SISSA
Scuola
Internazionale
Superiore di
Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Formulae of Jeffrey-Kirwan type in enumerative geometry

Candidate:
Riccardo Ontani

Advisor:
Jacopo Stoppa

Academic Year 2023-24

MA PER SEGUIR VIRTUTE E CANOSCENZA • SISSA •

Abstract

This thesis explores various aspects of the Jeffrey-Kirwan localisation formula, a powerful tool in computing integrals on quotients of smooth varieties by reductive group actions. Initially developed by Jeffrey and Kirwan in the symplectic context, this formula has seen various adaptations and extensions scattered throughout the literature. The primary goal of this thesis is to provide a fully algebraic proof of the Jeffrey-Kirwan localisation formula, building on the work of Lerman, Guillemin, and Kalkman. Furthermore, the thesis extends the formula to the equivariant setting, enabling the computation of equivariant integrals with respect to additional torus actions on the quotient. It also aims to clarify the relations among different versions of this formula found in the literature. In addition, the thesis explores some applications of these localisation techniques, specifically in deriving residue formulae for virtual invariants of critical loci in quotients of linear spaces, such as quiver varieties.

Contents

Chapter 1

Introduction.

1.1 A brief history of the subject.

There are many techniques that, in the literature, go under the name of *Jeffrey-Kirwan localisation formula*. In essence, they are all formulae to compute integrals on quotients of smooth varieties by reductive group actions. The original version of this formula, proved by Jeffrey and Kirwan in [JK95], lives in the symplectic category and it's of a very analytic nature: it expresses the integral on the quotient in terms of some inverse Laplace transforms of functions defined on the Lie algebra of the group. This version found a great application in the work [JK98], where the same authors computed intersection numbers on the moduli space of stable vector bundles on Riemann surfaces of genus $g \geqslant 2$.

Later, Brion and Vergne in [BV99] expressed the inverse Laplace transforms appearing in the original localisation formula in terms of a newly defined linear operator, which they called *Jeffrey-Kirwan residue*. Using this result Szenes and Vergne [SV04] proved an important variation of the formula in the case where the variety of interest is the quotient of a linear space by a torus action (namely a toric variety). In this case, they expressed the Jeffrey-Kirwan residue in a combinatorial way, in terms of iterated residues computed with respect to flags in the dual Lie algebra of the torus. This version achieved a lot of success by being the key ingredient in the proof of the toric mirror symmetry conjecture of Batyrev and Materov [BM02], given by Szenes and Vergne in the same paper. At this point of the story, there was no purely algebraic way of proving these localisation formulae, which relied on deep analytic results.

In a parallel direction, Lerman [Ler95] studied an interesting construction in symplectic geometry: the *symplectic cut*. Using this tool, he noticed that one could produce a formula for computing integrals on the quotient of a circle action in terms of residues of functions defined on the Lie algebra of the circle. This construction was iterated by Guillemin and Kalkman in [GK96] to produce residue formulae for integrals on quotients of symplectic varieties by Hamiltonian actions of tori. The case of a nonabelian reductive connected group G can be reduced to the case of its maximal torus T by a result of Martin [Mar00]: this expresses integrals over the G-quotient in terms of integrals over the T-quotient. This construction is much more geometric and can be easily translated into algebraic terms by simply replacing symplectic reduction with geometric invariant theory.

First aim of the thesis.

We follow this second path initiated by Lerman to give a fully algebraic proof of this Jeffrey-Kirwan localisation formula. The first step, where the group is $\mathbb{C}^*$, was already considered by Edidin and Graham in [EG98a]. We will also show how to relate this version of the localisation formula to the one described by Szenes and Vergne. This is the content of Section 4. Let's quickly discuss the content of this formula in the version of Szenes and Vergne in the case of a torus action (for a more precise description, see Section 4.5).

Given a torus T acting on a linear space V we can consider the weights of the action, namely the set $\mathfrak{A}$ of characters $\rho \in \chi(T)$ that appear as eigenvalues of the action. Given a suitably regular linearisation ξ, an equivariant cohomology class $\alpha \in A_T^*(V)$ defines a cohomology class $r(\alpha)$ on the quotient $V/\!/T$ and we are interested in computing $\int_{V/\!/T} r(\alpha)$. Since the equivariant cohomology ring $A_T^*(V)$ can be identified with the ring of polynomial functions on $\chi(T)^\vee$, we can think of the fraction $\frac{\alpha}{e^T(T_V)}$ as of a rational function on $\chi(T)^\vee$. The formula of Szenes and Vergne reads

$$\int_{V/\!/T} r(\alpha) = \mathrm{JK}_\xi^{\mathfrak{A}}\left(\frac{\alpha}{e_T(T_V)}\right)$$

where $\mathrm{JK}_\xi^{\mathfrak{A}}$ is a residue operation which essentially uses the linearisation ξ to select some "stable" ordered bases from $\mathfrak{A}$, which are then used to compute iterated residues (see Section 4.1.6 for more details).

1.2 The equivariant version.

In many cases, interesting moduli spaces can be built as quotients of the form $V/\!/G$, where V is a representation of a reductive connected group G. These moduli spaces are often noncompact, and one wants to compute *equivariant integrals* with respect to an additional torus action on $V/\!/G$ having compact fixed locus. This is the motivation behind:

The second aim of the thesis.

We prove a version of the JK localisation formula (in the form of Szenes and Vergne) that works in this equivariant context. This will be done in Section 5. Let's quickly discuss how this is different from the nonequivariant formula (for a more precise description see Section 5.1.5). If an additional action of a torus S on V is added to the picture, we can consider the corresponding weights of the total $(T \times S)$-action. They are the couples $(\rho, \nu) \in \chi(T) \times \chi(S)$ that appear as eigenvalues of the action. For every value $s \in \chi(S)^\vee$ of the equivariant parameter of the S-action, we can consider the hyperplane arrangement $\mathcal{H}_s$ in $\chi(T)^\vee$ given by the hyperplanes $\{\rho + \nu(s) = 0\}$. The equivariant version of the Szenes-Vergne formula will read

$$\int_{V/\!/T} r(\alpha)(s) = \sum_{\substack{P \text{ zero dimensional} \\ \text{intersection in } \mathcal{H}_s}} \mathrm{JK}_{\xi,P}^{\mathfrak{A}_P}\left(\frac{\alpha}{e^{T\times S}(T_V)}\right)$$

where $\mathfrak{A}_P$ is a subset of $\mathfrak{A}$ depending on P and the additional subscript P in the JK residue denotes the fact that we are computing such residue at the point P and not at the origin of $\chi(T)^\vee$. To be precise, the sum is not over all the zero dimensional intersections in $\mathcal{H}_s$ but only over some "stable" ones selected by the linearisation ξ.

1.3 Formulae from physics.

As we discussed, these formulae have found successful applications in several areas of mathematics, but the field in which they experienced the most popularity is theoretical physics. For example, Benini, Hori, Eager and Tachikawa in [Ben+15] recovered formulae for computing integrals over complete intersections in GIT quotients of linear spaces, such as products of Grassmannians. Later, Beaujard, Mondal and Pioline [BMP19] and Córdova and Shao [CS16] applied these formulae to study invariants of critical loci in quiver varieties. Other applications came from the work of Nekrasov and Piazzalunga [NP19], where they used these localisation techniques to compute (for low values of n) virtual invariants of the Hilbert scheme of n points on $\mathbb{A}^4$.

The third aim of the thesis.

We provide an algebro-geometric proof of the formulae of [Ben+15], [BMP19] and [CS16] appearing in the physics literature, and to discuss the application of JK localisation to the case of $\mathrm{Hilb}^n(\mathbb{A}^4)$. This will be the content of Section 6.

Let's informally summarise the content of the main result (for more details see Section 6.1.4). Assume we are given a quotient of the form $V /\!/ G$ together with an additional action of a torus S and an S-equivariant superpotential $\varphi : V /\!/ G \to \mathbb{C}$. Let $X := V(d\varphi)$ be the critical locus of φ. This carries a natural S-equivariant perfect obstruction theory, from which one can define S-equivariant invariants, such as the DT invariant, the virtual Hirzebruch genus and the virtual elliptic genus, via virtual localisation if the fixed locus X^S is proper. We will describe three meromorphic functions Z_{DT}, Z_χ and Z_{Ell} on $\chi(T \times S)^\vee_{\mathbb{C}}$ so that the invariants above can be computed from these functions by extracting residues:

$$\mathrm{DT}(X)(s) = \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P}\left(Z_{\mathrm{DT}}(-,s)\right),$$

$$\mathrm{ch}^S \chi(X)(2\pi i s) = \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P}\left(Z_{\chi}(-,s)\right),$$

$$\mathrm{ch}^S \mathrm{Ell}(X)(e^{2\pi i \tau})(2\pi i s) = \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P}\left(Z_{\mathrm{Ell}}(-,s,\tau)\right),$$

where W is the Weyl group of G. This result will be later specialised to the case where $V /\!/ G$ is a quiver variety in Section 6.2.4.

About the Hilbert scheme of points in $\mathbb{A}^4$, we will recall its construction as a vanishing locus of a section of a vector bundle in a quotient of the form $V /\!/ G$, via the generalised Atiyah-Drinfeld-Hitchin-Manin construction [Nek20]. By using this presentation, we will push the computation of its invariants to the smooth ambient space $V /\!/ G$ and write the JK formula for these quantities. For example, the $(\mathbb{C}^*)^4$-equivariant integral of 1 over the virtual class of $\mathrm{Hilb}^n(\mathbb{A}^4)$ can be extracted from the function

$$\left(\frac{\epsilon_{12}\epsilon_{13}\epsilon_{23}}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}\right)^n \prod_{i \neq j} \frac{(u_1 - u_j)\prod_{1 \leqslant a < b \leqslant 3}(u_i - u_j + \epsilon_{ab})}{\prod_{c=1}^4 (u_i - u_j + \epsilon_c)} \prod_{k=1}^n \frac{1}{u_k}$$

by taking JK residues with respect to the u variables. This is far from being an explicit computation of the invariants (which has already been achieved by Kool and Rennemo) but recovers an intermediate formula used by Nekrasov and Piazzalunga, clarifying a little bit the picture.

Chapter 2

Equivariant intersection theory.

In this section we review some of the core aspects of equivariant intersection theory, as masterfully described by Edidin and Graham in their sequence of works [EG98b; EG98c; Edi10; EG99]. The main ideas in this field come from merging Fulton's approach to Chow groups, key to doing intersection theory in algebraic geometry, with the classical theory of equivariant cohomology originally developed in algebraic topology. In this introduction we quickly recall some features of these two fields to motivate some of the constructions that will be discussed in this section.

In his book on intersection theory [Ful13], Fulton develops a purely algebro-geometric approach to homology and cohomology in the setting of schemes. The Chow groups, the analogues of the classical singular homology groups, of a variety X are generated by subvarieties up to a notion of continuous deformation called rational equivalence. As long as the variety is smooth, these groups carry a canonical ring structure called *intersection product*. One of the aims of enumerative geometry is, in very abstract and imprecise terms, to study the results of multiplications of interesting classes in the Chow groups.

In the topological world, this corresponds to computing integrals over smooth manifolds and equivariant cohomology has proven many times to be an extremely useful tool (see for example this paper by Atiyah and Bott [AB84]) for this kind of computations. The equivariant cohomology of a topological space X endowed with a G-action is defined as the singular cohomology of $X \times_G EG$, where EG is the infinite dimensional *classifying principal G-bundle*, whose quotient $BG = EG/G$ is called *classifying space* of G.

The infinite dimensionality of EG is the main obstruction to directly defining the equivariant Chow groups as Chow groups of the mixed product $X \times_G EG$, since this classifying bundle is not defined in the category of schemes. Equivariant Chow groups are instead defined in [EG98b] by using finite dimensional approximations of EG by schemes due to Totaro [Tot99]. In this section we will discuss how these definitions allow to use equivariant techniques to do intersection theory on a variety with a group action.

Contents of the section:

The structure of this section is organized as follows:

- We recall the definition of the equivariant Chow groups by finite dimensional approximations of EG.
- We introduce the change of group homomorphism $A^G(X) \to A^H(X)$ associated to a group homomorphism $H \to G$. We prove that this morphism is well defined and collect some of its basic properties.
- We go through some of the most important points of [EG98b]. We especially focus on the description of the Chow ring of a quotient by G in terms of the G-equivariant Chow group of the original variety. Additionally, we will discuss an equivariant version of this description.
- We will recall some key concepts in geometric invariant theory and adapt the results about the Chow rings of quotients from the previous sections to the case where the quotient is constructed through GIT.
- We discuss the analogous properties of equivariant Chow cohomology.
- We recall the content of the Atiyah-Bott localisation formula, originally developed in [AB84], and present it within the context of Chow groups as described in [EG98c].
- We recall the basics of equivariant K-theory.

2.1 Equivariant Chow groups.

In this part we recall the definitions of equivariant Chow groups and their basic properties. We also discuss some interesting morphisms that will be useful in the study of Chow groups of quotients.

Notation. Many properties of these groups that we are going to explore only hold true when the coefficients are taken in $\mathbb{Q}$. This is the reason why, from now on, all the Chow groups, Chow rings and K-groups are with coefficients in the field $\mathbb{C}$. We could use $\mathbb{Q}$ but at some point we will want to work with meromorphic functions, so we take $\mathbb{C}$ since the beginning. To make everything more readable, when working with Chow groups we will omit the star subscript by simply writing $A^G(X)$ for $A^G_*(X)$. Analogously, in the case where G is trivial, we will write $A(X)$ for $A_*(X)$. When we refer to the cohomology $A^*_G(X)$ of X (see section 2.3), we will always write the star so that no ambiguity should appear.

Consider an n-dimensional quasiprojective variety X over $\mathbb{C}$ together with the action of a reductive connected algebraic group G of dimension g.

2.1.1 Equivariant Chow homology.

Edidin and Graham construct in [EG98b] the G-equivariant Chow groups of X. Here we review the construction, which goes by approximation of the classifying bundle EG by schemes. Fixed $k > 0$ we can consider a representation V of G, having dimension v and admitting an open subset $U \subset V$ where the G-action is free and $V \backslash U$ has codimension bigger than $n - k$. Then the k-th G-equivariant Chow (homology) group of X is defined as

$$A^G_k(X) := A_{k+v-g}\left(\frac{X \times U}{G}\right)$$

and it's independent on V up to canonical isomorphism (Definition-Proposition 1 in [EG98b]).

Remark 1. Everywhere in this thesis we will denote with $X_{G,l}$ the scheme $(X \times U)/G$, where we assume that U is an open subset of a representation V over which the action of G is free and such that the codimension of $V \backslash U$ is bigger than l. Thus by definition $A^G(X) = A(X_{G,n-k})$.

These equivariant Chow groups enjoy the same functoriality properties (with respect to G-equivariant maps) of ordinary Chow groups. Notice that G-invariant k-dimensional subvarieties $Z \subseteq X$ induce degree k equivariant homology classes: we define $[Z]_G$ to be the class of the subvariety $Z_{G,n-k} \subseteq X_{G,n-k}$. In particular, if X is irreducible, then it possesses an equivariant fundamental class $[X]_G \in A^G_n(X)$.

Example 2.1.1 (Chow groups of points.)**.** Given an algebraic reductive group G we will denote with

$$\chi(G) := \mathrm{Hom}(G, \mathbb{C}^*) \quad , \quad \chi(G)^\vee := \mathrm{Hom}(\mathbb{C}^*, G)$$

the lattices of *characters* and *cocharacters* of G (we will write $\chi(G)_{\mathbb{C}}$ and $\chi(G)_{\mathbb{C}}^{\vee}$ for the corresponding $\mathbb{C}$-linear spaces obtained by tensoring with the field of complex numbers). If $T \subseteq G$ is a maximal subtorus, we can consider the Weyl group $W := N(T)/T$ acting on T. Given the trivial G-action on a point, Edidin and Graham [EG98b, Section 3.2] prove that the equivariant Chow group of a point with complex coefficients is

$$A^G(\mathrm{pt}) \simeq \mathrm{Sym}(\chi(T)_{\mathbb{C}})^W$$

where the second ring can be interpreted as the ring of Weyl-invariant polynomial functions on $\chi(T)_{\mathbb{C}}^{\vee}$.

2.1.2 Change of groups in Chow homology.

We shall now describe how the Chow homology changes when we change groups via an homomorphism $H \to G$. Assume that G acts on X and that H acts on X via the group homomorphism above. Fixed k consider a representation V of G having an open subset $U \subset V$ so that the complement $V \backslash U$ has codimension greater than $n-k$ in V and G acts freely on U. Analogously consider a representation V' of H having $U' \subset V'$ so that H acts freely on U' and $V' \backslash U'$ has codimension greater than $n-k$. Clearly H acts on U too and we have the morphism

$$\Phi : \frac{X \times U \times U'}{H} \to \frac{X \times U}{G} \tag{2.1}$$

induced from the projection $X \times U \times U' \to X \times U$. Notice that $V \times V'$ is a representation of H and $U \times U'$ is an open subset over which H acts freely. The complement of this open set is $(V \backslash U) \times V' \cup V \times (V' \backslash U')$ which has codimension greater than $n-k$. Since both the composition

$$X \times U \times U' \xrightarrow{H\text{-quotient}} \frac{X \times U \times U'}{H} \xrightarrow{\Phi} \frac{X \times U}{G}$$

and the H-quotient map are faithfully flat (being quotients by free group actions) we get that Φ is flat, hence we can consider the pullback morphism.

Definition 2.1.1. The pullback

$$(H \to G)^* : A_k^G(X) \to A_k^H(X).$$

through the morphism Φ described above is called *change of group homomorphism.*

Similar change of group homomorphisms have been considered by Krishna [Kri14, Section 2.2] in the case where H is a subgroup of G. The proof of the following statement, which ensures that this morphism doesn't depend on the choice of V and V', is just a routine check.

Lemma 2.1.1. *The morphism $(H \to G)^*$ defined above is independent of the choice of V and V'.*

Proof. Consider two other such representations:

- $G \curvearrowright \tilde{V}$ with an invariant open subset $\tilde{U}$ such that G acts freely on $\tilde{U}$ and $\operatorname{codim}_{\tilde{V}}(\tilde{V} \backslash \tilde{U}) > n - k$.
- $H \curvearrowright \tilde{V}'$ with an invariant open subset $\tilde{U}'$ such that H acts freely on $\tilde{U}'$ and $\operatorname{codim}_{\tilde{V}'}(\tilde{V}' \backslash \tilde{U}') > n - k$.

Then we construct a further couple of representations:

- $G \curvearrowright V \times \tilde{V}$. This has an invariant open subset W, containing both $V \times \tilde{U}$ and $U \times \tilde{V}$, such that G acts freely on W. Moreover the codimension of the complement of W in $V \times \tilde{V}$ is clearly greater than $n - k$.
- $H \curvearrowright V' \times \tilde{V}'$. This has an invariant open subset W', containing both $V' \times \tilde{U}'$ and $U' \times \tilde{V}'$, such that H acts freely on W'. Moreover the codimension of the complement of W' in $V' \times \tilde{V}'$ is clearly greater than $n - k$.

We can now relate the change of group homomorphisms defined through V, V' to the ones defined via the two representations we just defined. The corresponding morphisms of varieties fit into the commutative diagram

$$\begin{tikzcd}
\frac{X \times W \times W'}{H} \arrow[r] & \frac{X \times W}{G} \\
\frac{X \times U \times U' \times \tilde{V} \times \tilde{V}'}{H} \arrow[r] \arrow[u, hook] \arrow[d, "\text{vector bundle}"] & \frac{X \times U \times \tilde{V}}{G} \arrow[u, hook] \arrow[d, "\text{vector bundle}"] \\
\frac{X \times U \times U'}{H} \arrow[r] & \frac{X \times U}{G}
\end{tikzcd}$$

Notice that the vertical open embeddings in the first diagram induce isomorphisms of the Chow groups of the relevant dimensions; for example the arrow on the right induces via pushforward an isomorphism

$$A_{k+\dim(W)-g}\left(\frac{X \times U \times \tilde{V}}{G}\right) \simeq A_{k+\dim(W)-g}\left(\frac{X \times W}{G}\right)$$

since the codimension of the complement of $U \times \tilde{V}$ in W is greater than $n-k$. By composing with the flat pullbacks coming from the two vertical maps in the second diagrams we obtain the canonical identifications of the Chow groups discussed by Edidin and Graham in [EG98b, Definition/Proposition 1]. For example, for the two vertical arrows on the right of the diagram we obtain the isomorphism

$$A_{k+\dim(V)-g}\left(\frac{X \times U}{G}\right) \simeq A_{k+\dim(W)-g}\left(\frac{X \times W}{G}\right)$$

which is the canonical identification of the G-equivariant Chow groups of X defined through the representations V and $V \times \tilde{V}$. This shows that the change of group homomorphism defined by V and V' is related to the change of group morphism defined through $V \times \tilde{V}$ and $V' \times \tilde{V}'$ by the canonical isomorphisms between the equivariant Chow groups of X defined through these representations. This argument can be applied reversing the roles of (V, V') and $(\tilde{V}, \tilde{V}')$ completing the proof. □

The following results are an immediate consequence of the definition and of functoriality of flat pullbacks:

Lemma 2.1.2. *Given a group homomorphism $H \to G$, the following statements hold true:*

1. *(functoriality in the groups) for every other group homomorphism $K \to H$, the change of group homomorphisms satisfy $(K \to H)^* \circ (H \to G)^* = (K \to G)^*$.*
2. *(functoriality in the varieties) Given a G-equivariant flat morphism of varieties $f : Y \to X$, then $(H \to G)^* \circ f^* = f^* \circ (H \to G)^*$.*
3. *Given a G-invariant subvariety Z of X, then $(H \to G)^* ([Z]_G) = [Z]_H$.*

In particular we can study the case where X is a point:

Example 2.1.2. Given a group homomorphism $h : H \to G$, consider the trivial action of G on a point. The induced change of group homomorphism

$$(H \to G)^* : A^G(\mathrm{pt}) \to A^H(\mathrm{pt})$$

is, if we denote with $T_H \subseteq H$ and $T_G \subseteq G$ the maximal subtori and with W_H, W_G the Weyl groups, the morphism

$$\mathrm{Sym}\,(\chi(T_G)_{\mathbb{C}})^{W_G} \to \mathrm{Sym}\,(\chi(T_H)_{\mathbb{C}})^{W_H}$$

that sends the function $f : \chi(T_G)^{\vee}_{\mathbb{C}} \to \mathbb{C}$ to the function $g : \chi(T_H)^{\vee}_{\mathbb{C}} \to \mathbb{C}$ given by $g(\mu) := f(\phi \circ \mu)$ for every $\mu \in \chi(T_H)^{\vee}$.

Example 2.1.3. A particular example that will be useful later is the following. Assume we have a m-dimensional torus $\mathbb{T}$ acting on a point and m distinct rank 1 subtori $\lambda_i \subseteq \mathbb{T}$ so that $\lambda_1 \times \cdots \times \lambda_m \to \mathbb{T}$ is a surjection (necessarily with finite kernel). In this case the λ_i form a $\mathbb{C}$-basis of $\chi(\mathbb{T})_{\mathbb{C}}$ and the induced change of group homomorphism is

$$\mathrm{Sym}\,(\chi(\mathbb{T})_{\mathbb{C}}) \to \mathbb{C}[s_1, \ldots, s_m]$$

given by considering $s_1, \ldots, s_m$ as the elements of the dual basis to λ. Concretely, it sends a character ϕ into $\sum_{i=1}^{m} \langle \lambda_i, \phi \rangle s_i$.

2.1.3 Chow groups of quotient stacks.

In this section we recall the properties of Chow homology of Deligne-Mumford quotient stacks following the works of Edidin-Graham.

Notation. In this section the distinction between Chow groups with integral and rational (or complex) coefficients is important to appreciate so, only for this section, we will denote with $A(X)$ the integral Chow group of X.

Assume that G is a reductive algebraic group acting on a quasiprojective variety X so that the action is locally proper.

Definition 2.1.2. The action of a reductive group G on a scheme X is called *proper* if the action map $G \times X \to X \times X$ is proper. It's said to be *locally proper* if there is an invariant open cover of X so that for every open U in the cover the action $G \curvearrowright U$ is proper.

Remark 2. Here are some useful facts:

1. Since G is reductive then it is affine by definition, hence if the action is locally proper then *the stabilisers are finite.*

2. Notice that if the G-action is free then $G \times X \to X \times X$ is a closed embedding, hence the action is proper.

3. The action of G is locally proper if X admits an affine invariant open cover (see [EG98b, remark 3, pag. 18]). In particular, this is the case if X is the semistable locus for a linearised action on a closed subvariety of $\mathbb{P}^a \times \mathbb{A}^b$, if the semistable locus coincides with the stable locus. This is the case we are going to study later.

Definition 2.1.3. Given a G-invariant subvariety Z of X we denote with $\sigma_G(Z)$ the order of the G-stabiliser at a general point of Z.

The quotient stack $\mathcal{X} := [X/G]$ is a Deligne-Mumford stack and its integral Chow group is the equivariant Chow group of X [EG98b, Section 5.3]:

$$A_*(\mathcal{X}) = A^G_{*+g}(X).$$

Let M be a geometric quotient (in the sense of [MFK94]) for the action of G on X. Then M is a coarse moduli scheme for the stack $\mathcal{X}$ by [EG98b, Corollary 4.26] (and the converse is also true by [EG98b, Corollary 4.33]). The following important result of Edidin and Graham ([EG98b, Theorem 3] and [Edi10, Proposition 4.42]) allows to relate the Chow homology of $\mathcal{X}$ with the one of its coarse moduli space:

Theorem 2.1.1. *Assume that G is a reductive algebraic group acting on a quasiprojective variety X so that the action is locally proper and let M be a geometric quotient. There is an isomorphism of graded $\mathbb{Q}$-linear spaces*

$$\hat{\pi}_G : A_*(M)_{\mathbb{Q}} \xrightarrow{\sim} A^G_{*+g}(X)_{\mathbb{Q}}. \tag{2.2}$$

Consider the quotient map $\pi : X \to M$. Then the isomorphism $\hat{\pi}_G$ maps the class of a subvariety $[V]$ into the class $\sigma_G(\pi^{-1}(V)) \cdot [\pi^{-1}(V)]_G$. In particular $\hat{\pi}_G[M] = \sigma_G(X)[X]_G$.

Remark 3. Notice that a DM stack doesn't admit a scheme as a coarse moduli space in general. Here we are assuming $\mathcal{X}$ does.

A simple manipulation of this result shows that it also holds equivariantly. This is basically the content of [Kri13, Proposition 3.1] which we prove here to remain a bit more self-contained and to establish notation.

Theorem 2.1.2. *Assume that G is a reductive group of dimension g acting in a locally proper way on a quasiprojective variety X and let $\pi : X \to M$ be a geometric quotient. Let H be another reductive group of dimension h acting on X and commuting with G, so that the action descends to the quotient M. Then there is an isomorphism of graded $\mathbb{Q}$-linear spaces*

$$\hat{\pi}_{G\times H,H} : A^H_*(M)_{\mathbb{Q}} \xrightarrow{\sim} A^{G\times H}_{*+g}(X)_{\mathbb{Q}}$$

Given three groups G, H, K acting on X so that

1. *The action of* $G \times H$ *is locally proper,*

2. G, H, K *commute with each other,*

3. *there is a geometric quotient* $X \to M$ *for the* G*-action and one* $M \to N$ *for the induced* H*-action, so that together they give a geometric quotient* $X \to N$ *for the* $(G \times H)$*-action,*

then the compatibility condition

$$\hat{\pi}_{G\times H\times K, H\times K} \circ \hat{\pi}_{H\times K, K} = \hat{\pi}_{G\times H\times K, K}$$

holds true. Moreover $\hat{\pi}_{G,1} = \hat{\pi}_G$ *as described in the previous Theorem 2.1.1.*

Proof. All the Chow groups in this proof are with rational coefficients. First of all we define $\hat{\pi}_{G\times H,H}$. Fixed $k \in \mathbb{N}$ we can pick a representation V_H of H having an open subset U_H over which H acts freely and such that the codimension of $V_H \backslash U_H$ is greater than $n-k$. Then, by the definition of equivariant Chow groups, we notice that $A_j^{G\times H}(X) \simeq A^{G\times H}_{j+\dim(V_H)}(X \times U_H)$ for all j. Now the action of $G\times H$ on $X \times U_H$ is with finite stabilisers and we have a geometric quotient in $\frac{M\times U_H}{H}$, which by Theorem 2.1.1 gives the isomorphism appearing as the lower horizontal arrow in the diagram

$$\begin{tikzcd}
A_k^H(M) \arrow[rr, "\hat{\pi}_{G\times H,H}"] \arrow[d, "\text{canonical iso}"'] & & A^{G\times H}_{k+g}(X) \arrow[d, "\text{flat pullback}"] \\
A_{k+\dim(U_H)-h}\left(\frac{M\times U_H}{H}\right) \arrow[r, "\hat{\pi}_{G\times H}"] & A^{G\times H}_{k+\dim(U_H)+g}(X \times U_H) &
\end{tikzcd} \tag{2.3}$$

We define $\hat{\pi}_{G\times H,H}$ to fit in the diagram above, which makes sense since all the other arrows are isomorphisms. It's easy to show that this doesn't depend on the choice of the representation V_H and that $\hat{\pi}_{G,1} = \hat{\pi}_G$. First we prove the compatibility in the case where $K = 1$, where the relation we want to prove reads

$$\hat{\pi}_{G\times H,H} \circ \hat{\pi}_H = \hat{\pi}_G. \tag{2.4}$$

Here we write X/G for M and $X/(G \times K)$ for N to make the argument easier to follow. Consider the following diagram

$$\begin{tikzcd}
 & A^{G\times H}(X) \arrow[r, "\substack{\text{flat}\\ \text{pullback}}"] & A^{G\times H}(X\times U_H) \\
A(\frac{X}{G\times H}) \arrow[ur, "\hat{\pi}_{G\times H}"] \arrow[dr, "\hat{\pi}_H"'] & & \\
 & A^H(X/G) \arrow[uu, dashed, "\hat{\pi}_{G\times H}"] \arrow[r, "\substack{\text{canonical}\\ \text{iso}}"] & A(\frac{(X/G)\times U_H}{H}) \arrow[uu, "\hat{\pi}_{G\times H}"']
\end{tikzcd}$$

where the right square, whose arrows are isomorphisms, commutes by definition, while we want to prove the commutativity of the left triangle. We can do this by proving the commutativity of the full diagram, and we start by giving names to the corresponding morphisms

$$\begin{tikzcd}
 & X \arrow[dl, "\pi_{G\times H}"] & X\times U_H \arrow[l] \arrow[dd, "p"] \\
\frac{X}{G\times H} & & \\
 & X/G \arrow[ul, "\pi_H"] & \frac{(X/G)\times U_H}{H}
\end{tikzcd}$$

Now notice that by the explicit description of the maps $\hat{\pi}$ given in Theorem 2.1.1, we are left to proving that given a subvariety W of $X/(G\times H)$ the following equality holds true:

$$p^{-1}\left(\frac{\pi_H^{-1}(W)\times U_H}{H}\right) = \pi_{G\times H}^{-1}(W)\times U_H,$$

which is obvious. The general case, with $K\neq 1$, follows by considering the action $G\times H\times K \curvearrowright X\times U_K$ and applying (2.4) for the groups G, $H\times K$. $\square$

Remark 4. Notice that, if X is smooth, this result endows the equivariant Chow ring of the possibly singular geometric quotient M with a product induced by $\hat{\pi}_{G\times H,H}$. We will call this the *stacky ring structure* on $A^H(M)$. Notice this is not intrinsic of M but depends on its presentation as a geometric quotient.

The maps we just discussed will play a fundamental role in what follows, so we give them a name:

Definition 2.1.4. Assume that $G \times H$ acts on X so that G acts with finite stabilisers and let M be a geometric quotient for the G-action. The morphism $\hat{\pi}_{G\times H,H}$ of Theorem 2.1.2 is called *H-equivariant ascent map* for the action. Its inverse, denoted with $\hat{d}_{G\times H,H}$, is called *descent map.*

These maps satisfy many good properties. For example they behave well with respect to proper pushforwards:

Proposition 2.1.1. *Let X, Y be quasiprojective varieties with a $G \times H$ action so that G acts in a locally proper way. Assume they admit geometric quotients $X \to M$, $Y \to N$ and consider a proper G-equivariant morphism $f : X \to Y$ inducing a proper morphism $\overline{f} : M \to N$. The following diagram commutes:*

$$\begin{tikzcd}
A_*^{G\times H}(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A_*^{G\times H}(Y)_{\mathbb{Q}} \\
A_*^{H}(M)_{\mathbb{Q}} \arrow[r, "\overline{f}_*"] \arrow[u, "\hat{\pi}_{G\times H,H}"] & A_*^{H}(N)_{\mathbb{Q}}. \arrow[u, "\hat{\pi}_{G\times H,H}"']
\end{tikzcd}$$

Proof. This is the content of Proposition 11 in [EG98b], which can be extended to the H-equivariant case by applying it to the products $X \times U_H$ and $Y \times U_H$. □

They are also compatible with the change of group homomorphisms:

Proposition 2.1.2. *Assume that G is a reductive group acting on a quasiprojective variety X in a locally proper way and let $\pi : X \to M$ be a geometric quotient. Let H be another reductive group of dimension h acting on X and commuting with X, so that the action descends to the quotient M. Given a group homomorphism $K \to H$ the following relation holds true*

$$\hat{\pi}_{G\times K,K} \circ (G \times K \to G \times H)^* = (K \to H)^* \circ \hat{\pi}_{G\times H,H}.$$

Proof. The proof follows by the commutativity of the diagram

$$\begin{tikzcd}
A\left(\frac{(X/G)\times U_H}{H}\right)_{\mathbb{Q}} \arrow[r, "\hat{\pi}_{G\times H}"] \arrow[dd, "\Phi^*"] & A^{G\times H}(X \times U_H)_{\mathbb{Q}} \arrow[d, "(G\times K\to G\times H)^*"] \\
 & A^{G\times K}(X \times U_H)_{\mathbb{Q}} \arrow[d, "\text{flat pullback}"] \\
A\left(\frac{(X/G)\times U_H\times U_K}{K}\right)_{\mathbb{Q}} \arrow[r, "\hat{\pi}_{G\times K}"] & A^{G\times K}(X \times U_H \times U_K)_{\mathbb{Q}}
\end{tikzcd}$$

where Φ is just the flat morphism (2.1) used to define the change of group homomorphism. The commutativity is immediately checked by using the explicit description of the morphisms $\hat{\pi}$ given by Theorem 2.1.1 and the description of the change of group homomorphism given in point 3 of Lemma 2.1.2. By fitting this diagram in the middle of the two diagrams (2.3) defining $\hat{\pi}_{G\times H,H}$ and $\hat{\pi}_{G\times K,K}$ we immediately see that the composition of the vertical arrows on the left is $(K \to H)^*$, while the composition of the vertical arrows on the right is $(G \times K \to G \times H)^*$ by using point 1 in Lemma 2.1.2. □

Finally, we can use the descent isomorphism $\hat{d}_{G\times H,H}$ to define the degree operation on DM stacks of the form $\mathcal{X} = [X/G]$:

Definition 2.1.5. Assume that G is a reductive group acting in a locally proper way on a quasiprojective variety X and let $\pi : X \to M$ be a proper geometric quotient. Let H be another reductive group of dimension h acting on X and commuting with X, so that the action descends to the quotient M. Given a class $z \in A^H_*(\mathcal{X}) = A^{G\times H}_{*+g}(X)$ we define its *H-equivariant degree* as $\deg_{\mathcal{X}}(z) := \deg(\hat{d}_{G\times H,H}(z))$.

2.2 Some geometric invariant theory.

References for more background on GIT are the original book [MFK94] by Mumford, Fogarty and Kirwan and the notes [Hos15] of Hoskins and [Tho05] of Thomas, which also explore the relations with symplectic reduction.

Consider a reductive algebraic group G acting on a quasiprojective variety X. A G-equivariant line bundle $\mathcal{L}$ on X is called a *linearisation*.

Definition 2.2.1. A point $x \in X$ is called *semistable* if there is an invariant section $s \in H^0(X, \mathcal{L}^{\otimes n})^G$ for some $n > 0$ so that $s(x) \neq 0$ and the open subscheme $\{s \neq 0\}$ is affine. It is called *stable* if the stabiliser G_x is finite and there is such invariant section s so that $s(x) \neq 0$, $\{s \neq 0\}$ is affine and $G \cdot x$ is closed in $\{s \neq 0\}$. We will denote the semistable locus with $X(G)^{\mathrm{ss}}$ and the stable locus with $X(G)^s$. If the semistable locus coincides with the stable locus we will say that the linearisation $\mathcal{L}$ is *regular*.

Notice that this condition on $\{s \neq 0\}$ being affine is always satisfied, for example, in the case of ample line bundles on projective varieties or for the trivial line bundle on an affine variety. Given such data, Mumford [MFK94] showed that there is a good quotient

$$X^{\mathrm{ss}} \to X/\!/G$$

with quasiprojective target (see for example [Hos15, Section 5.5]). We will only focus on the case in which X is a closed subvariety of $\mathbb{P}^a \times \mathbb{A}^b$ and the linearisation $\mathcal{L}$ is the restriction of $\mathcal{O}(1)$ carrying some equivariant structure. Notice that this bundle is ample, meaning all nonvanishing loci of sections are affine. We are now interested in a particular version of the Hilbert-Mumford numerical criterion for checking if a point $x \in X$ is semistable. As kindly pointed out by Johan Martens, this is a particular case of the main result of [GHH15]. We keep the proof here as it is quite brief and recaps the techniques that King used to prove the analogous result for affine varieties:

Proposition 2.2.1. *Assume that X admits a G-equivariant closed embedding in $\mathbb{P}^a \times \mathbb{A}^b$ for some $a, b \geqslant 0$. Let $\mathcal{O}(1)$ be a linearisation of the action on $\mathbb{P}^a$. A point $x \in X$ is semistable with respect to the pullback of $\mathcal{O}(1)$ if and only if, for every homomorphism $\lambda : \mathbb{C}^\times \to G$ so that the limit $\overline{x} := \lim_{t\to 0} t \cdot x$ exists, the weight of the $\mathbb{C}^\times$-representation $\mathcal{O}(1)_{|\overline{x}}$ given by λ, denoted with $\langle \lambda, \mathcal{O}(1)_{|\overline{x}} \rangle$, is non-negative.*

Proof. Since the embedding of X in the ambient space is closed, in order to check the semistability of $x \in X$ we can check the semistability of $x \in \mathbb{P}^a \times \mathbb{A}^b$. Here the proof of King's analogous result for affine varieties [Kin94, Section 2] goes through without modifications. More precisely, let $\hat{x} \in \mathbb{A}^{a+1} \times \mathbb{A}^b$ be a point lying over x. From the definition and using the fact that G is geometrically reductive (more precisely [Hos15, Lemma 4.29]) it's easy to see that x is semistable if and only if $\overline{G \cdot \hat{x}}$ doesn't intersect $O \times \mathbb{A}^b$. By the fundamental theorem [Kem78, Theorem 1.4], saying that any closed G-invariant subset of a representation meeting the closure of a G-orbit also meets the closure of the orbit of a 1-parameter subgroup λ of G, we obtain that x is semistable if and only if, for every 1-parameter subgroup $\lambda \in \chi(G)^\vee$ so that $\lim_{t\to 0}\lambda(t) \cdot \hat{x}$ exists, then this limit is not in $O \times \mathbb{A}^b$. Then the conclusion follows as in the classical Hilbert-Mumford theorem, as we now show. Consider a point $x = (y, z) \in \mathbb{P}^a \times \mathbb{A}^b$ and pick a lift $\hat{x} = (\hat{y}, z) \in \mathbb{A}^{a+1} \times \mathbb{A}^b$.

- First we assume that x is semistable. Let λ be a 1-parameter subgroup and assume that the limit $\lim_{t\to 0} \lambda(t)(y, z)$ exists. Diagonalise the action on the projective space so that λ acts on $\mathbb{A}^{a+1}$ as $(t^{d_0} y_0, ..., t^{d_a} y_a)$. Consider the limit $\lim_{t\to 0}\lambda(t) \cdot \hat{y}$. We have three possibilities:

  1. This limit doesn't exist, which means that there is an index i so that $\hat{y}_i \neq 0$ and $d_i < 0$. Since the Hilbert-Mumford weight is the opposite of the minimum of the d_j so that $y_j \neq 0$, we find that the Hilbert-Mumford weight is positive.

2. This limit exist and it is nonzero, which means that there all indices i so that $\hat{y}_i \neq 0$ satisfy $d_i = 0$. Here we find that the Hilbert-Mumford weight is zero.
3. The limit exists and it is zero. This is impossible since otherwise the limit $\hat{x} = (\hat{y}, z)$ would be in $O \times \mathbb{A}^b$, contradicting the semistability of x by the first part of the proof.

- The converse is completely analogous.

□

Notice that this result is not true for general quasiprojective varieties. Luckily, the class of varieties we consider is closed under the operation of taking quotients:

Lemma 2.2.1. *Let T, S be two tori acting on $\mathbb{P}^a \times \mathbb{A}^b$ and consider a $T \times S$-equivariant structure on the pullback of $\mathcal{O}(1)$. The quotient $(\mathbb{P}^a \times \mathbb{A}^b)//T$ by T embeds into the product of a projective variety and an affine space.*

Proof. For these varieties, by definition the GIT quotient is built as

$$(\mathbb{P}^a \times \mathbb{A}^b)//T \simeq \operatorname{Proj}\left(\bigoplus_{n \geqslant 0} H^0(\mathbb{P}^a \times \mathbb{A}^b, \mathcal{O}(n))^T\right)$$

and the graded ring $\bigoplus_{n\geqslant 0} H^0(\mathbb{P}^a \times \mathbb{A}^b, \mathcal{O}(n))^T$ is finitely generated over $\mathbb{C}$, being the invariant functions on $\mathcal{O}(-1) \times \mathbb{A}^b$, or in other words invariant functions on the corresponding representation $\mathbb{A}^{a+1} \times \mathbb{A}^b$. In particular, this ring is finitely generated as an algebra over its degree zero part $H^0(\mathbb{A}^b, \mathcal{O}_{\mathbb{A}^b})^T$ and we can pick generators $s_0, \dots, s_n$ (we can even assume they are of the same degree, since Veronese subrings induce the same projective scheme). Then we have the following surjection given by evaluation of x_i at s_i

$$H^0(\mathbb{A}^b, \mathcal{O}_{\mathbb{A}^b})^T \otimes \mathbb{C}[x_0, \dots, x_n] \to \bigoplus_{n \geqslant 0} H^0(\mathbb{P}^a \times \mathbb{A}^b, \mathcal{O}(n))^T$$

which in turn induces the closed embedding of the quotient variety into the product $\mathbb{P}^n \times \operatorname{Spec}(H^0(\mathbb{A}^b, \mathcal{O}_{\mathbb{A}^b})^T)$. □

2.2.1 Kirwan maps.

Here we express the results on descent maps, especially those on the Chow groups of quotient varieties, in the framework of geometric invariant theory. Consider a reductive algebraic group G acting on a variety X via a closed embedding into $\mathbb{P}^a \times \mathbb{A}^b$, together with a regular linearisation $G \curvearrowright \mathcal{L}$. In particular, the action of G on the semistable locus $X(G)^{ss}$ is locally proper by point 3 in Remark 2.

Definition 2.2.2. The composition

$$r_G : A^G(X) \xrightarrow{i^*} A^G(X(G)^{ss}) \xrightarrow{\hat{d}_G} A(X/\!/G)$$

is called the *Kirwan map* for $(G, X, \mathcal{L})$. The second arrow is the descent map $\hat{d}_G = \hat{\pi}_G^{-1}$ of Definition 2.1.4.

Assume that an additional group H acts on X, commutes with G and extends to an action on the linearisation $\mathcal{L}$. Then we have the *H-equivariant Kirwan map*

$$r_{G\times H,H} : A^{G\times H}(X) \xrightarrow{i^*} A^{G\times H}(X(G)^{ss}) \xrightarrow{\hat{d}_{G\times H,H}} A^H(X/\!/G).$$

Remark 5. Notice that since the actions of G and H commute the G-semistable locus is H-invariant. Assume indeed we have a G-invariant section $s \in H^0(X, \mathcal{L}^{\otimes n})^G$ and that $s(x) \neq 0$ for some point $x \in X$. Then, for every $h \in H$ we can consider the section $h^{-1} \cdot s$, which is G-invariant and doesn't vanish on $h \cdot x$ since $h \cdot s(h \cdot x) = h \cdot s(x) \neq 0$.

Finally, assume that $G \times H$ acts on X, via an embedding in $\mathbb{P}^a \times \mathbb{A}^b$, and that $\mathcal{L}$ is a regular linearisation for the action. Assume that K is an additional group acting on X whose action lifts to $\mathcal{L}$. The $(G \times H)$-linearisation descends to the intermediate quotient by G and we obtain $H \curvearrowright \mathcal{L}/G$, where $\mathcal{L}/G$ is the induced bundle on the quotient $X/\!/G$. We have the following straightforward compatibility results:

Lemma 2.2.2. *There is*

- *compatibility of linearisations, namely $(X/\!/G)(H)^{ss} = X(G \times H)^{ss}/\!/G$ as open subschemes of $X/\!/G$ and there is a canonical isomorphism $X/\!/(G \times H) \simeq (X/\!/G)/\!/H$.*
- *compatibility of Kirwan maps, namely*

$$r_{G\times H\times K,K} = r_{H\times K,K} \circ r_{G\times H\times K,H\times K}.$$

Proof. The first result is just the fact that quotients can be built in two steps. We show the second statement in the case of $K = 1$, the general case is proven in an analogous way. Consider the following diagram:

$$\begin{tikzcd}
A^{G\times H}(X) \arrow[r, "i^*"] \arrow[dr, "r_{G\times H,H}"'] & A^{G\times H}\left(X(G)^{\mathrm{ss}}\right) \arrow[r, "j^*"] & A^{G\times H}\left(X(G\times H)^{\mathrm{ss}}\right) & \\
& A^H\left(\frac{X(G)^{\mathrm{ss}}}{G}\right) \arrow[u, "\hat{\pi}_{G\times H,H}"] \arrow[r, "j^*"] \arrow[dr, "r_H"'] & A^H\left(\frac{X(G\times H)^{\mathrm{ss}}}{G}\right) \arrow[u, "\hat{\pi}_{G\times H,H}"'] \\
& & A\left(\frac{X(G\times H)^{\mathrm{ss}}}{G\times H}\right) \arrow[u, "\hat{\pi}_H"']
\end{tikzcd}$$

where the horizontal maps are pullbacks via open embeddings. Commutativity of the triangles is true by definition while the square commutes by the explicit description of the ascent maps $\hat{\pi}$ given in Theorem 2.1.1. Then Theorem 2.1.2 ensures that the composition of the arrows in the first row and of the inverses of the arrows in the last column coincides with $r_{G\times H}$. □

Remark 6. Since we are allowing the actions to have nontrivial (but finite) stabilisers on the semistable locus, it's possible that the linearisation $\mathcal{L}$ only induces a bundle on the intermediate quotient stack $[X(H)^{\mathrm{ss}}/H]$ and not on the quotient scheme $X/\!/H$. Luckily, in this case there is a positive integer $n \in \mathbb{N}$ so that $\mathcal{L}^n$ descends to a line bundle on the GIT quotient $X/\!/H$. Notice that taking tensor products of the linearisation doesn't change anything: the semistable locus is the same and the quotient is the same too.

2.3 Equivariant Chow cohomology.

The Chow cohomology groups $A^k_G(X)$ are constructed from the equivariant Chow homology groups exactly as the classical Chow cohomology is built from Chow homology ([Ful13], Definition 17.3). Elements $c \in A^k_G(X)$ are collections of homomorphisms $c(t) : A^G_*(Y) \to A^G_{*-k}(Y)$, one for each G-equivariant morphism $t : Y \to X$, which are compatible under flat pullback, lci pullback, proper pushforward etc. As for classical Chow cohomology, $A^*_G(X)$ is naturally a ring with respect to the product $\cup$ induced by composition. Moreover, there is a clear action of the Chow cohomology on Chow homology, and we denote it by

$$\cap : A^k_G(X) \otimes A^G_l(X) \to A^G_{l-k}(X) \quad : \quad c \cap Z := c(\mathbb{1}_X)(Z).$$

In particular this ensures that G-equivariant Chow homology groups $A_*^G(X)$ are modules over $A_G^*(X)$ and therefore over $A_G^*(\mathrm{pt})$. Corollary 2 in [EG98b] ensures that, under the hypotheses we are working with (X separated and in characteristic zero), the Chow cohomology groups can be computed straight from the "homotopy quotient approximations" we introduced above: $A_G^k(X) \simeq A^k(X_{G,k})$. As a corollary of this and of classical Poincaré duality ([Ful13], Corollary 17.4) applied to $X_{G,k}$, G-equivariant Poincaré duality holds true:

Theorem 2.3.1. *Let X be a smooth quasiprojective variety of dimension n with a G-action and let $k \in \mathbb{Z}$. The Poincaré homomorphism*

$$A_G^k(X) \to A_{n-k}^G(X) \quad : \quad c \mapsto c \cap [X]_G$$

is an isomorphism.

Almost all the constructions we performed in the context of Chow homology can be translated in the context of Chow cohomology. For example, pulling back along the same morphism (2.1) defines a change of group homomorphism

$$(H \to G)^* : A_G^*(X) \to A_H^*(X)$$

in cohomology which still satisfies all the relevant functoriality properties with respect to group homomorphisms and equivariant morphisms of varieties, giving an analogous of Lemma 2.1.2. Edidin-Graham's result on the homology of quotients still holds in cohomology [EG98b, Theorem 4]:

Theorem 2.3.2. *Assume that G is a reductive algebraic group acting on a quasiprojective variety X so that the action is locally proper and let M be a geometric quotient. There is an isomorphism of $\mathbb{C}$-algebras*

$$\pi^* : A^*(M) \xrightarrow{\sim} A_G^*(X) \tag{2.5}$$

which satisfies $\hat{\pi}(c \cap y) = \pi^* c \cap \hat{\pi} m$ for all $c \in A^*(M)$ and $y \in A_*(M)$.

With the aid of this result one can prove that if X is smooth then the Poincaré duality morphism for M, namely $c \to c \cap [M]$ which is well defined being M irreducible, is an isomorphism of $\mathbb{C}$-linear spaces even if M is singular! This endows $A_*(M)_{\mathbb{C}}$ with a canonical ring structure, which is independent from the realisation of M as a coarse moduli space. We make this more explicit with the following

Lemma 2.3.1. *If $\sigma_G(X)$ is the order of the stabiliser of G on X at a general point, the following diagram of $\mathbb{C}$-linear spaces is commutative*

$$\begin{array}{ccc} A^*(M) & \xrightarrow{\pi^*} & A^*_G(X) \\ {\scriptstyle c\mapsto c\cap[M]}\downarrow & & \downarrow{\scriptstyle c\mapsto\sigma_G(X)\cdot(c\cap[X]_G)} \\ A_{dim(M)-*}(M) & \xrightarrow{\hat{\pi}} & A^G_{dim(X)-*}(X) \end{array}$$

In particular the Poincaré duality morphism for M is an isomorphism over $\mathbb{C}$ (not over $\mathbb{Z}$!).

Proof. By the Theorems 2.1.1 and 2.3.2 we have that for all $c \in A^*(M)$

$$\hat{\pi}\,(c \cap [M]) = \pi^* c \cap \hat{\pi}[M] = \sigma_G(X)(\pi^* c \cap [X]_G).$$

□

Remark 7. Assume that a scheme M appears as a geometric quotient of a smooth quasiprojective variety X by a locally proper action of a reductive group G. There are two ring structures on $A_*(M)$ that we can consider. (1) The *canonical* ring structure induced by the one on $A^*(M)$ via the Poincaré duality isomorphism $c \mapsto c \cap [M]$, which is well defined by the previous lemma. If M is smooth then this is the usual ring structure given by the intersection product. (2) The *stacky* ring structure of Remark 4, induced by the isomorphism $\hat{\pi}$, which is extrinsic to M and depends on the presentation as a geometric quotient. The previous result shows that these two structures are related by a twist by $\sigma_G(X)$.

2.3.1 Equivariant cohomology of quotients and Kirwan maps.

We start with the following

Remark 8. The morphism π^* of Theorem 2.3.2 is defined by Edidin and Graham at the beginning of the proof of Theorem 4 [EG98b, Page 25]. After unravelling the definitions, this coincides with the composition

$$A^*(M) \xrightarrow{(G\to 1)^*} A^*_G(M) \xrightarrow{\pi^*} A^*_G(X)$$

described by using the change of group homomorphism and the G-equivariant pull-back through the quotient map.

Assume now that G is a reductive group acting in a locally proper way on a quasiprojective variety X and let $\pi : X \to M$ be a geometric quotient. Let H be another reductive group acting on X and commuting with G, so that the action descends to the quotient M.

In the same way we did in Theorem 2.1.2, by using Theorem 2.3.2 applied to the mixed spaces $X \times_H U$, we can prove that if H is another group acting on X so that the action commutes with the one of G, then the composition

$$\pi^*_{G\times H,H} : A^*_H(M) \xrightarrow{(G\times H\to H)^*} A^*_{G\times H}(M) \xrightarrow{\pi^*} A^*_{G\times H}(X)$$

is an isomorphism, compatible with the one in homology via the H-equivariant version of the Poincaré duality diagram of Lemma 2.3.1:

$$\begin{tikzcd} A^*_H(M) \arrow[r, "\pi^*_{G\times H,H}"] \arrow[d, "c\mapsto c\cap[M]_H"'] & A^*_{G\times H}(X) \arrow[d, "c\mapsto \sigma_G(X)\cdot(c\cap[X]_{G\times H})"] \\ A^H_{\dim(M)-*}(M) \arrow[r, "\hat{\pi}_{G\times H,H}"] & A^{G\times H}_{\dim(X)-*}(X) \end{tikzcd} \tag{2.6}$$

In this way we can also define the inverse, called again *descent map*

$$d_{G\times H,H} : A^*_{G\times H}(X) \xrightarrow{\sim} A^*_H(M).$$

In the situation of section 2.2.1, where G acts on X with a regular linearisation $\mathcal{L}$, we can define *Kirwan maps*

$$r_{G\times H,H} : A^*_{G\times H}(X) \to A^*_{G\times H}(X(G)^{\mathrm{ss}}) \xrightarrow{d_{G\times H,H}} A^*_H(X/\!/G)$$

by precomposing the descent map with the restriction to the semistable locus. By using the previous definition of degree of an equivariant class in the Chow group of $\mathcal{X} = [X/G]$, we can recall the definition of integral of an equivariant Chow cohomology class:

Definition 2.3.1. Assume that G is a reductive group acting in a locally proper way on a quasiprojective variety X and let $\pi : X \to M$ be a proper geometric quotient. Let H be another reductive group acting on X and commuting with G, so that the action descends to the quotient M. Let $\mathcal{X} := [X/G]$ be the quotient stack for the action. We define the *integral* of a class $\alpha \in A^*_H(\mathcal{X}) = A^*_{G\times H}(X)$ to be

$$\int_{\mathcal{X}} \alpha := \deg_{\mathcal{X}}(\alpha \cap [X]_{G\times H}) = \deg\left(\hat{d}(\alpha \cap [X]_{G\times H})\right).$$

2.3.2 Localisation for torus actions.

Consider a torus T acting on a quasiprojective variety X. The localisation theorem of [EG98c] shows that the pushforward along the inclusion of the fixed locus X^T is an isomorphism of T-equivariant Chow groups after localisation.

Theorem 2.3.3. *Consider the inclusion of the fixed locus $i : X^T \hookrightarrow X$ and consider the multiplicative system $S \subset A^T * (pt)$ of homogeneous elements of positive degree. Once we set $\mathcal{Q} := S^{-1}A^*_T(pt)$, the pushforward*

$$i_* : A^T(X^T) \otimes_{A^*_T(pt)} \mathcal{Q} \to A^T(X) \otimes_{A^*_T(pt)} \mathcal{Q}$$

is an isomorphism.

Remark 9. Notice that this is an isomorphism of modules over the equivariant cohomology of the point. This is a polynomial ring by Example 2.1.1 and equivariant Poincaré duality.

Remark 10. The formal reason why tensoring by $\mathcal{Q}$ is necessary comes from the map i_* fitting into the long exact sequence of higher Chow groups:

$$\cdots \to A^T(X\backslash X^T, 1) \to A^T(X^T) \xrightarrow{i_*} A^T(X) \to A^T(X\backslash X^T) \to 0.$$

According to Proposition 3 in [EG98c], for each m, there exists an element $f_m \in S$ that annihilates the module $A^T(X\backslash X^T, m)$. This implies that the isomorphism remains valid without having to invert the entirety of S; it suffices to invert just the product $f_0 \cdot f_1$. This means that we can always think of $\mathcal{Q}$ as the localisation $A^*_T(\mathrm{pt})_f$ for some appropriate f.

This theorem has an important corollary in the case where X is a smooth variety, since in this case the fixed locus X^T is regularly embedded and there is an explicit way to invert the morphism i_*:

Theorem 2.3.4. *Assume that T acts on a smooth quasiprojective variety X. The Euler class $\mathrm{e}^T(\mathcal{N}_{X^T/X})$ is an invertible element of $A^T(X^T) \otimes_{A^*_T(pt)} \mathcal{Q}$. Then the pullback along the regular embedding $i : X^T \hookrightarrow X$ satisfies $i^* i_* \alpha = \alpha \cdot \mathrm{e}^T(\mathcal{N}_{X^T/X})$ and therefore it is a (ring) isomorphism.*

Remark 11. The quantity on the right-hand side of the previous equality should be read as a sum over the connected components of the fixed locus.

We will see later in Section 4.3 that it is not strictly necessary for X to be smooth in order to explicitly invert the isomorphism i_*. If X is the geometric quotient of a smooth variety by a reductive group action, this approach still works, even if the fixed locus is not regularly embedded.

2.4 Equivariant K-theory.

Given a variety X with the action of an algebraic group G we will denote with $K_G(X)$ the Grothendieck group of G-equivariant vector bundles on X. We will also denote with $K^G(X)$ the Grothendieck group of G-equivariant coherent sheaves on X.

Remark 12. Notice that the equivariant K-theory *is not* the K-theory of the mixed spaces $X \times_G U$. In this, K-theory differs from cohomology.

Notice that $K_G(X)$ forms a ring under direct sum and tensor product, while $K^G(X)$ is a $K_G(X)$-module. Notice that if X is smooth then the two coincide, since all coherent sheaves are perfect on smooth varieties.

Example 2.4.1. The G-equivariant K-theory of a point is isomorphic to the ring $R(G)$ of representations of G.

As in the nonequivariant case there is the *Chern class map* $c^G : K_G(X) \to A^*_G(X)$. In particular we have the following

Definition 2.4.1. Let E be a G-equivariant vector bundle on X. For every k, the k-th equivariant Chern class of E is defined as the k-th Chern class of the bundle

$$E \times_G U \to X \times_G U$$

where U is a suitable representative for the classifying bundle of G.

In [EG98b, Section 2.4] it is proven that the one above is a vector bundle over $X \times_G U$ and that the resulting class doesn't depend on the choice of U.

In [EG99], Edidin and Graham describe the *Chern character*

$$\mathrm{ch}^G : K_G(X) \to \prod_{k=0}^{\infty} A^i_G(X) \qquad \mathrm{ch}^G(E) := \sum_{i=1}^{\mathrm{rk}(E)} e^{x_i}$$

and the *Todd class*

$$\mathrm{Td}^G : K_G(X) \to \prod_{k=0}^{\infty} A^i_G(X) \qquad \mathrm{Td}^G(E) := \prod_{i=1}^{\mathrm{rk}(E)} \frac{x_i}{1 - e^{-x_i}}$$

where $x_1, \ldots, x_r \in A^1_G(X)$ are the equivariant Chern roots of E. If X is a smooth projective variety they satisfy the equivariant Hirzebruch Riemann-Roch theorem

$$\mathrm{ch}^G\left(\chi^G(X, E)\right) = \int_X \mathrm{ch}^G(E) \cup \mathrm{Td}^G(T_X) \tag{2.7}$$

as shown in [EG99, Corollary 3.1]. Sometimes we will use the notation $\mathrm{Td}^G(X)$ to denote the Chow homology class $\mathrm{Td}^G(T_X) \cap [X]_G$.

2.4.1 Descent maps and Kirwan maps.

Assume that $G \times H$ is a reductive group acting so that G acts freely on X and there is a geometric quotient $X/\!/G$. Exactly as in the case of cohomology, the composition

$$K_H(X/\!/G) \xrightarrow{(G\times H \to H)^*} K_{G\times H}(X/\!/G) \xrightarrow{\pi^*} K_{G\times H}(X)$$

defines an isomorphism, where π^* is the pullback through the quotient map and $(G \times H \to H)^*$ is defined, at the level of vector bundles, by considering the action of $G \times H$ on a H-equivariant bundle E induced by $G \times H \to H$. The inverse is the the *descent* map in K-theory

$$d_{G\times H,H} : K_{H\times G}(X) \xrightarrow{\sim} K_H(X/\!/G).$$

It satisfies the following compatibility conditions with the descent maps in (co)-homology:

Lemma 2.4.1. *Let E be a $(G\times H)$-equivariant vector bundle on X. Given a number k, set $d := d_{G\times H,H}$ we have*

$$c_k^H(d(E)) = d(c_k^{G\times H}(E))$$

and, for every $x \in A^{G\times H}(X)$,

$$\hat{d}(c_k^{G\times H}(E) \cap x) = c_k^H(d(E)) \cap \hat{d}(x).$$

Proof. The first part follows from the statement regarding the Chern character in [Kri14, Lemma 5.5]. The second statement follows from the first one and Poincaré duality (2.6). □

Indeed d is the inverse of the pullback through the quotient map and the property follows from the functoriality of pullbacks and Theorem 2.3.2. The following is the equivariant version of [Edi12, Equation (8)]:

Lemma 2.4.2. *For a K-theory class $E \in K_{G\times H}(X)$*

$$ch^H\left(\chi^H(X/\!/G, d(E))\right) = \int_{X/\!/G} d\left(ch^H(E)\,Td^H(T_X - \mathfrak{g})\right).$$

Proof. By using the equivariant Hirzebruch-Riemann-Roch formula (2.7) we can write

$$\begin{aligned}\mathrm{ch}^H\left(\chi^H(X/\!\!/G, d(E))\right) &= \int_{X/\!\!/G} \mathrm{ch}^H(d(E))\mathrm{Td}^H(T_{X/\!\!/G}) \\ &= \int_{X/\!\!/G} \mathrm{ch}^H(d(E))\mathrm{Td}^H(d(T_X - \mathfrak{g})) \\ &= \int_{X/\!\!/G} d\left(\mathrm{ch}^H(E)\mathrm{Td}^H(T_X - \mathfrak{g})\right).\end{aligned}$$

Here we have used that $T_{X/\!\!/G} = d(T_X - \mathfrak{g})$ where $\mathfrak{g}$ is the trivial bundle on X having as fibre the adjoint representation of G, which follows from the short exact sequence

$$0 \to \mathfrak{g} \to T_X \to \pi^* T_{X/\!\!/G} \to 0$$

where π is the quotient map. □

In the GIT framework of section 2.2.1 we can define the K-theoretic Kirwan map

$$r : K_{G\times H}(X) \to K_{G\times H}(X(G)^{\mathrm{ss}}) \xrightarrow{d} K_H(X/\!\!/G)$$

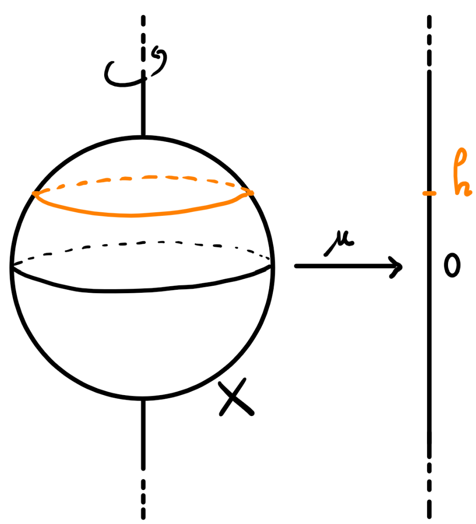
as the composition of the restriction to the semistable locus and the descent map.

Chapter 3

The algebraic cut.

The algebraic cut, introduced by Edidin and Graham in [EG98a], translates the technique of symplectic cutting of Lerman [Ler95] in the algebraic geometry framework. Given the role played by this construction in the geometric proof of the localisation formula by Jeffrey and Kirwan, which is a central focus of this thesis, we will devote significant attention to understanding this geometric construction. The algebraic cutting technique is more intuitive to understand within its original symplectic context. Here, we'll provide an informal overview of this construction to lay the ground for understanding the technical aspects that will be discussed later on in the section.

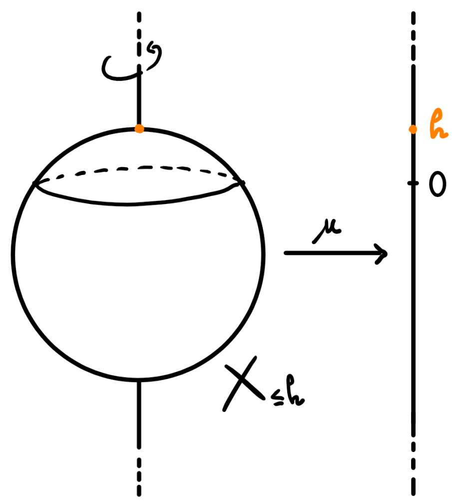
Consider a symplectic manifold X with a Hamiltonian S^1 action and an associated moment map μ. A classic example is the sphere S^2 rotating around the z-axis, where the μ is the height function:



Choose a value $h \in \mathrm{Im}\mu$ such that the action is free on the level set $\mu^{-1}(h)$. We are interested in the quotient $X_h := \mu^{-1}(h)/S^1$, known as the *Hamiltonian reduction of* X *at* h, which parameterises all orbits at the given height h. The symplectic cut construction provides a method to embed this quotient into a specially constructed ambient space. The process involves:

1. considering the manifold X,
2. cutting it at the level h to obtain $\mu^{-1}((-\infty, h])$ and
3. taking the quotient of the boundary $\mu^{-1}(h)$ by S^1.

The resulting space $X_{\leqslant h}$ is the *symplectic cut* of X at the value h and remains a symplectic manifold with a residual Hamiltonian action of S^1:



In this new space, the connected components of the fixed locus are:

1. the embedded symplectic reduction X_h and
2. the components $F \subset X^{S^1}$ of the fixed locus of X that map, via μ, to values smaller than h.

By applying the Atiyah-Bott localisation formula [AB84] to this master space—as we will do algebraically in Section 4—we can *express integrals on the symplectic reduction* X_h *in terms of integrals on the connected components of the fixed locus of* X.

In this section we recall how to see the symplectic cutting from a purely algebraic point of view.

Contents of the section.

The section is structured as follows

- We first recall some basic results on the momentum polytope of a closed subvariety of $\mathbb{P}^a \times \mathbb{A}^b$ endowed with a torus action and a linearisation.

- In particular, we will discuss the wall and chamber structure of this polytope.

- We study the construction of the algebraic cut in the case where the torus is of rank one. We describe the hypotheses that make this space projective and smooth and we study its fixed loci under its canonical $\mathbb{C}^*$-action.

- We then specialise the results on the algebraic cut to a particular situation. Consider a torus $\mathbb{T}$ acting on a smooth variety Y, via an equivariant closed embedding into $\mathbb{P}^a \times \mathbb{A}^b$, together with a regular linearisation $\mathcal{L}$. We will study the properties of space obtained by selecting a splitting of $\mathbb{T} \simeq T \times \mathbb{C}^*$ and applying the algebraic cut construction to the induced action $\mathbb{C}^* \curvearrowright Y/\!/T$ on the intermediate quotient.

3.1 The momentum polytope.

Consider a variety X, admitting a closed embedding in a product $\mathbb{P}^a \times \mathbb{A}^b$, with the action of a torus T together with a linearisation $\mathcal{L}$. In this section, we will examine some fundamental results concerning the structure of the momentum polytope associated with this setup. A comprehensive reference for this discussion is [DH98], which provides an in-depth analysis of the case for projective varieties. Another relevant reference is [Sja98], where the algebraic approach to the momentum polytope (which we will follow) is related to the standard symplectic one, for which many of the following results are classical. However, since the essential results we require can be readily derived from the Hilbert-Mumford criterion, we will present all necessary proofs to ensure the discussion remains self-contained.

Given a character $\psi \in \chi(T) := \mathrm{Hom}(T, \mathbb{C}^*)$ (thought as a 1-dimensional representation of T, hence an equivariant structure on the trivial line bundle on X), we can consider the twisted linearisation $\mathcal{L} \otimes \psi$. We denote with $X(T)^{\psi\text{-ss}}$ the semistable locus with respect to $\mathcal{L} \otimes \psi$.

Definition 3.1.1. We call *momentum polytope* Δ for the data $(X, \mathcal{L})$ the subset of the character space $\chi(T)_{\mathbb{Q}} := \chi(T) \otimes \mathbb{Q}$ given by characters ψ such that the following

Hilbert-Mumford type of inequality holds true:

$$\exists x \in X \ : \ \forall \lambda \in \chi(T)^\vee \text{ such that } \overline{x} := \lim_{t \to 0} \lambda(t) \cdot x \text{ exists, } \langle \lambda, \mathcal{L}_{|\overline{x}} \rangle \geqslant \langle \lambda, \psi \rangle.$$

In this case we say that x is *ψ-semistable* with respect to $\mathcal{L}$.

Notice that if ψ is an integral character, then ψ belongs to Δ if and only if the semistable locus $X(T)^{\psi\text{-ss}}$ is nonempty, since the condition above is exactly the Hilbert-Mumford criterion. We now characterise in a similar way the rational points of Δ:

Lemma 3.1.1. *Let $\psi \in \chi(T)_{\mathbb{Q}}$ be a character and let $n > 0$ be a natural number so that $n\psi$ is integral. Then, for every x in X, the point x is ψ-semistable with respect to $\mathcal{L}$ if and only if it is $n\psi$-semistable with respect to $\mathcal{L}^{\otimes n}$. In particular, the character ψ belongs to Δ if and only if there is $x \in X$ that is $n\psi$-semistable with respect to the linearisation $\mathcal{L}^{\otimes n}$.*

Proof. We can check this on $\mathbb{P}^a \times \mathbb{A}^b$ with a fixed linearisation induced by an action on the total space of the pullback of $\mathcal{O}(-1)$:

$$t \cdot (x_0, \dots, x_a, y_1, \dots, y_b) = (\phi_0(t)x_0, \dots, \phi_a(t)x_a, \nu_1(t)y_1, \dots, \nu_b(t)y_b), \tag{3.1}$$

where $\phi_i, \nu_j \in \chi(T)$ are characters. This induces an equivariant structure on $\mathcal{O}(n)$, given by dualising the action on $H^0(\mathbb{P}^a, \mathcal{O}(n)) \times \mathbb{A}^b$ such that, if $y_{i_0,\dots,i_a}$ is the coordinate corresponding to the monomial $x_0^{i_0} \cdots x_a^{i_a}$, then $t \cdot y_{i_0,\dots,i_a} = \phi_0(t)^{i_0} \cdots \phi_a(t)^{i_a} y_{i_0,\dots,i_a}$ while the action on $\mathbb{A}^b$ stays the same. Fixed a point x, Hilbert-Mumford ensures that its ψ-semistability with respect to $\mathcal{O}(1)$ is equivalent to the condition

$$\forall \lambda \in \chi(T)^\vee \text{ such that } \overline{x} := \lim_{t \to 0} \lambda(t) \cdot x \text{ exists, } \min_{i : x_i \neq 0} \langle \lambda, \phi_i \rangle \leqslant \langle \lambda, \psi \rangle.$$

By multiplication by n this is equivalent to

$$\forall \lambda \in \chi(T)^\vee \text{ such that } \overline{x} := \lim_{t \to 0} \lambda(t) \cdot x \text{ exists, } \min_{i : x_i \neq 0} \langle \lambda, n\phi_i \rangle \leqslant \langle \lambda, n\psi \rangle \tag{3.2}$$

and notice that

$$\min_{i : x_i \neq 0} \langle \lambda, n\phi_i \rangle = \min \left(\langle \lambda, \sum_{i : x_i \neq 0} k_i \phi_i \rangle \qquad \text{over all } k \text{ such that } \sum_{i : x_i \neq 0} k_i = n \right).$$

This shows that (3.2) is the Hilbert-Mumford numerical criterion for $n\psi$-semistable point with respect to $\mathcal{O}(n)$, concluding the proof. ∎

One immediate and important property of the momentum polytope is its convexity. This was proven in the symplectic context by Atiyah [Ati82] and Guillemin and Sternberg [GS82] in the compact setting. The algebraic analogue for projective varieties was considered in [Bri87]. The case of quasiprojective varieties was studied in the symplectic category in [HNP94].

Lemma 3.1.2. *The momentum polytope Δ is convex in $\chi(T)_{\mathbb{Q}}$.*

Proof. A simple application of the Hilbert-Mumford criterion. Pick a point $x \in X$ which is ψ-semistable and $(\psi + m\phi)$-semistable. If $\lambda \in \chi(T)^\vee$ is such that the limit $\overline{x} := \lim_{t\to 0} \lambda(t) \cdot x$ exists, we have that for all $k \in [0, m]$, if we denote with c the quantity $\langle \lambda, \mathcal{L}_{|\overline{x}}\rangle + \langle \lambda, \psi\rangle$, then the Hilbert-Mumford weight $c + k\langle \lambda, \phi\rangle$ is always contained in between the numbers c and $c + m\langle \lambda, \phi\rangle$ which are both non-negative by the semistability hypothesis. □

How does the momentum polytope change when we restrict to a smaller torus?

Lemma 3.1.3. *Let $\mathbb{T}$ act on X with linearisation $\mathcal{L}$ and consider a subtorus $T \hookrightarrow \mathbb{T}$. A point $x \in X$ is T-semistable with respect to $\mathcal{L}$ if and only if there is a character $\xi \in \chi(\mathbb{T})_{\mathbb{Q}}$ orthogonal to T such that x is ξ-semistable with respect to $\mathcal{L}$:*

$$X(T)^{ss} = \bigcup_{\substack{\xi \in \chi(\mathbb{T})_{\mathbb{Q}} \\ \xi_{|T} = 0}} X(\mathbb{T})^{\xi\text{-}ss}$$

In particular, the momentum polytope Δ for the T-action and the induced linearisation coincides with the restriction to T of the momentum polytope $\mathbb{\Delta}$ for the $\mathbb{T}$-action. In other words, if we denote with $\cdot_{|T}$ the morphism $\chi(\mathbb{T})_{\mathbb{Q}} \to \chi(T)_{\mathbb{Q}}$ we get that $\Delta = \mathbb{\Delta}_{|T}$.

Proof. It's enough to prove the statement for $X = \mathbb{P}^a \times \mathbb{A}^b$ and $\mathbb{T} \simeq T \oplus \mathbb{C}^*$ acting diagonally. Clearly if $x \in X(\mathbb{T})^{\xi\text{-ss}}$ then it also belongs to $X(T)^{\text{ss}}$ by Hilbert-Mumford. On the other hand pick a point $(x, y) \in X(T)^{\text{ss}}$. Then there is a section $X_0^{d_0} \cdots X_a^{d_a} \in H^0(\mathbb{P}^a, \mathcal{O}(n))$ and a monomial $y_1^{e_1} \cdots y_b^{e_b}$ in the coordinates of $\mathbb{A}^b$ such that

$$f(X, y) := (X_0^{d_0} \cdots X_a^{d_a})(y_1^{e_1} \cdots y_b^{e_b})$$

is a T-invariant function on the total space of the pullback of $\mathcal{O}(-n)$ not vanishing on a lift of (x, y). Being f monomial, f is also $\mathbb{C}^*$-equivariant of some weight k: $f(s \cdot X, s \cdot y) = s^k f(X, y)$. This in particular shows that $f \in H^0(X, \mathcal{O}(n) \otimes k\xi)^{\mathbb{T}}$ where $\xi : \mathbb{T} \to \mathbb{C}^*$ is the character given by the splitting, hence $(x, y) \in X(\mathbb{T})^{k\xi\text{-ss}}$. □

We get, as an immediate consequence, the following

Corollary 3.1.1. *Assume that $\mathcal{L}$ is a linearisation for $T \rightharpoonup X$ and that the corresponding momentum polytope is contained in $\lambda^\perp$ for some $\lambda \in \chi(T)$. Then λ acts trivially on X. In particular, if $\mathcal{L}$ admits a stable point then the corresponding polytope has nonempty interior.*

Proof. We can assume that the origin belongs to Δ, otherwise we can simply translate by changing $\mathcal{L}$. First of all, it is immediate to show that if $\mathbb{C}^*$ acts on $\mathbb{P}^a \times \mathbb{A}^b$ with a linearisation $\mathcal{O}(1)$ such that the corresponding polytope has only one point, then the action is trivial. By taking the image $\lambda(\mathbb{C}^*)$ and the connected component of the identity, we can assume that λ determines a subgroup $\lambda : \mathbb{C}^* \hookrightarrow T$, and therefore an action $\mathbb{C}^* \hookrightarrow X$ that carries the induced linearisation from $\mathcal{L}$. Lemma 3.1.3 ensures that the momentum polytope for this action is $\Delta_{|\mathbb{C}^*}$, which we know coincides with the origin, hence this action is trivial. Assume that $\mathcal{L}$ admits a stable point. Then, being Δ convex, if it has empty interior then it is contained in a hyperplane and hence there is a $\mathbb{C}^*$ acting trivially on X, which is a contradiction. □

We can also prove a result in the converse direction, obtaining the following

Proposition 3.1.1. *Let S be subtorus of T and let $S^\perp \subseteq \chi(T)_\mathbb{Q}$ be the set of characters vanishing on S. Then S acts trivially on X if and only if the momentum polytope of X is contained in $\phi + S^\perp$ for some $\phi \in \chi(T)$.*

Proof. We can assume, up to twisting $\mathcal{L}$ with a character, that the origin belongs to Δ. In this case we can prove the result for $\phi = 0$. The "if" part follows from an iterated application of the previous Corollary. On the other hand if S acts trivially on X we can consider a character ψ that doesn't vanish on S. First of all notice that since $0 \in \Delta$ we have that the weight of the S-action on all the fibres of $\mathcal{L}$ is trivial, so $\langle \lambda, \mathcal{L}_{|x} \rangle = 0$ for all $\lambda \in \chi(S)^\vee$ and $x \in X$. Assume by contradiction x is a ψ-semistable point of X. For all $\lambda \in \chi(S)^\vee$, we clearly have the existence of the limit $\lim_{t\to 0} \lambda(t) \cdot x$ which is x itself. The Hilbert-Mumford weight for this is the pairing $\langle \lambda, \psi \rangle$ and by semistability this is non-negative. Consider the other cocharacter $-\lambda$ for which we obtain the weight $-\langle \lambda, \psi \rangle$ which is again non-negative by semistability, thus proving $\langle \lambda, \psi \rangle = 0$. Since ψ doesn't vanish on S, this is a contradiction. □

Here we introduce one important open subscheme of X which will be useful later. We work in the case of rank 1 where $T = \mathbb{C}^*$ and we identify $\chi(\mathbb{C}^*)$ with $\mathbb{Z}$ via the standard $n \mapsto t^n$.

Definition 3.1.2. Assume $\mathbb{C}^*$ acts on X with a linearisation $\mathcal{L}$. We denote with X^- the open subscheme

$$X^- := \bigcup_{\substack{\psi \in \Delta \\ \psi \leqslant 0}} X(\mathbb{C}^*)^{\psi\text{-ss}}.$$

We can give a characterisation of this scheme in terms of limits. We need an auxiliary lemma first:

Lemma 3.1.4. *Let $\mathbb{C}^\times$ act on X with linearisation $\mathcal{L}$. For every fixed point $p \in X^{\mathbb{C}^\times}$ denote with $w_p \in \mathbb{Z}$ the character of the action on the fibre $\mathcal{L}_{|p}$. For every point $x \in X$ admitting both limits*

$$\overline{x} := \lim_{t \to 0} t \cdot x \quad \textit{and} \quad \underline{x} := \lim_{t \to 0} t^{-1} \cdot x$$

the inequality $w_{\overline{x}} \geqslant w_{\underline{x}}$ holds true.

Proof. The points in $\mathbb{P}^a \times \mathbb{A}^b$ that can have both limits with respect to λ are such that the whole λ-orbit has constant components in $\mathbb{A}^b$, so we can check this condition in the projective case. Consider $\mathbb{P}^a$ with linearisation given by $\mathcal{O}(1)$. Indeed consider the linearisation given by dualising the action on $\mathcal{O}(-1)$ given by $t \cdot (x_0, \ldots, x_a) := (t^{\lambda_0} x_0, \ldots, t^{\lambda_a} X_x)$ for $\lambda_0, \ldots, \lambda_a \in \mathbb{Z}$. Then given $x = [x_0 : \cdots : x_a] \in \mathbb{P}^a$ we can consider the limit $\overline{x}$ for t going to zero. If the coordinate $\overline{x}_i$ is nonzero then λ_i is the minimum among all λ_j such that $x_j \neq 0$:

$$\overline{x}_i \neq 0 \Rightarrow \lambda_i = \overline{\lambda} := \min \left(\lambda_j \quad : \quad x_j \neq 0 \right).$$

Analogously for the limit $t^{-1} \to 0$ we have

$$\underline{x}_i \neq 0 \Rightarrow \lambda_i = \underline{\lambda} := \max \left(\lambda_j \quad : \quad x_j \neq 0 \right).$$

This immediately shows that $\mathbb{C}^*$ acts on the fibre $\mathcal{O}(-1)_{\overline{x}}$ by the character $\overline{\lambda}$ and on $\mathcal{O}(-1)_{\underline{x}}$ by the character $\underline{\lambda}$. By dualizing we find that $w_{\overline{x}} = -\overline{\lambda} \geqslant -\underline{\lambda} = w_{\underline{x}}$. □

Proposition 3.1.2. *A point $x \in X$ belongs to X^- if and only if, whenever the limit*

$$\lim_{t \to 0} t \cdot x = \overline{x}$$

exists, the weight of the action of $\mathbb{C}^*$ on the fibre $\mathcal{L}_{|\overline{x}}$ of the linearisation is non-negative.

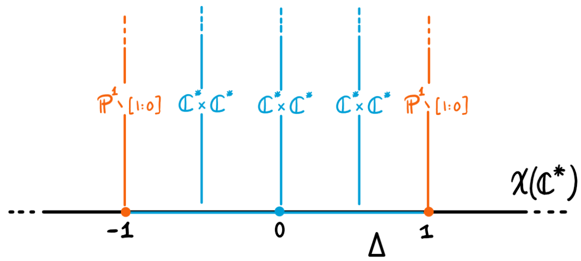


Figure 3.1: The momentum polytope and the semistable locus.

Proof. ($\Rightarrow$) Assume that $x \in X(\mathbb{C}^*)^{\psi\text{-ss}}$ for some $\psi < 0$. Then by the Hilbert-Mumford criterion we have that if the limit $\overline{x}$ exists then the weight of the linearisation $\mathcal{L}_{|\overline{x}} \otimes \psi$ is non-negative. This is the weight of $\mathcal{L}_{|\overline{x}}$ plus the integer ψ, hence the first summand is positive. ($\Leftarrow$) The semistability of x is determined by the other limit

$$\underline{x} := \lim_{t \to 0} t^{-1} \cdot x.$$

If this limit doesn't exist then the point is semistable for $\mathcal{L}$. If it exists then the weight of t^{-1} on $\mathcal{L}_{|\underline{x}}$ is either non-negative or negative. If it's non-negative then x is semistable for $\mathcal{L}$. If it's negative, denote this weight by ψ and , via Lemma 3.1.4, it's immediate to check that x is semistable for $\mathcal{L} \otimes \psi$. $\square$

Example 3.1.1. Consider the action of $\mathbb{C}^*$ on the projective line $X := \mathbb{P}^1$ by $t \cdot [x_0 : x_1] := [t^2 x_0 : x_1]$ and consider the linearisation $\mathcal{L} = \mathcal{O}(1)$ dual to the action on the tautological bundle $\mathcal{O}(-1)$ described by $(tx_0, t^{-1}x_1)$. Notice that the weight of the action on the fibre of $\mathcal{O}(-1)$ over the point $[1 : 0]$ is by the character t, while the action over $\mathcal{O}(-1)_{|[0:1]}$ is by the character t^{-1}. Dually we see that $\mathbb{C}^*$ acts with weight -1 on $\mathcal{L}_{|[1:0]}$ and with weight 1 over $\mathcal{L}_{|[0:1]}$. There are many ways to study the momentum polytope for such linearisation, but the simplest trick is to notice that, by the Hilbert-Mumford criterion, there is a semistable point for $\mathcal{L} \otimes \psi$ if and only if the weights of the actions on the fibre above the two fixed points, namely $-1 + \psi$ and $1 + \psi$, are either of different signs or one of the two is zero. In this way we see that the momentum polytope is $\Delta = [-1, 1] \subset \mathbb{Q} \simeq \chi(\mathbb{C}^*)_{\mathbb{Q}}$ (Figure 3.1). Moreover, we can describe the subset $(\mathbb{P}^1)^-$. In this case all the points different from $[1 : 0]$ flow, via the action of $t \to 0$, to the point $[0 : 1]$. The action on the fibre of $\mathcal{L}$ over $[0 : 1]$ is with positive weight, hence we see that $(\mathbb{P}^1)^- = \mathbb{P}^1 \backslash [1 : 0]$ as in Figure 3.2.

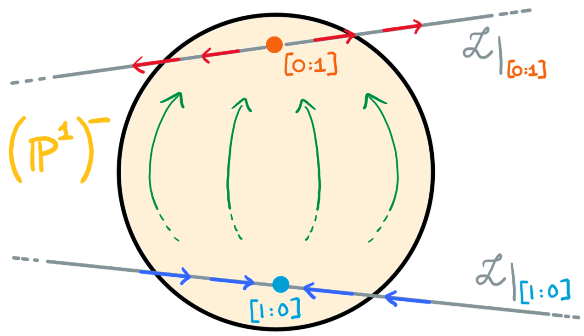


Figure 3.2: The example of $(\mathbb{P}^1)^- = \mathbb{P}^1 \backslash [1:0]$.

3.2 The stratification of the momentum polytope.

In this section we will work with the action of an algebraic torus T on a closed subvariety $X \subseteq \mathbb{P}^a \times \mathbb{A}^b$ admitting a linearisation $\mathcal{L}$. We will also assume that there is no subtorus of positive rank acting trivially on X. The momentum polytope has an important stratification:

Definition 3.2.1. Let $k \in \{0, \dots, \dim(T)\}$. The *codimension-$k$ stratum of* Δ is the subset $\partial_k \Delta \subseteq \Delta$ given by characters $\psi \in \chi(T)_{\mathbb{Q}}$ such that there is a point $x \in X^{\psi\text{-ss}}$ having stabiliser of dimension greater or equal to k.

Notice that the complement in Δ of the 1-codimensional stratum $\partial_1 \Delta$ only contains, by definition, regular linearisations. Pick $\lambda \in \chi(T)^\vee$ and consider the subvariety X^λ of X fixed by λ.

Definition 3.2.2. Let $T' \hookrightarrow T$ be a subtorus of rank k. Consider the fixed locus $X^{T'}$, which is a union of smooth irreducible subvarieties of X. Consider the momentum polytopes for the action of T on these connected components with respect to the linearisation induced by restricting $\mathcal{L}$. If the momentum polytope for one such component is of codimension k, we call this a *wall of codimension* k in Δ. We will denote with $\mathrm{Wall}_k(\Delta)$ the set of all walls of codimension k of Δ.

Remark 13. Notice that by Proposition 3.1.1 the momentum polytope of such connected components is always contained in a hyperplane of codimension k, which is a translation of the hyperplane of characters vanishing on T'. It defines a wall if it is precisely of codimension k, or in other words if the action of T/T' on the corresponding component of $X^{T'}$ doesn't admit any nontrivial subtorus that acts trivially.

A straightforward application of Luna's étale slice theorem allows to prove an interesting property of torus actions:

Lemma 3.2.1. *Assume that a torus T acts on X so that $X \neq X^T$. Every connected component F of X^T is strictly contained in the connected component X^S for some codimension 1 subtorus $S \subset T$.*

Proof. By Luna's slice theorem [Dré04, Lemma 5.1] there is an affine open subscheme $V \subseteq X$ containing x and a T-equivariant morphism $\phi : V \to T_x X$ that is étale. Consider the weight space decomposition

$$T_x X \simeq \bigoplus_{\rho \in \chi(T)} (T_x X)_\rho$$

where the T-action on $(T_x X)_\rho$ is by the character ρ. Notice that at least one nonzero character $\tilde{\rho}$ gives a nonzero weight space $(T_x X)_{\tilde{\rho}}$, otherwise the T-action on V (and hence on X) is trivial being ϕ equivariant and étale. Finally consider the closed subscheme $\phi^{-1}((T_x X)_{\tilde{\rho}})$ of X. This is fixed by the codimension 1 subtorus $S := \ker(\tilde{\rho})$ and has a positive dimensional connected component C passing through x. Clearly this connected component can't be fixed by T being ϕ equivariant, hence the connected component F' of X^S containing both F and C satisfies the claim. □

Corollary 3.2.1. *For every $k \in \{0, \ldots, dim(T) - 1\}$, every wall of codimension $k+1$ is contained in a wall of codimension k.*

Proof. Assume we are given a subtorus $T' \subseteq T$ of rank $k+1$ such that there is a connected component $X_y^{T'}$ of $X^{T'}$ with momentum polytope of codimension $k+1$. By considering the T'-action on X we can invoke the previous Lemma 3.2.1 to find T'' so that ${X^{T'}}_y \subset X_y^{T'}{}'$. The momentum polytope of $X_y^{T''}$ contains the previous one but, additionally, it is not contained in $(T')^\perp$ by Proposition 3.1.1.Then its momentum polytope has codimension $k+1$ and defines a wall of codimension $k+1$. □

Corollary 3.2.2. *The codimension k stratum of Δ is the union of all codimension k walls:*

$$\partial_k \Delta = \bigcup_{w \in Wall_k(\Delta)} w_k$$

Proof. Clearly a point in a codimension k wall is contained in $\partial_k \Delta$ by definition. Assume now that ψ is a point of $\partial_k \Delta$. Let $\overline{k}$ be the biggest number such that $\psi \in \partial_{\overline{k}} \Delta$. Then there is $y \in X$ which is ψ-semistable and fixed by a subtorus T'

of dimension $\overline{k}$. Consider the connected component$X_y^{T'}$ of $X^{T'}$ containing y and look at its momentum polytope. By Proposition 3.1.1 and by maximality of $\overline{k}$, the momentum polytope of $X_y^{T'}$ is $\overline{k}$-codimensional, hence it defines a wall w of Δ of codimension $\overline{k}$. Every wall of codimension $\overline{k}$ is contained in one of codimension k, concluding the proof. □

3.3 Algebraic cuts.

Here we consider the construction of the algebraic cut, originally described in symplectic geometry by Lerman [Ler95], due to Edidin and Graham [EG98a] and we study it in the cases of varieties with a closed embedding into a product of projective and affine space. Assume $\mathbb{C}^*$ acts on a variety X via an embedding into $\mathbb{P}^a \times \mathbb{A}^b$ and let $\mathcal{L}$ be a regular linearisation of the action. If $\mathbb{A}^1$ is the 1-dimensional representation of $\mathbb{C}^*$ with weight 1, we consider the additional action

$$\mathbb{C}^* \curvearrowright X \times \mathbb{A}^1 \quad : \quad s \cdot (x, z) := (s \cdot x, sz).$$

Consider the linearisation we obtain by pulling back $\mathcal{L}$ via the $\mathbb{C}^*$-equivariant projection $X \times \mathbb{A}^1 \to X$. We'll keep denoting this equivariant line bundle with $\mathcal{L}$.

Definition 3.3.1. The *algebraic cut* of X at $\mathcal{L}$ is the GIT quotient of $X \times \mathbb{A}^1$ with respect to the linearisation $\mathcal{L}$. We denote it with $X_c := (X \times \mathbb{A}^1)/\!/\mathbb{C}^*$.

We start the study of X_c by describing the semistable locus $(X \times \mathbb{A}^1)^{\mathrm{ss}}$.

Lemma 3.3.1. $(X \times \mathbb{A}^1)^{ss} = (X^- \times \mathbb{C}^*) \cup (X^{ss} \times 0)$. *In particular, $\mathcal{L}$ is a regular linearisation.*

Proof. We can use the Hilbert-Mumford criterion to study the semistable locus. First of all notice that the condition for the points of the form $(x, 0) \in X \times \mathbb{A}^1$ to be semistable coincides with the condition of x being semistable in X, hence $(X \times \mathbb{A}^1)^{\mathrm{ss}} \cap X \times 0 = X^{\mathrm{ss}} \times 0$. On the other hand, a point (x, z) is semistable if and only if, whenever the limit

$$(\overline{x}, \overline{z}) := \lim_{t \to 0} t \cdot (x, z)$$

exists, then the character of the action on $\mathcal{L}_{|(\overline{x}, \overline{z})}$ is non-negative (notice we don't care about the limit for $t^{-1} \to 0$ since it can never converge being $z \neq 0$). The existence of this limit is equivalent to the existence of the limit $\overline{x}$ for $t \cdot x$, and the condition on the weight is equivalent to the character for $\mathcal{L}_{|\overline{x}}$ being non-negative. This, together with Proposition 3.1.2, shows that (x, z) is semistable if and only if $x \in X^-$. To complete the proof we have to show that all stabilisers are finite at the semistable points. On $X \times \mathbb{C}^*$ the action is free by construction, while on $X^{ss} \times 0$ this holds true since $\mathcal{L}$ is regular. □

This shows that in our hypotheses X_c has at most finite quotient singularities. From GIT we have the projective morphism

$$X_c \to \operatorname{Spec}\left(H^0(X \times \mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1})^{\mathbb{C}^*}\right). \tag{3.3}$$

This gives us a simple criterion to check projectivity of the algebraic cut:

Proposition 3.3.1. *The following statements are equivalent:*

1. *the algebraic cut X_c is projective.*
2. *the only global functions $f : X \to \mathbb{C}$ scaling with non positive weights are the constants. In other words, every function $f \in H^0(X, \mathcal{O}_X)$ satisfying*

   $$\exists a \leqslant 0 \quad : \quad f(t \cdot x) = t^a f(x) \quad \forall t \in \mathbb{C}^*,\ \forall x \in X$$

   is constant.
3. *the quotient $X /\!/ \mathbb{C}^*$ is projective and, for every point $x \in X$, the limit $lim_{t\to 0} t \cdot x$ exists.*
4. *the quotient $X /\!/ \mathbb{C}^*$ is projective and the momentum polytope Δ is bounded from below, namely $\exists n \in \mathbb{Z}$ such that $\Delta \subset \mathbb{Z}_{>n}$*

Proof. $(1 \Rightarrow 2)$ If f scales with non-positive weight $-k$ then $F(x, z) := f(x)z^k$ is a regular function on $X \times \mathbb{A}^1$ which is $\mathbb{C}^*$-invariant and nonconstant. Clearly it stays not constant on the semistable locus (being it open) and by universal property of the GIT quotient it factors as

$$(X \times \mathbb{A}^1)^{\mathrm{ss}} \to X_c \to \mathbb{C},$$

which defines a nonconstant function on X_c which therefore is not projective. $(2 \Rightarrow 1)$ We start by studying the ring $H^0(X \times \mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1})^{\mathbb{C}^*}$. Clearly $H^0(X \times \mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}) \simeq H^0(X, \mathcal{O}_X)[z]$ and a function $F(x, z) = \sum_{k=0}^{d} f_k(x)z^k$ is $\mathbb{C}^*$-invariant if and only if

$$\sum_{k=0}^{d} \left(f_k(x) - t^k f_k(t \cdot x)\right) z^k = 0 \qquad \forall x \in X,\ z \in \mathbb{A}^1,\ t \in \mathbb{C}^*.$$

This implies that all the f_k are equivariant functions scaling with non-positive weight. Thus condition 2 is equivalent to $H^0(X\times\mathbb{A}^1,\mathcal{O}_{X\times\mathbb{A}^1})^{\mathbb{C}^*}\simeq\mathbb{C}$. This shows that the base of the projective morphism (3.3) is a point. $(2\Rightarrow 3)$ By GIT we have a projective morphism $X/\!/\mathbb{C}^*\to\operatorname{Spec}(H^0(X,\mathcal{O}_X)^{\mathbb{C}^*})$ and the base is a point, so the domain is projective. Moreover, consider the closed embedding

$$X\hookrightarrow\mathbb{P}^a\times\mathbb{A}^b$$

and the two projections $\pi_{\mathbb{P}}$ and $\pi_{\mathbb{A}}$ to the projective and the affine spaces. Notice that given $x\in X$ the limit of $t\cdot x$ for t tending to zero exists in $\mathbb{P}^a\times\mathbb{A}^b$ if and only if both limits

$$\lim_{t\to 0}t\cdot\pi_{\mathbb{P}}(x)\quad\text{and}\quad\lim_{t\to 0}t\cdot\pi_{\mathbb{A}}(x)$$

exist. Notice that the first limit always exists being $\mathbb{P}^a$ projective, while the second limit exists and is equal to zero since all the coordinate functions scale with positive weight. Since X is closed in this ambient space, the limit exists in X too. $(3\Rightarrow 2)$ Since $X/\!/\mathbb{C}^*$ is projective we have that $H^0(X,\mathcal{O}_X)^{\mathbb{C}^*}\simeq\mathbb{C}$. Assume by contradiction that there is a nonconstant function f on X that scales with negative weight $-k$. Then pick $x\in X$ so that $f(x)\neq 0$. Then the limit must satisfy

$$f\left(\lim_{t\to 0}t\cdot x\right)=\lim_{t\to 0}t^{-k}f\left(x\right)$$

but the limit on the right doesn't exist, giving a contradiction. $(3\Rightarrow 4)$ $F_1,\dots,F_l$ be the connected components of the fixed locus of the action on X. Let $\psi\in\chi(\mathbb{C}^*)$ be such that the weight of the linearisation $\mathcal{L}_{|F_i}$ is smaller than ψ for all i. Then, for all $\tilde{\psi}<-\psi$, the linearisation $\mathcal{L}\otimes\tilde{\psi}$ is such that the weight of the $\mathbb{C}^*$-action on the fibre above every fixed point is negative. Since all limits for $t\to 0$ exist, then there is no $\tilde{\psi}$-semistable point. $(4\Rightarrow 3)$ Assume by contradiction that x is a point such that the limit for $t\to 0$ of $t\cdot x$ doesn't exist. Then, given $\psi\in\chi(\mathbb{C}^*)$, the ψ-semistability of x is determined by the limit $\underline{x}=\lim_{t\to 0}t^{-1}\cdot x$. If the limit doesn't exist then x is ψ-semistable for all ψ, contradicting the hypothesis of point 4. If it exists, the ψ-semistability of x is equivalent to the weight of $\mathcal{L}_{|\underline{x}}\otimes\psi$ being non-positive, which clearly happens for all ψ sufficiently negative, again contradicting the hypothesis 4. □

Notice that the algebraic cut has a residual $\mathbb{C}^*$-action coming from the action on X. Indeed we can consider the action of the 2-dimensional torus

$$\mathbb{C}^*_1\times\mathbb{C}^*_2\curvearrowright X\times\mathbb{A}^1\quad:\quad(s_1,s_2)\cdot(x,z):=(s_1\cdot x,s_1s_2^{-1}z)$$

and realise that X_c is the quotient by $\mathbb{C}_1^*$, so that $\mathbb{C}_2^*$ still acts on it. Denote with $(X \times \mathbb{A}^1)_{\mathrm{opp}}$ the variety $X \times \mathbb{A}^1$ endowed with the action of $\mathbb{C}_1^* \times \mathbb{C}_2^*$ obtained by exchanging the roles of s_1 and s_2 in the definition above. We can explicitly describe the action on X_c in terms of the original one on X:

Proposition 3.3.2. *There is a closed embedding $i : X/\!/\mathbb{C}^* \hookrightarrow X_c$ (possibly non regular if stabilisers are nontrivial) induced by the $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariant embedding $I : X^{ss} \times 0 \hookrightarrow (X \times \mathbb{A}^1)^{ss}$. The resulting divisor is a fixed locus of X_c and its complement is given by the open embedding $\lambda : X^- \hookrightarrow X_c$ induced by the $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariant embedding*

$$\Lambda : (X^- \times \mathbb{C}^*)_{opp} \xrightarrow{K} X^- \times \mathbb{C}^* \hookrightarrow (X \times \mathbb{A}^1)^{ss}$$

where $K(x, z) := (z^{-1} \cdot x, z^{-1})$.

Proof. By Lemma 3.3.1 the $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariant closed embedding I exists and hence induces a closed embedding between the quotients by the action of $\mathbb{C}_1^*$. The complement of the closed embedding I is given, thanks to the same Lemma, by the equivariant open embedding $X^- \times \mathbb{C}^* \hookrightarrow (X \times \mathbb{A}^1)^{\mathrm{ss}}$, inducing an open embedding $U \hookrightarrow X_c$ of the quotients. Notice that the morphism K is a $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariant isomorphism, hence it induces a $\mathbb{C}_2^*$-equivariant isomorphism on the quotients by $\mathbb{C}_1^*$, but clearly $X^- \simeq (X^- \times \mathbb{C}^*)_{\mathrm{opp}}/\mathbb{C}_1^*$ so we obtain $X^- \xrightarrow{\sim} U$. □

3.4 Geometry of the algebraic cut.

Consider a torus $\mathbb{T}$ acting on a smooth variety Y, via an equivariant closed embedding into $\mathbb{P}^a \times \mathbb{A}^b$, together with a regular linearisation $\mathcal{L}$. In this section we study the properties of the algebraic cut obtained by selecting a splitting of $\mathbb{T} \simeq T \times \mathbb{C}^*$ and applying the algebraic cut construction to the intermediate quotient $Y/\!/T$.

Lemma 3.4.1. *A primitive character $\phi \in \chi(\mathbb{T})$ defines a splitting of the form $\mathbb{T} \simeq T \oplus \mathbb{C}^*$ where T is the connected component of $ker(\phi)$ containing 1 and $\mathbb{C}^*$ is the image of any $\lambda \in \chi(\mathbb{T})^\vee$ such that $\langle \lambda, \phi \rangle > 0$. Conversely, any such splitting determines a primitive character by projecting on the second factor.*

Consider a character ϕ inducing a splitting of $\mathbb{T} \simeq T \oplus \mathbb{C}^*$. We can consider the intermediate quotient $Y/\!/T$, which is endowed with a residual $\mathbb{C}^*$-action and linearisation. Notice that the further quotient $(Y/\!/T)/\!/\mathbb{C}^*$ coincides with $Y/\!/\mathbb{T}$. We would now like to address the following question: what is the condition that we have to impose on the character ϕ, corresponding to the fixed splitting of $\mathbb{T}$, to ensure the projectivity of the algebraic cut of $Y/\!/T$?

Proposition 3.4.1. *Assume that $Y/\!/\mathbb{T}$ is projective and that the momentum polytope Δ of $\mathbb{T} \curvearrowright Y$ doesn't contain the negative ray of the line $\mathbb{Q} \cdot \phi$. Then the algebraic cut of $Y/\!/T$ is projective.*

Proof. Assume by contradiction that the algebraic cut is not projective. Then, by Proposition 3.3.1 we have a nonzero regular function $f \in H^0(Y/\!/T, \mathcal{O}_{Y/\!/T})$ scaling with negative weight $-k$ with respect to the residual $\mathbb{C}^*$-action. Since $H^0(Y/\!/T, \mathcal{O}_{Y/\!/T}) \simeq H^0(Y, \mathcal{O}_Y)^T$ by geometric invariant theory, we can consider the corresponding regular function $F : Y \to \mathbb{C}$. Since its restriction to the semistable locus factors as $Y^{\mathbb{T}\text{-ss}} \to Y/\!/T \to \mathbb{C}$, for every $\lambda \in \chi(\mathbb{T})^\vee$ the function F scales as $F(\lambda(t) \cdot y) = t^{-k\langle\lambda,\phi\rangle}F(y)$. The function F is nonzero, so we can take a semistable point $y \in Y^{\mathbb{T}-ss}$ for $\mathcal{L}$ so that $F(y) \neq 0$. Given $m > 0$ the Hilbert-Mumford criterion says that y is $(-m\phi)$-semistable if and only if, whenever $\lambda \in \chi(\mathbb{T})^\vee$ is such that the limit $\overline{y} := \lim_{t\to 0}\lambda(t) \cdot y$ exists, the weight $\langle\lambda, \mathcal{L}_{|\overline{y}} \otimes (-m\phi)\rangle = \langle\lambda, \mathcal{L}_{|\overline{y}}\rangle - m\langle\lambda, \phi\rangle$ is non-negative. Since $F(y) \neq 0$, the limit can't exist for λ satisfying $\langle\lambda, \phi\rangle > 0$. On the other hand, for any λ such that $\langle\lambda, \phi\rangle \leqslant 0$ we have that, if the limit exists, then

$$\langle\lambda, \mathcal{L}_{|\overline{y}}\rangle - m\langle\lambda, \phi\rangle \geqslant \langle\lambda, \mathcal{L}_{|\overline{y}}\rangle \geqslant 0$$

where the last equality follows from y being semistable for the linearisation $\mathcal{L}$. □

Example 3.4.1. Consider the following simple example where the torus $\mathbb{T} := (\mathbb{C}^*)^2$, with character space canonically isomorphic to $\chi(\mathbb{T})_{\mathbb{Q}} \simeq \mathbb{Q}^2$, acts on the 2-dimensional affine space $\mathbb{A}^2$ diagonally. Assume that the linearisation is given by the trivial bundle with action on the fibre given by the character t_1t_2. We now show that the character $\phi_1 := t_1$ yields a projective algebraic cut, while the opposite $\phi_2 = t_1^{-1}$ doesn't. First of all we recognise that $T := \ker(\phi)$ is the same for both characters and equals the subtorus $\mathbb{C}^* \ni t \to (1, t)$. The induced T-action on $\mathbb{A}^2$ is given by $t \cdot (x, y) = (x, ty)$ and the linearisation is by the character t. This shows that the GIT quotient $\mathbb{A}^2/\!/T$ is the one dimensional affine space $\mathbb{A}^1$ with quotient map given by

$$\mathbb{A}^1 \times \mathbb{C}^* \to \mathbb{A}^1 \quad : \quad (x, y) \mapsto x.$$

Now we consider the algebraic cuts for the two different characters ϕ_1 and ϕ_2:

1. In the case of ϕ_1 the splitting $\mathbb{T} \simeq T \times \mathbb{C}^*$ is given by the choice of the second factor as $\mathbb{C}^* \ni s \mapsto (s, 1) \in \mathbb{T}$. Notice that indeed this subgroup pairs positively with ϕ_1. The induced action on the intermediate quotient $\mathbb{A}^1$ is given by $s \cdot x = sx$ and the induced linearisation is given by the character s. The algebraic cut is built, by definition, as the GIT quotient for the action $\mathbb{C}^* \hookrightarrow (\mathbb{A}^2/\!/T) \times \mathbb{A}^1$

given by $s \cdot (x, z) := (sx, sz)$ with respect to the linearisation given by s. This quotient is immediately seen to be

$$\mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1 \quad : \quad (x, z) \mapsto [x : z],$$

hence the algebraic cut is $\mathbb{P}^1$, which is projective as expected.

2. In the case of ϕ_2 the splitting $\mathbb{T} \simeq T \times \mathbb{C}^*$ is given by the choice of the second factor as $\mathbb{C}^* \ni s \mapsto (s^{-1}, 1) \in \mathbb{T}$. Notice that indeed this subgroup pairs positively with ϕ_2. The induced action on the intermediate quotient $\mathbb{A}^1$ is given by $s \cdot x = s^{-1}x$ and the induced linearisation is given by the character s^{-1}. The algebraic cut is built, by definition, as the GIT quotient for the action $\mathbb{C}^* \hookrightarrow (\mathbb{A}^2/\!/T) \times \mathbb{A}^1$ given by $s \cdot (x, z) := (s^{-1}x, sz)$ with respect to the linearisation given by s^{-1}. This quotient is immediately seen to be

$$\mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1 \quad : \quad (x, z) \mapsto xz,$$

hence the algebraic cut is $\mathbb{A}^1$ and it is not projective.

This suggests us to consider momentum polytopes that are *strictly convex*, namely polytopes that don't contain any line of $\chi(\mathbb{T})_{\mathbb{Q}}$. If we work under this condition, for every character ϕ, at least one among the splittings induced by ϕ and $-\phi$ produce a projective algebraic cut. Is there a simple way to describe the fixed locus on the algebraic cut in terms of the data on Y? Consider $\lambda \in \chi(\mathbb{T})^\vee$ defining a wall w_λ in Δ. Then we can consider the union Y_{w_λ} of those irreducible components of Y^λ whose momentum polytope coincides with the wall w_λ.

Lemma 3.4.2. *Fix $\lambda \in \chi(\mathbb{T})$ and a wall w_λ in the momentum polytope Δ. Assume that the line $\mathbb{Q} \cdot \phi$ meets w_λ away from a wall of codimension higher than 1. Then $\mathcal{L}$ is a regular linearisation for Y_{w_λ} and for every connected component K of Y_{w_λ} the quotient $K/\!/T$ is a nonempty, $\mathbb{C}^*$-fixed irreducible subvariety of $Y/\!/T$. Moreover, if $\lambda, \mu \in \chi(\mathbb{T})^\vee$ are not multiples, then $Y_{w_\lambda}/\!/T$ and $Y_{w_\mu}/\!/T$ are disjoint for any choice of the walls w_λ and w_μ.*

Proof. We write down the proof assuming that Y_{w_λ} is connected. The same proof works in general by replacing Y_{w_λ} with each connected component K. We know that $T = \ker(\phi)$, so if we consider the action of T on Y the momentum polytope is given by projecting along the direction ϕ:

$$\Delta = \mathbb{A}_{|T}.$$

In particular Lemma 3.1.3 above shows that

$$Y(T)^{\mathrm{ss}} = \bigcup_{q \in \mathbb{Q}} Y(\mathbb{T})^{q\phi\text{-ss}}.$$

Since there is a $q \in \mathbb{Q}$ such that $q\phi \in w_\lambda$, for that value of q the semistable locus $Y_{w_\lambda}(T)^{\mathrm{ss}} = Y_{w_\lambda}(\mathbb{T})^{q\phi\text{-ss}}$ is nonempty. Since Y_{w_λ} is a closed invariant subvariety of Y, it descends to a closed subvariety of $Y/\!/T$ after taking the quotient. Notice that, for every $y \in Y_{w_\lambda}(T)^{\mathrm{ss}}$, there is no $\mu \in \chi(T)^\vee$ fixing y, since otherwise the 2-dimensional subtorus generated by μ and λ (which are independent since $\phi(\mu) = 0$ and $\phi(\lambda) \neq 0$) would fix y, but $q\phi$ is not in a codimension-2 wall and this would be a contradiction. In particular, $\mathcal{L}$ is a regular linearisation for the action of T on Y_{w_λ}. Finally, λ fixes all points in $Y_{w_\lambda}(T)^{\mathrm{ss}}$ and thus $Y_{w_\lambda}/\!/T$ is fixed in $Y/\!/T$. The final claim follows from the fact that if $Y_{w_\lambda}(T)^{\mathrm{ss}} \cap Y_{w_\mu}(T)^{\mathrm{ss}} \neq \varnothing$ then the points in the intersection are fixed by the 2-dimensional subtorus generated by μ and λ, which contradicts the fact that $q\phi$ is not a higher codimension wall in Δ. □

This allows to prove the following result characterising the fixed locus on $Y/\!/T$:

Proposition 3.4.2. *Assume that the character ϕ is such that the line $\mathbb{Q} \cdot \phi$ doesn't intersect $\partial_2\Delta$. The fixed locus for the residual $\mathbb{C}^*$-action on $Y/\!/T$ is*

$$(Y/\!/T)^{\mathbb{C}^*} = \bigcup_{\substack{w \in \mathit{Wall}_1(\Delta) \\ w \cap \mathbb{Q}\phi \neq \varnothing}} Y_w/\!/T,$$

where the union is over all 1-codimensional walls of Δ which intersect the line $\mathbb{Q}\phi$. Moreover, $Y_w/\!/T$ is contained in $(Y/\!/T)^-$ if and only if w intersects the negative ray of the line $\mathbb{Q}\phi$. Lastly, the linearisation $\mathcal{L}$ restricted to Y_w is regular.

Proof. We know that the subvarieties of the form $Y_w/\!/T$ appearing in the right-hand side of the equality are all nonempty, fixed and disjoint from each other. On the other hand consider a point $[y] \in Y/\!/T$ fixed by $\mathbb{C}^*$. Then there is a $\lambda \in \chi(\mathbb{T})^\vee$ fixing $y \in Y$ and consider the fixed subvariety Y^λ. Let Y^λ_y be the connected component containing y. By Proposition 3.1.3 there is $q \in \mathbb{Q}$ so that $y \in Y(\mathbb{T})^{q\phi\text{-ss}}$, hence $q\phi \in \partial_1\Delta \backslash \partial_2\Delta$. This means that Y^λ_y must have momentum polytope of codimension 1, hence it must define a wall w_λ containing $q\phi$, hence $[y] \in Y_{w_\lambda}/\!/T$. This concludes the first part of the proof. Consider a point $y \in Y_w/\!/T$ and let $\lambda \in \chi(\mathbb{T})^\vee$ be such that $w = w_\lambda$ and $\langle \lambda, \phi \rangle \geqslant 1$, so that the action induced by λ on the quotient $Y/\!/T$ is a positive multiple of the residual $\mathbb{C}^*$-action. Let $q\phi$ be the intersection of w with the line $\mathbb{Q}\phi$. Since y is $q\phi$-semistable and fixed by λ, the Hilbert-Mumford criterion gives that

$$\langle \lambda, \mathcal{L}_{|y} \rangle = -q\langle \lambda, \phi \rangle.$$

which allows to conclude that $y \in (Y/\!/T)^-$ if and only if $q \leqslant 0$ by Proposition 3.1.2. $\square$

Chapter 4

Jeffrey-Kirwan localisation.

In this section, we will describe how the algebraic cut construction can be employed to derive residue formulae for the degrees of cycles in quotients of smooth varieties by reductive connected group actions. These formulae will be related to the so-called Jeffrey-Kirwan localisation formula.

To begin, we address a common misunderstanding that often arises when discussing this formula. The key feature of the Jeffrey-Kirwan localisation formula is that it allows to compute degrees, or equivalently integrals, on *quotient varieties* by localising to fixed loci of the action *upstairs*, used to take the quotient. This approach contrasts with the Atiyah-Bott localisation formula of Theorem 2.3.3, which computes degrees of classes by localizing at fixed loci on the *same* variety. This should show why the Jeffrey-Kirwan localization is not merely a non-abelian counterpart to Atiyah-Bott's formula: even when the group is abelian, the two formulae are fundamentally different and serve distinct purposes.

The literature features numerous formulae under the name of "Jeffrey-Kirwan localisation formula", each a variation on the same theme. However, understanding how they are related to each other can be challenging. We will consider three specific localisation formulae:

1. The original formula by Jeffrey and Kirwan [JK95], which is analytic in nature and expresses the relevant intersection number in terms of *inverse Laplace transforms.*

2. The formula by Guillemin and Kalkman [GK96], which expresses the intersection number as a sum of certain residues, referred to in this thesis as *Guillemin-Kalkman residues.*

3. The formula by Brion, Szenes, and Vergne [BV99; SV04], that can be used when the variety we take the quotient of is a linear space. This uses a different residue operation known as the *Jeffrey-Kirwan residue*.

These formulae are interrelated as follows: Brion and Vergne prove in [BV99] that the inverse Laplace transforms in the Jeffrey-Kirwan formula can be expressed via a residue operation they term Jeffrey-Kirwan residue. Consequently, the formula in point (3) is a direct corollary of the original formula in point (1). The formula in point (2), however, has a completely different proof and is initially unrelated to the other two. Notably, it features an additional aspect: it is not a single formula but a family of formulae, each corresponding to a combinatorial object called a *dendrite*, for which there is generally no canonical choice. In [JK05], Jeffrey and Kogan derive the localisation formula (1) from (2) in the case of a torus action. We will instead derive the formula of Szenes-Vergne from the one of Guillemin-Kalkman, giving a fully algebraic proof of the formula (3).

Contents of the section.

The section is structured as follows:

- We first describe the various residue operations that we are going to consider. We will discuss their properties and how these residues are related to each other.
- We will outline the main steps in the proof of the Guillemin-Kalkman localisation formula in a concrete example, in order to give an idea of the structure of the proof.
- We describe in detail the first step of the proof of this localisation formula. This expresses the intersection number we are interested in in terms of intersection numbers defined on varieties obtained as quotients by smaller dimensional tori.
- We describe how to iterate the previous step, obtaining the Guillemin-Kalkman localisation formula (Theorem 4.4.1).
- We specialise to the case of actions on linear spaces and recover the localisation formula of Szenes and Vergne (Theorem 4.5.1) from the one of Guillemin and Kalkman.

- Prove the nonabelian version of these formulae by using a result of Martin (Theorem 4.6.1) relating integrals on quotients by nonabelian groups to integral over the corresponding quotient by the maximal subtorus.

4.1 Residues.

In this section we will describe different ways of taking the residues of a meromorphic function on $\mathbb{C}^n$. We will only describe residues at the origin, since this is the only interesting case up to translation.

4.1.1 Iterated residues

First of all, given a ring R, consider the following big ring of power series

$$A := R[\![x_n]\!]_{x_n} \cdots [\![x_1]\!]_{x_1}.$$

Its elements will be Laurent power series in x_1 with coefficients in $R[\![x_n]\!]_{x_n} \cdots [\![x_2]\!]_{x_2}$.

Remark 14. Notice that, by rearranging the sums, we have an isomorphism

$$R[\![x_1, \dots, x_n]\!] \simeq R[\![x_n]\!] \cdots [\![x_1]\!]$$

with the formal power series ring in n variables. Notice, however, that this just extends to an inclusion

$$R[\![x_1, \dots, x_n]\!]_{x_1,\dots,x_n} \hookrightarrow R[\![x_n]\!]_{x_n} \cdots [\![x_1]\!]_{x_1}$$

and not to an isomorphism.

Example 4.1.1. Let $n = 2$ and denote the variables with $x_1 = x, x_2 = y$. Then the element $x + y$ is invertible in $R[\![y]\!]_y[\![x]\!]_x$, where the inverse is

$$(x+y)^{-1} = \sum_{k=0}^{\infty} \left(y^{-1-k}\right) x^k.$$

by the geometric series expansion. It's immediate to check that the series on the right-hand side is not a Laurent power series, hence $R[\![x, y]\!]_{x,y}$ is only a subring of $R[\![y]\!]_y[\![x]\!]_x$.

We have well defined residue maps on these spaces:

Definition 4.1.1. The R-linear homomorphism

$$\mathrm{res}_{x=0} : R[\![x]\!]_x \longrightarrow R$$

picking the coefficient of x^{-1} is called the *residue map*.

Clearly, a morphism of R-modules $B \to C$ induces a morphism $B[\![x]\!]_x \to C[\![x]\!]_x$. Therefore, the homomorphism

$$\mathrm{res}_{x_i=0} : R[\![x_n]\!]_{x_n} \cdots [\![x_i]\!]_{x_i} \to R[\![x_n]\!]_{x_n} \cdots [\![x_{i+1}]\!]_{x_{i+1}}$$

induces a morphism

$$A \to R[\![x_n]\!]_{x_n} \cdots [\![x_{i+1}]\!]_{x_{i+1}} [\![x_{i-1}]\!]_{x_{i-1}} \cdots [\![x_1]\!]_{x_1}$$

Definition 4.1.2. Given $i \in \{1, ..., n\}$, the map

$$\mathrm{res}_{x_i=0} : A \to R[\![x_n]\!]_{x_n} \cdots [\![x_{i+1}]\!]_{x_{i+1}} [\![x_{i-1}]\!]_{x_{i-1}} \cdots [\![x_1]\!]_{x_1}$$

defined as above by taking the residue with respect to x_i is called the i^{th} *residue map*. The composition of all the residue maps

$$R[\![x_n]\!]_{x_n} ... [\![x_1]\!]_{x_1} \xrightarrow{\mathrm{Res}_{x_1=0}} R[\![x_n]\!]_{x_n} ... [\![x_2]\!]_{x_2} \xrightarrow{\mathrm{Res}_{x_2=0}} ...$$
$$... \xrightarrow{\mathrm{Res}_{x_{n-1}=0}} R[\![x_n]\!]_{x_n} \xrightarrow{\mathrm{Res}_{x_n=0}} R$$

gives a R-linear morphism

$$\mathrm{IR}_0 : R[\![x_n]\!]_{x_n} ... [\![x_1]\!]_{x_1} \to R$$

called *iterated residue map*.

Remark 15. Notice that the the order of the variables at which we take the residue doesn't change the definition of the iterated residue.

In a completely analogous way we can define the map that takes the coefficient of x_i and iterating it for all the variables, obtaining the *iterated derivative map*

$$\mathrm{ID}_0 : R[\![x_n]\!]_{x_n} ... [\![x_1]\!]_{x_1} \to R.$$

As the iterated residue, also ID_0 doesn't depend on the choice of the ordering of the variables.

4.1.2 Szenes-Vergne residues

Definition 4.1.3. Let $f : \mathbb{C}^n \dashrightarrow \mathbb{C}$ be a meromorphic function. We define the Szenes-Vergne residue in the following way. Notice that the ring homomorphism

$$\sigma_1 : \mathrm{Hol}(\mathbb{C}^n) \to \mathrm{Mer}(\mathbb{C}^{n-1})[\![x_1]\!] \quad : \quad \sigma_1 f := \sum_{j=0}^{+\infty} \left(\frac{\partial}{\partial z_1}\right)^j f(0, z_2, ..., z_n) x_1^j$$

is injective, hence it defines an extension of the corresponding fields of fractions

$$\sigma_1 : \mathrm{Mer}(\mathbb{C}^n) \to \mathrm{Mer}(\mathbb{C}^{n-1})[\![x_1]\!]_{x_1}$$

Thus we can consider the injection

$$\sigma_n \cdots \sigma_1 : \mathrm{Mer}(\mathbb{C}^n) \to \mathbb{C}[\![x_n]\!]_{x_n} \cdots [\![x_1]\!]_{x_1}$$

The composition $\mathrm{SV}(f) := \mathrm{IR}_0 \circ \sigma_n \cdots \sigma_1 : \mathrm{Mer}(\mathbb{C}^n) \to \mathbb{C}$ is called *Szenes-Vergne* residue.

Remark 16. This is the operation that one would usually call *residue* of a meromorphic function with respect to the ordered coordinates $x_1, \ldots, x_n$.

If $y_1, \ldots, y_n$ is another set of coordinates on $\mathbb{C}^n$ we denote with $\mathrm{SV}_{y_1,\ldots,y_n}$ the residue with respect to these new coordinates.

Definition 4.1.4. Let $\mathcal{H}$ be a hyperplane arrangement centered at the origin of $\mathbb{C}^n$. A meromorphic function f defined on an open neighborhood of the origin is called $\mathcal{H}$-meromorphic if, locally around the origin, its poles lie on the union of the hyperplanes belonging to $\mathcal{H}$. In this case we write $f \in \mathrm{Mer}_{\mathcal{H}}$.

Szenes [Sze98, Proposition 3.1] has a useful integral characterisation of the residue in this case:

Lemma 4.1.1. *Let $\mathcal{H}$ be a hyperplane arrangement of $\mathbb{C}^n$ centered at the origin. There is an oriented compact n-dimensional subtorus $Z \subset \mathbb{C}^n$ so that, for every $f \in Mer_{\mathcal{H}}$,*

$$SV(f) = \int_Z f dz_1 \wedge \cdots \wedge dz_n.$$

This immediately shows that these residues behave well under linear changes of coordinates as pointed out in [SV04, Pag. 12]:

Lemma 4.1.2. *Consider two bases* $\{\beta_1, ..., \beta_n\}$ *and* $\{\gamma_1, ..., \gamma_n\}$ *of* $\mathbb{C}^n$ *such that the following conditions hold:*

1. *span* $(\beta_1, ..., \beta_k) =$ *span* $(\gamma_1, ..., \gamma_k)$ *for all* $k = 1, ..., n$.

2. $\{\beta_1, ..., \beta_n\}$ *and* $\{\gamma_1, ..., \gamma_n\}$ *are oriented in the same way.*

Then, given a meromorphic function f *on* $\mathbb{C}^n$ *with hyperplanes as poles,*

$$\left|\frac{\gamma_1 \wedge \cdots \wedge \gamma_n}{\beta_1 \wedge \cdots \wedge \beta_n}\right| SV_{\beta_1,...,\beta_n}(f) = SV_{\gamma_1,...,\gamma_n}(f). \tag{4.1}$$

Another useful corollary of the integral representation is the behavior of these residues under uniform convergence of sequences.

Lemma 4.1.3. *Let* $D \subset \mathbb{C}^n$ *be an open neighborhood of the origin. Consider a hyperplane arrangement* $\mathcal{H}$ *and a sequence of meromorphic functions* $f_k : D \dashrightarrow \mathbb{C}$ *with poles on* $\mathcal{H}$. *Assume that there is a compact subset* $K \subset D$ *so that* f_k *converges uniformly to* $f \in Mer_{\mathcal{H}}$ *on* $D\backslash K$. *Then*

$$\lim_{k\to\infty} SV(f_k) = SV(f).$$

Proof. Clearly we can find a representative $Z' \subset D\backslash K$ of the homology class of the torus Z of Lemma 4.1.1 corresponding to $\mathcal{H}$. We want to prove that for the torus $Z \subset D$ we have the convergence of integrals

$$\lim_{k\to\infty} \int_{Z'} f_k dz_1 \wedge \cdots \wedge dz_n = \int_{Z'} f dz_1 \wedge \cdots \wedge dz_n$$

which follows by uniform convergence. □

4.1.3 Guillemin-Kalkman residues

Definition 4.1.5. Consider the inclusion of integral domains

$$\mathbb{C}[x_1, ..., x_n] \hookrightarrow \mathbb{C}[x_2, ..., x_n][\![x_1^{-1}]\!]_{x_1^{-1}}$$

This descends to a morphism of the respective fields of fractions

$$\delta_1 : \mathbb{C}(x_1, ..., x_n) \hookrightarrow \mathbb{C}(x_2, ..., x_n)[\![x_1^{-1}]\!]_{x_1^{-1}} \tag{4.2}$$

Iterating this procedure we obtain the following fields extension

$$\delta_n \cdots \delta_1 : \mathbb{C}(x_1, ..., x_n) \hookrightarrow \mathbb{C}[\![x_n^{-1}]\!]_{x_n^{-1}} \cdots [\![x_1^{-1}]\!]_{x_1^{-1}}$$

and the composition

$$\mathrm{GK}_{x_1,...,x_n} : \mathbb{C}(x_1, ..., x_n) \xrightarrow{\delta_n \cdots \delta_1} \mathbb{C}[\![x_n^{-1}]\!]_{x_n^{-1}} \cdots [\![x_1^{-1}]\!]_{x_1^{-1}} \xrightarrow{\mathrm{ID}_0} \mathbb{C}$$

is called *Guillemin-Kalkman residue map.*

Remark 17. Notice that x_i^{-1} has degree 1 in this ring of power series, so ID_0 picks the coefficient of the factor $\prod_i x_i^{-1}$.

If we are given another set of coordinates $y_1, \ldots, y_n$ on $\mathbb{C}^n$ we denote with $\mathrm{GK}_{y_1,...,y_n}$ the Guillemin-Kalkman residue with respect to these variables.

4.1.4 SV = GK when poles are linear spaces.

There is a nice situation in which these two notions of residue coincide, that is when f is a rational function having its only poles on hyperplanes through the origin. Let $\mathcal{S} \subset \mathbb{C}[x_1, ..., x_n]$ be the multiplicatively closed subset of the form

$$\mathcal{S} := \left\{ \prod_{s \in S} \sum_{i=1}^{n} a_s^i x_i \qquad : S \text{ finite set and } a_s \in \mathbb{C}^n \backslash \{O\} \right\}$$

using the convention that the product over $S = \varnothing$ gives 1 as result. Then such f lives in the localisation $R := \mathcal{S}^{-1}\mathbb{C}[x_1, ..., x_n]$.

Proposition 4.1.1. *The equality of residues $SV_{x_1,...,x_n}(f) = GK_{x_n,...,x_1}(f)$ holds true for every $f \in R$.*

Remark 18. Notice that the order of the variables used to extract the SV residue is opposite to the one used for the GK residue!

Let's start with a simple example:

Example 4.1.2. Consider the rational function

$$f(x, y) = \frac{1}{x(y - x)}.$$

In order to compute $\mathrm{SV}_{x,y}(f)$ we have to expand f in x first, then in y and finally take the residue:

$$f = \frac{1}{xy}\sum_{k=0}^{\infty} y^{-k}x^k \quad \Rightarrow \quad \mathrm{SV}_{x,y}(f) = 1.$$

On the other hand, to compute $\mathrm{GK}_{y,x}(f)$ we have to expand f in y^{-1} first, then in x^{-1} and finally take the residue:

$$f = x^{-1}y^{-1}\sum_{k=0}^{\infty}(x^{-1})^{-k}y^{-k} \quad \Rightarrow \quad \mathrm{GK}_{y,x}(f) = 1.$$

As expected, we obtained the same result from these residue operations.

Remark 19. Notice that, outside the subring R of functions having poles on hyperplanes through the origin, these notions of residue don't coincide in general. The simplest case in which this happens is for the rational function $f := (1-x)^{-1}$. Notice that $\mathrm{SV}(f) = 0$ is the usual complex-analytical residue, while we have $\mathrm{GK}(f) = \mathrm{res}_{x=0}\left(-\sum_{k=1}^{\infty} x^{-k}\right) = -1$.

Proof. The proof follows these lines. First we show that the two different expansions of the rational function factor through a common first step, namely a morphism to a common ring T. Then, we show that the iterated residue and iterated derivative operations agree on these elements. Consider the following ring

$$T := \mathbb{C}[\![x_{\frac{n-1}{n}}]\!]_{x_{\frac{n-1}{n}}} \cdots [\![x_{\frac{1}{n}}]\!]_{x_{\frac{1}{n}}}[x_n]_{x_n}.$$

which embeds into the relevant fields:

$$\alpha : T \hookrightarrow \mathbb{C}[\![x_n]\!]_{x_n} \cdots [\![x_1]\!]_{x_1}$$

is defined by $x_{i/n} \mapsto x_n^{-1}x_i$ and $x_n \mapsto x_n$, while

$$\beta : T \hookrightarrow \mathbb{C}[\![x_1^{-1}]\!]_{x_1^{-1}} \cdots [\![x_n^{-1}]\!]_{x_n^{-1}}$$

is defined by $x_{i/n} \mapsto x_n^{-1}(x_i^{-1})^{-1}$ and $x_n \mapsto (x_n^{-1})^{-1}$. It's immediate to check that these two morphisms of rings are well defined. Now notice that the inclusion

$$I : \mathbb{C}[x_1, \dots, x_n] \hookrightarrow T \quad : \quad x_i \mapsto x_{\frac{i}{n}}x_n \text{ and } x_n \mapsto x_n$$

extends to R since all the elements of $\mathcal{S}$ are mapped to invertible elements of T. To see this, it's enough to prove that all linear polynomials $\sum_{i=1}^{n} a_i x_i$ are mapped to invertible elements. We can do this by induction on the number of coefficients a_i that are nonzero. If there is only one nonzero coefficient the statement is trivial. On the other hand let $j < n$ be the smallest coefficient appearing. By the geometric series we find

$$\frac{1}{a_j x_{\frac{j}{n}} + \sum_{i\neq j,n} a_i x_{\frac{i}{n}} + a_n} = \frac{1}{\sum_{i\neq j,n} a_i x_{\frac{i}{n}} + a_n} \sum_{k=0}^{\infty} \left(- \frac{a_j}{\sum_{i\neq j,n} a_i x_{\frac{i}{n}} + a_n} \right)^k x_{\frac{j}{n}}^k$$

and the right-hand side is a well defined expression in T by inductive hypothesis. Notice that the diagram

$$\begin{tikzcd}
\mathbb{C}[[x_n]]_{x_n} \cdots [[x_1]]_{x_1} & T \arrow[l, "\alpha"'] \arrow[r, "\beta"] & \mathbb{C}[[x_1^{-1}]]_{x_1^{-1}} \cdots [[x_n^{-1}]]_{x_n^{-1}} \\
& R \arrow[ul, "\sigma_n \cdots \sigma_1"] \arrow[u, "I"] \arrow[ur, "\delta_1 \cdots \delta_n"'] &
\end{tikzcd}$$

commutes since the arrows are ring homomorphsims and they agree on $x_1, \ldots, x_n$. Finally, it's easy to check that $\mathrm{IR}_0 \circ \alpha = \mathrm{ID}_0 \circ \beta$, since they are linear maps agreeing on monomials, namely

$$\mathrm{IR}_0(x_1^{d_1} \cdots x_{n-1}^{d_{n-1}} x_n^{d_n - \sum_{i<n} d_i}) = \mathrm{ID}_0((x_1^{-1})^{-d_1} \cdots (x_{n-1}^{-1})^{-d_{n-1}} (x_n^{-1})^{-d_n + \sum_{i<n} d_i}).$$

$\square$

4.1.5 Generalised GK residues

As a last notion we describe a generalised version of the GK residue where the base ring is not $\mathbb{C}$. In our applications, this will be the Chow ring of some smooth variety. Given a ring S, we can consider the multiplicatively closed subset $I_S^k \subset S[t_1, ..., t_k]$ given by all the possible finite products of the elements of the set

$$\left\{ s + \sum_{i=1}^{k} a_i t^i \quad s \in S \text{ and } a \in \mathbb{Z}^k \backslash \{O\} \right\} \cup \{1\}$$

we can define the map

$$\delta_{t_n} : (I_S^n)^{-1} S[t_1, ..., t_n] \hookrightarrow (I_S^{n-1})^{-1} S[t_1, ..., t_{n-1}][[t_n^{-1}]]_{t_n^{-1}} \tag{4.3}$$

extending the identity on $S[t_1, ..., t_n]$ since, whenever $a_n \neq 0$,

$$\frac{1}{s + \sum_{i=1}^{n} a_i t_i} = \frac{1}{a_n t_n}\left(1 + \frac{s + \sum_{i=1}^{n-1} a_i t_i}{a_n t_n}\right)^{-1} = \frac{1}{a_n t_n}\sum_{k=0}^{\infty}\left(-\frac{s + \sum_{i=1}^{n-1} a_i t_i}{a_n t_n}\right)^{k}.$$

We can iterate this construction to obtain

$$\delta_n \circ \cdots \circ \delta_1 : (I_S^n)^{-1} S[t_1, ..., t_n] \hookrightarrow S[\![t_n^{-1}]\!]_{t_n^{-1}} \cdots [\![t_1^{-1}]\!]_{t_1^{-1}}$$

And we can finally extract the coefficient of $t_1^{-1} \cdots t_n^{-1}$ by applying ID_0, obtaining the *S-valued GK residue*

$$\mathrm{GK}^{S}_{t_1,\dots,t_n} : (I_S^n)^{-1} S[t_1, ..., t_n] \to S \tag{4.4}$$

We will often omit writing the S in GK^S when it's clear from the context. The following Lemma is just a technical tool that will be handy in the evaluation of a residue that we will encounter at a later stage.

Lemma 4.1.4. *Consider a morphism of rings $S \to S'[t']$ and the induced morphism $f : S[t] \to S'[t, t']$. Then, for every $\alpha \in S[t]$ and for every $e \in I^1_S$, $e' \in I^1_{S'}$ we have that*

$$GK^{S'}_{t,t'}\left(\frac{f(\alpha)}{f(e)e'}\right) = GK^{S'}_{t'}\left(\frac{1}{e'} f\left(GK^{S}_{t}\left(\frac{\alpha}{e}\right)\right)\right) \tag{4.5}$$

Proof. Clearly f commutes with both the operations of applying δ and taking the residue with respect to t, hence

$$f\left(\mathrm{GK}^{S}_{t}\left(\frac{\alpha}{e}\right)\right) = \mathrm{GK}^{S'[t']}_{t}\left(\frac{f(\alpha)}{f(e)}\right)$$

Now e' is an element of $S'[t']$, hence the expression for $\frac{1}{e'}$ doesn't contain t, so

$$\frac{1}{e'}\mathrm{GK}^{S'[t']}_{t}\left(\frac{f(\alpha)}{f(e)}\right) = \mathrm{GK}^{S'[t']}_{t}\left(\frac{f(\alpha)}{f(e)e'}\right)$$

Finally, the thesis follows by the equality

$$\mathrm{GK}^{S'}_{t,t'} = \mathrm{GK}^{S'}_{t'} \circ \mathrm{GK}^{S'[t']}_{t}$$

which is clear by definition. □

4.1.6 Jeffrey-Kirwan residues.

Let $\mathfrak{a}$ be a n-dimensional $\mathbb{Q}$-linear space and assume that the dual space $\mathfrak{a}^\vee$ is endowed with a lattice Γ of full rank. Let $\mathfrak{A}$ be a *projective* finite subset of Γ, namely a set whose positive linear span doesn't contain a line (or in other words the positive span is a strictly convex cone).

Definition 4.1.6. An element $\xi \in \mathfrak{a}^\vee$ is called a *regular stability* if there is no subset $S \subset \mathfrak{A}$ of cardinality $n-1$ so that $\xi \in \mathrm{Span}_{\mathbb{Q}_{\geqslant 0}}(S)$. If we set

$$\Sigma\mathfrak{A} := \left\{ \sum_{w \in S} w \quad : \quad S \subset \mathfrak{A} \right\},$$

we say that ξ is *sum-regular* if there is no subset $S \subset \Sigma\mathfrak{A}$ of cardinality $n-1$ so that $\xi \in \mathrm{Span}_{\mathbb{Q}_{\geqslant 0}}(S)$. If ξ is a regular stability and $\tilde{\xi}$ is a sum regular stability so that the segment between them in $\mathfrak{a}^\vee$ is entirely made of regular stabilities, we say that $\tilde{\xi}$ is a *sum-regular perturbation of* ξ.

Remark 20. It's easy to check that every regular stability admits a sum-regular perturbation.

Example 4.1.3. Assume that a torus $\mathbb{T}$ of dimension n acts effectively on a complex linear space V, let $\mathfrak{A} \subset \chi(\mathbb{T})$ be the set of weights (the characters with which the torus acts on V) and choose a linearisation $\xi \in \chi(\mathbb{T})_{\mathbb{Q}}$. We can consider the linear space $\chi(\mathbb{T})_{\mathbb{Q}}^\vee$, whose dual $\chi(\mathbb{T})_{\mathbb{Q}}$ is endowed with the full rank lattice $\Gamma := \chi(\mathbb{T})$. It's well known [Dol03] that

1. $\mathfrak{A}$ is a projective subset of $\chi(\mathbb{T})_{\mathbb{Q}}$ if and only if there is a linearisation such that $V /\!/ T$ is projective.

2. The stability ξ is regular in the sense of Definition 4.1.6 if and only if the corresponding linearisation is regular in the usual sense, namely semistability and stability agree.

In this context, the character ξ is sum-regular if and only if there is no set of $n-1$ elements of $\Sigma\mathfrak{A}$ so that ξ belongs to their positive span.

Pick any integral basis of the lattice Γ and define $d\mu$ as the top form on $\mathfrak{a}$ given by the wedge product of the elements of the basis (this is well defined up to sign). Given a flag F spanned by some elements of $\mathfrak{A}$, we can define some elements $\kappa_1, \ldots, \kappa_n \in \Gamma$ as

$$\kappa_i := \sum_{w \in \mathfrak{A} \cap F_i} w. \tag{4.6}$$

Definition 4.1.7. If these elements form a basis $\kappa := \{\kappa_1, ..., \kappa_n\}$ of $\mathfrak{a}^\vee$ we say that the flag F is *proper*. Fixed a sum regular stability ξ, if $\xi \in \mathrm{span}_{\mathbb{Q}_{>0}}(\kappa)$ we say that the flag is *stable*. We denote with $\mathcal{F}(\mathfrak{A}, \xi)$ the set of all proper stable flags spanned by elements of $\mathfrak{A}$.

There is a residue operation induced by every such flag:

Definition 4.1.8. Let $F \in \mathcal{F}(\mathfrak{A}, \xi)$ be a proper stable flag. The *flag residue* of a meromorphic function f on $\mathfrak{a} \otimes_{\mathbb{Q}} \mathbb{C}$ is, up to a constant, the Szenes-Vergne residue computed with respect to κ:

$$\mathrm{Res}_F(f) := \left| \frac{d\mu}{\kappa_1 \wedge \cdots \wedge \kappa_n} \right| \mathrm{SV}_{\kappa_1,\dots,\kappa_n}(f). \tag{4.7}$$

Finally we can define the Jeffrey-Kirwan residue operation:

Definition 4.1.9. Fix a finite projective set $\mathfrak{A} \subset \Gamma$ and a sum-regular stability $\xi \in \mathfrak{a}^\vee$. The Jeffrey-Kirwan residue of a meromorphic function $f : \mathfrak{a} \otimes_{\mathbb{Q}} \mathbb{C} \dashrightarrow \mathbb{C}$ is

$$\mathrm{JK}^{\mathfrak{A}}_{\xi}(f) := \sum_{F \in \mathcal{F}(\mathfrak{A},\xi)} \mathrm{Res}_F(f).$$

Given $a \in \mathfrak{a} \otimes_{\mathbb{Q}} \mathbb{C}$, we will denote the residue at this point with

$$\mathrm{JK}^{\mathfrak{A}}_{\xi,a}(f) := \mathrm{JK}^{\mathfrak{A}}_{\xi}(f \circ \tau_a),$$

where τ_a is translation by a.

Here is an important remark that will allow us to work with regular stabilities ξ which are not sum-regular:

Remark 21 (Definition for regular stabilities). Consider the hyperplane arrangement $\mathcal{H}$ in $\mathfrak{a}^\vee$ defined by the hyperplanes spanned by elements of $\mathfrak{A}$. From the description in [SV04, Equation (2.1)] the residue $\mathrm{JK}^{\mathfrak{A}}_{\tilde{\xi}}(f)$ of a function $f \in \mathrm{Mer}_{\mathcal{H}}$ doesn't depend on the particular sum-regular stability $\tilde{\xi}$ but only on the chamber of the hyperplane arrangement defined by the elements of $\mathfrak{A}$. In particular, given a regular but not sum-regular stability ξ, every sum-regular perturbation $\tilde{\xi}$ gives the same result. Thus, whenever ξ is only regular and $f \in \mathrm{Mer}_{\mathcal{H}}$, we define

$$\mathrm{JK}^{\mathfrak{A}}_{\xi}(f) := \mathrm{JK}^{\mathfrak{A}}_{\tilde{\xi}}(f)$$

where $\tilde{\xi}$ is any fixed sum-regular perturbation of ξ.

The following is an immediate corollary of Lemma 4.1.3

Lemma 4.1.5. *Let $D \subset \mathfrak{a} \otimes \mathbb{C}$ be an open neighborhood of the origin. Consider a hyperplane arrangement $\mathcal{H}$ and a sequence of meromorphic functions $f_k : D \dashrightarrow \mathbb{C}$ with poles on $\mathcal{H}$. Assume that there is a compact subset $K \subset D$ so that f_k converges uniformly to $f \in Mer_{\mathcal{H}}$ on $D \backslash K$. Then*

$$\lim_{k \to \infty} JK_{\xi}^{\mathfrak{A}}(f_k) = JK_{\xi}^{\mathfrak{A}}(f).$$

Proof. Write the JK residue as a sum of SV residues and use Lemma 4.1.3. □

At some point we will also need the following simple fact which immediately follows from the definition of JK residue as sum of iterated residues and from the fact that $\mathrm{res}_{x=0} f(\lambda x) = \lambda^{-1} \mathrm{res}_{x=0} f(x)$ for any $\lambda \in \mathbb{C}^*$.

Lemma 4.1.6. *Given a meromorphic function $f : \mathfrak{a} \otimes \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, then $JK_{\xi}^{\mathfrak{A}}(f(\lambda x)) = \lambda^{-dim(\mathfrak{a})} JK_{\xi}^{\mathfrak{A}}(f(x))$.*

Example 4.1.4. Here we discuss a simple computation of a JK residue. Consider the linear space $\mathfrak{a} \simeq \mathbb{Q}^2$. In the dual space $\mathfrak{a}^\vee \simeq \mathbb{Q}^2$ consider the lattice $\mathbb{Z}^2$ and let $t_1, t_2 \in \mathfrak{a}^\vee$ be the coordinates on $\mathfrak{a}$. Set $\mathfrak{A} = \{t_1, t_2\}$ and consider the regular stability $t_1 + t_2 \in \mathfrak{a}^\vee$. This is regular, since it is contained neither in the span of t_1 nor in the span of t_2. Unfortunately, ξ is not sum-regular being in the span of $t_1 + t_2$, so we pick a sum-regular perturbation $\tilde{\xi} = t_1 + (1+\epsilon) t_2$ with $\epsilon > 0$ small. Assume we want to compute the JK residue of the following rational function

$$f : \mathfrak{a} \otimes \mathbb{C} \dashrightarrow \mathbb{C} \quad : \quad f = \left(\frac{t_1 - t_2}{t_1 t_2} \right)^2 .$$

First of all we must consider the set $\mathcal{F}(\mathfrak{A}, \tilde{\xi})$ of proper stable flags we can extract from $\mathfrak{A}$. There are two flags generated by elements of $\mathfrak{A}$:

$$0 \subset \mathrm{span}(t_1) \subset \mathfrak{a}^\vee \quad \text{and} \quad 0 \subset \mathrm{span}(t_2) \subset \mathfrak{a}^\vee .$$

For the first flag the corresponding vectors (4.6) are $\kappa_1 = t_1$, $\kappa_2 = t_1 + t_2$, hence the flag is proper (since κ_1, κ_2 form a basis) but not stable (since $\tilde{\xi}$ is not in the positive span of κ_1, κ_2). On the other hand for the second flag we have $\kappa_1 = t_2$, $\kappa_2 = t_1 + t_2$ and the flag is proper and stable. The flag residue corresponding to this flag F is

$$\left| \frac{t_1 \wedge t_2}{t_1 \wedge t_2} \right| \mathrm{res}_{t_2=0} \mathrm{res}_{t_1=0}(f) = \mathrm{res}_{t_2=0} \mathrm{res}_{t_1=0} \left(\frac{1}{t_1^2} - \frac{2}{t_1 t_2} + \frac{1}{t_2^2} \right) = -2$$

and this is $\mathrm{JK}_{\xi}^{\mathfrak{A}}(f)$.

Example 4.1.5. Here we give another example which will be used in a later geometric situation. Consider the linear space $\mathfrak{a} \simeq \mathbb{Q}^2$, whose coordinates we denote with u_1, u_2 and the projective set $\mathfrak{A} := \{u_1, u_2\} \subset \mathfrak{a}^\vee$, spanning a lattice Γ defining the top form $d\mu := u_1 \wedge u_2$ on $\mathfrak{a}$. Let $\xi \in \mathfrak{a}^\vee$ be the regular stability $\xi = u_1 + u_2$. Fixed a generic parameter $s \in \mathbb{C}$, we want to compute $\mathrm{JK}_\xi^{\mathfrak{A}}(Z)$ for the meromorphic function on $\mathfrak{a} \otimes \mathbb{C}$ given by

$$\begin{aligned} Z(u_1, u_2) = &\left(\frac{\pi}{\sin(\pi s)}\right)^2 \left(\frac{\sin(\pi(s-u_1))}{\sin(\pi u_1)}\right)^4 \left(\frac{\sin(\pi(s-u_2))}{\sin(\pi u_2)}\right)^4 \\ &\times \frac{\sin(\pi(4u_1+4u_2))}{\sin(\pi(s-4u_1-4u_2))} \frac{\sin(\pi(u_2-u_1))}{\sin(\pi(s+u_2-u_1))} \frac{\sin(\pi(u_1-u_2))}{\sin(\pi(s+u_1-u_2))}. \end{aligned}$$

Notice that around the origin Z has poles only on the hyperplanes generated by elements of $\mathfrak{A}$, so the Jeffrey-Kirwan residue is well defined even if ξ is just a regular (and not sum-regular) stability by Remark 21. In order to compute the Jeffrey-Kirwan residue we have to perturb ξ into a sum-regular stability $\tilde{\xi}$. We chose $\tilde{\xi} := \frac{1}{10}(11u_1 + 9u_2)$. There are two possible flags of $\mathfrak{a}^\vee$ that we can generate with the two vectors $\{u_1, u_2\}$ and here we compute their contribution to the JK residue:

- Consider first the flag $F_1 := 0 \subset \mathrm{span}_\mathbb{C}(u_1) \subset \mathfrak{a}^\vee$. The corresponding basis κ is given by $\kappa_1 = u_1$ and $\kappa_2 = u_1 + u_2$, so

  $$\tilde{\xi} = \frac{2}{10}\kappa_1 + \frac{9}{10}\kappa_2$$

  and the flag is stable. The contribution of this flag is by definition

  $$\begin{aligned} \left|\frac{d\mu}{\kappa_1 \wedge \kappa_2}\right| \mathrm{Res}_{\kappa_2=0}\mathrm{Res}_{\kappa_1=0} Z(u_1, u_2) &= \mathrm{Res}_{\kappa_2=0}\mathrm{Res}_{\kappa_1=0} Z(\kappa_1, \kappa_2 - \kappa_1) \\ &= \mathrm{Res}_{u_2=0}\mathrm{Res}_{u_1=0} Z(u_1, u_2), \end{aligned}$$

  where the second equality follows by Lemma 4.1.2. After some computations, taking the residue first at $u_1 = 0$ and then at $u_2 = 0$ gives

  $$352\sin(\pi s)\left(\cos^2(\pi s)\cot(\pi s) + \sin(\pi s)\cos(\pi s)\right)$$

  as result.

- The second flag is $F_2 := 0 \subset \mathrm{span}_\mathbb{C}(u_2) \subset \mathfrak{a}^\vee$. The corresponding basis κ is given by $\kappa_1 = u_2$ and $\kappa_2 = u_1 + u_2$, so

  $$\tilde{\xi} = -\frac{2}{10}\kappa_1 + \frac{11}{10}\kappa_2$$

  and the flag is unstable, hence it gives no contribution to the JK residue.

We have finally shown that

$$\mathrm{JK}_{\xi}^{\{u_1,u_2\}}(Z) = 352\sin(\pi s)\left(\cos^2(\pi s)\cot(\pi s) + \sin(\pi s)\cos(\pi s)\right).$$

4.2 An introductory example.

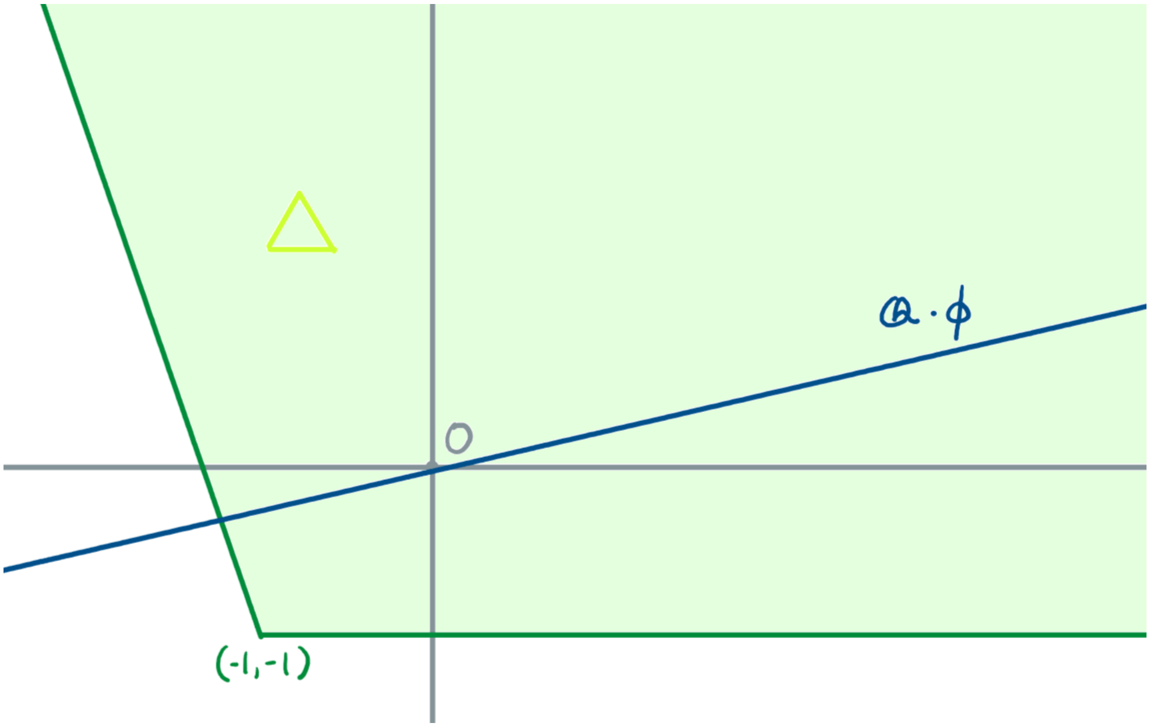
This part of the thesis might be a bit technical, so let's start with a concrete example which shows the procedure we are going to follow. Assume we are interested in the quotient for the action of $\mathbb{T} := \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{A}^2$ given by $(t_1,t_2)\cdot(x,y) := (t_1^{-1}t_2^3 x, t_1 y)$ with respect to the linearisation given by the character $\xi := t_1 t_2$. Notice that the stacky quotient of the semistable locus is $[(\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{T}] = [1/\mu_3]$, hence we expect that

$$\deg(r_{\mathbb{T}}([\mathbb{A}^2])) = \deg([1/\mu_3]) = \frac{1}{3}.$$

Keep in mind that, with this example, our aim is not to just compute this particular intersection number, which is an easy task due to the simple geometry involved. We want to show how, following a prescription that can be applied in general, we can express this intersection number, originally defined in terms of data on the quotient of $\mathbb{A}^2$ by $\mathbb{T}$, as a sum of intersection numbers which can be computed on quotients of subvarieties $Y_w \subset \mathbb{A}^2$ by smaller dimensional tori $T \subset \mathbb{T}$. The reason behind this strategy lies in the possibility to recursively apply this procedure. In the following sections, we'll demonstrate how this iterative approach results to an expression for this intersection number in terms of degrees of cycles defined on proper subschemes (points!) of the original space $\mathbb{A}^2$.

4.2.1 The polytope and the first algebraic cut.

In Lemma 4.5.1 the momentum polytope will be shown to be the cone spanned by the elements $(-1,3)$ and $(1,0)$ of the linear space $\chi(T)_{\mathbb{Q}} \simeq \mathbb{Q}^2$ (translated so that the vertex of the cone is at $-\xi$). To start we pick a character, for example $\phi = t_1^N t_2$ with $N > 1$, so that the ray $\mathbb{Q}_{<0}\cdot\phi$ is not entirely contained in Δ and it meets the codimension 1 stratum in smooth points:



The wall w met by this ray has, as associated subvariety, $Y_w := \mathbb{A}^1 \times O$. The choice of such character induces a splitting of the torus $\mathbb{T}$ in kernel plus complement; if we denote with T the subtorus of elements of the form (t^{-1}, t^N) and with $\mathbb{C}_1^*$ the subtorus of elements of the form $(1, s)$ we have the splitting $\mathbb{T} \simeq T \times \mathbb{C}_1^*$. The main idea behind the following localisation procedure is that the original quotient $[1/\mu_3]$ we want to study is the quotient of $\mathbb{A}^2/\!/T$ by the residual action of $\mathbb{C}_1^*$, hence it is embedded as a fixed locus in the algebraic cut for $\mathbb{C}_1^* \curvearrowright \mathbb{A}^2/\!/T$. Let's describe the quotient $\mathbb{A}^2/\!/T$. The induced action is $t \cdot (x, y) = (t^{3N+1}x, t^{-1}y)$ and the linearisation is given by the character t^{N-1}, hence the quotient is

$$\mathbb{A}^2/\!/T \simeq \operatorname{Proj}\left(\mathbb{C}[xy^{3N+1}][x]\right) \simeq \mathbb{A}^1$$

where xy^{3N+1} has degree zero and x has positive degree. The quotient map is

$$\mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1 \quad (x, y) \mapsto xy^{3N+1}.$$

Notice that the residual $\mathbb{C}_1^*$-action is given by $s \cdot z = s^3 z$ and comes with the induced linearisation corresponding to the character s (notice that the quotient by this action is indeed $[1/\mu_3]$). We can now consider the algebraic cut of $\mathbb{A}^2/\!/T \simeq \mathbb{A}^1$. This is the quotient of $\mathbb{C}_1^* \curvearrowright \mathbb{A}^1 \times \mathbb{A}^1$ given by $s \cdot (z, w) = (s^3 z, sw)$ with respect to the linearisation given by the character s. We immediately see that

$$(\mathbb{A}^1)_c = \mathbb{A}^1 \times \mathbb{A}^1 /\!/ \mathbb{C}_1^* \simeq \operatorname{Proj}\left(\mathbb{C}[z, w^3]\right) \simeq \mathbb{P}^1$$

and that the quotient map is

$$\mathbb{A}^2 \backslash O \to \mathbb{P}^1 \quad : \quad (z, w) \mapsto [z : w^3].$$

4.2.2 Localisation on the first algebraic cut.

Notice that our original quotient of $\mathbb{A}^2$ by $\mathbb{T}$ corresponds to the quotient of $\mathbb{C}^* \times 0 \subset \mathbb{A}^2 \backslash O$ and hence to the point $[1 : 0] \in \mathbb{P}^1$. On the other hand, the point $[0 : 1]$ corresponds to the origin in $\mathbb{A}^1 \simeq \mathbb{A}^2/\!/T$ and hence to the locus Y_w in the original $\mathbb{A}^2$. They are fixed points for the residual $\mathbb{C}_1^*$-action on $(\mathbb{A}^1)_c$ induced by the action on $\mathbb{A}^2/\!/T$, which is $s \cdot [x_0 : x_1] = [s^3 x_0 : x_1]$, hence the localisation Theorem 2.3.3 ensures that

$$[\mathbb{P}^1] = i_* \frac{[1:0]}{e^{\mathbb{C}^*}(\mathcal{N}_{[1:0]/\mathbb{P}^1})} + j_* \frac{[0:1]}{e^{\mathbb{C}^*}(\mathcal{N}_{[0:1]/\mathbb{P}^1})} = i_* \frac{[1:0]}{-3s} + j_* \frac{[0:1]}{3s}. \tag{4.8}$$

Now the trick is to try to write the right-hand side of the previous equality in terms of the data of the original action on $\mathbb{A}^2$. For example, with the aid of Theorem 2.1.1, we can immediately rewrite the equality above as

$$[\mathbb{P}^1] = i_* \frac{r_{\mathbb{T}}([\mathbb{A}^2])}{-s} + (3N+1) j_* \frac{r_T([Y_w])}{3s},$$

where the coefficient $3N+1$ comes from the cardinality of the T-stabiliser of Y_w. By applying the pushforward to a point and the degree map we obtain an equality of the form

$$\text{polynomial in } (s) = \frac{\deg\left(r_{\mathbb{T}}([\mathbb{A}^2])\right)}{-s} + (3N+1) \frac{\deg(r_T[Y_w])}{3s}.$$

By applying the residue map $\operatorname{res}_{s=0}$ we obtain

$$\deg\left(r_{\mathbb{T}}([\mathbb{A}^2])\right) = \frac{3N+1}{3} \deg\left(r_T\left([Y_w]\right)\right) \tag{4.9}$$

which reaches our goal to express the degree of our intersection number in terms of degrees of intersection numbers computed in quotients by smaller dimensional tori.

4.2.3 The second algebraic cut, and localisation there.

By further considering the algebraic cut of Y_w with respect to the action of T we would obtain $(Y_2)_c \simeq \mathbb{P}^1$ with quotient map

$$Y_w \times \mathbb{A}^1 \dashrightarrow \mathbb{P}^1 \quad : \quad (x, z) \mapsto [x, z^{3N+1}].$$

The induced $\mathbb{C}^*$-action inherited from $T \curvearrowright Y_w$ is the action $t \cdot [x_0 : x_1] = [t^{3N+1}x_0 : x_1]$. Moreover, $Y_w /\!/ T$ is embedded as the fixed point $[1 : 0]$ while the origin $O \in Y_w$ is embedded as $[0 : 1]$. By applying the localisation formula 2.3.3, pushing forward to a point and taking the degree as done in the previous step, we obtain the equality

$$\deg\left(r_T\left([Y_w]\right)\right) = \frac{1}{3N+1} \deg\left([O]\right).$$

Putting this together with (4.9) we obtain

$$\deg\left(r_{\mathbb{T}}([\mathbb{A}^2])\right) = \frac{1}{3} \deg\left([O]\right)$$

which finally expresses the intersection number on the quotient of $\mathbb{A}^2$ by $\mathbb{T}$ in terms of an intersection number computed at the origin (the fixed locus) of $\mathbb{A}^2$. In particular, since the degree of the point $[O]$ is 1, we recover the expected result 1/3.

4.3 The first step of localisation.

Assume that a torus $\mathbb{T}$ acts on a smooth variety Y via an equivariant embedding in $\mathbb{P}^a \times \mathbb{A}^b$ and let $\mathcal{L}$ be a regular linearisation so that $Y /\!/ \mathbb{T}$ is projective. Given a class $\alpha \in A_*^{\mathbb{T}}(Y)$ we are interseted in finding a residue formula for the number

$$\deg(r_{\mathbb{T}}(\alpha)). \tag{4.10}$$

We start by choosing a nonzero primitive character $\phi \in \chi(\mathbb{T})$. We can pick a splitting of the form $\mathbb{T} \simeq T \times \mathbb{C}_1^*$ where T is the connected component of $\ker(\phi)$ containing 1 and $\mathbb{C}^* \simeq \mathbb{C}_1^* \subseteq \mathbb{T}$ is a subtorus of rank 1 satisfying $\phi_{|\mathbb{C}^*} = \mathrm{id}$. We consider an additional copy $\mathbb{C}_2^* \simeq \mathbb{C}^*$ and the action

$$T \times \mathbb{C}_1^* \times \mathbb{C}_2^* \curvearrowright Y \times \mathbb{A}^1 \quad : \quad (t, s_1, s_2) \cdot (x, z) := (t \cdot s_1 \cdot x, s_1 s_2^{-1} z).$$

Notice that the GIT quotient of $Y \times \mathbb{A}^1$ by $\mathbb{T}$, via the regular linearisation $\mathcal{L}$ pulled back from Y, is the algebraic cut $(Y /\!/ T)_c$ for the residual action $\mathbb{C}_1^* \curvearrowright Y /\!/ T$. We can easily describe the semistable locus for this new action on $Y \times \mathbb{A}^1$:

Lemma 4.3.1. *The $\mathbb{T}$-semistable locus of $Y \times \mathbb{A}^1$ is the open subscheme*

$$(Y \times \mathbb{A}^1)(\mathbb{T})^{ss} = (Y(\mathbb{T})^{ss} \times 0) \cup (Y^- \times \mathbb{C}^*),$$

where Y^- is the inverse image of $(Y/\!/T)^-$ along the quotient map $\pi : Y(T)^{ss} \to Y/\!/T$.

Proof. It's a straightforward consequence of the compatibility of semistability with taking quotients of Lemma 2.2.2. Let $\tilde{\pi} : (Y \times \mathbb{A}^1)(T)^{\mathrm{ss}} \to Y/\!/T \times \mathbb{A}^1$ be the quotient map for the T-action. Then we have that

$$\begin{aligned}(Y \times \mathbb{A}^1)(\mathbb{T})^{\mathrm{ss}} &= \tilde{\pi}^{-1}\left((Y/\!/T \times \mathbb{A}^1)(\mathbb{C}^*)^{\mathrm{ss}}\right) \\ &= \tilde{\pi}^{-1}\left((Y/\!/T)(\mathbb{C}^*)^{\mathrm{ss}} \times 0\right) \cup \tilde{\pi}^{-1}\left((Y/\!/T)^- \times \mathbb{C}^*\right)\end{aligned}$$

where the second equality is given by Lemma 3.3.1. Since $\tilde{\pi}$ acts as π on the first factor and as the identity on the second factor, this is equal to

$$\begin{aligned}&= \left(\pi^{-1}((Y/\!/T)(\mathbb{C}^*)^{\mathrm{ss}}) \times 0\right) \cup \left(Y^- \times \mathbb{C}^*\right) \\ &= (Y(\mathbb{T})^{\mathrm{ss}} \times 0) \cup \left(Y^- \times \mathbb{C}^*\right).\end{aligned}$$

□

In order to obtain a well behaved algebraic cut $(Y/\!/T)_c$ we want to choose the character ϕ such that:

- the negative ray of the line $\mathbb{Q} \cdot \phi$ is not entirely contained in the momentum polytope Δ. By Proposition 3.4.1 this implies that the algebraic cut $(Y/\!/T)_c$ is projective.

- the line $\mathbb{Q} \cdot \phi$ intersects the singular variety $\partial_1\Delta$ in its smooth points, away from the codimension 2 walls. In this case Proposition 3.4.2, together with Lemma 3.3.2, describes completely the fixed locus of the algebraic cut $(Y/\!/T)_c$. One connected component is given by the quotient we want to study, namely $i : Y/\!/\mathbb{T} \hookrightarrow (Y/\!/T)_c$, via the quotient of the $\mathbb{T}$-equivariant morphism

  $$I : Y(\mathbb{T})^{\mathrm{ss}} \times 0 \hookrightarrow (Y \times \mathbb{A}^1)(\mathbb{T})^{\mathrm{ss}}.$$

  For every intersection of the ray $\mathbb{Q}_{<0} \cdot \phi$ with a wall w of Δ, the subvariety $j_w : Y_w/\!/T \hookrightarrow (Y/\!/T)_c$ given by the quotient of the $\mathbb{T}$-equivariant morphism

  $$J_w : (Y_w(T)^{\mathrm{ss}} \times \mathbb{C}^*)_{\mathrm{opp}} \hookrightarrow (Y^- \times \mathbb{C}^*)_{\mathrm{opp}} \overset{\Lambda}{\hookrightarrow} (Y \times \mathbb{A}^1)(\mathbb{T})^{\mathrm{ss}} \tag{4.11}$$

  is fixed. The subscript "opp" denotes the fact that the actions of the two rank 1 tori $\mathbb{C}^*_1$ and $\mathbb{C}^*_2$ are exchanged and the morphism Λ is the equivariant open embedding $\Lambda(y, z) := (z^{-1} \cdot y, z^{-1})$. Moreover, these are all the fixed loci of the algebraic cut.

The main idea is to use the localisation formula of Theorem 2.3.3 to express the intersection number (4.10), computed in $Y/\!/\mathbb{T}$, in terms of intersection numbers that can be computed on the ambient algebraic cut $(Y/\!/T)_c$ and on the fixed loci $Y_w/\!/T$. The first goal that we need to accomplish, in order to use the localisation theorem in an effective way, is to invert the localisation isomorphism.

Lemma 4.3.2. *Since I is a regular embedding, it defines a pullback I^* of equivariant Chow groups. This descends to a morphism i^* defined by the following diagram*

$$\begin{tikzcd} A^{\mathbb{T}\times\mathbb{C}_2^*}((Y\times\mathbb{A}^1)(\mathbb{T})^{ss}) \arrow[r, "I^*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y(\mathbb{T})^{ss}) \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\ A^{\mathbb{C}_2^*}((Y/\!/T)_c) \arrow[r, "i^*"] & A^{\mathbb{C}_2^*}(Y/\!/\mathbb{T}). \end{tikzcd}$$

The following self-intersection formula holds true:

$$i^*i_*\beta = (\hat{d}_{\mathbb{T}}(s_1) - s_2)\beta$$

where s_1, s_2 are the equivariant variables for $\mathbb{C}_1^*$, $\mathbb{C}_2^*$ and $\hat{d}_{\mathbb{T}} : A^{\mathbb{T}}(Y(\mathbb{T})^{ss}) \to A(Y/\!/\mathbb{T})$ is the descent map for $\mathbb{T} \curvearrowright Y$.

Proof. By Proposition 2.1.1 the square on the left of

$$\begin{tikzcd} A^{\mathbb{T}\times\mathbb{C}_2^*}(Y(\mathbb{T})^{\mathrm{ss}}) \arrow[r, "I_*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}((Y\times\mathbb{A}^1)(\mathbb{T})^{\mathrm{ss}}) \arrow[r, "I^*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y(\mathbb{T})^{\mathrm{ss}}) \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\ A^{\mathbb{C}_2^*}(Y/\!/\mathbb{T}) \arrow[r, "i_*"] & A^{\mathbb{C}_2^*}((Y/\!/T)_c) \arrow[r, "i^*"] & A^{\mathbb{C}_2^*}(Y/\!/\mathbb{T}) \end{tikzcd}$$

is commutative, and notice that here i_* is the honest pushforward by the embedding i. The composition of the two horizontal arrows in the first line is the multiplication by the equivariant Euler class of the normal bundle, hence by $s_1 - s_2$. This means that the composition of the two lower horizontal arrows is the multiplication by $\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}(s_1 - s_2) = \hat{d}_{\mathbb{T}}(s_1) - s_2$. □

We prove the same kind of statement for the other fixed loci:

Lemma 4.3.3. *Let w be a wall of Δ which is met by the negative ray of the line $\mathbb{Q}\cdot\phi$. Since J_w is a regular embedding, it defines a pullback J_w^* of equivariant Chow* *groups. This descends to a morphism j_w^* defined by the following diagram*

$$\begin{tikzcd} A^{\mathbb{T}\times\mathbb{C}_2^*}((Y\times\mathbb{A}^1)(\mathbb{T})^{ss}) \arrow[r, "J_w^*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}((Y_w(T)^{ss}\times\mathbb{C}^*)_{opp}) \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\ A^{\mathbb{C}_2^*}((Y/\!/T)_c) \arrow[r, "j_w^*"] & A^{\mathbb{C}_2^*}(Y_w/\!/T). \end{tikzcd}$$

The following self-intersection formula holds true:

$$j_w^* j_{w*}\beta = \beta\cdot ev_{s_1=s_2}\left(\hat{d}_{\mathbb{T},\mathbb{C}_1^*}\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)\right).$$

Proof. As in the proof of the previous proposition we consider the same diagram composed by the two squares defined by J_{w*} and J_w^*. Consider the regular closed embedding $K_w : Y_w \hookrightarrow Y$. If we set $Z := Y_w(T)^{\mathrm{ss}}\times\mathbb{C}^*$ we can consider the induced regular closed embedding

$$K_w : Z \hookrightarrow Y^-\times\mathbb{C}^*$$

which we denote with the same letter. Since J_w factors as K_w followed by an open embedding, the diagram defined by J_w reduces to

$$\begin{tikzcd} A^{\mathbb{T}\times\mathbb{C}_2^*}(Z_{\mathrm{opp}}) \arrow[r, "K_{w*}"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}((Y^-\times\mathbb{C}^*)_{\mathrm{opp}}) \arrow[r, "K_w^*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Z_{\mathrm{opp}}) \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\ A^{\mathbb{C}_2^*}((Y_w/\!/T)_{\mathrm{opp}}) \arrow[r, "k_{w*}"] & A^{\mathbb{C}_2^*}((Y/\!/T)^-_{\mathrm{opp}}) \arrow[r, "k_w^*"] & A^{\mathbb{C}_2^*}((Y_w/\!/T)_{\mathrm{opp}}). \end{tikzcd}$$

and the composition of the lower horizontal arrows is $j_w^* j_{w*}$. By inverting the role of $\mathbb{C}_1^*$ and $\mathbb{C}_2^*$ we find that $\mathrm{ev}_{s_1=s_2} j_w^* j_{w*} \mathrm{ev}_{s_2=s_1}$ is the composition of the horizontal arrows at the bottom of

$$\begin{tikzcd} A^{\mathbb{T}\times\mathbb{C}_2^*}(Z) \arrow[r, "K_{w*}"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_1^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y^-\times\mathbb{C}^*) \arrow[r, "K_w^*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_1^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Z) \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_1^*}"] \\ A^{\mathbb{C}_1^*}(Y_w/\!/T) \arrow[r, "k_{w*}"] & A^{\mathbb{C}_1^*}((Y/\!/T)^-) \arrow[r, "k_w^*"] & A^{\mathbb{C}_1^*}(Y_w/\!/T). \end{tikzcd}$$

By compatibility of descent maps we can split this diagram in two by taking the quotient with respect to the action of $\mathbb{C}_2^*$, which now only acts on the second factor

$\mathbb{C}^*$:

$$\begin{tikzcd}
A^{\mathbb{T}\times\mathbb{C}_2^*}(Z) \arrow[r, "K_{w*}"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{T}}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y^- \times \mathbb{C}^*) \arrow[r, "K_w^*"] \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{T}}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Z) \arrow[d, "\hat{d}_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{T}}"] \\
A^{\mathbb{T}}(Y_w(T)^{\mathrm{ss}}) \arrow[r, "K_{w*}"] \arrow[d, "\hat{d}_{\mathbb{T},\mathbb{C}_1^*}"] & A^{\mathbb{T}}(Y^-) \arrow[r, "K_w^*"] \arrow[d, "\hat{d}_{\mathbb{T},\mathbb{C}_1^*}"] & A^{\mathbb{T}}(Y_w(T)^{\mathrm{ss}}) \arrow[d, "\hat{d}_{\mathbb{T},\mathbb{C}_1^*}"] \\
A^{\mathbb{C}_1^*}(Y_w/\!\!/T) \arrow[r, "k_{w*}"] & A^{\mathbb{C}_1^*}((Y/\!\!/T)^-) \arrow[r, "k_w^*"] & A^{\mathbb{C}_1^*}(Y_w/\!\!/T).
\end{tikzcd}$$

Notice that composition of the horizontal arrows in the middle row is the multiplication by $e^{\mathbb{T}}(Y_w/Y)$ (to be precise, by the restriction of this class to the semistable locus), and therefore the self-intersection formula follows by the commutativity of the two lower squares. □

Now that we have the self-intersection formulae we can invert the localisation isomorphism. Recall that $\sigma_G(Z)$ denotes the order of the G-stabiliser of a general point in Z.

Proposition 4.3.1. *The following equation holds true in the localised equivariant Chow group $A^{\mathbb{C}_2^*}((Y/\!\!/T)_c)_{s_2}$:*

$$\frac{1}{\sigma_{\mathbb{T}}(Y\times\mathbb{A}^1)}[(Y/\!\!/T)_c] =$$
$$\frac{1}{\sigma_{\mathbb{T}}(Y)} i_*\left(\frac{[Y/\!\!/\mathbb{T}]}{\hat{d}_{\mathbb{T}}(s_1)-s_2}\right) + \sum_{\substack{w\in Wall_1(\Delta)\\ w\cap\mathbb{Q}_{<0}\cdot\phi\neq\varnothing}} \frac{1}{\sigma_T(Y_w)} j_{w*}\left(\frac{[Y_w/\!\!/T]}{ev_{s_1=s_2}\left(r_{\mathbb{T},\mathbb{C}_1^*}\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)\right)}\right)$$

Proof. The localisation Theorem2.3.3 shows that there are classes α and β_w such that

$$[(Y/\!\!/T)_c] = i_*\alpha + \sum_{\substack{w\in \mathrm{Wall}_1(\Delta)\\ w\cap\mathbb{Q}_{<0}\cdot\phi\neq\varnothing}} j_{w*}\beta_w \tag{4.12}$$

By applying i^* (defined in Lemma 4.3.2) we have that $i^*[(Y/\!\!/T)_c] = \frac{\sigma_{\mathbb{T}}(Y\times\mathbb{A}^1)}{\sigma_{\mathbb{T}}(Y)}[Y/\!\!/\mathbb{T}]$ by Theorem 2.1.1. Since $I^*\circ J_{w*} = 0$ we have that $i^*j_{w*} = 0$ and thus

$$\frac{\sigma_{\mathbb{T}}(Y\times\mathbb{A}^1)}{\sigma_{\mathbb{T}}(Y)}[Y/\!\!/\mathbb{T}] = i^*i_*\alpha = (\hat{d}_{\mathbb{T}}(s_1)-s_2)\alpha.$$

Since $\hat{d}_{\mathbb{T}}(s_1) - s_2$ is of the form "nilpotent + unit" we can invert it with the geometric series, so we have found α. Analogously, by applying j_w^* to (4.12) we find

$$\frac{\sigma_{\mathbb{T}}(Y \times \mathbb{A}^1)}{\sigma_{\mathbb{T}}(Y_w \times \mathbb{A}^1)}[Y_w /\!/ T] = \mathrm{ev}_{s_1 = s_2}\left(r_{\mathbb{T},\mathbb{C}_1^*}\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)\right)\beta_w$$

which again identifies β_w. The claim follows by noticing that $\sigma_{\mathbb{T}}(Y_w \times \mathbb{A}^1) = \sigma_T(Y_w)$. $\square$

We have to find a way to use this relation of classes in the Chow group of the cut to produce a residue formula for intersection numbers on $Y /\!/ \mathbb{T}$. The first step is to find a way to extend classes from $Y /\!/ \mathbb{T}$ to the whole algebraic cut $(Y /\!/ T)_c$:

Lemma 4.3.4. *There is a morphism*

$$s : A^{\mathbb{T}}(Y) \to A^{\mathbb{C}_2^*}((Y /\!/ T)_c)$$

satisfying the following conditions:

- $i^* \circ s = r_{\mathbb{T}}$. *In particular, classes of the form* $i^* \circ s(\alpha)$ *have trivial* $\mathbb{C}_2^*$-*equivariance.*
- *For every wall* $w \in Wall_1(\Delta)$ *meeting the ray* $\mathbb{Q}_{<0} \cdot \phi$, *consider the regular closed embedding* $K_w : Y_w \hookrightarrow Y$. *The equality* $j_w^* \circ s = ev_{s_1 = s_2}\, r_{\mathbb{T},\mathbb{C}_1^*} \circ K_w^*$ *holds true.*

Proof. First of all we define s as the composition

$$s : A^{\mathbb{T}}(Y) \xrightarrow{(\mathbb{T}\times\mathbb{C}_2^* \to \mathbb{T})^*} A^{\mathbb{T}\times\mathbb{C}_2^*}(Y) \xrightarrow{P^*} A^{\mathbb{T}\times\mathbb{C}_2^*}(Y \times \mathbb{A}^1) \xrightarrow{r_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}} A^{\mathbb{C}_2^*}((Y /\!/ T)_c),$$

where $\mathbb{T} \times \mathbb{C}_2^* \to \mathbb{T}$ is the homomorphism $(t, s_2) \mapsto t$ and $P : Y \times \mathbb{A}^1 \to Y$ is the projection $(y, z) \mapsto y$, which is flat and therefore induces a pullback on Chow groups. To prove the first claim we consider the diagram

$$\begin{tikzcd}
A^{\mathbb{T}}(Y) \arrow[r, "(\mathbb{T}\times\mathbb{C}_2^* \to \mathbb{T})^*"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y) \arrow[r, "P^*"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y \times \mathbb{A}^1) \arrow[r, "I^*"] \arrow[d, "r_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y \times 0) \arrow[d, "r_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\
& & A^{\mathbb{C}_2^*}((Y /\!/ T)_c) \arrow[r, "i^*"] & A^{\mathbb{C}_2^*}(Y /\!/ \mathbb{T}).
\end{tikzcd}$$

Notice that $P \circ I$ is the identity on Y, hence $i \circ s = r_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_2^*} \circ (\mathbb{T} \times \mathbb{C}_2^* \to \mathbb{T})^* = (\mathbb{C}_2^* \to 1)^* \circ r_{\mathbb{T}}$ where the last equality follows by Proposition 2.1.2. Notice that the change of groups homomorphism relative to $\mathbb{C}_2^* \to 1$ just corresponds to the inclusion

$A(Y/\!\!/\mathbb{T}) \hookrightarrow A^{\mathbb{C}_2^*}(Y/\!\!/\mathbb{T})$, which completes the proof of the first claim. For the second claim the proof is similar. First of all notice that s fits into the commutative diagram

$$\begin{tikzcd}
A^{\mathbb{T}}(Y) \arrow[r, "(\mathbb{T}\times\mathbb{C}_2^*\to\mathbb{T})"] \arrow[d, "r_{\mathbb{T},\mathbb{C}_1^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y) \arrow[r, "P^*"] \arrow[d, "r_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_1^*\times\mathbb{C}_2^*}"] & A^{\mathbb{T}\times\mathbb{C}_2^*}(Y\times\mathbb{A}^1) \arrow[d, "r_{\mathbb{T}\times\mathbb{C}_2^*,\mathbb{C}_1^*\times\mathbb{C}_2^*}"] \\
A^{\mathbb{C}_1^*}(Y/\!\!/T) \arrow[r, "(\mathbb{C}_1^*\times\mathbb{C}_2^*\to\mathbb{C}_1^*)"'] & A^{\mathbb{C}_1^*\times\mathbb{C}_2^*}(Y/\!\!/T) \arrow[r, "p^*"'] & A^{\mathbb{C}_1\times\mathbb{C}_2^*}(Y/\!\!/T\times\mathbb{A}^1) \arrow[d, "r_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\
 & & A^{\mathbb{C}_2^*}((Y/\!\!/T)_c)
\end{tikzcd} \tag{4.13}$$

as the composition of the upper row with the column to the right. Denote with $\tilde{s} : A^{\mathbb{C}_1^*}(Y/\!\!/T) \to A^{\mathbb{C}_2^*}((Y/\!\!/T)_c)$ the composition of the lowest row with the last arrow of the column to the right. We claim that $j_w \circ \tilde{s} = \mathrm{ev}_{s_2=s_1} k_w^*$, for the closed embedding $k_w : Y_w/\!\!/T \hookrightarrow (Y/\!\!/T)^-$. Notice that, as seen before, k_w might not be regular but the pullback k_w^* is still well defined, being induced from K_w^*. We have the commutative diagram

$$\begin{tikzcd}
A^{\mathbb{C}_1^*\times\mathbb{C}_2^*}(Y_w/\!\!/T) \arrow[r, "p^*"] & A^{\mathbb{C}_1^*\times\mathbb{C}_2^*}(Y_w/\!\!/T\times\mathbb{C}^*) \arrow[dr, "\lambda^*"] & \\
A^{\mathbb{C}_1^*\times\mathbb{C}_2^*}((Y/\!\!/T)^-) \arrow[r, "p^*"] \arrow[u, "k_w^*"] & A^{\mathbb{C}_1^*\times\mathbb{C}_2^*}((Y/\!\!/T)^-\times\mathbb{C}^*) \arrow[r, "k_w^*\circ\lambda^*"] \arrow[u, "k_w^*"] \arrow[d, "\hat{d}_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] & A^{\mathbb{C}_1^*\times\mathbb{C}_2^*}((Y_w/\!\!/T\times\mathbb{C}^*)_{\mathrm{opp}}) \arrow[d, "\hat{d}_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*}"] \\
 & A^{\mathbb{C}_2^*}((Y/\!\!/T)_c) \arrow[r, "j_w^*"] & A^{\mathbb{C}_2^*}(Y_w/\!\!/T)
\end{tikzcd}$$

where $\lambda : ((Y/\!\!/T)^- \times \mathbb{C}^*)_{\mathrm{opp}} \to (Y/\!\!/T \times \mathbb{A}^1)^{\mathrm{ss}}$, given by $\lambda(y,z) := (z^{-1}\cdot y, z^{-1})$, is the morphism induced on the T-quotient by the Λ in (4.11). Notice that $\tilde{s}$ (composed with the restriction to the open subscheme $u : (Y/\!\!/T)^- \hookrightarrow Y/\!\!/T$) is the morphism one obtains by following the lowest path in the diagram. By commutativity of the diagram

$$j_w^* \circ \tilde{s} \circ u_* = \hat{d}_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*} \circ (p\circ\lambda)^* \circ k_w^*. \tag{4.14}$$

On the other hand, the composition of the two morphisms at the top of the diagram is the pullback along the flat morphism

$$p\circ\lambda : (Y_w/\!\!/T\times\mathbb{C}^*)_{\mathrm{opp}} \to Y_w/\!\!/T \quad : \quad p\circ\lambda(y,z) = z^{-1}\cdot y = y$$

(where we have used that $Y_w//T$ is fixed by $\mathbb{C}_2^*$) and therefore, for every closed subvariety Z of $Y_w//T$

$$\hat{d}_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*}\circ(p\circ\lambda)^*[Z] = \hat{d}_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*}[Z\times\mathbb{C}^*] = [Z]$$

where the last equality follows by Theorem 2.1.1. Moreover $\hat{d}_{\mathbb{C}_1^*\times\mathbb{C}_2^*,\mathbb{C}_2^*}\circ(p\circ\lambda)^*(s_1) = s_2$. This shows that equation (4.14) can be rewritten as

$$j_w^*\circ\tilde{s}\circ u_* = \mathrm{ev}_{s_1=s_2}k_w^*.$$

By applying u^* to the right of this equality we find

$$j_w^*\circ\tilde{s}\circ u_*u^* = \mathrm{ev}_{s_1=s_2}(u\circ k_w)^*.$$

It's immediate from the definition of $\tilde{s}$ that $j_w^*\tilde{s}([Z]) = 0$ for every subvariety Z contained in $(Y//T)\backslash(Y//T)^-$, hence $j_w^*\circ\tilde{s} = \mathrm{ev}_{s_1=s_2}(u\circ k_w)^*$. From diagram (4.13) we conclude that $j_w^*\circ s = \mathrm{ev}_{s_1=s_2}(u\circ k_w)^*\circ r_{\mathbb{T},\mathbb{C}_1^*} = \mathrm{ev}_{s_1=s_2}\circ r_{\mathbb{T},\mathbb{C}_1^*}K_w^*$. □

Putting together Proposition 4.3.1 and Lemma 4.3.4 we have the following

Proposition 4.3.2. *Let $\alpha\in A^{\mathbb{T}}(Y)$. The intersection number $deg(r_{\mathbb{T}}(\alpha))$ is equal to*

$$\sum_{\substack{w\in Wall_1(\Delta)\\ w\cap\mathbb{Q}_{<0}\cdot\phi\neq\varnothing}}\langle\lambda,\phi\rangle\cdot deg\,\pi_{Y_w//T*}\left(r_T\left(res_{s_\lambda=0}\,(T\times\lambda\to\mathbb{T})^*\frac{\alpha_{|Y_w}}{e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})}\right)\right). \tag{4.15}$$

where, given a wall w, the morphism $r_T : A^T(Y_w)\to A(Y_w//T)$ is the Kirwan map for the T-action on Y_w and the residue operation is defined as follows. Given the rank 1 subtorus $\lambda\in\chi(\mathbb{T})^\vee$ fixing Y_w and such that $\langle\lambda,\phi\rangle>0$, consider the surjection $T\times\lambda\to\mathbb{T}$. The homomorphism $(T\times\lambda\to\mathbb{T})^*$ is the change of group homomorphism with respect to this surjection. Then $A^{\mathbb{T}}(Y_w)\simeq A^T(Y_w)[s_\lambda]$, where s_λ is the equivariant variable corresponding to λ, and the residue is the generalised Guillemin-Kalkman residue GK_{s_λ} for the ring $A^T(Y_w)$ and the variable s_λ as defined in (4.4).

Proof. We intersect the fundamental class $[(Y//T)_c]$ with $s(\alpha)$ using the stacky ring structure on $A^{\mathbb{C}_2^*}((Y//T)_c)_{s_2}$ induced from the isomorphism with $A^{\mathbb{T}\times\mathbb{C}_2^*}((Y\times\mathbb{A}^1)(\mathbb{T})^{\mathrm{ss}})$. By the localisation formula of Proposition 4.3.1, the projection formula and Lemma 4.3.4 we get

$$\frac{1}{\sigma_{\mathbb{T}}(Y\times\mathbb{A}^1)}s(\alpha)\cap[(Y//T)_c] =$$

$$\frac{1}{\sigma_{\mathbb{T}}(Y)}i_*\left(\frac{r_{\mathbb{T}}(\alpha)\cap[Y//\mathbb{T}]}{\hat{d}_{\mathbb{T}}(s_1)-s_2}\right)+\mathrm{ev}_{s_1=s_2}\sum_{\substack{w\in\mathrm{Wall}_1(\Delta)\\ w\cap\mathbb{Q}_{<0}\cdot\phi\neq\varnothing}}\frac{1}{\sigma_T(Y_w)}j_{w*}\left(\frac{r_{\mathbb{T},\mathbb{C}_1^*}(\alpha_{|Y_w})\cap[Y_w//T]}{r_{\mathbb{T},\mathbb{C}_1^*}\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)}\right)$$

where $\cap$ denotes the stacky product of Remark 4. Since this product satisfies

$$s(\alpha) \cap [(Y/\!/T)_c] = \sigma_{\mathbb{T}}(Y \times \mathbb{A}^1) s(\alpha)$$

(and analogously for the other terms), we apply the pushforward to a point and the degree map obtaining the following equality in $\mathbb{C}[s_2]_{s_2}$:

$$\begin{aligned} &\deg \pi_{(Y/\!/T)_c *}(s(\alpha)) = \\ &\deg \pi_{Y/\mathbb{T}*}\left(\frac{r_{\mathbb{T}}(\alpha)}{\hat{d}_{\mathbb{T}}(s_1) - s_2}\right) + \mathrm{ev}_{s_1 = s_2} \sum_{\substack{w \in \mathrm{Wall}_1(\Delta) \\ w \cap \mathbb{Q}_{<0}\cdot\phi \neq \varnothing}} \deg \pi_{Y_w/\!/T*}\left(\frac{r_{\mathbb{T},\mathbb{C}_1^*}(\alpha_{|Y_w})}{r_{\mathbb{T},\mathbb{C}_1^*}\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)}\right). \end{aligned}$$

Notice that $\deg \pi_{(Y/\!/T)_c *}(s(\alpha)) \in \mathbb{C}[s_2]$ being the degree of an equivariant class on $(Y/\!/T)_c$. By taking the residue with respect to s_2 we immediately obtain

$$\deg \pi_{Y/\mathbb{T}*}(r_{\mathbb{T}}(\alpha)) = \sum_{\substack{w \in \mathrm{Wall}_1(\Delta) \\ w \cap \mathbb{Q}_{<0}\cdot\phi \neq \varnothing}} \mathrm{res}_{s_1=0} \deg \pi_{Y_w/\!/T*}\left(\frac{r_{\mathbb{T},\mathbb{C}_1^*}(\alpha_{|Y_w})}{r_{\mathbb{T},\mathbb{C}_1^*}\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)}\right). \tag{4.16}$$

Finally we want to simplify the residue computation for every wall. Given a wall w we can consider the one dimensional subtorus $\lambda \in \chi(T)^\vee_{\mathbb{Q}}$ such that λ acts trivially on Y_w and such that $\langle \lambda, \phi \rangle = 1$. Once we consider the surjection $T \times \lambda \to \mathbb{T}$ and the corresponding morphism $\xi_w : T \times \lambda \to \mathbb{T} \xrightarrow{\sim} T \times \mathbb{C}_1^*$, inducing $\overline{\xi}_w : \lambda \to \mathbb{C}_1^*$ on the second factors, we have the diagram

$$\begin{array}{ccc} A^{T \times \mathbb{C}_1^*}(Y_w) & \xrightarrow{r_{\mathbb{T},\mathbb{C}_1^*}} & A^{\mathbb{C}_1^*}(Y_w/\!/T) \\ \big\downarrow{\scriptstyle \xi_w^*} & & \big\downarrow{\scriptstyle \overline{\xi}_w^*} \\ A^{T\times\lambda}(Y_w) & \xrightarrow{r_{T\times\lambda,\lambda}} & A^{\lambda}(Y_w/\!/T) \end{array}$$

by Proposition 2.1.2. Notice that since $Y_w/\!/T$ is fixed by both $\mathbb{C}_1^*$ and λ the vertical arrow on the left is just the homomorphism of polynomial rings

$$A(Y_w/\!/T)[s_1] \to A(Y_w/\!/T)[s_\lambda] \quad : \quad s_1 \mapsto \langle \lambda, \phi \rangle s_\lambda,$$

where s_λ denotes the equivariant variable for the action of λ. Since λ acts trivially on Y_w, the map $r_{T\times\lambda,\lambda}$ can be computed as

$$\begin{array}{ccc} A^{T\times\lambda}(Y_w) & \xrightarrow{r_{T\times\lambda,\lambda}} & A^{\lambda}(Y_w/\!/T) \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \sim} \\ A^{T}(Y_w)[s_\lambda] & \xrightarrow{r_T} & A(Y_w/\!/T)[s_\lambda] \end{array}$$

and shows that the residues we want to compute on the right-hand side of (4.16) are of the form

$$\deg \pi_{Y_w//T*}\left(r_T\left(\mathrm{GK}_{\langle\lambda,\phi\rangle s_\lambda}\left(\frac{\xi_w^*\left(\alpha_{|Y_w}\right)}{\xi_w^*\left(e^{\mathbb{T}}(\mathcal{N}_{Y_w/Y})\right)}\right)\right)\right).$$

□

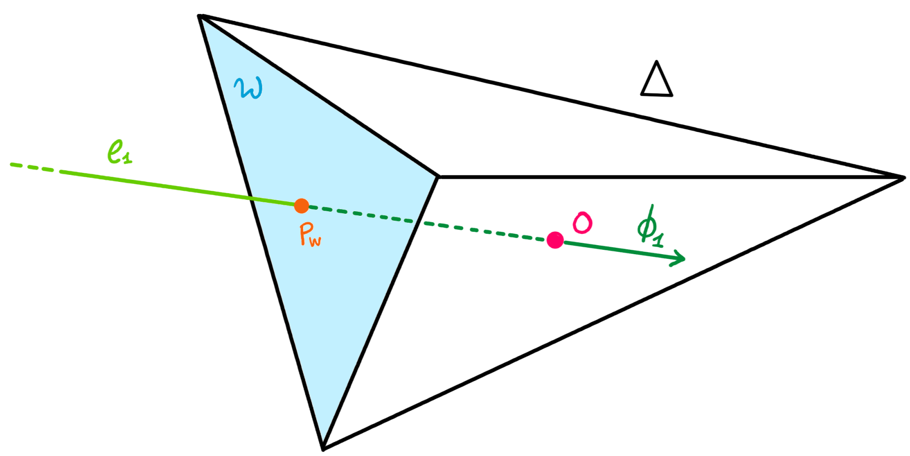
We have finally achieved our aim of having a residue formula for the intersection number (4.10). Still, this seems very complicated. We have reduced the original intersection number on $Y//\mathbb{T}$ to a sum of intersection numbers which are computed over $Y_w//T$, which are quotients by a torus of smaller dimension. By iterating this construction, we can find a simpler formula.

4.4 Guillemin-Kalkman localisation.

Consider the action of a dimension m torus $\mathbb{T}$ on a smooth closed subvariety of $\mathbb{P}^a \times \mathbb{A}^b$ and let $\alpha \in A^{\mathbb{T}}(Y)$. Our aim in this section is to find a combinatorial formula for $\deg(r_{\mathbb{T}}(\alpha))$ by an iterated application of Proposition 4.3.2 of the previous section. First of all, we show how to keep track of the combinatorial structure underlying this iteration. Given a ray $l \in \chi(\mathbb{T})_{\mathbb{Q}}$ (not necessarily centered at the origin) and a number $d > 0$, we say that l satisfies the condition (C_d) if

- the ray l is not entirely contained in the momentum polytope Δ of $\mathbb{T} \curvearrowright Y$,
- the ray l meets $\partial_d\Delta$ in its smooth points, away from walls of codimension $d+1$. (C_d)

In Proposition 4.3.2, we showed that given a primitive character $\phi_1 \in \chi(\mathbb{T})$ (or equivalently a ray $l_1 = \mathbb{Q}_{<0} \cdot \phi_1$ through the origin in $\chi(\mathbb{T})_{\mathbb{Q}}$) satisfying (C_1), then the intersection number on $[Y/\mathbb{T}]$ can be computed in terms of intersection numbers on $[Y_w/T_1]$, where $w \in \mathrm{Wall}_1(\Delta)$ are the 1-codimensional walls of Δ met by the ray l_1 and T_1 is the connected component of the identity in the kernel of ϕ (hence a codimension 1 subtorus of $\mathbb{T}$).

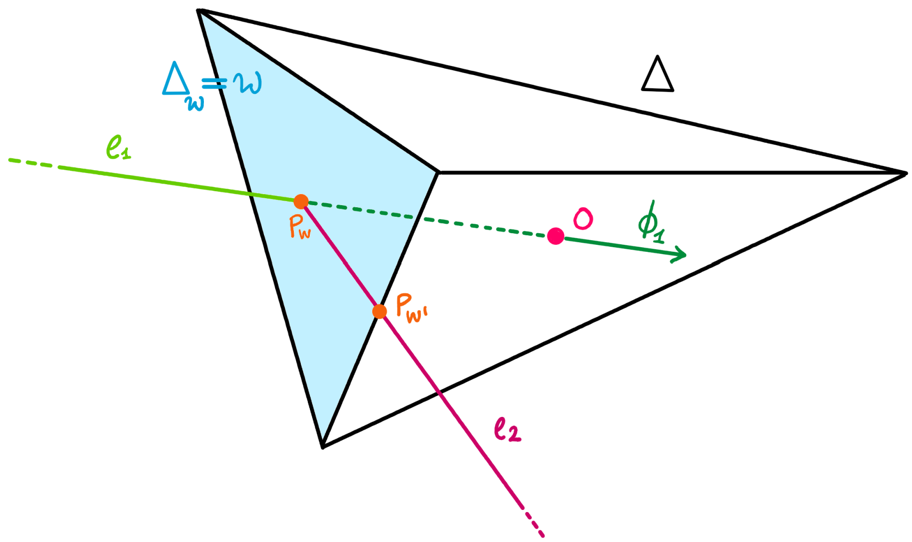


What happens if we try to apply the same formula of Proposition 4.3.2 to compute the contribution of $[Y_w/T_1]$? We need to start by picking another character $\phi_2 \in \chi(T_1)$ so that the negative ray $l_2 := \mathbb{Q}_{<0} \cdot \phi_2$ satisfies (C_1) with respect to the momentum polytope Δ_w for the action $T_1 \curvearrowright Y_w$. If $\lambda \in \chi(\mathbb{T})^\vee$ is the subtorus acting trivially on Y_w, through the following surjection of groups and the corresponding isomorphism of character spaces

$$T_1 \times \lambda \to \mathbb{T} \quad , \quad \chi(\mathbb{T})_\mathbb{Q} \xrightarrow{\sim} \chi(T_1)_\mathbb{Q} \times \chi(\lambda)_\mathbb{Q}$$

we can identify the momentum polytope Δ_w of $T_1 \curvearrowright Y_w$ with the wall w in Δ. The ray l_2, seen as a ray in $\chi(\mathbb{T})_\mathbb{Q}$ by this isomorphism, intersects Δ along the wall w and originates from the intersection point $p_w = l_1 \cap w$. For such a ray, the conditions

(C_1) with respect to Δ_w are equivalent to the conditions (C_2) with respect to Δ.



The character $\phi_2 \in \chi(T_1)$ defines a 1-codimensional subtorus $T_2 \subset T_1$ and by applying the formula of Proposition 4.3.2 we can compute the intersection number on $[Y_w/T_1]$ in terms of intersection numbers computed on $[Y_{w'}/T_2]$, where w' ranges over codimension 2 walls $w' \in \mathrm{Wall}_2(\Delta)$ met by the ray l_2. Then we can keep iterating this construction, and we now introduce the object that keeps track if the combinatorics involved.

Definition 4.4.1. A *dendrite* for a strictly convex momentum polytope Δ is a set of rays in $\chi(\mathbb{T})_{\mathbb{Q}}$ of the following form:

- There is a *level 1 ray* l_1 satisfying (C_1).
- Fixed $d > 1$, for every intersection p of a ray l_{d-1} of level $d-1$ with a $(d-1)$-codimensional wall w of Δ, there is a *ray of level* d l_d originating by p and intersecting Δ inside the wall w which satisfies condition (C_d).

Definition 4.4.2. Let D be a dendrite in Δ. A *path* in D is a polygonal line in Δ starting from the origin and reaching a wall of maximal codimension in Δ by only moving along rays appearing in the dendrite D and passing, for each level k by exactly only one ray at that level.

Notice that, fixed a dendrite D in Δ, a path determines a sequence of rays $l_1, \ldots, l_{\dim(\mathbb{T})}$ in $\chi(\mathbb{T})_{\mathbb{Q}}$ and hence a filtration of tori

$$\mathbb{T} = T_0 \supset T_1 \supset \cdots \supset T_m = 1 \tag{4.17}$$

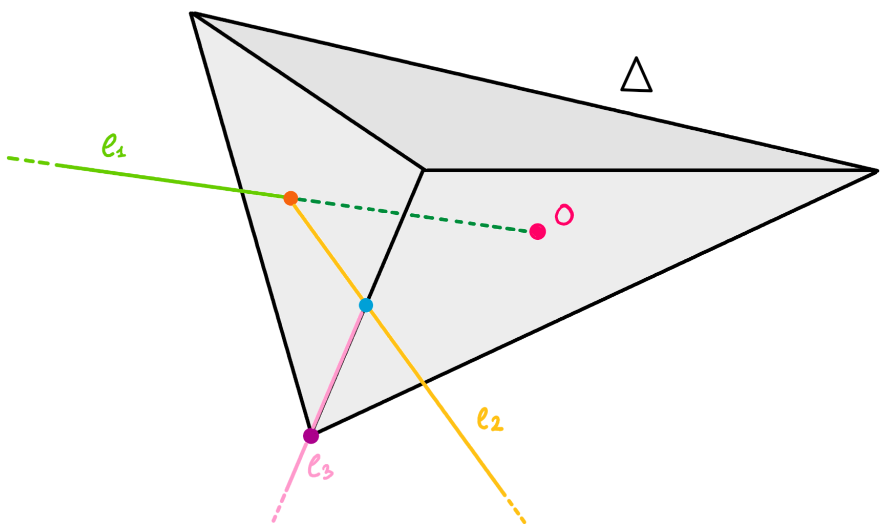


Figure 4.1: A 3-dimensional dendrite.

so that $T_i := \ker\left(l_{i|_{T_{i-1}}}\right)$. In other words, T_i is the subtorus of T_{i-1} given by the elements which vanish once evaluated at a nonzero character belonging to l_i (or rather the connected component of the identity of this subgroup). We also have a sequence of walls

$$\Delta = w_0 \supset w_1 \supset \cdots \supset w_m \tag{4.18}$$

of strictly increasing codimension, where w_i is the wall at which the path changes direction for the i-th time. This determines a sequence of smooth closed $\mathbb{T}$-invariant subvarieties

$$Y = Y_0 \supset Y_1 \supset \cdots \supset Y_m = F \tag{4.19}$$

so that Y_i is the subvariety of Y whose momentum polytope coincides with the wall w_i. Notice that, by Proposition 3.1.1, Y_i is a fixed locus for the subtorus of $\mathbb{T}$ orthogonal to the wall w_i, in particular F is a fixed locus for the $\mathbb{T}$-action on Y. Notice that the path identifies m different 1-dimensional subtori

$$\lambda_i \subseteq T_{i-1} \tag{4.20}$$

characterised by $Y_i = Y_{i-1}^{\lambda_i}$ and oriented so that $\langle \lambda_i, l_i \rangle < 0$.

Definition 4.4.3. Let P be a path in the dendrite D for Δ. Given a class $\alpha \in A^{\mathbb{T}}(Y)$ we can consider the corresponding fixed locus $F_P \subset Y$ and the induced class $\mathrm{res}_P(\alpha) \in A(F_P)$ defined as

$$\mathrm{res}_P(\alpha) := \left|\frac{\lambda_1 \wedge \cdots \wedge \lambda_m}{e_1 \wedge \cdots \wedge e_m}\right| \mathrm{GK}_{s_1,\ldots,s_m}\left((\lambda_1 \times \cdots \times \lambda_m \to \mathbb{T})^* \frac{\alpha_{|F_P}}{e^{\mathbb{T}}(\mathcal{N}_{F_P/Y})}\right), \quad (4.21)$$

where λ_i are the subtori (4.20) defined by the path and

1. the number $\left|\frac{\lambda_1 \wedge \cdots \wedge \lambda_m}{e_1 \wedge \cdots \wedge e_m}\right|$ is computed with respect to an integral basis $e_1, \ldots, e_m$ of $\chi(\mathbb{T})^\vee$. This is the absolute value of the determinant of the matrix defined by the coefficients of the λ_i expressed in terms of the basis e_j.

2. The change of group homomorphism with respect to the surjection $\lambda_1 \times \cdots \times \lambda_m \to \mathbb{T}$ is, by Example 2.1.3, the isomorphism

$$A(F_P) \otimes \mathrm{Sym}\left(\chi(\mathbb{T})_{\mathbb{C}}\right) \to A(F_P)[s_1, \ldots, s_m]$$

   given by expressing characters of $\mathbb{T}$ as linear combinations of $s_1, \ldots, s_m$, thought as elements of the basis of $\chi(\mathbb{T})_{\mathbb{C}}$ dual to λ.

3. GK denotes the Guillemin-Kalkman residue (4.4) with respect to the ring $A(F_P)$ and the variables $s_1, \ldots, s_m$.

Remark 22. Let $\phi_i \in \chi(T)$ be the generator for the ray l_i which restricts to a primitive element of $\chi(T_{i-1})$. Then $T_i = \ker\phi_i$ for every i, which implies that $\phi_1, \ldots, \phi_n$ form an integral basis of $\chi(T)$.

Theorem 4.4.1 (Guillemin-Kalkman localisation.)**.** *Consider an action of a torus $\mathbb{T}$ on a smooth variety Y with a regular linearisation $\mathcal{L}$ induced from an equivariant closed embedding $Y \hookrightarrow \mathbb{P}^a \times \mathbb{A}^b$. Assume that $Y/\!/\mathbb{T}$ is projective and that the momentum polytope Δ is strictly convex. Given a dendrite D for Δ and a class $\alpha \in A^{\mathbb{T}}(Y)$, the following equality*

$$deg(\pi_{Y/\!/\mathbb{T}*} r_{\mathbb{T}}(\alpha)) = \sum_{P \in Path(D)} deg\left(\pi_{F_P*}\left(res_P(\alpha)\right)\right)$$

holds true, where the sum is over all possible paths in the dendrite D and the morphism $\pi_{F_P} : F_P \to pt$ is the map to a point.

Remark 23. Notice that the conditions on Δ being strictly convex and $Y/\!/\mathbb{T}$ being projective imply that the fixed loci F_P we encounter are projective, so we can pushforward to a point.

Proof. By applying iteratively the formula of Proposition 4.3.2 we get that $\deg(r_{\mathbb{T}}(\alpha))$ is a sum over the paths in the dendrite D. We compute the contribution of a path P as follows. For every i let $\phi_i \in \chi(T_{i-1})$ be the primitive character generating the ray l_i of the path, or in other words so that $l_i = \mathbb{Q}_{<0} \cdot \phi_i$. Set $\alpha_0 := \alpha$ and, for every $i \in \{1, ..., m\}$, define

$$\alpha_i = \langle \lambda_i, \phi_i \rangle \cdot \mathrm{GK}_{s_i} \left((T_i \times \lambda_i \to T_{i-1})^* \frac{\alpha_{i-1|Y_i}}{e^{T_{i-1}}(\mathcal{N}_{Y_i/Y_{i-1}})} \right) \in A^{T_i}(Y_i),$$

where GK is the Guillemin-Kalkman residue (4.4) for the ring $A^{T_i}(Y_{w_i})$ with respect to the variable s_i corresponding to the 1-dimensional subtorus λ_i in T_{i-1} fixing Y_i. The contribution of the path P to the intersection number is given by $\deg \pi_{F_P *}(\alpha_n)$. Now notice that

$$\begin{aligned}
&\prod_{j=1}^{2} \langle \lambda_j, \phi_j \rangle^{-1} \cdot \alpha_2 \\
=&\mathrm{GK}_{s_2} \left((T_2 \times \lambda_2 \to T_1)^* \frac{1}{e^{T_1}(\mathcal{N}_{Y_2/Y_1})} i^*_{Y_2} \mathrm{GK}_{s_1} \left((T_1 \times \lambda_1 \to \mathbb{T})^* \frac{\alpha_{|Y_1}}{e^{\mathbb{T}}(\mathcal{N}_{Y_1/Y})} \right) \right) \\
=&\mathrm{GK}_{s_1,s_2} \left((T_2 \times \lambda_1 \times \lambda_2 \to \mathbb{T})^* \frac{\alpha_{|Y_2}}{e^{\mathbb{T}}(\mathcal{N}_{Y_2/Y})} \right)
\end{aligned}$$

by Lemma 4.1.4, and by iterating this procedure we find that

$$\alpha_m = \prod_{j=1}^{m} \langle \lambda_i, \phi_i \rangle \cdot \mathrm{GK}_{s_1,\ldots,s_m} \left((\lambda_1 \times \cdots \times \lambda_m \to \mathbb{T})^* \frac{\alpha_{|F_P}}{e^{\mathbb{T}}(\mathcal{N}_{F_P/Y})} \right).$$

By simple linear algebra (as shown in the auxiliary Lemma 4.4.1 below) it follows that

$$\prod_{j=1}^{n} \langle \lambda_j, \phi_j \rangle = \left| \frac{\lambda_1 \wedge \cdots \wedge \lambda_m}{e_1 \wedge \cdots \wedge e_m} \right|,$$

for an integral basis $e_1, \ldots, e_m$ of $\chi(\mathbb{T})^\vee$, concluding the proof. □

In order to complete the proof we have to prove the following linear algebra statement:

Lemma 4.4.1. *Consider a filtration of lattices*

$$0 \subset \mathbb{Z} \subset \mathbb{Z}^2 \subset \cdots \subset \mathbb{Z}^m$$

with rank 1 subquotients. Assume there are homomorphisms $\phi_i : \mathbb{Z}^i \to \mathbb{Z}$ *so that*

- *$\mathbb{Z}^i = ker(\phi_i)$ and*

- *ϕ_i is an integral and primitive morphism of lattices.*

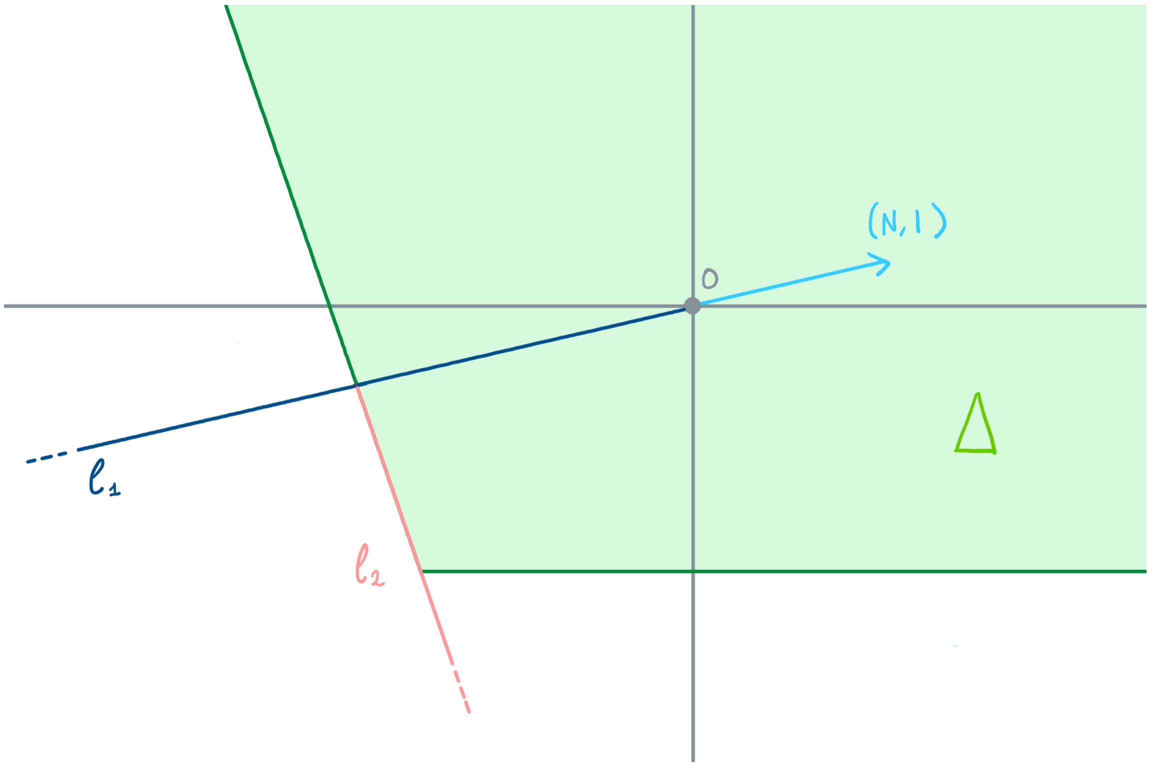
Then there are $\tilde{\phi}_1, \ldots, \tilde{\phi}_m : \mathbb{Z}^m \to \mathbb{Z}$ extending the functions above to the whole $\mathbb{Z}^m$ and forming an integral basis of this lattice. In particular, for every $\lambda_1 \in \mathbb{Z}, \ldots, \lambda_m \in \mathbb{Z}^m$ satisfying $\langle \lambda_i, \phi_i \rangle \geqslant 0$ we have that

$$\prod_{j=1}^{m} \langle \lambda_j, \phi_j \rangle = \left| \frac{\lambda_1 \wedge \cdots \wedge \lambda_m}{e_1 \wedge \cdots \wedge e_m} \right|$$

where $e_1, \ldots, e_m$ is an integral basis of $\mathbb{Z}^m$.

Proof. The fact that ϕ_i can be extended to the whole space is obvious: since ϕ_m is primitive it is surjective and from its exact sequence we find a splitting $\mathbb{Z}^m \simeq \mathbb{Z}^{m-1} \times \mathbb{Z}$. But ϕ_{m-1} is a primitive morphism on $\mathbb{Z}^{m-1}$, hence we can split the lattice as $\mathbb{Z}^m \simeq \mathbb{Z} \times \cdots \times \mathbb{Z}$ where $\mathbb{Z}^i$ is the product of the first i copies of $\mathbb{Z}$. Then it is obvious to extend the functions to the whole space by setting them to zero on the other components. The proof of these extension being an integral basis is straightforward by using induction, the base case $m = 1$ being trivial. The final equality holds by choosing as $e_1, \ldots, e_m$ the dual basis to $\phi_1, \ldots, \phi_m$ to compute both terms. □

Example 4.4.1. Consider the situation of Example 4.2. Here the momentum cone in $\mathbb{Q}^2$ is spanned by the characters $(-1, 3)$ and $(1, 0)$ with vertex at $(-1, -1)$. We can choose the dendrite given by the two rays l_1 and l_2 in the picture:



This dendrite only has one path $P = \{l_1, l_2\}$, of which we compute the contribution to $\deg(r_{\mathbb{T}}([\mathbb{A}^2]_{\mathbb{T}}))$. The first ray, whose direction is given by the character $(-N, -1)$, identifies the first subtorus $T_1 = l_1^{\perp}$ given by

$$\mathbb{C}^* \hookrightarrow \mathbb{T} \quad : \quad t \mapsto (t^{-1}, t^N).$$

Consider the second ray l_2, whose direction in $\mathbb{T}$ is given by the character $(1, -3)$. This restricts to the ray in $\chi(T_1)$ spanned by the character $-(3N+1)$ and hence it identifies the trivial subtorus $T_2 = l_2^{\perp} = 1 \subset T_1 \subset \mathbb{T}$. The filtration by of tori (4.17) is, in this case,

$$\mathbb{C}^* \times \mathbb{C}^* \supset \{t_1^N t_2 = 0\} \supset 1.$$

The sequence of walls (4.18) is given by

$$\Delta \supset \{(-1 - x, 3x - 1) \ : \ x \geqslant 0\} \supset (-1, -1)$$

and the corresponding sequence of subvarieties (4.19) is

$$\mathbb{A}^2 \supset \mathbb{A} \times 0 \supset O.$$

The subtorus λ_1 of $\mathbb{T}$ fixing $\mathbb{A} \times 0$ and pairing negatively with the ray l_1 is

$$\mathbb{C}^* \hookrightarrow \mathbb{T} \quad : \quad t \mapsto (t^3, t),$$

while the subtorus λ_2 of T_1 fixing the origin of $\mathbb{A}^2$ and pairing negatively with the direction of l_2 is T_1 itself. Notice that the $\mathbb{T}$-equivariant Euler class of the normal bundle to the origin in $\mathbb{A}^2$ is the polynomial expression in the characters given by

$$e^{\mathbb{T}}(\mathcal{N}_{O/\mathbb{A}^2}) = (3t_2 - t_1)t_1$$

which can be reexpressed in terms of the dual basis to λ as

$$e^{\lambda_1 \times \lambda_2}(\mathcal{N}_{O/\mathbb{A}^2}) = (3N + 1)s_2(3s_1 - s_2).$$

The Guillemin-Kalkman localisation formula of Theorem 4.4.1 states that the degree $\deg(r_{\mathbb{T}}([\mathbb{A}^2]))$ is equal to

$$\left|\det \begin{pmatrix} -1 & N \\ 3 & 1 \end{pmatrix}\right| \mathrm{GK}_{s_1,s_2}\left(\frac{1}{(3N+1)s_2(3s_1 - s_2)}\right) = \mathrm{GK}_{s_1,s_2}\left(\frac{1}{s_2(3s_1 - s_2)}\right).$$

In order to compute this Guillemin-Kalkman residue we have to first expand with respect to s_1^{-1} and take the residue, then do the same with respect to s_2:

$$\mathrm{GK}_{s_1,s_2}\left(\frac{1}{s_2(3s_1 - s_2)}\right) = \mathrm{GK}_{s_1,s_2}\left(\frac{1}{3s_1 s_2}\sum_{k>0}\left(\frac{s_2}{3s_1}\right)^k\right) = \mathrm{GK}_{s_2}\left(\frac{1}{3s_2}\right) = \frac{1}{3}$$

as we expected, since the quotient stack $[(\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{T}]$ is $[1/\mu_3]$ and we are computing the degree of its fundamental class.

4.5 Szenes-Vergne localisation.

In the previous section we have seen how we can compute intersection numbers on $Y/\!\!/\mathbb{T}$ by looking at a particular combinatorial object called dendrite. In particular, different choices of the dendrite in the momentum polytope Δ of Y give different localisation formulae for the same intersection number. Here we show how, in the case where $Y = V$ is a linear space, there is a canonical choice for the dendrite and this produces a famous version of the Jeffrey-Kirwan localisation formula. We start by giving an explicit description of the momentum polytope and of its walls:

Lemma 4.5.1. *Let $\rho_1, \ldots, \rho_n \in \chi(\mathbb{T})$ be characters so that V has a decomposition*

$$V \simeq \bigoplus_{i=1}^{n} V_i$$

where the $\mathbb{T}$-action on V_i is given by multiplication with ρ_i. We will call them weights of the action. Consider the trivial linearisation $\mathcal{L}_0$ whose $\mathbb{T}$-action on the fibres is trivial. Then

1. *The momentum polytope Δ for the action is the cone spanned by the characters $\rho_1, ..., \rho_n$.*
2. *The codimension-k walls of Δ are the cones spanned by $dim(\mathbb{T}) - k$ linearly independent weights $\rho_{i_1}, \ldots, \rho_{i_{dim(\mathbb{T})-k}}$.*

We omit the proof of this straightforward fact.

Remark 24. Notice that, since line bundles on affine spaces are trivial, every other linearisation differs from the one with trivial action on the fibre via the twist by a character. This just produces a translation of the momentum polytope, so that the polytope for an arbitrary linearisation $\mathcal{L}$ is the translation of the polytope we just described for $\mathcal{L}_0$ so that the vertex lies on the point $-\xi \in \chi(\mathbb{T})$, where ξ denotes the character of the $\mathbb{T}$-action on the fibre above the origin of the linearisation $\mathcal{L}$.

We will always work under the following conditions:

Lemma 4.5.2. *The linear space V has no nontrivial fixed subspace, i.e. $V^{\mathbb{T}} = O$, and that the momentum polytope is strictly convex if and only if, for every character $\xi \in \chi(\mathbb{T})_{\mathbb{Q}}$, the quotient $V//T$ built with respect to the linearisation $\mathcal{L}_0 \otimes \xi$ is projective.*

Proof. By picking a basis we can assume $V \simeq \mathbb{A}^m$ and that $\mathbb{T}$ acts diagonally. We know that $V//\mathbb{T}$ is projective if and only if there are non nonconstant $\mathbb{T}$-invariant functions on V. Assume that $V^{\mathbb{T}} = O$ and that Δ is strictly convex. Assume by contradiction that $f : V \to \mathbb{C}$ is a nonconstant invariant function. Then we have a nonconstant monomial $x_{i_1}^{d_1} \cdots x_{i_k}^{d_k}$ which is invariant, so the corresponding characters $\rho_{i_1}, \ldots, \rho_{i_k}$ must satisfy

$$\sum_{j=1}^{k} d_j \rho_{i_j} = 0.$$

This either means that all the weights ρ_{i_j} are zero (and so there are nontrivial invariant subspaces) or that Δ contains a line (hence it is not strictly convex). Conversely if $V^{\mathbb{T}}$ is a nontrivial subspace, the corresponding coordinate functions define nonconstant invariant functions. Moreover if Δ is not strictly convex one can find a relation among the weights of the form $\sum_{i=1}^{n} d_i\rho_i = 0$ with $d_i \geqslant 0$ not all zero and hence the corresponding monomial $\prod_{i=1}^{n} x_i^{d_i}$ is invariant. □

In this case, if we choose a character $\xi \in \chi(\mathbb{T})_{\mathbb{Q}}$, we can consider the corresponding quotient $V/\!/\mathbb{T}$ with respect to the linearisation $\mathcal{L} := \mathcal{L}_0 \otimes \xi$, namely the one whose action on the fibres is given by ξ. We want to apply the localisation formula of Theorem 4.4.1 to compute intersection numbers of the form $\deg(r_{\mathbb{T}}(\alpha))$, where $\alpha \in A^{\mathbb{T}}(V)$ and $r_{\mathbb{T}} : A^{\mathbb{T}}(V) \to A(V/\!/\mathbb{T})$ is the Kirwan map. The first step to apply this localisation formula is to pick a dendrite in Δ; here we show how to do this in a canonical way once ξ is *sum-regular* in the sense of Example 4.1.3. Recall that the momentum polytope Δ for this action is the translation of the one described in Lemma 4.5.1 by the character $-\xi$ as discussed in Remark 24.

Definition 4.5.1. Let $\mathfrak{A} \subseteq \chi(\mathbb{T})$ be the set of weights of the torus action on the linear space V. Consider the following dendrite, called *canonical dendrite* for the sum-regular linearisation $\mathcal{L}$:

- (level 1) Consider the element $\nu_1 := \sum_{\rho\in\mathfrak{A}} \rho$ and choose the ray $l_i := \mathbb{Q}_{<0} \cdot \nu_1$.
- (level d) Given an intersection point p of a level $d-1$ ray with a $(d-1)$-codimensional wall w of Δ we can set $\nu_d := \sum_{\rho\in\mathfrak{A}\cap(w+\xi)} \rho$ and consider the ray $p + \mathbb{Q}_{<0} \cdot \nu_d$.

Lemma 4.5.3. *The rays above define a dendrite in Δ.*

Proof. Notice that every ν_d is nonzero since Δ is assumed to be strictly convex. It is clear that these rays are not entirely contained in Δ, which is the positive span of the weights. Moreover, since ξ is sum-regular by hypothesis, we have that the ray of level d doesn't meet the $(d-2)$-dimensional skeleton of Δ, otherwise we could write ξ as a positive linear combination of $m-1$ elements of $\chi(\mathbb{T})$ defined as sums of weights, contradicting sum-regularity. Thus the rays l_d satisfy the conditions (C_d) and therefore define a dendrite for Δ. □

From a sum-regular character ξ we have constructed a dendrite and we can consider the associated formula (4.4.1). We start by giving a different combinatorial description of paths in the canonical dendrite in terms of flags in the linear space $\chi(\mathbb{T})_{\mathbb{Q}}$ of characters:

Proposition 4.5.1. *The paths in the canonical dendrite are in bijection with the set of flags $\mathcal{F}(\mathfrak{A}, \xi)$ introduced in Definition 4.1.7.*

Proof. In this proof we set $m := \dim(\mathbb{T})$ to shorten the notation. Consider a path P for the canonical dendrite D. We can consider the associated decreasing chain of walls of Δ

$$\Delta = w_0 \supset w_1 \supset \cdots \supset w_m = -\xi$$

and define the corresponding flag in $\chi(\mathbb{T})_{\mathbb{Q}}$ as

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = \chi(\mathbb{T})_{\mathbb{Q}}$$

via $F_j := \mathrm{span}_{\mathbb{Q}}(w_{m-j} + \xi)$. We claim that this flag is in $\mathcal{F}(\mathfrak{A}, \xi)$. The ray at level d of the path is generated by the vector $\nu_d = \kappa_{m-d+1}$ by definition (4.6), hence we have that

$$F_d := \mathrm{span}_{\mathbb{Q}}(w_{m-d} + \xi) = \mathrm{span}_{\mathbb{Q}}(\nu_m, ..., \nu_{m-d+1}) = \mathrm{span}_{\mathbb{Q}}(\kappa_1, ..., \kappa_d)$$

for every d, and in particular the flag is proper. Moreover, since ξ is connected to the origin (the unique vertex of the momentum cone) via the rays of the dendrite we have that $\xi \in \mathrm{span}_{\mathbb{Q}_{>0}}(\kappa_1, \ldots, \kappa_m)$, hence the flag is stable and thus it belongs to $\mathcal{F}(\mathfrak{A}, \xi)$. On the other hand, given a flag $F \in \mathcal{F}(\mathfrak{A}, \xi)$, notice that the translations by $-\xi$ of the linear spaces F_j in the flag intersect Δ in walls (of dimension j). Since F is proper we can write $F_j = \mathrm{span}_{\mathbb{Q}}(\kappa_1, \ldots, \kappa_j)$ for all j. By definition the flag F is stable and hence we can write

$$\xi = \sum_{j=1}^{m} c_j \kappa_j$$

for some $c_1, \ldots, c_n > 0$. Therefore

$$\xi_{m-1} := \sum_{j=1}^{m-1} c_j \kappa_j$$

belongs to the intersection of the ray $\xi + \mathbb{Q}_{<0} \cdot \kappa_m$ with F_{m-1}. In the same way we see that

$$\xi_{m-2} := \sum_{j=1}^{m-2} c_j \kappa_j$$

belongs to $(\xi_{m-1} + \mathbb{Q}_{<0} \cdot \kappa_{m-1}) \cap F_{m-2}$ and so on. By subtracting ξ we have shown that the points $O, \xi_{m-1} - \xi, ..., \xi_1 - \xi, -\xi$ define a path for the canonical dendrite D. It's immediate to check that these two constructions are one the inverse of the other. $\square$

Remember that, given a flag in $\chi(\mathbb{T})_{\mathbb{Q}}$, flag residues (4.7) are computed with respect to the basis $\kappa_1, \dots, \kappa_{\dim(\mathbb{T})}$ induced by the flag. The following result relates the path residue (4.21) with respect to a path P to the flag residue corresponding to the associated flag:

Proposition 4.5.2. *The path residue of $\alpha \in A^{\mathbb{T}}(V)$ can be computed as a flag residue with respect to the corresponding flag:*

$$res_P(\alpha) = res_F\left(\frac{\alpha}{e^{\mathbb{T}}(T_V)}\right)$$

Proof. Set $m := \dim(\mathbb{T})$. Consider the 1-dimensional subtori $\lambda_1, \dots, \lambda_m$ associated to the path P and let $s_1, \dots, s_m$ be the corresponding dual basis of $\chi(\mathbb{T})$ We have to prove that, for the characters $\kappa_1, \dots, \kappa_m$ associated to the flag F, the following equality holds true:

$$\begin{aligned} &\left|\frac{\lambda_1 \wedge \cdots \wedge \lambda_m}{e_1 \wedge \cdots \wedge e_m}\right| \mathrm{GK}_{s_1,\dots,s_m}\left(\frac{\alpha_{|O}}{e^{\mathbb{T}}(T_V)}\right) \\ = &\left|\frac{e_1^\vee \wedge \cdots \wedge e_m^\vee}{\kappa_1 \wedge \cdots \wedge \kappa_m}\right| \mathrm{SV}_{\kappa_1,\dots,\kappa_m}\left(\frac{\alpha_{|O}}{e^{\mathbb{T}}(T_V)}\right) \end{aligned}$$

where $e_1, ..., e_m$ is an integral basis of $\chi(\mathbb{T})^\vee$ and $e_1^\vee, ..., e_m^\vee$ is the dual basis of $\chi(\mathbb{T})$. Notice that the dual bases satisfy

$$\left|\frac{\lambda_1 \wedge \cdots \wedge \lambda_m}{e_1 \wedge \cdots \wedge e_m}\right| = \left|\frac{e_1^\vee \wedge \cdots \wedge e_m^\vee}{s_1 \wedge \cdots \wedge s_m}\right|,$$

so we can focus on proving

$$\left|\frac{\kappa_1 \wedge \cdots \wedge \kappa_m}{s_1 \wedge \cdots \wedge s_m}\right| \mathrm{GK}_{s_1,\dots,s_m}\left(\frac{\alpha_{|O}}{e^{\mathbb{T}}(T_V)}\right) = \mathrm{SV}_{\kappa_1,\dots,\kappa_m}\left(\frac{\alpha_{|O}}{e^{\mathbb{T}}(T_V)}\right).$$

By Proposition 4.1.1 we see that the symbol $\mathrm{GK}_{s_1,\dots,s_m}$ on the left-hand side can be replaced with $\mathrm{SV}_{s_m,\dots,s_1}$ since the class $e^{\mathbb{T}}(T_V)$ is a product of linear factors. Finally, we claim that the resulting equality holds true by virtue of Lemma 4.1.2. In order to apply this lemma we just have to show that the two ordered bases $\{s_m, \dots, s_1\}$ and $\{\kappa_1, \dots, \kappa_m\}$ are such that

1. $\mathrm{span}(s_m, \dots, s_{m-i}) = \mathrm{span}(\kappa_1, \dots, \kappa_i)$ for all i,

2. they are oriented in the same way.

1) Clearly span$(\kappa_1, \ldots, \kappa_i)$ is the span of the i-dimensional wall of the path. Recall that the equations of this linear space, by Proposition 3.1.1, are $\lambda_j = 0$ for $j > m-i$, and $s_1, \ldots, s_{m-i}$ satisfy those. Being linearly independent they span the linear space.
2) Being the i^{th} ray defined by the direction $-\kappa_{m-i}$, the inequality $\langle \lambda_i, \kappa_i \rangle > 0$ holds true for every i. This, together with point 1, ensures that the two bases are oriented in the same way as can be seen by expressing the κ_i in terms of the s_j. □

Finally, we can restate the Guillemin-Kalkman localisation formula in the case of the canonical dendrite:

Theorem 4.5.1 (Szenes-Vergne Localization Formula)**.** *Let $\mathbb{T} \curvearrowright V$ be a representation of an algebraic torus having trivial fixed part $V^{\mathbb{T}} = O$. Fixed a regular linearisation $\mathcal{L}$ corresponding to a character $\xi \in \chi(\mathbb{T})$, let Δ be the associated momentum polytope and assume it is strictly convex. For every $\alpha \in A^{\mathbb{T}}(V)$ we have the equality*

$$deg\left(\pi_{V/\!/\mathbb{T}*}\left(r_{\mathbb{T}}(\alpha)\right)\right) = JK_{\xi}^{\mathfrak{A}}\left(\frac{\alpha}{e_T(T_V)}\right) \tag{4.22}$$

where $\frac{\alpha}{e_T(T_V)}$ is thought as a rational function on $\chi(\mathbb{T})^{\vee}_{\mathbb{C}}$.

4.6 Nonabelian localisation.

Until now we have only discussed degree computations on quotients by torus actions. Assume we are interested in formulae for degrees of cycles in quotients of the form $Y/\!/G$, where G is a reductive connected algebraic group acting on a smooth variety Y with a linearisation $\mathcal{L}$. In this section we will work in the case where, denoted with T a maximal subtorus of G, the G and T-actions on the respective semistable loci are free, so that the quotients $Y/\!/G$ and $Y/\!/T$ (with respect to the same linearisation $\mathcal{L}$) are smooth. We also assume these quotients are projective.

Remark 25. I am convinced that these smoothness hypotheses are not necessary for the results of this section to hold, even though the arguments presented here rely on them. In particular I expect that a purely algebraic analogue of Martin's result below holds true, so having "semistable = stable" for the linearisation $\mathcal{L}$ should be enough.

The way to obtain formulae for nonabelian quotients of the form $Y/\!/G$ is to relate them to abelian the abelian quotients $Y/\!/T$ through the following diagram:

$$\begin{tikzcd} \frac{Y(G)^{\mathrm{ss}}}{T} \arrow[r, hook, "j"] \arrow[d, "\pi"] & Y/\!/T \\ Y/\!/G \end{tikzcd} \tag{4.23}$$

where j is the open embedding of the G-semistable locus in the T-semistable one and π is the residual $G/\!/T$ fibration.

Remark 26. Notice that if $B \subset G$ is a Borel subgroup containing T then π factors as an affine bundle g followed by G/B-fibration f (which is, in particular, projective)

$$Y(G)^{\mathrm{ss}}/\!/T \xrightarrow{g} Y(G)^{\mathrm{ss}}/\!/B \xrightarrow{f} Y/\!/G$$

as discussed in [ES89, Section 2.5]

Example 4.6.1. Here we describe this picture explicitly in the case of a Grassmannian. Consider the action of $G = \mathrm{GL}_2$ on $V := \mathrm{Mat}_{2\times n}(\mathbb{C})$ by left multiplication and let $\mathcal{L}$ be the linearisation corresponding to the character $\xi = \det$. The ring of invariant sections of powers of $\mathcal{L}$ is generated by the 2×2 minors, which are the Plücker coordinates for the embedding of $\mathrm{Gr}(2, n) \hookrightarrow \mathbb{P}(\wedge^2\mathbb{C}^n)$.

Consider the maximal subtorus $T \simeq (\mathbb{C}^*)^2$ given by diagonal matrices. The induced action $(t_1, t_2)\cdot M$ is by multiplication of the first row of M with t_1 and of the second row with t_2. The induced linearisation corresponds to the character $\xi_{|T} = t_1t_2$ and the quotient is the product of projective spaces $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ corresponding to the rows of the matrix.

By our discussion of the G-invariant sections of $\mathcal{L}^{\otimes n}$, the T-quotient of the G-unstable locus is cut out by the equations defining the 2×2 minors, and therefore it corresponds to the diagonal:

$$(V\backslash V(G)^{\mathrm{ss}})/\!/T = \Delta \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

This describes completely the diagram (4.23) in this example, which is

$$\begin{tikzcd} (\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})\backslash\Delta \arrow[r, hook, "j"] \arrow[d, "\pi"] & \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \\ \mathrm{Gr}(2, n) \end{tikzcd}$$

with the vertical morphism π being

$$\pi([x],[y]) \mapsto [x_i y_j - x_j y_i]$$

geometrically mapping the two distinct lines $[x]$ and $[y]$ of $\mathbb{C}^n$ to the plane they generate.

The factorisation of π of Remark 26 can be seen as follows. Let B be the subgroup of upper triangular matrices. Then the quotient $V(G)^{\mathrm{ss}}/\!/B$ coincides with the flag variety of (1,2)-dimensional flags in $\mathbb{C}^n$ and π factors as

$$(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})\backslash\Delta \xrightarrow{g} \mathrm{F}(1,2,n) \xrightarrow{f} \mathrm{Gr}(2,n)$$

where the first map sends two lines $([x],[y])$ in the flag given by $[x]$ and the 2-dimensional subspace containing them, while f is the $\mathbb{P}^1$-bundle over the Grassmannian whose fibre above a point is the set of lines contained in the corresponding plane.

The main technical tool is the following formula of Martin [Mar00]:

Theorem 4.6.1. *Let $\alpha \in H^*(Y/\!/G)$ and $\beta \in H^*(Y/\!/T)$ be such that $\pi^*\alpha = j^*\beta$. Then*

$$\int_{Y/\!/G} \alpha = \frac{1}{|W|}\int_{Y/\!/T} \beta \cup e(\mathcal{R}),$$

where W is the Weyl group of G and $\mathcal{R}$ is the roots bundle of $Y/\!/T$, namely the bundle obtained by descending to the quotient the T-equivariant vector bundle

$$X \times \mathfrak{g}/\mathfrak{h} \to X,$$

where $\mathfrak{g} := T_1G$ and $\mathfrak{h} := T_1T$, with action on the fibre induced by the adjoint representation.

Notice that Theorem 4.6.1 directly transposes into Chow groups by virtue of the *cycle class maps* [Ful13, Chapter 19]

$$\mathrm{cl} : A_*(Z) \to H^{2\dim(Y)-2*}(Z).$$

Notice that, in principle, the target should be the Borel-Moore homology of Z, but we will only consider smooth Z so Poincaré duality allows us to map to singular cohomology. We can translate the theorem above in the following way:

Theorem 4.6.2. *Let $\alpha \in A_*(Y/\!/G)$ and $\beta \in A_*(Y/\!/T)$ be such that $\pi^*\alpha = j^*\beta$. Then*

$$deg(\pi_{Y/\!/G*}(\alpha)) = \frac{1}{|W|} deg\left(\pi_{Y/\!/G*}\left(\beta \cup e(\mathcal{R})\right)\right).$$

Proof. Since cycle maps commute with flat pullbacks we see that $\pi^*\mathrm{cl}(\alpha) = j^*\mathrm{cl}(\beta)$, hence by Martin's formula

$$\int_{Y/\!/G} \mathrm{cl}(\alpha) = \frac{1}{|W|}\int_{Y/\!/T} \mathrm{cl}(\beta) \cup e(\mathcal{R}).$$

Moreover cycle maps are ring homomorphisms and are well behaved with respect to Chern classes, so

$$\int_{Y/\!/G} \mathrm{cl}(\alpha) = \frac{1}{|W|}\int_{Y/\!/T} \mathrm{cl}\left(\beta \cup e(\mathcal{R})\right).$$

Finally, cycle maps commute with proper pushforwards (which in cohomology correspond to integration) and hence we obtain the wanted result. □

This result, paired with the abelian localisation formulae of the previous section, immediately translates to localisation formulae for nonabelian quotients:

Theorem 4.6.3 (Nonabelian Guillemin-Kalkman localisation.)**.** *Consider the action of a reductive connected algebraic group on a smooth subvariety Y of $\mathbb{P}^a \times \mathbb{A}^b$ and let $\mathcal{L}$ be a linearisation. If T is a maximal subtorus of G, assume that the T and G actions on the corresponding semistable loci are free and that $Y/\!/G$ and $Y/\!/T$ are projective. Let $\alpha \in A^G(Y)$ be a G-equivariant class, $\alpha_T := (T \hookrightarrow G)^*\alpha \in A^T(Y)$ be the corresponding T-equivariant class and consider the T-equivariant vector bundle on Y with fibre $\mathfrak{g}/\mathfrak{h}$. Then, fixed a dendrite D for the momentum polytope of the T-action, the equality*

$$deg(\pi_{Y/\!/G*} r_G(\alpha)) = \frac{1}{|W|}\sum_{P \in Path(D)} deg\left(\pi_{F_P*}\left(res_P(\alpha_T \cdot e^T(\mathfrak{g}/\mathfrak{h}))\right)\right)$$

holds true, where W is the Weyl group of G and the sum is over all possible paths in the dendrite D.

Proof. The proof simply follows by combining Theorem 4.6.2 above and the abelian Guillemin-Kalkman formula of Theorem 4.4.1. The only thing we have to notice is that $\pi^* r_G = j^* r_T$ which immediately follows from Theorem 2.1.1. □

Remark 27. Notice that if $Y/\!/T$ is projective then $Y/\!/G$ is projective too, since all G-invariant functions on Y are also T-invariant. This means that we can just require the first condition in the hypotheses of the previous theorem.

Analogously we have a nonabelian version of Szenes-Vergne localisation.

Theorem 4.6.4 (Nonabelian Szenes-Vergne localisation.)**.** *Consider a representation V of a reductive connected algebraic group G with maximal subtorus T and assume that $V^T = O$ and that the momentum polytope for the T-action is strictly convex. Let $\mathcal{L}$ be a linearisation so that the G and T-actions on the respective semistable loci are free. Given an G-equivariant class $\alpha \in A^G(V)$ and the corresponding T-equivariant one $\alpha_T := (T \hookrightarrow G)^*\alpha \in A^T(V)$, the equality*

$$deg(\pi_{Y/\!/G*} r_G(\alpha)) = \frac{1}{|W|} JK_{\xi}^{\mathfrak{A}} \left(\frac{\alpha_T \cdot e^T(\mathfrak{g}/\mathfrak{h})}{e^T(T_V)} \right)$$

holds true, where

- *W is the Weyl group of G.*
- *$\xi \in \chi(G) = \chi(T)^W$ is the character of G so that the linearisation $\mathcal{L}$ has action on the fibre given by ξ.*
- *$\mathfrak{A} \subset \chi(T)$ is the set of characters of the action of T on V.*

Example 4.6.2. Let's build on the previous Example 4.6.1 and assume we want to compute the degree of a point in the Grassmannian $\mathrm{Gr}(2,4)$. By Theorem 2.1.1 we know that we want to compute

$$\deg\left(\pi_{\mathrm{Gr}(2,4)*} r_G\left([\mathbb{C}^4]_G\right)\right)$$

where $\mathbb{C}^4$ is, for example, the subspace of 2×4 matrices of the form

$$\begin{pmatrix} x_1 & x_2 & 0 & 0 \\ y_1 & y_2 & 0 & 0 \end{pmatrix}.$$

Notice that, via the pullback along $V \to \mathrm{pt}$, we have the isomorphism $A^G_{*+8}(V) \simeq A^G_*(\mathrm{pt}) \simeq \mathbb{Q}[t_1, t_2]^{\mathfrak{S}_2}$ by Example 2.1.1. Since $\mathbb{C}^4$ is cut as the zero locus of a section of the rank 4 vector bundle $E \to C$ having as fibre the G-representation $\mathrm{Mat}_{2\times 2}(\mathbb{C})$, $[\mathbb{C}^4]_G$ is the Euler class of E, namely

$$[\mathbb{C}^4]_G = e^G(E) = t_1^2 t_2^2$$

and we want to compute

$$\deg\left(\pi_{\mathrm{Gr}(2,4)*} r_G(t_1^2 t_2^2)\right).$$

Analogously we see that $e^T(\mathfrak{g}/\mathfrak{h}) = (t_1 - t_2)(t_2 - t_1)$ and $e^T(T_V) = t_1^4 t_2^4$, so the Szenes-Vergne localisation formula of Theorem 4.6.4 reads

$$\deg\left(\pi_{\mathrm{Gr}(2,4)*} r_G(t_1^2 t_2^2)\right) = -\frac{1}{2}\mathrm{JK}^{\mathfrak{A}}_{t_1+t_2}\left(\left(\frac{t_1 - t_2}{t_1 t_2}\right)^2\right).$$

Here $\mathfrak{A}$ is the set of weights for the action of T on $V = \mathrm{Mat}_{2\times 4}(\mathbb{C})$, hence $\mathfrak{A} = \{t_1, t_2\}$. The JK residue on the right-hand side was computed in Example 4.1.4 and shown to be -2, thus we find that the degree we wanted to compute is 1.

Chapter 5

Equivariant and K-theoretic localisation.

From now on we are going to focus on the formula of Szenes and Vergne, since it's the one that has found more applications in the recent years. A completely analogous discussion can be carried on for the version of Guillemin and Kalkman.

We have described how this localisation formula can be used to compute degrees of Chow classes on quotients of linear spaces. In this section we are going to show how we can extend this to the equivariant setting. Similar to the Atiyah-Bott formula in the classical setting, this new formula simplifies the residue computation (analogous to the integral in the classical setting) but introduces increased combinatorial complexity in enumerating the many points at which residues must be computed (analogous to the number of fixed loci). These analogies are not coincidental; they arise from the proof of this equivariant formula as a corollary of Atiyah-Bott's one.

Additionally, we will discuss an extension of this localization formula to equivariant K-theory via the Hirzebruch-Riemann-Roch theorem.

Notation. From now on we will use with the integral sign $\int_X$ also in Chow homology, to denote the composition of the degree map with the pushforward through the projection $\pi : X \to \mathrm{pt}$ to a point. In more explicit terms, given a cycle $Z \in A_*(X)$ we will use the notation

$$\int_X Z := \deg(\pi_* Z)$$

and we will do the same in the equivariant case. This will make the formulae much easier to read in what follows.

Contents of the section.

The section is structured as follows.

- We will discuss the equivariant generalisation of the Szenes-Vergne localisation formula (Theorem 5.1.1 in the abelian setting and Theorem 5.1.3 in the non-abelian one). The new feature of this equivariant formula is the role played by a hyperplane arrangement defined in the character space of the torus used to build the quotient. The result will be expressed as a sum of Jeffrey-Kirwan residues, one for each isolated intersection of the arrangement.

- We will also study a K-theoretic version of this formula (Theorem 5.2.1), useful to compute equivariant Euler characteristics of K-theory classes.

5.1 Equivariant Szenes-Vergne localisation.

In this section we are going to describe a generalisation of this formula to the equivariant setting, so that it computes degrees of *equivariant* cycles on quotients of linear spaces. This will be done in two steps:

1. First, we apply the classical Atiyah-Bott localisation formula to the degree we want to compute. This splits up the degree in several contributions arising from the fixed loci *in the quotient variety.*

2. We then realise that all these fixed loci can be described as GIT quotient themselves. Then the non-equivariant version of the Szenes-Vergne formula can be applied to compute the contribution of each fixed locus.

3. We pack together all this data in a nice combinatorial description of the fixed loci and of their contributions in terms of intersections in a hyperplane arrangement and of residues computed at this points.

Different equivariant versions of the Jeffrey-Kirwan localisation formula were studied in [Zie18] and [Mar08]. The strength of the version that we are interested in lies in the fact that it produces very explicit computations and that it is widely applied in physics, as we will see in later sections.

5.1.1 The fixed locus.

Consider a linear space V together with the action of two tori T and S and a regular T-linearisation $\mathcal{L}$ given by a character $\xi \in \chi(T)$. We also assume that the T-action commutes with the S-action, which therefore descends to the quotient $V/\!/T$. We have the weight-space decomposition

$$V \simeq \bigoplus_{\substack{\rho \in \chi(T) \\ \nu \in \chi(S)}} V_{\rho,\nu} \tag{5.1}$$

where $V_{\rho,\nu} \subset V$ is the subspace over which T and S acts via the characters ρ and ν respectively. The following lemma will be useful to study the fixed locus $(V/\!/T)^S$:

Lemma 5.1.1. *The following conditions are equivalent for a vector $v \in V(T)^{ss}$:*

1. *The vector v defines a fixed point in $V/\!/T$.*
2. *The stabiliser $G \subseteq T \times S$ of v is of dimension dim(S).*
3. *Denoted with I the set of $(\rho, \nu) \in \chi(T \times S)$ so that $v_{\rho,\nu} \neq 0$ in (5.1), the quotient of the subspace $\bigoplus_{(\rho,\nu)\in I} V_{\rho,\nu}$ by T is fixed by S.*
4. *Denoted with I the set of $(\rho, \nu) \in \chi(T \times S)$ so that $v_{\rho,\nu} \neq 0$ in (5.1), the intersection of hyperplanes $U := \bigcap_{(\rho,\nu)\in I}\{\rho + \nu = 0\}$ of $\chi(T \times S)^\vee_{\mathbb{C}}$ is of dimension dim(S).*

Proof. ($1 \Rightarrow 2$) Being the image of v through the quotient map fixed by S, the projection of G to S must be surjective, so $\dim(G) \geqslant \dim(S)$. On the other hand the dimension can't be greater, otherwise the fibre of G above $1 \in S$ must be positive dimensional, giving a $\mathbb{C}^* \subseteq T$ acting trivially on v and contradicting semistability. ($2 \Rightarrow 3$) Notice that being v semistable, the fibre of $G \to S$ above the identity is finite, hence the kernel is finite and G maps surjectively on S by dimensional reasons. This implies that, if G acts trivially on $v' \in V(T)^{\text{ss}}$, then the image of v' through the quotient map is fixed. Notice that G acts on $V_{\rho,\nu}$ via the character $G \hookrightarrow T \times S \xrightarrow{\rho \cdot \nu} \mathbb{C}^*$. Since G acts trivially on v, for every $(\rho, \nu) \in I$ we have that the character of G defined above is trivial, hence G acts trivially on $V_{\rho,\nu}$. ($3 \Rightarrow 1$) is trivial. ($4 \Rightarrow 2$) Notice that 4 is the infinitesimal version of 2. Given such U we can consider the subgroup of $G \subset T \times S$ having U as space of cocharacters. In other words we can find a $\mathbb{C}$-basis of U given by integral elements $\lambda_1, \ldots, \lambda_{\dim(S)}$, which define a subgroup G of dimension $\dim(S)$. It's clear that this fixes the vector v since $(\lambda_i \cdot v)_{\rho,\nu} = \rho(\lambda_i)\nu(\lambda_i)v_{\rho,\nu} = v_{\rho,\nu}$ whenever $\rho, \nu \in I$ by construction. Finally assume that the stabiliser of v is of bigger dimension. Then its space of cocharacters would be a bigger dimensional space contained in the intersection U, contradicting the hypothesis on the dimension of U. $(2 \Rightarrow 4)$ Clearly $\chi(G)^\vee_{\mathbb{C}}$ is contained in U, so U is at least $\dim(S)$-dimensional. If it were higher dimensional, we could construct a bigger G fixing v as in the previous point, contradicting the hypothesis on the dimension of the stabiliser. $\square$

This allows us to give a combinatorial description of the S-fixed locus on $V/\!/T$ in terms of the following hyperplane arrangement:

Definition 5.1.1. We will denote with $\mathcal{H}$ the hyperplane arrangement in the linear space $\chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}}$ given by the hyperplanes of the form $\{\rho + \nu = 0\}$ for every choice of $\rho \in \chi(T)$ and $\nu \in \chi(S)$ so that $V_{\rho,\nu} \neq 0$ in (5.1). Given a subspace $U \subset \chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}}$ we will denote with $\mathcal{H}_U$ the set of hyperplanes of the arrangement containing U.

Notation. Given characters $\rho \in \chi(T)$ and $\nu \in \chi(S)$ we will write $\rho + \nu \in \mathcal{H}_U$ if the corresponding hyperplane $\{\rho + \nu = 0\}$ belongs to $\mathcal{H}_U$. By abusing notation, we will also write $\rho \in \mathcal{H}_U$ is there if a ν so that $\rho + \nu \in \mathcal{H}_U$.

Definition 5.1.2. Given an intersection U of hyperplanes in $\mathcal{H}$, we say that it is *stable* if it is $\dim(S)$-dimensional and ξ belongs to the positive span of the set $\{\rho \,|\, \rho \in \mathcal{H}_U\}$. For such a U define the corresponding subspace of V

$$V_U := \bigoplus_{\rho+\nu \in \mathcal{H}_U} V_{\rho,\nu}.$$

Proposition 5.1.1. *The function*

$$\begin{aligned} \{\textit{stable intersections of } \mathcal{H}\} &\to \left\{\textit{connected components of } (V/\!/T)^S\right\} \\ U &\mapsto V_U/\!/T \end{aligned}$$

is a well-defined bijection.

Proof. First of all, notice that since U is stable the T-semistable locus of V_U is nonempty by Lemma 4.5.1, hence the quotient is nonempty. The quotient is fixed by Lemma 5.1.1 and obviously connected. Notice that given two different stable intersections U and U' the corresponding quotient varieties don't intersect. Assume in fact that $v \in V$ belongs to the intersection of the corresponding linear spaces. Then the intersection of hyperplanes corresponding to v given in point 4 of Lemma 5.1.1 contains both U and U', so it is of dimension strictly bigger than $\dim(S)$. The same argument used in the previous Lemma ensures that the dimension of the stabiliser is bigger than $\dim(S)$ and therefore v can't be stable. Finally let $v \in V(T)^{ss}$ be a point that descends to a fixed point on the quotient. By Lemma 5.1.1, point 4 ensures that the corresponding intersection U is of the correct dimension. Moreover, U is stable by Lemma 4.5.1. □

Example 5.1.1. Consider the simplest case possible, namely the case of $\mathbb{P}^1$. Let $T = S = \mathbb{C}^*$ and consider the action on $\mathbb{A}^2$ given by

$$T \times S \curvearrowright \mathbb{A}^2 \quad : \quad (t,s) \cdot (x,y) := (tsx, ty). \tag{5.2}$$

If we consider the T-linearisation given by the character $\xi := t$, we immediately see that the quotient $\mathbb{A}^2/\!/T$ is the projective line $\mathbb{P}^1$.

Proposition 5.1.1 gives a description of the fixed locus of the residual S-action on $\mathbb{P}^1$ in terms of a hyperplane arrangement. Here we show this correspondence explicitly.

The hyperplane arrangement in the space of cocharacters $\chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}} \simeq \mathbb{C} \times \mathbb{C}$ is given by the weights of the $(T \times S)$-action, hence it is the arrangement given by the two lines $t + s = 0$ and $t = 0$. Each of these two lines is a stable intersection since ξ is in the positive span of t. The first stable intersection $\{t + s = 0\}$ corresponds to the subspace $\mathbb{A}^1 \times 0 \subset \mathbb{A}^2$, hence our Proposition 5.1.1 predicts that its quotient $[1 : 0] \in \mathbb{P}^1$ is a fixed locus for the induced S-action. Analogously the second stable intersection $\{t = 0\}$ corresponds to the subspace $0 \times \mathbb{A}^1 \subset \mathbb{A}^2$, hence our proposition predicts that its quotient $[0 : 1] \in \mathbb{P}^1$ is a fixed locus. Indeed, these are the only two fixed loci for the induced S-action on $\mathbb{P}^1$, which is $s \cdot [x : y] = [sx : y]$ as can be seen from (5.2).

This gives a combinatorial description of the fixed locus $(V/\!/T)^S$ in terms of a hyperplane arrangement. We now show that this arrangement is not just an ad-hoc gadget, but it's an intrinsic invariant of $V/\!/T$, namely *the spectrum of its homology ring*.

5.1.2 The spectrum of homology.

This and the following sections are not strictly needed for the proof of equivariant JK localisation, but they provide more context and a nice interpretation of the hyperplane arrangements that appear in the formula. We start by considering the more general context of a torus S acting on a smooth quasiprojective variety $S \curvearrowright X$ with a regular linearisation $\mathcal{L}$. The first observation is that, since the Atiyah-Bott localisation formula of Theorem 2.3.4 ensures that the pullback defines an isomorphism of $A_S^*(\mathrm{pt})_f$-algebras

$$i^* : A^S(X)_f \to A^S(X^S)_f$$

for some $f \in A_S^*(\mathrm{pt}) \simeq \mathrm{Sym}\left(\chi(S)_{\mathbb{C}}\right)$ vanishing at the origin. By taking Spec we find an isomorphism of schemes over the open subscheme $\{f \neq 0\}$ of $\chi(S)_{\mathbb{C}}^{\vee}$:

$$\begin{tikzcd} \mathrm{Spec}\left(A^S(X^S)_f\right) \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Spec}\left(A^S(X)_f\right) \arrow[dl] \\ & \{f \neq 0\} & \end{tikzcd}$$

This helps us studying the spectrum of the equivariant homology of X:

Proposition 5.1.2. *Consider the decomposition $X^S = \bigsqcup_{F \subseteq X^S} F$ of the fixed locus in its connected components. Then the ring $A^S(X)_f$ is isomorphic to $\bigoplus_F A^S(F)_f$ and*

$$\mathit{Spec}\left(A^S(X)_f\right) \simeq \bigsqcup_{F \subseteq X^S} \mathit{Spec}\left(A^S(F)_f\right)$$

is the decomposition in connected components, and each component is a fattening of the base $\{f \neq 0\} \subset \chi(S)_{\mathbb{C}}^{\vee}$.

Proof. The first part of the claim is an immediate corollary of Atiyah-Bott's formula. Notice that for every connected component of the fixed locus $A^S(F)_f \simeq A(F) \otimes_{\mathbb{C}} A^S(\mathrm{pt})_f$. Moreover $A(F)$ is nilpotent, being the Chow group of a smooth quasiprojective variety (there is nothing below degree zero), hence

$$\mathrm{Spec}\left(A^S(F)_f\right) \simeq \mathrm{Spec}(A(F)) \times \{f \neq 0\},$$

where $\mathrm{Spec}(A(F))$ is a fat point. □

In the case where X is a geometric quotient of the form $Y/\!/T$ we can say more about $A^S(X)$:

Proposition 5.1.3. *$\mathit{Spec}\left(A^S(Y/\!/T)\right)$ is a closed subscheme of $\mathit{Spec}\left(A^{T\times S}(Y)\right)$ and its ideal sheaf is the image of the $(T \times S)$-equivariant pushforward j_*, where j is the inclusion of the non-semistable locus $j : Y\backslash Y(T)^{ss} \hookrightarrow Y$.*

Proof. Consider the exact sequence of equivariant Chow groups [EG98b, Lemma 4]

$$A^{T\times S}(Y\backslash Y(T)^{\mathrm{ss}}) \xrightarrow{j_*} A^{T\times S}(Y) \xrightarrow{u^*} A^{T\times S}(Y(T)^{\mathrm{ss}}) \to 0 \tag{5.3}$$

where u^* is a flat pullback and hence a ring homomorphism. Notice that $A^{T\times S}(Y(T)^{\mathrm{ss}})$ is isomorphic to $A_*^S(Y/\!/T)$ by Theorem 2.1.2. □

5.1.3 Intersections in the hyperplane arrangement as homology schemes.

Let's go back at our case of interest, where $T \times S \curvearrowright V$ with a regular T-linearisation $\mathcal{L}$. We have seen that the spectrum of $A^S(V/\!\!/T)$ is a closed subscheme of $\chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}}$ and (away from a hypersurface) it coincides with the disjoint union of the spectra of the cohomology of fixed loci. We have shown how connected components of $(V/\!\!/T)^S$ are indexed by stable intersections of $\mathcal{H}$. We now want to prove that the fixed locus corresponding to the stable intersection U has U itself as reduced spectrum of its equivariant homology:

Proposition 5.1.4. *Let U be a stable intersection of $\mathcal{H}$. Then the support of the cohomology of $V_U/\!\!/T$ coincides with U:*

$$supp\left(Spec(A^S(V_U/\!\!/T))\right) = U.$$

Scheme-theoretically, this subscheme is cut inside $\chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}}$ by the equations

$$\prod_{\rho \in J} \prod_{\rho+\nu \in \mathcal{H}_U} (\rho + \nu)^{dim(V_{\rho,\nu})}$$

where J ranges over the minimal subsets of $\{\rho \,|\, \rho \in \mathcal{H}_U\}$ such that ξ is not in the positive span of the complement J^c.

Proof. First of all, notice that the T-unstable locus of V_U is a union of T-invariant linear spaces since the ring of invariant sections

$$\bigoplus_{n \geqslant 0} H^0(V_U, \mathcal{L}^{\otimes n})^T \simeq \bigoplus_{n \geqslant 0} H^0(V_U, \mathcal{O}_{V_U} \otimes n\xi)^T$$

is generated by monomials. Moreover notice that a linear subspace is T-invariant if and only if it is a direct sum of eigenspaces for the T-action. By Lemma 4.5.1, this means that the unstable locus in V_U is the union, indexed over the maximal subsets $I \subset \{\rho \,|\, \rho \in \mathcal{H}_U\}$ so that ξ is not in the positive span of I, of the subspaces $V_I := \bigoplus_{\rho \in I} V_\rho$:

$$V_U \backslash V_U(T)^{\mathrm{ss}} = \bigcup_I V_I.$$

Consider the exact sequence (5.3) specialised to $Y = V_U$; the equations we seek are given by the generators for the image of the pushforward map

$$A^{T \times S}\left(V_U \backslash V_U(T)^{\mathrm{ss}}\right) \xrightarrow{j_*} A^{T \times S}(V_U).$$

We can consider the surjection

$$\bigsqcup_I V_I \to V_U \backslash V_U(T)^{\mathrm{ss}}$$

inducing a surjection of Chow groups by pushforward. So the image we want to study coincides with the image of

$$\bigoplus_I A^{T\times S}(V_I) \xrightarrow{j_*} A^{T\times S}(V_U),$$

which is the ideal generated by the images of the various $A^{T\times S}(V_I) \xrightarrow{j_*} A^{T\times S}(V_U)$. Notice that, since both V_I and V_U are linear spaces, their Chow groups are canonically isomorphic to the Chow group of a point via the flat pullback through the projection. This makes j_* an endomorphism of $A^S(\mathrm{pt}) \simeq \mathrm{Sym}(\chi(T)_{\mathbb{C}} \times \chi(S)_{\mathbb{C}})$. The pullback through the regular embedding j is an isomorphism, so we can easily compute j_* by the projection formula

$$j^* j_* \alpha = e^{T\times S}(\mathcal{N}_{V_I/V_U}) \cdot \alpha.$$

Since that this Euler class is

$$\prod_{\rho\in I^c} \prod_{\rho+\nu\in\mathcal{H}_U} (\rho+\nu)^{\dim(V_{\rho,\nu})} = 0$$

the morphism j_* is the multiplication by this function. This discussion shows that $\mathrm{Spec}(A^S(V_U /\!\!/ T))$ is the zero locus of the functions above, and by setting $J = I^c$ we complete the proof of the scheme-theoretic statement. Set-theoretically, U is contained in the vanishing locus of those functions by construction. On the other hand, assume that a point $x \in \chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}}$ belongs to that zero set. The set of characters

$$I := \{\rho \in \mathcal{H}_U \mid \exists \nu \text{ so that } \rho+\nu \in \mathcal{H}_U \text{ and } (\rho+\nu)(x) = 0\}$$

forms a set of generators of $\chi(T)_{\mathbb{C}}$, otherwise ξ would not be contained in the positive span of I but the function

$$\prod_{\rho\in I^c} \prod_{\rho+\nu\in\mathcal{H}_U} (\rho+\nu)^{\dim(V_{\rho,\nu})} \tag{5.4}$$

wouldn't vanish on x, causing a contradiction. Hence the functions $\rho+\nu$ that vanish on x cut a subspace of dimension $\dim(S)$, which must then be precisely U by dimensional reasons, so $x \in U$. □

This result shows that the stable intersections of the hyperplane arrangement $\mathcal{H}$ are an intrinsic invariant of $V/\!/T$, namely they describe the spectrum of its S-equivariant cohomology!

Example 5.1.2. Let's continue Example 5.1.1. We show that the hyperplane arrangement we studied before, given by the two lines $\{t+s=0\}$ and $\{t=0\}$, coincides with $\mathrm{Spec}(A^S(\mathbb{P}^1))$. Indeed $A^S(\mathbb{P}^1) \simeq A^{T\times S}(\mathbb{A}^2\backslash O)$ fits into the exact sequence (5.3)

$$A^{T\times S}(O) \xrightarrow{j_*} A^{T\times S}(\mathbb{A}^2) \xrightarrow{u^*} A^S(\mathbb{P}^1) \to 0$$

which, by identifying the first two Chow groups with the Chow group of a point via the flat pullback along the projection to a point, coincides with

$$\mathbb{C}[t,s] \xrightarrow{\times(t+s)t} \mathbb{C}[t,s] \xrightarrow{u^*} A^S(\mathbb{P}^1) \to 0$$

finally giving $A^S(\mathbb{P}^1) \simeq \mathbb{C}[t,s]/((t+s)t)$.

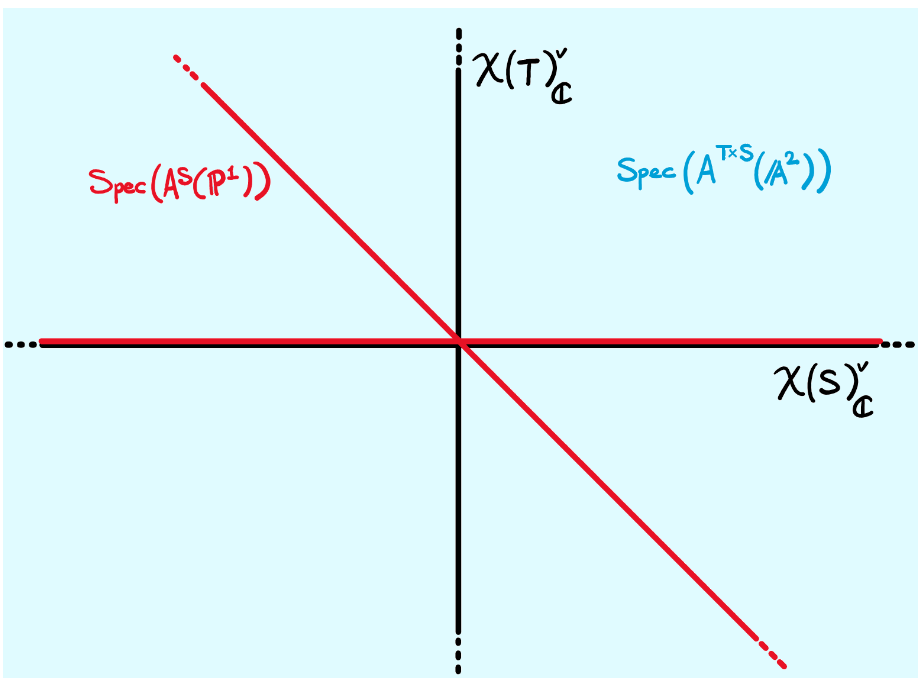


Figure 5.1: The spectrum of $A^S(\mathbb{P}^1)$ as a subscheme of $\mathrm{Spec}(A^{T\times S}(\mathbb{A}^2))$.

5.1.4 A remark on hyperplane arrangements.

Consider the weight space decomposition (5.1) of V for the action of $T \times S$:

$$V \simeq \bigoplus_{\substack{\rho\in\chi(T)\\ \nu\in\chi(S)}} V_{\rho,\nu}.$$

We described the fixed loci on $V/\!/T$ in terms of the stable intersections of the hyperplane arrangement $\mathcal{H}$ in $\chi(T\times S)^\vee_\mathbb{C}$. This was the arrangement of hyperplanes of the form

$$\{\rho+\nu=0\} \qquad : \quad V_{\rho,\nu}\neq 0.$$

Consider a subspace U of the same dimension as S obtained by intersecting some hyperplanes in $\mathcal{H}$ and set

$$\mathfrak{A}_U := \{\rho\in\chi(T) \quad | \quad U\subset\{\rho+\nu=0\} \text{ and } V_{\rho,\nu}\neq 0\}.$$

Such subspace is called a stable intersection if the stability ξ is contained in the positive span of $\mathfrak{A}_U$. For every $s\in\chi(S)^\vee_\mathbb{C}$ we consider the point

$$\zeta_U(s) := U\cap\left(\chi(T)^\vee_\mathbb{C}\times\{s\}\right).$$

The equivariant localisation formulae of the next section will describe the relevant intersection number, evaluated at a generic $s\in\chi(S)^\vee_\mathbb{C}$, as a sum of contributions, one for each stable intersection U, of the form

$$\sum_{\substack{U \text{ stable}\\ \text{intersection in } \mathcal{H}}} \mathrm{JK}^{\mathfrak{A}_U}_{\xi,\zeta_U(s)}(\dots). \tag{5.5}$$

Our aim is now to simplify a little bit this residue operator.

Definition 5.1.3. . Fixed $s\in\chi(S)^\vee_\mathbb{C}$ we can consider the hyperplane arrangement in $\chi(T)^\vee_\mathbb{C}$

$$\mathcal{H}_s := \mathcal{H}\cap\left(\chi(T)^\vee_\mathbb{C}\times\{s\}\right)$$

explicitly given by the affine hyperplanes of the form

$$\{\rho+\nu(s)=0\} \qquad : \quad V_{\rho,\nu}\neq 0.$$

A point in $\chi(T)^\vee_\mathbb{C}$ is called an *isolated intersection* of $\mathcal{H}_s$ if it's the intersection of $\dim(T)$ independent hyperplanes. Given such intersection P consider

$$\mathfrak{A}_P := \{\rho\in\chi(T) \quad | \quad \exists\nu \text{ s.t. } P\subset\{\rho+\nu(s)=0\} \text{ and } V_{\rho,\nu}\neq 0\}.$$

We say that P is a *stable isolated intersection* if ξ is contained in the positive span of $\mathfrak{A}_P$.

For generic s the stable intersections of U of $\mathcal{H}$ are in bijection, via

$$U \mapsto \zeta_U(s) := U \cap \left(\chi(T)^\vee_{\mathbb{C}} \times \{s\}\right),$$

with the *stable isolated intersections* of $\mathcal{H}_s$. Notice that this correspondence satisfies $\mathfrak{A}_U = \mathfrak{A}_{\zeta_U(s)}$. This means that, for a generic s, the operator (5.5) is equal to

$$\sum_{\substack{P \text{ stable isolated} \\ \text{intersection in } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P}(\dots)\,. \tag{5.6}$$

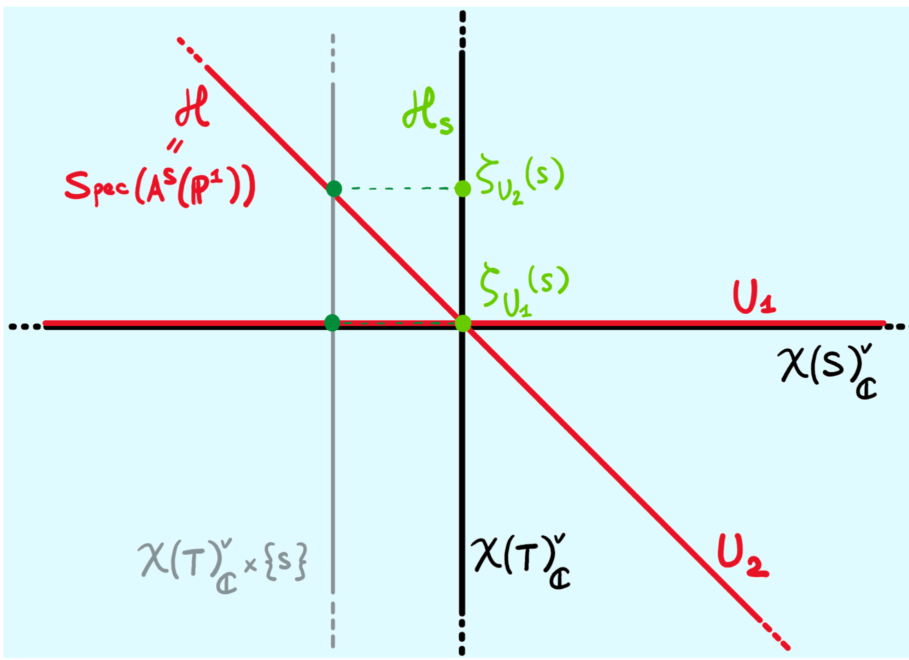


Figure 5.2: The hyperplane arrangement $\mathcal{H}_s$ in the case of Example 5.1.2. It is the arrangement in $\chi(T)^\vee_{\mathbb{C}}$ realised by intersecting $\mathcal{H}$ with $\chi(T)^\vee_{\mathbb{C}} \times \{s\}$.

5.1.5 Equivariant abelian localisation.

In this section we wish to prove the S-equivariant version of the abelian Szenes-Vergne localisation formula:

Theorem 5.1.1 (Abelian equivariant localisation)**.** *Consider a linear space V together with the action of two commuting tori T and S and a regular T-linearisation $\mathcal{L}$*

given by a character $\xi \in \chi(T)$. Assume that the momentum polytope for the T-action is strictly convex and that $V^T = O$. Given $\alpha \in A^{T\times S}(V)$, consider the induced class $r(\alpha) \in A^S(V/\!/T)$, where $r := r_{T\times S,S}$ is the S-equivariant Kirwan map for the action of $T \times S$ on V. For a generic cocharacter $s \in \chi(S)^\vee_{\mathbb{C}}$, its degree can be computed as

$$\int_{V/\!/T} r(\alpha)(s) = \sum_{\substack{P \text{ stable isolated} \\ \text{intersection in } \mathcal{H}_s}} JK^{\mathfrak{A}_P}_{\xi,P}\left(\frac{\alpha}{e^{T\times S}(T_V)}\right),$$

where $\mathcal{H}_s$ and $\mathfrak{A}_P$ are those of Definition 5.1.3 and the argument of the JK residue is evaluated at s, so it defines a rational function on $\chi(T)^\vee_{\mathbb{C}}$.

Remark 28. This formula computes the value of the equivariant degree at a generic cocharacter $s \in \chi(S)^\vee_{\mathbb{C}}$. Notice that this is not restrictive at all, since this degree is a polynomial function on $\chi(S)^\vee_{\mathbb{C}}$ and knowing it generically is enough to describe it everywhere. Moreover, the computation for "the generic s" can be done in one step by just treating s as a formal variable not satisfying any relation that would make it non-generic.

Proof. In this proof all character spaces are with complex coefficients. For simplicity, we will not reflect this in the notation. In Proposition 5.1.1 we have seen that the connected components of $(V/\!/T)^S$ are described by stable intersections in $\mathcal{H}$, using the notation of Section 5.1.4.By applying Atiyah-Bott localisation (Theorem 2.3.3) with respect to the S-action on the quotient $V/\!/T$, we can write

$$\int_{V/\!/T} r(\alpha) = \sum_{\substack{U \text{ stable} \\ \text{intersection in } \mathcal{H}}} \int_{V_U/\!/T} \frac{r(\alpha_{|V_U})}{r(e^{T\times S}(\mathcal{N}_{V_U/V}))} \tag{5.7}$$

exactly as we did in Section 4.3. Consider the morphism r, which restricted to V_U becomes

$$r : \mathrm{Sym}(\chi(T)) \otimes \mathrm{Sym}(\chi(S)) \to A(V_U/\!/T) \otimes \mathrm{Sym}(\chi(S)).$$

We can express this map in terms of the non-equivariant Kirwan map $r_T : A^T(V_U) \to A(V_U/\!/T)$ with the following

Lemma 5.1.2. *Given stable intersection U of $\mathcal{H}$ consider the $\mathbb{C}$-linear morphism $\zeta_U : \chi(S)^\vee \to \chi(T)^\vee$ so that U is the graph. Given a polynomial function $f : \chi(T)^\vee \to \mathbb{C}$ we can consider the associated function*

$$\tilde{f} : \chi(T)^\vee \times \chi(S)^\vee \to \mathbb{C} \quad : \quad \tilde{f}(t,s) := f(t + \zeta_U(s)).$$

If we split the contributions of T and S as $\tilde{f}(t,s) = \sum_{k=0}^{d} f_k(t) g_k(s)$, then

$$r(f(t) \otimes h(s)) = \sum_{k=0}^{d} r_T(f_k) \otimes g_k(s) h(s).$$

for every polynomial function $h : \chi(S) \to \mathbb{C}$.

Proof. Consider the subtorus $G \subseteq T \times S$ fixing V_U, whose cocharacter space satisfies $\chi(G)^\vee = U \subset \chi(T)^\vee \times \chi(S)^\vee$ by the proof of Lemma 5.1.1. The finite morphism $T \times G \to T \times S$ gives a change of group homomorphism fitting in the diagram

$$\begin{tikzcd}
A^{T\times S}(V_U) \arrow[r, "r"] \arrow[d, "(T\times G\to T\times S)^*"'] & A^S(V_U/\!/T) \arrow[d, "(G\to S)^*"] \\
A^{T\times G}(V_U) \arrow[r, "r_{T\times G,G}"] & A^G(V_U/\!/T)
\end{tikzcd}$$

which coincides with the diagram

$$\begin{tikzcd}
\mathrm{Sym}(\chi(T)) \otimes \mathrm{Sym}(\chi(S)) \arrow[r, "r"] \arrow[d, "\substack{\text{composition with} \\ \chi(T)^\vee \times U \to \chi(T)^\vee \times \chi(S)^\vee}"'] & A(V_U/\!/T) \otimes \mathrm{Sym}(\chi(S)) \arrow[d, "\substack{\text{composition with} \\ U \to \chi(S)^\vee}"] \\
\mathrm{Sym}(\chi(T)) \otimes \mathrm{Sym}(U^\vee) \arrow[r, "r_T \otimes \mathrm{Id}"] & A(V_U/\!/T) \otimes \mathrm{Sym}(U^\vee)
\end{tikzcd}$$

We know that U projects isomorphically onto $\chi(S)^\vee$, so we can pick the inverse and attach to the bottom of the diagram the following clearly commutative square

$$\begin{tikzcd}
\mathrm{Sym}(\chi(T)) \otimes \mathrm{Sym}(U^\vee) \arrow[r, "r_T \otimes \mathrm{Id}"] \arrow[d, "\substack{\text{composition with} \\ \chi(T)^\vee \times \chi(S)^\vee \to \chi(T)^\vee \times U}"'] & A(V_U/\!/T) \otimes \mathrm{Sym}(U^\vee) \arrow[d, "\substack{\text{composition with} \\ \chi(S)^\vee \to U}"] \\
\mathrm{Sym}(\chi(T)) \otimes \mathrm{Sym}(\chi(S)) \arrow[r, "r_T \otimes \mathrm{Id}"] & A(V_U/\!/T) \otimes \mathrm{Sym}(\chi(S)).
\end{tikzcd}$$

The wanted equality follows from the commutativity of this big diagram, since the composition of the vertical arrows on the right is the identity while the composition of the vertical arrows on the left is the endomorphism of $\mathrm{Sym}(\chi(T)) \otimes \mathrm{Sym}(\chi(S))$ given by the composition with

$$\chi(T)^\vee \times \chi(S)^\vee \to \chi(T)^\vee \times \chi(S)^\vee \quad : \quad (t,s) \mapsto (t + \zeta_U(s), s).$$

$\square$

Notice that (5.7) is an equality in $\mathrm{Sym}(\chi(S))_f$ for some polynomial function f. In particular it becomes an equality of numbers whenever we evaluate it at a cocharacter $s \in \chi(S)^\vee$ where the Euler classes don't vanish. If we fix such $s \in \chi(S)^\vee$, thanks to the lemma we just proved, the contribution of the stable intersection U is

$$\int_{V_U /\!/ T} \frac{r_T\left(\alpha(t+\zeta_U(s), s)\right)}{r_T\left(E(t+\zeta_U(s), s)\right)}.$$

where we have denoted with $E : \chi(T)^\vee \times \chi(S)^\vee \to \mathbb{C}$ the $T \times S$-equivariant Euler class of $\mathcal{N}_{V_U/V}$. Notice that, having nonzero constant term, $E(t+\zeta_U(s), s)$ is invertible in the ring of formal power series in t by means of the geometric series, hence

$$\frac{1}{E(t+\zeta_U(s), s)} = \sum_{k=0}^{\infty} \phi_k(t)$$

where ϕ_k is an homogeneous function of degree k on $\chi(T)^\vee$ (notice that if we tensor $\chi(T)^\vee$ with $\mathbb{C}$ this is the power series expansion of the holomorphic function $E(t+\zeta_U(s), s)^{-1}$ at the origin). Then we can write the integral above as

$$\int_{V_U /\!/ T} r_T \left(\alpha(t+\zeta_U(s), s) \sum_{k=0}^{\infty} \phi_k(t) \right),$$

where the sum can be truncated at every k bigger than the dimension of $V_U /\!/ T$. By nonequivariant Szenes-Vergne localisation we find that this integral coincides with the residue

$$\mathrm{JK}_{\tilde{\xi}}^{\mathfrak{A}_U} \left(\frac{\alpha(t+\zeta_U(s), s) \sum_{k=0}^{\infty} \phi_k(t)}{e^T(T_{V_U})} \right) = \mathrm{JK}_{\tilde{\xi}}^{\mathfrak{A}_U} \left(\frac{\alpha(t+\zeta_U(s), s)}{E(t+\zeta_U(s), s) e^T(T_{V_U})} \right),$$

where the second equality holds true by Lemma 4.1.5. To conclude we just have to notice that

$$e^T(T_{V_U})(t) = e^{T\times S}(T_{V_U})(t+\zeta_U(s), s),$$

which is obvious being

$$e^{T\times S}(T_{V_U})(t, s) = \prod_{\rho+\nu \in \mathcal{H}_U} (\rho(t) + \nu(s))^{\dim(V_{\rho,\nu})}$$

hence

$$\begin{aligned}
e^{T\times S}(T_{V_U})(t+\zeta_U(s),s) &= \prod_{\rho+\nu\in\mathcal{H}_U} (\rho(t)+\rho(\zeta_U(s))+\nu(s))^{\dim(V_{\rho,\nu})} \\
&= \prod_{\rho+\nu\in\mathcal{H}_U} (\rho(t))^{\dim(V_{\rho,\nu})} \\
&= \prod_{\rho\in\mathfrak{A}_\rho} \rho(t)^{\dim(V_\rho)} = e^T(T_{V_U})(t).
\end{aligned}$$

since $(\zeta_U(s),s)$ belongs to U, where all the $\rho+\nu\in\mathcal{H}_U$ vanish by definition. Finally, we use the fact that the operators (5.5) and (5.6) coincide. □

Example 5.1.3. Let's continue with our series of examples on $\mathbb{P}^1$ started in Example 5.1.1. The T-equivariant Kirwan map $r_T : A^T(\mathbb{A}^2)\to A(\mathbb{P}^1)$ sends the equivariant class t into the class of the point h. In particular this shows that the S-equivariant Kirwan map r sends the class $at+bs$ into some S-equivariant lift of $ah\in A(\mathbb{P}^1)$. In particular we should expect

$$\int_{\mathbb{P}^1} r(at+bs) = a. \tag{5.8}$$

Let's use the formula of Theorem 5.1.1 to show this. We already showed that the hyperplane arrangement in $\chi(T)^\vee_{\mathbb{C}}\times\chi(S)^\vee_{\mathbb{C}}\simeq\mathbb{C}\times\mathbb{C}$ is

$$\mathcal{H}=\{\{t+s=0\},\{t=0\}\}$$

and both the lines are stable intersections. For the first line the morphism

$$\zeta_{\{t+s=0\}}:\mathbb{C}\to\mathbb{C}$$

maps x into $-x$. Thus, for a generic value of s, the contribution of this intersection to the S-equivariant integral (5.8) is

$$\mathrm{JK}^{\{t\}}_{t,-s}\left(\frac{at+bs}{(t+s)t}\right)=\mathrm{JK}^{\{t\}}_{t}\left(\frac{at+(b-a)s}{t(t-s)}\right)=a-b,$$

where the last equality holds true since, in this case, the JK residue is simply the usual complex analytic residue at $t=0$. Analogously the contribution of the line $\{t=0\}$ is

$$\mathrm{JK}^{\{t\}}_{t}\left(\frac{at+bs}{(t+s)t}\right)=b,$$

so their sum is a as expected.

5.1.6 Quasiprojective quotients.

In the previous section we have discussed the case where $V/\!/T$ is projective, or equivalently by Lemma 4.5.2 the case of Δ strictly convex and $V^T = O$. If $V/\!/T$ is not projective we can still *define* the equivariant degree $\int_{V/\!/T} r(\alpha)$ by localisation, as long as the fixed locus $(V/\!/T)^S$ is proper. In this case, the formula of Theorem 5.1.1 still works by the same exact proof.

We can translate the condition on the properness of the fixed locus in the following condition on the stable intersections in the hyperplane arrangement $\mathcal{H}$:

Proposition 5.1.5. *Assume that $V^T = O$ and let U be a stable intersection of the hyperplane arrangement $\mathcal{H}$. The fixed variety $V_U/\!/T$ is fixed if and only if the subset $\mathfrak{A}_U$ spans a strictly convex cone. In particular, the fixed locus $(V/\!/T)^S$ is projective if and only if, for every isolated intersection U in the hyperplane arrangement $\mathcal{H}$, the set $\mathfrak{A}_U$ spans a strictly convex cone.*

Proof. This follows at once by Lemma 4.5.1 ensuring that the momentum cone of V_U is spanned by $\mathfrak{A}_U$ and by Lemma 4.5.2. □

Another sufficient, and usually easy to check, condition for properness of $(V/\!/T)^S$ is given by the following:

Proposition 5.1.6. *Assume that there is no nonconstant $(T \times S)$-invariant function on V. Then $(V/\!/T)^S$ is projective.*

Proof. The action of S on V commuting with the one of T induces an S-action on $H^0(V, \mathcal{O}_V)^T$. This is a finitely generated algebra and we can take the generators to be S-equivariant. This is because we know this algebra is generated by some elements of degree bounded by some number d, and the subspace $W_d \subset H^0(V, \mathcal{O}_V)^T$ of nonconstant functions of degree at most d is a finite dimensional subrepresentation of S. Then we can split this subrepresentation in 1-dimensional representations and take generators $f_1, ..., f_N$ for all of them. This shows that the projective morphism

$$V/\!/T \to \operatorname{Spec}(H^0(V, \mathcal{O}_V))^T \subseteq \mathbb{A}^N$$

given by GIT is S-equivariant with respect to an S-action on $\mathbb{A}^N$ having the origin as its only fixed point. Then $(V/\!/T)^S$, being fixed, is mapped to the origin and therefore it is contained in a fibre of the projective morphism, so it is projective. □

5.1.7 Equivariant nonabelian localisation.

The two ingredients we used to pass from the abelian version to the nonabelian one are the cycle class map and the Theorem 4.6.1 of Martin. Both keep working in the same way in the equivariant context. In particular a reference for equivariant cycle maps is [EG98b, Section 2.8], where it's shown they enjoy all the functoriality properties of the usual cycle maps. The following equivariant version of Martin's formula holds true by the same proof of Martin:

Theorem 5.1.2. *Let G be a reductive connected algebraic group acting on a smooth quasiprojective variety Y with a linearisation $\mathcal{L}$. Denoted with T a maximal subtorus of G, assume that the G and T-actions on the respective semistable loci are free and that the quotients $Y//G$ and $Y//T$ are projective. Let S be another torus action on Y commuting with the action of G. Let $\alpha \in H_S^*(Y//G)$ and $\beta \in H_S^*(Y//T)$ be such that $\pi^*\alpha = j^*\beta$. Then*

$$\int_{Y//G} \alpha = \frac{1}{|W|} \int_{Y//T} \beta \cup e^S(\mathcal{R}),$$

where W is the Weyl group of G and $\mathcal{R}$ is the roots bundle of $Y//T$, namely the bundle obtained by descending to the quotient the $T \times S$-equivariant vector bundle

$$Y \times \mathfrak{g}/\mathfrak{h} \to X,$$

where $\mathfrak{g} := T_1G$ and $\mathfrak{h} := T_1T$, with action on the fibre induced by the adjoint T-action and trivial S-action.

The equivariant version of the nonabelian Szenes-Vergne formula follows from the abelian one exactly as in the nonequivariant case:

Theorem 5.1.3 (Equivariant Szenes-Vergne localisation)**.** *Consider a linear space V with the action of a reductive connected group G and a torus S. Let $T \subseteq G$ be a maximal subtorus and consider a linearisation $\mathcal{L}$ for the G action, given by a character $\xi \in \chi(G)$. Assume that*

1. *The actions of G and S commute.*
2. *The actions of T and G are free on the respective semistable loci.*
3. *The S-fixed loci $(V//T)^S$ and $(V//G)^S$ are proper.*

Given $\alpha \in A^{G\times S}(V)$, consider the induced class $r(\alpha) \in A^S(V/\!/G)$, where $r := r_{G\times S,S}$ is the S-equivariant Kirwan map for the action of $G \times S$ on V. For a generic cocharacter $s \in \chi(S)^\vee_{\mathbb{C}}$, its degree can be computed as

$$\int_{V/\!/G} r(\alpha)(s) = \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection in } \mathcal{H}_s}} JK^{\mathfrak{A}_P}_{\xi,P}\left(\frac{\alpha \cdot e^T(\mathfrak{g}/\mathfrak{h})}{e^{T\times S}(T_V)}\right),$$

where

- *The argument of the JK residue is evaluated at s, so it defines a rational function on $\chi(T)^\vee_{\mathbb{C}}$.*
- *W is the Weyl group of G.*
- *$\mathfrak{g}$ is the adjoint representation of G and $\mathfrak{h}$ is the Lie algebra of T.*
- *$\mathcal{H}_s, \mathfrak{A}_P$ are those of Definition 5.1.3.*

Remark 29 (Projectivity of the fixed loci). Notice that, by Proposition 5.1.6, if $H^0(V, \mathcal{O}_V)^{T\times S} = \mathbb{C}$ then $(V/\!/T)^S$ is proper. Clearly, since G-invariant functions are T-invariant too, we obtain that $H^0(V, \mathcal{O}_V)^{G\times S} = \mathbb{C}$ and hence that $(V/\!/G)^S$ by the same proof of Proposition 5.1.6. This shows that the condition

$$H^0(V, \mathcal{O}_V)^{T\times S} \simeq \mathbb{C} \tag{5.9}$$

implies that both $(V/\!/T)^S$ and $(V/\!/G)^S$ are projective.

5.1.8 Integrating other classes.

Assume we are in the context of the previous section and that we want to compute the integral of a class $\alpha \in A^S(V/\!/G)$. Assume we also don't know how to express it in the form $r(\beta)$ for some $\beta \in A^{G\times S}(V)$. Maybe we know that there are two classes $\beta, \gamma \in A^{G\times S}(V)$ so that $r(\gamma)$ is invertible on $(V/\!/G)^S$ and

$$\alpha_{|(V/\!/G)^S} = \frac{r(\beta)_{|(V/\!/G)^S}}{r(\gamma)_{|(V/\!/G)^S}}.$$

Then the same exact argument used in the previous sections ensures that, in the notation of the previous Theorem 5.1.3,

$$\int_{V/\!/T} \alpha(s) = \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection in } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P}\left(\frac{\beta \cdot e^T(\mathfrak{g}/\mathfrak{h})}{\gamma \cdot e^{T\times S}(T_V)}\right). \tag{5.10}$$

Notice that it is crucial that $r(\gamma)$ is invertible on $(V/\!/G)^S$ for this equality to hold true. Here we consider a very important example of such classes:

Lemma 5.1.3. *Let E be a $(G \times S)$-equivariant vector bundle on V so that its equivariant Euler class $e^{T\times S}(E)$, thought as a polynomial function on $\chi(T \times S)^\vee_{\mathbb{C}}$, doesn't vanish entirely on any stable intersection of the hyperplane arrangement $\mathcal{H}$ associated to V (see Definition 5.1.1). Then $r(e^{T\times S}(E))$ is invertible on $(V/\!/T)^S$ and hence $r(e^{G\times S}(E))$ is invertible on $(V/\!/G)^S$.*

Proof. By splitting the $(T\times S)$-representation E in subrepresentations we can assume that E is 1-dimensional, so $e^{T\times S}(E) = \phi + \psi$ for some $\phi \in \chi(T), \psi \in \chi(S)$. Every fixed locus on $V/\!/T$ is of the form $V_U/\!/T$ for some stable intersection U of $\mathcal{H}$ as seen in Proposition 5.1.1. Moreover, notice that $r(\phi + \psi) \in A^S_1(V_U/\!/T)$ is invertible if and only if it is not constant in the equivariant parameter $s \in \chi(S)^\vee_{\mathbb{C}}$. Then, as described in Lemma 5.1.2, for every $s \in \chi(S)^\vee$ we have

$$r(\phi + \psi)(s) = r(\phi) + \phi(\zeta_U(s)) + \psi(s) = r(\phi) + e^{T\times S}(E)(\zeta_U(s), s)$$

which doesn't depend on s if and only if $e^{T\times S}(E)(\zeta_U(s), s) = 0$ for all s, or in other words $e^{T\times S}(E)$ vanishes on U. □

5.2 K-theoretic version.

In this section we prove a version of the Szenes-Vergne localisation formula for Euler characteristics by using the previous formula and a the Hirzebruch-Riemann-Roch theorem. For another approach to K-theoretic localisation formulae of Jeffrey-Kirwan type see [AFO18, Appendix A].

Definition 5.2.1. Given a formal variable y and a K-theory class represented by a vector bundle $E \in K_G(X)$ we will denote with $\Lambda_{-y}E$ the class $\sum_{k=1}^{\mathrm{rk}(E)} y^k \Lambda^k E \in K_G(X)[y]$. In particular $\Lambda_{-1}E = \sum_{k=1}^{\mathrm{rk}(E)} (-1)^k \Lambda^k E \in K_G(X)$.

Theorem 5.2.1 (K-theoretic Szenes-Vergne localisation.)**.** *Consider a linear space V with the action of a reductive connected group G and a torus S. Let $T \subseteq G$ be a maximal subtorus and consider a linearisation $\mathcal{L}$ for the G action, given by a character $\xi \in \chi(G)$. Assume that*

1. *The actions of G and S commute.*

2. *The actions of T and G are free on the respective semistable loci.*

3. The S-fixed loci $(V/\!/T)^S$ and $(V/\!/G)^S$ are proper.

Given $E \in K_{G\times S}(V)$, denoting with $r(E)$ the S-equivariant K-theory class induced on $V/\!/G$, the following formula holds true for a generic cocharacter $s \in \chi(S)^\vee_{\mathbb{C}}$:

$$\begin{gathered} ch^S\left(\chi^S\left(V/\!/G, r(E)\right)\right)(s) = \\ \frac{1}{|W|}\sum_{\substack{P \text{ stable isolated} \\ \text{intersection in } \mathcal{H}_s}} JK^{\mathfrak{A}_P}_{\tilde{\xi},P}\left(ch^{T\times S}\left(\frac{E\otimes\Lambda_{-1}(\mathfrak{g}/\mathfrak{h})}{\Lambda_{-1}\Omega_V}\right)\right), \end{gathered}$$

where

- *The argument of the JK residue is evaluated at s, so it defines a meromorphic function on $\chi(T)^\vee_{\mathbb{C}}$.*
- *W is the Weyl group of G.*
- *$\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{h}$ is the Lie algebra of T.*
- *$\mathcal{H}_s$ and $\mathfrak{A}_P$ are those of Definition 5.1.3.*

Proof. The Lemma 2.4.2 above shows that

$$\mathrm{ch}^S\left(\chi^S(V/\!/G, r(E))\right) = \int_{V/\!/G} r\left(\mathrm{ch}^H(E)\mathrm{Td}^H(T_V - \mathfrak{g}) \cap [V]_{G\times H}\right).$$

The thesis follows by applying the formula of Theorem 5.1.3 to this latter integral, noticing that

$$\mathrm{Td}^{G\times H}(F) = \frac{e^{G\times H}(F)}{\mathrm{ch}^{G\times H}(\Lambda_{-1}F^\vee)}$$

for every $F \in K_{G\times H}(V)$. □

5.2.1 Computing with other classes.

Exactly as in the cohomological case, we might be interested in computing Euler characteristics of classes E so that there are $A, B \in K_{G\times S}(V)$, with $r(B)$ is invertible over $(V/\!/G)^S$, satisfying

$$E_{|(V/\!/G)^S} = \frac{r(A)_{|(V/\!/G)^S}}{r(B)_{|(V/\!/G)^S}}.$$

If this holds then we can use the same formula of Theorem 5.2.1 (under the same hypotheses) to write

$$\begin{aligned}&\mathrm{ch}^S\left(\chi^S\left(V/\!\!/G, E\right)\right)(s) = \\ &\frac{1}{|W|}\sum_{\substack{P \text{ stable isolated} \\ \text{intersection in } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P}\left(\mathrm{ch}^{T\times S}\left(\frac{A\otimes\Lambda_{-1}(\mathfrak{g}/\mathfrak{h})}{B\otimes\Lambda_{-1}\Omega_V}\right)\right).\end{aligned} \tag{5.11}$$

As in Lemma 5.1.3, we can see that given an equivariant bundle $E \in K_{G\times S}(V)$, $\Lambda_{-1}E$ is invertible on $(V/\!\!/T)^S$ if and only if the characters of the representation E don't vanish on the stable intersections of the hyperplane arrangement $\mathcal{H}$ associated to V.

Chapter 6

Applications.

In this section we will explore various corollaries of the localisation formulae discussed in the previous sections.

Contents of the section.

We will consider the following applications:

- First of all, we will study the case of invariants of critical loci cut inside varieties of the form $V/\!/G$. This virtual invariants will be the integral of 1 over the virtual fundamental class and its K-theoretic and elliptic analogues. We will provide an interpretation for these invariants in simple cases and use the JK formulae of the previous sections to give a general formula in Theorem 6.1.1. The origins of this result lie in the work of the Benini-Hori-Eager-Tachikawa [Ben+15] in theoretical physics.

- We will specialise to the case of critical loci in quiver varieties, obtaining Theorem 6.2.1 computing the invariants above in this context. We will discuss the concrete formula that one finds in the case of the Hilbert scheme of points on $\mathbb{A}^3$.

- We will then focus on $\mathrm{Hilb}^n(\mathbb{A}^4)$, a scheme which is cut inside a variety of the form $V/\!/G$ by a section of a vector bundle, but which is not a critical locus. We will show how a direct application of the JK localisation formula recovers some interesting formulae for its invariants already used in physics.

6.1 Invariants of critical loci.

In this section we are going to compute, via the Szenes-Vergne localisation formula described before, invariants of critical loci in quotients of linear space by actions of reductive connected groups. Many interesting spaces are of this form, for example critical loci in quiver varieties. Classical examples are $\mathrm{Hilb}^n(\mathbb{A}^2)$ and $\mathrm{Hilb}^n(\mathbb{A}^3)$, but also quot schemes [FMR21].

Notation. The *Dedekind eta function* is the formal power series $\eta \in q^{\frac{1}{24}} \cdot \mathbb{Z}[\![q]\!]$ given by

$$\eta(q) = q^{\frac{1}{24}} \prod_{n \geqslant 1} (1 - q^n).$$

The *Jacobi theta function* is the power series $\theta \in q^{\frac{1}{8}} y^{-\frac{1}{2}} \cdot \mathbb{C}[\![y, q]\!]_y$

$$\theta(q; y) := -iq^{\frac{1}{8}} (y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n \geqslant 1} (1 - q^n)(1 - yq^n)(1 - y^{-1} q^n).$$

We also denote with the same Greek letters the functions

$$\eta(\tau) := \eta(e^{2\pi i \tau}), \qquad \theta(\tau | z) := \theta(e^{2\pi i \tau}, e^{2\pi i z}),$$

which enjoy nice modular properties [Mum07].

6.1.1 Which invariants?

A reference for this section is the work [BF97] of Behrend and Fantechi.

Consider a smooth quasiprojective variety $\mathcal{A}$, which we will call *ambient space*. Given a *superpotential*, namely a regular function $\varphi \in H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$, we can consider the corresponding critical locus $X := V(d\varphi) \subseteq \mathcal{A}$. This locus is often singular, its classical invariants are hard to define/compute and they depend on the specific φ.

Instead, we are going to focus on a kind of invariants which are easier to compute and deformation invariant, which in this case means that they do not depend on φ. Of course, they will not be invariants of just the scheme X, but they will also remember that we used a section of $\Omega_{\mathcal{A}}$ to construct it. More precisely, they will be invariants of the scheme X endowed with the *perfect obstruction theory* $\mathbb{E} \in \mathrm{D}^{[-1,0]}_{\mathrm{perf}}(X)$ dual to the *virtual tangent bundle*

$$T_X^{\mathrm{vir}} := \left[(T_{\mathcal{A}})_{|X} \xrightarrow{dd\varphi} (\Omega_{\mathcal{A}})_{|X} \right] \tag{6.1}$$

where $dd\varphi$ is the *vertical derivative* of the differential $d\varphi$. In more explicit terms, the differential of φ can be thought as a morphism of schemes $d\varphi : \mathcal{A} \to \Omega_{\mathcal{A}}$ and its differential defines a morphism of vector bundles on $\mathcal{A}$

$$dd\varphi : T_{\mathcal{A}} \to (d\varphi)^* T_{\Omega_{\mathcal{A}}}.$$

Since $d\varphi^* T_{\Omega_{\mathcal{A}}}$, once it's restricted to $X = V(d\varphi)$, splits as $T_{\mathcal{A}} \oplus \Omega_{\mathcal{A}}$, we can consider the composition of the morphism above with the projection to the second component and call it vertical derivative. Once we have this perfect obstruction theory we can recover the building blocks for the invariants we want to consider. They are the *virtual fundamental class* $[X]^{\mathrm{vir}} \in A_0(X)$ and the virtual structure sheaf $\mathcal{O}_X^{\mathrm{vir}} \in K_0(X)$, which are defined from the p.o.t. via the intrinsic normal cone construction (see [BF97]). From these two classes we can define many interesting invariants via integration.

In this context, the interesting case is where X is not proper (hence $\mathcal{A}$ is only quasiprojective) and we define invariants by localisation with respect to an additional torus action. This situation will be explored in later sections. Here, as a warm-up, we consider the simple case where $\mathcal{A}$ is proper and there is no equivariance involved.

Definition 6.1.1. Assume that X is proper. The *DT invariant* of X is the number

$$\mathrm{DT}(X) := \int_{[X]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The *virtual Hirzebruch genus* of X is

$$\chi(X) := \chi\left(X, \sqrt{K_X^{\mathrm{vir}}} \otimes \mathcal{O}_X^{\mathrm{vir}}\right) \in \mathbb{Z}$$

where $K_X^{\mathrm{vir}} \in K^0(X)$ is the determinant of the virtual tangent bundle (6.1) and its square root is $(K_{\mathcal{A}})_{|X}$ (as can be seen directly from (6.1)).

Another interesting invariant is expressed in terms of the following equivariant K-theory class defined in [FMR21]: for every vector bundle E on X set

$$\mathcal{E}_{1/2}(E) := \bigotimes_{n \geqslant 1} \mathrm{Sym}_{q^n}\left(E \oplus E^\vee\right) \in 1 + qK^0(X)[\![q]\!]$$

having constant term in q equal to 1. Notice that $\mathcal{E}_{1/2}$ defines a group homomorphism between $(K^0(X), +)$ and $(1 + q \cdot K^0(X)[\![q]\!], \otimes)$.

Definition 6.1.2. If X is proper, the *virtual chiral elliptic genus* of X is the Euler characteristic

$$\mathrm{Ell}(X)(q) := \chi\left(X, \mathcal{E}_{1/2}(T_X^{\mathrm{vir}}) \otimes \sqrt{K_X^{\mathrm{vir}}} \otimes \mathcal{O}_X^{\mathrm{vir}}\right)$$

which belongs to $\mathbb{Z}[\![q]\!]$.

The three invariants DT, χ and Ell are all equal in this simple context:

Proposition 6.1.1. *If X is proper, since $[X]^{vir}$ is of dimension zero, then*

$$DT(X) = \chi(X) = Ell(X)(q).$$

Proof. By the virtual Hirzebruch-Riemann-Roch theorem [FG10], for every $E \in K^0(X)$

$$\chi(X, E \otimes \mathcal{O}_X^{\mathrm{vir}}) = \int_{[X]^{\mathrm{vir}}} \mathrm{ch}(E)\mathrm{Td}(T_X^{\mathrm{vir}}).$$

Now notice that $[X]^{\mathrm{vir}}$ is of dimension zero, hence only the component in $A^0(X)$ of the class under the integral sign will contribute to the Euler characteristic. Notice that by definition of Chern character and Todd class this constant part coincides with the rank of E, hence

$$\chi(X, E \otimes \mathcal{O}_X^{\mathrm{vir}}) = \mathrm{rk}(E)\mathrm{DT}(X).$$

The class $\sqrt{K_X^{\mathrm{vir}}}$ must be of rank 1 since squares to a line bundle, so $\chi(X) = \mathrm{DT}(X)$. On the other hand, since $\mathcal{E}_{1/2}(E) = \mathcal{E}_{1/2}(E^\vee)$ and $\mathcal{E}_{1/2}$ is a group homomorphism, we see from (6.1) that $\mathcal{E}_{1/2}(T_X^{\mathrm{vir}}) = 1$. □

Furthermore, if $\mathcal{A}$ is proper too, then φ is constant and the critical locus is $X = \mathcal{A}$. The virtual tangent bundle (6.1) has $dd\varphi = 0$, and the virtual fundamental class is $e(\Omega_{\mathcal{A}}) \cap [\mathcal{A}]$. In particular, we can see that the DT invariant is the topological Euler characteristic up to sign:

$$\mathrm{DT}(\mathcal{A}) = \int_{\mathcal{A}} e(\Omega_{\mathcal{A}}) = (-1)^{\dim(\mathcal{A})}\chi_{\mathcal{A}}. \tag{6.2}$$

Assume that $\mathcal{A} = V/\!/G$ for an action of a reductive connected group G on a linear space V with a linearisation given by a character $\xi \in \chi(G)$ so that

- there is a maximal subtorus $T \subseteq G$ so that the actions of T on $V(T)^{\mathrm{ss}}$ and of G on $V(G)^{\mathrm{ss}}$ are free.
- $V/\!/T$ is proper, hence so is $V/\!/G$.

The DT invariant (6.2) is an integral over $V/\!/G$, so we can compute it by Szenes-Vergne localisation. Consider the weight-space decomposition of V for the action of T

$$V \simeq \bigoplus_{\rho \in \chi(T)} V_\rho$$

and let $\mathfrak{A} \subset \chi(T)$ be the set of characters ρ so that $V_\rho \neq 0$.

Proposition 6.1.2. *Assume that $V/\!/T$ is proper. Then*

$$DT(\mathcal{A}) = JK_\xi^{\mathfrak{A}} \left(\frac{c^T(\Omega_V)}{e^T(T_V)} \frac{e^T(\mathfrak{g}/\mathfrak{h})}{c^T(\mathfrak{g}/\mathfrak{h})} \right).$$

where $c^T(E) := \sum_{k=1}^{rk(E)} c_k^T(E)$ denotes the equivariant total Chern class. In particular, if we denote with $\Phi \subset \chi(T)$ the set of roots of G, we can explicitly write the rational function on $\chi(T)_{\mathbb{C}}^\vee$ that is the argument of the JK residue:

$$DT(\mathcal{A}) = JK_\xi^{\mathfrak{A}} \left(\prod_{\rho \in \mathfrak{A}} \left(\frac{1-\rho}{\rho} \right)^{dim V_\rho} \prod_{\alpha \in \Phi} \frac{\alpha}{1+\alpha} \right).$$

Proof. Notice that the class we want to integrate over $V/\!/G$ is

$$c(\Omega_{V/\!/G}) = d\left(\frac{c^G(\Omega_V)}{c^G(\mathfrak{g}^\vee)} \right)$$

since we have the short exact sequence on $V(G)^{\mathrm{ss}}$

$$0 \to \mathfrak{g} \to T_V \to \pi^* T_{V/\!/G} \to 0.$$

By Theorem 4.6.4 we obtain the expression

$$\mathrm{JK}_\xi^{\mathfrak{A}} \left(\frac{c^T(\Omega_V)}{e^T(T_V)} \frac{e^T(\mathfrak{g}/\mathfrak{h})}{c^T(\mathfrak{g}^\vee)} \right)$$

for this integral. By definition of roots of G, we have the weight-space decomposition for the adjoint action of T

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where each α is 1-dimensional and T acts on it via α. This, together with the weight-space decomposition of V above allows to compute the equivariant classes in the formula. Notice that, since roots of G come in positive-negative pairs, $c^T(\mathfrak{g}) = c^T(\mathfrak{g}^\vee)$. Moreover, $\mathfrak{h}$ doesn't contribute to $c^T(\mathfrak{g})$ since the T-action is trivial on it. □

6.1.2 The equivariant setting.

In the previous section we have seen that the only thing we could compute was the signed Euler characteristic of $\mathcal{A}$, since if $\mathcal{A}$ is proper and X is a critical locus, then $X = \mathcal{A}$ and

$$(-1)^{\dim(X)}\chi_X = DT(X) = \chi(X) = \mathrm{Ell}(X)(q). \tag{6.3}$$

The situation becomes much more interesting once we allow an additional torus S to act on $\mathcal{A}$ and we perform the same argument equivariantly with respect to S. This will allow us to work with nonproper ambient spaces $\mathcal{A}$ (as long as $(\mathcal{A})^S$ is proper) and therefore with nonconstant superpotentials φ, giving nontrivial critical loci X. Somehow surprisingly, even in the case where $\mathcal{A} = X$ is proper, the equivariant version of the invariants above will contain much more information on the variety $\mathcal{A}$: the equality (6.3) will stop being true and $\mathrm{DT}(X), \chi(X)$ and $\mathrm{Ell}(X)$ will compute important classical invariants of $\mathcal{A}$, namely the *Euler characteristic*, the *Hirzebruch genus* and the *Elliptic genus*.

Consider a 1-dimensional representation $\mathfrak{s}$ of the torus S, corresponding to a character $\psi \in \chi(S)$, and a *S-equivariant superpotential*, namely a S-equivariant function

$$\varphi : \mathcal{A} \to \mathfrak{s}.$$

The differential $d\varphi$ defines an invariant section of the S-equivariant bundle $\Omega_\mathcal{A} \otimes \mathfrak{s}$. Let X be the corresponding critical locus $X := V(d\varphi) \subseteq \mathcal{A}$, which is clearly a S-invariant subscheme. This is endowed with the S-equivariant perfect obstruction theory dual to the virtual tangent bundle

$$T_X^{\mathrm{vir}} := \left[(T_\mathcal{A})_{|X} \xrightarrow{dd\varphi} (\Omega_\mathcal{A} \otimes \mathfrak{s})_{|X} \right] \tag{6.4}$$

which again defines a S-equivariant virtual fundamental class and a S-equivariant virtual structure sheaf:

$$[X]^{\mathrm{vir}} \in A_0^S(X) \quad , \quad \mathcal{O}_X^{\mathrm{vir}} \in K^S(X).$$

The definitions of the invariants $\mathrm{DT}(X)$, $\chi(X)$ and $\mathrm{Ell}(X)$ are exactly the same as those given in the previous section. The only difference is that they live in the S-*equivariant* homology and K-theory of the point and, if X is nonproper, they are defined through the virtual localisation formula of [GP97]:

Definition 6.1.3. The S-equivariant *DT-invariant* of X is defined as

$$\mathrm{DT}(X) := \int_{[X]^{\mathrm{vir}}} 1 = \int_{[X^S]^{\mathrm{vir}}} \frac{1}{e^S(\mathcal{N}^{\mathrm{vir}}_{X^S/X})}$$

and it is a rational function of degree zero on $\chi(S)^\vee_{\mathbb{C}}$, meaning that $\mathrm{DT}(X)(\lambda s) = \mathrm{DT}(X)(s)$ for every $\lambda \in \mathbb{C}^*$.

Remark 30. First of all, if X is proper then $\mathrm{DT}(X)$ must belong to $\mathbb{Z}$. If X is not proper, then the DT invariant defined by virtual localisation should in principle belong to $A^S(\mathrm{pt})_f \simeq \mathrm{Sym}(\chi(S))_f$ for some homogeneous polynomial function f on $\chi(S)^\vee_{\mathbb{C}}$. Notice that $R := A^*_S(X^S)_f$ is a graded ring being the localisation of a graded ring by a homogeneous element, and the homogeneous component of degree d is given by

$$R_d \simeq \bigoplus_{\substack{p,q,u \in \mathbb{N} \\ q+u-p\deg(f)=d}} f^{-p} \cdot \mathrm{Sym}^q\left(\chi(S)_{\mathbb{C}}\right) \otimes A^u(X^S)$$

By definition of virtual class of the fixed locus and of virtual normal bundle, the degree of $[X^S]^{\mathrm{vir}}$ and the rank of the virtual normal bundle sum to the virtual dimension of X, namely zero. This means that there is a d so that $[X^S] \in A^S_d(X^S)$ and $e^S(\mathcal{N}^{\mathrm{vir}}) \in R_{-d}$. This shows that $e^S(\mathcal{N}^{\mathrm{vir}})^{-1} \in R_d$ and therefore

$$\frac{1}{e^S(\mathcal{N}^{\mathrm{vir}})} \cap [X^S]^{\mathrm{vir}} \in \bigoplus_{p \in \mathbb{N}} f^{-p} \cdot \mathrm{Sym}^{p \cdot \deg(f)}\left(\chi(S)_{\mathbb{C}}\right) \otimes A_0(X^S),$$

so its degree is a rational function of degree zero on $\chi(S)^\vee_{\mathbb{C}}$. In particular notice that DT is constant if S is of rank 1.

Definition 6.1.4. The S-*equivariant virtual Hirzebruch genus* of X is

$$\chi(X) := \chi^S\left(X, \sqrt{K_X^{\mathrm{vir}}} \otimes \mathcal{O}_X^{\mathrm{vir}}\right) = \chi^S\left(X^S, \frac{\sqrt{K_X^{\mathrm{vir}}} \otimes \mathcal{O}_X^{\mathrm{vir}}}{\Lambda_{-1}(\mathcal{N}^{\mathrm{vir}})^\vee}\right)$$

where the *virtual canonical bundle* $K_X^{\mathrm{vir}} \in K^0(X)$ is the determinant of the equivariant virtual tangent bundle (6.1) and its square root is the one described in Lemma 6.1.1 below. It is an element of $R(S)_f[\sqrt{\mathfrak{s}}]$ for some $f \in R(S)$.

The following simple lemma is the equivariant version of [FMR21, Proposition 3.2]

Lemma 6.1.1. *Up to formally adding to $K_S^0(X)$ the square root of the representation $\mathfrak{s}$, the scheme X possesses a canonical (once we fix a presentation as critical locus) square root of its S-equivariant virtual canonical bundle.*

Proof. By directly computing the determinant of (6.4) we find

$$K_X^{\mathrm{vir}} \simeq K_{V/\!/T} \otimes K_{V/\!/T} \otimes \mathfrak{s}^{\dim(X)}$$

giving as square root $\sqrt{K_X^{\mathrm{vir}}} = K_{V/\!/T} \otimes \mathfrak{s}^{\frac{\dim(X)}{2}}$. □

Remark 31. Notice that if X is proper then $\chi(X)$ belongs to $R(S)[\sqrt{\mathfrak{s}}]$.

Analogously we define the equivariant chiral elliptic genus

Definition 6.1.5. The *virtual chiral elliptic genus* of X is the Euler characteristic

$$\begin{aligned}
\mathrm{Ell}(X)(q) :&= \chi^S\left(X, \mathcal{E}_{1/2}(T_X^{\mathrm{vir}}) \otimes \sqrt{K_X^{\mathrm{vir}}} \otimes \mathcal{O}_X^{\mathrm{vir}}\right) \\
&= \chi^S\left(X^S, \frac{\mathcal{E}_{1/2}(T_X^{\mathrm{vir}}) \otimes \sqrt{K_X^{\mathrm{vir}}} \otimes \mathcal{O}_X^{\mathrm{vir}}}{\Lambda_{-1}(\mathcal{N}^{\mathrm{vir}})^\vee}\right)
\end{aligned}$$

which belongs to $R(S)_f[\sqrt{\mathfrak{s}}][\![q]\!]$ for some $f \in R(S)$.

Remark 32. Again, if X is proper this belongs to $R(S)[\sqrt{\mathfrak{s}}][\![q]\!]$.

Let's specialise to the case of a proper ambient space. This will help to clarify the names some of these invariants.

Proposition 6.1.3. *Assume that $\mathcal{A}$ is proper. The only S-equivariant function is the zero one (or constants if $\mathfrak{s}$ is trivial) and $X = \mathcal{A}$. Then, if we denote with d the dimension of $\mathcal{A}$,*

$$\begin{aligned}
&DT(X) = (-1)^d \chi_{\mathcal{A}} \\
&\chi(X) = (-\sqrt{\mathfrak{s}})^{-d} \chi\left(X, \Lambda_{-\mathfrak{s}} \Omega_X\right) \\
&Ell(X)(q) = (-\sqrt{\mathfrak{s}})^{-d} \chi\left(X, \mathcal{E}_{1/2}((1-\mathfrak{s}^{-1})T_X) \otimes \Lambda_{-\mathfrak{s}} \Omega_X\right).
\end{aligned}$$

Remark 33. Notice that if we pick the trivial representation $\mathfrak{s} = 1$ then we recover the equality (6.3) of the previous section, since $\mathcal{E}_{1/2}(0) = 1$ and $\chi^S(X, \Lambda_{-1}\Omega_X) = \int_X e^S(T_X) = \chi_X$ by Hirzebruch-Riemann-Roch.

Remark 34. The quantities on the right are the evaluations of classical invariants at the representation $\mathfrak{s}$. Indeed, the *Hirzebruch* χ_y *genus* of a smooth projective variety X is a Laurent polynomial in the variable y

$$\chi_y(X) := \chi(X, \Lambda_y \Omega_X) \tag{6.5}$$

while the *elliptic genus* is

$$\mathrm{Ell}_{y,q}(X) := \chi\left(X, \mathcal{E}_{1/2}((1-y^{-1})T_X) \otimes \Lambda_{-y}\Omega_X\right) \tag{6.6}$$

as they are described in [HBJ92, Page 175]. It's a classical fact that these invariants are *rigid*, namely that if they are computed G-equivariantly with respect to the action of a connected group G on X they display no equivariance at all, they don't depend on equivariant parameters. This is easy to see for the Hirzebruch genus. By Hodge theory this G-equivariant genus is the character for the G-action on the cohomology groups $H^{p,0}(X)$, but G acts on X by biholomorphisms homotopic to the identity, so the action on cohomology is trivial. The fact that the G-equivariant elliptic genus of a smooth projective variety coincides with the classical elliptic genus is the rigidity theorem conjectured by Witten [Wit88], proven for spin manifolds by Bott and Taubes [BT89] and in the general case by Hirzebruch (theorem at page 181 of [HBJ92]).

We will need the following preliminary lemma

Lemma 6.1.2. *Let E be an S-equivariant vector bundle on X. Then*

$$\Lambda_{-1}E = -det(E) \otimes \Lambda_{-1}E^\vee.$$

Proof. This is an immediate application of the splitting principle. If $E = L$ is a line bundle then

$$\Lambda_{-1}E = 1 - L = -L \otimes (1 - L^\vee) = -\det(E) \otimes \Lambda_{-1}E^\vee.$$

If E is a direct sum of line bundles the proof is the same, hence it follows for all vector bundle by the splitting principle. □

Proof of Proposition 6.1.3. In this case we have that $[X]^{\mathrm{vir}} = e^S(\Omega_{\mathcal{A}})$, $\mathcal{O}_X^{\mathrm{vir}} = \Lambda_{-1}(T_X \otimes \mathfrak{s}^\vee)$, $T_S^{\mathrm{vir}} = T_X - \Omega_X \otimes \mathfrak{s}$ and $\sqrt{K_X^{\mathrm{vir}}} = K_X \otimes \mathfrak{s}^{d/2}$, so

$$\begin{aligned}
&\mathrm{DT}(X) = \int_X e^S(\Omega_X \otimes \mathfrak{s})\\
&\chi(X) = \chi^S\left(X, K_X \otimes \mathfrak{s}^{d/2} \otimes \Lambda_{-1}(T_X \otimes \mathfrak{s}^\vee)\right)\\
&\mathrm{Ell}(X)(q) = \chi^S\left(X, \mathcal{E}_{1/2}((1-\mathfrak{s}^\vee)T_X) \otimes K_X \otimes \mathfrak{s}^{d/2} \otimes \Lambda_{-1}(T_X \otimes \mathfrak{s}^\vee)\right)
\end{aligned}$$

By using the equality of Lemma 6.1.2 above specialised to $E := \Omega_X \otimes \mathfrak{s}$

$$K_X \otimes \Lambda_{-1}(T_X \otimes \mathfrak{s}^\vee) = (-\mathfrak{s}^\vee)^d \otimes \Lambda_{-1}(\Omega_X \otimes \mathfrak{s}),$$

we can write the two genera as

$$\begin{aligned}
&\chi(X) = (-\sqrt{\mathfrak{s}})^{-d}\chi^S\left(X, \Lambda_{-1}(\Omega_X \otimes \mathfrak{s})\right)\\
&\mathrm{Ell}(X)(q) = (-\sqrt{\mathfrak{s}})^{-d}\chi^S\left(X, \mathcal{E}_{1/2}((1-\mathfrak{s}^\vee)T_X) \otimes \Lambda_{-1}(\Omega_X \otimes \mathfrak{s})\right)
\end{aligned}$$

where we have used that $\chi^S(X, E \otimes \mathfrak{r}) = \mathfrak{r} \otimes \chi^S(X, E)$ for every S-representation $\mathfrak{r} \in R(S)$, which is just the projection formula in equivariant K-theory. Finally notice that $\Lambda_{-1}(\Omega_X \otimes \mathfrak{s}) = \Lambda_{-\mathfrak{s}}\Omega_X$ and hence we can write our invariants as

$$\begin{aligned}
&\mathrm{DT}(X) = \int_X e^S(\Omega_X \otimes \mathfrak{s})\\
&\chi(X) = (-\sqrt{\mathfrak{s}})^{-d}\chi^S\left(X, \Lambda_{-\mathfrak{s}}\Omega_X\right)\\
&\mathrm{Ell}(X)(q) = (-\sqrt{\mathfrak{s}})^{-d}\chi^S\left(X, \mathcal{E}_{1/2}((1-\mathfrak{s}^\vee)T_X) \otimes \Lambda_{-\mathfrak{s}}\Omega_X\right)
\end{aligned}$$

Notice that the DT invariant is necessarily the nonequivariant integral of $e(\Omega_X)$ by dimensional reasons, so it coincides with the signed Euler characteristic. Since χ^S commutes with taking products with $\mathfrak{s}$ we can consider $\mathfrak{s}$ as a formal variable, which we can rename as y. Then our expressions for $\chi(X)$ and $\mathrm{Ell}(X)$ coincide with the classical ones of Remark 34, apart from the fact that here we are taking equivariant Euler characteristics instead of nonequivariant ones, which doesn't affect the computation by the rigidity of these invariants as described in the same remark. □

At some point we will need the following variant of the morphism $\mathcal{E}_{1/2}$

Definition 6.1.6. Given an equivariant K-theory class E so that $\Lambda_{-1}E^\vee$ is invertible, consider the class

$$\hat{\mathcal{E}}_{1/2}(E) := \frac{\mathcal{E}_{1/2}(E) \otimes \sqrt{\det(E^\vee)}}{\Lambda_{-1}E^\vee}. \tag{6.7}$$

Notice that since all the classes in its definition are multiplicative, $\hat{\mathcal{E}}_{1/2}$ sends sums into products:

$$\hat{\mathcal{E}}_{1/2}(E+F) = \hat{\mathcal{E}}_{1/2}(E) \otimes \hat{\mathcal{E}}_{1/2}(F).$$

6.1.3 Pushing forward.

Let's go back to the general case where $\mathcal{A}$ can be nonproper as long as $\mathcal{A}^S$ is projective and $\varphi : \mathcal{A} \to \mathfrak{s}$ is an equivariant function whose critical locus is X.

In this simple case, since X is globally the zero locus of a section of a vector bundle on a smooth ambient space, we can easily compute the pushforwards of the relevant classes of X to the smooth ambient space $\mathcal{A}$ so that computations are easier to perform. Since X is globally cut by a section of $\Omega_{\mathcal{A}} \otimes \mathfrak{s}$ we have

$$i_*[X]^{\mathrm{vir}} = e^S(\Omega_{\mathcal{A}} \otimes \mathfrak{s}) \qquad \text{and} \qquad i_*\mathcal{O}_X^{\mathrm{vir}} = \Lambda_{-1}(T_{\mathcal{A}} \otimes \mathfrak{s}^\vee).$$

so that, by projection formula, the invariants we want to compute are

$$\begin{aligned}
&\mathrm{DT}(X) = \int_{\mathcal{A}} e^S(\Omega_{\mathcal{A}} \otimes \mathfrak{s}) \\
&\chi(X) = (-\sqrt{\mathfrak{s}})^{-d}\chi^S\left(\mathcal{A}, \Lambda_{-\mathfrak{s}}\Omega_{\mathcal{A}}\right), \\
&\mathrm{Ell}(X)(q) = \chi^S\left(\mathcal{A}, \mathcal{E}_{1/2}(T_{\mathcal{A}}^{\mathrm{vir}}) \otimes \sqrt{\det(T_{\mathcal{A}}^{\mathrm{vir}})^\vee} \otimes \Lambda_{-\mathfrak{s}^\vee} T_{\mathcal{A}}\right).
\end{aligned}$$

where we have set $T_{\mathcal{A}}^{\mathrm{vir}} := T_{\mathcal{A}} - \Omega_{\mathcal{A}} \otimes \mathfrak{s} \in K_S^0(\mathcal{A})$, which satisfies $T_X^{\mathrm{vir}} = i^* T_{\mathcal{A}}^{\mathrm{vir}}$. W have also used Lemma 6.1.2 to simplify the formula for the Hirzebruch genus.

Remark 35. Notice that the reason behind us being able to push the computation to $\mathcal{A}$ is that we know how to push forward $[X]^{\mathrm{vir}}$ and $\mathcal{O}_X^{\mathrm{vir}}$, and all the other classes that appear are given in terms of T_X^{vir}, which is a pullback from $\mathcal{A}$ by definition (6.4). Thus we can use the projection formula to express the invariants of X in terms of computations on $\mathcal{A}$.

Remark 36. In case $\mathcal{A}$ is nonproper, a little care is required for checking what we are doing is compatible with the definition of the integral by localisation to the proper fixed locus. In this case we are first localising to X^S by means of the virtual localisation formula, then thinking of X^S as cut in $\mathcal{A}^S$ by the invariant part of $d\varphi$ and pushing forward the computation on $\mathcal{A}^S$.

6.1.4 The result.

Consider a reductive connected group acting on a linear space V together with a linearisation $\mathcal{L}$ given by a character $\xi \in \chi(G)$. Assume that

1. there is a maximal subtorus $T \subset G$ so that the actions of T on $V(T)^{\mathrm{ss}}$ and of G on $V(G)^{\mathrm{ss}}$ are free.

2. there is an additional torus S acting on V so that the action commutes with the one of G. Assume that the fixed loci $(V/\!/T)^S$ and $(V/\!/G)^S$ are proper.

3. we have fixed a 1-dimensional representation $\mathfrak{s}$ of S, corresponding to a character $\psi \in \chi(S)$. Let $\varphi : V/\!/G \to \mathfrak{s}$ be an S-equivariant function and let X be its critical locus, endowed with the corresponding S-equivariant perfect obstruction theory.

Then we can consider the weight-space decomposition of V

$$V \simeq \bigoplus_{\substack{\rho \in \chi(T) \\ \nu \in \chi(S)}} V_{\rho,\nu}$$

where $V_{\rho,\nu}$ is the subspace over which $T \times S$ acts by $(t,s) \cdot v = \rho(t)\nu(s)v$. For every $s \in \chi(S)^\vee_{\mathbb{C}}$, this defines a hyperplane arrangement $\mathcal{H}_s$ in $\chi(T)^\vee_{\mathbb{C}}$ given by

$$\{\rho + \nu(s) = 0\} \qquad : \qquad V_{\rho,\nu} \neq 0.$$

Let $\Phi \subset \chi(T)$ be the set of roots of G, namely the weights α of the adjoint representation of G

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Lie algebra of T. We will need the following additional condition

4. No stable isolated intersection P of the hyperplane arrangement $\mathcal{H}_s$ (in the sense of Definition 5.1.3) is contained in a hyperplane of the form $\alpha + \psi(s) = 0$, where $\alpha \in \Phi$.

The main result of this section is expressed in terms of the following meromorphic functions on $\chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}}$ having poles on hyperplanes:

$$\begin{aligned}
Z_{\mathrm{DT}} &= \psi^{-\dim T} \prod_{\rho,\nu} \left(\frac{\psi - \nu - \rho}{\rho + \nu} \right)^{\dim V_{\rho,\nu}} \prod_{\alpha \in \Phi} \frac{\alpha}{\psi + \alpha}, \\
Z_\chi &= \left(\frac{\pi}{\sin(\pi\psi)} \right)^{\dim T} \prod_{\rho,\nu} \left(\frac{\sin\left(\pi(\psi - \rho - \nu)\right)}{\sin\left(\pi(\rho + \nu)\right)} \right)^{\dim V_{\rho,\nu}} \prod_{\alpha \in \Phi} \frac{\sin(\pi\alpha)}{\sin\left(\pi(\alpha + \psi)\right)}. \\
Z_{Ell} &= \left(\frac{2\pi\eta(\tau)^3}{\theta(\tau|\psi)} \right)^{\dim T} \prod_{\rho,\nu} \left(\frac{\theta\left(\tau|\psi - \rho - \nu\right)}{\theta\left(\tau|\rho + \nu\right)} \right)^{\dim V_{\rho,\nu}} \prod_{\alpha \in \Phi} \frac{\theta(\tau|\alpha)}{\theta\left(\tau|\alpha + \psi\right)}.
\end{aligned} \tag{6.8}$$

Given $s \in \chi(S)^\vee_{\mathbb{C}}$ and $\tau \in \mathbb{C}$ not belonging to the poles of these functions, we will denote with $Z_{\mathrm{DT}}(-, s), Z_\chi(-, s)$ and $Z_{\mathrm{Ell}}(-, s, \tau)$ the meromorphic functions $\chi(T)^\vee_{\mathbb{C}} \dashrightarrow \mathbb{C}$ obtained by restricting the functions we just defined.

Theorem 6.1.1. *For a generic $s \in \chi(S)^\vee_{\mathbb{C}}$, the invariants of X can be computed as sums of residues at the stable intersections of the hyperplane arrangement $\mathcal{H}_s$:*

$$\begin{aligned}
DT(X)(s) &= \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} JK^{\mathfrak{A}_P}_{\xi,P}\left(Z_{DT}(-, s)\right), \\
ch^S\chi(X)(2\pi i s) &= \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} JK^{\mathfrak{A}_P}_{\xi,P}\left(Z_\chi(-, s)\right), \\
ch^S Ell(X)(e^{2\pi i\tau})(2\pi i s) &= \frac{1}{|W|} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} JK^{\mathfrak{A}_P}_{\xi,P}\left(Z_{Ell}(-, s, \tau)\right).
\end{aligned}$$

where W is the Weyl group of G and

$$\mathfrak{A}_P := \left\{\rho \in \chi(T) \quad | \quad \exists \nu \text{ s.t. } P \subset \{\rho + \nu(s) = 0\} \text{ and } V_{\rho,\nu} \neq 0\right\}.$$

for every stable isolated intersection P.

6.1.5 Szenes-Vergne localisation.

Notice that in $K^0_S(V/\!/G)$ we have the equalities

$$\begin{aligned}
\Omega_{V/\!/G} \otimes \mathfrak{s} &= r\left(\Omega_V \otimes \mathfrak{s} - \mathfrak{g} \otimes \mathfrak{s}\right), \\
T^{\mathrm{vir}}_{V/\!/G} &= r\left(T_V - \Omega_V \otimes \mathfrak{s} - \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{s}\right)
\end{aligned}$$

where $r = r_{G\times S,S}$ is the S-equivariant K-theoretic Kirwan map for the G-action on V. Notice that we can't directly use this to give a description of the classes we want to compute in terms of the Kirwan map. For example, we can't write

$$e^S(\Omega_{V/\!/G} \otimes \mathfrak{s}) = r\left(\frac{e^{T\times S}(\Omega_V \otimes \mathfrak{s})}{e^{T\times S}(\mathfrak{g} \otimes \mathfrak{s})}\right)$$

since the quantity inside the bracket doesn't make sense in $A^{T\times S}(V)$. On the other hand, since the hypothesis 4 holds true, we can use Lemma 5.1.3 to show that $r(e^{G\times S}(\mathfrak{g}/\mathfrak{s}))$ is invertible once restricted to the fixed locus $(V/\!/G)^S$, and hence the variation (5.10) of Theorem 5.1.3 ensures that for a generic cocharacter $s \in \chi(S)^\vee_{\mathbb{C}}$ the invariant $\mathrm{DT}(X)(s)$ is

$$\frac{1}{|W|} \sum_{\substack{U \text{ stable} \\ \text{intersection of } \mathcal{H}}} \mathrm{JK}^{\mathfrak{A}_U}_{\xi,\zeta_U(s)}\left(\frac{e^{T\times S}(\Omega_V \otimes \mathfrak{s})}{e^{T\times S}(T_V)} \frac{e^{T\times S}(\mathfrak{g}/\mathfrak{h})}{e^{T\times S}(\mathfrak{g} \otimes \mathfrak{s})}\right)$$

where $\mathcal{H}$ is the hyperplane arrangement in $\chi(S\times T)^\vee_{\mathbb{C}}$ of Definition 5.1.1. Analogously, the version (5.11) of the K-theoretic localisation formula of Theorem 5.2.1, gives the following expressions, where $\mathcal{H}$ is the same hyperplane arrangement: $\mathrm{ch}^S\chi(X)(s)$ is equal to

$$\frac{(-e^{\psi/2})^{-d}}{|W|} \sum_{\substack{U \text{ stable} \\ \text{intersection in } \mathcal{H}}} \mathrm{JK}^{\mathfrak{A}_U}_{\tilde{\xi},\zeta_U(s)}\left(\mathrm{ch}^{T\times S}\left(\frac{\Lambda_{-\mathfrak{s}}\Omega_V}{\Lambda_{-1}\Omega_V} \otimes \frac{\Lambda_{-1}\mathfrak{g}/\mathfrak{h}}{\Lambda_{-\mathfrak{s}}\mathfrak{g}}\right)\right)$$

and $\mathrm{ch}^S\mathrm{Ell}(X)(q)(s)$ is

$$\frac{1}{|W|} \sum_{\substack{U \text{ stable} \\ \text{intersection in } \mathcal{H}}} \mathrm{JK}^{\mathfrak{A}_U}_{\tilde{\xi},\zeta_U(s)}\left(\mathrm{ch}^{T\times S}\left(\frac{\hat{\mathcal{E}}_{1/2}(T_V)}{\hat{\mathcal{E}}_{1/2}(\Omega_V \otimes \mathfrak{s})} \otimes \frac{\hat{\mathcal{E}}_{1/2}(\mathfrak{g} \otimes \mathfrak{s})}{\hat{\mathcal{E}}_{1/2}(\mathfrak{g}/\mathfrak{h})} \otimes \frac{1}{\mathcal{E}_{1/2}(\mathfrak{h})}\right)\right)$$

where $\hat{\mathcal{E}}_{1/2}$ is the class of Definition 6.1.6.

6.1.6 Setting up the computation.

In order to make computations a bit simpler to follow, in this section we will adopt the following notation: given a character $\rho \in \chi(T)$ we will denote with $t^\rho \in R(T)$ the corresponding 1-dimensional representation. We will do the same with S, so for example $\mathfrak{s} = s^\psi$, where ψ was the character with which S acts on $\mathfrak{s}$. Notice that this defines an isomorphism of $K_{T\times S}(\mathrm{pt})$ with the group algebra over the character lattice $\chi(T\times S)$.

The first step is to consider the weight-space decomposition of V:

$$V \simeq \bigoplus_{\substack{\rho\in\chi(T)\\ \nu\in\chi(S)}} V_{\rho,\nu}$$

where the action of $T\times S$ on $V_{\rho,\nu}$ is given by $(t,s)\cdot v = \rho(t)\nu(s)v$. The hyperplane arrangement $\mathcal{H}$ appearing in the previous section is the arrangement in $\chi(T\times S)^\vee_{\mathbb{C}}$ of hyperplanes of the form

$$\{\rho+\nu=0\} \qquad : \quad V_{\rho,\nu}\neq 0.$$

Then we can describe the representations involved in our previous formulae:

$$T_V = \sum_{\rho,\nu} \dim(V_{\rho,\nu}) t^\rho s^\nu \qquad \Omega_V = \sum_{\rho,\nu} \dim(V_{\rho,\nu}) t^{-\rho} s^{-\nu}.$$

Notice that, if $\Phi\subset\chi(T)$ is the set of roots of G,

$$\mathfrak{g} = \dim(\mathfrak{h}) + \sum_{\alpha\in\Phi} t^\alpha.$$

We have the following two lemmas, useful to compute the various classes in the previous formulae:

Lemma 6.1.3. *Let E be a representation of $T\times S$:*

$$E = \sum_{j=1}^m t^{w_j} s^{z_j}.$$

with trivial fixed part. Then

$$ch^{T\times S}\left(\frac{\Lambda_{-1}(E\otimes\mathfrak{s})}{\Lambda_{-1}E}\right) = (e^{\frac{\psi}{2}})^{rkE} \prod_{j=1}^m \frac{\sinh(\frac{w_j+z_j+\psi}{2})}{\sinh(\frac{w_j+z_j}{2})}.$$

If W is a trivial, 1-dimensional representation, then

$$\Lambda_{-1}(W\otimes\mathfrak{s}) = -2e^{\psi/2}\sinh(\psi/2).$$

Proof. Notice that the Chern character is a ring homomorphism and

$$\Lambda_{-1}E = \prod_{j=1}^{m}(1 - t^{w_j}s^{z_j}),$$

hence the Chern character we want to compute is the product over j of

$$\frac{1 - e^{w_j+z_j+\psi}}{1 - e^{w_j+z_j}} = e^{\psi/2}\frac{e^{-\psi/2} - e^{w_j+z_j+\psi/2}}{1 - e^{w_j+z_j}} = e^{\psi/2}\frac{e^{-\frac{w_j+z_j+\psi}{2}} - e^{\frac{w_j+z_j+\psi}{2}}}{e^{-\frac{w_j+z_j}{2}} - e^{\frac{w_j+z_j}{2}}}$$

which is precisely the expression in the hyperbolic sine function that we wanted to find. The second statement follows in the same way. □

Lemma 6.1.4. *Given a torus T, consider an element of $K_T(pt)$ of the form*

$$E = \sum_{j=0}^{m} t^{w_j} - \sum_{k=0}^{n} t^{z_k}.$$

with trivial fixed part. Then

$$\hat{\mathcal{E}}_{1/2}(E) = \left(-iq^{\frac{1}{12}}\eta(q)\right)^{rkE} \frac{\prod_{k=0}^{n}\theta(q;t^{z_k})}{\prod_{j=0}^{m}\theta(q;t^{w_j})}.$$

If W is a trivial, 1-dimensional representation of T:

$$\mathcal{E}_{1/2}(W) = \frac{q^{\frac{1}{12}}}{\eta(q)^2}$$

Proof. This is an immediate consequence of $\mathrm{Sym}_q(t^w) = (1 - qt^w)^{-1}$ and of the definition of the eta and theta functions. □

6.1.7 The proof for DT.

We just have to explicitly write the rational function $\chi(T \times S)^{\vee}_{\mathbb{C}} \dashrightarrow \mathbb{C}$

$$\frac{e^{T\times S}(\Omega_V \otimes \mathfrak{s})}{e^{T\times S}(T_V)} \frac{e^{T\times S}(\mathfrak{g}/\mathfrak{h})}{e^{T\times S}(\mathfrak{g}\otimes\mathfrak{s})}$$

in terms of the weights ρ, ν, of the roots α and of the character ψ. Notice that we can split $\mathfrak{g}\otimes\mathfrak{s}$ as the sum of $(\mathfrak{g}/\mathfrak{h})\otimes\mathfrak{s}$ and $\mathfrak{h}\otimes\mathfrak{s}$, obtaining

$$\psi^{-\dim T}\prod_{\rho,\nu}\left(\frac{\psi-\rho-\nu}{\rho+\nu}\right)^{\dim V_{\rho,\nu}}\prod_{\alpha\in\Phi}\frac{\alpha}{\alpha+\psi}.$$

6.1.8 The proof for the Hirzebruch genus.

Again, we have to rewrite the following meromorphic function on $\chi(T\times S)^\vee_{\mathbb{C}}$

$$(-e^{\psi/2})^{-d}\mathrm{ch}^{T\times S}\left(\frac{\Lambda_{-\mathfrak{s}}\Omega_V}{\Lambda_{-1}\Omega_V}\otimes\frac{\Lambda_{-1}\mathfrak{g}/\mathfrak{h}}{\Lambda_{-\mathfrak{s}}\mathfrak{g}}\right)$$

Splitting $\Lambda_{-\mathfrak{s}}\mathfrak{g}$ as the product of $\Lambda_{-\mathfrak{s}}\mathfrak{g}/\mathfrak{h}$ and $\Lambda_{-\mathfrak{s}}\mathfrak{h}$ we find, thanks to Lemma 6.1.3, the following function

$$(-1)^{d+\mathrm{dim}T}\left(\frac{1}{2\sinh(\psi/2)}\right)^{\mathrm{dim}T}\prod_{\rho,\nu}\left(\frac{\sinh\left(\frac{\psi-\rho-\nu}{2}\right)}{\sinh\left(\frac{-\rho-\nu}{2}\right)}\right)^{\mathrm{dim}V_{\rho,\nu}}\prod_{\alpha\in\Phi}\frac{\sinh(\frac{\alpha}{2})}{\sinh\left(\frac{\alpha+\psi}{2}\right)}$$

By our localisation theorem, for a generic $s\in\chi(S)^\vee_{\mathbb{C}}$ this computes $\mathrm{ch}^S\chi(X)(s)$. The expression becomes slightly simpler if we consider $\mathrm{ch}^S\chi(X)(2\pi is)$ instead. In this case we can also use the fact that scaling the coordinates that we use to take the residue simply changes the residue by multiplication by the inverse scaling factor (Lemma 4.1.6) to rescale the coordinates on $\chi(T)^\vee_{\mathbb{C}}$ by $2\pi i$, obtaining the following expression for $\mathrm{ch}^S\chi(X)(2\pi is)$:

$$\left(\frac{\pi}{\sin(\pi\psi)}\right)^{\mathrm{dim}T}\prod_{\rho,\nu}\left(\frac{\sin\left(\pi(\psi-\rho-\nu)\right)}{\sin\left(\pi(\rho+\nu)\right)}\right)^{\mathrm{dim}V_{\rho,\nu}}\prod_{\alpha\in\Phi}\frac{\sin(\pi\alpha)}{\sin\left(\pi(\alpha+\psi)\right)}.$$

Notice that the minus sign in front disappeared as we now discuss. We changed the sign in the $-\rho-\nu$ denominator, so that the global sign before the product would be $(-1)^{d+\mathrm{dim}T+\mathrm{dim}V}$, but notice that the exponent is congruent to $\mathrm{dim}G-\mathrm{dim}T$ modulo 2 , which is always even by representation theory.

6.1.9 The proof for the elliptic genus.

This proof is completely analogous to the one given for the Hirzebruch genus. we have to rewrite the following meromorphic function on $\chi(T\times S)^\vee_{\mathbb{C}}$

$$\mathrm{ch}^{T\times S}\left(\frac{\hat{\mathcal{E}}_{1/2}(T_V)}{\hat{\mathcal{E}}_{1/2}(\Omega_V\otimes\mathfrak{s})}\otimes\frac{\hat{\mathcal{E}}_{1/2}(\mathfrak{g}\otimes\mathfrak{s})}{\hat{\mathcal{E}}_{1/2}(\mathfrak{g}/\mathfrak{h})}\otimes\frac{1}{\mathcal{E}_{1/2}(\mathfrak{h})}\right)$$

Splitting $\hat{\mathcal{E}}_{1/2}(\mathfrak{g}\otimes\mathfrak{s})$ as the product of $\hat{\mathcal{E}}_{1/2}(\mathfrak{g}/\mathfrak{h}\otimes\mathfrak{s})$ and $\hat{\mathcal{E}}_{1/2}(\mathfrak{h}\otimes\mathfrak{s})$ we find, thanks to Lemma 6.1.4, the following function

$$\left(-\frac{i\eta(q)^3}{\theta(q;e^{\psi})}\right)^{\mathrm{dim}T}\prod_{\rho,\nu}\left(\frac{\theta\left(q;e^{\psi-\rho-\nu}\right)}{\theta\left(q;e^{\rho+\nu}\right)}\right)^{\mathrm{dim}V_{\rho,\nu}}\prod_{\alpha\in\Phi}\frac{\theta(q;e^{\alpha})}{\theta\left(q;e^{\alpha+\psi}\right)}$$

By our localisation theorem, for a generic $s \in \chi(S)^\vee_{\mathbb{C}}$ this computes $\mathrm{ch}^S\mathrm{Ell}(X)(q)(s)$. The expression becomes slightly simpler if we consider $\mathrm{ch}^S\mathrm{Ell}(X)(q)(2\pi i s)$ instead. As for the Hirzebruch genus, we rescale the coordinates on $\chi(T)^\vee_{\mathbb{C}}$ by $2\pi i$. Moreover we formally set $q := e^{2\pi i \tau}$ obtaining the following expression for $\mathrm{ch}^S\mathrm{Ell}(X)(e^{2\pi i\tau})(2\pi i s)$:

$$\left(\frac{2\pi\eta(\tau)^3}{\theta(\tau|\psi)}\right)^{\mathrm{dim}T}\prod_{\rho,\nu}\left(\frac{\theta\left(\tau|\psi-\rho-\nu\right)}{\theta\left(\tau|\rho+\nu\right)}\right)^{\mathrm{dim}V_{\rho,\nu}}\prod_{\alpha\in\Phi}\frac{\theta(\tau|\alpha)}{\theta\left(\tau|\alpha+\psi\right)}.$$

6.1.10 Complete intersections in GIT quotients.

As a first application we show how to use Theorem 6.1.1 to compute classical invariants of complete intersections in GIT quotients of linear spaces. First of all, we show how to recover these from virtual invariants of critical loci.

Consider a smooth projective variety Y together with a locally free sheaf $\mathcal{E}$ admitting a transversal section. Let $E = \mathrm{Spec}(\mathrm{Sym}\mathcal{E}^\vee)$ be the vector bundle built as total space of $\mathcal{E}^\vee$ and consider the $\mathbb{C}^*$-action on E that scales the fibres. If we consider the $\mathbb{C}^*$-representation $\mathfrak{s}$ of weight one, the virtual invariants of the corresponding critical locus compute the classical invariants of the zero locus of a transversal section of $\mathcal{E}$.

Proposition 6.1.4. *Let $s \in H^0(Y,\mathcal{E})$ be a transversal section and let $Z := V(s)$ be its zero locus, whose dimension we denote with d. Let X be a critical locus of a S-equivariant function $\varphi : E \to \mathfrak{s}$. Then*

$$\begin{aligned} DT(X) &= (-1)^d\chi_Z,\\ \chi(X) &= (-\sqrt{\mathfrak{s}})^{-d}\chi_{-\mathfrak{s}}(Z),\\ Ell(X)(q) &= (-\sqrt{\mathfrak{s}})^{-d}Ell_{\mathfrak{s},q}(Z)\end{aligned}$$

where the invariants on the right-hand side are the Euler number, the Hirzebruch χ_y genus (6.5) and the elliptic genus (6.6) of the smooth projective variety Z.

Proof. The key is to use the short exact sequence

$$0 \to \mathcal{E}\otimes\mathfrak{s} \to (\Omega_Y\otimes\mathfrak{s})_{|Z} \to \Omega_X\otimes\mathfrak{s} \to 0$$

together with Poincaré duality, namely the fact that

$$\int_Z \alpha_{|Z} = \int_Y \alpha\cdot e(\Omega_E) \quad \text{and} \quad \chi(Z, A_{|Z}) = \chi(Y, A\otimes\Lambda_{-1}\mathcal{E})$$

for all $\alpha \in A^*_S(Y)$ and $A \in K_S(Y)$. □

Now assume that Y is isomorphic to the GIT quotient of a linear space V_1 by the action of a reductive connected algebraic group G with respect to a linearisation given by a character ξ so that the action of G and of a maximal subtorus T are free on the semistable loci. Suppose moreover that there is some other G-representation V_2 so that, pulling back the linearisation ξ to $V := V_1 \oplus V_2$, the quotient $V/\!/G$ is the vector bundle E. Finally assume that, if we let $S := \mathbb{C}^*$ act trivially on V_1 and by scalar multiplication on V_2, the induced action on E is the one scaling the fibres.

Remark 37. Notice that in thise context the hypotheses of Theorem 6.1.1 are satisfied. The S-fixed locus coincides with Y, which is proper. The condition (4) is always satisfied, since there is only one stable intersection of the hyperplane arrangement $\mathcal{H}$ (since there is only one connected component of the fixed locus, namely Y) and it coincides with the subspace $\{0\} \times \chi(S)^\vee_{\mathbb{C}}$, which clearly satisfies (4).

Then the formulae of Theorem 6.1.1 compute the classical invariants Z by virtue of Proposition 6.1.4 above.

Example 6.1.1. Consider the embedding of $G(2,4)$ via the Plücker embedding, so that $\mathcal{O}(1) \simeq \det(S)^\vee$, where S is the tautological subbundle. By adjunction, a generic section of $\mathcal{O}(4)$ cuts a smooth Calabi-Yau threefold Z. We wish to compute its classical Hirzebruch genus by means of Theorem 6.1.1. The total space of the bundle $\mathcal{O}(-4)$ can be built as the quotient of

$$V := \mathrm{Mat}_{2\times 4}(\mathbb{C}) \times \mathbb{C}$$

by the action of $G := GL_2(\mathbb{C})$ given by

$$g \cdot (M, z) := (gM, \det(g)^{-4} z)$$

with respect to the linearisation given by the character $\xi = \det$. The easiest way to see this is probably to notice that the tautological subbundle can be, as in the case of the projective space, built as the quotient of $\mathrm{Mat}_{2\times 4}(\mathbb{C}) \oplus \mathrm{Mat}_{2\times 2}(\mathbb{C})$ with action given by $g \cdot (M, N) := (gM, Ng^{-1})$. This quotient has a map $[M, N] \mapsto ([M], [NM])$, defining a closed embedding into $\mathrm{Gr}(2,4) \times \mathrm{Mat}_{2\times 4}(\mathbb{C})$, which allows to directly check that it coincides with the tautological subbundle S. Moreover, the $\mathbb{C}^*$-action scaling the fibres of this bundle is induced by the action on V which is trivial on the first summand and by scalar multiplication on the second summand $\mathbb{C}$.

Let's set up the computation of the formula for the Hirzebruch genus in Theorem 6.1.1. The maximal subtorus is $G \supset T \simeq (\mathbb{C}^*)^2$ has cocharacter space $\chi(T)^\vee_{\mathbb{C}} \simeq \mathbb{C}^2$ and we denote its coordinates with $u_1, u_2 \in \chi(T)$. The Weyl group is $W \simeq \mathfrak{S}_2$ and acts by exchanging the coordinates. In this notation the linearisation is given by the Weyl-invariant character $\xi := u_1 + u_2$. The set $\mathfrak{A}$ of weights for the G-action on V is

$$\{u_1, u_2, -4u_1 - 4u_2\}$$

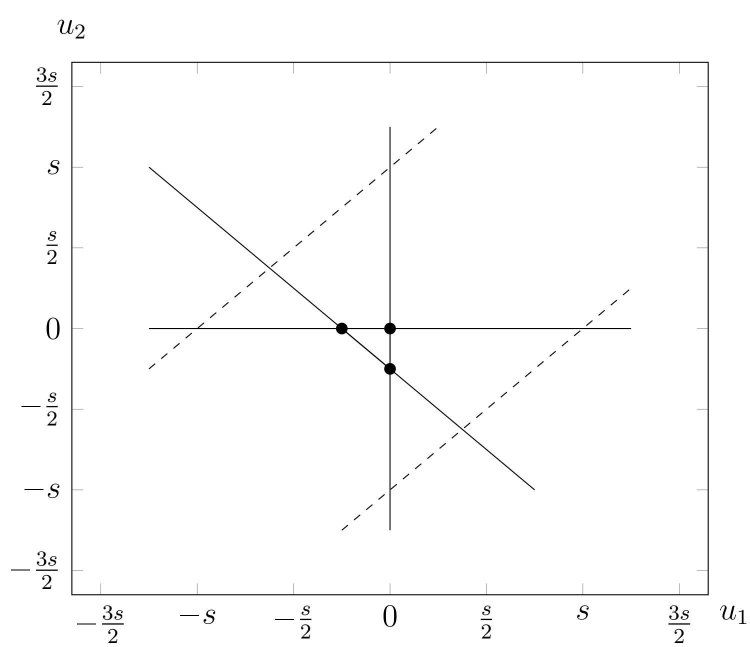
and the corresponding weight-space decomposition is

$$V \simeq \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}.$$

The roots of G are the functionals $\pm(u_1 - u_2)$. Moreover the cocharacter space of $S = \mathbb{C}^*$ is isomorphic to $\mathbb{C}$ and we consider its coordinate function $s \in \chi(S)$. Once we fix the representation $\mathfrak{s}$ over which $\mathbb{C}^*$ acts by scalar multiplication (hence $\psi = s$ in the notation of Theorem 6.1.1) we can explicitly write the function Z_χ as

$$Z_\chi(u_1, u_2, s) = \left(\frac{\pi}{\sin(\pi s)}\right)^2 \left(\frac{\sin(\pi(s - u_1))}{\sin(\pi u_1)}\right)^4 \left(\frac{\sin(\pi(s - u_2))}{\sin(\pi u_2)}\right)^4 \\ \times \frac{\sin(\pi(4u_1 + 4u_2))}{\sin(\pi(s - 4u_1 - 4u_2))} \frac{\sin(\pi(u_2 - u_1))}{\sin(\pi(s + u_2 - u_1))} \frac{\sin(\pi(u_1 - u_2))}{\sin(\pi(s + u_1 - u_2))}$$

Fixed a generic $s \in \mathbb{C}$, the hyperplane arrangement $\mathcal{H}_s$ is drawn in the following picture: the full lines correspond to the poles coming from the weights while the dashed ones come from the roots, for which we have to check condition 4:



As it's clear from the picture there are 3 isolated intersections of the hyperplane arrangement defined by the weights:

$$\{\text{isolated intersections of } \mathcal{H}_s\} = \left\{(0,0)\,,\left(\frac{s}{4},0\right),\left(0,\frac{s}{4}\right)\right\}.$$

Notice that, as shown in the picture, no hyperplane defined by the roots passes from these points, hence condition (4) is satisfied. Let's check which one of these intersections is stable.

- The weights corresponding to the hyperplanes vanishing at the origin $(0,0)$ are u_1 and u_2. Since $\xi = u_1 + u_2$ is in the positive span of these vectors, the origin is stable.

- The weights corresponding to the hyperplanes vanishing at the point $(\frac{s}{4},0)$ are $-4u_1 - 4u_2$ and u_2. Since

$$\xi = u_1 + u_2 = -\frac{1}{4}(-4u_1 - 4u_2),$$

  this point is not stable.

- The weights corresponding to the hyperplanes vanishing at the point $(0,\frac{s}{4})$ are $-4u_1 - 4u_2$ and u_1. Since

$$\xi = u_1 + u_2 = -\frac{1}{4}(-4u_1 - 4u_2),$$

  again this point is not stable.

We have finally shown that, as expected, there is only one stable isolated intersection in the origin. Hence, for a critical locus X of an equivariant function $V/\!/G \to \mathfrak{s}$ we find

$$\mathrm{ch}^S\chi(X)(2\pi i s) = \frac{1}{2}\mathrm{JK}^{\{u_1,u_2\}}(Z_\chi(-,s),\tilde{\xi})$$

by Theorem 6.1.1. This JK residue was computed in Example 4.1.5 earlier, and by using the result we obtained we see that

$$\mathrm{ch}^S\chi(X)(2\pi i s) = 176\sin(\pi s)\left(\cos^2(\pi s)\cot(\pi s) + \sin(\pi s)\cos(\pi s)\right). \tag{6.9}$$

Now $\chi(X) \in R(\mathbb{C}^*) \simeq \mathbb{C}[\mathfrak{s}]_{\mathfrak{s}}$ is a virtual representation of $\mathbb{C}^*$ and can be written as $\chi(X) = \sum_{k\in\mathbb{Z}} a_k \mathfrak{s}^k$ for some $a_k \in \mathbb{C}$. By definition of Chern character we find that

$$\mathrm{ch}^S\chi(X)(2\pi i s) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k s}.$$

This shows that $\chi(X)$ is the evaluation of (6.9) at $s = \frac{\log(\mathfrak{s})}{2\pi i}$, which gives

$$\chi(X) = 88\left(\frac{1+\mathfrak{s}}{\sqrt{\mathfrak{s}}}\right).$$

as result. Proposition 6.1.4 ensures that the classical Hirzebruch genus of Z is

$$\chi_{-\mathfrak{s}}(Z) = -\sqrt{\mathfrak{s}}^3\chi(X) = -88(\mathfrak{s}+\mathfrak{s}^2).$$

This result can be checked by using the description of Z as a complete intersection of a quadric and a quartic hypersurfaces in $\mathbb{P}^5$ (by following the Plücker embedding) [IIM19, Table 1, row 1]. Then, if we apply Hirzebruch-Riemann-Roch to compute the genus we obtain the same result.

6.2 Invariants of critical loci in quiver varieties.

Given a quiver Q, the corresponding quiver varieties are defined as the GIT quotients of the spaces of representations by actions of products of general linear groups. Here we recall how to build these varieties, for more details see the survey [Rei08]. Let Q be a connected quiver with finitely many arrows and nodes. It can be with or without oriented cycles, with or without loops. The set of nodes of the quiver is denoted with Q_0, while the set of arrows with Q_1. We have two functions, called *head* and *tail*

$$h, t : Q_1 \to Q_0,$$

which send an arrow into the node corresponding to its head or tail.

6.2.1 The representation theoretic setup.

Given a *dimension vector* $D \in \mathbb{N}^{Q_0}$, we consider the space of D-dimensional representations

$$V := \bigoplus_{\beta\in Q_1} \mathrm{Mat}_{D_{h(\beta)}\times D_{t(\beta)}}(\mathbb{C}). \tag{6.10}$$

There is a group $\prod_{v\in Q_0} \mathrm{GL}_{D_v}(\mathbb{C})$ acting on the representation space by

$$\prod_{v\in Q_0} \mathrm{GL}_{D_v}(\mathbb{C}) \curvearrowright V \qquad : \qquad (M\cdot\Phi)_\beta := M_{h(\beta)}\Phi_\alpha M^{-1}_{t(\beta)}. \tag{6.11}$$

The diagonal $\Delta \simeq \mathbb{C}^*$ inside this group acts trivially so, in order to work with an effective action, we consider the action of the projectivized group[1]:

$$G := \left(\prod_{v \in Q_0} \mathrm{GL}_{D_v}(\mathbb{C}) \right) / \Delta.$$

The maximal subtorus of G is the quotient of the group of tuples of diagonal matrices by Δ:

$$T = \left(\prod_{v \in Q_0} (\mathbb{C}^*)^{D_v} \right) / \Delta.$$

The Lie algebras $\chi(T)^\vee_{\mathbb{C}} \simeq \mathfrak{h} \subseteq \mathfrak{g}$ are the quotients of

$$\bigoplus_{v \in Q_0} \mathbb{C}^{D_v} \subseteq \bigoplus_{v \in Q_0} \mathfrak{gl}_{D_v}(\mathbb{C})$$

by the diagonal subspace $\mathrm{span}_{\mathbb{C}}(\mathbb{1})$. Let u_i^v be the coordinate functions on $\bigoplus_{v \in Q_0} \mathbb{C}^{D_v}$. Once we have fixed a couple $\overline{v} \in Q_0$ and $\overline{i} \in \{1, \ldots, D_v\}$, we obtain an isomorphism of $\chi(T)^\vee_{\mathbb{C}}$ with the codimension 1 subspace where the coordinate u_i^v vanishes:

$$\chi(T)^\vee_{\mathbb{C}} \simeq V(u_{\overline{i}}^{\overline{v}}) \qquad \text{and} \qquad \chi(T) \simeq \bigoplus_{(v,i) \neq (\overline{v}, \overline{i})} \mathbb{Z} \cdot u_i^v.$$

This isomorphism is fixed in the following discussion, so every time $u_{\overline{i}}^{\overline{v}}$ appears anywhere it must be set to zero.

The Weyl group of G is

$$W = \prod_{v \in Q_0} \mathfrak{S}_{D_v}$$

and acts on $\mathfrak{h}$ by permuting the components in each piece $\mathbb{C}^{D_v}$. We omit the proof of the following two straightforward lemmas.

Lemma 6.2.1. *Then roots of G are the characters of the form*

$$\alpha_{j,i}^v = u_j^v - u_i^v$$

where $v \in Q_0$ and $i, j \in \{1, ..., D_v\}$.

[1]Notice this is still reductive since Δ is the center and hence normal.

It's also easy to describe the weights for the action of T on V:

Lemma 6.2.2. *The weights of the T-representation V, whose set we denote with $\mathfrak{A}$, are the characters of the form*

$$\rho^{\beta}_{j,i} = u_j^{h(\beta)} - u_i^{t(\beta)}$$

where $\beta \in Q_1$, $i \in \{1, ..., D_{t(\beta)}\}$ and $j \in \{1, ..., D_{h(\beta)}\}$. Notice that two arrows define the same weights if and only if they share the same head and tail.

6.2.2 Linearisations and stabilities.

Since V is an affine space, the linearisations for this actions corresponds to choices of a character of G, namely an element of

$$\chi(G) \simeq \chi(T)^W.$$

Clearly the only such characters of G are of the form

$$G \to \mathbb{C}^* \quad : \quad g \mapsto \det{}^{\xi_v}(g_v).$$

where $\xi \in \mathbb{Z}^{Q_0}$ satisfies $\sum_{v \in Q_0} D_v \xi_v = 0$. As shown in [Rei08], if ξ is a regular stability the action on the semistable locus is free and the corresponding GIT quotient $V/\!/G$ is smooth. In the literature, this is called a *quiver variety* and it's denoted with $\mathcal{M}_D^{\xi\text{-ss}}(Q)$.

6.2.3 The additional torus action.

Let S be another torus. We can endow V with an action of S by choosing a set of characters $R \in \chi(S)^{Q_1}$ (called the R-charge in physics) and writing

$$(s \cdot \Phi)_\beta := R_\beta(s)\Phi_\beta \qquad \forall \beta \in Q_1.$$

Notice that, since G acts linearly on each irreducible piece $\mathrm{Mat}_{D_{h(\beta)} \times D_{t(\beta)}}(\mathbb{C})$ of V, then the actions of G and S commute. Now we want to study under which conditions the fixed subvarieties $(V/\!/T)^S$ and $(V/\!/G)^S$ are proper. We have a simple description, proven in [LP90], of the T-invariant functions on the space of representations: they are all generated by monomials of the form

$$\prod_{\beta \in \text{cycle}} (\Phi_\beta)_{j_\beta, i_\beta}$$

where the product is over arrows belonging to a fixed oriented cycle in the quiver and the indices $j \in \{1,\dots,D_{h(\beta)}\}$, $i \in \{1,\dots,D_{t(\beta)}\}$ satisfy $i_{\beta_2} = j_{\beta_1}$ for consecutive arrows $\cdots \xrightarrow{\beta_1} \bullet \xrightarrow{\beta_2} \cdots$ in the cycle.

The description of the generators of the ring of invariant functions given by the theorem above, together with the characterisation (5.9) of properness of $(V/\!/T)^S$ and $(V/\!/G)^S$, allows us to find a condition to impose on the R-charge in order to have a projective fixed subvariety:

Proposition 6.2.1. *Assume that there is a strictly convex cone C in $\chi(S)$ so that*

$$\sum_{\beta\in\gamma} R_\beta \in C \textit{ for every minimal oriented cycle } \gamma \textit{ in } Q. \tag{6.12}$$

Then the fixed loci $(V/\!/T)^S$ and $(V/\!/G)^S$ are projective.

6.2.4 The formula.

Let's specialise the statement of Theorem 6.1.1 to the setting of quiver varieties we described in the previous sections. Fixed $s \in \chi(S)^\vee_{\mathbb{C}}$, consider the hyperplane arrangement $\mathcal{H}_s$ of $\chi(T)^\vee_{\mathbb{C}} \simeq \mathfrak{h}$ defined by the weights of the T-action and the R-charge:

$$H^\beta_{i,j} := \left\{u^{h(\beta)}_j - u^{t(\beta)}_i + R_\beta(s) = 0\right\},$$

indexed by $\beta \in Q_1$, $i \in \{1,\dots,D_{t(\beta)}\}$ and $j \in \{1,\dots,D_{h(\beta)}\}$. Consider the set of points $P \in \chi(T)^\vee_{\mathbb{C}}$ at which at least $\dim T$ independent hyperplanes vanish. For each such isolated intersection of $\mathcal{H}_s$, consider the set of characters vanishing on it:

$$\mathfrak{A}_P := \left\{u^{h(\alpha)}_j - u^{t(\alpha)}_i \in \chi(T) \quad \middle| \quad P \in H^\beta_{i,j}\right\}.$$

Then P is a table isolated intersection of $\mathcal{H}_s$ (in the sense of Definition 5.1.3) if and only if ξ belongs to the cone spanned by $\mathfrak{A}_P$ in $\chi(T)$. Denoting with $|D|$ the total dimension vector $\sum_{v\in Q_0} D_v$ of the quiver and given a character $\psi \in \chi(S)$, consider the following meromorphic functions $\chi(T)^\vee_{\mathbb{C}} \times \chi(S)^\vee_{\mathbb{C}} \dashrightarrow \mathbb{C}$:

- The function Z_{DT} computing the DT invariant

$$\begin{aligned} Z_{\mathrm{DT}} =& \psi^{1-|D|} \prod_{\beta\in Q_1} \prod_{i=1}^{D_{t(\beta)}} \prod_{j=1}^{D_{h(\beta)}} \frac{\psi - R_\beta + u^{t(\beta)}_i - u^{h(\beta)}_j}{u^{h(\beta)}_j - u^{t(\beta)}_i + R_\beta} \\ &\times \prod_{v\in Q_0} \prod_{\substack{i,j=1\\ i\neq j}}^{D_v} \frac{u^v_j - u^v_i}{\psi + u^v_j - u^v_i}. \end{aligned}$$

- The function Z_χ computing the Hirzebruch genus

$$Z_\chi = \left(\frac{\pi}{\sin(\pi\psi)}\right)^{|D|-1} \prod_{\beta\in Q_1}\prod_{i=1}^{D_{t(\beta)}}\prod_{j=1}^{D_{h(\beta)}} \frac{\sin\left(\pi(\psi - R_\beta + u_i^{t(\beta)} - u_j^{h(\beta)})\right)}{\sin\left(\pi(u_j^{h(\beta)} - u_i^{t(\beta)} + R_\beta)\right)}$$
$$\times \prod_{v\in Q_0}\prod_{\substack{i,j=1\\ i\neq j}}^{D_v} \frac{\sin(\pi(u_j^v - u_i^v))}{\sin\left(\pi(u_j^v - u_i^v + \psi)\right)}.$$

- The function Z_{Ell}, depending on an additional formal parameter τ, computing the elliptic genus

$$Z_{Ell} = \left(\frac{2\pi\eta(\tau)^3}{\theta(\tau|\psi)}\right)^{|D|-1} \prod_{\beta\in Q_1}\prod_{i=1}^{D_{t(\beta)}}\prod_{j=1}^{D_{h(\beta)}} \frac{\theta\left(\tau|\psi - R_\beta + u_i^{t(\beta)} - u_j^{h(\beta)}\right)}{\theta\left(\tau|u_j^{h(\beta)} - u_i^{t(\beta)} + R_\beta\right)}$$
$$\times \prod_{v\in Q_0}\prod_{\substack{i,j=1\\ i\neq j}}^{D_v} \frac{\theta(\tau|u_j^v - u_i^v)}{\theta\left(\tau|u_j^v - u_i^v + \psi\right)}.$$

The following result is the specialisation of Theorem 6.1.1 to the case of quivers:

Theorem 6.2.1. *Consider the critical locus X of a regular S-equivariant function on the quiver moduli space*

$$\varphi : \mathcal{M}_D^{\xi ss}(Q) \to \mathfrak{s},$$

where $\mathfrak{s}$ is a 1-dimensional representation of S corresponding to a character $\psi \in \chi(S)$. Let $D \in \mathbb{N}^{Q_0}$ be a dimension vector for a quiver Q. Assume the stability ξ is regular and that the R-charge $R \in \chi(S)^{Q_1}$ satisfies the condition (6.12). Assume that, for a generic $s \in \chi(S)^\vee_{\mathbb{C}}$, for every stable isolated intersection P of the hyperplane arrangement $\mathcal{H}_s$ the inequality

$$u_i^v(P) - u_j^v(P) + \psi(s) \neq 0 \tag{6.13}$$

holds true for every $v \in Q_0$ and $i, j \in \{1, ..., D_v\}$. For a generic $s \in \chi(S)^\vee_{\mathbb{C}}$ the *invariants X can be computed by*

$$
\begin{aligned}
DT(X)(s) &= \frac{1}{\prod_{v \in Q_0}(D_v!)} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} JK_{\xi,P}^{\mathfrak{A}_P}\left(Z_{DT}(-,s)\right), \\
ch^S\chi(X)(2\pi i s) &= \frac{1}{\prod_{v \in Q_0}(D_v!)} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} JK_{\xi,P}^{\mathfrak{A}_P}\left(Z_{\chi}(-,s)\right), \\
ch^S Ell(X)(e^{2\pi i \tau})(2\pi i s) &= \frac{1}{\prod_{v \in Q_0}(D_v!)} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} JK_{\xi,P}^{\mathfrak{A}_P}\left(Z_{Ell}(-,s,\tau)\right).
\end{aligned}
$$

This result appeared for the first time in the physics literature in the works of Beaujard, Mondal and Pioline [BMP19] and Córdova and Shao [CS16].

Remark 38. It's worth remarking that in the case of quiver varieties, the procedure of pulling back integrals from $V//G$ onto $V//T$ through Martin's formula (Theorem 5.1.2) corresponds to pulling back integrals from the variety corresponding to the quiver Q to the one corresponding to the quiver $\hat{Q}$ obtained by "abelianising" the nodes. This procedure consists in replacing a node of dimension vector d with d distinct nodes of dimension 1 and connecting two nodes in $\hat{Q}$ with an arrow if and only if the original nodes were connected in Q. For example, if we start from the quiver Q

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, bend left] \arrow[r, bend right] & \mathbb{C}^3 \arrow[r] & \mathbb{C}
\end{tikzcd},
$$

the corresponding $\hat{Q}$ is

$$
\begin{tikzcd}
& \mathbb{C} \arrow[dr] & \\
\mathbb{C} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \mathbb{C} \arrow[r] & \mathbb{C} \\
& \mathbb{C} \arrow[ur] &
\end{tikzcd}.
$$

6.2.5 Example: DT invariants of $\mathbb{A}^3$.

The rigorous application of Jeffrey-Kirwan localisation techniques to the problem of *instanton counting* (the computation of integrals on the Hilbert scheme of points of affine spaces) is due to Martens [Mar08]. Here we consider the quot scheme of quotient sheaves of $\mathcal{O}_{\mathbb{A}^3}^{\oplus r}$ having finite length n:

$$X_r^n := \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n).$$

As shown in [FMR21], this scheme is closely related to the moduli space of representations of the quiver

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop above, "\alpha_1"] \arrow[loop left, "\alpha_2"] \arrow[loop below, "\alpha_3"] & \mathbb{C} \arrow[l, bend right, "\beta_1"'] \arrow[l, "\cdots"] \arrow[l, bend left, "\beta_r"] \end{tikzcd}$$

The space of representations is

$$\mathrm{Rep}(Q) \simeq \mathrm{Mat}_{n\times n}(\mathbb{C})^{\oplus 3} \oplus (\mathbb{C}^n)^{\oplus r}$$

and the gauge group $G := \mathrm{GL}_n(\mathbb{C})$ acts on it by

$$g \cdot (A_1, A_2, A_3, b_1, \dots, b_r) := (gA_1g^{-1}, gA_2g^{-1}, gA_3g^{-1}, gb_1, \dots, gb_r).$$

The stability $\xi := 1 \in \mathbb{Z} \simeq \chi(G)$ is regular and the corresponding quotient is a smooth quasiprojective variety $\mathcal{A} := \mathrm{Rep}(Q)/\!/G$. The G-invariant function

$$\tilde{\varphi} : \mathrm{Rep}(Q) \to \mathbb{C} \quad : \quad \tilde{\varphi}(A_1, A_2, A_3, b_1, ..., b_r) := \mathrm{Tr}\,(A_1[A_2, A_3])$$

defines a regular function $\varphi : \mathcal{A} \to \mathbb{C}$ whose critical locus is the quot scheme:

$$X_r^n = \mathrm{Crit}(\varphi) \subseteq \mathcal{A}.$$

We can enrich the picture with an action of $S := (\mathbb{C}^*)^3$ on $\mathrm{Rep}(Q)$ given by

$$(s_1, s_2, s_3) \cdot (A_1, A_2, A_3, b_1, ..., b_r) := (s_1A_1, s_2A_2, s_3A_3, b_1, ..., b_r).$$

Condition (6.12) is clearly satisfied and ensures that the fixed locus $\mathcal{A}^S$ is projective. Moreover, φ equivariant once we let S act on the target $\mathbb{C}$ with the character $\psi := s_1s_2s_3$. The equivariant DT invariant of X_r^n is called n^{th} *degree zero cohomological DT invariant of rank* r of $\mathbb{A}^3$:

$$\mathrm{DT}_r^n(\mathbb{A}^3) := \int_{[X_r^n]^{\mathrm{vir}}} 1.$$

Our formula can, in principle, be used to compute these invariants (this approach to compute the invariants of $\mathbb{A}^3$ is present in the physics literature, in particular in [Ben+19]). By identifying the maximal subtorus $T \subset G$ with diagonal matrices there is an isomorphism $\chi(T)^\vee_{\mathbb{C}} \simeq \mathbb{C}^n$ and let $u_1, \dots, u_n \in \chi(T)$ be the coordinate functions. Analogously, we can identify $\chi(S)^\vee_{\mathbb{C}}$ with $\mathbb{C}^3$ and the characters s_1, s_2, s_3

are the coordinate functions. The function $Z_{\mathrm{DT}} : \mathbb{C}^n \dashrightarrow \mathbb{C}$ of which we have to extract the residues is

$$Z_{\mathrm{DT}}(u_1, ..., u_n, s_1, s_2, s_3) := \psi^{-n} \prod_{k=1}^{n} \left(\frac{\psi - u_k}{u_k} \right)^r \prod_{l=1}^{3} \prod_{a,b=1}^{n} \frac{\psi - s_l - u_b + u_a}{s_l + u_b - u_a} \times \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{u_i - u_j}{\psi + u_i - u_j},$$

where $\psi = s_1 + s_2 + s_3$. The hyperplane arrangement $\mathcal{H}_s$ is given by the hyperplanes of the following two types

$$H_k := \{u_k = 0\} \qquad \text{and} \qquad H^l_{a,b} := \{s_l + u_b - u_a = 0\}$$

and it's easy to see that for a generic choice of $(s_1, s_2, s_3) \in \mathbb{C}^3$ the *stable* isolated intersections in this arrangement are not contained in any hyperplane of the form $\{u_i - u_j + s_1 + s_2 + s_3 = 0\}$, thus condition (6.13) is satisfied and Theorem 6.2.1 shows

$$\mathrm{DT}^n_r(\mathbb{A}^3) = \frac{1}{n!} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_s}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P} \left(Z_{\mathrm{DT}}(-, s) \right)$$

It is combinatorially challenging (even though doable with some effort) to enumerate the many stable isolated intersections of this hyperplane arrangement. Basically, they correspond to the same plane partitions that enumerate the fixed points on the quot scheme. In practice some additional stable intersection appears but still gives a trivial JK residue (see Remark 39 below for an explanation of this phenomenon). By checking numerically the result for small values of n, the formula confirms equality

$$\sum_{n=0}^{\infty} DT^n_r(\mathbb{A}^3) q^n = M((-1)^r q)^{-r \frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}},$$

which has been proven with classical virtual localisation techniques in [FMR21]. Here M is the *MacMahon function*, the generating functions of plane partitions:

$$M(q) = \prod_{k=1}^{\infty} \frac{1}{(1 - x^k)^k}.$$

Remark 39. By unraveling the proof of this localisation formula, notice that in this case Szenes-Vergne localisation consists in the following steps:

1. Push the computation forward from the quot scheme to the smooth "noncommutative" quot scheme $\mathcal{A}$.

2. Do Atiyah-Bott localisation on $\mathcal{A}$.

3. Compute the contribution of each fixed locus as a JK residue.

It seems that by pushing the computation to $\mathcal{A}$ one is adding some fixed loci that only belong to $\mathcal{A}$ and not to the actual quot scheme, but notice that their contribution to the formula is zero since the Euler class $e^S(\Omega_{\mathcal{A}})$ vanishes there, as the section used to cut X is nonvanishing over that fixed component.

In this specific example one doesn't gain much from this procedure of pushing forward to the smooth ambient variety $\mathcal{A}$. In some other cases, as we will see in the next section, this technique might be very helpful.

6.3 The case of $\mathrm{Hilb}^n(\mathbb{A}^4)$.

In [OT23], a theory of algebraic virtual classes for moduli spaces of sheaves on Calabi-Yau fourfolds is developed. The simplest example of such moduli space is $\mathrm{Hilb}^n\mathbb{A}^4$. In [NP19] Nekrasov and Piazzalunga conjecture a formula (now proven in full generality by Kool and Rennemo using Oh-Thomas localisation), for the equivariant integral of 1 over this Hilbert scheme. They check the conjecture for small values of n by using a Jeffrey-Kirwan type of formula motivated by physical arguments. Here we show how to mathematically recover this JK formula from the formalism of Oh-Thomas. This doesn't give a new proof of Nekrasov's formula for $\int_{\mathrm{Hilb}^n\mathbb{A}^4} 1$, since the JK formula we are going to show still requires a lot of hard combinatorics to be made before reaching the result of Kool-Rennemo, and we don't know how to perform such computations.

Remark 40. In practice, Nekrasov and Piazzalunga consider K-theoretic invariants (see Remark 41 below). We will focus on the cohomological invariant $\int_{[X]^{\mathrm{vir}}} 1$ for simplicity, but the K-theoretic discussion is completely analogous.

6.3.1 Recap on the Oh-Thomas virtual cycle.

The reference for this section is the work [OT23] of Oh and Thomas. The theory of virtual classes for moduli spaces of sheaves on fourfolds is based on the following local model: the relevant moduli space $\mathcal{M}$ is build as the zero locus, in some smooth ambient space $\mathcal{A}$, of an isotropic section s of an orthogonal vector bundle $\mathcal{E}$ with nondegenerate bilinear form

$$q : \mathcal{E} \otimes \mathcal{E} \to \mathcal{O}_{\mathcal{A}}.$$

Denoted with r is the rank of $\mathcal{E}$, an *orientation* is needed, namely a trivialization

$$o : \mathcal{O}_{\mathcal{A}} \xrightarrow{\sim} \det\mathcal{E}$$

so that $(-1)^{\frac{r(r-1)}{2}} o^{\otimes 2}$ is the inverse to the isomorphism $(\det\mathcal{E})^{\otimes 2} \simeq \mathcal{O}_{\mathcal{A}}$ defined by q. Up to passing to a cover $p : \tilde{\mathcal{A}} \to \mathcal{A}$, the bundle $\tilde{\mathcal{E}} := p^*\mathcal{E}$ admits a maximal isotropic subbundle $\tilde{\Lambda} \hookrightarrow \tilde{\mathcal{E}}$, hence we have a short exact sequence induced by q:

$$0 \to \tilde{\Lambda} \hookrightarrow \tilde{\mathcal{E}} \xrightarrow{\tilde{q}} \tilde{\Lambda}^\vee \to 0.$$

We can always find (on the cover $\tilde{\mathcal{A}}$) this $\tilde{\Lambda}$ so that it is *positive*, namely the isomorphism

$$\mathcal{O}_{\tilde{\mathcal{A}}} \xrightarrow{\tilde{o}} \det\tilde{\mathcal{E}} \simeq \det\tilde{\Lambda} \otimes \det\tilde{\Lambda}^\vee \simeq \mathcal{O}_{\tilde{\mathcal{A}}}$$

sends i^n to 1. A virtual cycle $[\mathcal{M}]^{\mathrm{vir}} \in A_*(\mathcal{M})$ can be built from this data and, if $\mathcal{E}$ admits a maximal isotropic positive subbundle Λ on $\mathcal{A}$, the virtual cycle pushes forward along $i : \mathcal{M} \hookrightarrow \mathcal{A}$ as

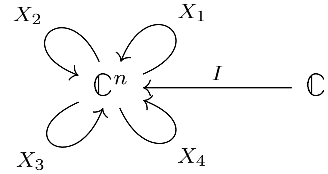
$$i_*[\mathcal{M}]^{\mathrm{vir}} = e(\Lambda).$$

We can do everything equivariantly: if a torus S acts on $\mathcal{A}$, the bundle $\mathcal{E}$ has a S-equivariant structure, s is an invariant section of $\mathcal{E}$ and $\lambda \hookrightarrow \mathcal{E}$ is an invariant positive maximal isotropic subbundle, then the virtual cycle is equivariant $[\mathcal{M}]^{\mathrm{vir}} \in A_*^S(\mathcal{M})$ and pushes forward to

$$i_*[\mathcal{M}]^{\mathrm{vir}} = e^S(\Lambda).$$

6.3.2 Building the Hilbert scheme.

The first step is to realize the relevant moduli space as the zero locus of an isotropic section of an orthogonal bundle. This is all standard, for example see [Nek20]. Consider the quiver



The corresponding representations space is

$$V \simeq \mathrm{End}(\mathbb{C}^n)^{\oplus 4} \oplus \mathbb{C}^n.$$

which is acted upon by $G := \mathrm{GL}_n$ as

$$g \cdot (X_1, X_2, X_3, X_4, I) := (gX_1g^{-1}, gX_2g^{-1}, gX_3g^{-1}, gX_4g^{-1}, gI).$$

Fixing the stability $\xi := 1 \in \mathbb{Z} \simeq \chi(G)$ we see by the "generalised AHDM construction" that $\mathrm{Hilb}^n\mathbb{A}^4$ is the zero locus of a section on the quiver variety $\mathcal{A} := V/\!/GL_n$:

$$\begin{tikzcd} & \mathcal{E} \arrow[d, two heads] \\ \mathrm{Hilb}^n\mathbb{A}^4 \simeq V(s) \arrow[r, hook, "i"] & \mathcal{A} := V/\!/\mathrm{GL}_n \arrow[u, bend right, "s"'] \end{tikzcd}$$

The bundle $\mathcal{E}$ is induced on the quotient by the G-equivariant bundle on V having as fibre the GL_n-representation

$$\mathfrak{gl}_n \otimes \wedge^2\mathbb{C}^4$$

where the action on the second factor is trivial while on first factor it is given by the adjoint representation. If we denote with $X_i \wedge X_j$, $i < j$ the elements of a basis of the 6-dimensional linear space $\wedge^2\mathbb{C}^4$, the section s is induced by the G-invariant section

$$s = \sum_{1 \leqslant i < j \leqslant 4} [X_i, X_j] \otimes X_i \wedge X_j.$$

Lemma 6.3.1. *The bundle $\mathcal{E}$ has a natural structure of orthogonal bundle given by the product of the nondegenerate bilinear forms*

$$\mathfrak{gl}_n \times \mathfrak{gl}_n \xrightarrow{tr} \mathbb{C} \quad : \quad tr(A, B) := tr(AB)$$

and

$$\wedge^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^4 \xrightarrow{\wedge} \wedge^4\mathbb{C}^4 \simeq span_{\mathbb{C}}(X_1 \wedge X_2 \wedge X_3 \wedge X_4) \simeq \mathbb{C}.$$

Moreover, the section s is isotropic.

Proof. The only thing to prove is that $q(s,s)=0$. Clearly

$$q(s,s) = \mathrm{Tr}\left(\sum_{\sigma\in\mathfrak{S}_4}(-1)^\sigma[X_{\sigma(1)},X_{\sigma(2)}][X_{\sigma(3)},X_{\sigma(4)}]\right).$$

Notice that for every a,b,c,d the sum over the order 4 cyclic subgroup generated by the cycle $(1,2,3,4)\in\mathfrak{S}_4$

$$\sum_{\sigma\in\langle(1,2,3,4)\rangle}(-1)^\sigma[X_{\sigma(a)},X_{\sigma(b)}][X_{\sigma(c)},X_{\sigma(d)}]$$

is trace-free, since the trace $\mathrm{tr}:\mathfrak{gl}_n\to\mathbb{C}$ is linear and invariant under cyclic permutations in a product. If we pick a representative in $\mathfrak{S}_4$ for each class in the set $\mathfrak{S}_4/\langle(1,2,3,4)\rangle$, we can write $q(s,s)$ as the trace of

$$\sum_{[\eta]\in\frac{\mathfrak{S}_4}{\langle(1,2,3,4)\rangle}}\ \sum_{\sigma\in\langle(1,2,3,4)\rangle}(-1)^\sigma[X_{\sigma(\eta(1))},X_{\sigma(\eta(2))}][X_{\sigma(\eta(3))},X_{\sigma(\eta(4))}]$$

which is trace-free by the above discussion. □

Notice that the bundle E on V has a maximal isotropic subbundle L given by the sections $X\wedge Y$, $X\wedge Z$ and $Y\wedge Z$. This induces a maximal isotropic subbundle $\Lambda\hookrightarrow\mathcal{E}$ on $V/\!/G$.

6.3.3 The torus action.

We can enrich the picture with an action of the four dimensional torus $S:=(\mathbb{C}^*)^4$ on V by

$$s\cdot(X_1,X_2,X_3,X_4,I) := (s_1X_1,s_2X_2,s_3X_3,s_4X_4,I).$$

Notice that the action of S commutes with the action of G, hence $\mathbb{T}$ acts on $V/\!/G$. We can endow E with a S-equivariant structure induced from the action on the fibre by

$$S \curvearrowright \mathfrak{gl}_n\otimes_{\mathbb{C}}\wedge^2\mathbb{C}^4 \qquad : \qquad s\cdot M\otimes X_i\wedge X_j := s_is_jM\otimes X_i\wedge X_j.$$

and notice that s is S-invariant. Notice that in particular the subbundle $\Lambda\subset\mathcal{E}$ is S-invariant. This means we are in the framework of Oh-Thomas and hence

$$\int_{[\mathrm{Hilb}^n\mathbb{A}^4]^{\mathrm{vir}}}1 = \int_{V/\!/G}e^S(\Lambda) = \int_{V/\!/G}r(e^{G\times S}(L)).$$

The last expression for this invariant is precisely the kind of integral that the localisation formula of Theorem 5.1.3 can help to compute.

6.3.4 The hyperplane arrangement and the localisation formula.

Consider the maximal subtorus $T \subset G$ given by diagonal matrices, which gives the isomorphism $\chi(T)^\vee_{\mathbb{C}} \simeq \mathbb{C}^n$ and denote the coordinate functions with $u_1, \dots, u_n \in \chi(T)$. Analogously $\chi(S)^\vee_{\mathbb{C}} \simeq \mathbb{C}^4$ and consider the coordinate functions $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \chi(S)$ (these would be denoted s_1, s_2, s_3, s_4 in the notation of the previous sections, but we use the letter ϵ to match the notation of Nekrasov). Fixed $\epsilon \in \chi(S)^\vee_{\mathbb{C}}$ the hyperplane arrangement $\mathcal{H}_\epsilon$ in $\chi(T)^\vee_{\mathbb{C}}$ is given by the hyperplanes of the form

$$H^c_{i,j} := \{u_i - u_j + \epsilon_c\} \qquad \text{and} \qquad H_k := \{u_k = 0\}$$

where $1 \leqslant c \leqslant 4$ and $1 \leqslant i, j, k \leqslant n$. Theorem 5.1.3 shows that, for a generic ϵ, this integral is

$$\int_{[\mathrm{Hilb}^n \mathbb{A}^4]^{\mathrm{vir}}} 1 = \frac{1}{n!} \sum_{\substack{P \text{ stable isolated} \\ \text{intersection of } \mathcal{H}_\epsilon}} \mathrm{JK}^{\mathfrak{A}_P}_{\xi,P} \left(\frac{e^{T\times S}(L) e^T(\mathfrak{gl}_n/\mathfrak{h})}{e^{T\times S}(T_V)} \right).$$

Here the argument of the JK residue is readily computed as

$$\left(\frac{\epsilon_{12}\epsilon_{13}\epsilon_{23}}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}\right)^n \prod_{i\neq j} \frac{(u_1 - u_j)\prod_{1\leqslant a<b\leqslant 3}(u_i - u_j + \epsilon_{ab})}{\prod_{c=1}^4 (u_i - u_j + \epsilon_c)} \prod_{k=1}^n \frac{1}{u_k}$$

where we used the notation $\epsilon_{ab} := \epsilon_a + \epsilon_b$. This is the content of equations (2.24) and (2.25) in the work of Nekrasov and Piazzalunga [NP19].

Remark 41. To be precise, Nekrasov and Piazzalunga are doing a *K-theoretic* computation with some tautological insertions (namely insertions built from K-theory classes on $\mathbb{A}^4$ by pulling back to the universal family and pushing down to the Hilbert scheme). More precisely, they are computing the so called *Nekrasov genus* described in [CKM22, Definition 0.2]. The one we studied is the cohomological limit discussed at the end of [CKM22, Appendix A], but the Jeffrey-Kirwan approach to the original K-theoretic computation is completely analogous.

Remark 42. While discussing the case of $\mathbb{A}^3$ in Section 6.2.5, we noted that there is not much to gain from pushing the computation forward to the smooth ambient noncommutative Hilbert scheme $\mathcal{A}$. In the case of $\mathbb{A}^4$ the situation is radically different: doing localisation directly on $\mathrm{Hilb}^n(\mathbb{A}^4)$ is much harder than in the 3d case. The technical reason for this is that it is hard to keep track of the sign in the square root of $e^S(\mathcal{E})$ when localising to the fixed points. On the other hand, by pushing the computation forward to the ambient space one recovers the usual localisation formula on the smooth variety $\mathcal{A}$ which completely forgets about the square root problem.

Bibliography

[AB84] M. F. Atiyah and R. Bott. "The moment map and equivariant cohomology". In: *Topology* 23 (1984), pp. 1–28.

[AFO18] M. Aganagic, E. Frenkel, and A. Okounkov. "Quantum q-Langlands correspondence". In: *Transactions of the Moscow Mathematical Society* 79 (2018), pp. 1–83.

[Ati82] M. F. Atiyah. "Convexity and Commuting Hamiltonians". In: *Bulletin of The London Mathematical Society* 14 (1982), pp. 1–15. URL: https://api.semanticscholar.org/CorpusID:18142550.

[Ben+15] F. Benini et al. "Elliptic genera of 2d $\mathcal{N}=2$ Gauge theories". In: *Communications in Mathematical Physics* 333.3 (2015), p. 1241.

[Ben+19] F. Benini et al. "Elliptic non-Abelian Donaldson-Thomas invariants of C3". In: *Journal of High Energy Physics* (July 2019).

[BF97] K. Behrend and B. Fantechi. "The intrinsic normal cone". In: *Inventiones Mathematicae* 128 (1997), p. 45.

[BM02] V. V. Batyrev and E. N. Materov. "Toric residues and mirror symmetry". In: *Moscow Mathematical Journal* 2.3 (2002), pp. 435–475.

[BMP19] G. Beaujard, S. Mondal, and B. Pioline. "Quiver indices and Abelianization from Jeffrey-Kirwan residues". In: *Journal of High Energy Physics* 2019.10 (2019), p. 184.

[Bri87] M. Brion. "Sur l'image de l'application moment". In: *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin*. Ed. by M.-P. Malliavin. Berlin, Heidelberg: Springer Berlin Heidelberg, 1987, pp. 177–192. ISBN: 978-3-540-48081-5.

[BT89] R. Bott and C. Taubes. "On the Rigidity Theorems of Witten". In: *Journal of the American Mathematical Society* 2 (1989), p. 137.

[BV99] M. Brion and M. Vergne. "Arrangement of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue". In: *Annales scientifiques de l'Ecole normale supérieure*. Vol. 32. 5. 1999, p. 715.

[CKM22] Y. Cao, M. Kool, and S. Monavari. "K-theoretic DT/PT correspondence for toric Calabi–Yau 4-folds". In: *Communications in Mathematical Physics* 396.1 (2022), pp. 225–264.

[CS16] C. Córdova and S.-H. Shao. "An index formula for supersymmetric quantum mechanics". In: *J. Singul. 15* (2016), p. 14.

[DH98] I. V. Dolgachev and Y. Hu. "Variation of geometric invariant theory quotients". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 87 (1998), pp. 5–51.

[Dol03] I. Dolgachev. *Lectures on Invariant Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.

[Dré04] J.-M. Drézet. *Luna's slice theorem and applications*. 2004.

[Edi10] D. Edidin. "Equivariant geometry and the cohomology of the moduli space of curves". In: *arXiv preprint arXiv:1006.2364* (2010).

[Edi12] D. Edidin. "Riemann-Roch for Deligne-Mumford stacks". In: *arXiv preprint arXiv:1205.4742* (2012).

[EG98a] D. Edidin and W. Graham. "Algebraic cuts". In: *Proceedings of the American Mathematical Society* 126.3 (1998), p. 677.

[EG98b] D. Edidin and W. Graham. "Equivariant intersection theory". In: *Inventiones mathematicae* 131 (1998), p. 595.

[EG98c] D. Edidin and W. Graham. "Localization in equivariant intersection theory and the Bott residue formula". In: *American Journal of Mathematics* 120.3 (1998), pp. 619–636.

[EG99] D. Edidin and W. Graham. "Riemann-Roch for equivariant Chow groups". In: *Duke Mathematical Journal* 102 (1999), p. 567.

[ES89] G. Ellingsrud and S. A. Strømme. "On the Chow Ring of a Geometric Quotient". In: *Annals of Mathematics* 130 (1989), pp. 159–187.

[FG10] B. Fantechi and L. Göttsche. "Riemann-Roch theorems and elliptic genus for virtually smooth schemes". In: *Geometry & Topology* 14 (2010), p. 83.

[FMR21] N. Fasola, S. Monavari, and A. T. Ricolfi. "Higher rank K-theoretic Donaldson-Thomas Theory of points". In: *Forum of Mathematics, Sigma* 9 (2021).

[Ful13] W. E. Fulton. *Intersection theory*. Vol. 2. Springer Science & Business Media, 2013.

[GHH15] M. G. Gulbrandsen, L. H. Halle, and K. Hulek. “A relative Hilbert–Mumford criterion”. In: *Manuscripta Mathematica* 148 (2015), pp. 283–301.

[GK96] V. Guillemin and J. Kalkman. “The Jeffrey-Kirwan localization theorem and residue operations in equivariant cohomology.” In: *Journal für die reine und angewandte Mathematik* 470 (1996), pp. 123–142.

[GP97] T. B. Graber and R. Pandharipande. “Localization of virtual classes”. In: *Inventiones mathematicae* 135 (1997), pp. 487–518.

[GS82] V. Guillemin and S. Sternberg. “Convexity properties of the moment mapping”. In: *Inventiones Mathematicae* 67 (Oct. 1982), pp. 491–513. DOI: 10.1007/BF01398933.

[HBJ92] F. Hirzebruch, T. Berger, and R. Jung. *Manifolds and modular forms*. Aspects of Mathematics. Springer, 1992.

[HNP94] J. Hilgert, K.-H. Neeb, and W. Plank. “Symplectic convexity theorems and coadjoint orbits”. eng. In: *Compositio Mathematica* 94.2 (1994), pp. 129–180. URL: http://eudml.org/doc/90332.

[Hos15] V. Hoskins. “Moduli problems and geometric invariant theory”. In: *Lecture notes 2016* 7.28 (2015), p. 12.

[IIM19] D. Inoue, A. Ito, and M. Miura. “Complete intersection Calabi–Yau manifolds with respect to homogeneous vector bundles on Grassmannians”. In: *Mathematische Zeitschrift* 292 (2019), p. 677.

[JK05] L. Jeffrey and M. Kogan. “Localization theorems by symplectic cuts”. In: *The Breadth of Symplectic and Poisson Geometry: Festschrift in Honor of Alan Weinstein* (2005), pp. 303–326.

[JK95] L. C. Jeffrey and F. C. Kirwan. “Localization for nonabelian group actions”. In: *Topology* 34.2 (1995), p. 291.

[JK98] L. C. Jeffrey and F. C. Kirwan. “Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface”. In: *Annals of Mathematics* 148.1 (1998), pp. 109–196.

[Kem78] G. R. Kempf. “Instability in invariant theory”. In: *Annals of Mathematics* 108.2 (1978), p. 299.

[Kin94] A. King. "Moduli of representations of finite dimensional algebras". In: *The Quarterly Journal of Mathematics* 45.4 (1994), p. 515.

[Kri13] A. Krishna. "Higher Chow groups of varieties with group action". In: *Algebra and Number Theory* 7.2 (2013), pp. 449–506.

[Kri14] A. Krishna. "Riemann–Roch for equivariant K-theory". In: *Advances in Mathematics* 262 (2014), pp. 126–192.

[Ler95] E. Lerman. "Symplectic cuts". In: *Mathematical Research Letters* 2 (1995), pp. 247–258.

[LP90] L. Le Bruyn and C. Procesi. "Semisimple representations of quivers". In: *Trans. Amer. Math. Soc.* 317.2 (1990), p. 585.

[Mar00] S. Martin. "Symplectic quotients by a nonabelian group and by its maximal torus". arXiv:math/0001002. 2000.

[Mar08] J. Martens. "Equivariant volumes of non-compact quotients and instanton counting". In: *Communications in mathematical physics* 281.3 (2008), pp. 827–857.

[MFK94] D. Mumford, J. Fogarty, and F. C. Kirwan. *Geometric invariant theory.* 34. Springer-Verlag, 1994.

[Mum07] D. Mumford. *Tata Lectures on Theta I.* Modern Birkhäuser Classics. Birkhäuser, 2007.

[Nek20] N. Nekrasov. "Magnificent four". In: *Annales de l'Institut Henri Poincaré D* 7.4 (2020), pp. 505–534.

[NP19] N. Nekrasov and N. Piazzalunga. "Magnificent four with colors". In: *Communications in Mathematical Physics* 372.2 (2019), pp. 573–597.

[OT23] J. Oh and R. P. Thomas. "Counting sheaves on Calabi-Yau 4-folds, I". In: *Duke Mathematical Journal* 172.7 (2023), p. 1333.

[Rei08] M. Reineke. "Moduli of representations of quivers". In: *Trends in Representation Theory of Algebras and Related Topics* (2008). D.O.I. 10.4171/062-1/14, p. 589.

[Sja98] R. Sjamaar. "Convexity Properties of the Moment Mapping Re-examined". In: *Advances in Mathematics* 138.1 (1998), pp. 46–91. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.1998.1739. URL: https://www.sciencedirect.com/science/article/pii/S000187089891739X.

[SV04] A. Szenes and M. Vergne. "Toric reduction and a conjecture of Batyrev and Materov". In: *Inventiones Mathematicae* 158.3 (2004), p. 453.

[Sze98] A. Szenes. “Iterated residues and multiple Bernoulli polynomials”. In: *International Mathematics Research Notices* 1998.18 (1998), pp. 937–956.

[Tho05] R. P. Thomas. “Notes on GIT and symplectic reduction for bundles and varieties”. In: *arXiv preprint math/0512411* (2005).

[Tot99] B. Totaro. “The Chow ring of a classifying space”. In: *Proceedings of symposia in pure mathematics*. Vol. 67. Providence, RI; American Mathematical Society; 1998. 1999, pp. 249–284.

[Wit88] E. Witten. “The index of the dirac operator in loop space”. In: *Elliptic Curves and Modular Forms in Algebraic Topology*. Springer, 1988, p. 161.

[Zie18] M. Zielenkiewicz. “Residue formulas for push-forwards in equivariant cohomology: a symplectic approach”. In: *Journal of Symplectic Geometry* 16.5 (2018), pp. 1455–1480.