



On a Conjecture About Higgs Bundles and Some Inequalities

Ugo Bruzzo, Beatriz Graña Otero and Daniel Hernández Ruipérez

Abstract. We briefly review an open conjecture about Higgs bundles that are semistable after pulling back to any curve, and prove it in the rank 2 case. We also prove some results in higher rank under suitable additional assumptions. Moreover, we establish a set of inequalities holding for H-nef Higgs bundles that generalize some of the Fulton–Lazarsfeld inequalities for numerically effective vector bundles.

Mathematics Subject Classification. 14H60, 14J60.

Keywords. Higgs bundles, semistability, numerical effectiveness.

1. Introduction

The progenitor of the results discussed in this paper may be traced back to a theorem by Miyaoka [20], which characterizes the semistability of a vector bundle E on a smooth projective curve X in terms of the nefness of a numerical class in the projectivized bundle $\mathbb{P}E$: if

$$\lambda(E) = c_1(\mathcal{O}_{\mathbb{P}E}(1)) - \frac{1}{r}\pi_1^*(c_1(E)) \in N^1(\mathbb{P}E) \otimes \mathbb{Q}, \quad (1)$$

where $\pi_1: \mathbb{P}E \rightarrow X$ is the projection, and $r = \text{rk } E$, then E is semistable if and only if $\lambda(E)$ is nef (note that $r\lambda(E)$ is the relative anticanonical class of $\mathbb{P}E$ over X).

U.B.’s research is partly supported by Bolsa de Produtividade 313333/2020-3 of Brazilian CNPq, by PRIN “Birational geometry and moduli problems” and INdAM-GNSAGA. B.G.O. and D.H.R.’s research is partly supported by Grant PID2021-128665NB-I00 funded by MCIN/AEI/ 10.13039/501100011033, by “ERDF A way of making Europe,” and Universidad de Salamanca through Programa XIII.

1.1. Curve Semistable (Higgs) Bundles

The following theorem was proved in [21] and rediscovered in [6] in a slightly different and seemingly stronger, albeit equivalent form. It may be regarded as a higher dimensional generalization of Miyaoka’s theorem. Let X be an n -dimensional smooth connected complex projective variety. For any coherent \mathcal{O}_X -module F of positive rank define its discriminant as

$$\Delta(F) = c_2(F) - \frac{\text{rk } F - 1}{2 \text{rk } F} c_1(F)^2 \in H^4(X, \mathbb{Q}).$$

Moreover, if E is a vector bundle on X , the class $\lambda(E)$ is defined as in Eq. (1).

Theorem 1.1. *Let E be a vector bundle on X . The following conditions are equivalent:*

- (i) E is semistable with respect to some polarization H , and $\Delta(E) = 0$;
- (ii) for any morphism $f: C \rightarrow X$, where C is a smooth projective curve, the vector bundle f^*E is semistable;
- (iii) the class $\lambda(E)$ is nef.

(In Nakayama the condition on the discriminant was $\Delta(E) \cdot H^{n-2} = 0$, but via Theorem 2 in [24] this is readily shown to be equivalent to $\Delta(E) = 0$ whenever E is semistable with respect to H .) We shall call *curve semistable* the vector bundles satisfying condition (ii). It may be natural to wonder if Theorem 1.1 also holds true for Higgs bundles. We recall that a Higgs sheaf is a pair $\mathfrak{F} = (F, \phi)$, where F is a coherent \mathcal{O}_X -module, and $\phi: F \rightarrow F \otimes \Omega_X^1$ is an \mathcal{O}_X -linear morphism such that the composition

$$\phi \wedge \phi: F \xrightarrow{\phi} F \otimes \Omega_X^1 \xrightarrow{\phi \times \text{id}} F \otimes \Omega_X^1 \otimes \Omega_X^1 \rightarrow F \otimes \Omega_X^2$$

is zero. A Higgs bundle is a Higgs sheaf with F locally free. Semistability and stability are defined as for vector bundles but only with reference to ϕ -invariant subsheaves. Curve semistability is defined as for vector bundles. So the Higgs bundle version of Theorem 1.1 is the following conjecture:

Conjecture 1.2. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on X . The following conditions are equivalent:*

- (i) \mathfrak{E} is semistable with respect to some polarization H , and $\Delta(E) = 0$;
- (ii) for any morphism $f: C \rightarrow X$, where C is a smooth projective curve, the Higgs bundle $f^*\mathfrak{E}$ is semistable.

(We shall state the condition generalizing the nefness of the class $\lambda(E)$ later on.) The fact that condition (i) implies condition (ii) was proved in [6]. A motivation for expecting that the opposite implication may hold true is Bogomolov inequality [14]: if E is a vector bundle on an n -dimensional smooth projective variety, semistable with respect to a polarization H , then $\Delta(E) \cdot H^{n-2} \geq 0$. The underlying vector bundle E of a semistable Higgs bundle $\mathfrak{E} = (E, \phi)$ satisfies the same inequality, even when E itself is not semistable [23]; i.e., semistability is a sufficient but non-necessary condition for the non-negativity of the quantity $\Delta(E) \cdot H^{n-2}$, and one can imagine the same happens for the vanishing of $\Delta(E)$ for curve semistable bundles.

We conjecture that the reverse implication holds true for any smooth projective variety. In this paper we prove this when \mathfrak{E} has rank two; actually,

we prove the implication in any rank when the Grassmannian of Higgs quotients of some rank (to be defined later) has a component that is a divisor in the full Grassmannian and surjects onto X . Then we prove that such a component always exists in rank two.

1.2. Higgs Varieties

One easily shows that a curve semistable Higgs bundle is semistable with respect to any polarization. So the nontrivial content of the conjecture is the following statement:

A curve semistable Higgs bundle has vanishing discriminant.

Here curve semistability for Higgs bundles is defined as in condition (ii) of Conjecture 1.2. Waiting for the conjecture to be eventually settled in the positive or negative, it makes sense to prove it for specific classes of varieties. The authors of [8] defined a *Higgs variety* X as one on which the conjecture holds. The easiest case is that of varieties with slope-semistable cotangent bundle of nonnegative degree, simply because in this situation the underlying vector bundle E of a curve semistable Higgs bundle is itself curve semistable. Starting from this one can identify other Higgs varieties, such as:

- Rationally connected varieties;
- Abelian varieties;
- Fibrations over a Higgs variety whose fibers are rationally connected;
- Bases of finite étale covers whose total space is a Higgs variety;
- Varieties of dimension ≥ 3 containing an effective ample divisor which is a Higgs variety;
- Varieties with nef tangent bundle (in dimension 2 and 3 these were classified in [10]);
- Varieties birational to a Higgs variety.

Moreover, in [7] it was shown that algebraic K3 surfaces are Higgs varieties, and this was extended, using different techniques, to simply connected Calabi–Yau varieties in [3]. Some results in the case of elliptic surfaces are proved in [9]. A review of this problem updated to 2017 can be found in [18].

1.3. Contents

The main tool we use in this paper is the *Higgs Grassmannian* of a Higgs bundle $\mathfrak{E} = (E, \phi)$, a notion that some of us introduced in [6]. This object is defined in Sect. 2, where some of its basic properties are studied. It seems quite difficult to find general results about the Higgs Grassmannian, but its structure is quite clear in the case $\text{rk } E = 2$, and this is indeed the key to the proof of the conjecture in the rank 2 case that we give in Sect. 4.2. Actually in Sect. 3 we prove the conjecture assuming that the rank d Higgs Grassmannian $\text{Gr}_d(\mathfrak{E})$ has a component that is a divisor in the full Grassmann bundle $\text{Gr}_d(E)$ which surjects onto X . Such a divisor always exists in the rank 2 case, due to the fact that the Higgs Grassmannian of a rank 2 Higgs bundle over a curve is never empty, thus providing a full proof of the conjecture in the rank 2 case.

The Higgs Grassmannian allows one to introduce a notion of numerical effectiveness for Higgs bundles, a notion that “feels” the Higgs field. This was studied in [4, 5]. In the final Sect. 5 of this paper we show that Higgs bundles that are numerically effective in this sense satisfy some inequalities which generalize some of the Fulton–Lazarsfeld inequalities for numerically effective vector bundles ([13], see also [10]).

Notation and conventions. All varieties and schemes are over the complex numbers, and, unless otherwise stated, all varieties are supposed to be connected. A “sheaf” on a scheme X will be a coherent \mathcal{O}_X -module.

2. The Higgs Grassmannian

The *Higgs Grassmannian* is an object that parameterizes locally free Higgs quotients of a Higgs bundle

exactly as the usual Grassmann bundle parameterizes locally free quotients of a vector bundle. This was introduced in [6]. We recall here its definition and some of its properties.

2.1. Definition of the Higgs Grassmannian

Let X be a smooth variety over \mathbb{C} . For a given rank r vector bundle E on X , and for every d in the range $0 < d < r$, we denote the Grassmann bundle of rank d locally free quotients of E as $\text{Gr}_d(E)$. Since $\text{Gr}_1(E) = \mathbb{P}E$ we shall use the latter notation. One has the universal exact sequence

$$0 \rightarrow S_d \rightarrow \pi_d^*E \rightarrow Q_d \rightarrow 0$$

of vector bundles on $\text{Gr}_d(E)$, where Q_d is the rank d universal quotient bundle, S_d is the corresponding kernel, and $\pi_d: \text{Gr}_d(E) \rightarrow X$ is the projection. If $\mathfrak{E} = (E, \phi)$ is a Higgs bundle, we form the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_d & \xrightarrow{a_d} & \pi_d^*E & \longrightarrow & Q_d \longrightarrow 0 \\
 & & & & \downarrow \pi_d^*\phi & & \\
 0 & \longrightarrow & S_d \otimes \Omega_{\text{Gr}_d(E)}^1 & \longrightarrow & \pi_d^*E \otimes \Omega_{\text{Gr}_d(E)}^1 & \xrightarrow{b_d} & Q_d \otimes \Omega_{\text{Gr}_d(E)}^1 \longrightarrow 0
 \end{array} \tag{2}$$

The d -th Higgs Grassmannian of \mathfrak{E} , denoted $\text{Gr}_d(\mathfrak{E})$, is the subscheme of $\text{Gr}_d(E)$ defined by the zero locus of the composition $b_d \circ \pi_d^*\phi \circ a_d$. By construction, the restrictions of the bundles S_d and Q_d to $\text{Gr}_d(\mathfrak{E})$ carry Higgs fields induced by $\pi_d^*\phi$, so that we have an exact sequence of Higgs bundles on $\text{Gr}_d(\mathfrak{E})$

$$0 \rightarrow \mathfrak{S}_d \rightarrow \rho_d^*\mathfrak{E} \rightarrow \mathfrak{Q}_d \rightarrow 0.$$

The scheme $\text{Gr}_d(\mathfrak{E})$ may be singular, reducible, nonreduced, non-equidimensional. On the positive side, it enjoys the analogous universal property of the usual Grassmann bundles: if $f: Y \rightarrow X$ is a scheme morphism,

and \mathfrak{G} is a rank d locally free Higgs quotient of $f^*\mathfrak{E}$, there is a morphism $g: Y \rightarrow \text{Gr}_d(\mathfrak{E})$ such that $\mathfrak{G} = g^*\mathfrak{Q}_d$, and the diagram

$$\begin{array}{ccc}
 & & \text{Gr}_d(\mathfrak{E}) \\
 & \nearrow g & \downarrow \rho_d \\
 Y & \xrightarrow{f} & X
 \end{array}$$

commutes.

Now assume that X is projective. Given a rank r Higgs bundle $\mathfrak{E} = (E, \phi)$ on X , for every $0 < d < r$ we define the following classes in $N^1(\text{Gr}_d(\mathfrak{E})) \otimes \mathbb{Q}$

$$\theta_d(\mathfrak{E}) = c_1(Q_d) - \frac{d}{r} \rho_d^*(c_1(E)). \tag{3}$$

It was proved in [6] (see also [5]) that \mathfrak{E} is curve semistable if and only if *all* classes $\theta_d(\mathfrak{E})$ are nef. Note that $\theta_1(\mathfrak{E})$ is the restriction of the class $\lambda(E) \in N^1(\mathbb{P}E) \otimes \mathbb{Q}$ to $\text{Gr}_1(\mathfrak{E})$. Here one can note a different behavior of Higgs bundles as opposed to vector bundles: while in the latter case the condition that the class $\lambda(E)$ is nef is equivalent to curve semistability, in the Higgs case one needs the nefness of *all* classes $\theta_d(\mathfrak{E})$; see [6] for an example of a rank 3 Higgs bundle on a curve with $\theta_1(\mathfrak{E})$ nef, $\theta_2(\mathfrak{E})$ not nef, which is not semistable.

2.2. Higgs Numerical Effectiveness

In [5] by means of the Higgs Grassmannians a notion of numerical effectiveness for Higgs bundles was introduced. It is a definition based on recursion on the rank of the successive universal quotient bundles. Since we are going to use this definition later on, we recall it here.

Definition 2.1. A Higgs bundle $\mathfrak{E} = (E, \phi)$ of rank one on a smooth projective variety is said to be Higgs-numerically effective (for short, H-nef) if the underlying vector bundle E is numerically effective in the usual sense. If $\text{rk } E \geq 2$ we require that:

- (i) All bundles \mathfrak{Q}_k are Higgs numerically effective;
- (ii) The line bundle $\det(E)$ is nef.

If both \mathfrak{E} and \mathfrak{E}^* are Higgs-numerically effective, \mathfrak{E} is said to be Higgs-numerically flat (H-nflat).

3. The Conjecture in any Rank

3.1. A Push-Forward Formula

We recall from [16] a push-forward formula for the Segre classes of the universal quotient bundle over Grassmann bundles $\pi_d: \text{Gr}_d(E) \rightarrow X$. Here X will be a smooth projective variety of dimension n and E a rank $r > 1$ vector bundle.

Greek letters such as λ, μ will denote a *partition*, i.e., a finite non-increasing sequence of natural numbers. We let

$$|\lambda| = \sum_{i=1}^q \lambda_i,$$

where $\lambda = (\lambda_1, \dots, \lambda_q)$, while π_* will denote the push-forward of Chow groups

$$\pi_* : A^k(\text{Gr}_d(E)) \rightarrow A^{k-d(r-d)}(X).$$

Moreover, we define the Segre classes of the vector bundle F on a variety X by the formula

$$s(F) = \sum_{i=0}^{\dim X} (-1)^i s_i(F) = \frac{1}{c(F)}$$

where $c(F)$ is the total Chern class of F (we follow the normalization of [12], hence the minus signs).

Lemma 3.1. [15,16] *Let Q be the rank d universal quotient bundle of a rank r vector bundle E over X . The following push-forward formula holds:*

$$\pi_* \Delta_\mu(s(Q)) = \Delta_{\mu-\varepsilon}(s(E)).$$

Here

- (i) ε is the partition of length d whose elements are all $r - d$;
- (ii) for every $c \in A^\bullet(X)$, $\Delta_\lambda(c)$ is the Schur polynomial associated with λ computed on the components of c in $A^\bullet(X)$, that is,

$$\Delta_\lambda(c) = \det[c_{\lambda_i + j - i}]_{1 \leq i, j \leq n}.$$

Corollary 3.2. *Let $\chi = c_1(Q)$. For N in the range $d(r - d) \leq N \leq d(r - d) + n$ one has*

$$\pi_* \chi^N = \sum_{|\lambda|=N-d(r-d)} f^{\lambda+\varepsilon} \Delta_\lambda(s(E))$$

where f^λ is the number of standard Young tableaux of shape λ .¹

Some notation and facts:

- A natural number k is regarded as a partition of length 1. For every k , $\Delta_k(c) = c_k$, i.e., the degree k term of c . In particular,

$$\Delta_k(c(E)) = c_k(E), \quad \Delta_k(s(E)) = s_k(E).$$

- Conjugate partitions: given a partition λ , let $\bar{\lambda}$ be the conjugate partition, i.e., the partition which describes the conjugate Young tableau of λ (the one obtained by flipping it with respect to its diagonal.) Then

$$\Delta_{\bar{\lambda}}(c(E)) = \Delta_\lambda(s(E)).$$

¹A standard Young tableau [2] is a Young tableau (say with b boxes) whose boxes are labelled with the integers from 1 to b , in such a way that all rows and columns contain increasing sequences of integers. To “have shape $\lambda = (\lambda_1, \dots, \lambda_q)$ ” means that the i -th row has λ_i boxes. Then f^λ is the number of ways a Young tableau of shape λ can be labelled.

- We shall denote by $p(k)$ the partition made up by k 1's. Then k and $p(k)$ are conjugate partitions, so that

$$\Delta_{p(k)}(s(E)) = c_k(E).$$

- An explicit formula for f^λ is the following [11, 19]. Let λ be a partition of length $q > 1$. Then

$$f^\lambda = \frac{|\lambda|!}{\prod_{i=1}^q \ell_i!} \prod_{1 \leq i < j \leq q} (\ell_i - \ell_j), \quad \ell_i = \lambda_i + q - i$$

We note that

$$f^{\varepsilon+1} = \frac{md}{r} f^\varepsilon, \quad f^{\varepsilon+p(2)} = \frac{m(m+1)d(d-1)}{2r(r-1)} f^\varepsilon$$

$$f^{\varepsilon+2} = \frac{m(m+1)d(d+1)}{2r(r+1)} f^\varepsilon$$

where $m = d(r - d) + 1$. So, setting $n = 2$, Corollary 3.2 yields

$$\pi_* \chi^{m-1} = f^\varepsilon, \quad \pi_* \chi^m = \frac{md}{r} f^\varepsilon c_1(E) \tag{4}$$

$$\pi_* \chi^{m+1} = \frac{m(m+1)d(d+1)}{2r(r+1)} f^\varepsilon (c_1(E)^2 - c_2(E))$$

$$+ \frac{m(m+1)d(d-1)}{2r(r-1)} f^\varepsilon c_2(E). \tag{5}$$

3.2. The result

In this Section we prove the main result of this paper.

Theorem 3.3. *Given a curve semistable Higgs bundle $\mathfrak{E} = (E, \phi)$ on a surface X , if, for some d in the range $0 < d < r = \text{rk } E$, the Higgs Grassmannian $\text{Gr}_d(\mathfrak{E})$ has an irreducible component Z which is a divisor in $\text{Gr}_d(E)$ and surjects onto X , then $\Delta(E) = 0$.*

Proof. Using the Leray-Hirsch Theorem we define the classes $\beta_i \in A^i(X)$ by

$$[Z] = \chi \pi^* \beta_0 + \pi^* \beta_1 \in A^1(\text{Gr}_d(E)),$$

where $[Z]$ is the class of Z in $A^1(\text{Gr}_d(E))$ (for a version of the Leray-Hirsch Theorem for Chow groups which applies to the present case see [17]). Recalling equation (3),

a rather lengthy computation yields²

$$\begin{aligned} \int_Z \theta_d(\mathfrak{E})^m &= \int_{\text{Gr}_d(E)} \theta_d(\mathfrak{E})^m \smile [Z] = \int_{\text{Gr}_d(E)} \left[\chi - \frac{d}{r} \pi^* c_1(E) \right]^m (\chi \pi^* \beta_0 + \pi^* \beta_1) \\ &= \int_{\text{Gr}_d(E)} \left[\beta_0 \chi^{m+1} + \left(\beta_1 - \beta_0 \frac{md}{r} c_1(E) \right) \chi^m \right. \\ &\quad \left. + \left(-\frac{md}{r} \beta_1 c_1(E) + \frac{m(m-1)d^2}{2r^2} \beta_0 c_1(E)^2 \right) \chi^{m-1} \right] \\ &= -\frac{m(m+1)d(r-d)}{r(r+1)(r-1)} f^\varepsilon \beta_0 \Delta(E) = -\frac{m(m+1)(m-1)}{r(r+1)(r-1)} f^\varepsilon \beta_0 \Delta(E). \end{aligned}$$

In the last line, by integrating over X , we think of β_0 and $\Delta(E)$ as integers. β_0 is positive by the following argument. Denote by π_Z the restriction of π_d to Z , and by Z_x its fiber at a point $x \in X$. We also denote $F_x = \pi_d^{-1}(x)$. We assume that π_Z is surjective so that by [1, Lemma 29.28.2] every irreducible component of its fibers has either dimension $d(r-d)$ or $d(r-d)-1$ (to apply that result we need Z to be integral but we can achieve that by replacing it with its reduced subscheme if needed). On the other hand, since π_Z is proper, by the semicontinuity of the fiber dimension (see e.g. [1, Lemma 37.29.5]) the locus in X where the fiber Z_x has dimension $d(r-d)-1$ is open. It is nonempty because if all fibers Z_x have dimension $d(r-d)$ then $\text{Gr}_d(\mathfrak{E})$ would coincide with $\text{Gr}_d(E)$, a situation which we may exclude. So the generic fiber of π_Z has dimension $d(r-d)-1$, the “expected dimension”. Hence for generic x , Z_x determines a class in $A^1(F_x)$. Since the restriction χ_x of χ to F_x is ample,

$$0 < [Z_x] \cdot \chi_x^{m-2} = \beta_0 \chi_x^{m-1},$$

so that $\beta_0 > 0$. So we get

$$\Delta(E) \leq 0.$$

Now by the Bogomolov inequality $\Delta(E) \geq 0$ we obtain $\Delta(E) = 0$. □

Note the “miraculous disappearance” of β_1 !

4. The Conjecture in Rank Two

4.1. A Non Emptiness Result

The following result is a key to our proof of the Conjecture in rank two.

Theorem 4.1. *Let X be a smooth curve, which may be projective or affine, and let $\mathfrak{E} = (E, \phi)$ be a rank 2 Higgs bundle on X . The Higgs Grassmannian $\text{Gr}_1(\mathfrak{E})$ of rank one Higgs quotients of \mathfrak{E} is not empty.*

Lemma 4.2. *Let X be a smooth curve (projective or affine), and let $\mathfrak{E} = (E, \phi)$ be a rank 2 Higgs bundle on X . Assume that the Higgs Grassmannian*

²From the second line we omit to write the pullbacks. We use the fact that integration on $\text{Gr}_d(E)$ is the push-forward to X followed by integration on X . The last line follows from the insertion of Eqs. (4), (5) and some fractional calculus gimmickry.

$\text{Gr}_1(\mathfrak{E})$ of rank one Higgs quotients of \mathfrak{E} is empty. Then the Higgs field ϕ induces a splitting of the exact sequence

$$0 \rightarrow \pi_1^* \Omega_X^1 \rightarrow \Omega_{\mathbb{P}E}^1 \rightarrow \Omega_{\mathbb{P}E/X}^1 \rightarrow 0. \tag{6}$$

Note that since $\text{Gr}_1(\mathfrak{E})$ is assumed to be empty, the Higgs field ϕ is necessarily nonzero.

Proof. We refer to diagram (2). Note that $Q_1^\vee \otimes S_1 \simeq \Omega_{\mathbb{P}E/X}^1$. Define a map $s: \Omega_{\mathbb{P}E/X}^1 \rightarrow \Omega_{\mathbb{P}E}^1$ by letting

$$s = (b_1 \circ \pi_1^* \phi \circ a_1) \otimes \text{id}_{Q_1^\vee}.$$

Since $\Omega_{\mathbb{P}E/X}^1$ is a line bundle, and $\Omega_{\mathbb{P}E}^1$ is locally free, the morphism s is either zero or is injective; but if it were zero, since the Higgs Grassmannian $\text{Gr}_1(\mathfrak{E})$ is the zero locus of the composition

$$b_1 \circ \pi_1^* \phi \circ a_1 = s \otimes \text{id}_{Q_1^\vee},$$

the Higgs Grassmannian $\text{Gr}_1(\mathfrak{E})$ would be the entire $\mathbb{P}E$, and therefore would not be empty. So we have an exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}E/X}^1 \xrightarrow{s} \Omega_{\mathbb{P}E}^1 \rightarrow R \rightarrow 0$$

where R is by definition the quotient, which has rank one.

As the Higgs Grassmannian is empty, s has no zeroes, so that R is locally free. We form the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega_{\mathbb{P}E/X}^1 & \xrightarrow{s} & \Omega_{\mathbb{P}E}^1 & \xrightarrow{r} & R \longrightarrow 0 \\
 & & \uparrow h & & \parallel & \nearrow g & \\
 0 & \longrightarrow & \pi_1^* \Omega_X^1 & \xrightarrow{i} & \Omega_{\mathbb{P}E}^1 & \xrightarrow{p} & \Omega_{\mathbb{P}E/X}^1 \longrightarrow 0
 \end{array} \tag{7}$$

We show that the morphism $g = r \circ i: \pi_1^* \Omega_X^1 \rightarrow R$ cannot be zero. Indeed if it were zero we would have a morphism $h: \pi_1^* \Omega_X^1 \rightarrow \Omega_{\mathbb{P}E/X}^1$ which is not zero as $i = s \circ h$. However since the fiber degree of $\Omega_{\mathbb{P}E/X}^1$ is -2 , the restriction of h to each fiber of π is zero, i.e., $h = 0$, which is a contradiction.

Thus, g is nonzero, hence is injective. We prove it is an isomorphism. We have an exact sequence

$$0 \rightarrow \pi_1^* \Omega_X^1 \xrightarrow{g} R \rightarrow N \rightarrow 0$$

where N has rank zero. For any fiber F of π_1 , by a standard argument, we have an exact sequence

$$0 \rightarrow \mathcal{O}_F \rightarrow R|_F \rightarrow N|_F \rightarrow 0.$$

Since R has fiber degree 0, $R|_F$ is isomorphic to \mathcal{O}_F , so that $N|_F = 0$. As this holds for every fiber, $N = 0$, hence g is an isomorphism.

Now we have a diagram

$$\begin{array}{ccccccc}
 & & R & & & & \\
 & & \downarrow g^{-1} & \swarrow r & & & \\
 0 & \longrightarrow & \pi^* \Omega_X^1 & \xrightarrow{i} & \Omega_{\mathbb{P}E}^1 & \longrightarrow & \Omega_{\mathbb{P}E/X}^1 \longrightarrow 0
 \end{array}$$

which shows that the sequence (6) splits. □

Proof of Theorem 4.1.. Note that the first line in diagram (7) splits as $i \circ g^{-1}$ is a section of r . Let t' be a retraction of the morphism s . Then $t = t' \otimes \text{id}_Q$ is a morphism $Q_1 \otimes \Omega_{\mathbb{P}E/X}^1 \rightarrow S_1$. Define $\zeta : \pi_1^* E \rightarrow S_1$ as

$$\zeta = t \circ b_1 \circ \pi_1^* \phi.$$

Then

$$\zeta \circ a_1 = t \circ b_1 \circ \pi_1^* \phi \circ a_1 = (t' \otimes \text{id}_{Q_1}) \circ (s \otimes \text{id}_{Q_Y}) = \text{id}_{S_1}$$

so that the first line in diagram (2) splits. But this is impossible as on each fiber of π_1 that sequence reduces to the Euler exact sequence. □

Corollary 4.3. *Let $\mathfrak{E} = (E, \phi)$ be a rank two Higgs bundle on a smooth n -dimensional projective variety X . The Higgs Grassmannian $\text{Gr}_1(\mathfrak{E})$ has a component of dimension at least n which surjects onto X .*

Proof. If $\text{Gr}_1(\mathfrak{E})$ does not have such a component, let Y be its image in X (actually taking its reduced subscheme if it happens to be nonreduced), let C be a curve in X not contained in Y , and let C' be C minus its intersection points with Y , and minus its possible singular points. Then $\mathfrak{E}|_{C'}$ has an empty Higgs Grassmannian, a contradiction to Corollary 4.1. □

Remark 4.4. The splitting of the exact sequence (6) means that E is projectively flat, i.e., $\mathbb{P}E$ comes from a projective representation $\pi_1(X) \rightarrow \text{PGL}_2(\mathbb{C})$ of the fundamental group of X . This agrees with the result in [22], whose authors, as a particular case of their equivalence of categories, prove that semistable Higgs bundles on a curve are projectively flat. Note indeed that if the Higgs Grassmannian is empty, the Higgs bundle is stable.

4.2. The proof

We start with the case $\dim X = 2$, i.e., X is smooth projective surface. From Corollary 4.3 we get that $\text{Gr}_1(\mathfrak{E})$ has a component Z of dimension 2, and we are in the hypotheses of Sect. 3.2, so that $\Delta(E) = 0$.

This can be extended to the higher dimensional case $\dim X = n$. Let H be the class of an ample line bundle $L = \mathcal{O}_X(D)$ and let Y be the intersection of $n - 2$ generic divisors in the linear system $|mD|$ for $m \gg 0$. The result for dimension 2 implies that

$$\Delta(E) \cdot H^{n-2} = \frac{1}{m^{n-2}} \Delta(E|_Y) = 0.$$

So we have proved:

Theorem 4.5. *Let $\mathfrak{E} = (E, \phi)$ be a rank two Higgs bundle on an n -dimensional smooth projective variety. Then the following conditions are equivalent:*

- (i) \mathfrak{E} is curve semistable;
- (ii) \mathfrak{E} is semistable with respect to some polarization and the class $\theta_1(\mathfrak{E})$ is nef;
- (iii) \mathfrak{E} is semistable with respect to a polarization H , and

$$\Delta(E) \cdot H^{n-2} = 0.$$

Proof. We know from [6] that (i) and (ii) are equivalent. The implication (ii) \Rightarrow (iii) was just proved. To prove (iii) \Rightarrow (ii) one can easily adapt the proof of one of the directions of Theorem 1.3 in [6]. □

Actually condition (iii) can be strengthened to $\Delta(E) = 0$. To see that we need some preliminary results. In particular, we shall prove that rank 2 H-nflat Higgs bundles have vanishing Chern classes (H-nflat Higgs bundles were defined in Definition 2.1).

Lemma 4.6. *If $\mathfrak{E} = (E, \phi)$ is an H-nflat Higgs bundle of any rank over a smooth n -dimensional projective variety X and $\Delta(E) \cdot H^{n-2} = 0$, then all Chern classes of E vanish.*

Proof. One has $c_1(E) = 0$ as $\det(E)$ is numerically flat and then the condition on the discriminant implies $\text{ch}_2(E) \cdot H^{n-2} = 0$. By Theorem 2 in [24] \mathfrak{E} has a filtration in (stable) Higgs bundles with zero Chern classes, whence the claim follows (note that H-nflat Higgs bundles are semistable with respect to any polarization, see [5]). □

Theorem 4.7. *If $\mathfrak{E} = (E, \phi)$ is a rank two H-nflat Higgs bundle over a smooth projective surface X then all Chern classes of E vanish.*

Proof. Since H-nflat bundles are curve semistable by Theorem 4.5, if H is an ample class in X , we have $\Delta(E) \cdot H^{n-2} = 0$. Then Lemma 4.6 implies the claim. □

We can now strengthen Theorem 4.5 in the following form.

Theorem 4.8. *Let $\mathfrak{E} = (E, \phi)$ be a rank two Higgs bundle on an n -dimensional smooth projective variety. Then the following conditions are equivalent:*

- (i) \mathfrak{E} is curve semistable;
- (ii) \mathfrak{E} is semistable with respect to some polarization and the class $\theta_1(\mathfrak{E})$ is nef;
- (iii) \mathfrak{E} is semistable with respect to some polarization and $\Delta(E) = 0$.

Proof. We only need to show that if (ii) holds then $\Delta(E) = 0$. If H is an ample class in X , from Theorem 4.5 we have $\Delta(E) \cdot H^{n-2} = 0$, which is equivalent to $\Delta(\text{End}(E)) \cdot H^{n-2} = 0$. So the Higgs bundle $\text{End}(\mathfrak{E})$ is curve semistable, and since $c_1(\text{End}(E)) = 0$, the Higgs bundle $\text{End}(\mathfrak{E})$ is H-nflat [5]. By Lemma 4.6 $\Delta(\text{End}(E)) = 0$, that is, $\Delta(E) = 0$. □

Actually we have a stronger result.

Theorem 4.9. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on an n -dimensional smooth projective variety, such that the Higgs Grassmannian $\text{Gr}_d(\mathfrak{E})$ has an irreducible component Z which is a divisor in $\text{Gr}_d(E)$ that surjects onto X . Then the following conditions are equivalent:*

- (i) \mathfrak{E} is curve semistable;
- (ii) \mathfrak{E} is semistable with respect to some polarization and the class $\theta_d(\mathfrak{E})$ is nef;
- (iii) \mathfrak{E} is semistable with respect to some polarization and $\Delta(E) = 0$.

Note that, contrary to the general situation, and due to the assumption that $\text{Gr}_d(\mathfrak{E})$ has a divisorial component that surjects onto X , the nefness of the class $\theta_d(\mathfrak{E})$ is enough to have curve semistability (but assuming a priori that \mathfrak{E} is semistable).

5. Inequalities for H-nef Higgs Bundles

We know from [10, 13] that a nef vector bundle E on an n -dimensional smooth projective variety satisfies, for every ample class H and for every $1 \leq k \leq n$, the inequalities

$$s_k(E) \cdot H^{n-k} \geq 0,$$

where $s_k(E)$ are the Segre classes of E . In this Section, we prove a version of these inequalities for H-nef Higgs bundles.

Theorem 5.1. *Let $\mathfrak{E} = (E, \phi)$ be a rank r H-nef Higgs bundle over an n -dimensional smooth projective variety. Assume that Z is an irreducible component of the Higgs Grassmannian $\text{Gr}_1(\mathfrak{E})$, and write its class in $A^N(\mathbb{P}E)$, where N is the codimension of Z in $\mathbb{P}E$, as*

$$[Z] = \sum_{i=0}^N \pi_1^* \beta_i \cdot \xi^{N-i},$$

where $\beta_i \in A^i(X)$. Then for every $1 \leq k \leq n$ we have the inequality

$$\sum_{i=0}^{r-1} \beta_i \cdot s_{k-i}(E) \cdot H^{n-k} \geq 0$$

for any ample class H in X .

Proof. We first prove the inequality for $k = n$. Since \mathfrak{E} is H-nef, the hyperplane class ξ of $\mathbb{P}E$ is nef on Z . We shall use the identity [12, Sect. 3.1]

$$\pi_{1*} \xi^{r+i-1} = s_i(E)$$

for $0 \leq i \leq n = \dim X$. Now we have

$$\begin{aligned} 0 \leq \xi^{r+n-1-N} \frown [Z] &= \int_{\mathbb{P}E} \xi^{r+n-1-N} \frown \sum_{i=0}^N \pi_1^* \beta_i \cdot \xi^{N-i} \\ &= \sum_{i=0}^N \int_{\mathbb{P}E} \xi^{r+n-i-1} \frown \pi_1^* \beta_i = \sum_{i=0}^{r-1} \beta_i \cdot s_{n-i}(E) \end{aligned}$$

For $k < n$ we have

$$0 \leq (\pi_1^* H^{n-k} \frown \xi^{r+k-1-N}) \frown [Z] = \sum_{i=0}^{r-1} \beta_i \cdot s_{k-i}(E) \cdot H^{n-k}.$$

□

Since $1 \leq N \leq r - 1$, the last summation in both equations may contain terms with $i > N$ but these are zero as $\beta_i = 0$ in that range.

Remark 5.2. In the non-Higgs case we have $\beta_0 = 1$, $\beta_i = 0$ for $i > 0$ and we recover the identities of [10, 13] when the Schur polynomial is a Segre class.

Author contributions All authors equally contributed to the writing of this paper.

Funding Open access funding provided by Scuola Internazionale Superiore di Studi Avanzati - SISSA within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare no competing interests.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] AA.VV.: *The Stacks Project*. <https://stacks.math.columbia.edu/>
- [2] Adin, R., Roichman, Y.: *Standard Young tableaux*, in Handbook of enumerative combinatorics, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, pp. 895–974 (2015)
- [3] Bruzzo, U., Capasso, A.: Filtrations of numerically flat Higgs bundles and curve semistable Higgs bundles on Calabi–Yau manifolds. *Adv. Geom.* **23**, 215–222 (2023)
- [4] Bruzzo, U., Graña Otero, B.: Metrics on semistable and numerically effective Higgs bundles. *J. Reine Angew. Math.* **612**, 59–79 (2007)
- [5] Bruzzo, U., Graña Otero, B.: Semistable and numerically effective principal (Higgs) bundles. *Adv. Math.* **226**, 3655–3676 (2011)
- [6] Bruzzo, U., Hernández Ruipérez, D.: *Semistability vs. nefness for (Higgs) vector bundles*, *Differ. Geom. Appl.*, 24, 403–416 (2006)

- [7] Bruzzo, U., Lanza, V., Lo Giudice, A.: Semistable Higgs bundles on Calabi-Yau manifolds. *Asian J. Math.* **23**, 905–918 (2019)
- [8] Bruzzo, U., Lo Giudice, A.: Restricting Higgs bundles to curves. *Asian J. Math.* **20**, 399–408 (2016)
- [9] Bruzzo, U., Peragine, V.: Semistable Higgs bundles on elliptic surfaces. *Adv. Geom.* **22**, 151–169 (2022)
- [10] Demailly, J.-P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.* **3**, 295–345 (1994)
- [11] Frobenius, F.G.: *Über die Charaktere der symmetrischer Gruppe*, Berlin, 1900. Reprinted in *Ges. Abh. III* (J.-P. Serre, ed.), Springer-Verlag, Berlin, 148–166 (1968)
- [12] Fulton, W.: *Intersection theory*, vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, Springer-Verlag, Berlin, second ed., (1998)
- [13] Fulton, W., Lazarsfeld, R.: Positive polynomials for ample vector bundles. *Ann. Math.* **2**(118), 35–60 (1983)
- [14] Huybrechts, D., Lehn, M.: *The geometry of moduli spaces of sheaves*, *Aspects of Mathematics*, E31. Friedr. Vieweg & Sohn, Braunschweig (1997)
- [15] Józefiak, T., Lascoux, A., Pragacz, P.: Classes of determinantal varieties associated with symmetric and skew-symmetric matrices. *Math. USSR Izvestija* **18**, 575–586 (1982)
- [16] Kaji, H., Terasoma, T.: Degree formula for Grassmann bundles. *J. Pure Appl. Algebra* **219**, 5426–5428 (2015)
- [17] Krishna, A.: Higher Chow groups of varieties with group action. *Algebra Number Theory* **7**, 449–507 (2013)
- [18] Lanza, V., Lo Giudice, A.: Bruzzo’s conjecture. *J. Geom. Phys.* **118**, 181–191 (2017)
- [19] MacMahon, P.A.: *Memoir on the theory of the partitions of numbers—Part IV*. *Phil. Trans. R. Soc. Lond. (A)* **209**, 153–175 (1909)
- [20] Miyaoka, Y.: *The Chern classes and Kodaira dimension of a minimal variety*, in *Algebraic geometry*, Sendai, 1985, vol. 10 of *Adv. Stud. Pure Math.*, North-Holland, Amsterdam, 449–476 (1987)
- [21] Nakayama, N.: *Normalized tautological divisors of semi-stable vector bundles*, *Sūrikaiseikikenkyūsho Kōkyūroku*. Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998)
- [22] Pan, C.-P., Zhang, C.-J., Zhang, X.: *Projectively flat bundles and semi-stable Higgs bundles*. [arXiv:1911.03593](https://arxiv.org/abs/1911.03593) [math.DG]
- [23] Simpson, C.T.: Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *J. Am. Math. Soc.* **1**, 867–918 (1988)
- [24] Simpson, C.T.: Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.* **75**, 5–95 (1992)

Ugo Bruzzo
SISSA (International School for Advanced Studies)
Via Bonomea 265
34136 Trieste
Italy

and

Departamento de Matemática
Universidade Federal da Paraíba
João Pessoa PB
Brazil

and

INFN (Istituto Nazionale di Fisica Nucleare)
Sezione di Trieste
Trieste
Italy

and

IGAP (Institute for Geometry and Physics)
Trieste
Italy
e-mail: bruzzo@sissa.it

Beatriz Graña Otero and Daniel Hernández Ruipérez
Departamento de Matemáticas and IUFFYM (Instituto de Física Fundamental y Matemáticas)
Universidad de Salamanca
Plaza de la Merced 1-4
37008 Salamanca
Spain
e-mail: beagra@usal.es

Daniel Hernández Ruipérez
e-mail: ruiperez@usal.es

Daniel Hernández Ruipérez
Real Academia de Ciencias Exactas
Físicas y Naturales
Madrid
Spain

Received: May 6, 2023.

Revised: August 1, 2023.

Accepted: August 12, 2023.