheless, if interactions allow an overlap between the time evolution of a positive norm state and a ghost, the total probability associated to the subspace U cannot be conserved, so the time evolution is not unitary.

Path integral quantization

As an alternative to canonical quantization, one can define a path integral for the Pais-Uhlenbeck oscillator. We introduce the auxiliary variable r and we rewrite the action (3.1.15) as

$$L = \frac{\omega_1^2+\omega_2^2}{2}\dot{q}^2 - \frac{\omega_1^2\omega_2^2}{2}q^2 + r\ddot{q} + \frac{1}{2}r^2 \ . \tag{3.1.37}$$

the equation of motion of r is $r = -\ddot{q}$, hence on-shell we recover the initial Lagrangian. With the redefinitions

$$X_1 = \frac{1}{\sqrt{\omega_1^2-\omega_2^2}}\left(\omega_2^2 q - r\right) \ , \tag{3.1.38}$$

$$X_2 = \frac{1}{\sqrt{\omega_1^2-\omega_2^2}}\left(\omega_1^2 q - r\right) \ , \tag{3.1.39}$$

and their inverse expressions

$$q = \frac{X_2 - X_1}{\sqrt{\omega_1^2-\omega_2^2}} \ , \tag{3.1.40}$$

$$r = \left(\frac{X_2}{\omega_1^2} - \frac{X_1}{\omega_2^2}\right)\frac{\omega_1^2\omega_2^2}{\sqrt{\omega_1^2-\omega_2^2}} \ , \tag{3.1.41}$$

also the action (3.1.37) can be written as two harmonic oscillators with opposite signs:

$$L = \frac{1}{2}\left(\dot{X}_2^2 - \omega_2^2 X_2^2\right) - \frac{1}{2}\left(\dot{X}_1^2 - \omega_1^2 X_1^2\right) \ . \tag{3.1.42}$$

The transition amplitudes between an initial state $|x_{1(i)}, x_{2(i)}\rangle$ at time t_i and a final state $|x_{1(f)}, x_{2(f)}\rangle$ at time t_f is defined in path integral quantization as

$$\langle x_{1(f)}, x_{2(f)}|\, \eta e^{iH(t_f-t_i)}\, |x_{1(i)}, x_{2(i)}\rangle = \int_{x_j=x_{j(i)}}^{x_j=x_{j(f)}} Dx_1 Dx_2 e^{i\int_{t_i}^{t_f} dtL} \ . \tag{3.1.43}$$

To make it convergent, we have to add a small imaginary part to the Lagrangian, and there are two ways to do it. If we send $\omega_2^2 \to \omega_2^2 - i\epsilon$ and $\omega_1^2 \to \omega_1^2 + i\epsilon$ in (3.1.42), the path integral is convergent. The Lagrangian can be analytically extended to imaginary time by Wick-rotating the time variables of the two harmonic oscillators in different directions in the complex plane ($t_2 \to -i\tau_2$ and $t_1 \to i\tau_1$), the result is the Euclidean Lagrangian

$$L_E = \frac{1}{2}\left(X_2'^2 + \omega_2^2 X_2^2\right) + \frac{1}{2}\left(X_1'^2 + \omega_1^2 X_1^2\right) \ , \tag{3.1.44}$$

where primes stand for derivatives respect to τ_i. Notice that the action built from it is positive definite, however such independent rotation is possible only in noninteracting theories. If the two oscillators interact with each other, there is only one time coordinate, so a simple rotation of t to the imaginary axis without including any poles in the contour integral is impossible. Moreover, the direct consequence of the different regularization prescriptions between the two oscillators is that causality relations are inverted in the two systems. Hence, the price to pay for a converging functional integral is that the arrow of time is inverted for the harmonic oscillator with frequency ω_1 respect to the one with ω_2 [38]. This is equivalent to the presence of negative energies in the spectrum, exactly as in canonical quantization with vacuum annihilated by both a_i operators.

On the other hand, with the different prescriptions $\omega_2^2 \to \omega_2^2 - i\epsilon$ and $\omega_1^2 \to \omega_1^2 - i\epsilon$, the path integral of the oscillator with frequency ω_1 does not converge, so the path integral is not well defined. This occurrence is equivalent to the nonrenormalizabilty of the vacuum state defined as (3.1.27). With this convention, there exists a unique analytical Wick rotation allowed, that is the usual $t \to -i\tau$ and leads to the euclidean action

$$L_E = \frac{1}{2}\left(X_2'^2 + \omega_2^2 X_2^2\right) - \frac{1}{2}\left(X_1'^2 + \omega_1^2 X_1^2\right) \ . \tag{3.1.45}$$

At first sight it might seem that in this case the Euclidean path integral is not convergent, since the two harmonic oscillator have opposite signs. However, writing it in terms of q one finds

$$L_E = \frac{1}{2}q''^2 + \frac{\omega_1^2 + \omega_2^2}{2}q'^2 + \frac{\omega_1^2\omega_2^2}{2}q^2 \ , \tag{3.1.46}$$

the result of the same Wick rotation acting on (3.1.15). This expression is a sum of squares of real variables, hence positive.

3.1.3 Higher derivative quantum field theory

The Ostrogradskij construction can be easily extended to quantum field theory [39]. Starting from a Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi)$ depending in a nondegenerate way on the second time derivative of the field, the Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta\phi} - \frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} + \frac{\partial^2}{\partial x^\mu\partial x^\nu}\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi} = 0 \tag{3.1.47}$$

is a partial differential equation containing the fourth order time derivative of the field. So, we need 4 fields to identify a trajectory in the phase space. Hence, we introduce two fields

$$\phi_1 = \phi \ , \tag{3.1.48}$$
$$\phi_2 = \dot{\phi} \tag{3.1.49}$$

and their conjugated momenta

$$\pi_1 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\ddot{\phi}} \ , \tag{3.1.50}$$
$$\pi_2 = \frac{\partial\mathcal{L}}{\partial\ddot{\phi}} \ . \tag{3.1.51}$$

Thanks to the nondegeneracy of the Lagrangian, we can invert the last definition and write $\dot{\phi}_2 = \ddot{\phi}$ as a function of ϕ_1, ϕ_2 and π_2. With these canonical variables we construct the Hamiltonian density by a double Legendre transform

$$\mathcal{H}(\phi_1, \phi_2, \pi_1, \pi_2) = \pi_1\dot{\phi}_1 + \pi_2\dot{\phi}_2(\phi_1, \phi_2, \pi_2) - \mathcal{L}(\phi_1, \phi_2, \pi_2) \ . \tag{3.1.52}$$

The resulting Hamilton equations

$$\dot{\phi}_i = \frac{\delta\mathcal{H}}{\delta\pi_i} \ , \tag{3.1.53}$$
$$\dot{\pi}_i = -\frac{\delta\mathcal{H}}{\delta\phi_i} \tag{3.1.54}$$

are equivalent to the field equation (3.1.47). Again, $\mathcal{H}$ is linear in π_1, so we have an instability.

Free field canonical quantization

If we consider the free higher derivative scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[-(m_1^2+m_2^2)\partial_\mu\phi\partial^\mu\phi - m_1^2m_2^2\phi\phi - \Box\phi\Box\phi\right] \ , \tag{3.1.55}$$

canonical quantization proceeds in a way similar to 2-derivative QFTs: one has to impose canonical commutation relations

$$[\phi_i,\pi_j] = i\delta^3(\vec{x})\delta_{ij} \ , \quad [\phi_i,\phi_j]=0 \ , \quad [\pi_i,\pi_j]=0 \tag{3.1.56}$$

on the canonical variables

$$\phi_1 = \phi \ , \qquad \pi_1 = (m_1^2+m_2^2)\dot{\phi} + \Box\dot{\phi} \ , \tag{3.1.57}$$

$$\phi_2 = \dot{\phi} \ , \qquad \pi_2 = -\Box\phi \ , \tag{3.1.58}$$

that is equivalent to replace the momenta π_i with $-i\frac{\delta}{\delta\phi_i}$. Then we can observe that, if we write the field ϕ in terms of its spatial Fourier transform as

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}}\phi(\vec{p},t) \ , \tag{3.1.59}$$

the Euler-Lagrange equation

$$\left[m_1^2m_2^2 - (m_1^2+m_2^2)\Box + \Box^2\right]\phi = 0 \tag{3.1.60}$$

reduces to

$$\left[m_1^2m_2^2 - (m_1^2+m_2^2)\vec{p}^{\,2} + \vec{p}^{\,4} - (m_1^2+m_2^2+2\vec{p}^{\,2})\partial_t^2 + \partial_t^4\right]\phi = 0 \ , \tag{3.1.61}$$

that is the equation of motion of a free Pais-Uhlenbeck oscillator (3.1.16) with fancy frequencies. After the spatial Fourier transform, in standard 2-derivative theories the equation of motion coincides for each mode $\vec{p}$ with a harmonic oscillator, and quantization proceeds by introducing a set of creation and annihilation operators for each of them. Analogously, here we will have two harmonic oscillators with opposite signs and two couples of creation and annihilation operators for each value of spatial momentum, the one with negative energy creating and destructing ghost particles. That means that the discussion about the negative spectrum of the Hamiltonian and unitarity can also be applied to higher derivative QFT.

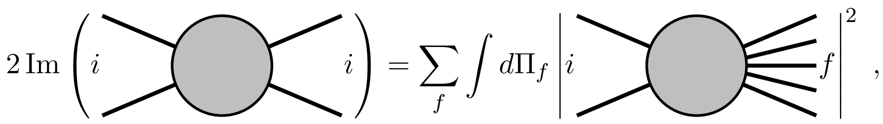
The unitarity of time evolution in a quantum field theory is usually rephrased in terms of unitary of the S-matrix and this feature translates in the well-known optical theorem. If we write the scattering matrix as

$$S = 1 + iT \ , \tag{3.1.62}$$

requiring S to be unitary, i.e. $S^\dagger S = 1$, implies that $T - T^\dagger = T^*T$. If we focus on diagonal matrix elements of the interacting part of the S-matrix $\langle i|T|i\rangle$, with i a physical state of the theory, their imaginary part must be equal to the sum over all physical states $|f\rangle$ states of the modulus square of the transition amplitude between $|i\rangle$ and $|f\rangle$, namely [2]

$$2\,\mathrm{Im}\,\langle i|T|i\rangle = \sum_f |\langle f|T|i\rangle|^2 \ . \tag{3.1.63}$$

This equation can be diagrammatically depicted as



(3.1.64)

where Π_f is the integration measure of the phase space integral of the state f and the summation runs over the particle content of the state.

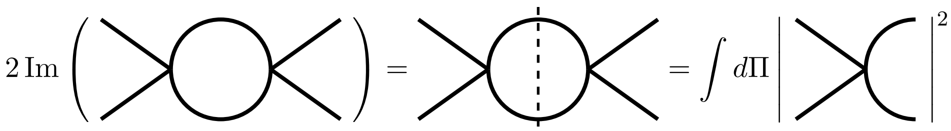
This theorem is usually exploited via the so-called Cutkosky rules, which state that the imaginary part on the left can be computed in three simple steps:

- separate loop diagrams in two disconnected pieces not mixing in and out states via a cut through internal lines that allows the cut propagators to be simultaneously on-shell;

- substitute cut lines with on-shell Dirac delta via the rule $\frac{1}{p^2+m^2-i\epsilon} \to 2\pi i \delta(p^2+m^2)$ and perform the loop integrals;

- sum over all the possible ways to execute such cut.

For example, in the $\lambda\phi^4$ scalar theory considered in chapter 2, the Cutkosky rules for the $2 \to 2$ scattering can be diagrammatically depicted as



(3.1.65)

The result of this procedure matches with the right hand side of the optical theorem if, for each cut, the state composed by on-shell particles from cut propagators is an element of the set of physical states over which the sum in the right hand side is carried on. As stated before, if we use a quantization prescription that admits negative norm ghosts, we have to exclude these states from the Hilbert space of physical states. However, if on-shell ghosts are generated inside loops on the left hand side, the matrix element will acquire a virtual part that cannot be produced on the right hand side, since such intermediate states are absent from the summation. So, the optical theorem is violated in presence of negative norm ghosts. In mathematical terms, if we separate the states $|j\rangle$ into positive norm states $|j_+\rangle$ and negative norm states $|j_-\rangle$ and we reduce the space of physical states to $|j_+\rangle$, the optical theorem would require

$$2\,\mathrm{Im}\,\langle i_+|\,T\,|i_+\rangle = \sum_{f_+} |\,\langle f_+|\,T\,|i_+\rangle\,|^2 \ . \tag{3.1.66}$$

Nevertheless, what we obtain from Cutkosky rules and explicit calculations of the left hand side is

$$2\,\mathrm{Im}\,\langle i_+|\,T\,|i_+\rangle = \sum_{f_+} |\,\langle f_+|\,T\,|i_+\rangle\,|^2 + \sum_{f_-} |\,\langle f_-|\,T\,|i\rangle\,|^2 \ , \tag{3.1.67}$$

hence the S-matrix is not unitary.

Path integral quantization

In quantum field theory the path integral quantization is often preferred to the canonical one, due to its explicit covariance and the possibility to properly quantize gauge theories. The Lorentzian path integral associated with the Lagrangian (3.1.55) is

$$Z[J] = \int D\phi \, e^{i\int d^4x\,[\mathcal{L}+\phi J]} \,. \tag{3.1.68}$$

Also in this case we expect to need two fields to properly describe the theory, so, analogously to the quantum mechanical path integral, we want to introduce an auxiliary field η that, when integrated out, gives back the starting Lagrangian [39]. The action

$$S[\phi,\eta] = \int d^4x \left[-\frac{1}{2}(m_1^2+m_2^2)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_1^2m_2^2\phi^2 - \eta\Box\phi + \frac{1}{2}\eta^2\right] \tag{3.1.69}$$

performs the requested task, so we can build with it the equivalent partition function

$$Z[J] = \int D\phi D\eta \, e^{iS[\phi,\eta]+i\int d^4x\,\phi J} \,. \tag{3.1.70}$$

With a redefinition in the field variables

$$\phi = \frac{\chi-\psi}{\sqrt{m_1^2-m_2^2}} \,, \tag{3.1.71}$$

$$\eta = \left(\frac{\chi}{m_1^2} - \frac{\psi}{m_2^2}\right)\frac{m_1^2m_2^2}{\sqrt{m_1^2-m_2^2}} \tag{3.1.72}$$

and $J \to J\sqrt{m_1^2-m_2^2}$, the path integral reduces to the product of the partition functions of two free particles with opposite signs in front of the relative actions

$$\begin{aligned} Z[J] = &\int D\chi \, e^{i\int d^4x\,[-\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m_2^2\chi^2+\chi J]} \\ \times &\int D\psi \, e^{-i\int d^4x\,[-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m_1^2\psi^2+J\psi]} \,. \end{aligned} \tag{3.1.73}$$

At this point we have to introduce a small imaginary mass to make the integral convergent and impose a contour prescription to the propagators. Similarly to the quantum mechanical case, it is exactly at this stage that the analogue of the choice between a positive definite spectrum and a unitary time evolution we had to make in canonical quantization comes into play. If we add a term $i\epsilon\phi^2$ to the Lagrangian (3.1.55), or equivalently $i\epsilon(\chi^2-\psi^2)$ to the path integral (3.1.73), the functional integral converges, but the field ψ, corresponding to the ghost particle, has a different causal direction. If we add an interaction that prevents us from factorizing the two functional integrals, there is a unique time variable and the ghost has negative energy modes propagating forward in time and positive energy modes propagating backward. This choice is equivalent to allowing negative energy states in the spectrum. On the other hand, if we use the same prescription for both particles, i. e. we add $i\epsilon(\chi^2+\psi^2)$, the partition function for ψ diverges and the ghost has a negative pole in the propagator, equivalent to a negative norm.

This can be made even clearer by looking at the propagators [40]. The propagator of (3.1.55) can be decomposed in two partial fractions

$$\frac{-i}{p^4+(m_1^2+m_2^2)p^2+m_1^2m_2^2} = \frac{-i}{(p^2+m_1^2)(p^2+m_2^2)} = \frac{-i}{m_1^2-m_2^2}\left[\frac{1}{p^2+m_2^2} - \frac{1}{p^2+m_1^2}\right] \,, \tag{3.1.74}$$

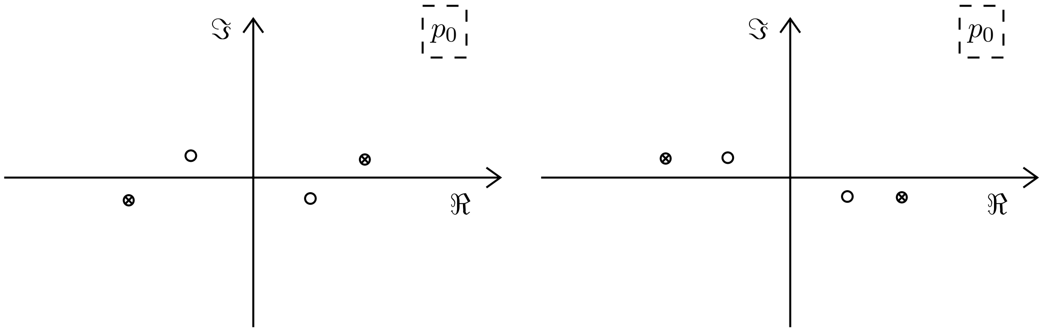


Figure 3.1: Poles structure with prescription (3.1.75) on the left and with prescription (3.1.76 on the right in the complex p_0 plane. Poles of the normal particle are represented by simple circles, while Ghost poles are represented by crossed circles.

the first part corresponding to the positive energy mode and the second corresponding to the ghost. The prescription $i\epsilon\phi$ gives

$$\frac{-i}{m_1^2 - m_2^2}\left[\frac{1}{p^2 + m_2^2 - i\epsilon} - \frac{1}{p^2 + m_1^2 + i\epsilon}\right] , \tag{3.1.75}$$

so the pole of the ghost propagator in the complex energy plane with negative real part sits below the real axis, while that with positive real part stands above the real axis, as depicted in the picture on the left of figure 3.1. That means that if we do a time Fourier transform, negative frequency modes contribute to the retarded propagator, while those with positive energy contribute to the advanced one, the opposite of normal particles with Feynman prescription. The second prescription $i\epsilon(\chi^2 + \psi^2)$ is equivalent to use the usual Feynman prescription for both particles:

$$\frac{-i}{m_1^2 - m_2^2}\left[\frac{1}{p^2 + m_2^2 - i\epsilon} - \frac{1}{p^2 + m_1^2 - i\epsilon}\right] . \tag{3.1.76}$$

The pole structure in this case is depicted in the picture on the right of figure 3.1. With this choice, all positive energy modes propagate forward in time, however the price to pay is a negative residue associated with the poles in the ghost propagator. The norm of a one-particle state $|p\rangle$ is proportional to this residue, hence in this prescription ghost states have negative norms, with all the issues related to unitarity already discussed.

Moreover, it must be stressed that the good properties of higher derivative theories concerning renormalization are, from a technical point of view, the result of a cancellation between the two propagators in the high-energy limit, where $m_i^2 \ll p^2$, that leaves $\frac{1}{p^4}$ as the leading term. This cancellation takes place in the Euclidean version of the theory, however the former prescription (3.1.75) is not an analytical continuation of the Euclidean theory because of poles in the first quadrant of the complex plane for p_0. Only the latter choice (3.1.76) can be analytically Wick rotated and vice versa, hence it is necessary to obtain also in Lorentzian signature the goal for which higher derivative theories are usually introduced, namely to improve the UV behaviour of a nonrenormalizable theory. This can be also seen in the following way [34]: if we assign two different prescriptions ϵ and ϵ' to the two partial propagators, the total propagator can be written as

$$\begin{aligned}&\frac{-i}{m_1^2 - m_2^2}\left[\frac{1}{p^2 + m_2^2 - i\epsilon} - \frac{1}{p^2 + m_1^2 - i\epsilon'}\right]\\ &\quad = \frac{-i}{(p^2 + m_1^2 - i\epsilon)(p^2 + m_2^2 - i\epsilon)} - \frac{\pi\delta}{m_1^2 - m_2^2}(p^2 + m_1^2)[\text{sign}(\epsilon') - \text{sign}(\epsilon)] ,\end{aligned} \tag{3.1.77}$$

where we have added and subtracted $\frac{-i}{m_1^2-m_2^2}\frac{1}{p^2+m_1^2-i\epsilon}$ to the left hand side and used the well-known relation

$$\lim_{\epsilon\to 0}\frac{1}{x-i\epsilon}=P\frac{1}{x}+i\pi\delta(x)\ , \tag{3.1.78}$$

with P denoting the principal value. To have a quartic propagator we need to remove the Dirac delta, hence we need $\text{sign}(\epsilon')=\text{sign}(\epsilon)$.

Massless case

In quantum field theory, the degenerate limit $m_1=m_2\neq 0$ is not much interesting, since the main reason of interest in higher derivative quantum field theories is gravity, and the graviton is massless. Hence we will briefly discuss only the $m_1=m_2=0$ case, that can be seen as a toy model for conformal gravity. The massless version of (3.1.55) is

$$\mathcal{L}=-\frac{1}{2}\Box\phi\Box\phi \tag{3.1.79}$$

and the associated equation of motion is

$$\Box^2\phi=0\ . \tag{3.1.80}$$

By a spatial Fourier transform it can be written as

$$(\partial_t^4+2\vec{p}^{\,2}\partial_t^2+\vec{p}^{\,4})\phi=0\ , \tag{3.1.81}$$

so one obtains a degenerate Pais-Uhlenbeck oscillator for each mode $\vec{p}$. On the other hand, in coordinate space, it has two family of solutions:[41, 42]

$$\phi_k=e^{ik_\mu x^\mu}\quad\text{and}\quad \phi'_{k,N}=x^\mu N_\mu e^{ik_\mu x^\mu} \tag{3.1.82}$$

where N_μ is a generic vector and $k^2=0$. It could seem that the family of solutions $\phi'_{k,N}$ is much wider than ϕ_k, because of the freedom in the choice of N, but actually a solution ϕ'_k is linearly independent of the plain wave solutions only if $N^\mu k_\mu\neq 0$. This can be easily seen from the fact that

$$\phi'_{k,N}=N_\mu\partial_{k_\mu}\phi_k=\lim_{\epsilon\to 0}\frac{\phi_{k+\epsilon N}-\phi_k}{\epsilon} \tag{3.1.83}$$

The last expression is a linear combination of two solutions ϕ_k, $\phi_{k'}$ with $k'=k+\epsilon N$ only if $k'^2=0$, that implies, at linear order in ϵ, $N^\mu k_\mu=0$. The space of vectors orthogonal to a light-like vector has dimension three, hence we can choose N_μ non zero only in the time direction.

$$\phi(x)=\int\frac{d^3\vec{k}}{(2\pi)^3}c_k\phi_k+c'_k\phi'_k\quad\text{with now}\quad \phi'_k=te^{ik_\mu x^\mu} \tag{3.1.84}$$

One could be tempted to remove the modes ϕ' from the space of physical states [43], since they grow linearly with time, so they are not well defined asymptotic states. This is equivalent to consider only scattering processes between bunches of ϕ_k modes. However, this would provoke problems with unitarity and, even worse, ϕ_k and ϕ'_k are not scalars under Poincarè group because they are mixed up by non-purely spatial transformations [41][42]. Considering a superposition of static and growing modes, it is possible to define a Poincarè invariant subspace that could furnish a good space of physical states [42], but the free theory (3.1.79) has a gauge symmetry $\phi\to\phi+\Lambda$ with $\Box\Lambda=0$ that does not leave this reduced space invariant. That means that the free massless higher derivative theory does not contain any meaningful physical state [42, 44, 45]. By introducing interactions that break the symmetry, one could a priori obtain a meaningful theory, however this would require a nonperturbative approach. There is the possibility that such a field can never exists as an asymptotic state, but only as a virtual particle inside interaction diagrams.

3.2 Beyond Ostrogradskij

Considering all these problems, one could be tempted to discard higher derivative theories from the landscape of physically viable theories. However, in recent years, it has been shown both numerically and analytically that the interacting Pais-Uhlenbeck oscillator can have stable trajectories in the presence of particular potentials. In fact, some models admit islands of stability, which are regions of the phase space where trajectories remain confined [36, 37], while others have been shown analytically to be globally stable [46]. Moreover, quadratic gravity itself seems to be classically metastable below the Planck scale [47]. These surprisingly good features of some classical higher derivative theories despite of Ostrogradskij theorem motivated physicists in the quest for a consistent formulation of the quantum version of higher derivative theories. In this Section we will briefly present two possible general approaches to solve the related problems of the unbounded spectrum of the Hamiltonian and the ill-defined notion of probability.

3.2.1 Imaginary variables

The first approach consists in complexifying canonical variables in order to remove negative energies from the spectrum of the Hamiltonian. This modification can be introduced alternatively as an attempt to apply in a meaningful way the ideas of quantum mechanics to spaces with an indefinite inner product, or as a particular case of the implementation of quantum mechanics in theories with a non-Hermitan, but $\mathcal{PT}$ symmetric, Hamiltonian.

Pauli-Dirac quantization

The Pauli-Dirac quantization is a quantization prescription different from the canonical one introduced by Pauli [48] as a prosecution of a previous work by Dirac [49] and later developed in [34, 50, 51]. This prescription is meant to treat theories with an inner product admitting negative norm states, as the Pais-Uhlenbeck oscillator quantized with the vacuum defined in (3.1.27). The main criticalities that emerged in that case were the non-normalizable wave function of the vacuum and the absence of a well-defined notion of probability. Pauli-Dirac quantization admits Hermitian operators with purely imaginary spectrum. Consider a generic operator $\hat{x}$ with

$$\hat{x}\,|x\rangle = ix\,|x\rangle \tag{3.2.1}$$

with x real, we will have $\langle x'|\,\hat{x}\,|x\rangle = ix\,\langle x'|x\rangle$. At the same time, we want $\hat{x}$ to be Hermitian, hence $ix\,\langle x'|x\rangle = \langle x|\,\hat{x}\,|x'\rangle^* = -ix'\,\langle x'|x\rangle$, that can be reduced to the condition $(x+x')\,\langle x'|x\rangle = 0$. Up to an arbitrary normalization, this equation is solved by

$$\langle x'|x\rangle = \delta(x+x')\ . \tag{3.2.2}$$

We can introduce the operator η that acts on the eigenstates as $\eta\,|x\rangle = |-x\rangle$ and use it to write the resolution of the identity in terms of the eigenstates of this imaginary operator

$$\int dx\,|x\rangle\,\langle -x| = 1 = \int dx\,|x\rangle\,\langle x|\,\eta \quad \text{and} \quad \int dx\,|x\rangle\,\langle x| = \eta\ . \tag{3.2.3}$$

In order to preserve the canonical commutation relations, also the conjugated variable $\hat{p}$ must have purely imaginary eigenvalues and must be quantized à la Pauli-Dirac. It acts on the wavefunction $\phi(x) = \langle x|\,\eta\,|\phi\rangle$ as $\frac{d}{dx}\psi(x)$ and we have

$$\hat{p}\,|p\rangle = ip\,|p\rangle\ ,\quad \langle p'|p\rangle = \delta(p'+p)\ ,\quad \int dp\,|p\rangle\,\langle p| = \eta\ . \tag{3.2.4}$$

Summarizing, this new prescription is equivalent to applying the canonical transformation $x \to ix$ and $p \to -ip$ on variable quantized as usual. With this new tool, we can return to the Pais-Uhlenbeck oscillator. We observed that the vacuum defined in (3.1.27) cannot be normalized to one, since the modulus square of its wave function (3.1.28) is divergent in x_2; nevertheless, if we use Dirac-Pauli quantization on x_2 and its conjugated momentum p_2 and the canonical prescription on x_1 and p_1, the vacuum wave function becomes

$$\psi(x_1, x_2) = \exp\left[\frac{-x_1^2\omega_1\omega_2 - x_2}{2}(\omega_1 + \omega_2) + x_1x_2\omega_1\omega_2\right] \tag{3.2.5}$$

and is now normalizable, as can be seen by solving simple Gaussian integrals in x_1 and x_2.

This is not a completely arbitrary prescription, since it is coherent with the properties of canonical variables under time reversal symmetry T, that sends $t \to -t$. While x_1 is even, i.e. $Tx_1T^{-1} = x_1$, $x_2 = \dot{q}$ is odd, that means $Tx_2T^{-1} = x_2$, because it contains a time derivative. On the other hand, being T anti-unitary, $x_2 = i\dot{q}$ is even under time reversal like x_1.

In a theory with purely imaginary operators, one can define also a meaningful Euclidean path integral starting from a newly defined transition amplitude

$$\langle x_{1(f)}, x_{2(f)}| \eta e^{-H(\tau_f - \tau_i)} |x_{1(i)}, x_{2(i)}\rangle \tag{3.2.6}$$

and introducing as usual a large number of identities $1 = \int dx\, |x\rangle \langle x| \eta$ at infinitesimal time steps. In the continuous limit, thanks to the η insertions, one finds

$$\langle x_{1(f)}, x_{2(f)}| \eta e^{-H(\tau_f - \tau_i)} |x_{1(i)}, x_{2(i)}\rangle = \int Dx_1Dx_2Dp_1Dp_2 e^{\int d\tau\, ip_1x_1' + ip_2x_2' - \bar{H}} \ , \tag{3.2.7}$$

where

$$\begin{aligned}\bar{H} &= \frac{\langle p_1, p_2| H |x_1, x_2\rangle}{\langle p_1, p_2|x_1, x_2\rangle} \\ &= ip_1x_2 + \frac{1}{2}p_2^2 + \frac{\omega_1^2 + \omega_2^2}{2}x_2^2 + \frac{\omega_1^2\omega_2^2}{2}x_1^2\end{aligned} \tag{3.2.8}$$

is a new Hamiltonian different from the one defined by Ostrogradskij (3.1.19). We will briefly discuss how to write this complex Hamiltonian in the form (3.1.23) later on. Now we can solve the functional integral in p_1, that forces via a Dirac delta $x_2 = x_1'$, and also the one in p_2, that can be done exactly too, hence we obtain

$$\int Dq\, e^{-\int d\tau L_E} \tag{3.2.9}$$

with

$$L_E = \frac{1}{2}q''^2 + \frac{\omega_1^2 + \omega_2^2}{2}q'^2 + \frac{\omega_1^2\omega_2^2}{2}q^2 \ . \tag{3.2.10}$$

The latter is the Euclidean Lagrangian (3.1.46) and defines a finite euclidean path integral. It is important to stress that a potential $V(q = x_1)$ can be inserted in the action without major issues concerning the reality of the spectrum of the Hamiltonian, since it depends only on the real operator $\hat{x}_1$. This is not true if one complexifies the variable X_1 and P_1 instead of x_2 and p_2, as suggested in [52], since in this case the appearance of a complex part in the potential cannot be excluded in general. In fact, the inverse transformation of (3.1.21) for x_1 is

$$q = x_1 = \frac{X_2 - P_1/\omega_1}{\sqrt{\omega_1^2 - \omega_2^2}} \ , \tag{3.2.11}$$

so a generic potential $V(q = x_1)$ would acquire an imaginary part if P_1 were taken complex. With this choice in quantization, requiring a real spectrum for the Hamiltonian would impose a strict constraint on admissible interactions.

In Pauli Dirac quantization of higher derivative theories, the Lorentzian path integral is defined as the standard Wick rotation of the Euclidean one, i. e. $\tau \to it$, which returns the starting Lagrangian (3.1.15) and leads to the prescription (3.1.76) in propagators. Moreover, in the classical limit that corresponds to first order in stationary phase approximation, one recovers the classical equation of motion (3.1.16). The Hamilton equations can also be extracted from the action in equation (3.2.8) by minimizing it with respect to p_i, x_i and x_i'. If we consider the variation with respect to p_2 we get

$$ix_2' = \frac{\partial \bar{H}}{\partial p_2} \tag{3.2.12}$$

and, by imposing $x_2 = x_1'$, it becomes

$$-x_1'' = -q'' = \frac{\partial \bar{H}}{-i\partial p_2} \ . \tag{3.2.13}$$

If we now return to real time, we have $x_2 = -i\dot{q}$ and the new Hamilton equation is

$$\ddot{q} = \frac{\partial \bar{H}}{-i\partial p_2} \ , \tag{3.2.14}$$

consistently with the complexification of x_2 and p_2. For an extension to quantum field theories and in particular to quadratic gravity, see [51].

Up to this point, we are still missing a proper notion of probability. For a generic observable associated to an operator U with a complete set of eigenstates u_i such that $\mathrm{sign}(U\,|u_i\rangle) = \mathrm{sign}(\langle u_i|u_i\rangle)$ and none of its degenerate eigenspaces contains mixed norm states, we can uniquely define the Hermitian operator P_U via the matrix elements

$$\langle u_i|\, P_U \,|u_j\rangle = \delta_{ij} \tag{3.2.15}$$

and use it to introduce the positive definite inner product

$$\langle \phi_1|\phi_2\rangle_U = \langle \phi_1|\, P_U \,|\phi_2\rangle \ . \tag{3.2.16}$$

The new Born rule given by this product

$$P_{\phi,u_i} = \frac{|\,\langle \phi|u_i\rangle_U\,|^2}{\langle \phi|\phi\rangle_U \,\langle u_i|u_i\rangle_U} \tag{3.2.17}$$

is hence positive definite and probabilities sum up to one:

$$\sum_i \frac{|\,\langle \phi|u_i\rangle_U\,|^2}{\langle \phi|\phi\rangle_U \,\langle u_i|u_i\rangle_U} = \frac{\langle \phi|\, P_U}{\sqrt{\langle \phi|\phi\rangle_U}} \left(\sum_i \frac{|u_i\rangle \langle u_i|\, P_a}{\langle u_i|\, P_U \,|u_j\rangle} \right) \frac{|\phi\rangle}{\sqrt{\langle \phi|\phi\rangle_U}} = 1 \tag{3.2.18}$$

because

$$\sum_i \frac{|u_i\rangle \langle u_i|\, P_a}{\langle u_i|\, P_U \,|u_j\rangle} = 1 \ . \tag{3.2.19}$$

Despite this promising definition, it must be stressed that not all operators respect the requirements necessary to introduce P_U. For example $\hat{x}_2$ does not admit a well-defined measure of probability in this sense, since the norm of its eigenstates is zero. However, in this particular case it has been suggested in [34] to use the inner product $\langle .|.\rangle_\eta = \langle .|\,\eta\,|.\rangle$. With it, one has

$$\langle x_1', x_2'|\,\eta\,|x_1, x_2\rangle = \delta(x_1' - x_1)\delta(x_2' - x_2) \tag{3.2.20}$$

and it can be used to define the probability of a measure of $\hat{x}_1 \otimes \hat{x}_2$. On the other hand, if the eigenspace of a particular eigenvalue is degenerate and contains both states with positive and negative norm, the associated inner product is not unique. It remains to be clarified whether this should be seen as a limit of the quantization method or a constraint on physical observables [53]. With such a notion of probability, we do not need anymore to exclude negative norm states from the set of physical states, hence the time evolution is unitary.

$\mathcal{PT}$ symmetric quantization

An alternative approach also leading to the complexification of x_2 and p_2 is the so-called $\mathcal{PT}$-symmetric quantization, see [54] for a review for the general definition of quantum mechanics with non-Hermitian Hamiltonians, [40, 55] for its application to higher derivative theories and [56] for a particular focus on quadratic gravity. One starts from the observation that the commutation relation $[x_i, p_i] = i$ is meaningful only when the commutator acts on states, however we know that with the prescription (3.1.27) the vacuum state is non-normalizable and the same holds for the complete tower of states (3.1.29). Anyway, we have already seen that the wave function (3.1.28) has a converging norm if x_2 and p_2 take purely imaginary values via the similarity transformation

$$x_2 \to e^{\frac{\pi}{2}p_2 x_2} x_2 e^{-\frac{\pi}{2}p_2 x_2} = i x_2 \ , \qquad p_2 \to e^{\frac{\pi}{2}p_2 x_2} p_2 e^{-\frac{\pi}{2}p_2 x_2} = -i p_2 \ , \tag{3.2.21}$$

that preserves canonical commutation relations. The starting Hamiltonian (3.1.19) is Hermitian in the sense that $H = H^{*T}$ and $\mathcal{PT}$-symmetric. After the complexification, expression (3.2.8) is no more Hermitian, but can be shown to be still $\mathcal{PT}$-symmetric: since the time reversal and parity operators must also be transformed accordingly to (3.2.21) before acting on $\bar{H}$, the Hamiltonian holds this invariance [57]. It has been shown that non-Hermitian but $\mathcal{PT}$-symmetric operators admit a complete basis of left and right eigenstates, respectively $\langle n_L|$ and $|n_R\rangle$, with real eigenvalues. Moreover, a positive definite metric can be associated to these eigenstates. Up to a null measure set of singular theories, there always exists an Hermitian operator $\mathcal{Q}$ such that

$$\langle m_L|\, e^{-\mathcal{Q}}\, |n_R\rangle = \delta_{m,n} \ , \qquad \sum_n |n_R\rangle \langle n_L|\, e^{-\mathcal{Q}} = 1 \ . \tag{3.2.22}$$

Moreover, an Hermitian Hamiltonian $\bar{H}$ can be associated to the non-Hermitian one via the transformation

$$\tilde{H} = e^{-\mathcal{Q}/2} \bar{H} e^{\mathcal{Q}/2} \ . \tag{3.2.23}$$

In the Pais-Uhlenbeck oscillator, the new Hamiltonian $\tilde{H}$ can be put in the form of two harmonic oscillators [40]

$$\tilde{H} = \omega_1(\tilde{a}_1^\dagger \tilde{a}_1) + \omega_2(\tilde{a}_2^\dagger \tilde{a}_2) + \frac{1}{2}(\omega_2 + \omega_1) \tag{3.2.24}$$

thanks to a new set of creation and annihilation operators with eigenstates $|\tilde{n}\rangle$ related to those of $\bar{H}$ via the identity $|\tilde{n}\rangle = e^{-\mathcal{Q}/2}\,|n_L\rangle$. Notice that the positive definite inner product introduced above (3.2.22) is the standard inner product for states $|\tilde{n}\rangle$ and is basically identical to the measure introduced in the Pauli-Dirac formalism (3.2.16) in the particular case where U is the Hamiltonian [53].

The $\mathcal{PT}$-symmetric quantization in the degenerate limit of the Pais-Uhlenbeck oscillator was described in [58], and gives a radically different result: the complexification of canonical variable is still possible, however the complex Hamiltonian $\bar{H}$ cannot be diagonalized, since $\mathcal{Q}$ is singular in this limit. Since the Hamiltonian is not diagonalizable, the two sets of eigenstates, one for the oscillator with frequency ω_1 and one for the oscillator with frequency ω_2, reduce to only one set eigenstates of an anharmonic oscillator with doubled energy and a set of nonstatic solutions of the Schrödinger equation. The eigenstates correspond to the degenerate limits of sums of eigenstates of both the oscillators, so they all have zero norm. These nonstationary solutions are the $\mathcal{PT}$ quantized versions of the growing modes that we observed in the discussion of the purely quartic quantum field theory in Section 3.1.3. Also in this case, a meaningful notion of probability can be recovered only when considering as asymptotic states a superposition of both stationary and nonstationary solutions of the Schrödinger equation, however the physical implications of these mixtures are not clear.

3.2.2 Reduced physical Hilbert space

The second idea we are going to present aims to reduce the space of physical states to positive norm states on both sides of the identity in the optical theorem, allowing for a well-defined Born rule and a unitary time evolution. While in gauge theories the exclusion of Faddeev-Popov ghosts from the space of physical asymptotic states is obtained via the BRST cohomology [59, 60], there are no symmetries allowing to do the same with ghosts in higher derivative theories. Ghosts can be removed from the asymptotic Fock space either by switching on an interaction that forces ghosts to decay in short time or by introducing a new ad-hoc prescription in ghosts propagators that makes them purely virtual particles. Both of these ideas can be realized only if the ghost has a nonzero mass bigger that the mass of the healthy mode, because this separation is necessary both to have a decay of the ghost and to separate the two propagators via the partial fraction decomposition.

Lee-Wick quantization

In the introduction of this Section we discussed how some interactions permit to avoid runaway trajectories that seemed to be unavoidable in the free theory. The Lee-Wick quantization [61, 62] is based on the idea that interactions could heal the ghost also at quantum level. If the two harmonic oscillators in the Pais-Uhlenbeck model quantized with the prescription (3.1.27) interact with each other, the ghost modes, which are related to higher energy levels with respect to the positive norm ones, can decay in the eigenstates of the oscillator with frequency $\omega_2 < \omega_1$. That means that all states with negative norm forms couples of eigenstates with complex conjugated eigenvalues which are no more part of the real part of the spectrum of the interacting Hamiltonian. When the same idea is applied to quantum field theories for each Pais-Uhlenbeck oscillator associated to a 3-momentum configuration, the would be negative norm asymptotic states corresponding to free propagating massive ghosts disappear from the space of eigenvectors with real eigenvalues, and we remain with a Hilbert space composed only by the massless (or lighter) positive norm states. Hence, thanks to the decay of massive ghosts, negative norm states disappear spontaneously from the space of asymptotic physical states used to define the S-matrix. In the optical theorem (3.1.67), due to the nonzero decay rate of massive ghosts, $|f_-\rangle$ cannot be the final state of a scattering process, implying $\langle f_-|T = 0$. Thus, the equation reduces to

$$2\,\mathrm{Im}\,\langle i_+|\,T\,|i_+\rangle = \sum_{f_+} |\,\langle f_+|\,T\,|i_+\rangle\,|^2 \ , \tag{3.2.25}$$

that is equal to the desired expression (3.1.66). So, the time evolution in presence of unstable ghosts is unitary.

The same idea of considering an interacting theory instead of its ill-defined free version has been recently applied to path integral quantization [39, 63, 64]. We start from the propagator 3.1.74 and add to the theory an interaction with normal particles with mass m, such that $m_2 \ll m \ll m_1$. The propagator will receive quantum corrections, that we can resum as a self-energy insertion $\Sigma(p^2)$ in the denominator

$$iD(p^2) = \frac{-i}{p^4 + (m_1^2 + m_2^2)p^2 + m_1^2 m_2^2 + \Sigma(p^2)} \ . \tag{3.2.26}$$

Suppose that, due to the decay of the ghost in two of these particles, above the energy $4m^2$ the self-energy acquires an imaginary part and can be written in the form

$$\Sigma(p^2) = (m_1^2 - m_2^2)\left[\delta M^2(p^2) + i\theta(q^2 - 4m^2)\gamma(q^2)\right] \ , \tag{3.2.27}$$

where δM^2 real and $\gamma(q^2) > 0$. The positivity of the imaginary part γ is a consequence of the optical theorem, that is problematic in higher derivative theories, however we will see that this assumption permits to satisfy the optical theorem at all loops in Lee-Wick theories, hence it is consistent. The propagator has a pole at

$$p^2 = -M_2^2 = -\frac{1}{2}\left\{m_1^2 + m_2^2 - \sqrt{(m_1^2 - m_2^2)\left[m_1^2 - m_2^2 - \delta M(M_2^2)^2\right]}\right\} \ , \tag{3.2.28}$$

that reduces to $M_2^2 \sim m_2^2 + \delta M(M_2^2)^2$ if $m_1^2 \gg \delta M^2$, as expected in the weak coupling regime. So we can write the momentum square as $p^2 = -M_2^2 + (M_2^2 + p^2)$ and find

$$iD(p^2) \sim \frac{-i}{m_1^2(p^2 + M_2^2 - i\epsilon)} \ , \tag{3.2.29}$$

that is the standard pole corresponding to the light particle with an added $-i\epsilon$ term which implements the Feynman prescription. The real part of the other pole is at

$$Re(p^2) = -M_1^2 = -\frac{1}{2}\left\{m_1^2 + m_2^2 + \sqrt{(m_1^2 - m_2^2)\left[m_1^2 - m_2^2 - \delta M(M_1^2)^2\right]}\right\} \sim m_1^2 - \delta M(M_1^2)^2 \ . \tag{3.2.30}$$

Near the pole, i.e. taking $p^2 = -M_1^2 + (M_1^2 + p^2)$, the propagator looks like

$$iD(p^2) \sim \frac{i}{m_1^2[p^2 + M_1^2 + i\gamma(M_1^2)]} \ . \tag{3.2.31}$$

Notice that there are two sign differences with respect to the low energy pole (3.2.29): the overall sign in the numerator and the sign in front of the imaginary term in the denominator. That means that we are in a situation similar to prescription (3.1.75), with the ghost propagating backward in time [38, 64]. However, contrary to the free theory, the imaginary part is given by γ, not by ϵ, hence it remains there even in the limit $\epsilon \to 0$. Consequently, ghost modes associated to this pole have an exponential decay instead of a purely oscillatory behaviour and should not be considered as asymptotic states in an interacting theory. Moreover, the propagator cannot go on-shell in loops with real kinematical variables. In 1963 Veltman [65] showed that, in a theory containing normal

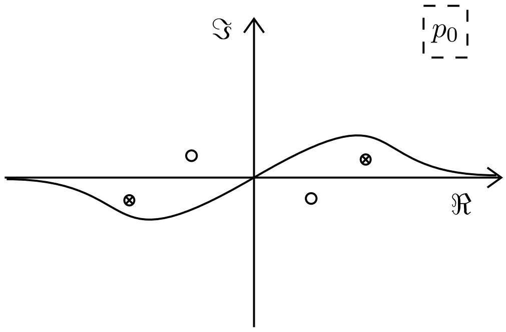


Figure 3.2: Lee-Wick contour. Poles of the normal particle are represented by simple circles, while Ghost poles are represented by crossed circles.

(non-ghost) unstable particles, their propagator should not be cut in Cutkosky rules to respect the optical theorem. The logic is the following: the cutting rules work because the imaginary part of loop diagrams is generated by on-shell propagators via relation (3.1.78). Anyway, when the particle is unstable, the imaginary part of the loops comes from γ, that is proportional to a power of the coupling constant associated to the interaction vertex responsible of the decay of the particle. That means that its contribution to the imaginary part will be of higher order in the small coupling expansion with respect to the contribution of a cut through a stable particle. In fact, if one considers the higher order diagram where a self-energy loop is inserted in the unstable propagator, a cut through stable particles that can be produced by the decay process exactly reproduces the contribution to the imaginary part due to γ. Hence, order by order in loop expansion, only cuts through stable particles must be taken in account. The imaginary part due to unstable particle propagators enters in the optical theorem at higher order in loop expansion, when the self-energy loop of stable particles that produced γ in the unstable propagator is cut through. The same argument can be applied to unstable ghosts [63] and permits to verify the optical theory and show the unitarity of the S-matrix.

When the decay rate is small with respect to the time scale taken in account, the unstable particle can be seen as a stable particle in the so-called narrow width (NW) approximation [66, 67]. In this case γ is treated as the ϵ of stable particles and long lasting resonances are added to the asymptotic states and are cut through in the optical theorem. If one wants to reproduce the NW approximation in Lee-Wick theories, something peculiar happens. One would be tempted to use as a propagator for the long-lasting ghost the expression (3.2.31) with the substitution $\gamma \to \epsilon$

$$iD(p^2) \sim \frac{i}{m_1^2[p^2 + M_1^2 + i\epsilon]} \ , \tag{3.2.32}$$

that would lead to prescription (3.1.75). However, by doing explicit computations, the $\gamma \to 0$ limit of amplitudes between stable states is recovered in NW approximation only via a peculiar integration contour called Lee-Wick contour (see figure 3.2). It consists in passing above the pole with positive real and imaginary part in the q_0 integral in loops and is essentially equivalent to adopting prescription (3.1.76). In this limit, the problems related to negative norm states and transition probabilities come back, however, since ghost modes propagate backward in time, it seems logical to expect to lose the concept of conditional probability usually adopted in quantum physics at the time scale where the ghost can be seen as a stable particle. Only at larger scales, at which the ghost cannot survive, the notion of causality, and hence of transition probability, can emerge.

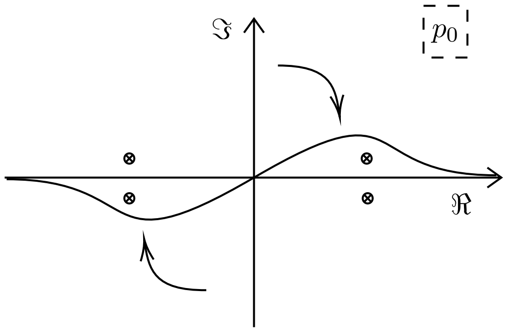


Figure 3.3: Fakeon poles structure and Lee-Wick contour

While being well-defined at one loop, Lee-Wick contour can become much more convolute at higher loops and a good definition within this framework is still missing.

Purely virtual particles

Another way to reduce the Hilbert space of physical states and recover both unitarity and a well-defined probability consists in introducing a new type of purely virtual particles called fakeons. Instead of using prescription (3.1.75), that does not allow the cancellations necessary for renormalizability of higher derivative theories, or (3.1.76), that produces a ill-defined notion of probability or equivalently a violation of unitarity, one can take a free-particle propagator

$$\frac{\mp i}{p^2+m^2} = \mp i \frac{p^2+m^2}{(p^2+m^2)^2} \tag{3.2.33}$$

and avoid the singularity on the real axis via the new prescription [68, 69]

$$\mp i \frac{p^2+m^2}{(p^2+m^2)^2-\varepsilon^4} = \frac{1}{2}\left(\frac{\mp i}{p^2+m^2-i\varepsilon^2}+\frac{\mp i}{p^2+m^2+i\varepsilon^2}\right) . \tag{3.2.34}$$

This new propagator vanishes on-shell, namely at $p^2 = -m^2$, hence it does not represent a proper propagating particle. These purely virtual particles are also called fakeons. At tree level this prescription is equivalent to use only the principal value of the propagator and drop the on-shell pole

$$\frac{\mp i}{p^2+m^2} \to P\frac{\mp i}{p^2+m^2} , \tag{3.2.35}$$

however the full expression (3.2.34) is necessary in loop integrals [70]. To do calculations, one has to evaluate Feynman diagrams with fakeons propagators in place of ghosts' in Euclidean time, and then analytically continue the result to real time. Due to the presence of poles in the first quadrant of the complex plane for q_0, the Wick rotated result does not correspond to the Lorentzian integral along the real axis, but is actually equal to do the integration along the Lee-Wick contour (see figure 3.3). At this point, the final amplitude can be found by sending $\epsilon \to 0$ in standard propagators and, at a later time, $\varepsilon \to 0$ in fakeon propagators.

Since the propagator of a purely virtual particle does not have a pole along the real axis, no imaginary part is produced by loop integrals when the propagator goes on-shell. That means fakeon propagators must not be cut in computing the imaginary part of the left hand side of the optical theorem and, if one sums only over positive norm states on the right hand side, the identity is respected, similarly to what happens with unstable ghosts in Lee-wick quantization.

Also with this prescription, the price to pay for unitarity is a violation of causality on time scales smaller than the mass of the virtual particle [69]. In presence of pure virtual particles, the classical theory must be defined as the classical limit of the renormalized quantum theory. Following this procedure, due to violation of microcausality by fakeons, the effective classical theory is expected to differ from the theory identified by the initial bare Lagrangian. In particular, nonlocal effects at scales similar to that of acausal behaviours emerge in the classical limit of the quantum effective action. In the particular case of quadratic gravity, such modifications could potentially generate observable corrections to general relativity [71, 72].

Chapter 4

Nonperturbative flow between Gaussian theories

Perturbative methods are powerful tools to study the properties of quantum or statistical field theories in the neighborhood of a fixed point (FP), but they do not say much about the global properties of the theory space. For example, one would like to know which FP can be joined by an RG trajectory to another FP. Such questions can sometimes be answered, for example by the c-theorem in two dimensions or the a-theorem in four. Another possibility is to simply solve the RG equations. This is impossible in the full theory space, but it can be done within approximations. For example, in the $\mathbb{Z}_2$-invariant scalar theory in three dimensions, one can find trajectories that join the (free) Gaussian FP in the UV to the Wilson-Fisher (WF) FP in the IR.

In higher derivative theories, a peculiar situation arises when trying to visualize the theory space: to each kinetic term there corresponds a different noninteracting Gaussian theory that is a fixed point of the RG flow. More generally, one can think of infinitely many Gaussian FP's corresponding to the Lagrangians $\phi\,\Box^n\phi$. We will refer to them as GFP_n. Each of them can be viewed as sitting in the origin of theory space, but then all the others are nowhere to be seen. Which GFP we see is related to which GFP we take as the starting point of a perturbative expansion, and hence to the canonical dimension of the field. For example (in four dimensions) GFP_1 is in the origin of a theory space for a field of canonical dimension one, GFP_2 in the origin of theory space when the field is dimensionless. In this way it would almost seem that for each choice of field dimension we have a different theory space, and that these spaces are unrelated to each other. There is some physical basis for this point of view, because different GFP's have different numbers of propagating degrees of freedom. One could view the theory space where a given GFP is in the origin as describing the interactions of a particular set of physical degrees of freedom. For example, whereas in Minkowski space GFP_1 describes a single propagating scalar degree of freedom, GFP_2 describes two. When GFP_2 is infinitesimally deformed by adding a term of the form $\phi\,\Box\phi$, one of the two fields is massive and the other is massless. By integrating out the massive degree of freedom one remains with the free massless one. Thus there should be an RG trajectory joining GFP_2 in the UV to GFP_1 in the IR.

There is one obvious trajectory that does this: it consists of "generalized free theories"[73, 74] with Lagrangians of the form [1]

$$\frac{1}{2}\phi\left(Z_1\Box + Z_2\Box^2\right)\phi\ . \tag{4.0.1}$$

[1] This type of RG flows has been considered before in [75, 76].

In the RG one has to parametrize the theory space with dimensionless coordinates. If we choose for example the field to have canonical dimension of mass, Z_1 is already dimensionless, and the other direction is parametrized by $\tilde{Z}_2 = Z_2 k^2$, where k is some external "renormalization group scale", that in the present situation we can identify with the momentum p. In these free theories, Z_1 and Z_2 do not run, so $\tilde{Z}_2$ is negligible at low energy, but dominant at high energy. Note that choosing a different dimension for the field does not change this conclusion. For example, if the field is dimensionless, Z_2 is already dimensionless and the other direction has to be parameterized by $\hat{Z}_1 = Z_1/k^2$. So, again, Z_1 is dominant at low energy and negligible at high energy. In both cases, this "classical RG" just tells us that the four derivative term dominates over the two-derivative term at high energy.

The independence of the physical implications of the RG flow from the convention adopted for the field dimensionality is expected to hold also in interacting theories. Indeed, the fact that the physical predictions of the FRG are in general independent of the choice of the mass dimension of the field can be shown in the following way. Assume that the effective action is a quasi-local functional of the field of the form:

$$\Gamma_k[\phi] = \sum_i g_i \mathcal{O}_i(\phi) \tag{4.0.2}$$

For the sake of power counting, the operators $\mathcal{O}_i$ have the general form

$$\mathcal{O}_i(\phi) = \int d^d x \, \partial^{m_i} \phi^{n_i} \ , \tag{4.0.3}$$

where the integrand stands for any scalar constructed with m_i derivatives and n_i fields. Assuming that the field has dimension $[\phi] = d_\phi$, $\mathcal{O}_i$ has dimension $[\mathcal{O}_i] = -d + m_i + n_i d_\phi \equiv -d_i$ and the coupling g_i has dimension d_i.

Now let us change variable from ϕ to a new field ϕ' of dimension d'_ϕ:

$$\phi' = \phi k^{\Delta d_\phi} \ , \tag{4.0.4}$$

with $\Delta d_\phi = d'_\phi - d_\phi$. The effective action of the new field is related to that of the old field by

$$\Gamma'_k[\phi'] = \Gamma_k[\phi] \ . \tag{4.0.5}$$

This means that, while the two functionals have numerically the same values when the fields are related as in (4.0.4), Γ'_k is a different functional of its argument from Γ_k. In particular, writing

$$\Gamma'_k[\phi'] = \sum_i g'_i \mathcal{O}_i(\phi') \ , \tag{4.0.6}$$

we find that

$$g'_i = g_i k^{-n_i \Delta d_\phi} \ . \tag{4.0.7}$$

At this point, it is important to understand that once the functional Γ'_k has been defined by (4.0.5), we are free to attribute all the k-dependence to the couplings g'_i and to think of the field ϕ' as being k-independent. If we do so, we can use (4.0.7) as a change of coordinates, but not to calculate the k-dependence of g'_i. In fact, when we extract the beta functions from the generating functionals $\partial_t \Gamma_k$ and $\partial_t \Gamma'_k$, where $t = \log k$, in both cases we will keep the field fixed. In this way, we arrive at the following relation between the beta functions:

$$\partial_t g'_i = \partial_t g_i k^{-n_i \Delta d_\phi} \ . \tag{4.0.8}$$

The apparent contradiction with (4.0.7) is due to our keeping ϕ' fixed. Thus the missing contribution is compensated by the fact that at the same time we also ignore the k-term in (4.0.4). Equation (4.0.8) expresses the fact that the calculation of the loop corrections is the same, for any dimension of the field, up to an overall factor of k that accounts for the different dimensions.

In the discussion of RG flows and fixed points we must use the dimensionless variables

$$\tilde{g}_i = g_i k^{-d_i} \ , \qquad \tilde{g}'_i = g'_i k^{-d'_i} \ . \tag{4.0.9}$$

However, using (4.0.8), one finds that the beta functions of these dimensionless variables are different, namely

$$\partial_t \tilde{g}'_i = \partial_t g'_i k^{-d'_i} - d'_i \tilde{g}'_i = \partial_t \tilde{g}_i + n_i \Delta d_\phi \tilde{g}_i \ . \tag{4.0.10}$$

This does not happen if we properly take into account the normalization of the field. Among the couplings g_i there is the wave function renormalization constant Z or Z', that has dimension $d_Z = d - 2 - 2d_\phi$ or $d'_Z = d - 2 - 2d'_\phi$ respectively. Let us therefore define

$$\tilde{g}_i = g_i Z^{-n_i/2} k^{-d_i + n_i d_Z/2} \ , \qquad \tilde{g}'_i = g'_i Z'^{-n_i/2} k^{-d'_i + n_i d'_Z/2} \ . \tag{4.0.11}$$

Note that

$$-d'_i + n_i d'_Z/2 = -d + m_i + n_i \frac{d-2}{2} = -d_i + n_i d_Z/2 \ , \tag{4.0.12}$$

and then, if we use (4.0.7), $\tilde{g}'_i = \tilde{g}_i$. Equation (4.0.8) implies that $\partial_t Z' = \partial_t Z k^{-2\Delta d_\phi}$ and therefore

$$\partial_t \log Z' = \partial_t \log Z \ . \tag{4.0.13}$$

So, using the preceding formulae,

$$\begin{aligned} \partial_t \tilde{g}'_i &= \partial_t g'_i Z'^{-n_i/2} k^{-d'_i + n_i d'_Z/2} + \left(-d + m_i + n_i \frac{d - 2 - \partial_t \log Z'}{2} \right) \tilde{g}'_i \\ &= \partial_t g_i Z^{-n_i/2} k^{-d_i + n_i d_Z/2} + \left(-d + m_i + n_i \frac{d - 2 - \partial_t \log Z}{2} \right) \tilde{g}_i = \partial_t \tilde{g}_i \ . \end{aligned} \tag{4.0.14}$$

Thus the flows of the dimensionless *and* canonically normalized couplings is the same, independently of the dimension of the field. This underlines the importance of including the redundant wave function renormalization constants in the definition of the coordinates on theory space.

In the following, we will analyze the nonperturbative RG flow of some simple higher derivative scalar theories using the functional renormalization group equation and we will try to map the flow between different Gaussian fixed points and other eventual interacting fixed points.

For most of the rest of this thesis we will ignore the issue of the ghosts and study the renormalization of Euclidean theories. When considering scattering amplitudes in Chapter 5, we will assume the prescription (3.1.76), that is the result of the usual Wick rotation of the euclidean amplitude.

4.1 The shift invariant scalar model

In this Section we will focus on a shift-symmetric ($\phi \to \phi + \text{constant}$) and $\mathbb{Z}_2$-symmetric scalar field. These symmetries restrict the Euclidean free energy, or effective action, to have the form

$$\Gamma[\phi] = \int d^4x \left[\frac{1}{2}Z_1(\partial\phi)^2 + \frac{1}{2}Z_2(\Box\phi)^2 + \frac{1}{4}g((\partial\phi)^2)^2 + \dots\right] , \tag{4.1.1}$$

that is power counting renormalizable despite the derivative interaction term. The higher derivative interactions are similar to those which appear in gravitational theories, moreover the inclusion of both two and four derivatives in the kinetic energies is typical of many applications of the FRG in which operators of different dimensions appear. The restriction over possible interaction vertices due to shift symmetry is crucial in order to have a renormalaizable theory, since, with a field with zero canonical dimension, operators with infinite powers of the field can be associated with a marginal or even relevant coupling. That means power counting is not a sufficient criterion for renormalizability, because the number of parameters to be experimentally fixed can be infinite even in a theory without irrelevant operators. In a theory that contains only one scalar field and a finite number of operators polynomial in the field and its derivatives, shift invariance is a necessary condition for the theory to be renormalizable [77], while in more complicated theories other symmetries can constrain the theory space in an efficient way, as does, for example, diffeomorphism invariance in quadratic gravity. Superficially, this theory appears to be pathological. First, theories with higher derivative kinetic energies contain a ghost with squared mass $m^2 = \frac{Z_1}{Z_2}$. This can be seen from the propagator in the full theory with Z_1 normalized to 1 and hence $Z_2 = 1/m^2$)

$$iD(p^2) = \frac{-i}{p^2 + \frac{p^4}{m^2}} = -i\left[\frac{1}{p^2} - \frac{1}{p^2 + m^2}\right] . \tag{4.1.2}$$

The signs in the original Lagrangian have been chosen to have the massive pole at timelike momentum, because the opposite sign would have the pole being tachyonic. Second, as we shall discuss, the theory is asymptotically free, in the sense that the coupling runs logarithmically to zero in the UV limit, but only for negative coupling. In this it is reminiscent of Symanzik's observation that ordinary $\lambda\Phi^4$ theory is asymptotically free for $\lambda < 0$ [78]. In spite of this, we can study the renormalization of this model and draw from it some useful lessons.

Besides being an interesting case study for several aspects of renormalization theory, our model is also of independent interest and has appeared recently in various different contexts. Without the higher derivative kinetic term, it is a textbook example of Effective Field Theory (EFT), being the low energy description of a $U(1)$-invariant linear sigma model in the Higgs phase [79, 80]. With the higher derivative kinetic term, it is the low energy EFT for the higher derivative version of the same model. As a CFT, the higher-derivative model has been discussed in [81]. In the context of Asymptotic Safety, it has been presented as a type of matter interaction that would necessarily have to be present if gravity has a nontrivial fixed point [82, 83]. Finally, it has been studied recently by Tseytlin [43] and by Holdom [84], who found evidence that the model may be less pathological than would first appear.

In perturbation theory, one normally decides to normalize the field either with Z_1 or Z_2 and treats the other kinetic term as a perturbation. The limiting cases $Z_1 = 0$ and $Z_2 = 0$ have been discussed recently in [81] as elements of an infinite family of shift invariant theories, while in Section 5.1 we will study the perturbative behaviour of the theory with a mixed propagator. Beyond perturbation theory, the action (4.1.1) and its generalizations containing higher derivative terms, are part of a single "theory space", where all terms can be present simultaneously. One is then interested in understanding the mutual relations between different fixed points. In particular, the question we shall investigate is whether there exist nontrivial RG trajectories joining them.

The tool we shall use is the Wetterich-Morris form of the non-perturbative RG equation for the 1-PI effective action, a.k.a. the effective average action (EAA) discussed in Section 2.2.1. The EAA is a functional of the fields depending on an external scale k that acts as an IR cutoff. By making an ansatz for the EAA of the form (4.1.1), the constants Z_1, Z_2 and g become k-dependent running couplings. Inserting the ansatz in the RG equation one can read off the beta functions and anomalous dimensions. As usual in FRG computations, we will consider quantum field theories in Euclidean signature, so we will not care of problems related to unitarity and ghosts in general.

We shall calculate the RG flow based on two different choices of field dimension, which are in turn related to different Gaussian FP's, and show how these flows are related by a coordinate transformation in theory space. This yields a global picture of the flow where both GFP's are simultaneously present.

In the neighborhood of a Gaussian fixed point, the anomalous dimension must be small. If, following the RG flow, we end at another FP, we can in principle calculate the anomalous dimension of the field at this endpoint. Such calculations are always based on some approximations and therefore the calculated anomalous dimension is generally not exact. Remarkably, we shall see that in the case of flows between Gaussian FPs the result is exact, in the following sense: the canonical dimension of the field at the UV FP plus the calculated anomalous dimension gives exactly the canonical dimension of the field at the IR FP.

4.1.1 The flow from GFP_2 to GFP_1.

Implicit in the definition of free particle states is the choice of a Gaussian FP. Then, it is natural to give the field the canonical dimension that pertains to that free theory. When one contemplates flows interpolating between different FP's, the choice of dimension of the field is no longer so natural. In this section we will discuss flows joining GFP_1 and GFP_2, where the fields have canonical dimension one and zero respectively. We will therefore exhibit the flow equations in both cases. The power counting in the two cases is very different, but the calculation of the loop contributions via the Wetterich-Morris equation is essentially the same. The differences arise from the choice of dimensionless coordinates for theory space, that come natural in the neighborhood of each FP. For each GFP_i ($i = 1, 2$) we will therefore define a chart, consisting of an open subset of theory space U_i and suitable coordinate functions. We will then discuss the transformation between the two charts and give a global picture of the RG flow.

In order to write an explicit RG equation we have to choose a form for the cutoff (or "regulator") function R_k that suppresses the low momentum modes in the path integral. We choose:

$$R_k^{(24)} = Z_1(k^2 - q^2)\theta(k^2 - q^2) + Z_2(k^4 - q^4)\theta(k^4 - q^4) \ . \qquad (4.1.3)$$

The presence of the couplings Z_1 and Z_2 makes it an "adaptive" cutoff (in contrast to a "non-adaptive" or "pure" cutoff [85]). Normally only one of the two terms is considered, but for our purposes this choice is preferable, because it treats the two possible kinetic terms on an equal footing. Additionally, it leads to the simplest beta functions, among the choices we have tried. We discuss in Appendix A different choices of the cutoff.

Dimensionful field and the chart U_1

If in the action (4.1.1) we choose the term with two derivatives to define the propagator, the field ϕ has canonical dimension of mass. Then Z_1 is dimensionless, Z_2 has dimension of inverse mass squared and g of inverse mass to power 4. the power counting is that of a non-renormalizable theory. The flows of g, Z_1 and Z_2 are extracted from the functional RG equation [10, 11]

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} k\frac{dR_k}{dk} \tag{4.1.4}$$

by inserting the Ansatz (4.1.1) for Γ_k, evaluating the functional trace Tr (which for a scalar in flat space is a momentum integral) and extracting from it the coefficients of the three operators $(\partial\phi)^2$, $(\Box\phi)^2$ and $((\partial\phi)^2)^2$. the beta functions are

$$\partial_t Z_1 = -\frac{(8-\eta_1)Z_1 + 16k^2 Z_2}{128\pi^2(Z_1+k^2Z_2)^2}\, gk^4 \ , \tag{4.1.5}$$

$$\partial_t Z_2 = 0 \ , \tag{4.1.6}$$

$$\partial_t g = \frac{(10-\eta_1)Z_1 + 20k^2 Z_2}{64\pi^2(Z_1+k^2Z_2)^3}\, g^2k^4 \ . \tag{4.1.7}$$

It is natural to parametrize theory space by

$$\tilde{g} = \frac{gk^4}{Z_1^2} \ , \qquad \tilde{Z}_2 = \frac{Z_2k^2}{Z_1} \ . \tag{4.1.8}$$

The powers of k make the couplings dimensionless and the powers of Z_1 are such that if these definitions are inserted in the action, Z_1 can be set to one by rescaling the field. This makes it clear that Z_1 is a redundant coupling. The anomalous dimension $\eta_1 = -\partial_t \log Z_1$ (where $t = \log(k/k_0)$) and the beta functions of $\tilde{g}$ and $\tilde{Z}_2$ are obtained from equations (4.1.5), (4.1.6) and (4.1.7).

We find

$$\eta_1 = \frac{8\tilde{g}\left(1+2\tilde{Z}_2\right)}{\tilde{g}+128\pi^2(1+\tilde{Z}_2)^2} \tag{4.1.9}$$

and

$$\beta_{\tilde{g}} \equiv \partial_t\tilde{g} = (4+2\eta_1)\tilde{g} + \frac{10+20\tilde{Z}_2-\eta_1}{64\pi^2(1+\tilde{Z}_2)^3}\tilde{g}^2 \ . \tag{4.1.10}$$

where (4.1.9) has to be used. The beta function of the dimensionful Z_2 (4.1.6) vanishes, which implies

$$\beta_{\tilde{Z}_2} \equiv \partial_t\tilde{Z}_2 = (2+\eta_1)\tilde{Z}_2 \ . \tag{4.1.11}$$

These beta functions have some nontrivial zeroes. We see from (4.1.11) that $\beta_{\tilde{Z}_2}$ can vanish in two ways: one is by having $\tilde{Z}_2 = 0$, the other by $\eta_1 = -2$. Besides GFP$_1$ there are two FPs of the first type, occurring at $\tilde{g} = 128\pi^2(2\sqrt{6}-5)$ and $\tilde{g} = 128\pi^2(-2\sqrt{6}-5)$ and one FP of the second type at $\tilde{Z}_2 = -3/5$, $\tilde{g} = -512\pi^2/5$. The properties of these FPs are summarized in Table 4.1. Some of these FPs had already been observed in [82, 83, 86].

We note that the beta functions have singularities for $\tilde{g} = -128\pi^2(1+\tilde{Z}_2)^2$ and for $\tilde{Z}_2 = -1$. The fixed points GFP$_1$, NGFP$_1$ are on one side of the singularities, while the

FP	$\tilde{Z}_{2*}$	$\tilde{g}_*$	η_{1*}	θ_1	θ_2
GFP_1	0	0	0	-4	-2
$NGFP_1$	0	-127.6	-0.90	4.40	-1.10
$NGFP_2$	0	-12505	8.90	43.60	-10.90
$NGFP_3$	-0.6	-1011	-2	-13.84	10.84

Table 4.1: Properties of the finite FPs seen with dimensionful field. The second and third columns give the values of the couplings (4.1.8), the fourth the anomalous dimension and the last two the scaling exponents.

other two are on the other side. Thus for the purpose of studying the flows that can start/end at GFP_1, the area with $\tilde{Z}_2 < -1$ or $\tilde{g} < -128\pi^2(1+\tilde{Z}_2)^2$ is unphysical.

One can get a general overview picture of the flow in the chart U_1 by defining

$$\tilde{Z}_2 = \tan u \tag{4.1.12}$$

$$\tilde{g} = 128\pi^2(2\sqrt{6}-5)\tan v \ . \tag{4.1.13}$$

The rescaling factor has been chosen in such a way that $NGFP_1$ is at $u=0$, $v=\pi/4$, while the singularity of the flow is at $u=-\pi/4$. The result is depicted on the left side of figure (4.1).

If we study the function η_1 in the bottom right quadrant, we find that the condition $\eta_1 = -2$ is satisfied asymptotically for $\tilde{Z}_2 \to \infty$ and

$$\tilde{g} \sim -16\pi^2 \tilde{Z}_2 \ . \tag{4.1.14}$$

This leads us to suspect the existence of another FP in the bottom right corner, outside the domain of this chart. We also note the existence of a "separatrix": the RG trajectory that arrives at $NGFP_1$ from the irrelevant direction, corresponding to the eigenvector with components

$$\left(\frac{5(11\sqrt{6}-27)}{256\pi^2(505\sqrt{6}-1237)}\epsilon, -\epsilon\right) \approx (0.0143\epsilon, -\epsilon) \ . \tag{4.1.15}$$

This trajectory can be found numerically and it has the asymptotic behavior (4.1.14). In fact all other trajectories in the fourth quadrant that end at GFP_1 have this same asymptotic behavior, as we shall show later. If we follow these trajectories in the sense of increasing k or t, those trajectories that emerge from GFP_1 at a steeper angle reach this behavior sooner, while those that come out nearly horizontally only reach this regime at higher k. We shall discuss the meaning of these facts later.

Dimensionless field and the chart U_2

We start again from (4.1.1), but we assume that the propagator is given by the four derivative kinetic term, so the field is canonically dimensionless. In order not to confuse the couplings of this case with those of the previous section, we shall use a different notation for the effective action:

$$F[\varphi] = \int d^4x \left[\frac{1}{2}\zeta_1(\partial\varphi)^2 + \frac{1}{2}\zeta_2(\Box\varphi)^2 + \frac{1}{4}\gamma((\partial\varphi)^2)^2 + \dots\right] \tag{4.1.16}$$

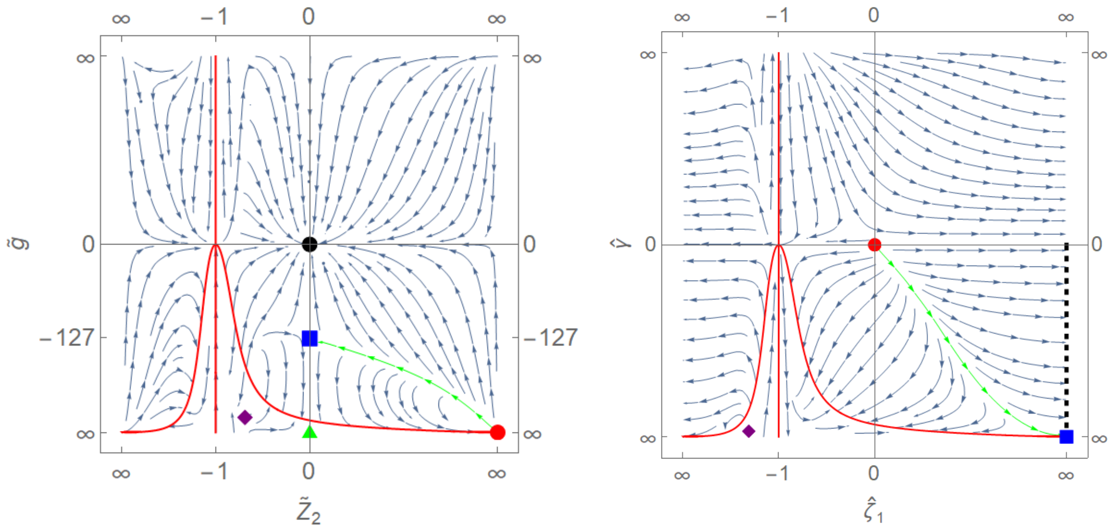


Figure 4.1: The flow in the chart U_1 (left) and in the chart U_2 (right). The black dot (on the left) and the dashed black line (on the right) mark GFP$_1$, the red dot marks GFP$_2$, the blue square marks NGFP$_1$. The separatrix is the green flow line. The continuous red lines are singularities of the flow.

It can be obtained from (4.1.1) by changing the dimension of the field and redefining the couplings

$$\phi = k\varphi \ , \quad Z_1 = k^{-2}\zeta_1 \ , \quad Z_2 = k^{-2}\zeta_2 \ , \quad g = k^{-4}\gamma \ . \tag{4.1.17}$$

Now the wave function renormalization constant ζ_2 is dimensionless, while ζ_1 has dimension of mass squared, and the coupling γ is dimensionless. The power counting is that of a renormalizable theory, with ζ_1 having the meaning of a mass. As mentioned in the introduction to this chapter, the beta functions of the original, generally dimensionful, parameters in the Lagrangian are related as in (4.0.7). This is because the calculation of the loop contributions is the same, up to a redefinition of the dimensions. We can see this explicitly in the case of the shift-symmetric theory. So, with dimensionless field and action (4.1.16), the dimensionful beta functions are the same of equations (4.1.5), (4.1.6) and (4.1.7), with the replacements $Z_i \to \zeta_i$, $g \to \gamma$, and we can use these equations to find the RG flow of the parameters of this map of the theory space.

The natural variables for the parametrization of theory space are

$$\hat{\zeta}_1 = \frac{\zeta_1}{\zeta_2 k^2} \ , \qquad \hat{\gamma} = \frac{g}{\zeta_2^2} \ . \tag{4.1.18}$$

Inserting these definitions in the action, we see that ζ_2 is a redundant coupling, because it can be set to one by a field rescaling.

As in the previous section, $\partial_t \zeta_2 = 0$, so the anomalous dimension $\eta_2 = -\partial_t \log Z_2$ is trivial

$$\eta_2 = 0 \ , \tag{4.1.19}$$

while the beta functions are

$$\beta_{\hat{\zeta}_1} = -2\hat{\zeta}_1 - \frac{8\hat{\gamma}(2+\hat{\zeta}_1)}{\hat{\gamma} + 128\pi^2(1+\hat{\zeta}_1)^2} \tag{4.1.20}$$

FP	$\hat{\zeta}_{1*}$	$\hat{\gamma}_*$	η_{2*}	θ_1	θ_2
GFP_2	0	0	0	2	0
$NGFP_3$	-1.67	-2807	0	-13.84	10.84

Table 4.2: Properties of the finite FPs seen with dimensionless field. The second and third columns give the values of the couplings (4.1.18), the fourth the anomalous dimension and the last two the scaling exponents.

and

$$\beta_{\hat{\gamma}} = \frac{(2+\hat{\zeta}_1)(\hat{\gamma}+640\pi^2(1+\hat{\zeta}_1)^2)}{32\pi^2(1+\hat{\zeta}_1)^3\left(\hat{\gamma}+128\pi^2(1+\hat{\zeta}_1)^2\right)}\hat{\gamma}^2 \ . \tag{4.1.21}$$

The fixed points of these beta functions are listed in Table 4.2. We note that the nontrivial FP has the same scaling exponents as $NGFP_3$ and has been labeled accordingly. We shall soon understand this identification better.

By expanding the beta functions in $\hat{\zeta}_1$ and $\hat{\gamma}$, and demanding that they form in the $(\hat{\zeta}_1, \hat{\gamma})$ plane a vector pointing towards the origin, we find that the only direction by which one can tend to GFP_2 is $(\epsilon, -16\pi^2\epsilon)$.

Now we can introduce a new set of coordinates analogous to (4.1.12) and (4.1.13) in order to compactify the configuration space. The result is the right hand side of figure (4.1). Also in this case in the fourth quadrant there is a separatrix. It distinguishes curves for which $\hat{\zeta}_1$ tends to infinity in the limit $k \to 0$ from those for which $\hat{\zeta}_1$ reaches a maximum and then turns down again. For large $\hat{\zeta}_1$ the separatrix has the asymptotic form

$$\hat{\gamma} = -128\pi^2(2\sqrt{6}-5)\hat{\zeta}^2 \ . \tag{4.1.22}$$

This limit corresponds to GFP_1.

Global properties of the flows

We shall now show that the two flows described in the previous subsections are merely coordinate transformation of each other, outside the two Gaussian FP's, and derive various physical properties of the system.

The chart U_1 contains GFP_1 but not GFP_2, and vice-versa. In order to understand the flows from one Gaussian FP to the other, we must understand that in each chart "the other" FP is a limiting set. To this end, we need the coordinate transformation. From the relations (4.1.8), (4.1.17) and (4.1.18) the two sets of coordinates for theory space are related by

$$\hat{\zeta}_1 = \frac{1}{\tilde{Z}_2} \ , \qquad \hat{\gamma} = \frac{\tilde{g}}{\tilde{Z}_2^2} \qquad \text{or conversely} \quad \tilde{g} = \frac{\hat{\gamma}}{\hat{\zeta}_1^2} \ . \tag{4.1.23}$$

From here we see that in the chart U_2, taking the limit $\hat{\zeta}_1 \to \infty$ for any fixed and finite $\hat{\gamma}$, gives $\tilde{Z}_2 = 0$ and $\tilde{g} = 0$. Therefore, all these limit points correspond to GFP_1. Conversely in the chart U_1, if we take the limit $\tilde{Z}_2$ and $\tilde{g} \to \infty$ with relation (4.1.14), we find $\hat{\zeta}_1 = 0$, $\hat{\gamma} = 0$, which corresponds to GFP_2. While mathematically clear, these statements may sound a bit puzzling: from the point of view of the chart U_2, how can it be that the theory becomes free in the IR even as the coupling $\hat{\gamma}$ remains constant? Even more dramatically, from the point of view of the chart U_1, how can it be that the theory becomes free in the UV even as the coupling $\tilde{g}$ diverges? We shall gain a better understanding of these statements by studying the properties of the RG trajectories. After the picture in dimensionless variables has been clarified, we will discuss the picture in dimensionful variables.

In the chart U_2, Z_1 is a mass squared and therefore there is an obvious mass threshold located where $\hat{\zeta}_1 = 1$, with the UV located to its left and the IR to its right. By the same token, in the chart U_1 the mass threshold is at $\tilde{Z}_2 = 1$, with the IR to its left and the UV to its right. It is somehow natural to use the chart U_2 for energy scales above the threshold and the chart U_1 for energy scales below it, even though the validity of both charts extends far below this point.

We have shown in Section 4.1.1 that all trajectories emerge from GFP_2 with $\hat{\gamma} = -16\pi^2\hat{\zeta}_1$. Applying the transformation (4.1.23) this implies that in the chart U_1 all the trajectories have the asymptotic behavior (4.1.14), as mentioned in Section 4.1.1.

Next note that the lines with $\tilde{g} = 0$ (in the chart U_1) and $\hat{\gamma} = 0$ (in the chart U_2) correspond to the classical trajectory (4.0.1), joining GFP_2 in the UV to GFP_1 in the IR, and consisting entirely of free theories. At the other extreme, the separatrix is in some sense the "strongest interacting" trajectory. In the chart U_1 it consists of two segments: first the line going from GFP_2 in the UV to NGFP_1 in the IR, and then the trajectory with $\tilde{Z}_2 = 0$, joining NGFP_1 in the UV to GFP_1 in the IR. [2] We will be interested in the infinitely many trajectories that are contained between these two extremes, see Figure 4.1.

There are trajectories that remain entirely in the perturbative domain, i.e. are close to the classical trajectory. This is not obvious when one works in a fixed chart, because both $\tilde{g}$ and $\hat{\gamma}$ do not go to zero at both ends of the trajectory. Flowing out of GFP_2 in the chart U_2 they are the ones for which $\hat{\gamma}$ is small at least down to the mass threshold $\hat{\zeta}_1 = 1$. Eventually, if one proceeds further towards the IR, $\hat{\gamma}$ remains constant. However, around $\hat{\zeta}_1 = 1$, one can change chart: at that point $\tilde{g} = \hat{\gamma}$ is small and following the flow towards the IR in the chart U_1, the coupling $\tilde{g}$ tends to zero. The qualitative behavior of the trajectories is the same also when the coupling in mid-flow is strong.

We can now see the automatic change of dimensionality of the field along the flow. In the chart U_1, that is more appropriate to describe the low energy physics, the field has dimension of mass and Z_1 is dimensionless. Near GFP_1 the anomalous dimension is small and negative. However, if we follow any RG trajectory towards the UV, as discussed in Sect 4.1.1, the anomalous dimension grows and eventually tends to -2. Recalling that the canonically normalized field

$$\tilde{\phi} = \sqrt{Z_1}\phi \tag{4.1.24}$$

has scaling dimension $(d-2+\eta_1)/2$, this means that the field is effectively dimensionless in the UV limit. This is indeed the natural choice for the field at GFP_2 in the chart U_2. We observe that this automatic adjustment of the dimension is a consequence of the form (4.1.11) of the beta function of $\tilde{Z}_2$. The fact that in the UV limit $\eta_1 \approx -2$ also means that the wave function renormalization constant scales like $Z_1 \sim k^2$ in that limit. Thus in going from GFP_1 to GFP_2, Z_1 gets multiplied by an infinite factor.

One can arrive at the same conclusions in the chart U_2. Here the RG equation to be observed is not the one for the anomalous dimension η_2, which is identically zero, but the running of the mass $\hat{\zeta}_1$. In fact eq. (4.1.20) can also be written in the form

$$\beta_{\hat{\zeta}_1} = -(2+\eta_1)\hat{\zeta}_1 \ , \tag{4.1.25}$$

where

$$\eta_1 = \frac{8\hat{\gamma}(2+\hat{\zeta}_1)}{\hat{\gamma}+128\pi^2(1+\hat{\zeta}_1)^2}\frac{1}{Z_1} \ . \tag{4.1.26}$$

[2]Strictly speaking this should be seen as two separate trajectories, since each one takes infinite RG time, but it can be seen as the limit of trajectories joining directly GFP_2 to GFP_1.

and one recovers $\eta_1 = -2$ in the fixed point GFP$_2$.

The picture in dimensionful variables

Even though the charts U_1 and U_2 are defined only for the dimensionless coordinates on theory space, when we consider the flow in the dimensionful parameters appearing in the Lagrangian, there is still a vestige of these coordinates in the choice of the dimension of the field.

Because terms with fewer derivatives dominate at low energy, it is natural to describe the IR physics in terms of the dimensionful field ϕ with two-derivative kinetic term. In the limit $k \to 0$ both Z_1 and g become constants, see Equations (4.1.5,4.1.7), and recall that Z_2 is also a constant. Therefore the effective action is (4.1.1) with arbitrary fixed coefficients. This does not look like a free theory, but if we identify the scale k with a characteristic external momentum p, by mere momentum counting the first term is the dominant one in the IR limit. The interaction is of order gp^4 and goes to zero much faster, and the same happens for the higher-derivative kinetic term. [3] It is noteworthy that in order to identify the $k \to 0$ limit as a free theory it is necessary to identify k as a physical momentum scale.

On the other hand, using the asymptotic behavior (4.1.14), and solving the flow equation for $\tilde{Z}_2$, we find that the behavior for large k is $\tilde{Z}_2 \sim \frac{11}{4} \log k$ and therefore [4]

$$Z_1 = \frac{4 Z_2 k^2}{11 \log k} \ , \tag{4.1.27}$$

where Z_2 is fixed and arbitrary. Then using (4.1.14) one obtains

$$g = -\frac{64\pi^2 Z_2^2}{11 \log k} \ . \tag{4.1.28}$$

Thus for $k \to \infty$, g goes to zero and we remain with a free theory. Identifying again k with the momentum in the two-point function, the four-derivative kinetic term has an overall momentum-dependence p^4 whereas the two-derivative one goes like $p^4/\log p$. Thus in the UV limit the four-derivative kinetic term is the dominant one, but only logarithmically.

For large momentum it is natural to redefine the field as in (4.1.17), thus absorbing in the field the power in the running of Z_1 at high energy. Then we find that $\hat{\zeta}_1 = 4/(11 \log k)$ and

$$\zeta_1 = \frac{4 \zeta_2 k^2}{11 \log k} \ , \tag{4.1.29}$$

where ζ_2 is an arbitrary dimensionless constant that can be set to one. Thus the "mass squared" ζ_1 has the expected power behavior, with a logarithmic correction. For the coupling γ we get

$$\gamma = -\frac{64\pi^2 \zeta_2^2}{11 \log k} \ , \tag{4.1.30}$$

which is the expected behavior of a renormalizable coupling and demonstrates asymptotic freedom at high energy.

If we now look at the IR limit using the field φ we find that ζ_1, ζ_2 and γ become all constants and we recover the previous statement that the (free) two-derivative term is the dominant one. Once again, the understanding of this limit as a free theory hinges on identifying the RG scale k with a characteristic external momentum p.

[3] The identification $k = p$ is unambiguous for the two point function, but may require further qualifications for more complicated physical processes.

[4] This gives the anomalous dimension $-2 + 1/\log k$.

The mass of the ghost

In this section we think of the theory in Minkowski space, where the classical action differs from (4.0.1) by an overall sign. It describes two propagating particles: a normal massless scalar and a massive scalar ghost with mass $m^2_{gh} = Z_1/Z_2$ (Notice that this statement is independent of the dimension of the field). If $g = 0$, Z_1 and Z_2 do not run and the statement holds *verbatim* also at the quantum level. If we now switch on g, the value of the physical (pole) mass will receive quantum corrections. The two-point function of the theory is defined as the limit for $k \to 0$ of the two point function of the EAA. In general, the dependence of the n-point functions on the external momenta and on the parameter k are not interchangeable, but in the case of the two-point function, given that the external momentum p has the effect of an IR cutoff in the integrations over the loop momenta, the k-dependence is a good proxy for the p-dependence. We can therefore reliably calculate the pole mass from the running of the parameters Z_1 and Z_2 with the cutoff scale k.

In the presence of a running (renormalized) mass $m^2_R(k)$, the pole mass can be defined by

$$m^2_{pole} = m^2_R(k = m_{pole}) \tag{4.1.31}$$

and corresponds to the threshold discussed in the previous section. There is an old argument that if m_R grows sufficiently fast, there may be no pole at all. [5] This would remove the unwanted ghost state.

Working in the chart U_1, the location of the pole is defined by

$$\tilde{Z}_2 = 1 \ . \tag{4.1.32}$$

Since all the trajectories run from $\tilde{Z}_2 = \infty$ to $\tilde{Z}_2 = 0$, they inevitably hit the pole and the argument mentioned above cannot apply. However, the pole may be shifted to arbitrarily high scale.

To see this we start by setting $Z_1 = 1$ in the IR limit. Then on the "classical" RG trajectory (4.0.1), $Z_1 = 1$ everywhere and the pole mass is at

$$k^2_P = \frac{Z_1}{Z_2} = \frac{1}{Z_2} \ . \tag{4.1.33}$$

Let us now switch on the interaction. The anomalous dimension η_1 is negative and therefore Z_1 becomes larger than one. Thus the pole is shifted to a larger value, compared to the "classical" trajectory. This effect becomes stronger as one considers trajectories that are further away from the classical one. In the limit, consider a trajectory that is close to the separatrix. Already for small k, $\tilde{g}$ becomes quickly very negative, until one gets close to NGFP$_1$. There the running of $\tilde{g}$ almost stops, but Z_1 grows like

$$Z_1(k) \sim k^{0.90} \ . \tag{4.1.34}$$

This behavior can last for many orders of magnitude of k. By the time the RG trajectory finally leaves the vicinity of NGFP$_1$ and reaches $\tilde{Z}_2 = 1$, Z_1 can be arbitrarily large. Thus the mass of the ghost grows continuously from $1/\sqrt{Z_2}$ to infinity as one moves from the classical RG trajectory to the separatrix.

Redundant couplings and the essential RG

One says that a coupling in the Lagrangian is "redundant" or "inessential" *at a specified FP*, if it can be removed from the Lagrangian by means of a local field redefinition [88,

[5] See e.g. [87].

89]. There has been recently an interesting discussion of the "essential RG", which is a way of simplifying the RG flow by eliminating all redundant couplings [90].

The prime example of a redundant coupling is the wave function renormalization: the parameter Z_1 is redundant at GFP_1 and ζ_2 is redundant at GFP_2, because they can be fixed to 1 by rescaling the field. We have already taken this into account by putting suitable powers of Z_1 or ζ_2 in the definition of the coordinates in theory space. It has been shown before that doing so is necessary if we demand that the beta functions are independent of the dimension of the field.

However, also the parameter Z_2 is redundant at GFP_1. Indeed, if $\tilde{Z}_2$ is infinitesimal, it can be removed by an infinitesimal field redefinition of the form

$$\delta\phi = \frac{Z_2}{2Z_1}\Box\phi \ . \tag{4.1.35}$$

One could therefore eliminate also Z_2 and get the essential flow equation for the single (in our approximation) coupling $\tilde{g}$: in Figure 4.1, left panel, it would be a flow in the vertical direction only, and would lead to different properties of NGFP_1. Similar considerations can be used to prove, in a much more general setting than a mere scalar theory, that if the kinetic term is the standard one containing two derivatives, then in perturbation theory one will never generate higher-derivative kinetic terms [91].

On the other hand when one considers GFP_2, ζ_1 is not redundant there because it cannot be removed by a local field redefinition. We conclude that by studying only the essential RG at GFP_1 we would not realize the possibility of flowing to GFP_2 in the UV, which would imply an increase in the number of propagating degrees of freedom, but we would still have the possibility of flowing to the non-Gaussian FP [90].

4.1.2 Discussion

We now review our main findings and then comment on possible extensions.

The general theory space contains all possible kinetic terms and none of them plays an a priori preferred role. It is only when one considers the perturbative expansion around a Gaussian fixed point that the corresponding kinetic term acquires a special meaning. One then has a clear choice for the canonical dimensionality of the field. [6] Otherwise, the choice of the dimension of the field is essentially arbitrary: physical conclusions are independent of this choice. However, the picture of the flow that follows from different choices can be quite different. For a fixed $n \geq 0$ the choice of kinetic term $Z_n\phi\Box^n\phi$ dictates that the field has canonical dimension $(d-n)/2$ and this fixes the dimension of all the couplings g_i in the Lagrangian. When suitably rescaled by powers of n and Z_n, these couplings define coordinates on an open subdomain U_n of theory space. Thus theory space is a manifold covered by infinitely many charts. In the origin of the chart U_n there sits GFP_n, while all the other Gaussian FPs are outside this chart, but in its closure.

We have discussed mainly the RG trajectories joining GFP_2 to GFP_1. They describe the unfamiliar situation of interacting theories that are free both in the UV and in the IR. [7] Starting in the perturbative regime near GFP_1 at low energy, the coupling $\tilde{g}$ grows without bound as one goes towards the UV. Superficially one may conclude that the theory does not have a good UV limit. However, one has to take into account the infinite amount of running of the wave function renormalization constants: while $\tilde{g}$ increases, $\tilde{Z}_2$ also increases at a similar rate, in such a way that the combination $\hat{\gamma}$ goes to zero, as

[6] In the case of "generalized free theories" (4.0.1), this is not the case.

[7] For examples of gauge couplings in semisimple gauge theories that have this behavior see [92].

seen from (4.1.23). At the same time, the anomalous dimension also becomes large and tends to -2, which is a sign that the scaling dimension of the field becomes exactly zero, as appropriate to GFP_2. Thus, at some point, it becomes natural to move to the chart U_2 where one sees again a perturbative theory, this time governed by the four-derivative kinetic term.

It is natural to conjecture the existence of flows between GFP_n and GFP_ℓ with $n > \ell$. However, theories with large n have negative canonical field dimension and infinitely many relevant couplings, a problematic situation. In spacetime dimension $d = 4$ this already occurs for $n = 3$, and this is the reason why this case has not been discussed here. One may think of restricting the number of relevant operators by imposing higher order symmetries of the form

$$\delta^{(n)}\phi = c^0 + c^1_\mu x^\mu + \dots c^{n-1}_{\mu_1 \dots \mu_{n-1}} x^{\mu_1} \dots x^{\mu_{n-1}} \ , \tag{4.1.36}$$

which is the symmetry of the kinetic term $\phi\Box^n\phi$, for $n > 0$. More generally, this will be a symmetry for Lagrangians where every field appears under at least n derivatives. By choosing the cutoff appropriately one can obtain flows that respect the symmetry and therefore remain in the symmetric subspaces of theory space, which would acquire a complicated stratified structure. However, in the last chapter we stated that, in the purely quartic scalar theory, due to gauge invariance under shifts in the field by harmonic functions, no physical content remains to be observed. So, a similar situation could be found also here, after imposing such symmetries.

All theories with $n > 1$ have ghosts at perturbative level, but we have shown that in this case the mass of the ghost mode depends on the trajectory and there are trajectories where it is arbitrarily high. The limiting case is the trajectory connecting GFP_1 in the IR to NGFP_1 in the UV, which is free of ghosts. Another pathology is that the coupling must be negative, leading to negative interaction energy. This was already well known in the case of Symanzik's asymptotically free scalar theory, and it is generally agreed that in spite of the coupling going asymptotically to zero, this is an unphysical feature [93]. Thus these theories are probably not very useful, even as statistical models, but we think that they still offer some interesting lessons in quantum field theory.

Finally, let us discuss the limitations of our analysis, starting in the chart U_1. We have found that when we run the RG towards the UV, all the trajectories above the separatrix in the fourth quadrant tend to GFP_2. However, going beyond the truncation (4.1.1), infinitely many other irrelevant terms will be turned on. For generic initial conditions near GFP_1, these couplings will go to infinity in the UV, signaling that these are just effective field theories. It is only for a very special subset of trajectories that UV completeness can be achieved. This is best seen by running the RG in the other direction, starting from GFP_2. Also in this case all other couplings compatible with the symmetries will be generated when one looks beyond linear order. As an example, working in the chart U_2, one can consider the coupling γ_2 that multiplies the six-derivative operator $(\Box\phi)^2(\partial\phi)^2$. The beta function of the dimensionless $\hat{\gamma}_2 = \gamma_2 k^2/\zeta_2^2$ is

$$\partial_t\hat{\gamma}_2 = 2\hat{\gamma}_2 + \frac{1024\hat{\gamma}^2(2+\hat{\zeta}_1)}{3(1+\hat{\zeta}_1)(\hat{\gamma}+128\pi^2(1+\hat{\zeta}_1)^2)} + O(\hat{\gamma}_2^2) \ . \tag{4.1.37}$$

As soon as $\hat{\gamma}$ is turned on, this beta function becomes nonzero and $\hat{\gamma}_2$ starts to grow. However, assuming that $\hat{\gamma}_2$ does not change too much the behavior of the other two couplings, in the IR $\hat{\zeta}_1$ goes to infinity and suppresses the loop term, while the classical term remains. Thus $\hat{\gamma}_2$ is expected to go again to zero in the IR. This is confirmed by numerically solving the equations. Similarly, all the other local couplings will be generated, but they are irrelevant for GFP_2 and even more so for GFP_1. Thus, one expects that they all go to zero as one flows towards the IR.

In the recent paper [86], the shift-invariant scalar theory has been studied in a truncation involving potentially infinitely many terms, all powers of $(\partial\phi)^2$. This gives more insight in the flow along the axis $\tilde{Z}_2 = 0$, in particular on the properties of $NGFP_1$. However, we observe that the term $(\Box\phi)^2$ will be generated by quantum fluctuations: first, one loop effects of the coupling $\tilde{g}$ will generate the coupling γ_2 as indicated above (this happens independently of the form of the kinetic term and of the cutoff) and then one loop effects involving γ_2 will give a nonzero beta function for Z_2. [8] Since all the additional terms $((\partial\phi)^2)^n$, $n > 2$, are irrelevant at GFP_2, our conclusions will not be modified by the inclusion of such terms in the truncation, except for changes in the properties of the trajectories at strong coupling, and in particular near the fixed point $NGFP_1$. Another work [95], suggested that the $NGFP_1$ is not physical, but just an effect of the finite truncation. It could be interesting to explore whether the same is true even after the inclusion of the higher derivative kinetic term.

4.2 ϕ^4 theory

Now we will consider the case of a theory with an action of the form

$$S = \int d^4x \left[\frac{1}{2} Z_2 (\Box\phi)^2 + \frac{1}{2} Z_1 (\partial\phi)^2 + \frac{1}{2} Z_0 \phi^2 + \frac{1}{4!} \lambda \phi^4 + ... \right] . \tag{4.2.1}$$

There are two potential terms that do not have shift symmetry. The coefficient of the quadratic term without derivatives, that is normally viewed as a mass squared, has been called Z_0 since it is a member of the family of free Lagrangians. If we momentarily set $Z_2 = 0$, we have the well known ϕ^4 scalar theory, so we will start our analysis precisely from this particular case.

4.2.1 The flow from GFP_1 to GFP_0

We have already discussed the $\lambda\phi^4$ theory in our general overview of the renormalization group, however we would like to reconsider it from a different point of view. The action is

$$S = \int d^4x \left[\frac{1}{2} Z_1 (\partial\phi)^2 + \frac{1}{2} Z_0 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right] \tag{4.2.2}$$

and the theory is power counting renormalizable. As in the case of the shift symmetric theory, there are two natural choices of coordinates. In the standard approach the field is assigned dimension of mass, in which case the wave function renormalization Z_1 is redundant and the coordinates on theory space are

$$\tilde{Z}_0 = \frac{Z_0}{k^2 Z_1} , \qquad \tilde{\lambda} = \frac{\lambda}{Z_1^2} . \tag{4.2.3}$$

This is the same chart U_1 considered above, extended to a shift-non-invariant interaction, and it has GFP_1 in the origin. Using the cutoff $R_k(z) = Z_1(k^2 - z)\theta(k^2 - z)$, the beta

[8] If one gives up $\mathbb{Z}_2$ symmetry, Z_2 is generated by quantum fluctuations involving the interaction $\partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi$ [94].

functions have the familiar form

$$\beta_{\tilde{Z}_0} = -2\tilde{Z}_0 - \frac{\tilde{\lambda}}{32\pi^2(1+\tilde{Z}_0)^2} \;, \tag{4.2.4}$$

$$\beta_{\tilde{\lambda}_0} = \frac{3\tilde{\lambda}^2}{16\pi^2(1+\tilde{Z}_0)^3} \;. \tag{4.2.5}$$

The beta function of Z_1 is zero, and so is the anomalous dimension $\eta_1 = -\partial_t \log Z_1$. There are no FPs in this chart except for GFP_1. Using the rescaling $u = \tan(\tilde{Z}_0)$ and $v = \tan(\tilde{\lambda})$, the flow lines have the form shown in Figure 4.2. We recognize that λ is asymptotically free for $\lambda < 0$, as was noticed by Symanzik [78]. The fact that the flow lines asymptote horizontally is due to the decoupling effect of the denominators: for sufficiently small k, the running of λ stops whereas $\tilde{Z}_0$ continues to run due to the classical term.

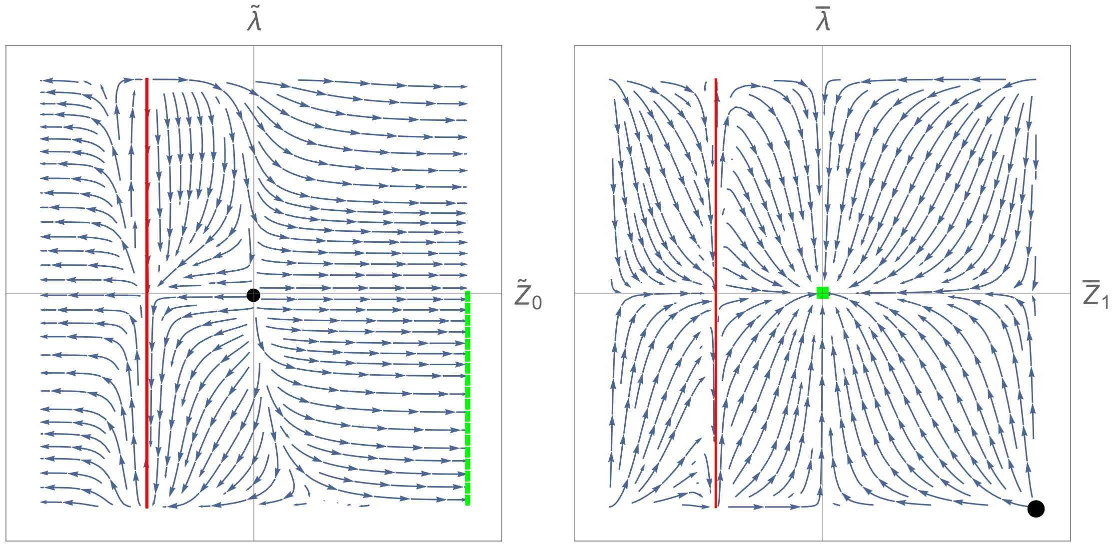


Figure 4.2: The flow in the chart U_1 (left) and in the chart U_0 (right). Flow lines in the lower right quadrant go from GFP_1 (black) to GFP_0 (green). The vertical red lines are singularities of the flow.

Another chart U_0 is centered on the fixed point GFP_0, that is a free conformal field theory where the field ϕ has canonical dimension of mass squared. This is sometimes called a trivial fixed point, because in Minkowski signature it has no propagating degrees of freedom, whereas in the Euclidean theory the correlation length at the fixed point is zero. In this case the "squared mass" Z_0 is actually dimensionless and redundant, Z_1 is an irrelevant coupling of dimension -2 while λ has dimension -4. The coordinates on theory space are

$$\bar{Z}_1 = \frac{Z_1 k^2}{Z_0} \;, \qquad \bar{\lambda} = \frac{\lambda k^4}{Z_0^2} \;. \tag{4.2.6}$$

The running of Z_0 is described by the anomalous dimension

$$\eta_0 = -\partial_t \log Z_0 = -\frac{\bar{\lambda}\bar{Z}_1}{32\pi^2(1+\bar{Z}_1)^2} \tag{4.2.7}$$

whereas the beta functions are

$$\partial_t \bar{Z}_1 = (2+\eta_0)\bar{Z}_1 \ , \tag{4.2.8}$$

$$\partial_t \bar{\lambda} = (4+2\eta_0)\bar{\lambda} + \frac{3\bar{\lambda}^2 \bar{Z}_1}{16\pi^2(1+\bar{Z}_1)^3} \ . \tag{4.2.9}$$

Again, there are no nontrivial finite fixed points. The beta function of $\bar{Z}_1$ can vanish either because $\bar{Z}_1 = 0$, or because $\eta_0 = -2$, which is satisfied asymptotically for $\bar{\lambda} \sim -64\pi^2 \bar{Z}_1$ and $\bar{Z}_1 \to \infty$. These asymptotes correspond to GFP$_1$.

We note that, aside from the absence of other FP's, the picture of the flow is very similar to the one of the shift-symmetric theory. In the chart U_1, the origin GFP$_1$ is the source of all flow lines with $\tilde{\lambda} < 0$ and GFP$_0$ corresponds to all points with $\tilde{\lambda} < 0$ finite and $\tilde{Z}_0 \to \infty$, so all the RG flow lines that are visible in the fourth quadrant joint GFP$_1$ to GFP$_0$. The same lines are visible in the chart U_0, where GFP$_1$ is in the bottom right corner and GFP$_0$ in the center.

The coordinate transformation between the charts U_0 and U_1 is

$$\bar{Z}_1 = \frac{1}{\tilde{Z}_0} \ , \qquad \bar{\lambda} = \frac{\tilde{\lambda}}{\tilde{Z}_0^2} \qquad \text{or conversely} \quad \tilde{\lambda} = \frac{\bar{\lambda}}{\bar{Z}_1^2} \ . \tag{4.2.10}$$

and the beta functions transform as vectors under this transformation.

We observe that also in this case the kinetic term of the UV fixed point (GFP$_1$), which gives rise to a propagating degree of freedom, is a redundant operator from the point of view of the IR fixed point (GFP$_0$), where nothing propagates. In fact, every local perturbation of GFP$_0$ is redundant.

4.2.2 The higher derivative theory

After this curious exercise, we would like to move to the higher derivative theory, from which we could learn something new. If we consider also the $\Box^2$ kinetic term, in the map centered in GFP$_1$, we can reintroduce the rescaled coordinate $\hat{Z}_2$ as in (4.1.8) and use the FRG equation. With the cutoff $R^{(24)}$ from (4.1.3), the β functions are

$$\beta \hat{Z}_2 = 2\hat{Z}_2 \tag{4.2.11}$$

$$\beta_{\hat{\lambda}} = \frac{3\lambda^2(8\hat{Z}_2+4)}{128\pi^2\left(\hat{Z}_0+\hat{Z}_2+1\right)^3} \tag{4.2.12}$$

$$\beta_{\hat{Z}_0} = -\frac{\lambda(2\hat{Z}_2+1)}{64\pi^2\left(\hat{Z}_0+\hat{Z}_2+1\right)^2} - 2\hat{Z}_0 \tag{4.2.13}$$

while the anomalous dimension η_1 is zero.

In the U_2 map the field is dimensionless and the potential is marginal. The running is driven by the following β functions:

$$\beta_{\tilde{Z}_1} = -2\tilde{Z}_1 \tag{4.2.14}$$

$$\beta_{\tilde{Z}_0} = -\frac{\tilde{\lambda}(\tilde{Z}_1+2)}{64\pi^2\left(\tilde{Z}_0+\tilde{Z}_1+1\right)^2} - 4\tilde{Z}_0 \tag{4.2.15}$$

$$\beta_{\tilde{\lambda}} = \frac{\tilde{\lambda}^2(12\tilde{Z}_1 + 24)}{128\pi^2\left(\tilde{Z}_0 + \tilde{Z}_1 + 1\right)^3} - 4\tilde{\lambda} \tag{4.2.16}$$

with $\eta_2 = 0$.

The perturbative version of the beta functions written here will furnish us an interesting example for our discussion in next chapter.

Chapter 5

Perturbative renormalization group of higher derivative theories

In the last chapter we were able to analyze the nonperturbative renormalization group flow of a simple scalar theory, resulting in an intriguing scenario where asymptotic freedom is recovered both in the UV and in the IR by moving between two different Gaussian fixed points. In this chapter we will study to what extent this picture is physically meaningful and compare how the choice of a renormalization scheme affects the RG flow in higher derivative theories.

The perspective that a flow between free theories could happen also in some more complicated renormalizable theory as quadratic gravity is very attractive, however we concluded Chapter 2 with a big question mark concerning universality of the renormalization group in presence of a quartic kinetic term. Before spending time and energies in attempting to map the complicated structure of the nonperturbative RG flow of gravitational theories, it would be wise to fully understand whether the beta functions from the FRG actually reproduce the momentum dependence of correlation functions in higher derivative theories. We will start analyzing the two simple theories considered in Chapter 4, and than we will try to extract a general rule. Then, in Chapter 6, we will consider theories living in a curved space, including gravitational ones.

5.1 The shift invariant scalar model

As a first example, we return to the shift invariant model. In this section, we will assume that the field ϕ has mass dimension one, which is the natural choice when we interpret the two-derivative term as defining the propagator. Then, Z_2 and g have dimension of inverse mass to power 2 and 4 respectively. It is thus natural to consider the Lagrangian

$$\mathcal{L} = -\frac{Z_1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{Z_1}{2m^2}\Box\phi\Box\phi - \frac{Z_1^2 g}{4M^4}(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi) \tag{5.1.1}$$

as a reparametrization of the Ansatz (4.1.1), where $m^2 = \frac{Z_1}{Z_2}$. We have defined the coupling constant with an explicit factor of the mass M in order to make g dimensionless. The value of this somewhat redundant notation is that it facilitates the use of dimensional analysis by showing the mass factors explicitly. The notation is natural when one views this as the low energy limit of the $U(1)$ linear sigma model, in which case the masses m and M are parametrically independent. Even though here we shall consider the theory as being potentially UV complete in itself, without the radial mode, we shall retain this notation. One can set $M = m$ without loss of generality.

In order to see this as a toy model for gravity, we recall that the action of Quadratic Gravity is schematically of form $m_P^2 R + \frac{1}{\xi}C^2$ (with C the Weyl tensor), so if we rescale the metric fluctuation by $m_P \sim 1/\sqrt{G}$, the action contains, among other terms,

$$(\partial h)^2 + \frac{1}{\xi m_P^2}(\Box h)^2 + \frac{1}{\xi m_P^4}(\partial h)^4 \ .$$

Recalling that the mass of the ghost is $m = \xi m_P^2$, this becomes essentially the same as (5.1.1) with $Z_1 = 1$, $M = m$ and $g = \xi$. We will discuss the gravitational case in more detail in Section 6.3.

Irrespective of the notation, it is important to keep in mind that the Lagrangian contains two mass scales: the mass of the ghost, m, and the scale $M/\sqrt[4]{g}$ at which tree level unitarity is violated and above which one would appear to be in a strongly interacting regime, due to the E^4 derivative interaction. In this paper we will always assume that $m < M/\sqrt[4]{g}$, in such a way that the massive ghosts can propagate and still be weakly interacting. Depending on the characteristic scale of the process, we thus have three energy regions which have different behaviors. Let us name these:

- **Low Energy (LE):** This region is defined by energies small compared to the ghost mass m. The heavy ghost is not dynamically active and can be integrated out.
- **Intermediate Energy (IE):** This corresponds to energies above the mass m, but below the apparent strongly-interacting regime. Here the heavy ghost is dynamically active. We will also briefly comment on an intermediate case where $s \sim -u \gg m^2$ but $t \ll m^2$.
- **High Energy (HE):** This occurs when the energy is high enough that $gE^4/M^4 > 1$. At these energies, perturbation theory would seem to break down.

In this paper we will study the scattering amplitude of this theory in the first two regimes. The four-point vertex in momentum space is

$$-\frac{2ig^2}{M^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_4 \cdot p_2) + (p_1 \cdot p_4)(p_3 \cdot p_2)\right] \ , \tag{5.1.2}$$

hence at tree level the $2 \to 2$ amplitude between massless on-shell particles is given by

$$-\frac{1}{2}g(s^2 + t^2 + u^2) \ , \tag{5.1.3}$$

where we have used the Mandel'stam variables introduced in (2.1.7). At low energy, quantum corrections generate new effective interactions with six or eight derivatives. This is the expected behavior of a nonrenormalizable theory, treated with standard EFT methods. Somewhat unexpectedly, the higher dimension operators cancel above the mass threshold for the production of ghosts, leaving us with a theory that looks renormalizable, with a logarithmically running coupling.

By comparing the loop corrections with the two- and four-point amplitudes, we will identify the physical running (or lack of running) of the parameters. The results differ in general from those given by other methods, but agree in some limits.

We will proceed as follows. We begin by describing the model in the absence of the higher derivative kinetic term. It is useful to have this description because the full theory reduces to this EFT in the low energy limit. Then, the calculation of the two-point function and the four-point scattering amplitude in the full theory are presented. The final results are given in Eq. (5.1.14) for the two point quantum correction, and in Eq (5.1.20) for the full four point correction. We will see how to match the four-point amplitude at low energy to the previously obtained EFT results, and also consider the remarkable simplifications that occur in the high energy limit, Eqs. (5.1.28) and (5.1.33). Finally, we compare the physical beta functions derived from the amplitude to the beta functions obtained from the FRG and other definitions.

5.1.1 Effective Field Theory at Low Energy

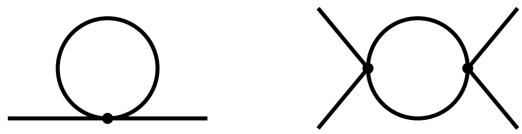


Figure 5.1: The diagrams giving corrections to the two- and four-point functions.

In generating an Effective Field Theory (EFT) one needs to know the low energy degrees of freedom and the symmetries. The massless mode is the only one which is dynamical at low energy. The symmetry is the same as that of the full theory, which in this case consists of the shift and reflection symmetry. One then writes out a normal theory with only the massless particle, consistent with these symmetries. In general this may have higher derivative nonrenormalizable interactions. By this procedure we arrive at the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{M^4}(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi) + \mathcal{L}_6 + \mathcal{L}_8 + ... \ . \tag{5.1.4}$$

Here $\mathcal{L}_6$ and $\mathcal{L}_8$ are Lagrangians with six and eight derivatives, which will be described more fully below. In principle one might consider a notation where the coupling strength g differs from that of the original theory. However we will see that the coupling in the effective theory is identified with the coupling of the full theory when the latter is renormalized at low energy.

At one loop, wavefunction renormalization would arise from the tadpole diagram with two external legs, as shown in the two-point diagram of Figure 5.1. However this vanishes, because the tadpole integral

$$\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu}{k^2} \tag{5.1.5}$$

is a scale-less integral which vanishes in dimensional regularization. This sets $Z_1 = 1$ in the effective field theory limit.

For the scattering amplitude, the one loop amplitude arises at order E^8 or equivalently it is described by a Lagrangian with eight derivatives. This can be seen dimensionally from the factor of g^2/M^8 which arises from two factors of the fundamental interaction. In dimensional regularization there are no other mass scales in the theory, and so the numerator factors arising from the one loop amplitude must be powers of the external energies. There will be a divergence in this amplitude and the coefficients at order E^8 will need to be renormalized. Along with the renormalization will come the usual logs, and because this is a mass independent renormalization, these must be factors of $\log s/\mu^2$ or similar logarithms. This tells us that the coefficients at order E^8 can be interpreted as "physically running" couplings. These logarithms will be finite and are predictions of the effective field theory.

In contrast, there will be no renormalization of the coefficients at order E^4 or E^6, as seen by the power counting described in the previous paragraph. This implies that there also will not be any logarithms generated. The couplings at order E^4 and E^6 will not be running in the physical sense.

Let us see this in explicit detail. In $\phi+\phi \to \phi+\phi$ scattering, there are a limited number of kinematic invariants involved consistent with the symmetry under the exchange of identical bosonic particles of the amplitude. This limits the number of effective Lagrangians involved. At dimension six and eight, these can be taken to be

$$\begin{aligned}\mathcal{L}_6 &= \frac{g_6}{4M^6}\partial_\mu\phi\partial^\mu\phi\Box\left(\partial_\nu\phi\partial^\nu\phi\right) + \frac{g_6'}{4M^6}\partial_\mu\phi\partial_\nu\phi\Box\left(\partial^\mu\phi\partial^\nu\phi\right)\\ \mathcal{L}_8 &= -\frac{g_8}{4M^8}\partial_\mu\phi\partial^\mu\phi\Box^2\left(\partial_\nu\phi\partial^\nu\phi\right) - \frac{g_8'}{4M^8}\partial_\mu\phi\partial_\nu\phi\Box^2\left(\partial^\mu\phi\partial^\nu\phi\right)\end{aligned} \tag{5.1.6}$$

We have calculated the one loop scattering amplitude in this theory. From the explicit calculation, the s channel gives

$$\begin{aligned}\delta\mathcal{M}_s =& \frac{ig^2s^2\left(41s^2+t^2+u^2\right)}{1920\pi^2M^8\epsilon}\\ &-\frac{ig^2s^2\left(15(\log(\frac{-s}{4\pi\mu^2})+\gamma_E)\left(41s^2+t^2+u^2\right)-1301s^2-46t^2-46u^2\right)}{28800\pi^2M^8}+O\left(\epsilon^1\right)\ ,\end{aligned} \tag{5.1.7}$$

while channels t and u can be found thanks to crossing symmetry. Channel t is given by the substitution $s\to t$ and $t\to s$ and u corresponds to the cyclic permutation $s\to u$, $t\to s$, $u\to t$. the total one-loop quantum correction to the four-point amplitude is

$$\begin{aligned}\delta\mathcal{M} =& \frac{g^2\left(41(s^4+t^4+u^4)+2(s^2t^2+t^2u^2+u^2s^2)\right)}{1920\pi^2M^8\epsilon}\\ &-\frac{g^2}{28800\pi^2M^8}\Bigg\{15\Bigg[s^2\left(41s^2+t^2+u^2\right)\log\left(\frac{-s}{4\pi\mu^2}\right)\\ &\qquad +t^2\left(s^2+41t^2+u^2\right)\log\left(\frac{-t}{4\pi\mu^2}\right)+u^2\left(s^2+t^2+41u^2\right)\log\left(\frac{-u}{4\pi\mu^2}\right)\Bigg]\\ &-(1301-615\gamma_E)(s^4+t^4+u^4)-2(46-15\gamma)(s^2t^2+t^2u^2+u^2s^2)\Bigg\}+O\left(\epsilon^1\right)\end{aligned} \tag{5.1.8}$$

Because the field here is massless, the logarithms can only involve kinematic factors of $s,\ t,\ u$.

The divergence in this expression can be absorbed into the renormalization of the dimension 8 coefficients in the effective Lagrangian. When renormalized at a scale $s =$

$t = u = \mu_R^2$, the amplitude has the form

$$\begin{aligned}\mathcal{M} = &-\frac{g}{2M^4}(s^2+t^2+u^2) \\ &+\frac{g_6}{2M^6}(s^3+t^3+u^3)+\frac{g_6'}{4M^6}(s^2t+s^2u+t^2u+t^2s+u^2s+u^2t) \\ &-\frac{g_8(\mu_R)}{2M^8}(s^4+t^4+u^4)-\frac{g_8'(\mu_R)}{2M^8}(s^2t^2+s^2u^2+t^2u^2) \\ &-\frac{g^2}{1920\pi^2M^8}\left[41s^4\log\left(\frac{-s}{\mu_R^2}\right)+41t^4\log\left(\frac{-t}{\mu_R^2}\right)+41u^4\log\left(\frac{-u}{\mu_R^2}\right)\right. \\ &\left.+s^2(t^2+u^2)\log\left(\frac{-s}{\mu_R^2}\right)+t^2(s^2+u^2)\log\left(\frac{-t}{\mu_R^2}\right)+u^2(t^2+s^2)\log\left(\frac{-u}{\mu_R^2}\right)\right]\end{aligned} \tag{5.1.9}$$

The values of $g_6,\ g_6',\ g_8(\mu_R),\ g_8'(\mu_R)$ are not predictions of the effective field theory and must be determined by either measurement or by matching to the full theory. We will explicitly perform the matching below, using the amplitude of the full theory.

The "physical" beta functions of the various couplings can be read off from the amplitude. These are

$$\begin{aligned}\beta_g &= 0 \\ \beta_{g_6} &= 0 \\ \beta_{g_6'} &= 0 \\ \beta_{g_8} &= \frac{41g^2}{480\pi^2} \\ \beta_{g_8'} &= \frac{g^2}{240\pi^2}\end{aligned} \tag{5.1.10}$$

These beta functions are predictions of the effective field theory.

The expected maximum limit of the effective field theory treatment of this matrix element occurs when

$$\frac{gE^4}{M^4} \sim 1 \tag{5.1.11}$$

where E^4 here represents any of the kinematic invariants $E^4 \sim s^2,\ t^2,\ u^2$. At these energies the interaction strength becomes large and the EFT treatment fails. All of the terms in the derivative expansion become relevant, with unknown coefficients. The actual limit of the EFT will either be when new degrees of freedom become dynamically active or at the energy implied by Eq. (5.1.11), which ever is lower.

The key elements of this section are that in the effective field theory treatment: 1) The original coupling g is not renormalized and does not run in the physical sense and 2) We need to renormalize the couplings of the eight derivative Lagrangian, and these couplings are running couplings.

5.1.2 The higher derivative theory

In this section, we continue to use the notation of Eq. (5.1.1). The Feynman rules are given by the propagator

$$-\frac{i}{p^2+\frac{1}{m^2}p^4} \tag{5.1.12}$$

At one loop, the quantum corrections to the two point function are given by the (tadpole) integral

$$-\frac{g}{Z_1 M^4}\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+\frac{1}{m^2}q^4}\left[p^2q^2+2(p\cdot q)^2\right] \tag{5.1.13}$$

In the absence of the four-derivative kinetic term (i.e. for $m\to\infty$) this is quartically divergent and is zero in dim reg.

In the general case, setting $d=4-2\epsilon$, it is equal to

$$i\frac{3}{2}\frac{g}{Z_1}\left(\frac{m}{M}\right)^4 p^2\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+\log 4\pi-\gamma-\log\frac{m^2}{\mu^2}+\frac{7}{6}+O(\epsilon)\right) \tag{5.1.14}$$

At one loop, only Z_1 receives quantum corrections, since there are no terms proportional to p^4. However, the μ-dependence in (5.1.14) does not correspond to a logarithmic p-dependence of the 2-point function. This means the μ dependence of the field renormalization can be reabsorbed once for all without producing any large logs with the physical energy scale of the scattering process. For example setting $\mu=m$ the logarithm disappears altogether.

Let us now compute the corrections to the four point function. In this case, one has to consider three Feynman diagrams that correspond to the s, t and u, channels. These are related by crossing symmetry. In the s channel, the integral one has to evaluate is

$$\frac{2g^2\mu^{4-d}}{Z_1^2M^8}\int\frac{d^dq}{(2\pi)^d}\frac{N}{(q^2+\frac{1}{m^2}q^4)((q+p)^2+\frac{1}{m^2}(q+p)^4)} \tag{5.1.15}$$

where $p=p_1+p_2$ and the numerator is

$$\begin{aligned}N=&\left[(p_1\cdot p_2)(q\cdot(q+p))+(p_1\cdot q)(p_2\cdot(q+p))+(q\cdot p_2)(p_1\cdot(q+p))\right]\\&\times\left[(p_3\cdot p_4)(q\cdot(q+p))+(p_3\cdot q)(p_4\cdot(q+p))+(q\cdot p_4)(p_3\cdot(q+p))\right]\ ,\end{aligned} \tag{5.1.16}$$

The other channels only differ by permutations of the external momenta.

Using (4.1.2), the fourth order propagators in the integral can be decomposed in a massless second order propagator and a massive ghost propagator. This is equivalent to replacing the quartic propagators in the diagrams either with the massless or the massive ones, and summing over all the possible combinations. In this way, for each channel the correction to the scattering amplitude becomes

$$\delta\mathcal{M}=\mathcal{M}_1-\mathcal{M}_2-\mathcal{M}_3+\mathcal{M}_4\ ,$$

where $\mathcal{M}_1$ contains only the contributions of the massless particles, $\mathcal{M}_4$ that of the massive ghosts and the other two mixed contributions with one massive and one massless propagator. In each partial amplitude we introduce a Feynman parameter, such that the denominators become (for the s channel)

$$\begin{aligned}\frac{1}{q^2(q+p)^2}&=\int_0^1 dx\frac{1}{(q'^2+\Delta_1)^2}\quad\text{with}\quad\Delta_1=x(1-x)p^2\ ;\\\frac{1}{q^2[(q+p)^2+m^2]}&=\int_0^1 dx\frac{1}{(q'^2+\Delta_2)^2}\quad\text{with}\quad\Delta_2=x(1-x)p^2+xm^2\ ;\\\frac{1}{(q^2+m^2)(q+p)^2}&=\int_0^1 dx\frac{1}{(q'^2+\Delta_3)^2}\quad\text{with}\quad\Delta_3=(1-x)(xp^2+m^2)\ ;\\\frac{1}{(q^2+m^2)[(q+p)^2+m^2]}&=\int_0^1 dx\frac{1}{(q'^2+\Delta_4)^2}\quad\text{with}\quad\Delta_4=x(1-x)p^2+m^2\ .\end{aligned}$$

and $q' = q + xp$. After some manipulations the numerators can be written as

$$N = N_0 + N_1(q')^2 + N_2(q')^4 \ ,$$

where

$$\begin{aligned}
N_0 &= x^2(1-x)^2 s^4 \ , \\
N_1 &= -\frac{1}{d}\left[(6+d)(x^2-x)+1\right] s^3 \ , \\
N_2 &= \frac{1}{4d(d+2)}\left[(d^2+6d+12)s^2 + 4(t^2+u^2)\right] \ .
\end{aligned} \tag{5.1.17}$$

Thus the partial corrections to the amplitude in d dimensions are

$$\begin{aligned}
\mathcal{M}_\ell &= 2g^2 \frac{1}{Z_1^2}\int \frac{d^d q'}{(2\pi)^d}\int_0^1 dx \frac{N}{((q')^2+\Delta_\ell)^2} \\
&= \frac{1}{(4\pi)^{d/2}}\frac{2g^2}{Z_1^2}\int_0^1 dx \Big[\Gamma\left(2-\tfrac{d}{2}\right)\Delta_\ell^{(d-4)/2} N_0 \\
&\quad + \tfrac{d}{2}\Gamma\left(1-\tfrac{d}{2}\right)\Delta_\ell^{(d-2)/2} N_1 + \tfrac{d(d+2)}{4}\Gamma\left(-\tfrac{d}{2}\right)\Delta_\ell^{d/2} N_2\Big] \ .
\end{aligned} \tag{5.1.18}$$

Finally performing the x-integration, we obtain for the s channel, without making any assumptions on the relative size of s, m, M,

$$\begin{aligned}
\delta\mathcal{M}_s = &\frac{g^2 m^4\left(13s^2+t^2+u^2\right)}{192\pi M^8\epsilon} - \frac{g^2}{5760\pi^2 s^3 M^8}\Bigg\{ -3s^5\left(41s^2+t^2+u^2\right)\log\left(-\frac{m^2}{s}\right) \\
&-6m^4(-s+m^2)^3\Big[\left(s^2+t^2+u^2\right) - 2\frac{s}{m^2}\left(-9s^2+t^2+u^2\right) \\
&\qquad + \frac{s^2}{m^4}\left(41s^2+t^2+u^2\right)\Big]\log\left(\frac{m^2}{m^2-s}\right) \\
&+s^2m^6\Big[-2\frac{s}{m^2}\left(352s^2+37(t^2+u^2)-15\gamma_E\left(13s^2+t^2+u^2\right)\right) \\
&\qquad -3\left(-31s^2+9\left(t^2+u^2\right)\right) + 6\frac{m^2}{s}\left(s^2+t^2+u^2\right)\Big] \\
&+6s^{5/2}m^4\sqrt{4m^2-s}\left(16(6s^2+t^2+u^2) - 8\frac{s}{m^2}(16s^2+t^2+u^2) + \frac{s^2}{m^4}(41s^2+t^2+u^2)\right) \\
&\times \operatorname{arccot}\sqrt{\frac{4m^2}{s}-1} - 30s^3m^4\left(13s^2+t^2+u^2\right)\log\left(\frac{4\pi\mu^2}{m^2}\right)\Bigg\} \ .
\end{aligned} \tag{5.1.19}$$

In this expression we can find a diverging contribution to the $(\partial\phi)^4$ operator, but again the scale parameter μ appears only in logs divided by the ghost mass m, hence the most convenient choice is to set $\mu = m$ for each value of the kinematic variables s, t and u. The scattering amplitude gains an imaginary part both from the on-shell loops of the massless modes thanks to $\log\left(-\frac{m^2}{s}\right)$ and from the on-shell ghosts in loops when $s > m^2$ in $\log\left(\frac{m^2}{m^2-s}\right)$.

The total quantum correction to the four point amplitude is

$$
\begin{aligned}
&\frac{5g^2m^4\left(s^2+t^2+u^2\right)}{64\pi^2M^8\epsilon}+\frac{g^2}{5760\pi^2M^8}\Bigg\{\\
&+\frac{m^4}{s^2}\Big[-6m^4\left(s^2+t^2+u^2\right)+3sm^2\left(-31s^2+9\left(t^2+u^2\right)\right)\\
&\qquad\quad +2s^2\left((352-195\gamma_E)s^2-(15\gamma_E-37)\left(t^2+u^2\right)\right)\Big]\\
&+6s^{-1/2}m^4\sqrt{4m^2-s}\big[16m^4(6s^2+t^2+u^2)\\
&\qquad\quad -8sm^2(16s^2+t^2+u^2)+s^2(41s^2+t^2+u^2)\big]\operatorname{arccot}\sqrt{\frac{4m^2}{s}-1}\\
&+\frac{m^4}{t^2}\Big[-6m^4\left(s^2+t^2+u^2\right)+3tm^2\left(-31t^2+9(s^2+u^2)\right)\\
&\qquad\quad +2t^2\left((352-195\gamma_E)t^2-(15\gamma_E-37)\left(s^2+u^2\right)\right)\Big]\\
&+6t^{-1/2}m^4\sqrt{4m^2-t}\big[16m^4(s^2+6t^2+u^2)\\
&\qquad\quad -8tm^2(s^2+16t^2+u^2)+t^2(s^2+41t^2+u^2)\big]\operatorname{arccot}\sqrt{\frac{4m^2}{t}-1}\\
&+\frac{m^4}{u^2}\Big[-6m^4\left(s^2+t^2+u^2\right)+3m^2\left(-31u^2+9\left(s^2+t^2\right)\right)\\
&\qquad\quad +2u^2\left((352-195\gamma_E)u^2-(15\gamma_E-37)\left(s^2+t^2\right)\right)\Big]\\
&+6u^{-1/2}m^4\sqrt{4m^2-u}\big[16m^4(s^2+t^2+6u^2)\\
&\qquad\quad -8um^2(s^2+t^2+16u^2)+u^2(s^2+t^2+41u^2)\big]\operatorname{arccot}\sqrt{\frac{4m^2}{u}-1}\\
&+3s^2\left(41s^2+t^2+u^2\right)\log\left(-\frac{m^2}{s}\right)\\
&+3t^2\left(s^2+41t^2+u^2\right)\log\left(-\frac{m^2}{t}\right)\\
&+3u^2\left(s^2+t^2+41u^2\right)\log\left(-\frac{m^2}{u}\right)\\
&+\frac{6(u-m^2)^3}{u^3}\log\left(\frac{m^2}{m^2-u}\right)\left[m^4\left(s^2+t^2+u^2\right)-2um^2\left(s^2+t^2-9u^2\right)+u^2\left(s^2+t^2+41u^2\right)\right]\\
&+\frac{6(t-m^2)^3}{t^3}\log\left(\frac{m^2}{m^2-t}\right)\left[m^4\left(s^2+t^2+u^2\right)-2tm^2\left(s^2-9t^2+u^2\right)+t^2\left(s^2+41t^2+u^2\right)\right]\\
&+\frac{6(s-m^2)^3}{s^3}\log\left(\frac{m^2}{m^2-s}\right)\left[m^4\left(s^2+t^2+u^2\right)-2sm^2\left(-9s^2+t^2+u^2\right)+s^2\left(41s^2+t^2+u^2\right)\right]\\
&+450m^4\left(s^2+t^2+u^2\right)\log\left(\frac{4\pi\mu^2}{m^2}\right)\Bigg\}
\end{aligned}
\tag{5.1.20}
$$

The arccots can be rewritten as logs, using

$$
\operatorname{arccot}\sqrt{x-1}=\frac{i}{2}\left[\log\left(1-\frac{i}{\sqrt{x-1}}\right)-\log\left(1+\frac{i}{\sqrt{x-1}}\right)\right]\,. \tag{5.1.21}
$$

The $Z_1 = 0$ case

It will be instructive to consider the case when there is no two-derivative kinetic term. Clearly in this case we cannot assume the canonical normalization $Z_1 = 1$. In fact, we want to consider the limit when Z_1, m and M all go to zero at the same rate. As in the previous section, by defining the dimensionless field φ and the coupling γ by

$$\frac{Z_1}{m^2}\phi^2 = \varphi^2 \ , \qquad \frac{Z_1^2}{M^4}\phi^4 = \gamma\varphi^4 \ ,$$

the action (5.1.1) becomes

$$\mathcal{L} = \frac{1}{2}m^2\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\Box\varphi\Box\varphi - \frac{\gamma}{4}(\partial_\mu\varphi\partial^\mu\varphi)(\partial_\nu\varphi\partial^\nu\varphi) \tag{5.1.22}$$

where the field is now canonically normalized with respect to the four-derivative kinetic term. Now we can simply set $m = 0$.

The calculation of the amplitude follows the steps of the general case but is much simpler. The s channel brings the following quantum correction:

$$\begin{aligned}\delta\mathcal{M}_s =& \frac{\gamma^2\left(13s^2+t^2+u^2\right)}{192\pi^2\epsilon} \\ &+ \frac{\gamma^2\left(3\left(13s^2+t^2+u^2\right)\left(\log\left(\frac{4\pi\mu^2}{-s}\right)-\gamma_E\right)+32s^2+5\left(t^2+u^2\right)\right)}{576\pi^2} + O\left(\epsilon^1\right) .\end{aligned} \tag{5.1.23}$$

Defining the renormalized coupling at the scale $s = t = u = \mu_R^2$ by the formula

$$\gamma(\mu_R) = \gamma - \frac{\gamma^2}{16\pi^2}\left[\frac{5}{2}\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi\mu^2}{\mu_R^2}\right) - \gamma_E\right) + \frac{7}{3}\right] \ , \tag{5.1.24}$$

(where the couplings in the r.h.s. are the bare ones) and exploiting crossing symmetry, we obtain the complete 4-point amplitude

$$\begin{aligned}\mathcal{M} =& -\frac{\gamma(\mu_R)}{2}\left(s^2+t^2+u^2\right) + \frac{\gamma^2}{192\pi^2}\left[\log\left(\frac{\mu_R^2}{-s}\right)\left(13s^2+t^2+u^2\right)\right. \\ &\left.+\log\left(\frac{\mu_R^2}{-t}\right)\left(s^2+13t^2+u^2\right)+\log\left(\frac{\mu_R^2}{-u}\right)\left(s^2+t^2+13u^2\right)\right] + O\left(\epsilon^1\right) \ .\end{aligned} \tag{5.1.25}$$

This agrees with [43].

In this case the μ_R-dependence is always associated to the dependence on the kinematic variables s, t, u, hence the physical beta function is

$$\mu_R\frac{\partial\gamma}{\partial\mu_R} = \frac{5\gamma^2}{16\pi^2} \ . \tag{5.1.26}$$

Understanding the General Amplitude

We will study the scattering amplitude in this theory and identify the physical running (or lack of running) of the parameters. The results differ from those given by usual methods. The amplitude calculation is also an instructive example of effective field theory when treated at low energy. Finally we identify a novel (as far as we know) phenomenon of the disappearance of certain operators as one increases the energy.

Here we will discuss the general case where we start with Z_1 and Z_2 in principle different from zero. For this section, we will revert to the notation of Eq. 5.1.1, where $Z_2 = 1/m^2$ and g is rescaled by a factor of M^4.

At one loop there is no renormalization of $Z_2 = 1/m^2$, as the one loop tadpole diagram only has two factors of the external momentum. We have seen in (5.1.14) that the one-loop contribution to Z_1 is independent of the momentum. Therefore we can renormalize to $Z_1 = 1$, and this result will be valid for all energies. From this we see that Z_1 is not a running parameter in the amplitude analysis and therefore

$$\beta_{Z_1} = 0 \tag{5.1.27}$$

for all energies. For the rest of this section we set $Z_1 = 1$.

The full result simplifies in the low energy limit. The logarithms involving mass factors can be Taylor expanded in the momentum, so that the only logarithms remaining are of the form $\log -s,\ \log -t,\ \log -u$.

For $Z_2 s \ll Z_1$, we find that the quantum correction is given by

$$\begin{aligned}\delta\mathcal{M} = \frac{5g^2m^4\left(s^2+t^2+u^2\right)}{64\pi^2M^8\epsilon} - \frac{g^2}{11520\pi^2M^8}\Bigg\{ &-900m^4\left(s^2+t^2+u^2\right)\log\left(\frac{4\pi\mu^2}{m^2}\right)\\ +30(30\gamma_E-11)m^4\left(s^2+t^2+u^2\right) &+ 6\Bigg[s^2\left(41s^2+t^2+u^2\right)\log\left(\frac{-s}{m^2}\right)\\ +t^2\left(s^2+41t^2+u^2\right)\log\left(\frac{-t}{m^2}\right) &+ u^2\left(s^2+t^2+41u^2\right)\log\left(\frac{-u}{m^2}\right)\Bigg]\\ -3\left(79(s^4+t^4+u^4)+6(s^2t^2+t^2u^2+u^2s^2)\right) &-760m^2(s^3+t^3+u^3)\Bigg\}\end{aligned} \tag{5.1.28}$$

One can see that the logarithm which is proportional to the original interaction, i.e. $s^2+t^2+u^2$, involves $\log(\mu^2/m^2)$ and is independent of the kinematic variables. This means that we can define a renormalized value of the coupling g by collecting all of the factors which multiply the invariant $s^2+t^2+u^2$ and identifying it with the coupling measured at low energy using the fundamental interaction. Then, we find

$$g(\mu) = g_B - \frac{5g^2m^4}{32\pi^2M^4}\left[\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{4\pi\mu^2}{m^2}\right) + \frac{11}{30}\right] \ . \tag{5.1.29}$$

Here g_B is the original unrenormalized coupling. Now, if we define the beta function by the usual recipe of deriving with respect to μ we find

$$\beta_g^\mu = \mu\frac{\partial g(\mu)}{\partial\mu} = \frac{5g^2m^4}{16\pi^2M^4} \ . \tag{5.1.30}$$

However, g does not depend on the energy so the physical beta function is

$$\beta_g = 0 \tag{5.1.31}$$

in the LE region.

The remainder of the amplitude involves powers of energy at order $E^6 \sim s^3,\ s^2t, ...$ and at order $E^8 \sim s^4,\ s^2t^2$. Those of order E^6 do not involve any logarithms, while there are logarithms at order E^8. A bit of inspection shows that the amplitude is exactly that of the effective field theory given in Eq. 5.1.9, with the identification

$$
\begin{aligned}
g &= g \\
g_6 &= -\frac{53g^2m^2}{384\pi^2M^2} \\
g_6' &= -\frac{7g^2m^2}{516\pi^2M^2} \\
g_8(\mu_R) &= \frac{79g^2}{1920\pi^2} + \frac{41g^2}{960\pi^2}\log\frac{\mu_R^2}{m^2} \\
g_8'(\mu_R) &= \frac{3g^2}{320\pi^2} + \frac{g^2}{480\pi^2}\log\frac{\mu_R^2}{m^2} \quad .
\end{aligned} \tag{5.1.32}
$$

Whereas in the effective field theory by itself these parameters were unknown, here we see that they are predicted by the full theory. This procedure is referred to as *matching* the EFT to the full theory.

We see that in this region the heavy ghost is not dynamically active and the one loop calculation amounts to integrating it out of the full theory to one loop order. The result is described by an effective field theory, with specific values of the coupling. This is an instructive example of effective field theory reasoning.

On the other hand, if all of the kinematic invariants are greater that m^2 in magnitude, i.e. $(s,\ |t|,\ |u|) \gg m^2$, another limit is recovered. If we use the definition (5.1.29) of the renormalized coupling defined below the mass threshold, the amplitude is finite and can be written in the form

$$
\begin{aligned}
\mathcal{M} = &-\frac{g}{2M^4}\left[1-\frac{17gm^4}{192\pi^2M^4}\right](s^2+t^2+u^2) - \frac{g^2m^4}{192\pi^2M^8}\left[\log\left(\frac{-s}{m^2}\right)(13s^2+t^2+u^2)\right. \\
&\left.+\log\left(\frac{-t}{m^2}\right)(s^2+13t^2+u^2)+\log\left(\frac{-u}{m^2}\right)(s^2+t^2+13u^2)\right] \quad .
\end{aligned} \tag{5.1.33}
$$

We can instead define the coupling at the (off-shell) renormalization point $s = t = u = \mu_R^2$ by making the finite renormalization

$$
\bar{g}(\mu_R) = g + \frac{5g^2m^4}{32\pi^2M^4}\left[\log\left(\frac{\mu_R^2}{m^2}\right) - \frac{17}{30}\right] \ , \tag{5.1.34}
$$

in which case the amplitude becomes

$$
\begin{aligned}
\mathcal{M} = &-\frac{\bar{g}(\mu_R)}{2M^4}(s^2+t^2+u^2) - \frac{\bar{g}^2m^4}{192\pi^2M^8}\left[\log\left(\frac{-s}{\mu_R^2}\right)(13s^2+t^2+u^2)\right. \\
&\left.+\log\left(\frac{-t}{\mu_R^2}\right)(s^2+13t^2+u^2)+\log\left(\frac{-u}{\mu_R^2}\right)(s^2+t^2+13u^2)\right] \quad .
\end{aligned} \tag{5.1.35}
$$

It agrees with the one calculated in the limit $Z_1 = 0$, eq. (5.1.25). This is understandable because at high energy the quartic terms in the propagator will dominate over the quadratic terms, and simply ignoring the quadratic terms yields the correct result.

There are a couple of striking observations which can be made from this result. The first is that all of the terms of order E^8 and E^6 have disappeared from the result. Because the general amplitude of Eq. (5.1.20) has many such terms, this requires special cancellations which we will discuss below. The second is that here we *can* define a running coupling which removes the potentially large logarithms of the form $\log s/m^2$. We consider the (off-shell) renormalization point $s = t = u = \mu_R^2$.

For $(s,\ |t|,\ |u|) \gg m^2$ this captures an important part of the quantum correction. There are still logarithms left over, but they are not large. This corresponds to a beta function

$$\beta_{\bar{g}} = \frac{5\bar{g}^2 m^4}{16\pi^2 M^4} \ . \tag{5.1.36}$$

The 17/30 in the formula for Eq. (5.1.34) amounts to an optional threshold correction matching the amplitude above and below the threshold.

The disappearance of the E^8 and E^6 terms appears initially surprising. There are many such terms with factors such as $s^4,\ t^4, ...$ and $s^3,\ t^3, ...$ in the general result. However, we can begin to see that there are cancelations by looking at the logarithmic type terms which arise at the highest order, E^8. We recall the result of Passarino and Veltman that all one loop diagrams can be expressed in terms of factors of the scalar tadpole, bubble, triangle and box diagrams. Here only the tadpoles and bubbles contribute. The tadpoles do not depend on the external momenta and do not give kinematic logs. These kinematic factors inside the logarithms come from the scalar bubble diagrams, which have the form

$$I_2(m_1,\ m_2,\ p^2) = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + \gamma - \log 4\pi - \int_0^1 dx \log\left(\frac{x m_1^2 + (1-x)m_2^2 - p^2 x(1-x)}{\mu^2}\right)\right] . \tag{5.1.37}$$

The logarithmic integral has the form

$$\begin{aligned} \int_0^1 dx \log\left(\frac{x m_1^2 + (1-x)m_2^2 - p^2 x(1-x)}{\mu^2}\right) &= \\ &= \log\left(-\frac{p^2}{\mu^2}\right) - 2 \ , \qquad\qquad m_1 = m_2 = 0 \\ &= \log\frac{m^2}{\mu^2} + \left(1 - \frac{m^2}{p^2}\right)\log\left(1 - \frac{p^2}{m^2}\right) - 2 \ , \qquad m_1 = 0,\ m_2 = m \\ &= \log\frac{m^2}{\mu^2} + \sqrt{1 - \frac{4m^2}{p^2}}\log\left(\frac{\sqrt{1-4m^2/p^2}+1}{\sqrt{1-4m^2/p^2}-1}\right) - 2 \ , \ m_1 = m_2 = m \ . \end{aligned} \tag{5.1.38}$$

The reader can see these logarithmic factors in the general amplitude. Moreover, one can see that there are common factors preceding these logs and the result involves the combination

$$I_2(0,\ 0,\ p^2) - 2I_2(0,\ m,\ p^2) + I_2(m,\ m,\ p^2) \ . \tag{5.1.39}$$

The divergences and the factor of $\log\mu^2$ cancel with this combination leaving a finite result as observed. In the low energy region, the latter two components of this expression go to constants, and only $I_2(0, 0, p^2)$ gives kinematic logarithms. This leads to the log dependence found in the LE/EFT limit in Eq. 5.1.9. However at high energy each of the components involves equal factors of $\log(-p^2)$, and the leading energy dependence of this combination will cancel. This leads to the vanishing of the terms of order E^8 at high energy. It requires detailed work to verify that the remaining terms of order E^6 also cancel, but the general idea is the same. The amplitude that starts out containing orders $E^4,\ E^6,\ E^8$ at low energy ends up at order E^4 only at high energy.

The latter results hold for all energy scales such that the mass can be ignored. However, at the beginning of Section 5.1 we distinguished an intermediate from a high energy scale. In the high energy region a puzzling situation now presents itself. From equations (5.1.26) or (5.1.36) one sees that for $g < 0$ the coupling is asymptotically free (in agreement with earlier calculation [43]). However, it appears that a focus on the coupling constant is insufficient. Even if the coupling constant is running logarithmically to an asymptotically free fixed point, the amplitude itself is blowing up with energy.

At high enough energy, the one-loop scattering amplitude will become greater than unity. This occurs when the kinematic invariants s, t, $u \sim E^2$ are of order

$$\frac{g(E)E^4}{M^4} \sim 1 \ . \tag{5.1.40}$$

A logarithmic decrease in the coupling is not enough to offset the power-law growth. This behavior puts the notion of asymptotic freedom in question. A solution to this problem concerning the definition of asymptotic safety could be given by considering inclusive initial and final states [84]. In a process with only the stable massless mode as asymptotic states, the total tree level cross section goes like gs/m^4, hence a logarithmic decrease in the coupling does not imply a noninteracting high energy limit. However, it could be that actual very high-energy processes happen between off shell virtual quanta of the full field ϕ, similar to what we see in the parton model in QCD. These processes can be represented, as a first approximation, as scattering processes between on-shell initial and final inclusive states which are a superposition of massless and ghost particles. In such cross sections, other cancellations take place between the sum over external single particle states at fixed transferred momentum s and scattering angle θ, thanks to the negative norm of the ghosts. Surprisingly, the final result is still a positive cross section, but the first two orders in s are suppressed, leaving a total cross section proportional to gm^2/s. Using a generalized version of the one-loop scattering amplitude (5.1.20) that admits ghost particles as external states, we managed to show the same cancellation also at one-loop. In fact, at leading order, the effect of quantum corrections reduce to the substitution of the coupling with its renormalized version. So, these inclusive scattering processes are asymptotically free at one-loop.

Another point of view on the problem of asymptotic freedom in theories with derivative interactions start from the idea that the actual higher derivative theory we are interested in is quadratic gravity, a gauge theory due to diffeomorphisms invariance. Hence, both the graviton and the ghost are gauge dependent states and cannot be proper physical asymptotic states, in the same way as no one expects to directly observe a gluon. Since matter in quadratic gravity interacts with a mixture of the massless graviton and the spin 2 ghost (the analogue of the field ϕ in our toy model), the same cancellations could take place in all interactions involving internal gravitons. A clear example of how this mechanism could work is given by the gravitational scattering between two scalar particles in quadratic gravity, that has a constant scattering amplitude in the high energy limit, and so its total cross section goes like $1/s$ [96, 97].

Peripheral scattering

The peripheral scattering limit is that of large s and $t \sim 0$. In this case, the s and u channels give the similar contributions to the scattering amplitude, since on shell $u = -s$ and the Mandel'stam variables appear only quadratically or in the Log in the dominant terms in the high-energy limit

$$\frac{7g^2m^4s^2}{48M^8\pi^2\epsilon} + \frac{g^2m^4}{576M^8\pi^2}\left\{(74 - 84\gamma_E)s^2 + 42s^2\left[\log\left(\frac{4\pi\mu^2}{s}\right) + \log\left(-\frac{4\pi\mu^2}{s}\right)\right]\right\} + O\left(\epsilon^1\right) \ . \tag{5.1.41}$$

On the other hand, the t channel is a bit more subtle, since the terms with powers of t in the denominator could give some divergences. However, if we expand (5.1.19) with t and

s exchanged at small t, all the divergent terms are actually zero and we obtain

$$\begin{aligned}
&\frac{g^2m^4s^2}{96\pi^2M^8\epsilon}+\frac{g^2m^4s^2\left[6\log\left(\frac{4\pi\mu^2}{m^2}\right)-(6\gamma-5)\right]}{576\pi^2M^8}\\
&+\frac{g^2m^2st}{96\pi^2M^8\epsilon}+\frac{g^2m^2st\left[7s+12m^2\log\left(\frac{4\pi\mu^2}{m^2}\right)+2m^2(-6\gamma+5)\right]}{1152\pi^2M^8}\\
&+\frac{7g^2t^2m^4}{96\pi^2M^8\epsilon}+\frac{g^2t^2\left[-6s^2\log\left(-\frac{t}{m^2}\right)+9s^2+35m^2s+420m^4\log\left(\frac{4\pi\mu^2}{m^2}\right)-m^4(420\gamma+140)\right]}{5760\pi^2M^8}\\
&+O\left(t^3\right)\ . \qquad (5.1.42)
\end{aligned}$$

The first line is clearly dominant, hence the one loop quantum corrections in peripheral scattering are

$$\begin{aligned}
\delta\mathcal{M}=&\frac{5g^2m^4s^2}{32\pi^2\epsilon M^8}+\frac{g^2m^4s^2}{576M^8\pi^2}\{(79-90\gamma_E)\\
&+42\left[\log\left(\frac{4\pi\mu^2}{s}\right)+\log\left(-\frac{4\pi\mu^2}{s}\right)\right]+6\log\left(\frac{4\pi\mu^2}{m^2}\right)\bigg\}+O\left(t\right)\ . \qquad (5.1.43)
\end{aligned}$$

After renormalization, the amplitude has the form

$$\mathcal{M}=\frac{gs^2}{M^4}\left[1+\frac{7gm^4}{96\pi^2M^4}\left(\log\left(\frac{-s}{m^2}\right)+\log\left(\frac{s}{m^2}\right)\right)+\frac{79gm^4}{576\pi^2M^4}\right]+\mathcal{O}(st) \qquad (5.1.44)$$

in the present notation. For this process one can defined a running coupling

$$\tilde{g}(\mu_R)=g+\frac{7g^2m^4}{48\pi^2M^4}\log\frac{\mu_R^2}{m^2}+\frac{79g^2m^4}{576\pi^2M^4}$$

when renormalizing at the scale $s=\mu_R^2$, where again the $79/576\pi^2$ factor is optional. This removes the potentially large logarithms, and carries the beta function

$$\beta_{\tilde{g}}=\frac{7g^2m^4}{24\pi^2M^4}\quad .$$

It is interesting that one can define a physical beta function in this region, yet it is different from that found when all the kinematic variables are large.

We can understand this in the following way. The universal beta functions that one calculates from perturbation theory are only universal as long as one considers processes that depend on a single momentum scale. This is the case, for example, for $2\to 2$ scattering at a fixed angle: the ratios of the Mandel'stam variables are fixed and the amplitude depends just on s. In the case of peripheral scattering we are changing the scattering angle together with the energy, and the amplitude is not a function of s alone. While there is no guarantee that a running coupling can be defined in this setting, it appears possible in the one loop calculation.

5.1.3 Comparing different definitions of running parameters

Here we return to our introductory point that there are different flavors of renormalization group techniques. In turn, we address the three which we highlighted: the physical running the cutoff or μ-running and the FRG.

To make the comparison meaningful, some clarification concerning the beta functions produced by the FRG needs to be done. In Section 4.1 the Lagrangian was parametrized as in (4.1.1) and the running of Z_1, Z_2 and g was calculated using the full FRG. The coupling then depend on a scale k that has the meaning of an IR cutoff. This calculation goes beyond the one loop approximation, because the couplings in the r.h.s. of the FRG equation are treated as running couplings. This kind of "RG improvement" amounts to a resummation of infinitely many diagrams. In order to compare with the amplitude calculation, we have to downgrade those results to the one loop approximation. This is easily achieved by neglecting the RG improvement.

Assuming that the field has dimension of mass, one arrives at the following beta functions for the dimensionful couplings

$$k\partial_k Z_1 = -\frac{Z_1 + 2k^2 Z_2}{16\pi^2 (Z_1 + k^2 Z_2)^2}\, g k^4 \tag{5.1.45}$$

$$k\partial_k Z_2 = 0 \tag{5.1.46}$$

$$k\partial_k g = \frac{5(Z_1 + 2k^2 Z_2)}{32\pi^2 (Z_1 + k^2 Z_2)^3}\, g^2 k^4 \tag{5.1.47}$$

With dimensionless field the beta functions are the same, but of course the dimension of the couplings is different and so are the powers of k in the r.h.s. These one loop beta functions differ from the full ones of Section 4.1, but the qualitative features of the RG flow remain the same.

We are now ready to compare the physical running with the results of the FRG and the μ-running.

We begin by comparing the physical running of g to the μ-running. We will see that this coupling shows the typical threshold behaviour discussed in Section 2.3.1. At low energy, below the mass m, the amplitude does not give a physical running for g:

$$\beta_g = 0, \qquad E \ll m\ . \tag{5.1.48}$$

This is in disagreement with the $\log \mu$ derivative approach, which predicts a logarithmic running, see eq.(5.1.30). This β function comes out from the $\log m^2/\mu^2$ in (5.1.29). Here μ is an unphysical parameter which disappears from all physical reactions after renormalization. The apparent running of the coupling g arises from taking the negative logarithmic derivative of correction (5.1.29) with respect to μ. This often is appropriate in other settings because in mass independent renormalization schemes the logarithmic factor is $\log p^2/\mu^2$ (where p is some kinematic energy factor) so that taking the derivative with respect to μ reveals the dependence of the amplitude on the kinematic variables $\log p^2$. However here there is no dependence on any kinematic variable. If we perform renormalization at any kinematic scale below the mass threshold, it remains that value as long as the ghosts stay frozen. Of the two definitions of running given in dimensional regularization, the physical one matches the results from the EFT, where we did not observe any quantum corrections to the coupling g.

On the other hand, there is agreement in the running of g in the high energy region for energies above the mass m. Here the beta function describing the running coupling in the amplitude is

$$\beta_g = \frac{5g^2 m^4}{16\pi^2 M^4}, \qquad E \gg m\ . \tag{5.1.49}$$

This indeed agrees with equation (5.1.30), and does not change if we add a finite piece as in (5.1.34).

We now compare the physical running of g to the FRG results. At low energy the FRG running is power-law:

$$\beta_g = \frac{5(Z_1 + 2k^2/m^2)}{32\pi^2(Z_1 + k^2/m^2)^3}\frac{g^2k^4}{M^4} \to \frac{5g^2k^4}{32\pi^2M^4}, \qquad \text{for } k \ll m \quad . \tag{5.1.50}$$

This beta function rapidly runs to zero at lower energies, asymptoting to the constant value of g found in the amplitude calculation.

On the other hand, at high energy the FRG gives

$$\beta_g = \frac{5(Z_1 + 2k^2/m^2)}{32\pi^2(Z_1 + k^2/m^2)^3}\frac{g^2k^4}{M^4} \to \frac{5g^2m^4}{16\pi^2M^4}, \qquad k \gg m \quad . \tag{5.1.51}$$

which agrees both with the physical running and the μ-running.

Between these two limits, the behavior of the amplitude is more complicated, and if one tries to define a running coupling as in (5.1.34), namely isolating the coefficient of $s^2 + t^2 + u^2$ in the full amplitude (5.1.20) and setting $s = t = u = \mu_R^2$, it turns out to be impossible to unambiguously identify it. Hence the definition of a physical running is only meaningful in the asymptotic regions, where different power-laws are clearly separated. The standard, conventional way of joining them is to assume that g does not run all the way up to the mass m and to match this to the high-energy logarithmic behavior (5.1.35) via (5.1.34). This is shown by the black dashed line in Fig.5.2.

We note that whereas the beta function becomes universal (scheme-independent) at high energy, the relation between the low-energy value of the coupling and its high-energy behavior is not. By choosing a different constant in the bracket in (5.1.34) we can change the offset between the low- and high-energy parts of the curve in Fig.5.2 and shift up or down the part of the curve above the threshold $k/m = 1$.

The same effect can also be obtained by using a different renormalization point. In the UV, the non-polynomial dependence on the kinematical variables of the terms of the amplitude proportional to s^2, t^2 and u^2 is given by $\log(-s/\mu)$, $\log(-t/\mu)$ or $\log(-u/\mu)$. Thus, if we choose to renormalize at $s = at = bu = \mu_R^2$ with a and b fixed constants, the coupling gets shifted by $-\log(ab)$, at the price of having $\log\left(\frac{-ta}{\mu_R^2}\right)$ and $\log\left(\frac{-ub}{\mu_R^2}\right)$ in eq. (5.1.35). In this work we have used the symmetric point $a = b = 1$ and the definition (5.1.34), because these best capture the behavior of the amplitude and are best suited for comparison to the FRG, but we stress that these are arbitrary choices.

In on-shell configurations, choosing the parameters a and b is equivalent to fixing the scattering angle. This angle should be held fixed along the running from the IR to the UV regime, otherwise one could observe different beta functions and consequently different runnings of the coupling. If one allows the scattering angle to also depend on s, the amplitude is no longer described by the universal running coupling, as demonstrated by the example of peripheral scattering.

The FRG gives a continuous interpolation for the running of the quartic coupling, and can separately account for the six- and eight-derivative couplings that will inevitably be generated.

In order to compare the RG trajectory of the coupling in the FRG with the trajectory of the physical coupling, we have to make an identification of the argument of the former, which is an arbitrary cutoff scale k, with the argument of the latter, which at the symmetric point is $\sqrt{s}$. If we just put $k^2 = s$, and we adjust the initial conditions so that the two trajectories have the same IR limit $g(0)$, then in the UV limit they differ by a small offset. This can be fixed by choosing $k = \sqrt{s}/\xi$, where $\xi = e^{25/40-17/60} \approx 1.4$. This is illustrated again in Fig.5.2.

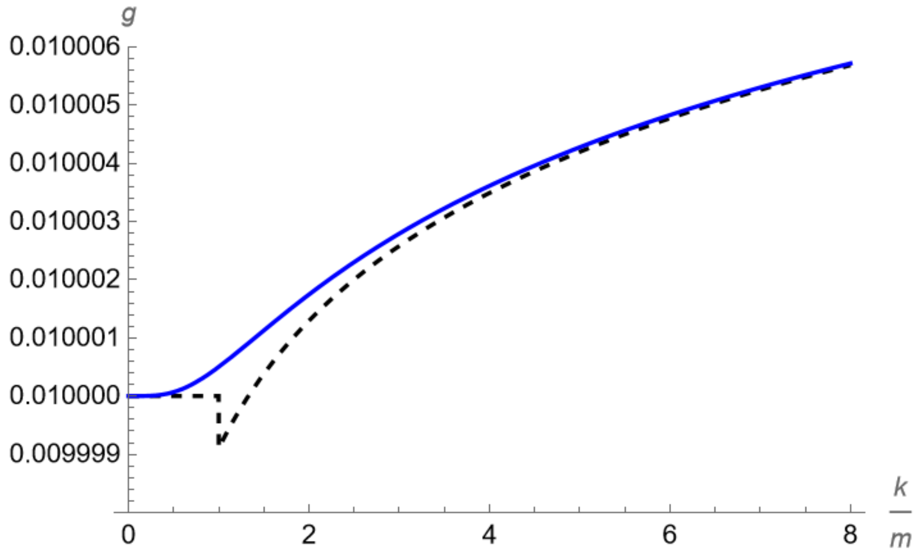


Figure 5.2: The running coupling calculated from the FRG (blue continuous curve) and the one obtained by matching the low- and high-energy physical running (black dashed). They have been calculated here for the same low energy limit $g = 0.01$.

Summarizing, at low energy the power-law running of g found in the FRG is not directly observed in the amplitude. It is an aspect of the threshold behavior, interpolating between a constant in the IR limit and the logarithmic behavior at the high energy. The threshold behavior of the couplings in FRG is not universal, and in any case there is no definition of physical running to compare with in that regime.

At high energy, the dependence on $\log k^2$ mirrors correctly the dependence of the amplitude on $\log E^2$, and gives the correct beta function. This is due to the fact that for fixed ratios t/s and u/s, and in the limit when $m/s \to 0$, the amplitude depends only on a single mass scale s, which enters in the denominators of the loop intergrals in a way that is reminiscent of an IR cutoff. Thus, in this regime, the k-dependence of the running coupling correctly reflects the s-dependence of the amplitude. At energies close to m, the amplitude becomes a complicated function of s and m and no RG calculation exactly reproduces the amplitude.

The other running parameter within the FRG is Z_1. In the notation of this section, the general expression was

$$\beta_{Z_1} = -\frac{Z_1 + 2k^2/m^2}{16\pi^2(Z_1 + k^2/m^2)^2} \frac{gk^4}{M^4} \ . \tag{5.1.52}$$

This is in disagreement with the amplitude calculation, for which Z_1 does not run at all energies

$$\beta_{Z_1} = 0 \tag{5.1.53}$$

If we had defined the running of Z_1 not by the dependence on energy or on renormalization scale, but by the dependence of the counterterm on the unphysical parameter μ which appears in dimensional regularization, we would have identified

$$\beta^{\mu}_{Z_1} = \frac{3}{16\pi^2} \frac{gm^4}{M^4} \ . \tag{5.1.54}$$

We noted that the asymptotic form at large k of the FRG result was

$$\beta_{Z_1} = \frac{gm^2k^2}{M^4} + \frac{3}{16\pi^2}\frac{gm^4}{M^4} + ... \tag{5.1.55}$$

which, if one disregards the power-law running, would agree on the logarithmic part with the μ-running. So in this case the issue concerning the proper definition of the beta function is not limited to a given kinematical domain, but whether considering Z_1 a physically running coupling at all.

The one-loop correction to the kinetic energy term was found to be

$$\frac{3gm^4}{16\pi^2M^4}p^2\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m^2}{\mu^2} + \frac{7}{6}\right] \quad . \tag{5.1.56}$$

The portion to focus on is again the $\log m^2/\mu^2$ and what is going on is very similar to the low energy regime of g. If we perform wavefunction renormalization at any kinematic scale, setting $Z_1 = 1$, it remains that value at any other scale. Taking the derivative with respect to μ does not give us physical information in this case.

This is not a significant issue, since, if we choose the field to be dimensionless, probably a more fitting description for the high-energy limit, Z_1 becomes the mass m and its positive classical dimension will overcome any quantum correction in the UV limit of the weakly interacting theory, exactly as observed for the mass of the ϕ^4 theory in Chapter 2.

The other issue is that of power-law corrections found within the FRG. At one-loop, in the case of Z_1 this is a less significant issue than for g, since Z_1 is a redundant coupling and is not associated directly with any scattering process. Another way to say this is that in a two-point function the only invariant scale is p^2, which on shell is just equal to the pole mass m_2. Nevertheless, we can interpret the difference between the beta functions computed here and those coming from the FRG as follows. We have perturbed around a generic free theory containing both kinetic terms, which is not a fixed point in general: only the theories with $Z_1 = 0$ or $Z_2 = 0$ are fixed points. There is a trivial running with p that goes from the one to the other, since the quartic term dominates in the UV and the quadratic one dominates in the IR, but the dimension of the field remains fixed and does not enter in any of our conclusions. However, the canonical dimension of the field at a free fixed point is fixed: it is one at the two-derivative fixed point and zero at the four-derivative one. In the FRG this is correctly taken into account. Within the context of the FRG, the power running of Z_1 with scale k is necessary to correctly interpolate between the low-energy and high-energy Gaussian fixed points.

In conclusion, the scattering amplitude reveals what we are calling the "physical" running, as it describes the running parameters seen in physical processes. This differs from some other definitions of running couplings using different methods, and we have used explicit calculations to illustrate these differences.

Some of the lessons from this example can be summarized as follows:

1. Physical running couplings can only be defined far from mass thresholds, and there are different patterns of running above and below the threshold. In our case, the coupling g does not run below the threshold and runs logarithmically above it. Effective Field Theory is useful in understanding the low energy region.

2. Power-law running is not seen in the physical amplitudes. Instead, in the EFT regime, the effects which depend on higher powers of the kinematic invariants are organized as higher order operators in an effective Lagrangian. These higher order operators disappear altogether above the mass threshold (operator "melting").

3. Alternate methods of defining running couplings using $\Lambda\frac{\partial}{\partial\Lambda}$, $k\frac{\partial}{\partial k}$ or $\mu\frac{\partial}{\partial\mu}$ (where Λ, k, μ refer to UV cutoffs, IR cutoffs or the dimensional regularization auxiliary scale) sometimes yield running behavior which is not seen in physical processes. This happens for the coupling Z_1. The culprit is factors of $\log m^2/\Lambda^2$ etc, which does not involve any of the kinematic invariants and hence does not change with the energy scale of the physical reaction.

The disappearance of the higher order operators of the low energy EFT is expected when the model is UV completed in a linear $U(1)$ sigma model, but surprisingly also happens when the four-derivative kinetic term becomes important. This offers a glimpse of how, in a derivatively coupled theory, one could transition from the low energy EFT regime to an asymptotically free (and possibly asymptotically safe) regime. In principle, this could provide an alternative UV completion to the $U(1)$ linear sigma model, mentioned in the Introduction.

This kind of behavior may be extended also to gravitational theories. For example, it raises the possibility that at least some of these higher order operators, such as those of order R^3 should not be used above certain thresholds, because the coefficients of the higher order operators vanish.

Our model seems to enter a strong coupling regime at very high energy. This is because the powers of momentum of the interaction overwhelm the logarithmic decrease of the coupling. We have briefly discussed here how proper asymptotic freedom could be recovered in scattering processes with inclusive asymptotic states, or gauge invariant ones in the case of gauge theories.

5.2 ϕ^4 theory

For the higher derivative version of ϕ^4 scalar theory, we can study the one-loop self-energy correction and the $2 \to 2$ scattering amplitude and compute the contribution of this interaction vertex to its own beta function and to those of Z_i. The calculation is quite similar to the one described for the shift invariant model, so we will go directly to the results. The tadpole diagram is logarithmically divergent because of the quartic propagator and is equal to

$$\frac{\lambda}{32\pi(m_1^2 - m_2^2)}\left[m_1^2\log\left(\frac{\mu^2}{m_1^2}\right) - m_2^2\log\left(\frac{\mu^2}{m_2^2}\right)\right] , \tag{5.2.1}$$

where we have introduced the masses of the ghost and the healthy mode in analogy with Section 3.1.3, i. e. $Z_0/Z_2 = m_1^2 m_2^2$ and $Z_1/Z_2 = m_1^2 + m_2^2$. This diagram gives a correction to the mass therm Z_0, since the result does not contain powers of the transferred momentum. It induces a cutoff or μ-running in this coupling even if there is no corresponding physical running, because tadpole diagrams cannot produce logarithms of momenta. At a first glance the renormalization of the mass seems very similar to the 2-derivative theory, since in both cases we expect the classical running induced by the positive mass dimension of the mass term to prevail over quantum corrections at high energies. However, in this case the divergence is purely logarithmical instead of power-law, so the quantum correction in dim reg does not arrive from a subleading term in the $m_i^2/p^2 \ll 1$ expansion and it is not multiplied by positive powers of the mass. On the other hand, the bubble integral is not divergent and the s channel is

$$-\frac{\lambda^2}{32(m_1^2-m_2^2)^2\pi^2(-s)^{3/2}}\Bigg[2s\sqrt{4m_1^2-s}\,\text{arctanh}\sqrt{\frac{-s}{4m_1^2-s}}+2s\sqrt{4m_2^2-s}\,\text{arctanh}\sqrt{\frac{-s}{4m_2^2-s}}$$
$$+2\sqrt{-s}\sqrt{m_1^4+(m_2^2-s)^2-2m_1^2(m_2^2+s)}\,\text{arctanh}\left[\frac{m_1^2-m_2^2-s}{\sqrt{m_1^4+(m_2^2-s)^2-2m_1^2(m_2^2+s)}}\right]$$
$$-2\sqrt{-s}\sqrt{m_1^4+(m_2^2-s)^2-2m_1^2(m_2^2+s)}\,\text{arctanh}\left[\frac{m_1^2-m_2^2+s}{\sqrt{m_1^4+(m_2^2-s)^2-2m_1^2(m_2^2+s)}}\right]$$
$$-2\sqrt{-s}(m1^2-m2^2)\big(\log m_1-\log m_2\big)\Bigg]\ . \tag{5.2.2}$$

The other channels can be easily found using crossing symmetry. In the high energy limit $s \gg m_i^2$, the total amplitude reduces to

$$\mathcal{M}=\lambda+\lambda^2\frac{m_2^2\log\left(\frac{-s}{m_2^2}\right)-m_1^2\log\left(\frac{-s}{m_1^2}\right)}{16(m_1^2-m_2^2)\pi^2s^2}+(s\to t)+(s\to u)\ , \tag{5.2.3}$$

while, with s below m_1^2, the ghost freezes out and we recover the amplitude (2.1.9) with m_1 acting as the UV cutoff of the EFT instead of Λ. Whereas in the low-energy regime we observe exactly what we expected, the high-energy limit is quite surprising: despite the absence of UV divergences, there are logarithms of the external momenta. Luckily, they are harmless at large s thanks to the s^2 in the denominator, so there are no problems involving perturbativity in this case.

Using the functional renormalization group, the one-loop version of the β functions of the dimensionful couplings with $[\phi]=1$ are

$$k\partial_k Z_0 = -\frac{\lambda(2k^2Z_2+Z_1)}{64\pi^2k^2\left(k^{-2}Z_0+k^2Z_2+Z_1\right)^2}\ , \tag{5.2.4}$$

$$k\partial_k Z_1 = 0\ , \tag{5.2.5}$$

$$k\partial_k Z_2 = 0\ , \tag{5.2.6}$$

$$k\partial_k \lambda = \frac{3\lambda^2(8k^2Z_2+4Z_1)}{128\pi^2\left(k^{-2}Z_0+k^2Z_2+Z_1\right)^3}\ . \tag{5.2.7}$$

As in the μ-running, the only operator quadratic in the field to run is Z_0 in the FRG beta functions. In the high-energy limit the classical scaling prevails over quantum corrections, as can be seen from the powers of k in the numerator. Hence, the gap with respect to the physical scheme is closed at high-energies.

In the picture where the field has dimension one, we can set $Z_1=1$ and consequently

$$Z_2=\frac{1}{m_1^2+m_2^2}\ ,\qquad Z_0=\frac{m_1^2m_2^2}{m_1^2+m_2^2}\ . \tag{5.2.8}$$

So, when we consider the running of λ, we observe three phases:

- when $k \gg m_1$, the beta function is dominated by the k^2Z_2 terms and the running is suppressed by a k^4 in the denominator;
- when k is comprised between m_1 and m_2, both k^2Z_2 and $k^{-2}Z_0$ are small and the part proportional to Z_1 gives the beta function of the theory with the 2-derivative kinetic term;

- at low energies, $k \ll m_2$, there is no running because $k^{-2}Z_0$ in the denominator kills the beta function.

In the end, also in this case the overall behaviour mimics quite well the physical running, so, there seems not to be a problem for the universality of the renormalization group.

5.3 Non-universality of perturbative beta functions

Despite of the reassuring conclusion of the analysis of the ϕ^4 theory, in its one-loop quantum corrections we can observe two features that can become dangerous in other more general cases: on one hand, the tadpole diagram is logarithmically divergent without needing any mass, so it is no more granted in general that at high energies the logarithms of the cutoff from these diagrams affect only relevant couplings, as in two derivative theories. On the other hand, the bubble diagram shows large logarithms of the kinematical variables not related to the UV regulator, which is not there at all, since the integral is finite. Such an effect is related to the fact that the massless version of the higher derivative bubble integral is infrared divergent in ϕ^4 theory, hence masses act as infrared regulators and come in logs with momenta. Since the interaction is a pure potential, the vertex cannot produce additional powers of the momenta in the amplitude. Thus, the tadpole ends up renormalizing a relevant operator negligible in the UV, while in the bubble (5.2.3) an s^2 term in the numerator overcomes the large logarithms, so high-energy universality is restored.

One might wonder whether a marginal derivative interaction could break this fragile equilibrium and lead to a failure in the one-loop universality of the renormalization group. It turns out that the only marginal interactions containing four fields in a theory with only one higher derivative scalar field are [77]

$$((\partial_\mu\phi)^2)^2 \ , \qquad (\partial_\mu\phi)^2\phi\Box\phi \ , \qquad (\phi\Box\phi)^2 \ . \tag{5.3.1}$$

We have already studied the first vertex in the shift invariant model and it gave a good matching between the different running in the high energy limit, since shift invariance protects from infrared divergences. The other two are equal to zero on-shell in the same regime, because the $\Box$ operator is zero when acting on external legs. One can consider only the diagrams where the $\Box$ acts exclusively on internal lines. If we focus on $2 \to 2$ scattering, in the case of $(\phi\Box\phi)^2$ these diagrams are IR finite as in the shift invariant case, because there is at least one $\Box$ operator acting on each virtual particle, while with the vertex $(\partial_\mu\phi)^2\phi\Box\phi$ there is one diagram potentially divergent in the IR, the one schematically depicted in figure 5.3.

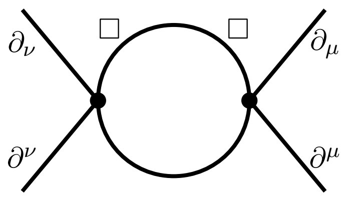


Figure 5.3: the infrared divergent bubble diagram with the interaction vertex $(\partial_\mu\phi)^2\phi\Box\phi$. Derivatives are placed on the propagators of particles they act on.

We notice from the disposition of partial derivatives over the external legs that, summing this diagram with its cross-symmetric equivalents in channels u and t, the total result is proportional to the tree level shift invariant interaction. In the massless limit, the propagator of the internal line over which the two box operators act is exactly canceled, so we remain with a tadpole integral:

$$\int d^4q \, \frac{q^4}{q^4(q+p)^4} = \int d^4q \, \frac{1}{q^4} \ . \tag{5.3.2}$$

As already stated, it depends on the UV cutoff, but at the same time it is independent of the transferred momentum s, t or u. The one-loop quantum correction from these three diagrams in a cutoff renormalization scheme is

$$\frac{1}{8\pi^2}\left(s^2+t^2+u^2\right)\log\left(\frac{\Lambda^2}{k^2}\right) \ , \tag{5.3.3}$$

hence they contribute to the beta function of g, but they cannot produce any physical running. This is a serious threat to RG universality and we are going to discuss this issue from a more general point of view in the coming pages.

We have already hinted how infrared divergences can disrupt RG universality in higher derivative theories in the end of chapter 2. Now we will resume that discussion and describe how different regularization methods behave in such a situation.

The IR behaviour of the generic one-loop integral in higher derivative theories presented in Section 2.3.3

$$\int_k^\Lambda d^4q \, \frac{N(q,p_i)}{q^4(q+p_1)^4 \times \cdots \times (q+p_1+\cdots+p_{n-1})^4} \tag{5.3.4}$$

actually depends on the numerator N. If the bare action is local in fields and their derivatives, N can be written as a sum of monomials of entire positive powers of momenta running on internal propagators forming the loop and powers of momenta of external particles (notice that some p_i can be associated to a vertex with more than one incoming external particle, meaning $p_i = \sum_j p_{i,j}$)

$$N(q,p_{i,j}) = \sum_\alpha a_\alpha \prod_{m=1}^{n}\left(q+\sum_{i=1}^{m} p_{i-1}\right)^{b_{\alpha,m}} \prod_j p_{m,j}^{c_{\alpha,m,j}} \ , \tag{5.3.5}$$

where by convention $p_0 = 0$. The integral associated to one of these monomials has a logarithmic divergence in $q = -\sum_{i=1}^m p_{i-1}$ if $b_{\alpha,m}$ is zero. Hence, integral (5.3.4) is IR finite for arbitrary off-shell external momenta only if $b_{\alpha,m} \neq 0 \ \forall m$. This is the case, for example, of shift invariant theories.

To better understand where these divergences come from and how they are treated in different renormalization schemes, we will focus on the tadpole and the bubble diagrams built with a generic interaction vertex $\mathcal{O}$, equivalent respectively to the case $n=1$ and $n=2$ of (5.3.4). We will start from the purely quartic propagator and only at a later time we will include the two derivative kinetic term and a mass.

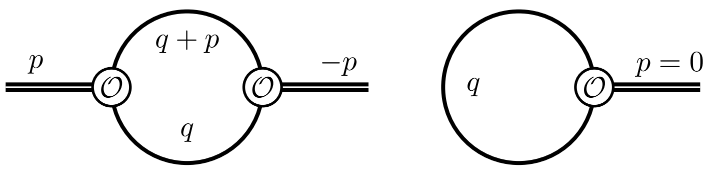


Figure 5.4: The bubble Diagram (on the left) and the tadpole diagram (on the right)

The tadpole integral is simply

$$\int d^4q \, \frac{\mathcal{O}}{q^4} \ , \tag{5.3.6}$$

with $p = \sum k_j = 0$ due to overall momentum conservation. If $\mathcal{O}$ is independent of momenta, the integral is divergent both in the UV and the IR. If $\mathcal{O}$ is a derivative interaction, derivatives acting on the virtual particle running in the loop regularize the IR and make the UV diverge with a power-law. Also in higher derivative theories, tadpoles cannot produce a physical running, because the total momentum entering in the loop is zero. On the other hand, the bubble integral is

$$\int d^4q \, \frac{\mathcal{O}^2}{q^4(q+p)^4} \ . \tag{5.3.7}$$

This diagram is convergent in the UV if $\mathcal{O}$ is less than quadratic in q, since the denominator is q^8 in the high energy limit, but, if $q(q+p)$ cannot be factorized out of $\mathcal{O}^2$, it is IR divergent. In fact, in the $q \ll p$ and $q \to -p$ limits, it reduces to

$$\frac{\mathcal{O}^2}{p^4} \int d^4q \, \frac{1}{q^4} \ . \tag{5.3.8}$$

With a cutoff regularization, Λ in the UV and k in the IR, and $\mathcal{O}$ independent of q, the tadpole diagram is

$$\mathcal{O} \int_k^\Lambda d^4q \, \frac{1}{q^4} = 2\pi^2 \mathcal{O} \log\left(\frac{\Lambda}{k}\right) \ , \tag{5.3.9}$$

while the bubble gives

$$\mathcal{O}^2 \int_k^\Lambda d^4q \, \frac{1}{q^4(q+p)^4} = \frac{2\pi^2\mathcal{O}^2}{p^4}\left[\log\left(\frac{p^2}{k^2}\right)\right] + \dots \ , \tag{5.3.10}$$

where dots stand for finite terms. One can immediately see that the tadpole can potentially contribute to the Wilsonian running, however there cannot be any contribution to the physical beta function. In the bubble integral, on the contrary, there are no logarithms of the UV cutoff that can generate a Wilsonian running, however we have large logs of the momentum p coupled with logarithms of the infrared cutoff k that can be relevant in the physical scheme. If $\mathcal{O}$ is a potential term, i.e. it is independent of momenta, the $\frac{1}{p^4}$ factor in front of the logarithms ends up suppressing the contribution of this diagram in the high-energy limit, so the effect of the infrared large logarithm is negligible; however, if $\mathcal{O} \sim p^4$, the loop correction can potentially contribute to the physical running of $\mathcal{O}$ itself, since

$$\frac{\mathcal{O}^2}{p^4} \sim \mathcal{O} \tag{5.3.11}$$

in terms of power counting. Thus, in higher derivative theories the Wilsonian beta functions in general can be inequivalent to the physical ones and can introduce a dependence of the running coupling on the cutoff that does not reflect the true behaviour of the scattering amplitudes and does not allow us to reabsorb all large logarithms of momenta in the high energy limit.

If we use dimensional regularization and $\overline{MS}$ prescription instead of cutoffs, the tadpole integral is automatically set to 0, since it does not contain any dimensionful parameter (this can also be seen from 2.3.5 by continuing the dimension to $d = 4+2\varepsilon$ instead of $4-2\varepsilon$ and taking the limit $\Delta \to 0$). This is in accordance with the physical scheme. Moving to the bubble integral, we observe that, while dimensional regularization is usually exploited to regularize the UV part of momentum integrals, by simply changing the analytical continuation to $d = 4 + 2\varepsilon$ we can regularize the IR divergent part of the integral. If we wait until the last step in the calculation before taking the limit $\varepsilon \to 0$, we obtain

$$-\frac{2\pi^2\mathcal{O}^2}{p^4\varepsilon} + \frac{2\pi^2\mathcal{O}^2}{p^4}\left[-1 + \gamma_E - \log(4\pi) + \log\left(\frac{p^2}{\mu^2}\right)\right] . \tag{5.3.12}$$

The logarithm of the momentum is associated to $\log\mu$, hence also in this case the $\overline{MS}$ scheme will give a beta function consistent with the physical scheme at high energy. From this rapid overview, $\overline{MS}$ prescription continues to be a good proxy for the logarithmic dependence of scattering amplitude on external momenta in the massless case.

We have already observed that the massless limit is problematic from the point of view of the definition of the free theory. Moreover, the presence of these IR divergences is a big threat to the definition of the low-energy limit of the theory, that should match in some sense with our everyday experience. So we will add to the propagator a two derivative kinetic term,

$$\frac{1}{q^4} \to \frac{1}{q^4 + m^2q^2} . \tag{5.3.13}$$

In this way the infrared divergences are substituted by terms like $\log\left(\frac{p^2+m^2}{m^2}\right)$ that give the same large logs as before in the limit $p^2 \gg m^2$, but go to zero below the mass threshold. In the high-energy limit, the only effect of the mass with cutoff regularization is that m substitutes k in expressions (5.3.9) and (5.3.10). On the other hand, in dim reg, the new mass will regulate the infrared part of the integral instead of ε. As a result, the tadpole integral will give something like

$$\mathcal{O}\int d^4q\,\frac{1}{q^4 + m^2q^2} \sim 2\pi^2\mathcal{O}\left[\frac{1}{\varepsilon} + \gamma_E - \log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right)\right] , \tag{5.3.14}$$

since the new mass is the dimensional parameter needed for a nonzero result. The fact that the purely quartic tadpole is zero in dim reg can be seen as an exact cancellation between the UV divergence and the IR one. If one of the two is missing due to other regularization mechanisms, the integral starts to depend on μ in the same way as the cutoff regulated integral depends on Λ. On the other hand, the bubble will be independent of the parameter μ, because it is UV finite and the IR part of the momentum integral is regulated by the mew mass parameter. So the limit $\varepsilon \to 0$ is smooth and no traces of μ remain in the final result

$$\int d^4q\,\frac{\mathcal{O}^2}{(q^4 + q^2m^2)\left[(q+p)^4 + m^2(q+p)^2\right]} \xrightarrow{p^2 \gg m^2} \frac{2\pi^2\mathcal{O}^2}{p^4}\log\left(\frac{p^2}{m^2}\right) . \tag{5.3.15}$$

In the end, in the presence of a mass scale regulating the infrared part of the integral, $\overline{MS}$ regularization scheme ends up having the same problems of the Wilsonian one.

The failure of one-loop universality of the logarithmic running of couplings can be schematically described in the following way. Consider a hypothetical amplitude of the form

$$\mathcal{M}(p) = \lambda(\mu) + a\lambda^2(\mu)\log\left(\frac{m^2}{\mu^2}\right) + b\lambda^2(\mu)\log\left(\frac{p^2}{\mu^2}\right) + c\lambda^2(\mu)\log\left(\frac{p^2}{m^2}\right) , \tag{5.3.16}$$

where μ comes from dimensional regularization and m is either a mass that is present in the theory or an IR regulator. Such an amplitude could arise, for example, from the application of the $\overline{\text{MS}}$ scheme. The first term represents the contribution of the tadpole, the other two are the contributions of the UV and IR part of the bubble. We can also rewrite the amplitude as

$$\mathcal{M}(p) = \lambda(\mu) + (a-c)\lambda^2(\mu)\log\left(\frac{m^2}{\mu^2}\right) + (b+c)\lambda^2(\mu)\log\left(\frac{p^2}{\mu^2}\right) \ . \tag{5.3.17}$$

From the μ-independence of the amplitude we obtain the "μ-beta function", as we did in equation (2.3.12),

$$\beta_{\lambda(\mu)} \equiv \mu\frac{d}{d\mu}\lambda(\mu) = 2(a+b)\lambda^2 \ . \tag{5.3.18}$$

In this way the μ-dependence of the coupling contains a spurious part (the one proportional to a) that does not reflect a momentum dependence in the amplitude, and misses the momentum dependence of the term proportional to c. This mismatch has the effect that this definition of running coupling does not solve the problem of the large logarithms that arises when p becomes large, which is the main reason for the use of the renormalizaton group in perturbation theory. Indeed, if we choose $\mu \approx p$ in order to make the second logarithm in (5.3.17) small, the first logarithm will generically be large: with this choice, we can rewrite the amplitude as

$$\mathcal{M}(p) = \lambda(p) + (a-c)\lambda^2(p)\log\left(\frac{m^2}{p^2}\right) \ . \tag{5.3.19}$$

The problem is solved by using the physical scheme, where we define the renormalized coupling by identifying it with the measured interaction strength at the scale μ_R

$$\lambda(\mu_R) = \mathcal{M}(\mu_R) = \lambda(\mu) + (a-c)\lambda^2(\mu)\log\left(\frac{m^2}{\mu^2}\right) + (b+c)\lambda^2(\mu)\log\left(\frac{\mu_R^2}{\mu^2}\right) \ . \tag{5.3.20}$$

In this way, we absorb the first logarithm of (5.3.17) and all the μ dependent part of the amplitude in the definition of the renormalized coupling

$$\lambda(\mu) \to \lambda(\mu_R) = \lambda(\mu) - (a-c)\lambda^2(\mu)\log\left(\frac{m^2}{\mu^2}\right) - (b+c)\lambda^2(\mu)\log\left(\frac{\mu_R^2}{\mu^2}\right) \ , \tag{5.3.21}$$

so the amplitude reads

$$\mathcal{M}(p) = \lambda(\mu_R) + (b+c)\lambda^2(\mu_R)\log\left(\frac{p^2}{\mu_R^2}\right) \ . \tag{5.3.22}$$

From the requirement that this must be μ_R-independent, one gets what we call the physical beta function

$$\beta_\lambda \equiv \mu_R\frac{d}{d\mu_R}\lambda(\mu_R) = 2(b+c)\lambda^2 \ . \tag{5.3.23}$$

Now the μ_R-dependence of the coupling faithfully tracks the momentum-dependence of the amplitude, and with this definition of running the problem of the large logarithms is solved. That Green functions could exhibit different dependence on μ and on momentum, when higher derivatives are present, has been observed also in [98] and the possibility of large IR enhancements far from the soft region in higher derivative theories was already suggested in [99].

The introduction of mass scales in the propagator does not only permit us to have a low energy limit free of off-shell infrared divergences, but also allows us to shed more light on the origin of such infrared large logarithms. Via partial fraction decomposition, higher derivative loops can be recast as sums of diagrams with two-derivatives propagators, as we already did to calculate the scattering amplitudes of the shift invariant theory. At first glance, there seems to be a contradiction, since in chapter 2 we stated that in two derivative theories there are no off-shell large logarithms regulated by infrared regulators like, for example, a mass, however a sum of loops with two derivatives propagators produces such terms in higher derivative theories. Let us take the decomposed propagator (3.1.74) and call $I_2(m_a, m_b, p)$ the bubble diagram with two derivatives propagators with different masses, the higher derivative bubble will be

$$I_4(m_1, m_2, p) = (m_1^2 - m_2^2)^2 \left[I_2(m_1, m_1, p) - I_2(m_1, m_2, p) - I_2(m_2, m_1, p) + I_2(m_2, m_2, p)\right] . \tag{5.3.24}$$

The integral I_2 is a slight variation of the bubble integral that gives the one-loop correction to the four point function in the scalar $\lambda\phi^4$ theory. In the relevant limits of large and small momentum p, it is

$$I_2(m_1, m_2, p) \sim \begin{cases} 2\mathcal{O}^2 \left[\log\left(-\frac{p^2}{\Lambda^2}\right) + \sum_i \frac{m_i^2}{p^2} \log\left(-\frac{m_i}{p^2}\right) \right. & \\ \qquad \left. + \frac{m_1^2 m_2^2}{p^4} \sum_i \log\left(-\frac{m_i}{p^2}\right) + O\left(\frac{p^4}{m_i^4}\right)\right] & \text{if } p^2 \gg m_i^2 \, ; \\ -1 + \frac{2m_2^2(\log m_1 - \log m_2)}{m_1^2 - m_2^2} + \log\left(\frac{m_1^2}{\Lambda^2}\right) + O\left(\frac{p}{m_i}\right) & \text{if } p^2 \ll m_i^2 \, . \end{cases} \tag{5.3.25}$$

As expected, the integral is UV divergent. Infrared divergences in two derivative theories can arrive only from the small momentum square limit, but in this case it is finite. We notice that, at leading order in the high energy limit, we recover the well-known $\log\left(\frac{p^2}{\Lambda^2}\right)$, however in the subleading terms $\log\left(\frac{p^2}{m_i^2}\right)$ starts to appear. When we move to I_4, the leading and subleading terms of the high energy limit cancel with each other and we remain with

$$I_4(m_1, m_2, p) \sim \begin{cases} \mathcal{O}^2 \frac{1}{m_1^2 - m_2^2} \sum_i (-1)^i \frac{m_i^2}{p^2} \log\left(\frac{m_i}{p^2}\right) + O\left(\frac{p^4}{m_i^4}\right) & \text{if } p^2 \gg m_i^2 \, ; \\ 2 - 2\frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \log\left(\frac{m_1}{m_2}\right) + O\left(\frac{p}{m_i}\right) & \text{if } p^2 \ll m_i^2 \, . \end{cases} \tag{5.3.26}$$

It is exactly thanks to these type of cancellations that the E^8 and E^6 orders disappeared in the high energy scattering amplitude of the shift invariant theory and higher derivative theories in general have a better UV behaviour than standard ones. However, if a 2-derivative diagram is not divergent "enough", the cancellations will produce a IR divergent higher derivative diagram. The infrared large logarithms in higher derivative theories do not actually come from infrared divergences regulated by the masses in the two-derivative terms of the sum (5.3.24), but form their sub-subleading terms in the high-energy limit. The peculiar way these infrared logarithms emerge in the massive theories and the nontrivial limit between them and the massless case, (which could actually be ill-defined), explains why the Kinoshita-Lee-Nauberg theorem fails in higher derivative quantum field theories.

In renormalizable higher derivative theories with only one scalar field, such infrared divergences are impossible, because shift symmetry is a necessary condition in this class [77]. However, including more fields with various spins, a new landscape of possibilities opens in front of us. In the next chapter we will study the effects of these new large logarithms in higher derivative theories in curved spacetime.

Chapter 6

Higher derivative theories in curved spacetime

6.1 Technical tools

In the last chapter we have discussed how different renormalization schemes can lead to different beta functions in higher derivative theories and not all of them are capable of reproducing the high-energy behaviour of scattering amplitudes. We already mentioned in chapter 3 that the most interesting application of higher derivative theories is related to gravity, hence in this chapter we will see the effects of such a scheme dependence in a higher derivative quantum field theory in curved spacetime and in gravitational theories.

To do this, we first have to introduce some fundamental concepts and tools that are very useful to do RG computations in curved space. In this chapter we will mainly think in terms of dimensional regularization and consider spaces with Euclidean signature.

6.1.1 The quantum effective action

The renormalization of quantum field theories in curved spacetime is usually studied from the point of view of the effective action, since it permits to deal with manifestly covariant expressions, while, in order to define a scattering amplitude, we always have to choose a preferred reference frame and a set of coordinates. We already quickly introduced the effective action in Section 2.2.1, however, to use it on a curved background and in presence of gauge symmetries, as in the case of gravity, we need to put more attention on how this object is introduced and on some of its features. In doing this excursus, we will mainly base our discussion on the book by Buchbinder, Odintsov and Shapiro [100].

We start from the partition function

$$Z[J] = \int D\phi \, e^{-S[\phi]+\int dx\, \phi(x)J(x)} \ , \tag{6.1.1}$$

that generates the n-point Green functions of the theory through the relation

$$G^n(x_1, x_2, ..., x_n)|_J = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)} \ . \tag{6.1.2}$$

From it, one can define the generating functional of the connected Green functions, the free energy $W[J]$, as

$$Z[J] = e^{W[J]} \ . \tag{6.1.3}$$

The expectation value of the field, or mean field, is given by the relation

$$\varphi(x) = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} \tag{6.1.4}$$

and depends on the external source configuration J. This relation can be inverted to write the source as a functional of the mean field, namely $J(x) = J_\varphi(x)$. Using this expression, the effective action is introduced as the Legendre transform of the free energy with respect to φ and J

$$\Gamma[\varphi] = -W[J] + \int dx\, \varphi(x) J_\varphi(x) \; . \tag{6.1.5}$$

This new functional is the generator of 1-point irreducible Green functions

$$G^n_{\rm 1PI}(x_1, x_2, ..., x_n) = \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)...\delta\varphi(x_n)} \; . \tag{6.1.6}$$

Its first variation with respect to the mean field

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} = J \tag{6.1.7}$$

really resemble the classical equation of motion of a field theory coupled with an external source, where the role of the classical action has been taken by the effective action. Moreover, in the same way as functional derivatives of the classical action generate classical interaction vertices to be inserted in Feynman diagrams, the 1-point irreducible Green functions generated by $\Gamma[\varphi]$ are interpreted as the effective interaction vertices in the full quantum theory. Thus, this functional plays, for the expectation value of the quantum field, the same role of the classical action in a classical field theory, already incorporating all quantum corrections. This is the reason why it is usually named quantum effective action.

The effective action in general is a complicated nonlocal functional, but it can be rewritten as a perturbative expansion in $\hbar$ around the classical action S. By reintroducing the Planck constant in (6.1.1) and (6.1.5), we find

$$e^{-\frac{1}{\hbar}\Gamma[\varphi] - \frac{1}{\hbar}\int dx\, \varphi(x)J(x)} = \int D\phi\, e^{-\frac{1}{\hbar}S[\phi] + \frac{1}{\hbar}\int dx\, \phi(x)J(x)} \; . \tag{6.1.8}$$

Now, we redefine the integration variable as $\phi \to \varphi + \phi$, where ϕ represents the quantum fluctuation and φ is the mean field that acts as an external field on ϕ, so we obtain

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \int D\phi\, e^{-\frac{1}{\hbar}S[\varphi+\phi] + \frac{1}{\hbar}\int dx\, \phi(x)\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}} \; . \tag{6.1.9}$$

We expand S around the mean field by introducing the quantities

$$S_{(n)}[\varphi]\phi^n = \int dx_1 dx_2 ... dx_n \; \left.\frac{\delta^n S}{\delta\phi(x_1)\phi(x_2)...\phi(x_n)}\right|_{\phi=\varphi} \phi(x_1)\phi(x_2)...\phi(x_n) \tag{6.1.10}$$

and analogously

$$\phi\Gamma_{(1)} = \int dx\, \phi(x)\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} \; , \tag{6.1.11}$$

so, after a rescaling of ϕ as $\phi = \hbar^{1/2}\phi$, we end up with

$$e^{-\frac{1}{\hbar}\Gamma[\varphi]} = e^{-\frac{1}{\hbar}S[\varphi]} \int D\phi\, e^{-\frac{1}{2}S_{(2)}[\varphi]\phi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_{(n)}[\varphi]\phi^n + \frac{1}{\sqrt{\hbar}}\phi(\Gamma_{(1)}[\varphi] - S_{(1)}[\varphi])} \; . \tag{6.1.12}$$

The functional integral on the right hand side is equal to the logarithm of the difference between the effective action and the classical action and can be expanded in powers of $\hbar$, so we can write

$$\Gamma[\varphi] - S[\varphi] = \sum_{n=1}^{\infty} \hbar^n \Gamma^{n-\text{loop}}[\varphi] \ , \tag{6.1.13}$$

where we have observed that the $\hbar$ expansion coincides with the loop expansion[1]. By substituting this expression in the latter, we recover the one-loop approximation of the effective action written in (2.2.15):

$$e^{-\frac{1}{\hbar}\Gamma^1[\varphi]} = e^{-\frac{1}{\hbar}S[\varphi]-\Gamma^{1-\text{loop}}[\varphi]} = e^{-\frac{1}{\hbar}S[\varphi]} \int D\phi \, e^{-\frac{1}{2}S_{(2)}[\varphi]\phi^2} \ , \tag{6.1.14}$$

that implies, after a Gaussian integration,

$$\Gamma^{1-\text{loop}}[\varphi] = \frac{1}{2}\log \det S_{(2)}[\varphi] = \frac{1}{2}\text{tr}\log S_{(2)}[\varphi] \ . \tag{6.1.15}$$

In this way the calculation of one-loop corrections of the effective action is reduced to the calculation of the functional trace of the logarithm of the Hessian of the classical action S_2.

Effective action in gauge theories

Up to now, we have considered a simple scalar field, however things get more involved when we try to define the effective action of a gauge theory.

Let us consider a generic field ϕ^i transforming as $\phi^i \to \phi^i + \delta\phi^i$ with

$$\delta\phi^i = R^i_\alpha \xi^\alpha \tag{6.1.16}$$

under the action of an element of a Lie group G with infinitesimal parameter ξ^α. The theory defined by the classical action $S[\phi]$ has a gauge symmetry with gauge group G if it is invariant off-shell under the action of elements of the group, i.e.

$$\delta_\xi S[\phi] = \frac{\delta S[\phi]}{\delta\phi^i(x)} R^i_\alpha \xi^\alpha(x) = 0 \quad \forall \ \xi^\alpha(x) \ , \tag{6.1.17}$$

that implies

$$\frac{\delta S[\phi]}{\delta\phi^i(x)} R^i_\alpha = 0 \ . \tag{6.1.18}$$

If one tries to define a partition function for this theory as in (6.1.1), one will incur in an overcounting of field configurations, since the entire orbit determined by the action of G on a field configuration $\phi(x)$ corresponds to only one physical state. For this reason, one would like to have a way to count only once each inequivalent field configuration. The problem is overcome by the Faddeev-Popov procedure, which permits to integrate over fields after having fixed a particular section over the orbits of the gauge group.

Suppose there exists a functional $\chi^\alpha[\phi]$, called gauge, such that the hypersurface

$$\chi^\alpha[\phi] - l^\alpha = 0 \tag{6.1.19}$$

[1]this equivalence is true only if the classical action is taken independent of $\hbar$, however it is not the only possible convention [101]. Sometimes this particular choice is not the right one in order to recover classical results in the $\hbar \to 0$ limit [102]. Anyway, the loopwise expansion is independent of which parameters are taken dependent or independent of $\hbar$, so the identification 6.1.15 is true in general.

is crossed only once by the G orbit of all inequivalent field configurations. In order to count only once physical configurations, we introduce the identity

$$1 = \Psi[\phi] \int Dg\, \delta(\chi^\alpha[{}^g\phi] - l^\alpha) \ , \tag{6.1.20}$$

where the integration is done over elements of the gauge group g and ${}^g\phi$ is the action of g on ϕ. It can be shown that the functional $\Psi[\phi]$ not only exists, but is also gauge invariant and equal to $\det\Delta_{\rm gh}$, with

$$\Delta_{gh}{}^\alpha_\beta = \delta_\beta \chi^\alpha = \frac{\delta\chi^\alpha[\phi]}{\phi^i} R^i_\beta \ . \tag{6.1.21}$$

Inserting this peculiar form of the identity in the formal functional integral we get

$$\int D\phi\, e^{-S[\phi]} = \int DgD\phi\, e^{-S[\phi]} \delta(\chi^\alpha[{}^g\phi] - l^\alpha) \det\Delta_{\rm gh} \ . \tag{6.1.22}$$

Via a change of variable $\phi \to {}^{g^{-1}}\phi$, the integrand becomes independent of g

$$\int D\phi\, e^{-S[\phi]} = \int D\phi\, e^{-S[\phi]} \delta(\chi^\alpha[\phi] - l^\alpha) \det\Delta_{\rm gh} \int Dg \ , \tag{6.1.23}$$

hence the integral over the gauge group simply gives a multiplicative factor. One gets $\int Dg = {\rm Vol}_G$, that is the volume of G, and it can be removed by an overall normalization. Moreover, the delta function in (6.1.23) grants that only one field configuration for each orbit of G is accounted for in the functional integration, as we desired.

Now we would like to rewrite the integrand in a more familiar exponential form, so we proceed in two steps: first we introduce a Gaussian integral over the chosen gauge condition l^α through the identity

$$1 = \det{}^{\frac{1}{2}} Y_{\alpha\beta}[\phi] \int Dl\, e^{-\frac{1}{2} l^\alpha Y_{\alpha\beta}[\phi] l^\beta} \ , \tag{6.1.24}$$

where the spatial integral over x in the exponent is implicit for brevity, and we get

$$\int D\phi\, e^{-S[\phi] - \frac{1}{2}\chi^\alpha Y_{\alpha\beta}[\phi]\chi^\beta} \det\Delta_{\rm gh} \det{}^{\frac{1}{2}} Y \ . \tag{6.1.25}$$

As second step, we introduce three anticommuting fields $\bar{C}_\alpha$, C^β and b^α to rewrite the determinants as Gaussian integrals over Grassmann variables. The final expression is

$$\int D\phi D\bar{C} DC Db\, e^{-S[\phi] - \frac{1}{2}\chi^\alpha[\phi] Y_{\alpha\beta}[\phi] \chi^\beta[\phi] - \bar{C}_\alpha \Delta_{\rm gh}{}^\alpha_\beta C^\beta - \frac{1}{2} b^\alpha Y_{\alpha\beta}[\phi] b^\beta} \ . \tag{6.1.26}$$

The fields $\bar{C}_\alpha$ and C^β are known as Faddeev-Popov ghosts, while b is usually known as the third ghost or Nakanishi-Lautrup ghost. These fields have negative norms, however they do not gives problems related to unitarity like ghosts particles of higher derivative theories because they are excluded from the Hilbert space of physical states by BRST cohomology [59, 60]. In fact, they are necessary to remove from the left hand side of the optical theorem (3.1.63) the loop contributions containing pure gauge virtual modes, which are absent from the sum over intermediate physical states on the right hand side. This functional integral can be used to introduce a partition function for a gauge theory as

$$Z[J] = \int D\phi D\bar{C} DC Db\, e^{-S[\phi] - S_{\rm gf}[\phi] - S_{\rm gh}[\phi, \bar{C}, C, b] - \int dx\, \phi^i(x) J_i(x)} \ . \tag{6.1.27}$$

Notice that only the fields ϕ^i have received an external source, since they are the only physical particles that can appear in a Green function. In many gauge theories $Y_{\alpha\beta}[\phi]$ is taken equal to $\delta_{\alpha\beta}$, in order to avoid the third ghost, however, a nontrivial choice turns out to be useful in higher derivative theories and in particular in quadratic gravity, so we will keep it.

To define the partition function we had to choose a gauge functional $\chi^\alpha[\phi]$, so one could wonder how this decision affects the resulting functional. Without introducing the auxiliary fields, the partition function defined in the gauge χ is

$$Z_\chi[J] = \int D\phi\, e^{-S[\phi]-\int dx\, \phi^i(x)J_i(x)}\delta(\chi^\alpha[\phi] - l^\alpha)\Psi_\chi[\phi] \ . \tag{6.1.28}$$

If we take $J = 0$, we have the starting integral

$$\int D\phi\, e^{-S[\phi]} \ , \tag{6.1.29}$$

which is manifestly independent of χ. To explicitly show this, we can introduce another identity (6.1.20) written in term of another gauge functional χ'

$$1 = \Psi_{\chi'}[\phi] \int Dg\, \delta(\chi'^\alpha[{}^g\phi] - l'^\alpha) \ , \tag{6.1.30}$$

so we obtain

$$Z_\chi[0] = \int DgD\phi\, e^{-S[\phi]}\delta(\chi^\alpha[\phi] - l^\alpha)\delta(\chi'^\alpha[{}^g\phi] - l'^\alpha)\Psi_\chi[\phi]\Psi_{\chi'}[\phi] \ . \tag{6.1.31}$$

Via the change of variables $\phi \to {}^{g^{-1}}\phi$ and $g^{-1} \to g$, only the delta function containing χ depends on g, hence we can use relation (6.1.20) to write

$$Z_\chi[0] = \int DgD\phi\, e^{-S[\phi]}\delta(\chi'^\alpha[{}^g\phi] - l'^\alpha)\Psi_{\chi'}[\phi] = Z_{\chi'}[0] \ . \tag{6.1.32}$$

Thus, with zero source, the partition function is independent of the gauge fixing. However, the same cannot be done with $J \neq 0$: in this case the field ϕ in the source term also depends on g after the change of variable, so the integral over G does not factorize. In fact, the partition function with nonzero source depends on the gauge choice, and the same holds for the Green functions it generates. The independence from the gauge is recovered only in on-shell S-matrix elements.

From the partition function (6.1.27), one can construct a free energy functional and an effective action via a Legendre transform with respect to the mean field $\varphi^i = \frac{\delta W[J]}{\delta J^i}$,

$$e^{-\Gamma[\varphi]} = \int D\phi D\bar{C}DCDb\, e^{-S[\varphi+\phi]-S_{\rm gf}[\varphi+\phi]-S_{\rm gh}[\varphi+\phi,\bar{C},C,b]-\int dx\, \phi^i(x)\frac{\delta\Gamma[\varphi]}{\delta\varphi^i(x)}} \tag{6.1.33}$$

An expansion in $\hbar$ analogue to the one accomplished for the simpler case considered before leads to

$$\Gamma[\varphi] - S[\varphi] - S_{\rm gf}[\varphi] = \sum_{n=1}^{\infty} \hbar^n \Gamma^{n-{\rm loop}}[\varphi] \tag{6.1.34}$$

and

$$\Gamma^{1-{\rm loop}}[\varphi] = \frac{1}{2}{\rm tr}\log\frac{\delta^2(S[\varphi] + S_{\rm gf}[\varphi])}{\delta\varphi^i\delta\varphi^j} - {\rm tr}\log\Delta_{\rm gh}[\varphi] - \frac{1}{2}{\rm tr}\log Y[\varphi] \ , \tag{6.1.35}$$

that can be used to find the one-loop beta functions of the theory.

The effective action (6.1.33) inherits the dependence on the gauge choice from the partition function. It can be shown that, when the gauge functional is linear in ϕ ($\chi^\alpha = t^\alpha_i \phi^i$ with t independent of ϕ), its dependence on Y and t can be expressed as

$$\frac{\delta\Gamma[\varphi]}{\delta Y_{\alpha\beta}} = \frac{\delta\Gamma[\varphi]}{\delta\varphi^i} \langle R^i_\gamma \Delta_{\text{gh}}^{-1}{}^\gamma_\delta Y^{\delta(\alpha} t^{\beta)}_j \phi^j \rangle \ , \tag{6.1.36}$$

$$\frac{\delta\Gamma[\varphi]}{\delta t^\alpha_i} = \frac{\delta\Gamma[\varphi]}{\delta\varphi^j} \langle R^j_\beta \Delta_{\text{gh}}^{-1}{}^\beta_\alpha \phi^i \rangle \ . \tag{6.1.37}$$

Clearly both these expressions are zero on-shell, when $\frac{\delta\Gamma[\varphi]}{\delta\varphi^i} = 0$. Even worse, Γ in general is not even invariant under gauge transformations of φ^i. To avoid this inconvenience, we can use the dependence of the effective action on the gauge fixing parameters to define a G invariant effective action without affecting the S-matrix, that is independent of the gauge choice.

If we restrict ourself to gauge Lie groups which act linearly on the field, namely $\frac{\delta^2 R^i_\alpha}{\delta\phi^j\delta\phi^k} = 0$, there exists a class of gauges, called background field gauges, that allow to define an effective action invariant with respect to transformations of φ^i [103]. Instead of writing the effective action as in (6.1.33), where the gauge fixing and ghost terms depend only on the combination $\phi + \varphi$, we introduce a background field $\bar\varphi$ and take a gauge functional $\chi^\alpha[\bar\varphi, \varphi, \phi] = t^\alpha_i[\bar\varphi](\varphi^i + \phi^i)$ and an operator $Y_{\alpha\beta}$ depending only on $\bar\varphi$ and such that they transform under gauge transformations of the background field $\delta\bar\varphi^i = R^i_\alpha[\bar\varphi]\xi^\alpha$ as

$$\delta t^\alpha_i = \left(f^\alpha_{\beta\gamma} t^\gamma_i - t^\alpha_j \frac{\delta R^j_\beta[\bar\varphi]}{\delta\bar\varphi^i} \right) \xi^\beta \ , \tag{6.1.38}$$

$$\delta Y_{\alpha\beta} = -\left(Y_{\alpha\delta} f^\delta_{\gamma\beta} + Y_{\delta\beta} f^\delta_{\alpha\gamma} \right) \xi^\gamma \ , \tag{6.1.39}$$

where $f^\alpha_{\beta,\gamma}$ are the structure constants of the Lie group. The effective action takes the form

$$e^{-\Gamma[\varphi,\bar\varphi]} = \int D\phi D\bar C DC Db \times e^{-S[\varphi+\phi] - \frac{1}{2}(\varphi^i+\phi^i) t^\alpha_i[\bar\varphi] Y_{\alpha\beta}[\bar\varphi] t^\beta_j[\bar\varphi](\varphi^j+\phi^j) - \bar C_\alpha t^\alpha[\bar\varphi] R^i_\beta[\varphi+\phi] C^\beta - \frac{1}{2} b^\alpha Y_{\alpha\beta}[\bar\varphi] b^\beta - \phi^i \frac{\delta\Gamma[\bar\varphi,\varphi]}{\delta\varphi^i}} \ . \tag{6.1.40}$$

This effective action depends on two fields in general, φ and $\bar\varphi$, but we can choose $\bar\varphi = \varphi$ and rename $\Gamma[\varphi, \bar\varphi]|_{\bar\varphi=\varphi} = \Gamma[\varphi]$[2]. It can be shown that, in this particular gauge depending only on the mean field, the quantity

$$\tilde\Gamma[\varphi] = \Gamma[\varphi] - \frac{1}{2}\varphi^i t^\alpha_i[\varphi] Y_{\alpha\beta}[\varphi] t^\beta_j[\varphi]\varphi^j = \Gamma[\varphi] - S_{\text{gf}}[\varphi] \tag{6.1.41}$$

is invariant under gauge transformations of the background or mean field. Hence, all terms in the loop expansion are also invariant thanks to eq. (6.1.34), since the classical action S is gauge invariant from the beginning. The background gauge assures us that all possible local divergences emerging from loop corrections can be reabsorbed in bare couplings associated to covariant terms in the classical Lagrangian. $\tilde\Gamma[\varphi]$ is called the gauge invariant effective action, but from now on we will call it simply effective action and refer to it as $\Gamma[\varphi]$ when discussing gauge theories.

[2] In the EAA one cannot simply remove the dependence on one of the two fields in this way, because the regulator ΔS_k depends on $\bar\varphi$ and $(\varphi + \phi)$ in different ways [12].

Different runnings from the effective action

In general, the regularized effective action can be written as the classical bare action plus a finite part and a UV divergent one

$$\Gamma = S + \Gamma_{\text{div}} + \Gamma_{\text{fin}} \tag{6.1.42}$$

The divergent part, proportional to $\frac{1}{\epsilon}$ in dimensional regularization, is always local if the theory contains an energy scale m regulating the infrared, so, in a renormalizable theory, it can be reabsorbed as usual by a redefinition of bare quantities in the classical action. What remains, is a renormalilzed effective action

$$\Gamma_{\text{ren}} = \Gamma_{\text{loc}} + \Gamma_{\text{nl}} \ , \tag{6.1.43}$$

where Γ_{loc} is the local part and Γ_{nl} the nonlocal one. Γ_{loc} is composed by all the local operators of the classical action associated to renormalized couplings, while Γ_{nl} produces all the terms nonanalytical in momenta that appear in effective vertices via nonlocal operators, even the large logarithms that forced us to introduce the physical running.

In momentum space, the nonlocal part of the effective action can be expanded in a sum of monomials of mean fields multiplied by functions of their momenta, usually called form factors

$$\int d^d p_1 \int d^d p_2 ... \int d^d p_n \varphi_1(p_1)\varphi_2(p_2)...\varphi_n(p_n) f(p_1, p_2, ..., p_{n-1}, \mu, m)\delta(p_1 + p_2 + ... + p_n) \ . \tag{6.1.44}$$

When the scales of the theory are well separated, i.e. $m^2 \ll p^2$, we can expand the part analytical in momenta of form factors in a power series

$$f = \sum_i f_i(p_1, p_2, ..., p_{n-1}, \mu, m) \prod_{r=1}^{n-1} p_r^{\gamma_{i,r}} \ , \tag{6.1.45}$$

with f_i representing the residual nonanalytical part. If we sum each term of the series with the renormalized coupling g_i in Γ_{loc} associated with the same powers of momenta and fields, we obtain a high-energy expansion of the 1-point irreducible n-point functions. As already stated, the latter must be independent of the scale μ, hence the renormalized couplings g_i must depend on the unphysical scale μ to cancel the dependence of f_i on it. The general structure is usually reduced to

$$g_i(\mu) + f_i(p^2; \mu^2, m^2) = g_i(\mu) + b_i \log(\mu^2/m^2) + c_i \log(p^2/m^2) \ , \tag{6.1.46}$$

that is analog to (5.3.17). Hence, all the reasoning concerning the physical running and the μ running in scattering amplitudes can be applied to this expression. The coefficients of the two logarithms in μ and p are generally different, $c_i \neq b_i$, so the integration of the μ-running, given by b_i, does not resum the large logarithms in p^2, unless $b_i = c_i$. If $c_i \neq b_i$, in order to have a running that resums the large logarithms, the term with b_i in (6.1.46) can be absorbed in g through a finite subtraction. The result is a coupling renormalized at the scale m that is independent of μ, since the absorbed logarithm removes the old dependence. The residual form factor depends only on momenta and masses, in this case through $\log(p^2/m^2)$. After that, by moving the renormalization point again from m to $\bar{p}$ close to p, we can carry out the desired task. Identifying $p = |p|$ with the scale of some physical process, the logarithmic derivative with respect to p of (6.1.46) gives the physical running, which, by construction, depends only on c_i.

Effective action in curved spacetime

In the discussion of renormalization and running couplings in the effective action of quantum field theories in curved spacetime, we will distinguish two main cases: theories of matter in a curved background described by an external metric and quantum field theories of gravity, where the metric itself is treated as a quantum dynamical field.

In the former case, the effective action will have the form $\Gamma[\bar{g}_{\mu\nu}, \varphi^A]$, where φ^A represents a generic matter field with arbitrary spacetime and internal indices, and $\bar{g}_{\mu\nu}$ is the background metric. In the following, a bar always indicates a geometric quantity calculated from the background metric. If we start from a covariant classical action, as required by general relativity, we can write

$$\frac{\delta S}{\delta \bar{g}_{\mu\nu}} R_{\mu\nu,\alpha} + \frac{\delta S}{\delta \phi^A} R^A_\alpha = 0 \ , \tag{6.1.47}$$

where $R_{\mu\nu,\alpha}$ is the generator of coordinate transformations acting on the metric tensor and R^A_α the one acting on the matter field ϕ^A. For an infinitesimal change of coordinates $x^\mu \to x^\mu + \xi^\mu(x)$, we have

$$R_{\mu\nu,\alpha}\xi^\alpha = \left(\bar{g}_{\mu\alpha}\partial_\nu + \bar{g}_{\nu\alpha}\partial_\mu + \partial_\alpha \bar{g}_{\mu\nu}\right)\xi^\alpha \tag{6.1.48}$$

for the metric, while the matter transformation has the general structure $R^A_{B\alpha}\phi^B$ with $R^A_{B\alpha}$ dependent on the type of matter field but such that $\frac{\delta R^A_{B\alpha}}{\delta \phi^C} = 0$.

With these conditions, the free energy defined by

$$e^{W[\bar{g}_{\mu\nu}, J]} = \int D\phi \, e^{-S[\bar{g}_{\mu\nu},\phi] + \phi^A J_A} \tag{6.1.49}$$

transforms as

$$e^{W[\bar{g}_{\mu\nu}+\delta\bar{g}_{\mu\nu}, J]} = \int D\phi \, e^{-S[\bar{g}_{\mu\nu},\phi] + \phi^A J_A + R^A_{B\alpha}\phi^B J_A} \tag{6.1.50}$$

under diffeomorphisms, then the effective action

$$\Gamma[\bar{g}_{\mu\nu}, \varphi] = -W[\bar{g}_{\mu\nu}, J] + \varphi^A J_A \ , \tag{6.1.51}$$

with $\varphi^A = \frac{\delta W}{\delta J_A}$, is diffeomorphism invariant like the classical action.

One can compute the perturbative expansion in loops of the effective action in the same way as in the flat spacetime case and quantum corrections at all loops orders will be covariant. Loops contributions will take the form of all the scalar objects that can be built up with the field φ^A, the metric, the covariant derivative and the curvature tensors associated with the metric itself, compatibly with power counting and other symmetries of the theory. The local UV divergences will be of three types: the covariantized version of divergences already present in flat spacetime, where partial derivatives have been replaced by covariant one, nonminimal couplings between matter fields and curvature tensors, which are zero in flat spacetime, and divergent terms containing only the background metric, totally independent of the field φ. In a power counting renormalizable theory in 4 spacetime dimensions, the only admissible local and covariant UV divergences that can be generated at one loop in the vacuum effective action, namely $\Gamma[\bar{g}_{\mu\nu}] = \Gamma[\bar{g}_{\mu\nu}, \varphi]|_{\varphi=0}$, are

$$\Gamma[\bar{g}_{\mu\nu}]_{\text{div}} \sim \mu^{d-4} \int d^4x \sqrt{g} \left\{ a_1 + a_2 \bar{R} + a_3 \bar{R}^2 + a_4 \bar{C}^{\mu\nu\rho\sigma}\bar{C}_{\mu\nu\rho\sigma} + a_5 \bar{E} + a_6 \bar{\Box}\bar{R} \right\} \ , \tag{6.1.52}$$

where

$$\bar{C}^{\mu\nu\rho\sigma} = \bar{R}^{\mu\nu\rho\sigma} - \frac{1}{2}\left(\bar{g}^{\mu\rho}\bar{R}^{\nu\sigma} - \bar{g}^{\mu\sigma}\bar{R}^{\nu\rho} - \bar{g}^{\nu\rho}\bar{R}^{\mu\sigma} + \bar{g}^{\nu\sigma}\bar{R}^{\mu\rho}\right) + \frac{1}{6}\bar{R}(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} - \bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}) \tag{6.1.53}$$

is the Weyl tensor and

$$\bar{E} = \bar{R}^2_{\mu\nu\alpha\beta} - 4\bar{R}^2_{\mu\nu} + \bar{R}^2 \tag{6.1.54}$$

is the Euler density scalar, which is a topological term in $d = 4$. Due to this kind of divergences, we are forced to add the same terms depending only on the background metric in the bare action, in order to reabsorb infinities with counterterms

$$S_{\text{ext}}[\bar{g}_{\mu\nu}] = \int d^4x\,\sqrt{g}\left\{\Lambda_B - G_{N,B}\bar{R} + \frac{1}{\xi_B}\bar{R}^2 + \frac{1}{2\lambda_B}\bar{C}^{\mu\nu\rho\sigma}\bar{C}_{\mu\nu\rho\sigma} - \frac{1}{\rho_B}\bar{E} + \tau_B\bar{\Box}\bar{R}\right\}\,. \tag{6.1.55}$$

On the other hand, when we consider gravity as a dynamical field, we have to treat the metric as the gauge field associated to diffeomorphisms invariance. Since $R_{\mu\nu,\alpha}$ is linear in $g_{\mu\nu}$, as can be seen from (6.1.48), all the discussion around the effective action of gauge theories can be applied to gravity. Hence, we can split the metric in a background part and a quantum fluctuation $h_{\mu\nu}$ via $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, use the background gauge to obtain a covariant effective action and finally compute one-loop corrections by calculating the functional traces in (6.1.35). In pure gravity we will produce a partition function $\Gamma[\bar{g}_{\mu\nu}]$ similar to the vacuum effective action of the matter case, while in a theory describing both matter and gravity we will find $\Gamma[\bar{g}_{\mu\nu}, \varphi]$, potentially containing all pure matter, pure gravity and non-minimal gravity-matter terms. Unfortunately, we do not inherit from the discussion of gauge theories only the machinery necessary to not overcount gauge degrees of freedom, but also the problem of the dependence of the effective action on the gauge fixing. Only the part of the effective action that survives on-shell is gauge independent and it must be checked case by case in different gravitational theories. For example, in pure Einstein gravity, the one-loop divergences computed in [30] are zero on-shell, and in fact there exists a particular gauge such that the off-shell divergences are zero too [104]. A gauge invariant notion of effective action was suggested by Vilkovisky in [105] and later on analysed and modified by De Witt in [106, 107], where a set of Riemaniann normal coordinates is introduced in the space of field configurations and used to define a unique effective action. However, the so-called supermetric [108] on the space of metric configurations

$$G^{\mu\nu\alpha\beta} = \sqrt{\bar{g}}\left(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\nu\alpha}\bar{g}^{\mu\beta} + a\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}\right) \tag{6.1.56}$$

has a degree of arbitrariness due to the free parameter a, so even the "unique" effective action is not actually unique in gravity.

In an asymptotically flat spacetime, in the hypothesis $\nabla\nabla\bar{R} \gg \bar{R}^2$, the leading non-trivial pure gravity form factors are [109]

$$\begin{aligned}\Gamma_{\text{nl}} = \int d^4x\sqrt{\bar{g}}\Big\{&\bar{C}_{\mu\nu\alpha\beta}f_\lambda(\bar{\Box};\mu^2,m^2)\bar{C}^{\mu\nu\alpha\beta}\\ &+\bar{R}f_\xi(\bar{\Box};\mu^2,m^2)\bar{R} + O(\bar{R}^3)\Big\}\,,\end{aligned} \tag{6.1.57}$$

and we will focus on them in the rest of the chapter. It is known also through explicit computations that two-derivative scalars and vectors, as well as spinorial fields, give effective actions in curved space for which the dependence on μ of the gravitational form factors is a good proxy of their dependence on momenta, in the same way as in flat spacetime.

See for example Refs. [110] and references therein. However, the same may not be true in higher derivative theories.

In higher derivative theories without irrelevant operators, the leading terms at high-energy in the gravitational local effective action are

$$\Gamma_{\rm loc} = \int d^4x\sqrt{\bar{g}}\Big\{\frac{1}{2\lambda}\bar{C}^2 + \frac{1}{\xi}\bar{R}^2\Big\} \tag{6.1.58}$$

at all loops, due to power counting, while the high-energy expansion of the nonlocal effective action looks like

$$\begin{aligned}\Gamma_{\rm nl}[\bar{g}] = b_\lambda \log\left(\frac{\mu^2}{m^2}\right)\bar{C}^{\mu\nu\rho\sigma}\bar{C}_{\mu\nu\rho\sigma} + b_\xi \log\left(\frac{\mu^2}{m^2}\right)\bar{R}^2 \\ + c_\lambda \bar{C}^{\mu\nu\rho\sigma}\log\left(\frac{\bar{\Box}}{m^2}\right)\bar{C}_{\mu\nu\rho\sigma} + c_\xi \bar{R}\log\left(\frac{\bar{\Box}}{m^2}\right)\bar{R}\ .\end{aligned} \tag{6.1.59}$$

The μ-running of the couplings is

$$\beta^\mu_\lambda = 2\lambda^2\mu\frac{df_\lambda}{d\mu} = 4\lambda^2 b_\lambda\,, \quad \beta^\mu_\xi = \xi^2\mu\frac{df_\xi}{d\mu} = 2\xi^2 b_\xi\ , \tag{6.1.60}$$

while the coefficients c_λ and c_ξ induce the physical running of the couplings λ and ξ in the vacuum effective action (6.1.55). The associated beta functions are

$$\beta_\lambda = -4\lambda^2 c_\lambda\ , \quad \beta_\xi = -2\xi^2 c_\xi\ . \tag{6.1.61}$$

Once assessed how to treat properly the RG flow of quantum field theories in curved spacetime, in order to compute the one-loop beta functions we just have to understand how to treat functional traces like those appearing in equations (6.1.15) and (6.1.34).

6.1.2 Computing functional traces

The most popular technique to compute functional traces of quadratic differential operators is the Heat Kernel (HK). It was originally introduced in physics by Schwinger and DeWitt [111, 112] and there are many textbooks and reviews approaching the subject, see for instance [12, 100, 113–117]. We will briefly review how it works and then we will study its application to fourth order operators. We will see that it is not suitable to compute the physical running of higher derivative theories and then we will propose an alternative method.

The heat kernel

The heat kernel permits to isolate the divergent part of functional traces of differential operators while keeping manifest covariance throughout the whole procedure.

Let us consider a self-adjoint differential operator acting on a given field ϕ^A

$$\mathcal{O} = -\bar{\Box} + \mathcal{E}\ , \tag{6.1.62}$$

where the covariant derivatives in $\bar{\Box}$ can contain both spacetime and gauge connections, while $\mathcal{E}$ is an endomorphism in the space of field configurations. Suppose that $\mathcal{O}$ is the Hessian with respect to the field ϕ^A of the action of a particular theory and we are interested in its one-loop quantum corrections, hence we need the quantity

$$\mathrm{tr}\ln\mathcal{O}\ . \tag{6.1.63}$$

To study this object, it is convenient to introduce the heat kernel

$$K(t,x,y;\mathcal{O}) = \langle x|\, e^{-t\mathcal{O}}\, |y\rangle \ , \tag{6.1.64}$$

which is a formal expression for the solution of the heat diffusion equation with respect to a new auxiliary proper time coordinate t

$$(\partial_t + \mathcal{O})K(t,x,y;\mathcal{O}) = 0 \tag{6.1.65}$$

with initial conditions

$$K(0,x,y;\mathcal{O}) = \delta(x,y) \ . \tag{6.1.66}$$

Being $\mathcal{O}$ an Hermitian operator, it has a complete basis of eigenfunctions ϕ_n^A with eigenvalues λ_n and we can rewrite the trace (6.1.63) as

$$\mathrm{tr}\log\mathcal{O} = \sum_n \log\lambda_n \ . \tag{6.1.67}$$

Using the well known relation $\log x = -\frac{d}{ds}x^{-s}|_{s=0}$, we can write it as a derivative of the zeta function of the operator $\mathcal{O}$

$$\zeta_{\mathcal{O}}(s) = \sum_n \lambda_n^{-s} \ , \tag{6.1.68}$$

a generalization of the Riemann's zeta function $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$. In fact we have

$$\mathrm{tr}\log\mathcal{O} = \sum_n -\frac{d}{ds}\lambda^{-s}|_{s=0} = -\frac{d}{ds}\zeta_{\mathcal{O}}(s)|_{s=0} \ . \tag{6.1.69}$$

On the other hand, the heat kernel in the eigenstates basis is

$$K(t,x,y;\mathcal{O}) = \sum_n \phi_n^{\dagger A}(x)\phi_{An}(y)e^{-t\lambda_n} \ , \tag{6.1.70}$$

hence its trace at coincident points reduces to

$$\mathrm{tr}K_{\mathcal{O}}(t) = \int d^dx\sqrt{g}K_{\mathcal{O}}(t,x,x;\mathcal{O}) = \sum_n e^{-t\lambda_n} \ . \tag{6.1.71}$$

The trace of the heat kernel is related to the zeta function via a Mellin transform

$$\zeta_{\mathcal{O}}(s) = \Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1}\mathrm{tr}K_{\mathcal{O}}(t) \ . \tag{6.1.72}$$

The last relation can be shown with the help of the integral representation of the Euler gamma function

$$\Gamma(s) = \int_0^\infty dt\, t^{s-1}e^{-t} \ , \tag{6.1.73}$$

that permits to write the identity

$$\lambda^{-s} = \Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1}e^{-\lambda t} \tag{6.1.74}$$

for each eigenvalue of the differential operator. Putting together expressions (6.1.69) and (6.1.72), we manage to express the trace of the logarithm of $\mathcal{O}$ in terms of its heat kernel

$$\mathrm{tr}\log\mathcal{O} = -\frac{d}{ds}\left[\Gamma(s)^{-1}\int_0^\infty dt\, t^{s-1}\mathrm{tr}K_{\mathcal{O}}(t)\right]_{s=0} = \int_0^\infty dt\, t^{-1}\mathrm{tr}K_{\mathcal{O}}(t) \ , \tag{6.1.75}$$

where in the last step we have used the fact that $\Gamma(s)^{-1} \sim s$ for $s \to 0$. In general, such an integral can be divergent both in the limiting regions $t \to 0$ and $t \to \infty$, where the first one corresponds to the sum over UV modes, and the latter to the sum over those in the IR.

The reason why this expression is useful is that the heat kernel $K_{\mathcal{O}}(t)$ has a very nice series expansion at small t. If the infrared divergences are regularized either by an upper cutoff T over the proper time or by a mass term m^2 in $\mathcal{O}$ that introduces an exponential suppression at large t via a factor e^{-tm^2} in (6.1.75), all divergences determining the cutoff or μ-running can be extracted thanks to such local, or small proper time, expansion.

In flat spacetime, with $\mathcal{O} = -\partial_\mu \partial^\mu$, the heat kernel is exactly

$$K(t, x, y; -\partial_\mu \partial^\mu) = (4\pi t)^{-\frac{2}{n}} e^{-\frac{(x-y)^2}{4t}} \ . \tag{6.1.76}$$

Since all differentiable manifolds are locally flat, the behaviour at $t = 0$ of the trace of $K(t, x, y; -\partial_\mu \partial^\mu)$ is common to all covariant operators of the type (6.1.62) and one can expand the heat kernel at coincident points $y = x$ as

$$K(t, x, x, \mathcal{O}) = K(t, x, x, -\partial_\mu \partial^\mu)[1 + tb_2(x) + t^2 b_4(x) + ...] \ . \tag{6.1.77}$$

Here b_i are all covariant local polynomials of the background metric, other background fields and their derivatives. These coefficients have been computed for large classes of operators and can be found in the literature [112, 114, 118–120].

In this local expansion, the one loop correction to the effective action is

$$\Gamma^{1-\text{loop}} = \frac{1}{2} \text{tr} \log \mathcal{O} = \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{g} \int_0^\infty dt \left[t^{-\frac{d}{2}-1} + b_2 t^{-\frac{d}{2}} + \cdots + b_d t^{-1} + \ldots \right] \ . \tag{6.1.78}$$

Contributions from b_i with $i > d$ are finite, while those with $i \leq d$ can be regularized by means of any procedure: an hard cutoff $t = 1/\Lambda$ in the t integral, dimensional regularization, or the introduction of a infrared cutoff in the operator $\mathcal{O}$ as in the FRG. To each of these regularization methods obviously corresponds different renormalization schemes and beta functions, as discussed in Chapter 2.

The heat kernel of higher derivative operators

The same approach can be applied also to higher derivative operators, as explained in [114]: the functional trace of the logarithm of an operator $\bar{\Box}^2 + A$ can be approximated by

$$\text{tr} \log \left(\bar{\Box}^2 + A \right) \approx \text{tr} \left[2 \log \bar{\Box} + A \frac{1}{\bar{\Box}^2} - \frac{1}{2} A \frac{1}{\bar{\Box}^2} A \frac{1}{\bar{\Box}^2} + \cdots \right] \tag{6.1.79}$$

and, in the quite general case

$$\mathcal{O} = \bar{\Box}^2 \mathbb{I} + \mathbb{V}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \mathbb{N}^\mu \bar{\nabla}_\mu + \mathbb{U} \ , \tag{6.1.80}$$

the logarithmic divergences, or equivalently the $1/\epsilon$ poles in dimensional regularization, are proportional to

$$\begin{aligned} \frac{1}{2} \frac{1}{(4\pi)^2} \int \mathrm{d}^d x \sqrt{g} b_4 = \frac{1}{32\pi^2} \int d^4 x \, \text{tr} \Big[& \frac{\mathbb{I}}{90} \left(\bar{R}^2_{\rho\lambda\sigma\tau} - \bar{R}^2_{\rho\lambda} + \frac{5}{2} \bar{R}^2 \right) + \frac{1}{6} \mathbb{R}_{\rho\lambda} \mathbb{R}^{\rho\lambda} \\ & - \frac{\bar{R}_{\rho\lambda} \mathbb{V}^{\rho\lambda} - \frac{1}{2} \bar{R} \mathbb{V}^\rho{}_\rho}{6} + \frac{\mathbb{V}_{\rho\lambda} \mathbb{V}^{\rho\lambda} + \frac{1}{2} \mathbb{V}^\rho{}_\rho \mathbb{V}^\lambda{}_\lambda}{24} - \mathbb{U} \Big] \ , \end{aligned} \tag{6.1.81}$$

where boundary terms have been neglected and $\mathbb{R}_{\rho\lambda}$ is defined as the action of the commutator $[\nabla_\rho, \nabla_\lambda]$ on ϕ^A. From these divergences one can define the μ-running of the effective action as explained in section (6.1.1). We would like to understand if the μ-beta functions match with the physical running given by the part nonanalytical in momenta of the form factors.

The operator (6.1.80) has the same structure of the one acting on the scalars in the higher derivative sigma models discussed in [121], so also its analysis is quite similar. The terms in the first line of (6.1.81) are the ones that we would get for $\mathcal{O} = \bar{\Box}^2$. Using the formula $\mathrm{Tr}\log\bar{\Box}^2 = 2\mathrm{Tr}\log\bar{\Box}$, one can conclude that the μ-running generated by these divergences has a counterpart in the physical running, since in second order operators the two schemes are equivalent.

On the other hand consider $\mathrm{tr}\mathbb{U}$. This contribution comes from the functional trace $\mathrm{Tr}\mathbb{U}/\bar{\Box}^2$ in the second term of the right hand side of (6.1.79). Clearly the most divergent part of this expression, that is present even with the flat propagator, is a tadpole. If $\mathbb{U}$ is quadratic in curvature, it will contribute to the μ beta functions of operators quadratic in curvatures, however, as already observed with tadpoles in flat space, there is no way to produce a nonlocal term like those in (6.1.59) from $\mathrm{tr}\mathbb{U}$.

Thus, we cannot rely on the μ-running to deduce the high-energy behaviour of n-point functions. The latter can only be obtained from the nonlocal finite part of the one-loop corrections to the effective action, however the local expansion is blind with respect to this nonlocal part, since the coefficients b_i are all local. Some attempts have been made in the literature to resum the local expansion in order to reproduce the nonlocal part of the effective action [109, 122–126], however they apply only to second order operators of the form (6.1.62). Hence, if we want to obtain all the contributions of the form (6.1.59) in $\Gamma^{1-\mathrm{loop}}$ for an higher derivative theory, we need a method different from the heat kernel.

An alternative approach

In flat space, we were able to compute the full nonanalytical part of scattering amplitude thanks to integrals over momenta. However, in curved space, the eigenfunctions of the free propagator do not have in general a simple representation in Fourier space. In order to use Feynman diagrams with momentum integrals in non-flat space, we have to consider all interactions between the fields and the background metric that induce the curvature of the manifold as perturbations of the free theory in flat space. This idea corresponds to go back to the first perturbative approaches to the problem of renormalization of gravity by t'Hooft-Veltman [30] and Julve-Tonin [127], however, while they focused only on the UV divergent parts, we will take in account the complete loop integral, like in Section 5.3. We assume that the background metric is a small deformation of flat space $\bar{g}_{\mu\nu} = \delta_{\mu\nu} + f_{\mu\nu}$. In this way, the Christoffel symbol can be written as a power series in $f_{\mu\nu}$ with the leading order linear in the perturbation. The same holds for curvature tensors, so, in detail

$$\bar{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}\delta^{\lambda\rho}(\partial_\mu f_{\rho\nu} + \partial_\nu f_{\rho\mu} - \partial_\rho f_{\mu\nu}) + O(f^2) \tag{6.1.82}$$

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu\partial_\sigma f_{\rho\nu} - \partial_\mu\partial_\rho f_{\sigma\nu} - \partial_\nu\partial_\sigma f_{\rho\mu} + \partial_\nu\partial_\rho f_{\sigma\mu}) + O(f^2)\ , \tag{6.1.83}$$

$$\bar{R}_{\mu\nu} = \frac{1}{2}(-\partial_\rho\partial^\rho f_{\mu\nu} + \partial_\mu\partial_\rho f^\rho{}_\nu + \partial_\nu\partial_\rho f^\rho{}_\mu - \partial_\nu\partial_\mu f^\rho{}_\rho) + O(f^2)\ , \tag{6.1.84}$$

$$\bar{R} = -\partial^\mu\partial_\mu f^\rho{}_\rho + \partial_\mu\partial_\nu f^{\mu\nu} + O(f^2)\ . \tag{6.1.85}$$

Expanding around flat space, the operator (6.1.80) takes the form

$$\mathcal{O} = \Box^2 + D^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma + C^{\mu\nu\rho}\partial_\mu\partial_\nu\partial_\rho + V^{\mu\nu}\partial_\mu\partial_\nu + N^\mu\partial_\mu + U\ , \tag{6.1.86}$$

where $\boxdot^2$ stands for the free propagator in flat space and all operators D, C, V, N and U are at least linear in $f_{\mu\nu}$ or other background fields. The functional trace of the logarithm of the operator $\mathcal{O}$ that contributes to $\Gamma^{1-\text{loop}}$ is expanded as

$$
\begin{aligned}
\operatorname{tr}\log(\mathcal{O}) \approx \operatorname{tr}\log\left(\boxdot^2\right) + \operatorname{tr}\Bigg[(D_{\rho\lambda\alpha\beta}\partial^\rho\partial^\lambda\partial^\alpha\partial^\beta + \cdots)\frac{1}{\boxdot^2} - \\
\frac{1}{2}(D_{\rho\lambda\alpha\beta}\partial^\rho\partial^\lambda\partial^\alpha\partial^\beta + \cdots)\frac{1}{\boxdot^2}(D_{\rho\lambda\alpha\beta}\partial^\rho\partial^\lambda\partial^\alpha\partial^\beta + \cdots)\frac{1}{\boxdot^2} + \cdots\Bigg] . \qquad (6.1.87)
\end{aligned}
$$

In the background gauge the effective action is covariant at all loop orders and $\bar{C}^{\mu\nu\rho\sigma}\bar{C}_{\mu\nu\rho\sigma}$ and $\bar{R}^2$ are the only covariant expressions that contain terms with 2 metric perturbations and 4 momenta and are not total derivatives. So, the coefficients in (6.1.59) are unequivocally determined by the terms containing two fluctuations f and four momenta in the Fourier transform of the expansion around flat space of $\Gamma^{1-\text{loop}}$ and they can be read from the two-point function of the background fluctuation $f_{\mu\nu}$. This correlation function is entirely given by the first two terms in the expansion (6.1.87) represented graphically by the diagrams in Figure 6.1, because terms with more vertices are at least cubic in background fields.

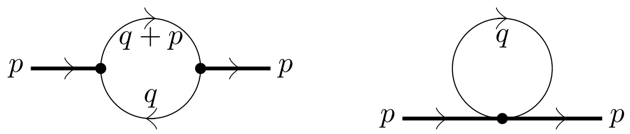


Figure 6.1: Diagrams contributing to the two-point function: bubbles (left) and tadpoles (right). The thin line represents the propagator of a quantum particle, the thick line is the f propagator, with momentum p.

Contributions from bubbles are built up with the part of D, C, V, N and U linear in the metric perturbation, while for the tadpole to contribute to the two-point function one has to expand the vertices to second order in f. Being logarithmically divergent, the tadpole contributes to the μ-running, but not to the p-dependence that we are interested in. Thus, the bulk of the calculation of the physical beta functions consists of working out the Feynman integrals for each of the 15 possible bubble diagrams that one can build with the Fourier transforms of the interaction vertices D, C, V, N and U, which differ from each other only by the numerator. The contribution from bubble diagrams to the one-loop effective action can be schematically written as

$$UU+NN+VV+CC+DD+2(UV+UN+UC+UD+NV+NC+ND+VC+VD+CD). \qquad (6.1.88)$$

Once we have computed them in the high energy limit, we have to reconstruct the covariant form of the nonlocal effective action. The leading order in $f_{\mu\nu}$ of $\bar{C}^{\mu\nu\rho\sigma}\bar{C}_{\mu\nu\rho\sigma}$ is

$$
\begin{aligned}
\bar{C}^{\mu\nu\rho\sigma}\bar{C}_{\mu\nu\rho\sigma} = &\frac{1}{2}p^4 f_{\mu\nu}f^{\mu\nu} - \frac{1}{6}p^4 f^\mu{}_\mu f^\nu{}_\nu - p^2 f_\mu{}^\nu f_{\nu\rho}p^\mu p^\rho \\
&+\frac{1}{3}p^2 f^{\mu\nu}f^\rho{}_\rho p^\mu p^\nu + \frac{1}{3}f_{\mu\nu}f_{\rho\sigma}p^\mu p^\nu p^\rho p^\sigma + O(f^3) , \qquad (6.1.89)
\end{aligned}
$$

where p^μ is the external momentum of the background metric perturbation, while the expansion of $\bar{R}^2$ gives

$$\bar{R}^2 = p^4 f^\mu{}_\mu f^\nu{}_\nu - 2p^2 f^{\mu\nu}f^\rho{}_\rho p^\mu p^\nu + f_{\mu\nu}f_{\rho\sigma}p^\mu p^\nu p^\rho p^\sigma + O(f^3) . \qquad (6.1.90)$$

We have just to replace these expressions with the covariant one to identify the contributions to the form factors f_λ and f_ξ respectively. In particular, the part proportional to $\log p^2$ will produce the physical running. All the considerations concerning the higher derivative bubble diagrams we did in Section 5.3 hold also in this case. In dimensional regularization, the $\log \mu$ terms always appear together with the $1/\epsilon$ pole, so the μ-running just traces the logarithmic divergences of the theory. in presence of a mass or a 2 derivative kinetic term, we have checked that putting together all the bubble *and* tadpole diagrams one reconstructs the covariant expression (6.1.81). If we just drop the tadpoles, the resulting function of f is not the linearization of a covariant expression. Thus, there are other contributions responsible of the difference between the physical and the $\overline{MS}$ schemes, namely the infrared logarithms from the bubbles. There are then two ways to find them. One is removing all mass scales from the flat propagator $\boxdot$ and continue to use dimensional regularization to regulate also the IR divergences. In this case *all* the logs are again of the form $\log p^2/\mu^2$, but in addition to the UV logs there are now also IR logs, that change the beta function. Alternatively, one can keep the existing masses or introduce one by hand as an IR regulator if $\boxdot^2 = (\partial_\mu \partial^\mu)^2$. The presence of the regulator mass leads to terms of the form $\log p^2/m^2$ and $\log \mu^2/m^2$, and we have to collect all the $\log p^2$ effects. Of course, there are various way one can introduce a mass in the flat propagator, the most general one being

$$\frac{1}{p^4} \to \frac{1}{p^4 + m_1 p^2 + m_2^4} \ . \tag{6.1.91}$$

In the outcome, the coefficients of the $\log(p^2)$ contributions to the form factors are independent of the particular choice of m_i. We have checked that both procedures lead exactly to the same result.

The IR divergences always appear with powers of the external momentum p in the denominator (see Section 5.3) and therefore they apparently give rise to nonlocal covariant terms containing $1/\Box$ or $1/\Box^2$. However, as can be deduced from (6.1.82-6.1.85), many of the interaction vertices involving only one background metric perturbation as external fields contain derivatives. Thus, negative powers of the derivatives can be offset by an equal number of powers of p in the numerator. Let us see how it happens in detail. In the high-energy limit we can neglect all couplings with a positive mass dimension. Since all terms in (6.1.86) have mass dimension 4, the vertices are schematically, at linear order in f in Fourier space,

$$U \sim p^4 f \,, \quad N \sim p^3 f \,, \quad V \sim p^2 f \,, \quad C \sim pf \,, \quad D \sim f \ . \tag{6.1.92}$$

On the other hand, the generic bubble integral with 4-derivative propagators and two of these vertices has mass dimension 4. In the two-point function there is only one background momentum the result of the loop integral can depend on, so all operators generated by off-shell infrared divergences with negative powers of momenta are actually of the type

$$\frac{p^8 f^2}{p^4} = p^4 f^2 \ . \tag{6.1.93}$$

The simplification is always possible, since f^2 has only 4 indices that can be nontrivially contracted with momenta, leaving always at least 4 momenta in the numerator contracted with each others. Notice that the right hand side is local and has the same structure of the linearizations of $\bar{C}^2$ (6.1.89) and $\bar{R}^2$ (6.1.90).

Such a localization process, that happens spontaneously at order f^2 in the one-loop effective action, in the covariant form of the loop corrections is due to differential relations such as

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{R}^{\mu\nu\rho\sigma} \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{R}^{\alpha\beta}{}_{\rho\sigma} = \bar{R}_{\mu\nu} \bar{\Box}^2 \bar{R}^{\mu\nu} - \frac{1}{4} \bar{R} \bar{\Box}^2 \bar{R} + O(\bar{R}^3) \ , \tag{6.1.94}$$

Which are a consequence of the Bianchi identities. The importance of similar identities in the definition of running couplings in gravitational theories was already noticed in [128]. In this way also the logs of infrared origin can affect the running of local operators. It is important to highlight that the IR logarithms found in this way are not in general by themselves the linearization of a covariant expression, in the same way the UV logarithms from bubbles, given by the total UV divergences (6.1.81) minus the tadpoles, were not assured to be covariant. It is only when one adds the two together that they give rise to a covariant expression as in (6.1.59). Both types of logs, UV and IR, are physical and both are needed to maintain general covariance.

After this general discussion on running couplings in higher derivative theories in curved space, we will consider some concrete examples and study their physical perturbative RG flow.

6.2 Higher-derivative scalar field

In this section, we regard the metric as a nondynamical field, which could be seen simply as the source of the energy-momentum tensor. For this reason, we will temporarily omit the bar distinguishing between the dynamical metric g and the background one $\bar{g}$. The most general quadratic action of a dimensionless scalar field coupled to the metric in four dimensions is

$$\begin{aligned} S_{\rm hds}[\phi] &= \frac{1}{2} \int d^4x \sqrt{g} \phi \mathcal{D}_4 \phi \,, \\ \mathcal{D}_4 \phi &= \Box^2 \phi + \nabla_\mu \Big(\big(\xi_1 R^{\mu\nu} + \xi_2 g^{\mu\nu} R \big) \nabla_\nu \phi \Big) + \mathcal{E} \phi \,, \\ \mathcal{E} &= \lambda_1 C^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R^2 + \lambda_4 \Box R \,, \end{aligned} \tag{6.2.1}$$

where $C^2 = C_{\mu\nu\rho\theta} C^{\mu\nu\rho\theta}$ is the square of Weyl tensor. The differential operator $\mathcal{D}_4$ is written in such a way that it is manifestly self-adjoint. In the limit $\lambda_i = 0$ the action $S_{\rm hds}[\phi]$ is shift-invariant, i.e., invariant under $\phi \to \phi + c$ for constant c, which can be seen easily integrating by parts the first two terms. Another relevant limit is $\lambda_i = 0$, $\xi_1 = 2$ and $\xi_2 = -\frac{2}{3}$, for which $\mathcal{D}_4$ becomes the Weyl covariant Paneitz operator and the action itself becomes conformal invariant [129–131]. In the following, the action with this particular choice of coefficients will be called S_c. In the conformal limit, the trace of the classical variational energy-momentum tensor of $S_{\rm hds}[\phi]$ is zero when one goes on-shell with the equations of motion of ϕ (i.e., $\mathcal{D}_4 \phi = 0$), in agreement with the Noether identities of Weyl symmetry.

The vacuum effective action is obtained by integrating ϕ in the path integral. The effective action depends only on the metric and at one loop it is

$$\Gamma = S_{\rm hds} + \frac{1}{2} {\rm tr} \log \frac{\delta^2 S_{\rm hds}}{\delta \phi^2} \,. \tag{6.2.2}$$

Since ϕ appears quadratically in $S_{\rm hds}[\phi]$, the Hessian can be trivially read off from the action (6.2.1) and has the structure (6.1.80) with

$$\mathbb{V}^{\mu\nu} = \xi_1 R^{\mu\nu} + \xi_2 g^{\mu\nu} R \ , \tag{6.2.3}$$

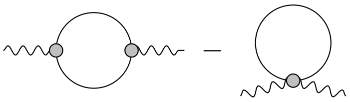
$$\mathbb{N}^{\nu} = \nabla_\mu \left(\xi_1 R^{\mu\nu} + \xi_2 g^{\mu\nu} R \right) \ , \tag{6.2.4}$$

$$\mathbb{U} = \mathcal{E} \ . \tag{6.2.5}$$

The explicit computation of the form factors can be performed in two equivalent ways. The first approach is the one discussed in the former section, in which the background metric is expanded as $g_{\mu\nu} = \delta_{\mu\nu} + f_{\mu\nu}$ and the covariant expression of Γ is reconstructed from its perturbative expansion at $O(f^2)$. We report in table B.1 the high-energy limit of the part proportional to $\log(p^2)$ in all the possible bubble diagrams in (6.1.88) for a scalar field. The terms with powers of momentum in the numerator are the new potential large logarithms produced by infrared effects.

A second approach follows a different strategy, used in Ref. [126] for a similar computation, and involves the comparison of the one-loop graviton-graviton two point function

$$\Gamma^{1-\text{loop}}_{(2)} = -\frac{1}{2}\Big\{2$$



$$\Big\} \qquad (6.2.6)$$

with the projection of the second variation of (6.1.57) using the decomposition in spin-projectors in momentum space. Denoting the transverse spin-1 projectors as $P_{\mu\nu} = \delta_{\mu\nu} - p^{-2}\, p_\mu p_\nu$, we define the transverse-traceless (TT) and scalar spin-2 projectors in $d = 4$ as

$$H^{\alpha\beta}_{\mu\nu} = P^{\alpha}_{(\mu}P^{\beta}_{\nu)} - \frac{1}{3}P_{\mu\nu}P^{\alpha\beta}\,, \quad S^{\alpha\beta}_{\mu\nu} = \frac{1}{3}P_{\mu\nu}P^{\alpha\beta}\,. \tag{6.2.7}$$

The complete decomposition includes two more projectors that we do not need for this presentation [31]. The second variation of (6.2.2) with respect to the metric in the flat space limit and in momentum space is expressed using the projectors as

$$\left.\frac{\delta^2\Gamma}{\delta g_{\mu\nu}\delta g_{\alpha\beta}}\right|_{\text{flat}} = 2f_\lambda H_{\mu\nu\alpha\beta} + 12 f_\xi S_{\mu\nu\alpha\beta} + \cdots\,, \tag{6.2.8}$$

where the dots hide the other spin-projectors. It follows that Eq. (6.2.8) can be compared with the explicit computation of Eq. (6.2.6) to extract the form factors f_λ and f_ξ as functions of p^2. The comparison allows for the separate determination of the coefficients of the dimensional poles b_i as well as the c_i that multiply the $\log(p^2)$ terms in (6.1.59).

6.2.1 Beta functions

The renormalized couplings of Eq. (6.1.58) can be arranged to resum either the $\log(\mu^2)$ or the $\log(p^2)$ logarithms, but not both. As a consequence, we have the two different notions of RG running, defined by relations (6.1.60) and (6.1.61). The difference between the two runnings can only be due to the operators λ_i. This has two explanations: on the one hand, the tadpole diagram is logarithmically divergent only if there are no derivatives of the fluctuating field in the interaction vertex; on the other hand, infrared divergences in the bubble integral occur only if in the numerator there are no powers of the momentum carried by both propagators. These two conditions for the appearance of $\log(m^2)$ terms in the one-loop corrected effective action are equivalent to require $U \neq 0$ in (6.1.86) and consequently $\mathcal{E} \neq 0$ in the action (6.2.1), because the covariant derivative of a scalar is equal to its partial derivative, hence the expansion near flat space is trivial and cannot produce any endomorphism term. The operators related to λ_1, λ_2 and λ_3 are quadratic in curvatures, so they can contribute at quadratic order in the metric fluctuation only to the μ-running via a tadpole diagram. In contrast, $\Box R$ is linear in curvature, so it can contribute via bubbles to the physical running, but, if inserted in a tadpole, it will give a total derivative. Then, $\mathcal{E} = 0$ means that the fluctuating field has no effective mass and in this case no discrepancy in the two definitions of running coupling is expected.

The explicit computation is consistent with the above expectations. It reveals for the μ-running

$$
\begin{aligned}
(4\pi)^2\beta^{s,\mu}_{\lambda} &= -\lambda^2\left[\frac{1}{30}+\frac{\xi_1}{24}(\xi_1-4)-2\lambda_1-\lambda_2\right], \\
(4\pi)^2\beta^{s,\mu}_{\xi} &= -\xi^2\left[\frac{1}{18}+\frac{\xi_1}{18}+\frac{5\xi_1^2}{72}+\frac{\xi_2}{3}+\frac{\xi_1\xi_2}{3}+\xi_2^2-\frac{2\lambda_2}{3}-2\lambda_3\right],
\end{aligned}
\tag{6.2.9}
$$

which agrees with Ref. [114], and for the physical running

$$
\begin{aligned}
(4\pi)^2\beta^{s}_{\lambda} &= -\lambda^2\left[\frac{1}{30}+\frac{\xi_1}{24}(\xi_1-4)\right], \\
(4\pi)^2\beta^{s}_{\xi} &= -\xi^2\left[\frac{1}{18}+\frac{\xi_1}{18}+\frac{5\xi_1^2}{72}+\frac{\xi_2}{3}+\frac{\xi_1\xi_2}{2}+\xi_2^2-2\lambda_4^2\right].
\end{aligned}
\tag{6.2.10}
$$

A consistency check shows that the difference between the two runnings is always equal to the logarithmic derivative with respect to the regulating mass m^2, in agreement with the expression given in Eq. (6.1.46).

6.2.2 Conformal limit and the trace-anomaly

The variational energy-momentum tensor of the conformal action S_c is defined as $T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_c}{\delta g_{\mu\nu}}$. Using the equation of motion $\mathcal{D}_4\phi = 0$ of the scalar, it is easy to show that diffeomorphisms invariance implies that it is conserved, $\nabla_\mu T^{\mu\nu} = 0$. In the conformal limit, $\lambda_i = 0$, $\xi_1 = 2$ and $\xi_2 = -\frac{2}{3}$, the variational energy-momentum tensor of S_c is also traceless, $T = T^{\mu}{}_{\mu} = 0$. Quantum mechanically we have that the path integral induces an anomaly which in $d = 4$ has the general form

$$
\langle T\rangle = \frac{1}{(4\pi)^2}\Big\{bC^2 + aE\Big\}, \tag{6.2.11}
$$

as dictated by the Wess-Zumino integrability condition [132, 133]. This implies, for example, that there is no independent R^2 term, besides the one in E. In the above formula, we have discarded a "trivial" $\Box R$ anomaly, which can be eliminated by including R^2 in $S_{\rm hds}$ when the metric is *not* dynamical and there are no self-interactions of the field ϕ.

Using the Callan-Symanzik equation of Γ and the fact that $\langle T^{\mu\nu}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}}$ for either RG scale, a general argument relates the coefficients of the anomaly with the beta functions of the couplings [132]. We expect that, in the conformal limit,

$$
\langle T\rangle = \frac{1}{2}\beta_{\frac{1}{\lambda}}C^2 + \beta_{\frac{1}{\xi}}R^2 + \frac{a}{(4\pi)^2}E, \tag{6.2.12}
$$

where $\beta_{\frac{1}{\lambda}}$ and $\beta_{\frac{1}{\xi}}$ are equal respectively to $-\frac{1}{\lambda^2}\beta_\lambda$ and $-\frac{1}{\xi^2}\beta_\xi$ and the coefficient a is not determined by our computation, given that we are working with two-point functions in asymptotically flat spacetime.

The expressions (6.2.11) and (6.2.12) for $\langle T\rangle$ should be compatible in the overlapping regimes of validity. Taking into account the fact that we compute the RG on asymptotically flat spacetimes, that is, $E = 0$, we have that compatibility requires that $\beta_{\frac{1}{\xi}} = 0$ in the conformal limit. Fortunately, this is verified by both the μ-running and the physical beta functions given above in Eqs. (6.2.9) and (6.2.10), respectively.

Furthermore, the trace anomaly is an observable, in the sense that we can construct identities among renormalized n-point functions that are constrained by the form of $\langle T\rangle$.

Since the anomaly is an observable, it may be tempting to ask whether the observable coefficients should be determined by the μ-running or by the physical beta functions, given that the Callan-Symanzik equation can be formulated with either.

The answer is actually simpler: in the conformal limit, the μ-running and the physical running do coincide, in agreement with the expectation that the conformal limit should be completely scaleless. In fact,

$$(4\pi)^2\beta^{\rm s,conf}_{\frac{1}{\lambda}} = -\frac{2}{15}\,, \qquad (4\pi)^2\beta^{\rm s,conf}_{\frac{1}{\xi}} = 0\,. \tag{6.2.13}$$

Combining everything together we have that the conformal higher derivative scalar has the anomaly

$$\langle T\rangle = \frac{1}{(4\pi)^2}\Big\{-\frac{1}{15}C^2 + aE\Big\}\,, \tag{6.2.14}$$

where the so-called b-anomaly coefficient of C^2 agrees with the literature [134, 135], while the a-anomaly coefficient is not determined by our procedure.[3] In order to compute a from physical correlators, it would be necessary to use either 3- or 4-point functions, depending on the approach [132, 136]. However, we know already that $a = \frac{7}{90}$ from standard covariant methods using the heat-kernel expansion [134].

One final observation is that the μ- and physical runnings given in Eqs. (6.2.9) and (6.2.10) do coincide in the more general limit $\lambda_i = 0$ in which the higher derivative action (6.2.1) is shift-invariant [81]. Notice that the action requires an integration by parts in order to be manifestly shift-invariant (i.e., to depend only on $\partial\phi$). These models admit shift-invariant interactions and, while naively nonunitary, they have received renewed attention in attempts to generalize the notion of unitarity [43, 84]. Furthermore, shift-symmetry plays a role in the construction of a natural virial current which is a signature of a theory that is *scale*-but-not-conformal invariant [137], which could explain why the running coincide.

6.3 Quadratic gravity

Quadratic gravity is an extension of Einstein's theory whose action contains terms quadratic in curvature [12, 31, 34, 100, 138–141]. To discuss its general properties, we will momentarily go to Lorentzian signature, however we will return to a Riemaniann manifold later on to study its renormalization. In signature $(-,+,+,+)$, the most general local gravitational action containing up to mass dimension 4 operators reads

$$S_{\rm qg} = \int d^4x\sqrt{|g|}\left[\frac{m_P^2}{2}(R-2\Lambda) - \alpha R^2 - \beta R_{\mu\nu}R^{\mu\nu} - \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \tau\Box R\right]\,, \tag{6.3.1}$$

where $m_P = 1/\sqrt{8\pi G_N}$ is the reduced Planck mass and Λ is the cosmological constant. Notice that the last term is a total derivative, so we will ignore it from now on. This is not the unique way it can be written. thanks to the identity

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{6.3.2}$$

[3] In fact, in the flat space limit and for nondynamical metric Eq. (6.2.6) is precisely renormalizing $\int d^4x\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle e^{ip\cdot x}$, where $T_{\mu\nu}$ is seen as the composite operator sourced by $g_{\mu\nu}$. The coefficients of the anomaly can be related to it in both broken and unbroken phases of conformal symmetry [136].

and the definition of the Euler density (6.1.54), it can be recast in the alternative basis

$$S_{\rm qg} = \int d^4x\sqrt{|g|}\left[\frac{m_P^2}{2}(R-2\Lambda) - \frac{1}{2\lambda}C^2 - \frac{1}{\xi}R^2 + \frac{1}{\rho}E - \tau\Box R\right] \ . \tag{6.3.3}$$

The couplings in the two basis are related via the transformations

$$\lambda = \frac{1}{\beta+4\lambda}, \qquad \rho = \frac{2}{\beta+2\gamma}, \qquad \xi = \frac{3}{3\alpha+\beta+\gamma} \tag{6.3.4}$$

and their inverses

$$\alpha = -\frac{1}{\rho} + \frac{1}{\xi} + \frac{1}{6\lambda}, \qquad \beta = \frac{4}{\rho} - \frac{1}{\lambda}, \qquad \gamma = -\frac{1}{\rho} + \frac{1}{2\lambda} \ . \tag{6.3.5}$$

We will call (6.3.1) the Riemann frame and (6.3.3) the Weyl frame. Since E is locally a total derivative, we observe that only two couplings actually affect the equations of motion.

The Euler-Lagrange equations are

$$\frac{m_P^2}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda\right) - \alpha E^{(1)}_{\mu\nu} - \beta E^{(2)}_{\mu\nu} - \gamma E^{(3)}_{\mu\nu} = 0 \ , \tag{6.3.6}$$

where

$$E^{(1)}_{\mu\nu} = 2RR_{\mu\nu} - 2\nabla_\mu\nabla_\nu R + g_{\mu\nu}\left(2\Box R - \frac{1}{2}R^2\right) \ , \tag{6.3.7}$$

$$E^{(2)}_{\mu\nu} = 2R_{\mu\lambda}R^{\lambda}{}_{\nu} - 2\nabla^\lambda\nabla_{(\mu}R_{\nu)\lambda} + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(\Box R - R_{\rho\sigma}R_{\rho\sigma}\right) \ , \tag{6.3.8}$$

$$E^{(3)}_{\mu\nu} = 2R_{\mu\rho\lambda\sigma}R_{\nu}{}^{\rho\lambda\sigma} + 4\nabla_{(\rho}\nabla_{\lambda)}R_{\mu}{}^{\rho}{}_{\nu}{}^{\lambda} - \frac{1}{2}g_{\mu\nu}R_{\lambda\tau\rho\sigma}R^{\lambda\tau\rho\sigma} \ . \tag{6.3.9}$$

Taking $\Lambda = 0$ and considering a perturbation with respect to a flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, they reduce to

$$\left[H_{\mu\nu\alpha\beta}\left(\frac{m_P^2}{8} - \left(\frac{\beta}{4}+\gamma\right)\Box\right)\Box + S_{\mu\nu\alpha\beta}\left(-\frac{m_P^2}{4} + (3\alpha+\beta+\gamma)\Box\right)\Box\right]h^{\alpha\beta} = 0 \ , \tag{6.3.10}$$

where the spin projectors (6.2.7) are now written in the coordinate space in terms of partial derivatives. As expected, it actually depends on only two combinations of the couplings α, β and γ and, if we use the Weyl basis, we get

$$\left[H_{\mu\nu\alpha\beta}\left(\frac{m_P^2}{8} - \frac{1}{4\lambda}\Box\right)\Box - S_{\mu\nu\alpha\beta}\left(\frac{m_P^2}{4} + \frac{3}{\xi}\Box\right)\Box\right]h^{\alpha\beta} = 0 \ . \tag{6.3.11}$$

The higher derivative nature of the theory is now manifest and we see that the dynamical degrees of freedom of the theory are given by a spin two part, whose higher derivative kinetic term is given by C^2, and a spin zero part, whose dynamics is generated by the R^2 term. To write a propagator for the gravitational perturbation we have to fix a gauge, since the vector and the second scalar modes are pure gauge and are in the kernel of the Hessian of the action. Anyway, we can write the propagators for the physical sectors. For the spin-2 we have

$$\frac{-i4\lambda}{p^4 + \frac{1}{2}p^2 m_p^2\lambda} = \frac{-i8}{m_p^2}\left(\frac{1}{p^2} - \frac{1}{p^2 + \frac{1}{2}\lambda m_p^2}\right) \ , \tag{6.3.12}$$

where we see the massless graviton of general relativity plus a spin-2 ghost with mass $\frac{1}{2}\lambda m_p^2$ that becomes a tachyon if $\lambda < 0$. In the scalar sector identified by the $S^{\mu\nu\rho\sigma}$ projector, the propagator is

$$\frac{\xi}{3}\frac{i}{p^4 - \frac{1}{12}p^2 m_p^2 \xi} = \frac{-i4}{m_p^2}\left(-\frac{1}{p^2} + \frac{1}{p^2 - \frac{1}{12}\xi m_p^2}\right) . \tag{6.3.13}$$

In this case we have a massless ghost, which can be gauged away as in GR, hence does not actually propagate [12], and a massive scalar with mass $-\frac{1}{12}\xi m_p^2$. The latter is a tachyon if $\xi > 0$.

The theory is power counting renormalizable, unlike general relativity, and this can be schematically shown in the perturbative regime around flat spacetime. Since the theory is nonlinear, in GR we have schematically

$$S_{\text{HE}} \sim \frac{m_P^2}{4}\int d^4x \left[-\frac{1}{2}(\partial h)^2 + h(\partial h)^2 + h^2(\partial h)^2 + \dots\right] , \tag{6.3.14}$$

were fluctuations $h_{\mu\nu}$ and partial derivatives can be variously contracted. To have a canonically normalized kinetic term, we redefine the field as $h \to h' = m_p h/2$ and we obtain

$$S_{\text{HE}} \sim \int d^4x \left[-\frac{1}{2}(\partial h')^2 + \frac{2}{m_p}h'(\partial h')^2 + \frac{4}{m_p^2}h'^2(\partial h')^2 + \dots\right] . \tag{6.3.15}$$

all the interaction terms with more that two gravitons acquire a coupling constant with negative mass dimension, hence they produce a power counting nonrenormalizabe theory. In quadratic gravity, the dominant terms in the UV are

$$\begin{aligned} S_{\text{qg}} \sim \frac{1}{2\lambda}\int d^4x &\left[-\frac{1}{2}(\Box h)^2 + h(\Box h)^2 + h^2(\Box h)^2 + \dots\right] \\ +\frac{1}{\xi}\int d^4x &\left[-\frac{1}{2}(\Box h)^2 + h(\Box h)^2 + h^2(\Box h)^2 + \dots\right] . \end{aligned} \tag{6.3.16}$$

The canonically normalized field is now only divided by $\sqrt{\lambda}$ or $\sqrt{\xi}$, which are dimensionless parameters, so the metric perturbations remain dimensionless. After the rescaling, all interaction vertices are multiplied by half-integer powers of the couplings ξ and λ, so when they go to zero we have a free theory. Moreover, all vertices generated by operators quadratic in curvatures have mass dimension 4, hence they are marginal in this parametrization, while those generated by the Einstein-Hilbert action are relevant, because of the factor m_p^2 in front. Thus, the theory is power counting renormalizable, that means only operators with mass dimension ≤ 4 are generated at all orders in loop expansion.

In the background gauge, all loop corrections are covariant. That implies the theory is multiplicatively renormalizable, because divergent terms can only assume the form of the operators already included in the classical action (6.3.3). Multiplicative renormalizability in arbitrary gauge can be shown tanks to Slavonov-Taylor identities (the Ward-Takahashi identities produced by the global BRST symmetry common to all gauge theories) as explained in [31, 100, 142].

6.3.1 Physical running of couplings in quadratic gravity

To calculate the RG flow of the theory we move to Euclidean space. The Euclidean action is equal to the one written in (6.3.1) and (6.3.3), up to a global minus sign. The first attempt to compute beta functions for this theory was made by Julve and Tonin in [127], but that work missed the contribution of the Nakanishi-Lautrup ghosts. This was corrected in [143] and then, with some further corrections, in [144].

Calculations of the beta functions so far have been based on the background field method, expanding $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ around a general background $\bar{g}$.

The effective action of quadratic gravity is hence obtained by integrating out $h_{\mu\nu}$ in the path integral. At one loop it is given by

$$\Gamma^1 = S_{\rm qg}[\bar{g}] + \frac{1}{2}{\rm tr}\log S_{\rm qg+gf\,(2)}|_{\bar{g}} - {\rm tr}\log \Delta_{\rm gh} - \frac{1}{2}{\rm tr}\log Y \ , \tag{6.3.17}$$

in accordance with (6.1.35). The second variation of the Einstein Hilbert action gives [12]

$$\begin{aligned}\frac{m_P^2}{4}\int d^4x\sqrt{\bar{g}}h^{\mu\nu}\ \Big[&-\frac{1}{2}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\bar{\Box} + \bar{g}_{\nu\beta}\bar{\nabla}_\mu\bar{\nabla}_\alpha - \bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta + \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\bar{\Box} + \bar{g}_{\mu\nu}\bar{R}_{\alpha\beta} \\ &-\bar{g}_{\mu\alpha}\bar{R}_{\nu\beta} - \bar{R}_{\mu\alpha\nu\beta} + \frac{2\Lambda - \bar{R}}{2}\left(\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\right)\Big]h^{\alpha\beta}\ ,\end{aligned} \tag{6.3.18}$$

while the second variation of terms quadratic in curvature in the Riemann basis gives [145, 146]

$$\begin{aligned}\alpha\int d^4x\sqrt{\bar{g}}h^{\mu\nu}\Big[&\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\nabla}_\alpha\bar{\nabla}_\beta - 2\bar{g}_{\mu\nu}\bar{\Box}\bar{\nabla}_\alpha\bar{\nabla}_\beta + \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\bar{\Box}^2 - \bar{g}_{\nu\beta}\bar{R}\bar{\nabla}_\mu\bar{\nabla}_\alpha \\ &-2\bar{R}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta + \bar{g}_{\mu\nu}\bar{R}\bar{\nabla}_\alpha\bar{\nabla}_\beta + 2\bar{g}_{\mu\nu}\bar{R}_{\alpha\beta}\bar{\Box} + \frac{1}{2}(\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} - \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta})\bar{R}\bar{\Box} \\ &-\bar{g}_{\mu\nu}\bar{R}\bar{R}_{\alpha\beta} - \frac{1}{4}J_{\mu\nu\alpha\beta}\bar{R}^2 + \bar{g}_{\nu\beta}\bar{R}\bar{R}_{\mu\alpha} + \bar{R}_{\mu\nu}\bar{R}_{\alpha\beta} + \bar{R}\bar{R}_{\mu\alpha\nu\beta} \\ &+2\bar{g}_{\mu\nu}\bar{\Box}\bar{R}_{\alpha\beta} + 2\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{R} - \bar{g}_{\nu\beta}\bar{\nabla}_\mu\bar{\nabla}_\alpha\bar{R} + \frac{1}{4}\left(3\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} + \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\right)\bar{\Box}\bar{R} \\ &+\bar{g}_{\nu\beta}\bar{\nabla}_\mu\bar{R}\bar{\nabla}_\alpha + 4\bar{g}_{\mu\nu}\bar{\nabla}_\rho\bar{R}_{\alpha\beta}\bar{\nabla}^\rho + 2\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{R}\bar{\nabla}_\beta\Big]h^{\alpha\beta},\end{aligned} \tag{6.3.19}$$

from the Riemann scalar square,

$$\begin{aligned}\beta\int d^4x\sqrt{\bar{g}}h^{\mu\nu}\Big[&\frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\nabla}_\alpha\bar{\nabla}_\beta - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}\bar{\nabla}_\alpha\bar{\nabla}_\beta - \frac{1}{2}\bar{g}_{\nu\beta}\bar{\nabla}_\mu\bar{\Box}\bar{\nabla}_\alpha + \frac{1}{4}(\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} + \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta})\bar{\Box}^2 \\ &+\frac{1}{2}\bar{R}_{\mu\alpha}\bar{\nabla}_\nu\bar{\nabla}_\beta - 2\bar{g}_{\nu\beta}\bar{R}^\rho_\mu\bar{\nabla}_\rho\bar{\nabla}_\alpha + \bar{g}_{\mu\nu}\bar{R}^\rho_\alpha\bar{\nabla}_\rho\bar{\nabla}_\beta + \bar{R}_{\mu\alpha\nu\beta}\bar{\Box} + \frac{1}{2}J_{\mu\nu\alpha\beta}\bar{R}^{\rho\lambda}\bar{\nabla}_\rho\bar{\nabla}_\lambda \\ &+\frac{1}{2}\bar{g}_{\nu\beta}\bar{R}_{\mu\rho}\bar{R}^\rho_\alpha + \frac{1}{2}\bar{R}_{\mu\alpha}\bar{R}_{\nu\beta} + \bar{R}^\rho_\mu\bar{R}_{\rho\alpha\nu\beta} - \bar{g}_{\mu\nu}\bar{R}^{\rho\sigma}\bar{R}_{\rho\alpha\sigma\beta} + \bar{R}_{\rho\mu\sigma\nu}\bar{R}^\rho{}_\alpha{}^\sigma{}_\beta - \frac{1}{4}J_{\mu\nu\alpha\beta}\bar{R}_{\rho\lambda}\bar{R}^{\rho\lambda} \\ &+\frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}\bar{R}_{\alpha\beta} + \frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{R} + \frac{1}{8}J_{\mu\nu\alpha\beta}\bar{\Box}\bar{R} + 2\bar{g}_{\nu\beta}\bar{\nabla}_\mu\bar{R}_{\alpha\rho}\bar{\nabla}^\rho - \bar{g}_{\nu\beta}\bar{\nabla}_\rho\bar{R}_{\mu\alpha}\bar{\nabla}^\rho \\ &+\bar{\nabla}_\alpha\bar{R}_{\mu\nu}\bar{\nabla}_\beta - \frac{1}{2}\bar{\nabla}_\mu\bar{R}_{\nu\beta}\bar{\nabla}_\alpha + (\bar{\nabla}_\alpha\bar{R}_{\mu\beta} - \bar{\nabla}_\mu\bar{R}_{\alpha\beta})\bar{\nabla}_\nu + \frac{1}{2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{R}\bar{\nabla}_\beta + \bar{g}_{\mu\nu}\bar{\nabla}_\rho\bar{R}_{\alpha\beta}\bar{\nabla}^\rho\Big]h^{\alpha\beta}.\end{aligned} \tag{6.3.20}$$

from the Ricci square and

$$\gamma\int d^4x\sqrt{\bar g}h^{\mu\nu}\Big[\bar\nabla_\mu\bar\nabla_\nu\bar\nabla_\alpha\bar\nabla_\beta-2\bar g_{\nu\beta}\bar\nabla_\mu\bar\Box\bar\nabla_\alpha+\bar g_{\mu\alpha}\bar g_{\nu\beta}\bar\Box^2+3\bar R_{\mu\alpha\nu\beta}\bar\Box-2\bar g_{\nu\beta}\bar R_{\mu\alpha}\bar\Box$$
$$-2\bar g_{\mu\nu}\bar R_{\alpha\rho\beta\sigma}\bar\nabla^\rho\bar\nabla^\sigma+2\bar g_{\nu\beta}\bar R_{\mu\rho\alpha\sigma}\bar\nabla^\rho\bar\nabla^\sigma+4\bar R_{\alpha\mu\rho\nu}\bar\nabla^\rho\bar\nabla_\beta+4\bar R_{\mu\alpha}\bar\nabla_\nu\bar\nabla_\beta-4\bar g_{\nu\beta}\bar R_{\mu\rho}\bar\nabla^\rho\bar\nabla_\alpha$$
$$+\bar g_{\mu\alpha}\bar g_{\nu\beta}\bar R_{\rho\sigma}\bar\nabla^\rho\bar\nabla^\sigma-\bar g_{\mu\nu}\bar R_{\alpha\lambda\rho\sigma}\bar R_\beta{}^{\lambda\rho\sigma}+2\bar R_{\mu\alpha}\bar R_{\nu\beta}-2\bar g_{\nu\beta}\bar R_{\mu\rho}\bar R^\rho{}_\alpha+2\bar g_{\nu\beta}\bar R_{\mu\lambda\rho\sigma}\bar R_\alpha{}^{\lambda\rho\sigma}$$
$$-\frac{1}{4}J_{\mu\nu\alpha\beta}\bar R_{\rho\sigma\lambda\tau}\bar R^{\rho\sigma\lambda\tau}+5\bar R_{\mu\rho\alpha\sigma}\bar R_\nu{}^\rho{}_\beta{}^\sigma-4\bar R_{\mu\alpha\rho\sigma}\bar R_{\nu\beta}{}^{\rho\sigma}-3\bar R_{\mu\rho\nu\sigma}\bar R_\alpha{}^\rho{}_\beta{}^\sigma+3\bar R_{\mu\rho}\bar R_{\nu\alpha}{}^\rho{}_\beta$$
$$+8(\bar\nabla_\alpha\bar R_{\mu\nu}-\bar\nabla_\mu\bar R_{\alpha\nu})\bar\nabla_\beta+8\bar\nabla_\alpha(\bar\nabla_\beta\bar R_{\mu\nu}-\bar\nabla_\mu\bar R_{\beta\nu})+3\bar\nabla_\rho\bar R_{\mu\alpha\nu\beta}\bar\nabla^\rho+2\bar\nabla_\alpha\bar R_{\beta\mu\rho\nu}\bar\nabla^\rho$$
$$+2\bar g_{\nu\beta}\bar\nabla_\mu\bar R_{\alpha\rho}\bar\nabla^\rho-4\bar g_{\nu\beta}\bar\nabla_\rho\bar R_{\alpha\mu}\bar\nabla^\rho+\frac{1}{2}\bar g_{\mu\alpha}\bar g_{\nu\beta}\bar\nabla_\rho\bar R\nabla^\rho\Big]h^{\alpha\beta}. \tag{6.3.21}$$

from the Riemann tensor square, where the tensor J is defined by

$$J_{\mu\nu\alpha\beta}=\delta_{\mu\nu,\alpha\beta}-\frac{1}{2}\bar g_{\mu\nu}\bar g_{\alpha\beta}\ , \tag{6.3.22}$$

with

$$\delta_{\mu\nu,\alpha\beta}=\frac{1}{2}(\bar g_{\mu\alpha}\bar g_{\nu\beta}+\bar g_{\mu\beta}\bar g_{\nu\alpha})\equiv\mathbb{I} \tag{6.3.23}$$

being the identity in the space of symmetric tensors.

We choose the background gauge

$$\chi_\mu=\bar\nabla^\lambda h_{\lambda\mu}+\mathbf{b}\bar\nabla_\mu h\,, \tag{6.3.24}$$

where $h=h^\mu{}_\mu$, and enforce it by adding to the action the gauge fixing term and the action of the Faddeev-Popov ghost C_μ

$$S_{\text{gf+FP}}=-\frac{1}{2\mathbf{a}}\int d^4x\sqrt{\bar g}\Big\{\chi_\mu Y^{\mu\nu}\chi_\nu$$
$$+i\bar C_\mu Y^{\mu\nu}[\bar g_{\nu\rho}\bar\Box+(2\mathbf{b}+1)\bar\nabla_\nu\bar\nabla_\rho+\bar R_{\nu\rho}]C^\rho\Big\}. \tag{6.3.25}$$

With this choice,

$$\Delta_{\text{gh}}=-\delta^\mu_\nu\bar\Box-(2\mathbf{b}+1)\bar\nabla^\mu\bar\nabla_\nu-\bar R^\mu{}_\nu\,. \tag{6.3.26}$$

In higher derivative theories it is more convenient to use a higher derivative gauge-fixing condition, so we take

$$Y_{\mu\nu}=-\bar g_{\mu\nu}\bar\Box-\mathbf{c}\bar\nabla_\mu\bar\nabla_\nu+\mathbf{d}\bar\nabla_\nu\bar\nabla_\mu\,. \tag{6.3.27}$$

The fourth order part of the kinetic term of the graviton can be reduced to the minimal form $\bar\Box^2$ by a smart choice of gauge fixing parameters [145]

$$\mathbf{a}=\frac{1}{\beta+4\gamma}\,,\qquad \mathbf{b}=-\frac{4\alpha+\beta}{4(\gamma-\alpha)}\,,\qquad \mathbf{c}=\frac{2(\gamma-\alpha)}{\beta+4\gamma}\,,\qquad \mathbf{d}=1\,. \tag{6.3.28}$$

leading to an order h^2 expansion of the action

$$S_{\text{qg+gf}\,(2)}h^2=\int d^4x\sqrt{|\bar g|}h_{\alpha\beta}\mathcal{H}^{\alpha\beta,\gamma\delta}h_{\gamma\delta} \tag{6.3.29}$$

such that the propagation of gravitons has the form (suppressing the indices),

$$\mathcal{H}=\bar\Box^2\mathbb{K}+\mathbb{J}^{\mu\nu}\bar\nabla_\mu\bar\nabla_\nu+\mathbb{L}^\mu\bar\nabla_\mu+\mathbb{W} \tag{6.3.30}$$

and $\mathbb{K}$, $\mathbb{J}$, $\mathbb{L}$, $\mathbb{W}$ are matrices in the space of symmetric tensors, depending on $\bar{R}$ and its covariant derivatives. In particular

$$\mathbb{K} = \frac{1}{4\lambda}\mathbb{P}_{\rm tl} + \frac{9}{4(3\xi - 2\lambda)}\mathbb{P}_{\rm tr} \tag{6.3.31}$$

where $\mathbb{P}_{\rm tr}^{\alpha\beta,\gamma\delta} = \frac{1}{4}\bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta}$ is the projector on the trace part and $\mathbb{P}_{\rm tl} = \mathbb{I} - \mathbb{P}_{\rm tr}$ the projector on the traceless part. In flat space, $\mathbb{K}$ can be viewed as a tensorial wave function renormalization constant that gives different weights to the spin-2 and spin-0 components of h. As usual, it is convenient to canonically normalize the fields by redefining $h \to \sqrt{\mathbb{K}^{-1}}h$, so that the action can be rewritten as

$$S_{{\rm qg+gf}\,(2)}h^2 = \int d^4x\sqrt{|\bar{g}|}h_{\alpha\beta}\mathcal{O}^{\alpha\beta,\gamma\delta}h_{\gamma\delta}\ , \tag{6.3.32}$$

with structure (6.1.80) and $\mathbb{V} = \sqrt{\mathbb{K}^{-1}}\mathbb{J}\sqrt{\mathbb{K}^{-1}}$ etc. Now $\mathbb{V}$ contains terms proportional to $\bar{R}$ and m_P^2, $\mathbb{N}$ contains terms proportional to $\bar{\nabla}\bar{R}$, whereas $\mathbb{U}$ contains terms proportional to $\bar{R}^2$, $\bar{\nabla}^2\bar{R}$, $m_P^2\bar{R}$ and $m_P^2\Lambda$.

Using the formula (6.1.81) with

$$\mathbb{R}_{\rho\lambda}\mathbb{R}^{\rho\lambda} = -6\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} \tag{6.3.33}$$

for a spin 2 field in 4 dimensions, and adding to it the ghosts' contributions

$$\mathrm{tr}\log Y[\bar{g}]_{\rm div} = \frac{1}{(4\pi)^2}\int d^4x\sqrt{\bar{g}}\Big[-\frac{1}{15}\bar{R}_{\mu\nu\rho\lambda}\bar{R}^{\mu\nu\rho\lambda} + \frac{29}{60}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - \frac{1}{8}\bar{R}^2\Big], \tag{6.3.34}$$

$$\mathrm{tr}\log \Delta_{\rm gh}[\bar{g}]_{\rm div} = \frac{1}{(4\pi)^2}\int d^4x\sqrt{\bar{g}}\Big[-\frac{1}{15}\bar{R}_{\mu\nu\rho\lambda}\bar{R}^{\mu\nu\rho\lambda} + \frac{67+44\mathbf{b}-8\mathbf{b}^2}{34560(\mathbf{b}+1)^2}\bar{R}^2_{\mu\nu} + \frac{41+86\mathbf{b}+100\mathbf{b}^2+40\mathbf{b}^3}{144(\mathbf{b}+1)^2(2\mathbf{b}+1)}\bar{R}^2\Big], \tag{6.3.35}$$

the final result gives the following beta functions

$$(4\pi)^2\beta_\lambda = -\frac{133}{10}\lambda^2\ , \tag{6.3.36}$$

$$(4\pi)^2\beta_\xi = -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{36}\ , \tag{6.3.37}$$

These expressions have been confirmed in several calculations using different techniques [147–151].

The flow lines around the free fixed point $\lambda = \xi = 0$ are shown in Figure 6.2. The flow of λ is independent of ξ and the fixed point $\lambda = 0$ is attractive for $\lambda > 0$ and repulsive for $\lambda < 0$. Hence, asymptotic freedom is possible only in the sector of the theory without spin 2 tachyons. The flow for ξ is more complicated: there are three separatrices, along which the motion is purely radial. Getting close to the line $\lambda = 0$ from positive λ, the point $\xi = 0$ is UV attractive for $\xi > 0$ and UV repulsive for $\xi < 0$; the line s_1 is defined by

$$\xi \approx 131\lambda \tag{6.3.38}$$

and is attractive for $\lambda > 0$ and repulsive for $\lambda < 0$. The line s_2 is defined by

$$\xi \approx 0.548\lambda \tag{6.3.39}$$

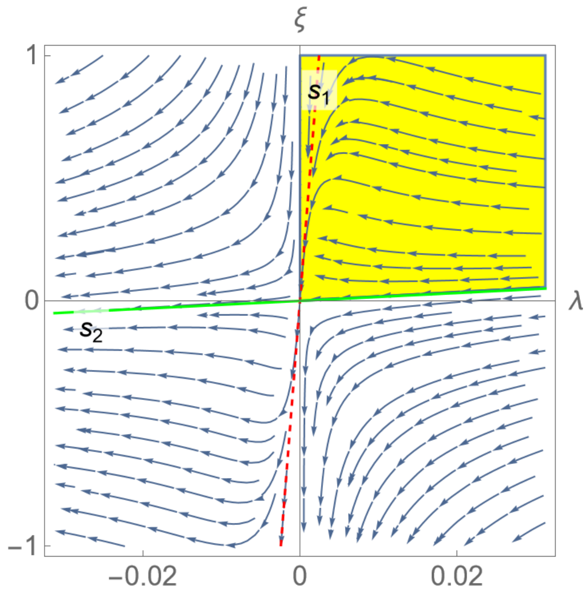


Figure 6.2: Flowlines of the beta functions (6.3.46,6.3.47). The red dashed line corresponds to (6.3.38) and the green line to (6.3.39). Initial points in the shaded area are asymptotically free. In the two left quadrants the massive spin 2 state is a tachyon, in the two upper quadrants the massive spin 0 state is a tachyon.

and is repulsive for $\lambda > 0$ and attractive for $\lambda < 0$. Thus the region that is attracted towards the free fixed point is totally included in the upper right quadrant. Recall that absence of tachyons requires $\lambda > 0$ and $\xi < 0$, so, With these beta functions, full asymptotic freedom can only be obtained for the case of a tachyonic coupling $\xi > 0$.

The beta functions (6.3.36,6.3.37) give the dependence of the renormalized λ and ξ on the renormalization scale μ. We called this the μ-running. However, what one is really interested in is the dependence of the running couplings on external momenta, that we called physical running. We observe that with our choice of gauge, both the Faddeev-Popov ghost operator and the third ghost operator are of second order in derivatives, hence none of these can produce a discrepancy between the two runnings. So, their contribution can be taken from traditional heat kernel calculations. On the other hand, $\mathcal{O}$ is a fourth order differential operator and $\mathbb{U} \neq 0$ (the parts of $\mathcal{O}$ relevant for the μ-running can be found in [145, 147]), this is enough to conclude that the standard beta functions (6.3.36) and (6.3.37) may be different from the physical ones.

To find the physical running we follow the procedure described in the last part of Section 6.1.2. $\mathcal{O}$ expands to

$$\mathcal{O} = \Box^2\mathbb{I} + \mathcal{D}^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma + \mathcal{C}^{\mu\nu\rho}\partial_\mu\partial_\nu\partial_\rho + \mathcal{V}^{\mu\nu}\partial_\mu\partial_\nu + \mathcal{N}^\mu\partial_\mu + \mathcal{U} \tag{6.3.40}$$

and the contribution from bubble diagrams to the one-loop effective action reads

$$\mathcal{UU}+\mathcal{NN}+\mathcal{VV}+\mathcal{CC}+\mathcal{DD}+2(\mathcal{UV}+\mathcal{UN}+\mathcal{UC}+\mathcal{UD}+\mathcal{NV}+\mathcal{NC}+\mathcal{ND}+\mathcal{VC}+\mathcal{VD}+\mathcal{CD}), \tag{6.3.41}$$

Now each vertex $\mathcal{A}$ in (6.3.40) has 4 indices to be contracted with quantum metric fluctuations. We introduce a generalized index notation for symmetric rank-2 tensors, $h_A := h_{\mu\nu}$ and $\mathbb{I} = \delta_{AB}$. With this notation, the operator $\mathcal{O}$ and all its terms look like e.g. $\mathcal{V}^{\mu\nu AB}$.

As a check for the procedure, taking as $\mathcal{O}$ the operator $\Box^2$ expanded at first order in f with respect to the perturbed metric $\bar{g}_{\mu\nu} = \delta_{\mu\nu} + f_{\mu\nu}$, one finds

$$-\frac{11}{48\pi^2}\bar{R}^{\mu\nu}\bar{R}_{\mu\nu} + \frac{7}{96\pi^2}\bar{R}^2 \ . \tag{6.3.42}$$

Since $\operatorname{tr}\log(\Box^2) = 2\operatorname{tr}\log\Box$, we would expect the result to be equal to

$$2\frac{1}{16\pi^2}b_4(\Box) = -\frac{R^{\mu\nu}R_{\mu\nu}}{144\pi^2} - \frac{R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}}{18\pi^2} + \frac{5R^2}{288\pi^2} \tag{6.3.43}$$

(where b_4 is the heat kernel coefficient). This expression is indeed equal to (6.3.42) if one takes $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 4R^{\mu\nu}R_{\mu\nu} - R^2$, which is true, up to a total derivative term.

The actual calculation can be simplified by neglecting the terms proportional to m_P, if we are interested only in leading terms in the $p^2 \gg m_p^2$ expansion. This is justified in the UV limit, as seen explicitly in the case of the simple shift-invariant scalar model in Section 5.1. Such a simplification allows us to choose between regulating the massless theory with dimensional regularization both in the UV and in the IR, or introducing an artificial mass via a simpler two derivative kinetic term $h^{\mu\nu}m^2\partial^\mu\partial_\mu h_{\mu\nu}$, which gives the same mass to each component of the metric fluctuation and avoids the complex tensorial structure of (6.3.18). They correspond respectively to taking $\boxdot^2 = (\partial^\mu\partial_\mu)^2$ or $\boxdot^2 = (\partial^\mu\partial_\mu)^2 - m^2\partial^\mu\partial_\mu$ and in both cases the calculation of the relevant Feynman integrals becomes straightforward. The terms proportional to $\log p^2$ from each of the possible bubble diagrams are reported in Table B.2.

How different schemes get contributions from different types of vertices can be seen easily by considering the vertex $\mathbb{U}$. As in the scalar higher derivative model, it enters the heat kernel calculation linearly, see (6.1.81), corresponding to a tadpole. It is only the part of $\mathbb{U}$ quadratic in curvature that contributes to the beta functions (6.3.36), (6.3.37) describing μ running, since a term $\bar{\nabla}\bar{\nabla}\bar{R}$ contributes with a total derivative to the one-loop effective action. By contrast, in our calculation we need two powers of $\mathbb{U}$ to build a bubble diagram and hence only the part proportional to $\bar{\nabla}\bar{\nabla}\bar{R}$ contributes at order f^2. These $\mathbb{U}$-$\mathbb{U}$ bubbles are UV finite but contain $\log p^2/m^2$ contributions coming from the IR region. These come with a factor p^4 in the denominator, from the propagators, but also p^4 in the numerator from the vertices, thanks to derivatives acting on curvatures. It is always possible to make these powers of momenta cancel each other up to nonlocal term at least cubic in curvature thanks to Bianchi identities and relations like (6.1.94) descending from them. Thus they also contribute to the coefficients c_i in (6.1.59).

Something similar happens with the operator $\mathbb{N}^\mu$: from dimensional analysis we know that the only divergence it can produce in the μ running is $\nabla_\mu\mathbb{N}^\mu$, that is a total derivative, so it is usually neglected. On the other hand, in the physical running, $\mathbb{N}^\mu$ represents an important component of $\mathcal{N}^\mu$ and from Table (B.2) we see that there is a new large logarithm that potentially arises from the bubble $\mathcal{N}\mathcal{U}$. Even in this case, the p^4 factor in the denominator is canceled by an equal factor in the numerator.

The part proportional to $\log(p^2)$ and quadratic in $f_{\mu\nu}$ of the trace of $\log\mathcal{O}$ can be covariantized in the expression

$$-\left[-6\alpha^3+\alpha^2(4\beta+34\gamma)+\alpha\left(17\beta^2+182\beta\gamma+438\gamma^2\right)\right.$$
$$\left.+6\beta^3+73\beta^2\gamma+246\beta\gamma^2+206\gamma^3\right]\frac{\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}}{48\pi^2(3\alpha+\beta+\gamma)(\beta+4\gamma)^2}$$
$$-\left[2496\alpha^3+52\alpha^2(49\beta+52\gamma)+\alpha\left(877\beta^2+1812\beta\gamma+704\gamma^2\right)\right.$$
$$\left.+99\beta^3+275\beta^2\gamma+104\beta\gamma^2-144\gamma^3\right]\frac{\bar{R}^2}{384\pi^2(3\alpha+\beta+\gamma)(\beta+4\gamma)^2}$$
$$+\frac{3(7\beta+32\gamma)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}}{128\pi^2(\beta+4\gamma)} \tag{6.3.44}$$

Putting it together with the ghosts' contributions, our final result is

$$\Gamma^1 \supset \frac{1617\lambda-20\xi}{5760\pi^2\lambda}\bar{C}_{\mu\nu\rho\sigma}\log(\bar{\Box})\bar{C}^{\mu\nu\rho\sigma}-\frac{2520\lambda^2+36\lambda\xi-\lambda\xi^2}{1152\pi^2\lambda\xi^2}\bar{R}\log(\bar{\Box})\bar{R}\,, \tag{6.3.45}$$

that leads to the beta functions in the Weyl basis

$$(4\pi)^2\beta_\lambda = -\frac{(1617\lambda-20\xi)\lambda}{90}\,, \tag{6.3.46}$$
$$(4\pi)^2\beta_\xi = -\frac{\xi^2-36\lambda\xi-2520\lambda^2}{36}\,, \tag{6.3.47}$$

The flowlines around the free fixed point $\lambda=\xi=0$ are shown in Figure 6.3.

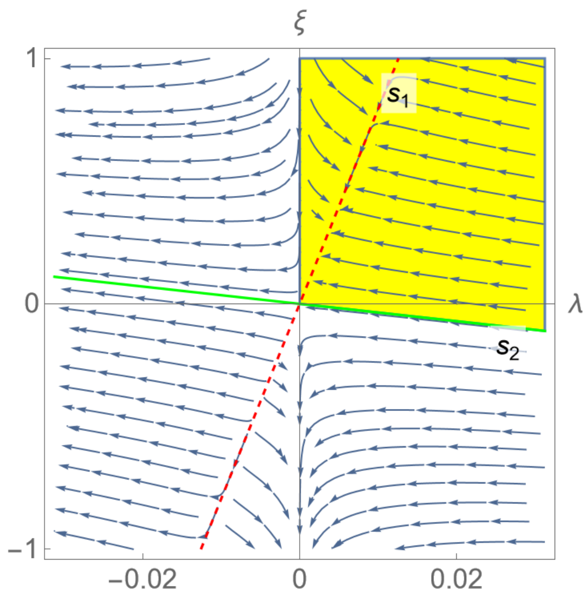


Figure 6.3: Flowlines of the beta functions (6.3.46) and (6.3.47). The red dashed line corresponds to (6.3.48) and the green line to (6.3.49). Initial points in the shaded area are asymptotically free. In the two left quadrants the massive spin 2 state is a tachyon, in the two upper quadrants the massive spin 0 state is a tachyon.

Now both beta functions depend on both couplings. There are again three separatrices, along which the motion is purely radial. The line $\lambda=0$ is now UV repulsive for $\xi>0$ and UV attractive for $\xi<0$; the line s_1 is defined by

$$\xi=\frac{569+\sqrt{386761}}{15}\lambda\approx 79.4\lambda \tag{6.3.48}$$

and is attractive for $\lambda > 0$ and repulsive for $\lambda < 0$, while the line s_2 is defined by

$$\xi = \frac{569 - \sqrt{386761}}{15}\lambda \approx -3.53\lambda \tag{6.3.49}$$

and has moved to the second-fourth quadrants. It is repulsive for $\lambda > 0$ and attractive for $\lambda < 0$. Thus, the region that is attracted towards the free fixed point is the upper right quadrant, plus a triangular slice of the lower right quadrant that lies above the separatrix s_2.

There is a unique trajectory that is asymptotically free and lies entirely in the tachyon-free area: it is the separatrix s_2. This behavior is to be contrasted with the flow given by μ running in Eqns. (6.3.36) and (6.3.37), for which the analog of the separatrix s_2 (6.3.39) delimits the asymptotically free trajectory region, but with a positive slope, with the result that the asymptotic free sector lies entirely in the tachyonic region. Conversely, the physical running couplings allow asymptotic freedom without tachyons. Moreover, there may be an additional possibility. One can have asymptotically free trajectories where the coupling ξ changes sign, as long as it is negative at the momenta where the pole in the propagator occurs, thus avoiding a tachyonic state. One can see these trajectories that lie above s_2 and thus have $\xi > 0$ in the far UV but eventually cross into $\xi < 0$ when one goes towards the IR. For these, it could be sufficient to demand that $\xi < 0$ at momenta corresponding to the poles of the propagators.

Up to now, we did not discuss about the gauge dependence of beta functions in quadratic gravity. It is well known [142, 152] that the μ-beta functions of λ, ξ, ρ and τ are gauge independent, however the same is not true for the dimensionful couplings m_p and Λ. In fact, there exists a particular gauge where the Newton constant does not run at all. The gauge dependence of the physical running has not yet been studied. We hope to address this problem in a future work.

Another aspect that should be stressed, is that the computation method adopted here needs an asymptotically flat background to be consistent. That means the result could be different in presence of a cosmological constant. On the other hand, we observed that the general structure of momentum logarithms generated in the infrared is not affected by the particular choice of the infrared regulator. Hence, this result could be unaffected by $\Lambda \neq 0$.

6.4 Higher-derivative conformal gravity

In this Section, we study the physical beta functions of conformal gravity. In this theory, the graviton is the only field taken into account and the action reads

$$S_{\rm cg}[g_{\mu\nu}] = \int d^4x\sqrt{|g|}\Big\{-\frac{1}{2\lambda}C^2 + \frac{1}{\rho}E\Big\} \tag{6.4.1}$$

in $(-,+,+,+)$ signature, which is the most general gravitational action invariant under Weyl transformations. In the Riemann basis, it is equivalent to take $3\alpha + \beta + \gamma = 0$. Unlike quadratic gravity, only the transverse-traceless (TT) part of the metric fluctuations propagates here, because there is no R^2 to give a dynamics to $h^\mu{}_\mu$.

To define a quantum effective action, we follow the same procedure outlined for quadratic gravity in the last section. The only difference is that the gauge arbitrariness due to Weyl invariance can be explicitly fixed by projecting the quantum fluctuation on its traceless part. Then, we have to treat the gauge freedom coming from diffeomorphisms invariance. A convenient set of gauge parameters that reduce the kinetic part of the graviton fluctuation to the minimal form is

$$\mathbf{a} = \lambda\,, \qquad \mathbf{b} = -\frac{1}{4}\,, \qquad \mathbf{c} = -2\,, \qquad \mathbf{d} = 1\,. \tag{6.4.2}$$

The μ-running of conformal gravity was originally calculated in Refs. [153], and it results in

$$(4\pi)^2 \beta_\lambda^{\mathrm{cg},\mu} = -\frac{199}{15}\lambda^2\,, \qquad (4\pi)^2 \beta_\rho^{\mathrm{cg},\mu} = \frac{137}{60}\rho^2\,. \tag{6.4.3}$$

With the gauge (6.4.2), the beta function of ξ is equal to zero, since that choice falls into the class of conformal background gauges [143, 154]. However, this is not true in general: a gauge fixing that does not preserve the conformal invariance of the theory will introduce divergences proportional to $\bar{R}^2$ at one-loop. Due to non-zero beta functions, we expect the conformal symmetry to be broken and the theory to be multiplicatively nonrenormalizable at higher loops [155]. Anyway, this model can still be regarded as a good approximation of a hypothetical high-energy regime [154].

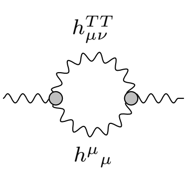
Moving to the physical running, since the structure of the gauge fixing and Faddeev-Popov ghost terms is the same as in quadratic gravity, also in this case the operators Δ_{gh} and Y are second order, hence their effect on the physical running coincides with the μ-running. On the other hand, in this gauge $S_{\mathrm{cg+gf}(2)}|_{\bar{g}}$ has the structure (6.1.80) with $\mathbb{I}$ replaced by $\mathbb{P}_{\mathrm{tl}}$. It is a higher derivative operator, so we expect a difference between its contribution to the two runnings. We calculate the one-loop corrections to the two-point functions of the background metric perturbation $f_{\mu\nu}$ and match them with the second order perturbation of C^2 around flat spacetime to reconstruct the covariant structure of the one-loop effective action. The physical beta function for λ turns out to be

$$(4\pi)^2 \beta_\lambda^{\mathrm{cg}} = -\frac{93}{5}\lambda^2\,. \tag{6.4.4}$$

One can immediately see that in this case, despite the conformal symmetry, the two runnings are different, but their qualitative behavior is preserved. In particular, we have asymptotic freedom for $\lambda > 0$, which depends only on the sign of the beta function.

6.4.1 Further properties of the physical running

It was observed for the first time in Ref. [153] that the difference in the μ-running of λ between quadratic gravity and conformal gravity is equal to the contribution of two free scalar field or, equivalently, of one higher derivative free scalar. This discrepancy is due to the conformal mode of the graviton, which is not dynamical in conformal gravity [156]. The difference may be seen also as coming from a partial gauge fixing of Weyl invariance [157, Sect. V]. Comparing the result (6.4.4) with the physical running of quadratic gravity given by equations (6.3.46) and (6.3.47), one immediately sees that the difference is no longer equal to the contribution of two free scalar fields. In fact, β_λ receives more contributions from infrared large logarithms. For example, the mixed diagram containing one traceless and one scalar fluctuation generates corrections to the term $\bar{C}^2$ independent of ξ in the beta function, schematically

$$\Delta\beta_{\frac{1}{\lambda}}\Big|_{\xi=0}\ \log(p^2/m^2)\ \subset$$ 

, (6.4.5)

which explains why the difference is actually expected.

Moving on to another point, recall that the trace-anomaly as considered in Sect. 6.2.2 is well-defined only for a nondynamical metric which acts as a source to the energy-momentum tensor, in which case the anomaly has to satisfy appropriate integrability conditions [132]. In the case of a dynamical metric for the conformal invariant theory, there is no natural notion of energy-momentum tensor, unless one considers a pseudotensor. Using $SU(N)$ Yang-Mills gauge theories as guidance, in practice we need to ensure that there is no anomaly at RG fixed points, or else the gauge-invariance of the theory is broken by quantum effects other than the RG. In the case of conformal gravity, we have that $\lambda \to 0^+$ is an asymptotically free fixed point for either runnings (6.4.3) and (6.4.4). We thus expect that the theory is conformal in the UV, similarly to gauge theories, and argue that there is no inconsistency between (6.4.3) and (6.4.4), as long as both runnings result in asymptotic freedom. Notice that, in the case of gauge theories, to formally prove conformal invariance it is necessary to work with local couplings, or with the parametrization of the action such that the asymptotically free coupling appears as interaction, rather than as global normalization of $F_{\mu\nu}^2$ [158]. We may expect the same for conformal higher derivative gravity, with the additional complication of having to deal with an energy-momentum pseudotensor.

Chapter 7

Renormalization and running in $d = 2$

A phenomenon similar to what we observed in higher derivative theories in $d = 4$ can potentially happen also in $d = 2$ with ordinary rank 2 kinetic terms. The general one-loop integral depicted in 2.2 is

$$\int_k^\Lambda d^2q \, \frac{N(q, p_i)}{q^2(q+p_1)^2 \times \cdots \times (q+p_1+\cdots+p_{n-1})^2} \; , \tag{7.0.1}$$

that, in the integration region near $q = 0$, reduces to

$$\int_k^{\lambda' \ll p_i} d^2q \, \frac{N(0, p_i)}{q^2(p_1)^2 \times \cdots \times (p_1+\cdots+p_{n-1})^2} \; , \tag{7.0.2}$$

that is potentially IR divergent for $k \to 0$. Hence, all the discussion in Section (5.3) can be easily readapted. However, if the theory is unitary and well defined, there must be a set of IR safe observables the theory can be described in terms of. In this case, we expect the infrared divergences to cancel in quantum corrections to expectation values of quantities built up with these observables.

As an example, we calculate the scattering amplitude in the two dimensional $CP(1)$ model in a regularization scheme independent approach. The physical running of the coupling with renormalization scale arises from a UV finite Feynman integral in all regularizations. Even though the pathway to obtaining the beta function can be different in different renormalization schemes, we always reproduce the usual result with asymptotic freedom.

7.1 $CP(1)$ model

The two dimensional $CP(1)$ nonlinear sigma model [1] is defined by the Lagrangian

$$\mathcal{L} = \frac{\partial_\mu \phi^* \partial^\mu \phi}{(1 + \frac{g^2}{2}\phi^*\phi)^2} \; . \tag{7.1.1}$$

[1] A few words on terminology. Historically, the $CP(1)$ model was formulated with two complex scalars and a U(1) gauge invariance. Indeed, early references for CP($N-1$) models talk about this formulation [159–167]. In this work we will use the formulation as given in Ref. [168]. In the end such formulations must be equivalent because $CP(1) = S^2$. Indeed, what we do here – the sphere in stereographic coordinates – can legitimately be called $CP(1)$ model, but long before that it was also referred to as $O(3)$ model [169].

Despite its nonlinear nature, with the Lagrangian containing all powers of the fields, it is renormalizable. In fact, renormalizing the two particle scattering amplitude renormalizes all amplitudes due to the integrability of the theory, a feature which also implies that the scattering should be elastic [170]. It is also interesting because the coupling is asymptotically free.

Our focus here is the regularization scheme dependence. Previous calculations have been done using cutoffs, with the tadpole loop diagram playing a prominent role in obtaining asymptotic freedom. However, the tadpole diagram vanishes in dimensional regularization, which initially appears puzzling. Moreover, the tadpole diagram does not contain any momentum dependence, so it is not initially clear how the scattering amplitudes of the theory would manifest asymptotic freedom in their kinematic dependence. Therefore we will calculate a scattering amplitude in various regularization schemes. The result has some interesting features. Because the calculation itself is relatively simple, we will use this introduction also as a summary of the main features of the result.

When naively using a cutoff regularization, one finds loop effects which violate the symmetry of the theory. For example, with the scattering diagrams shown in Figure 7.1, the tadpole diagram is obtained from the interaction $\frac{3g^4}{4}\partial_\mu\phi^*\partial^\mu\phi(\phi^*\phi)^2$. The contraction of the two fields involving derivatives leaves behind an interaction with no factors of momentum which is different from any term in the original Lagrangian. Moreover it involves the quadratically divergent integral

$$\int^\Lambda \frac{d^2p}{(2\pi)^2}\frac{p^2}{p^2+i\varepsilon} \sim \Lambda^2 \tag{7.1.2}$$

In Section 7.1.1 we describe the origin of Feynman rules following from the path integral measure which have the effect of removing these symmetry violating quadratic divergences from the theory. This modification is not required in dimensional regularization because the scaleless integral of Eq. 7.1.2 is set equal to zero.

When renormalizing the theory one is confronted with both UV divergences and IR divergences. Somewhat surprisingly, in all $2d$ sigma models with $O(N)$ symmetry, including the $CP(1)$ model, $O(N)$ invariant observables are known to be IR finite [171]. While the result in Ref [171] refers to $O(N)$ invariant observables and we are calculating a specific process, both are governed by the same coupling constant. In Sec. 7.1.2, we use the background field method to renormalize the 4-point vertex of this theory and to show the IR finiteness of our particular observable.

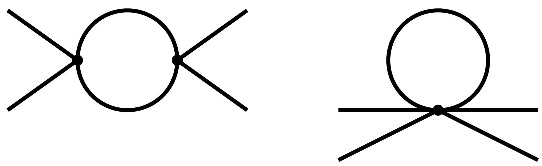


Figure 7.1: Diagrams associated with one-loop scattering.

The calculation of the amplitude is given in Sec 7.1.3. We use the Passarino-Veltman reduction method to calculate the scattering amplitude for the reaction $\phi_1+\phi_1\to\phi_2+\phi_2$ using real fields defined by $\phi=(\phi_1+i\phi_2)/\sqrt{2}$. In any renormalization scheme we find

$$\begin{aligned}\mathcal{M} = g_0^2 s - \frac{g_0^4 s}{4}\left[I(t)+I(u)\right]\\ +\frac{g_0^4}{4}(u-t)[I(t)-I(u)]\end{aligned} \tag{7.1.3}$$

where $I(q^2)$ is a specific combination of the scalar tadpole and scalar bubble diagrams

$$I(q^2) = 2T - q^2 B(q^2) \tag{7.1.4}$$

with the two dimensional tadpole and bubble diagrams defined by

$$\begin{aligned} T &= -i \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + i\varepsilon} \\ B(q^2) &= -i \int \frac{d^2p}{(2\pi)^2} \frac{1}{[p^2 + i\varepsilon][(p-q)^2 + i\varepsilon]} \ . \end{aligned} \tag{7.1.5}$$

The combination $I(q^2)$ is the unique combination of the scalar tadpole and the scalar bubble diagrams which is IR finite. Given the IR finiteness of the coupling, this must be the combination which appears in all amplitude calculations.

If we define the coupling at the renormalization point $s = t = u = \mu_R^2$, we can define the renormalized coupling as

$$g_r^2(\mu_R) = g_0^2 - \frac{g_0^4}{2} I(\mu_R^2) \ . \tag{7.1.6}$$

In any renormalization scheme we have

$$I(q^2) - I(\mu_R^2) = \frac{1}{2\pi} \log(\mu_R^2/q^2) \tag{7.1.7}$$

so that the renormalized amplitude becomes

$$\mathcal{M} = g_r^2(\mu_R^2)s - \frac{g_r^4\, s}{8\pi}\left(\log(-t/\mu_R^2) + \log(-u/\mu_R^2)\right) - \frac{g_r^4}{8\pi}(t-u)\log(t/u) \ . \tag{7.1.8}$$

This form implies the beta function

$$\beta_g = \mu_R \frac{\partial g_R(\mu_R)}{\partial \mu_R} = -\frac{g^3}{4\pi} \ , \tag{7.1.9}$$

which is the usual answer implying asymptotic freedom.

While this result is satisfying and perhaps not surprising, the way that this emerges in specific schemes is remarkably different. We here discuss three possible schemes.

- Cutoff regularization. In this scheme the tadpole diagram has both UV and IR sensitivity

$$T = -\frac{1}{4\pi} \log \frac{\Lambda^2}{k^2} \ , \tag{7.1.10}$$

  where Λ is a UV cutoff and k is an IR cutoff. The scalar bubble diagram is UV finite

$$q^2 B(q^2) = -\frac{1}{2\pi} \log \frac{-q^2}{k^2} \ . \tag{7.1.11}$$

- Dimensional regularization. Here the tadpole diagram is scaleless and hence vanishes

$$T = 0 \ . \tag{7.1.12}$$

  The scalar bubble diagram is again UV finite but has an IR divergence. It becomes convergent in $d < 2$. with the result

$$q^2 B(q^2) = \frac{1}{2\pi}\left[\frac{1}{\epsilon} - \log \frac{-q^2}{\mu^2}\right] \ , \tag{7.1.13}$$

  where $\epsilon = (2-d)/2$.

- Hybrid regularization. Here we use an IR cutoff k, but use dimensional regularization in the UV. In this case the tadpole is convergent if $\epsilon > 0$ and it gives

$$T = -\frac{1}{4\pi}\left[\frac{1}{\epsilon} - \log\frac{k^2}{\mu^2}\right] \ . \tag{7.1.14}$$

On the other hand, the bubble diagram is the same as in the cutoff regularization

$$q^2 B(q^2) = -\frac{1}{2\pi}\log\frac{-q^2}{k^2} \ . \tag{7.1.15}$$

Notice that, both in the cutoff and in the hybrid regularization, an artificial mass term can be equivalently used as an infrared cutoff instead of k. One simply has to replace k^2 with m^2 in the expressions for T and B.

In calculations in the literature (see e.g. [168]), the running has been calculated using cutoff regularization by following the UV cutoff Λ. We can call this cutoff running. In the amplitude for the process $\phi_1 + \phi_1 \to \phi_2 + \phi_2$ the bubble is finite and Λ appears uniquely in the tadpole diagram. In contrast, in dimensional regularization the tadpole diagrams vanish. The UV divergence of the tadpole is replaced by a factor of $1/\epsilon$ of IR origin in the bubble diagram, with the accompanying factor of $\log\mu$. In the hybrid scheme, the UV divergence of $1/\epsilon$ reappears in the tadpole diagram. In these cases, one can follow the running by following $\log\mu$, which can be called μ running. Again, since both Λ and μ disappear from renormalized amplitudes, the running that is observed in practice involves the behavior of the amplitudes with the kinematic variables. We show how all schemes lead to the same physical running.

For this theory, the equivalence of all schemes is basically due to dimensional analysis. There are no explicit dimensional factors in the Lagrangian, and it is also important that there is no sensitivity to a potential infrared cutoff. Therefore the logarithms must involve $\log(\Lambda^2/E^2)$ or $\log(\mu^2/E^2)$ (where E is a common energy scale in the amplitude), and the Λ or μ behavior tracks the physical running.

A somewhat surprising aspect of this result is that the running of the amplitude with the physical momentum comes from the UV finite bubble diagram in all schemes. In four dimensions with mass-independent renormalization, we are used to have the bubble diagram providing both the UV divergence and the related factor of $\log E^2$ which produces the physical running. In two dimensions the bubble is still the only source of the dependence on the physical momenta, but it is UV finite and IR sensitive. A focus only on the divergent diagrams may be dangerous unless a full calculation is performed. We have found this lesson also in our treatment of physical running in Quadratic Gravity 6.3.1.

7.1.1 Regularization scheme and Feynman rules

In theories with Goldstone bosons, the symmetry requires that the interaction terms involve derivatives. This forbids mass terms and also enforces the Adler zeroes for scattering amplitudes. However in such theories, there can be loops where the derivatives act on the loop particles leaving no momentum factors for the external particles. Besides the example given in the introduction, the $CP(1)$ theory could generate a mass term from the interaction $\partial_\mu\phi^*\partial^\mu\phi(\phi^*\phi)$ if the two fields with derivatives are contracted in a tadpole loop. This problem was known for sigma models in the 1960's and the resolution is a contribution to the Feynman rules from the path integral measure, which cancels the offending diagrams and preserves the symmetry[172, 173]. For our purposes the simplest way to obtain this factor is to follow the derivation of the Lagrangian of Eq. (7.1.1) from the constrained $O(3)$ version. The $CP(1)$ model describes a spin system in $2d$

$$S = \frac{1}{2g^2}\int d^2x\ \partial_\mu \mathbf{S}(x)\cdot\partial^\mu \mathbf{S}(x) \tag{7.1.16}$$

with $\mathbf{S} = (S_1, S_2, S_3)$, subject to the constraint

$$\mathbf{S}(x)\cdot\mathbf{S}(x) = 1\ \ . \tag{7.1.17}$$

In order to change to the unconstrained version, one identifies

$$\begin{aligned} S_1 &= \frac{g\phi_1}{1+\frac{g^2}{4}(\phi_1^2+\phi_2^2)}\ , \\ S_2 &= \frac{g\phi_2}{1+\frac{g^2}{4}(\phi_1^2+\phi_2^2)}\ , \\ S_3 &= \frac{1-\frac{g^2}{4}(\phi_1^2+\phi_2^2)}{1+\frac{g^2}{4}(\phi_1^2+\phi_2^2)}\ , \end{aligned} \tag{7.1.18}$$

which reproduces the Lagrangian of Eq. 7.1.1 with $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. However, the transformation has a Jacobian $J_{ij}[\phi] = \frac{\delta S_i}{\delta\phi_j}$, where i runs over two of the three components of $\mathbf{S}$ chosen as independent. So, the path integrals in the two coordinate frames are related by

$$\int [d\mathbf{S}] e^{iS[\mathbf{S}]} = \int [d\phi]\mathrm{det}J[\phi] e^{iS[\phi]}\ . \tag{7.1.19}$$

Following Honerkamp and Meetz [172] (see also [173–178]) we are led to the invariant measure

$$\int [d\phi]\mathrm{det}J[\phi] = \int \frac{[d\phi]}{1+\frac{g^2}{2}\phi^*\phi}\ . \tag{7.1.20}$$

This can be converted into an interaction by exponentiation

$$e^{\delta^2(0)\int d^2x \log(1+\frac{g^2}{2}\phi^*\phi)}\ . \tag{7.1.21}$$

The leading $\delta^2(0)$ vanishes in dimensional regularization because it is a scaleless integral. However, in cutoff regularization this cancels the Λ^2 contributions found in the Feynman diagrams, preserving the symmetry. Note that one can also obtain the invariant measure in a somewhat more cumbersome way using canonical quantization through a modification in the canonical momenta in theories with higher derivatives [173].

7.1.2 Background field renormalization and IR finiteness

In this section we use a hybrid renormalization scheme to renormalize the four-point vertex in the background field method. This will show the infrared finiteness of the vertex and we will obtain the running coupling using what we refer to as μ running. The renormalization scheme uses an infrared cutoff during intermediate steps, and regularizes the UV physics with dimensional regularization. This feature permits to easily distinguish the two types of divergences.

Let's consider a generic NLSM in Euclidean space of dimension $d = 2$

$$S[\varphi] = \frac{1}{2g^2}\int d^2x\, h(\varphi)_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\ . \tag{7.1.22}$$

The background field method applied to NLSM has been widely discussed in literature [179–181]. The second variation of the action can be put in a covariant form with respect to coordinate transformation in the target space using a nonlinear split. The quantum field $\xi^\alpha(x)$ is a vector field, related to the total field $\varphi^\alpha(x)$ and the background field $\bar{\varphi}^\alpha(x)$ by the exponential map

$$\varphi(x) = \exp_{\bar{\varphi}(x)}(\xi(x)) \ . \tag{7.1.23}$$

With this choice, the action up to order ξ^2 is

$$S[\bar{\varphi}]+\frac{1}{g^2}\int d^2x\, h(\bar{\varphi})_{\alpha\beta}\partial_\mu\bar{\varphi}^\alpha D_\mu\xi^\beta+\frac{1}{2g^2}\int d^2x\,\xi^\alpha\left(-h(\bar{\varphi})_{\alpha\beta}D_\mu D^\mu+\partial_\mu\bar{\varphi}^\gamma\partial^\mu\bar{\varphi}^\delta R_{\alpha\gamma\beta\delta}\right)\xi^\beta \ , \tag{7.1.24}$$

where the covariant derivative $D_\mu\xi^\alpha$ is defined as

$$D_\mu\xi^\alpha = \partial_\mu\xi^\alpha + \partial_\mu\bar{\varphi}^\gamma\Gamma_\gamma{}^\alpha{}_\beta\xi^\beta \tag{7.1.25}$$

and Γ and R are respectively the Christoffel symbols and the Riemann tensor of the metric $h_{\alpha\beta}(\bar{\varphi})$ on the target space. The linear term gives the classical equations of motion

$$\partial_\mu\partial^\mu\varphi^\alpha + \partial_\mu\varphi^\gamma\partial^\mu\varphi^\beta\Gamma_\gamma{}^\alpha{}_\beta = 0 \ , \tag{7.1.26}$$

while the quadratic term can be written as $\frac{1}{2g^2}\int\xi^\alpha\mathcal{O}_{\alpha\beta}\xi^\beta$ and used to compute the one-loop correction to the quantum effective action

$$\Gamma[\bar{\phi}] = S[\bar{\phi}] + \frac{1}{2}\mathrm{tr}\log\mathcal{O} \ . \tag{7.1.27}$$

The propagator in (7.1.24) has a nonstandard form due to the metric $h_{\alpha\beta}(\bar{\varphi})$, that forbids an easy Fourier transform and diagrammatic computations. A way around consists in defining the vielbein $e^a{}_\alpha$, the vector field $\xi^a = \xi^\alpha e^a{}_\alpha$ and the spin connection $\omega_\gamma{}^{ab}$. The new covariant derivative is

$$D_\mu\xi^a = \partial_\mu\xi^a + \partial_\mu\bar{\varphi}^\gamma\omega_\gamma{}^a{}_b\xi^b \tag{7.1.28}$$

and the second order term in the ξ expansion of the action turns into

$$S^{(2)} = \frac{1}{2g^2}\int d^2x\,\xi^a\left(-D_\mu D^\mu\delta_{ab} + \partial_\mu\bar{\varphi}^\gamma\partial_\mu\bar{\varphi}^\delta R_{a\gamma b\delta}\right)\xi^b \ , \tag{7.1.29}$$

with $R_{a\gamma b\delta} = R_{\alpha\gamma\beta\delta}e_a{}^\alpha e_b{}^\beta$. In this way $\mathcal{O}$ can be written in the form

$$\mathcal{O}_{ab} = \delta_{ab}\partial_\mu\partial^\mu + \mathcal{N}^\mu_{ab}\partial_\mu + \mathcal{U}_{ab} \ . \tag{7.1.30}$$

If $\mathcal{N}$ and $\mathcal{U}$ are small with respect to the kinetic term, one can perturbatively expand $\mathrm{tr}\log\mathcal{O}$ and find

$$\frac{1}{2}\mathrm{tr}\log\mathcal{O} = \frac{1}{2}\mathrm{tr}\left[\frac{\mathcal{U}}{\Box} - \frac{1}{2}\left(\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{U} + \frac{1}{\Box}\mathcal{N}\partial\frac{1}{\Box}\mathcal{N}\partial + 2\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{N}\partial\right)\right] \ , \tag{7.1.31}$$

where $\frac{1}{\Box}\mathcal{N}\partial$ has been neglected because it contributes only with a boundary term. In the hybrid renormalization scheme,

$$\mathrm{tr}\frac{\mathcal{U}}{\Box} = -\frac{1}{4\pi}\mathcal{U}_{aa}\left[-\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{m^2}\right)\right] \ , \tag{7.1.32}$$

$$\mathrm{tr}\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{U} = -\frac{1}{4\pi}(\mathcal{U}_{ab} + \mathcal{U}_{ba})\log\left(-\frac{\Box}{m^2}\right)\frac{1}{\Box}\mathcal{U}^{ab} \ , \tag{7.1.33}$$

$$\mathrm{tr}\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{N}\partial = -\frac{1}{4\pi}(\mathcal{U}_{ab} + \mathcal{U}_{ba})\log\left(-\frac{\Box}{m^2}\right)\frac{1}{\Box}\partial_\mu\mathcal{N}^{\mu ab} \ , \tag{7.1.34}$$

$$\begin{aligned}\mathrm{tr}\frac{1}{\Box}\mathcal{N}\partial\frac{1}{\Box}\mathcal{N}\partial = & \frac{1}{16\pi}\mathcal{N}_{\mu ab}\left(-\frac{1}{\epsilon} - \log\left(-\frac{\Box}{\mu^2}\right)\right)\left(\mathcal{N}^{\mu ab} - \mathcal{N}^{\mu ba}\right) \\ & + \frac{1}{8\pi}\mathcal{N}_{\mu ab}\log\left(-\frac{\Box}{m^2}\right)\frac{\partial_\mu\partial_\nu}{\Box}\mathcal{N}^{\nu ab} \ ,\end{aligned} \tag{7.1.35}$$

up to finite terms. $\mathcal{U}$ is symmetric in vielbein indices, while $\mathcal{N}$ is antisymmetric, hence the $\mathcal{U}\mathcal{N}$ bubble is identically zero.

In the particular case of the $CP(1)$ nonlinear sigma model the metric is simply

$$h_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{[1 + \frac{1}{4}(\varphi^\alpha\varphi^\alpha)]^2}$$

with indices running from 1 to 2. The curvatures tensors and connections can be computed using standard differential geometry. Notice that with a simple rescaling $\phi = \frac{1}{g}\varphi$ we get exactly the action (7.1.1).

Since we are interested in the one loop correction to the four point function, we can use the small ϕ expansion

$$S[\phi] = \frac{1}{2}\int d^2x\, \delta_{\alpha\beta}\partial_\mu\phi^\alpha\partial^\mu\phi^\beta\left[1 - \frac{g^2}{2}\phi^\gamma\phi^\gamma + \frac{3g^4}{16}(\phi^\gamma\phi^\gamma)^2\right] + O(\phi^8). \tag{7.1.36}$$

The classical equations of motion are

$$\partial_\mu\partial^\mu\phi^\alpha = \frac{g^2}{2}\partial_\mu\phi^\gamma\partial^\mu\phi^\beta(\delta^\alpha_\beta\phi_\gamma + \delta^\alpha_\gamma\phi_\beta - \delta_{\gamma\beta}\phi^\alpha) + O(\phi^5) \tag{7.1.37}$$

and we just need

$$\mathcal{N}^\mu{}_{ab} = -g^2\partial^\mu\bar{\phi}^\gamma(\delta^a_\gamma\bar{\phi}^b - \delta^b_\gamma\bar{\phi}^a) + O(\bar{\phi}^4) \tag{7.1.38}$$

$$\begin{aligned}\mathcal{U}_{ab} = & g^2\partial_\mu\bar{\phi}^\gamma\partial^\mu\bar{\phi}^\delta(1 - \frac{g^2}{2}\bar{\phi}^\gamma\bar{\phi}^\gamma)(\delta_{ab}\delta_{\gamma\delta} - \delta_{a\delta}\delta_{b\gamma}) + \frac{g^2}{4}\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\alpha(\bar{\phi}_a\partial^\mu\bar{\phi}_b + \bar{\phi}_b\partial^\mu\bar{\phi}_a) \\ & - \frac{g^2}{4}\partial_\mu\bar{\phi}_a\partial^\mu\bar{\phi}_b\bar{\phi}^\alpha\bar{\phi}^\alpha - \frac{g^2}{4}\partial_\mu\bar{\phi}^\alpha\partial^\mu\bar{\phi}^\alpha\bar{\phi}_a\bar{\phi}_b + O(\bar{\phi}^6)\end{aligned} \tag{7.1.39}$$

to compute (7.1.31) at order $\bar{\phi}^4$. So the bubble contributions are

$$\mathrm{tr}\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{U} \approx -\frac{g^4}{2\pi}\partial_\mu\bar{\phi}^\gamma\partial_\mu\bar{\phi}^\delta\log\left(-\frac{\Box}{m^2}\right)\frac{1}{\Box}\partial_\nu\bar{\phi}^\gamma\partial_\nu\bar{\phi}^\delta \tag{7.1.40}$$

$$\begin{aligned}\mathrm{tr}\frac{1}{\Box}\mathcal{N}\partial\frac{1}{\Box}\mathcal{N}\partial \approx & \frac{g^4}{4\pi}\left[\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\left(-\frac{1}{\epsilon} - \log\left(-\frac{\Box}{\mu^2}\right)\right)(\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta - \bar{\phi}^\beta\partial_\mu\bar{\phi}^\alpha)\right] \\ & + \frac{g^4}{4\pi}\left[\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\log\left(-\frac{\Box}{m^2}\right)\frac{\partial_\mu\partial_\nu}{\Box}(\bar{\phi}^\alpha\partial_\nu\bar{\phi}^\beta - \bar{\phi}^\beta\partial_\nu\bar{\phi}^\alpha)\right] \ .\end{aligned} \tag{7.1.41}$$

In this approximation, thanks to the equations of motions, we can write

$$\Box(\bar{\phi}^\alpha\bar{\phi}^\beta) = 2\partial_\mu\bar{\phi}^\alpha\partial^\mu\bar{\phi}^\beta + O(\bar{\phi}^4) \tag{7.1.42}$$

and the bubble diagrams reduce to

$$\mathrm{tr}\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{U} \approx -\frac{g^4}{4\pi}\bar{\phi}^\gamma\bar{\phi}^\delta\log\left(-\frac{\Box}{m^2}\right)\partial_\mu\bar{\phi}^\gamma\partial_\mu\bar{\phi}^\delta \tag{7.1.43}$$

$$\mathrm{tr}\frac{1}{\Box}\mathcal{N}\partial\frac{1}{\Box}\mathcal{N}\partial \approx \frac{g^4}{4\pi}\left[\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\left(-\frac{1}{\epsilon}-\log\left(-\frac{\Box}{\mu^2}\right)\right)(\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta-\bar{\phi}^\beta\partial_\mu\bar{\phi}^\alpha)\right] . \tag{7.1.44}$$

By an integration by parts we can move the partial derivative from one side to the other of $\log(-\Box)$, obtaining

$$\begin{aligned}\mathrm{tr}\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{U}+\mathrm{tr}\frac{1}{\Box}\mathcal{N}\partial\frac{1}{\Box}\mathcal{N}\partial \approx{}& \frac{g^4}{2\pi}\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\log(-\Box)\bar{\phi}^\beta\partial_\mu\bar{\phi}^\alpha+\frac{g^4}{4\pi}\log(m^2)\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\alpha\bar{\phi}^\beta\partial_\mu\bar{\phi}^\beta\\ &+\frac{g^4}{4\pi}\left[-\frac{1}{\epsilon}+\log\left(\mu^2\right)\right]\left(\bar{\phi}^\alpha\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\partial_\mu\bar{\phi}^\beta-\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\alpha\bar{\phi}^\beta\partial_\mu\bar{\phi}^\beta\right) .\end{aligned} \tag{7.1.45}$$

On the other hand, the tadpole at order $\bar{\phi}^4$ is

$$\mathrm{tr}\frac{\mathcal{U}}{\Box} \approx -\frac{g^2}{4\pi}\left[\partial_\mu\bar{\phi}^\beta\partial^\mu\bar{\phi}^\beta(1-g^2\bar{\phi}^\alpha\bar{\phi}^\alpha)+\frac{g^2}{2}(\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\alpha)^2\right]\left[-\frac{1}{\epsilon}+\log\left(\frac{\mu^2}{m^2}\right)\right], \tag{7.1.46}$$

then

$$\begin{aligned}\frac{1}{2}\mathrm{tr}\log\mathcal{O} \approx{}& -\frac{g^2}{8\pi}\partial_\mu\bar{\phi}^\beta\partial^\mu\bar{\phi}^\beta(1-\frac{g^2}{2}\bar{\phi}^\alpha\bar{\phi}^\alpha)\left[-\frac{1}{\epsilon}+\log\left(\mu^2\right)\right]\\ &+\frac{g^2}{8\pi}\partial_\mu\bar{\phi}^\beta\partial^\mu\bar{\phi}^\beta(1-g^2\bar{\phi}^\alpha\bar{\phi}^\alpha)\log\left(m^2\right)-\frac{g^4}{8\pi}\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\log(-\Box)\bar{\phi}^\beta\partial_\mu\bar{\phi}^\alpha .\end{aligned} \tag{7.1.47}$$

Again, the equations of motion implies on-shell

$$\bar{\phi}^\alpha\partial_\mu\partial^\mu\bar{\phi}^\alpha \approx -g^2\partial_\mu\bar{\phi}^\beta\partial^\mu\bar{\phi}^\beta\bar{\phi}^\alpha\bar{\phi}^\alpha , \tag{7.1.48}$$

so the term proportional to $\log m^2$ in (7.1.47) can be set to zero via integration by parts. That is the clear sign that IR divergences cancel out in the four-point amplitude. At the same time, the rest of the one-loop correction to the effective action reduces to

$$\frac{1}{2}\mathrm{tr}\log\mathcal{O} \approx -\frac{g^4}{16\pi}\partial_\mu\bar{\phi}^\beta\partial^\mu\bar{\phi}^\beta\bar{\phi}^\alpha\bar{\phi}^\alpha\left[-\frac{1}{\epsilon}+\log\left(\mu^2\right)\right]-\frac{g^4}{8\pi}\bar{\phi}^\alpha\partial_\mu\bar{\phi}^\beta\log(-\Box)\bar{\phi}^\beta\partial_\mu\bar{\phi}^\alpha . \tag{7.1.49}$$

Comparing the last expression with (7.1.36), one can renormalize the coupling in the following way

$$g^2(\mu) = g_B^2 - \frac{g_B^4}{4\pi}\log\left(\mu^2\right) , \tag{7.1.50}$$

from which we recover the μ-running

$$\beta_\mu(g) = -\frac{1}{4\pi}g^3 . \tag{7.1.51}$$

The term containing $\log(-\Box)$ generates at tree level the one loop amplitude (7.1.3), hence the physical running of the four-point function is equal to the μ-running.

From [171], the expectation value of $\mathcal{L}$ should be independent of $\log(m^2)$. However, the well known expression in the literature for the one loop effective action [168]

$$\mathcal{L}^0 + \mathcal{L}^{1-loop} = 2\left[\frac{1}{g_0^2} - \frac{1}{4\pi}\log\left(\frac{\Lambda^2}{m^2}\right)\right]\frac{\partial_\mu\phi^*\partial^\mu\phi}{(1+\phi^*\phi)^2} \tag{7.1.52}$$

clearly depends on m. In the small coupling (or small field) expansion $\langle\mathcal{L}\rangle$ should be independent of m order by order, but the tree level four-point amplitude from (7.1.52) is

$$s\left[\frac{1}{g_0^2} - \frac{1}{4\pi}\log\left(\frac{\Lambda^2}{m^2}\right)\right]^{-1} . \tag{7.1.53}$$

The usual solution consists in identifying m with the running scale μ and forget about the actual origin of $\log(\mu)$ terms. In our calculation we have seen that this substitution is a nontrivial effect due to the presence of non-local operators in the one-loop effective action. These terms, generated by the $p \ll q$ regions of Feynman integrals, cancel all m dependence from the four-point amplitude and allow us to reproduce at tree level the correct momentum structure near the renormalization point.

The generalization to the $O(N)$ NLSM for any N does not require much effort. The only change is that $\delta_{\alpha\alpha} = N-1$, because there are $N-1$ active particles, giving

$$\mathrm{tr}\frac{1}{\Box}\mathcal{U}\frac{1}{\Box}\mathcal{U} \approx -\frac{g^4}{4\pi}\left[(N-3)\bar\phi^\alpha\bar\phi^\alpha\log\left(-\frac{\Box}{m^2}\right)\partial_\mu\bar\phi^\beta\partial^\mu\bar\phi^\beta + \bar\phi^\gamma\bar\phi^\delta\log\left(-\frac{\Box}{m^2}\right)\partial_\mu\bar\phi^\gamma\partial_\mu\bar\phi^\delta\right] \tag{7.1.54}$$

and

$$\mathrm{tr}\frac{\mathcal{U}}{\Box} \approx -\frac{g^2}{4\pi}\left[\partial_\mu\bar\phi^\beta\partial^\mu\bar\phi^\beta(N-2)(1-g^2\bar\phi^\alpha\bar\phi^\alpha) + \frac{g^2}{2}(\bar\phi^\alpha\partial_\mu\bar\phi^\alpha)^2\right]\left[-\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{m^2}\right)\right] . \tag{7.1.55}$$

Also in this case, the part proportional to $\log(m^2)$ is equal to zero on-shell at order $\bar\phi^4$.

7.1.3 Passarino-Veltman reduction and the scattering amplitude

In two dimensions, the Passarino-Veltman reduction technique says that all one loop integrals can be reduced to momentum dependent factors times the scalar bubble and tadpole diagrams, given in Eq. (7.1.5). In calculating the amplitude, we need only one such example, which is the tensor bubble diagram given by

$$\begin{aligned} B_{\mu\nu}(q) &= -i\int\frac{d^dp}{(2\pi)^d}\frac{p_\mu p_\nu}{[p^2+i\epsilon][(p-q)^2+i\epsilon]} \\ &= \frac{2T}{4(d-1)}\left(\eta^{\mu\nu} + \frac{(d-2)q^\mu q^\nu}{q^2}\right) - \frac{q^2B(q)}{4(d-1)}\left(\eta_{\mu\nu} - d\frac{q_\mu q_\nu}{q^2}\right) \end{aligned} \tag{7.1.56}$$

in any dimension and any renormalization scheme. Only the scalar bubble carries momentum dependence. This reduction makes it simple to calculate the scattering amplitude.

Let us first give the calculation for $\phi_1+\phi_1 \to \phi_2+\phi_2$ in pure dimensional regularization. The tadpole diagram vanishes because it is scaleless. The relevant amplitudes are

$$-i\mathcal{M}(\phi_1(p_1)\phi_1(p_2) \to \phi_2(p_3)\phi_2(p_4)) = ig_0^2[p_1\cdot p_2 + p_3\cdot p_4] \to ig_0^2 s \tag{7.1.57}$$

and

$$\begin{aligned} &-i\mathcal{M}(\phi_1(p_1)\phi_1(p_2) \to \phi_1(p_3)\phi_1(p_4)) = \\ &\qquad ig_0^2[p_1\cdot p_2 + p_3\cdot p_4 + p_1\cdot p_3 + p_2\cdot p_4 + p_1\cdot p_4 + p_3\cdot p_2] \to ig_0^2[s+t+u] = 0 , \end{aligned}$$

where the second form in each case is the on-shell amplitude. When combining these into loop amplitudes of Fig 1, we find that the s-channel loop vanishes, while the t-channel and u-channel loops are non-vanishing [2]. Using the reduction technique, we find

$$\mathcal{M} = g_0^2 s + \frac{g_0^4 s}{4}\left[tB(t) + uB(u)\right] - \frac{g_0^4}{4}(u-t)\left[tB(t) - uB(u)\right] \qquad (7.1.58)$$

in terms of the scalar bubble diagram. The renormalization and running of this amplitude is analyzed in the Introduction. We also note that this amplitude can be constructed directly using unitarity based methods, in which the cuts are calculated in all channels and the logarithms are related to the value of the on-shell cut amplitude. The vanishing of the s-channel loop is directly related to the vanishing of Eq. (7.1.58) on-shell.

Unitarity methods also make it simple to generalize to the $O(N)$ case. In that situation the case of similar particles vanishes on shell as in Eq. (7.1.58) and $2 \to 2$ amplitudes with three or more types of particles are zero. Hence, the only relevant process is $\phi_i + \phi_i \to \phi_j + \phi_j$ with $i \neq j$ and those related to it via crossing symmetry. In the t-channel and u-channel only bubble loops with both virtual particles having index $k = i$ or $k = j$ are non-zero, so they give the same contribution calculated in the $CP(1)$ model. In the s-channel, the new $N-3$ bubbles with both virtual particles having index $k \neq i, j$ are non-zero and give the contribution

$$(N-3)\frac{g_0^4 s}{2} sB(s) \; . \qquad (7.1.59)$$

In the symmetric point, the three channels lead to an overall factor of $(N-2)$ in the beta function of g, which is the dual Coxeter number for $O(N)$ as identified in Ref. [182].

We can readily convert these results to any renormalization scheme by using the knowledge that the result is independent of an infrared regulator, as confirmed in the previous section. This is because the only combination of the scalar tadpole and scalar bubble which is IR independent is the combination (7.1.4). The complete result is then of the form of Eq. (7.1.3).

In the early days of quantum field theory, Landau, Pomeranchuk and collaborators studied the running couplings in many theories and concluded that all quantum field theories have Landau poles – i.e. are not asymptotically free [183–185]. This was famously overcome by the proof of asymptotic freedom in Yang-Mills theory, and this is often attributed to the extra degrees of freedom found in the non-Abelian gauge theory. The $2d$ $CP(1)$ model is another interesting counterexample to the Landau argument: The $CP(1)$ model shows that non-gauge theories may also exhibit asymptotic freedom.

[2]Here and throughout this chapter we are focusing only on the divergences and the logarithms, and hence drop terms of order ϵ

Chapter 8

Conclusions and outlooks

Higher derivative theories could furnish the simplest renormalizable UV completions to many nonrenormalizable theories, the most relevant one being gravity. Quadratic gravity is renormalizable and, thanks to the R^2 term, could potentially include Starobinsky inflation, one important inflationary model compatible with experimental observations.

However, it is still not clear whether this class of theories is actually well-defined. This is due to Ostrogradskij instability and related problems with negative energy states and unitarity emerging in the quantization process. We reviewed some of the most popular attempts to address this problem. More in detail, we saw that the possible approaches can be separated in two classes, one introducing complex variables in the canonical space, in order to make positive definite the spectrum of the quantum Hamiltonian and preserve at the same time a meaningful concept of probability; the other consisting in removing ghost states from the Hilbert space of physical states.

Since higher derivative theories are mainly interesting as UV completions of 2-derivative nonrenormalizable effective actions, a full understanding of how the renormalization group works in this kind of theories is crucial.

We reviewed how the renormalization group is usually introduced in high-energy physics and in other branches of theoretical physics. We noticed that these definitions have a good degree of universality in quantum field theories with two derivatives in the kinetic term in 4 spacetime dimensions, however that is not always true in other situations, including higher-derivative theories in $d = 4$.

We studied the nonperturbative RG flow of a shift-invariant higher derivative scalar theory. We observed that, because of the presence of two free field theories in the theory space, it cannot be completely mapped by only one chart with values in $\mathbb{R}^n$. Since the field normalization is a redundant coupling, physical information can be extracted only from the couplings adequately rescaled. In presence of more than one free fixed point, we have different possible choice for the field normalization. Which one is the most suitable can change in different regimes. In the presence of nonperturbative fixed points, a complicated structure connecting it with different free theories can emerge in the RG flow. Moreover, it is possible to avoid the ghost pole in the propagator if the fixed point value of the ratio between the dimensionless coupling associated with the two-derivative kinetic term and the one associated with the four-derivative term is larger than one. In this way, the value of the pole grows like p^2 and is never reached in the UV. This mechanism could explain how the ghost could not manifests itself near the Reuter fixed point [186], even when higher derivative operators are taken into account in the truncation of the effective average action, as, for example, in [145, 148, 187].

At the perturbative level, different implementations of the renormalization group give inequivalent beta functions. In scattering amplitudes of higher derivative theories, large logarithms containing physical momenta are not produced only by UV divergences, but also at the threshold scale determined by the ghost mass. If this mass were sent to zero, the logarithms of momenta would be associated to IR divergences. It is important to observe that these "IR" effects that emerge above the ghost mass are off-shell. This feature clearly distinguishes them from the usual IR divergences in massless 2-derivative theories like QED or QCD. In those cases IR divergences emerge only in soft or collinear on-shell momenta configurations and can be removed by considering IR-safe inclusive observables as initial and final states. Only momentum subtraction renormalization schemes take in account these new contributions and produce what we have called the physical beta functions. The universality of the one loop beta functions with respect of the renormalization scheme is recovered in shift-invariant theories, where such IR threshold effects cannot occur.

Moving to quantum field theories in curved spacetime, we have discussed three four-derivative models that corroborate the idea that the physical RG running differs from what we have called the μ-running.

The first model that we have considered is a general quadratic scalar coupled to a nondynamical metric, which could be regarded as a toy-model that is simple to dissect. In this model, it is possible to discuss with relative simplicity the role of each interaction in shaping the difference between the two runnings. We have also confirmed the fact that in the conformal limit, in which the action becomes Weyl invariant, the infrared divergences cancel [188, 189], making the two runnings coincide and produce the standard charges when used to determine the trace anomaly. One interesting feature that we have noticed is that the two runnings coincide for a more general, shift-invariant, subset of theories also in presence of a curved background. If this is a manifestation of a general fact, it would be interesting to explore the general implication that shift-invariance has in ensuring that different notions of RG are the same at one-loop. A possible route would be to explore the "trace-anomaly" of a nonconformal theory in the sense discussed in Refs. [190–192].

The second model we considered is quadratic gravity. In this case the metric is dynamical and the theory has a gauge symmetry, given by diffeomorphisms invariance. We find that the form factors associated to the Weyl tensor square and to the Riemann scalar square in the one-loop quantum effective action have a different dependence on the Laplacian than on the UV regulator. In particular, the physical running defined using form factors suggests the presence of a UV asymptotically free sector of the theory space free of tachyonic modes, in contradiction with the beta functions obtained with the heat kernel, which predict asymptotic freedom only in presence of a scalar tachyon.

The third one is Weyl invariant gravity, in which only the transverse-traceless spin-2 mode is propagating. In this case, we have observed that, although different, both runnings have negative beta functions and lead to asymptotic freedom. Importantly, From the fact that the conformal anomaly is not well defined in presence of a dynamical metric, we have argued that the two runnings can be different, even if the gravitational theory is Weyl invariant, as long as they both give rise to asymptotic freedom.

In conclusion, our findings strengthen the position of quadratic gravity as a possible UV completion of Einstein general relativity, with a behaviour in the far UV that could be better understood via an analogy with QCD, as suggested in [138], thanks to the common feature of asymptotic freedom.

However, there are still open questions related to the results presented in this thesis. To substantiate our findings in gravitational theories, it would be important to verify whether the physical beta functions are actually gauge-independent. A possible approach could be to repeat the computation with free gauge parameters. This would imply losing the minimal structure (6.1.80) in the Hessian of the action with respect to the metric fluctuations. Even if this implies an increase in the computational complexity of the problem, it does not seem to be an insuperable problem. Another possibility consists in integrating the quantities (6.1.37) and (6.1.36) along a path in the space of gauge parameters, and track back the changes in the physical running, as done in [152] for the beta functions obtained from the heat kernel.

Scheme dependent beta functions similar to those just described in higher derivative theories can potentially emerge also in 2-derivative theories in $d = 2$. However, if the massless theory is well-defined and unitary, the NLK theorem applies and it is possible to define a set of IR safe observables which are not affected by infrared divergences. We explicitly saw that, in the $CP(1)$ or $O(3)$ NLSM, due to the absence of IR divergences, $a = c$ in equation (5.3.17), when applied to the $2 \to 2$ scattering amplitude. The logarithms of the energy variables can be found by taking the unitarity cuts in all channels. The analysis using either cutoff regularization or dimensional regularization is consistent with the physical running.

At this point, one could wonder whether a set of IR safe observables, that do not coincide with the n-point functions of the quantum field could exists also in higher derivative theories. If pure 4-derivative theories without any mass parameters, as, for example, conformal gravity, would be shown to be unitary, we expect from the NLK theorem the existence of such infrared safe observables, and all their correlation functions should depend on the energy scale in the way suggested by the μ-running, with all the IR effects contributing to our physical running canceling out in a nontrivial way. On the other hand, most of the approaches described in Chapter 3 to solve the problems related to Ostrogradskij instability require a non-zero ghost mass to work. If the scaleless theory is intrinsically ill-defined and non-unitary, the NLK theorem would not hold and there would not be any IR safe observables in the high-energy limit. In this scenario, higher derivative theories make sense only in presence of a nonzero ghost mass that regulates IR divergences. Below this mass scale, we recover all the good features of the corresponding 2-derivative effective field theory, with only eventual soft and collinear divergences that can be treated as usual. Above the ghost mass, the threshold large logarithms we discussed in this thesis appear and affect the high-energy behaviour of scattering amplitudes, producing physical beta functions different from those determined by the μ-running.

A possible way to discriminate between these two alternatives in quadratic gravity could be to consider on-shell matrix elements of the S-matrix, because in this way we include only the large logarithms that are actually relevant for physics and remove all possible gauge dependencies from the beta functions extracted from the effective action. The tree level basic scattering amplitudes in quadratic gravity can be found in many references, for example [193–195], but a proper computation of their one-loop version is still missing. Moreover, the graviton itself is gauge dependent as an asymptotic state, so it does not furnish a good physical gauge invariant initial and final state. The computation of the one-loop cross section of the gravitational scattering of two scalar particles minimally coupled to quadratic gravity via a 2-derivative kinetic term could give the correct answer to this question.

The same process could give a hint on the problem of asymptotic freedom in presence of derivative interactions. Even if a dimensionless coupling goes to zero logarithmically, it does not ensures in general that all cross sections of processes mediated by that coupling vanishes in the UV. This happens in Yang-Mills theories, where the three point vertex and the ghost vertex contain one derivative, however it is not sure in Quadratic gravity, where interaction vertices can contain up to four derivatives. We observed that at tree level the exceeding powers of momenta cancel out in this particular cross section [96, 97], so, if the only effect of quantum corrections is a renormalization of λ and ξ, the couplings going to zero logarithmically would be enough for asymptotic freedom. However, the cancellations we referred to before must extend up to the fourth order in the high-energy expansion of the cross section, hence subleading finite terms absent in the classical action must be taken into account together with the dominant C^2 and R^2 terms.

For these reasons, the computation of one-loop physical scattering amplitudes and cross sections will be a crucial step in the understanding of quadratic gravity and higher derivative theories in general.

In this thesis, we furnished a method to compute the physical running in higher derivative theories in asymptotically flat spacetimes. However, we know that our universe is in an accelerated expansion phase at the moment, driven by what really resembles a cosmological constant. For this reason, it would be very useful a technique allowing to define a physical running also if $\Lambda \neq 0$. A possible way consists in extending the nonlocal resummation of the heat kernel [109, 122–126] to higher derivative operators. The bigger difficulty in this path is the fact that the free heat kernel in flat space does not have anymore a Gaussian structure like (6.1.76), but is given by a much more complicated expression [196].

Finally, it would be interesting to study at a deeper level the relation between the physical running and the FRG, and in particular whether the integration of k to zero produces an effective action that depends on momenta as predicted by the physical running and whether there exists a modification of the FRG capable of reproducing the physical RG in a way similar to how the k running is obtained.

Appendix A

Other cutoffs

We considered also cutoffs different from $R_k^{(24)}$ in (4.1.3): either the two-derivative cutoff:

$$R_k^{(2)} = Z_1(k^2 - q^2)\theta(k^2 - q^2) \ , \tag{A.0.1}$$

which is the standard choice, or the four-derivatives cutoff:

$$R_k^{(4)} = Z_2(k^4 - q^4)(k^4 - q^4) \ , \tag{A.0.2}$$

which is sometimes used when the kinetic term has four derivatives.

A.1 Dimensionful fields and two-derivative cutoff

With the field dimension set to 1 and the cutoff $R_k^{(2)}$, we find

$$\eta_1 = \frac{-6\tilde{Z}_2 + 6\sqrt{\tilde{Z}_2}(1+\tilde{Z}_2)\arctan(\sqrt{\tilde{Z}_2})}{(1+\tilde{Z}_2)\left(64\pi^2\tilde{Z}_2^2 + 3\tilde{g}\sqrt{\tilde{Z}_2}\arctan(\sqrt{\tilde{Z}_2}) - 3\tilde{g}\log(1+\tilde{Z}_2)\right)}\tilde{g} \ , \tag{A.1.1}$$

while the beta function is

$$\beta_{\tilde{g}} = (4+2\eta_1)\tilde{g} + \left[\frac{5\left(4\tilde{Z}_2^2 - (3+5\tilde{Z}_2+2\tilde{Z}_2^2)\eta_1\right)}{128\pi^2\tilde{Z}_2^2(1+\tilde{Z}_2)^2} + \frac{15\eta_1}{128\pi^2\tilde{Z}_2^{5/2}}\arctan\sqrt{\tilde{Z}_2}\right]\tilde{g}^2 \ . \tag{A.1.2}$$

where (A.1.1) has to be used. These are real only if $\tilde{Z}_2 \geq 0$, hence this set of flow equation has a restricted domain. Concerning the beta function of Z_2, we have $\partial_t Z_2 = 0$, hence the equation is independent of the cutoff and we ave again

$$\beta_{\tilde{Z}_2} \equiv \partial_t \tilde{Z}_2 = (2+\eta_1)\tilde{Z}_2 \ . \tag{A.1.3}$$

In the limit $\tilde{Z}_2 \to 0$, we recover the same nontrivial NGFP_1 and NGFP_2 of the cutoff $R_k^{(24)}$, while NGFP_3 is in the half-plane where the RG equations are complex. In the right half-plane the condition $\eta_1 = -2$ is satisfied on the curve

$$\tilde{g} = -\frac{64}{3}\pi\tilde{Z}_2^{3/2}. \tag{A.1.4}$$

The separatrix is not a straight line anymore, but we still have $\hat{\gamma} = 0$ after the coordinate transformation from U_1 to U_2, hence a region with a flow between GFP_1 and GFP_2 exists also with this cutoff.

A.2 Dimensionful field and four-derivative cutoff

If we use the cutoff (A.0.2), the anomalous dimension is

$$\eta_1 = \frac{3\tilde{g}\tilde{Z}_2\left(1+2\tilde{Z}_2\right)}{8\pi^2(1+\tilde{Z}_2)} - \frac{3\tilde{g}\tilde{Z}_2^2}{4\pi^2}\log\left(\frac{1+\tilde{Z}_2}{\tilde{Z}_2}\right) \ . \tag{A.2.1}$$

and the beta function is

$$\beta_{\tilde{g}} = 4\tilde{g} + \frac{16+21\tilde{Z}_2(3+2\tilde{Z}_2)}{8\pi^2(1+\tilde{Z}_2)^2}\tilde{g}^2\tilde{Z}_2 - \frac{42}{8\pi^2}\tilde{g}^2\tilde{Z}_2^2\log\left(\frac{1+\tilde{Z}_2}{\tilde{Z}_2}\right) \ . \tag{A.2.2}$$

where (A.2.1) has been used. The beta function of Z_2 is still given by (A.1.3). The $\log\left(\frac{1+\tilde{Z}_2}{\tilde{Z}_2}\right)$ is ill-defined or complex in the region $-1 < \tilde{Z}_2 \leq 0$, therefore the nontrivial FPs can not be found and the only finite FP of these beta functions is GFP_1.

The beta functions are influenced by the logs on $\tilde{Z}_2 = 0$, but, up to a small neighborhood of this axis, the general behaviour of the RG flow in the fourth quadrant is very similar to the one described in section 4.1.1. The condition $\eta_1 = -2$ is satisfied for

$$\tilde{g} \approx -16\pi^2\tilde{Z}_2 \ . \tag{A.2.3}$$

giving exactly the same asymptotic behaviour as $R_k^{(24)}$.

A.3 Dimensionless field and two-derivative cutoff

The dimensionless field with the cutoff $R_k^{(2)}$ gives

$$\beta_{\hat{\zeta}_1} = -2\hat{\zeta}_1 + \frac{6\hat{\gamma}\hat{\zeta}_1\left(\sqrt{\hat{\zeta}_1} - (1+\hat{\zeta}_1)\,\mathrm{arccot}\sqrt{\hat{\zeta}_1}\right)}{(1+\hat{\zeta}_1)\left(64\pi^2\sqrt{\hat{\zeta}_1} - 3\hat{\gamma}\sqrt{\hat{\zeta}_1}\log\left(\frac{1+\hat{\zeta}_1}{\hat{\zeta}_1}\right) + 3\hat{\gamma}\,\mathrm{arccot}\sqrt{\hat{\zeta}_1}\right)} \tag{A.3.1}$$

The beta function of $\hat{\gamma}$ is

$$\beta_{\hat{\gamma}} = \frac{5\hat{\gamma}^2\sqrt{\hat{\zeta}_1}\left[128\pi^2 + \hat{\gamma}\left(6+9\hat{\zeta}_1 - 6\sqrt{\hat{\zeta}_1}(4+3\hat{\zeta}_1)\,\mathrm{arccot}\sqrt{\hat{\zeta}_1} + 9\sqrt{\hat{\zeta}_1}(1+\hat{\zeta}_1)^2\mathrm{arccot}^2\sqrt{\hat{\zeta}_1} - 6\log\left(\frac{1+\hat{\zeta}_1}{\hat{\zeta}_1}\right)\right)\right]}{64\pi^2(1+\hat{\zeta}_1)^2\left(64\pi^2\sqrt{\hat{\zeta}_1} - 3\hat{\gamma}\sqrt{\hat{\zeta}_1}\log\left(\frac{1+\hat{\zeta}_1}{\hat{\zeta}_1}\right) + 3\hat{\gamma}\,\mathrm{arccot}\sqrt{\hat{\zeta}_1}\right)} \tag{A.3.2}$$

while the anomalous dimension, which does not receive any contributions from one loop diagrams, is still zero, as in section 4.1.1. Also with $[\varphi] = 0$, the cutoff $R_k^{(2)}$ introduces some $\sqrt{\hat{\zeta}_1}$ that reduce the domain of reality of the beta functions to the half plane $\hat{\zeta}_1 \geq 0$. Moreover now there are also some logs which may give problems in the interval $-1 < \hat{\zeta}_2 \leq 0$. The qualitative behaviour in the bottom right quadrant is the same of the regulator $R_k^{(24)}$, with a separatrix leading to FP_1 at $\hat{\zeta}_1 \to \infty$ and delimiting the attractive basin of GFP_1. The main difference is its trajectory near to GFP_2, which is not linear, as could be expected from the different asymptotic behaviour observed in U_1 for large $\hat{Z}_2$.

A.4 Dimensionless field and four-derivative cutoff

Using $R_k^{(4)}$, we have again $\eta_2 = 0$. Then,

$$\beta_{\hat{\zeta}_1} = -2\hat{\zeta}_1 + \frac{3\hat{\gamma}}{8\pi^2\hat{\zeta}_1^3}\left[-\frac{\hat{\zeta}_1(2+\hat{\zeta}_1)}{1+\hat{\zeta}_1} + 2\log(1+\hat{\zeta}_1)\right] \tag{A.4.1}$$

$$\beta_{\hat{\gamma}} = \frac{5\hat{\gamma}^2}{8\pi^2\hat{\zeta}_1^4}\left[\frac{\hat{\zeta}_1(6+9\hat{\zeta}_1+2\hat{\zeta}_1^2)}{(1+\hat{\zeta}_1)^2} - 6\log(1+\hat{\zeta}_1)\right] \tag{A.4.2}$$

The terms $\log(1+\hat{\zeta}_1)$ becomes complex for $\zeta_1 < -1$, so these beta functions are real only in the region $\zeta_1 > -1$. The RG flow is similar to $R_k^{(24)}$ near GFP_2, but the behaviour for large $\hat{\zeta}_1$ is different: one can still observe a region where there is a flow from GFP_2 in the UV to GFP_1 at infinity in the IR, however there is no clear separatrix, because with this regulator there is no NGFP_1 in the chart U_1.

A.5 Vanishing regulator

We may put a prefactor a in front of the regulator (4.1.3):

$$R_k^{(24)} = aZ_1(k^2-q^2)\theta(k^2-q^2) + aZ_2(k^4-q^4)\theta(k^4-q^4)\ .$$

The beta functions of dimensionful couplings in the action depend on this parameter, but the qualitative features of the flow are the same for a large range of values of a. For our discussion the main point to observe is that the value of $\tilde{g}$ at NGFP_1 decreases when a increases, and increases when a decreases.

The limit of vanishing regulator $a \to 0$ is interesting because it is related to dimensional regularization [197, 198]. In this limit NGFP_1 goes to $-\infty$, the point in the middle of the bottom side in Figure 1, and the separatrix also disappears at infinity. The trajectories we have been discussing now fill up the bottom right quadrant and the conclusions regarding the mass of the ghost remain valid.

Appendix B

Loop integrals

vertices	numerator	$\log(p^2)$ term
UU	$-\frac{1}{2}UU$	$-\frac{UU}{16\pi^2 p^4}$
UN	$-\frac{i}{2}UN^\mu q_\mu$	$i\frac{UN^\mu p_\mu}{32\pi^2 p^4}$
UV	$\frac{1}{2}UV^{\mu\nu}q_\mu q_\nu$	$\frac{UV^{\mu\nu}p_\mu p_\nu}{32\pi^2 p^4}$
UC	$\frac{i}{2}UC^{\mu\nu\rho}q_\mu q_\nu q_\rho$	$-i\frac{UC^{\mu\nu\rho}p_\mu p_\nu p_\rho}{32\pi^2 p^4}$
UD	$-\frac{1}{2}UD^{\mu\nu\rho\sigma}q_\mu q_\nu q_\rho q_\sigma$	$-\frac{1}{32\pi^2}\left(\frac{1}{p^4}UD^{\mu\nu\rho\sigma}p_\mu p_\nu p_\rho p_\sigma - \frac{1}{8}UD^{\mu\ \nu}_{\ \mu\ \nu}\right)$
NN	$-\frac{1}{2}N^\mu N^\nu q_\mu(q+p)_\nu$	0
NV	$-\frac{i}{2}N^\mu V^{\nu\rho}(p+q)_\mu q_\nu q_\rho$	0
NC	$\frac{1}{2}N^\sigma C^{\mu\nu\rho}q_\mu q_\nu q_\rho(q+p)_\sigma$	$-\frac{N^\mu C_{\mu\ \nu}^{\ \nu}}{256\pi^2}$
ND	$\frac{i}{2}N^\lambda D^{\mu\nu\rho\sigma}q_\mu q_\nu q_\rho q_\sigma(q+p)_\lambda$	$\frac{i}{128\pi^2}\left(p_\lambda N_\mu D^{\mu\lambda\nu}{}_\nu - \frac{p_\lambda N^\lambda D^{\mu\ \nu}_{\ \mu\ \nu}}{4}\right)$
VV	$-\frac{1}{2}V^{\mu\nu}V^{\rho\sigma}q_\mu q_\nu(q+p)_\rho(q+p)_\sigma$	$\frac{1}{384\pi^2}\left(V^{\mu\nu}V_{\mu\nu} + \frac{V^\mu_\mu V^\nu_\nu}{2}\right)$
VC	$-\frac{i}{2}V^{\sigma\lambda}C^{\mu\nu\rho}q_\mu q_\nu q_\rho(q+p)_\sigma(q+p)_\lambda$	$-\frac{i}{256\pi^2}\left(-p_\mu V^\mu_\nu C^{\nu\rho}{}_\rho + \frac{p_\mu V^\nu_\nu C^{\mu\rho}{}_\rho}{2} + p_\mu V_{\nu\rho}C^{\mu\nu\rho}\right)$
VD	$\frac{1}{2}V^{\lambda\delta}D^{\mu\nu\rho\sigma}q_\mu q_\nu q_\rho q_\sigma\times$ $(q+p)_\lambda(q+p)_\delta$	$\frac{1}{160\pi^2}\Big(p_\mu p_\nu V^\mu_\rho D^{\nu\rho\sigma}{}_\sigma - \frac{3V^{\mu\nu}p_\mu p_\nu D^{\rho\ \sigma}_{\ \rho\ \sigma}}{16} - \frac{3p_\mu p_\nu V^\rho_\rho D^{\mu\nu\sigma}{}_\sigma}{8}$ $-\frac{3p_\mu p_\nu V_{\rho\sigma}D^{\mu\nu\rho\sigma}}{4} + \frac{p^2 V^\mu_\mu D^{\nu\ \rho}_{\ \nu\ \rho}}{16} + \frac{p^2 V_{\mu\nu}D^{\mu\nu\rho}{}_\rho}{4}\Big)$
CC	$-\frac{1}{2}C^{\mu\nu\rho}C^{\sigma\lambda\delta}q_\mu q_\nu q_\rho\times$ $(q+p)_\sigma(q+p)_\lambda(q+p)_\delta$	$-\frac{1}{640\pi^2}\Big(\frac{C^{\mu\nu\rho}C_{\mu\nu\rho}p^2}{2} + \frac{3C^{\mu\ \nu}_{\ \nu}C_{\mu\ \rho}^{\ \rho}p^2}{4} +$ $3p_\mu p_\nu C^{\mu\rho\sigma}C^\nu{}_{\rho\sigma} + \frac{3p_\mu p_\nu C^{\mu\rho}{}_\rho C^{\nu\sigma}{}_\sigma}{2} - \frac{9p_\mu p_\rho C^{\mu\rho}{}_\nu C^{\nu\sigma}{}_\sigma}{2}\Big)$
CD	$-\frac{i}{2}C^{\lambda\delta\alpha}D^{\mu\nu\rho\sigma}q_\mu q_\nu q_\rho q_\sigma\times$ $(q+p)_\lambda(q+p)_\delta(q+p)_\alpha$	$-\frac{i}{320\pi^2}\Big(-\frac{3C^{\rho\sigma\delta}D^{\mu\nu}{}_{\sigma\delta}p_\mu p_\nu p_\rho}{2} - \frac{3C^{\rho\sigma}{}_\sigma D^{\mu\nu\delta}{}_\delta p_\mu p_\nu p_\rho}{4} +$ $C^\sigma{}_{\sigma\delta}D^{\mu\nu\rho\delta}p_\mu p_\nu p_\rho + \frac{3C^{\mu\nu}{}_\sigma D^{\rho\sigma\delta}{}_\delta p_\mu p_\nu p_\rho}{2} - \frac{C^{\mu\nu\rho}D^{\sigma\ \delta}_{\ \sigma\ \delta}p_\mu p_\nu p_\rho}{4} +$ $\frac{3p^2 p_\mu C^{\mu\nu\rho}D_{\nu\rho\ \delta}^{\ \ \ \delta}}{4} + \frac{3p^2 p_\mu C^{\mu\nu}{}_\nu D^{\rho\ \sigma}_{\ \rho\ \sigma}}{16} - \frac{3p^2 p_\mu C^{\nu\ \rho}_{\ \nu}D^{\mu\ \delta}_{\ \rho\ \delta}}{4} -$ $\frac{p^2 p_\mu C_{\nu\rho\sigma}D^{\mu\nu\rho\sigma}}{2}\Big)$
DD	$-\frac{1}{2}D^{\mu\nu\rho\sigma}D^{\lambda\delta\alpha\beta}q_\mu q_\nu q_\rho q_\sigma\times$ $(q+p)_\lambda(q+p)_\delta(q+p)_\alpha(q+p)_\beta$	$-\frac{1}{140\pi^2}\Big(-\frac{9D^{\mu\nu}{}_{\alpha\beta}D^{\rho\sigma\alpha\beta}p_\mu p_\nu p_\rho p_\sigma}{16} - \frac{9D^{\mu\nu\lambda}{}_\lambda D^{\rho\sigma\delta}{}_\delta p_\mu p_\nu p_\rho p_\sigma}{32} +$ $D^{\mu\nu\rho\delta}D^{\sigma\ \lambda}_{\ \delta\ \lambda}p_\mu p_\nu p_\rho p_\sigma - \frac{5D^{\mu\nu\rho\sigma}D^{\lambda\ \delta}_{\ \lambda\ \delta}p_\mu p_\nu p_\rho p_\sigma}{32} -$ $\frac{3p^2 p_\mu p_\nu D^{\mu\rho\sigma\lambda}D^\nu{}_{\rho\sigma\lambda}}{8} - \frac{9p^2 p_\mu p_\nu D^{\mu\rho\ \lambda}_{\ \ \rho}D^{\nu\ \sigma}_{\ \lambda\ \sigma}}{16} +$ $\frac{3p^2 p_\mu p_\nu D^{\mu\nu\rho\sigma}D_{\rho\sigma\ \lambda}^{\ \ \ \lambda}}{4} + \frac{3p^2 p_\mu p_\nu D^{\mu\nu\rho}{}_\rho D^{\sigma\ \lambda}_{\ \sigma\ \lambda}}{16} -$ $\frac{3p^4 D^{\mu\nu\rho\sigma}D_{\mu\nu\rho\sigma}}{128} - \frac{9p^4 D^{\mu\ \rho\sigma}_{\ \mu}D_{\rho\sigma\ \nu}^{\ \ \ \nu}}{128} - \frac{9p^4 D^{\mu\ \nu}_{\ \mu\ \nu}D^{\rho\ \sigma}_{\ \rho\ \sigma}}{1024}\Big)$

Table B.1: Scalar bubble loops.

vertices	numerator	$\log(p^2)$ term
$\mathcal{U}\mathcal{U}$	$-\frac{1}{2}\mathcal{U}_{AB}\mathcal{U}^{BA}$	$-\frac{\mathcal{U}^{AB}\mathcal{U}_{BA}}{16\pi^2 p^4}$
$\mathcal{U}\mathcal{N}$	$-\frac{i}{2}\mathcal{U}_{AB}\mathcal{N}^{\mu BA}q_\mu$	$i\frac{\mathcal{U}^{AB}\mathcal{N}^{\mu}_{BA}p_\mu}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{V}$	$\frac{1}{2}\mathcal{U}_{AB}\mathcal{V}^{\mu\nu BA}q_\mu q_\nu$	$\frac{\mathcal{U}^{AB}\mathcal{V}^{\mu\nu}_{BA}p_\mu p_\nu}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{C}$	$\frac{i}{2}\mathcal{U}_{AB}\mathcal{C}^{\mu\nu\rho BA}q_\mu q_\nu q_\rho$	$-i\frac{\mathcal{U}^{AB}\mathcal{C}^{\mu\nu\rho}_{BA}p_\mu p_\nu p_\rho}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{D}$	$-\frac{1}{2}\mathcal{U}_{AB}\mathcal{D}^{\mu\nu\rho\sigma BA}q_\mu q_\nu q_\rho q_\sigma$	$-\frac{1}{32\pi^2}\left(\frac{\mathcal{U}^{AB}\mathcal{D}^{\mu\nu\rho\sigma}_{BA}p_\mu p_\nu p_\rho p_\sigma}{p^4} - \frac{\mathcal{U}^{AB}\mathcal{D}^{\mu}{}_{\mu}{}^{\nu}{}_{\nu BA}}{8}\right)$
$\mathcal{N}\mathcal{N}$	$-\frac{1}{2}\mathcal{N}^{\mu}_{AB}\mathcal{N}^{\nu BA}q_\mu(q+p)_\nu$	0
$\mathcal{N}\mathcal{V}$	$-\frac{i}{2}\mathcal{N}^{\mu}_{AB}\mathcal{V}^{\nu\rho BA}q_\nu q_\rho(p+q)_\mu$	0
$\mathcal{N}\mathcal{C}$	$\frac{1}{2}\mathcal{N}^{\sigma}_{AB}\mathcal{C}^{\mu\nu\rho BA}q_\mu q_\nu q_\rho(q+p)_\sigma$	$-\frac{\mathcal{N}^{\sigma AB}\mathcal{C}_{\sigma}{}^{\mu}{}_{\mu BA}}{256\pi^2}$
$\mathcal{N}\mathcal{D}$	$\frac{i}{2}\mathcal{N}^{\lambda}_{AB}\mathcal{D}^{\mu\nu\rho\sigma BA}q_\mu q_\nu q_\rho q_\sigma(q+p)_\lambda$	$-\frac{i}{128\pi^2}\left(\frac{\mathcal{N}^{\lambda}_{AB}p_\lambda\mathcal{D}^{\mu}{}_{\mu}{}^{\nu}{}_{\nu}{}^{BA}}{4} - \mathcal{N}^{\rho}_{AB}p_\mu\mathcal{D}^{\mu}{}_{\rho}{}^{\nu}{}_{\nu}{}^{BA}\right)$
$\mathcal{V}\mathcal{V}$	$-\frac{1}{2}\mathcal{V}^{\mu\nu}_{AB}\mathcal{V}^{\rho\sigma BA}q_\mu q_\nu(q+p)_\rho(q+p)_\sigma$	$\frac{1}{384\pi^2}\left(\mathcal{V}^{\mu\nu}_{AB}\mathcal{V}_{\mu\nu}{}^{BA} + \frac{\mathcal{V}^{\mu}{}_{\mu AB}\mathcal{V}^{\nu}{}_{\nu}{}^{BA}}{2}\right)$
$\mathcal{V}\mathcal{C}$	$-\frac{i}{2}\mathcal{V}^{\sigma\lambda}_{AB}\mathcal{C}^{\mu\nu\rho BA}q_\mu q_\nu q_\rho(q+p)_\sigma(q+p)_\lambda$	$-\frac{i}{256\pi^2}\left(\frac{p_\mu\mathcal{V}^{\nu}{}_{\nu AB}\mathcal{C}^{\mu\rho}{}_{\rho}{}^{BA}}{2} + p_\mu\mathcal{V}_{\nu\rho AB}\mathcal{C}^{\mu\nu\rho BA}\right.$ $\left.-p_\mu\mathcal{V}^{\mu\nu}_{AB}\mathcal{C}_{\nu}{}^{\rho}{}_{\rho}{}^{BA}\right)$
$\mathcal{V}\mathcal{D}$	$\frac{1}{2}\mathcal{V}^{\lambda\delta}_{AB}\mathcal{D}^{\mu\nu\rho\sigma BA}q_\mu q_\nu q_\rho q_\sigma(q+p)_\lambda(q+p)_\delta$	$\frac{1}{160\pi^2}\left(-\frac{3\mathcal{V}^{\mu\nu}_{AB}p_\mu p_\nu\mathcal{D}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma}{}^{BA}}{16} + p_\mu p_\nu\mathcal{V}^{\mu}{}_{\rho AB}\mathcal{D}^{\nu\rho\sigma}{}_{\sigma}{}^{BA} -\right.$ $\frac{3p_\mu p_\nu\mathcal{V}^{\rho}{}_{\rho AB}\mathcal{D}^{\mu\nu\sigma}{}_{\sigma}{}^{BA}}{8} - \frac{3p_\mu p_\nu\mathcal{V}_{\rho\sigma AB}\mathcal{D}^{\mu\nu\rho\sigma BA}}{4} +$ $\left.\frac{p^2\mathcal{V}^{\mu}{}_{\mu AB}\mathcal{D}^{\nu}{}_{\nu}{}^{\rho}{}_{\rho}{}^{BA}}{16} + \frac{p^2\mathcal{V}_{\mu\nu AB}\mathcal{D}^{\mu\nu\rho}{}_{\rho}{}^{BA}}{4}\right)$
$\mathcal{C}\mathcal{C}$	$-\frac{1}{2}\mathcal{C}^{\mu\nu\rho}_{AB}\mathcal{C}^{\sigma\lambda\delta BA}q_\mu q_\nu q_\rho\times$ $(q+p)_\sigma(q+p)_\lambda(q+p)_\delta$	$\frac{1}{640\pi^2}\left(-3p_\mu p_\nu\mathcal{C}^{\mu\rho\sigma}_{AB}\mathcal{C}^{\nu}{}_{\rho\sigma}{}^{BA} - \frac{3p_\mu p_\nu\mathcal{C}^{\mu\rho}{}_{\rho AB}\mathcal{C}^{\nu\sigma}{}_{\sigma}{}^{BA}}{2} -\right.$ $\left.\frac{p^2\mathcal{C}^{\mu\nu\rho}_{AB}\mathcal{C}_{\mu\nu\rho}{}^{BA}}{2} - \frac{3p^2\mathcal{C}^{\mu}{}_{\mu}{}^{\nu}{}_{AB}\mathcal{C}_{\nu}{}^{\rho}{}_{\rho}{}^{BA}}{4} + \frac{9p_\mu p_\nu\mathcal{C}^{\mu\nu\sigma}_{AB}\mathcal{C}_{\sigma}{}^{\rho}{}_{\rho}{}^{BA}}{2}\right)$
$\mathcal{C}\mathcal{D}$	$-\frac{i}{2}\mathcal{C}^{\lambda\delta\alpha}_{AB}\mathcal{D}^{\mu\nu\rho\sigma BA}q_\mu q_\nu q_\rho q_\sigma\times$ $(q+p)_\lambda(q+p)_\delta(q+p)_\alpha$	$\frac{i}{320\pi^2}\left(\frac{3\mathcal{C}^{\rho\sigma\delta}_{AB}\mathcal{D}^{\mu\nu}{}_{\sigma\delta}{}^{BA}p_\mu p_\nu p_\rho}{2} + \frac{3\mathcal{C}^{\rho\sigma}{}_{\sigma AB}\mathcal{D}^{\mu\nu\delta}{}_{\delta}{}^{BA}p_\mu p_\nu p_\rho}{4} -\right.$ $\frac{3\mathcal{C}^{\mu\nu}{}_{\sigma AB}\mathcal{D}^{\rho\sigma\delta}{}_{\delta}{}^{BA}p_\mu p_\nu p_\rho}{2} + \frac{\mathcal{C}^{\mu\nu\rho}_{AB}\mathcal{D}^{\sigma}{}_{\sigma}{}^{\delta}{}_{\delta}{}^{BA}p_\mu p_\nu p_\rho}{4} -$ $\mathcal{C}^{\sigma}{}_{\sigma\delta AB}\mathcal{D}^{\mu\nu\rho\delta BA}p_\mu p_\nu p_\rho - \frac{3p^2 p_\mu\mathcal{C}^{\mu\nu\rho}_{AB}\mathcal{D}_{\nu\rho}{}^{\delta}{}_{\delta}{}^{BA}}{4} -$ $\frac{3p^2 p_\mu\mathcal{C}^{\mu\nu}{}_{\nu AB}\mathcal{D}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma}{}^{BA}}{16} + \frac{p^2 p_\mu\mathcal{C}^{\nu\rho\sigma}_{AB}\mathcal{D}^{\mu}{}_{\nu\rho\sigma}{}^{BA}}{2} +$ $\left.\frac{3p^2 p_\mu\mathcal{C}^{\nu}{}_{\nu}{}^{\sigma}{}_{AB}\mathcal{D}^{\mu}{}_{\sigma}{}^{\delta}{}_{\delta}{}^{BA}}{4}\right)$
$\mathcal{D}\mathcal{D}$	$-\frac{1}{2}\mathcal{D}^{\mu\nu\rho\sigma}_{AB}\mathcal{D}^{\lambda\delta\alpha\beta BA}q_\mu q_\nu q_\rho q_\sigma\times$ $(q+p)_\lambda(q+p)_\delta(q+p)_\alpha(q+p)_\beta$	$-\frac{1}{140\pi^2}\left(-\frac{9\mathcal{D}^{\mu\nu}{}_{\alpha\beta AB}\mathcal{D}^{\rho\sigma\alpha\beta BA}p_\mu p_\nu p_\rho p_\sigma}{16} -\right.$ $\frac{9\mathcal{D}^{\mu\nu\lambda}{}_{\lambda AB}\mathcal{D}^{\rho\sigma\delta}{}_{\delta}{}^{BA}p_\mu p_\nu p_\rho p_\sigma}{32} + \mathcal{D}^{\mu\nu\rho\delta}_{AB}\mathcal{D}^{\sigma}{}_{\delta}{}^{\lambda}{}_{\lambda}{}^{BA}p_\mu p_\nu p_\rho p_\sigma -$ $\frac{10\mathcal{D}^{\mu\nu\rho\sigma}_{AB}\mathcal{D}^{\lambda}{}_{\lambda}{}^{\delta}{}_{\delta}{}^{BA}p_\mu p_\nu p_\rho p_\sigma}{64} - \frac{3p^2 p_\mu p_\nu\mathcal{D}^{\mu\rho\sigma\lambda}_{AB}\mathcal{D}^{\nu}{}_{\rho\sigma\lambda}{}^{BA}}{8} -$ $\frac{9p^2 p_\mu p_\nu\mathcal{D}^{\mu\rho}{}_{\rho}{}^{\lambda}{}_{AB}\mathcal{D}^{\nu}{}_{\lambda}{}^{\sigma}{}_{\sigma}{}^{BA}}{16} + \frac{3p^2 p_\mu p_\nu\mathcal{D}^{\mu\nu\rho\sigma}_{AB}\mathcal{D}_{\rho\sigma}{}^{\lambda}{}_{\lambda}{}^{BA}}{4} +$ $\frac{3p^2 p_\mu p_\nu\mathcal{D}^{\mu\nu\rho}{}_{\rho AB}\mathcal{D}^{\sigma}{}_{\sigma}{}^{\lambda}{}_{\lambda}{}^{BA}}{16} - \frac{3p^4\mathcal{D}^{\mu\nu\rho\sigma}_{AB}\mathcal{D}_{\mu\nu\rho\sigma}{}^{BA}}{128} -$ $\left.\frac{9p^4\mathcal{D}^{\mu}{}_{\mu}{}^{\rho\sigma}{}_{AB}\mathcal{D}_{\rho\sigma}{}^{\nu}{}_{\nu}{}^{BA}}{128} - \frac{9p^4\mathcal{D}^{\mu}{}_{\mu}{}^{\nu}{}_{\nu AB}\mathcal{D}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma}{}^{BA}}{1024}\right)$

Table B.2: Bubble loops for the graviton field.

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