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Destabilising Subvarieties for Partial Differential Equations in Complex Geometry

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All that you have read has made you a scholar,
Yet your own heart, you have never read!

— Bulleh Shah

Abstract

In this thesis, we consider the set of destabilising subvarieties associated to various geometric partial differential equations (PDEs) of Monge-Ampère type arising in complex geometry, including the J-equation, the deformed Hermitian Yang-Mills equation and certain generalised Monge-Ampère equations. Each of these PDEs has an associated Nakai-Moishezon type criterion characterising their solvability in terms of a certain stability condition involving subvarieties. We show that the set of subvarieties that violate this criterion is finite under certain mild hypotheses of positivity, which are always satisfied on compact Kähler surfaces. We use the results to show that the locus of stable (or solvable) PDEs is open in the locus of all the PDEs, and admits a locally finite wall-chamber structure whose walls are cut out by equations involving certain rigid subvarieties.

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Introduction

In mathematics, as in other fields of human knowledge, an indication of the merit of an idea is its being discovered by disparate and seemingly unrelated lines of enquiry. The notion of *stability* has proven central to the study of at least two distinct areas of mathematics: the construction of moduli spaces of geometric objects associated to a smooth projective variety, and the solvability of many different classes of partial differential equations on compact Kähler manifolds arising in geometry. Both of these lines of enquiry have benefited one from the other and thereby propelled the field forward. From this point of view, the study of stability conditions for their own sake has emerged as a fruitful and promising endeavor. This thesis aims to study the *destabilising subvarieties* in the context of stability conditions attached to certain geometric PDEs on compact Kähler manifolds.

The idea that in order to construct moduli spaces one should impose a stability condition arose in algebraic geometry already in the theory of constructing geometric orbit spaces for algebraic group actions, that is, *geometric invariant theory.* From there, the idea was taken up by geometers into other contexts, most notably in constructing moduli spaces of vector bundles on algebraic varieties, where the relevant stability condition was identified as *slope stability.* If E is a vector bundle on a compact Kähler manifold X of dimension $\dim_{\mathbb{C}} X = n$ with fixed Kähler class α, then E is said to be *slope stable* if, for all proper non-zero coherent subsheaves $S \subseteq E$, we have

$$\mu_\alpha(S) = \frac{\int_X c_1(S) \cdot \alpha^{n-1}}{\operatorname{rank}(S)} < \mu_\alpha(E) = \frac{\int_X c_1(E) \cdot \alpha^{n-1}}{\operatorname{rank}(E)}.$$

Here, two seemingly distant threads of geometry become entwined in a beautiful correspondence called the *Hitchin-Kobayashi correspondence*, established by the theorems of Donaldson and Uhlenbeck-Yau.

Theorem (Donaldson [1], Uhlenbeck-Yau [2])**.** *Suppose E is simple, that is, the space $H^0(X, \mathrm{End}E)$ of global holomorphic endomorphisms of E is isomorphic to $\mathbb{C}$. Then, the following are equivalent.*

1. *For any choice of Kähler form $\omega \in \alpha$, there exists a Hermitian metric h on E whose curvature form $\frac{\sqrt{-1}}{2\pi} F_h$ satisfies the Hermitian Yang-Mills equation*

$$\left(\frac{\sqrt{-1}}{2\pi} F_h\right) \wedge \omega^{n-1} = \frac{\mu_\alpha(E)}{\int_X \alpha^n} \omega^n \otimes \mathrm{Id}_E.$$

2. *The vector bundle E is slope stable.*

Thus, the slope stability of a simple vector bundle is both necessary and sufficient for it to admit a solution of the *Hermitian Yang-Mills equation.* Other examples of this phenomenon include the important *Yau-Tian-Donaldson conjecture.* This conjecture, which has been central to the subject of complex differential geometry, asserts that the existence of a *constant scalar curvature Kähler (cscK) metric* in a given Kähler class is equivalent to a certain algebro-geometric stability condition called *K-polystability.*

The cscK equation or the Hermitian Yang-Mills equation are just two examples of a whole host of partial differential equations (PDEs) that arise in complex differential geometry and whose solvability is conjectured or known to be equivalent to a numerical criterion defining stability. An important subclass of these equations are those whose associated numerical criterion involves checking that a certain inequality holds for intersection numbers attached to all proper subvarieties. The most prototypical example of this kind of correspondence is the complex Monge-Ampère equation. We explain how this correspondence arises as the result of two important theorems in complex differential geometry. Let X be a smooth projective variety of dimension n and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a cohomology class on X. Let $\Omega \in \mathcal{A}^{n,n}(X)$ be a smooth volume form on X such that

$$\int_X \tau^n = \int_X \Omega.$$

Then, Yau's solution of the Calabi conjecture can be stated as follows.

Theorem (Yau [3])**.** *The following conditions are equivalent.*

1. *The cohomology class α is a Kähler class.*
2. *There exists a unique smooth Kähler form $\omega \in \alpha$ such that*

$$\omega^n = \Omega.$$

On the other hand, if X is smooth and projective, then a theorem of Demailly-Paŭn gives us a characterisation of the set of Kähler classes in terms of a numerical criterion involving subvarieties.

Theorem (Nakai-Moishezon, Demailly-Paŭn [4])**.** *The following conditions are equivalent.*

1. *The cohomology class α is a Kähler class.*
2. *For every proper irreducible subvariety V of X of positive dimension, we have*

$$\int_V \alpha^{\dim_{\mathbb{C}} V} > 0.$$

In this thesis, we shall be interested in equations which admit a similar Nakai-Moishezon criterion. Among these, we shall focus mostly on the *J-equation*, certain cases of the *deformed Hermitian Yang-Mills equation*, the *Z-critical equations* and *generalised Monge-Ampère equations.*

These equations arise naturally in complex geometry and their study is central to many different aspects of the subject. These include, but are not limited to, the *constant scalar curvature* equation in Kähler geometry and the associated theory of *moduli of K-stable varieties*, the study of *mirror symmetry*, the theory of *Bridgeland stability conditions* and the associated theory of *moduli of coherent sheaves.* From the point of view of the present thesis, however, all of these equations fit into the same setup, which we describe now.

Overview

For the purpose of explaining the picture in generality, we deliberately avoid getting too notationally precise or mathematically specific at this stage.

In each case, the PDE is specified by choosing a particular value Ω from a continuous family of admissible parameters. What is sought as a solution of the PDE is a smooth form $\omega \in \alpha$ in a fixed cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$, which is often Kähler. Whether or not we can find the sought-after ω is characterised by certain inequalities involving intersection numbers of cohomology classes $\vartheta_p(\Omega), p = 1, \ldots, n-1$ depending only on Ω, say

$$\int_V \vartheta_p(\Omega) > 0$$

for each p-dimensional subvariety V of X. Our primary interest lies in the following question.

Question. Suppose the equation associated to the parameter value Ω cannot be solved. What are the possible subvarieties V that violate the numerical criterion for solvability, that is, for which subvarieties V (which we shall call *destabilising subvarieties*) do we have

$$\int_V \vartheta_p(\Omega) \leq 0?$$

This question is quite well motivated. By understanding the geometric properties of these subvarieties, we might be able, in certain cases, to rule out the existence of *any* destabilising subvarieties and therefore conclude that the equation is solvable. By understanding the cardinality of the set of destabilising subvarieties, we might get more precise information about the solvability of the equations and the nature of *optimal destabilisers*, as we continuously vary the parameters Ω defining the equation.

We shall see that, in all our cases of interest, under certain assumptions on the value of the parameters Ω and cohomology class α (involving concepts of positivity), the subvarieties violating the numerical criterion (which we shall call *destabilising subvarieties*) lie in the non-Kähler locus of certain big cohomology classes $\tau_p(\Omega), p = 1, 2, \ldots, n-1$ determined by Ω. In the case of $n = 2$, we use the Zariski decomposition of big classes to conclude that the set of destabilising subvarieties is a finite set of curves of negative self-intersection. In the case of $n = 3$, we use similar ideas of positivity to conclude that the *union* of the destabilising subvarieties is an analytic subset V_Ω of X, each one of whose irreducible components is rigid in a precise sense.

There is another motivation to study the set of all destabilising subvarieties, which comes from the abstract theory of stability. We now briefly explain this point of view.

The PDEs mentioned above should be understood as the 'rank one' case of families of PDEs coming from stability conditions, that is, we should think of α as the first Chern class $c_1(L)$ of a holomorphic line bundle on X, and think of the specific equations as being the rank one case of a family of equations defined for each holomorphic vector bundle E of arbitrary rank, or, indeed, any coherent sheaf $\mathcal{F}$ on X.

One such family of PDEs, the *Z-critical equation*, has been introduced by Dervan-McCarthy-Sektnan [5] as a differential-geometric analogue of the notion of a *Bridgeland stability condition* on $D^b\mathrm{Coh}(X)$. From this point of view, one should expect features of the theory of Bridgeland stability to arise also in the study of these Z-critical equations. One such feature is the existence of a locally finite *wall-chamber* decomposition. We briefly explain this now.

The set of all stability conditions $\mathcal{S}$ has a natural topology which makes $\mathcal{S}$ into a smooth manifold. Then, the locally-finite wall-chamber decomposition is the following phenomenon. For each stability condition $Z \in \mathcal{S}$, there exists an open neighbourhood U of Z together with finitely many closed submanifolds $W_i \subseteq U$ of (real) codimension one (the 'walls') having the following property. If U_i is any connected component (a 'chamber') of the complement of the union of the W_i in U, then the stability of an object E with respect to any element $Z_1 \in U_i$ is equivalent to the stability of E with respect to any other element $Z_2 \in U_i$.

If something similar occurs for the Z-critical equation, then it is good evidence that the analogy with Bridgeland stability conditions works well. In this thesis, we find that under a natural hypothesis, the 'rank one' case of some of the equations under consideration have this property. We deduce this by observing that the analogue of the 'walls' are cut out by the irreducible components of the set V_Ω, which vary in a finite set when Ω varies in a small open set.

Statements of results

We now state our main results more carefully in the special case of the J-equation. For the sake of brevity and readability, we avoid making the analogous statements about all the other PDEs at this stage, but precise statements will be made and proved in the main body of the thesis.

Thus, let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$. Let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be Kähler classes on X. Then, for every Kähler form $\theta \in \beta$, the J-equation seeks a smooth Kähler form $\omega \in \alpha$ such that

$$n\omega^{n-1} \wedge \theta = \mu_{\alpha,\beta}\omega^n \tag{1}$$

where the *J-slope* $\mu_{\alpha,\beta} = \mu_{\alpha,\beta}(X)$ is a topological constant, and $\mu_{\alpha,\beta}(V)$ for any analytic subvariety V of X is defined to be

$$\mu_{\alpha,\beta}(V) = (\dim_{\mathbb{C}} V)\frac{\alpha^{\dim_{\mathbb{C}} V-1} \cdot \beta \cdot [V]}{\alpha^{\dim_{\mathbb{C}} V} \cdot [V]}.$$

The numerical criterion for stability in the case of the J-equation is due to the works of Gao Chen [6], Datar-Pingali [7], and Song [8], and says that (1) is solvable precisely when $\mu_{\alpha,\beta}(V) < \mu_{\alpha,\beta}(X)$ for all proper analytic subvarieties V of positive dimension. We call V a *destabiliser* if $\mu_{\alpha,\beta}(V) \geq \mu_{\alpha,\beta}(X)$ and an *optimal destabiliser* if it is a destabiliser and

$$\mu_{\alpha,\beta} - \mu_{\alpha,\beta}(V) = \inf_Z \left(\mu_{\alpha,\beta} - \mu_{\alpha,\beta}(Z)\right).$$

The triple (X, α, β) is called *J-semistable* (respective *J-stable*) if $\mu_{\alpha,\beta}(V) \leq \mu_{\alpha,\beta}$ (respectively $\mu_{\alpha,\beta}(V) < \mu_{\alpha,\beta}$). Our results about the cardinality of the set of destabilising subvarieties can then be summarised as follows.

Theorem A. *Suppose X is a smooth projective variety and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ are Kähler classes and $\theta \in \beta$ is a Kähler form.*

1. *Suppose $\dim_{\mathbb{C}} X = 2$. Then, there are only finitely many destabilising curves, each of which is a curve of negative self-intersection.*

2. *Suppose* $\dim_{\mathbb{C}} X = 3$ *and* $\tau_2(\alpha, \beta) = \mu_{\alpha,\beta}\alpha - 2\beta$ *is a big cohomology class. Then, there exists a proper analytic subset* $V_{\alpha,\beta}$ *such that each irreducible component of* $V_{\alpha,\beta}$ *is a destabilising subvariety and all destabilising subvarieties are contained in* $V_{\alpha,\beta}$*. If moreover* (X, α, β) *is J-semistable, then there exist only finitely many destabilising subvarieties.*

3. *Suppose* $\tau_p(\alpha, \beta) = \mu_{\alpha,\beta}\alpha - p\beta$ *is a* $(p+1)$*-modified Kähler class for each* $p = 1, 2, \ldots, \dim_{\mathbb{C}} X - 1$*. Then, there exist only finitely many destabilising subvarieties.*

Part 1 of Theorem A is the content of Theorem 2.9, Part 2 is Theorems 2.11 and 2.10 and Part 3 is Theorem 2.15.

For each Part of Theorem A, we have an accompanying statement about the rigidity of the destabilising subvarieties.

Theorem B. *Let* (X, α, β) *and* θ *be as in Theorem A.*

1. *Suppose* $\dim_{\mathbb{C}} X = 2$*. Then, each destabilising curve is the unique effective analytic cycle representing its homology class.*

2. *Suppose* $\dim_{\mathbb{C}} X = 3$ *and* $\tau_2(\alpha, \beta) = \mu_{\alpha,\beta}\alpha - 2\beta$ *is a big cohomology class. Then, each irreducible component* V *of the set* $V_{\alpha,\beta}$ *is rigid in the following sense. If* $V = S$ *is an irreducible surface, it is the unique effective analytic cycle representing its homology class. If* $V = C$ *is a curve, then for each surface* S *containing* C*, either* C *is an irreducible component of the singular locus of* S *or (the strict transform of)* C *is a curve of negative self-intersection in (any resolution of singularities of)* S*.*

3. *Suppose* $\tau_{n-1}(\alpha, \beta) = \mu_{\alpha,\beta}\alpha - (n-1)\beta$ *is a big cohomology class. Then, each destabilising divisor is the only effective analytic cycle representing its homology class.*

Part 1 of Theorem B is contained in Theorem 2.9, Part 2 is Theorem 2.14, while Part 3 is contained in Proposition 2.16.

Finally, each Part of Theorems A and B have an accompanying statement about the wall-chamber structure.

Theorem C. *Let* X *be as in Theorem A and let* $\alpha_1, \ldots, \alpha_s \in H^{1,1}(X, \mathbb{R})$ *be a finite collection of Kähler classes.*

1. *If* $\dim_{\mathbb{C}} X = 2$*, let* $\mathcal{S} \subseteq H^{1,1}(X, \mathbb{R})$ *be the set of Kähler classes.*

2. *If* $\dim_{\mathbb{C}} X = 3$*, let* $\mathcal{S} \subseteq H^{1,1}(X, \mathbb{R})$ *be the set comprising Kähler classes* β *such that* $\mu_{\alpha_i,\beta}\alpha_i - 2\beta$ *is a big cohomology class for each* $i = 1, \ldots, s$*.*

3. *If* $\dim_{\mathbb{C}} X \geq 4$*, let* $\mathcal{S} \subseteq H^{1,1}(X, \mathbb{R})$ *be the set comprising Kähler classes* β *such that* $\mu_{\alpha_i,\beta}\alpha_i - p\beta$ *is a* $(p+1)$*-modified Kähler class for each* $i = 1, \ldots, s$ *and* $p = 1, \ldots \dim_{\mathbb{C}} X - 1$*.*

In each case, the open set $\mathcal{S} \subseteq H^{1,1}(X, \mathbb{R})$ *enjoys the following property. For each* $\beta_0 \in \mathcal{S}$*, there exists an open neighbourhood* U *containing* β_0 *and finitely many closed submanifolds* $W_1, \ldots, W_r$ *of* U *of codimension one such that for each connected component* U_j *of*

$$U \setminus \bigcup_k W_k$$

and each $i = 1, \ldots, s$*, the triple* (X, α_i, β) *is J-stable for some* $\beta \in U_j$ *if and only if* (X, α_i, β) *is J-stable for all* $\beta \in U_j$*.*

Part 1 of Theorem A is contained in 3.5, Part 2 is Theorem 3.10 while Part 3 follows from a special case of the more general Theorem 3.12.

We emphasise once again that this thesis contains analogues of Theorems A, B, and C for the deformed Hermitian Yang-Mills equations, Z-critical equations and generalised Monge-Ampère equations.

Outlook

A few remarks about the results and techniques of this thesis are in order. The use of Zariski decompositions (rather than the Siu Decomposition Theorem, as in [9, Proposition 4.5], for example) and non-Kähler loci of big classes is a novelty. This allows us to control the destabilising subvarieties while varying the cohomological data of the PDEs and thereby enables us to deduce results about wall-chamber structure. The results obtained are the first of their kind on the purely differential geometric side.

Along with obtaining some completely new results, the techniques also allow us to clarify some previous results. For example, for the case of the J-equation, it was early observed in the work of Song-Weinkove [9, Theorem 1.4] that on compact Kähler surfaces, a certain geometric flow associated to the J-equation develops singularities along a finite union of certain curves, confirming an expectation of Donaldson [10, Section 4.3]. The same was observed in subsequent works about geometric flows associated to the deformed Hermitian Yang-Mills equation on surfaces. In fact, our results show that the destabilising curves are among this union, and each member of a minimal such union is a curve of negative self-intersection. This is discussed in [11, Section 6].

The results of the thesis also relate to many open problems in the subject. Here, we briefly highlight three important problems in complex differential and algebraic geometry to which our results especially pertain.

The first is the problem of finding singular solutions to these PDEs when no smooth solutions may exist, generalising the results of Guedj-Eyssidieux-Guedj-Zeriahi [12] for the complex Monge-Ampère to a wider class of PDEs. A related problem is to study the geometric flows associated to some of these equations following [9] or the recent work of [13]. In both of these related problems, the destabilising subvarieties will likely play an important role, as evidenced by previous results of these authors.

The second important problem is extending these results to 'higher rank', that is, we should consider the vector bundle analogues of each of the PDEs. Here, the main difficulty is that destabilising objects may now come in two different forms. In addition to there being destabilising subvarieties, there might also be destabilising subsheaves of the associated holomorphic vector bundle. However, some of the same considerations might apply to this much more complicated setting. A special case of this setup has recently been considered by Keller-Scarpa [14]. In their work, they term what we call stability as *positivity* and reserve the term *stability* for the higher rank condition involving subsheaves (or conditions that involve a mixture of both subsheaves and subvarieties). They propose general conjectures that relate positivity, stability and the existence of solutions of the relevant PDEs.

A related problem to the above is clarifying the relationship of Bridgeland stability conditions with the solvability of, for example, the deformed Hermitian Yang-Mills equation. The correspondence here is not expected to be as exact as for the Hermitian Yang-Mills equation, but there are interesting partial results by Collins-Shi [15], and Collins-Lo-Shi-Yau [16].

The third important problem is understanding the role played by mirror symmetry.

The solutions of the deformed Hermitian Yang-Mills equation can be seen as the mirrors of special Lagrangian submanifolds in symplectic geometry on a Calabi-Yau manifold, under the mirror map given by the Strominger-Yau-Zaslow [17] vision of mirror symmetry. It would be interesting to understand if there exists any interpretation of the solutions of more general PDEs as certain calibrated submanifolds of the mirror Calabi-Yau manifold.

Organisation of the thesis

We briefly explain the layout of the material in the thesis, as well as explain any overlap with existing or future work, to avoid any potential issues of self-plagiarism.

Chapter 1 is mostly background about concepts of positivity in complex geometry, with a view toward the case of Kähler manifolds. In Section 1.1 some important foundational results are recalled, mostly without proof, but effort has been made to give precise references to proofs available in the literature. Section 1.2, still mostly background, contains the statements and gives complete proofs of a few key lemmas, probably all of them well-known to experts, but for which no reference in the literature is readily available.

Chapter 2 is the heart of the thesis and contains all the main results regarding destabilising subvarieties. Section 2.1 gives a brief overview of the various PDEs which are treated, and also explains some of their various connections with each other and with important problems in complex geometry, and recalls the associated Nakai-Moishezon criteria. Section 2.2 deals with the important base case of surfaces. In Section 2.3, we treat the case of the J-equation in higher dimensions. In Section 2.4 we introduce a certain subclass of PDEs to which all of our considerations in Section 2.3 can be applied.

Chapter 3 is about the important application of the results of Chapter 2 to the problem of describing wall-chamber structure. Once again, we have decided to present the case of surfaces separately in Section 3.1. Higher dimensional results are then presented in Section 3.2.

Sections 2.2 and 3.1, have significant overlap with [11], and Sections 2.3, 2.4, 3.2 are likely to have significant overlap with [18], which is currently in preparation. Although all original results are joint with Sjöström Dyrefelt, the exposition of the material is meant to emphasise the interests and the contribution of the author.

Contents

Chapter 1

Positive currents in complex geometry

In this Chapter, we recall some basic notions concerning positivity on compact Kähler manifolds. Most of the material here is explained in great detail in [19] and [20], as well as in [21] and the references therein. However, in Section 1.2 we take care to give complete proofs of certain basic results that we require in our later arguments and for which no direct statement is easily available in the literature.

1.1 Background on positivity

1.1.1 The sheaf of differential forms

Let X be a complex manifold of (complex) dimension n. Given an open neighbourhood $U \subseteq X$ the space of *degree k (complex) smooth differential forms over U* is denoted $\mathcal{A}^k(U)$. We have a natural splitting

$$\mathcal{A}^k(U) \cong \bigoplus_{p+q=k} \mathcal{A}^{p,q}(U).$$

Here $\mathcal{A}^{p,q}(U)$ is the space of *degree (p,q) forms over U*, that is, those forms $\alpha \in \mathcal{A}^k(U)$ which can locally be written as

$$\sum_{I,J} \alpha_{I\overline{J}} dz_I \wedge d\overline{z}_J$$

where α_{IJ} are complex-valued smooth functions. In the above expression $I = \{i_1 < \cdots < i_p\}$, and $J = \{j_1 < \cdots < j_q\}$ are indexing subsets of $\{1, \ldots, n\}$, and

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

$$dz_i = dx_i + \sqrt{-1}dy_i, \quad d\overline{z}_i = dx_i - \sqrt{-1}dy_i, \quad x_i = \mathrm{Re}(z_i), y_i = \mathrm{Im}(z_i)$$

for any choice $z_1, \cdots, z_n$ of holomorphic local coordinates. By expressing

$$dx_i = \frac{1}{2}(dz_i + d\overline{z_i}), \quad dy_i = \frac{1}{2\sqrt{-1}}(dz_i - d\overline{z}_i),$$

we see that there is an injective map

$$\mathcal{A}^k_{\mathbb{R}}(U) \to \mathcal{A}^k(U)$$

from the space $\mathcal{A}^k_{\mathbb{R}}(U)$ of *real-valued degree k smooth differential forms over U*. The image of this map comprises precisely those forms $\alpha \in \mathcal{A}^k(U)$ that are invariant under the *complex conjugation map* given locally by

$$\alpha = \sum_{I,J} \alpha_{I\overline{J}} dz_I \wedge d\overline{z}_J \mapsto \overline{\alpha} = \sum_{I,J} \overline{\alpha_{I\overline{J}}} d\overline{z}_I \wedge dz_J.$$

1.1.2 The sheaf of currents

Let $U \subseteq X$ be an open subset of a smooth, oriented manifold of real dimension $\dim_{\mathbb{R}} X = n$ and K a compact subset of U. Denote by $\mathcal{A}^k(U;K) \subseteq \mathcal{A}^k(U)$ the subspace of smooth differential forms of degree k on U with support contained in K. Moreover, let $\mathcal{A}^k_c(U)$ denote the union of $\mathcal{A}^k(U;K)$ as K ranges over all compact subsets of U. The subspace $\mathcal{A}^k_c(U)$ comprises precisely the smooth forms with compact support. If $V \subseteq U$ is an open coordinate chart with coordinates $t = (t_1, \dots, t_n)$, for each compact subset $K \subseteq V$ and $\alpha \in \mathcal{A}^k(U)$, set

$$\nu_{t,K}(\alpha) = \sup_{x\in K} \sup_{|I|=k} |\alpha_I(x)|, \text{ where } \sum_{|I|=k} \alpha_I dt_I = \alpha|_V.$$

Each $\nu_{t,K}$ defines a seminorm on $\mathcal{A}^k_c(U)$ and we denote the collection of such seminorms for all possible t and K by $\mathrm{SN}(U)$. Moreover, for each tuple $s = (s_1, \dots, s_n)$ of non-negative integers, let $\partial^s_t \alpha \in \mathcal{A}^k(V)$ denote the form given by

$$\partial^s_t \alpha = \sum_{|I|=k} \frac{\partial^{|s|}\alpha_I}{\partial t_1^{s_1} \partial t_2^{s_2} \dots \partial t_n^{s_n}} dt_I.$$

The following definition is due to de Rham, based on the concept of distributions introduced by Schwartz.

Definition 1.1. A *current of dimension k and degree $n-k$ on U* is a linear map $T : \mathcal{A}^k_c(U) \to \mathbb{C}$ which is continuous in the following sense. For all compact subsets $K \subseteq U$ and all sequences $\{\alpha_j\}$ belonging to $\mathcal{A}^k_c(U;K)$ if $\nu(\partial^s_t \alpha_j) \to 0$ for all $\nu \in \mathrm{SN}(U)$ and for all $s \in \mathbb{Z}^n_{\geq 0}$, then we have $T(\alpha_j) \to 0$. If moreover $\nu(\partial^s_t \alpha_j) \to 0$ for all $\nu \in \mathrm{SN}(U)$ and merely for $|s| = s_1 + \dots + s_n \leq d$ already implies $T(\alpha_j) \to 0$, then T is said to be of *order d*. A current is of *finite order* if it is of order d for some $d \geq 0$. The set of currents of dimension k (or degree $n-k$) is denoted by $\mathcal{D}_k(U)$ (or $\mathcal{D}^{n-k}(U)$). A current of dimension zero is called a *distribution*. A distribution of order zero is called a *measure*.

Remark 1. If a current $T \in \mathcal{D}_k(U)$ is of order s, then T admits a unique extension to continuous linear functional $T : C^s_c(U, \bigwedge^k(T^*X \otimes \mathbb{C}) \to \mathbb{C}$ where $C^s_c(U, \bigwedge^k(T^*X \otimes \mathbb{C})$ is the space of compactly supported k-forms with s-times continuously differentiable coefficients. In particular, if T is a measure, then it is a continuous linear functional on $C^0_c(U, \mathbb{C})$. By the *Riesz representation theorem* (see, for example, [22, Theorem 1.38]), for every coordinate chart $V \subseteq U$ carrying local coordinates t, there exists a unique *complex Radon measure μ* (that is, a set-map in the usual sense of measure theory) on V such that

$$T(f) = \int_V f d\mu \quad \text{for } f \in C^0_c(V, \mathbb{C}).$$

This justifies the term *measure* for distributions of order 0. Sometimes, the two (closely corresponding) senses of the word measure are deliberately confused, and one may write

$T(A)$ for a Borel set A to mean the measure of the set A with respect to the measure determined by the distribution T.

Remark 2. As an immediate consequence of the definition, we see that a current is locally a differential form with distribution coefficients. Indeed, if V is a coordinate chart and $T \in \mathcal{D}^p(V)$, then we can write

$$T = \sum_{|I|=p} T_I dx_I$$

where T_I is determined by $T \wedge dx_{I^c} = T_I \wedge dx_I \wedge dx_{I^c}$ with I^c the set of complementary indices to I. Now T_I is a current of degree 0 but it can be identified with the distribution $\tilde{T}_I(f) = T_I(f dx_1 \wedge \cdots \wedge dx_n)$. Then T is of order s, smooth etc. if and only if $\tilde{T}_I$ is of order s, smooth etc. for all I and all coordinate open neighbourhoods V.

If $U \subseteq V \subseteq X$ are open subsets, then it is clear that $\mathcal{A}_c^k(U) \subseteq \mathcal{A}_c^k(V)$ in a canonical way. Thus, if T is a current of dimension k defined on V, then its restriction to $\mathcal{A}_c^k(U)$ defines a current of dimension k on U. By a straightforward argument using partitions of unity with compact support, one sees that the assignments $U \mapsto \mathcal{D}_k(U)$ together with the respective restriction maps define a sheaf on X, denoted $\mathcal{D}_k$ or $\mathcal{D}^{n-k}$. For each open set $U \subseteq X$, we equip the space $\mathcal{D}^p(U)$ with the topology of *weak convergence*, that is, a sequence of currents $T_i \in \mathcal{D}^p(U)$ converges to $T \in \mathcal{D}^p(U)$ precisely when, for each $\alpha \in \mathcal{A}_c^{n-p}(U)$, the sequence of complex numbers $T_i(\alpha)$ converges to $T(\alpha)$, and in this case, we say T_i *converges weakly to* T.

Since X is assumed oriented, every $\beta \in \mathcal{A}^{n-k}(U)$ (or, more generally, any differential form β whose coefficients are locally integrable functions in each coordinate chart) defines a current T_β by the formula

$$T_\beta(\alpha) = \int_U \beta \wedge \alpha \quad \text{for } \beta \in \mathcal{A}_c^k(U).$$

The map $\beta \mapsto T_\beta$ defines an injective map of sheaves $\mathcal{A}^r \to \mathcal{D}^r$. When $T = T_\alpha$ for some smooth form α, one often says that T is *smooth*. A convenient abuse of notation (which we shall often employ) will be to denote the current T_α also by α.

There is also a map of sheaves $\mathcal{D}^r \otimes \mathcal{A}^\ell \to \mathcal{D}^{\ell+r}, T \otimes \alpha \mapsto T \wedge \alpha$ given by

$$(T \wedge \alpha)(\beta) = T(\alpha \wedge \beta)$$

and this is compatible with the above inclusion. It is also convenient to set $\alpha \wedge T = (-1)^{r\ell} T \wedge \alpha$. We can also extend the exterior differential to currents $d : \mathcal{D}^r \to \mathcal{D}^{r+1}, T \mapsto dT$ by the formula

$$dT(\alpha) = (-1)^{r+1} T(d\alpha)$$

and with this definition, we have

$$d^2 = 0, \quad dT_\alpha = T_{d\alpha}, \quad d(\alpha \wedge T) = (d\alpha) \wedge T + (-1)^k \alpha \wedge T \quad \text{for } \alpha \in \mathcal{A}^k(U), T \in \mathcal{D}^r(U).$$

Note that if T is of order d, then dT is of order at most $d + 1$. We say T is *normal* if both T and dT are of order zero. We say T is *closed* (respectively *exact*) if $dT = 0$ (respectively $T = dS$ for some current S). The current T_α associated to a smooth form α is always normal and closed or exact according as α is closed or exact.

Another important example of a normal current is the *current of integration along an oriented, closed submanifold with boundary.* Let $Y \subseteq U$ be a closed subset that is also an oriented submanifold of dimension k (thus, Y may have non empty boundary ∂Y). Let $[Y]$ denote the assignment given by

$$[Y](\beta) = \int_Y \beta|_Y \quad \text{for } \beta \in \mathcal{A}_c^k(U).$$

This defines a current of order zero, and in fact by Stokes' Theorem, we have $d[Y] = (-1)^{n-k+1}[\partial Y]$.

1.1.3 Cohomology of currents

Let X be an oriented, smooth manifold. The exterior derivative defined on currents makes $(\mathcal{D}^\bullet(X), d)$ into a co-chain complex. Let us denote the cohomology groups thus obtained by $H^k_{\mathcal{D}R}(X, \mathbb{C})$. Observe that there is a map of co-chain complexes $(\mathcal{A}^\bullet(X), d) \to (\mathcal{D}^\bullet(X), d), \alpha \mapsto T_\alpha$, which induces a map $H^k_{dR}(X, \mathbb{C}) \to H^k_{\mathcal{D}R}(X, \mathbb{C})$, where $H^k_{dR}(X, \mathbb{C})$ denotes the usual *de Rham cohomology of* X, that is, the cohomology of $(\mathcal{A}^\bullet(X), d)$. This map is in fact an isomorphism. This is a consequence of the following *Poincaré lemma* for currents.

Theorem 1.2 (Poincaré Lemma)**.** *Suppose U is any open subset of X which is diffeomorphic to a star-shaped open subset of $\mathbb{R}^n$, and $T \in \mathcal{D}^k(U)$ is a closed current.*

1. *If $k > 0$, then T is exact, that is, $T = dS$ for some $S \in \mathcal{D}^k(U)$. Moreover, S can be chosen smooth if T is smooth.*
2. *If $k = 0$, then $T = T_c$ is the current associated to a constant function c.*

Proof. See, for example, [20, Chapter I, Theorem 1.22 and Section 2.D.4]. □

Indeed, this lemma implies that the sheaf $\underline{\mathbb{C}}$ of locally constant complex valued functions is resolved by the two different acyclic complexes of sheaves $(\mathcal{A}^k, d)$ and $(\mathcal{D}^k, d)$. This, in turn, means that the canonical map $H^k_{dR}(X, \mathbb{C}) \to H^k_{\mathcal{D}R}(X, \mathbb{C})$ is the composition of the two isomorphism $H^k_{dR}(X, \mathbb{C}) \to H^k(X, \underline{\mathbb{C}}) \to H^k_{\mathcal{D}R}(X, \mathbb{C})$. (See, for example, [23, Section 4.4].)

If X is a complex manifold of dimension $\dim_{\mathbb{C}} X = m$. Corresponding to the decomposition

$$\mathcal{A}_c^k = \bigoplus_{p+q=k} \mathcal{A}_c^{p,q}$$

on X we get a direct sum decomposition of sheaves

$$\mathcal{D}^k = \bigoplus_{p+q=k} \mathcal{D}^{p,q}.$$

In the same way as above, we can define the maps $\partial : \mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}, T \mapsto \partial T$ and $\overline{\partial} : \mathcal{D}^{p,q} \mapsto \mathcal{D}^{p,q+1}, T \mapsto \overline{\partial} T$ in such a way that $\partial^2 = \overline{\partial}^2 = 0, d = \partial + \overline{\partial}$. Just as in the case of the sheaves $\mathcal{A}^{p,q}$, the operator $\overline{\partial}$ makes $(\mathcal{D}^{p,\bullet}(X), \overline{\partial})$ into a chain complex with cohomology groups $H^{p,q}_{\overline{\mathcal{D}}}(X)$. The following result, called the *Grothendieck-Dolbeault Lemma*, is the analogue of the Poincaré Lemma in this case.

Theorem 1.3 (Grothendieck-Dolbeault Lemma)**.** *Suppose U is an open subset of X which is biholomorphic to a polydisc in $\mathbb{C}^n$ and $T \in D^{p,q}(U)$ is a $\overline{\partial}$ closed current.*

1. *If $q > 0$, then there exists a current $S \in D^{p,q-1}(U)$ such that $T = \overline{\partial} S$. If T is smooth, then S can be chosen to be smooth.*

2. *If $q = 0$, then $T = T_\alpha$ is the current associated to a* holomorphic form α*, that is,* $\alpha \in \mathcal{A}^{p,0}(U)$ *and* $\overline{\partial}\alpha = 0$.

Proof. See, for example, [20, Chapter I, Theorem 3.29]. □

For entirely analogous reasons as for $H^k_{dR}(X, \mathbb{C})$ we therefore obtain that $H^{p,q}_{\overline{\mathcal{D}}}(X) \cong H^{p,q}_{\overline{\partial}}(X)$, the usual Dolbeault cohomology groups, via the obvious map.

Thus, we can (and henceforth do) think of d-closed (respectively $\overline{\partial}$-closed) currents as representing de Rham (respectively Dolbeault) cohomology classes.

1.1.4 Positive currents on a complex manifold

Let X be a complex manifold and let $U \subseteq X$ be an open subset of X. There is a complex conjugation map $\mathcal{A}^{n-q,n-p}_c(U) \to \mathcal{A}^{n-p,n-q}_c(U), \alpha \mapsto \overline{\alpha}$ which induces a conjugate-linear map of sheaves $\mathcal{D}^{p,q} \to \mathcal{D}^{q,p}, T \mapsto \overline{T}$. We shall say that a current $T \in \mathcal{D}^{p,p}(U)$ is *real* if $T = \overline{T}$. The following definition is due to Lelong.

Definition 1.4. A current $T \in \mathcal{D}^{p,p}(U)$ is called *positive* if for every choice $\alpha_1, \ldots, \alpha_p \in \mathcal{A}^{1,0}(U)$, the distribution

$$T \wedge (\sqrt{-1}\alpha_1 \wedge \overline{\alpha_1}) \wedge \cdots \wedge (\sqrt{-1}\alpha_p \wedge \overline{\alpha_p}) \in \mathcal{D}^{n,n}(U)$$

is a *positive distribution*, that is, a distribution which assigns real non-negative values to real-valued non-negative functions with compact support. If this is the case, we write $T \geq 0$ and write $T_1 \geq T_2$ if $T_1 - T_2 \geq 0$. A smooth form α is *positive* if the associated current T_α is so, and is *strongly positive* if $T \wedge \alpha$ is positive for all positive currents T. A current T is *strongly positive* if $T \wedge \alpha$ is positive for all positive forms α.

Remark 3. Every positive current is necessarily real, and as the terminology suggests, strongly positive forms and currents are also positive, but not conversely in general, except when their degree is $(0,0), (1,1), (n-1,n-1)$ or (n,n). Moreover, with this definition, volume forms that define the same orientation as the canonical orientation of X as a complex manifold are positive of degree (n,n), and (the $(1,1)$-forms associated to) Hermitian forms are positive of degree $(1,1)$.

Remark 4. Every positive current $T \in \mathcal{D}^{p,p}(U)$ is necessarily of order 0, that is, what we have called a measure. To see this, we can assume T is a distribution, meaning $p = n$. Now observe that if K is any compact subset of U then we can find a compact subset K' of U and a real-valued function ψ with support contained in K' with the following properties: firstly, that K lies in the interior of K', and secondly, that ψ has support contained in K' with $0 \leq \psi \leq 1$ on U and $\psi(x) = 1$ for $x \in K$. Then, it is clear that for any real-valued smooth function f with support in K, we have $f \leq (\sup_{x\in K} |f(x)|)\psi$, and therefore, by the positivity of T, we have

$$T(f) \leq C \sup_{x\in K} |f(x)|, \quad C = T(\psi).$$

Thus, if $\{f_k\}$ is any sequence of smooth functions with support contained in K which converges uniformly to zero, then the above shows that the real and imaginary parts of the sequence $\{T(f_k)\}$ also converge to zero. This in particular shows that every closed positive current is normal.

One of the most important examples of a closed positive current is given by the current of integration along a closed analytic subset $Z \subseteq U$ of dimension p. This is defined by

$$[Z](\alpha) = \int_{Z^{\mathrm{reg}}} \alpha|_{Z^{\mathrm{reg}}} \text{ for } \alpha \in \mathcal{A}_c^{p,p}(U)$$

where Z^{reg} denotes the smooth locus of Z. This integral is well-defined essentially because the singularities of Z are of finite type, that is, they can locally be desingularised by a finite-sheeted branched covering. (This is certainly not the case for arbitrary immersed submanifolds of U.) This defines $[Z]$ as a current of order 0 and degree (p,p). A straightforward calculation shows that in fact $[Z]$ is a positive current. Moreover, it is easy to verify that the restriction of $[Z]$ on $U \setminus Z^{\mathrm{sing}}$ is d-closed (where Z^{sing} denotes the singular locus of Z). It is a theorem of Lelong that $[Z]$ is d-closed on all of U. (See, for example [19, Theorem 1.18].)

Another important example of a closed positive current is the degree $(1,1)$ current associated to a *plurisubharmonic function*. We briefly recall the definition.

Definition 1.5. Let $U \subseteq \mathbb{C}^n$ be a connected open subset. An upper-semicontinuous function $u : U \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if u is not identically $-\infty$ and for all $x \in U$,

$$u(x) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + e^{\sqrt{-1}\theta}\zeta) d\theta$$

whenever $\zeta \in \mathbb{C}^n \setminus \{0\}$ is such that $\overline{B(x, |\zeta|)} \subseteq U$.

A plurisubharmonic function is also subharmonic, that is, it satisfies the mean value inequality. From this, it follows that that every plurisubharmonic function is in fact locally integrable with respect to the Lebesgue volume form $d\lambda_z$ on U, that is, for every compact subset $K \subseteq U$, the integral

$$\int_K |u(z)| d\lambda_z < \infty.$$

Thus, one can define the current $T_u \in \mathcal{D}^0(U)$ of order 0 and degree 0 given by

$$T_u(f d\lambda_z) = \int_U u(z) f(z) d\lambda_z, \quad \text{for } f \in C_c^\infty(U, \mathbb{C}).$$

Lemma 1.6. *Let $U \subseteq \mathbb{C}^n$ be a connected open subset, $T \in \mathcal{D}^0(U)$ a degree zero current on U. Then, the following are equivalent.*

1. *$\sqrt{-1}\partial\bar{\partial}T$ is a positive current.*
2. *There exists a plurisubharmonic function u on U such that $T = T_u$ is the current associated to u.*

Proof. See, for example, [20, Chapter I, Section 5]. □

In fact, the condition of plurisubharmonicity is invariant under holomorphic change of coordinates, and thus the notion of a plurisubharmonic function makes sense on any complex manifold. Moreover, the above result, by the help of the Grothendieck-Dolbeault lemma, immediately implies the following fact about closed positive currents of degree $(1,1)$.

Corollary. *Let X be a complex manifold and $S \in \mathcal{D}^{1,1}(X)$ be a d-closed positive current. Then, for every $x \in X$ there exists an open neighbourhood U containing x such that $S|_U = \sqrt{-1}\partial\bar{\partial}T_u$ for a plurisubharmonic function u on U.*

1.1.5 Intersection numbers on compact complex manifolds

An important application of closed positive currents is a straightforward definition of *intersection numbers* on a compact complex manifold X. Given closed analytic subvarieties Z_i of X of (complex) codimension p_i the closed positive currents of integration $[Z_i]$ define cohomology classes $\tau_i \in H^{2p_i}_{dR}(X, \mathbb{R})$.

Definition 1.7. The *intersection number* $Z_1 \cdot Z_2 \cdot \dots \cdot Z_s$ is defined by

$$Z_1 \cdot Z_2 \cdot \dots \cdot Z_s = \int_X \theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_s$$

where $\theta_i \in \mathcal{A}^{2p_i}(X)$ are smooth representatives of the cohomology classes τ_i.

Remark 5. Note that if $\sum_i p_i \neq \dim_{\mathbb{C}} X$ in the notation of the above definition, then the intersection number is by definition zero. Moreover, although it is not manifestly obvious with this definition, $Z_1 \cdot Z_2 \cdot \dots \cdot Z_s$ is in fact geometric, that is, if the intersection of the Z_i is transverse and therefore comprises k isolated points of multiplicity 1, then $Z_1 \cdot Z_2 \cdot \dots \cdot Z_s = k$, and therefore this definition agrees, in the case X is a smooth projective, with the intersection number as defined using algebraic geometry. In particular, if X is a compact complex surface, and C, D are distinct irreducible curves in X, then $C \cdot D \geq 0$.

Remark 6. Given cohomology classes α_i and a closed analytic subvariety Z, we will often use the notations

$$\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_s \cdot [Z] = \int_Z \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_s = \int_Z \theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_s$$

where θ_i are smooth representatives of the α_i.

Remark 7. Suppose $Z \subseteq X$ is an analytic subvariety of X, and $f : \tilde{X} \to X$ is any bimeromorphic holomorphic map which admits a partial inverse defined on a Zariski open subset U of X. If the smooth locus of Z is contained in U, then defining $\tilde{Z}$ to be the closure of $f^{-1}(U \cap Z)$ in $\tilde{X}$, we see directly from the definition that

$$\int_{\tilde{Z}} f^*\alpha_1 \cdot \dots \cdot f^*\alpha_s = \int_Z \alpha_1 \cdot \ldots \alpha_s.$$

These conditions are satisfied in any resolution of singularities of Z. This is a special case of the *projection formula*.

1.1.6 Lelong numbers and generic multiplicities

One should think of a closed positive current as a singular analogue of a closed positive form. In fact, the singularities of a closed positive current should be deemed as an important feature of this theory, and carry geometric information about X. This is captured by the concept of Lelong numbers, first introduced by Lelong.

Definition 1.8. Let $T \in \mathcal{D}^{p,p}(X)$ be a closed positive current on a complex manifold X and let $x \in X$. Pick an open coordinate neighbourhood V of x carrying holomorphic coordinates $z = (z_1, \dots, z_n)$ centred at x. Let λ_z denote the Lebesgue measure on V with respect to the coordinates z_i and $\sigma_{T,z}$ denote the positive measure on V given by

$$\sigma_{T,z} = \frac{1}{(n-p)!} T \wedge \left(\frac{\sqrt{-1}}{2} \partial\overline{\partial}|z|^2 \right)^{n-p} = \frac{1}{(n-p)!} T \wedge \left(\frac{\sqrt{-1}}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \right)^{n-p}.$$

The *Lelong number* $\nu(T,x)$ of T at x is the non-negative real number given by

$$\lim_{r\to 0^+} \frac{\sigma_{T,z}(B_z(x,r))}{\lambda_z(B_z(x,r))}$$

where $B_z(x,r)$ is the open ball of radius r around x in V, that is, the set $\{x \in V : \sum |z_k(x)|^2 < r\}$.

Remark 8. Here, we have abused notation in writing $\sigma_{T,z}(B_z(x,r))$ as explained at the end of Remark 1. Note that the limit in the above definition always exists because the ratio of $\sigma_{T,z}(B_z(x,r))$ and $\lambda_z(B_z(x,r))$ is a decreasing function of r.

The following theorem (due to Siu) highlights the intimate connection of closed positive currents with complex analytic geometry and is foundational.

Theorem 1.9 (Siu)**.** *Let X be a complex manifold and $T \in \mathcal{D}^{p,p}(X)$ a d-closed positive current. Then, the following statements hold.*

1. *The Lelong number $\nu(T,x)$ of T at the point $x \in X$ does not depend on the choice of holomorphic coordinates around x.*
2. *For every $c > 0$ the superlevel set $E_c(T)$ comprising the points $x \in X$ such that $\nu(T,x) \geq c$ is a closed analytic subset of X of codimension at least p.*

Remark 9. We will denote by $E_+(T)$ the union of all the superlevel sets $E_c(T)$ as $c > 0$. Thus, by the above Theorem, it follows that $E_+(T)$ is a union of at most countably many distinct irreducible analytic subvarieties of X.

Remark 10. It is a result of Lelong (see [19, Theorem 2.8(b)] that if $T = \sqrt{-1}\partial\bar{\partial}u$ is a closed positive current of degree $(1,1)$ associated to a plurisubharmonic function u then $\nu(T,x)$ is the supremum of $\nu \geq 0$ such that the function $z \mapsto u(z) - \nu \log|z-x|$ remains bounded in a neighbourhood of x. This in particular shows that $\nu([Z_f],x) = \mathrm{ord}_x(f)$ for holomorphic f, where Z_f is the zero locus of f and $\mathrm{ord}_x(f)$ is the order of vanishing of f at x. Moreover, $\nu(T,x) = 0$ whenever T is smooth in a neighbourhood of x.

Given $T \in \mathcal{D}^{p,p}(X)$ a closed positive current, let us set

$$\nu(T,V) = \inf_{x\in V} \nu(T,x)$$

for V any analytic subvariety. If V has codimension smaller than p, then Siu's Theorem 1.9 of course implies that $\nu(T,V) = 0$. If the codimension of V is exactly p, then it turns that $T \geq \nu(T,V)[V]$ and so $T - \nu(T,V)[V] \in \mathcal{D}^{p,p}(X)$ is again a closed positive current. One can therefore 'subtract off' the p-codimensional components of $E_+(T)$. The *Siu decomposition theorem* says that this process can be carried out until no components of codimension p remain.

Theorem 1.10 (Siu Decomposition Theorem)**.** *Let $T \in \mathcal{D}^{p,p}(X)$ be a closed positive current. Then, the series*

$$S(T) = \sum_{\substack{\nu(T,Z)>0 \\ \mathrm{codim} Z = p}} \nu(T,Z)[Z]$$

converges weakly. Moreover, the remainder $R(T) = T - S(T)$ is a closed positive current such that all the irreducible components of $E_c(R)$ for $c > 0$ have codimension strictly greater than p.

1.1.7 The $\partial\bar{\partial}$-lemma

From now on, assume that X is a compact Kähler manifold. This is the case in which we shall be chiefly interested. Thus, the cohomology groups $H^{p,q}(X,\mathbb{C})$ are all finite-dimensional and the Hodge decomposition

$$H^k_{dR}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X)$$

holds. Moreover, the $\partial\bar{\partial}$-lemma continues to hold for currents. More precisely, we have the following.

Lemma 1.11 ($\partial\bar{\partial}$-lemma)**.** *Suppose X is a compact Kähler manifold and $T \in \mathcal{D}^{p,q}(X)$ is a current of pure bidegree (p,q) which is d-exact, that is, $T = dS$ for some $S \in \mathcal{D}^{p+q-1}(X)$. Then, there exists a current $u \in \mathcal{D}^{p-1,q-1}(X)$ such that $\sqrt{-1}\partial\bar{\partial}u = T$.*

Proof. We only recall the proof for a real current $T \in \mathcal{D}^{1,1}(X)$ of pure bidegree $(1,1)$. (This is the only case which will be pertinent to the present thesis.) Let $S \in \mathcal{D}^1(X)$ be such that $T = dS$. Then, we also have $T = d\bar{S}$, and so we can assume that S is real. Decompose $S = S^{(1,0)} + S^{(0,1)}$ into its bidegree components. Then, we have $\overline{S^{(1,0)}} = S^{(0,1)}, \bar{\partial}S^{(0,1)} = \partial S^{(1,0)} = 0$. By the Grothendieck-Dolbeault lemma for currents, there exists an open cover U_j of X such that on U_j we have $S^{(0,1)} = \bar{\partial}v_j$ for some $v_j \in \mathcal{D}^0(U_j)$. Clearly we then have (by conjugation) that $\partial\bar{v}_j = S^{(1,0)}$ on U_j. Thus, on U_j, we obtain

$$dS = d(S^{(1,0)} + S^{(0,1)}) = d(\partial\bar{v}_j + \bar{\partial}v_j) = \bar{\partial}\partial\bar{v}_j + \partial\bar{\partial}v_j = \sqrt{-1}\partial\bar{\partial}(2\mathrm{Im}v_j).$$

Let us pick a partition of unity ρ_j subordinate to U_j and define $u = \sum \rho_j u_j$ where $u_j = 2\mathrm{Im}v_j$. Then, on U_j we can calculate

$$T - \sqrt{-1}\partial\bar{\partial}u = dS - \sqrt{-1}\partial\bar{\partial}u = \sqrt{-1}\partial\bar{\partial}\sum_k \rho_k(u_j - u_k).$$

Let x be a point in U_j. Then, if $x \in U_k$ we have $\sqrt{-1}\partial\bar{\partial}(u_j - u_k) = dS - dS = 0$. This means that $u_j - u_k$ is a pluriharmonic function on $U_j \cap U_k$, and therefore $\rho_k(u_j - u_k)$ is smooth around x. On the other hand, if $x \notin U_k$, then ρ_k vanishes in a neighbourhood of x, and so $\rho_k(u_j - u_k)$ is once again smooth around x. Therefore, $\sum_k \rho_k(u_j - u_k)$ is in fact smooth on each U_j, and so the current $T - \sqrt{-1}\partial\bar{\partial}u$ is a smooth form, say α. But then we have

$$\alpha = T - \sqrt{-1}\partial\bar{\partial}u = d(S + \sqrt{-1}\bar{\partial}u).$$

But this implies in fact that $\alpha = d\beta$ for a smooth form β. Indeed, this claim is equivalent to the injectivity of the natural map $H^2_{dR}(X,\mathbb{C}) \to H^2_{\mathcal{D}R}(X,\mathbb{C})$, but this map is even an isomorphism. Now, since X is compact and Kähler, every smooth d-exact form is $\partial\bar{\partial}$-exact by the usual $\partial\bar{\partial}$-lemma for smooth forms. (See, for example, [24, Corollary 3.2.10].) Thus, $\alpha = \sqrt{-1}\partial\bar{\partial}f$ for some smooth function f. This means that we finally obtain

$$T = \sqrt{-1}\partial\bar{\partial}u + \alpha = \sqrt{-1}\partial\bar{\partial}(u + f).$$

This proves the claim. □

Remark 11. Note that the above proof works for any complex manifold X which satisfies the following property: if a closed, real form $\alpha \in \mathcal{A}^{1,1}(X)$ satisfies $\alpha = d\beta$ for some $\beta \in \mathcal{A}^1(X)$, then $\alpha = \sqrt{-1}\partial\bar{\partial} f$ for some smooth function f. More generally, one can drop the Kähler hypothesis and work instead throughout with the group $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$, namely the group of d-closed real currents of bidegree $(1, 1)$ modulo the image under $\sqrt{-1}\partial\bar{\partial}$ of the group of currents of degree zero. However, since most of our considerations in the later Chapters exclusively involve examining PDEs whose solution is a Kähler form or whose auxiliary data include Kähler forms, we do not sacrifice any generality by imposing this hypothesis. Many of the results that are stated in this Chapter nevertheless remain true if the group $H^{1,1}(X, \mathbb{R})$ is replaced with $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ throughout.

Let us denote by $H^{p,p}(X, \mathbb{R})$ the image of closed, real currents in $H^{p,p}_{\bar{\partial}}(X) \cap H^{2p}_{dR}(X, \mathbb{C})$. The above $\partial\bar{\partial}$-lemma 1.11 is very convenient from the point of view of parametrising closed currents in a given cohomology class $\tau \in H^{1,1}(X, \mathbb{R})$ that possess a very flexible form of positivity.

Definition 1.12. A real $(1, 1)$ current $T \in \mathcal{D}^{1,1}(X)$ is called *almost positive* if $T \geq \alpha$ for a smooth, real $(1, 1)$ form $\alpha \in \mathcal{A}^{1,1}(X)$.

Let $\tau \in H^{1,1}(X, \mathbb{R})$ be a cohomology class and θ a smooth representative of τ. Then, every closed current $T \in \tau$ can be written as $T = \theta + \sqrt{-1}\partial\bar{\partial} u$ for some degree 0 current $u \in \mathcal{D}^0(X)$. Now, if $T \geq \alpha$ is almost positive, then on each small coordinate chart U with holomorphic coordinate $z = (z_1, \ldots, z_n)$, there exists some $C > 0$ such that, on U, we have

$$\alpha + C\sqrt{-1}\partial\bar{\partial}(|z_1|^2 + \cdots + |z_n|^2) \geq 0,$$

and also $\theta = \sqrt{-1}\partial\bar{\partial} f$ for a smooth function f on U. Then, $T + C\sqrt{-1}\partial\bar{\partial}|z|^2 = \sqrt{-1}\partial\bar{\partial}(f + u + C|z|^2) \geq 0$. By Lemma 1.6, it follows that $u + f + C|z|^2$ is a plurisubharmonic function on U. From this, we conclude that u is in fact locally integrable and can be locally written as a sum of a smooth function and a plurisubharmonic function. This motivates the following definition.

Definition 1.13. An *almost plurisubharmonic function* (or *quasi-plurisubharmonic function*) is a locally integrable function which can locally be expressed as the sum of a smooth function and a plurisubharmonic function. If θ is a smooth representative of $\tau \in H^{1,1}(X, \mathbb{R})$, we denote by $\mathrm{PSH}(X, \theta)$ the set comprising those almost plurisubharmonic functions u such that $\theta + \sqrt{-1}\partial\bar{\partial} u$ is a closed positive current.

Remark 12. Thus, every almost positive current T in the cohomology class $\tau \in H^{1,1}(X, \mathbb{R})$ can be written as $T = \theta + \sqrt{-1}\partial\bar{\partial} u$, where θ is a smooth representative of τ and u is an almost plurisubharmonic function on X. If $T = \theta' + \sqrt{-1}\partial\bar{\partial} u'$ is another such representation of T, then $u - u'$ is a globally defined smooth function on X. This is a consequence of the (proof of the) $\partial\bar{\partial}$-lemma 1.11. The almost plurisubharmonic function u is sometimes called the *almost plurisubharmonic potential of T with respect to θ*.

Given a closed almost positive current T in the cohomology class $\tau \in H^{1,1}(X, \mathbb{R})$, we can extend to T the concept of Lelong numbers by setting

$$\nu(T, x) = \nu(\sqrt{-1}\partial\bar{\partial} u, x)$$

where u is any plurisubharmonic function defined on a neighbourhood of x such that $T = \sqrt{-1}\partial\bar{\partial}(u + f)$ for some smooth function f on U. Similarly, we can define $\nu(T, Z)$ for any analytic subvariety of Z. One checks directly that this definition is independent of any choices.

1.1.8 Singularities of currents

The language of almost plurisubharmonic functions is convenient from the point of view of studying the singularities of almost positive currents. This is made precise by the following definition.

Definition 1.14. Let X be a compact Kähler manifold and u_1, u_2 almost plurisubharmonic functions on X. We say that u_1 is *less singular than* u_2 and write $u_1 \preceq u_2$ if there exists $C > 0$ such that $u_1 + C \geq u_2$. Let T_1, T_2 be closed almost positive currents on X. We say that T_1 is *less singular* than T_2 and write $T_1 \preceq T_2$ if, whenever we can write $T_i = \theta_i + \sqrt{-1}\partial\bar{\partial}u_i$ with θ_i smooth and closed, and u_i almost plurisubharmonic, we have $u_1 \preceq u_2$.

Remark 13. The pre-order relation $\preceq$ on closed almost positive currents in a given cohomology class generates an equivalence relation. If T_1, T_2 belong to the same equivalence class under this relation, we say T_1 and T_2 have the same *singularity class.* Two closed almost positive currents in the same singularity class have identical Lelong numbers. This is a consequence of a theorem of Siu mentioned in Remark 10.

Given $\tau \in H^{1,1}(X, \mathbb{R})$ and a real $\alpha \in \mathcal{A}^{1,1}(X)$, let us denote by $\tau[\alpha]$ the set of closed almost positive currents T in the cohomology class τ satisfying $T \geq \alpha$. The utility of the above Definition 1.14 is in the following lemma.

Lemma 1.15. *Let X be a compact Kähler manifold, $\tau \in H^{1,1}(X, \mathbb{R})$ and $\alpha \in \mathcal{A}^{1,1}(X)$ a real, smooth $(1,1)$ form. Then, for any (nonempty) family $T_j, j \in J$ of closed almost positive currents in $\tau[\alpha]$, there exists an almost positive current $\inf_j T_j$ which is an infimum with respect to the pre-order relation $\preceq$ and satisfies $\inf T_j \geq \alpha$. Moreover, the singularity class of any such infimum is unique and for every $x \in X$ we have*

$$\nu(\inf_j T_j, x) = \inf_j \nu(T_j, x).$$

Proof. See [21, Section 2.8]. □

By taking the family to be all of $\tau[\alpha]$, we obtain the following useful corollary.

Corollary. *Let X be a compact Kähler manifold, $\tau \in H^{1,1}(X, \mathbb{R})$ a cohomology class and $\alpha \in \mathcal{A}^{1,1}(X)$ a smooth, real form. Then, if $\tau[\alpha]$ is non-empty, there exists a current $T_{\min,\alpha} \in \tau$ such that $T_{\min,\alpha} \geq \alpha$ and $T_{\min,\alpha} \preceq T$ for all $T \in \tau[\alpha]$. Any other T' satisfying the same properties as $T_{\min,\alpha}$ lies in the same singularity class as $T_{\min,\alpha}$.*

Another useful idea is the concept of *analytic singularities.*

Definition 1.16. A closed almost positive current T is said to have *analytic singularities* if, whenever $T = \theta + \sqrt{-1}\partial\bar{\partial}u$ with θ a smooth closed form and u almost plurisubharmonic, then there exists a $c > 0$ such that on any sufficiently small open neighbourhood, the restriction of u lies in the same singularity class as the function

$$\frac{c}{2\pi}\log(|f_1|^2 + |f_2|^2 + \cdots + |f_s|^2)$$

for some local holomorphic functions $f_1, \ldots, f_s$. The local functions f_i generate an ideal sheaf $\mathcal{I}$ called a *sheaf of singularities* of T, and T is said to have *analytic singularities of type* $(\mathcal{I}, c)$.

1.1.9 Regularisation theorem of Demailly

Here we briefly recall a regularisation result due to Demailly that allows us to approximate any closed almost positive current by a sequence of almost positive currents in the same cohomology class whose singularities are much better behaved.

Theorem 1.17 (Demailly Regularisation Theorem)**.** *Let X be a compact Kähler manifold and let ω be a Kähler form on X. Suppose, $T \in \mathcal{D}^{1,1}(X)$ is a closed almost positive current in the cohomology class $\tau \in H^{1,1}(X, \mathbb{R})$ satisfying the lower bound $T \geq \alpha$ for some smooth real form $\alpha \in \mathcal{A}^{1,1}(X)$. Then, the following statements hold.*

1. *There exists a sequence $\theta_k \in \mathcal{A}^{1,1}(X)$ of smooth, real forms in the cohomology class τ, and continuous functions λ_k on X such that θ_k converge weakly to T, $\lambda_k(x)$ decrease to $\nu(T, x)$ and $\theta_k \geq \alpha - C\lambda_k\omega$. Here, $C > 0$ is a constant depending only on the curvature of (T_X, ω) as a Hermitian vector bundle.*

2. *There exists a sequence of closed currents T_k in the cohomology class τ, and $\varepsilon_k > 0$ such that T_k converges weakly to T, ε_k decreases to zero, $T_k \geq \alpha - \varepsilon_k\omega$ for each k, and $\nu(T_k, \bullet)$ converges uniformly to $\nu(T, \bullet)$.*

1.2 Positive cones in $H^{1,1}(X, \mathbb{R})$

In this thesis, we shall be concerned with positivity conditions that require certain cohomology classes associated to geometric PDEs to lie in various positive cones in $H^{1,1}(X, \mathbb{R})$. Here, we recall the definitions of all of these cones and prove some basic properties which will be crucial to our later considerations. Thus, let X, be a compact Kähler manifold. The cohomology classes that can be represented by Kähler forms comprise an open convex cone called the *Kähler cone*, denoted $\mathcal{K}_X$, or simply $\mathcal{K}$ when no confusion is likely to occur.

Definition 1.18. Let $\tau \in H^{1,1}(X, \mathbb{R})$ be a cohomology class. We say τ is

- *pseudoeffective* if τ can be represented by a closed positive current;
- *nef* if for all $\varepsilon > 0$, τ can be represented by a smooth form θ_ε such that $\theta_\varepsilon \geq -\varepsilon\omega$;
- *big* if there exists some $\varepsilon > 0$ such that τ can be represented by a closed current T with $T \geq \varepsilon\omega$. Such a current is called a *Kähler current*.

The pseudoeffective, nef and big classes comprise, respectively, the *pseudoeffective, nef and big cones*, denoted $\mathcal{E}_X, \mathcal{N}_X$ and $\mathcal{B}_X$ respectively.

Remark 14. Since any two Kähler forms mutually commensurate each other, the definitions above do not depend on the choice of ω. This will be true for all the definitions of positive cones that follow. It also follows easily from the definitions that $\mathcal{E}$ is the closure of the open cone $\mathcal{B}$ and $\mathcal{N}$ is the closure of the open cone $\mathcal{K}$.

Definition 1.19. Given a pseudoeffective class $\tau \in H^{1,1}(X, \mathbb{R})$, the *minimal multiplicity* $\nu(\tau, x)$ of τ at x is given by

$$\nu(\tau, x) = \sup_{\varepsilon > 0} \nu(T_{\min,\varepsilon}, x)$$

where $T_{\min,\varepsilon} = T_{\min,-\varepsilon\omega}$ is a closed $(1,1)$ *current of minimal singularities* in $\tau[-\varepsilon\omega]$. If V is any irreducible analytic subvariety of X, we set

$$\nu(\tau, V) = \inf_{x \in V} \nu(\tau, x).$$

Remark 15. Note that $\tau[-\varepsilon\omega]$ is non-empty for all $\varepsilon > 0$ whenever τ is a pseudoeffective class. Also, if $0 < \varepsilon < \varepsilon'$, then it is clear that $T_{\min,\varepsilon} \in \tau[-\varepsilon'\omega]$, so $T_{\min,\varepsilon'} \preceq T_{\min,\varepsilon}$ and therefore $\nu(T_{\min,\varepsilon}, x) \geq \nu(T_{\min,\varepsilon'}, x)$ by Lemma 1.15. Thus, the supremum in Definition 1.19 is always well-defined and can be replaced with a supremum over any small open interval with endpoint zero. Moreover, when τ is big, the supremum can even be replaced by the formula $\nu(\tau, x) = \nu(T_{\min}, x)$, where $T_{\min}$ is any closed positive current of minimal multiplicity in τ. (See [21, Proposition 3.6].)

The following definition is due to Wu [25] and generalises the notion of a *modified nef class* defined by Boucksom in [21].

Definition 1.20. Let X be a compact Kähler manifold of dimension n. A pseudoeffective $(1,1)$ class τ on X is said to be *nef in codimension* q or $(n-q)$*-modified nef* if the minimal multiplicity $\nu(\tau, Z) = 0$ for any irreducible analytic subvariety of codimension $k \leq q$. These classes comprise a closed cone $\mathcal{M}^q\mathcal{N} = \mathcal{M}_{n-q}\mathcal{N}$, which we shall call the $(n-q)$*-modified nef cone* and its interior $\mathcal{M}^q\mathcal{K} = \mathcal{M}_{n-q}\mathcal{K}$ the $(n-q)$*-modified Kähler cone*. A class $\tau \in \mathcal{M}_p\mathcal{K}$ is called a p*-modified Kähler class*.

Remark 16. We have an obvious inclusion of cones

$$\mathcal{N} = \mathcal{M}_0\mathcal{N} \subseteq \mathcal{M}_1\mathcal{N} \subseteq \ldots \subseteq \mathcal{M}_n\mathcal{N} = \mathcal{E}$$

and similarly for $\mathcal{M}_i\mathcal{K}$. In particular, note that that $\mathcal{M}_n\mathcal{K} = \mathcal{B}$ is the big cone. In fact, as a corollary of a result of Paŭn (see Lemma 2 in [25]), we also have $\mathcal{M}^{n-1}\mathcal{N} = \mathcal{M}_1\mathcal{N} = \mathcal{N}$. In [21], the cone $\mathcal{M}_{n-1}\mathcal{N}$ is called the *modified nef cone*, where it is denoted simply $\mathcal{MN}$. Note, however, that our terminology of n-modified nef for the pseudoeffective cone contrasts with the usage of the words 'modified nef' (especially in the sense of [21, Proposition 2.3]). However, whenever convenient, we will call $\mathcal{E}$, respectively $\mathcal{B}$ the n-modified nef, respectively n-modified Kähler cones.

Definition 1.21. The *non-Kähler locus* $E_{nK}(\tau)$ of a big class τ is given by

$$E_{nK}(\tau) = \bigcap_{T \in \tau} E_+(T)$$

where the intersection ranges over all Kähler currents T representing τ. Here $E_+(T)$ is the set comprising $x \in X$ such that $\nu(T, x) > 0$.

The following result is contained in [21] and plays a central role in obtaining finiteness results about the set of destabilising subvarieties.

Theorem 1.22 (Boucksom)**.** *Let X be a compact Kähler manifold and $\tau \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class on X. Then, the non-Kähler locus $E_{nK}(\tau)$ is an analytic subset of X and there exists a Kähler current $T \in \tau$ with analytic singularities such that $E_+(T) = E_{nK}(\tau)$.*

Proof. See [21, Theorem 3.17(ii)]. □

1.2.1 The non-Kähler loci of big classes

A straightforward consequence of standard results and the above definitions that we shall use is the following result.

Lemma 1.23. *Let X be a compact Kähler manifold of dimension n and let $\tau \in \mathcal{M}_n\mathcal{K}$ be a big class on X. Then, $\tau \in \mathcal{M}_p\mathcal{K}$ on X if and only if every irreducible component of $E_{nK}(\tau)$ has dimension strictly less than p.*

Proof. Fix a Kähler form ω on X. First suppose $\tau \in \mathcal{M}_p\mathcal{K}$. Then, we can find $\varepsilon > 0$ so small that $\tau - 2\varepsilon[\omega] \in \mathcal{M}_p\mathcal{N}$. Thus, for any $V \subseteq X$ of dimension at least p, we have by definition

$$\nu(\tau - 2\varepsilon[\omega], V) = 0.$$

In particular, we have

$$\nu(T_{\min,\varepsilon}, V) = 0$$

where $T_{\min,\varepsilon}$ is a current of minimal singularities in $\tau - 2\varepsilon[\omega]$ satisfying $T_{\min,\varepsilon} \geq -\varepsilon\omega$. But then, the current

$$T = T_{\min,\varepsilon} + 2\varepsilon\omega$$

lies in the class τ and satisfies $T \geq \varepsilon\omega$. Thus, T is a Kähler current in τ and satisfies $\nu(T, V) = 0$ for all $V \subseteq X$ of dimension at least p. Since $E_{nK}(\tau) \subseteq E_+(T)$, we see that no such V can be contained in $E_{nK}(\tau)$.

Conversely, suppose that $E_{nK}(\tau)$ does not contain any irreducible analytic subset of dimension greater than or equal to p. Then, because τ is big, by [21, Theorem 3.17(ii)] there exists a Kähler current $T \in \tau$ such that $E_+(T) = E_{nK}(\tau)$. Because T is a Kähler current, we can find $\varepsilon > 0$ so that $T \geq 2\varepsilon\omega$ and $\tau - \varepsilon[\omega]$ is a big class. In particular, $\nu(T - \varepsilon\omega, V) = 0$ for all $V \subseteq X$ of dimension at least p. Let $T_{\min}$ be any current of minimal singularities in $\tau - \varepsilon[\omega]$ satisfying $T_{\min} \geq 0$. Then, by minimality, we have

$$\nu(T_{\min}, x) \leq \nu(T - \varepsilon\omega, x)$$

for all $x \in X$. Thus, $\nu(T_{\min}, V) = 0$ for all V of dimension at least p. Now, by [21, Proposition 3.6(ii)], we have $\nu(T_{\min}, x) = \nu(\tau - \varepsilon[\omega], x)$, and so we obtain $\nu(\tau - \varepsilon[\omega], V) = 0$ whenever V is of dimension at least p. Thus, $\tau - \varepsilon[\omega] \in \mathcal{M}_p\mathcal{N}$. Since this holds for an arbitrary Kähler class $[\omega]$, we obtain that $\tau \in \mathcal{M}_p\mathcal{K}$ is a p-modified Kähler class. □

We shall use Lemma 1.23 in conjunction with the following straightforward consequence of results of Boucksom and Demailly.

Lemma 1.24. *Let τ be a big cohomology class on a compact Kähler manifold X, and $\omega_1, \omega_2, \ldots, \omega_{p-1}$ be smooth Kähler forms on X. Then, for any p-dimensional subvariety V satisfying $V \not\subseteq E_{nK}(\tau)$, we have*

$$\int_V \tau \cdot [\omega_1] \cdot [\omega_2] \cdot \ldots \cdot [\omega_{p-1}] > 0.$$

Proof. By Boucksom's Theorem 1.22, there exists a Kähler current with analytic singularities $T \in \tau$ such that $E_+(T) = E_{nK}(\tau)$. By rescaling ω_1 if necessary, we may assume that $T \geq \omega_1$. By the regularisation thorem of Demailly (Theorem 1.17) there exist smooth forms θ_k representing τ and continuous functions $\lambda_k : X \to \mathbb{R}$ such that θ_k converge weakly to T, $\theta_k + \lambda_k\omega_1 \geq \omega_1$ in the sense of currents and $\lambda_k(x)$ decrease to $\nu(T, x)$. Since X is compact, we can find smooth functions $\rho_k : X \to \mathbb{R}$ such that for all $x \in X$ we have

$$0 \leq \rho_k(x) - \lambda_k(x) \leq 2^{-k}.$$

Then, we obtain
$$\theta_k + \rho_k\omega_1 \geq (1+2^{-k})\omega_1.$$
From this, it follows that
$$(\theta_k + \rho_k\omega_1) \wedge \omega_1 \wedge \ldots \wedge \omega_{p-1} \geq (1+2^{-k})\omega_1^2 \wedge \ldots \wedge \omega_{p-1}$$
as smooth measures on the regular part of V. Thus, we have
$$\int_V \tau \cdot [\omega_1] \cdot \ldots \cdot [\omega_{p-1}] + \int_V \rho_k \omega_1^2 \wedge \omega_2 \wedge \ldots \wedge \omega_{p-1} \geq (1+2^{-k}) \int_V \omega_1^2 \wedge \omega_2 \wedge \ldots \wedge \omega_{p-1}.$$
Now $\rho_k(x)$ also converges to $\nu(T,x)$, and thus the sequence ρ_k of smooth functions converges to zero almost everywhere on the regular part of V with respect to the measure $\omega_1^2 \wedge \omega_2 \wedge \ldots \wedge \omega_{p-1}$. Thus, taking the limit and applying the bounded convergence theorem, we obtain that
$$\int_V \tau \cdot [\omega_1] \cdot \ldots \cdot [\omega_{p-1}] \geq \int_V [\omega_1]^2 \cdot [\omega_2] \cdot \ldots \cdot [\omega_{p-1}] > 0.$$
□

We shall also need the following statement, which will be used to establish results related to wall-chamber decompositions.

Lemma 1.25. *Let X be a compact Kähler manifold and τ a $(p+1)$-modified Kähler class on X. Then, there exists an open neighbourhood U of τ in $H^{1,1}(X,\mathbb{R})$ and a finite set $S = \{V_1, \ldots, V_r\}$ of irreducible p-dimensional subvarieties of X such that for all Kähler forms $\omega_1, \ldots, \omega_{p-1}$ and all $\tau' \in U$ whenever we have*
$$\int_V \tau \cdot [\omega_1] \cdot \ldots [\omega_{p-1}] \leq 0$$
then $V \in S$.

Proof. By the openness of the $(p+1)$-modified Kähler cone $\mathcal{M}_{p+1}\mathcal{K}$, we can find $\tau_1, \ldots, \tau_\ell$ such that
$$\tau \in U = \mathrm{Int}(\mathrm{conv}(\tau_1, \ldots, \tau_\ell)) \neq \varnothing.$$
Here, $\mathrm{Int}(\mathrm{conv}(\tau_1, \ldots, \tau_\ell))$ denotes the interior of the convex hull of the classes $\tau_1, \ldots, \tau_\ell$. Since τ_i is $(p+1)$-modified Kähler, $E_{nK}(\tau_i)$ contains no $(p+1)$-dimensional irreducible subvarieties of X. Let S be the union of all the p-dimensional irreducible subvarieties of X contained in $E_{nK}(\tau_i)$ for $i = 1, \ldots, \ell$. Now, let $\tau' \in U$ and $\omega_1, \ldots, \omega_{p-1}$ be Kähler forms on X. Suppose that we have
$$\int_V \tau' \cdot [\omega_1] \cdot \cdots \cdot [\omega_{p-1}] \leq 0$$
for some p-dimensional irreducible subvariety V of X. Then, since $\tau' \in U$, we can write
$$\tau' = \sum_{i=1}^{\ell} a_i \tau_i$$
where $a_i > 0$ and $\sum_i a_i = 1$. From this, it follows that
$$\int_V \tau_i \cdot [\omega_1] \cdot \cdots \cdot [\omega_{p-1}] \leq 0,$$
for some i, and this in turn implies (by Lemma 1.24) that V lies in $E_{nK}(\tau_i)$, and therefore $V \in S$. □

1.2.2 The divisorial Zariski decomposition

A classical theorem of Zariski [26] states that any effective divisor D on a surface can be uniquely decomposed as a sum of $\mathbb{Q}$-divisors $D = Z + N$ where Z is nef, N has negative-definite intersection matrix and $Z \cdot N = 0$. In [21] Boucksom generalised this decomposition to arbitrary pseudoeffective classes on compact complex manifolds. We briefly go over the construction.

Definition 1.26. Let $\tau \in \mathcal{E}$ be a pseudoeffective class. The *negative part* of τ is the closed positive current given by

$$N(\tau) = \sum_D \nu(\tau, D)[D]$$

where the sum runs over all irreducible divisors (analytic subvarieties of codimension one). The *positive part* of τ is the cohomology class

$$Z(\tau) = \tau - [N(\tau)].$$

The decomposition $\tau = Z(\tau) + [N(\tau)]$ is called the *Zariski decomposition* of τ.

Remark 17. The negative part $N(\tau)$ is well-defined, since it is at worst a convergent series, being dominated by $S(T_{\min})$ (in the notation of the Siu Decomposition Theorem 1.10) where $T_{\min}$ is any positive current of minimal singularities in τ. What is remarkable, however, is that this power series is actually a finite sum, and hence (the current of integration along) a $\mathbb{R}$-divisor. In fact, $N(\tau)$ is the unique positive current in its cohomology class, and, for this reason, we will often confuse $[N(\tau)]$ with $N(\tau)$, preferring to write the latter even when we mean the cohomology class $[N(\tau)]$.

The properties of the Zariski decomposition are summarised as follows. (See [21, Sections 3.2, 3.3].)

Theorem 1.27 (Boucksom)**.** *Let X be a compact Kähler manifold of dimension n and $\tau \in H^{1,1}(X, \mathbb{R})$ a pseudoeffective class. Then, the Zariski decomposition $\tau = Z(\tau) + N(\tau)$ of τ possesses the following properties.*

1. *The postive part $Z(\tau) \in \mathcal{M}_{n-1}\mathcal{N}_X$ is an $(n-1)$-modified nef class.*

2. *The maps Z and N are projections, that is, $Z \circ Z = Z, N \circ N = N, Z \circ N = N \circ Z = 0$.*

3. *The negative part $N(\tau)$ is equal to the current of integration along an effective $\mathbb{R}$-divisor, that is,*

$$N(\tau) = \sum_{i=1}^{\ell} a_i [E_i]$$

for some $a_i > 0$ and $E_i \subseteq X$ analytic subvarieties of codimension one. The current $N(\tau)$ is the unique closed positive current in its cohomology class. Moreover, the classes $[E_i]$ are linearly independent in $H^{1,1}(X, \mathbb{R})$. In particular, $\ell \leq \rho(X) \leq \dim_{\mathbb{R}} H^{1,1}(X, \mathbb{R})$.

4. *The map $\tau \mapsto N(\tau)$ is convex.*

1.2.3 Big classes on compact Kähler surfaces

Suppose X is a compact Kähler surface. Then, the nef cone $\mathcal{N}_X$ and the modified nef cone $\mathcal{M}_{2-1}\mathcal{N}_X$ coincide. (See, for example, [21, Theorem 4.1].) Moreover, the prime divisors E_i (in this case, curves) appearing in Theorem 1.27 (3) have a negative-definite intersection matrix and this property characterises the negative parts of pseudoeffective classes on X. (See [21, Theorem 4.5].)

In fact, the various different notions of positivity are much better behaved on surfaces. In particular, we have the following theorem of Lamari [27, Theorem 5.3] characterising the Kähler cone $\mathcal{K}_X$ of a compact Kähler surface.

Theorem 1.28 (Lamari)**.** *Suppose X is a compact Kähler surface and $\alpha, \beta \in H^{1,1}(X,\mathbb{R})$ are cohomology classes such that α can be represented by a smooth, closed non-zero form ω with $\omega \geq 0$. Then, $\beta \in \mathcal{K}_X$ if and only if the following three conditions are satisfied.*

1. $\int_X \alpha \cdot \beta > 0$,
2. $\int_X \beta^2 > 0$.
3. *We have $\int_C \beta > 0$ for every irreducible curve C in X of negative self-intersection.*

Moreover, the *intersection pairing* on $H^{1,1}(X,\mathbb{R})$ given by

$$(\alpha, \beta) \mapsto \int_X \alpha \cdot \beta$$

has signature $(1, \dim_{\mathbb{R}} H^{1,1}(X,\mathbb{R}) - 1)$. This is a consequence of the *Hodge Index Theorem.* (See, for example, [28, Theorem 6.33].) This fact allows one to give a sufficient numerical criterion for a class $\alpha \in H^{1,1}(X,\mathbb{R})$ to be big, which we now recall.

Let $\mathcal{P}_X^+$ denote the open set of classes $\tau \in H^{1,1}(X,\mathbb{R})$ such that $\int_X \tau^2 > 0$ and $\int_X \tau \cdot \alpha > 0$ for some Kähler class $\alpha \in \mathcal{K}_X$. It is clear that $\mathcal{K}_X \subseteq \mathcal{P}_X^+$. In fact, the set $\mathcal{P}_X^+$ does not depend on the choice of α. This is a consequence of the Hodge Index Theorem once again. Indeed, if τ satisfies

$$\int_X \tau^2 > 0, \quad \int_X \tau \cdot \alpha_1 > 0, \quad \int_X \tau \cdot \alpha_2 \leq 0$$

for some Kähler classes $\alpha_1, \alpha_2 \in \mathcal{K}_X$, then we would have $\int_X \tau \cdot (r\alpha_1 + (1-r)\alpha_2) = 0$ for some $r \in [0,1]$. This would contradict the fact that the intersection pairing has only one positive eigenvalue, for then τ and $r\alpha_1 + (1-r)\alpha_2$ would be two different orthogonal positive vectors for the intersection pairing. The same argument yields the following.

Lemma 1.29. *The open set $\mathcal{P}_X^+ \subseteq H^{1,1}(X,\mathbb{R})$ is an open convex cone.*

Proof. It is immediate that if $\tau \in \mathcal{P}_X^+$, then so is $r\tau$ for any $r > 0$. It only remains to prove that if $\tau_1, \tau_2 \in \mathcal{P}_X^+$, then so is $\tau_1 + \tau_2$. We clearly have

$$\int_X (\tau_1 + \tau_2) \cdot \alpha > 0.$$

Now

$$\int_X (\tau_1 + \tau_2)^2 > \int_X 2\tau_1 \cdot \tau_2$$

and we claim that $\int_X \tau_1 \cdot \tau_2 > 0$. Indeed, if $\int_X \tau_1 \cdot \tau_2 \leq 0$, then we would have,

$$\int_X \tau_1 \cdot (r\tau_2 + (1-r)\alpha) = 0$$

for some $r \in (0,1]$. But clearly

$$\int_X (r\tau_2 + (1-r)\alpha)^2 > 0$$

so this contradicts the Hodge Index Theorem once again. □

Thus, we get an inclusion of closed cones

$$\mathcal{N}_X = \overline{\mathcal{K}_X} \subseteq \overline{\mathcal{P}_X^+}$$

and so by duality

$$\left(\overline{\mathcal{P}_X^+}\right)^* \subseteq \mathcal{N}_X^*.$$

Now it can be verified, by explicit calculation after picking a basis of any vector space with a pairing of signature $(1, \dim_{\mathbb{R}} V - 1)$, that $\overline{\mathcal{P}_X^+}$ is actually self-dual. On the other hand, it follows by Lamari's Theorem 1.28 that $\mathcal{N}_X^* = \mathcal{E}_X = \overline{\mathcal{B}_X}$. (See, for example, [21, proof of Theorem 4.1].) Thus, we have $\mathcal{P}_X^+ \subseteq \mathcal{B}_X$ and we obtain the following sufficient criterion for big classes on surfaces.

Lemma 1.30. *Let X be a compact Kähler surface and $\tau \in H^{1,1}(X, \mathbb{R})$ be a cohomology class satisfying*

$$\int_X \tau^2 > 0.$$

Then, either $\tau \in \mathcal{B}_X$ or $-\tau \in \mathcal{B}_X$.

Proof. If α is any Kähler class, then by the Hodge Index Theorem, $\int_X \tau \cdot \alpha \neq 0$. Thus, either $\tau \in \mathcal{P}_X^+$ or $-\tau \in \mathcal{P}_X^+$. □

Remark 18. It should be remarked that the above sufficient criterion for a class τ to be big is far from necessary. Indeed, if $p : X \to \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ in a point with exceptional divisor E, the cohomology class τ of the closed positive current $\varepsilon\omega + [E]$ is clearly big for any Kähler form ω on X and any $\varepsilon > 0$. However, $\int_X \tau^2 < 0$ for $\varepsilon > 0$ small enough, because $E^2 = -1$. On the other hand $\int_X \tau \cdot [\omega] > 0$. So $\pm\tau \notin \mathcal{P}_X^+$.

We shall make use of the following proposition when studying the set of destabilising curves on surfaces.

Proposition 1.31. *Let X be a compact Kähler surface with $\tau \in \mathcal{B}_X \subseteq H^{1,1}(X, \mathbb{R})$ a big cohomology class that is not Kähler. Then, there exists a (non-empty) finite set of irreducible curves $\{E_1, \cdots, E_\ell\}$ such that*

$$\int_E \tau \leq 0 \text{ precisely when } E = E_i \text{ for some } i = 1, \cdots, \ell. \tag{1.1}$$

Moreover, the intersection matrix $(E_i \cdot E_j)$ of the curves E_i is negative-definite and their classes $[E_i] \in H^{1,1}(X, \mathbb{R})$ are linearly independent. In particular, $\ell \leq \rho(X) \leq h^{1,1}(X)$ where $\rho(X) = \dim_{\mathbb{R}} NS(X)_{\mathbb{R}}$. Moreover, if X is projective, then $\ell \leq \rho(X) - 1$.

Proof. Let us first assume that we are in the special case whereby the big class τ is not nef. The Zariski decomposition of τ can be written as

$$\tau = Z(\tau) + N(\tau)$$

where $Z(\tau)$ is a nef class and

$$N(\tau) = \sum_{i}^{s} a_i[E_i]$$

for a unique non-zero effective $\mathbb{R}$-divisor

$$D = \sum_{i}^{s} a_i E_i$$

(with prime components $E_j \subseteq X$ and $a_j > 0$) such that the intersection matrix $(E_i \cdot E_j)$ is negative-definite and the classes $[E_i]$ are linearly independent in $NS(X)_{\mathbb{R}} \subseteq H^{1,1}(X,\mathbb{R})$. Let $E \subseteq X$ be any closed irreducible curve such that $\int_E \tau < 0$. But if $E \neq E_j$ for any $j = 1, \cdots, s$ then we have

$$\int_E \tau = \int_E Z(\tau) + \sum_{i=1}^{\ell} a_i E \cdot E_i \geq \int_E Z(\tau) \geq 0$$

which contradicts our assumption on E. So we must have that $E = E_j$ for some $j = 1, \cdots, s$, and since the classes $[E_i]$ are linearly independent in $NS(X)_{\mathbb{R}} \subseteq H^{1,1}(X,\mathbb{R})$, we also have $s \leq \rho(X)$. Moreover, if X is projective, then $NS(X)_{\mathbb{R}}$ contains at least one positive eigenvector of the intersection form, so in that case $\ell \leq \rho(X) - 1$. To summarise, we have proven that if τ is not nef, then there exist at most $\rho(X)$ (and in case X is projective, at most $\rho(X) - 1$) distinct curves $E \subseteq X$ such that $\int_E \tau < 0$, all of which occur as prime components of $N(\tau)$.

Now suppose, in full generality of the proposition, that τ is not Kähler. Fix a Kähler form ω on X. Then, $\tau - \varepsilon[\omega]$ is not nef but still contained in the big cone $\mathcal{B}_X$ for any $\varepsilon > 0$ small enough. Now if $\int_E \tau \leq 0$ for some curve $E \subseteq X$, we have $\int_E (\tau - \varepsilon[\omega]) < 0$, so the set of such curves is contained among the prime components of $N(\tau - \varepsilon[\omega])$, say $E_1, \cdots, E_s$. After relabelling if necessary, we may suppose $\int_{E_j} \tau \leq 0$ precisely for $j = 1, \cdots, \ell$. Then clearly we have $\ell \leq s \leq \rho(X)$ (and if X is projective, $s \leq \ell \leq \rho(X) - 1$). Finally, the claim about the intersection matrix follows because any submatrix of a negative-definite matrix is itself negative-definite. □

Lemma 1.32. *Let $\mathcal{C}$ be an open convex cone in a finite-dimensional vector space. Then, any compact subset $K \subseteq \mathcal{C}$ is contained in the convex hull of finitely many points of $\mathcal{C}$.*

Proof. Without loss of generality, assume that K is connected. We may then choose an open covering of K by means of open cubes C_i whose closure is in $\mathcal{C}$. By compactness there is a finite open subcover

$$\bigcup_{i \in I, |I| < +\infty} C_i.$$

Taking the convex hull of enough of the (finitely many) vertices of C_ν, $\nu \in I$, this set clearly contains K. □

As an immediate consequence, we obtain the following fact.

Lemma 1.33. *Let X be a compact Kähler surface and K be any subset of $\mathcal{B}_X$ such that K lies in the positive cone generated by the convex hull of finitely many classes $\tau_1, \tau_2, \ldots, \tau_s \in \mathcal{B}_X$. Then, there exists a finite set S (depending only on K) of curves of negative self-intersection such that for every $\tau \in K$, if $\int_E \tau \leq 0$, then $E \in S$. The cardinality of S is bounded above by $s\rho(X)$ and by $s\rho(X) - s$ if X is projective.*

Proof. Suppose that $\tau = r\sum_{i=1}^{k} a_i\tau_i, \quad a_i > 0, r > 0$. Then $\int_C \tau \leq 0$ implies $\int_C \tau_i \leq 0$ for some i. Hence C is in the set given by 1.31. □

Chapter 2

Destabilising subvarieties

2.1 Overview of the PDEs

Our aim in this Section is to give a brief overview of the various PDEs that we shall treat, and to introduce terminology and notation that will be used in the subsequent Sections. To each PDE is associated, under certain hypotheses, a Nakai-Moishezon type criterion (involving intersection numbers attached to subvarieties) which characterises when a solution exists. We state, in each case, this criterion and briefly explain all the necessary hypotheses.

2.1.1 The J-equation

Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$, and $\alpha, \beta \in \mathcal{K}_X$ be two Kähler classes with $\omega \in \alpha, \theta \in \beta$ fixed smooth Kähler forms. Let $\mathcal{H}(\omega)$ denote the set of *smooth Kähler potentials*, that is, the set comprising those $\psi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X, \mathbb{R})$ such that $\omega_\psi = \omega + \sqrt{-1}\partial\bar{\partial}\psi$ is a Kähler form. Then, the *J-equation* seeks a smooth function $\psi \in \mathcal{H}(\omega)$ such that

$$n\omega_\psi^{n-1} \wedge \theta = \mu_{\alpha,\beta}\omega_\psi^n, \tag{2.1}$$

where the cohomological constant $\mu_{\alpha,\beta}$ is given by

$$\mu_{\alpha,\beta} = n\frac{\alpha^{n-1} \cdot \beta \cdot [X]}{\alpha^n \cdot [X]}.$$

Another formulation of the J-equation, equivalent to the above (2.1), is in terms of the $C^\infty(X, \mathbb{C})$-linear endomorphism $\mathcal{R}_\psi : T^{1,0}X \to T^{1,0}X$ determined by

$$\omega_\psi(\mathcal{R}_\psi u, \overline{v}) = \theta(u, \overline{v}) \quad \text{for every } u, v \in C^\infty(X, T^{1,0}X). \tag{2.2}$$

The endomorphism $\mathcal{R}_\psi$ defined by (2.2) is sometimes also denoted $\omega_\psi^{-1}\theta$. It can be checked easily that $\mathcal{R}_\psi$ is self-conjugate with respect to the Hermitian metric on $T^{1,0}X$ induced by ω_ψ, and as such, has real eigenvalues at each point. Another way to state (2.1) is then given by

$$\mathrm{Tr}(\mathcal{R}_\psi) = \mu_{\alpha,\beta}. \tag{2.3}$$

The J-equation is closely linked to one of the main problems in Kähler geometry: determining which Kähler classes $\alpha \in \mathcal{K}_X$ admit a *constant scalar curvature Kähler (cscK) metric*, that is, a Kähler form $\omega \in \alpha$ for which the function $S(\omega)$ determined by

$$n\mathrm{Ric}(\omega) \wedge \omega^{n-1} = S(\omega)\omega^n \tag{2.4}$$

is identically equal to the (cohomological) constant

$$\hat{S}(\alpha) = \frac{nc_1(X) \cdot \alpha^{n-1} \cdot [X]}{\alpha^n \cdot [X]}.$$

We briefly recall this connection of the J-equation with the cscK equation. For our fixed smooth Kähler representative $\omega \in \alpha$, the cscK equation seeks a smooth function $\psi \in \mathcal{H}(\omega)$ such that

$$S(\omega_\psi) = \hat{S}(\alpha). \tag{2.5}$$

A solution ψ of this equation is a critical point of the *Mabuchi energy functional* $M : \mathcal{H}(\omega) \to \mathbb{R}$, which can be expressed as

$$M(\psi) = H_\omega(\psi) + J^\omega_{-\mathrm{Ric}(\omega)}(\psi)$$

where $H_\omega, J^\omega_\theta : \mathcal{H}(\omega) \to \mathbb{R}$ are defined by

$$H_\omega(\psi) = \frac{1}{n!}\int_X \log\left(\frac{\omega_\psi^n}{\omega^n}\right)\omega_\psi^n,$$

$$J^\omega_\theta(\psi) = \frac{1}{n!}\sum_{k=0}^{n-1}\int_X \psi\omega_\psi^k \wedge \omega^{n-k-1} \wedge \theta - \frac{\mu_{\alpha,\beta}}{(n-1)!}\sum_{k=0}^{n}\int_X \psi\omega_\psi^k \wedge \omega^{n-k}.$$

The functional H_ω is called the *entropy* and J^ω_θ is referred to as *Donaldson's J-functional*. Solutions of the J-equation (2.1) are critical points of J^ω_θ.

If $F : \mathcal{H}(\omega) \to \mathbb{R}$ is any functional, we say F is *coercive* if there exist $\delta > 0$ and a constant C such that

$$F(\psi) \geq \delta J^\omega_\omega(\psi) - C \text{ for every } \psi \in \mathcal{H}(\omega).$$

One of the major breakthroughs in the study of the cscK equation is the result of Chen-Cheng [29] which says that (in the case X has a discrete automorphism group) the cscK equation (2.5) admits a unique solution precisely when the Mabuchi energy M is coercive. (In fact, they also proved that an adapted version of coercivity still implies the existence of a cscK metric, even if the automorphism group of X is not discrete.)

It follows that α admits a cscK metric if $J^\omega_{-\mathrm{Ric}(\omega)}$ is coercive, since it is easy to see that for any $\psi \in \mathcal{H}(\omega)$, $H_\omega(\psi) \geq 0$ in general. (In fact, it can be shown that H_ω is in fact even coercive in general.)

On the other hand Collins-Székelyhidi proved [30, Propositions 21, 22] that the solvability of the J-equation (2.1) is equivalent to the coercivity of J^ω_θ. Therefore, whenever $-c_1(X)$ is a Kähler class, the solvability of the J-equation for the triple $(X, \alpha, -c_1(X))$ implies the existence of a cscK metric in α. This is one way of producing many examples of compact Kähler manifolds X and Kähler classes α such that α admits a cscK metric. In fact, one can also go further than this, to the case when $-c_1(X)$ is assumed to be merely nef (a case of considerable interest in the minimal model programme of birational geometry), as in the works of Jian-Shi-Song [31] and Sjöstrom Dyrefelt [32].

In [6], Gao Chen proved a uniform version of the following Nakai-Moishezon criterion, first conjectured (and proved in the toric case) by Lejmi-Székelyhidi [33], whose resolution in general is due to the work of Datar-Pingali [7] (when X is a smooth projective variety) and Song [8] (in full generality). Before stating this result, we introduce some notation. In the spirit of slope stability, and following [13], we set

$$\mu_{\alpha,\beta}(Z) = (\dim_{\mathbb{C}} Z)\frac{\alpha^{\dim_{\mathbb{C}} Z - 1} \cdot \beta \cdot [Z]}{\alpha^{\dim_{\mathbb{C}} Z} \cdot [Z]}.$$

and call $\mu_{\alpha,\beta}(Z)$ the *slope* (or J-slope) of Z with respect to (X, α, β). We will abbreviate (as we have already been doing) $\mu_{\alpha,\beta}(X) = \mu_{\alpha,\beta}$. The triple (X, α, β) is called *J-stable* (respectively *J-semistable*) if for all proper irreducible analytic subvarieties Z of X, we have

$$\mu_{\alpha,\beta}(Z) < \mu_{\alpha,\beta} \quad \text{(respectively } \mu_{\alpha,\beta}(Z) \leq \mu_{\alpha,\beta}),$$

and *J-unstable* if it is not J-semistable. We will often drop the prefix and refer to these notions as simply stable or semistable.

Remark 19. Note that the term *J-stability* was first used in the literature for a different (though not unrelated) notion involving test configurations, as in [33]. What we call J-stability has been termed J-slope stability by Datar-Mete-Song in [13]. However, for the sake of consistency with the rest of the thesis and other PDEs, we shall call this notion J-stability, bearing in mind that it is the *correct* notion from the point of view of characterising the solvability of the J-equation, as indicated by the following Theorem.

Theorem 2.1 (Gao Chen, Datar-Pingali, Song)**.** *Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$ and $\alpha, \beta \in \mathcal{K}_X$ Kähler classes on X. Fix Kähler forms $\omega \in \alpha, \theta \in \beta$. Then, the following are equivalent.*

1. *There exists a smooth $\psi \in \mathcal{H}(\omega)$, unique up to additive constants, solving the J-equation*
$$n\omega_\psi^{n-1} \wedge \theta = \mu_{\alpha,\beta}\omega_\psi^n.$$
2. *The triple (X, α, β) is J-stable.*

2.1.2 The deformed Hermitian Yang-Mills equation

Let X be a compact Kähler manifold as before, with $\beta \in \mathcal{K}_X$ a Kähler class and $\alpha \in H^{1,1}(X, \mathbb{R})$ a cohomology class (not necessarily Kähler). Fix $\theta \in \beta$ a Kähler form, and $\omega \in \alpha$ a smooth real $(1,1)$ form. Then, the *deformed Hermitian Yang-Mills (dHYM) equation* seeks a smooth $\psi \in C^\infty(X, \mathbb{R})$ such that the function $F_\psi \in C^\infty(X, \mathbb{C})$ determined by

$$(\theta + \sqrt{-1}\omega_\psi)^n = F_\psi \theta^n$$

is nowhere zero and has constant argument. This can only be the case if

$$Z_\beta(\alpha) = \int_X (\beta + \sqrt{-1}\alpha)^n \neq 0,$$

in which case $\arg F_\psi = \varphi_\beta(\alpha)$ where $\varphi_\beta(\alpha) = \arg Z_\beta(\alpha)$ is the *cohomological angle*. The dHYM equation is usually written in this notation as

$$\operatorname{Im}\left(e^{-\sqrt{-1}\varphi_\beta(\alpha)}(\theta + \sqrt{-1}\omega_\psi)^n\right) = 0. \tag{2.6}$$

The dHYM equation was first derived by Leung-Yau-Zaslow in [34] as the counterpart under mirror symmetry of the special Lagrangian condition on a Calabi-Yau manifold, using the SYZ formalism of mutually dual torus fibrations. Thus, the dHYM equation is of special interest to both mathematics and theoretical physics, and has attracted considerable interest in both fields. (See [35] for a survey.)

The dHYM equation is also related to the aforementioned J-equation. To explain this, we reformulate the equation in a different way. Let $\mathcal{T}_\psi$ be the $C^\infty(X,\mathbb{C})$-linear endomorphism of $T^{1,0}X$ determined by the condition

$$\theta(\mathcal{T}_\psi u, \overline{v}) = \omega_\psi(u, \overline{v}) \quad \text{for every } u, v \in C^\infty(X, T^{1,0}X).$$

Then, $\mathcal{T}_\psi$ is self-conjugate with respect to the Hermitian metric induced on $T^{1,0}X$ by θ and as such has real eigenvalues λ_i at each point. When α is a Kähler class and ω_ψ is a Kähler form, then it is clear that $\mathcal{T}_\psi = \mathcal{R}_\psi^{-1}$, where $\mathcal{R}_\psi$ is as defined by (2.2). An easy calculation using holomorphic normal coordinates shows that

$$F_\psi = \prod_{i=1}^{n}(1+\sqrt{-1}\lambda_i)$$

and so we see that $|F_\psi| \geq 1$, and therefore F_ψ is in fact nowhere zero in general. Moreover, it follows that

$$\arg F_\varphi = \sum_{i=1}^{n} \arctan(\lambda_i) \mod 2\pi.$$

Note that the function Φ_ψ defined by

$$\Phi_\psi = \sum_{i=1}^{n} \arctan(\lambda_i)$$

is a smooth, real-valued function on X taking values in $(-n\pi/2, n\pi/2)$. (This is so because the assignment mapping a conjugate-symmetric matrix A to the sum of the arctangents of its eigenvalues is a unitary invariant function, and restricts to a smooth symmetric function on the space of diagonal matrices. See, for example, [36, proof of Theorem 7.13].) When we wish to emphasise the dependence of F_ψ, Φ_ψ etc. on θ, we shall write $F_\psi(\theta), \Phi_\psi(\theta)$ etc.

It is clear that if ψ solves (2.6), then we must have

$$\Phi_\psi = \varphi_\beta(\alpha) + 2\pi k \tag{2.7}$$

for some integer k. It is a result of Jacob-Yau [37, Theorem 1.1] that the solutions of the dHYM equation are unique, and so the quantity $\hat{\varphi}_\beta(\alpha) = \varphi_\beta(\alpha) + 2\pi k \in (-n\pi/2, n\pi/2)$, called the *lifted angle*, is uniquely determined whenever a solution ψ of (2.6) exists. Without assuming that a solution ψ exists, we can still define the lifted angle whenever there exists some smooth function ψ_0 for which F_{ψ_0} takes values in some open half-plane. If this happens, the lifted angle is by definition the unique representative $\hat{\varphi}_\beta(\alpha)$ of $\varphi_\beta(\alpha)$ modulo 2π that lies in the range of Φ_{ψ_0}.

This always uniquely determines $\hat{\varphi}_\beta(\alpha)$ in the following sense: if $\psi, \psi' \in C^\infty(X,\mathbb{R})$ and $\theta, \theta' \in \beta$ are such that both $F_\psi(\theta), F_{\psi'}(\theta')$ take values in some (potentially different) half-planes, then $\Phi_\psi(\theta) - \arg F_\psi(\theta) = \Phi_{\psi'}(\theta') - \arg F_{\psi'}(\theta')$ identically on X, so the two lifted angles agree.

Definition 2.2. The triple (X, α, β) is said to have *supercritical phase* (respectively, *hypercritical phase*) if the lifted angle $\hat{\varphi}_\beta(\alpha)$ is defined and lies in the interval $((n-2)\pi/2, n\pi/2)$ (respectively, $((n-1)\pi, n\pi/2)$).

Note that if there exists a $\psi \in C^\infty(X,\mathbb{R})$ such that Φ_ψ takes values in the interval $((n-1)\pi/2, n\pi/2)$, then it is clear that (X,α,β) has hypercritical phase. Moreover, in this case α is a Kähler class, and a solution of the dHYM equation (if it exists) will necessarily be a Kähler form.

The connection with the J-equation arises as follows. Suppose both α and β are Kähler classes, with θ a fixed Kähler form in β and ω_ψ a Kähler form for some smooth $\omega \in \alpha$ and $\psi \in C^\infty(X,\mathbb{R})$. Then, for any $\varepsilon > 0$, we have $\mathcal{T}_\psi(\varepsilon\theta) = \varepsilon^{-1}\mathcal{T}_\psi(\theta)$, and so the (positive) eigenvalues of $\mathcal{T}_\psi(\varepsilon\theta)$ all converge uniformly to $+\infty$ as $\varepsilon \to 0$. This means the triple $(X,\alpha,\varepsilon\beta)$ has supercritical phase for $\varepsilon > 0$ small enough. Moreover, a simple calculation shows that for small values of $\varepsilon > 0$, the lifted angle $\hat{\varphi}_{\varepsilon\beta}(\alpha)$ satisfies

$$\lim_{\varepsilon\to 0}\left(\hat{\varphi}_{\varepsilon\beta}(\alpha) - \left(\frac{n\pi}{2} - \varepsilon\mu_{\alpha,\beta}\right)\right) = 0.$$

On the other hand, for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
\Phi_\psi(\varepsilon\theta) &= \sum_{i=1}^{n} \arctan(\varepsilon^{-1}\lambda_i) \\
&= \sum_{i=1}^{n}\left(\frac{\pi}{2} - \arctan\left(\frac{\varepsilon}{\lambda_i}\right)\right) \\
&= \frac{n\pi}{2} - \varepsilon\mathrm{Tr}(\mathcal{T}_\psi(\theta)^{-1}) + \frac{\varepsilon^3}{3}\mathrm{Tr}(\mathcal{T}_\psi(\theta)^{-3}) - \dots
\end{aligned}$$

where we have used the identity $\arctan(t) = \pi/2 - \arctan(t^{-1})$ and used the power series expansion of arctan around zero. Recalling that $\mathcal{T}_\psi(\theta)^{-1} = \mathcal{R}_\psi(\theta)$, it follows that, uniformly on X we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}(\Phi_\psi(\varepsilon\theta) - \hat{\varphi}_{\varepsilon\beta}(\alpha)) = \mu_{\alpha,\beta} - \mathrm{Tr}(\mathcal{R}_\psi(\theta)).$$

Thus, the family of differential operators

$$\psi \mapsto \varepsilon^{-1}(\Phi_\psi(\varepsilon\theta) - \hat{\varphi}_{\varepsilon\beta}(\alpha))$$

associated to the dHYM equations for the triples $(X,\alpha,\varepsilon\beta)$ converges to the differential operator

$$\psi \mapsto \mu_{\alpha,\beta} - \mathrm{Tr}(\mathcal{R}_\psi(\theta)$$

associated to the J-equation for the triple (X,α,β). This is sometimes expressed by saying that the J-equation is the *small volume limit* of the dHYM equation. This observation is due to Collins-Jacob-Yau [38].

We now state the Nakai-Moishezon criterion for the supercritical dHYM equation. For this, it is convenient to define the *complementary lifted angle* by $\hat{\phi}_\beta(\alpha) = n\pi/2 - \hat{\varphi}_\beta(\alpha)$. Note that (X,α,β) has supercritical phase precisely when $\hat{\phi}_\beta(\alpha) \in (0,\pi)$. In this notation, the deformed Hermitian Yang-Mills equation seeks a smooth ψ such that

$$\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\phi}_\beta(\alpha)}(\omega_\psi + \sqrt{-1}\theta)^n\right) = 0. \tag{2.8}$$

Under these assumptions, Chu-Lee-Takahashi [39] prove the following numerical criterion characterising solvability of the dHYM equation.

Theorem 2.3 (Chu-Lee-Takahashi)**.** *Let X be a smooth projective variety, β a Kähler class on X and α a $(1,1)$-cohomology class. Suppose (X, α, β) has supercritical phase. Then, the following are equivalent.*

1. *For every Kähler form $\theta \in \beta$, there exists a smooth ψ solving the deformed Hermitian Yang-Mills equation (2.8).*

2. *For all proper irreducible subvarieties $V \subseteq X$, we have*

$$\int_V \left(\mathrm{Re}(\alpha + \sqrt{-1}\beta)^{\dim_{\mathbb{C}} V} - \cot \hat{\phi}_\beta(\alpha) \, \mathrm{Im}(\alpha + \sqrt{-1}\beta)^{\dim_{\mathbb{C}} V} \right) > 0.$$

2.1.3 Z-critical equations

In [5, Section 2.1] Dervan-McCarthy-Sektnan introduced the notion of a *Z-critical connection* on a holomorphic vector bundle $E \to X$ on a compact Kähler manifold X of dimension n, where Z is a choice of a *polynomial central charge*, as a differential geometric analogue (in the case of vector bundles) of the abstract notion of a Bridgeland stability condition (introduced in [40]). In their setup, the choice of a polynomial central charge $Z = Z_\Omega$ is the data of a triple $\Omega = (\beta, \rho, U)$ where β is a Kähler class, $\rho = (\rho_0, \rho_1, \cdots, \rho_n) \in \mathbb{C}^{n+1}$ is a *stability vector* such that ρ_d/ρ_{d+1} and ρ_n lie in the upper half-plane, and $U \in \bigoplus_i H^{i,i}(X, \mathbb{R})$ is *unipotent cohomology class*, that is, a mixed-degree cohomology class whose degree zero part is equal to 1. With this notation, for a fixed Kähler form $\theta \in \beta$ and a fixed smooth representative $\tilde{U} \in U$, set $\tilde{Z}_\Omega(E, h)$ to be the degree (n, n) part of the complex endomorphism-valued form

$$\left(\sum_{d=0}^{n} \rho_i \theta^d \right) \wedge \tilde{U} \wedge \mathrm{ch}(E, h)$$

where h is a Hermitian metric on E and $\mathrm{ch}(E, h)$ is the Chern-Weil representative of the total Chern character $\mathrm{ch}(E)$ of E with respect to h. Then, the *Z-critical equation* is written

$$\mathrm{Im}\left(e^{-\sqrt{-1}\varphi(E)} \tilde{Z}_\Omega(E, h) \right) = 0 \tag{2.9}$$

where the metric h is understood as the desired solution. Here, $e^{-\sqrt{-1}\varphi(E)}$ is the unique cohomological constant that, modulo 2π, satisfies

$$\varphi(E) = \arg \left(\int_X \left(\tilde{Z}_\Omega(E, h) \right) \right)$$

and the imaginary part is understood as the anti-self adjoint component of the form with respect to the metric h. More precisely, there is a map $(\cdot)^\dagger : \mathcal{A}^{n,n}(\mathrm{End}\, E) \to \mathcal{A}^{n,n}(\mathrm{End}\, E)$ that sends any (complex) endomorphism-valued (n, n)-form to its Hermitian conjugate. Then the imaginary part of a form Θ is simply

$$\mathrm{Im}(\Theta) = \frac{1}{2\sqrt{-1}}(\Theta - \Theta^\dagger).$$

As mentioned above, the Z-critical equation is meant to capture, in the context of differential geometry, the abstract notion of a Bridgeland stability condition on the derived category of coherent sheaves of a smooth complex projective variety. From this perspective, one should expect to see features of the theory of Bridgeland stability arise purely in terms of the differential geometry of the Z-critical equation. One such feature is a locally finite wall-chamber structure of the manifold parametrising stability data (see, for example [41, Section 9]). Our main interest in the Z-critical equation is from this point of view.

2.1.4 Generalised Monge-Ampère equations

Many different examples of PDEs arising in complex geometry (including the J-equation) can be treated along similar lines in terms of their analytic properties, and admit similar criteria for solvability. In an attempt to capture this wide class of PDEs, Pingali introduced the term *generalised Monge-Ampère equations*. In the present thesis, we shall treat what is technically only a special case of this fairly general class, which was treated by Datar-Pingali [7]. The more general case of these equations also has been recently treated by Fang-Ma [42].

Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = n$ and $\alpha, \beta \in \mathcal{K}_X$ be Kähler classes, with $\omega \in \alpha, \theta \in \beta$ be fixed Kähler forms. Let $c_1, \ldots, c_{n-1} \geq 0$ be constants and $f \in C^\infty(X, \mathbb{R})$ be a smooth function. Moreover, assume that the data satisfy cohomological condition

$$\int_X \frac{\alpha^n}{n!} = \sum_{k=1}^{n-1} \frac{c_k}{(n-k)!} \int_X \beta^k \cdot \alpha^{n-k} + \int_X f\theta^n. \tag{2.10}$$

Then the *generalised Monge-Ampère equation (gMA)* seeks a smooth function $\psi \in \mathcal{H}(\omega)$ such that

$$\frac{\omega_\psi^n}{n!} = \sum_{k=1}^{n-1} \frac{c_k}{(n-k)!} \theta^k \wedge \omega_\psi^{n-k} + f\theta^n. \tag{2.11}$$

It is more convenient to express the above equation using slightly different notation. For any smooth $(1,1)$ form η, set

$$\exp(\eta) = \sum_{k=0}^{n} \frac{1}{k!} \eta^k$$

and for any multi-degree form Ω, let $\Omega^{[k,k]}$ denote the degree (k,k) part of Ω. Then, the gMA equation can be reformulated as

$$\exp(\omega_\psi)^{[n,n]} = (\exp(\omega_\psi) \wedge \Theta)^{[n,n]} \tag{2.12}$$

where we denote by Θ the multi-degree form

$$\Theta = \sum_{k=1}^{n-1} c_k \theta^k + f\theta^n.$$

In this notation, we will say that (2.12) is the gMA equation associated to the triple (X, α, Θ), only implicitly keeping in mind that Θ actually depends on a choice of c_k, θ and f.

The gMA equation (2.12) captures a lot of PDEs that arise naturally in complex geometry. For example, for $\Theta = \mu_{\alpha,\beta}^{-1}\theta$, (2.12) reduces to the J-equation, while for $\Theta = f\theta$, $f > 0$, it reduces to the complex Monge-Ampère equation. If $\Theta = \kappa\theta^p$ for an appropriate cohomological constant κ satisfying (2.10), it reduces to the *inverse Hessian equations*.

The relevant Nakai-Moishezon criterion in this setting is due to Datar-Pingali [7, Theorem 1.1] when X is smooth and projective. Recently, Fang-Ma [42, Theorem 1.10] have obtained a more general result for X any compact Kähler manifold. Both of these results hold under the assumption of a certain positivity condition on the data (X, α, Θ). A sufficient positivity condition is stated precisely in [7, (1.2)], which we shall not recall explicitly. However, it is worth remarking that the positivity condition, which is always satisfied if $f \geq 0$ in in the expression for Θ, is also satisfied in some cases when $f(x) < 0$ for some $x \in X$.

Theorem 2.4 (Datar-Pingali)**.** *Let X be a smooth projective variety of dimension n, $\alpha, \beta \in \mathcal{K}_X$ Kähler classes on X, and $\omega \in \alpha, \theta \in \beta$ fixed Kähler forms. Let c_k, $k = 1, \dots, n-1$ and $f \in C^\infty(X, \mathbb{R})$ be such that $(X, \alpha, [\Theta])$ satisfies* (2.10) *and the positivity conditions [7, (1.2)]. Then, the following are equivalent.*

1. *There exists a smooth $\psi \in \mathcal{H}(\omega)$ that solves the gMA equation*
$$\exp(\omega_\psi)^{[n,n]} = (\exp(\omega_\psi) \wedge \Theta)^{[n,n]}. \tag{2.13}$$

2. *For any proper subvariety V of X, we have*
$$\int_V \exp(\alpha) \cdot (1 - [\Theta]) > 0.$$

2.2 The case of surfaces

In this Section, we state and prove our results in the cases of the dHYM equation, J-equation and the Z-critical equations on compact Kähler surfaces. The common theme in all these cases is that all of these equations reduce to the complex Monge-Ampère equation, and we are able to use the divisorial Zariski decomposition, which is very well-behaved for surfaces. Except for minor changes in notation and terminology, this Section is contained in [11].

2.2.1 The deformed Hermitian Yang-Mills equation

Let X denote a compact Kähler surface, and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ cohomology classes with $\beta \in \mathcal{K}_X$ a Kähler class. Fix a Kähler form $\theta \in \beta$ and a smooth representative $\omega \in \alpha$. Then, the cohomological angle $\varphi_\beta(\alpha)$ for the dHYM equation associated with the triple (X, α, β) is given by
$$\varphi_\beta(\alpha) = \arg\left(\int_X (\beta + \sqrt{-1}\alpha)^2\right) = \arg\left(\int_X (\beta^2 - \alpha^2) + 2\sqrt{-1}\int_X \alpha \cdot \beta\right)$$

Clearly, if $\alpha = 0$, then we can always solve (2.6), so assume $\alpha \neq 0$. Let us first note that if we have $\int_X \alpha \cdot \beta = 0$, then we can also always solve (2.6) for the triple (X, α, β). Indeed, $\int_X \alpha \cdot \beta = 0$ implies, by the Hodge Index Theorem, that $\int_X \alpha^2 < 0$. But then, $\int_X (\beta^2 - \alpha^2) > 0$, so $\varphi_\beta(\alpha) = 0$. From this, we see that (2.6) reduces to
$$2\omega_\psi \wedge \theta = 0$$
or, equivalently,
$$2(\sqrt{-1}\partial\bar{\partial}\psi) \wedge \theta = -2\omega \wedge \theta. \tag{2.14}$$
Let $g \in C^\infty(X, \mathbb{R})$ be given by $2\omega \wedge \theta = g\theta^2$. Then, (2.14) is equivalent to solving $\Delta_\theta \psi = -g$, which is just the Poisson equation, and can always be solved because
$$\int_X g\theta^2 = \int_X 2\omega \wedge \theta = 2\int_X \alpha \cdot \beta = 0.$$
(See, for example, [43, Theorem 2.12].) Let us therefore assume from now on that $\int_X \alpha \cdot \beta \neq 0$. Then, the angle $\varphi_\beta(\alpha)$ satisfies
$$\cot \varphi_\beta(\alpha) = \frac{\int_X (\beta^2 - \alpha^2)}{2\int_X \alpha \cdot \beta}.$$

In this case, the dHYM equation (2.6) is equivalent to solving

$$(\omega_\psi + \cot(\varphi_\beta(\alpha))\theta)^2 = \left(1 + \cot(\varphi_\beta(\alpha))^2\right)\theta^2.$$

which is a complex Monge-Ampère equation for the class $\alpha + \cot(\varphi_\beta(\alpha))\beta$. Thus, in this Subsection, we shall write

$$\tau(\alpha, \beta) = \alpha + \cot(\varphi_\beta(\alpha))\beta.$$

Theorem 2.5 ([11]). *Let X be a compact Kähler surface and let*

$$K \subseteq \left\{(\alpha, \beta) \in H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X \ : \ \int_X \alpha \cdot \beta \neq 0\right\}$$

be a compact, connected subset. Then, there exists a non-negative integer $\ell \geq 0$ and curves of negative self-intersection $E_1, \cdots, E_\ell$ on X and a sign $s = \pm 1$ (depending only on K) such that for all $(\alpha, \beta) \in K$ the following are equivalent.

1. *For any choice of Kähler form $\theta \in \beta$ and any smooth representative $\omega \in \alpha$ there exists a smooth $\psi \in C^\infty(X, \mathbb{R})$, unique up to the additive constants, which is a solution to the deformed Hermitian Yang-Mills equation*

$$\mathrm{Im}\left(e^{-\sqrt{-1}\varphi_\beta(\alpha)}(\theta + \sqrt{-1}\omega_\psi)^2\right) = 0.$$

2. *For every curve $E \subseteq X$, we have*

$$s\int_E \tau(\alpha, \beta) > 0.$$

3. *For $i = 1, \cdots, \ell$, we have*

$$s\int_{E_i} \tau(\alpha, \beta) > 0.$$

Proof. From the discussion preceding the statement of the Theorem, we see that for any $(\alpha, \beta) \in K$, we have

$$\int_X \tau(\alpha, \beta)^2 = (1 + \cot^2(\varphi_\beta(\alpha))) \int_X \beta^2 > 0.$$

Moreover, by the Hodge Index Theorem, we also must then have

$$\int_X \tau(\alpha, \beta) \cdot \beta \neq 0.$$

and so $\pm\tau(\alpha, \beta) \in \mathcal{P}_X^+$. If $-\tau(\alpha, \beta) \in \mathcal{P}_X^+$, set $s = -1$, otherwise, set $s = 1$. Since K is connected by assumption, s does not depend on the choice of $(\alpha, \beta) \in K$. Now, the continuous map

$$\left\{(\alpha, \beta) \in H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X \ : \ \int_X \alpha \cdot \beta \neq 0\right\}, \quad (\alpha, \beta) \mapsto s\tau(\alpha, \beta)$$

takes K onto a compact subset $\tilde{K}$ of $\mathcal{P}_X^+$. Now, by Lemma 1.32, $\tilde{K}$ is contained in the convex hull of finitely many points of $\mathcal{P}_X^+$ and by Lemma 1.33, there exists a non-negative integer $\ell \geq 0$ and finitely many curves of negative self-intersection $E_1, \cdots, E_\ell$ such that a class $\tau \in \tilde{K}$ is Kähler if and only

$$\int_{E_i} \tau > 0$$

for $i = 1, \cdots, \ell$. The proof that these curves satisfy the conclusion of the Theorem now proceeds via a standard argument. For the sake of completeness, we recall this simple argument.

Suppose $(\alpha, \beta) \in K$ is such that the dHYM equation (2.6) admits a smooth solution ψ for a choice of Kähler form $\theta \in \beta$ and smooth representative ω. Then, from the above discussion, we see that

$$(\omega_\psi + \cot(\varphi_\beta(\alpha)\theta)^2 = (1 + \cot^2(\varphi_\beta(\alpha)))\theta^2.$$

From this equality of forms, it follows that the $(1,1)$ form $(\omega_\psi + \cot(\varphi_\beta(\alpha)))$ or its negative is a Kähler form. Indeed, the equality obviously implies that $\omega + \cot(\varphi_\beta(\alpha))\theta$ is non-degenerate and defines the same orientation as the Kähler form θ. On a surface, this condition is equivalent to definiteness. But the topological fact that the class $s\tau(\alpha, \beta)$ is in $\mathcal{P}_X^+$ ensures that the form $s(\omega + \cot(\varphi_\beta(\alpha))$ is positive-definite. This shows that the class $s\tau(\alpha, \beta)$ is a Kähler class, and proves that (1) implies (2).

It is obvious that (2) implies (3). Finally, let there be given any choice of Kähler form $\theta \in \beta$ and a smooth representative $\omega \in \alpha$. Then (3) implies, by our choice of the curves E_i, that the class $s\tau(\alpha, \beta)$ is a Kähler class. By Yau's solution of the Calabi conjecture [3], this implies that we can find a smooth $\psi \in C^\infty(X, \mathbb{R})$, unique up to additive constants, such that

$$(\omega + \cot(\varphi_\beta(\alpha)\theta + \sqrt{-1}\partial\bar{\partial}\psi)^2 = (1 + \cot^2(\varphi_\beta(\alpha)))\theta^2.$$

This shows that (3) implies (1). □

Corollary ([11])**.** *Suppose X is a compact Kähler surface that admits no curves of negative self-intersection. Then, the deformed Hermitian Yang-Mills equation* (2.6) *always admits a solution for any pair $(\alpha, \beta) \in H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$, and any choice of Kähler form $\theta \in \beta$.*

Proof. If $\int_X \alpha \cdot \beta = 0$, then we have already seen that we can solve the Poisson equation to which (2.6) is in that case equivalent. Otherwise, take $K = \{(\alpha, \beta)\}$ in Theorem 2.5 and observe that we must have $\ell = 0$, so the third condition is vacuously satisfied. □

Remark 20. The above Corollary might not be entirely new, as it can already be derived (with slightly different hypotheses) from [44, Theorem 1.4], although in their work, the emphasis is more on the study of a dHYM flow. We also remark that the above Theorem 2.5 is a special case of Theorem 2.8 below, but for the sake of clarity of exposition, we have chosen to present it separately.

The following definition is natural in light of Theorem 2.5.

Definition 2.6. Suppose X is a compact Kähler surface and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ are cohomology class with $\beta \in \mathcal{K}_X$ a Kähler class. Assume that $\int_X \alpha \cdot \beta \neq 0$. Set $s(\alpha, \beta)$ to be the sign of the non-zero real number $\int_X \tau(\alpha, \beta) \cdot \beta$. Then, the triple (X, α, β) is called *(dHYM) stable* (respectively, *semistable*) if for all irreducible curves $C \subseteq X$, we have

$$s(\alpha, \beta) \left(\tan \left(\arg \left(\int_C (\beta + \sqrt{-1}\alpha) \right) \right) + \cot(\varphi_\beta(\alpha)) \right) > 0, \quad \text{(respectively } \geq 0). \quad (2.15)$$

An irreducible curve C that does not satisfy the strict inequality in (2.15) is called a *(dHYM) destabilising curve* for the triple (X, α, β).

Remark 21. It is clear from the proof of Theorem 2.5, that (X, α, β) is stable if and only if (2.6) admits a solution. Sometimes, we shall still call (X, α, β) (dHYM) stable if $\int_X \alpha \cdot \beta = 0$, even though in this case (2.6) degenerates to the Poisson equation. Since the latter always admits a solution in our case, this abuse of terminology is relatively harmless in this naive sense. Moreover, it remains harmless when considering families of dHYM equations. See Lemma 3.1 below.

Moreover, we obtain the following Corollary.

Corollary ([11])**.** *Suppose X is a compact Kähler surface and $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ are cohomology class with $\beta \in \mathcal{K}_X$ a Kähler class. Assume that $\int_X \alpha \cdot \beta \neq 0$. Then, (X, α, β) has only finitely many (dHYM) destabilising curves, each of which is a curve of negative self-intersection. Moreover, the number of these curves is bounded above by $\rho(X)$, the Picard rank of X. If X is projective, then the number of destabilising curves is bounded above by $\rho(X) - 1$.*

Proof. From the proof of the Theorem, we can see that $s(\alpha, \beta)\tau(\alpha, \beta) \in \mathcal{P}_X^+ \subseteq \mathcal{B}_X$. Moreover, observe that each destabilising curve C satisfies

$$\int_C s(\alpha, \beta)\tau(\alpha, \beta) \leq 0.$$

Thus, we conclude by applying Proposition 1.31. □

2.2.2 Z-critical equation

In this Section, our aim is to prove our main result about the Z-critical equation on surfaces. Before embarking on the proof, we briefly fix some notation. Thus, let $\Omega = (\beta, \rho, U)$ be the data of a *polynomial central charge* on a compact Kähler surface. Namely, $\rho = \rho(t) = \rho_0 + \rho_1 t + \rho_2 t^2$ is a polynomial with non-zero complex coefficients subject to the constraints

$$\mathrm{Im}(\rho_2) > 0, \quad \mathrm{Im}\left(\frac{\rho_1}{\rho_2}\right) > 0, \quad \mathrm{Im}\left(\frac{\rho_0}{\rho_1}\right) > 0, \tag{2.16}$$

and β and U are cohomology classes with $\beta \in \mathcal{K}_X$ a Kähler class, and $U = 1 + U_1 + U_2 \in \oplus_i H^{i,i}(X, \mathbb{R})$ a unipotent cohomology class (with graded components $U_i \in H^{i,i}(X, \mathbb{R})$). Then, for any holomorphic line bundle $L \to X$, we shall write

$$Z_\Omega(L) = \int_X \rho(\beta) \cdot U \cdot \mathrm{ch}(L)$$

where $\mathrm{ch}(L) = 1 + c_1(L) + c_1(L)^2/2 \in \oplus_i H^{i,i}(X, \mathbb{R})$ denotes the Chern character of L. We shall moreover always assume that our choice of Ω is such that $Z_\Omega(L)$ lies in the upper-half plane for the holomorphic line bundle L under consideration. (In this case, we shall informally say that Ω *defines a polynomial central charge.* This can always be achieved, for example, by scaling $\theta \mapsto t\theta$ for $t > 0$ very large for any fixed L.) Then, the *phase* or *Z_Ω-phase* $\varphi(L) = \varphi_\Omega(L)$ of L (with respect to $Z = Z_\Omega$) is given by

$$\varphi(L) = \arg Z_\Omega(L).$$

The Z_Ω-critical equation is specified once a *lift* $\tilde{\Omega}$ of Ω is fixed. Concretely, this means that we fix a choice of Kähler form $\theta \in \beta$ and smooth representative $\tilde{U}_i \in U_i$. Then we define $\tilde{Z}_\Omega(L, \tilde{\Omega}, h)$ as the degree (2,2) part of the form

$$\rho(\theta) \wedge \tilde{U} \wedge \mathrm{ch}(L, h)$$

where $\tilde{U} = 1 + \tilde{U}_1 + \tilde{U}_2$ and $\mathrm{ch}(L, h)$ is the Chern-Weil representative of $\mathrm{ch}(L)$ with respect to a Hermitian metric h; namely

$$\mathrm{ch}(L, h) = \exp\left(\frac{\sqrt{-1}}{2\pi}F_h\right) = 1 + \frac{\sqrt{-1}}{2\pi}F_h + \frac{1}{2}\left(\frac{\sqrt{-1}}{2\pi}F_h\right)^2$$

with F_h the curvature (1,1)-form of the Chern connection associated to h. Then the Z_Ω-critical equation takes the form

$$\mathrm{Im}\left(e^{-\sqrt{-1}\varphi(L)}\tilde{Z}_\Omega(L, \tilde{\Omega}, h)\right) = 0. \tag{2.17}$$

This is a second order fully non-linear equation for the metric h.

In [5, Section 2.3] the authors derive the notion of a subsolution for the Z-critical equation. (See [5, Definition 2.33].) They then prove that in our present special case where X is a smooth projective surface, $E = L \to X$ is a line bundle, and the lifted data $(\theta, \rho, \tilde{U})$ satisfy the so-called *volume form hypothesis*, the existence of a subsolution is equivalent to the existence of a solution. More precisely, define the forms $\tilde{\eta} = \tilde{\eta}(\tilde{\Omega}, L)$ and $\tilde{\gamma} = \tilde{\gamma}(\tilde{\Omega}, L)$ by the formula

$$\mathrm{Im}\left(e^{-\sqrt{-1}\varphi(L)}\tilde{Z}_\Omega(L, h)\right) = c(\chi_h^2 + \chi_h \wedge \tilde{\eta} + \tilde{\gamma}) \tag{2.18}$$

where $c \in \mathbb{R}$ is a (non-zero) normalisation constant and $\chi_h = \frac{\sqrt{-1}}{2\pi}F_h$ is the curvature form of (the Chern connection associated to) the Hermitian metric h. (This formula implicitly assumes that $\varphi(L) \neq \arg(\pm\rho_0)$ so the χ_h^2 term does not vanish in the expansion. See the proof of Lemma 2.7.) As the authors observe in [5] after completing the square, the equation is equivalent to solving

$$\left(\chi_h + \frac{1}{2}\tilde{\eta}\right)^2 = \frac{1}{4}\tilde{\eta}^2 - \tilde{\gamma}.$$

Then the (lifted) data $\tilde{\Omega} = (\theta, \rho, \tilde{U})$ are said to satisfy the *volume form hypothesis for L* if the $(2, 2)$-form

$$\frac{1}{4}\tilde{\eta}^2 - \tilde{\gamma} \in \mathcal{A}^{2,2}_{\mathbb{R}}(X)$$

is a volume form on X.

Lemma 2.7. *Let $\Omega = (\beta, \rho, U)$ be a choice of stability data defining a polynomial central charge on a projective surface X, with a fixed lift*

$$\tilde{\Omega} = \left(\theta, \rho_0 + \rho_1 t + \rho_2 t^2, 1 + \tilde{U}_1 + \tilde{U}_2\right),$$

and let $L \to X$ be a holomorphic line bundle on X such that $\varphi(L) \neq \arg(\pm\rho_0)$. Then the forms $\tilde{\eta}(\tilde{\Omega}, L)$ and $\tilde{\gamma}(\tilde{\Omega}, L)$ are given by

$$\tilde{\eta}(\tilde{\Omega}, L) = \frac{2}{c_0}\left(c_0\tilde{U}_1 + c_1\theta\right), \tag{2.19}$$

$$\tilde{\gamma}(\tilde{\Omega}, L) = \frac{2}{c_0}\left(c_0\tilde{U}_2 + c_1\theta \wedge \tilde{U}_1 + c_2\theta^2\right), \tag{2.20}$$

where $c_k = \mathrm{Im}(\rho_k)\cot\varphi(L) - \mathrm{Re}(\rho_k)$.

Proof. This is a straightforward computation. Writing χ_h for the curvature (1,1) form $\frac{\sqrt{-1}}{2\pi}F_h$, we first note that $Z_{\tilde{\Omega}}(L,h)$, that is, the $(2,2)$ part of the form

$$\left(\rho_0+\rho_1\theta+\rho_2\theta^2\right)\wedge \mathrm{ch}(L,h)\wedge\left(1+\tilde{U}_1+\tilde{U}_2\right),$$

is given by

$$\frac{1}{2}\rho_0\chi_h^2+\left(\rho_0\tilde{U}_1+\rho_1\theta\right)\wedge\chi_h+\rho_0\tilde{U}_2+\rho_1\theta\wedge\tilde{U}_1+\rho_2\theta^2.$$

Let us write $e^{\sqrt{-1}\varphi(L)}=A+\sqrt{-1}B$ where A,B are real numbers. By the hypothesis that Ω defines a polynomial central charge, $B>0$. Therefore, we can write $e^{-\sqrt{-1}\varphi(L)}=B^{-1}(\cot\varphi(L)-\sqrt{-1})$. Now observe that

$$\mathrm{Im}(e^{-\sqrt{-1}\varphi(L)}\tilde{Z}_{\tilde{\Omega}}(L,h))=B^{-1}\left(\frac{1}{2}c_0\chi_h^2+(c_0\tilde{U}_1+c_1\theta)\wedge\chi_h+c_0\tilde{U}_2+c_1\theta\wedge\tilde{U}_1+c_2\theta^2\right)$$

where $c_k=\mathrm{Im}(\rho_k)\cot\varphi(L)-\mathrm{Re}(\rho_k)$. Now, if $\varphi(L)\neq\arg(\pm\rho_0)$ then $c_0\neq 0$ and the result follows by comparing with (2.18). ∎

Let us therefore set

$$\eta(\Omega,L)=\frac{2}{c_0}\left(c_0U_1+c_1\beta\right)\in H^{1,1}(X,\mathbb{R}),\tag{2.21}$$

$$\gamma(\Omega,L)=\frac{2}{c_0}\left(c_0U_2+c_1\beta\wedge U_1+c_2\beta^2\right)\in H^{2,2}(X,\mathbb{R}).\tag{2.22}$$

Corollary. *Let X,L and Ω be as in the Lemma. If*

$$V(\Omega,L)=\int_X\frac{1}{4}\eta(\Omega,L)^2-\gamma(\Omega,L)>0$$

then there exists a choice of lift $\tilde{\Omega}$ that satisfies the volume form hypothesis for L.

Proof. Clearly, if the numerical inequality $V(\Omega,L)>0$ is satisfied, then the class

$$\frac{1}{4}\eta(\Omega,L)^2-\gamma(\Omega,L)\in H^4(X,\mathbb{R})$$

contains a volume form v. Fix any lift $\tilde{\Omega}_0=(\theta,\rho,1+\tilde{U}_1+\tilde{U}_2)$ of Ω. Then, by the $\partial\bar{\partial}$-lemma, there exists a real valued $(1,1)$ form ζ on X such that

$$v=\frac{1}{4}\tilde{\eta}(\tilde{\Omega}_0,L)^2-\tilde{\gamma}(\tilde{\Omega}_0,L)+\sqrt{-1}\partial\bar{\partial}\zeta.$$

Setting $\tilde{U}_1'=\tilde{U}_1$ and

$$\tilde{U}_2'=\tilde{U}_2-\frac{\sqrt{-1}}{2}\partial\bar{\partial}\zeta,$$

we see immediately from (2.19) that if $\tilde{\Omega}=(\omega,\rho,1+\tilde{U}_1'+\tilde{U}_2')$ then

$$v=\frac{1}{4}\tilde{\eta}(\tilde{\Omega},L)^2-\tilde{\gamma}(\tilde{\Omega},L)$$

is a volume form. ∎

We are now in a position to state and prove the main result for Z-critical equations on surfaces.

Theorem 2.8 ([11]). *Let X be a projective surface and $L \to X$ a holomorphic line bundle on X. Suppose $K \subseteq \mathcal{K}_X \times (\mathbb{C}^*)^3 \times \bigoplus_i H^{i,i}(X, \mathbb{R})$ be a compact subset such that each $\Omega \in K$ defines a polynomial central charge Z_Ω on X. Moreover, assume that for each $\Omega \in K$, we have $V(\Omega, L) > 0$ and $\varphi(L) \neq \arg(\pm\rho_0)$. Then, there exists a non-negative integer $\ell \geq 0$ and finitely many curves $E_1, \cdots, E_\ell$ on X, of negative self-intersection, such that the following are equivalent.*

1. *For every $\Omega \in K$ and every lift $\tilde{\Omega}$ satisfying the volume form hypothesis at L, the Z_Ω-critical equation admits a solution.*
2. *For $i = 1, \cdots, \ell$, we have*

$$s(\Omega, L)\left(\int_{E_i} c_1(L) + \frac{1}{2}\eta(\Omega, L)\right) > 0$$

where $s(\Omega, L)$ is the sign of the non-zero real number

$$\int_X \left(c_1(L) + \frac{1}{2}\eta(\Omega, L)\right) \cdot \beta \in \mathbb{R}.$$

Proof. Let a compact subset $K \subseteq \mathcal{K}_X \times (\mathbb{C}^*)^3 \times \bigoplus_i H^{i,i}(X, \mathbb{R})$ be given such that each element $\Omega = (\beta, \rho, U) \in K$ defines a valid polynomial central charge with $\varphi_\Omega(L) \neq \arg(\pm\rho_0)$ and such that $V(\Omega, L) > 0$. Recall that the topological constant $\varphi(L) = \varphi_\Omega(L)$ is chosen precisely so that

$$\int_X \operatorname{Im}\left(e^{-\sqrt{-1}\varphi_\Omega(L)} Z_\Omega(L)\right) = 0.$$

In our notation, this is equivalent to

$$0 = \int_X \left(c_1(L)^2 + c_1(L) \cdot \beta(\Omega, L) + \gamma(\Omega, L)\right) = \int_X \left(c_1(L) + \frac{1}{2}\eta(\Omega, L)\right)^2 - V(\Omega, L).$$

Therefore, the inequality $V(\Omega, L) > 0$ implies that the class $\sigma(\Omega, L) = c_1(L) + \frac{1}{2}\eta(\Omega, L)$ has positive self-intersection. So, by the Hodge-Index Theorem, either $\sigma(\Omega, L)$ or its negative is in the cone $\mathcal{P}_X^+$. Let $s(\Omega, L) \in \{\pm 1\}$ be defined by the condition that

$$\tau_Z(\Omega, L) = s(\Omega, L)\tau(\Omega, L) \in \mathcal{P}_X^+.$$

Clearly, we have

$$s(\Omega, L) = \operatorname{sign}\left(\int_X \sigma(\Omega, L) \cdot \beta\right).$$

Thus, the map

$$\mathcal{K}_X \times (\mathbb{C}^*)^3 \times \bigoplus_i H^{i,i}(X, \mathbb{R}) \to H^{1,1}(X, \mathbb{R}), \quad \Omega \mapsto \tau_Z(\Omega, L)$$

is continuous and takes K onto a compact subset $\tilde{K} = \tau_Z(K)$ of $\mathcal{P}_X^+$. By Lemma 1.32, $\tilde{K}$ is contained in the convex hull of finitely many points of $\mathcal{P}_X^+$. Now, by Lemma 1.33, there exist finitely many curves $E_1, \cdots, E_\ell$ of negative self-intersection such that for each $\tau \in \tilde{K}$, τ is Kähler if and only if

$$\int_{E_i} \tau > 0$$

for $i = 1, \cdots, \ell$. Finally, according to [5, Section 2.3.3] this is equivalent to the existence of a subsolution for the Z_Ω-critical equation, and by [5, Theorem 2.45], equivalent to the existence of a solution. This completes the proof. □

Corollary ([11])**.** *Let X and L be as in the Theorem 2.8. Suppose X does not admit any curves of negative self-intersection. Let Ω be a choice of stability data defining a polynomial central charge Z_Ω and such that $\varphi(L) \neq \arg(\pm\rho_0)$ and $V(\Omega, L) > 0$. Then, for any lift $\tilde{\Omega}$ satisfying the volume form hypothesis, the Z_Ω-critical equation admits a solution on L.*

Proof. In Theorem 2.8, take $K = \{\Omega\}$ and note that necessarily $\ell = 0$. Thus, the second of the two equivalent conditions is vacuously true. □

2.2.3 The J-equation

Much like the dHYM equation, the J-equation also reduces to the complex Monge-Ampère equation on surfaces, and we can argue in a very similar way as for the dHYM equation. In fact, the argument is simpler, since we do not have to worry about any degenerate cases.

Let X be a compact Kähler surface and let $\alpha, \beta \in \mathcal{K}_X$ be Kähler classes, with $\omega \in \alpha, \theta \in \beta$ fixed Kähler forms. Then, the J-equation seeks a smooth $\psi \in C^\infty(X, \mathbb{R})$ such that

$$2\omega_\psi \wedge \theta = \mu_{\alpha,\beta}\omega_\psi^2. \tag{2.23}$$

We recall that for any irreducible curve C of X,

$$\mu_{\alpha,\beta}(C) = \frac{\int_C \beta}{\int_C \alpha},$$

is the slope of C and

$$\mu_{\alpha,\beta} = \frac{2\int_X \alpha \cdot \beta}{\int_X \alpha^2}.$$

Recall also that C is called a destabilising curve for (X, α, β) if $\mu_{\alpha,\beta}(C) \geq \mu_{\alpha,\beta}$.

By completing the square in ω_ψ, we can re-write (2.23) as

$$(\mu_{\alpha,\beta}\omega_\psi - \theta)^2 = \theta^2. \tag{2.24}$$

In this Subsection, let us therefore define

$$\tau(\alpha, \beta) = \mu_{\alpha,\beta}\alpha - \beta.$$

Then, (2.24) is a complex Monge-Ampère equation for the class $\tau(\alpha, \beta)$. Moreover, we can directly verify that

$$\int_X \tau(\alpha, \beta)^2 = \int_X \beta^2 > 0, \quad \int_X \tau(\alpha, \beta) \cdot \alpha = \int_X \alpha \cdot \beta > 0.$$

Thus, $\tau(\alpha, \beta) \in \mathcal{P}_X^+$.

Theorem 2.9 ([11])**.** *Let X be a compact Kähler surface and let $K \subseteq \mathcal{K}_X \times \mathcal{K}_X$ be a compact subset. Then, there exists a non-negative integer $\ell \geq 0$ and curves of negative self-intersection $E_1, \cdots, E_\ell$ on X (depending only on K) such that for all $(\alpha, \beta) \in K$ the following are equivalent.*

1. *For any choice of Kähler form $\theta \in \beta$ and any smooth representative $\omega \in \alpha$ there exists a smooth $\psi \in \mathcal{H}(\omega)$, unique up to additive constants, which is a solution to the J-equation*
$$2\theta \wedge \omega_\psi = \mu_{\alpha,\beta}\omega_\psi^2. \tag{2.25}$$
2. *For every curve $E \subseteq X$, we have*
$$\int_E \tau(\alpha, \beta) > 0.$$
3. *For $i = 1, \cdots, \ell$, we have*
$$\int_{E_i} \tau(\alpha, \beta) > 0.$$

Proof. The discussion preceding the statement of the Theorem shows that the continuous map
$$\mathcal{K}_X \times \mathcal{K}_X \to H^{1,1}(X, \mathbb{R}), \quad (\alpha, \beta) \mapsto \tau(\alpha, \beta)$$
maps the compact subset K onto a compact subset $\tilde{K}$ of $\mathcal{P}_X^+$. The rest of the argument is identical to the proof of Theorem 2.5, except for the claim that the solution ψ should be in $\mathcal{H}(\omega)$. But this is obvious from (2.25), as $\mu_{\alpha,\beta} > 0$. □

Corollary. *Suppose X is a compact Kähler surface that admits no curves of negative self-intersection. Then, the J-equation* (2.25) *always admits a solution for any pair $(\alpha, \beta) \in \mathcal{K}_X \times \mathcal{K}_X$, and any choice of Kähler form $\theta \in \beta$.*

Proof. Take $K = \{(\alpha, \beta)\}$ in the Theorem. Then, $\ell = 0$ and condition 3 is vacuously satisfied. □

Remark 22. The above Corollary is not new. In fact, it was already observed by Donaldson in [10, Section 4.3], who first introduced the J-equation from the point of view of a moment map. An interesting consequence of this (see [29, Corollary 1.7] is the fact that any compact Kähler surface X with $c_1(X) < 0$ and no curves of negative self-intersection admits a cscK metric in every Kähler class. The interesting problem of classifying surfaces which admit a cscK metric in every Kähler class is still open in general.

Corollary ([11])**.** *Suppose X is a compact Kähler surface and $\alpha, \beta \in \mathcal{K}_X$ are Kähler classes. Then, (X, α, β) has only finitely many destabilising curves, each of which is a curve of negative self-intersection. Moreover, the number of these curves is bounded above by the $\rho(X)$. If X is projective, then the number of destabilising curves is bounded above by $\rho(X) - 1$.*

Proof. The proof of Theorem 2.9 shows that $\tau(\alpha, \beta) \in \mathcal{P}_X^+ \subseteq \mathcal{B}_X$. Moreover, a curve C is destabilising if and only if $\mu_{\alpha,\beta}(C) \geq \mu_{\alpha,\beta}$, but this is equivalent to
$$\int_C \tau(\alpha, \beta) \leq 0.$$
The conclusion then follows from Proposition 1.31. □

Remark 23. Let (X, α, β) be as in the Corollary. An interesting (though trivial) consequence of these results is that we always have *optimal destabilisers*, that is, curves C that achieve the infimum

$$\mu_{\alpha,\beta} - \mu_{\alpha,\beta}(C) = \inf_{C'}(\mu_{\alpha,\beta} - \mu_{\alpha,\beta}(C').$$

Since both $\mu_{\alpha,\beta}(C)$ and $\mu_{\alpha,\beta}$ are linear in β, the same C achieves this infimum if we replace β by $\beta_t = t\beta + (1-t)\alpha$. Moreover, it is a result of Sjöström-Dyrefelt [45, Theorem 5] that this infimum computes, under certain hypotheses, the *coercivity threshold* of Donaldson's J-functional. This has some applications for the cscK equation. (See, in particular, [45, Corollary 3].)

2.3 The J-equation in higher dimensions

Ideally, one would like to prove an analogue of Theorem 2.9 in all dimensions. However, if $\dim_{\mathbb{C}} X \geq 3$, the J-equation does not reduce to a complex Monge-Ampère equation. Of course, this in itself is not enough to show that the analogous statement does not hold in higher dimensions. However, there are examples where the set of destabilising subvarieties is not finite. One such example is given in this Section as Example 2.17. Nevertheless, we aim to show that under certain positivity conditions, we can still get closely analogous results about destabilising subvarieties.

2.3.1 Finiteness of destabilisers in three dimensions

Theorem 2.10 ([18])**.** *Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = 3$ and let α, β be Kähler classes on X such that the triple (X, α, β) is J-semistable and the class $\mu_{\alpha,\beta}\alpha - 2\beta$ is big. Then, there exist at most finitely many irreducible subvarieties $Z \subseteq X$ such that*

$$\mu_{\alpha,\beta}(Z) = \mu_{\alpha,\beta}.$$

Proof. Observe that (X, α, β) is semistable but not stable precisely if there exists an irreducible proper subvariety Z of X satisfies $\mu_{\alpha,\beta}(Z) = \mu_{\alpha,\beta}$, that is, if

$$(\mu_{\alpha,\beta}\alpha - (\dim_{\mathbb{C}} Z)\beta) \cdot \alpha^{\dim_{\mathbb{C}} Z - 1} \cdot [Z] = 0.$$

Because $\mu_{\alpha,\beta}\alpha - 2\beta$ (and hence also $\mu_{\alpha,\beta}\alpha - \beta$) is big, by Lemma 1.24, such a Z necessarily satisfies $Z \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - (\dim_{\mathbb{C}} Z)\beta)$. Thus, if an irreducible surface S satisfies $\mu_{\alpha,\beta}(S) = \mu_{\alpha,\beta}$, then $S \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - 2\beta)$ implies necessarily that S must be one of the finitely many irreducible surface components of $E_{nK}(\mu_{\alpha,\beta}\alpha - 2\beta)$.

Similarly, if C is an irreducible curve satisfying $\mu_{\alpha,\beta}(C) = \mu_{\alpha,\beta}$, then $C \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$. If C is not one of the finitely many irreducible components of this analytic set, then $C \subseteq S$ for some irreducible surface in $E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$. Suppose first that S is a smooth surface. Then, we have

$$(\mu_{\alpha,\beta}(S)\alpha - \beta)^2 \cdot [S] = \beta^2 \cdot [S] > 0,$$

and

$$(\mu_{\alpha,\beta}(S)\alpha - \beta) \cdot \alpha \cdot [S] = \alpha \cdot \beta \cdot [S] > 0.$$

From this, it follows that the class $(\mu_{\alpha,\beta}(S)\alpha - \beta)|_S$ is big on S and therefore by the Proposition 1.31 there exist only finitely many curves C on S such that $(\mu_{\alpha,\beta}(S)\alpha - \beta) \cdot [C] \leq 0$. Moreover, by semistability of (X, α, β) we have $\mu_{\alpha,\beta}(S) \leq \mu_{\alpha,\beta}$, so $(\mu_{\alpha,\beta}\alpha - \beta) \cdot [C] \leq 0$ for only finitely many curves $C \subset S$, as desired.

In case S is singular, let $f : \tilde{X} \to X$ be any bimeromorphic morphism such that the proper transform $\tilde{S}$ of S in $\tilde{X}$ is smooth and an isomorphism away from the singular set of S. (Such a morphism always exists by Hironaka's Theorems on Resolution of Singularities [46]. More precisely, we need the complex analytic version of Hironaka's Embedded Desingularisation Theorem due to Włodarczyk [47, Theorem 2.0.2].) Then, $f_*[\tilde{S}] = [S]$ as currents and by the projection formula (see Remark 7), we have

$$\mu_{f^*\alpha, f^*\beta}(\tilde{S}) = \frac{2f^*(\alpha \cdot \beta) \cdot [\tilde{S}]}{f^*(\alpha^2) \cdot [\tilde{S}]} = \frac{2\alpha \cdot \beta \cdot [S]}{\alpha^2 \cdot [S]} = \mu_{\alpha,\beta}(S).$$

From this, we obtain (after another application of the projection formula) that

$$(\mu_{f^*\alpha, f^*\beta}(\tilde{S}) f^*\alpha - f^*\beta)^2 \cdot [\tilde{S}] = f^*(\mu_{\alpha,\beta}(S)\alpha - \beta)^2 \cdot [\tilde{S}] = f^*\beta^2 \cdot [\tilde{S}] = \beta^2 \cdot [S] > 0.$$

Moreover, since α is a Kähler class, $f^*\alpha$ can be represented by a semipositive form on $\tilde{X}$ (and hence also on $\tilde{S}$) and we have

$$(\mu_{f^*\alpha, f^*\beta}(\tilde{S}) f^*\alpha - f^*\beta) \cdot f^*\alpha \cdot [\tilde{S}] = f^*(\alpha \cdot \beta) \cdot [\tilde{S}] = \alpha \cdot \beta \cdot [S] > 0.$$

This implies, by Lemma 1.30 that the class $f^*(\mu_{\alpha,\beta}(S)\alpha - \beta)$ is a big class on $\tilde{S}$. As before, it follows from semistability of (X, α, β) that

$$(\mu_{\alpha,\beta} f^*\alpha - f^*\beta) - (\mu_{\alpha,\beta}(\tilde{S}) f^*\alpha - f^*\beta) = (\mu_{\alpha,\beta} - \mu_{\alpha,\beta}(S)) f^*\alpha$$

is nef on $\tilde{X}$, therefore also on $\tilde{S}$, and in particular, the class $f^*(\mu_{\alpha,\beta}\alpha - \beta)|_{\tilde{S}}$ is a big class on $\tilde{S}$. Hence we conclude from the Proposition 1.31 that there are finitely many curves $\tilde{C}$ in $\tilde{S}$ such that $\mu_{\alpha,\beta}(\tilde{C}) = \mu_{\alpha,\beta}$. But this implies that if $C \subseteq S$ does not lie entirely in the singular locus of S, then its proper transform can only be one of the finitely many $\tilde{C}$ in $\tilde{S}$, and this implies that there are at most finitely many destabilising curves $C \subseteq S$.

We have therefore proven that the destabilising subvarieties are among the finite list of

1. irreducible curve components C of $E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$
2. irreducible surface components S of $E_{nK}(\mu_{\alpha,\beta}\alpha - 2\beta)$
3. the irreducible curve components of the singular locus of S as S ranges over the surface components of $E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$
4. for every irreducible surface component S of $E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$ the curves $C \subseteq S$ not lying entirely in the singular locus of S whose strict transform, under any resolution of singularities $f : \tilde{X} \to X$ of S as above, occurs as an irreducible component of the negative part of the Zariski decomposition of the class $f^*(\mu_{\alpha,\beta}\alpha - \beta)$ on $\tilde{S}$.

This concludes the proof. □

Remark 24. We note that the choice of resolution $f : \tilde{X} \to X$ of the irreducible surface $S \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$ does not affect the set of potential destabilisers described in the proof. More precisely, let $f_i : \tilde{X}_i \to X$ be two resolutions of S coming from Hironaka's proof, that is, the strict transform $\tilde{S}_i$ of S_i in $\tilde{X}_i$ is smooth and f_i is a composition of blowing up repeatedly in points or smooth curves contained in the singular locus of (the strict transforms of) S. Then, by resolving the indeterminacy of the rational map $f_2^{-1} \circ f_1$ we get a common resolution $g : \tilde{X} \to X$ with morphisms $g_i : \tilde{X} \to \tilde{X}_i$ with $f_i \circ g_i = g$ for

$i = 1, 2$. If we denote by $\tilde{S}$ the strict transform of S in $\tilde{X}$, then by what we said above about Hironaka's proof, $g_i : \tilde{S} \to S_i$ is a composition of blowups in points of the smooth surface S_i. Now the formulas (see [21, Lemma 5.8])

$$(g_i)_* N(g_i^* f_i^*(\mu_{\alpha,\beta}\alpha - \beta)) = N(f_i^*(\mu_{\alpha,\beta}\alpha - \beta))$$

imply that the support of the negative part $N(g^*(\mu_{\alpha,\beta}\alpha - \beta))$ comprises the strict transforms of the irreducible components of $N(f_i^*(\mu_{\alpha,\beta}\alpha - \beta))$ plus a possible union of a divisor supported on the exceptional locus of g_i. Since this is true for g_1 and g_2, it follows that for each curve $C \subseteq S$, if the strict transform $\tilde{C}_1 \subseteq \tilde{S}_1$ under f_1 occurs as a component of $N(f_1^*(\mu_{\alpha,\beta}\alpha - \beta))$, then its strict transform $\tilde{C}_2 \subseteq \tilde{S}_2$ under f_2 also occurs as a component of $N(f_2^*(\mu_{\alpha,\beta}\alpha - \beta))$, and vice versa.

Away from semistability, it may happen that there are infinitely many destabilising subvarieties. However, in this case we still have adequate control over the destabilising subvarieties in the following precise sense:

Theorem 2.11 ([18])**.** *Suppose X is a compact Kähler manifold of* $\dim_{\mathbb{C}} X = 3$ *and α, β are Kähler classes on X such that $\mu_{\alpha,\beta}\alpha - 2\beta$ is a big class. Let $V_{\alpha,\beta}$ be the union of all irreducible subvarieties Z of X with $\mu_{\alpha,\beta}(Z) \geq \mu_{\alpha,\beta}$. Then, $V_{\alpha,\beta}$ is a proper analytic subset of X.*

Remark 25. In particular, $V_{\alpha,\beta}$ has finitely many irreducible components.

Proof of Theorem 2.11. If S is any destabilising irreducible surface in X, then we have

$$\int_S (\mu_{\alpha,\beta}\alpha - 2\beta) \cdot \alpha \leq 0$$

so by Lemma 1.24 we have

$$S \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - 2\beta).$$

Since by assumption $\mu_{\alpha,\beta}\alpha - 2\beta$ is big, there are always at most finitely many destabilising surfaces. Let V_2 be the union of all of these.

We must prove that all but finitely many destabilising curves C are contained in V_2. But a destabilising curve C likewise satisfies $C \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$. There are only finitely many irreducible components of $E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$ of dimension one. Assume therefore that C is not an irreducible component of $E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$. Then C lies instead in a surface $S \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - \beta)$. If $S \subseteq V_2$, there is nothing to check. So, suppose S is not contained in V_2, that is, $\mu_{\alpha,\beta}(S) < \mu_{\alpha,\beta}$. Then, by the same argument as in the proof of Theorem 2.10 above, there exist at most finitely many curves $C \subseteq S$ with $\mu_{\alpha,\beta}(C) \geq \mu_{\alpha,\beta}(S)$ and therefore also only finitely many curves C with $\mu_{\alpha,\beta}(C) \geq \mu_{\alpha,\beta} > \mu_{\alpha,\beta}(S)$. This proves the claim. □

The proof of the theorem in particular yields the following useful corollary:

Corollary ([18])**.** *Let (X, α, β) be as in the Theorem. Then, for each surface $S \subseteq V_{\alpha,\beta}$ satisfying $\mu_{\alpha,\beta}(S) \leq \mu_{\alpha,\beta}$ there are only finitely many curves $C \subseteq S$ such that $\mu_{\alpha,\beta}(C) \geq \mu_{\alpha,\beta}$.*

Proof. Observe that $\mu_{\alpha,\beta}(S) \leq \mu_{\alpha,\beta}$ is all one needs to conclude that the class $\mu_{\alpha,\beta}\alpha - \beta$ is big on S. Then, if S is smooth, one applies Proposition 1.31 directly. Otherwise, one proceeds as in the proof of Theorem 2.11. □

Lemma 2.12. *Let $\alpha, \beta_i, i = 1, 2$ be Kähler classes on a compact Kähler 3-fold X such that $\mu_{\alpha,\beta_i}\alpha - 2\beta_i$ is a big class for $i = 1, 2$. Then for any $t \in [0, 1]$ we have*

$$V_{\alpha,\beta_1} \cap V_{\alpha,\beta_2} \subseteq V_{\alpha,t\beta_1+(1-t)\beta_2} \subseteq V_{\alpha,\beta_1} \cup V_{\alpha,\beta_2}.$$

Proof. This is an immediate consequence of

$$\mu_{\alpha,t\beta_1+(1-t)\beta_2}(Z) = t\mu_{\alpha,\beta_1}(Z) + (1-t)\mu_{\alpha,\beta_2}(Z). \tag{2.26}$$

□

Proposition 2.13 ([18])**.** *Let α, β_i be Kähler classes for $i = 1, \cdots, s$ on a compact Kähler manifold X of dimension $\dim_{\mathbb{C}} X = 3$ such that $\mu_{\alpha,\beta_i}\alpha - 2\beta_i$ is a big class for each i. Then as β ranges over the convex hull $\mathrm{conv}(\beta_1, \ldots, \beta_s)$ of $\beta_1, \ldots, \beta_s$, the collection of irreducible subvarieties of X occurring as irreducible components $V_{\alpha,\beta}$ is a finite set.*

Proof. By the lemma, we have that

$$V_{\alpha,\beta} \subseteq \bigcup_{i=1}^{s} V_{\alpha,\beta_i} \tag{2.27}$$

as β ranges over $\mathrm{conv}(\beta_1, \cdots, \beta_2)$. This implies that the irreducible surface components of $V_{\alpha,\beta}$ are among the finitely many irreducible surface components of V_{α,β_i} for $i = 1, \cdots, s$. Therefore, we only need to prove that the irreducible curve components of $V_{\alpha,\beta}$ only take values in a finite set.

Denote by $S_1, \cdots, S_n$ the finitely many irreducible surfaces occurring as irreducible components of all the V_{α,β_i} for $i = 1, \cdots, s$, and let C be an irreducible curve component of some $V_{\alpha,\beta}$ for $\beta \in \mathrm{conv}(\beta_1, \cdots, \beta_s)$. Then, by (2.27) we have $C \subseteq V_{\alpha,\beta_i}$ for some i. If C is not contained in any S_i, then C can only be one of the finitely many irreducible curve components of V_{α,β_i}. Thus, all the curves C that occur as irreducible components of some $V_{\alpha,\beta}$ not contained in one of the S_i are among the finitely many irreducible curve components of V_{α,β_i} for $i = 1, \cdots, s$. It remains to treat the case of curves lying in some S_i.

For each $i = 1, \cdots, n$, consider the following partition

$$K = \mathrm{conv}(\beta_1, \cdots, \beta_s) = K_1^{(i)} \cup K_2^{(i)},$$

where

$$K_1^{(i)} = \{\beta \in K \mid \mu_{\alpha,\beta}(S_i) \le \mu_{\alpha,\beta}\}, \quad K_2^{(i)} = \{\beta \in K \mid \mu_{\alpha,\beta}(S_i) > \mu_{\alpha,\beta}\}.$$

Then, it is clear that S_i occurs as an irreducible component of $V_{\alpha,\beta}$ for each $\beta \in K_2^{(i)}$. On the other hand, $\beta \mapsto \mu_{\alpha,\beta}(S_i) - \mu_{\alpha,\beta}$ is a linear map. Thus, the closed set $K_1^{(i)}$ is also the convex hull of finitely many points, say $\beta_1^{(i)}, \cdots, \beta_{r_i}^{(i)}$. Now, if $\beta \in K_1^{(i)}$ and C is an irreducible component of $V_{\alpha,\beta}$ lying in S_i, then, by definition, we have $\mu_{\alpha,\beta}(C) \ge \mu_{\alpha,\beta}$. But then (2.26) implies that

$$\mu_{\alpha,\beta_j^{(i)}}(C) \ge \mu_{\alpha,\beta_j^{(i)}}$$

for some $1 \le j \le r_i$. But we also have that $\mu_{\alpha,\beta_j^{(i)}}(S_i) \le \mu_{\alpha,\beta_j^{(i)}}$ and thus, by the Corollary to Theorem 2.11, there are only finitely many curves C in S_i such that $\mu_{\alpha,\beta_j^{(i)}}(C) \ge \mu_{\alpha,\beta_j^{(i)}}$. Therefore, the irreducible components of $V_{\alpha,\beta}$ for $\beta \in K$ are among the following finite list:

1. the irreducible curve components of $E_{nK}(\mu_{\alpha,\beta_i}\alpha - \beta_i)$ for $i = 1, \cdots, s$.
2. the irreducible surface components $S_1, \cdots, S_n$ of $E_{nK}(\mu_{\alpha,\beta_i}\alpha - 2\beta_i)$ for $i = 1, \cdots, s$.
3. the finitely many curves C in S_i such that

$$\mu_{\alpha,\beta_j^{(i)}}(C) \geq \mu_{\alpha,\beta_j^{(i)}}$$

for $i = 1, \cdots, n$, and $j = 1, \cdots, r_i$.

□

2.3.2 Ridigity of destabilisers in three dimensions

Aside from the production of a finite list of subvarieties that destabilise (or optimally destabilise) the J-equation, it is also desirable to note that they have certain properties in common. Most notably, we have the following rigidity statement in the case of threefolds.

Theorem 2.14 ([18]). *Let (X, α, β) and $V_{\alpha,\beta}$ be as in the statement of Theorem 2.11. Then every irreducible component of $V_{\alpha,\beta}$ is rigid. More precisely, every irreducible surface component of $V_{\alpha,\beta}$ is the unique effective cycle representing its homology class, and every irreducible curve component C of $V_{\alpha,\beta}$ satisfies the following: for every irreducible surface S containing C, either C is an irreducible component of the singular locus of S or the strict transform of C under any resolution of singularities of S is a curve of negative self-intersection.*

Proof. Let S be a surface component of $V_{\alpha,\beta}$. Then the conclusion about S follows immediately from Boucksom's Theorem 1.27 noting that S is then an irreducible component of the negative part of the divisorial Zariski decomposition of $\mu_{\alpha,\beta}\alpha - 2\beta - \varepsilon\beta$ for any $\varepsilon > 0$ small enough, and hence exceptional. (See the proof of Proposition 2.16 below for this last claim.) Now let C be a curve component of $V_{\alpha,\beta}$. Then, by the definition of $V_{\alpha,\beta}$, it follows that if S' is any irreducible surface that contains C, then $\mu_{\alpha,\beta}(S') < \mu_{\alpha,\beta}$. Since X is projective, we can always find some S' with this property. But then the proof of Theorem 2.11 shows that either C is contained in the singular locus of S or the proper transform of C is an irreducible component in the negative part of the Zariski decomposition of the class $\mu_{\alpha,\beta}(S')\alpha - \beta$ on any resolution of singularities of S'. Thus, C has negative self-intersection as a divisor in any resolution of singularities of S'. □

2.3.3 Finiteness and rigidity in arbitrary dimensions

The same argument as for 3-folds can still be carried out on compact Kähler manifolds of arbitrary dimension. However, the price to pay is that we must assume several positivity conditions at once.

Theorem 2.15 ([18]). *Let X be a compact Kähler manifold and α, β Kähler classes on X. Suppose moreover that $\mu_{\alpha,\beta}\alpha - p\beta \in \mathcal{M}_{p+1}\mathcal{K}$ is $(p+1)$-modified Kähler on X for every $p = 1, 2, \ldots, n-1$. Then, there exist at most finitely many irreducible subvarieties $Z \subseteq X$ such that*

$$\mu_{\alpha,\beta}(Z) \geq \mu_{\alpha,\beta}(X) = \mu_{\alpha,\beta}.$$

Moreover, each divisor D such that $\mu_{\alpha,\beta}(D) \geq \mu_{\alpha,\beta}$ is the unique effective analytic cycle representing its homology class.

Proof. Let $1 \leq p \leq n-1$ and Z an irreducible analytic subvariety of X of dimension p. Note that $\mu_{\alpha,\beta}(Z) \geq \mu_{\alpha,\beta}$ is equivalent to

$$\int_Z (\mu_{\alpha,\beta}\alpha - p\beta) \cdot \alpha^{p-1} \leq 0,$$

which implies, by Lemma 1.24 that $Z \subseteq E_{nK}(\mu_{\alpha,\beta}\alpha - p\alpha)$. But the class $\mu_{\alpha,\beta}\alpha - p\beta \in \mathcal{M}_{p+1}\mathcal{K}$ is a $(p+1)$-modified Kähler class. Thus, Z must be one of the finitely many p-dimensional irreducible components of the proper analytic subset $E_{nK}(\mu_{\alpha,\beta}\alpha - p\beta)$ because this set does not contain any irreducible subvarieties of dimension greater than or equal to $p+1$ thanks to Lemma 1.23. □

Finally, we point out the useful fact that in arbitrary dimension, we still have a rigidity statement about destabilising divisors.

Proposition 2.16. *Let X be a compact Kähler manifold and $\alpha, \beta \in \mathcal{K}_X$ be Kähler classes, with $\theta \in \beta$ a fixed Kähler form. Suppose that $\tau_{n-1}(\alpha,\beta) = \mu_{\alpha,\beta}\alpha - (n-1)\beta \in \mathcal{B}_X$ is a big class. Then, there are only finitely many destabilising divisors D of (X, α, β), that is, divisors D satisfying $\mu_{\alpha,\beta}(D) \geq \mu_{\alpha,\beta}$.*

Proof. If D is a divisor such that $\mu_{\alpha,\beta}(D) \geq \mu_{\alpha,\beta}$, then D must lie in the non-Kähler locus of $\tau = \tau_{n-1}(\alpha,\beta) = \mu_{\alpha,\beta}\alpha - (n-1)\beta$. Now, let $\varepsilon > 0$ be so small that $\tau - \varepsilon\beta$ is still a big class. We claim that $\nu(\tau - \varepsilon\beta, D) > 0$. Indeed, by Remark 15, if $T_{\min}$ is any closed positive current of minimal singularities in $\tau - \varepsilon\beta$, then we have $\nu(\tau - \varepsilon\beta, D) = \nu(T_{\min}, D)$. If $\nu(T_{\min}, D) = 0$, then also $\nu(T_{\min} + \varepsilon\theta, D) = 0$ because θ is a smooth Kähler form. But $T_{\min} + \varepsilon\theta$ is a Kähler current in τ, and therefore $E_{nK}(\tau) \subseteq E_+(T_{\min} + \varepsilon\theta)$, so D cannot be in the non-Kähler locus of τ. This is a contradiction. Thus, $\nu(\tau - \varepsilon\beta, D) > 0$ and so D is an irreducible component of the negative part $N(\tau - \varepsilon\beta)$ of the Zariski decomposition of $\tau - \varepsilon\beta$. The result then follows by Boucksom's Theorem 1.27. □

Example 2.17. Here, we recall an example (a special case of an example found in [25]) of a manifold X of dimension n such that on X the cones $\mathcal{M}_p\mathcal{K}$ of p-modified Kähler classes are all pairwise distinct (except, of course, $\mathcal{M}_1\mathcal{K} = \mathcal{M}_0\mathcal{K} = \mathcal{K}$). To begin with, we proceed in slightly more generality than we need. Let Y be a smooth projective variety and L a very ample line bundle on Y. Let E denote the vector bundle

$$E = \mathcal{O}_Y \oplus (L^{\otimes a_1})^{\oplus b_1} \oplus \cdots \oplus (L^{\otimes a_r})^{\oplus b_r}$$

on Y, where $a_0 = 0 < a_1 < a_2 < \cdots < a_r$ is a strictly increasing sequence of positive integers and each b_i is a positive integer. We shall denote by $X = \mathbb{P}(E^\vee)$ the projective bundle of one-dimensional subspaces of the fibres of $E \to Y$ and $\pi : X \to Y$ the associated projection map. Then (for $r \geq 1$) we have

$$H^{1,1}(X, \mathbb{R}) = \mathbb{R}H \oplus \pi^* H^{1,1}(Y, \mathbb{R})$$

where $H = \mathcal{O}_{\mathbb{P}(E)}(1)$ is the dual of the tautological sub-line bundle of π^*E. For $i, j = 1, \ldots, r$, denote by D_{ij} the divisor associated to the (dual of the) surjection

$$E \twoheadrightarrow \mathcal{O}_Y \oplus (L^{\otimes a_1})^{\oplus b_1} \oplus \cdots \oplus \widehat{(L^{\otimes a_i})} \oplus \cdots \oplus (L^{\otimes a_r})^{\oplus b_r}.$$

where $\widehat{(L^{\otimes a_i})}$ means we omit the j-th summand equal to $L^{\otimes a_i}$. Note that the class of D_{ij} is equal to $H - a_i L$ (where we denote by L again the pullback π^*L). We can apply [25, Proposition 4] to deduce that the class $aL + bH$ is pseudoeffective (respectively nef) on X if and only if $b \geq 0$ and $a + a_r b \geq 0$ (respectively $b \geq 0$ and $a \geq 0$). Note that if $aL + bH$ is pseudoeffective, then we can write

$$aL + bH = (a + a_r b)L + bD_{rj} \text{ for any } j = 1, \dots, b_r$$

whence we see (since $(a + a_r b)L$ is nef) that the non-nef locus of every pseudoeffective class of the form $aL + bH$ satisfies

$$E_{nn}(aL + bH) \subseteq \bigcap_{j=1}^{b_r} D_{rj}.$$

In fact, the non-nef loci of pseudoeffective classes are much more constrained on X. Indeed, whenever $a + a_p b \geq 0$ for any $p = 0, 1, \dots, r$, we can write the decomposition

$$aL + bH = (a + a_p b)L + bD_{pj} \text{ for any } j = 1, \dots, b_p.$$

This shows that $\nu(aL + bH, x) = 0$ for $x \notin V_p$ where

$$V_p = \bigcap_{i \geq p} \bigcap_{j=1}^{b_i} D_{ij}.$$

In other words, if $a + a_p b \geq 0$ and $b \geq 0$, the non-nef locus of $aL + bH$ is contained in V_p, which is a d_p-dimensional smooth subvariety of X, with $d_p := n + b_1 + \cdots + b_p$. This proves that

$$\mathcal{M}_{d_p+1}\mathcal{K} \supseteq \{aL + bH \mid a + a_p b > 0, b > 0\}.$$

On the other hand, let $aL + bH$ be a big class, that is, $a + a_r b > 0$ and $b > 0$. Then, according to the proof of [25, Proposition 5] we get that

$$\nu(aL + bH, V_{p+1}) \geq \min\{t \geq 0 \mid 0 \in (a + t[a_{p+1}, a_r] + (b - t)[0, a_p])\}.$$

This implies that if $a + a_p b < 0$ then we have

$$\nu(aL + bH, V_{p+1}) \geq -\frac{a + a_p b}{a_r - a_p} > 0.$$

Since $d_{p+1} = \dim_{\mathbb{C}} V_{p+1} \geq d_p + 1$, this shows that $aL + bH \notin \mathcal{M}_{d_p+1}\mathcal{K}$ if $a + a_p b < 0$. Thus, we have shown that for $p = 1, \dots, n + r$ we have

$$\operatorname{span}_{\mathbb{R}}(L, H) \cap \mathcal{M}_p\mathcal{K} = \{aL + bH \mid a + a_{q-1} b > 0, b > 0\}$$

where q is the unique integer satisfying $d_{q-1} < p \leq d_q$. It is instructive to examine special cases of this construction.

1. Let Y be a Riemann surface, $r = n - 1$, L any line bundle on Y of degree $d > 0$. Set $b_1 = b_2 = \cdots = b_{n-1} = 1$, then $X = \mathbb{P}(\mathcal{O}_Y \oplus (L^\vee)^{\otimes a_1} \oplus \cdots \oplus (L^\vee)^{\otimes a_{n-1}})$ is an n-dimensional projective manifold. We see that the cones $\mathcal{M}_p\mathcal{K}$ are given by

$$\mathcal{M}_p\mathcal{K} = \{aL + bH \mid a + a_{p-1} b > 0, b > 0\} \tag{2.28}$$

and hence are all pairwise distinct for all $p = 1, \cdots, n$. Intersection theory on X is given by the formulas

$$H^n = d \sum_{i=1}^{n-1} a_i, \quad H^{n-1} \cdot L = d, \quad H^{n-i} \cdot L^i = 0 \text{ for } i \geq 2. \tag{2.29}$$

2. In the above special case, let $Y = \mathbb{P}^1, L = \mathcal{O}_{\mathbb{P}^1}(1), r = 2$ and $a_1 = 1, a_2 = 3$. Then, $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3))$. Let $\alpha = L + H$ and $\beta = L + bH$ for $b > 0$. Let us examine all the possible destabilising subvarieties for the J-equation on (X, α, β). Denote by S and C respectively the subvarietes

$$S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \subseteq X, \quad C = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}) \subseteq X.$$

Making use of (2.28) and (2.29), one can show that the class $\mu_{\alpha,\beta}\alpha - 2\beta$ is big if and only if $b > 1/15$. In the notation of Theorem 2.11 we then have

$$V_{\alpha,\beta} = \begin{cases} S \text{ for } 1/15 < b \leq 5/26 \\ C \text{ for } 5/26 < b \leq 2/9 \\ \varnothing \text{ for } b > 2/9. \end{cases}$$

In fact, if we let $\mathrm{Dest}_{\alpha,\beta}$ denote the set of destabilisers, we have

$$\mathrm{Dest}_{\alpha,\beta} = \begin{cases} \{S, C\} \text{ for } 1/15 < b \leq 5/26 \\ \{C\} \text{ for } 5/26 < b \leq 2/9 \\ \varnothing \text{ for } b > 2/9, \end{cases}$$

whereas for $b \leq 1/29$, $\mathrm{Dest}_{\alpha,\beta}$ contains infinitely many curves.

Remark 26. Let Y be a compact Kähler manifold of dimension n, L an ample line bundle on Y, $r = 1$ and $b_1 = m + 1$. Then $X = \mathbb{P}(\mathcal{O}_Y \oplus (L^\vee)^{\oplus(m+1)})$ has dimension $n + m + 1$. Then, for any big class of the form $aL + bH$, the non-nef locus is empty if and only if $a + b \geq 0, b > 0$, and equal to $P = \mathbb{P}(\mathcal{O}_Y) \subseteq X$ if and only if $a + b < 0, b > 0$. Thus, (writing $\mathcal{M}_p{}'\mathcal{K} = \mathcal{M}_p\mathcal{K} \cap \mathrm{span}(H, L)$) we have

$$\mathcal{M}_0'\mathcal{K} = \mathcal{M}_1'\mathcal{K} = \cdots = \mathcal{M}_{n-1}'\mathcal{K} \subsetneq \mathcal{M}_n'\mathcal{K} = \mathcal{M}_{n+1}'\mathcal{K} = \cdots = \mathcal{M}_{n+m+1}'\mathcal{K}.$$

In this situation, Datar-Mete-Song [13, Theorem 1.6] have shown that if $\alpha = L + aH$, $\beta = L + bH$, for $a, b > 0$, and ω_0, θ are fixed Kähler forms in α, β respectively satisfying a *Calabi-ansatz*, then, in the case that (X, α, β) is J-unstable, the J-flow with initial form ω_0 converges smoothly away from P. It therefore seems natural to ask whether, under suitable hypotheses, in the setting of Theorems 2.11,2.10 and 2.15 the J-flow should always converge smoothly away from the union of some collection of destabilising subvarieties.

2.4 Factorisable equations

The nature of the arguments in the previous Section indicates that we should be able to apply them to many more PDEs whose associated numerical criterion 'factorises' in an appropriate sense, so that we can apply Lemma 1.24. In practice, this actually happens for many geometric PDEs. Our aim in this Section is to make this notion of 'factorisability' precise. In order to proceed in sufficient generality to illustrate the ideas, we choose the setting of the generalised Monge-Ampère equations.

2.4.1 Factorisable generalised Monge-Ampère equations

Suppose X is a smooth projective variety of dimension $\dim_{\mathbb{C}} X = n$, $\alpha, \beta \in \mathcal{K}_X$ are Kähler classes with $\theta \in \beta$ a fixed Kähler form. Let (X, α, Θ) define a gMA equation, as explained in Subsection 2.1.4, with

$$\Theta = \sum_{k=1}^{n-1} c_k \theta^k + f\theta^n$$

for a fixed Kähler form $\theta \in \beta \in \mathcal{K}_X$. In the light of Theorem 2.4 we introduce the following natural definition.

Definition 2.18. Suppose (X, α, Θ) satisfy the cohomological constraint (2.10) and the positivity condition given in [7, (1.2)]. Then, we say that (X, α, Θ) is *(gMA) semistable (resp. stable)* if for every proper irreducible analytic subvariety $V \subseteq X$ we have

$$\int_V (\exp(\alpha) \wedge (1 - [\Theta]))^{[\dim_{\mathbb{C}} V, \dim_{\mathbb{C}} V]} \geq 0 \quad (\text{resp.} \ > 0).$$

If (X, α, Θ) is not semistable, we say that it is unstable. If V violates the above strict inequality, we say that it *(gMA) destabilises* the triple (X, α, Θ).

In order to apply our techniques, we need to give precise meaning to the notion of 'factorisability'. To this end, let us define the polynomials

$$P(y) = \sum_{k=1}^{n-1} c_k y^k$$

and for $p = 1, \ldots, \dim_{\mathbb{C}} X - 1$,

$$Q_p(x, y) = (\exp(x)(1 - P(y)))^{[p]} \tag{2.30}$$

where by $R(x, y)^{[p]}$ we mean the degree p homogeneous part of the power series $R(x, y)$. We note that the polynomials $Q_p(x, y)$ only depend on the coefficients $c_1, \cdots, c_{n-1}$ and not on θ, f or α.

Definition 2.19. Let (X, α, Θ) satisfy the cohomological constraint (2.10) and the positivity condition given in [7, (1.2)]. We say that the triple (X, α, Θ) defines a *factorisable* gMA equation if for each $p = 1, \ldots, \dim_{\mathbb{C}} X - 1$, the polynomial $Q_p(x, y)$ can be factorised as

$$Q_p(x, y) = (x - r_p y)\tilde{Q}_p(x, y)$$

where

1. the constant $r_p \geq 0$,
2. the polynomial $\tilde{Q}_p(x, y)$ is a polynomial with non-negative real coefficients.

The cohomology class $\tau_p(\alpha, [\theta]) = \alpha - r_p\beta$ is called the *associated factor class at dimension* p (or *of degree* p).

Example 2.20. For $k \geq 1$, let $\Theta = \kappa\theta^k$ where κ is the uniquely determined cohomological constant

$$\kappa = \frac{n! \int_X [\theta]^k \cdot \alpha^{n-k}}{(n-k)!} > 0.$$

This choice of Θ defines an *inverse Hessian equation*. Then, for $p < k$ we have

$$Q_p(x, y) = \frac{1}{p!}x^p$$

and for $p \geq k$ we have

$$Q_p(x, y) = \frac{1}{p!}x^p - \frac{\kappa}{(p-k)!}x^{p-k}y^k = \left(x - \kappa_p^{1/k}y\right)\left(\frac{1}{p!}\sum_{r=0}^{p-1}\kappa_p^{r/k}x^{p-1-r}y^r\right),$$

where $\kappa_p = p!\kappa/(p-k)! > 0$. Thus we see that all inverse Hessian equations are factorisable, with the associated factor class at each dimension $p = k, \ldots, \dim_{\mathbb{C}} X - 1$ given by

$$\tau_p(\alpha, [\theta]) = \alpha - \kappa_p^{1/k}[\theta].$$

When $k = 1$ this corresponds to the J-equation.

Example 2.21. Let $\dim_{\mathbb{C}} X = 3$ and define

$$\Theta = c\theta + d\theta^2 + f\theta^3$$

where $c, d \geq 0$ are real constants and f is a smooth function such that the triple (X, α, Θ) satisfies the cohomological constraint (2.10) and the positivity condition given in [7, (1.2)]. Then, we have

$$Q_2(x, y) = \frac{1}{2}x^2 - cxy - dy^2 = \frac{1}{2}(x + (-c + \sqrt{c^2 + 2d})y)((x + (-c - \sqrt{c^2 + 2d})y)$$

and

$$Q_1(x, y) = x - cy.$$

Thus, we see that the triple (X, α, Θ) always defines a factorisable gMA equation with associated factor class at dimension two equal to

$$\tau_2(\alpha, [\theta]) = \alpha + (-c - \sqrt{c^2 + 2d})[\theta].$$

Proposition 2.22 ([18])**.** *Suppose X is a smooth projective variety, $\alpha, \beta \in \mathcal{K}_X$ are Kähler classes with $\theta \in \beta$ a fixed Kähler form. Let $c_k \geq 0$ and $f \in C^\infty(X, \mathbb{R})$ be such that (X, α, Θ) satisfies the cohomological constraint* (3.2) *and the positivity condition given in [7, (1.2)]. Then, (X, α, Θ) defines a factorisable gMA equation.*

Proof. We must prove that if $Q_p(x, y)$ is defined by (2.30), then it can be factorised as

$$Q_p(x, y) = (x - r_p)\tilde{Q}_p(x, y)$$

where $r_p \geq 0$ and all the coefficients of $\tilde{Q}_p(x, y)$ are non-negative. It suffices to prove that if $h_p(x) = Q_p(x, 1)$, then each $h_p(x)$ can be factorised as

$$h_p(x) = (x - r_p)g_p(x)$$

with $r_p \geq 0$ and $g_p(x)$ is a polynomial with non-negative coefficients, for then the claim will follow by homogenising $h_p(x)$ and $g_p(x)$.

Now note that

$$h_p(x) = \frac{x^p}{p!} - \sum_{k=1}^{p} c_k \frac{x^{p-k}}{(p-k)!}.$$

It is clear that $h_p(x) \to +\infty$ as $x \to +\infty$, and $h(\varepsilon) \leq 0$ for $\varepsilon > 0$ small enough, with strict inequality if some $c_k > 0$ for $k = 1, \dots p$. Therefore, $h_p(x)$ admits a non-negative real root. Let r_p be the largest non-negative real root of $h_p(x)$. Then, $r_p = 0$ if and only if $c_k = 0$ for all $k = 1, \dots, p$. Now, we may write

$$h_p(x) = (x - r_p)g_p(x)$$

where $g_p(x)$ is a polynomial with real coefficients. We must prove that all the coefficients of $g_p(x)$ are non-negative. Clearly, if $r_p = 0$, then all the $c_k = 0$ for $k = 1, \dots, p$, so in that case $g_p(x) = x^{p-1}/p!$ certainly has non-negative coefficients. So we may assume that $r_p > 0$. Write $g_p(x) = P_p(x) - N_p(x)$, where both $P_p(x)$ and $N_p(x)$ have non-negative coefficients. If $N_p(x) \neq 0$, then let k_0 be the smallest integer such that the coefficient of x^{k_0} in $N_p(x)$ is strictly positive. Since $g_p(x)$ has degree $p - 1$, $k_0 < p$. Then, we have the identity

$$\frac{x^p}{p!} - \sum_{k=0}^{p-1} \frac{c_{p-k}}{k!} x^k = xP_p(x) + r_p N_p(x) - (xN_p(x) + r_p P_p(x)).$$

The monomial x^{k_0} occurs with non-zero coefficient on the right-hand-side of this identity only in the terms $r_p N_p(x)$ (where it has a strictly positive coefficient) and (possibly) in $xP_p(x)$ (where it has a non-negative coefficient). On the other hand, all the coefficients of x^k with $k < p$ have non-positive coefficients on the left-hand-side of this identity. This is a contradiction. Hence, $N_p(x) = 0$ and $g_p(x) = P_p(x)$ has non-negative coefficients. □

Therefore, the factor classes $\tau_p(\alpha, [\theta])$ are always well-defined for any triple (X, α, Θ), and the notion of factorisable gMA equations is not restrictive, at least at the level of generality allowed by gMA equations of the form (2.11). Another key feature of the factor classes $\tau_p(\alpha, [\theta])$ is that they are descending in p. More precisely, we have the following result.

Lemma 2.23 ([18])**.** *Let the triple (X, α, Θ) define a factorisable gMA equation. Then, for each $p = 1, \dots, n-2$, we have $\tau_p(\alpha, [\theta]) \geq \tau_{p+1}(\alpha, [\theta])$ with strict inequality whenever $\tau_{p+1}(\alpha, [\theta]) \neq \alpha$.*

Proof. Since (X, α, Θ) defines a factorisable gMA equation, the polynomials $Q_p(x, y)$ factorise as

$$Q_p(x, y) = (x - r_p y)\tilde{Q}_p(x, y)$$

where $r_p \geq 0$ and $Q_p(x, y)$ has non-negative coefficients. Note that this implies that $x \mapsto Q_p(x, 1)$ has at most one non-negative real root, namely r_p. But it is easy to verify by straightforward computation that

$$\frac{\partial}{\partial x} Q_{p+1}(x, y) = Q_p(x, y).$$

If $r_{p+1} = 0$, or equivalently, $\tau_{p+1}(\alpha, [\theta]) = \alpha$, then the this implies that $\tau_p(\alpha, [\theta]) = \alpha$ also, and so $\tau_p(\alpha, [\theta]) \geq \tau_{p+1}(\alpha, [\theta])$. So, suppose $r_{p+1} > 0$. Then, the factorisation of $Q_{p+1}(x, y)$ above implies that the polynomial function $x \mapsto Q_{p+1}(x, 1)$ has exactly one zero on the positive real axis, namely r_{p+1}. But since $Q_p(x, 1) = \frac{d}{dx} Q_{p+1}(x, 1)$ we see immediately $Q_p(r_{p+1}, 1) > 0$. Thus, $r_p < r_{p+1}$ because $x \mapsto Q_p(x, 1)$ has at most one non-negative real root. This shows that $\tau_p(\alpha, [\theta]) > \tau_{p+1}(\alpha, [\theta])$. □

Proposition 2.24 ([18])**.** *Let X be a smooth projective variety of* $\dim_{\mathbb{C}} X = n$*. Suppose* (X, α, Θ) *satisfies the cohomological constraint* (2.10) *and the positivity hypothesis given in [7, (1.2)], and defines a factorisable gMA equation whose associated factor classes* $\tau_p(\alpha, [\theta]) \in \mathcal{M}_{p+1}\mathcal{K}$ *at dimension* p *belong to the* $(p+1)$*-modified Kähler cone, for* $p = 1, \ldots, \dim_{\mathbb{C}} X - 1$*. Then, there exist at most finitely many subvarieties that (gMA) destabilise* (X, α, Θ)*.*

Proof. Any destabiliser V of dimension p satisfies

$$\int_V \tau_p(\alpha, [\theta]) \cdot \tilde{Q}_p(\alpha, [\theta]) \leq 0.$$

Since $\tilde{Q}_p(\alpha, [\theta])$ is a non-negative linear combination of powers of Kähler classes α and $[\theta]$, by Lemma 1.24, V must therefore be contained in $E_{nK}(\tau_p(\alpha, [\theta]))$. This is a proper analytic subset not containing any $(p+1)$-dimensional subvarieties, so V must be one its finitely many irreducible components. □

Remark 27. The above theorem applies to a subclass of Z-critical equations, given appropriate choice of stability datum. We explain a particular example of this more concretely (though other examples are also possible), in the notation of Subsection 2.1.3. Let L be a holomorphic line bundle on a compact Kähler manifold X of dimension n. Let us consider unipotent cohomology classes of the form $U = \exp(A)$ where $A \in H^{1,1}(X, \mathbb{R})$ is any cohomology class such that $A + c_1(L) \in \mathcal{K}_X$. Then, for any stability datum $\Omega = (\beta, \rho, \exp(A))$ where the stability vector $\rho = (\rho_0, \rho_1, \ldots, \rho_n)$ satisfies

$$b_k = \frac{\operatorname{Im}\left(\overline{Z_\Omega(L)}\rho_k\right)}{\operatorname{Im}\left(\overline{Z_\Omega(L)}\rho_0\right)} < 0, \quad k = 1, 2, \ldots, n,$$

we obtain a gMA equation satisfying the positivity conditions given in [7, (1.2)] for the Kähler class $\alpha = A + c_1(L)$ and

$$\Theta = \sum_{k=1}^{n} b_k \theta^k$$

for any Kähler form $\theta \in \beta$. Here, we have written

$$Z_\Omega(L) = \int_X \tilde{Z}_\Omega(L, h)$$

for some (hence any) choice of hermitian metric h on L.

We finally also point out that we get a much better result when $\dim_{\mathbb{C}} X = 3$, just as in the special case of the J-equation.

Theorem 2.25 ([18])**.** *Suppose X is a compact Kähler manifold of dimension* $\dim_{\mathbb{C}} X = 3$*,* $\alpha, \beta \in \mathcal{C}_X$ *Kähler classes, with* $\omega \in \alpha, \theta \in \beta$ *Kähler forms. Let* $\Theta = c_1\theta + c_2\theta^2 + f\theta^3$ *be such that* (X, α, Θ) *satisfies the cohomological constraint* (2.10) *and the positivity condition given in [7, (1.2)]. Suppose that the factor class* $\tau_2(\alpha, \beta)$ *of degree* 2 *associated to* (X, α, Θ) *is big, and* (X, α, Θ) *is gMA-semistable. Then, there are finitely many curves and surfaces that (gMA)-destabilise the triple* (X, α, Θ)*.*

Proof. The argument is very similar to the proof of Theorem 2.10. We sketch the parts that are identical and elaborate on the parts that are different. Let us briefly recall the notation. The polynomials $Q_p(x, y)$ for $p = 1, 2$ are given by

$$Q_2(x, y) = \frac{1}{2}x^2 - c_1xy - c_2y^2, \quad Q_1(x, y) = x - c_1y.$$

Observe that if $c_1 = c_2 = 0$, then the equation reduces to the complex Monge-Ampère equation and there are no destabilisers in this case. So we may assume that $c_1 > 0$ or $c_2 > 0$. When the triple (X, α, Θ) is (gMA)-semistable, a destabilising surface S (respectively, a destabilising curve C) satisfies

$$\int_S Q_2(\alpha, \beta) = 0, \qquad \text{(respectively, } \int_C Q_1(\alpha, \beta) = 0).$$

By what was said in Example 2.17, we can write

$$Q_2(\alpha, \beta) = \frac{1}{2}(\alpha - r_2\beta) \cdot (\alpha + s\beta)$$

where

$$r_2 = c_1 + \sqrt{c_1^2 + 2c_2}, \quad s = -c_1 + \sqrt{c_1^2 + 2c_2}.$$

It is clear that $r_2 > 0, s \geq 0$. Recall also that $\tau_2(\alpha, \beta) = \alpha - r_2\beta$. Now, if S is a (gMA)-destabilising surface, then by Lemma 1.24 we must have $S \subseteq E_{nK}(\tau_2(\alpha, \beta))$, and there are only finitely many surfaces S which are contained in $E_{nK}(\tau_2(\alpha, \beta))$. Suppose C is a (gMA)-destabilising curve. Then, we have

$$\int_C \alpha - c_1\beta = 0.$$

Now $\alpha - c_1\beta \geq \alpha - r_2\beta$ (either by direct observation, or by Lemma 2.23), so $\alpha - c_1\beta$ is a big class as well. Once again, by Lemma 1.24 this implies that $C \subseteq E_{nK}(\alpha - c_1\beta)$. Thus, C is either an irreducible curve component of $E_{nK}(\alpha - c_1\beta)$ or C is contained in an irreducible surface component S of $E_{nK}(\alpha - c_1\beta)$. It therefore remains to show that for each of the finitely many surface components S of $E_{nK}(\alpha - c_1\beta)$, there are only finitely many curves $C \subseteq S$ such that

$$\int_C \alpha - c_1\beta = 0.$$

But now we note that

$$\frac{1}{2}\int_S (\alpha - c_1\beta)^2 = \int_S \left(\frac{1}{2}\alpha^2 - c_1\alpha \cdot \beta + c_1^2\beta^2\right) = \int_S Q_2(\alpha, \beta) + \frac{1}{2}(c_1^2 + 2c_2)\int_S \beta^2 > 0,$$

and

$$\frac{1}{2}\int_S (\alpha - c_1\beta) \cdot (\alpha + s\beta) = \int_S Q_2(\alpha, \beta) + \sqrt{c_1^2 + 2c_2}\int_S \beta \cdot (\alpha + s\beta) > 0.$$

This shows that, if S is smooth, then $(\alpha - c_1\beta)|_S$ is a big class on S. Now we argue exactly as in the proof of 2.10 to conclude. □

Remark 28. In fact, with exactly the same strategy of proof, one can also obtain the more general analogues of Theorems 2.11 and 2.14 for the generalised Monge-Ampère equations when $\dim_{\mathbb{C}} X = 3$, making necessary adjustments as in the proof of Theorem 2.25.

2.4.2 The supercritical deformed Hermitian Yang-Mills equation

Let X be a smooth projective variety of dimension $\dim_{\mathbb{C}} X = n$ and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be cohomology classes on with $\beta \in \mathcal{K}_X$ a Kähler class. Fix a Kähler form $\theta \in \beta$.

In view of the Nakai-Moishezon type criterion given by Theorem 2.3, we introduce the following natural definitions.

Definition 2.26. Suppose (X, α, β) satisfies the supercritical phase hypothesis. Then we say that (X, α, β) is *(dHYM-)semistable (resp. stable)* if for every closed, proper, irreducible subvariety $V \subseteq X$ we have

$$\int_V \left(\operatorname{Re}(\alpha + \sqrt{-1}\beta)^{\dim_{\mathbb{C}} V} - \cot \hat{\phi}_\beta(\alpha) \operatorname{Im}(\alpha + \sqrt{-1}\beta)^{\dim_{\mathbb{C}} V} \right) \geq 0 \ \ (\text{resp. } > 0).$$

If (X, α, β) is not semistable, we say that it is unstable. If V violates the above strict inequality, we say that it *(dHYM-)destabilises* the triple (X, α, β).

From the point of view of the Theorem 2.3, we are naturally led to consider the following family of polynomials. Fix $\hat{\phi} \in (0, \pi)$ and $p \geq 0$, and define

$$Q^{\mathrm{dHYM}}_{p,\hat{\phi}}(x, y) = \operatorname{Re}(x + iy)^p - (\cot \hat{\phi}) \operatorname{Im}(x + iy)^p.$$

Lemma 2.27. *The polynomials $Q^{\mathrm{dHYM}}_{p,\hat{\phi}}(x, y)$ admit the factorisation*

$$Q^{\mathrm{dHYM}}_{p,\hat{\phi}}(x, y) = \prod_{l=0}^{p-1} \left(x - \cot \left(\frac{\hat{\phi} + l\pi}{p} \right) y \right). \tag{2.31}$$

Proof. A straightforward calculation shows that $Q^{\mathrm{dHYM}}_{p,\hat{\phi}}(x, y)$ admits p distinct roots given by

$$(x_l, y_l) = \left(\cos \left(\frac{\hat{\phi} + l\pi}{p} \right), \sin \left(\frac{\hat{\phi} + l\pi}{p} \right) \right)$$

for $l = 0, \dots, p - 1$. Now $\hat{\phi} \in (0, \pi)$ implies that $y_l > 0$ and the result follows. □

The above factorisation allows us to deduce the following analogue of Theorem 2.10.

Theorem 2.28 ([18])**.** *Let X be a smooth projective variety of dimension $n \leq 4$, β a Kähler class on X and α a real $(1,1)$-cohomology class. Suppose (X, α, β) has supercritical phase, that is $\hat{\phi}_\beta(\alpha) \in (0, \pi)$. If $\dim_{\mathbb{C}} X = 4$ suppose moreover that $\hat{\phi}_\beta(\alpha) \in (\pi/2, \pi)$. If the classes*

$$\tau^{\mathrm{dHYM}}_p(\alpha, \beta) = \alpha - \cot \left(\frac{\hat{\phi}_\beta(\alpha)}{p} \right) \beta$$

for $p = 1, \dots, \dim_{\mathbb{C}} X - 1$ lie in the $(p+1)$-modified Kähler cone, then there exist at most finitely many (dHYM) destabilising proper irreducible subvarieties $V \subseteq X$.

Proof. By hypothesis, the class $\tau^{\mathrm{dHYM}}_p(\alpha, \beta)$ is $(p + 1)$-modified Kähler, and the classes

$$\alpha - \cot \left(\frac{\hat{\phi}_\beta(\alpha) + \pi l}{p} \right) \beta$$

for $l = 1, \ldots, p-1$ are also Kähler. Indeed, $\hat{\phi}_\beta(\alpha) \in (0, \pi)$ implies that $\cot((\hat{\phi}_\beta(\alpha)+\pi)/2) < 0$, and $\hat{\phi}_\beta(\alpha) \in (\pi/2, \pi)$ implies that

$$\cot\left(\frac{\hat{\phi}_\beta(\alpha)+\pi}{3}\right) < 0, \quad \cot\left(\frac{\hat{\phi}_\beta(\alpha)+2\pi}{3}\right) < 0.$$

Thus, in view of the factorisation (2.31) and Lemma 1.24, any p-dimensional irreducible subvariety V such that

$$\int_V \left(\mathrm{Re}(\alpha+\sqrt{-1}\beta)^p - \cot\hat{\phi}_\beta(\alpha)\,\mathrm{Im}(\alpha+\sqrt{-1}\beta)^p\right) \leq 0.$$

must satisfy

$$V \subseteq E_{nK}(\tau_p^{\mathrm{dHYM}}(\alpha,\beta)),$$

but the latter set contains at most finitely many irreducible subvarieties of dimension p, since it does not contain any $(p+1)$-dimensional subvariety and is a proper analytic subset of X. Thus, V must be one of the finitely many irreducible components of $E_{nK}(\tau_p^{\mathrm{dHYM}}(\alpha,\beta))$. □

Just as for the J-equation, we obtain a slightly sharper result in the three dimensional setting, albeit under the condition that α is a Kähler class.

Theorem 2.29 ([18]). *Let X be a smooth projective variety of dimension $\dim_{\mathbb{C}} X = 3$. Let α, β be Kähler classes on X such that (X, α, β) satisfies the supercritical phase hypothesis and is (dHYM) semistable. Suppose the class $\tau_2^{\mathrm{dHYM}}(\alpha,\beta)$ is big. Then, there are only finitely many (dHYM) destabilising subvarieties of X.*

Proof. If S is any destabilising surface in X, then the proof of the above theorem shows that $S \subseteq E_{nK}(\tau_2^{\mathrm{dHYM}}(\alpha,\beta))$ and so there are always at most finitely many destabilising surfaces.

A destabilising curve C must satisfy

$$C \subseteq E_{nK}(\tau_1^{\mathrm{dHYM}}(\alpha,\beta)) = E_{nK}(\alpha - \cot\hat{\phi}_\theta(\alpha)\beta).$$

If $\hat{\phi} = \hat{\phi}_\beta(\alpha) \in [\pi/2, \pi)$, then $\tau_1^{\mathrm{dHYM}}(\alpha,\beta)$ is a Kähler class, so there are no such curves. So, suppose $\hat{\phi} \in (0, \pi/2)$, i.e. $\cot\hat{\phi} > 0$. Now observe that by semistability, for any surface S, we have

$$0 \leq \int_S (\alpha - \cot(\hat{\phi}/2)\beta)\cdot(\alpha - \cot((\hat{\phi}+\pi)/2)\beta) = \int_S (\alpha - \cot(\hat{\phi})\beta)^2 - (\csc(\hat{\phi}))^2 \int_S \alpha\cdot\beta,$$

where we have used the identity $\cot x = \cot 2x + \csc 2x$. Thus, we have

$$\int_S (\alpha - \cot(\hat{\phi})\beta)^2 \geq (\csc\hat{\phi})^2 \int_S \alpha\cdot\beta > 0, \quad \int_S (\alpha - \cot(\hat{\phi})\beta)\cdot\alpha \geq \cot(\hat{\phi}) \int_S \alpha\cdot\beta > 0.$$

Thus, if S is any surface component of $E_{nK}(\tau_1^{\mathrm{dHYM}}(\alpha,\beta))$, and S is smooth, the class $\tau_1^{\mathrm{dHYM}}(\alpha,\beta)$ restricts to a big class on S and therefore we conclude as in the proof of Theorem 2.10 above that S contains only finitely many destabilising curves. If S is singular, we again conclude as in the proof of Theorem 2.10 above. □

Corollary ([18]). *Let X be as in the Theorem. Let α, β be Kähler classes on X such that (X, α, β) has supercritical phase. Suppose the class $\tau_2^{\mathrm{dHYM}}(\alpha,\beta)$ is big. Then, the union $V_{\alpha,\beta}^{\mathrm{dHYM}}$ of all subvarieties of X that (dHYM) destabilise the triple (X, β, α) is a proper analytic subset of X.*

Proof. This argument is very nearly identical to the proof of Theorem 2.11 above after making appropriate straightforward changes. □

Chapter 3

Wall-chamber decompositions

One key motivation behind results like Theorem 2.5 and Proposition 2.13 is to understand the nature of stability (and hence solvability) when we vary the PDEs in continuous families. This motivation comes from analogous questions in algebraic geometry, where one finds that if certain numerical invariants are fixed, the space of all stability data admits a locally finite wall-chamber decomposition, with the notion of stability of any given object being the same for each element in any given chamber. By analogy, one should therefore expect something similar to happen in the case of PDEs. In this Chapter, we show that analogous results do indeed hold in the case of surfaces for the J-equation, dHYM equation, and, under mild and natural hypotheses, the Z-critical equations. In higher dimensions, we show that under the positivity conditions described in the previous Chapter, we are still able to get closely analogous results. These results are the first in the literature of their kind.

3.1 The case of surfaces

As in the previous Chapter, we single out the case of surfaces, both for clarity and because the results are much cleaner.

3.1.1 The deformed Hermitian Yang-Mills equation

Let X be a compact Kähler surface and let D denote the 'degenerate' locus

$$\mathrm{D} = \left\{(\alpha, \beta) \in H^{1,1} \times (X, \mathbb{R}) \times \mathcal{K}_X \ : \ 2\int_X \alpha \cdot \beta = 0\right\}$$

where the equation (2.6) for the triples (X, α, β) reduces to the Poisson equation. Let us recall from the previous Chapter that (2.6) is always solvable in this degenerate case. We wish to prove that in fact, this remains true in an open neighbourhood U of D in $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$. In this Subsection, we shall once again write

$$\tau(\alpha, \beta) = \alpha + \frac{\int_X (\beta^2 - \alpha^2)}{2\int_X \alpha \cdot \beta}\beta$$

whenever $2\int_X \alpha \cdot \beta \neq 0$.

Lemma 3.1. *Let X be a compact Kähler surface and D be as defined above. Then, there exists an open subset $U \subseteq H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ containing D such that for every $(\alpha, \beta) \in U$ with $2\int_X \alpha \cdot \beta \neq 0$, the triple (X, α, β) is (dHYM) stable.*

Proof. It suffices to prove that for each $(\alpha_0, \beta_0) \in \mathrm{D}$, there exists an open subset U containing (α_0, β_0) with the desired property. Therefore, let $(\alpha_0, \beta_0) \in \mathrm{D}$ be given. Then, because β_0 is a Kähler class, there exists $M > 0$ such that both $\alpha_0 + M\beta_0$ and $-\alpha_0 + M\beta_0$ are Kähler classes. Moreover, by the Hodge Index Theorem, since $\int_X \alpha_0 \cdot \beta_0 = 0$, it follows that $\int_X \alpha_0^2 \leq 0$, so we must also have that

$$N = \int_X (\beta_0^2 - \alpha_0^2) > 0.$$

Let us define

$$U_0 = \left\{ (\alpha, \beta) : \left| \int_X (\beta^2 - \beta_0^2) \right| < \frac{N}{4}, \left| \int_X (\alpha^2 - \alpha_0^2) \right| < \frac{N}{4}, \left| \int_X \alpha \cdot \beta \right| < \frac{N}{4M} \right\}.$$

This U_0 is clearly an open subset of $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ and contains (α_0, β_0). Now, if $(\alpha, \beta) \in U_0$ with $2 \int_X \alpha \cdot \beta \neq 0$ then we have

$$\int_X (\beta^2 - \alpha^2) > \frac{N}{2} > 0$$

and thence

$$\left| \frac{\int_X (\beta^2 - \alpha^2)}{2 \int_X \alpha \cdot \beta} \right| > \frac{N}{4 \left| \int_X \alpha \cdot \beta \right|} > M.$$

Now if we set

$$U = \{ (\alpha, \beta) \in U_0 \ : \ (\pm \alpha + M\beta) \in \mathcal{K}_X \}$$

then U is still an open neighbourhood of (α_0, β_0) and for any choice of $(\alpha, \beta) \in U \setminus \mathrm{D}$ the class

$$\tau(\alpha, \beta) = \alpha + \frac{\int_X (\beta^2 - \alpha^2)}{2 \int_X \alpha \cdot \beta} \beta$$

is either a Kähler class or the negative of a Kähler class. From the proof of Theorem 2.5, it follows that the dHYM equation (2.6) is solvable for the triple (X, α, β), or equivalently, that (X, α, β) is (dHYM) stable. □

Let $\mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$ be the subset of $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ comprising those pairs (α, β) such that for any fixed Kähler form $\theta \in \beta$ and smooth representative $\omega \in \alpha$, (2.6) admits a smooth solution ψ, unique up to additive constants. Note that D is disjoint from the boundary $\partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$, and in fact lies in the interior of $\mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$.

Theorem 3.2 ([11])**.** *The boundary $\partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$ is a locally finite union of smooth submanifolds W of $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ of (real) codimension one, each one of them cut out by an equation of the form*

$$\int_C \tau(\alpha, \beta) = 0$$

for some curve of negative self-intersection C in X.

Proof. Let $(\alpha_0, \beta_0) \in \partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$ be a boundary point. Pick any open neighbourhood V of (α_0, β_0) whose closure K is a compact subset of $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ and which does not meet D. The existence of such a V is guaranteed by Lemma 3.1. Now, we apply Theorem 2.5 to K and obtain a sign $s \in \{\pm 1\}$ and finitely many curves $E_1, \ldots, E_\ell$ of negative self-intersection such that for every $(\alpha, \beta) \in K$, and hence for every $(\alpha, \beta) \in V$, the triple

(X, α, β) admits a solution of the dHYM equation (2.6) if and only if for every $i = 1, \dots, \ell$ we have

$$s \int_{E_i} \tau(\alpha, \beta) > 0.$$

This shows that $V \cap \partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$ has boundary in V given by the finitely many loci

$$W_i = \left\{ (\alpha, \beta) \in V \;:\; \int_{E_i} \tau(\alpha, \beta) = 0 \right\}.$$

It only remains to prove that any locus W of this kind is a smooth submanifold of V of real codimension one. Note that if

$$W = \left\{ (\alpha, \beta) \in V \;:\; \int_{E} \tau(\alpha, \beta) = 0 \right\}$$

for any curve E, then W is the zero set of the function $F : V \to \mathbb{R}$ given by

$$F(\alpha, \beta) = \int_E \left(\alpha + \frac{\int_X (\beta^2 - \alpha^2)}{2 \int_X \alpha \cdot \beta} \beta \right).$$

It is easy to verify that

$$\left. \frac{d}{dr} \right|_{r=0} F(\alpha, (1+r)\beta) = \frac{\int_X \beta^2}{\int_X \alpha \cdot \beta} \int_E \beta > 0$$

because β is a Kähler class. Hence zero is a regular value of F, and so W is a smooth submanifold of codimension one. □

Corollary. *Let X be a compact Kähler surface. Then, $\mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$ is an open submanifold of $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ and contains* D *as a closed submanifold.*

Proof. This is an immediate consequence of the Theorem. □

With this Theorem in hand, we can finally state and prove our result about the locally finite wall-chamber structure associated to the dHYM equations.

Theorem 3.3 ([11])**.** *Let X be a compact Kähler surface and $\alpha_1, \dots, \alpha_s \in H^{1,1}(X, \mathbb{R})$ be a finite collection of cohomology classes. For each $\beta_0 \in \mathcal{K}_X$, there exists an open subset $U \subseteq \mathcal{K}_X$ containing β_0 and finitely many closed submanifolds $W_1, \dots, W_r$ of U of codimension one such that for each connected component*

$$U_j \subseteq U \setminus \bigcup_k W_k$$

and each $i = 1, \dots, s$, the triple (X, α_i, β) is (dHYM) stable for some $\beta \in U_j$ if and only if (X, α_i, β) is (dHYM) stable for all $\beta \in U_j$.

Proof. Let $\beta_0 \in \mathcal{K}_X$ be given. If any $(\alpha_i, \beta_0) \notin \partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$, then there exists an open neighbourhood V_i of (α_i, β_0) in $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ such that (X, α, β) is either always dHYM stable or dHYM unstable for each $(\alpha, \beta) \in V_i$. Thus, we may discard any such α_i without affecting the conclusion of the Theorem. Assume therefore that each $(\alpha_i, \beta_0) \in \partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$. By applying Theorem 3.2, we obtain open subsets $U_i \subseteq H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X$ containing (α_i, β_0) such that $U_i \cap \partial \mathcal{S}(X, \mathrm{dHYM})^{\mathrm{Stab}}$ has boundary in U_i given by finitely many walls $\tilde{W}_{ij}$ cut out by curves of negative self-intersection E_{ij}, and such that for each $(\alpha, \beta) \in U_i$ the triple (X, α, β) is (dHYM) stable if and only if

$$s_i \int_{E_{ij}} \tau(\alpha, \beta) > 0$$

for all j. Then, we may take U to be the common intersection of the images of the slices $(\{\alpha_i\} \times \mathcal{K}_X) \cap U_i$ under the projection $H^{1,1}(X, \mathbb{R}) \times \mathcal{K}_X \to \mathcal{K}_X$. This is an open neighbourhood of β_0 in $\mathcal{K}_X$. The walls W_{ij} are given by the same process: we take the slice $(\{\alpha_i\} \times \mathcal{K}_X) \cap \tilde{W}_{ij}$ and project it to $\mathcal{K}_X$. Then, for $\beta \in U$ the stability of (X, α_i, β) is determined by the signs of continuously varying quantities

$$\beta \mapsto s_i \int_{E_{ij}} \tau(\alpha_i, \beta)$$

which are the same on each connected component of $U \setminus \cup W_{ij}$.

It only remains to prove that each W_{ij} is a smooth submanifold of $\mathcal{K}_X$. But this follows easily from the proof of Theorem 3.2: W_{ij} is the zero locus of the function $G : U \to \mathbb{R}$ given by

$$G(\beta) = F(\alpha_i, \beta) = \int_{E_{ij}} \tau(\alpha_i, \beta)$$

where F is as defined in the proof of Theorem 3.2 and zero is a regular value of G for the same reason as that for F. □

3.1.2 The J-equation

As we saw in the previous Chapter, the J-equation, being the small volume limit of the dHYM equation, admits a very similar analysis. For the sake of avoiding repetition, we shall therefore not give complete arguments, as they are nearly identical to the ones given in the last Subsection.

Let X be a compact Kähler surface and let $\mathcal{S}(X, J)^{\mathrm{Stab}}$ denote the subset of $\mathcal{K}_X \times \mathcal{K}_X$ comprising (α, β) such that the triple (X, α, β) is J-stable. In this Subsection, we shall once again write

$$\tau(\alpha, \beta) = \mu_{\alpha,\beta}\alpha - \beta$$

for $\alpha, \beta \in \mathcal{K}_X$.

Theorem 3.4 ([11])**.** *The boundary $\partial\mathcal{S}(X, J)^{\mathrm{Stab}}$ is a locally finite union of smooth submanifolds W of $\mathcal{K}_X \times \mathcal{K}_X$ of (real) codimension one, each one of them cut out by an equation of the form*

$$\int_C \tau(\alpha, \beta) = 0$$

for some curve of negative self-intersection C in X.

Proof. The argument is almost identical to the proof of Theorem 3.2, applying Theorem 2.9 instead of Theorem 2.5. In the last part, we should define F by

$$F(\alpha, \beta) = \int_E (\mu_{\alpha,\beta}\alpha - \beta).$$

Then, a straightforward calculation yields

$$\left.\frac{d}{dr}\right|_{r=0} F(\alpha, \beta + r\alpha) = \int_E \alpha > 0.$$

Thus, the loci W are smooth submanifolds of real codimension one. □

Corollary. *Let X be a compact Kähler surface. Then, $\mathcal{S}(X,J)^{\mathrm{Stab}}$ is an open submanifold of $\mathcal{K}_X \times \mathcal{K}_X$.*

In the same vein, we obtain the result about the locally finite wall-chamber structure.

Theorem 3.5 ([11]). *Let X be a compact Kähler surface and $\alpha_1, \ldots, \alpha_s \in \mathcal{K}_X$ be a finite collection of Kähler classes. For each $\beta_0 \in \mathcal{K}_X$, there exists an open subset $U \subseteq \mathcal{K}_X$ containing β_0 and finitely many closed submanifolds $W_1, \ldots, W_r$ of U of codimension one such that for each connected component*

$$U_j \subseteq U \setminus \bigcup_i W_i$$

and each $i = 1, \ldots, s$, the triple (X, α_i, β) is J-stable for some $\beta \in U_j$ if and only if (X, α_i, β) is stable for all $\beta \in U_j$.

Proof. The argument is very nearly identical to the proof of Theorem 3.3. We omit the details. □

3.1.3 Z-critical equations

The following result can be seen as a first analogue of a locally finite wall-chamber structure on the side of differential geometry for a wide family of PDEs associated to central charges coming from the study of stability. (We draw the reader's attention to [41, Proposition 9.4], which gives a closely analogous result in the case of K3 surfaces to Theorem 3.6 below.)

Theorem 3.6 ([11]). *Fix a finite set S of holomorphic line bundles $L_i \to X$ (for $i = 1, \cdots, k$) on a compact Kähler surface X. Let V_S denote the set of triples $\Omega = (\beta, \rho, U)$ in $\mathcal{K}_X \times (\mathbb{C}^*)^3 \times \bigoplus H^{i,i}(X, \mathbb{R})$ such that $Z_\Omega(L_i)$ lies in the upper half-plane, $\varphi_\Omega(L_i) \neq \arg(\pm\rho_0)$ and $V(\Omega, L_i) > 0$ for each $i = 1, \cdots, k$. Let $\mathcal{U}_i$ denote the subset of V_S comprising those Ω such that for any choice of lift $\tilde{\Omega}$ of Ω satisfying the volume form hypothesis for L_i, the Z-critical equation (2.17) admits a solution. Then, $\mathcal{U}_i$ is an open subset of V_S and for any compact subset K of V_S, the set $\mathcal{U}_i \cap K$ is cut out by the finitely many real algebraic inequalities*

$$W_{ij}(\Omega) = \int_{E_{ij}} \tau_Z(\Omega, L_i) > 0, \quad j = 1, \cdots, \ell_i$$

where $E_{i1}, \cdots, E_{i\ell_i}$ are the curves appearing in Theorem 2.8. In particular, if C is any connected component of

$$K \setminus \bigcup_{i,j} \{W_{ij}(\Omega) = 0\}$$

then for any $i_0 \in \{1, \cdots, k\}$, we have $\Omega \in \mathcal{U}_{i_0}$ for some $\Omega \in C$ if and only if $\Omega \in \mathcal{U}_{i_0}$ for every $\Omega \in C$.

Proof. The first claim is immediate from the proof of Theorem 2.8. The only thing that needs justification is the last sentence. But if C is a connected component of

$$K \setminus \bigcup_{i,j} \{W_{ij}(\Omega) = 0\}$$

then, for each value of i, j, the continuous assignment $\Omega \mapsto W_{ij}(\Omega)$ must be non-zero and take the same sign for every $\Omega \in C$. Now fix a value i_0 for i and note that for the finitely many values of i, j, the signs of W_{ij} on C are all positive or not all positive according as the class $\tau_Z(\Omega, L_{i_0})$ is Kähler or not, according as $\Omega \in \mathcal{U}_{i_0}$ or not, and this proves the claim. □

Finally, to justify the terminology of 'wall-chamber' decomposition that we have used we explain why the loci

$$W_{ij}(\Omega) = 0$$

in the set $\mathcal{U}_i$ (in the notation of Corollary 3.6) are real codimension one loci. More precisely, we have the following proposition.

Proposition 3.7 ([11]). *Let L be a holomorphic line bundle and let V denote the set comprising stability data $\Omega = (\beta, \rho, U) \in \mathcal{K}_X \times (\mathbb{C}^*)^3 \times \bigoplus H^{i,i}(X, \mathbb{R})$ for which $Z_\Omega(L)$ lies in the upper half-plane, $\varphi_\Omega(L) \neq \arg(\pm\rho_0)$ and $V(\Omega, L) > 0$. Let E be any curve on X such that*

$$\int_E \tau_Z(\Omega, L) = 0$$

for some $\Omega \in V$. Then the locus

$$W_E = \left\{ \Omega \in V \ : \ \int_E \tau_Z(\Omega, L) = 0 \right\}$$

is a real codimension one submanifold of V.

Proof. Recall that the class $\tau_Z(\Omega, L)$ is given, up to a sign, by

$$\tau_Z = \pm \left(c_1(L) + \frac{1}{2}\eta(\Omega, L) \right),$$

where

$$\eta(\Omega, L) = \frac{2}{c_0}\left(c_0 U_1 + c_1 \beta\right) = 2\left(U_1 + \frac{c_1}{c_0}\beta\right)$$

and $c_k = c_k(\Omega, L) = \operatorname{Im}(\rho_k)\cot\varphi_\Omega(L) - \operatorname{Re}(\rho_k)$. Recall that $c_0 \neq 0$; this is a consequence of the hyphotesis that $\varphi_\Omega(L) \neq \arg(\pm\rho_0)$. Fix

$$\Omega = (\beta, \rho_0 + \rho_1 t + \rho_2 t^2, U)$$

such that

$$\int_E \tau_Z(\Omega, L) = 0$$

for some curve E and consider the family of stability data given by

$$\Omega_\varepsilon = \left(\beta, \rho_0 + \rho_1 t + \rho_2 t^2, U + \frac{\varepsilon}{\int_X \beta^2}\beta^2 \right)$$

for $\varepsilon \in \mathbb{R}$ small. Then, we have

$$Z_{\Omega_\varepsilon}(L) = Z_\Omega(L) + \frac{\varepsilon}{\int_X \beta^2}\int_X \beta^2 \cdot (\rho_0 + \rho_1\beta + \rho_2\beta^2) \cdot ch(L) = Z_\Omega(L) + \varepsilon\rho_0.$$

Writing $a = \operatorname{Re}(\rho_0), b = \operatorname{Im}(\rho_0)$, we get that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \cot\varphi_{\Omega_\varepsilon}(L) = \frac{a \operatorname{Im} Z_\Omega(L) - b \operatorname{Re} Z_\Omega(L)}{(\operatorname{Im} Z_\Omega(L))^2} = -\frac{c_0}{\operatorname{Im} Z_\Omega(L)}$$

recalling that $c_0 = b\cot\varphi_\Omega(L) - a \neq 0$. (Throughout, we write $c_k = c_k(\Omega, L)$ for brevity, reserving the more elaborate notation $c_k(\Omega_\varepsilon, L)$ for the perturbed constants.) A simple calculation now shows that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{c_1(\Omega_\varepsilon, L)}{c_0(\Omega_\varepsilon, L)} = -\frac{1}{c_0 \operatorname{Im} Z_\Omega(L)} (c_0 \operatorname{Im}\rho_1 - c_1 \operatorname{Im}\rho_0) = -\frac{1}{c_0 \operatorname{Im} Z_\Omega(L)} \operatorname{Im}\left(\frac{\rho_0}{\rho_1}\right) \neq 0$$

the last inequality following from our hypotheses and the very definition of a stability datum. This shows that the function

$$f(\varepsilon) = \int_E \tau_Z(\Omega_\varepsilon, L)$$

satisfies

$$f'(0) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(\int_E c_1(L) + U_1 + \frac{c_1(\Omega_\varepsilon, L)}{c_0(\Omega_\varepsilon, L)}\beta\right) = -\frac{1}{c_0 \operatorname{Im} Z_\Omega(L)} \operatorname{Im}\left(\frac{\rho_0}{\rho_1}\right) \int_E \beta \neq 0$$

and hence zero is a regular value of the map

$$\Omega \mapsto \int_E \tau_Z(\Omega, L).$$

□

Remark 29. It is worth remarking that this wall-chamber decomposition cannot in general be *globally* finite. Indeed, there exist Kähler surfaces which admit infinitely many distinct curve classes with negative self-intersection. (For example, the blowup of $\mathbb{P}^2$ in nine points in general position has infinitely many smooth rational curves of self-intersection -1.) If X is any such surface, and E is any curve on X with negative self-intersection, then consider the family of stability data given by

$$\Omega_r = \left(\beta, \frac{r}{E^2}\sqrt{-1} - \frac{1}{\int_E \beta} t + \sqrt{-1}t^2, 1 + [E]\right)$$

where β is any Kähler class on X with $\int_X \beta^2 = 1$ and $r > 0$ is a positive constant whose value we shall vary in the range $r \in (0, 2]$. (In particular, we have set $U_2 = 0$.) We wish to consider the Z-critical equations associated to this family of stability data on the *trivial* line bundle $L = \mathcal{O}_X$. One verifies quite easily that

$$Z_{\Omega_r}(L) = -1 + \sqrt{-1}$$

independently of r and therefore $\varphi_{\Omega_r}(L) = \frac{3\pi}{4} \neq \arg(\pm\frac{2}{3E^2}\sqrt{-1}) = \pm\frac{\pi}{2}$. A straightforward calculation then shows that

$$V(\Omega_r, L) = \frac{E^2(r-2)}{r} + \left(\frac{E^2}{r\int_E \beta}\right)^2 > 0$$

as $r \in (0, 2]$. In other words, all the hypotheses of Theorem 2.8 are satisfied. However, another straightforward calculation shows that

$$\int_E \tau_Z(\Omega_r, L) = \frac{E^2(1-r)}{r},$$

so the assignment $\Omega_r \mapsto \int_E \tau_Z(\Omega_r, L)$ changes sign as r crosses the value $r = 1$. In other words, the stability data Ω_r cross the wall

$$W_E = \left\{ \Omega \ : \ \int_E \tau_Z(\Omega, L) = 0 \right\}$$

defined by the curve E. Thus, every curve of negative self-intersection gives rise to a wall which has non-empty intersection with the space of admissible stability data for the trivial bundle, and there are therefore infinitely many distinct walls whenever there are infinitely many distinct curve classes with negative self-intersection.

Example 3.8 ([11])**.** We wish to present a concrete illustration of the results of this Section (in particular the wall-and-chamber decomposition of Theorem 3.6) in a simple case. To this end, we consider the classical example of the blowup $\pi : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ in two distinct points $p_1, p_2 \in \mathbb{P}^2$. Let H denote the pullback of a line and $E_i = \pi^{-1}(p_i), i = 1, 2$ the exceptional curves of the blowup. Then $H^{1,1}(X, \mathbb{R})$ is spanned by (the classes of) H, E_1, E_2. Moreover, it is easy to check that X admits precisely three curves of negative self-intersection, namely E_1, E_2 and T, the strict transform of the unique line passing through p_1 and p_2, whose class in cohomology is equal to $H - E_1 - E_2$. (By the usual abuse of notation, we identify E_i with $c_1(\mathcal{O}_X(E_i))$ etc.) For s a complex number lying in the upper half-plane, let Ω_s be the stability datum given by the triple

$$\Omega_s = (\beta, U, \rho_s(t)) = \left(3H - E_1 - E_2, 1, 1 - st + \frac{s^2 t^2}{2} \right).$$

This stability datum satisfies all the requirements to define a polynomial central charge, except perhaps the condition $\operatorname{Im}(s^2) > 0$, but this is merely a normalisation condition. After fixing a choice of Kähler form $\omega \in \beta$, one checks that solving the Z_{Ω_s}-critical equation on a line bundle L on X is equivalent to solving the equation

$$\left(\chi_h - a\omega + \frac{b^2\beta^2 - (c_1(L) - a\beta)^2}{2\beta \cdot (c_1(L) - a\beta)} \omega \right)^2 = b^2 \left(1 + \left(\frac{b^2\beta^2 - (c_1(L) - a\beta)^2}{2\beta \cdot (c_1(L) - a\beta)} \right)^2 \right) \omega^2 \qquad (3.1)$$

for a Hermitian metric h on L. Here $a = \operatorname{Re}(s)$, $b = \operatorname{Im}(s)$, and χ_h is the curvature form $\frac{\sqrt{-1}}{2\pi} F_h$ of the metric h. (We will furthermore require that $\beta \cdot (c_1(L) - a\beta) > 0$ for all line bundles L under consideration.) In particular, we see immediately that the volume form hypothesis is always satisfied. In fact, the equation (3.1) is equivalent to a dHYM equation for the class $c_1(L) - a\beta$ with auxiliary Kähler form $b\omega$.

Let $L_1 = \mathcal{O}_X(E_1)$ and $L_2 = \mathcal{O}_X(T)$. Then, one checks that for $a < \frac{1}{7}$, we have $\beta \cdot (c_1(L_i) - a\beta) > 0$ for $i = 1, 2$. Moreover, from (3.1) we see that we can solve the Z_{Ω_s}-critical equation on L_i if and only if

$$\tau(L_i, \Omega_s) = c_1(L_i) - a\beta + \frac{b^2\beta^2 - (c_1(L_i) - a\beta)^2}{2\beta \cdot (c_1(L_i) - a\beta)} \beta$$

is a Kähler class. (The class $\tau(L_i, \Omega_2)$ can never be the negative of a Kähler class, because for b large and positive, it is clearly Kähler and $s \mapsto \tau(L_i, \Omega_s)$ is a continuous mapping into the disjoint union $\pm\mathcal{P}_X^+$.) A straightforward calculation (using the fact that E_1, E_2 and T are the only curves of negative self-intersection) then shows that for $i = 1$ this happens precisely when

$$W_1(\Omega_s) = \int_{E_1} \tau(L_1, \Omega_s) = \frac{7b^2 + 8a}{1 - 7a} > 0$$

and for $i = 2$ precisely when

$$W_2(\Omega_s) = \int_T \tau(L_2, \Omega_s) = \frac{7b^2 + 4(a+1)^2 - 8}{5 - 11a} > 0.$$

This gives us a two-dimensional local slice of the wall-and-chamber decomposition which is shown in the figure below.

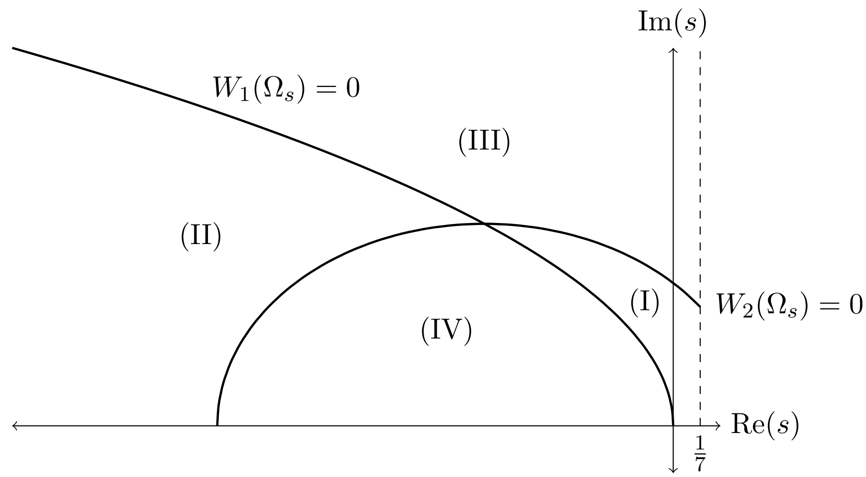

Figure 3.1: (I): only L_1 is Z_{Ω_s}-stable. (II): only L_2 is Z_{Ω_s}-stable. (III): both L_1 and L_2 are Z_{Ω_s}-stable. (IV): neither L_1 nor L_2 is Z_{Ω_s}-stable.

One can make many different choices for varying the stability datum Ω, and carry out a more general analysis of the above decomposition without much additional difficulty, as long as one understands the boundary of the nef cone of X.

3.2 Higher-dimensional results

3.2.1 The J-equation in three dimensions

Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = 3$ and let $\alpha, \beta \in \mathcal{K}_X$ be Kähler classes. Recall that all our results in three dimensions about the set of destabilising subvarieties for the J-equation associated with the triple (X, α, β) hold under the assumption that the class $\mu_{\alpha,\beta}\alpha - 2\beta$ be a big class. Under the same assumption, we can prove an analogue of Theorem 3.2.

Let $\mathcal{S}_+(X, J)$ denote the subset of $\mathcal{K}_X \times \mathcal{K}_X$ comprising those (α, β) such that $\mu_{\alpha,\beta}\alpha - 2\beta$ is a big class. We shall denote by $\mathcal{S}_+(X, J)^{\mathrm{Stab}} \subseteq \mathcal{S}_+(X, J)$ the subset comprising (α, β) such that (X, α, β) is J-stable. Note that $\mathcal{S}_+(X, J)$ is a non-empty open subset of $\mathcal{K}_X \times \mathcal{K}_X$.

Theorem 3.9 ([18])**.** *The boundary $\partial\mathcal{S}_+(X, J)^{\mathrm{Stab}}$ of $\mathcal{S}_+(X, J)^{\mathrm{Stab}}$ in $\mathcal{S}_+(X, J)$ is a locally finite union of closed smooth submanifolds W of $\mathcal{S}_+(X, J)$ of (real) codimension one, each one of them cut out by an equation of the form*

$$\int_C (\mu_{\alpha,\beta}\alpha - \beta) = 0$$

or

$$\int_S (\mu_{\alpha,\beta}\alpha - 2\beta) \cdot \alpha = 0$$

for some curve C or some surface S, rigid in the sense of Theorem 2.14.

Proof. Let $(\alpha_0, \beta_0) \in \partial\mathcal{S}_+(X,J)^{\mathrm{Stab}}$ be a point in the boundary. Pick an open neighbourhood $U \subseteq \mathcal{S}_+(X,J)$ of (α_0,β_0) whose closure K is a compact subset of $\mathcal{S}_+(X,J)$. By Proposition 2.13, there exist finitely many irreducible surfaces S_i and curves C_j such that, for each $(\alpha,\beta) \in K$, and hence each $(\alpha,\beta) \in U$, each irreducible component of $V_{\alpha,\beta}$ (in the notation of Theorem 2.11) is among the finitely many surfaces S_i or curves C_j. This shows that if (X,α,β) is not J-stable, then $\mu_{\alpha,\beta}(S_i) \geq \mu_{\alpha,\beta}$ for some S_i or $\mu_{\alpha,\beta}(C_j) \geq \mu_{\alpha,\beta}$ for some C_j. Thus, we have

$$\mathcal{S}_+(X,J)^{\mathrm{Stab}} \cap U = \{(\alpha,\beta) \in U \ : \ \mu_{\alpha,\beta}(S_i) < \mu_{\alpha,\beta},\ \mu_{\alpha,\beta}(C_j) < \mu_{\alpha,\beta} \text{ for every } i,j\}.$$

This shows that $\mathcal{S}_+(X,J)^{\mathrm{Stab}}$ is an open submanifold of $\mathcal{S}_+(X,J)$. Now note that each component W of the boundary is the zero locus of a function $F : U \to \mathbb{R}$ of the form

$$F(\alpha,\beta) = \int_V (\mu_{\alpha,\beta}\alpha - (\dim_{\mathbb{C}} V)\beta) \cdot \alpha^{\dim_{\mathbb{C}} V - 1}.$$

But zero is a regular value of F in view of the fact

$$\left.\frac{d}{dr}\right| F(\alpha, \beta + r\alpha) = (3 - \dim_{\mathbb{C}} V) \int_V \alpha^{\dim_{\mathbb{C}} V} > 0$$

as $\dim_{\mathbb{C}} V = 1$ or 2, and α is a Kähler class. This shows that each W is a codimension one smooth, closed submanifold of U. □

Corollary ([18])**.** *Let X be a compact Kähler manifold of $\dim_{\mathbb{C}} X = 3$. Then, $\mathcal{S}_+(X,J)^{\mathrm{Stab}}$ is an open submanifold of $\mathcal{S}_+(X,J)$.*

Theorem 3.10 ([18])**.** *Let X be a compact Kähler manifold of dimension $\dim_{\mathbb{C}} X = 3$ and $\alpha_1, \dots, \alpha_s \in \mathcal{K}_X$ be a finite collection of Kähler classes. For each $\beta_0 \in \mathcal{K}_X$ such that $\mu_{\alpha_i,\beta_0}\alpha_i - 2\beta_0$ is a big cohomology class for each i, there exists an open subset $U \subseteq \mathcal{K}_X$ containing β_0 and finitely many closed submanifolds $W_1, \dots, W_r$ of U of codimension one such that for each connected component*

$$U_j \subseteq U \setminus \bigcup_k W_k$$

and each $i = 1, \dots, s$, the triple (X, α_i, β) is J-stable for some $\beta \in U_j$ if and only if (X, α_i, β) is stable for all $\beta \in U_j$.

Proof. Just as in the proof of Theorem 3.5, the argument is very nearly identical to the proof of Theorem 3.3, and therefore we omit the details. □

3.2.2 Factorisable generalised Monge-Ampère equations

As a final application of these ideas, we illustrate how our arguments for the J-equation can be generalised to a wider class of PDEs. As in the previous Chapter, we choose the setting of generalised Monge-Ampère equations, all of which satisfy the condition of factorisable introduced in Definition 2.19.

Let X be a smooth projective variety, with $\alpha, \beta \in \mathcal{K}_X$ Kähler classes and $\theta \in \beta$ a fixed Kähler form. Recall that a gMA equation (2.12) is given by data (X, α, Θ) where

$$\Theta = \sum_{k=1}^{n-1} c_k \theta^k + f\theta^n$$

with the $c_k \geq 0$ and f satisfying the cohomological condition (2.10) and the the required positivity condition given in [7, (1.2)]. Recall moreover that the numerical criterion for gMA equations says that the equation (2.12) is solvable precisely when

$$\int_V \exp(\alpha) \cdot (1 - [\Theta]) > 0$$

for all proper irreducible subvarieties of X.

For the sake of clarity, we make another simplifying assumption. We assume that $f = c_n$ is a constant. In this case, the positivity condition [7, (1.2)] simplifies considerably, and is equivalent to demanding that the c_k are not all zero. This includes the case of all inverse Hessian equations.

The space of gMA equations with $f = c_n$ a constant is in a natural way a codimension one closed submanifold of $\mathbb{R}_+[y]_n \times \mathcal{K}_X \times \mathcal{K}_X$ where $\mathbb{R}_+[y]_n$ denotes the set of degree n non-zero real polynomials in one variable y with zero constant term and all of whose coefficients are either zero or strictly positive. (We note that $\mathbb{R}_+[y]_n \times \mathcal{K}_X \times \mathcal{K}_X$ is in a natural way a manifold with corners.) We now explain this correspondence. Let $(P(y), \alpha, \beta) \in \mathbb{R}_+[y]_n \times \mathcal{K}_X \times \mathcal{K}_X$ satisfy the cohomological condition (2.10) given by

$$\int_X \exp(\alpha) \cdot (1 - P(\beta)) = 0.$$

This locus of triples $(P(y), \alpha, \beta)$ is a smooth submanifold because it is the zero locus of the function $F(P(y), \alpha, \beta) = \int_X \exp(\alpha) \cdot (1 - P(\beta))$ and we have

$$\left.\frac{d}{dr}\right|_{r=0} F(P(y) - ry^k, \alpha, \beta)) = \frac{1}{(n-k)!} \int_X \alpha^{n-k} \cdot \beta^k > 0$$

where k is any positive integer such that the coefficient of y^k in $P(y)$ is non-zero. Such a k always exists by assumption. Then, for any choice of Kähler forms $\omega \in \alpha, \theta \in \beta$ the triple $(P(y), \alpha, \beta)$ corresponds to the gMA equation that seeks a smooth $\psi \in \mathcal{H}(\omega)$ such that

$$\exp(\omega_\psi)^{[n,n]} = (\exp(\omega_\psi) \wedge P(\theta))^{[n,n]}, \tag{3.2}$$

where $P(\theta)$ is the multi-degree form given by substituting θ for y in the polynomial $P(y)$. Let us denote by $\mathcal{S}(X, \text{gMA})$ the set of triples $(P(y), \alpha, \beta) \in \mathbb{R}_+[y]_n \times \mathcal{K}_X \times \mathcal{K}_X$ that satisfy the cohomological condition (3.2). Recall that by Proposition 2.22, each triple $(P(y), \alpha, \beta) \in \mathcal{S}(X, \text{gMA})$ defines a factorisable gMA equation. Let us denote by $\tau_p(\alpha, \beta, P(y))$ the associated factor class of degree p, as given by Definition 2.19. We shall denote by $\mathcal{S}_+(X, \text{gMA})$ the subset of those triples $(P(y), \alpha, \beta)$ such that the associated factor classes $\tau_p(\alpha, \beta, P(y))$ of degree p are $(p+1)$-modified Kähler classes. Our aim is to prove a wall structure type statement for the set $\mathcal{S}_+(X, \text{gMA})$.

Lemma 3.11. *For each $p = 1, \ldots, \dim_{\mathbb{C}} X - 1$, the assignment*

$$T : \mathcal{S}(X, \text{gMA}) \to (H^{1,1}(X, \mathbb{R}))^{n-1}, \quad (P(y), \alpha, \beta) \mapsto (\tau_p(\alpha, \beta, P(y)))_{p=1}^{n-1}$$

is continuous.

Proof. Recall that $\tau_p(\alpha, \beta, P(y)) = \alpha - r_p\beta$, where $x - r_p y$ is the linear form given by the factorisation

$$Q_p(x, y) = (x - r_p y)\tilde{Q}_p(x, y).$$

Here $Q_p(x,y)$ is the degree p homogeneous part of the power series $\exp(x)(1-P(y))$, and $\tilde{Q}_p(x,y)$ has non-negative coefficients. By the proof of Proposition 2.22, it follows that r_p is uniquely determined as the largest real root of $h_p(x) = Q_p(x,1)$. But the roots of a polynomial depend continuously on the coefficients. □

Corollary ([18]). *The subset $\mathcal{S}_+(X, \mathrm{gMA})$ is an open subset of the manifold with corners $\mathcal{S}(X, \mathrm{gMA})$.*

Proof. This follows immediately from the Lemma and the fact that $\mathcal{M}_p\mathcal{K}_X$ are open (by definition) in $H^{1,1}(X,\mathbb{R})$. □

Now, let $(P(y), \alpha, \beta) \in \mathcal{S}_+(X, \mathrm{gMA})$ be given. Using Lemma 1.25, we can find open neighbourhoods $U_p \subseteq \mathcal{M}_{p+1}\mathcal{K}_X$ of $\tau_p(\alpha, \beta, P(y))$ and finite sets S_p of p-dimensional irreducible subvarieties of X such that for all $\tau' \in U_p$, all irreducible p-dimensional subvarieties V and all Kähler forms $\omega_1, \ldots, \omega_{p-1}$ whenever we have

$$\int_V \tau_p(\alpha, \beta, P(y)) \cdot [\omega_1] \cdot \cdots \cdot [\omega_{p-1}] \leq 0$$

then $V \in S_p$. Let $U = T^{-1}(U_1 \times U_2 \times \cdots \times U_{n-1})$. Then U is an open neighbourhood of $(P(y), \alpha, \beta)$ in $\mathcal{S}_+(X, \mathrm{gMA})$. In fact, for any Kähler form $\theta \in \beta$, if V is any (gMA) destabiliser for the triple $(X, \alpha, P(\theta))$ then we have

$$\int_V \exp(\alpha) \cdot (1 - P(\beta)) = \int_V \tau_p(\alpha, \beta, P(y)) \cdot \tilde{Q}_p(\alpha, \beta)) \leq 0,$$

but since $\tilde{Q}_p(\alpha, \beta)$ is a non-negative linear combination of products of powers of Kähler classes α and β, we must have

$$\int_V \tau_p(\alpha, \beta) \cdot \alpha^r \cdot \beta^{r-p-1} \leq 0$$

for some $0 \leq r \leq p-1$. But since $\tau_p(\alpha, \beta, P(y)) \in U_p$ we get immediately that this implies that $V \in S_p$.

Let us denote by $\mathcal{S}_+(X, \mathrm{gMA})^{\mathrm{Stab}}$ the locus of those $(P(y), \alpha, \beta)$ such that $(X, \alpha, P(\theta))$ is (gMA) stable for any choice of Kähler form $\theta \in \beta$.

Theorem 3.12 ([18]). *The boundary $\partial\mathcal{S}_+(X, \mathrm{gMA})^{\mathrm{Stab}}$ of $\mathcal{S}_+(X, \mathrm{gMA})^{\mathrm{Stab}}$ in $\mathcal{S}_+(X, \mathrm{gMA})$ is a locally finite union of closed submanifolds W of $\mathcal{S}_+(X, \mathrm{gMA})$ of (real) codimension one, each one of them cut out by an equation of the form*

$$\int_V \exp(\alpha) \cdot (1 - P(\beta)) = 0.$$

Proof. The local finiteness follows from the discussion preceding the statement of the Theorem. The only claim that needs justification is that the boundary loci are submanifolds of codimension one. But they are the zero locus of a function F of the form $F(P(y), \alpha, \beta)) = \int_V \exp(\alpha) \cdot (1 - P(\beta))$. One can easily show that zero is a regular value of this function. □

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