

Almost global existence for some Hamiltonian PDEs on manifolds with globally integrable geodesic flow

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Abstract

In this paper we prove an abstract result of almost global existence for small and smooth solutions of some semilinear PDEs on Riemannian manifolds with globally integrable geodesic flow. Some examples of such manifolds are Lie groups (including flat tori), homogeneous spaces and rotational invariant surfaces. As applications of the abstract result we prove almost global existence for a nonlinear Schrödinger equation with a convolution potential and for a nonlinear beam equation. We also prove H^s stability of the ground state in NLS equation. The proof is based on a normal form procedure and the combination of the arguments used in Bambusi and Langella (2022 arXiv:2202.04505) to

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bound the growth of Sobolev norms in linear systems and a generalization of the arguments in Bambusi *et al* (2024 *Commun. Math. Phys.* **405** 253–85).

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1. Introduction

Given a Hamiltonian PDE on a compact Riemannian manifold, a classical problem is the determination of the time of existence of its solutions and the possible existence of solutions enforcing transfer of energy from low to high frequency modes. Here we deal with the problem of proving that, given an initial datum with s -Sobolev norm of size $\epsilon \ll 1$, the corresponding solution exists for times of order ϵ^{-r} , $\forall r$, provided that s is large enough. Furthermore its s -Sobolev norm remains of order ϵ over this time scale. Such results are usually known as results of ‘almost global existence’.

It is known since almost 20 years [Bam03, BG06] how to prove almost global existence in semilinear Hamiltonian PDEs in *space dimension 1* (see also [Gré07, Bam08, FG13, BMP20, FM23]), and recently also the theory of quasilinear Hamiltonian PDEs in one space dimension has achieved a satisfactory form [Del12, BD18, BMM24] (see also [FI21]). On the other hand, in space dimension larger than 1, it is only known how to deal with equations having small divisors with the same structure as in dimension 1, like on Zoll Manifolds [BDGS07, Del15] or in the case of the NLS on square tori [BG03, BG06] (see also [FGL13]) and the quantum harmonic oscillator [GIP09]. Recently some of the results of [BG03, BG06] have been extended to general tori [BFM24] on which the small divisors have the structure generally expected in higher dimensional PDEs. Concerning more general manifolds, one can find many partial results in which existence and smoothness of small solutions are proven for times of order ϵ^{-r} with *some* $r > 1$, typically smaller or equal than 2 (see [FZ10, Ime16, DI17, IP19, BFGI21, BFF21, FM22, FIM22, BFP23, FGI23]).

In the present paper we investigate the possibility of extending the result of [BFM24] to general manifolds: in particular we aim at stepping away from tori and Zoll manifolds. As a result, we prove almost global existence for some equations on a class of manifolds in which the Laplacian is a ‘globally integrable quantum system’ [BL22] (see section 2 for a precise definition). Some concrete examples of these manifolds are flat tori, compact Lie groups, homogeneous spaces, rotation invariant surfaces and also products of these kinds of manifolds. Indeed, we prove an abstract theorem of almost global existence for semilinear PDEs on these manifolds and subsequently we apply it to some concrete models. Precisely, we prove almost global existence for NLS with a ‘potential’ which is a spectral multiplier and for the beam equation. We also apply our abstract theorem to the problem of H^s stability of the ground state in NLS and prove that it is stable for times of order $\epsilon^{-r} \forall r$. We emphasize that all these results were only known for equations on tori [BFM24]. As explained below, in the present paper we combine some of the key ideas of [BFM24] and of [DS06, BL22] with some other new ingredients in order to get the result.

A related line of research, parallel to the one pursued here, is the so-called KAM theory for PDEs, which aims to construct quasiperiodic solutions of the considered equations. As for almost global existence, KAM theory for PDEs is nowadays quite well understood, for both semilinear and quasilinear equations, in space dimension 1. Without trying to be exhaustive, we mention, among others, the seminal works [Kuk87, Kuk93, Way90, Po96, BBM14, BBM16] for a first reading in this wide field. Concerning the higher dimensional domains, which are our main focus, only the cases of tori [Bou04, EK10, GXY11, BB13, PP15, EGK16] and compact Lie groups or homogeneous space [BP11, BCP15] have been completely treated. We think that techniques similar to the ones developed in this work, combined with the so-called multiscale analysis [Bou04], could be used to prove KAM-type results for rotation invariant surfaces. This is left for future developments.

Finally, we recall the results in [Bou96, Bou05, BMP21, BMP23, NG07] and references therein (see also [BGR24]). In these works some invariant Lagrangian tori filled by almost

periodic solutions have been constructed for some semilinear PDEs. It would be very interesting to generalize these results to PDEs on manifolds with globally integrable geodesic flow.

Scheme of the proof. As already mentioned, the starting point of the present paper is [BFM24]. So we first recall the strategy of [BFM24]. The idea is to look for a canonical transformation that puts the system in a normal form which is a variant of the classical Birkhoff normal form. More in detail, the idea of [BFM24] is to split the Fourier modes into low and high frequency modes, to work in the standard way on the low modes and, concerning the high modes to exploit a cluster decomposition of \mathbb{Z}^d (indexes of the Fourier modes) introduced by Bourgain. Such a decomposition has the property that if $k, l \in \mathbb{Z}^d$ belong to different clusters then

$$||k|^2 - |l|^2| + |k - l| \geq C (|k|^\delta + |l|^\delta) ,$$

for some $C > 0$ and $\delta \in (0, 1)$. This property allows to construct a normal form which ensures that there is no exchange of energy among different clusters of modes of high frequency. In [BFM24] this was exploited altogether with a nonresonance assumption on the frequencies which is typically fulfilled in any space dimension, this allows us to show also that the low modes do not exchange energy among them and with the high modes and to conclude that one can control the Sobolev norm of the solution up to the considered times.

The main difficulty in generalizing the above idea to general manifolds is that in general manifolds Fourier series are not available: one has to identify some basis of $L^2(M)$ which has properties similar to the Fourier basis.

Such a basis was identified in [BL22] for some manifolds in which the Laplacian is a ‘globally integrable quantum system’, namely it is a function of some commuting operators which play the role of quantum actions. In turn the quantum actions I_j are operators whose spectrum is of the form $\{n + \kappa\}_{n \in \mathbb{Z}}$ with some $\kappa \in [0, 1)$ (see section 2 for a more precise definition). Thus the spectrum of the Laplacian turns out to be given by

$$\sigma(-\Delta_g) = \{h_L(a^1, \dots, a^d) \mid (a^1, \dots, a^d) \in \Lambda \subset \mathbb{Z}^d\} ,$$

with h_L a function. Correspondingly, the eigenfunctions of the Laplacian are labelled by discrete indexes $(a^1, \dots, a^d) \in \Lambda \subset \mathbb{Z}^d$. Such eigenfunctions are the objects that substitute the exponentials of the torus.

Once the basis is identified, one has to study its properties. It is known that the main property needed to develop perturbation theory is a bilinear estimate involving the expansion of the product of two eigenfunctions on the basis of the eigenfunctions themselves. The main idea is that we do not require such an expansion to *decay with the difference of the corresponding eigenvalues, but with the distance of the indexes labeling them!* Once one realizes this fact, it is easy to obtain the wanted estimate by working as in [DS06, Bam08].

After proving the estimate of the product of eigenfunctions, we have to study the composition properties of the polynomials obtained by expanding the nonlinearity on the eigenfunctions. This is very similar to what is done in [DS06, Bam08]. However, these computations are technically quite heavy in our setting, since we are dealing with a vectorial labelling of eigenfunctions instead of a scalar one. All the needed estimates are performed in appendix A.

After this is done one still has to verify a nonresonance condition which is more or less standard: this is done following the ideas of [DS04]. Finally one concludes the proof as in [BFM24].

Before closing this section, we remark that at present our method is restricted to systems whose linear parts have frequencies $\{\omega_a\}_{a \in \Lambda}$, with $\omega_a \sim |a|^\beta$ and $\beta > 1$ in the sense that there are no examples with $\beta \leq 1$ in which Bourgain partition is known to exist. Thus, the case of the

wave equation is not covered. Moreover, our techniques apply only to the case of semilinear equations, namely, perturbations involving derivatives are not allowed by our theory. We think that this second issue can be overcome by developing some tools of the kind of those used in [BD18, BL22], while the first requires new ideas.

2. Statement of the abstract theorem

Let (M, g) be a compact Riemannian manifold of dimension n , denote by $\Psi^m(M)$ the space of pseudodifferential operators *à la* Hörmander of order m on M (see [Hö85]).

Let $H^s(M) = H^s(M; \mathbb{C})$, $s \geq 0$ be the standard Sobolev spaces on M . We shall study equations of the form

$$i\dot{u} = H_L u + \nabla_{\bar{u}} P(u, \bar{u}), \tag{2.1}$$

where H_L is a linear selfadjoint operator on which we are going to make several assumptions and P is a nonlinear functional that we describe below. Here $\nabla_{\bar{u}}$ is the gradient of P with respect to the variable \bar{u} and the L^2 metric.

We remark that the system (2.1) is Hamiltonian with Hamiltonian function

$$H = H_0 + P,$$

where

$$H_0(u) := \int_M \bar{u} H_L u \, dx. \tag{2.2}$$

2.1. Assumptions on the linear system

We are going to assume that H_L is a *globally integrable quantum system*, as introduced in [BL22]. Roughly speaking, it is a linear operator which is a function of some first-order pseudodifferential operators that we will call *quantum actions*. The precise definitions are recalled below.

Definition 2.1 (System of Quantum Actions). Let $\{I_j\}_{j=1, \dots, d}$ be d selfadjoint pseudodifferential operators of order 1, fulfilling

- i. $I_j \in \Psi^1(M)$, for any $j = 1, \dots, d$;
- ii. $[I_i, I_j] = 0$, for any $i, j = 1, \dots, d$;
- iii. there exists a constant $c_1 > 0$ such that $c_1 \sqrt{1 - \Delta_g} \leq \sqrt{1 + \sum_{j=1}^d I_j^2}$ in the sense of quadratic forms;
- iv. there exists $\kappa \in [0, 1)^d$ such that the joint spectrum Λ of the operators I_1, \dots, I_d fulfills

$$\Lambda \subset \mathbb{Z}^d + \kappa. \tag{2.3}$$

We refer to (I_1, \dots, I_d) as the *quantum actions*.

Remark 2.2. By iii., the operator $\mathbf{1} + \sum_{j=1}^d I_j^2$ has compact inverse, therefore the spectrum $\sigma(I_j)$ of each one of the I_j 's is pure point and formed by a sequence of eigenvalues.

Remark 2.3. We recall that the joint spectrum Λ of the operators I_j is defined as the set of the $a = (a^1, \dots, a^d) \in \mathbb{R}^d$ s.t. there exists $\psi_a \in \mathcal{H}$ with $\psi_a \neq 0$ and

$$I_j \psi_a = a^j \psi_a, \quad \forall j = 1, \dots, d. \tag{2.4}$$

Definition 2.4 (Globally integrable quantum system). A linear selfadjoint operator H_L will be said to be the Hamiltonian of a *globally integrable quantum system* if there exists a function $h_L : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$H_L = h_L(I_1, \dots, I_d),$$

where the operator function is spectrally defined.

Remark 2.5. Systems fulfilling definition 2.4 with the further property that the multiplicity of common eigenvalues (a^1, \dots, a^d) of the actions is 1 were called toric integrable quantum systems in [TZ02].

If $H_L = h_L(I_1, \dots, I_d)$ is a globally integrable quantum system, then its eigenvalues are

$$\omega_a := h_L(a^1, \dots, a^d) \equiv h_L(a), \quad a \equiv (a^1, \dots, a^d) \in \Lambda.$$

We denote by $\Sigma = \{\omega_a\}_{a \in \Lambda}$ the spectrum of H_L ; in our setting, it coincides with the set of the frequencies.

We assume the following Hypotheses on H_L :

Hypothesis L.0 (Integrability of H_L). H_L is a globally integrable quantum system.

Hypothesis L.1 (Asymptotics). There exist positive constants C_1, C_2 and β s.t.

$$C_1 |a|^\beta \leq |\omega_a| \leq C_2 |a|^\beta, \quad \forall a \in \Lambda \tag{2.5}$$

where, for a vector $a \in \mathbb{R}^d$, we denote $|a| := \sqrt{\sum_{j=1}^d (a^j)^2}$ its Euclidean norm.

Let $\{\Sigma_n\}_{n \geq 1}$ be a sequence of closed real intervals with the property that $\max\{x \in \Sigma_n\} < \min\{x \in \Sigma_{n+1}\}$ for any n and

$$\left\{ \omega_a^{1/\beta} \right\}_{a \in \Lambda} \subset \bigcup_{n \geq 1} \Sigma_n, \quad C_3 n \leq \max\{x \in \Sigma_n\} \leq C_4 n, \tag{2.6}$$

with some positive constants C_3, C_4 .

We are now going to assume a nonresonance condition which allows to use normal form theory in order to eliminate from the Hamiltonian the terms enforcing exchanges of energy among modes labelled by indexes belonging to different intervals Σ_n .

First, in order to keep into account that the system depends both on u and \bar{u} , we extend the space of the indexes $a \in \Lambda$, and consider

$$\Lambda_e := \Lambda \times \{\pm 1\} \ni (a, \sigma) \equiv A. \tag{2.7}$$

Then, given a sequence of intervals Σ_n as above, we define what we mean by set of resonant indexes.

Definition 2.6. A multi-index $\mathbf{A} \equiv (A_1, \dots, A_r) \in \Lambda_e^r$, $A_j \equiv (a_j, \sigma_j)$ is said to be *resonant* if r is even and there exists a permutation τ of $(1, \dots, r)$ and a sequence $n_1, \dots, n_{r/2}$ s.t.

$$\forall j = 1, \dots, r/2, \quad \omega_{a_{\tau(j)}}^{1/\beta}, \omega_{a_{\tau(j+r/2)}}^{1/\beta} \in \Sigma_{n_{\tau(j)}} \quad \text{and} \quad \sigma_{\tau(j)} = -\sigma_{\tau(j+r/2)}.$$

In this case, we will write $\mathbf{A} \in W$.

Hypothesis L.2 (Non-resonance). There exists a sequence of disjoint intervals Σ_n fulfilling (2.6) with the following property: for any $r \geq 3$, there are constants $\gamma, \tau > 0$ such that for any multi-index $\mathbf{A} = (A_1, \dots, A_r) \in \Lambda_e^r \setminus W$ one has

$$\left| \sum_{j=1}^r \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1, \dots, r} |a_j|)^\tau}.$$

Finally, we assume a clustering property of Bourgain’s type.

Hypothesis L.3 (Bourgain clusters). There exists a partition

$$\Lambda = \bigcup_{\alpha \in \mathfrak{A}} \Omega_\alpha$$

with the following properties.

- i. Each Ω_α is dyadic, in the sense that there exists a constant $C > 1$, independent of α , such that

$$\sup_{a \in \Omega_\alpha} |a| \leq C \inf_{a \in \Omega_\alpha} |a|.$$

- ii. There exist $\delta, C_\delta > 0$ such that, if $a \in \Omega_\alpha$ and $b \in \Omega_\beta$ with $\alpha \neq \beta$, then

$$|a - b| + |\omega_a - \omega_b| \geq C_\delta (|a|^\delta + |b|^\delta).$$

2.2. Assumption on the nonlinearity and statement

Concerning P we assume the following Hypothesis:

Hypothesis P. (1) P has the structure

$$P(u, \bar{u}) = \int_M F(N(u, \bar{u}), u(x), \bar{u}(x), x) dx, \tag{2.8}$$

where

$$N(u, \bar{u}) := \int_M u(x) \bar{u}(x) dx$$

and $F \in C^\infty(\mathcal{U} \times \mathcal{U} \times \mathcal{U} \times M; \mathbb{C})$ is a smooth function and $\mathcal{U} \subset \mathbb{C}$ is an open neighborhood of the origin.

- (2) if \bar{u} is the complex conjugate of u , then $F(N, u, \bar{u}, x) \in \mathbb{R}$.
- (3) P has a zero of order at least three at $u = 0$.

Our main result is the following:

Theorem 2.7. Consider the Hamiltonian system (2.1). Assume hypotheses L.0–L.3 and P, then for any integer $r \geq 3$, there exists $s_r \in \mathbb{N}$ such that, for any $s \geq s_r$, there are constants $\epsilon_0 > 0$, $c > 0$ and C for which the following holds: if the initial datum $u_0 \in H^s(M, \mathbb{C})$ fulfills

$$\epsilon := \|u_0\|_s < \epsilon_0,$$

then the Cauchy problem has a unique solution $u \in C^0((-T_\epsilon, T_\epsilon), H^s(M, \mathbb{C}))$ with $T_\epsilon > c\epsilon^{-r}$. Moreover, one has

$$\|u(t)\|_s \leq C\epsilon, \quad \forall t \in (-T_\epsilon, T_\epsilon). \tag{2.9}$$

2.3. Remark on the abstract theorem

We conclude the presentation of our main abstract results with a couple of comments.

The assumptions on H_L and P are deeply connected with the specific equation one wants to consider and with the geometry of the manifold on which the Equation is posed. More specifically:

- (Assumptions on P) Assumption P on the nonlinearity is not strictly necessary. Actually we prove the abstract result in a slightly more general setting, where P is a function with ‘localized coefficients’ (according to definition 4.15). This is done in theorem 5.1. Then theorem 5.2 guarantees that nonlinearities fulfilling hypothesis P belong to the class of functions with localized coefficients, and theorem 2.7 follows as a consequence of theorems 5.1 and 5.2.

Actually, as it will be clear from the proof, theorem 2.7 holds also for more general nonlinearities of the form $P(\mathcal{F}u, \mathcal{F}\bar{u})$, with \mathcal{F} a real bounded Fourier multiplier and P given by (2.8). This is relevant in order to deal with equations of second order in time.
- (Assumptions on H_L) In section 3.1 we will consider some Riemannian manifolds (M, g) on which the Laplace–Beltrami operator Δ_g is a globally integrable quantum system. Then, for these choices of the manifold M , in section 3.2 we will consider situations where H_L is a (parameter dependent) function of Δ_g . To fulfill the nonresonance relation L.2, in H_L in sections 3.2.1–3.2.3 we will suitable tune the parameters. The clustering hypothesis L.3 may follow directly from the spectral properties of Δ_g (this is the case for flat tori) or may follow from a ‘quantum Nekhoroshev theorem’ proved in [BL22], that extends the results in [BLM22b, BLM22a], according to which, if the function h_L (c.f. definition 2.4) is steep (see definition 2.9 below), then hypothesis L.3 is automatic. We state such a quantum Nekhoroshev theorem in the rest of this Section.

We first need some definitions:

Definition 2.8. A function $h_L \in C^\infty(\mathbb{R}^d; \mathbb{R})$ is said to be homogeneous of degree d at infinity if there exists an open ball $\mathcal{B}_r \in \mathbb{R}^d$, centered at the origin, such that

$$h_L(\lambda a) = \lambda^d h_L(a), \quad \forall \lambda > 1, \quad a \in \mathbb{R}^d \setminus \mathcal{B}_r.$$

For the sake of completeness, we now recall the definition of steepness [GCB16]. It is important to note that such a property is generic and it is implied by convexity. Moreover, steepness can be quite easily verified using the results of [Nie06]. For the examples treated in section 3.1 it was explicitly proved in [BL22].

Definition 2.9 (Steepness). Let $\mathcal{U} \subset \mathbb{R}^d$ be a bounded connected open set with nonempty interior. A function $h_L \in C^1(\mathcal{U})$ is said to be steep in \mathcal{U} with steepness radius τ , steepness

indices $\alpha_1, \dots, \alpha_{d-1}$ and (strictly positive) steepness coefficients B_1, \dots, B_{d-1} , if its gradient $w(a) := \frac{\partial h_L}{\partial a}(a)$ satisfies the following estimates: $\inf_{a \in \mathcal{U}} \|w(a)\| > 0$ and for any $a \in \mathcal{U}$ and for any s dimensional linear subspace $E \subset \mathbb{R}^d$ orthogonal to $w(a)$, with $1 \leq s \leq d-1$, one has

$$\max_{0 \leq \eta \leq \xi} \min_{u \in E: \|u\|=1} \|\Pi_M w(a + \eta u)\| \geq B_s \xi^{\alpha_s}, \quad \forall \xi \in (0, r],$$

where Π_E is the orthogonal projector on E . The quantities u and η are also subject to the limitation $a + \eta u \in \mathcal{U}$.

The following theorem holds:

Theorem 2.10. *Let $H_L = h_L(I_1, \dots, I_d)$ a globally integrable quantum system. If h_L is smooth and homogeneous at infinity of degree $\beta > 1$ (see definition 2.8) and steep (see definition 2.9), then hypothesis L.3 is satisfied.*

We will prove theorem 2.10 in the appendix C, as a Corollary of theorem 8.28 of [BL22].

3. Applications

In this section, we collect various applications which will be a consequence of the abstract theorem 2.7. As anticipated in the Introduction, we are interested in understanding in which manifolds one can prove results of almost global existence. We start by presenting in section 3.1 three examples of manifolds for which Hypothesis L.3 holds for the Laplace–Beltrami operator; then, in section 3.2, we consider some explicit PDEs on these manifolds. For these Equations, we prove almost global existence.

3.1. Manifolds

We now exhibit three specific examples of compact Riemannian manifolds (M, g) where the Laplace–Beltrami operator $-\Delta_g$ fulfills hypotheses L.0 and L.3, namely $-\Delta_g$ is a quantum integrable system with Bourgain clustering of the eigenvalues. In fact, all the examples treated in this section are globally integrable quantum systems in which H_L is a function of the Laplacian. For simplicity, we will use the notation $\Delta_g = \Delta$.

1. **Flat tori.** Given a basis u_1, \dots, u_d of \mathbb{R}^d we define a maximal dimensional lattice $\Gamma \subset \mathbb{R}^d$ by

$$\Gamma := \left\{ x = \sum_{j=1}^d m_j u_j, \quad m_j \in \mathbb{Z} \right\}$$

and the corresponding maximal dimensional torus $T_\Gamma^d := \mathbb{R}^d / \Gamma$. By using in \mathbb{R}^d the basis u_j , one is reduced to the standard torus \mathbb{T}^d endowed by a flat metric. Then the actions are given by the operators $-i\partial_j$. Bourgain’s clustering property holds for the eigenvalues of the Laplacian, as proved in [Bou04, BM19].

2. **Rotation invariant surfaces.** Consider a real analytic function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, invariant by rotations around the z axis, and assume it is a submersion at $f(x, y, z) = 1$. Denote by M the level surface $f(x, y, z) = 1$, assume it is diffeomorphic to a sphere and endow it by the metric g induced by the embedding. We need a further assumption that we are now going to specify. First introduce coordinates in M as follows: let N and S be the north and the south

poles (intersection of M with the z axis) and denote by $\theta \in [0, L]$ the curvilinear abscissa along the geodesic given by the intersection of M with the xz plane; we orient it as going from N to S and consider also the cylindrical coordinates (r, ϕ, z) of \mathbb{R}^3 , then the coordinates $(\theta, \phi) \in (0, L) \times (0, 2\pi)$ can be used as coordinates on M . Then on M the distance r from the z axis is a function of θ . We assume that the function $r(\theta)$ has only one critical point $\theta_0 \in (0, L)$. The fact that the Laplacian is a globally integrable quantum system was proved by Colin de Verdière [CdV80]. The property of Bourgain’s clustering was already proved by Delort [Del09] (see also [BL22]).

3. **Lie Groups and Homogeneous spaces.** If M is a compact, simply connected Lie group, then property L.0 of the Laplacian of M was proved in detail in [BL22]. Property L.3 was proved in [BP11, BCP15] (see also [BL22]). The extension to homogeneous spaces can be obtained by using the same tools used in [BL22] for compact Lie groups (see [BFM24]).

From now on we assume that M is such that Δ on M is a globally integrable quantum system.

3.2. Models

We now give three concrete examples of equations where M is one of the previous manifolds, and H_L is a function of the Laplace–Beltrami operator $-\Delta$ on M .

3.2.1. *Schrödinger equation with spectral multiplier.* Our first result concerns a nonlinear Schrödinger equation with a spectral multiplier. To define it, we start by defining the spectral projector.

Definition 3.1. For $a \equiv (a^1, \dots, a^d) \in \Lambda$,

$$\Pi_a := \Pi_{a^1} \dots \Pi_{a^d}$$

where Π_{a^j} is the orthogonal projector on the eigenspace of I_j with eigenvalue a^j .

Given $u \in L^2(M, \mathbb{C})$, consider its spectral decomposition,

$$u = \sum_{a \in \Lambda} \Pi_a u,$$

and let $V = \{V_a\}_{a \in \Lambda}$ be a real valued sequence; correspondingly, we define a spectral multiplier by

$$V_{\sharp} u := \sum_a V_a \Pi_a u. \tag{3.1}$$

In the following, we will assume that V belongs to the space

$$\mathcal{V}_n := \left\{ V = \{V_a\}_{a \in \Lambda} : V_a \in \mathbb{R}, |V_a| \langle a \rangle^n \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}, \tag{3.2}$$

for some $n > d/2$, where d is the dimension of M , and we endow \mathcal{V}_n by the product measure. Consider the Cauchy problem

$$\begin{cases} i\partial_t \psi = -\Delta \psi + V_{\sharp} \psi + f(x, |\psi|^2) \psi, & x \in M, \\ \psi(0) = \psi_0 \end{cases} \tag{3.3}$$

where $V \in \mathcal{V}_n$ and the nonlinearity f is of class $C^\infty(M \times \mathcal{U}, \mathbb{R})$, $\mathcal{U} \subset \mathbb{R}$ being a neighborhood of the origin, and fulfills $f(x, 0) = 0$.

Theorem 3.2. *There exists a set $\mathcal{V}^{(\text{res})} \subset \mathcal{V}_n$ of zero measure, s.t., if $V \in \mathcal{V}_n \setminus \mathcal{V}^{(\text{res})}$ the following holds. For any $r \in \mathbb{N}$, there exists $s_r > d/2$ such that for any $s > s_r$, there is $\epsilon_s > 0$ and $C > 0$ such that if the initial datum for (3.3) belongs to H^s and fulfills $\epsilon := |\psi_0|_s < \epsilon_s$ then*

$$\|\psi(t)\|_s \leq C\epsilon, \quad \text{for all } |t| \leq C\epsilon^{-r}.$$

3.2.2. Sobolev stability of ground state for NLS equation. Our second result is the long time Sobolev stability of the ground state solution of the nonlinear Schrödinger equation

$$i\dot{\psi} = -\Delta\psi + f(|\psi|^2)\psi, \tag{3.4}$$

with $f \in C^\infty(\mathcal{U}; \mathbb{R})$, $\mathcal{U} \subset \mathbb{R}$ an open neighborhood of the origin, and f having a zero of order at least one at the origin. It is well known that for any $p_0 \in \mathcal{U} \cap \mathbb{R}^+$ equation (3.4) has a solution given by a plane wave of the form

$$\psi_*(t) = \sqrt{p_0}e^{-i\nu t},$$

provided $\nu = f(p_0)$. Denote by $\bar{\lambda}$ the lowest non vanishing eigenvalue of $-\Delta$, then we will prove the following result.

Theorem 3.3. *Assume there exists $\bar{p}_0 > 0$ such that $\bar{\lambda} + 2f(p_0) > 0$ for any $p_0 \in (0, \bar{p}_0]$. Then there exists a zero measure set $\mathcal{P} \subset (0, \bar{p}_0]$ such that if $p_0 \in (0, \bar{p}_0] \setminus \mathcal{P}$ then for any $r \in \mathbb{N}$ there exists s_r for which the following holds. For any $s \geq s_r$, there exist constants ϵ_0 and C such that if the initial datum ψ_0 fulfills*

$$\|\psi_0\|_0^2 = p_0, \quad \inf_{\alpha \in \mathbb{T}} \|\psi_0 - \sqrt{p_0}e^{-i\alpha}\|_s = \epsilon \leq \epsilon_0,$$

then the corresponding solution fulfills

$$\inf_{\alpha \in \mathbb{T}} \|\psi(t) - \sqrt{p_0}e^{-i\alpha}\|_s \leq C\epsilon \quad \forall |t| \leq C\epsilon^{-r}.$$

Remark 3.4. Note that if f is a positive function (the so-called *defocusing* case) there is no restriction in the L^2 norm of the initial datum, since $\lambda + 2f(p_0) > 0$ for any p_0 .

3.2.3. Beam equation. The third result concerns the beam equation

$$\psi_{tt} + \Delta^2\psi + m\psi = -\partial_\psi F(x, \psi), \tag{3.5}$$

with $F \in C^\infty(M \times \mathcal{U})$, $\mathcal{U} \subset \mathbb{R}$ being a neighbourhood of the origin, and $m > 0$ a real positive parameter that we will call mass. We will assume F to have a zero of order at least 3 at $\psi = 0$. The precise statement of the main theorem is the following one.

Theorem 3.5. *There exists a set of zero measure $\mathcal{M}^{(\text{res})} \subset \mathbb{R}^+$ such that if $m \in \mathbb{R}^+ \setminus \mathcal{M}^{(\text{res})}$ then for all $r \in \mathbb{N}$ there exist $s_r > d/2$ such that the following holds. For any $s > s_r$ there exist $\epsilon_s, C > 0$ such that if the initial datum for (3.5) fulfills*

$$\epsilon := \left\| \left(\psi_0, \dot{\psi}_0 \right) \right\|_s := \|\psi_0\|_{H^{s+2}} + \|\dot{\psi}_0\|_{H^s} < \epsilon_s,$$

then the corresponding solution satisfies

$$\left\| \left(\psi(t), \dot{\psi}(t) \right) \right\|_s \leq C\epsilon, \quad \text{for } |t| \leq C\epsilon^{-r}.$$

Remark 3.6. The result holds also for the case of Hamiltonian nonlinearities which are functions also of the first and second derivatives of ψ .

4. Functional setting

4.1. Phase space

Let M be an arbitrary manifold and assume that there exists a system of quantum actions (I_1, \dots, I_d) (see definition 2.1); correspondingly we define the spectral projectors Π_a on points a of their joint spectrum as in definition 3.1.

Definition 4.1. For any $s \geq 0$, the space $\mathcal{H}^s := \mathcal{H}^s(M)$ is the space of the functions $u \in L^2(M, \mathbb{C})$ s.t.

$$\|u\|_s^2 := \sum_{a \in \Lambda} (1 + |a|)^{2s} \|\Pi_a u\|_0^2 < \infty. \tag{4.1}$$

For $s < 0$ \mathcal{H}^s is the completion of L^2 in the norm (4.1).

Remark 4.2. By (i.) and (iii.) of definition 2.1, for any s the norm (4.1) is equivalent to the norm

$$\left\| (\mathbf{1} - \Delta)^{\frac{s}{2}} u \right\|_{L^2(M, \mathbb{C})},$$

so that the spaces \mathcal{H}^s of definition 4.1 are equivalent to the standard Sobolev spaces $H^s(M, \mathbb{C})$.

It is also useful to introduce the spaces

$$\mathcal{H}^{-\infty} := \bigcup_s \mathcal{H}^s, \quad \mathcal{H}^{\infty} := \bigcap_s \mathcal{H}^s.$$

In particular we remark that, by Sobolev embedding theorems, $H^{\infty} = C^{\infty}(M)$. In the following, we will work in the complex extension of the phase space, which amounts to considering \bar{u} as a variable independent of u , where the bar denotes the complex conjugate.

Definition 4.3. For $s \in \mathbb{R}$, define $\mathcal{H}_e^s := \mathcal{H}_e^s(M) := \mathcal{H}^s(M) \times \mathcal{H}^s(M)$. For $u \equiv (u_+, u_-) \in \mathcal{H}_e^s$ we will use the norm

$$\|u\|_{\mathcal{H}_e^s(M)}^2 \equiv \|(u_+, u_-)\|_{\mathcal{H}_e^s(M)}^2 := \|u_+\|_s^2 + \|u_-\|_s^2.$$

Correspondingly, for $u \in \mathcal{H}_e^s$ and $A \equiv (a, \sigma) \in \Lambda_e$, we define

$$\Pi_A u := \Pi_{(a, \sigma)}(u_+, u_-) := \Pi_a u_{\sigma},$$

where Π_a is given in definition 3.1. For $u \in \mathcal{H}_e^s$ one has again the spectral decomposition

$$u = \sum_{A \in \Lambda_e} \Pi_A u.$$

For $A \in \Lambda_e$ we will denote

$$|A| \equiv |(a, \sigma)| := |a|.$$

Given an element in \mathcal{H}_e^s , we define theinvolution that we use in order to identify the subspace of ‘real functions’, on which $u_+ = \bar{u}_-$.

Definition 4.4. Let $u \equiv (u_+, u_-) \in \mathcal{H}_e^{-\infty}$, we define

$$I(u) := (\bar{u}_-, \bar{u}_+)$$

with the bar denoting the complex conjugate. If $I(u) = u$ we will say that u is *real*.

Correspondingly, it is useful to define, for $A \in \Lambda_e$,

$$\bar{A} \equiv \overline{(a, \sigma)} := (a, -\sigma),$$

so that one has

$$\Pi_A(I(u)) = \overline{\Pi_{\bar{A}}u}.$$

Definition 4.5. We will denote the ball centered at the origin of \mathcal{H}_e^s of radius R by

$$\mathcal{B}_R^s := \{u \in \mathcal{H}_e^s : \|u\|_s < R\}.$$

4.2. Hamiltonian structure

Given a function $H \in C^\infty(\mathcal{O}, \mathbb{C})$ with $\mathcal{O} \subset \mathcal{H}_e^s$ open for some s , we define the corresponding Hamiltonian vector field by

$$\dot{u}_+ = i\nabla_{u_-}H, \quad \dot{u}_- = -i\nabla_{u_+}H$$

where ∇_{u_\pm} is the L^2 -gradient with respect to u_\pm . Namely, it is defined by the following identity:

$$d_{u_+}Hh_+ = \langle \nabla_{u_+}H, h_+ \rangle, \quad \forall h_+ \in \mathcal{H}^\infty(M),$$

and similarly for ∇_{u_-} . We will denote the vector field associated to H by

$$X_H := (i\nabla_{u_-}H, -i\nabla_{u_+}H). \tag{4.2}$$

The Poisson brackets of two functions are defined as follows.

Definition 4.6. Given two functions $f, g \in C^\infty(\mathcal{O}, \mathbb{C})$, with \mathcal{O} as above, we define their Poisson brackets by

$$\{f, g\}(u) := df(u)X_g(u).$$

Remark 4.7. In general, the Poisson brackets of two functions can be ill-defined. In the framework developed in the next paragraph, this fact does not happen and Poisson brackets will always be well defined, as guaranteed by lemma 4.23.

4.3. Polynomials with localized coefficients

In order to prove the stability result of theorem 2.7, we shall work with nonlinearities P more general than those allowed by hypothesis P: they belong to a class which is a generalization of the *polynomials with localized coefficients* introduced in [DS06, Bam08]. The generalization rests in the fact that the indexes labeling the components of a polynomial are here vectors $a \in \Lambda$ instead of scalar coefficients. This complicates considerably the proof of the fundamental lemmas.

First, we recall that given a polynomial function P of degree r there exists a unique r -linear symmetric function \tilde{P} such that

$$P(u) = \tilde{P}(u, \dots, u).$$

In the following, we will measure the size of an index by the size of the corresponding frequency. Precisely we give the following definition.

Definition 4.8. For any $a \in \Lambda$, we define

$$\langle |a| \rangle := \omega_a^{\frac{1}{p}}. \tag{4.3}$$

Moreover, for $A \in \Lambda_e$ we define

$$\langle |A| \rangle = \langle |(a, \sigma)| \rangle := \langle |a| \rangle. \tag{4.4}$$

Remark 4.9. By the asymptotic property of the frequency, there exist $C_1, C_2 > 0$ such that, for any $\Lambda \ni a \neq 0$, one has the equivalence

$$C_1 |a| \leq \langle |a| \rangle \leq C_2 |a|.$$

As a consequence the quantity

$$\|u\|_s^2 := \sum_{a \in \Lambda} (1 + \langle |a| \rangle)^{2s} \|\Pi_a u\|_0^2$$

controls and is controlled by the s th Sobolev norm.

Remark 4.10. In the following, we often use the notation $a \lesssim b$ meaning that there exists a constant C , independent of the relevant parameters, such that $a \leq Cb$. Moreover, we will write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

If we need to specify that the constant depends on a parameter, say for example s , we will write $a \lesssim_s b$.

Definition 4.11. Consider $r \geq 1$ and a multi-index $\mathbf{a} = (a_1, \dots, a_r) \in \Lambda^r$.

(i) We denote by τ_{ord} the permutation of $(1, \dots, r)$ with the property that

$$\langle |a_{\tau_{\text{ord}}(j)}| \rangle \geq \langle |a_{\tau_{\text{ord}}(j+1)}| \rangle, \quad \forall j = 1, \dots, r-1.$$

(ii) We define

$$\begin{aligned} \mu(\mathbf{a}) &:= \langle |a_{\tau_{\text{ord}}(3)}| \rangle, \\ S(\mathbf{a}) &:= \mu(\mathbf{a}) + |a_{\tau_{\text{ord}}(1)} - a_{\tau_{\text{ord}}(2)}|. \end{aligned}$$

We also set $\mu(\mathbf{A}) := \mu(\mathbf{a})$ and $S(\mathbf{A}) := S(\mathbf{a})$.

We are now in the position to state the localization property fulfilled by the nonlinearity. The following definition will be satisfied in the applications thanks to theorem B.1.

Definition 4.12 (Polynomial with localized coefficients). (i) Let $\nu \in [0, +\infty)$, $N \geq 1$. We denote by $L_r^{\nu, N}$ the class of the polynomials F homogeneous of degree r , such that there exists C_N s.t.

$$\left| \tilde{F}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r) \right| \leq C_N \frac{\mu(\mathbf{A})^{\nu+N}}{S(\mathbf{A})^N} \|\Pi_{A_1} u_1\|_0 \dots \|\Pi_{A_r} u_r\|_0, \quad (4.5)$$

$$\forall u_1, \dots, u_r \in \mathcal{H}_e^\infty, \quad \forall \mathbf{A} \equiv (A_1, \dots, A_r) \in \Lambda_e^r.$$

The smallest possible constant C_N such that (4.5) holds defines a norm in $L_r^{\nu, N}$, precisely

$$\|F\|^{\nu, N} := \sup_{\|u_1\|_0=1, \dots, \|u_r\|_0=1} \sup_{A_1, \dots, A_r} |\tilde{F}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r)| \frac{S(\mathbf{A})^N}{\mu(\mathbf{A})^{N+\nu}}.$$

(ii) We say that a polynomial F has *localized coefficients* if there exists $\nu \in [0, +\infty)$ and N_0 , such that, for any $N \geq N_0$ one has $F \in L_r^{\nu, N}$. In this case we write $F \in L_r := \cup_{N_0 \geq 1} \cup_{\nu \geq 0} \cap_{N \geq N_0} L_r^{\nu, N}$.

For non homogeneous polynomials, we use the following notation.

Definition 4.13. Let r, \bar{r} be s.t. $r < \bar{r}$, then we define the space

$$L_{r, \bar{r}}^{\nu, N} := \bigoplus_{l=r}^{\bar{r}} L_l^{\nu, N}.$$

Definition 4.14. For $F \in L_r^{\nu, N}$ we define

$$\|F\|_R^{\nu, N} := \|F\|^{\nu, N} R^r.$$

For $F \in L_{r, \bar{r}}^{\nu, N}$, so that $F = \sum_{l=r}^{\bar{r}} F_l$ with $F_l \in L_l^{\nu, N}$, we define

$$\|F\|_R^{\nu, N} := \sum_{l=r}^{\bar{r}} \|F_l\|_R^{\nu, N}.$$

We also need the following definition:

Definition 4.15 (Function with localized coefficients). Let $s_0 > 0$ and consider a function $F \in C^\infty(\mathcal{O}; \mathbb{C})$, with $\mathcal{O} \subset \mathcal{H}_e^{s_0}$ an open neighborhood of the origin. F is said to have localized coefficients if both the following properties hold:

- all the monomials of the Taylor expansion of F at the origin have localized coefficients.
- For any $s > 0$ large enough there exists an open neighborhood of the origin $\mathcal{O}_s \subset \mathcal{H}_e^s$ s.t. X_F belongs to $C^\infty(\mathcal{O}_s, \mathcal{H}_e^s)$.

Definition 4.16. A function $F \in C^\infty(\mathcal{O}; \mathbb{C})$, with $\mathcal{O} \subset \mathcal{H}_e^{s_0}$ an open neighborhood of the origin will be said to be real for real u if $F(u) \in \mathbb{R}$ whenever $u = I(u)$ (see definition 4.4).

In theorem 5.2 we will show that a non-linearity of the form described in hypothesis P is a function with localized coefficients.

It is also useful to extend the above definition to polynomial maps taking value in \mathcal{H}_e^s . Given a polynomial map X of degree r , we still denote with \tilde{X} the unique r -linear symmetric function such that

$$X(u) = \tilde{X}(u, \dots, u).$$

Definition 4.17. Let $X : \mathcal{H}_e^\infty \rightarrow \mathcal{H}_e^{-\infty}$ be a polynomial map of degree r and let \tilde{X} be the associated multilinear form.

- (i) Let $\nu \in [0, +\infty)$ and $N \geq 1$. We denote by $M_r^{\nu, N}$ the space of r -homogeneous polynomial maps such that there exists $C_N > 0$ such that

$$\begin{aligned} \|\Pi_B \tilde{X}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r)\|_0 &\leq C_N \frac{\mu(B, \mathbf{A})^{\nu+N}}{S(B, \mathbf{A})^N} \|\Pi_{A_1} u_1\|_0 \dots \|\Pi_{A_r} u_r\|_0, \\ \forall u_1, \dots, u_r \in \mathcal{H}_e^\infty, \quad \forall (B, \mathbf{A}) \in \Lambda_e \times \Lambda_e^r. \end{aligned}$$

Here we denoted by (B, \mathbf{A}) the multi-index (B, A_1, \dots, A_r) . The smallest possible constant C_N defines a seminorm, namely

$$\|X\|^{\nu, N} = \sup_{\|u_1\|_0=1, \dots, \|u_r\|_0=1} \sup_{A_1, \dots, A_r, B} \|\tilde{X}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r)\|_0 \frac{S(\mathbf{A}, B)^N}{\mu(\mathbf{A}, B)^{N+\nu}}.$$

We also define

$$\|X\|_R^{\nu, N} := \|X\|^{\nu, N} R^r.$$

- (ii) We say that a map X has localized coefficients if there is $\nu \geq 0$ such that for any N one has $F \in M_r^{\nu, N}$, and we write $X \in M_r := \cup_{\nu \geq 0} \cap_{N \geq 1} M_r^{\nu, N}$.

The following result is proven in appendix A:

Lemma 4.18. Let $F \in L_{r+1}^{\nu, N}$, then $X_F \in M_r^{\nu, N}$. Furthermore,

$$\|X_F\|^{\nu, N} \leq r \|F\|^{\nu, N} \tag{4.6}$$

and therefore

$$\|X_F\|_R^{\nu, N} \leq \frac{r}{R} \|F\|_R^{\nu, N}.$$

Remark 4.19. By the very definition of the property of localization of coefficients it is clear that any (finite) linear combination of functions or maps with localized coefficients has localized coefficients.

4.3.1. General properties. In this subsection, we state the main properties of polynomials with localized coefficients introduced in definition 4.12. The proofs of all these results will be given in appendix A.

Theorem 4.20 (Tame estimate). *Let $X \in \mathcal{M}_r^{\nu,N}$ and fix $s > \frac{3}{2}d + \nu$. If $N > d + s$, for any $s_0 \in (\frac{3}{2}d + \nu, s)$ one has*

$$\left\| \tilde{X}(u_1, \dots, u_r) \right\|_s \lesssim_{s,s_0,N} \|X\|^{\nu,N} \sum_{j=1}^r \|u_j\|_s \prod_{k \neq j} \|u_k\|_{s_0} \quad \forall u_1, \dots, u_r \in \mathcal{H}_e^\infty. \quad (4.7)$$

Corollary 4.21. *Let F be a polynomial function with localized coefficients, then (4.7) holds for its Hamiltonian vector field.*

Corollary 4.22. *Let $X \in \mathcal{M}_r^{\nu,N}$ and let $s > \frac{3}{2}d + \nu$. If $N > d + s$, for any $s_0 \in (\frac{3}{2}d + \nu, s)$, one has*

$$\sup_{\|u\|_s \leq R} \|X(u)\|_s \lesssim_{s,s_0,N} \|X\|_R^{\nu,N}. \quad (4.8)$$

The following result ensures that the class of polynomials with localized coefficients is closed with respect to Poisson brackets:

Lemma 4.23 (Poisson brackets). *Given $F \in L_{r_1+1}^{\nu_1,N}$ and $X \in M_{r_2}^{\nu_2,N}$, we have*

$$dFX \in L_{r_1+r_2}^{\nu',N'}$$

with $N' = N - d - 1 - \max\{\nu_1, \nu_2\}$ and $\nu' = \nu_1 + \nu_2 + d + 1$. Moreover,

$$\|dFX\|^{\nu',N'} \lesssim \|F\|^{\nu_1,N} \|X\|^{\nu_2,N}.$$

Corollary 4.24. *Let $F \in L_{r_1,r_2}^{\nu,N}$ and $G \in L_{r'_1,r'_2}^{\nu',N'}$, then one has $\{F; G\} \in L_{r_1+r'_1-2, r_2+r'_2-2}^{\nu'',N''}$ with*

$$\|\{F; G\}\|_R^{\nu'',N''} \lesssim \frac{1}{R^2} \|F\|_R^{\nu,N} \|G\|_R^{\nu',N'},$$

$N'' = \min\{N, N'\} - d - 1 - \max\{\nu, \nu'\}$, and $\nu'' = \nu + \nu' + d + 1$.

4.3.2. High and low modes. In the definition of the normal form we will distinguish between low and high modes. To this end, we fix some large K with the following property:

$$\forall a, b \in \text{As.t. } \langle |a| \rangle < K \text{ and } \langle |b| \rangle \geq K, \quad \omega_a \in \Sigma_n \Rightarrow \omega_b \notin \Sigma_n \quad (4.9)$$

$\forall n \in \mathbb{N}$, where $\{\Sigma_n\}_n$ are the intervals of assumption L.2.

Definition 4.25. For $K > 1$, as above, we define

$$u^{\leq} = \Pi^{\leq} u := \sum_{\{A: \langle |A| \rangle \leq K\}} \Pi_A u,$$

$$u^\perp = \Pi^\perp u := \sum_{\{A: \langle |A| \rangle > K\}} \Pi_A u.$$

Definition 4.26. A polynomial F of degree r is of order $k \leq r$ in the high modes u^\perp if, $\forall \lambda > 0$ one has

$$F(u^\leq + \lambda u^\perp) = \lambda^k F(u^\leq + u^\perp). \quad (4.10)$$

Using theorem 4.20 and noticing that

$$\|\Pi^\perp u\|_{s_0} \leq \frac{1}{K^{s-s_0}} \|\Pi^\perp u\|_s,$$

one has the following result, which is proven in section A.2 of the appendix:

Corollary 4.27. Let $F \in L_{r+1}^{\nu, N}$.

i) If F is of order at least three in u^\perp then, for every $s_0 \in (\frac{3}{2}d + \nu, s)$, we have

$$\sup_{\|u\|_s \leq R} \|X_F(u)\|_s \lesssim \frac{\|F\|_R^{\nu, N}}{K^{s-s_0}} \frac{1}{R}.$$

ii) If F is at least of order two in u^\perp then, for every $s_0 \in (\frac{3}{2}d + \nu, s)$, we have

$$\sup_{\|u\|_s \leq R} \|\Pi^\leq X_F(u)\|_s \lesssim \frac{\|F\|_R^{\nu, N}}{K^{s-s_0}} \frac{1}{R}.$$

We also have the following simple, but important corollary, which is also proven in section A.2 of the appendix:

Corollary 4.28. Let $F \in L_{r+1}^{N, \nu}$ be of order 2 in u^\perp ; assume that

$$\tilde{F}(\Pi_{A_1} u_1, \dots, \Pi_{A_{r+1}} u_{r+1}) \neq 0 \implies |a_{\tau_{\text{ord}(1)}} - a_{\tau_{\text{ord}(2)}}| > K^\delta,$$

with τ_{ord} the ordering permutation defined in 4.11. Then, $\forall N' > N$ one has

$$\|F\|^{\nu, N} \leq \frac{\|F\|^{\nu, N'}}{K^{\delta(N'-N)}}, \quad (4.11)$$

and therefore, for any s large enough,

$$\sup_{\|u\|_s \leq R} \|X_F(u)\|_s \lesssim \frac{\|F\|_R^{\nu, N'}}{K^{\delta(N'-N)}} \frac{1}{R}. \quad (4.12)$$

5. The abstract theorem and the normal form lemma

In the following two Sections we prove the stability theorem 2.7. We shall deduce the result from the normal form theorem on a class of Hamiltonian functions discussed in section 5. In section 6 we deduce some dynamical consequences of the normal form and prove the main abstract result.

In this section, we prove theorem 2.7. We proceed as follows: first, we state an almost global existence result holding for nonlinearities P with localized coefficients, namely theorem 5.1. Then we prove that all nonlinearities P satisfying assumption P have localized coefficients (this

is theorem 5.2), and as a consequence we deduce the stability theorem 2.7 thanks to theorems 5.1 and 5.2.

Consider a Hamiltonian function of the form

$$H(u) = H_0(u) + P(u), \tag{5.1}$$

$$H_0(u) = \int_M u_- H_L u_+ dx = \sum_{a \in \Lambda} \omega_a \int_M \Pi_{(a,+)} u \Pi_{(a,-)} u dx. \tag{5.2}$$

We have the following:

Theorem 5.1. *Consider the Hamiltonian (5.1), assume that ω_a fulfill the hypotheses L.1–L.3, and assume that P is a function with localized coefficients, which is real for real states and that has a zero of order at least three at the origin, then for any integer $r \geq 3$, there exists $s_r \in \mathbb{N}$ such that, for any $s \geq s_r$, there are constants $\epsilon_0 > 0$, $c > 0$ and C for which the following holds: if the initial datum $u_0 \in \mathcal{H}^s(M, \mathbb{C})$ fulfills*

$$\epsilon := \|u_0\|_s < \epsilon_0,$$

then the Cauchy problem has a unique solution $u \in C^0((-T_\epsilon, T_\epsilon), H^s(M, \mathbb{C}))$ with $T_\epsilon > c\epsilon^{-r}$. Moreover, one has

$$\|u(t)\|_s \leq C\epsilon, \quad \forall t \in (-T_\epsilon, T_\epsilon). \tag{5.3}$$

Theorem 2.7 is a consequence of theorem 5.1, in view of the following:

Theorem 5.2. *A nonlinear functional of the form (2.8) fulfilling Hypothesis P is a function with localized coefficients.*

We postpone the proof of theorem 5.2 to appendix B and we now turn to the proof of theorem 5.1. All the remaining part of the present section, as long as the subsequent section 6, is devoted to the proof of theorem 5.1. We start with stating and proving the normal form results (see lemma 5.20 and proposition 5.8), which are the heart of the proof of theorem 5.1.

In order to define what we mean by normal form we first give the following definition.

Definition 5.3 (Support). Given a homogeneous polynomial

$$P_l(u) = \sum_{\mathbf{A}=(A_1, \dots, A_l) \in \Lambda_l^e} \tilde{P}(\Pi_{A_1} u, \dots, \Pi_{A_l} u),$$

we define the support of P_l as

$$\text{supp}(P_l) = \{ \mathbf{A} = (A_1, \dots, A_l) : \exists u_1, \dots, u_l \text{ s.t. } \tilde{P}_l(\Pi_{A_1} u_1, \dots, \Pi_{A_l} u_l) \neq 0 \}.$$

If $P = \sum_{l=3}^r P_l$ is a non homogeneous polynomial we define

$$\text{supp}(P) := \bigcup_l \text{supp}(P_l).$$

We now define what we mean by normal form.

Definition 5.4 (K-Block Resonant Normal Form). A non homogeneous polynomial Z_{BR} of degree r will be said to be in *K-Block Resonant Normal Form* if

$$Z_{BR} = Z_0 + Z_B,$$

with Z_0 and Z_B of order, respectively, 0 and 2 in u^\perp and

- i. $\mathbf{A} \in \text{Supp}(Z_0)$ implies $\mathbf{A} \in W$ (see definition 2.6);
- ii. $\mathbf{A} \in \text{Supp}(Z_B)$ implies that there exists a block Ω_α such that

$$a_{\tau_{\text{ord}}(1)}, a_{\tau_{\text{ord}}(2)} \in \Omega_\alpha, \text{ and } \sigma_{\tau_{\text{ord}}(1)} \sigma_{\tau_{\text{ord}}(2)} = -1,$$

where τ_{ord} is the ordering permutation defined in definition 4.11.

Definition 5.5 (Higher Order Normal Form). A non homogeneous polynomial Z_{HO} of degree r will be said to be in *Higher Order Normal Form* if it has the structure

$$Z_{HO} = Z_2 + Z_{\geq 3},$$

with $Z_{\geq 3}$ of order at least three in u^\perp and Z_2 of order 2 in u^\perp and s.t.:

$$\mathbf{A} \in \text{supp}(Z_2) \implies |a_{\tau_{\text{ord}}(1)} - a_{\tau_{\text{ord}}(2)}| > C_\delta K^\delta.$$

The terms in Z_0 are resonant in the standard sense of perturbation theory, namely they do not enforce the exchange of energy between modes pertaining to different sets Σ_n ; the terms in Z_B do not provoke the exchange of energy between modes pertaining to different blocks Ω_α and thus they conserve the total L^2 norm of the modes of a block Ω_α ; finally, according to corollaries 4.27 and 4.28, terms in Higher Order Normal Form will be shown to have a small vector field.

Definition 5.6. A polynomial which is the sum of polynomials in normal form according to definitions 5.4 and 5.5 will be said to be in normal form.

Definition 5.7. Given R and τ we define $K(R, \tau)$ as follows: let

$$n_* := \min \left\{ n : \min \{x \in \Sigma_n\} > R^{-\frac{1}{2\tau}} \right\},$$

then we put

$$K(R, \tau) := \frac{\max \{x \in \Sigma_{n_*-1}\} + \min \{x \in \Sigma_{n_*}\}}{2}. \tag{5.4}$$

The heart of the proof of theorem 5.1 is the following:

Proposition 5.8. For any $r \geq 0$, $\exists s_r$ and $\tau > 0$ such that, $\forall s \geq s_r$, $\exists R_{s,r} > 0$, with the property that $\forall R < R_{s,r}$ and a canonical transformation

$$\mathcal{T}^{(r)} : \mathcal{B}_{R/2^{2r}}^s \rightarrow \mathcal{B}_R^s \quad \text{with} \quad \left[\mathcal{T}^{(r)} \right]^{-1} : \mathcal{B}_{R/4^{2r}}^s \rightarrow \mathcal{B}_{R/2^{2r}}^s \tag{5.5}$$

s.t.

$$H \circ \mathcal{T}^{(r)} = H_0 + Z_0 + Z_B + \mathcal{R}^{(\bar{r})}$$

with Z_0 and Z_B as in K block-resonant normal form with $K = K(R, \tau)$, see definition 5.4, and

$$\|X_{\mathcal{R}^{(\tau)}}(u)\|_s \lesssim R^{r+2}, \quad \forall u \in \mathcal{B}_R^s.$$

Moreover, we have

$$\|\Pi^{\leq} X_{Z_B}(u)\|_s \lesssim R^{r+2}, \quad \forall u \in \mathcal{B}_R^s. \quad (5.6)$$

5.1. Lie Transform

The transformation $\mathcal{T}^{(r)}$ will be constructed by the composition of Lie transforms, so we start by studying the properties of the Lie transform.

Given $G \in C^\infty(\mathcal{H}_\sigma^s, \mathbb{C})$, we denote by Φ_G^t the flow generated by the Hamilton equation $\dot{u} = X_G(u)$. From theorem 4.20 one has the following result.

Lemma 5.9. *For \bar{r} , let $3 \leq r \leq \bar{r}$, $\nu \in [0, +\infty)$, $N \geq 1$ and $G \in L_{r, \bar{r}}^{\nu, N}$. $\forall s > \frac{3}{2}d + \nu$ there is a constant $C_{\bar{r}, N, s} > 0$ such that $\forall R > 0$ satisfying*

$$\|G\|_R^{\nu, N} \leq \frac{R^2}{C_{\bar{r}, N, s}} \quad (5.7)$$

the map $\Phi_G^t : \mathcal{B}_{R/2}^s \rightarrow \mathcal{B}_R^s$ is well defined for $|t| \leq 1$, and moreover

$$\sup_{\|u\|_s < R/2} \|\Phi_G^t(u) - u\|_s \lesssim_s |t| \frac{\|G\|_R^{\nu, N}}{R}.$$

Definition 5.10. We call $\Phi_G := \Phi_G^t|_{t=1}$ the Lie transform generated by G .

Lemma 5.11. *Let $G \in L_{r, \bar{r}}^{\nu, N}$, $3 \leq r \leq \bar{r}$ and let Φ_G be the Lie transform it generates. For any $s > \frac{3}{2}d + \nu$ there exists $R_s > 0$ such that for any $F \in C^\infty(\mathcal{B}_{R_s}^s)$ satisfying*

$$\sup_{\|u\|_s \leq R_s} \|X_F(u)\|_s < \infty,$$

one has

$$\sup_{\|u\|_s \leq R/2} \|X_{F \circ \Phi_G}(u)\|_s \leq 2 \sup_{\|u\|_s \leq R} \|X_F(u)\|_s, \quad \forall R < R_s.$$

Defining

$$Ad_G^0(P) := P \quad \text{and} \quad Ad_G^k(P) := \{Ad_G^{k-1}(P), G\} \text{ for } k \geq 1$$

we have the following standard lemma.

Lemma 5.12 (Lie transform). *Let $3 \leq r \leq \bar{r}$ and $G \in L_{r, \bar{r}}^{\nu, N}$. Assume (5.7), then for any $P \in C^\infty(\mathcal{B}_{2R_s}^s, \mathbb{C})$, we have,*

$$P(\Phi_G(u)) = \sum_{k=0}^n \frac{1}{k!} (Ad_G^k P)(u) + \frac{1}{n!} \int_0^1 (1-\tau)^n (Ad_G^{n+1} P)(\Phi_G^\tau(u)) d\tau \quad (5.8)$$

for any $n \in \mathbb{N}$ and $\forall u \in \mathcal{B}_{R/2}^s$.

From the estimate of Poisson brackets (see lemma 4.23), we deduce the following result.

Lemma 5.13. *Let $G \in L_{r_1+2, \bar{r}_1+2}^{\nu, N}$ and $F \in L_{r_2, \bar{r}_2}^{\nu, N}$, with $r_1 \leq \bar{r}_1$ and $r_2 \leq \bar{r}_2$. $\forall k \geq 0$ we have $Ad_G^k(F) \in L_{r_2+k r_1, \bar{r}_2+k \bar{r}_1}^{\nu_k, N_k}$ and*

$$\|Ad_G^k(F)\|_R^{\nu_k, N_k} \lesssim \left(\frac{\|G\|_R^{\nu, N}}{R^2}\right)^k \|F\|_R^{\nu, N}, \tag{5.9}$$

with $N_k = N - k(d + \nu)$ and $\nu_k = k(d + 2\nu)$.

Proof. We prove it by induction. For $k = 1$, the thesis follows from (4.24). Suppose the thesis is true at step k . Exploiting again (4.24), we get

$$\begin{aligned} \|Ad_G^{k+1}(F)\|_R^{\nu_{k+1}, N_{k+1}} &= \|\{Ad_G^k(F), G\}\|_R^{\nu_{k+1}, N_{k+1}} \\ &\lesssim \frac{1}{R^2} \left(\frac{\|G\|_R^{\nu, N}}{R^2}\right)^k \|F\|_R^{\nu, N} \|G\|_R^{\nu, N} = \left(\frac{\|G\|_R^{\nu, N}}{R^2}\right)^{k+1} \|F\|_R^{\nu, N}. \end{aligned}$$

□

It is useful for the perturbative iteration to summarize the last results in the following lemma.

Lemma 5.14. *Fix \bar{r} , let $P \in L_{r_1, \bar{r}_1}^{\nu, N}$ and $G \in L_{r_2+2, \bar{r}_2+2}^{\nu, N}$ with $r_1 < \bar{r}$, $r_2 + 2 < \bar{r}$. Let*

$$n = \frac{\bar{r} + 3 - r_1}{r_2}, \tag{5.10}$$

if the r.h.s. is integer, otherwise we define n to be the r.h.s. of (5.10) + 1. Let $s > \frac{3}{2}d + 2$ There exists $C_{\bar{r}, s, N} > 0$ such that if R fulfills (5.7), then the Lie transform $\Phi_G: \mathcal{B}_{R/2}^s \rightarrow \mathcal{B}_R^s$ is well defined. Moreover, one has

$$P \circ \Phi_G = P + P' + \mathcal{R}_{P, G}$$

and $\exists \nu', N'$ s.t., $P' \in L_{r_1+r_2, \bar{r}_1+n\bar{r}_2}^{\nu', N'}$. Furthermore $X_{\mathcal{R}_{P, G}} \in C^\infty(\mathcal{B}_R^s, \mathcal{H}_e^s)$ and one has

$$\|P'\|_R^{\nu', N'} \lesssim \|P\|_R^{\nu, N} \frac{\|G\|_R^{\nu, N}}{R^2} \tag{5.11}$$

$$\sup_{\|u\|_s \leq R/2} \|X_{\mathcal{R}_{P, G}}(u)\|_s \lesssim \|P\|_R^{\nu, N} \left(\frac{\|G\|_R^{\nu, N}}{R^2}\right)^{n+1} \frac{1}{R}, \tag{5.12}$$

so that $X_{\mathcal{R}_{P, G}}$ has a zero of order at least $\bar{r} + 2$ at the origin.

Proof. Define

$$P' := \sum_{k=1}^n \frac{1}{k!} Ad_G^k P$$

and $R_{P, G}$ be the integral term in (5.8). Then, by lemma 5.13, one has

$$\|P'\|_R^{\nu', N'} \lesssim \sum_{k=1}^n \frac{1}{k!} \left(\frac{\|G\|_R^{\nu, N}}{R^2}\right)^k \|P\|_R^{\nu, N} \lesssim \frac{\|G\|_R^{\nu, N}}{R^2} \|P\|_R^{\nu, N},$$

provided $\left(\frac{\|G\|^{p,N}}{R^2}\right) \leq 1/2$, which is ensured by (5.7). This gives (5.11). To get (5.12) just use the estimate (5.9), lemmas 4.18 and 5.11. \square

5.2. Solution of the homological equation

In this subsection, we write and solve the homological equation. We start defining the set of nonresonant indexes.

Definition 5.15 (Block K non resonant indexes). Let $K \gg 1$ as in (4.9). We say that a multi-index $\mathbf{A} = (A_1, \dots, A_r) \in \Lambda_e^r$ is Block- K -non-resonant, and we write $\mathbf{A} \in \mathcal{I}^K(r)$, if $\langle |A_{\tau_{\text{ord}}(3)}| \rangle < K$ (namely there are at most two large indexes) and one of the following holds:

- $\langle |A_{\tau_{\text{ord}}(1)}| \rangle < K$ (there are no indexes larger than K) and $\mathbf{A} \notin W$;
- $\langle |A_{\tau_{\text{ord}}(1)}| \rangle \geq K$ and $\langle |A_{\tau_{\text{ord}}(2)}| \rangle < K$ (there is exactly one large index);
- $\langle |A_{\tau_{\text{ord}}(2)}| \rangle \geq K$ and $\langle |A_{\tau_{\text{ord}}(3)}| \rangle < K$ (there are exactly two large indexes) and one of the following holds (recall hypothesis L.3):
 - $\exists \alpha$ s.t. $a_{\tau_{\text{ord}}(1)}, a_{\tau_{\text{ord}}(2)} \in \Omega_\alpha$ and $\sigma_{\tau_{\text{ord}}(1)} \sigma_{\tau_{\text{ord}}(2)} = +1$,
 - $\exists \alpha \neq \beta$ s.t. $a_{\tau_{\text{ord}}(1)} \in \Omega_\alpha, a_{\tau_{\text{ord}}(2)} \in \Omega_\beta$ and $|a_{\tau_{\text{ord}}(1)} - a_{\tau_{\text{ord}}(2)}| \leq K^\delta$.

Remark 5.16. By definitions 5.4 and 5.5, a polynomial supported only on multi-indexes $\mathbf{A} \notin \mathcal{I}^K(r)$ is in normal form according to definition 5.6

In the following lemma, we show that there are no resonant multi-indexes with just one large index.

Lemma 5.17. Fix K as in (4.9). If $\mathbf{A} \in \Lambda_e^r$ is s.t.

$$\langle |A_{\tau_{\text{ord}}(1)}| \rangle \geq K \quad \text{and} \quad \langle |A_{\tau_{\text{ord}}(2)}| \rangle < K, \tag{5.13}$$

then there exists a constant $\gamma'_r > 0$ such that

$$\left| \sum_{l=1}^r \sigma_l \omega_{a_l} \right| \geq \gamma'_r K^{-\tau},$$

where $\tau > 0$ is the constant appearing in hypotheses L.2.

Proof. For simplicity, we can suppose that $A_{\tau_{\text{ord}}(1)} = A_1$. We distinguish two cases.

Case 1 If $\langle |A_1| \rangle \geq K_1 = 2(r-1)^{\frac{1}{\beta}} K$, recalling $\omega_{a_j} = \langle |a_j| \rangle^\beta < K^\beta$ for any $j = 2, \dots, r$, we have that

$$\left| \sum_{l=2}^r \sigma_l \omega_{a_l} \right| \leq (r-1) K^\beta. \tag{5.14}$$

Then from (5.14) and $\langle |A_1| \rangle \geq K_1$ we deduce

$$\left| \sum_{l=2}^r \sigma_l \omega_{a_l} + \sigma_1 \omega_{a_1} \right| \geq K_1^\beta - (r-1) K^\beta \gtrsim 1,$$

that implies the thesis.

Case 2 If $\langle |A_1| \rangle < K_1$, we prove that $\mathbf{A} \notin W$. In fact, if by contradiction $\mathbf{A} \in W$, there should exist $A' \in \mathbf{A}$ with $A_1, A' \in \Sigma_n$ for some n . But then, from (4.9), it would follow that $\langle |A_{\tau_{\text{ord}}(2)}| \rangle \geq K$. This is in contradiction with (5.13).

Since $\mathbf{A} \notin W$, from Hypothesis (L.2) it follows that

$$\left| \sum_{j=1}^r \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1, \dots, r} |a_j|)^\tau} \gtrsim \frac{\gamma}{K_1^\tau} = \frac{\gamma'_r}{K^\tau},$$

with $\gamma'_r = \frac{\gamma}{2^{(r-1)\frac{\tau}{\beta}}}$. □

In the following lemma, we take care of multi-indexes with exactly two indexes with modulus larger than K .

Lemma 5.18. *If $\mathbf{A} \in \mathcal{I}^K(r)$ is s.t.*

$$\langle |A_{\tau_{\text{ord}}(2)}| \rangle \geq K \quad \text{and} \quad \langle |A_{\tau_{\text{ord}}(3)}| \rangle < K, \tag{5.15}$$

then there exist constants γ'_r and τ' such that

$$\left| \sum_{l=1}^r \sigma_l \omega_{a_l} \right| \geq \frac{\gamma'_r}{K^{\tau'}}.$$

Proof. For simplicity, suppose that

$$A_{\tau_{\text{ord}}(1)} = A_1 \quad \text{and} \quad A_{\tau_{\text{ord}}(2)} = A_2.$$

Case 1 In the case $\sigma_1 = \sigma_2$, the proof is a simple variant of the proof of lemma 5.17 and is omitted.

Case 2 Consider now the case $\sigma_1 \sigma_2 = -1$. It follows by definition of $\mathcal{I}^K(r)$ that $a_1 \in \Omega_\alpha$, $a_2 \in \Omega_\beta$ with $\alpha \neq \beta$ and moreover $|a_1 - a_2| \leq K^\delta$. From hypothesis L.3, it follows that

$$|\omega_{a_1} - \omega_{a_2}| \geq C_\delta |a_1|^\delta.$$

Case 2.i If $\langle |a_1| \rangle \geq K_2 := 2C_2 ((r-2)C_\delta^{-1})^{\frac{1}{\delta}} K^{\frac{\beta}{\delta}}$, with C_2 the constant in remark 4.9, we observe that $|a_1| \geq C_2^{-1} \langle |a_1| \rangle \geq C_2^{-1} K_2$ and then, recalling that $\omega_{a_j} = \langle |a_j| \rangle^\beta \leq K^\beta$ for $j = 2, \dots, r$,

$$\left| \sum_{l=3}^r \sigma_l \omega_{a_l} + \omega_{a_1} - \omega_{a_2} \right| \geq C_\delta |a_1|^\delta - (r-2)K^\beta \geq C_\delta C_2^{-\delta} K_2^\delta - (r-2)K^\beta \gtrsim 1,$$

and the thesis follows.

Case 2.ii If $\langle |a_1| \rangle < K_2$, one has that $\mathbf{A} \notin W$. To see this, remark that $|\omega_{a_1} - \omega_{a_2}| \geq C_\delta |a_1|^\delta > CK^\delta > 2$ but each Σ_n has a length smaller or equal to 2, so that a_1, a_2 belong to different set Σ_n and then the definition of W cannot be fulfilled. Then recalling hypothesis (L.2), we conclude that

$$\left| \sum_{j=1}^r \sigma_j \omega_{a_j} \right| \geq \frac{\gamma}{(\max_{j=1, \dots, r} |a_j|)^\tau} \gtrsim \frac{\gamma}{K_2^\tau} = \frac{\gamma'}{K^{\tau'}}$$

with $\gamma' = \frac{\gamma}{C(r-2)^{\frac{\tau}{\delta}}}$ and $\tau' = \frac{\beta}{\delta} \tau$.

□

In the next lemma, we solve the homological equation.

Lemma 5.19 (Homological equation). *Let \bar{r}, l be given in such a way that $\bar{r} \geq l \geq 3$, then for any $F \in L_{l, \bar{r}}^{\nu, N}$ there exist $G, Z \in L_{l, \bar{r}}^{\nu, N}$ which solve the homological equation*

$$\{H_0, G\} + F = Z. \tag{5.16}$$

Furthermore there exist $\tau = \tau(\bar{r})$ and $\gamma = \gamma(\bar{r})$ s.t.

$$\|G\|_R^{\nu, N} \leq \frac{K^\tau}{\gamma} \|F\|_R^{\nu, N}, \tag{5.17}$$

$Z \in L_{l, \bar{r}}^{\nu, N}$ is in normal form according to definition 5.6 and fulfills the estimate

$$\|Z\|_R^{\nu, N} \leq \|F\|_R^{\nu, N}.$$

Proof. Writing

$$F(u) = \sum_{l=3}^{\bar{r}} \sum_{\mathbf{A} \in \Lambda_l^i} \tilde{F}_l(\Pi_{A_1} u, \dots, \Pi_{A_l} u),$$

we define G, Z through their multilinear map. More precisely, recalling definition 5.15, we set, for $3 \leq l \leq \bar{r}$,

$$\tilde{Z}_l(\Pi_{A_1} u_1, \dots, \Pi_{A_l} u_l) := \begin{cases} \tilde{F}_l(\Pi_{A_1} u_1, \dots, \Pi_{A_l} u_l) & \text{if } (A_1, \dots, A_l) \notin \mathcal{I}^K(l) \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{G}_l(\Pi_{A_1} u_1, \dots, \Pi_{A_l} u_l) := \begin{cases} \frac{\tilde{F}_l(\Pi_{A_1} u_1, \dots, \Pi_{A_l} u_l)}{\sum_{j=1}^l \sigma_j \omega_{a_j}} & \text{if } (A_1, \dots, A_l) \in \mathcal{I}^K(l) \\ 0 & \text{otherwise.} \end{cases}$$

These polynomials solve the homological equation (5.16) since

$$\{H_0, G\}(u) = - \sum_{l=3}^{\bar{r}} \sum_{\mathbf{A} \in \mathcal{I}^K(l)} \tilde{F}(\Pi_{A_1} u, \dots, \Pi_{A_l} u).$$

In particular \tilde{Z} is in normal form (recall remark 5.16). The estimate (5.17) follows from lemmas 5.17 and 5.18. □

5.3. Proof of the normal form lemma

In this subsection, we complete the proof of proposition 5.8.

Fix $\bar{r} \geq 2$ and Taylor expand P at order $\bar{r} + 2$. Recalling (5.1) and definition 4.15, we have

$$P = P^{(0)} + \mathcal{R}_{T,0},$$

with $P^{(0)} \in L_{1,\bar{r}+2}^{N,\nu}$ for some positive N, ν . Furthermore $\mathcal{R}_{T,0}$ has a zero of order $\bar{r} + 2$ at $u = 0$. Moreover, since $X_{\mathcal{R}_{T,0}} \in C^\infty(\mathcal{B}_R^s, \mathcal{H}_e^s)$, one has, for s large enough and $R > 0$ small enough

$$\|X_{\mathcal{R}_T}(u)\|_s \lesssim R^{\bar{r}+2}, \quad \forall u \in \mathcal{B}_R^s,$$

In the following lemma, we describe the iterative step that proves theorem 5.21.

Lemma 5.20 (Iteration). *Fix $\bar{r} \geq 2$; for any $0 \leq k \leq \bar{r}$, there is a small constant $\mu_k > 0$ s.t., denoting*

$$\mu = \mu(R) := \frac{\|P^{(0)}\|_R^{N,\nu}}{R^2} K^\tau,$$

if R_k, K are s.t.

$$\mu(R_k) \leq \mu_k \tag{5.18}$$

then there exist $N_k, \nu_k > 0$ such that, $\forall s_0 > d/2 + \nu_k$ the following holds: there exists an invertible canonical transformation $T^{(k)} : \mathcal{B}_{R_k/2^k}^{s_0} \rightarrow \mathcal{B}_{R_k}^{s_0}$ such that

$$H^{(k)} = H \circ T^{(k)} = H_0 + Z^{(k)} + P^{(k)} + \mathcal{R}_T^{(k)}$$

and, $\exists R_k > 0$ s.t. one has

- $Z^{(k)} \in L_{3,\bar{r}_{k-1}+2}^{N_{k-1},\nu_{k-1}}$ is in normal form and

$$\|Z^{(k)}\|_R^{N_{k-1},\nu_{k-1}} \lesssim_k \|P^{(0)}\|_R^{N,\nu}, \quad \forall R < R_k, \tag{5.19}$$

- $P^{(k)} \in L_{k+3,\bar{r}_k+2}^{N_k,\nu_k}$ and

$$\|P^{(k)}\|_R^{N_k,\nu_k} \lesssim_k \mu^k \|P^{(0)}\|_R^{N,\nu}, \quad \forall R < R_k, \tag{5.20}$$

- $\forall s \geq s_0 \exists R_{s,k}$ s.t. $T^{(k)} \in C^\infty\left(\mathcal{B}_{\frac{R_{s,k}}{2^k}}^s; \mathcal{H}_e^s\right)$ and $[T^{(k)}]^{-1} \in C^\infty\left(\mathcal{B}_{\frac{R_{s,k}}{4^k}}^s; \mathcal{H}_e^s\right)$

$$T^{(k)}\left(\mathcal{B}_{R/2^k}^s\right) \subset \mathcal{B}_R^s, \quad [T^{(k)}]^{-1}\left(\mathcal{B}_{R/4^k}^s\right) \subset \mathcal{B}_{R/2^k}^s, \quad \forall R < R_{s,k}, \tag{5.21}$$

- $X_{\mathcal{R}_T^{(k)}} \in C^\infty(\mathcal{B}_{R_{s,k}/2^k}^s, \mathcal{H}_e^s)$ and, $\forall u \in \mathcal{B}_R^s$, with $R < R_{s,k}/2^k$ one has

$$\|X_{\mathcal{R}_T^{(k)}}(u)\|_s \lesssim_k R^2 (K^\tau R)^{\bar{r}}. \tag{5.22}$$

Proof. The result is true for $k = 0$ with $T^{(0)} = Id$ and $N_0 = N$, $\nu_0 = \nu$. We prove the inductive step $k \rightsquigarrow k + 1$.

We determine $G_{k+1}, Z_{k+1} \in L_{k+3, \bar{r}_{k+1}}^{N_k, \nu_k}$ solving the homological equation.

$$\{H_0, G_{k+1}\} + P^{(k)} = Z_{k+1}. \tag{5.23}$$

To this end, we remark that the maximal degree homogeneity of $P^{(k)}$ appears when $k = \bar{r} - 1$ and is smaller than $\bar{r}_\bar{r}$. So we take τ and γ in (5.17) to be $\tau(\bar{r}_\bar{r})$ and $\gamma(\bar{r}_\bar{r})$. Then we write

$$H^{(k+1)} := H^{(k)} \circ \Phi_{G_{k+1}}.$$

Exploiting (5.23), the Hamiltonian $H^{(k+1)}$ has the form,

$$\begin{aligned} H^{(k+1)} &= H_0 + Z^{(k)} + Z_{k+1} + \\ &+ H'_0 - \{H_0, G_{k+1}\} + (P^{(k)})' + (Z^{(k)})' + \\ &+ \mathcal{R}_{H_0, G_{k+1}} + \mathcal{R}_{P^{(k)}, G_{k+1}} + \mathcal{R}_{Z^{(k)}, G_{k+1}} + \mathcal{R}_{T, k} \circ \Phi_{G_{k+1}}, \end{aligned}$$

with the primed quantities defined as in lemma 5.14. Collecting the terms above, we define the following quantities:

$$\begin{aligned} Z^{(k+1)} &:= Z^{(k)} + Z_{k+1}, \\ P^{(k+1)} &:= H'_0 - \{H_0, G_{k+1}\} + (P^{(k)})' + (Z^{(k)})', \\ \mathcal{R}_{T, k+1} &:= \mathcal{R}_{H_0, G_{k+1}} + \mathcal{R}_{P^{(k)}, G_{k+1}} + \mathcal{R}_{Z^{(k)}, G_{k+1}} + \mathcal{R}_{T, k} \circ \Phi_{G_{k+1}}. \end{aligned}$$

First, we check the order of the polynomials, we have $(P^{(k)})' \in L_{2k+2, \bar{r}_k + n_1 \bar{r}_k}$, with a suitable n_1 . Similarly one has $H'_0 - \{H_0, G_{k+1}\} \in L_{2k+2, \bar{r}_k + n_2 \bar{r}_k}$, and $(Z^{(k)})' \in L_{3+k+1, \bar{r}_{k-1} + n_3 \bar{r}_k}$ with suitable n_2, n_3 . Defining

$$\bar{r}_{k+1} := \max \{ \bar{r}_{k-1} + n_3 \bar{r}_k, \bar{r}_k + n_2 \bar{r}_k, \bar{r}_k + n_1 \bar{r}_k \}$$

one gets the result on the order.

We come to the estimates. First we remark that

$$\mu \lesssim RK^\tau. \tag{5.24}$$

From lemma 5.19 and the inductive hypothesis, we get the estimates

$$\begin{aligned} \|G_{k+1}\|_R^{N_k, \nu_k} &\leq \frac{K^\tau}{\gamma} \|P^{(k)}\|_R^{N_k, \nu_k} \lesssim \frac{K^\tau}{\gamma} \mu^k \|P^{(0)}\|_R^{N, \nu} \lesssim \mu^{k+1} R^2, \\ \|Z_{k+1}\|_R^{N_k, \nu_k} &\lesssim \|P^{(k)}\|_R^{N_k, \nu_k} \lesssim \mu^k \|P^{(0)}\|_R^{N, \nu}. \end{aligned} \tag{5.25}$$

The estimate (5.19) follows from (5.25).

For the estimate (5.20), we prove it for $\tilde{H}'_0 - \{H_0, G_{k+1}\}$ applying lemma 5.13. Denoting $N' = N_{k+1} = N_k - n(\nu_k + d)$ and $\nu' = \nu_{k+1} = n(2\nu_k + d)$, and taking profit of (5.23), we have

$$\begin{aligned} \|H'_0 - \{H_0, G_{k+1}\}\|_R^{N_k, \nu_k} &= \left\| \sum_{l=1}^n \frac{1}{l!} Ad_{G_{k+1}}^l (\{H_0, G_{k+1}\}) \right\|_R^{N_k, \nu_k} \\ &= \left\| \sum_{l=1}^n \frac{1}{l!} Ad_{G_{k+1}}^l (Z_{k+1} - P^{(k)}) \right\|_R^{N_k, \nu_k} \\ &\lesssim \sum_{l=1}^n \frac{1}{l!} \left(\frac{\|G_{k+1}\|_R^{N_k, \nu_k}}{R^2} \right)^l \|P^{(k)}\|_R^{N_k, \nu_k} \lesssim \mu^k \|P^{(0)}\|_k^{N, \nu} \sum_{l=1}^n \frac{1}{l!} \mu^{(k+1)l} \\ &\lesssim \mu^k \|P^{(0)}\|_R^{N, \nu} \mu^{k+1} \sum_{l=0}^{\infty} \frac{1}{l!} \mu^{(k+1)l} \lesssim \mu^{2k+1} \|\hat{P}\|_R^{N, \nu}, \end{aligned}$$

that is (5.20) with k replaced by $k + 1$ (also in the case $k = 0$).

Concerning $(Z^{(k)})'$ we just use (5.11) which gives

$$\left\| (Z^{(k)})' \right\|_R^{N_k, \nu_k} \lesssim \|P^{(0)}\|_R^{N, \nu} \mu^{k+1}.$$

Similarly one gets the estimate of $(P^{(k)})'$.

We come to the remainders. Consider first $\mathcal{R}_{Z^{(k)}, G_{k+1}}$. Here one has $Z^{(k)} \in L_{3, \bar{r}_{k-1}}$, so that we apply (5.12) with $r_1 = k + 1$, $r_2 = 3$, which gives

$$n + 1 \geq \frac{\bar{r}}{k + 1},$$

so that the remainder is estimated by

$$\left\| X_{\mathcal{R}_{Z^{(k)}, G_{k+1}}} \right\| \lesssim \frac{\|Z^{(k)}\|_R^{N_k, \nu_k}}{R} (\mu^{k+1})^{n+1} \lesssim \frac{\|P^{(0)}\|_R^{N_k, \nu_k}}{R} \mu^{\bar{r}} \lesssim R^2 (RK^\tau)^{\bar{r}}.$$

Proceeding in the same way for the other terms one gets the thesis. □

We thus have the following:

Corollary 5.21 (Normal form). *Assume hypotheses L.1–L.3 for the Hamiltonian $H = H_0 + P$ of the form (5.1) and that P is a function with localized coefficients according to definition 4.15 and with a zero of order at least 3 at the origin. For any $\bar{r} > 3$ there exist τ and $s_{\bar{r}} \geq s_0 > d/2$ such that for any $s \geq s_{\bar{r}}$ the following holds. There exist constants $R_{\bar{r}, s}$ and $C_{\bar{r}, s}$ such that, if K and R fulfill*

$$RK^\tau \leq R_{\bar{r}, s},$$

then there exists an invertible canonical transformation

$$T^{(\bar{r})} : \mathcal{B}_{R/2^{\bar{r}}}^s \rightarrow \mathcal{B}_R^s, \quad [T^{(\bar{r})}]^{-1} : \mathcal{B}_{R/4^{\bar{r}}}^s \rightarrow \mathcal{B}_{R/2^{\bar{r}}}^s$$

such that

$$H^{(\bar{r})} = H \circ T^{(\bar{r})} = H_0 + Z^{(\bar{r})} + \mathcal{R}_T \tag{5.26}$$

where $Z^{(\bar{r})} \in L_{3, \bar{r}}^{\nu_r, N_r}$ is in normal form and $X_{\mathcal{R}_T} \in C^\infty(\mathcal{B}_{R/2^{\bar{r}}}^s, \mathcal{H}_e^s)$. Moreover, for any $u \in \mathcal{B}_R^s$, we have

$$\|X_{\mathcal{R}_T}(u)\|_s \lesssim R^2 (K^\tau R)^{\bar{r}}.$$

Proof of proposition 5.8. We apply corollary 5.21. Choosing according to definition 5.7

$$K := K(R, \tau) \implies RK^\tau \sim R^{\frac{1}{2}} \tag{5.27}$$

we have

$$(RK^\tau)^{\bar{r}} R^2 \simeq R^{\frac{\bar{r}+4}{2}}.$$

Choosing $\bar{r} := 2r$ we get

$$\|X_{\mathcal{R}_T}(u)\|_s \lesssim R^{r+2}.$$

From theorem 5.21, definitions 5.4 and 5.5, we can write

$$Z^{(\bar{r})} = Z_0 + Z_B + Z_2 + Z_{\geq 3}.$$

We have to show that Z_2 and $Z_{\geq 3}$ can be considered remainder terms of order R^{r+1} . From corollary 4.27, we have that, for any $u \in \mathcal{B}_{R/2^{\bar{r}}}^s$,

$$\|X_{Z_{\geq 3}}(u)\|_s \lesssim \frac{\|Z_{\geq 3}\|_R^{\nu_r, N_r}}{K^{s-s_0}} \frac{1}{R} \lesssim \frac{R^2}{K^{s-s_0}}.$$

Since (5.27) implies $K \sim R^{-\frac{1}{2\tau}}$, we have

$$\frac{R^2}{K^{s-s_0}} \sim R^{2+\frac{s-s_0}{2\tau}} \lesssim R^{r+1}$$

if $s > s_r = s_0 + 2\tau(r-1)$ and so

$$\|X_{Z_{\geq 3}}(u)\|_s \lesssim R^{r+1}.$$

It remains to consider X_{Z_2} . From lemma 4.28, it follows that, for any $u \in \mathcal{B}_R^s$,

$$\|X_{Z_2}(u)\|_s \lesssim \frac{\|Z_2\|_R^{\nu_r, N_r}}{K^{\delta(N'-N)}} \frac{1}{R} \lesssim \frac{R^2}{K^{\delta(N'-N)}},$$

for any $N' > N$. Denoting $N' = N + M_1$, we have

$$\frac{R^2}{K^{\delta M_1}} \sim R^{2+\frac{\delta M_1}{2\tau}} \sim R^{r+1}$$

for $M_1 = \frac{2\tau}{\delta}(r-2)$. We get the thesis denoting

$$\mathcal{R}^{(\bar{r})} := Z_2 + Z_{\geq 3} + \mathcal{R}_T.$$

The estimate (5.6) follows from lemma 4.27. □

6. From normal form to almost global existence

In this section, we conclude the proof of theorem 5.1. In particular, we use proposition 5.8 to study the dynamics corresponding to a smooth, small, real initial datum for the Hamilton equations of (5.1). Precisely, take some large s and assume that

$$\|u_0\|_s =: \epsilon < \frac{R}{2 \cdot 4^{2r}}, \tag{6.1}$$

with a small $R < R_{s,r}$ and $R_{r,s}$ from proposition 5.8. Denote

$$z_0 := \left(T^{(\bar{r})}\right)^{-1}(u_0),$$

and we consider the evolution in the z variables. Let $z = (z_+, z_-) \in \mathcal{H}^s \times \mathcal{H}^s \equiv \mathcal{H}_\epsilon^s$ and, for K defined in proposition 5.8, recall that

$$\begin{aligned} \Pi^{\leq} z = z^{\leq} &= \sum_{\{|A| \leq K\}} \Pi_A z, \\ \Pi^\perp z = z^\perp &= \sum_{\{|A| > K\}} \Pi_A z. \end{aligned}$$

Write the Hamilton equations of $H^{(\bar{r})}$ as a system:

$$\begin{cases} \dot{z}^{\leq} = X_{H_0}(z^{\leq}) + X_{Z_0}(z^{\leq}) + \Pi^{\leq} X_{Z_B}(z) + \Pi^{\leq} X_{R^{(\bar{r})}}(z), \\ \dot{z}^\perp = X_{H_0}(z^\perp) + \Pi^\perp X_{Z_B}(z) + \Pi^\perp X_{R^{(\bar{r})}}(z). \end{cases} \tag{6.2}$$

We start considering the dynamics of z^{\leq} . We remark first that $\Pi^{\leq} X_{Z_B}$ and $\Pi^{\leq} X_{R^{(\bar{r})}}$ are remainder terms of order R^{r+2} (see proposition 5.8). Then, it remains to analyze the role of Z_0 .

To this end, we define the set of indexes correspondent to each ‘band’ of the spectrum Σ_n , the correspondent projector and the correspondent ‘superaction’, namely

$$\begin{aligned} E_n &:= \left\{ a \in \Lambda : \omega_a^{1/\beta} \in \Sigma_n \right\}, & \Pi_n z &:= \sum_{\{A=(a,\sigma): a \in E_n\}} \Pi_A z, \\ J_n(z) &:= \sum_{a \in E_n} \int \Pi_{(a,+)} z \Pi_{(a,-)} z \, dx. \end{aligned}$$

In particular, if z is real (as we assumed)

$$J_n(z) = \sum_{a \in E_n} \|\Pi_{(a,+)} z\|_0^2 = \frac{1}{2} \sum_{a \in E_n} \|\Pi_{(a,+)} z\|_0^2 + \|\Pi_{(a,-)} z\|_0^2.$$

In the next lemma, we prove that J_n is preserved under the dynamics associated to Z_0 .

Lemma 6.1. *Let Z a polynomial supported on W , then*

$$\{Z, J_n\} = 0.$$

Proof. From definition 5.3 follows that \tilde{Z} is the sum of terms of the form

$$\int_M \Pi_{A_1} u_1 \dots \Pi_{A_l} u_l$$

with $\mathbf{A} = (A_1, \dots, A_l) \in W$. First, note that we have

$$X_{J_n}(u) = \mathbf{i} \sum_{a \in E_n} (\Pi_{(a,+)} u, -\Pi_{(a,-)} u) = \mathbf{i} \sum_{(a,\sigma)} \delta_{a \in \Sigma_n} (\delta_{\sigma,+} - \delta_{\sigma,-}) \Pi_{(a,\sigma)} u.$$

Moreover, for any homogeneous polynomial F , one has

$$dF(u)X = \tilde{F}(X, u, \dots, u) + \tilde{F}(u, X, \dots, u) + \dots + \tilde{F}(u, \dots, u, X).$$

Decompose $Z = \sum_l Z_l$ in homogeneous polynomials, we have

$$\begin{aligned} \{Z_l, J_n\}(u) &= dZ(u)X_{J_n}(u) \\ &= \mathbf{i} \sum_{A_1, \dots, A_l} \mathcal{Z}(\Pi_{A_1} u, \dots, \Pi_{A_l} u) \sum_{j=1}^n \delta_{a_j \in \Sigma_n} (\delta_{\sigma_j,+} - \delta_{\sigma_j,-}). \end{aligned}$$

We recall that Z is in normal form, which means that the sum can be restricted to multi-indexes belonging to W . So, fix one of the multi-indexes $A \in W$. The definition of W implies that there exists a permutation τ of $1, \dots, l$, and indexes n_1, \dots, n_l s.t. $a_{\tau(j)}, a_{\tau(j+1/2)} \in \Sigma_{n_j}$ and $\sigma_{\tau(j)} \sigma_{\tau(j+1/2)} = -1$. Thus consider the sum

$$\sum_{j=1}^n \delta_{a_j \in \Sigma_n} (\delta_{\sigma_j,+} - \delta_{\sigma_j,-}) = \sum_{j=1}^n \delta_{a_{\tau(j)} \in \Sigma_n} (\delta_{\sigma_{\tau(j)},+} - \delta_{\sigma_{\tau(j)},-}).$$

If $a_{\tau(j)} \in \Sigma_n$ with a sign, it means that also $a_{\tau(j+1/2)} \in \Sigma_n$ with the opposite sign. Thus the sum vanishes for any index in W . □

Corollary 6.2. *There exists a positive constant C_1 with the following property: assume that (6.1) holds and that there exists $T_e > 0$ s.t.*

$$\|z(t)\|_s \leq \frac{R}{2 \cdot 2^{2r}}, \quad \forall |t| \leq T_e \tag{6.3}$$

and some $R < R_{rs}$, then one has

$$\|z^{\leq}(t)\|_s^2 \leq C_1 \left(\|z^{\leq}(0)\|_s^2 + |t|R^{r+3} \right). \tag{6.4}$$

Proof. Define

$$a_{n,s} := \inf_{a \in E_n} (1 + \langle |a| \rangle)^{2s}$$

then there exist two constants C_3, C_4 such that, for any n , one has

$$C_3 \|\Pi_n z\|_s^2 \leq a_{n,s} J_n \leq C_4 \|\Pi_n z\|_s^2,$$

$$\|z^\leq\|_s^2 \simeq \sum_{\substack{\text{ns.t.} \\ \max E_n \leq K}} a_{n,s} J_n. \tag{6.5}$$

Then, by proposition 5.8 and lemma 6.1, one has

$$\frac{d}{dt} \sum_{\substack{\text{ns.t.} \\ \max E_n \leq K}} a_{n,s} J_n \lesssim \sum_n a_{n,s} \left\{ J_n, Z_2 + R^{(\bar{r})} \right\} \lesssim R^{r+3}, \tag{6.6}$$

which is valid for $|t| \leq T_e$. From (6.5) and (6.6) the thesis immediately follows. \square

Consider the dynamics of the high modes z^\perp given by (6.2). From proposition 5.8), $\Pi^\perp X_{R^{(r)}}$ is a remainder term of order R^{r+2} and Z_B is in Block Resonant Normal Form and of order 2 in z^\perp .

Recalling the dyadic partition $\Lambda = \bigcup_{\alpha \in \mathfrak{A}} \Omega_\alpha$, we define the correspondent projectors and the correspondent superactions. Since we are interested in the dynamics of the high modes, we consider only superactions defined on modes $\langle |a| \rangle \geq K$. This amount to consider a cutoff of the Bourgain’s blocks, that clearly do not break the dyadicity of the partition,

$$\begin{aligned} \Pi_\alpha z &:= \sum_{\{A=(a,\sigma) : a \in \Omega_\alpha, \langle |a| \rangle \geq K\}} \Pi_A z \\ J_\alpha(z) &:= \sum_{\{a \in \Omega_\alpha, \langle |a| \rangle \geq K\}} \int \Pi_{(a,+)} z \Pi_{(a,-)} z \, dx = \|\Pi_\alpha z\|_0^2. \end{aligned}$$

By definition of Block Resonant normal form (see definition 5.4) $\Pi^\perp X_{Z_B}$ is linear in z^\perp . Then, we show that the L^2 norm on each block is conserved along the dynamics associated to the normal form since Z_B is a real polynomial. Namely, the dynamics of the normal form Z_B enforces the exchange of energy only within high modes in the same block Ω_α . This is the content of the following lemma.

Lemma 6.3. *For any real $z \in \mathcal{H}_e^s$, we have that, for any $\alpha \in \mathfrak{A}$,*

$$\{J_\alpha, Z_B\}(z) = 0.$$

Proof. In this proof, we denote again $z_A := \Pi_A z$. Since Z_B is real and recalling definition 5.4, it is the sum of terms of the form

$$\begin{aligned} \tilde{Z}_\beta(z) &:= \int z_{(a,+)} z_{(b,-)} z_{A_3} \dots z_{A_r} + \int z_{(a,-)} z_{(b,+)} z_{A_3} \dots z_{A_r} \\ &= 2\text{Re} \left(\int z_{(a,+)} z_{(b,-)} z_{A_3} \dots z_{A_r} \right), \quad \forall z \text{ real} \end{aligned} \tag{6.7}$$

with $\mathbf{A} = ((a,+), (b,-), A_3, \dots, A_r) \in \Lambda_e^r$ such that $a, b \in \Omega_\beta$, $\langle |a| \rangle > K$ and $\langle |b| \rangle > K$ for some β . Here, for the sake of simplicity, we are considering multi-indexes for which the two largest indexes are the first and the second.

If $\beta \neq \alpha$, then $\{J_\alpha, Z_\beta\} = 0$. Otherwise, we have

$$\begin{aligned} \{J_\alpha, \tilde{Z}_\alpha\}(z) &= i \int z_{(a,-)} z_{(b,+)} \cdots z_{\bar{A}_r} - i \int z_{(a,+)} z_{(b,-)} \cdots z_{A_r} \\ &\quad + i \int z_{(b,-)} z_{(a,+)} \cdots z_{A_r} - i \int z_{(b,+)} z_{(a,-)} \cdots z_{\bar{A}_r} = 0, \end{aligned}$$

with $a, b \in \Omega_\alpha$. □

The following corollary is proved exactly in the same way as corollary 6.2

Corollary 6.4. *There exists a positive constant C_1 with the following property: assume there exists $T_e > 0$ s.t. (6.3) holds for some $R < R_{rs}$, then one has*

$$\|z^\perp(t)\|_s^2 \leq C_1 \left(\|z^\perp(0)\|_s^2 + |t| R^{r+3} \right). \tag{6.8}$$

Proof of theorem 5.1. We prove by a bootstrap argument that, if ϵ is small enough, the escape time T_e fulfills $1/R^{r+1} \lesssim T_e$. First we make the canonical transformation $\mathcal{T}^{(r)}$ and apply the estimate (5.5) with $R = 2^{-1} \epsilon 4^{2r}$ where $\epsilon := \|u_0\|$, getting that

$$\|z_0\|_s \leq \frac{R}{2 \cdot 2^{2r}}.$$

Then we can apply lemmas 6.2 and 6.4, since the assumptions on the initial datum $z(0)$ are fulfilled. Assume now, by contradiction, that there exists $t_* < T_e := R^{-r-1} (2 \cdot 2^{2r})^{-2}$ s.t. $z(t) \in \mathcal{B}_{R'_r}^s$ for all $|t| < t_*$ and $z(t_*) \in \partial \mathcal{B}_{R'_r}^s$, with $R'_r := \frac{\sqrt{2C_1}R}{2 \cdot 2^{2r}}$ and C_1 as in lemmas 6.4 and 6.2. For $|t| \leq t_*$, lemmas 6.2 and 6.4 give that

$$\left(\frac{\sqrt{2C_1}R}{2 \cdot 2^{2r}} \right)^2 = \|z(t_*)\|_s^2 \leq C_1 \left(\|z_0\|_s^2 + R^{r+3}|t| \right) \leq C_1 \left(\frac{R^2}{2^2 \cdot 2^{4r}} + \frac{R^2}{2^2 \cdot 2^{4r}} \right),$$

which is absurd. Going back to the variables u , changing $r + 1$ to r and adjusting the name of the constants one gets the result. □

7. Proof of the results on the applications

In this section, we prove the results stated in section 3, concerning applications of theorem 2.7. We recall that we are considering manifolds in which the Laplacian is a globally integrable quantum system. In particular, we denote by I_1, \dots, I_d the quantum actions of the Laplacian and by h_{L0} the function such that

$$-\Delta = h_{L0}(I_1, \dots, I_d).$$

Moreover, we denote by $\{\lambda_a\}_{a \in \Lambda}$ the eigenvalues of $-\Delta$, namely

$$\lambda_a = h_{L0}(a), \quad \forall a \in \Lambda.$$

As emphasized in section 3, they fulfill hypotheses L.0, L.1 and L.3.

From Weyl law it is also easy to prove the following lemma

Lemma 7.1. *There exists a sequence of intervals $\Sigma_n^0 = [c_n, d_n]$, $n \in \mathbb{N}, n \geq 1$ with the following properties:*

- $\frac{n}{2} \leq c_n < d_n < c_{n+1} < 3n$;
- $\sigma\left(\sqrt{-\Delta}\right) \subset \bigcup_n \Sigma_n^0$;
- $|d_n - c_n| \equiv |\Sigma_n^0| \leq 2$;
- $d(\Sigma_n^0, \Sigma_{n+1}^0) \equiv c_{n+1} - d_n \gtrsim n^{-d/2}$, where d is the dimension of M .

Such intervals will be the basis for the construction of the intervals Σ_n .

In the applications, we apply theorem 2.7 to an Hamiltonian with linear part H_L that is a relatively compact perturbation of $-\Delta$; for each of them we will check that they still fulfill hypotheses L.0, L.1 and L.3. We remark that considering small perturbations of the Laplacian is fundamental in order to verify the nonresonance assumption L.2.

Finally, we remark that all the nonlinear perturbations that we will meet are of the form (2.8), so they fulfill Hypothesis P. In other words, following theorem 5.2, they are functions with localized coefficients.

7.1. Schrödinger equation with convolution potentials

The nonlinear Schrödinger equation (3.3)

$$i \partial_t \psi = -\Delta \psi + V_{\sharp} \psi + f(x, |\psi|^2) \psi, \quad x \in M, \tag{7.1}$$

is Hamiltonian with Hamiltonian function

$$H = \int_M (\bar{\psi} (-\Delta \psi) + \bar{\psi} (V_{\sharp} \psi) + F(x, |\psi|^2)) \, dx$$

where F is such that $f(x, u) = \partial_u F(x, u)$.

The Hamiltonian of equation (7.1) fits the abstract settings (2.2) with H_L the globally integrable quantum system

$$H_L = -\Delta + V_{\sharp}.$$

The actions are I_1, \dots, I_d , namely they are the actions of the Laplacian; the associated function h_L is

$$\mathbb{R}^d \ni \xi \mapsto h_{L0}(\xi) + v(\xi),$$

where $v(\xi)$ is any C^∞ interpolation of V_{\sharp} on the lattice Λ , namely it is a function such that $v(\xi) = V_{\xi}, \forall \xi \in \Lambda$. Moreover, we remark that the frequencies are given by

$$\omega_a := \lambda_a + V_a.$$

In order to apply theorem 2.7 and prove theorem 3.2, it remains to verify that hypotheses L.1, L.3 and L.2 hold.

The intervals Σ_n are defined to be Σ_n^0 for n large, while they can be chosen in any arbitrary way for n in a bounded set. Due to the fact that $n > d/2$ they also partition the set $\left\{ \omega_a^{1/\beta} \right\}_{a \in \Lambda}$.

Hypotheses L.1 and L.3 clearly hold, since they hold for $\{\lambda_a\}_{a \in \Lambda}$ and the coefficients V_a have strong decay (see (3.2)). Indeed, for any $a, b \in \Lambda$,

$$|a - b| + |\lambda_a - \lambda_b| \geq C_\delta (|a|^\delta + |b|^\delta)$$

implies

$$\begin{aligned} |a - b| + |\omega_a - \omega_b| &\geq |a - b| + |\lambda_a - \lambda_b| - |V_a - V_b| \\ &\geq C_\delta (|a|^\delta + |b|^\delta) - 1/2 \geq C'_\delta (|a|^\delta + |b|^\delta). \end{aligned} \tag{7.2}$$

We come to prove the non-resonance condition L.2. Actually, we will prove a stronger condition, namely:

Lemma 7.2. *For any r there exists τ and a set $\mathcal{V}^{(\text{res})} \subset \mathcal{V}_n$ of zero measure, s.t., if $V \in \mathcal{V}_n \setminus \mathcal{V}^{(\text{res})}$ there exists $\gamma > 0$ s.t. for all $K \geq 1$ one has*

$$\begin{aligned} |\omega \cdot k| &\geq \frac{\gamma}{K^\tau}, \\ \forall k = (k_{a_1}, \dots, k_{a_r}) \text{ s.t. } |a_j| &\leq K \forall j = 1, \dots, r, |k|_{\ell^1} \leq r. \end{aligned} \tag{7.3}$$

First consider, for k fulfilling the second of (7.3),

$$\mathcal{V}(k, \gamma) := \{V \in \mathcal{V}_n : |\omega \cdot k| < \gamma\}.$$

We have the following lemma.

Lemma 7.3. *One has*

$$|\mathcal{V}(k, \gamma)| \leq 2\gamma K^n. \tag{7.4}$$

Proof. Actually, we prove that the Lemma is true for any arbitrary sequence λ_a , namely that the asymptotic behaviour is not important.

First, if $\mathcal{V}(k, \gamma)$ is empty there is nothing to prove. Assume that $\tilde{V} \in \mathcal{V}(k, \gamma)$. Since $k \neq 0$, there exists \bar{a} such that $k_{\bar{a}} \neq 0$ and thus $|k_{\bar{a}}| \geq 1$; so we have

$$\left| \frac{\partial \omega \cdot k}{\partial \hat{V}_{\bar{a}}} \right| \geq 1.$$

It means that if $\mathcal{V}(k, \gamma)$ is not empty it is contained in the layer

$$\left| \widehat{\hat{V}}_{\bar{a}}' - \hat{V}_{\bar{a}}' \right| \leq \gamma,$$

whose measure is $\gamma \langle \bar{a} \rangle^n \leq 2\gamma K^n$. This implies (7.4). □

Proof of lemma 7.2. From lemma 7.3 it follows that the measure of the set

$$\mathcal{V}^{(\text{res})}(\gamma) := \bigcup_{K \geq 1} \bigcup_k \mathcal{V}\left(k, \frac{\gamma}{K^{dr+2}}\right)$$

is estimated by a constant times γ . It follows that the set

$$\mathcal{V}^{(\text{res})} := \bigcap_{\gamma > 0} \mathcal{V}^{(\text{res})}(\gamma)$$

has zero measure and with this definition the lemma is proved. □

7.2. Stability of the ground state of NLS

The equation (3.4) is Hamiltonian with Hamiltonian

$$H(\psi, \bar{\psi}) = \int_M (\bar{\psi}(-\Delta\psi) + F(|\psi|^2)) dx, \tag{7.5}$$

where F is such that $F' = -f$. Following [FGL13], we introduce variables in which the ground state becomes an equilibrium point of a reduced system. To this end, we consider the space

$$L_0^2(M; \mathbb{C}) := \left\{ \varphi \in L^2(M; \mathbb{C}) : \int_M \varphi(x) dx = 0 \right\}$$

of the functions with vanishing average and we denote

$$N(\varphi) = \int_M |\varphi|^2 dx.$$

Then we consider the map

$$L_0^2(M, \mathbb{C}) \times \mathbb{R} \times \mathbb{T} \rightarrow L^2(M; \mathbb{C})$$

$$(\varphi, p, \theta) \mapsto \psi(\varphi, p, \theta) := e^{-i\theta} \left(\sqrt{p - \|\varphi\|^2} + \varphi(x) \right). \tag{7.6}$$

Lemma 7.4 (Faou, Lubich, Glouckler [FGL13]). *The map (7.6) defines a local coordinate system close to $\varphi = 0$. Furthermore, such coordinates are symplectic, namely the Hamilton Equations of a Hamiltonian function H have the form*

$$\dot{\theta} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial \theta}, \quad \dot{z} = -i\nabla_z H.$$

Then the Hamiltonian is just (7.5) with ψ given by (7.6). We now fix a value p_0 of p (which is an integral of motion) and expand in power series in φ getting (neglecting irrelevant terms independent of φ) a Hamiltonian of the form

$$H = H_0 + \hat{P}$$

with

$$H_0(\varphi) = \int_M \left[\bar{\varphi}(-\Delta\varphi) + 2f(p_0)\bar{\varphi}\varphi + \frac{f(p_0)}{2}(\varphi^2 + \bar{\varphi}^2) \right] dx$$

and $\hat{P} = \hat{P}(\varphi, \bar{\varphi})$ of the form (2.8). Thus we have just to verify the assumptions on the linear part.

Introducing the spectral decomposition relative to the quantum actions of the Laplacian, we get

$$H_0 = \sum_{a \in \Lambda} \int_M \left((\lambda_a + 2f(p_0)) \Pi_a \bar{\varphi} \Pi_a \varphi + \frac{f(p_0)}{2} (\Pi_a \varphi)^2 + \frac{f(p_0)}{2} (\Pi_a \bar{\varphi})^2 \right) dx,$$

which can be diagonalized through a symplectic change of coordinates of the form

$$\begin{pmatrix} w_a \\ \bar{w}_a \end{pmatrix} = S_a \begin{pmatrix} \varphi_a \\ \bar{\varphi}_a \end{pmatrix},$$

where the matrices S_a have norm uniformly bounded with respect to a . Such a change of coordinates of course does not change the nature of \hat{P} of having localized coefficients. In the new coordinates, one gets

$$H_0(w, \bar{w}) = \sum_a \omega_a \int_M \Pi_a \bar{w} \Pi_a w dx$$

with

$$\omega_a = \sqrt{\lambda_a^2 + 2f(p_0) \lambda_a}. \tag{7.7}$$

Hypotheses L.1 and L.3 hold for $\{\omega_a\}_{a \in \Lambda}$, since they hold for $\{\lambda_a\}_{a \in \Lambda}$ and one has

$$\omega_a = \sqrt{\lambda_a^2 + 2f(p_0) \lambda_a} = \lambda_a \left(1 + \frac{f(p_0)}{\lambda_a} + o\left(\frac{f(p_0)}{\lambda_a}\right) \right)$$

so that

$$|a - b| + |\lambda_a - \lambda_b| \geq C_\delta (|a|^\delta + |b|^\delta)$$

implies

$$|a - b| + |\omega_a - \omega_b| \geq C'_\delta (|a|^\delta + |b|^\delta),$$

with computations analogous to the ones in (7.2)

We prove now the non-resonance condition L.2.

First we define the intervals Σ_n working essentially as in [DS06]. Precisely, we consider the map $\mu \mapsto \nu(\mu) := (\mu^4 + 2f(p_0)\mu^2)^{1/4}$ and define $\Sigma_n := \nu(\Sigma_n^0)$. Then we follow the proof of Theorem 4.7 of [DS04] (see also [DS06]) which deals with the case of the wave equation on tori and Zoll Manifolds. In particular we apply the following theorem

Theorem 7.5. Theorem 5.1 of [DS04] *Let X be a closed ball \mathcal{B}_{R_0} in \mathbb{R}^p for some $R_0 > 0$ and by Y a compact interval in \mathbb{R} .*

Let $f : X \times Y \mapsto \mathbb{R}$ be a continuous subanalytic function, $\rho : X \mapsto \mathbb{R}$ a real analytic function, $\rho \neq 0$. Assume

- *f is real analytic on $\{x \in X; \rho(x) \neq 0\} \times Y$;*
- *for all $x_0 \in X$ with $\rho(x_0) \neq 0$, the equation $f(x_0, y) = 0$ has only finitely many solutions $y \in Y$.*

Then there are $N_0 \in \mathbb{N}, \alpha_0 > 0, \delta > 0, C > 0$, such that for any $\alpha \in (0, \alpha_0)$, any $N \geq N_0$, any $x \in X$ with $\rho(x) \neq 0$,

$$meas \{y \in Y: |f(x, y)| \leq \alpha |\rho(x)|^N\} \leq C \alpha^\delta |\rho(x)|^{N\delta}.$$

The strategy to deduce the existence of large sets of values p_0 corresponding to which assumption L.2 holds is nontrivial, but in our case, one can reproduce line by line the computations of section 5.2 of [DS04], just substituting the function f of that paper with the function

$$\mathfrak{f}(x, y) := \sum_{j=1}^l \sqrt{x_j^2 y + x_j} - \sum_{j=l+1}^p \sqrt{x_j^2 y + x_j},$$

which has the property that

$$m\mathbf{f}\left(\lambda_{a_j}, \frac{1}{m^2}\right)$$

with $m^2 = 2f(p_0)$ are exactly the small divisors one has to control. We also define, for $l \geq p/2$

$$\rho(x) := \prod_{\tau \in \mathfrak{S}_{p-l}} \left[\sum_{j=1}^l (x_j^2 - x_{\tau(j)+l}^2)^2 + \sum_{j=l+1}^{p-l} x_{\tau(j)+l}^2 \right], \tag{7.8}$$

and similarly for $l < p/2$. Here \mathfrak{S}_ℓ is the group of the permutations of the first ℓ integers.

The only nontrivial point is to prove that for any fixed value of $x \in [0, 1]^p$ the equation $\mathbf{f}(x, y) = 0$ has only isolated solutions in y . We just give a detailed proof of this fact.

First, we have to give a selection property for the sequences (x_1, \dots, x_p) : we will say that a sequence (x_1, \dots, x_p) , given an integer $l \leq p$ satisfy condition Z, if one of the following holds:

- p is odd
- p is even and $l \neq p/2$
- p is even, $l = p/2$ and for any permutation τ of $1, \dots, p/2$ exists j s.t. $x_{\tau(j)} \neq x_{p/2+j}$.

Lemma 7.6. *For any $(x_1, \dots, x_p) \in [0, 1]^p$ fulfilling condition Z, the equation $\mathbf{f}(x, y) = 0$ has a discrete set of solutions.*

Proof. Following [DS06] we remark that since \mathbf{f} is an analytic function of y , its roots can have accumulation points only if the function identically vanishes. We compute its Taylor expansion at $y = 0$ and show that it can be identically zero only if condition Z is violated. Denote

$$\nu_j = \sqrt{x_j^2 y + x_j},$$

by direct computation, we get

$$\frac{d^k \nu_j}{dy^k} = c_k \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j, \tag{7.9}$$

with suitable constants c_k . Thus we have

$$\frac{\partial^k \mathbf{f}}{\partial y^k} = c_k \left[\sum_{j=1}^l \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j - \sum_{j=l+1}^p \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j \right]. \tag{7.10}$$

Let consider the equivalence relation

$$x_i \sim x_j \iff (x_i = x_j \text{ and } (i, j \leq l \text{ or } i, j > l))$$

and denote by n_j the cardinality of the correspondent equivalence classes. So, we can write the condition $(7.10) = 0$ (renaming the indexes j), as

$$0 = \sum_{j=1}^{l_1} n_j \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j - \sum_{j=l_1+1}^{p_1} n_j \left(\frac{x_j}{\nu_j}\right)^{2k} \nu_j, \tag{7.11}$$

and now we have that $\forall i \neq j, x_i \neq x_j$. Remark that in our computation we have implicitly erased the terms with a plus sign which have a corresponding term with a minus sign, and that, since condition Z is fulfilled there must be at least a couple of indexes i, j with $i \leq l$ and $j > l$ such that $x_i \neq x_j$, so that not all the n_j 's vanish.

Now, (7.11) is a linear equation in the 'unknown' n_j : to have nontrivial solutions its determinant must vanish. However the determinant is essentially a Vandermonde determinant that can be explicitly computed giving

$$\nu_1 \dots \nu_{p_1} \prod_{1 \leq k < l \leq p_1} \left(\frac{x_l^2}{\nu_l^2} - \frac{x_k^2}{\nu_k^2} \right) \neq 0.$$

□

As we anticipated the rest of the proof goes exactly as in [DS06] and is omitted.

7.3. Beam equation

Consider the beam equation introduced in (3.5)

$$\psi_{tt} + \Delta_g^2 \psi + m\psi = -\partial_\psi P(x, \psi). \tag{7.12}$$

Introducing the variable $\varphi = \dot{\psi} \equiv \psi_t$, it is well known that (7.12) can be seen as an Hamiltonian system in the variables (ψ, φ) with Hamiltonian function

$$H(\psi, \varphi) := \int_M \left(\frac{\varphi^2}{2} + \frac{\psi (\Delta_g^2 + m) \psi}{2} + P(x, \psi) \right) dx.$$

To put the system in the form (2.2) introduce the variables

$$\begin{aligned} u(x) &:= \frac{1}{\sqrt{2}} \left((\Delta_g^2 + m)^{1/4} \psi + i (\Delta_g^2 + m)^{-1/4} \varphi \right), \\ \bar{u}(x) &:= \frac{1}{\sqrt{2}} \left((\Delta_g^2 + m)^{1/4} \psi - i (\Delta_g^2 + m)^{-1/4} \varphi \right), \end{aligned}$$

so that the Hamiltonian takes the form

$$H(u, \bar{u}) := \int_M \bar{u} (H_L u) + P(x, u, \bar{u}) dx,$$

with

$$H_L = \sqrt{-\Delta_g^2 + m}.$$

Then the verification of the hypotheses goes as in the previous case (but in a simpler way) and thus it is omitted.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Estimates on polynomials with localized coefficients

In this appendix, we will prove the results stated in section 4.3.

We denote $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}^0}$ and $\langle \cdot, \cdot \rangle_e = \langle \cdot, \cdot \rangle_{\mathcal{H}_e^0}$ and $\|\cdot\| := \|\cdot\|_0$.

Proof of lemma 4.18. For $B \in \Lambda_e$ and $\mathbf{A} = (A_1, \dots, A_r) \in \Lambda_e^r$, we want to bound

$$\left\| \Pi_B \tilde{X}_F(\Pi_{A_1} u, \dots, \Pi_{A_r} u) \right\|_0.$$

Suppose for simplicity that $B = (b, +)$, the case $B = (b, -)$ being totally analogous. We compute, exploiting self-adjointness of Π_B and the definition of the L^2 -gradient (4.2),

$$\begin{aligned} & \left\| \Pi_B \tilde{X}_F(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r) \right\|_0 \\ &= \sup_{h \in \mathcal{H}^\infty, \|h\|=1} \left| \langle \Pi_B \tilde{X}_F(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r), h \rangle_e \right| \\ &= \sup_{h \in \mathcal{H}^\infty, \|h\|=1} \left| \langle \tilde{X}_F(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r), \Pi_B h \rangle_e \right| \\ &\leq r \sup_{h \in \mathcal{H}^\infty, \|h\|=1} \left| \tilde{F}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r, \Pi_B h) \right| \\ &= r \|F\|_{r+1}^{\nu, N} \frac{\mu(\mathbf{A}, B)^{N+\nu}}{S(\mathbf{A}, B)^N} \|\Pi_{A_1} u_1\| \dots \|\Pi_{A_r} u_r\| \|\Pi_B h\|. \end{aligned}$$

From that, we deduce (4.6). □

A.1. Tame estimates

Lemma A.1. Fix $\nu > 0$. For $s > \nu + \frac{d}{2}$, we have

$$\sum_{A \in \Lambda_e} (1 + \langle |a| \rangle)^\nu \|\Pi_A u\|_0 \lesssim_s \|u\|_s. \tag{A.1}$$

Proof. We compute

$$\begin{aligned} \sum_{A \in \Lambda_e} (1 + \langle |a| \rangle)^\nu \|\Pi_A u\|_0 &\leq \sum_{A \in \Lambda_e} \frac{(1 + \langle |a| \rangle)^s}{(1 + \langle |a| \rangle)^{s-\nu}} \|\Pi_A u\|_0 \\ &\leq \sqrt{\sum_{A \in \Lambda_e} \frac{1}{(1 + \langle |a| \rangle)^{2(s-\nu)}}} \sqrt{\sum_{A \in \Lambda_e} (1 + \langle |a| \rangle)^{2s} \|\Pi_A u\|_0} \lesssim C_s \|u\|_s \end{aligned}$$

since the first sum converges for $2(s - \nu) > d$, that is $s > \nu + \frac{d}{2}$. □

Remark A.2. In the following computations, we will repeatedly compare $|a|$ and $\langle |a| \rangle$, taking profit of remark 4.9. In particular, we notice that for any constant $0 < K_2 < 1$ small enough there exists $K_1 > 0$ large enough such that

$$\langle |a| \rangle \geq K_1 \langle |b| \rangle \implies |a - b| \geq K_2 \langle |a| \rangle. \tag{A.2}$$

In fact we have, defining C_1, C_2 as in remark 4.9.

$$|a - b| \geq |a| - |b| \geq \frac{1}{C_2} \langle |a| \rangle - \frac{1}{C_1} \langle |b| \rangle \geq \left(\frac{1}{C_2} - \frac{1}{C_1 K_1} \right) \langle |a| \rangle.$$

Moreover $\exists C$ such that

$$\langle |a| \rangle \geq \langle |b| \rangle \implies |a - b| \leq C \langle |a| \rangle. \tag{A.3}$$

We are now ready to prove theorem 4.20:

Proof of theorem 4.20. We have

$$\|\tilde{X}(u_1, \dots, u_r)\|_s^2 \leq \sum_{B \in \Lambda} (1 + \langle |B| \rangle)^{2s} \left\| \sum_{A_1, \dots, A_r} \Pi_B \tilde{X}(\Pi_{A_1} u_1, \dots, \Pi_{A_r} u_r) \right\|_0^2.$$

Exploiting the definition of localized coefficients 4.17, the argument of the sum in B of the equality above is controlled by the square of

$$\|X\|^{\nu, N} \sum_{A_1, \dots, A_r} (1 + \langle |b| \rangle)^s \frac{\mu(\mathbf{A}, b)^{N+\nu}}{S(\mathbf{A}, b)^N} \|\Pi_{A_1} u_1\|_0 \dots \|\Pi_{A_r} u_r\|_0.$$

By symmetry, we can consider the case $\langle |a_1| \rangle \geq \langle |a_2| \rangle \geq \dots \geq \|a_r\|$. By definition of S and μ , we have

$$\begin{cases} \langle |b| \rangle \geq \langle |a_2| \rangle \implies S(\mathbf{A}, b) = |a_1 - b| + \langle |a_2| \rangle \geq |a_1 - b| \\ \langle |b| \rangle \leq \langle |a_2| \rangle \implies S(\mathbf{A}, b) \geq |a_1 - a_2| + \langle |b| \rangle. \end{cases}$$

We have, $\forall \kappa > d$,

$$\begin{aligned} \sum_{B \in \Lambda_e} S^{-\kappa} &\leq \sum_{\langle |b| \rangle \geq \langle |a_2| \rangle} S^{-\kappa} + \sum_{\langle |b| \rangle < \langle |a_2| \rangle} S^{-\kappa} \\ &\lesssim \sum_{B \in \Lambda_e} \frac{1}{(|a_1 - b| + 1)^\kappa} + \sum_{B \in \Lambda_e} \frac{1}{(|a_1 - a_2| + \langle |b| \rangle)^\kappa} \\ &\lesssim \sum_{B' \in \Lambda_e} \frac{1}{(|b'| + 1)^\kappa} + \sum_{B \in \Lambda_e} \frac{1}{(|a_1 - a_2| + |b|)^\kappa} < \infty \end{aligned}$$

where $B' = (b', \sigma_b)$, with $b' = b - a_1$. With a similar calculation, we obtain also

$$\sum_{A_1 \in \Lambda_e} S^{-\kappa} < \infty.$$

By Cauchy–Schwarz inequality, we estimate

$$\begin{aligned} \sum_{A_1, \dots, A_r} (1 + \langle |b| \rangle)^s \frac{\mu(\mathbf{A}, b)^{N+\nu}}{S(\mathbf{A}, b)^N} \|\Pi_{A_1} u_1\|_0 \dots \|\Pi_{A_r} u_r\|_0 &\leq \\ &\leq \left(\sum_{A_1, \dots, A_r} (1 + \langle |b| \rangle)^{2s} \frac{\mu(\mathbf{A}, b)^{2N+\nu-\kappa}}{S(\mathbf{A}, b)^{2N-\kappa}} \|\Pi_{A_1} u_1\|_0^2 \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}} \cdot \\ &\cdot \left(\sum_{A_1, \dots, A_r} \frac{\mu(\mathbf{A}, b)^{\nu+\kappa}}{S(\mathbf{A}, b)^\kappa} \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}}. \end{aligned}$$

Exploiting $\mu(\mathbf{A}, b) \leq \langle |a_2| \rangle$, the second term is estimated by

$$\begin{aligned} &\left(\sum_{A_1, \dots, A_r} \frac{\mu(\mathbf{A}, b)^{\nu+\kappa}}{S(\mathbf{A}, b)^\kappa} \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{A_1} S(\mathbf{A}, b)^{-\kappa} \sum_{A_2, \dots, A_r} \langle |a_2| \rangle^{\nu+\kappa} \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}} \lesssim \prod_{l=2}^r \|u_l\|_{s_0}^{\frac{1}{2}} \end{aligned}$$

for each $s_0 > \nu + \frac{d}{2} + \kappa$, where in the last passage we have used lemma A.1.

Consider now the first term (A.1). We claim that

$$\frac{\mu(\mathbf{A}, b)}{S(\mathbf{A}, b)} (1 + \langle |b| \rangle) \lesssim 1 + \langle |a_1| \rangle. \tag{A.4}$$

Indeed, (A.4) is trivial for $1 + \langle |b| \rangle \lesssim (1 + \langle |a_1| \rangle)$, since $\frac{\mu}{S} < 1$ by definition. On the other end, if $1 + \langle |b| \rangle \gtrsim (1 + \langle |a_1| \rangle)$, we have, from (A.2),

$$S(\mathbf{A}, b) \geq |b - a_1| \gtrsim \langle |b| \rangle$$

and then

$$\frac{\mu(\mathbf{A}, b)}{S(\mathbf{A}, b)} (1 + \langle |b| \rangle) \lesssim \mu(\mathbf{A}, b) \lesssim 1 + \langle |a_1| \rangle$$

that is (A.4). Then we can control the first term, provided that $N > s + \kappa$, with

$$\begin{aligned} & \sum_B \left(\sum_{A_1, \dots, A_r} (1 + \langle |b| \rangle)^{2s} \frac{\mu(\mathbf{A}, b)^{2N + \nu - \kappa}}{S(\mathbf{A}, b)^{2N - \kappa}} \|\Pi_{A_1} u_1\|_0^2 \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}} \\ & \lesssim \sum_B \left(\sum_{A_1, \dots, A_r} (1 + \langle |a_1| \rangle)^{2s} \frac{\mu(\mathbf{A}, b)^{\kappa + \nu}}{S(\mathbf{A}, b)^\kappa} \|\Pi_{A_1} u_1\|_0^2 \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}} \\ & \lesssim \sum_B \left(\sum_{A_1} \frac{(1 + \langle |a_1| \rangle)^{2s}}{S(\mathbf{A}, b)^\kappa} \|\Pi_{A_1} u_1\|_0^2 \sum_{A_2, \dots, A_r} \langle |a_2| \rangle^{\nu + \kappa} \|\Pi_{A_2} u_2\|_0 \dots \|\Pi_{A_r} u_r\|_0 \right)^{\frac{1}{2}} \\ & \lesssim \|u_1\|_s \prod_{l=2}^r \|u_l\|_{s_0}^{\frac{1}{2}}, \end{aligned}$$

with $s_0 \geq \kappa + \nu + \frac{d}{2}$. Summing over all the possible choices of the biggest index, we obtain the sum in the thesis. \square

A.2. High and low modes

Proof of corollary 4.27. Firstly recall the usual high modes estimate

$$\|u^\perp\|_{s_0}^2 = \sum_{\langle |A| \rangle > K} (1 + \langle |a| \rangle)^{2s_0} \|\Pi_A u\|_0^2 \lesssim \frac{1}{K^{2(s-s_0)}} \|u\|_s^2. \tag{A.5}$$

i) If P is at least of order 3 in u^\perp , X_P is at least of order two in u^\perp . Then

$$\begin{aligned} X_P(u) &= X(u^{\leq} + u^\perp) = \tilde{X}_P(u^{\leq} + u^\perp, \dots, u^{\leq} + u^\perp) \\ &= \sum_{l=2}^r \binom{r}{l} \tilde{X}_P \left(\underset{r-l\text{-times}}{u^{\leq}}, \dots, \underset{l\text{-times}}{u^\perp}, \dots, u^\perp \right). \end{aligned}$$

Applying the tame estimate in lemma 4.20, we get

$$\begin{aligned} \|X_P(u)\|_s &\leq \|P\|^{\nu, N} \sum_{l=2}^r \left(\|u^{\leq}\|_s \|u^{\leq}\|_{s_0}^{r-l-1} \|u^\perp\|_{s_0}^l + \|u^{\leq}\|_{s_0}^{r-l} \|u^\perp\|_{s_0}^{l-1} \|u^\perp\|_s \right) \\ &\stackrel{(A.5)}{\lesssim} \frac{\|P\|^{\nu, N}}{K^{s-s_0}} \|u\|_s^2 \|u\|_{s_0}^{r-2}. \end{aligned}$$

ii) We reason as in the previous case, since again $\Pi^{\leq} X_P$ is of order two in u^\perp . \square

Proof of Corollary 4.28. Writing $N' = N - M_1$, (4.11) amounts to show that

$$\begin{aligned} \frac{\mu(A_1, \dots, A_r)^{N+\nu}}{S(A_1, \dots, A_r)^N} &= \frac{\mu(A_1, \dots, A_r)^{N'+\nu+M_1}}{S(A_1, \dots, A_r)^{N'+M_1}} \leq \frac{\mu(A_1, \dots, A_r)^{N'+\nu+M_1}}{S(A_1, \dots, A_r)^{N'}} \frac{1}{|a_{\tau_{\text{ord}(1)}} - a_{\tau_{\text{ord}(2)}}|^{M_1}} \\ &\leq \frac{\mu(A_1, \dots, A_r)^{N'+\nu+M_1}}{S(A_1, \dots, A_r)^{N'}} \frac{1}{K^{\delta M_1}}. \end{aligned}$$

Then we apply directly lemmas 4.6 and 4.20 and we get (4.12). \square

A.3. Poisson brackets

We conclude this section by showing that our class is closed with respect to Poisson brackets. Before proving the result, we state the following useful lemma.

Lemma A.3. *Given a polynomial P and a polynomial map X , we have*

$$dPX(u) = \eta [\tilde{P}(u, \dots, u, X(u))],$$

where

$$\eta [\tilde{P}(u, \dots, u, h)] := \tilde{P}(h, u, \dots, u) + \tilde{P}(u, h, \dots, u) + \dots + \tilde{P}(u, \dots, u, h).$$

Moreover if P has degree $p + 1$ and X has degree q , the multilinear polynomial associated to dPX is given by

$$\widetilde{dPX}(u_1, \dots, u_{p+q}) = \eta [\tilde{P}(u_1, \dots, u_p, \tilde{X}(u_{p+1}, \dots, u_{p+q}))]. \tag{A.6}$$

Proof of lemma 4.23. Let $u_1, \dots, u_p, u_{p+1}, \dots, u_{p+q} \in \mathcal{H}_e^\infty$ and $\mathbf{A} = (A_1, \dots, A_{p+q}) \in \Lambda_e^{p+q}$. By lemma A.3, we have

$$\begin{aligned} \left| \widetilde{dFX}(\Pi_{A_1} u_1, \dots, \Pi_{A_{p+q}} u_{p+q}) \right| &= \left| \eta \left[\tilde{F} \left(\Pi_{A_1} u_1, \dots, \Pi_{A_p} u_p, \tilde{X} \left(\Pi_{A_{p+1}} u_{p+1}, \dots, \Pi_{A_{p+q}} u_{p+q} \right) \right) \right] \right| \\ &\leq (p + 1) \sum_{B \in \Lambda_e} \left| \tilde{F} \left(\Pi_{A_1} u_1, \dots, \Pi_{A_p} u_p, \Pi_B \tilde{X} \left(\Pi_{A_{p+1}} u_{p+1}, \dots, \Pi_{A_{p+q}} u_{p+q} \right) \right) \right| \\ &\leq (p + 1) \|F\|^{\nu_1, N} \times \sum_{B \in \Lambda_e} \frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p)^N} \left\| \Pi_B \tilde{X} \left(\Pi_{A_{p+1}} u_{p+1}, \dots, \Pi_{A_{p+q}} u_{p+q} \right) \right\|_0 \\ &\leq (p + 1) \|F\|^{\nu_1, N} \|X\|^{\nu_2, N} \times \sum_{B \in \Lambda_e} \frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p)^N} \frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_1, \dots, A_p)^N}, \end{aligned}$$

since F, X have localized coefficients. That is, we need to prove the following estimate

$$\sum_{B \in \Lambda_e} \frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p, b)^N} \frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_{p+1}, \dots, A_{p+q}, b)^N} \lesssim \frac{\mu(A_1, \dots, A_{p+q})^{N'+\nu'}}{S(A_1, \dots, A_{p+q})^{N'}}. \tag{A.7}$$

By symmetry, we can assume the following relations:

$$\langle |a_1| \rangle \geq \dots \geq \langle |a_p| \rangle, \quad \langle |a_{p+1}| \rangle \geq \dots \geq \langle |a_{p+q}| \rangle, \quad \langle |a_{p+1}| \rangle \leq \langle |a_1| \rangle.$$

Case 1 Assume $\langle |a_1| \rangle \geq \langle |a_{p+1}| \rangle \geq \langle |a_2| \rangle$.

In this case, we have

$$\begin{aligned} \mu(A_1, \dots, A_{p+q}) &= \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle), \\ S(A_1, \dots, A_{p+q}) &= |a_1 - a_{p+1}| + \mu(A_1, \dots, A_{p+q}), \end{aligned}$$

and

$$\begin{aligned} \mu(A_1, \dots, A_p, b) &\leq \mu(A_1, \dots, A_{p+q}), \\ \mu(A_{p+1}, \dots, A_{p+q}, b) &\leq \mu(A_1, \dots, A_{p+q}). \end{aligned} \tag{A.8}$$

Case 1.i For $\langle |b| \rangle > \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)$, we have

$$\begin{aligned} S(A_1, \dots, A_p, b) &= |a_1 - b| + \langle |a_2| \rangle, \\ S(A_{p+1}, \dots, A_{p+q}, b) &= |a_{p+1} - b| + \langle |a_{p+2}| \rangle, \\ |a_1 - a_{p+1}| &= |a_1 - b - a_{p+1} + b| \leq |a_1 - b| + |a_{p+1} - b|. \end{aligned} \tag{A.9}$$

From (A.8) and (A.9), we deduce

$$\begin{aligned} \frac{S(A_1, \dots, A_{p+q})}{\mu(A_1, \dots, A_{p+q})} &= 1 + \frac{|a_1 - a_{p+1}|}{\mu(A_1, \dots, A_{p+q})} \leq 1 + \frac{|a_1 - b|}{\mu(A_1, \dots, A_p, b)} + \frac{|a_{p+1} - b|}{\mu(A_{p+1}, \dots, A_{p+q}, b)} \\ &\leq \frac{S(A_1, \dots, A_p, b)}{\mu(A_1, \dots, A_p, b)} + \frac{S(A_{p+1}, \dots, A_{p+q}, b)}{\mu(A_{p+1}, \dots, A_{p+q}, b)} \\ &\leq 2 \max \left\{ \frac{S(A_1, \dots, A_p, b)}{\mu(A_1, \dots, A_p, b)}, \frac{S(A_{p+1}, \dots, A_{p+q}, b)}{\mu(A_{p+1}, \dots, A_{p+q}, b)} \right\}. \end{aligned} \tag{A.10}$$

Let us define

$$L_1 = \left\{ b \in \Lambda : \langle |b| \rangle > \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle), \frac{S(A_1, \dots, A_p, b)}{\mu(A_1, \dots, A_p, b)} \geq \frac{S(A_{p+1}, \dots, A_{p+q}, b)}{\mu(A_{p+1}, \dots, A_{p+q}, b)} \right\}.$$

Depending on the value of \mathbf{A} one could have $L_1 = \emptyset$, but this is irrelevant for the following. If $b \in L_1$, estimate (A.10) implies

$$\frac{\mu(A_1, \dots, A_p, b)}{S(A_1, \dots, A_p, b)} \leq 2 \frac{\mu(A_1, \dots, A_{p+q})}{S(A_1, \dots, A_{p+q})}. \tag{A.12}$$

We observe moreover that

$$\frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_{p+1}, \dots, A_{p+q}, b)^N} \leq \mu(A_{p+1}, \dots, A_{p+q}, b)^{\nu_2}. \tag{A.13}$$

Then, using (A.8), (A.12) and (A.13), we have

$$\begin{aligned} \sum_{B \in L_1} \frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p, b)^N} \frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_{p+1}, \dots, A_{p+q}, b)^N} &\stackrel{(A.8)}{\lesssim} \mu(A_1, \dots, A_{p+q})^{\nu_1+\nu_2} \\ &\times \sum_{B \in L_1} \frac{\mu(A_1, \dots, A_p, b)^{N-d-1}}{S(A_1, \dots, A_p, b)^{N-d-1}} \frac{\mu(A_1, \dots, A_p, b)^{d+1}}{S(A_1, \dots, A_p, b)^{d+1}} \\ &\lesssim \mu(A_1, \dots, A_{p+q})^{\nu_1+\nu_2} \sum_{b \in L_1} \frac{\mu(A_1, \dots, A_{p+q})^{N-d-1}}{S(A_1, \dots, A_{p+q})^{N-d-1}} \frac{\mu(A_1, \dots, A_{p+q})^{d+1}}{S(A_1, \dots, A_p, b)^{d+1}}, \\ &\stackrel{(A.12), (A.8)}{\lesssim} \frac{\mu(A_1, \dots, A_{p+q})^{N+\nu_1+\nu_2}}{S(A_1, \dots, A_{p+q})^{N-d-1}} \sum_{B \in L_1} \frac{1}{(|a_1 - b| + 1)^{d+1}}, \end{aligned}$$

that is (A.7), with $N' = N - d - 1$ and $\nu' = \nu_1 + \nu_2 + d + 1$. The case $b \in L_1^c$ is analogous.

Case 1.ii For $\langle |b| \rangle \leq \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)$, we remark that the index of the sum in (A.7) runs over a set with cardinality controlled by $\mu(A_1, \dots, A_{p+q})$. That is, it suffices to prove that

$$\frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p, b)^N} \frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_{p+1}, \dots, A_{p+q}, b)^N} \frac{S(A_1, \dots, A_{p+q})^{N'}}{\mu(A_1, \dots, A_{p+q})^{N'+\nu'}} \quad (\text{A.14})$$

is controlled by a constant independent of b .

Take $0 < K_2 < 1$ and let K_1 and C be as in (A.2) If $\langle |a_1| \rangle \geq K_1 \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)$, we get,

$$\begin{aligned} (\text{A.14}) &\leq \frac{\max(\langle |b| \rangle, \langle |a_3| \rangle)^{N+\nu_1}}{(K_2 \langle |a_1| \rangle)^N} \max(\langle |b| \rangle, \langle |a_{p+3}| \rangle)^{\nu_2} \\ &\quad \times \frac{(C \langle |a_1| \rangle + \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle))^{N'}}{\max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)^{N'+\nu'}} \quad (\text{A.15}) \\ &\lesssim \frac{\max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)^{N+\nu_1+\nu_2}}{\langle |a_1| \rangle^N} \frac{(\langle |a_1| \rangle)^{N'}}{\max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)^{N'+\nu'}} \lesssim 1, \end{aligned}$$

choosing $N' = N$ and $\nu' = \nu_1 + \nu_2$.

If $\langle |a_1| \rangle \leq K_1 \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)$, we get instead

$$\begin{aligned} (\text{A.14}) &\leq \max(\langle |b| \rangle, \langle |a_3| \rangle)^{\nu_1} \max(\langle |b| \rangle, \langle |a_{p+3}| \rangle)^{\nu_2} \\ &\quad \times \frac{(C \langle |a_1| \rangle + \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle))^{N'}}{\max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)^{N'+\nu'}} \quad (\text{A.16}) \\ &\lesssim \max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)^{\nu_1+\nu_2} \frac{(\max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle))^{N'}}{\max(\langle |a_2| \rangle, \langle |a_{p+2}| \rangle)^{N'+\nu'}} \lesssim 1, \end{aligned}$$

for $\nu' = \nu_1 + \nu_2$. This proves the claim in the case 1.ii.

Case 2 Assume $\langle |a_1| \rangle \geq \langle |a_2| \rangle \geq \langle |a_{p+1}| \rangle$. In this case, we have

$$\begin{aligned} \mu(A_1, \dots, A_{p+q}) &= \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle), \\ S(A_1, \dots, A_{p+q}) &= |a_1 - a_2| + \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle). \end{aligned}$$

Case 2.i Take K_2 as in (A.2) and determine the corresponding K_1 . For $\langle |b| \rangle > K_1 \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle)$, we have

$$\begin{aligned} S(A_1, \dots, A_p, b) &= |a_1 - \operatorname{argmax}(\langle |a_2| \rangle, \langle |b| \rangle)| + \min(\langle |a_2| \rangle, \langle |b| \rangle), \\ S(A_{p+1}, \dots, A_{p+q}, b) &= |b - a_{p+1}| + \langle |a_{p+2}| \rangle, \end{aligned}$$

and moreover,

$$|a_{p+1} - b| \geq K_2 \langle |b| \rangle.$$

Let us define

$$G_1 := \{b: \langle |b| \rangle > K_1 \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle), \langle |b| \rangle < \langle |a_2| \rangle\}.$$

For $b \in G_1$, we estimate

$$\begin{aligned}
 & \sum_{b \in G_1} \frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p, b)^N} \frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_{p+1}, \dots, A_{p+q}, b)^N} \\
 &= \sum_{b \in G_1} \frac{\langle |b| \rangle^{N+\nu_1}}{(|a_1 - a_2| + \langle |b| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{(|a_{p+1} - b| + \langle |a_{p+2}| \rangle)^N} \\
 &\lesssim \sum_{b \in G_1} \frac{\langle |b| \rangle^{N+\nu_1}}{(|a_1 - a_2| + \langle |b| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{\langle |b| \rangle^N} \\
 &\lesssim \sum_{\langle |b| \rangle > K_1 \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle)} \frac{\langle |b| \rangle^{\nu_1}}{(|a_1 - a_2| + \langle |b| \rangle)^N} \langle |a_{p+2}| \rangle^{N+\nu_2} \\
 &\lesssim \mu(A_1, \dots, A_{p+q})^{N+\nu_2} \frac{1}{(|a_1 - a_2| + \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle))^{N-\nu_1-d-1}} \\
 &\lesssim \frac{\mu(A_1, \dots, A_{p+q})^{N+\nu_2}}{S(A_1, \dots, A_{p+q})^{N-\nu_1-d-1}},
 \end{aligned} \tag{A.17}$$

where we are using the inequality

$$\sum_{|k| > A} \frac{|k|^l}{(|k| + B)^N} \leq \sum_{|k| > A} \frac{(|k| + B)^l}{(|k| + B)^N} \leq \frac{1}{(A + B)^{N-l-d-1}}.$$

This is the thesis with $N' = N - d - \nu_1 - 1$ and $\nu' = \nu_1 + \nu_2 + d$.

Let $G_2 := \{b: \langle |b| \rangle > K_1 \max(\langle |a_3| \rangle, \langle |a_{p+1}| \rangle), \langle |b| \rangle \geq \langle |a_2| \rangle\}$, then we have to estimate

$$\sum_{b \in G_2} \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{(|a_{p+1} - b| + \langle |a_{p+2}| \rangle)^N}. \tag{A.18}$$

We observe that there exist two constants K_3, K_4 such that

$$\langle |b| \rangle \leq K_4 \langle |a_1| \rangle \Rightarrow |a_1 - b| \geq K_3 |a_1 - a_2|. \tag{A.19}$$

Then we estimate (A.18) by

$$\begin{aligned}
 & \sum_{b \in G_2} \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{|a_{p+1} - b|^N} \\
 &\stackrel{(A.19)}{\lesssim} \sum_{b \in G_2, \langle |b| \rangle \leq K_4 \langle |a_1| \rangle} \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{\langle |b| \rangle^N} \\
 &\quad + \sum_{b \in G_2, \langle |b| \rangle \geq K_4 \langle |a_1| \rangle} \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{\langle |b| \rangle^N}.
 \end{aligned} \tag{A.20}$$

The first term in (A.20) is controlled by

$$\begin{aligned} & \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N+\nu_2}}{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N-\nu_1-d-1}} \sum_{b \in G_2} \frac{\langle |a_2| \rangle^{N+\nu_1} \langle |b| \rangle^{-N}}{(|a_1 - a_2| + \langle |a_2| \rangle)^{\nu_1+d+1}} \\ & \lesssim \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N+\nu_2}}{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N-\nu_1-d-1}} \sum_{b \in G_2} \frac{1}{\langle |b| \rangle^{d+1}} \\ & \lesssim \frac{\mu(A_1, \dots, A_{p+q})^{N'+\nu'}}{S(A_1, \dots, A_{p+q})^{N'}}. \end{aligned}$$

For the second term in (A.20) we claim that $\langle |b| \rangle \geq K_4 \langle |a_1| \rangle$ implies

$$\frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{\langle |b| \rangle^N} \lesssim \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}}{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}} \frac{1}{\langle |b| \rangle^{d+1}},$$

with $N' = N - \nu_1 - d - 1$ and $\nu' = \nu_1 + \nu_2 + d + 1$. Indeed, we have

$$\begin{aligned} & \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{\langle |b| \rangle^N} \frac{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \langle |b| \rangle^{d+1} \\ & \lesssim \langle |a_2| \rangle^{\nu_1} \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N+\nu_2} \frac{1}{\langle |b| \rangle^{N-d-1}} \frac{\langle |a_1| \rangle^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \\ & \stackrel{\langle |b| \rangle \geq K_4 \langle |a_2| \rangle}{\lesssim} \frac{\langle |a_1| \rangle^{N'}}{\langle |b| \rangle^{N-d-1-\nu_1}} \lesssim \frac{\langle |a_1| \rangle^{N'}}{\langle |a_1| \rangle^{N-d-1-\nu_1}} \lesssim 1, \end{aligned}$$

for any $N' \leq N - d - 1 - \nu_1$. Then, we estimate

$$\begin{aligned} & \sum_{b \in G_2, \langle |b| \rangle \geq K_4 \langle |a_1| \rangle} \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{\langle |b| \rangle^N} \\ & \lesssim \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}}{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}} \sum_{b \in G_2} \frac{1}{\langle |b| \rangle^{d+1}} \\ & \lesssim \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}}{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}}. \end{aligned}$$

Case 2.ii For $\langle |b| \rangle \leq K_1 \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)$, we argue as in Case 1.ii, and it is sufficient to bound from above

$$\frac{\mu(A_1, \dots, A_p, b)^{N+\nu_1}}{S(A_1, \dots, A_p, b)^N} \frac{\mu(A_{p+1}, \dots, A_{p+q}, b)^{N+\nu_2}}{S(A_{p+1}, \dots, A_{p+q}, b)^N} \frac{S(A_1, \dots, A_{p+q})^{N'}}{\mu(A_1, \dots, A_{p+q})^{N'+\nu'}} \quad (\text{A.21})$$

with a constant.

If $|a_1 - a_2| \leq 2 \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)$, we get

$$\begin{aligned} (\text{A.21}) & \leq \max(\langle |b| \rangle, \langle |a_3| \rangle)^{\nu_1} \mu(A_{p+1}, \dots, A_{p+q}, b)^{\nu_2} \\ & \quad \times \frac{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \\ & \lesssim \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{\nu_1+\nu_2} \frac{(\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \lesssim 1, \end{aligned}$$

for $\nu' \geq \nu_1 + \nu_2$.

If $|a_1 - a_2| \geq 2 \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)$, we consider separately the case $\langle |b| \rangle \leq \langle |a_2| \rangle$ and the case $\langle |b| \rangle > \langle |a_2| \rangle$.

If $\langle |b| \rangle \leq \langle |a_2| \rangle$, we have

$$\begin{aligned} \text{(A.21)} &\leq \frac{\max(\langle |b| \rangle, \langle |a_3| \rangle)^{N+\nu_1}}{|a_1 - a_2|^N} \mu(A_{p+1}, \dots, A_{p+q}, b)^{\nu_2} \\ &\quad \times \frac{(|a_1 - a_2| + \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle))^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \\ &\leq 2^{N+\nu_1+\nu_2} \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N+\nu_1+\nu_2}}{|a_1 - a_2|^N} \frac{(|a_1 - a_2| + \frac{1}{2}|a_1 - a_2|)^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \leq 3^{N'}, \end{aligned}$$

for $\nu' \geq \nu_1 + \nu_2$.

If $\langle |b| \rangle > \langle |a_2| \rangle$ and recalling that $\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle) \leq \frac{1}{2}|a_1 - a_2|$, we need to bound

$$\begin{aligned} \text{(A.21)} &\leq \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \frac{\langle |a_{p+2}| \rangle^{N+\nu_2}}{(|a_{p+1} - b| + \langle |a_{p+2}| \rangle)^N} \frac{(\frac{3}{2}|a_1 - a_2|)^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \\ &\lesssim \frac{\langle |a_2| \rangle^{N+\nu_1}}{(|a_1 - b| + \langle |a_2| \rangle)^N} \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{\nu_2} \frac{(\frac{1}{C}|a_1 - b| + \langle |b| \rangle)^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}}, \end{aligned} \tag{A.22}$$

where we are using the following triangular inequality,

$$|a_1 - a_2| \leq |a_1 - b| + |b - a_2| \leq |a_1 - b| + C\langle |b| \rangle,$$

and $\langle |a_2| \rangle < \langle |b| \rangle$.

If $|a_1 - b| \leq \langle |b| \rangle$ we conclude, recalling that $\langle |b| \rangle \leq K_1 \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)$, that

$$\begin{aligned} \text{(A.22)} &\lesssim \max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{\nu_1+\nu_2} \frac{\langle |b| \rangle^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \\ &\lesssim \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{\nu_1+\nu_2+N}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \lesssim 1, \end{aligned}$$

for $N = N'$ and $\nu' = \nu_1 + \nu_2$. If instead $|a_1 - b| \geq \langle |b| \rangle$, we conclude that

$$\text{(A.22)} \lesssim \frac{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N+\nu_1+\nu_2}}{|a_1 - b|^N} \frac{|a_1 - b|^{N'}}{\max(\langle |a_{p+1}| \rangle, \langle |a_3| \rangle)^{N'+\nu'}} \lesssim 1, \tag{A.23}$$

for $N \leq N'$ and $\nu' \geq \nu_1 + \nu_2$. □

Appendix B. Proof of theorem 5.2

In this section, we prove theorem 5.2.

The main step to prove it is the following.

Theorem B.1. *Let $g \in C^\infty(M)$, then there exists $\nu > 0$, depending on k and the dimension d , and $\forall N \in \mathbb{N}$ there exists a constant C_N such that, $\forall u_1, \dots, u_k \in \mathcal{H}^\infty$,*

$$\left| \int_M g \Pi_{a_1} u_1 \dots \Pi_{a_k} u_k \right| \leq C_N \frac{\mu(a_1, \dots, a_k)^{N+\nu}}{S(a_1, \dots, a_k)^N} \prod_{l=1}^k \|\Pi_{a_l} u_l\|_0,$$

for any $(a_1, \dots, a_k) \in \Lambda^k$, where Π_{a_i} are the projectors defined in (3.1).

The strategy of the proof is very similar to that of the corresponding results in [DS06, BDGS07, DI17]; the difference being that the indexes run over the set Λ related to the quantum actions. We just recall the main steps and we write the proof of the new lemmas.

First, given two linear operators P, B define, for any $N \in \mathbb{N}$,

$$Ad_P^N(B) = [Ad_P^{N-1}(B), P], \quad Ad_P^0(B) = B.$$

Then we recall the following Lemma from [DS06], (see the proof of proposition 1.2.1).

Lemma B.2. *There exists ν s.t. for any $P \in \Psi^1(M)$ and any $N \in \mathbb{N}$ there exist constants $C_N = C_N(P)$ with the property that for any $f \in C^\infty(M)$, denote by f also the operator of multiplication by f , then one has*

$$\|Ad_P^N f\|_{\mathcal{B}(L^2(M))} \leq C_N \|f\|_{N+\nu}.$$

Given $a = (a^1, \dots, a^d) \in \Lambda$, we denote with $l^*(a)$ the index for which $|a^l|$ is maximum. Namely,

$$l^*(a) := \operatorname{argmax}_{l=1, \dots, d} |a^l|. \quad (\text{B.1})$$

Lemma B.3. *Let $B \in \mathcal{B}(L^2(M))$ and $a, b \in \Lambda$. For any $N \geq 0$ and $u_1, u_2 \in \mathcal{H}^\infty$ we have*

$$|\langle B \Pi_a u_1, \Pi_b u_2 \rangle| \leq C_N \frac{\|ad_{I_{l^*}}^N(B)\|_{\mathcal{B}(L^2(M))}}{|a-b|^N} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0.$$

with $l^* = l^*(a-b)$.

Proof. First we claim that, for any $l = 1, \dots, d$,

$$|\langle B \Pi_a u_1, \Pi_b u_2 \rangle| \leq C_N \frac{\|ad_{I_l}^N(B) \Pi_a u_1\|_0}{|a^l - b^l|^N} \|\Pi_b u_2\|_0. \quad (\text{B.2})$$

For $N = 1$, recalling that the actions I_j are selfadjoint, we have

$$\begin{aligned} \langle Ad_{I_l}(B) \Pi_a u_1, \Pi_b u_2 \rangle &= |\langle B I_l \Pi_a u_1, \Pi_b u_2 \rangle - \langle I_l B \Pi_a u_1, \Pi_b u_2 \rangle| \\ &= |\langle B I_l \Pi_a u_1, \Pi_b u_2 \rangle - \langle B \Pi_a u_1, I_l \Pi_b u_2 \rangle| \\ &= |a^l \langle B \Pi_a u_1, \Pi_b u_2 \rangle - b^l \langle B \Pi_a u_1, \Pi_b u_2 \rangle| \\ &= |a^l - b^l| |\langle B \Pi_a u_1, \Pi_b u_2 \rangle|. \end{aligned}$$

Then, by induction on N , and arguing as in the case $N = 1$ with B replaced by $ad_{I_l}^{N-1}(B)$, we get

$$\begin{aligned} \langle Ad_{I_l}^N(B) \Pi_a u_1, \Pi_b u_2 \rangle &= \langle Ad_{I_l}(Ad_{I_l}^{N-1}(B)) \Pi_a u_1, \Pi_b u_2 \rangle \\ &= |a^l - b^l| \langle Ad_{I_l}^{N-1}(B) \Pi_a u_1, \Pi_b u_2 \rangle = |a^l - b^l|^N \langle B \Pi_a u_1, \Pi_b u_2 \rangle. \end{aligned}$$

This means that

$$|\langle B \Pi_a u_1, \Pi_b u_2 \rangle| = \frac{|\langle ad_{I_l}^N(B) \Pi_a u_1, \Pi_b u_2 \rangle|}{|a^l - b^l|^N}.$$

Then claim (B.2) holds by Cauchy-Schwartz inequality. Choosing $l = l^*(a - b)$ defined in (B.1) and exploiting the trivial inequality

$$|a| \leq \sqrt{a} |a^*|,$$

we get the thesis. □

Proof of theorem B.1. We first prove the theorem for $k = 3$. Without loss of generality, we can assume $\langle |a| \rangle \geq \langle |b| \rangle \geq \langle |c| \rangle$. We distinguish two cases.

Case 1 If $\langle |c| \rangle \leq 2|a - b|$, we apply lemma B.2 with $f = g \Pi_c u_3$ and $P = I_{l^*(a-b)}$ and we get

$$\|Ad_{I_{l^*}}^N(g \Pi_c u_3)\|_{\mathcal{B}(L^2(M))} \lesssim \|g \Pi_c u_3\|_{N+\nu} \lesssim \|g\|_{N+\nu} \|\Pi_c u_3\|_{N+\nu} \lesssim \langle |c| \rangle^{N+\nu} \|\Pi_c u_3\|_0,$$

where the unwritten constant depends on g . In order to get the thesis we apply lemma B.3. We have

$$\begin{aligned} \left| \int g \Pi_a u_1 \Pi_b u_2 \Pi_c u_3 \right| &\leq \frac{\|Ad_{I_{l^*}}^N(g \Pi_c u_3)\|_{\mathcal{B}(L^2(M))}}{|a - b|^N} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\ &\lesssim \frac{\langle |c| \rangle^{N+\nu}}{|a - b|^N} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \|\Pi_c u_3\|_0, \end{aligned}$$

where the non written constant depends on g , on I_{l^*} and on N . Finally, we observe that

$$\frac{\langle |c| \rangle^{N+\nu}}{|a - b|^N} \leq 3^N \frac{\langle |c| \rangle^{N+\nu}}{(\langle |c| \rangle + |a - b|)^N},$$

since $\langle |c| \rangle \leq 2|a - b|$.

Case 2 If $\langle |c| \rangle > 2|a - b|$, then

$$\frac{2}{3} \leq \frac{\mu(a, b, c)}{S(a, b, c)} \leq 1. \tag{B.3}$$

In this case, we take profit of the Sobolev embedding $H^{d/2+} \hookrightarrow L^\infty$ and we get

$$\begin{aligned} \left| \int g \Pi_a u_1 \Pi_b u_2 \Pi_c u_3 \right| &\leq \|g \Pi_c u_3\|_{L^\infty} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\ &\leq C_{s_0} \|g\|_{L^\infty} \|\Pi_c u_3\|_{s_0} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\ &\lesssim \langle |c| \rangle^{s_0} \|\Pi_c u_3\|_0 \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \\ &\stackrel{(B.3)}{\lesssim} \frac{\mu(a, b, c)^{N+\nu}}{S(a, b, c)^N} \|\Pi_a u_1\|_0 \|\Pi_b u_2\|_0 \|\Pi_c u_3\|_0, \end{aligned}$$

with $\nu \geq s_0$. The general case follows from lemma 4.23. □

Finally, in order to prove theorem 5.2, we need also the following lemma.

Lemma B.4. *Let P be a polynomial with localized coefficients, then also*

$$Q(u) := P(u) \int_M u \bar{u} dx \tag{B.4}$$

has localized coefficients.

Proof. Let r be the degree of P , then one has

$$\begin{aligned} & |\tilde{Q}(\Pi_{a_1} u_1, \Pi_{a_2} u_2, \Pi_{b_1} u_3, \dots, \Pi_{b_r} u_{r+2})| \\ &= \delta_{a_1, a_2} \left| \tilde{P}(\Pi_{b_1} u_3, \dots, \Pi_{b_r} u_{r+2}) \right| \|\Pi_{a_1} u_1\| \|\Pi_{a_2} u_2\|. \end{aligned} \tag{B.5}$$

Therefore, in order to get the thesis it is enough to show that

$$\delta_{a_1, a_2} \frac{\mu(\mathbf{b})^{N+\nu}}{S(\mathbf{b})^N} \lesssim \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N}. \tag{B.6}$$

For simplicity, we will also denote $a := a_1 = a_2$ and we consider, in full generality that $\langle |b_1| \rangle \geq \langle |b_2| \rangle \geq \langle |b_3| \rangle$ are the three largest indexes among b_1, \dots, b_r .

Case 1. If $\langle |a| \rangle \geq \langle |b_1| \rangle$ we have the trivial estimate

$$\begin{aligned} \frac{\mu(\mathbf{b})^{N+\nu}}{S(\mathbf{b})^N} &= \frac{\langle |b_3| \rangle^{\nu+N}}{(|b_1 - b_2| + \langle |b_3| \rangle)^N} \leq \langle |b_3| \rangle^\nu \\ &\leq \langle |b_1| \rangle^\nu = \frac{\langle |b_1| \rangle^{\nu+N}}{(|a_1 - a_2| + \langle |b_1| \rangle)^N} = \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N}, \end{aligned}$$

since $|a_1 - a_2| = 0$.

Case 2. If $\langle |b_1| \rangle > \langle |a| \rangle > \langle |b_2| \rangle$, we need to distinguish two cases.

Case 2.i. Consider first the case $\langle |b_1| \rangle > K_1 \langle |b_2| \rangle$, with $K_1 > 0$ so large that

$$|b_1 - b_2| \geq K_2 \langle |b_1| \rangle \tag{B.7}$$

for a constant $0 < K_2 < 1$; the existence of such constants is established in A.2. Then using $|b_1 - a| \lesssim \langle |b_1| \rangle + \langle |a| \rangle \lesssim \langle |b_1| \rangle$, we estimate

$$\frac{\langle |b_3| \rangle^{\nu+N}}{(|b_1 - b_2| + \langle |b_3| \rangle)^N} \leq \frac{\langle |b_3| \rangle^{\nu+N}}{(K_2 \langle |b_1| \rangle + \langle |b_3| \rangle)^N} \lesssim \frac{\langle |b_3| \rangle^{\nu+N}}{(|b_1 - a| + \langle |b_3| \rangle)^N}.$$

Since the function $f(x) = \frac{x^{\nu+N}}{(K+x)^N}$ is increasing for any $N, \nu > 0, K \geq 0$ and $x \geq 0$, the above quantity is bounded, up to a constant, by

$$\frac{\langle |a| \rangle^{\nu+N}}{(|b_1 - a| + \langle |a| \rangle)^N} = \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N}.$$

Case 2.ii. If $\langle |b_1| \rangle \leq K_1 \langle |b_2| \rangle$ we observe that

$$K_1 C \langle |b_2| \rangle \geq C \langle |b_1| \rangle \geq |b_1 - a|$$

and we estimate

$$\begin{aligned} \frac{\langle |b_3| \rangle^{\nu+N}}{(|b_1 - b_2| + \langle |b_3| \rangle)^N} &\leq \langle |b_2| \rangle^\nu = (K_1 C + 1)^N \frac{\langle |b_2| \rangle^{\nu+N}}{(K_1 C \langle |b_2| \rangle + \langle |b_2| \rangle)^N} \\ &\leq (K_1 C + 1)^N \frac{\langle |b_2| \rangle^{\nu+N}}{(|b_1 - a| + \langle |b_2| \rangle)^N}. \end{aligned} \tag{B.8}$$

Using again the monotonicity of the function $f(x) = \frac{x^{\nu+N}}{(K+x)^N}$, we get

$$\begin{aligned} (B.8) &\leq (K_1 C + 1)^N \frac{\langle |a| \rangle^{\nu+N}}{(|b_1 - a| + \langle |a| \rangle)^N} \\ &= (K_1 C + 1)^N \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N}. \end{aligned}$$

Case 3. If $\langle |b_2| \rangle \geq \langle |a| \rangle > \langle |b_3| \rangle$ we get

$$\frac{\mu(\mathbf{b})^{N+\nu}}{S(\mathbf{b})^N} = \frac{\langle |b_3| \rangle^{\nu+N}}{(|b_1 - b_2| + \langle |b_3| \rangle)^N} \leq \frac{\langle |a| \rangle^{\nu+N}}{(|b_1 - b_2| + \langle |a| \rangle)^N} = \frac{\mu(a_1, a_2, \mathbf{b})^{N+\nu}}{S(a_1, a_2, \mathbf{b})^N}.$$

Case 4. If $\langle |a| \rangle \leq \langle |b_3| \rangle$ the estimates (A.7) is obvious since it does not involve a . This concludes the proof. \square

Proof of theorem 5.2. The proof follows directly from lemma B.4 and theorem B.1, and from the definition of function with localized coefficients 4.15, since the Taylor expansion of a functional fulfilling Hypothesis P is the sum of terms P_m of the form (B.4). \square

Appendix C. The clusterization of the lattice. Proof of theorem 2.10.

We are going to deduce theorem 2.10 from theorem C.1 below, which was proved in [BL22]. Actually the following statement is obtained by looking at the construction of the extended blocks $E_{M,j}$ done for the proof of theorem 7.27 of [BL22]. In this section we will denote by

$$w(a) := \frac{\partial h_L}{\partial a}(a) \tag{C.1}$$

the frequencies of the classical Hamiltonian system with Hamiltonian h_L .

Theorem C.1. *Under the same assumptions of theorem 2.10, there exist $\rho_* > \beta - 1$ and $\mu_* > 0$ s.t. $\forall \rho, \mu$ fulfilling $\rho_* < \rho < \beta - 1, 0 < \mu < \mu_*$ there exists a partition*

$$\Lambda = \bigcup_{j \geq 0} E_j$$

with the following properties:

(i) *There exists $C > 0$ such that $a \in E_0$ implies $|a| < C$ and furthermore $\forall j \geq 1$*

$$\sup_{a \in E_j} |a| \leq C \inf_{a \in E_j} |a|,$$

(ii) Given $a, b \in \Lambda$,

$$|w(a) \cdot (b - a)| \leq |a|^\rho |b - a| \text{ and } |a - b| \leq |a|^\mu \tag{C.2}$$

imply that $\exists j$ s.t. $a, b \in E_j$.

Proof of theorem 2.10. We now modify the partition E_j as follows: fix a large R and define

$$\begin{aligned} \Omega_0 &:= \bigcup \{E_j \mid \exists a \in E_j \text{ s.t. } |a| \leq R\} \\ \Omega_j &= E_j \text{ if } |a| \geq R \quad \forall a \in E_j. \end{aligned}$$

The fact that such a new partition fulfills property (i) of hypothesis L.3 is a trivial consequence of point (i) of theorem C.1. We come to (ii). We are going to show that there exists $C_\delta > 0$ small enough such that, if $a, b \in \Lambda$ are such that (ii) is violated, namely

$$|a - b| + |\omega_a - \omega_b| < C_\delta (|a|^\delta + |b|^\delta), \tag{C.3}$$

then a and b belong to the same block.

First we remark that, if both $|a| \leq R$ and $|b| \leq R$, then a and b belong to the same block, thus let us suppose that $|a| \geq R$.

First of all, we observe that, by elementary computations there exists $c_\delta > 0$ such that, if $C_\delta \leq c_\delta$, then (C.3) implies

$$|a - b| \leq |a|^\delta. \tag{C.4}$$

We proceed now to show that, if (C.3) holds with C_δ small enough, one also has

$$|w(a) \cdot (b - a)| \leq |a|^\rho |a - b|. \tag{C.5}$$

Indeed, since h_L is an homogeneous function of order β , performing a Taylor development at order 2 one gets

$$\begin{aligned} \omega_b - \omega_a &= w(a) \cdot (b - a) + r_{a,b}, \\ |r_{a,b}| &\lesssim |a|^{\beta-2} |b - a|^2. \end{aligned} \tag{C.6}$$

Then, combining (C.6), (C.3) and (C.4), one obtains for $a \neq b$

$$\begin{aligned} |w(a) \cdot (b - a)| &\leq |\omega_b - \omega_a| + |r_{a,b}| \\ &\lesssim |a|^\delta + |b|^\delta + |a|^{\beta-2} |b - a|^2 \\ &\lesssim |a|^\delta + |a|^{\beta-2+\delta} |b - a| \\ &\lesssim \left(|a|^\delta + |a|^{\beta-2+\delta} \right) |b - a|. \end{aligned}$$

This proves (C.5), provided $\rho > \beta - 2 + \delta$, $\rho > \delta$ and taking $\delta < \mu$, which is possible since one can choose δ as small as one wants. But this is exactly (C.2), which implies that a and b belong to the same set. \square

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References

- [Bam03] Bambusi D 2003 Birkhoff normal form for some nonlinear PDEs *Commun. Math. Phys.* **234** 253–85
- [Bam08] Bambusi D 2008 A Birkhoff normal form theorem for some semilinear PDEs *Hamiltonian Dynamical Systems and Applications (Nato Sci. Peace Secur. Ser. B Phys. Biophys.)* (Springer) pp 213–47
- [BB13] Berti M and Bolle P 2013 Quasi-periodic solutions with Sobolev regularity of NLS on \mathbb{T}^d with a multiplicative potential *J. Eur. Math. Soc.* **15** 229–86
- [BBM14] Baldi P, Berti M and Montalto R 2014 KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation *Math. Ann.* **359** 471–536
- [BBM16] Baldi P, Berti M and Montalto R 2016 KAM for autonomous quasi-linear perturbations of KdV *Ann. Inst. Henri Poincaré C* **33** 1589–638
- [BCP15] Berti M, Corsi L and Procesi M 2015 An abstract Nash–Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds *Commun. Math. Phys.* **334** 1413–54
- [BD18] Berti M and Delort J M 2018 *Almost Global Solutions of Capillary-Gravity Water Waves Equations on the Circle (UMI Lecture Notes)* (Springer)
- [BDGS07] Bambusi D, Delort J-M, Grébert B and Szeftel J 2007 Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds *Commun. Pure Appl. Math.* **60** 1665–90
- [BFF21] Berti M, Feola R and Franzoi L 2021 Quadratic life span of periodic gravity-capillary water waves *Water Waves* **3** 85–115
- [BFGI21] Bernier J, Feola R, Grébert B and Iandoli F 2021 Long-time existence for semi-linear beam equations on irrational tori *J. Dyn. Differ. Equ.* **33** 1–36
- [BFM24] Bambusi D, Feola R and Montalto R 2024 Almost global existence for some Hamiltonian PDEs with small Cauchy data on general tori *Commun. Math. Phys.* **405** 253–85
- [BFP23] Berti M, Feola R and Pusateri F 2023 Birkhoff normal form and long time existence for periodic gravity water waves *Commun. Pure Appl. Math.* **76** 1416–94
- [BG03] Bambusi D and Grébert B 2003 Forme normale pour NLS en dimension quelconque *Comptes Rendus Math.* **337** 409–14
- [BG06] Bambusi D and Grébert B 2006 Birkhoff normal form for partial differential equations with tame modulus *Duke Math. J.* **135** 507–67
- [BGR24] Bernier J, Grébert B and Robert T 2024 Infinite dimensional invariant tori for nonlinear Schrödinger equations (arXiv:2412.11845)
- [BL22] Bambusi D and Langella B 2022 Growth of Sobolev norms in quasi integrable quantum systems (arXiv:2202.04505)
- [BLM22a] Bambusi D, Langella B and Montalto R 2022 Growth of Sobolev norms for unbounded perturbations of the Schrödinger equation on flat tori *J. Differ. Equ.* **318** 344–58
- [BLM22b] Bambusi D, Langella B and Montalto R 2022 Spectral asymptotics of all the eigenvalues of Schrödinger operators on flat tori *Nonlinear Anal.* **216** 112679
- [BM19] Berti M and Maspero A 2019 Long time dynamics of Schrödinger and wave equations on flat tori *J. Differ. Equ.* **267** 1167–200
- [BMM24] Berti M, Maspero A and Murgante F 2024 Hamiltonian Birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence *Ann. PDE* **10** 22
- [BMP20] Biasco L, Massetti J E and Procesi M 2020 An abstract Birkhoff normal form theorem and exponential type stability of the 1d NLS *Commun. Math. Phys.* **375** 2089–153
- [BMP21] Biasco L, Massetti J E and Procesi M 2021 Almost periodic invariant tori for the NLS on the circle *Ann. Inst. Henri Poincaré C* **38** 711–58

- [BMP23] Biasco L, Massetti J and Procesi M 2023 Small amplitude weak almost periodic solutions for the 1d NLS *Duke Math. J.* **172** 2643–2714
- [Bou96] Bourgain J 1996 Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations *Geom. Funct. Anal.* **6** 201–30
- [Bou04] Bourgain J 2004 *Green's Function Estimates for Lattice Schrödinger Operators and Applications* vol 158 (Princeton University Press)
- [Bou05] Bourgain J 2005 On invariant tori of full dimension for 1D periodic NLS *J. Funct. Anal.* **229** 62–94
- [BP11] Berti M and Procesi M 2011 Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces *Duke Math. J.* **159** 479–538
- [CdV80] Colin de Verdière Y 1980 Spectre conjoint d'opérateurs pseudo-différentiels qui commutent. II. Le cas intégrable *Math. Z.* **171** 51–74
- [Del09] Delort J M 2009 Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds *Int. Math. Res. Not.* **2010** 2305–28
- [Del12] Delort J M 2012 A quasi-linear Birkhoff normal forms method. Application to the quasi-linear Klein-Gordon equation on S^1 *Astérisque* **342** 1–112
- [Del15] Delort J-M 2015 Quasi-Linear perturbations of Hamiltonian Klein-Gordon equations on spheres *Mem. Am. Math. Soc.* **234** 80
- [DI17] Delort J-M and Imekraz R 2017 Long-time existence for the semilinear Klein-Gordon equation on a compact boundary-less Riemannian manifold *Commun. PDE* **42** 388–416
- [DS04] Delort J-M and Szeftel J 2004 Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres *Int. Math. Res. Not.* **2004** 1897–966
- [DS06] Delort J-M and Szeftel J 2006 Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds *Am. J. Math.* **128** 1187–218
- [EGK16] Eliasson L H, Grébert B and Kuksin S B 2016 KAM for the nonlinear beam equation *Geom. Funct. Anal.* **26** 1588–715
- [EK10] Eliasson L H and Kuksin S B 2010 KAM for the nonlinear Schrödinger equation *Ann. Math.* **172** 371–435
- [FG13] Faou E and Grébert B 2013 A Nekhoroshev-type theorem for the nonlinear Schrödinger equation on the torus *Anal. PDE* **6** 1243–62
- [FGI23] Feola R, Grébert B and Iandoli F 2023 Long time solutions for quasilinear Hamiltonian perturbations of Schrödinger and Klein-Gordon equations on tori *Anal. PDE* **16** 1133–203
- [FGL13] Faou E, Gauckler L and Lubich C 2013 Sobolev stability of plane wave solutions to the cubic nonlinear Schrödinger equation on a torus *Commun. PDE* **38** 1123–40
- [FI21] Feola R and Iandoli F 2021 Long time existence for fully nonlinear NLS with small Cauchy data on the circle *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **22** 109–82
- [FIM22] Feola R, Iandoli F and Murgante F 2022 Long-time stability of the quantum hydrodynamic system on irrational tori *Math. Eng.* **4** 1–24
- [FM22] Feola R and Montalto R 2022 Quadratic lifespan and growth of Sobolev norms for derivative Schrödinger equations on generic tori *J. Differ. Equ.* **312** 276–316
- [FM23] Feola R and Massetti J E 2023 Sub-exponential stability for the beam equation *J. Differ. Equ.* **356** 188–242
- [FZ10] Fang D and Zhang Q 2010 Long-time existence for semi-linear Klein-Gordon equations on tori *J. Differ. Equ.* **249** 151–79
- [GCB16] Guzzo M, Chierchia L and Benettin G 2016 The steep Nekhoroshev's theorem *Commun. Math. Phys.* **342** 569–601
- [GIP09] Grébert B, Imekraz R and Paturel E 2009 Normal forms for semilinear quantum harmonic oscillators *Commun. Math. Phys.* **291** 763–98
- [Gré07] Grébert B 2007 Birkhoff normal form and Hamiltonian PDEs *Partial Differential Equations and Applications (Sémin Congr.)* vol 15 (Soc. Math.) pp 1–46
- [GXY11] Geng J, Xu X and You J 2011 An infinite dimensional KAM theorem and its application to the two dimensional cubic Schrödinger equation *Adv. Math.* **226** 5361–402
- [Hö85] Hörmander L 1985 *The Analysis of Linear Partial Differential Operators I-III* (Springer)
- [Ime16] Imekraz R 2016 Long time existence for the semi-linear beam equation on irrational tori of dimension two *Nonlinearity* **29** 3067

- [IP19] Ionescu A D and Pusateri F 2019 Long-time existence for multi-dimensional periodic water waves *Geom. Funct. Anal.* **29** 811–70
- [Kuk87] Kuksin S B 1987 Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum *Funct. Anal. Appl.* **21** 192–205
- [Kuk93] Kuksin S B 1993 *Nearly Integrable Infinite-Dimensional Hamiltonian Systems (Lecture Notes in Mathematics)* vol 1556 (Springer)
- [NG07] Niu H and Geng J 2007 Almost periodic solutions for a class of higher-dimensional beam equations *Nonlinearity* **20** 2499–517
- [Nie06] Niederman L 2006 Hamiltonian stability and subanalytic geometry *Ann. Inst. Fourier* **56** 795–813
- [PP15] Procesi C and Procesi M 2015 A KAM algorithm for the resonant non-linear Schrödinger equation *Adv. Math.* **272** 399–470
- [Po96] Pöschel J 1996 A KAM-theorem for some nonlinear partial differential equations *Ann. Scuola Norm. Super. Pisa - Cl. Sci.* **23** 119–48 (Engl. transl.)
- [TZ02] Toth J A and Zelditch S 2002 Riemannian manifolds with uniformly bounded eigenfunctions *Duke Math. J.* **111** 97–132 (Engl. transl.)
- [Way90] Wayne C E 1990 Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory *Commun. Math. Phys.* **127** 479–528