



Topological Singularities in Periodic Media: Ginzburg–Landau and Core-Radius Approaches

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Abstract

We describe the emergence of topological singularities in periodic media within the Ginzburg–Landau model and the core-radius approach. The energy functionals of both models are denoted by $E_{\varepsilon, \delta}$, where ε represent the coherence length (in the Ginzburg–Landau model) or the core-radius size (in the core-radius approach) and δ denotes the periodicity scale. We carry out the Γ -convergence analysis of $E_{\varepsilon, \delta}$ as $\varepsilon \rightarrow 0$ and $\delta = \delta_\varepsilon \rightarrow 0$ in the $|\log \varepsilon|$ scaling regime, showing that the Γ -limit consists in the energy cost of finitely many vortex-like point singularities of integer degree. After introducing the scale parameter

$$\lambda = \min \left\{ 1, \lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \right\}$$

(upon extraction of subsequences), we show that in a sense we always have a separation-of-scale effect: at scales smaller than ε^λ we first have a concentration process around some vortices whose location is subsequently optimized, while for scales larger than ε^λ the concentration process takes place “after” homogenization.

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Introduction

Phase transitions mediated by the formation of topological defects characterize several physical phenomena such as superfluidity, superconductivity and plasticity (see [7, 30, 31, 35–38]). The study of such topological defects has become an extremely active research field in mathematics after the progress achieved in the analysis of the Ginzburg Landau (GL) energy functional in recent decades (see e.g. [13, 42]). In [3] it has been proved that the GL functional, originally introduced to model the phenomenology of phase transitions in Type-II superconductors through the formation of vortex singularities of a complex order parameter, provides a good variational description for the emergence of vortices in XY spin systems and of screw dislocations in crystal plasticity (see also [2, 4, 9, 24, 39]). The results obtained in [3] suggest to exploit the GL theory for a phenomenological alternative description of several material-dependent variational models, opening the way to a number of new mathematical problems involving the analysis of this functional. For instance, in the modeling of materials, one needs to suitably modify it to include the usual kinematic constraints and material constants which are specific of crystal structures. As a first step in this direction, here we study a variant of the GL energy functional to include heterogeneities of the medium.

The Ginzburg–Landau model. Before describing the case of heterogeneous media, we briefly recall the analysis in the homogeneous case. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set and let ε denote the *coherence length* of the GL energy (proportional to the length scale of the core of a screw dislocation in a plastic crystal or to the lattice spacing in a XY spin system). Let $a > 0$ and let $\mathcal{G}\mathcal{L}_\varepsilon : H^1(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}$ be the Ginzburg–Landau functional defined as

$$\mathcal{G}\mathcal{L}_\varepsilon(v) := a \int_\Omega |\nabla v(x)|^2 dx + \frac{1}{\varepsilon^2} \int_\Omega (1 - |v(x)|^2)^2 dx. \tag{0.1}$$

The asymptotic behavior of $\mathcal{G}\mathcal{L}_\varepsilon$ as $\varepsilon \rightarrow 0$ has been studied in order to give an energetic description of the onset of vortices (see for instance [13, 42]). A prototypical vortex of degree $z \in \mathbb{Z} \setminus \{0\}$ at a point $x_0 \in \Omega$ can be thought of as the point singularity of a vectorial order parameter $\bar{v}_\varepsilon : \Omega \rightarrow \mathbb{R}^2$ which, outside the ball of radius ε centered at x_0 , winds around the center as $(\frac{x-x_0}{|x-x_0|})^z$. The energy of \bar{v}_ε diverges at order $|\log \varepsilon|$ as $\varepsilon \rightarrow 0$. As a consequence, to detect the effective energy cost of finitely many vortex singularities, one needs to study the $\mathcal{G}\mathcal{L}_\varepsilon$ energy at a *logarithmic scaling*; that is, to consider the asymptotic behavior of functionals $\frac{\mathcal{G}\mathcal{L}_\varepsilon(v)}{|\log \varepsilon|}$. It has been proved in [1, 33] that a sequence $\{v_\varepsilon\}_\varepsilon$, along which these energy functionals are equi-bounded, has Jacobians Jv_ε that, up to a subsequence, converge in the flat sense (see Section 1) to an atomic measure $\mu = \sum_{i=1}^n z_i \delta_{x_i}$ whose support represents the position of the limiting vortices. The Γ -limit of $\frac{\mathcal{G}\mathcal{L}_\varepsilon}{|\log \varepsilon|}$ at μ with respect to this convergence is then given by $2\pi a \sum_{i=1}^n |z_i|$ (supposing

$x_i \neq x_j$ if $i \neq j$). This value can be rewritten as $2\pi a|\mu|(\Omega)$ and seen as a functional depending on the total variation $|\mu|(\Omega)$ of μ in Ω .

Now, if more in general Ω is regarded as a reference configuration of a heterogeneous material, described by periodic heterogeneities at a length scale δ_ε , we may consider the energies $GL_\varepsilon : H^1(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}$ defined as

$$GL_\varepsilon(v) := \int_\Omega a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla v(x)|^2 dx + \frac{1}{\varepsilon^2} \int_\Omega (1 - |v(x)|^2)^2 dx, \tag{0.2}$$

where $a : \mathbb{R}^2 \rightarrow [\alpha, \beta]$ ($0 < \alpha < \beta$) is a $(0, 1)^2$ -periodic function describing the material properties of the medium. Note that the energy GL_ε is controlled from (above and) below by a multiple of the GL energy \mathcal{GL}_ε (0.1). Therefore, setting

$$X(\Omega) := \left\{ \mu = \sum_{i=1}^n z_i \delta_{x_i} : n \in \mathbb{N}, z_i \in \mathbb{Z} \setminus \{0\}, x_i \in \Omega \right\},$$

the following compactness result holds true:

Theorem 0.1. *Let $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ be such that $GL_\varepsilon(v_\varepsilon) \leq C|\log \varepsilon|$. Then, there exists $\mu \in X(\Omega)$ such that, up to subsequences, $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$.*

Assuming that $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we expect the effective limiting energy at the vortex scaling to be a homogeneous energy combining both homogenization and concentration effects. As these effects depend on the mutual rate of convergence of the vanishing parameters ε and δ_ε , different regimes are possible. Heuristically, at some extreme regimes we will have ‘‘separation of scales’’. For illustrative purposes, assume that $\delta = \delta_\varepsilon = \varepsilon^\lambda$ for some $\lambda \in (0, 1)$. In order to estimate the contribution of the energy at scales between ε and δ , we can roughly assume that, first, $\varepsilon \rightarrow 0$ for δ fixed and, then, $\delta \rightarrow 0$. In such a case, the limit as $\varepsilon \rightarrow 0$ with δ fixed gives an energy of the form $2\pi \sum_{i=1}^n |z_i| a\left(\frac{x_i}{\delta}\right)$. The optimization of the location of vortices at minimum points for a (we may assume here that a is continuous), which tend to be dense as $\delta \rightarrow 0$, finally provides a limit of the form

$$2\pi \min a \sum_{i=1}^n |z_i|.$$

Conversely, at scales $\varepsilon \gg \delta$, we expect that the scaling factor ε can be considered as fixed and a homogenization process may be first performed with $\delta \rightarrow 0$. In this case, moreover, since the potential term in (0.2) forces v_ε to have modulus equal to one as $\varepsilon \rightarrow 0$, (neglecting for a moment the effect of singularities) we may regard the homogenization process to be restricted to the first part of the energy in (0.2), which can be written as

$$G_\delta(u) := \int_\Omega a\left(\frac{x}{\delta}\right) |\nabla u|^2 dx, \tag{0.3}$$

where u is the lifting of v ; i.e., $v = e^{iu}$. The homogenization of functionals of this form has been extensively studied in terms of Γ -convergence (see [19]) and it has

been shown that $G_\delta \xrightarrow{\Gamma} G_0$ as $\delta \rightarrow 0$, where

$$G_0(u) := \int_{\Omega} \langle \mathbb{A}^{\text{hom}} \nabla u, \nabla u \rangle \, dx, \tag{0.4}$$

and \mathbb{A}^{hom} is the two-by-two symmetric matrix defined by

$$\langle \mathbb{A}^{\text{hom}} \xi, \xi \rangle := \inf \left\{ \int_{(0,1)^2} a(y) |\xi + \nabla \varphi(y)|^2 \, dy : \varphi \in W_{\text{per}}^{1,\infty}((0,1)^2) \right\}. \tag{0.5}$$

At this point, the subsequent analysis involves the study of the Γ -limit as $\varepsilon \rightarrow 0$ of a homogeneous but anisotropic energy functional related to G_0 at scale $|\log \varepsilon|$. The validity of this separation of scales can be formalized by using a coarea formula-type argument, which shows that the Γ -convergence of GL_ε can be obtained working within another well-known framework in the analysis of topological singularities; i.e., the so-called *core-radius approach*. The latter approach consists in computing the gradient term in the energy outside small regions—the *cores*—around the singularities, and allows to directly work with \mathcal{S}^1 -valued order parameters (see e.g. [5, 13]), and thus reduces to considering energies (0.8) below. In this framework, we may describe the energy around a vortex of degree z at scales larger than δ_ε by an asymptotic formula of the type

$$\begin{aligned} \psi(z) = \lim_{\frac{R}{r} \rightarrow +\infty} \frac{1}{\log \frac{R}{r}} \min \left\{ \int_{B_R \setminus \overline{B}_r} \langle \mathbb{A}^{\text{hom}} \nabla u, \nabla u \rangle \, dx : u \in SBV^2(B_R \setminus \overline{B}_r), \right. \\ \left. e^{iu} \in H^1(B_R \setminus \overline{B}_r; \mathcal{S}^1), \deg(e^{iu}; B_r) = z \right\}, \end{aligned} \tag{0.6}$$

from which the Γ -limit is obtained by locally optimizing the degree (possibly approximating a vortex by more vortices). A computation eventually allows to conclude that the limit energy has the form

$$2\pi \sqrt{\det \mathbb{A}^{\text{hom}}} \sum_{i=1}^n |z_i|.$$

The final form of the Γ -limit for $\delta_\varepsilon = \varepsilon^\lambda$ with $\lambda \in (0, 1)$ is then

$$2\pi \left((1 - \lambda) \min a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) \sum_{i=1}^n |z_i|;$$

we prove actually that this is the case when

$$\lambda := \lim_{\varepsilon \rightarrow 0} \frac{|\log \delta|}{|\log \varepsilon|} \wedge 1. \tag{0.7}$$

Indeed, we have the following result which is proven in Section 6.2 in the case $\lambda = 1$ and in Section 7.2 for $\lambda \in [0, 1)$:

Theorem 0.2. *The following Γ -convergence result holds true:*

(i) (*Γ -liminf inequality*) Let $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ be such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ for some $\mu \in X(\Omega)$. Then

$$\liminf_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon(v_\varepsilon)}{|\log \varepsilon|} \geq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

(ii) (*Γ -limsup inequality*) For every $\mu \in X(\Omega)$, there exists a sequence $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ and

$$\limsup_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon(v_\varepsilon)}{|\log \varepsilon|} \leq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

Note that, since in the logarithmic regime, the GL_ε energies concentrate at any scale between ε and 1, their behavior is very different from that of the corresponding scalar version, the inhomogeneous Cahn–Hilliard functionals given (after scaling) by

$$CH_\varepsilon(u) := \varepsilon \int_\Omega a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla u(x)|^2 dx + \frac{1}{\varepsilon} \int_\Omega (1 - |u(x)|^2)^2 dx \quad u \in H^1(\Omega),$$

which concentrate at scale ε producing sharp-interface models. In that case separation of scale occurs for $\varepsilon \ll \delta_\varepsilon$ and $\delta_\varepsilon \ll \varepsilon$, while in the *critical regime* $\delta_\varepsilon \sim \varepsilon$ the effective surface tension is described by an optimal-profile problem depending on $K := \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon/\varepsilon$ (see [6]). In a sense, in the GL case we do not have a critical behavior and we always have separation of scales. The parameter λ above can be seen as describing a threshold scale above and below which the two types of separation of scales take place.

The core-radius approach. The analysis in Theorem 0.2, can be provided, and somehow deduced, from the corresponding analysis within the core-radius approach, which amounts to considering energies of the form

$$\mathcal{F}_{\varepsilon, \delta_\varepsilon}(\mu; w) := \int_{\Omega_\varepsilon(\mu)} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla w(x)|^2 dx \tag{0.8}$$

defined for $\mu = \sum_{i=1}^n z_i \delta_{x_i} \in X(\Omega)$ and $w \in \mathcal{AF}_\varepsilon(\mu)$, with

$$\begin{aligned} \mathcal{AF}_\varepsilon(\mu) := \{w \in H^1(\Omega_\varepsilon(\mu); \mathcal{S}^1) : \text{deg}(w, \partial B_\varepsilon(x_i)) = z_i \\ \text{for every } i = 1, \dots, n\}, \end{aligned} \tag{0.9}$$

where $\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{i=1}^n \overline{B}_\varepsilon(x_i)$. As the Jacobians in the GL theory, μ is the relevant parameter to keep track of energy concentration. Therefore, we let the functional depend only on μ by setting

$$\mathcal{F}_\varepsilon(\mu) := \inf_{w \in \mathcal{AF}_\varepsilon(\mu)} \mathcal{F}_{\varepsilon, \delta_\varepsilon}(\mu; w). \tag{0.10}$$

In order to make the core-radius functionals non-trivial, we define \mathcal{F}_ε only on the set

$$X_\varepsilon(\Omega) := \left\{ \mu = \sum_{i=1}^n z_i \delta_{x_i} \in X(\Omega) : \min_{i \neq j} \left\{ \frac{1}{2} |x_i - x_j|, \text{dist}(x_i, \partial\Omega) \right\} \geq 2\varepsilon \right\}.$$

(0.11)

In view of the classical results on the core-radius approach functional (see for instance [5, Theorem 3.2]), the functionals \mathcal{F}_ε satisfy compactness properties analogous to those established in Theorem 0.1.

Theorem 0.3. *Let $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ be such that $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ and that $\mathcal{F}_\varepsilon(\mu_\varepsilon) \leq C|\log \varepsilon|$. Then, there exists $\mu \in X(\Omega)$ such that, up to subsequences, $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$.*

The analog of Theorem 0.2 for the core-radius approach is provided by the next result, which is proven in Section 6.1 for $\lambda = 1$ and in Section 7.1 for $\lambda \in [0, 1)$.

Theorem 0.4. *The following statements hold true:*

- (i) (*Γ -liminf inequality*) *For any family $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ such that $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ and $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ with $\mu \in X(\Omega)$ we have*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \geq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

- (ii) (*Γ -limsup inequality*) *For every $\mu \in X(\Omega)$, there exists a sequence $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ with $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ such that $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \leq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

Within the core-radius approach, in the case $\lambda = 1$ treated in Section 6.1, we carry out the Γ -convergence analysis for more general quadratic functionals than the one in (0.8). Specifically, let $f : \mathbb{R}^2 \times \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$ be a Carathéodory function satisfying the following assumptions:

$$f(\cdot, M) \text{ is } (0, 1)^2\text{-periodic for every } M \in \mathbb{R}^{2 \times 2}; \tag{P}$$

there exist two constants α, β with $0 < \alpha \leq \beta$ such that (G)

$$\alpha|M|^2 \leq f(y, M) \leq \beta|M|^2, \quad \text{for every } M \in \mathbb{R}^{2 \times 2} \text{ and for almost every } y \in \mathbb{R}^2;$$

$$f(y, \cdot) \text{ is homogeneous of degree 2 for almost every } y \in \mathbb{R}^2. \tag{H}$$

We set

$$F_\delta(w; E) := \int_E f\left(\frac{x}{\delta}, \nabla w(x)\right) dx, \quad w \in H^1(E; \mathcal{S}^1), \tag{0.12}$$

and, with a little abuse of notation, we define

$$\mathcal{F}_\varepsilon(\mu) := \inf_{w \in \mathcal{AF}_\varepsilon(\mu)} F_{\delta_\varepsilon}(w; \Omega_\varepsilon(\mu)). \tag{0.13}$$

In [8] it has been proved that, as $\delta \rightarrow 0$, the functionals F_δ Γ -converge to the homogenized functional $F_{\text{hom}}(\cdot; E) : H^1(E; S^1) \rightarrow [0, +\infty)$ defined as

$$F_{\text{hom}}(w; E) := \int_E T f_{\text{hom}}(w(x), \nabla w(x)) \, dx.$$

In the formula above the energy density $T f_{\text{hom}}$ is the tangential homogenization of the function f (see formula (1.6)).

Note that, in view of assumption (G), the analog of Theorem 0.3 holds true also in this case. Next result concerns the asymptotic limit of the functionals \mathcal{F}_ε in (0.13).

Theorem 0.5. *If $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$, then the following Γ -convergence result holds true:*

(i) (*Γ -liminf inequality*) *For any family $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ such that $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ and $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ with $\mu \in X(\Omega)$ we have*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \geq \mathcal{F}_0(\mu).$$

(ii) (*Γ -limsup inequality*) *For every $\mu \in X(\Omega)$, there exists a sequence $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ with $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ such that $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \leq \mathcal{F}_0(\mu).$$

In the statement above $\mathcal{F}_0 : X(\Omega) \rightarrow [0, +\infty)$ is the functional defined as

$$\mathcal{F}_0(\mu) := \sum_{i=1}^n \Psi(z_i; T f_{\text{hom}}) \quad \text{for every } \mu = \sum_{i=1}^n z_i \delta_{x_i} \in X(\Omega), \quad (0.14)$$

where $\Psi(z; T f_{\text{hom}})$, introduced in (2.9), is the asymptotic energy cost of a singularity of degree z in a homogeneous medium whose energy is F_{hom} . The function $\Psi(z; T f_{\text{hom}})$ is obtained via an asymptotic cell-problem formula and a relaxation procedure. Loosely speaking, we first introduce the minimal F_{hom} energy in an annulus around a singularity with degree z and we show that such a quantity admits a finite limit, denoted by $\psi(z; T f_{\text{hom}})$, when the quotient of the radii goes to $+\infty$. Then $\Psi(\cdot; T f_{\text{hom}})$ is obtained as the relaxation of the function $\psi(\cdot; T f_{\text{hom}})$ on \mathbb{Z} (see formula (2.9)), accounting for the fact that a singularity of degree z can be approximated by a family of singularities of degree z_j with $\sum_j z_j = z$. In the simple case $f(x, M) = a(x)|M|^2$ with $a \in [\alpha, \beta]$, we actually prove that $\Psi(z; T f_{\text{hom}}) = 2\pi \sqrt{\det \mathbb{A}^{\text{hom}}}|z|$ (see Proposition 2.3).

Main technical issues. Although suggested by the heuristics, the computation of the Γ -limits described above is highly non-trivial and needs several new ideas in order to combine techniques from GL and homogenization theories. We briefly outline some of the most relevant technical issues.

Using a scaling argument, in Proposition 3.2 we show that $\psi(\cdot; Tf_{\text{hom}})$ is also the asymptotically minimal \mathcal{F}_ε energy on “fat” annuli around a vortex of degree z . Here, “fat” means thick enough to contain infinitely many δ_ε -periodicity cells. Such a property allows to apply the homogenization result in [8] that we show to hold even if the functionals are subject to a degree constraint (see Theorem 1.5 & Corollary 1.6).

A further technical aspect of our analysis is the use, in the proof of the lower bound, of a refinement of the celebrated *ball construction* introduced in [32,40]. This method allows us to find a one-parameter family $\mathcal{B}_\varepsilon(t)$ of growing and merging balls, that in turn identify a family of annuli where the energy concentrates. In our case, using a strategy similar to [25], we stop the process at an appropriate “time” t_ε at which the constructed family of annuli is “fat” enough to apply the analysis described above and to obtain the desired lower bound.

Conclusions and open questions. We conclude the introduction with a few comments and remarks about perspectives. A natural follow-up of our results is the extension of our analysis to GL energies with more general integrands in the leading term as those considered in the core-radius approach. A necessary first step in this direction is the proof of a homogenization result for energies defined on maps taking values in a tubular neighborhood of \mathcal{S}^1 . More specifically, one could relax the \mathcal{S}^1 -constraint in the functionals $F_\delta(\cdot; E)$ in (0.12), assuming the latter to be defined on $H^1(E; B_{1+\tau} \setminus B_{1-\tau})$ for some $\tau \in (0, 1)$, and then study their asymptotic behavior when both δ and τ tend to 0. Another possible extension of our model is the analysis of the case of energy density f satisfying mild coercivity assumptions. This would allow to analyze for instance the problem of topological singularities in presence of soft inclusions of the inhomogeneous material. In this respect, an analysis on the behavior of minimizers of GL functionals in perforated domains has been carried out in [12]. Another challenging issue is to look at a higher-order description of the functionals GL_ε and \mathcal{F}_ε , that in the homogeneous case leads to the so-called *renormalized energy* governing the dynamics of the singularities (see for instance [41] for the GL theory and [4] for discrete models exhibiting topological singularities). In our case of vanishing inhomogeneities we expect the corresponding renormalized energy to depend on the function Tf_{hom} . Furthermore, we believe that some of the techniques developed in this paper can also be used to make progress in studying stochastic homogenization problems in concentration theory, as for instance those in which the energy density f is replaced by a stationary random potential.

We finally note that inhomogeneities in the GL theory can also be introduced in the potential term; e.g., considering energies of the form

$$GL_\varepsilon(v) := \int_\Omega |\nabla v(x)|^2 dx + \frac{1}{\varepsilon^2} \int_\Omega \left(a\left(\frac{x}{\delta_\varepsilon}\right) - |v(x)|^2 \right)^2 dx. \quad (0.15)$$

For some homogenization results for energies (0.15) see [10,11,28,29] and the references therein. The results obtained in those papers differ from ours, since the energy in (0.15) describes a different physical system, namely Type II superconductors in presence of small impurities. Note that a complete study of energies of

the form (0.15) may require a very complex multi-scale analysis even in the scalar case (see e.g. [20,22,27]).

1. Notation and Preliminary Results

Basic notation. Given two vectors $x, y \in \mathbb{R}^2$, $x \cdot y$ denotes their scalar product. As usual, the norm of x is denoted by $|x| = \sqrt{x \cdot x}$. For every $r > 0$ and $x \in \mathbb{R}^2$, $B_r(x)$ denotes the open ball of radius r centered at x . For $x = 0$ we also write B_r in place of $B_r(0)$. S^1 denotes the boundary of B_1 , namely the unit circle in \mathbb{R}^2 . Given $a \in \mathbb{R}$, $\lfloor a \rfloor := \max\{z \in \mathbb{Z} : z \leq a\}$ and $\lceil a \rceil := \min\{z \in \mathbb{Z} : z \geq a\}$ denote the integer parts of a from below and from above, respectively. The imaginary unit is denoted by $\iota \in \mathbb{C}$ and the complex number $e^{ia} = \cos a + \iota \sin a \in \mathbb{C}$ is identified with the Euclidean vector $(\cos a, \sin a) \in \mathbb{R}^2$. The identification extends to all S^1 -valued maps that can be viewed as complex functions as well, if needed. In particular, for every $z \in \mathbb{Z}$, by $(x/|x|)^z$ we mean the complex function obtained by taking the z -th complex power of the function $x/|x|$. We say that a family $\{g_\eta\}_\eta$ converges to g_0 as $\eta \rightarrow 0$ in the topology \mathcal{T} , and we write $g_\eta \xrightarrow{\mathcal{T}} g_0$ whenever $g_{\eta_n} \xrightarrow{\mathcal{T}} g_0$ for any null sequence $\{\eta_n\}_{n \in \mathbb{N}}$. With a little abuse of terminology the family $\{g_\eta\}_\eta$ is still called a sequence. The letter C denotes a positive constant whose value may change each time we write it.

Weak star and flat convergence. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set with Lipschitz boundary. $C_c(\Omega)$ denotes the space of continuous functions compactly supported in Ω endowed with the supremum norm. We say that a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of measures converges weakly star in Ω to a measure μ , and we write $\mu_n \xrightarrow{*} \mu$ if for any $\varphi \in C_c(\Omega)$

$$\langle \mu_n, \varphi \rangle \rightarrow \langle \mu, \varphi \rangle \quad \text{as } n \rightarrow +\infty.$$

$C^{0,1}(\Omega)$ denotes the space of Lipschitz continuous functions on Ω endowed with the norm

$$\|\psi\|_{C^{0,1}} := \sup_{x \in \Omega} |\psi(x)| + \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|\psi(x) - \psi(y)|}{|x - y|},$$

and we let $C_c^{0,1}(\Omega)$ be its subspace of functions with compact support. The norm in the dual of $C_c^{0,1}(\Omega)$ will be denoted by $\|\cdot\|_{\text{flat}}$ and referred to as *flat norm*, while $\xrightarrow{\text{flat}}$ denotes the convergence with respect to this norm.

Jacobian, current, degree. Given $v = (v^1, v^2) \in H^1(\Omega; \mathbb{R}^2)$, the Jacobian Jv of v is the L^1 function defined as follows:

$$Jv := \det \nabla v.$$

For every $v \in H^1(\Omega; \mathbb{R}^2)$, we can interpret Jv as an element of the dual of $C_c^{0,1}(\Omega)$ by setting

$$\langle Jv, \psi \rangle := \int_{\Omega} Jv \psi \, dx, \quad \text{for any } \psi \in C_c^{0,1}(\Omega).$$

Notice that Jv can be written in a divergence form as $Jv = \operatorname{div}(v^1 v_{x_2}^2, -v^1 v_{x_1}^2)$, i.e., for any $\psi \in C_c^{0,1}(\Omega)$,

$$\langle Jv, \psi \rangle = - \int_{\Omega} \left(v^1 v_{x_2}^2 \psi_{x_1} - v^1 v_{x_1}^2 \psi_{x_2} \right) dx. \tag{1.1}$$

Equivalently, we have $Jv = \operatorname{curl}(v^1 \nabla v^2)$ and $Jv = \frac{1}{2} \operatorname{curl} j(v)$, where

$$j(v) := v^1 \nabla v^2 - v^2 \nabla v^1$$

is the so-called *current* associated to v .

Let $A \subset \Omega$ be an open set with Lipschitz boundary, and let $h \in H^{\frac{1}{2}}(\partial A; \mathbb{R}^2)$ with $|h| \geq c > 0$. The *degree* of h is defined as

$$\operatorname{deg}(h, \partial A) := \frac{1}{2\pi} \int_{\partial A} \frac{h}{|h|} \cdot \frac{\partial}{\partial \tau} \left(\frac{h_2}{|h|}, -\frac{h_1}{|h|} \right) d\mathcal{H}^1,$$

where τ is the tangent field to ∂A and the product in the above formula is understood in the sense of the duality between $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$. In [16,21] it is proven that the definition above is well-posed, it is stable with respect to the strong convergence in $H^{\frac{1}{2}}(\partial A; \mathbb{R}^2 \setminus B_c)$ and that $\operatorname{deg}(h, \partial A) \in \mathbb{Z}$. Moreover, if $v \in H^1(A; \mathbb{R}^2 \setminus B_c)$ then $\operatorname{deg}(v, \partial A) = 0$ (here and in what follows we identify v with its trace). Finally, if $v \in H^1(A; \mathbb{R}^2)$ and $|v| = 1$ on ∂A , by Stokes' theorem (and by approximating v with smooth functions) one has that

$$\int_A Jv \, dx = \frac{1}{2} \int_A \operatorname{curl} j(v) \, dx := \frac{1}{2} \int_{\partial A} j(v) \cdot \tau \, d\mathcal{H}^1 = \pi \operatorname{deg}(v, \partial A). \tag{1.2}$$

Note that any $v \in H^1(A; \mathbb{R}^2 \setminus B_c)$ can be written in polar coordinates as $v(x) = \rho(x)e^{iu(x)}$ on ∂A with $|\rho| \geq c$. The function u is said to be a *lifting* of v . By [14] (see also [15, Theorem 3 and Remark 3]), if A is simply connected, then $\operatorname{deg}(v, \partial A) = 0$ and the lifting can be selected in $H^{\frac{1}{2}}(\partial A)$ with the map $v \mapsto u$ continuous. For A not necessarily simply connected, if Γ is a connected component of ∂A and the degree of v on Γ is equal to $z \in \mathbb{Z}$, then the lifting jumps on Γ by $2\pi z$, but it can be locally selected to belong to $H^{\frac{1}{2}}$. For $0 < r < R$ and $\xi \in \mathbb{R}^2$, let $A_{r,R}(\xi) := B_R(\xi) \setminus \overline{B_r}(\xi)$ be the annulus of radii r and R centered at ξ , and let $v \in H^1(A_{r,R}(\xi); \mathcal{S}^1)$. Then for every cut L such that $A_{r,R}(\xi) \setminus L$ is a simply connected set, there exists a lifting $u \in H^1(A_{r,R}(\xi) \setminus L)$ of v . Hence, $j(v) = \nabla u$ and from (1.2) it follows that

$$\operatorname{deg}(v, \partial B_r(\xi)) = \frac{1}{2\pi} \int_{\partial B_r(\xi)} \nabla u \cdot \tau \, d\mathcal{H}^1.$$

We introduce a notion of modified Jacobian (a variant of the notion introduced in [1]), which we will use in our Γ -convergence results. Given $0 < \zeta < 1$ we define for $\rho \in [0, +\infty)$ the function $T_{\zeta}(\rho) := \min\{\frac{\rho}{\zeta}, 1\}$. If $v \in H^1(\Omega; \mathbb{R}^2)$ we set

$$v_{\zeta} := T_{\zeta}(|v|) \frac{v}{|v|} \quad \text{and} \quad J_{\zeta} v := Jv_{\zeta}. \tag{1.3}$$

Note that, for every $v := (v^1, v^2)$ and $w := (w^1, w^2)$ belonging to $H^1(\Omega; \mathbb{R}^2)$ it holds

$$Jv - Jw = \frac{1}{2}(J(v^1 - w^1, v^2 + w^2) - J(v^2 - w^2, v^1 + w^1)). \tag{1.4}$$

Gathering together (1.1) and (1.4) one deduces

Lemma 1.1. *There exists a universal constant $C > 0$ such that for any $v, w \in H^1(\Omega; \mathbb{R}^2)$ it holds that*

$$\|Jv - Jw\|_{\text{flat}} \leq C \|v - w\|_2 (\|\nabla v\|_2 + \|\nabla w\|_2).$$

As a corollary of Lemma 1.1 we obtain the following proposition:

Proposition 1.2. *Let $\{v_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\Omega; \mathbb{R}^2)$ such that $GL_\varepsilon(v_\varepsilon) \leq C |\log \varepsilon|$, and let $\eta \in (0, \frac{1}{2})$. Then there exists $C_\eta > 0$ such that*

$$\sup_{\zeta \in (\eta, 1-\eta)} \|Jv_\varepsilon - J_\zeta v_\varepsilon\|_{\text{flat}} \leq C_\eta \varepsilon |\log \varepsilon|, \quad \sup_{\zeta \in (\eta, 1-\eta)} |J_\zeta v_\varepsilon|(\Omega) \leq C_\eta |\log \varepsilon|.$$

Periodic homogenization of energies defined on S^1 -valued maps. In the following paragraph we state some useful propositions regarding periodic homogenization of energy functionals defined on maps from \mathbb{R}^2 to S^1 . The propositions below have been proven in [8] in the more general case of manifold-valued maps defined on \mathbb{R}^d with $d \in \mathbb{N}$. We specialize them here in the S^1 -version that we exploit in the following sections.

Let $f : \mathbb{R}^2 \times \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$ be a Carathéodory function satisfying assumptions **(P)** and **(G)**. For every $\delta > 0$ and for every open bounded set $E \subset \mathbb{R}^2$ we define the functional $F_\delta(\cdot, E) : L^2(E; \mathbb{R}^2) \rightarrow [0, +\infty]$ as

$$F_\delta(v; E) = \begin{cases} \int_E f\left(\frac{x}{\delta}, \nabla v\right) dx & \text{if } v \in H^1(E; S^1), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

For every $s = (s^1, s^2) \in S^1$ we set $s^\perp = (-s^2, s^1)$ and $\mathcal{T}_s(S^1) = \mathbb{R}s^\perp : \lambda \in \mathbb{R}$ denotes the tangent space of S^1 at the point s . We also introduce the set

$$\mathcal{TS}^1 := \{(s, M) : s \in S^1, M = s^\perp \otimes \xi, \xi \in \mathbb{R}^2\}$$

and for every $(s, M) = (s, s^\perp \otimes \xi) \in \mathcal{TS}^1$ we define

$$\begin{aligned} Tf_{\text{hom}}(s, M) &:= \liminf_{t \rightarrow +\infty} \inf \left\{ \frac{1}{t^2} \int_{tQ} f(y, M + \nabla \phi(y)) dy : \phi \in W_0^{1,\infty}(tQ; \mathcal{T}_s(S^1)) \right\} \\ &= \liminf_{t \rightarrow +\infty} \inf \left\{ \frac{1}{t^2} \int_{tQ} f(y, s^\perp \otimes (\xi + \nabla \varphi(y))) dy : \varphi \in W_0^{1,\infty}(tQ) \right\}, \end{aligned} \tag{1.6}$$

where $Q := (0, 1)^2$. By [8, Proposition 2.1], we have that the definition above is well-posed. The function Tf_{hom} is called the *tangential homogenization* of the function f .

The function Tf_{hom} is a *tangentially quasi-convex* function according to the following definition. We say that a Borel function $h : \mathcal{TS}^1 \rightarrow [0, +\infty)$ is tangentially quasi-convex if for all $(s, M) \in \mathcal{TS}^1$ and all $\varphi \in W_0^{1,\infty}(Q; \mathcal{T}_s(\mathcal{S}^1))$ it holds that

$$h(s, M) \leq \int_Q h(s, M + \nabla\varphi(y)) \, dy. \tag{1.7}$$

We note that the function Tf_{hom} satisfies the following property:

$$\alpha|M|^2 \leq Tf_{\text{hom}}(s, M) \leq \beta|M|^2 \quad \text{for every } (s, M) \in \mathcal{TS}^1. \tag{1.8}$$

Moreover, if $f(x, \cdot)$ is 2-homogeneous for almost every $x \in \mathbb{R}^2$; i.e., if $f(x, \lambda M) = \lambda^2 f(x, M)$ for almost every $x \in \mathbb{R}^2$ and every $M \in \mathbb{R}^{2 \times 2}$, $\lambda \in \mathbb{R}$, then also Tf_{hom} satisfies that

$$Tf_{\text{hom}}(s, \lambda M) = \lambda^2 Tf_{\text{hom}}(s, M) \quad \text{for every } (s, M) \in \mathcal{TS}^1, \lambda \in \mathbb{R}.$$

We define the functional $F_{\text{hom}}(\cdot; E) : L^2(E; \mathbb{R}^2) \rightarrow [0, +\infty]$ as

$$F_{\text{hom}}(v; E) = \begin{cases} \int_E Tf_{\text{hom}}(v(x), \nabla v(x)) \, dx & \text{if } v \in H^1(E; \mathcal{S}^1), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.9}$$

The following theorem has been proven in [8, Theorem 1.1]:

Theorem 1.3. *Let $E \subset \mathbb{R}^2$ be open and bounded and let $\{F_\delta(\cdot; E)\}_\delta$ be the sequence of functionals defined in (1.5). Then, as $\delta \rightarrow 0$, $\{F_\delta(\cdot; E)\}_\delta$ Γ -converge with respect to the strong convergence in $L^2(E; \mathbb{R}^2)$ to the functional $F_{\text{hom}}(\cdot; E)$ in (1.9).*

Remark 1.4. Note that if f is of the form

$$f(y, M) = a(y)|M|^2 \quad \text{for some } Q\text{-periodic measurable function } a : \mathbb{R}^2 \rightarrow [\alpha, \beta], \tag{1.10}$$

then for every $(s, M) = (s, s^\perp \otimes \xi) \in \mathcal{TS}^1$ we have that $f(y, M) = a(y)|\xi|^2$. Therefore, by (1.6) and by standard homogenization results of quadratic forms (see [19, Theorem 14.7]),

$$Tf_{\text{hom}}(s, M) = \langle \mathbb{A}^{\text{hom}} \xi, \xi \rangle,$$

where \mathbb{A}^{hom} is the symmetric matrix defined in (0.5).

For every $0 < r < R$ and for every $x \in \mathbb{R}^2$ we set $A_{r,R}(x) := B_R(x) \setminus \overline{B_r}(x)$ and $A_{r,R} := A_{r,R}(0)$. Moreover, for every $z \in \mathbb{Z} \setminus \{0\}$ we define

$$\mathcal{A}_{r,R}(z) := \{w \in H^1(A_{r,R}; \mathcal{S}^1) : \text{deg}(w, \partial B_r) = z\}.$$

Given $z \in \mathbb{Z} \setminus \{0\}$, for every $\delta > 0$ we define the functionals $F_\delta^z(\cdot; A_{r,R}) : H^1(A_{r,R}; \mathbb{R}^2) \rightarrow [0, +\infty]$ as

$$F_\delta^z(v; A_{r,R}) := \begin{cases} F_\delta(v; A_{r,R}) & \text{if } v \in \mathcal{A}_{r,R}(z), \\ +\infty & \text{otherwise,} \end{cases}$$

and $F_{\text{hom}}^z : H^1(A_{r,R}; \mathbb{R}^2) \rightarrow [0, +\infty]$ as

$$F_{\text{hom}}^z(v; A_{r,R}) := \begin{cases} F_{\text{hom}}(v; A_{r,R}) & \text{if } v \in \mathcal{A}_{r,R}(z). \\ +\infty & \text{otherwise,} \end{cases} \quad (1.11)$$

The next result is a consequence of Theorem 1.3.

Theorem 1.5. *Let $z \in \mathbb{Z} \setminus \{0\}$ and let $F_\delta^z(\cdot; A_{r,R})$ be the functional defined in (1.11)s. Then $F_\delta^z(\cdot; A_{r,R})$ Γ -converge with respect to the strong convergence in $L^2(A_{r,R}; \mathbb{R}^2)$ to the functional $F_{\text{hom}}^z(\cdot; A_{r,R})$ as $\delta \rightarrow 0$.*

Proof. By coercivity, the desired Γ -convergence is equivalent to the Γ -convergence in Theorem 1.3 performed with respect to the weak topology in $H^1(A_{r,R}; \mathbb{R}^2)$. It is then enough to prove that the constraint $\text{deg}(v, \partial B_r) = z$ is closed with respect to that convergence. Let $\{v_\delta\}_\delta \subset \mathcal{A}_{r,R}(z)$ be such that $v_\delta \rightharpoonup v_0$ in $H^1(A_{r,R}; \mathbb{R}^2)$ for some $v_0 \in H^1(A_{r,R}; S^1)$. By standard Fubini arguments, for almost every $r < \rho < R$, we have that the trace of w_δ on ∂B_ρ is bounded in $H^1(\partial B_\rho; S^1)$ and hence (up to a not relabeled subsequence) it weakly converges to a function g_ρ . Since $\|v_\delta - v_0\|_{L^2(A_{r,R}; \mathbb{R}^2)} \rightarrow 0$, we get that $g_\rho = v_0$ for a.e. $\rho \in (r, R)$. By the very definition of degree in (1.2), $\text{deg}(v_0, \partial B_r) = z$ and hence $v_0 \in \mathcal{A}_{r,R}(z)$. \square

The following corollary holds true as a consequence of (G), (1.8), Theorem 1.5 and thanks to the property of convergence of minima in Γ -convergence (see [17, 18, 23]):

Corollary 1.6. *Let $z \in \mathbb{Z} \setminus \{0\}$. Then, for every $0 < r < R$, it holds that*

$$\lim_{\delta \rightarrow 0} \inf_{w \in \mathcal{A}_{r,R}(z)} F_\delta(w; A_{r,R}) = \min_{w \in \mathcal{A}_{r,R}(z)} F_{\text{hom}}(w; A_{r,R}).$$

2. The Effective Energy of a Singularity

In this section we introduce and discuss the properties of the minimal energy cost $\Psi(z; h)$ of a vortex like singularity of degree z for a homogeneous quadratic functional of energy density h defined on S^1 -valued maps. The function $\Psi(\cdot; h)$ is crucial in order to determine the Γ -limits for both the cases $\delta_\varepsilon \lesssim \varepsilon$ and $\delta_\varepsilon \gg \varepsilon$, choosing $h = Tf_{\text{hom}}$, with Tf_{hom} defined in (1.6). On the one hand (see Section 7) for $\delta_\varepsilon \lesssim \varepsilon$, $\Psi(z; Tf_{\text{hom}})$ turns out to be the effective energy cost of a singularity of degree z (see Theorems 6.1 and 6.2). On the other hand (see Section 7) for $\delta_\varepsilon \gg \varepsilon$, recalling the definition of λ in (0.7), we have that $\lambda\Psi(z; Tf_{\text{hom}})$ is the effective energy cost of a singularity of degree z on scales of order between δ_ε and 1 (see Theorems 7.1 and 7.2).

Let $h : \mathcal{TS}^1 \rightarrow [0, +\infty)$ be a continuous function, tangentially quasi-convex according to (1.7), and such that

$$h(s, \lambda M) = \lambda^2 h(s, M), \quad \text{for every } (s, M) \in \mathcal{TS}^1 \text{ and } \forall \lambda \in \mathbb{R}. \quad (2.1)$$

Assume, moreover, that there exist α, β such that $0 < \alpha \leq \beta$ and

$$\alpha|M|^2 \leq h(s, M) \leq \beta|M|^2, \quad \text{for every } (s, M) \in \mathcal{TS}^1. \quad (2.2)$$

For every open bounded set $E \subset \mathbb{R}^2$ we define the functional $H(\cdot; E) : H^1(E; S^1) \rightarrow [0, +\infty)$ as

$$H(w; E) := \int_E h(w(x), \nabla w(x)) \, dx.$$

Given $z \in \mathbb{Z} \setminus \{0\}$ and $0 < r < R$ we set

$$\psi_{r,R}(z; h) := \frac{1}{\log \frac{R}{r}} \min_{w \in \mathcal{A}_{r,R}(z)} H(w; A_{r,R}). \tag{2.3}$$

Making the change of variable $y = \frac{x}{r}$ and considering the 2-homogeneity (2.1) of the function h we conclude that, for every $w \in H^1(A_{r,R}; S^1)$, the following relation holds:

$$\begin{aligned} H(w; A_{r,R}) &= \int_{A_{r,R}} h(w(x), \nabla w(x)) \, dx = \int_{A_{1,\frac{R}{r}}} h(\hat{w}(y), \nabla \hat{w}(y)) \, dy \\ &= H(\hat{w}; A_{1,\frac{R}{r}}). \end{aligned} \tag{2.4}$$

Here $\hat{w}(y) := w(ry)$. Gathering together (2.4) and (2.3) we deduce that

$$\psi_{r,R}(z; h) = \psi_{1,\frac{R}{r}}(z; h). \tag{2.5}$$

Proposition 2.1. *Let $h : TS^1 \rightarrow [0, +\infty)$ be a Carathéodory function satisfying (2.2) and (2.1). For $z \in \mathbb{Z} \setminus \{0\}$ and $0 < r < R$ let $\psi_{r,R}(z, h)$ be the function defined in (2.3). Then there exists the limit*

$$\psi(z; h) := \lim_{\frac{R}{r} \rightarrow +\infty} \psi_{r,R}(z; h). \tag{2.6}$$

Proof. In view of (2.5), it is enough to prove the inequality

$$\limsup_{R \rightarrow +\infty} \psi_{1,R}(z; h) \leq \liminf_{R \rightarrow +\infty} \psi_{1,R}(z; h). \tag{2.7}$$

For $\rho \in \mathbb{R}$ with $1 < \rho < R$ we define $K_{R,\rho} := \lfloor \frac{\log R}{\log \rho} \rfloor$ and note that

$$A_{1,R} \supset \bigcup_{k=1}^{K_{R,\rho}} A_{\rho^{k-1}, \rho^k}.$$

Denoting by w_R a minimizer of (2.3), letting $\bar{k} = \bar{k}_{R,\rho} \in \{1, \dots, K_{R,\rho}\}$ be such that

$$H(w_R; A_{\rho^{\bar{k}-1}, \rho^{\bar{k}}}) \leq H(w_R; A_{\rho^{k-1}, \rho^k}) \quad \text{for all } k = 1, \dots, K_{R,\rho},$$

and setting $\hat{w}_{\rho, \bar{k}}(y) := w_R(\rho^{\bar{k}-1}y)$, we obtain

$$\begin{aligned} \min_{w \in \mathcal{A}_{1,R}(z)} H(w; A_{1,R}) &\geq \sum_{k=1}^{K_{R,\rho}} H(w_R; A_{\rho^{k-1}, \rho^k}) \geq K_{R,\rho} H(w_R; A_{\rho^{\bar{k}-1}, \rho^{\bar{k}}}) \\ &= K_{R,\rho} H(\hat{w}_{\rho, \bar{k}}; A_{1,\rho}), \end{aligned}$$

where the last equality follows by (2.1). By the definition of $K_{R,\rho}$ we conclude that

$$\psi_{1,R}(z; h) \geq K_{R,\rho} \frac{\log \rho}{\log R} \psi_{1,\rho}(z; h) \geq \left(1 - \frac{\log \rho}{\log R}\right) \psi_{1,\rho}(z; h).$$

The inequality above yields (2.7) on taking first the limit as $R \rightarrow +\infty$ and then as $\rho \rightarrow +\infty$. \square

Note that

$$2\pi\alpha|z|^2 \leq \psi(z; h) \leq 2\pi\beta|z|^2 \quad \text{for every } z \in \mathbb{Z}, \tag{2.8}$$

where α and β are the constants appearing in (2.2). We define the function $\Psi(\cdot; h) : \mathbb{Z} \rightarrow [0, +\infty)$ as

$$\Psi(z; h) := \inf \left\{ \sum_{l=1}^L \psi(z_l; h) : \sum_{l=1}^L z_l = z, L \in \mathbb{N}, z_l \in \mathbb{Z} \right\}. \tag{2.9}$$

Remark 2.2. It follows from (2.8) that the infimum in problem (2.9) is actually a minimum and

$$2\pi\alpha|z| \leq \Psi(z; h) \leq 2\pi\beta|z| \quad \text{for every } z \in \mathbb{Z}. \tag{2.10}$$

Moreover, by definition, the function $\Psi(\cdot; h)$ is sub-additive; i.e.,

$$\Psi(z_1 + z_2; h) \leq \Psi(z_1; h) + \Psi(z_2; h) \quad \text{for every } z_1, z_2 \in \mathbb{Z}.$$

Such a property implies that the functional $\mathcal{F}(\cdot; h) : X(\Omega) \rightarrow [0, +\infty)$ defined by

$$\mathcal{F}(\mu; h) := \sum_{i=1}^n \Psi(z_i; h) \quad \text{for every } \mu = \sum_{i=1}^n z_i \delta_{x_i}$$

is lower semi-continuous with respect to the flat convergence, while (2.10) yields

$$2\pi\alpha|\mu|(\Omega) \leq \mathcal{F}(\mu; h) \leq 2\pi\beta|\mu|(\Omega).$$

In the next proposition we show that if f is of the form in (1.10), then $\Psi(z; Tf_{\text{hom}})$ equals $|z|$ up to a constant pre-factor.

Proposition 2.3. *Let $f : \mathbb{R}^2 \times \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$ satisfy (1.10). Then*

$$\Psi(z; Tf_{\text{hom}}) = 2\pi\sqrt{\det \mathbb{A}^{\text{hom}}}|z| \quad \text{for every } z \in \mathbb{Z}, \tag{2.11}$$

where \mathbb{A}^{hom} is defined in (0.5).

Proof. For $r, R \in \mathbb{R}, 0 < r < R$, let $L := \{(0, x_2) : -R \leq x_2 \leq -r\}$ be a cut of the annulus $A_{r,R}$. Then the domain $A_{r,R} \setminus L$ is simply connected. We set

$$\mathcal{A}_{r,R}^L(z) := \{u \in SBV^2(A_{r,R}) \cap H^1(A_{r,R} \setminus L) : e^{iu} \in \mathcal{A}_{r,R}(z)\}.$$

By Remark 1.4 and by (2.3) we have

$$\begin{aligned} \psi_{r,R}(z; T f_{\text{hom}}) &= \frac{1}{\log \frac{R}{r}} \min_{u \in \mathcal{A}_{r,R}^L(z)} \int_{A_{r,R}} \langle \mathbb{A}^{\text{hom}} \nabla u(x), \nabla u(x) \rangle \, dx \\ &= \frac{1}{\log \frac{R}{r}} \min_{u \in \mathcal{A}_{r,R}^L(z)} \int_{A_{r,R}} |\sqrt{\mathbb{A}^{\text{hom}}} \nabla u(x)|^2 \, dx, \end{aligned} \tag{2.12}$$

where the last equality follows from the fact that \mathbb{A}^{hom} is symmetric and hence so is $\sqrt{\mathbb{A}^{\text{hom}}}$. Setting $\hat{u}(y) := u(\sqrt{\mathbb{A}^{\text{hom}}}y)$, we have that $\nabla \hat{u}(y) := \sqrt{\mathbb{A}^{\text{hom}}} \nabla u(\sqrt{\mathbb{A}^{\text{hom}}}y)$. Thus the change of variables $x = \sqrt{\mathbb{A}^{\text{hom}}}y$ in (2.12) yields

$$\psi_{r,R}(z; T f_{\text{hom}}) = \frac{\sqrt{\det \mathbb{A}^{\text{hom}}}}{\log \frac{R}{r}} \min_{\hat{u} \in \mathcal{A}_{r,R}^L(z)} \int_{(\sqrt{\mathbb{A}^{\text{hom}}})^{-1}(A_{r,R})} |\nabla \hat{u}(y)|^2 \, dy, \tag{2.13}$$

where we have set

$$\begin{aligned} \mathcal{A}_{r,R}^L(z) &:= \left\{ \hat{u} \in SBV^2((\sqrt{\mathbb{A}^{\text{hom}}})^{-1}(A_{r,R})) \cap H^1((\sqrt{\mathbb{A}^{\text{hom}}})^{-1}(A_{r,R} \setminus L)) : \right. \\ &\quad \left. \hat{u} = u \circ \sqrt{\mathbb{A}^{\text{hom}}}, \quad u \in \mathcal{A}_{r,R}^L(z) \right\}. \end{aligned}$$

For sufficiently large R/r there exist $0 < \lambda < \Lambda$ that depend only on \mathbb{A}^{hom} and do not depend on r and R such that $A_{\lambda r, \Lambda R} \subset (\sqrt{\mathbb{A}^{\text{hom}}})^{-1}(A_{r,R})$ so that, by (2.13),

$$\begin{aligned} \psi_{r,R}(z; T f_{\text{hom}}) &= \frac{\sqrt{\det \mathbb{A}^{\text{hom}}}}{\log \frac{R}{r}} \min_{u \in \mathcal{A}_{\lambda r, \Lambda R}^L(z)} \int_{A_{\lambda r, \Lambda R}} |\nabla \hat{u}(y)|^2 \, dy + O\left(\frac{1}{\log \frac{R}{r}}\right) \\ &= 2\pi \sqrt{\det \mathbb{A}^{\text{hom}}} |z|^2 + O\left(\frac{1}{\log \frac{R}{r}}\right). \end{aligned}$$

It follows that $\psi(z) = 2\pi \sqrt{\det \mathbb{A}^{\text{hom}}} |z|^2$, whence (2.11) follows from the definition of Ψ in (2.9). \square

3. Asymptotic Analysis on Annuli

In this section we prove some auxiliary results on the asymptotic behavior of the minimal energy on an annulus when its inner and outer radii are powers of ε . Such results will be crucial in the proofs of the Γ -convergence theorems in Sections 7 and 7. The next lemma states that the minimum in (2.3) for $H = F_\delta$ changes by at most a multiple of z^2 if the competitors are chosen with fixed trace $(x/|x|)^z$ instead

of fixed degree z , thus belonging to a new appropriate set of admissible functions defined as

$$\tilde{\mathcal{A}}_{r,R}(z) := \left\{ w \in \mathcal{A}_{r,R}(z) : w(x) = \left(\frac{x}{|x|} \right)^z \text{ on } \partial B_r \cup \partial B_R \right\}. \tag{3.1}$$

Lemma 3.1. *Let $0 < 2r \leq R$ and let $F_\delta(\cdot, A_{r,R})$ be the functional defined in (1.5) for $E = A_{r,R}$ and with f satisfying condition (G). Then, there exists a constant $\bar{C} = \bar{C}(\alpha, \beta) > 0$ such that, for every $z \in \mathbb{Z} \setminus \{0\}$,*

$$\inf_{w \in \tilde{\mathcal{A}}_{r,R}(z)} F_\delta(w; A_{r,R}) \leq \inf_{w \in \mathcal{A}_{r,R}(z)} F_\delta(w; A_{r,R}) \leq \inf_{w \in \mathcal{A}_{r,R}(z)} F_\delta(w; A_{r,R}) + \bar{C}z^2. \tag{3.2}$$

Proof. The first inequality in (3.2) follows from the inclusion $\tilde{\mathcal{A}}_{r,R}(z) \subset \mathcal{A}_{r,R}(z)$. Hence, it is enough to show that for every $w \in \mathcal{A}_{r,R}(z)$ there exists $\widehat{w} \in \tilde{\mathcal{A}}_{r,R}(z)$ such that

$$F_\delta(\widehat{w}; A_{r,R}) \leq F_\delta(w; A_{r,R}) + \bar{C}z^2, \tag{3.3}$$

for some constant \bar{C} depending only on the constants α and β in (G). Set $K := \lfloor \frac{\log R - \log r}{\log 2} \rfloor$ and $A_k := A_{2^{k-1}r, 2^k r}$ for $k = 1, 2, \dots, K$. We have that

$$A_{r,R} = \bigcup_{k=1}^K A_k \cup A_{2^K r, R}.$$

Since $K \geq \frac{\log R - \log r}{\log 2} - 1$, in view of (G), we notice that

$$F_\delta\left(\left(\frac{x}{|x|}\right)^z; A_{2^K r, R}\right) \leq \beta z^2 \int_{A_{2^K r, R}} \frac{1}{|x|^2} dx = 2\pi\beta z^2 \log \frac{R}{2^K r} \leq 2\pi\beta z^2 \log 2. \tag{3.4}$$

We first consider the case where there is at most one annulus A_k such that

$$F_\delta\left(\left(\frac{x}{|x|}\right)^z; A_k\right) \geq F_\delta(w; A_k). \tag{3.5}$$

Then, in view of (G), we have

$$F_\delta\left(\left(\frac{x}{|x|}\right)^z; A_k\right) \leq \beta z^2 \int_{A_k} \frac{1}{|x|^2} dx = 2\pi\beta z^2 \log 2,$$

whence, using also (3.4), we deduce $F_\delta\left(\left(\frac{x}{|x|}\right)^z; A_{r,R}\right) \leq F_\delta(w; A_{r,R}) + \bar{C}z^2$, which proves (3.3) for $\widehat{w}(x) = (x/|x|)^z$.

From now on, we can assume that (3.5) is satisfied by at least two of the annuli A_k . We let k_1 and k_2 denote the smallest and the largest $k \in \{1, \dots, K\}$ satisfying (3.5). Let moreover $L := \{(0, x_2) : -R \leq x_2 \leq -r\}$ be a cut of the annulus $A_{r,R}$ such that the domain $A_{r,R} \setminus L$ is simply connected. By [14], there exists a lifting

$u \in H^1(A_{r,R} \setminus L; \mathbb{R})$ of w in $A_{r,R} \setminus L$. Moreover, since $\deg(w, \partial B_\rho) = z$ for every $\rho \in [r, R]$, we have that the function u jumps by $2\pi z$ across L . By the properties of the lifting,

$$\|\nabla u\|_{L^2(E; \mathbb{R}^2)} = \|\nabla w\|_{L^2(E; \mathbb{R}^{2 \times 2})} \quad \text{for every open set } E \subset A_{r,R}. \quad (3.6)$$

Furthermore, setting

$$\theta(x) := \begin{cases} \arctan \frac{x_2}{x_1} & \text{if } x_1 > 0, \\ \frac{\pi}{2} & \text{if } x_1 = 0, x_2 > 0, \\ \pi + \arctan \frac{x_2}{x_1} & \text{if } x_1 < 0, \\ \frac{3}{2}\pi & \text{if } x_1 = 0, x_2 < 0, \end{cases} \quad (3.7)$$

for every $x \in \mathbb{R}^2 \setminus \{0\}$, the function $z\theta \in SBV^2(A_{r,R})$ is a lifting of $(\frac{x}{|x|})^z$. Using the complex notation we set $\widehat{w} := e^{i\widehat{u}}$, where the lifting \widehat{u} is defined as

$$\widehat{u}(x) := \begin{cases} z\theta(x) & \text{if } r \leq |x| \leq 2^{k_1-1}r, \\ (1 - \sigma_1(|x|))z\theta(x) + \sigma_1(|x|)u(x) & \text{if } 2^{k_1-1}r \leq |x| \leq 2^{k_1}r, \\ u(x) & \text{if } 2^{k_1}r \leq |x| \leq 2^{k_2-1}r, \\ \sigma_2(|x|)z\theta(x) + (1 - \sigma_2(|x|))u(x) & \text{if } 2^{k_2-1}r \leq |x| \leq 2^{k_2}r, \\ z\theta(x) & \text{if } 2^{k_2}r \leq |x| \leq R. \end{cases} \quad (3.8)$$

In the formula above, for $i = 1, 2$ the function $\sigma_i : [2^{k_i-1}r, 2^{k_i}r] \rightarrow [0, 1]$ is defined by

$$\sigma_i(\rho) := \frac{1}{2^{k_i-1}r}(\rho - 2^{k_i-1}r)$$

and satisfies $\sigma'_i(\rho) = \frac{1}{2^{k_i-1}r}$ and $\|\sigma_i\|_{L^\infty} \leq 1$.

Note that $\widehat{w} \in \widetilde{\mathcal{A}}_{r,R}(z)$. By the definition of k_1 and by (3.4), we have that

$$\begin{aligned} F_\delta(\widehat{w}; A_{r,2^{k_1-1}r} \cup A_{2^{k_2}r,R}) &= F_\delta\left(\left(\frac{x}{|x|}\right)^z; A_{r,2^{k_1-1}r} \cup A_{2^{k_2}r,R}\right) \\ &\leq F_\delta(w; A_{r,2^{k_1-1}r} \cup A_{2^{k_2}r,R}) + Cz^2. \end{aligned}$$

Therefore, in view of (3.8) it is enough to prove that the energy of \widehat{w} on A_{k_1} and A_{k_2} is bounded from above by $\bar{C}|z|^2$ for some constant $\bar{C} > 0$ depending only on α and β . We prove this fact only for the annulus A_{k_1} , being the proof for A_{k_2} similar. To this end, note that in A_{k_1} one has that

$$\begin{aligned} |\nabla \widehat{w}|^2 &= |\nabla \widehat{u}|^2 \leq 3|\sigma'_1|^2|u - z\theta|^2 + 3|\sigma_1|^2|\nabla u - z\nabla\theta|^2 + 3z^2|\nabla\theta|^2 \\ &\leq \frac{C}{2^{2(k_1-1)}r^2}(|u|^2 + z^2|\theta|^2) + C(|\nabla u|^2 + z^2|\nabla\theta|^2). \end{aligned} \quad (3.9)$$

Set $l_{k_1} := \int_{A_{k_1}} u \, dx$. Up to adding an integer multiple of 2π , we can always assume that $|l_{k_1}| \leq 2\pi$ and estimate $\|l_{k_1}\|_{L^2(A_{k_1})}^2 \leq (2\pi)^2 2^{2k_1} r^2$. Hence

$$\begin{aligned} \|u\|_{L^2(A_{k_1})}^2 &\leq 2\|u - l_{k_1}\|_{L^2(A_{k_1})}^2 + 2\|l_{k_1}\|_{L^2(A_{k_1})}^2 \\ &\leq C2^{2k_1} r^2 \|\nabla u\|_{L^2(A_{k_1}; \mathbb{R}^2)}^2 + C2^{2k_1} r^2 \\ &\leq C2^{2k_1} r^2 z^2 \|\nabla \theta\|_{L^2(A_{k_1}; \mathbb{R}^2)}^2 + C2^{2k_1} r^2 \leq Cz^2 2^{2k_1} r^2, \end{aligned} \tag{3.10}$$

where the second inequality is a consequence of the Poincaré-Wirtinger inequality applied to the domain $A_{k_1} \setminus L$ (notice that this can be done up to passing to a diffeomorphic rectangle), and the third inequality follows on gathering together (3.5), (3.6), and (G). Note that all the constants appearing in (3.10) depend only on α and β . By integrating (3.9) and using (3.10), (3.5), (3.6) and (G), we deduce that

$$\begin{aligned} F_\delta(\widehat{w}; A_{k_1}) &\leq C\|\nabla \widehat{w}\|_{L^2(A_{k_1}; \mathbb{R}^{2 \times 2})}^2 \\ &\leq \frac{C}{2^{2(k_1-1)} r^2} \left(\|u\|_{L^2(A_{k_1})}^2 + z^2 \|\theta\|_{L^2(A_{k_1})}^2 \right) \\ &\quad + C \left(\|\nabla u\|_{L^2(A_{k_1}; \mathbb{R}^2)}^2 + z^2 \|\nabla \theta\|_{L^2(A_{k_1}; \mathbb{R}^2)}^2 \right) \\ &\leq \frac{C2^{2k_1} z^2}{2^{2(k_1-1)}} + Cz^2 \|\nabla \theta\|_{L^2(A_{k_1}; \mathbb{R}^2)}^2 =: \bar{C}z^2, \end{aligned}$$

thus concluding the proof of (3.3). □

In the next proposition we show that in the $|\log \varepsilon|$ regime, to some extent, the homogenization process commutes with the minimization process defining $\psi(z; Tf_{\text{hom}})$.

Proposition 3.2. *Let F_{δ_ε} be defined in (0.12) with f satisfying assumptions (P), (G), (H), and let Tf_{hom} be defined in (1.6). Then for any s_1 and s_2 such that $0 \leq s_1 < s_2 < 1$ and $\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon^{s_2}} = 0$ we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) = (s_2 - s_1)\psi(z; Tf_{\text{hom}}), \tag{3.11}$$

where $\psi(z; Tf_{\text{hom}})$ is the function defined in (2.6) with $h = Tf_{\text{hom}}$.

Proof. We first show that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) \geq (s_2 - s_1) \psi(z; T f_{\text{hom}}). \quad (3.12)$$

To this end, we fix $R > 1$, set $K_{\varepsilon, R} = \lfloor (s_2 - s_1) \frac{|\log \varepsilon|}{\log R} \rfloor$, and note that $A_{\varepsilon^{s_2}, \varepsilon^{s_1}} \supset \bigcup_{k=1}^{K_{\varepsilon, R}} A_{R^{k-1} \varepsilon^{s_2}, R^k \varepsilon^{s_2}}$. Let moreover $w_\varepsilon \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)$ be such that

$$F_{\delta_\varepsilon}(w_\varepsilon; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) \leq \inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) + C, \quad (3.13)$$

for some constant C (independent of ε) and let $\bar{k} = \bar{k}_{\varepsilon, R} \in \{1, \dots, K_{\varepsilon, R}\}$ be such that

$$F_{\delta_\varepsilon}(w_\varepsilon; A_{R^{\bar{k}-1} \varepsilon^{s_2}, R^{\bar{k}} \varepsilon^{s_2}}) \leq F_{\delta_\varepsilon}(w_\varepsilon; A_{R^{k-1} \varepsilon^{s_2}, R^k \varepsilon^{s_2}}), \quad \text{for all } k = 1, \dots, K_{\varepsilon, R}.$$

Therefore

$$F_{\delta_\varepsilon}(w_\varepsilon; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) \geq \sum_{k=1}^{K_{\varepsilon, R}} F_{\delta_\varepsilon}(w_\varepsilon; A_{R^{k-1} \varepsilon^{s_2}, R^k \varepsilon^{s_2}}) \geq K_{\varepsilon, R} F_{\delta_\varepsilon}(w_\varepsilon; A_{R^{\bar{k}-1} \varepsilon^{s_2}, R^{\bar{k}} \varepsilon^{s_2}}). \quad (3.14)$$

By the change of variable $y = R^{1-\bar{k}} \varepsilon^{-s_2} x$, $w'_{\varepsilon, \bar{k}}(y) := w_\varepsilon(R^{\bar{k}-1} \varepsilon^{s_2} y)$ and by property **(H)**, we have

$$F_{\delta_\varepsilon}(w_\varepsilon; A_{R^{\bar{k}-1} \varepsilon^{s_2}, R^{\bar{k}} \varepsilon^{s_2}}) = F_{\delta_\varepsilon R^{1-\bar{k}} \varepsilon^{-s_2}}(w'_{\varepsilon, \bar{k}}; A_{1, R}). \quad (3.15)$$

Therefore, since by assumption $\delta_\varepsilon R^{1-\bar{k}} \varepsilon^{-s_2} \leq \delta_\varepsilon \varepsilon^{-s_2} \rightarrow 0$ as $\varepsilon \rightarrow 0$, by using (3.13), (3.14), (3.15), and Corollary 1.6, we deduce that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) &\geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_\varepsilon; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \frac{K_{\varepsilon, R}}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{1, R}(z)} F_{\delta_\varepsilon R^{1-\bar{k}} \varepsilon^{-s_2}}(w; A_{1, R}) \\ &\geq \lim_{\varepsilon \rightarrow 0} \left(\frac{s_2 - s_1}{\log R} - \frac{1}{|\log \varepsilon|} \right) \liminf_{\varepsilon \rightarrow 0} \inf_{w \in \mathcal{A}_{1, R}(z)} F_{\delta_\varepsilon R^{1-\bar{k}} \varepsilon^{-s_2}}(w; A_{1, R}) \\ &= \frac{s_2 - s_1}{\log R} \min_{w \in \mathcal{A}_{1, R}(z)} F_{\text{hom}}(w; A_{1, R}) \\ &= (s_2 - s_1) \psi_{1, R}(z; T f_{\text{hom}}). \end{aligned}$$

Formula (3.12) follows from the estimate above as $R \rightarrow +\infty$ thanks to Proposition 2.1 applied to $h = T f_{\text{hom}}$. To conclude the proof of (3.11) we are left to show that

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) \leq (s_2 - s_1)\psi(z; T f_{\text{hom}}). \tag{3.16}$$

To this end, we take $R > 1$ and set $J_{\varepsilon, R} := \lceil (s_2 - s_1) \frac{|\log \varepsilon|}{\log R} \rceil$. We observe that

$$\inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) \leq \sum_{j=1}^{J_{\varepsilon, R}} \inf_{w \in \widetilde{\mathcal{A}}_{R^{j-1}\varepsilon^{s_2}, R^j\varepsilon^{s_2}}(z)} F_{\delta_\varepsilon}(w; A_{R^{j-1}\varepsilon^{s_2}, R^j\varepsilon^{s_2}}). \tag{3.17}$$

We also note that for every $R > 1$, thanks to Corollary 1.6, there exists a modulus of continuity ω such that

$$\inf F_\delta(w; A_{1, R}) \leq \min F_{\text{hom}}(w; A_{1, R}) + \omega(\delta).$$

We set $\delta_{\varepsilon, j} := \frac{\delta_\varepsilon}{R^{j-1}\varepsilon^{s_2}}$ for every $j = 1, \dots, J_{\varepsilon, R}$ and $\omega(\delta_{\varepsilon, j}) := \max_{j=1, \dots, J_{\varepsilon, R}} \omega(\delta_{\varepsilon, j})$. Note that $\delta_{\varepsilon, j} \leq \delta_{\varepsilon, 1} = \frac{\delta_\varepsilon}{\varepsilon^{s_2}} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, $\omega(\delta_{\varepsilon, j})$ depends only on ε and R and $\omega(\delta_{\varepsilon, j})$ vanishes as $\varepsilon \rightarrow 0$. For every $j = 1, \dots, J_{\varepsilon, R}$, using the change of variable $y = \frac{x}{R^{j-1}\varepsilon^{s_2}}$, applying Lemma 3.1 with $\delta = \delta_{\varepsilon, j}$ (see formula (3.2)) and Corollary 1.6, we have that

$$\begin{aligned} \inf_{w \in \widetilde{\mathcal{A}}_{R^{j-1}\varepsilon^{s_2}, R^j\varepsilon^{s_2}}(z)} F_{\delta_\varepsilon}(w; A_{R^{j-1}\varepsilon^{s_2}, R^j\varepsilon^{s_2}}) &= \inf_{w \in \mathcal{A}_{1, R}(z)} F_{\delta_{\varepsilon, j}}(w; A_{1, R}) \\ &\leq \inf_{w \in \mathcal{A}_{1, R}(z)} F_{\delta_{\varepsilon, j}}(w; A_{1, R}) + \bar{C}z^2 \\ &\leq \min_{w \in \mathcal{A}_{1, R}(z)} F_{\text{hom}}(w; A_{1, R}) + \omega(\delta_{\varepsilon, j}) \\ &\quad + \bar{C}z^2, \end{aligned} \tag{3.18}$$

where the constant $\bar{C} > 0$ is given in Lemma 3.1. By combining (3.18) with (3.17) we get that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{\varepsilon^{s_2}, \varepsilon^{s_1}}(z)} F_{\delta_\varepsilon}(w; A_{\varepsilon^{s_2}, \varepsilon^{s_1}}) &\leq \frac{s_2 - s_1}{\log R} \min_{w \in \mathcal{A}_{1, R}(z)} F_{\text{hom}}(w; A_{1, R}) + \frac{s_2 - s_1}{\log R} \bar{C}z^2 \\ &= (s_2 - s_1)\psi_{1, R}(z; T f_{\text{hom}}) + \frac{s_2 - s_1}{\log R} \bar{C}z^2, \end{aligned}$$

whence (3.16) follows by taking the limit as $R \rightarrow +\infty$ and using Proposition 2.1 with $h = T f_{\text{hom}}$. This concludes the proof of (3.11). \square

Remark 3.3. Note that (3.11) holds true also if the center of the annulus is a point ξ_ε depending on ε , since all the estimates in the previous proof do not depend on the center of the annulus.

4. The Ball Construction

In this section we present the so-called *ball construction* introduced in [32,40], which provides lower bounds of the Dirichlet energy in the presence of topological singularities. We slightly revisit it, following the approach by Sandier [40] and adopting the notation in [26] (see also [5]).

Let $\mathcal{B} = \{B_{r_1}(x_1), \dots, B_{r_n}(x_n)\}$ be a finite family of open balls in \mathbb{R}^2 with disjoint closure $\bar{B}_{r_i}(x_i) \cap \bar{B}_{r_j}(x_j) = \emptyset$ for $i \neq j$ and let $\mu = \sum_{i=1}^n z_i \delta_{x_i}$ with $z_i \in \mathbb{Z} \setminus \{0\}$.

Let moreover $\mathfrak{E}(\mathcal{B}, \mu, \cdot)$ be an increasing set-function defined on open subsets of \mathbb{R}^2 satisfying the following properties:

- (i) $\mathfrak{E}(\mathcal{B}, \mu, E_1 \cup E_2) \geq \mathfrak{E}(\mathcal{B}, \mu, E_1) + \mathfrak{E}(\mathcal{B}, \mu, E_2)$ for all E_1, E_2 open disjoint subsets of \mathbb{R}^2 ;
- (ii) for any annulus $A_{r,R}(x) = B_R(x) \setminus \bar{B}_r(x)$ with $A_{r,R}(x) \cap \bigcup_i \bar{B}_{r_i}(x_i) = \emptyset$, it holds

$$\mathfrak{E}(\mathcal{B}, \mu, A_{r,R}(x)) \geq 2\pi\alpha |\mu(B_r(x))| \log \frac{R}{r}, \tag{4.1}$$

for some constant $\alpha > 0$.

Remark 4.1. Let $w \in H^1_{loc}(\mathbb{R}^2 \setminus \bigcup_{B \in \mathcal{B}} B; \mathcal{S}^1)$ be such that $\mu = \sum_{B \in \mathcal{B}} \deg(w, \partial B) \delta_{x_B}$, where x_B is the center of B . Then, an explicit example of admissible functional $\mathfrak{E}(\mathcal{B}, \mu, \cdot)$ is given by

$$\mathfrak{E}(\mathcal{B}, \mu, A) := \alpha \int_{A \setminus \bigcup_{B \in \mathcal{B}} \bar{B}} |\nabla w|^2 dx,$$

for every open set $A \subset \mathbb{R}^2$. For further details see Remark 5.1.

For every ball $B \subset \mathbb{R}^2$, let $r(B)$ denote the radius of the ball B ; moreover, for every family \mathcal{B} of balls in \mathbb{R}^2 we set

$$\mathcal{R}ad(\mathcal{B}) := \sum_{B \in \mathcal{B}} r(B).$$

Proposition 4.2. *There exists a one-parameter family of open balls $\mathcal{B}(t)$ with $t \geq 0$ such that, setting $U(t) := \bigcup_{B \in \mathcal{B}(t)} B$, the following conditions are fulfilled:*

- (1) $\mathcal{B}(0) = \mathcal{B}$;
- (2) $U(t_1) \subset U(t_2)$ for any $0 \leq t_1 < t_2$;
- (3) the balls in $\mathcal{B}(t)$ are pairwise disjoint;
- (4) for any $0 \leq t_1 < t_2$ and for any open set $U \subset \mathbb{R}^2$,

$$\mathfrak{E}(\mathcal{B}, \mu, U \cap (U(t_2) \setminus \bar{U}(t_1))) \geq 2\pi\alpha \sum_{\substack{B \in \mathcal{B}(t_2) \\ B \subset U}} |\mu(B)| \log \frac{1+t_2}{1+t_1}; \tag{4.2}$$

- (5) $\mathcal{R}ad(\mathcal{B}(t)) \leq (1+t)\mathcal{R}ad(\mathcal{B})$.

Proof. In order to construct the family $\mathcal{B}(t)$, we closely follow the strategy of Sandier and Jerrard in [32,40]. It consists in letting the balls alternatively expand and merge into each other as follows. In the expansion phase the balls expand, without changing their centers, in such a way that, at each (artificial) time t the radius $r_i(t)$ of the ball centered at x_i satisfies

$$\frac{r_i(t)}{r_i} = 1 + t \quad \text{for all } i. \tag{4.3}$$

The first expansion phase stops at the first time T_1 when two balls bump into each other. Then the merging phase begins. It consists in identifying a suitable partition $\{S_j^1\}_{j=1, \dots, N_n}$ of the family $\{B_{r_i(T_1)}(x_i)\}$, and, for each subclass S_j^1 , in finding a ball $B_{r_j^1}(x_j^1)$ which contains all the balls in S_j^1 such that the following properties hold:

- (P1) $B_{r_j^1}(x_j^1) \cap B_{r_l^1}(x_l^1) = \emptyset$ for all $j \neq l$;
- (P2) $r_j^1 \leq \sum_{B \in S_j^1} r(B)$.

After the merging phase another expansion phase begins: we let the balls $\{B_{r_j^1}(x_j^1)\}$ expand in such a way that, for $t \geq T_1$, for every j we have that

$$\frac{r_j^1(t)}{r_j^1} = \frac{1 + t}{1 + T_1}. \tag{4.4}$$

Again note that $r_j^1(T_1) = r_j^1$. We iterate this procedure thus obtaining a *discrete* set of merging times $\{T_1, \dots, T_K\}$ with $K \leq n$ and a family $\mathcal{B}(t)$ for all $t \geq 0$. More precisely, $\mathcal{B}(t)$ is given by $\{B_{r_j(t)}(x_j)\}_j$ for $t \in [0, T_1)$; for $t \in [T_k, T_{k+1})$, $\mathcal{B}(t)$ can be written as $\{B_{r_j^k(t)}(x_j^k)\}_j$ for all $k = 1, \dots, K - 1$, while it consists of a single expanding ball for $t \geq T_K$. By construction, we clearly have properties (1), (2) and (3). Moreover, (5) is a consequence of (4.3), (4.4) and property (P2).

It remains to show property (4). We preliminarily note that, by (2), for every open set $U \subset \mathbb{R}^2$

$$\sum_{\substack{B \in \mathcal{B}(\tau_1) \\ B \subset U}} |\mu(B)| \geq \sum_{\substack{B \in \mathcal{B}(\tau_2) \\ B \subset U}} |\mu(B)| \quad \text{for any } 0 < \tau_1 < \tau_2. \tag{4.5}$$

Let $t_1 < \bar{t} < t_2$. In view of (4.5) and since \mathfrak{E} is an increasing set-function satisfying property (i), if we show that (4) holds true for the pairs (t_1, \bar{t}) and (\bar{t}, t_2) , then (4) follows also for t_1 and t_2 . Therefore, we can assume without loss of generality that $T_k \notin]t_1, t_2[$ for any $k = 1, \dots, K$.

Let $t_1 < \tau < t_2$ and let $B \in \mathcal{B}(\tau)$. Then there exists a unique ball $B' \in \mathcal{B}(t_1)$ such that $B' \subset B$. By construction, $\mu(B) = \mu(B')$ and by (4.1) we have that

$$\mathfrak{E}(\mathcal{B}, \mu, B \setminus \bar{B}') \geq 2\pi\alpha |\mu(B)| \log \frac{1 + \tau}{1 + t_1},$$

which, summing up over all $B \in \mathcal{B}(\tau)$ with $B \subset U$, and using (4.5), yields

$$\begin{aligned} \mathfrak{E}(\mathcal{B}, \mu, U \cap (U(t_2) \setminus \overline{U}(t_1))) &\geq 2\pi\alpha \sum_{\substack{B \in \mathcal{B}(\tau) \\ B \subset U}} |\mu(B)| \log \frac{1 + \tau}{1 + t_1} \geq 2\pi\alpha \\ &\sum_{\substack{B \in \mathcal{B}(t_2) \\ B \subset U}} |\mu(B)| \log \frac{1 + \tau}{1 + t_1}. \end{aligned}$$

Property (4) follows by letting $\tau \rightarrow t_2$. \square

We recall the following well-known lemma (see e.g., [26, Lemma 2.2]) for the reader’s convenience:

Lemma 4.3. *Let \mathcal{B} be a family of pairwise disjoint balls in \mathbb{R}^2 and let \mathcal{C} be the family of balls in \mathcal{B} which are contained in Ω . Let moreover ν_1, ν_2 be two Radon measures supported in Ω with*

$$\text{supp } \nu_1 \subset \bigcup_{B \in \mathcal{C}} B, \quad \text{supp } \nu_2 \subset \bigcup_{B \in \mathcal{B}} B \quad \text{and} \quad \nu_1(B) = \nu_2(B) \quad \text{for any } B \in \mathcal{C}.$$

Then, there exists a constant $C > 0$ such that

$$\|\nu_1 - \nu_2\|_{\text{flat}} \leq C \mathcal{R}ad(\mathcal{B})(|\nu_1| + |\nu_2|)(\Omega).$$

5. General Γ -Liminf Inequality

In this section we state and prove an asymptotic lower-bound estimate for general core-radius approach functionals (see Propositions 5.2 and 5.4); such results will be instrumental for the proofs of the Γ -liminf inequalities in Theorems 0.5, 0.4 and 0.2.

We introduce the increasing set-function \mathfrak{E} satisfying the assumptions (i) and (ii) in Section 4 as follows. Let $\mathcal{B} = \{B_{r_1}(x_1), \dots, B_{r_N}(x_N)\}$ be a finite family of open balls in \mathbb{R}^2 with $\overline{B}_{r_i}(x_i) \cap \overline{B}_{r_j}(x_j) = \emptyset$ for $i \neq j$, and let $\mu = \sum_{i=1}^n z_i \delta_{x_i}$ with $z_i \in \mathbb{Z} \setminus \{0\}$.

If $A_{r,R}(x)$ is an annulus that does not intersect any $B_{r_i}(x_i)$, we set

$$\mathfrak{G}(\mathcal{B}, \mu, A_{r,R}(x)) := 2\pi\alpha |\mu(B_r(x))| \log \left(\frac{R}{r} \right), \tag{5.1}$$

with α as in assumption (G). For every open set $A \subset \mathbb{R}^2$ we set

$$\mathfrak{E}(\mathcal{B}, \mu, A) := \sup \sum_j \mathfrak{G}(\mathcal{B}, \mu, A_j), \tag{5.2}$$

where the supremum is taken over all finite families of disjoint annuli $A_j \subset A$ that do not intersect any $B_{r_i}(x_i)$. Note that, if A is an annulus that does not intersect any $B_{r_i}(x_i)$, then $\mathfrak{E}(\mathcal{B}, \mu, A) = \mathfrak{G}(\mathcal{B}, \mu, A)$.

Remark 5.1. The convenience of introducing \mathfrak{E} in (5.2) to prove a lower-bound inequality for (an appropriate scaling of) the functional F_{δ_ε} in (0.12) will be clear in the following sections. However, the following simple observation already points in the right direction. Let $\Omega(\mathcal{B}) = \Omega \setminus \bigcup_{B \in \mathcal{B}} \overline{B}$, $w \in H^1(\Omega(\mathcal{B}); \mathcal{S}^1)$ and $\mu := \sum_{B \in \mathcal{C}} \deg(w, \partial B) \delta_{x_B}$ where \mathcal{C} denotes the family of balls in \mathcal{B} that are contained in Ω , and x_B is the center of B . Then, by Jensen’s inequality and by the lower bound in (G), we deduce that

$$\mathfrak{E}(\mathcal{B}, \mu, U) \leq \int_{U \cap \tilde{\Omega}} \alpha |\nabla w|^2 dx \leq F_{\delta_\varepsilon}(w; U \cap \Omega(\mathcal{B})) \tag{5.3}$$

for every open set $U \subset \Omega$.

For every $\mu \in X(\Omega)$ and for every family of pairwise disjoint balls \mathcal{B} such that $\text{supp } \mu \subset \bigcup_{B \in \mathcal{B}} B$, we set

$$\mathcal{AF}(\mu, \mathcal{B}) := \{w \in H^1(\Omega(\mathcal{B}); \mathcal{S}^1) : \deg(w, \partial B) = \mu(B) \text{ for every } B \in \mathcal{B}\}.$$

In addition, we set

$$\mathcal{F}_\varepsilon(\mu, \mathcal{B}) := \inf_{w \in \mathcal{AF}(\mu, \mathcal{B})} F_{\delta_\varepsilon}(w; \Omega(\mathcal{B})), \tag{5.4}$$

where F_{δ_ε} is defined in (0.12) and f satisfies (P), (G), (H).

Note that if $\mu = \sum_{i=1}^n z_i \delta_{x_i} \in X_\varepsilon(\Omega)$ for some $\varepsilon > 0$, setting $\mathcal{B}_\varepsilon = \{B_\varepsilon(x_i)\}_{i=1, \dots, n}$, we have that $\mathcal{AF}(\mu, \mathcal{B}_\varepsilon)$ coincides with the set $\mathcal{AF}_\varepsilon(\mu)$ defined in (0.9) and that $\mathcal{F}_\varepsilon(\mu, \mathcal{B}_\varepsilon) = \mathcal{F}_\varepsilon(\mu)$. We are now in a position to state the first main result of this section, concerning the case $|\log \delta_\varepsilon| \gtrsim |\log \varepsilon|$.

Proposition 5.2. *Let $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ be such that*

$$|\mu_\varepsilon|(\Omega) \leq C |\log \varepsilon| \tag{5.5}$$

and $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ for some $\mu \in X(\Omega)$. For every $\varepsilon > 0$ let \mathcal{B}_ε be a finite family of pairwise disjoint open balls such that $\text{supp } \mu_\varepsilon \subset \bigcup_{B \in \mathcal{B}_\varepsilon} B$ and

$$\mathcal{Rad}(\mathcal{B}_\varepsilon) \leq C \varepsilon |\log \varepsilon|. \tag{5.6}$$

If $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$, then

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon) \geq \mathcal{F}_0(\mu),$$

where \mathcal{F}_ε is defined in (0.12) and (5.4), \mathcal{F}_0 is defined in (0.14) and $\Psi(\cdot; Tf_{\text{hom}})$ is defined in (2.9) for $h = Tf_{\text{hom}}$.

Proof. For every $\varepsilon > 0$, let $w_\varepsilon \in \mathcal{AF}(\mu_\varepsilon, \mathcal{B}_\varepsilon)$ be such that

$$F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon) + C \tag{5.7}$$

for some constant C independent of ε . We can assume without loss of generality that

$$F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon) + C \leq C|\log \varepsilon|. \tag{5.8}$$

Moreover, by a standard localization argument in Γ -convergence, we can assume that $\mu = z_0\delta_{x_0}$ for some $z_0 \in \mathbb{Z} \setminus \{0\}$ and $x_0 \in \Omega$. We divide the proof into three steps. In Step 1, using the ball-construction procedure introduced in Section 4, we show that the sequence $\{\mu_\varepsilon\}_\varepsilon$ is flat-equivalent to a sequence $\{\mu_\varepsilon(p)\}_\varepsilon$ having uniformly bounded total variation. In Step 2, we show how to modify the functions w_ε in order to get rid of the balls containing “short” dipoles far from the limiting singularity. In such a way, we can bound $F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon))$ from below by the energy of the modified functions \hat{w}_ε , up to a bounded error. Therefore, referring again to the ball construction of Section 4, it is sufficient to consider the family of balls that contains the support of $\mu_\varepsilon(p)$, and estimate the energy of \hat{w}_ε in the complement to this family. This is done in Step 3, where the analysis developed in Section 3 is used in order to get the desired lower bound.

Step 1. In view of (5.8), by exploiting assumption (G) and by applying (5.3) with $U = \Omega$, we have that

$$\mathfrak{E}(\mathcal{B}_\varepsilon, \mu_\varepsilon, \Omega) \leq \alpha \int_{\Omega(\mathcal{B}_\varepsilon)} |\nabla w_\varepsilon|^2 dx \leq C|\log \varepsilon|, \tag{5.9}$$

where \mathfrak{E} is defined in (5.1) and (5.2). For every $\varepsilon > 0$, let $\mathcal{B}_\varepsilon(t)$ be a time-parametrized family of balls introduced as in Proposition 4.2, starting from $\mathcal{B}_\varepsilon =: \mathcal{B}_\varepsilon(0)$. For every $t \geq 0$, we set $\mathcal{R}_\varepsilon(t) := \text{Rad}(\mathcal{B}_\varepsilon(t))$, $\mathcal{C}_\varepsilon(t) := \{B \in \mathcal{B}_\varepsilon(t) : B \subset \Omega\}$ and $U_\varepsilon(t) := \bigcup_{B \in \mathcal{B}_\varepsilon(t)} B$. Moreover, for any $0 < p < 1$ we set

$$t_\varepsilon(p) := \frac{1}{\mathcal{R}_\varepsilon^{1-p}(0)} - 1 \quad \text{and} \quad \mu_\varepsilon(p) := \sum_{B \in \mathcal{C}_\varepsilon(t_\varepsilon(p))} \mu_\varepsilon(B)\delta_{x_B}.$$

By (5.9), applying (4.2) with $U = \Omega$, $t_1 = 0$ and $t_2 = t_\varepsilon(p)$, and using (5.6), we obtain

$$\begin{aligned} C|\log \varepsilon| &\geq \mathfrak{E}(\mathcal{B}_\varepsilon, \mu_\varepsilon, \Omega \cap (U_\varepsilon(t_\varepsilon(p)) \setminus U_\varepsilon(0))) \\ &\geq 2\pi\alpha \sum_{B \in \mathcal{C}_\varepsilon(t_\varepsilon(p))} |\mu_\varepsilon(B)|(1-p)|\log \mathcal{R}_\varepsilon(0)| \\ &= 2\pi\alpha(1-p)|\mu_\varepsilon(p)|(|\Omega| \log \mathcal{R}_\varepsilon(0)) \geq C(1-p)|\mu_\varepsilon(p)|(|\Omega| \log \varepsilon) \end{aligned}$$

for sufficiently small ε . Therefore, we have

$$|\mu_\varepsilon(p)|(|\Omega|) \leq C_p, \tag{5.10}$$

for some constant $C_p > 0$ depending on p (and independent of ε). By Proposition 4.2(5) and (5.6), we have that

$$\mathcal{R}_\varepsilon(t_\varepsilon(p)) \leq \mathcal{R}_\varepsilon^p(0) \leq C\varepsilon^p |\log \varepsilon|^p,$$

whence, by applying Lemma 4.3 with $v_1 = \mu_\varepsilon(p)$ and $v_2 = \mu_\varepsilon$, we deduce that

$$\begin{aligned} \|\mu_\varepsilon - \mu_\varepsilon(p)\|_{\text{flat}} &\leq C\mathcal{R}_\varepsilon(t_\varepsilon(p))(|\mu_\varepsilon| + |\mu_\varepsilon(p)|)(\Omega) \leq C\varepsilon^p |\log \varepsilon|^{1+p} \rightarrow 0 \\ &\text{as } \varepsilon \rightarrow 0. \end{aligned}$$

This relation, with (5.10) and the fact that $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$, yield that

$$\mu_\varepsilon(p) \xrightarrow{*} \mu = z_0 \delta_{x_0}, \quad \text{for every } 0 < p < 1. \tag{5.11}$$

Step 2. Let $c > 1$ be such that $\log c < \frac{p}{2} \frac{|\log \mathcal{R}_\varepsilon(0)|}{|\mu_\varepsilon|(\Omega)+1}$. Note that, since $|\log \mathcal{R}_\varepsilon(0)| \geq C|\log \varepsilon|$ and $|\mu_\varepsilon|(\Omega) \leq C|\log \varepsilon|$, we are allowed to take the constant c in the previous inequality independent of ε . By Lemma 5.3 below (applied with $p_1 = p$ and $p_2 = \frac{p}{2}$) there exist $t_\varepsilon(p) \leq \hat{t}_{\varepsilon,1} < \hat{t}_{\varepsilon,2} \leq t_\varepsilon(\frac{p}{2})$ with $(1 + \hat{t}_{\varepsilon,2}) = c(1 + \hat{t}_{\varepsilon,1})$ such that no merging occurs in the interval $[\hat{t}_{\varepsilon,1}, \hat{t}_{\varepsilon,2})$ and

$$\begin{aligned} \int_{\Omega \cap (U(\hat{t}_{\varepsilon,2}) \setminus \bar{U}(\hat{t}_{\varepsilon,1}))} |\nabla w_\varepsilon|^2 dx &\leq \frac{\log c \int_{\Omega(\mathcal{B}_\varepsilon)} |\nabla w_\varepsilon|^2 dx}{\frac{p}{2} |\log \mathcal{R}_\varepsilon(0)| - \log c (|\mu_\varepsilon|(\Omega) + 1)} \\ &\leq \frac{\frac{\log c}{\alpha} F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon))}{\frac{p}{2} (|\log \varepsilon| - \log |\log \varepsilon| + C) - \log c (C|\log \varepsilon| + 1)} \\ &\leq C, \end{aligned} \tag{5.12}$$

where the last but one inequality follows from (5.6) and (5.5), whereas the last inequality is a consequence of (5.8). We classify the balls in $\mathcal{C}_\varepsilon(\hat{t}_{\varepsilon,1})$ into two subclasses; namely,

$$\begin{aligned} \mathcal{C}_\varepsilon^{=0}(\hat{t}_{\varepsilon,1}) &:= \{B \in \mathcal{C}_\varepsilon(\hat{t}_{\varepsilon,1}) : \mu_\varepsilon(B) = 0\} \text{ and } \mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1}) \\ &:= \{B \in \mathcal{C}_\varepsilon(\hat{t}_{\varepsilon,1}) : \mu_\varepsilon(B) \neq 0\}. \end{aligned} \tag{5.13}$$

We first consider the balls in $\mathcal{C}_\varepsilon^{=0}(\hat{t}_{\varepsilon,1})$. For every such ball B we let \hat{B} denote the only ball in $\mathcal{C}_\varepsilon(\hat{t}_{\varepsilon,2})$ containing B . Note that, the center x_B of B is the same as the center of \hat{B} . By (5.12), we have that

$$\sum_{B \in \mathcal{C}_\varepsilon^{=0}(\hat{t}_{\varepsilon,1})} \int_{\hat{B} \setminus B} |\nabla w_\varepsilon|^2 dx \leq C.$$

Now we extend the function w_ε to a function $\hat{w}_\varepsilon \in H^1(\Omega(\mathcal{B}_\varepsilon) \cup \bigcup_{B \in \mathcal{C}_\varepsilon^{=0}(\hat{t}_{\varepsilon,1})} B; S^1)$ in such a way that

$$\|\nabla \hat{w}_\varepsilon\|_{L^2(\hat{B}; \mathbb{R}^{2 \times 2})} \leq \hat{C} \|\nabla w_\varepsilon\|_{L^2(\hat{B} \setminus B; \mathbb{R}^{2 \times 2})} \tag{5.14}$$

for every B and \hat{B} as above, for some universal constant \hat{C} . We consider $B = B_R(\xi)$ and $\hat{B} = B_{cR}(\xi)$ two balls as above. Since $\deg(w_\varepsilon, \partial B_R(\xi)) = \deg(w_\varepsilon,$

$\partial B_{cR}(\xi) = 0$, by arguing as in [14] (see also [15]), there exists a lifting $u_\varepsilon^{A_{R,cR}(\xi)} \in H^1(A_{R,cR}(\xi))$ of w_ε in $A_{R,cR}(\xi)$. Let $U_\varepsilon : A_{\frac{R}{c},R}(\xi) \rightarrow \mathbb{R}$ be the extension by reflection of the function $u_\varepsilon^{A_{R,cR}(\xi)}$ to the annulus $A_{\frac{R}{c},R}(\xi)$; i.e., $U_\varepsilon(x) := u_\varepsilon^{A_{R,cR}(\xi)}(\xi - c(x - \xi) + (1 + c)R\frac{x-\xi}{|x-\xi|})$. We let \bar{U}_ε denote the average of U_ε on $A_{\frac{R}{c},R}(\xi)$. Let $\eta : [\frac{R}{c}, R] \rightarrow \mathbb{R}$ be the cut-off function defined by $\eta(\rho) = \frac{c\rho - R}{R(c-1)}$. We define the function $\hat{u}_\varepsilon^{\hat{B}} : B_{cR}(\xi) \rightarrow \mathbb{R}$ as

$$\hat{u}_\varepsilon^{\hat{B}}(x) := \begin{cases} u_\varepsilon^{A_{R,cR}(\xi)} & \text{if } x \in A_{R,cR}(\xi), \\ \eta(|x - \xi|)U_\varepsilon(x) + (1 - \eta(|x - \xi|))\bar{U}_\varepsilon & \text{if } x \in A_{\frac{R}{c},R}(\xi), \\ \bar{U}_\varepsilon & \text{if } x \in B_{\frac{R}{c}}(\xi). \end{cases}$$

By the Poincaré–Wirtinger inequality and by the definition of U_ε we have that there exists a constant \hat{C} (independent of ε) such that

$$\begin{aligned} \int_{A_{\frac{R}{c},R}(\xi)} |\nabla \hat{u}_\varepsilon^{\hat{B}}|^2 dx &= \int_{A_{\frac{R}{c},R}(\xi)} |\nabla(\eta(|x - \xi|)(U_\varepsilon(x) - \bar{U}_\varepsilon))|^2 dx \\ &\leq 2 \frac{c^2}{R^2(c-1)^2} \int_{A_{\frac{R}{c},R}(\xi)} |U_\varepsilon(x) - \bar{U}_\varepsilon(x)|^2 dx \\ &\quad + 2 \int_{A_{\frac{R}{c},R}(\xi)} |\nabla U_\varepsilon|^2 dx \\ &\leq C \int_{A_{\frac{R}{c},R}(\xi)} |\nabla U_\varepsilon|^2 dx \leq \hat{C} \int_{A_{\frac{R}{c},R}(\xi)} |\nabla u_\varepsilon^{A_{R,cR}(\xi)}|^2 dx. \end{aligned}$$

Therefore, setting

$$\hat{w}_\varepsilon(x) := \begin{cases} e^{i\hat{u}_\varepsilon^{\hat{B}}(x)} & \text{if } x \in \hat{B} \text{ for some } \hat{B} \in \mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1}), \\ w_\varepsilon(x) & \text{elsewhere in } \Omega(\mathcal{B}_\varepsilon), \end{cases}$$

we have that $\hat{w}_\varepsilon \in H^1(\Omega(\mathcal{B}_\varepsilon) \cup \bigcup_{B \in \mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})} B; \mathcal{S}^1)$ and satisfies (5.14). Then from (5.12) and (5.14) we deduce that

$$\sum_{B \in \mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})} \int_B |\nabla \hat{w}_\varepsilon|^2 dx \leq C. \tag{5.15}$$

Step 3. We now focus on the balls in $\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})$. In view of the ball construction in Section 4 and of (5.10), we have that $\#\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1}) \leq C\rho$. Therefore, up to extracting a subsequence, we may assume that $\#\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1}) = L$ for every $\varepsilon > 0$ and for some $L \in \mathbb{N}$. For every $l = 1, \dots, L$, let x_ε^l be the center of the l -th ball B_ε^l in $\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})$. Up to a further subsequence, we can assume that the points x_ε^l converge to some points in the finite set $\{\xi_0 = x_0, \xi_1, \dots, \xi_{L'}\} \subset \bar{\Omega}$, where $L' \leq L$. Let $\rho > 0$

be such that $B_{2\rho}(x^0) \subset\subset \Omega$ and $B_{2\rho}(\xi_j) \cap B_{2\rho}(\xi_k) = \emptyset$ for all $j \neq k$. Then $x_\varepsilon^l \in B_\rho(\xi_j)$ for some $j = 1, \dots, L'$ and for ε small enough. We set

$$\tilde{\mu}_\varepsilon := \sum_{x_\varepsilon^l \in B_\rho(x_0)} \mu_\varepsilon(B_\varepsilon^l) \delta_{x_\varepsilon^l}.$$

By construction, we have that

$$|\tilde{\mu}_\varepsilon|(\Omega) \leq |\mu_\varepsilon(p)|(\Omega) \quad \text{and} \quad \|\tilde{\mu}_\varepsilon - \mu_\varepsilon(p)\|_{\text{flat}} \rightarrow 0, \quad (5.16)$$

which, in view of (5.10) and (5.11), implies that, up to a subsequence, $\tilde{\mu}_\varepsilon \xrightarrow{*} \mu = z_0 \delta_{x_0}$. Therefore, for sufficiently small ε ,

$$\tilde{\mu}_\varepsilon(B_{2\rho}(x_0)) = \sum_{x_\varepsilon^l \in B_\rho(x_0)} \mu_\varepsilon(B_\varepsilon^l) = z_0. \quad (5.17)$$

Thanks to (5.15) and the assumption (G), we have that

$$\begin{aligned} F_{\delta_\varepsilon}(w_\varepsilon; \Omega(B_\varepsilon)) &\geq \int_{\Omega(B_\varepsilon) \setminus \cup_{B \in \mathcal{C}_\varepsilon^0(\hat{t}_{\varepsilon,1})} B} f\left(\frac{x}{\delta_\varepsilon}, \nabla w_\varepsilon\right) dx \\ &\geq \int_{\Omega(B_\varepsilon) \cap B_{2\rho}(x_0)} f\left(\frac{x}{\delta_\varepsilon}, \nabla \hat{w}_\varepsilon\right) dx - C. \end{aligned} \quad (5.18)$$

It remains to prove the lower bound for the right-hand side of (5.18). To this end, we take $0 < p' < p$ such that $\mathcal{R}_\varepsilon(\hat{t}_{\varepsilon,1}) \leq \varepsilon^{p'}$ (note that such p' always exists since, by Lemma 5.3, $\mathcal{R}_\varepsilon(\hat{t}_{\varepsilon,1}) < \mathcal{R}_\varepsilon^{\frac{p}{2}}(0) \leq C\varepsilon^{\frac{p}{2}} |\log \varepsilon|^{\frac{p}{2}}$), choose $0 < \bar{p} < p'$ and let $g_\varepsilon : [\bar{p}, p'] \rightarrow \{1, \dots, L\}$ denote the function which associates to any $q \in [\bar{p}, p']$ the number $g_\varepsilon(q)$ of connected components of the set $\cup_{j=1}^L B_{\varepsilon^q}(x_\varepsilon^j)$. For every $\varepsilon > 0$, the function g_ε is monotonically non decreasing so that it can have at most $\hat{L} \leq L$ discontinuities. Let q_ε^j , for $j = 1, \dots, \hat{L}$, denote the discontinuity points of g_ε and assume that

$$\bar{p} \leq q_\varepsilon^1 < \dots < q_\varepsilon^{\hat{L}} \leq p'.$$

There exists a finite set $\Delta = \{q^1, q^2, \dots, q^{\hat{L}}\}$ with $q^i < q^{i+1}$ and $\tilde{L} \leq \hat{L}$ such that, up to a subsequence, $\{q_\varepsilon^j\}_\varepsilon$ converges to some point in Δ , as $\varepsilon \rightarrow 0$ for every $j = 1, \dots, \hat{L}$. Without loss of generality we may assume that $q^1 = \bar{p}$, and that $q^{\tilde{L}} = p'$. Let $\lambda > 0$ be such that $4\lambda < \min\{q^{i+1} - q^i : i \in \{1, 2, \dots, \tilde{L} - 1\}\}$ and let ε be so small that for every $j = 1, \dots, \hat{L}$, $|q_\varepsilon^j - q^i| < \lambda$ for some $q^i \in \Delta$. Then, the function g_ε is constant in the interval $[q^i + \lambda, q^{i+1} - \lambda]$, its value being denoted by M_ε^i . For every $i = 1, \dots, \tilde{L} - 1$ we construct a family of $M_\varepsilon^i \leq \tilde{L} - 1$ annuli that we let $C_\varepsilon^{i,m} := B_{\varepsilon^{q^i + \lambda}}(y_\varepsilon^m) \setminus \overline{B_{\varepsilon^{q^{i+1} - \lambda}}}(y_\varepsilon^m)$ with $y_\varepsilon^m \in B_\rho(x_0)$ and $m = 1, \dots, M_\varepsilon^i$. The annuli $C_\varepsilon^{i,m}$ can be taken pairwise disjoint for all i and m and such that

$$\bigcup_{x_\varepsilon^l \in B_\rho(x_0)} B_\varepsilon^l \subset \bigcup_{m=1}^{M_\varepsilon^i} B_{\varepsilon^{q^{i+1} - \lambda}}(y_\varepsilon^m)$$

for all $i = 1, \dots, \tilde{L} - 1$. Note that, for ε small enough, $C_\varepsilon^{i,m} \subset B_{2\rho}(x_0)$ for all i and m . By (5.16) we have that $|\mu_\varepsilon(B_{\varepsilon q^{i+1-\lambda}}(y_\varepsilon^m))| \leq C$ for every $i = 1, \dots, \tilde{L} - 1$ and $m = 1, \dots, M_\varepsilon^i$. Therefore, up to passing to a further subsequence, we can assume that $M_\varepsilon^i = M^i$ (since $M_\varepsilon^i \leq \tilde{L} - 1$) and that $\mu_\varepsilon(B_{\varepsilon q^{i+1-\lambda}}(y_\varepsilon^m)) = z_{i,m} \in \mathbb{Z} \setminus \{0\}$, with M^i and $z_{i,m}$ independent of ε . Finally, in view of (5.17), we have that

$$\sum_{m=1}^{M^i} z_{i,m} = z_0. \tag{5.19}$$

Observe that the assumption $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$ implies that $\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon q^{i+1-\lambda}} = 0$ for every i . Hence, we can apply Proposition 3.2 with $s_1 = q^i + \lambda < q^{i+1} - \lambda = s_2$ (see also Remark 3.3) to get that for every i and m there exists a modulus of continuity ω such that

$$\frac{1}{|\log \varepsilon|} \int_{C_\varepsilon^{i,m}} f\left(\frac{x}{\delta_\varepsilon}, \nabla \hat{w}_\varepsilon\right) dx \geq (q^{i+1} - q^i - 2\lambda)\psi(z_{i,m}; Tf_{\text{hom}}) - \omega(\varepsilon).$$

Summing the previous inequality over m and i and using (5.18) yields

$$\begin{aligned} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon)) &\geq \sum_{i=1}^{\tilde{L}-1} \sum_{m=1}^{M^i} (q^{i+1} - q^i - 2\lambda)\psi(z_{i,m}; Tf_{\text{hom}}) - \omega(\varepsilon) \\ &\geq \sum_{i=1}^{\tilde{L}-1} (q^{i+1} - q^i - 2\lambda)\Psi(z_0; Tf_{\text{hom}}) - \omega(\varepsilon) \\ &= (p' - \bar{p} - 2(\tilde{L} - 1)\lambda)\Psi(z_0; Tf_{\text{hom}}) - \omega(\varepsilon), \end{aligned} \tag{5.20}$$

where the second inequality follows from (5.19) and from the definition of Ψ in (2.9). Then, the claim follows by (5.20) taking the limits as $\varepsilon \rightarrow 0, \lambda \rightarrow 0, \bar{p} \rightarrow 0$, and $p, p' \rightarrow 1$ and using (5.7). \square

We turn to the technical lemma that has been exploited in the proof of Proposition 5.2 above (see formula (5.12)). For every $p \in (0, 1)$ let $t(p) := \frac{1}{\mathcal{R}ad^{1-p}(\mathcal{B})} - 1$.

Lemma 5.3. *Let $\mu \in X(\Omega)$ and let \mathcal{B} be a finite family of pairwise disjoint open balls such that $\text{supp } \mu \subset \bigcup_{B \in \mathcal{B}} B$ and $\mathcal{R}ad(\mathcal{B}) < 1$. Assume that $0 < p_2 < p_1 < 1$, and $c > 1$ be such that $\log c < (p_1 - p_2) \frac{|\log \mathcal{R}ad(\mathcal{B})|}{|\mu|(\Omega) + 1}$. Assume also that $\mathcal{B}(t)$ is a time-parametrized family of balls constructed as in Proposition 4.2, starting from $\mathcal{B} =: \mathcal{B}(0)$, and $U(t) := \bigcup_{B \in \mathcal{B}(t)} B$ for every $t \geq 0$. Then, there exist $\hat{t}_1, \hat{t}_2 \in [t(p_1), t(p_2))$ with $1 + \hat{t}_2 = c(1 + \hat{t}_1)$ such that no merging occurs in the interval $[\hat{t}_1, \hat{t}_2)$ and*

$$\int_{\Omega \cap (U(\hat{t}_2) \setminus \bar{U}(\hat{t}_1))} |\nabla w|^2 dx \leq \frac{\log c \int_{\Omega(\mathcal{B})} |\nabla w|^2 dx}{(p_1 - p_2) |\log \mathcal{R}ad(\mathcal{B})| - \log c(|\mu|(\Omega) + 1)},$$

for every $w \in \mathcal{AF}(\mu, \mathcal{B})$.

Proof. We set $J := \lfloor (p_1 - p_2) \frac{|\log \text{Rad}(\mathcal{B})|}{\log c} \rfloor$ and $\hat{t}^j := c^j (1 + t(p_1)) - 1$ for every $j = 0, 1, \dots, J$. Note that $\frac{1 + \hat{t}^{j+1}}{1 + \hat{t}^j} = c$ for every $j = 0, 1, \dots, J - 1$. We let \mathcal{J} denote the set of indices in $\{0, 1, \dots, J - 1\}$ for which no merging occurs in the interval $[\hat{t}^j, \hat{t}^{j+1})$. Since the number of merging times is bounded from above by $|\mu|(\Omega)$, we have $\#\mathcal{J} \geq J - |\mu|(\Omega) \geq (p_1 - p_2) \frac{|\log \text{Rad}(\mathcal{B})|}{\log c} - |\mu|(\Omega) - 1$. For every $j \in \mathcal{J}$ and for every ball $B(\hat{t}^j) \in \mathcal{B}(\hat{t}^j)$, $B(\hat{t}^{j+1})$ denotes the unique ball in $\mathcal{B}(\hat{t}^{j+1})$ such that $B(\hat{t}^j) \subset B(\hat{t}^{j+1})$. By the mean-value theorem, there exists $k \in \mathcal{J}$ such that

$$\begin{aligned} \int_{\Omega(\mathcal{B})} |\nabla w|^2 \, dx &\geq \sum_{j \in \mathcal{J}} \sum_{B(\hat{t}^j) \in \mathcal{B}(\hat{t}^j)} \int_{\Omega \cap (B(\hat{t}^{j+1}) \setminus \bar{B}(\hat{t}^j))} |\nabla w|^2 \, dx \\ &\geq \#\mathcal{J} \int_{\Omega \cap (U(\hat{t}^{k+1}) \setminus \bar{U}(\hat{t}^k))} |\nabla w|^2 \, dx \\ &\geq \left((p_1 - p_2) \frac{|\log \text{Rad}(\mathcal{B})|}{\log c} - |\mu|(\Omega) - 1 \right) \\ &\quad \int_{\Omega \cap (U(\hat{t}^{k+1}) \setminus \bar{U}(\hat{t}^k))} |\nabla w|^2 \, dx, \end{aligned}$$

from which the claim follows setting $\hat{t}_1 := \hat{t}^k$ and $\hat{t}_2 := \hat{t}^{k+1}$. \square

As for the general case (0.7), we restrict our analysis to functionals of the form

$$F_\delta(w; E) := \int_E a\left(\frac{x}{\delta}\right) |\nabla w|^2 \, dx, \quad w \in H^1(E; S^1). \tag{5.21}$$

In such a case the main result is

Proposition 5.4. *Let F_{δ_ε} , \mathcal{F}_ε be defined in (5.21), (5.4), respectively, where a is a measurable $(0, 1)^2$ -periodic function satisfying $a(x) \in [\alpha, \beta] \subset (0, +\infty)$ for a.e. $x \in \mathbb{R}^2$. Let $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ be such that*

$$|\mu_\varepsilon|(\Omega) \leq C |\log \varepsilon| \tag{5.22}$$

and $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ for some $\mu \in X(\Omega)$. For every $\varepsilon > 0$ let \mathcal{B}_ε be a finite family of pairwise disjoint open balls such that $\text{supp } \mu_\varepsilon \subset \bigcup_{B \in \mathcal{B}_\varepsilon} B$ and

$$\text{Rad}(\mathcal{B}_\varepsilon) \leq C \varepsilon |\log \varepsilon|. \tag{5.23}$$

If $\lambda := \lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} < 1$, then

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon) \geq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega),$$

where \mathbb{A}^{hom} is defined in (0.5).

Proof. The proof closely resembles the one of Proposition 5.2; here we only highlight the main changes that are needed to prove the different lower bound in the regime $\lambda < 1$.

Let $w_\varepsilon \in \mathcal{AF}_\varepsilon(\mu_\varepsilon, \Omega(\mathcal{B}_\varepsilon))$ be such that

$$F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon) + C \tag{5.24}$$

for some constant C independent of ε . By a standard localization argument in Γ -convergence, we can assume that $\mu = z_0\delta_{x_0}$ for some $z_0 \in \mathbb{Z} \setminus \{0\}$ and $x_0 \in \Omega$.

For every $\varepsilon > 0$, let $\mathcal{B}_\varepsilon(t)$ be a time-parametrized family of balls introduced as in Proposition 4.2, starting from $\mathcal{B}_\varepsilon =: \mathcal{B}_\varepsilon(0)$. For every $t \geq 0$, we set $\mathcal{R}_\varepsilon(t) := \text{Rad}(\mathcal{B}_\varepsilon(t))$, $\mathcal{C}_\varepsilon(t) := \{B \in \mathcal{B}_\varepsilon(t) : B \subset \Omega\}$ and $U_\varepsilon(t) := \bigcup_{B \in \mathcal{B}_\varepsilon(t)} B$. For every $0 < p < 1$ we set

$$t_\varepsilon(p) := \frac{1}{\mathcal{R}_\varepsilon^{1-p}(0)} - 1 \quad \text{and} \quad \mu_\varepsilon(p) := \sum_{B \in \mathcal{C}_\varepsilon(t_\varepsilon(p))} \mu_\varepsilon(B)\delta_{x_B}.$$

Fix $\lambda < p < 1$. By arguing as in the proof of (5.10) and (5.11), we have that

$$|\mu_\varepsilon(p)|(\Omega) \leq C_p \quad \text{and} \quad \mu_\varepsilon(p) \xrightarrow{*} \mu = z_0\delta_{x_0} \quad \text{as } \varepsilon \rightarrow 0. \tag{5.25}$$

Following the reasoning in the proof of Proposition 5.2 we have that for every $0 < \eta < p - \lambda$ there exists $t_\varepsilon(p) \leq \hat{t}_{\varepsilon,1} \leq t_\varepsilon(p - \eta)$ and a map $\hat{w}_\varepsilon \in H^1(\Omega(\mathcal{B}_\varepsilon) \cup \bigcup_{B \in \mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})} S^1)$ (with $\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})$ defined in (5.13)) satisfying (5.15).

Recalling the definition of $\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})$ in (5.13), we set $\mu(\hat{t}_{\varepsilon,1}) := \sum_{B \in \mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})} \mu_\varepsilon(B)\delta_{x_B}$. In view of the ball construction in Section 4 and of (5.25), we have that $\#\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1}) \leq C_p$. Therefore, up to extracting a subsequence we may assume that $\#\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1}) = L$ for every $\varepsilon > 0$ and for some $L \in \mathbb{N}$. For every $l = 1, \dots, L$, let x_ε^l be the center of the l -th ball B_ε^l in $\mathcal{C}_\varepsilon^{\neq 0}(\hat{t}_{\varepsilon,1})$. Up to a further subsequence, we can assume that the points x_ε^l converge to some points in the finite set $\{x_0 = \xi_0, \xi_1, \dots, \xi_{L'}\} \subset \bar{\Omega}$, where $L' \leq L$. Let $\rho > 0$ be such that $B_{2\rho}(x^0) \subset\subset \Omega$ and $B_{2\rho}(\xi_j) \cap B_{2\rho}(\xi_k) = \emptyset$ for all $j \neq k$. Then $x_\varepsilon^l \in B_\rho(\xi_j)$ for some $j = 1, \dots, L'$ and for ε small enough. Setting

$$\tilde{\mu}_\varepsilon := \sum_{x_\varepsilon^l \in B_\rho(x_0)} \mu_\varepsilon(B_\varepsilon^l)\delta_{x_\varepsilon^l},$$

by construction, we have that

$$|\tilde{\mu}_\varepsilon|(\Omega) \leq |\mu_\varepsilon(p)|(\Omega) \quad \text{and} \quad \|\tilde{\mu}_\varepsilon - \mu_\varepsilon(p)\|_{\text{flat}} \rightarrow 0, \tag{5.26}$$

which, in view of (5.25), implies that, up to a subsequence,

$$\tilde{\mu}_\varepsilon \xrightarrow{*} \mu = z_0\delta_{x_0}.$$

Therefore, for sufficiently small ε ,

$$\tilde{\mu}_\varepsilon(B_{2\rho}(x_0)) = \sum_{x_\varepsilon^l \in B_\rho(x_0)} \mu_\varepsilon(B_\varepsilon^l) = z_0, \tag{5.27}$$

and, by arguing as in the proof of (5.18), we obtain

$$F_{\delta_\varepsilon}(w_\varepsilon; \Omega(\mathcal{B}_\varepsilon)) \geq \int_{\Omega(\mathcal{B}_\varepsilon) \cap B_{2\rho}(x_0)} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla \hat{w}_\varepsilon|^2 dx - C. \tag{5.28}$$

It remains to prove the lower bound for the right-hand side of (5.28). To this end, we take $\lambda < p' < p$ such that $\mathcal{R}ad(\hat{t}_{\varepsilon,1}) \leq \varepsilon^{p'}$ (note that such p' always exists since $\mathcal{R}_\varepsilon(\hat{t}_{\varepsilon,1}) < \mathcal{R}_\varepsilon^{p-\eta}(0) \leq C\varepsilon^{p-\eta} |\log \varepsilon|^{p-\eta}$ for every $\eta < p - \lambda$), choose $0 < \bar{p} < \lambda < p'$ and let $g_\varepsilon : [\bar{p}, p'] \rightarrow \{1, \dots, L\}$ denote the function which associates to any $q \in [\bar{p}, p']$ the number $g_\varepsilon(q)$ of connected components of the set $\bigcup_{l=1}^L B_{\varepsilon^q}(x_\varepsilon^l)$. For every $\varepsilon > 0$, the function g_ε is monotonically non decreasing so that it can have at most $\hat{L} \leq L$ discontinuities. Let q_ε^j for $j = 1, \dots, \hat{L}_1$ and κ_ε^j for $j = 1, \dots, \hat{L}_2$ denote the discontinuity points of g_ε in $[\bar{p}, \lambda]$ and in $[\lambda, p']$, respectively. Assume that

$$\bar{p} \leq q_\varepsilon^1 < \dots < q_\varepsilon^{\hat{L}_1} \leq \lambda \leq \kappa_\varepsilon^1 \leq \dots \leq \kappa_\varepsilon^{\hat{L}_2} \leq p'.$$

There exists a finite set $\Delta_1 = \{q^1, \dots, q^{\tilde{L}_1}\}$ (resp., $\Delta_2 = \{\kappa^1, \dots, \kappa^{\tilde{L}_2}\}$) with $q^i < q^{i+1}$ (resp., $\kappa^i < \kappa^{i+1}$), and $\tilde{L}_1 \leq \hat{L}_1$ (resp., $\tilde{L}_2 \leq \hat{L}_2$) such that, up to a subsequence, $\{q_\varepsilon^j\}_\varepsilon$ converges to some point in Δ_1 , as $\varepsilon \rightarrow 0$ for every $j = 1, \dots, \hat{L}_1$ (resp., $\{\kappa_\varepsilon^j\}_\varepsilon$ converges to some point in Δ_2 , as $\varepsilon \rightarrow 0$ for every $j = 1, \dots, \hat{L}_2$). Without loss of generality we may assume that $q^1 = \bar{p}$, $q^{\tilde{L}_1} = \lambda = \kappa^1$, and $\kappa^{\tilde{L}_2} = p'$. Let $\eta > 0$ be such that $4\eta < \min\{q^{i+1} - q^i : i \in \{1, \dots, \tilde{L}_1\}\}$ and $4\eta < \min\{\kappa^{i+1} - \kappa^i : i \in \{1, \dots, \tilde{L}_2\}\}$ and let ε be so small that for every $j = 1, \dots, \hat{L}_1$, $|q_\varepsilon^j - q^i| < \eta$ for some $q^i \in \Delta_1$ and for every $j = 1, \dots, \hat{L}_2$, $|\kappa_\varepsilon^j - \kappa^i| < \eta$ for some $\kappa^i \in \Delta_2$. Then, the function g_ε is constant in the intervals $[q^i + \eta, q^{i+1} - \eta]$ and in the intervals $[\kappa^i + \eta, \kappa^{i+1} - \eta]$, and in both cases we let its value be denoted by M_ε^i . For every $i = 1, \dots, \tilde{L}_1 - 1$ (resp., $i = 1, \dots, \tilde{L}_2 - 1$) we construct a family of $M_\varepsilon^i \leq \tilde{L}_1 - 1$ (resp., $M_\varepsilon^i \leq \tilde{L}_2 - 1$) annuli that are denoted by $C_\varepsilon^{i,m} := B_{\varepsilon^{q^i+\eta}}(y_\varepsilon^m) \setminus \overline{B_{\varepsilon^{q^{i+1}-\eta}}(y_\varepsilon^m)}$ (resp., $C_\varepsilon^{i,m} := B_{\varepsilon^{\kappa^i+\eta}}(y_\varepsilon^m) \setminus \overline{B_{\varepsilon^{\kappa^{i+1}-\eta}}(y_\varepsilon^m)}$) with $y_\varepsilon^m \in B_\rho(x_0)$ and $m = 1, \dots, M_\varepsilon^i$. The annuli $C_\varepsilon^{i,m}$ can be taken pairwise disjoint for all i and m and such that

$$\bigcup_{x_\varepsilon^l \in B_\rho(x_0)} B_\varepsilon^l \subset \bigcup_{m=1}^{M_\varepsilon^i} B_{\varepsilon^{q^i+\eta}}(y_\varepsilon^m) \quad \text{and} \quad \bigcup_{x_\varepsilon^l \in B_\rho(x_0)} B_\varepsilon^l \subset \bigcup_{m=1}^{M_\varepsilon^i} B_{\varepsilon^{\kappa^i+\eta}}(y_\varepsilon^m).$$

Note that, for ε small enough, $C_\varepsilon^{i,m} \subset B_{2\rho}(x_0)$ for all i and m and, by (5.26), we get

$$|\mu_\varepsilon(B_{\varepsilon^{q^i+\eta}}(y_\varepsilon^m))| \leq C \quad \text{and} \quad |\mu_\varepsilon(B_{\varepsilon^{\kappa^i+\eta}}(y_\varepsilon^m))| \leq C.$$

Therefore, up to passing to a further subsequence, we can assume that $M_\varepsilon^i = M^i$ and that $\mu_\varepsilon(B_{\varepsilon^{q^i+\eta}}(y_\varepsilon^m)) = z_{i,m} \in \mathbb{Z} \setminus \{0\}$ and $\mu_\varepsilon(B_{\varepsilon^{\kappa^i+\eta}}(y_\varepsilon^m)) = z_{i,m} \in \mathbb{Z} \setminus \{0\}$, with M^i and $z_{i,m}$ independent of ε . Finally, in view of (5.27), we have that

$$\sum_{m=1}^{M^i} z_{i,m} = z_0. \tag{5.29}$$

For every $i = 1, \dots, \tilde{L}_2 - 1$ and for every $m = 1, \dots, M^i$ we have that

$$\begin{aligned} \int_{C_\varepsilon^{i,m}} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla \hat{w}_\varepsilon|^2 dx &\geq 2\pi \operatorname{ess\,inf} a(\kappa^{i+1} - \kappa^i - 2\eta) |\log \varepsilon| |z_{i,m}|^2 \\ &\geq 2\pi \operatorname{ess\,inf} a(\kappa^{i+1} - \kappa^i - 2\eta) |\log \varepsilon| |z_{i,m}| \end{aligned}$$

which, summing over m and over i , dividing by $|\log \varepsilon|$ and using (5.29), yields

$$\begin{aligned} \frac{1}{|\log \varepsilon|} \sum_{i=1}^{\tilde{L}_2-1} \sum_{m=1}^{M^i} \int_{C_\varepsilon^{i,m}} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla \hat{w}_\varepsilon|^2 dx &\geq 2\pi \operatorname{ess\,inf} a \sum_{i=1}^{\tilde{L}_2-1} (\kappa^{i+1} - \kappa^i - 2\eta) |z_0| \\ &\geq 2\pi \operatorname{ess\,inf} a(\kappa^{\tilde{L}_2} - \kappa^1 - 2\eta \tilde{L}_2) |z_0| \\ &= 2\pi \operatorname{ess\,inf} a(p' - \lambda - 2\eta \tilde{L}_2) |z_0|. \end{aligned} \tag{5.30}$$

Moreover, since $q^i + \eta < q^{\tilde{L}_1-1} + \eta < \lambda$ for every $i = 1, \dots, \tilde{L}_1 - 1$, by (7.1), we have that $\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon^{q^i + \eta}} = 0$ for every $i = 1, \dots, \tilde{L}_1 - 1$. Therefore, we can apply Proposition 3.2 with $s_1 = q^i + \eta < q^{i+1} - \eta = s_2$ (see also Remark 3.3) and Proposition 2.3 to get that for every i and m there exists a modulus of continuity ω such that

$$\begin{aligned} \frac{1}{|\log \varepsilon|} \int_{C_\varepsilon^{i,m}} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla \hat{w}_\varepsilon|^2 dx &\geq 2\pi (q^{i+1} - q^i - 2\eta) \sqrt{\det \mathbb{A}^{\operatorname{hom}}}|z_{i,m}|^2 - \omega(\varepsilon) \\ &\geq 2\pi (q^{i+1} - q^i - 2\eta) \sqrt{\det \mathbb{A}^{\operatorname{hom}}}|z_{i,m}|^2 - \omega(\varepsilon). \end{aligned}$$

Summing the previous inequality over m and i and using (5.29) yields

$$\begin{aligned} \frac{1}{|\log \varepsilon|} \sum_{i=1}^{\tilde{L}_1-1} \sum_{m=1}^{M^i} \int_{C_\varepsilon^{i,m}} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla \hat{w}_\varepsilon|^2 dx &\geq 2\pi \sqrt{\det \mathbb{A}^{\operatorname{hom}}}(q^{\tilde{L}_1} - \bar{p} - 2\eta \tilde{L}_1) |z_0| - \omega(\varepsilon) \\ &= 2\pi \sqrt{\det \mathbb{A}^{\operatorname{hom}}}(\lambda - \bar{p} - 2\eta \tilde{L}_1) |z_0| - \omega(\varepsilon). \end{aligned} \tag{5.31}$$

By (5.28), summing (5.30) and (5.31), the claim follows taking the limits as $\varepsilon \rightarrow 0$, $\eta \rightarrow 0$, $\bar{p} \rightarrow 0$ and $p, p' \rightarrow 1$ and using (5.24). \square

6. The Case $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$

This section is devoted to the proofs of Theorems 0.5 and 0.2 in the case $\lambda = 1$.

6.1. The Core-Radius Approach

For the reader’s convenience, we re-state Theorem 0.5 and recall that $X_\varepsilon(\Omega)$ is defined in (0.11).

Theorem 6.1. *Let F_{δ_ε} , \mathcal{F}_ε be defined in (0.12), (0.13), respectively, with f satisfying (P), (G), (H). Let moreover \mathcal{F}_0 be defined by formula (0.14) with $\Psi(\cdot; Tf_{\text{hom}})$ given by (2.9) and, in the latter formula, $h = Tf_{\text{hom}}$. If $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$, then the following statements hold true.*

(i) (Γ -liminf inequality) *For any $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ such that $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for each $\varepsilon > 0$ and $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ with $\mu \in X(\Omega)$ the following inequality holds:*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \geq \mathcal{F}_0(\mu).$$

(ii) (Γ -limsup inequality) *For every $\mu \in X(\Omega)$, there exists a sequence $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ with $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ such that $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \leq \mathcal{F}_0(\mu).$$

Proof of (i). For every $\varepsilon > 0$ we set $\mathcal{B}_\varepsilon := \{B_\varepsilon(x) : x \in \text{supp}(\mu_\varepsilon)\}$ and choose $w_\varepsilon \in \mathcal{AF}_\varepsilon(\mu_\varepsilon)$ in such a way that

$$F_{\delta_\varepsilon}(w_\varepsilon; \Omega_\varepsilon(\mu_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon) + C$$

for some constant C independent of ε . We can assume without loss of generality that

$$\alpha \int_{\Omega_\varepsilon(\mu_\varepsilon)} |\nabla w_\varepsilon|^2 dx \leq F_{\delta_\varepsilon}(w_\varepsilon; \Omega_\varepsilon(\mu_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon) + C \leq C|\log \varepsilon|, \tag{6.1}$$

where the first inequality is a consequence of assumption (G).

In view of (6.1), by applying (5.3) with $U = \Omega$, we have that

$$\mathfrak{E}(\mathcal{B}_\varepsilon, \mu_\varepsilon, \Omega) \leq C|\log \varepsilon|, \tag{6.2}$$

where \mathfrak{E} is defined in (5.1)–(5.2). By (6.2) and the Jensen inequality, considering the definition of $X_\varepsilon(\Omega)$, we get

$$C|\log \varepsilon| \geq \mathfrak{E}(\mathcal{B}_\varepsilon, \mu_\varepsilon, \Omega) \geq 2\pi\alpha \log 2 \sum_{B \in \mathcal{B}_\varepsilon} |\mu_\varepsilon|(B) = 2\pi\alpha \log 2 |\mu_\varepsilon|(\Omega), \tag{6.3}$$

whence we deduce that

$$\text{Rad}(\mathcal{B}_\varepsilon) \leq \varepsilon |\mu_\varepsilon|(\Omega) \leq C\varepsilon |\log \varepsilon| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{6.4}$$

The claim follows by Proposition 5.2 whose assumptions are fulfilled in view of (6.3) and (6.4).

Proof of (ii). Let $\tilde{\mathcal{F}}_0 : X(\Omega) \rightarrow [0, +\infty)$ be the functional defined by

$$\tilde{\mathcal{F}}_0(\mu) := \sum_{i=1}^n \psi(z_i; T f_{\text{hom}}) \quad \text{for every } \mu = \sum_{i=1}^n z_i \delta_{z_i} \in X(\Omega)$$

and note that the functional \mathcal{F}_0 in (0.14) is the lower semicontinuous envelope of $\tilde{\mathcal{F}}_0$ with respect to the flat convergence. Hence, given $\mu = \sum_{i=1}^n z_i \delta_{x_i} \in X(\Omega)$, it is enough to construct $w_\varepsilon \in \mathcal{AF}_\varepsilon(\mu)$ such that

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_\varepsilon; \Omega_\varepsilon(\mu)) \leq \tilde{\mathcal{F}}_0(\mu) = \sum_{i=1}^n \psi(z_i; T f_{\text{hom}}). \quad (6.5)$$

For this purpose, we take $\rho > 0$ such that $\bar{B}_{2\rho}(x_i) \subset \Omega$ for every $i = 1, \dots, n$, and $\bar{B}_{2\rho}(x_i) \cap \bar{B}_{2\rho}(x_j) = \emptyset$ for every $i, j = 1, \dots, n$ with $i \neq j$. Since, by assumption, $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$, then for any s with $0 < s < 1$ it holds true that

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon^s} = 0.$$

We set $\bar{\rho} := \min\{\rho, \frac{1}{2}\}$ and for every $i = 1, \dots, n$, we let $w_{\varepsilon,s}^i \in \tilde{\mathcal{A}}_{\varepsilon^s, 2\bar{\rho}}(z_i)$ (where $\tilde{\mathcal{A}}_{\varepsilon^s, 2\bar{\rho}}(z_i)$ is defined in (3.1)) be such that

$$F_{\delta_\varepsilon}(w_{\varepsilon,s}^i; A_{\varepsilon^s, 2\bar{\rho}}) \leq \inf_{w \in \tilde{\mathcal{A}}_{\varepsilon^s, 2\bar{\rho}}(z_i)} F_{\delta_\varepsilon}(w; A_{\varepsilon^s, 2\bar{\rho}}) + C, \quad (6.6)$$

for some constant C independent of ε . By arguing as in the proof of Lemma 3.1, we can write $w_{\varepsilon,s}^i = e^{u_{\varepsilon,s}^i}$ for some function $u_{\varepsilon,s}^i \in SBV^2(A_{\varepsilon^s, 2\bar{\rho}}(x_i))$ with $u_{\varepsilon,s}^i(\cdot) = z_i \theta(\cdot)$ on $\partial B_{\varepsilon^s} \cup \partial B_{2\bar{\rho}}$ (where θ is defined in (3.7)). Let furthermore $\sigma : [\bar{\rho}, 2\bar{\rho}] \rightarrow [0, 1]$ be the function defined by $\sigma(r) := \frac{1}{\bar{\rho}}(r - \bar{\rho})$ and set $\Theta(\cdot) := \sum_{k=1}^n z_k \theta(\cdot - x_k)$. We define the function $w_{\varepsilon,s} : \Omega_\varepsilon(\mu) \rightarrow \mathcal{S}^1$ as $w_{\varepsilon,s} := e^{u_{\varepsilon,s}}$ where

$$u_{\varepsilon,s}(x) = \begin{cases} z_i \theta(x - x_i) & \text{if } x \in A_{\varepsilon, \varepsilon^s}(x_i) \text{ for some } i, \\ u_{\varepsilon,s}^i(x) & \text{if } x \in A_{\varepsilon^s, \bar{\rho}}(x_i) \text{ for some } i, \\ (1 - \sigma(|x - x_i|)z_i \theta(x - x_i) + \sigma(|x - x_i|)\Theta(x)) & \text{if } x \in A_{\bar{\rho}, 2\bar{\rho}}(x_i) \text{ for some } i, \\ \Theta(x) & \text{elsewhere.} \end{cases} \quad (6.7)$$

The function $w_{\varepsilon,s}$ belongs to $\mathcal{AF}_\varepsilon(\mu)$. By property (G), for every $i = 1, \dots, n$ there exists a constant $C = C(\beta, \bar{\rho}, \Omega, \{z_i\}_i) > 0$ such that

$$F_{\delta_\varepsilon}(w_{\varepsilon,s}; A_{\varepsilon, \varepsilon^s}(x_i)) \leq \beta \int_{A_{\varepsilon, \varepsilon^s}(x_i)} |\nabla w_{\varepsilon,s}|^2 dx = 2\pi\beta |z_i|^2 (1 - s) |\log \varepsilon|, \quad (6.8)$$

$$F_{\delta_\varepsilon}(w_{\varepsilon,s}; \Omega_{2\bar{\rho}}(\mu)) \leq \beta \int_{\Omega_{2\bar{\rho}}(\mu)} |\nabla w_{\varepsilon,s}|^2 dx \leq C, \quad (6.9)$$

and

$$\begin{aligned}
 F_{\delta_\varepsilon}(w_{\varepsilon,s}; A_{\bar{\rho},2\bar{\rho}}(x_i)) &\leq \beta \int_{A_{\bar{\rho},2\bar{\rho}}(x_i)} |\nabla w_{\varepsilon,s}|^2 dx \\
 &\leq C \sum_{\substack{k=1 \\ k \neq i}}^n |z_k|^2 \int_{A_{\bar{\rho},2\bar{\rho}}(x_i)} |\sigma'(|x - x_k|)|^2 |\theta(x - x_k)|^2 dx \\
 &\quad + C \sum_{k=1}^n |z_k|^2 \int_{A_{\bar{\rho},2\bar{\rho}}(x_i)} |\nabla \theta(x - x_k)|^2 dx \leq C.
 \end{aligned}
 \tag{6.10}$$

In addition, since $2\bar{\rho} \leq 1$, by (6.6), Lemma 3.1 and Proposition 3.2, there exists a modulus of continuity ω such that, for every $i = 1, \dots, n$, we have

$$\begin{aligned}
 \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon,s}; A_{\varepsilon^s, \bar{\rho}}(x_i)) &\leq \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon,s}; A_{\varepsilon^s, 2\bar{\rho}}(x_i)) \\
 &\leq \frac{1}{|\log \varepsilon|} \inf_{w \in \mathcal{A}_{\varepsilon^s, 2\bar{\rho}}(x_i)} F_{\delta_\varepsilon}(w; A_{\varepsilon^s, 2\bar{\rho}}(x_i)) + \omega(\varepsilon) \\
 &= \left(s - \frac{|\log(2\bar{\rho})|}{|\log \varepsilon|} \right) \psi(z_i; Tf_{\text{hom}}) + \omega(\varepsilon).
 \end{aligned}
 \tag{6.11}$$

Finally, due to (6.8), (6.9), (6.10) and (6.11) we can choose ω in such a way that

$$\frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon,s}; \Omega_\varepsilon(\mu)) \leq s \sum_{i=1}^n \psi(z_i; Tf_{\text{hom}}) + 2\pi\beta(1-s) \sum_{i=1}^n |z_i|^2 + \omega(\varepsilon).
 \tag{6.12}$$

Suitably choosing $s_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$, we have that $w_\varepsilon = w_{\varepsilon,s_\varepsilon}$ satisfies the relations in (6.5). \square

6.2. The Ginzburg–Landau Model

This subsection is devoted to the proof of Theorem 0.2 in the case $\lambda = 1$. Here we prove such a result under slightly more general assumptions on the potential term. More specifically, we consider $W \in C^0([0, +\infty))$ such that $W(\tau) \geq 0$, $W^{-1}(0) = \{1\}$ and

$$\liminf_{\tau \rightarrow 1} \frac{W(\tau)}{(1 - \tau^2)} > 0, \quad \liminf_{\tau \rightarrow +\infty} W(\tau) > 0.$$

and, we define $GL_\varepsilon^W : H^1(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}$ as

$$GL_\varepsilon^W(v) := \int_\Omega a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla v(x)|^2 dx + \frac{1}{\varepsilon^2} \int_\Omega W(|v(x)|) dx.
 \tag{6.13}$$

We prove the following result, which is a slight generalization of Theorem 0.2 in the case $\lambda = 1$:

Theorem 6.2. Let GL_ε^W be defined in (6.13) where a is a measurable $(0, 1)^2$ -periodic function satisfying $a(x) \in [\alpha, \beta]$ for a.e. $x \in \mathbb{R}^2$. Let moreover \mathbb{A}^{hom} be the symmetric matrix defined in (0.5). If $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \geq 1$, then the following Γ -convergence result holds true:

- (i) (Compactness) Let $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ be such that $GL_\varepsilon^W(v_\varepsilon) \leq C|\log \varepsilon|$. Then, there exists $\mu \in X(\Omega)$ such that, up to subsequences, $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$.
- (ii) (Γ -liminf inequality) Let $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ be such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ for some $\mu \in X(\Omega)$. Then

$$\liminf_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^W(v_\varepsilon)}{|\log \varepsilon|} \geq 2\pi \sqrt{\det \mathbb{A}^{\text{hom}}} |\mu|(\Omega).$$

- (iii) (Γ -limsup inequality) For every $\mu \in X(\Omega)$, there exists a sequence $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ and

$$\limsup_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^W(v_\varepsilon)}{|\log \varepsilon|} \leq 2\pi \sqrt{\det \mathbb{A}^{\text{hom}}} |\mu|(\Omega). \tag{6.14}$$

Proof. Since $\alpha \leq a$ a.e., the compactness property (i) is a corollary of classical results in the variational analysis of the classical GL functional (see for instance [5, Theorem 4.1]).

Proof of (ii). The strategy of the proof is to bound from below $GL_\varepsilon^W(v_\varepsilon)$ with $\mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon)$ defined in (5.4) for a suitable choice of μ_ε and \mathcal{B}_ε satisfying the assumptions of Proposition 5.2. Without loss of generality we can assume that

$$GL_\varepsilon^W(v_\varepsilon) \leq C|\log \varepsilon| \tag{6.15}$$

and by the standard density arguments we can also assume that $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2) \cap C^1(\Omega; \mathbb{R}^2)$. For every $0 < \gamma_1, \gamma_2 < \frac{1}{2}$ and for every $\varepsilon > 0$ we set

$$K_{\varepsilon, \gamma_1, \gamma_2} := \{|v_\varepsilon| \leq 1 - \gamma_1\} \cup \{|v_\varepsilon| \geq 1 + \gamma_2\} \quad \text{and} \quad \Lambda_{\varepsilon, \gamma_1, \gamma_2} := \partial K_{\varepsilon, \gamma_1, \gamma_2} \setminus \partial\Omega.$$

Since $a \geq \alpha$ a.e., by Young’s inequality we have that

$$C|\log \varepsilon| \geq \int_\Omega \alpha |\nabla |v_\varepsilon||^2 + \frac{1}{\varepsilon^2} W(v_\varepsilon) \, dx \geq 2 \frac{\sqrt{\alpha}}{\varepsilon} \int_\Omega \sqrt{W(|v_\varepsilon|)} |\nabla |v_\varepsilon|| \, dx. \tag{6.16}$$

For every $t \in \mathbb{R}$ we set

$$h(t) = \int_t^1 \sqrt{W(s)} \, ds$$

and we define the function $\hat{v}_\varepsilon : \Omega \rightarrow \mathbb{R}^+$ as $\hat{v}_\varepsilon(x) = h(|v_\varepsilon(x)|)$. Note that $\hat{v}_\varepsilon \in H^1(\Omega)$ and that $|\nabla \hat{v}_\varepsilon| = \sqrt{W(|v_\varepsilon|)} |\nabla |v_\varepsilon||$, so that by (6.16), the coarea formula and the mean-value theorem, for every $\bar{\tau} \in (0, 1)$ there exists $\bar{\tau}_\varepsilon \in (\frac{\bar{\tau}}{2}, \bar{\tau})$ such that

$$C\varepsilon |\log \varepsilon| \geq \int_{\frac{\bar{\tau}}{2}}^{\bar{\tau}} \mathcal{H}^1(\{\hat{v}_\varepsilon = \tau\}) \, d\tau \geq \frac{\bar{\tau}}{2} \mathcal{H}^1(\{\hat{v}_\varepsilon = \bar{\tau}_\varepsilon\}). \tag{6.17}$$

We set $\gamma_1^\varepsilon := 1 - h^{-1}(\bar{\tau}_\varepsilon)$ and $\gamma_2^\varepsilon := h^{-1}(-\bar{\tau}_\varepsilon) - 1$ and note that, by construction, there exists $\gamma^{\bar{\tau}} \in (0, 1)$ such that $\gamma^{\bar{\tau}} \rightarrow 0$ as $\bar{\tau} \rightarrow 0$ and there exists a constant $0 < c < 1$ (independent of $\bar{\tau}$) such that $\gamma_1^\varepsilon, \gamma_2^\varepsilon \in (c\gamma^{\bar{\tau}}, \gamma^{\bar{\tau}})$. Moreover, we have that

$$\begin{aligned} \{\hat{v}_\varepsilon < \bar{\tau}_\varepsilon\} &= \{h^{-1}(\bar{\tau}_\varepsilon) < |v_\varepsilon| < h^{-1}(-\bar{\tau}_\varepsilon)\} = \{1 - \gamma_1^\varepsilon < |v_\varepsilon| < 1 + \gamma_2^\varepsilon\} \\ &= \Omega \setminus K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon} \end{aligned}$$

and from the regularity of the function v_ε it follows that

$$\{\hat{v}_\varepsilon = \bar{\tau}_\varepsilon\} = \partial\{\hat{v}_\varepsilon < \bar{\tau}_\varepsilon\} \setminus \partial\Omega = \partial(\Omega \setminus K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}) \setminus \partial\Omega = \partial K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon} \setminus \partial\Omega = \Lambda_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}.$$

Therefore, by (6.17), we obtain that

$$\mathcal{H}^1(\Lambda_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}) \leq C_{\bar{\tau}} \varepsilon |\log \varepsilon|. \tag{6.18}$$

By (6.15), we have that

$$C\varepsilon^2 |\log \varepsilon| \geq \int_{K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}} W(|v_\varepsilon|) \, dx \geq C_{\bar{\tau}} |K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}|. \tag{6.19}$$

As a result, thanks to the Lipschitz regularity of $\partial\Omega$ and to (6.18), we have that

$$\mathcal{H}^1(\partial K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}) \leq C_{\bar{\tau}} \mathcal{H}^1(\Lambda_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}) \leq C_{\bar{\tau}} \varepsilon |\log \varepsilon|. \tag{6.20}$$

Note that, by definition of Hausdorff measure, since $\partial K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}$ is compact, it is always contained in a finite union of balls $B_{r_i}(y_i)$ such that $\sum_i r_i \leq \mathcal{H}^1(\partial K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon})$. Moreover, after a merging procedure, we can always assume that such balls are disjoint. In view of (6.19), for ε small enough, we have that $K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}$ is contained in the union of such balls. Therefore thanks to the previous argument, by (6.20), we have proved that there exists a family of balls, that are denoted by \mathcal{B}'_ε , whose union contains $K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}$ and such that

$$\mathcal{R}ad(\mathcal{B}'_\varepsilon) \leq C_{\bar{\tau}} \mathcal{H}^1(\partial K_{\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon}) \leq C_{\bar{\tau}} \varepsilon |\log \varepsilon|. \tag{6.21}$$

For every $\varepsilon > 0$, let $\mathcal{B}'_\varepsilon(t)$ be a time parametrized family of balls constructed as in Proposition 4.2 starting from $\mathcal{B}'_\varepsilon(0) := \mathcal{B}'_\varepsilon$. Set $\mathcal{B}_\varepsilon := \mathcal{B}'_\varepsilon(1)$, $\mathcal{C}_\varepsilon := \{B \in \mathcal{B}_\varepsilon : B \subset \Omega\}$ and $\mu_\varepsilon := \sum_{B \in \mathcal{C}_\varepsilon} \deg(v_\varepsilon, \partial B) \delta_{x_B}$, where x_B denotes the center of the ball B . Note that

$$1 - \gamma^{\bar{\tau}} < |v_\varepsilon| < 1 + \gamma^{\bar{\tau}} \quad \text{in } \Omega(\mathcal{B}'_\varepsilon) \supset \Omega(\mathcal{B}_\varepsilon). \tag{6.22}$$

Now we consider the function $w_\varepsilon : \Omega(\mathcal{B}'_\varepsilon) \rightarrow \mathbb{R}^2$ defined as $w_\varepsilon(x) := \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$, and we note that $w_\varepsilon \in H^1(\Omega(\mathcal{B}'_\varepsilon); \mathcal{S}^1)$. Moreover, considering (1.10), (6.22), using the

relation $|\nabla v_\varepsilon|^2 = |v_\varepsilon|^2 \left| \nabla \frac{v_\varepsilon}{|v_\varepsilon|} \right|^2 + |\nabla |v_\varepsilon||^2$ and applying Proposition 4.2(4) with $t_1 = 0, t_2 = 1$, and $U = \Omega$, we obtain that

$$\begin{aligned} C |\log \varepsilon| &\geq \int_{\Omega(\mathcal{B}'_\varepsilon)} \alpha |\nabla v_\varepsilon|^2 dx \geq \int_{\Omega(\mathcal{B}'_\varepsilon)} \alpha |v_\varepsilon|^2 \left| \nabla \frac{v_\varepsilon}{|v_\varepsilon|} \right|^2 dx \geq \alpha (1 - \gamma^{\bar{\tau}})^2 \\ &\quad \int_{\Omega(\mathcal{B}'_\varepsilon)} |\nabla w_\varepsilon|^2 dx \\ &\geq 2\pi \alpha (1 - \gamma^{\bar{\tau}})^2 |\mu_\varepsilon|(\Omega) \log 2, \end{aligned}$$

from which we deduce that

$$|\mu_\varepsilon|(\Omega) \leq C_{\bar{\tau}} |\log \varepsilon|. \tag{6.23}$$

Furthermore, by Proposition 4.2(5) and by (6.21) it follows that

$$\mathcal{R}ad(\mathcal{B}_\varepsilon) \leq 2\mathcal{R}ad(\mathcal{B}'_\varepsilon) \leq C_{\bar{\tau}} \varepsilon |\log \varepsilon|. \tag{6.24}$$

Now we show that

$$\mu_\varepsilon \xrightarrow{\text{flat}} \mu. \tag{6.25}$$

By (1.2) $\deg(v_\varepsilon, \partial B) = \deg(w_\varepsilon, \partial B)$ for every $B \in \mathcal{C}_\varepsilon$. Hence, recalling the notion of modified Jacobian introduced in (1.3), we have that

$$(J_{1-\gamma_1^\varepsilon} v_\varepsilon - \pi \mu_\varepsilon)(B) = 0 \quad \text{for every } B \in \mathcal{C}_\varepsilon. \tag{6.26}$$

Using the triangle inequality, Proposition 1.2, (6.23), (6.24) and (6.26) we also have that

$$\begin{aligned} \|J v_\varepsilon - \pi \mu_\varepsilon\|_{\text{flat}} &\leq \|J v_\varepsilon - J_{1-\gamma_1^\varepsilon} v_\varepsilon\|_{\text{flat}} + \|J_{1-\gamma_1^\varepsilon} v_\varepsilon - \pi \mu_\varepsilon\|_{\text{flat}} \\ &\leq C_{\bar{\tau}} \varepsilon |\log \varepsilon| + 2 \sup_{\|\varphi\|_{C_c^{0,1}(\Omega)} \leq 1} \sum_{B \in \mathcal{B}_\varepsilon} |\mu_\varepsilon|(B) \text{osc}_B(\varphi) \\ &\leq C_{\bar{\tau}} \varepsilon |\log \varepsilon| + 2\mathcal{R}ad(\mathcal{B}_\varepsilon) |\mu_\varepsilon|(\Omega) \\ &\leq C_{\bar{\tau}} \varepsilon |\log \varepsilon| + 2C_{\bar{\tau}} \varepsilon |\log \varepsilon|^2. \end{aligned} \tag{6.27}$$

Eventually, (6.25) follows from (6.27) and from the assumption $J v_\varepsilon \xrightarrow{\text{flat}} \pi \mu$ applying the triangle inequality.

Thanks to (1.10) and (6.22), we get that

$$\begin{aligned} GL_\varepsilon^W(v_\varepsilon) &\geq \int_{\Omega(\mathcal{B}_\varepsilon)} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla v_\varepsilon|^2 dx \geq \int_{\Omega(\mathcal{B}_\varepsilon)} a\left(\frac{x}{\delta_\varepsilon}\right) |v_\varepsilon|^2 \left| \nabla \frac{v_\varepsilon}{|v_\varepsilon|} \right|^2 dx \\ &\geq (1 - \gamma^{\bar{\tau}})^2 \int_{\Omega(\mathcal{B}_\varepsilon)} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla w_\varepsilon|^2 dx \geq (1 - \gamma^{\bar{\tau}})^2 \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon), \end{aligned} \tag{6.28}$$

where \mathcal{F}_ε is defined in (5.4) with $f\left(\frac{x}{\delta_\varepsilon}, \nabla w(x)\right) = a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla w(x)|^2$. Thanks to (6.23), (6.24), (6.25) and (6.28) we are in a position to apply Proposition 5.2, obtaining

$$\liminf_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^W(v_\varepsilon)}{|\log \varepsilon|} \geq (1 - \gamma^{\bar{\tau}})^2 \mathcal{F}_0(\mu) = \sqrt{\det \mathbb{A}^{\text{hom}}} |\mu|(\Omega),$$

where the last equality follows by Proposition 2.3. The claim follows letting $\bar{\tau} \rightarrow 0$.

Proof of (iii). We prove the claim under more general assumptions on the functional GL_ε^W . Specifically, let f be a function satisfying **(P)**, **(G)**, **(H)** and define the energy functional $GL_\varepsilon^{W,f} : H^1(\Omega; \mathbb{R}^2) \rightarrow [0, +\infty)$ as

$$GL_\varepsilon^{W,f}(v) := \int_\Omega f\left(\frac{x}{\delta_\varepsilon}, \nabla v(x)\right) dx + \frac{1}{\varepsilon^2} \int_\Omega W(|v(x)|) dx.$$

We prove that for every $\mu = \sum_{i=1}^n z_i \delta_{x_i} \in X(\Omega)$ there exists a sequence $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ and

$$\limsup_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^{W,f}(v_\varepsilon)}{|\log \varepsilon|} \leq \mathcal{F}_0(\mu), \tag{6.29}$$

where \mathcal{F}_0 is defined in (0.14). In view of Proposition 2.3 we have that (6.14) is a consequence of (6.29).

By arguing as in the proof of Theorem 6.1 (iii), we may reduce to the case $\Psi(z_i; Tf_{\text{hom}}) = \psi(z_i; Tf_{\text{hom}})$ for every $i = 1, \dots, n$. Let $\rho > 0$ be such that $\bar{B}_{2\rho}(x_i) \subset \Omega$ for every $i = 1, \dots, n$ and $\bar{B}_{2\rho}(x_i) \cap \bar{B}_{2\rho}(x_j) = \emptyset$ for every $i, j = 1, \dots, n$ with $i \neq j$. For every $0 < s < 1$, since by assumption $\limsup_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon}$ is finite, we have that $\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon^s} = 0$. Finally, we set $\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{i=1}^n B_\varepsilon(x_i)$ and we let u_ε^s be the function defined in (6.7). For every $\varepsilon > 0$ we set

$$v_{\varepsilon,s}(x) := \begin{cases} e^{iu_\varepsilon^s(x)} & \text{if } x \in \Omega_\varepsilon(\mu), \\ \frac{|x-x_i|}{\varepsilon} \left(\frac{x-x_i}{|x-x_i|} \right)^{z_i} & \text{if } x \in B_\varepsilon(x_i) \text{ for some } i, \end{cases}$$

and we note that $v_{\varepsilon,s} \in H^1(\Omega; \mathbb{R}^2)$ and that $Jv_{\varepsilon,s} = \pi\mu$ for every $\varepsilon > 0$. In addition, for almost every $x \in \Omega_\varepsilon(\mu)$ we have that $|v_{\varepsilon,s}(x)| = 1$, hence $W(|v_{\varepsilon,s}(x)|) = 0$. The latter yields

$$\int_\Omega W(|v_{\varepsilon,s}|) dx = \sum_{i=1}^n \int_{B_\varepsilon(x_i)} W(|v_{\varepsilon,s}|) dx \leq nC\pi\varepsilon^2 \tag{6.30}$$

by the continuity of W . Furthermore, by the definition of $v_{\varepsilon,s}$, we have that

$$\sum_{i=1}^n \int_{B_\varepsilon(x_i)} f\left(\frac{x}{\delta_\varepsilon}, \nabla v_{\varepsilon,s}\right) dx \leq \beta \sum_{i=1}^n \int_{B_\varepsilon(x_i)} |\nabla v_{\varepsilon,s}|^2 dx \leq 2\beta \sum_{i=1}^n \pi(1 + |z_i|^2). \tag{6.31}$$

Gathering together (6.30), (6.31) and (6.12), we eventually obtain that

$$\begin{aligned} \frac{1}{|\log \varepsilon|} GL_\varepsilon^{W,f}(v_\varepsilon) &\leq \frac{1}{|\log \varepsilon|} \int_{\Omega_\varepsilon(\mu)} f\left(\frac{x}{\delta_\varepsilon}, \nabla v_{\varepsilon,s}\right) dx + o(1) \\ &\leq s \sum_{i=1}^n \psi(z_i; Tf_{\text{hom}}) + 2\pi\beta(1-s) \sum_{i=1}^n |z_i|^2 + o(1), \end{aligned}$$

which, suitably choosing $s_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$ and setting $v_\varepsilon := v_{\varepsilon,s_\varepsilon}$, gives (6.29). □

7. The Case $\lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \in [0, 1)$

This section is devoted to the proofs of Theorems 0.4 and of 0.2. We will prove the above Γ -convergence results under the assumption that

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = 0, \quad \lambda := \lim_{\varepsilon \rightarrow 0} \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \in [0, 1). \tag{7.1}$$

7.1. The Core-Radius Approach

For the reader’s convenience, we re-state Theorem 0.4 in the case $\lambda \in [0, 1)$ and we recall that $X_\varepsilon(\Omega)$ is defined in (0.11).

Theorem 7.1. *Let $F_{\delta_\varepsilon}, \mathcal{F}_\varepsilon$ be defined in (5.21), (0.13), respectively, with f of the form (1.10), where a is a measurable $(0, 1)^2$ -periodic function satisfying $a(x) \in [\alpha, \beta] \subset (0, +\infty)$ for a.e. $x \in \mathbb{R}^2$. Let moreover \mathbb{A}^{hom} be the matrix defined in (0.5). If (7.1) is satisfied, then the following statements hold true.*

(i) (Γ -liminf inequality) *For any family $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ such that $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ and $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ with $\mu \in X(\Omega)$ we have*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \geq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

(ii) (Γ -limsup inequality) *For every $\mu \in X(\Omega)$, there exists a sequence $\{\mu_\varepsilon\}_\varepsilon \subset X(\Omega)$ with $\mu_\varepsilon \in X_\varepsilon(\Omega)$ for every $\varepsilon > 0$ such that $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \leq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

Proof of (i). For every $\varepsilon > 0$ we set $\mathcal{B}_\varepsilon := \{B_\varepsilon(x) : x \in \text{supp}(\mu_\varepsilon)\}$ and choose $w_\varepsilon \in \mathcal{AF}_\varepsilon(\mu_\varepsilon)$ in such a way that

$$F_{\delta_\varepsilon}(w_\varepsilon; \Omega_\varepsilon(\mu_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon) + C \tag{7.2}$$

for some constant C independent of ε . We can assume without loss of generality that

$$\alpha \int_{\Omega_\varepsilon(\mu_\varepsilon)} |\nabla w_\varepsilon|^2 dx \leq F_{\delta_\varepsilon}(w_\varepsilon; \Omega_\varepsilon(\mu_\varepsilon)) \leq \mathcal{F}_\varepsilon(\mu_\varepsilon) + C \leq C |\log \varepsilon|. \tag{7.3}$$

By arguing as in the first part of the proof of Theorem 6.1(ii) we get that $|\mu_\varepsilon|(\Omega) \leq C |\log \varepsilon|$ and hence $\text{Rad}(\mathcal{B}_\varepsilon) \leq C\varepsilon |\log \varepsilon|$. Therefore, Proposition 5.4 yields the claim.

Proof of (ii). By standard density arguments in the Ginzburg–Landau theory, we can reduce to the case that $\mu = \sum_{i=1}^n z_i \delta_{x_i}$ with $|z_i| = 1$. We set $m := \text{ess inf } a$ and for every $\eta \in (0, 1)$ let

$$E_\eta := \{y \in [0, 1)^2 : a(y) \leq m + \eta\}.$$

By the very definition of ess inf we have that $|E_\eta| > 0$ for every η and there exists $y_\eta \in E_\eta$ having density 1 in E_η ; i.e.,

$$\lim_{r \rightarrow 0} \frac{|E_\eta \cap B_r(y_\eta)|}{r^2} = 1. \tag{7.4}$$

For every $i = 1, \dots, n$ we set

$$x_i^{\delta_\varepsilon, \eta} := \delta_\varepsilon \left\lfloor \frac{x_i}{\delta_\varepsilon} \right\rfloor + \delta_\varepsilon y_\eta. \tag{7.5}$$

Since $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ we have that $x_i^{\delta_\varepsilon, \eta} \rightarrow x_i$ as $\varepsilon \rightarrow 0$ for every $i = 1, \dots, n$. Therefore, setting

$$\mu_{\varepsilon, \eta} := \sum_{i=1}^n z_i \delta_{x_i^{\delta_\varepsilon, \eta}}, \tag{7.6}$$

we have that

$$\mu_{\varepsilon, \eta} \xrightarrow{\text{flat}} \mu \quad \text{as } \varepsilon \rightarrow 0. \tag{7.7}$$

Now we prove that for every $0 < s < 1$ there exists a function $w_{\varepsilon, \eta, s} \in \mathcal{AF}_\varepsilon(\mu_{\varepsilon, \eta})$ such that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon, \eta, s}; \Omega_\varepsilon(\mu_{\varepsilon, \eta})) &\leq 2\pi \left((1 - s\lambda)(m + \eta) + \lambda s \sqrt{\det \mathbb{A}^{\text{hom}}} \right) \\ &|\mu|(\Omega) \\ &+ 2\pi\beta(1 - s)\lambda|\mu|(\Omega). \end{aligned} \tag{7.8}$$

We fix $\rho > 0$ such that $\bar{B}_{2\rho}(x_i^{\delta_\varepsilon, \eta}) \subset \Omega$ for every $i = 1, \dots, n$ and $\bar{B}_{2\rho}(x_i^{\delta_\varepsilon, \eta}) \cap \bar{B}_{2\rho}(x_j^{\delta_\varepsilon, \eta}) \neq \emptyset$ for every $i, j = 1, \dots, n$ with $i \neq j$. Let $\bar{\rho} := \min\{\rho, \frac{1}{2}\}$.

Furthermore, for every $0 < s < 1$, we let $w_{\delta_\varepsilon, s}^i \in \mathcal{A}_{\delta_\varepsilon, 2\bar{\rho}}^s(z_i)$ be such that

$$F_{\delta_\varepsilon}(w_{\delta_\varepsilon, s}^i; A_{\delta_\varepsilon, 2\bar{\rho}}^s) \leq \inf_{w \in \mathcal{A}_{\delta_\varepsilon, 2\bar{\rho}}^s} F_{\delta_\varepsilon}(w; A_{\delta_\varepsilon, 2\bar{\rho}}^s) + C. \tag{7.9}$$

By arguing as in the proof of Lemma 3.1, we can write $w_{\delta_\varepsilon, s}^i = e^{u_{\delta_\varepsilon, s}^i}$ for some function $u_{\delta_\varepsilon, s}^i \in SBV^2(A_{\delta_\varepsilon, 2\bar{\rho}}^s(x_i))$ with $u_{\delta_\varepsilon, s}^i(\cdot) = z_i \theta(\cdot)$ on $\partial B_{\delta_\varepsilon} \cup \partial B_{2\bar{\rho}}$ (where θ is defined in (3.7)).

Let θ be the function defined in (3.7) and let $\Theta(\cdot) := \sum_{k=1}^n z_k \theta(\cdot - x_k)$. Let furthermore $\sigma : [\bar{\rho}, 2\bar{\rho}] \rightarrow [0, 1]$ be the function defined by $\sigma(r) := \frac{1}{\bar{\rho}}(r - \bar{\rho})$. For every $i = 1, \dots, n$ we set

$$u_{\varepsilon, \eta, s}^i(x) := \begin{cases} z_i \theta(x - x_i^{\delta_\varepsilon, \eta}) & \text{if } x \in A_{\varepsilon, \delta_\varepsilon}^s(x_i^{\delta_\varepsilon, \eta}), \\ u_{\delta_\varepsilon, s}^i(x) & \text{if } x \in A_{\delta_\varepsilon, \bar{\rho}}^s(x_i^{\delta_\varepsilon, \eta}), \\ (1 - \sigma(|x - x_i^{\delta_\varepsilon, \eta}|))z_i \theta(x - x_i^{\delta_\varepsilon, \eta}) + \sigma(|x - x_i^{\delta_\varepsilon, \eta}|)\Theta(x) & \text{if } x \in A_{\bar{\rho}, 2\bar{\rho}}(x_i^{\delta_\varepsilon, \eta}), \end{cases}$$

and we define the function $w_{\varepsilon,\eta,s} : \Omega_\varepsilon(\mu_{\varepsilon,\eta}) \rightarrow \mathcal{S}^1$ as $w_{\varepsilon,\eta,s} := e^{iu_{\varepsilon,\eta,s}}$ where

$$u_{\varepsilon,\eta,s}(x) = \begin{cases} u_{\varepsilon,\eta,s}^i(x) & \text{if } x \in A_{\varepsilon,2\bar{\rho}}(x_i^{\delta_\varepsilon,\eta}) \text{ for some } i \\ \Theta(x) & \text{elsewhere.} \end{cases} \tag{7.10}$$

Note that $w_{\varepsilon,\eta,s} \in \mathcal{AF}_\varepsilon(\mu_{\varepsilon,\eta})$.

Let $i = 1, \dots, n$. By (1.10), using the change of variable $x = \delta_\varepsilon y + \delta_\varepsilon \lfloor \frac{x_i}{\delta_\varepsilon} \rfloor$ and the 1-homogeneity of the function a , we have

$$\begin{aligned} F_{\delta_\varepsilon}(w_{\varepsilon,\eta,s}; A_{\varepsilon,\delta_\varepsilon^s}(x_i^{\delta_\varepsilon,\eta})) &= \int_{A_{\frac{\varepsilon}{\delta_\varepsilon},\delta_\varepsilon^{s-1}}(y_\eta)} \frac{a(y)}{|y - y_\eta|^2} dy \\ &= \int_{A_{\frac{\varepsilon}{\delta_\varepsilon},\delta_\varepsilon^{s-1}}(y_\eta) \cap E_\eta} \frac{a(y)}{|y - y_\eta|^2} dy \\ &\quad + \int_{A_{\frac{\varepsilon}{\delta_\varepsilon},\delta_\varepsilon^{s-1}}(y_\eta) \setminus E_\eta} \frac{a(y)}{|y - y_\eta|^2} dy \\ &\leq 2\pi(m + \eta)(|\log \varepsilon| - s|\log \delta_\varepsilon|) \\ &\quad + \beta \int_{A_{\frac{\varepsilon}{\delta_\varepsilon},\delta_\varepsilon^{s-1}}(y_\eta) \setminus E_\eta} \frac{1}{|y - y_\eta|^2} dy. \end{aligned} \tag{7.11}$$

We now estimate the last integral in (7.11). To this end, let $\gamma \in (0, 1)$. We note that

$$\int_{A_{\gamma,\delta_\varepsilon^{s-1}}(y_\eta) \setminus E_\eta} \frac{1}{|y - y_\eta|^2} dy \leq 2\pi \log \frac{\delta_\varepsilon^{s-1}}{\gamma} = 2\pi(1 - s)|\log \delta_\varepsilon| + 2\pi|\log \gamma|. \tag{7.12}$$

Let moreover $I := \lceil \frac{|\log \varepsilon| - |\log \delta_\varepsilon| - |\log \gamma|}{\log 2} \rceil$ and for every $i = 0, 1, \dots, I$ we set $r_i := 2^i \frac{\varepsilon}{\delta_\varepsilon}$; then, using (7.4), we get

$$\begin{aligned} \int_{A_{\frac{\varepsilon}{\delta_\varepsilon},\gamma}(y_\eta) \setminus E_\eta} \frac{1}{|y - y_\eta|^2} dy &\leq \sum_{i=1}^I \int_{A_{r_{i-1},r_i}(y_\eta) \setminus E_\eta} \frac{1}{|y - y_\eta|^2} dy \leq \sum_{i=1}^I 4 \frac{|B_{r_i}(y_\eta) \setminus E_\eta|}{r_i^2} \\ &\leq \left(\frac{|\log \varepsilon| - |\log \delta_\varepsilon| - |\log \gamma|}{\log 2} + 1 \right) C_\eta(\gamma), \end{aligned} \tag{7.13}$$

where $\lim_{\gamma \rightarrow 0} C_\eta(\gamma) = 0$ for every η .

By (7.11), (7.12) and (7.13), using (7.1), we deduce that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon,\delta_\varepsilon,s}; A_{\varepsilon,\delta_\varepsilon^s}(x_i^{\delta_\varepsilon,\eta})) \\ \leq 2\pi(1 - s\lambda)(m + \eta) + 2\pi\beta(1 - s)\lambda + \beta \frac{1 - \lambda}{\log 2} C_\eta(\gamma). \end{aligned} \tag{7.14}$$

In addition, since $2\bar{\rho} \leq 1$, by (7.9), Lemma 3.1, Proposition 3.2 (applied with $\varepsilon = \delta_\varepsilon$ and $s_2 = s$) and (7.1), there exist moduli of continuity ω_1, ω_2 such that for every $i = 1, \dots, n$ we have

$$\begin{aligned} & \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon, \eta, s}; A_{\delta_\varepsilon^s, \bar{\rho}}(x_i^{\delta_\varepsilon, \eta})) = \frac{|\log \delta_\varepsilon|}{|\log \varepsilon|} \frac{1}{|\log \delta_\varepsilon|} F_{\delta_\varepsilon}(w_{\delta_\varepsilon^s, s}^i, A_{\delta_\varepsilon^s, \bar{\rho}}(x_i^{\delta_\varepsilon, \eta})) \\ & \leq (\lambda + \omega_1(\varepsilon)) \frac{1}{|\log \delta_\varepsilon|} \inf_{w \in \mathcal{A}_{\delta_\varepsilon^s, 2\bar{\rho}}^s(x_i)} F_{\delta_\varepsilon}(w; A_{\delta_\varepsilon^s, 2\bar{\rho}}(x_i)) + \omega_2(\varepsilon) \tag{7.15} \\ & = (\lambda + \omega_1(\varepsilon)) \left(s - \frac{|\log(2\bar{\rho})|}{|\log \delta_\varepsilon|} \right) 2\pi \sqrt{\det \mathbb{A}^{\text{hom}}} + \omega_2(\varepsilon), \end{aligned}$$

where the last equality follows by (1.10) and (2.11). By (7.14), (7.15), recalling (6.9) and (6.10) we have that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_{\varepsilon, \eta, s}; \Omega_\varepsilon(\mu_\varepsilon, \eta)) & \leq 2\pi \left((1 - s\lambda)(m + \eta) + \lambda s \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega) \\ & \quad + \left(2\pi\beta(1 - s)\lambda + \beta \frac{1 - \lambda}{\log 2} C_\eta(\gamma) \right) |\mu|(\Omega), \end{aligned}$$

whence, suitably choosing $\gamma = \gamma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we get (7.8). Therefore, suitably choosing $s_\varepsilon \rightarrow 1$ and $\eta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, by (7.8) we get that $\mu_\varepsilon = \mu_{\varepsilon, \eta_\varepsilon} \xrightarrow{\text{flat}} \mu$ and $w_\varepsilon = w_{\varepsilon, \eta_\varepsilon, s_\varepsilon}$ satisfies

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} F_{\delta_\varepsilon}(w_\varepsilon; \Omega_\varepsilon(\mu_\varepsilon)) & \leq \limsup_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_\varepsilon)}{|\log \varepsilon|} \\ & \leq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega). \end{aligned}$$

□

7.2. The Ginzburg–Landau Model

Finally, we prove Theorem 0.2 for $\lambda \in [0, 1)$ in the more general setting introduced in Section 6.2.

Theorem 7.2. *Let GL_ε^W be defined in (6.13) where a is a measurable $(0, 1)^2$ -periodic function satisfying $a(x) \in [\alpha, \beta] \subset (0, +\infty)$ for a.e. $x \in \mathbb{R}^2$. Let moreover \mathbb{A}^{hom} be the symmetric matrix defined in (0.5). If (7.1) is satisfied, then the following Γ -convergence result holds true.*

(i) (Γ -liminf inequality) *Let $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ be such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ for some $\mu \in X(\Omega)$. Then*

$$\liminf_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^W(v_\varepsilon)}{|\log \varepsilon|} \geq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega).$$

(ii) (*Γ-limsup inequality*) For every $\mu \in X(\Omega)$, there exists a sequence $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2)$ such that $Jv_\varepsilon \xrightarrow{\text{flat}} \pi\mu$ and

$$\limsup_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^W(v_\varepsilon)}{|\log \varepsilon|} \leq 2\pi \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega). \tag{7.16}$$

Proof of (i). Without loss of generality we can assume that $GL_\varepsilon^W(w_\varepsilon) \leq C|\log \varepsilon|$ and by standard density arguments we can also assume that $\{v_\varepsilon\}_\varepsilon \subset H^1(\Omega; \mathbb{R}^2) \cap C^1(\Omega; \mathbb{R}^2)$.

Let $\bar{\tau} \in (0, 1)$. By arguing verbatim as in the proof of Theorem 6.2(ii), one can prove that for every $\varepsilon > 0$, there exist $\gamma^{\bar{\tau}} > 0$ with $\gamma^{\bar{\tau}} \rightarrow 0$ as $\bar{\tau} \rightarrow 0$, a family \mathcal{B}_ε of balls such that $\text{Rad}(\mathcal{B}_\varepsilon) \leq C\bar{\tau}\varepsilon|\log \varepsilon|$ and

$$1 - \gamma^{\bar{\tau}} < |v_\varepsilon| < 1 + \gamma^{\bar{\tau}} \quad \text{in } \Omega(\mathcal{B}_\varepsilon), \tag{7.17}$$

and a measure μ_ε with $\text{supp } \mu_\varepsilon \subset \bigcup_{B \in \mathcal{B}_\varepsilon} B$ such that $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$ as $\varepsilon \rightarrow 0$. For every $\varepsilon > 0$ we defined the function $w_\varepsilon \in H^1(\Omega(\mathcal{B}_\varepsilon); S^1)$ as $w_\varepsilon(x) := \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$. By (7.17) we get

$$\begin{aligned} GL_\varepsilon^W(v_\varepsilon) &\geq \int_{\Omega(\mathcal{B}_\varepsilon)} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla v_\varepsilon|^2 \, dx \geq (1 - \gamma^{\bar{\tau}}) \int_{\Omega(\mathcal{B}_\varepsilon)} a\left(\frac{x}{\delta_\varepsilon}\right) |\nabla w_\varepsilon|^2 \, dx \\ &\geq (1 - \gamma^{\bar{\tau}}) \mathcal{F}_\varepsilon(\mu_\varepsilon, \mathcal{B}_\varepsilon), \end{aligned} \tag{7.18}$$

where \mathcal{F}_ε is defined in (5.4) with F_δ defined in (5.21). Since the assumptions of Proposition 5.4 are satisfied, by (7.18) we have that

$$\liminf_{\varepsilon \rightarrow 0} \frac{GL_\varepsilon^W(v_\varepsilon)}{|\log \varepsilon|} \geq (1 - \gamma^{\bar{\tau}}) \left((1 - \lambda) \text{ess inf } a + \lambda \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega),$$

whence the claim follows letting $\bar{\tau} \rightarrow 0$.

Proof of (ii). By arguing as in the proof of Theorem 7.1 (iii), we may reduce to the case $|z_i| = 1$ for every $i = 1, \dots, n$. For every $0 < \eta, s < 1$, let $\mu_{\varepsilon, \eta}$ be defined as in (7.6) and let $u_{\varepsilon, \eta, s}$ be the function provided by (7.10). Setting $\Omega_\varepsilon(\mu_{\varepsilon, \eta}) := \Omega \setminus \bigcup_{i=1}^n B_\varepsilon(x_i^{\delta_\varepsilon, \eta})$ with $x_i^{\delta_\varepsilon, \eta}$ defined in (7.5), for every $\varepsilon > 0$ we define

$$v_{\varepsilon, \eta, s}(x) := \begin{cases} e^{iu_{\varepsilon, \eta, s}(x)} & \text{if } x \in \Omega_\varepsilon(\mu_{\varepsilon, \eta}), \\ \frac{|x-x_i|}{\varepsilon} \left(\frac{x-x_i}{|x-x_i|} \right)^{z_i} & \text{if } x \in B_\varepsilon(x_i^{\delta_\varepsilon, \eta}) \text{ for some } i. \end{cases}$$

We note that $v_{\varepsilon,\eta,s} \in H^1(\Omega; \mathbb{R}^2)$ and that $Jv_{\varepsilon,\eta,s} = \pi\mu_{\varepsilon,\eta}$ for every $\varepsilon > 0$. In addition, for almost every $x \in \Omega_\varepsilon(\mu_{\varepsilon,\eta})$ we have that $|v_{\varepsilon,\eta,s}(x)| = 1$, hence $W(|v_{\varepsilon,\eta,s}(x)|) = 0$. By arguing as in (6.30) and (6.31), and using (7.8), we thus obtain that

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} GL_\varepsilon^W(v_{\varepsilon,\eta,s}) \leq 2\pi \left((1 - s\lambda)(m + \eta) + \lambda s \sqrt{\det \mathbb{A}^{\text{hom}}} \right) |\mu|(\Omega) + 2\pi\beta(1 - s)\lambda|\mu|(\Omega).$$

Therefore, suitably choosing $s_\varepsilon \rightarrow 1$ and $\eta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ and setting $v_\varepsilon = v_{\varepsilon,\eta_\varepsilon,s_\varepsilon}$, by (7.7) we have that $Jv_\varepsilon \xrightarrow{\text{flat}} \mu$ as $\varepsilon \rightarrow 0$ and that $\{v_\varepsilon\}_\varepsilon$ satisfies (7.16).

Remark 7.3. Note that if δ_ε tends to zero much slower than ε in such a way that $\lambda = 0$ in (7.1), then, within the $|\log \varepsilon|$, scaling the homogenization process is not detected by the Γ -limit in Theorems 7.1 and 7.2, which in turns reduces to $2\pi \text{ess inf } a|\mu|(\Omega)$. This is the case if $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^p}{\delta_\varepsilon} = 0$ for all $p \in (0, 1]$; for example, if $\delta_\varepsilon = \frac{1}{|\log \varepsilon|}$.

Example 7.1. We can give an explicit example, choosing a piecewise constant on a checkerboard taking alternatively the values α and β . We have that $\mathbb{A}^{\text{hom}} = \sqrt{\alpha\beta} \mathbb{I}$ (see e.g. [34] Section 1.5), so that the corresponding Γ -limit is

$$2\pi \left((1 - \lambda)\alpha + \lambda\sqrt{\alpha\beta} \right) |\mu|(\Omega),$$

with λ given by (7.1). The limit has the same form if we choose a as a laminate taking only the values α and β with volume fraction $1/2$, whose homogenized matrix \mathbb{A}^{hom} has the eigenvalues $\frac{\alpha+\beta}{2}$ and $\frac{2\alpha\beta}{\alpha+\beta}$ (see e.g. [17, Section 12.2.2]).

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