SISSA

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The geometry of holomorphic submersions, their deformations and moduli

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Ὡς τοῦ άεὶ ὄντοσ γνώσεως, ἀλλὰ οὐ τοῦ ποτέ τι γιγνομένου καὶ ἀπολλυμένου. Εὐομολόγητον, ἔφη, τοῦ γὰρ ἀεὶ ὄντοσ ἡ γεωμετρικὴ γνῶσίς ἐστιν.

“Geometry is the knowledge of that which always is, and not of something which at some time comes into being and passes away."
“That is readily admitted, for geometry is the knowledge of the eternally existent."

Plato, Republic, VII, 527

ABSTRACT

Proper holomorphic submersions can be viewed as both generalising holomorphic vector bundles and as a way of studying families of smooth projective varieties. We consider submersions whose fibres are analytically K-semistable, thus they each admit a degeneration to a Kähler manifold with constant scalar curvature. On such holomorphic submersions, we introduce and study certain canonical relatively Kähler metrics, called *optimal symplectic connections*, which generalise Hermite-Einstein connections for vector bundles and are defined as solutions to a geometric partial differential equation.

Using optimal symplectic connections, we first give a general construction of extremal metric on the total space, in adiabatic classes, generalising results of Dervan-Sektnan, Fine, Hong. We then construct an analytic moduli space of holomorphic submersions admitting an optimal symplectic connection. To do so, we develop a deformation theory of holomorphic submersions and we combine techniques from geometric invariant theory with the study of the analytic properties of the optimal symplectic connection equation. We also show that the moduli space is a Hausdorff complex space which admits a Weil-Petersson type Kähler metric.

Contents

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Introduction

A fundamental result in the study of holomorphic vector bundles is the Hitchin-Kobayashi correspondence, which establishes an equivalence between the slope-stability of the vector bundle and the existence of a Hermite-Einstein connection. While the former is a purely algebro-geometric notion, the latter is a condition in the form of a geometric PDE involving the curvature of a connection. For vector bundles over a curve, the Hitchin-Kobayashi correspondence is a classical result of Narashiman and Seshadri [59], and it was extended to higher dimensional bases by Donaldson [18], Uhlenbeck and Yau [77].

Both slope stability and Hermite-Einstein metrics can be used to construct moduli spaces of vector bundles. Seshadri [69] and Newstead [60, 61] gave the first construction of a moduli space of stable vector bundles over a curve, while Mumford [56] introduced semistability in the sense of Geometric Invariant Theory (GIT) to study this moduli space and established its structure as a global GIT quotient. Fujiki and Schumacher [29] later directly constructed the moduli space of Hermite-Einstein vector bundles over a fixed compact Kähler manifold using analytic techniques. These moduli spaces remain a central object of study to this day, and we refer to Greb-Sibley-Toma-Wentworth [33] for recent work containing a discussion of analytic and algebraic compactifications.

Motivated by the Hitchin-Kobayashi correspondence, the Yau-Tian-Donaldson conjecture [78, 76, 21] predicts that an algebro-geometric notion of stability for polarised varieties, *K-stability*, should be equivalent to the existence of Kähler metrics with constant scalar curvature. While still open in full generality, the conjecture is known to be true for Fano varieties, due to Chen-Donaldson-Sun [9, 10, 11]. The fact that the existence of constant scalar curvature Kähler (cscK) metrics implies K-stability is also a theorem of Donaldson [21], Stoppa [71] and Berman-Darvas-Lu [2]. Both K-stability and cscK metrics lead to the existence of moduli spaces. An analytic moduli space of constant scalar curvature Kähler manifolds was constructed by Fujiki and Schumacher [31] in the discrete automorphism case, and extended by Dervan and Naumann in the presence of automorphisms [13] and by Inoue [45] to Fano manifolds with Kähler-Ricci solitons. On the other hand, moduli spaces of K-stable varieties, known as K-moduli, are an active area of research in algebraic geometry [73].

Our work falls into the general framework of studying how these two pictures interact in the context of proper holomorphic submersions. In this thesis, we focus on the analytic point of view and we study a generalisation of Hermite-Einstein connections on more general fibrations, called *optimal symplectic connections*. Throughout, by a fibration we always mean a proper holomorphic submersion $\pi_X : (X, H_X) \to (B, L)$ of a relatively polarised compact Kähler manifold onto a compact polarised base, and we assume that the fibres of π_X are analytically K-semistable. We explore the implications of having an optimal symplectic connection on the existence of special Kähler metrics on the total space and we construct the analytic moduli space of fibrations admitting an optimal symplectic connection. Indeed, one of the main goals of moduli theory besides parametrising certain geometric objects is to study their behaviour in families. From this point of view, our construction of the moduli space of holomorphic submersion is a step towards understanding how projective varieties vary in families.

The easiest and most instructive case to understand the ingredients involved is indeed the case of projectivised vector bundles. Hong [39] related the existence of a solution to the Hermite-Einstein equation on the vector bundle with the existence of a Kähler metric with constant scalar curvature on the total space. More precisely, any Hermitian metric on the vector bundle induces a fibrewise Fubini-Study metric on the projectivisation, and these metrics differ by an automorphism of the fibres. However, if the Hermitian metric satisfies the Hermite-Einstein condition, then it is uniquely determined, so that there is a canonical choice of Fubini-Study metric on the fibres of the projectivisation. This choice allowed Hong to construct constant scalar curvature Kähler metrics on the total space.

In the more general case of a polarised fibration with constant scalar curvature Kähler fibres, $\pi_X : (X, H_X) \to (B, L)$, Dervan and Sektnan [14] introduced optimal symplectic connections as analogous to the Hermite-Einstein connections for projectivised vector bundles. A relatively symplectic form ω on X is called a *symplectic connection* in the language of symplectic fibrations because it determines a splitting of the tangent bundle of X into a vertical and a horizontal part, where the horizontal vector bundle is defined using orthogonality with respect to ω. If one assumes that the fibres each have a cscK metric, then these metrics can be used to construct a relatively cscK metric ω on X, but such an ω is not unique if the fibres have non-trivial automorphisms. An optimal symplectic connection is then a *canonical* choice of ω, defined in terms of a solution to a second-order elliptic PDE.

We further extend their definition to the following setting. Let $(Y, H_Y) \to (B, L)$ be a holomorphic submersion and assume that the fibres are *analytically K-semistable*, i.e. they each admit a degeneration to a cscK manifold. We assume also that these degenerations vary holomorphically in B, so that we have a degeneration $(\mathcal{X}, \mathcal{H}) \to (B, L) \times S$ of $(Y, H_Y) \to (B, L)$ to a fibrewise cscK fibration $(X, H_X) \to (B, L)$ parametrised by $S \subseteq \mathbb{C}$. Using a relative version of Ehresmann's theorem (Proposition 2.19) we take the perspective of varying the complex structure of the underlying symplectic fibration, from a relatively cscK complex structure I to small compatible deformations J_s which keep π holomorphic. We say that ω is an *optimal symplectic connection* on (Y, H_Y) if it satisfies the geometric PDE

$$p_E(\Delta_{\mathcal{V}} \left(\Lambda_{\omega_B}(\gamma^* F_{\mathcal{H}})) + \Lambda_{\omega_B} \rho_{\mathcal{H}}\right) + \frac{\lambda}{2}\nu = 0. \tag{0.1}$$

In this expression $\gamma^* F_{\mathcal{H}}$ and $\rho_{\mathcal{H}}$ are curvature quantities which depend on ω, ν is a curvature quantity that depends on the infinitesimal change in the complex structure and $\lambda > 0$ is a constant. The left-hand side is a smooth function on Y, and the map p_E is the projection onto the global sections of the vector bundle $E \to B$ of fibrewise holomorphy potentials with respect to the relatively cscK complex structure of X. The vanishing of the first term is the condition for an optimal symplectic connection in the sense of [14], i.e. where all the fibres are cscK, so our notion generalises their notion.

In the following, we consider only integral Kähler classes, although this is not essential. Indeed all our results hold if $c_1(H_X)$ and $c_1(L)$ are replaced respectively by a relative Kähler class and a Kähler class that do not come from holomorphic line bundles. Moreover, the base B is considered fixed.

Summary of results

We make use of optimal symplectic connections to prove the existence of cscK and extremal metrics on the total space Y in *adiabatic classes*

$$c_1(H_X) + kc_1(L) \quad \text{for } k \gg 0.$$

To this end, we need to be able to choose an appropriate metric on the base manifold as follows. Let $\mathcal{M}^{cscK}$ be the moduli space of cscK manifolds and let $q : B \to \mathcal{M}^{cscK}$ be the moduli map induced by the central family $(X, H_X) \to B$, whose fibres are cscK. The space $\mathcal{M}^{cscK}$ can be endowed with a *Weil-Petersson type* Kähler metric, and we denote by α_{WP} the pull-back of it via q. This is a smooth semi-positive $(1,1)$-form on B.

We first consider the case where the group of automorphisms of (Y, H_Y) and of (B, L) which preserve the map q are discrete. Thus we require that the base admits a *twisted cscK metric* with twisting form α_{WP}:

$$\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha_{WP} = constant.$$

Theorem 0.1. *Assume that the automorphisms of (Y, H_Y) and of q are discrete. Let ω be an optimal symplectic connection and ω_B be a twisted cscK metric with twisting α_{WP}. Then there exists a constant scalar curvature Kähler metric on Y in the class $c_1(H_Y) + kc_1(L)$ for all $k \gg 0$.*

If we allow the moduli map q of the central fibration and the total space (Y, H_Y) to have automorphisms, the adiabatic limit method produces *extremal metrics* on the total space. In this case, we have to modify our hypotheses on ω and ω_B as follows: we require that ω_B is *twisted extremal*, i.e.

$$\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha_{WP}$$

is a holomorphy potential on B and that ω is an *extremal symplectic connection*, i.e.

$$p_E\left(\Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\gamma^* F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}\right) + \frac{\lambda}{2}\nu$$

is a holomorphy potential on Y. We also need some technical assumptions which we will explain in Section 3.5.2: the group of automorphisms of π_Y acts equivariantly on the family $\mathcal{X} \to B \times S$ and the extremal symplectic connection ω is invariant under the flow of the extremal vector fields.

Theorem 0.2. *Suppose that (B, L) admits a twisted extremal metric ω_B and (Y, H_Y) admits an extremal symplectic connection ω. Suppose also that all automorphisms of the moduli map q lift to (Y, H_Y). Then there exists an extremal metric on Y in the class $c_1(H_Y) + kc_1(L)$ for all $k \gg 0$.*

Our results generalise previous works by many authors who consider more special situations: we already mentioned Hong's paper [39] about cscK metrics on the projectivisation of stable holomorphic vector bundles, in the case of a discrete group of automorphisms. In the presence of automorphisms of the vector bundle, Brönnle [7] proved the existence of extremal metrics on the projectivisation of vector bundles given as direct sums of stable bundles. Fine [24] proved the existence of cscK metrics on the total space of a fibration where all the fibres and the base are Riemann surfaces of genus $g \geq 2$. In this case, the choice of a relatively Kähler metric on the total space falls naturally on the hyperbolic metric, and the optimal symplectic connection condition is vacuous. Dervan and Sektnan [14] proved that the optimal symplectic connection condition reduces to the Hermite-Einstein condition on projectivised vector bundles, thus being a genuine generalisation. Moreover, they prove Theorem 0.1 and Theorem 0.2 in the case of a relatively cscK fibration. Similarly, McCarthy [53] proved that on isotrivial fibrations, the optimal symplectic connection condition becomes the Hermite-Yang-Mills condition on an associated principal bundle.

The proof of Theorems 0.1 and 0.2 is carried out using the *adiabatic limit* technique, a strategy which originates in Kähler geometry in the work of Fine [24]. It consists of expanding the scalar curvature of $\omega + k\omega_B$ in inverse powers of k, with the idea that if k is large the base becomes very large and the curvature is concentrated in the vertical direction. In the easiest case of discrete automorphisms, the optimal symplectic connection condition and the twisted cscK equation on the base allow one to find a relatively Kähler metric which is constant to order k^{-1}. Then one proceeds inductively, adding at each step r a potential $i\partial\bar{\partial}\varphi_r$ in order to make the scalar curvature constant up to the k^{-r-1}-term. The implicit function theorem then allows one to deform the approximate solution to a genuine solution. Our approach is a version of the one just described, except with *two* parameters. We consider a degeneration $\mathcal{X} \to B \times S$ of the fibration $Y \to B$ to the relatively cscK fibration $X \to B$ and we expand the scalar curvature of the Kähler metric $(\omega + k\omega_B, J_s)$ in inverse powers of k and powers of s. Then we relate the parameters k and s by imposing $\lambda k^{-1} = s$, for some $\lambda > 0$.

Our notion of an optimal symplectic connection on a relatively K-semistable fibration should be the most general condition to ask in order to produce cscK or extremal metrics in adiabatic classes, provided all data is smooth and the aforementioned hypotheses on the lifting of the automorphism groups hold.

Allowing K-semistable fibres is essential for the construction of the moduli space of fibrations with an optimal symplectic connection. Indeed, when deforming a fibration with cscK fibres, one cannot expect that the fibres remain cscK. Analytic K-semistability, on the other hand, is an open condition, and this allows us to study the local behaviour of families of fibrations with an optimal symplectic connection. We prove the following result.

Theorem 0.3. *There exists a moduli space $\mathcal{M}$ that parametrises holomorphic submersions over a fixed base, with discrete relative automorphism group and which admit an optimal symplectic connection. The space $\mathcal{M}$ is a Hausdorff complex space which carries a Weil-Petersson type Kähler metric.*

We construct the moduli space $\mathcal{M}$ by gluing local charts around fibrations that admit an optimal symplectic connection. If $(Y, H_Y) \to (B, L)$ is such a fibration, the local moduli space around Y is given as the quotient
$$W_Y/\mathrm{Aut}(\pi_Y),$$
where W_Y is the complex space of all the deformations of Y that also admit an optimal symplectic connection, and we quotient by the action of the discrete group $\mathrm{Aut}(\pi_Y)$ of relative automorphisms, which is finite.

We explain in some more detail the definition of W_Y, which essentially involves two steps. The first step, explained in §4.1, consists of finding a locally closed analytic space which parametrises all small deformations of the complex structure of Y that admit a degeneration to a fibration with cscK fibres. To do so, we use the theory of deformations of cscK manifolds of Székelyhidi [74] and Brönnle [6] to develop a theory of deformations of fibrations. More precisely, we prove a fibration version of Kuranishi's deformation theorem (Theorem 2.21) that allows us to parametrise the compatible vertical deformations of the complex structure of X with a complex space V_π of harmonic $(0,1)$-forms with values in the $(1,0)$-vertical tangent bundle. Working locally in B, we then establish that the deformations of Y which degenerate to a relatively cscK fibration form a locally closed analytic subvariety V_π^+ of V_π. We explicitly construct the relatively cscK degeneration using techniques from GIT; although we do not use it directly, our construction is related to the Białinicki-Birula decomposition [4, 46]. This is the key new step in our construction, not present in other constructions of moduli spaces.

The second step, which is the topic of §4.2, consists of proving that, in a small open neighbourhood W_Y of the point associated to Y in V_π^+, all fibrations admit an optimal symplectic connection. The proof of this essentially relies on the implicit function theorem and employs the linearisation of the equation with respect to the complex structure. It is here that we use the assumption on the discreteness of the relative automorphism group.

In the definition of an optimal symplectic connection and in the construction of the space W_Y it is essential to assume that the connected component of the identity of the groups of automorphisms $\mathrm{Aut}_0(X_b, H_b)$ of the fibres of the relatively cscK degeneration are all isomorphic. This assumption is considered to be a smoothness assumption for our setting and a fixed datum in our construction of the moduli space.

Outlook

We have presented optimal symplectic connections as canonical choices of relatively Kähler metrics on relatively K-semistable fibrations. To genuinely call them *canonical*, optimal symplectic connections should be proven to be unique. In the relatively cscK case this is a theorem of Dervan, Sektnan [17] and Hallam [34]. We expect that uniqueness holds also in the relatively K-semistable case, up to the action of the group of automorphisms of the projection π_Y.

While our work concentrates on the analytic aspects of optimal symplectic connections and their moduli space, there are different algebro-geometric notions of stability that can be defined on fibrations and related to the existence of optimal symplectic connections. In particular, a fibration version of K-stability was developed by Dervan, Sektnan [16] and further studied by Hallam [34]. They prove that, on projectivised vector bundles, fibration semistability implies slope-semistability of the vector bundle and that the existence of an optimal symplectic connection implies stability, thus establishing first results in the direction of generalising the Hitchin-Kobayashi correspondence. Although they work on fibrations with K-polystable fibres, their definition of stability also makes sense in the relatively K-semistable case we treat.

It is natural to ask if it is possible to give an algebro-geometric construction of the moduli space of fibrations based on stability. Such a construction would lead to the structure of a variety on the moduli space rather than the structure of a complex space, and would naturally allow singular fibres. In particular, it would parametrise certain stable fibrations which degenerate to a fibration whose general fibre is K-polystable. Moreover, the automorphism group of the general fibre of the degeneration should be fixed.

Hattori later introduced two different notions of stability: $\mathfrak{f}$-stability [37], related to Dervan-Sekntan stability of fibrations, and adiabatic K-stability [37, 38]. The latter is a condition defined on a fibration for adiabatic classes and involves also the K-stability of the base. Hashizume and Hattori [36] have constructed a moduli space of adiabatically K-stable fibrations over a curve where the generic fibre is Calabi-Yau, using algebro-geometric techniques. In the special case where the fibres are all smooth, our moduli space constitutes an alternative construction of Hashizume-Hattori's moduli space.

In the construction of special Kähler metrics on the total space, open questions remain in the presence of singularities. In [25, §9] Fine explains a possible way to construct special Kähler metrics on the total space of holomorphic Lefschetz fibrations, where a finite number of fibres are singular, but the problem is still mostly open. Moreover, a key assumption in Theorem 0.2 is that all automorphisms of the moduli map on the base lift to the total space. When this assumption does not hold, existence results for special Kähler metrics were proved by Hong [40] in the case of projectivised vector bundles and extended by Lu-Seyyedali [50], but the problem is open on a general fibration, and even for projective bundles sharp results are not known.

Finally, Sektnan and Spotti [68] prove a similar result to our Theorem 0.2 on the total space of certain compactified test configurations, where the central fibre is cscK and the general fibre is just K-semistable. Such a test configuration can be viewed as a deformation of a compactified product test configuration for the central fibre. Their proof, however, does not require the extremal symplectic connection condition but requires that the vector bundle E of relatively cscK metrics is trivial. It is reasonable to expect that this in fact implies that the extremal symplectic connection condition is satisfied, thus relating the two constructions.

Outline

We describe briefly the contents of each chapter. In Chapter 1 we give preliminary definitions and results about Kähler geometry. In particular, in §1.1 we collect some basic properties of the scalar curvature equation and in §1.3 we describe the moment map interpretation of the scalar curvature. Then in §1.4 we describe the relevant definitions and results on deformations of a cscK manifold.

Chapter 2 is a description of Kähler geometry of holomorphic submersion. We describe relative Kähler metrics and the curvature quantities they induce. Then we discuss the notion of an optimal symplectic connection in the relatively cscK case following [14] and we extend it to the relatively K-semistable case. In §2.3 we extend the theory of deformations of a cscK manifold to the fibration setting and we prove a relative version of Kuranishi's Theorem.

In Chapter 3 we prove the existence of a cscK metric on the total space of the relatively K-semistable fibration: we derive the optimal symplectic connection equation by expanding the scalar curvature and we study its linearisation. Then we use the adiabatic limit strategy to prove the existence of cscK and extremal metrics on the total space.

In Chapter 4 we construct the moduli space of holomorphic submersions admitting an optimal symplectic connection. We then describe a Weil-Petersson type Kähler metric on the moduli space, along with a natural line bundle.

Chapters 2 and 3 are part of the author's article [64]. The results of Chapter 4 are contained in the author's preprint [63].

Notation

Throughout this work, we consider projective Kähler manifolds (M, L), where L is a fixed ample line bundle. Analogously, we work with fibrations $\pi_Y : (Y, H_Y) \to (B, L)$, where H_Y is a relatively ample line bundle and by “fibration" we will always mean a proper holomorphic submersion. Moreover, by “relatively" we refer to a property that holds fibrewise: for example, a relatively ample line bundle is a line bundle whose restriction to each fibre is ample, and a relatively Kähler metric is a closed two-form whose restriction to each fibre is Kähler.

When working in local coordinates we use the Einstein convention on repeated indexes. In particular, on fibrations, we denote by

> $\{w^1, \dots, w^m\}$ the vertical holomorphic coordinates; indices are denoted with the letters $a, b, c, \dots$;
>
> $\{z^1, \dots, z^n\}$ the holomorphic coordinates on the base; indices are denoted with the letters $i, j, k, \dots$;
>
> $\{\zeta^1, \dots, \zeta^{n+m}\}$ the holomorphic coordinates on the total space; indices are denoted with the letters $p, q, r, \dots$.

We also use the following notation convention:

$\mathfrak{h}_0$	the space of holomorphic vector fields which admit a holomorphy potential
$\overline{\mathfrak{h}}$	the space of holomorphy potentials
$\mathrm{grad}^\omega f$	symplectic gradient of the function f
$\nabla_g f$	Riemannian gradient of the function f
$\mathscr{J}_\pi$	the space of almost complex structures J on Y compatible with the relatively symplectic form and such that $d\pi \circ J = J_B \circ d\pi$
$\mathrm{Aut}(\pi_Y)$	the group of biholomorphisms of Y that lift to H_Y and preserve the projection
K_π	the group of fibrewise Hamiltonian isometries that preserve the projection

Chapter 1

Background

In this chapter, we describe the theory of special Kähler metrics and of compatible deformations of the complex structure. We then explain the infinite-dimensional moment map picture for the scalar curvature and a finite-dimensional reduction in the case of constant scalar curvature.

We begin by fixing the notation. Throughout, we always work with *projective* Kähler manifolds, that is smooth projective varieties endowed with an ample line bundle and denoted by (M, L). We call the pair (M, L) a *polarised Kähler manifold*, and the ample line bundle L a *polarisation* of M. We consider the polarisation to be a fixed datum of the various problems we describe.

Let ω be a Kähler form on M in the first Chern class of L and let J be the complex structure of M. We denote by $g = g(\omega, J)$ the Riemannian metric on M induced by J and ω, i.e.

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot).$$

We will often call either the pair (ω, J) or the Kähler form ω alone a *Kähler metric*.

Definition 1.1. The *Ricci curvature* of ω is the two-form

$$\mathrm{Ric}(\omega, J) = -\frac{i}{2\pi}\partial_J \bar{\partial}_J \log \omega^n.$$

The *scalar curvature* of the Kähler metric (ω, J) is a smooth function on M defined as the contraction of the Ricci curvature:

$$\mathrm{Scal}(\omega, J) := \Lambda_\omega \mathrm{Ric}(\omega, J).$$

We are interested in *special* Kähler metrics, where the scalar curvature is subject to certain constraints. Among those, we often consider Kähler metrics with constant scalar curvature, where the constant is given by the intersection product

$$\widehat{S} = \frac{n\, c_1(M) \cdot c_1(L)^{n-1}}{c_1(L)^n}.$$

In particular, $\widehat{S}$ is a *topological constant* fixed by the polarisation.

1.1 Extremal Kähler metrics

In this section, we recall some basic definitions and results on Kähler manifolds and the scalar curvature map; we refer to [75, Chapter 4] for an exhaustive discussion. Let (M, L) be a Kähler manifold and let (ω, J) be a Kähler metric. A smooth function h on M is called a *holomorphy potential* if the $(1,0)$-part of the Riemannian gradient of h, denoted by $\nabla_g^{1,0}h$, is a holomorphic vector field.

Definition 1.2. A Kähler metric (ω, J) on M is *extremal* if

$$\bar{\partial}\nabla_g^{1,0}\mathrm{Scal}(\omega, J) = 0,$$

i.e. if its scalar curvature is a *holomorphy potential*.

In particular, constant scalar curvature metrics are extremal. In the study of the existence of extremal and cscK metrics, it is essential to understand the linearisation of the scalar curvature, which can be written in terms of a differential operator called the *Lichnerowicz operator*. When linearising the scalar curvature function, we can either fix the complex structure J and vary the Kähler form ω or fix ω and vary J. In this section, we consider the complex structure J as fixed, and we describe the linearisation of $\mathrm{Scal}(\omega, J)$ when we vary ω in the cohomology class $c_1(L)$. In the next section we will describe the linearisation in the J-variable and the relation between the two. To avoid any confusion, in this section we write the scalar curvature as a function of ω alone, $\mathrm{Scal}(\omega)$. The set of Kähler metrics in the same Kähler class of ω with respect to J is

$$\mathcal{K}_J(\omega) = \left\{\omega' \in c_1(L) \mid \omega' = \omega + i\partial_J\bar{\partial}_J\varphi \text{ for some } \varphi \in C^\infty(M, \mathbb{R})\right\}. \tag{1.1}$$

Fixing a reference Kähler metric $\omega \in c_1(L)$, we can then describe the linearisation of $\mathrm{Scal}(\omega)$ at a Kähler potential φ.

Definition 1.3. Let $\mathcal{D} : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(T^{1,0}M)$ be the operator

$$\mathcal{D}(\varphi) = \bar{\partial}\nabla_g^{1,0}\varphi.$$

The Lichnerowicz operator is the composition $\mathcal{D}^*\mathcal{D}$, where $\mathcal{D}^*$ is the adjoint defined with respect to the $L^2(g)$-inner product.

It can be written explicitly as follows:

$$\mathcal{D}^*\mathcal{D}(\varphi) = \Delta_g^2(\varphi) + \langle \mathrm{Ric}(\omega), i\partial\bar{\partial}\varphi\rangle + \langle \nabla\mathrm{Scal}(\omega), \nabla\varphi\rangle.$$

The Lichnerowicz operator is a 4th-order elliptic operator. Its kernel, which by ellipticity is the kernel of $\mathcal{D}$, coincides with the space of holomorphy potentials on M. In particular, it is clear from the definition that ω is an extremal metric on M if and only if the scalar curvature of ω is in the kernel of $\mathcal{D}$. The linearisation of the scalar curvature at a Kähler potential φ can be written in terms of the Lichnerowicz operator as

$$-\mathcal{D}^*\mathcal{D}(\varphi) + \frac{1}{2}\langle \nabla\mathrm{Scal}(\omega), \nabla\varphi\rangle.$$

In particular, the linearisation at a constant scalar curvature metric is given exactly by the Lichnerowicz operator.

We next describe the linearisation of the scalar curvature at an extremal metric. We denote by $\overline{\mathfrak{h}}$ the space of holomorphy potentials and by $\mathfrak{h}_0$ the space of holomorphic vector fields which admit a holomorphy potential. Solving the extremal equation means searching a Kähler metric ω such that

$$\mathrm{Scal}(\omega) - f = 0$$

for some holomorphy potential f. If we change ω to $\omega + i\partial\bar{\partial}\varphi$, then the holomorphy potential f changes to $f + \frac{1}{2}\langle\nabla f, \nabla\varphi\rangle$. Therefore, an extremal metric in the Kähler class of ω is a zero of the operator

$$\begin{aligned} C^\infty(M,\mathbb{R}) \times \overline{\mathfrak{h}} &\to C^\infty(M,\mathbb{R}) \\ (\varphi, h) &\mapsto \mathrm{Scal}(\omega + i\partial\bar{\partial}\varphi) - \frac{1}{2}\langle\nabla f, \nabla\varphi\rangle - f. \end{aligned} \tag{1.2}$$

The linearisation $\mathcal{G}$ of this operator at a solution is given again by the Lichnerowicz operator itself: $\mathcal{G}(\varphi, 0) = -\mathcal{D}^*\mathcal{D}\varphi$.

We end this brief overview of extremal and cscK metrics with a description of their automorphism group. Let $\mathrm{Aut}(M, L)$ be the group of automorphisms of M which lift to L and let $\mathfrak{h}$ be its Lie algebra. Let $\mathrm{Isom}(M, \omega)$ be the group of holomorphic isometries of the Kähler metric (ω, J) and let $\mathfrak{k}$ be its Lie algebra. A well-known result of Matsushima and Lichnerowicz states that when ω is cscK, the group $\mathrm{Aut}(M, L)$ is reductive [52, 49].

Theorem 1.4. *Suppose that there exists a constant scalar curvature Kähler metric on M. Then*

$$\mathfrak{h}_0 = \mathfrak{k}_0 \oplus J\mathfrak{k}_0.$$

The above theorem is known as the *Matsushima criterion* or the *Cartan decomposition*. For an extensive discussion on the interplay between the existence of special Kähler metrics and the groups of automorphisms of the complex, Riemannian and symplectic structure we refer to [32, §3.4].

1.2 Moment maps and GIT-stability

In this section, we briefly describe the definition of a Hamiltonian action of a compact Lie group on a symplectic manifold. We explain how moment maps are used to take a quotient of a symplectic manifold with what is called the *symplectic reduction*. We also describe some features of Geometric Invariant Theory (GIT), which also allows us to take a quotient of a projective variety with respect to the action of a complex Lie group. The Kempf-Ness Theorem 1.12 relates the two constructions.

1.2.1 Hamiltonian actions

Let (M, ω) be a symplectic manifold.

Definition 1.5. A vector field η is *Hamiltonian* with respect to ω if there exists a function $h \in C^\infty(M, \mathbb{R})$ such that

$$\omega(\eta, \cdot) = -\mathrm{d}h.$$

We say that h is the *Hamiltonian function* of η.

On a Kähler manifold $\eta = J\nabla_g(h)$, where $\nabla_g h$ is the Riemannian gradient of h. A Hamiltonian vector field with Hamiltonian h is also called the *symplectic gradient* of h, and denoted by $\mathrm{grad}^\omega h$.

Let G be a Lie group that acts on M, and assume that the action preserves the symplectic form, i.e. for any $g \in G$, $g^*\omega = \omega$. Let $\mathfrak{g}$ be the Lie algebra of G. For any element $f \in \mathfrak{g}$, the *infinitesimal action* of f is the vector field

$$\widehat{f}_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp(-tf) \cdot x)$$

Definition 1.6. Let G act on (M, ω) by means of symplectomorphisms. We say that the action is Hamiltonian if there exists a *moment map*

$$\mu : M \to \mathfrak{g}^*$$

that is equivariant with respect to the G-action on M and the co-adjoint G-action on the dual Lie algebra $\mathfrak{g}^*$ and such that for each $x \in M$

$$\mathrm{d}_x\langle \mu, f \rangle = \omega(-, \widehat{f_x}),$$

i.e. $\langle \mu, f \rangle$ is a Hamiltonian function for the vector field $\widehat{f}$ on M.

It is clear from the definition of a Hamiltonian vector field that the Hamiltonian function is only unique up to a constant. The moment map then chooses a Hamiltonian function for the infinitesimal vector field.

An important feature of Hamiltonian actions is that they allow us to take a symplectic quotient of a projective manifold. Assume that K is a compact Lie group that acts on M and that the action is Hamiltonian. Since the moment map is equivariant, its level sets are also preserved by the group action. Moreover, the origin of $\mathfrak{k}^*$ is always fixed by the coadjoint action.

The symplectic quotient is defined as

$$\mu^{-1}(0)/K.$$

This orbit space was first considered by Marsden and Weinstein [51] and Meyer [55]. In fact, it is a theorem of Marsden and Weinstein that if the action on $\mu^{-1}(0)$ is free and proper the symplectic quotient carries a symplectic form. More generally, there is a stratified symplectic structure on $\mu^{-1}(0)/K$ [70] such that each leaf is a smooth symplectic manifold. The symplectic quotient is also called the *symplectic reduction* of M by the action of K, and denoted by $M /\!\!/^{\mathrm{red}} K$.

1.2.2 Geometric Invariant Theory

In this section, we briefly review some basic notions in Geometric Invariant Theory (GIT). We refer to the books of Mumford-Fogarty-Kirwan [57] and of Newstead [61] for extensive discussions and details. The main goal of Geometric Invariant Theory that we describe is to take a geometric quotient of a projective variety M with respect the action of a complex Lie group G, such that the quotient is again a projective variety. To do so, GIT introduces a notion of *stability* for the points of M and defines a quotient of stable orbits.

Let $M \subseteq \mathbb{CP}^d$ be a smooth projective variety. Let G be a complex reductive Lie group acting on M as a subgroup of $\mathrm{SL}(d+1, \mathbb{C})$, so that the projective embedding of M is G-equivariant. In particular, the action of G on $\mathbb{CP}^d$ lifts to an action on the line bundle $O(-1)$ and it restricts to an action on the affine cone $\widehat{M}$ of M. This lift of the action is called a *linearisation* of the action, and it determines the definition of stability and of the GIT quotient, as we now explain.

Let $\mathcal{I}_M$ be the homogeneous ideal of $\mathbb{C}[z_0, \ldots, z_d]$ defining M. Then the homogeneous coordinate ring of M is the graded ring

$$R(M) = \mathbb{C}[z_0, \ldots, z_d]/\mathcal{I}_M = \bigoplus_r H^0(M, O_M(r)).$$

Let $R(M)^G$ be the graded ring of G-invariant sections

$$R(M)^G = \bigoplus_r H^0(M, O_M(r))^G.$$

Nagata's theorem [58] (see also [61, Theorem 3.4]) guarantees that, since G is reductive, $R(M)^G$ is finitely generated. The GIT quotient of M by G is defined as

$$M /\!/ G := \operatorname{Proj} R(M)^G.$$

To understand the definition in a more geometric way, we introduce the notion of GIT stability.

Definition 1.7. Let $x \in M$ and $\widehat{x} \in O_M(1)$ be a lift of x. We say that is

1. *semistable* if there exists a non-constant homogeneous section $s \in H^0(M, O_M(r))^G$ such that $s(x) \neq 0$;
2. *polystable* if it is semistable and the orbit $G \cdot \widehat{x}$ is closed;
3. *stable* if it is semistable, its stabiliser is finite and the orbit $G \cdot \widehat{x}$ is closed.

A point that is not semistable is called *unstable*. The set of semistable points is denoted by M^{ss}.

In particular, the GIT quotient is the image of M under the birational map

$$M \dashrightarrow \mathbb{P}\left(H^0(M, O_M(r))^G\right)^*$$

for $r \gg 0$, and semistable points are the ones where the map is actually defined. We can then view the GIT quotient as parametrising the semistable orbits of the action of G on M. More precisely, we can think of $M /\!/ G$ as the quotient of the M^{ss} by the equivalence relation that identifies two points if and only if the closures of their orbits have non-empty intersection. The GIT quotient then may identify strictly semistable orbits, while it is a geometric quotient on stable orbits. We have the following uniqueness result for polystable points.

Lemma 1.8 ([43, Corollary 5.13])**.** *Let G be a reductive group acting linearly on $M \subseteq \mathbb{CP}^d$ and let $x \in M$ be a semistable point. Then the closure of the orbit $G \cdot x$ contains a unique polystable orbit.*

Although we have given all the definitions for a projective variety, we can take a GIT quotient also of an affine variety [61, §3] [43, §4.5, §4.6]. In this case, we give the following definition of stability that incorporates the choice of a standard linearisation.

Definition 1.9. Let $\mathbb{A}^d$ be an affine space of dimension d and G a reductive affine group acting on it. Let $(z_1, \ldots, z_d)$ be a system of coordinates on $\mathbb{A}^d$. The space $\mathbb{A}^d$ can be embedded in the projective space $\mathbb{P}^d$ as a coordinate chart with the map

$$(z_1, \ldots, z_d) \mapsto [1 : z_1 : \ldots : z_d].$$

We extend the action of G to an action on $\mathbb{P}^d$ by acting trivially on the first coordinate. Let $x \in \mathbb{A}^d$ and let $\widehat{x}$ be its image in $\mathbb{P}^d$. We say that x is *semistable*, *polystable* or *stable* if it is with respect to the trivial linearisation.

In particular, the polynomial $P(\mathbf{z}) = z_0$ is a G-invariant homogeneous polynomial that does not vanish at any point of $\mathbb{A}^d$. So every point of $\mathbb{A}^d$ is semistable.

Remark 1.10. The fact that every point is semistable holds because we defined the action on $\mathbb{P}^d$ to be trivial on the first homogenous coordinate. In principle, one can choose to extend the action in a non-trivial way and can still define stability as above, but unstable points might appear. We do not treat this case here, so we have included the trivial extension of the action in the definition of stability.

The Hilbert-Mumford criterion [57, Theorem 2.1] is a useful way of establishing the stability of a point by looking at the 1-parameter subgroups of G. If 1-parameter subgroup $\rho(t) : \mathbb{C}^* \subset G$ acts on a point x, then $\lim_{t\to 0} \rho(t)\cdot x$ is a fixed point for the action of $\rho(t)$, so $\rho(t)$ acts on the line $O_x(-1)$ that x represents. Moreover, $\rho(t)$ acts on the line by multiplication for t^a, where $a = a(\rho, x)$ is the weight of the action.

Theorem 1.11. *A point $x \in M$ is*

1. *semistable if and only if $a(\rho, x) \leq 0$ for all 1-parameter subgroups $\rho(t)$;*
2. *polystable if and only if $a(\rho, x) \leq 0$ for all 1-parameter subgroups $\rho(t)$ and $a(\rho, x) = 0$ holds if and only if $\lim_{t\to 0} \rho(t)\cdot x$ does not lie in the orbit $G \cdot x$;*
3. *stable if and only if $a(\rho, x) < 0$ for all 1-parameter subgroups $\rho(t)$.*

The GIT quotient is intimately related to the symplectic quotient and the theory of moment maps by the Kempf-Ness theorem [47], [57, Theorem 8.3]. Let $(M, O_M(1))$ be a polarised variety with a Kähler form $\omega \in c_1(O_M(1))$. Let G be a reductive Lie group acting on M such that the action of G lifts on $O_M(1)$. Let K be a maximal compact subgroup such that the action on M is Hamiltonian.

Theorem 1.12. *There exists a moment map μ for the K-action on M such that:*

1. *a G-orbit is semistable if and only if its closure contains a zero of the moment map; this zero is the unique polystable orbit in the closure of the semistable orbit;*
2. *the inclusion of $\mu^{-1}(0)$ in M^{ss} induces a homeomorphism*

$$M /\!\!/ G \simeq \mu^{-1}(0)/K.$$

The Kempf-Ness theorem is one of the first instances of what has become a guiding principle in the study of many geometric problems, including the one treated in this thesis, and it serves as a motivation for studying similar problems in the infinite-dimensional setting. When the group and the variety are infinite-dimensional, a moment map is often a differential operator, so finding a zero of the moment map means finding a solution of a geometric PDE. Conversely, proving that an equation is a moment map for the action of a group is then the first step towards establishing a relation with an algebro-geometric stability condition.

1.3 Scalar curvature as a moment map

In this section, we focus on the moment map interpretation of the scalar curvature, due to Fujiki [28] and Donaldson [19]. For details and proofs see also [65, Chapter 1]. Indeed, the scalar curvature function can be viewed as a moment map for the action of an infinite-dimensional Lie group on an infinite-dimensional space, as we now describe.

Let (M, ω) be a symplectic manifold, and consider the infinite-dimensional space of compatible almost complex structures:

$$\mathscr{J} = \left\{ J : TM \to TM \text{ almost complex structure compatible with } \omega \right\}.$$

The tangent space at a point J is given by

$$T_J\mathscr{J} = \{A : TM \to TM \mid JA + AJ = 0 \text{ and } \omega(u, Av) + \omega(Au, v) = 0\}.$$

To see this, it is enough to compute the derivative at $t = 0$ of $(J + tA)(J + tA) = -\mathbb{1}$.

Fix now $J \in \mathscr{J}$ integrable complex structure and $A \in T_J\mathscr{J}$. Consider the Riemannian metric g_J on M induced by J and ω by $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Using the symmetry of g_J we have

$$g_J(Au, v) = \omega(Au, Jv) = -\omega(u, AJv) = \omega(u, JAv) = g_J(u, Av),$$

so the bilinear form $(u, v) \mapsto g_J(Au, v)$ is symmetric. Moreover, since $AJ + JA = 0$, A maps $T^{1,0}M$ to $T^{0,1}M$ and $T^{0,1}M$ to $T^{1,0}M$, where the splitting is considered with respect to J. Since A is real, it is uniquely determined by one of the two restrictions, and we take $A : T^{0,1} \longrightarrow T^{1,0}$. This means that we may identify

$$T_J\mathscr{J} \longleftrightarrow T^{0,1}_J\mathscr{J} = \left\{\alpha \in \Omega^{0,1}(T^{1,0}M) \mid \omega(\alpha(u), v) + \omega(u, \alpha(x)) = 0\right\}.$$

Now, if $A \in T_J\mathscr{J}$, also $JA \in T_J\mathscr{J}$, so $\mathscr{J}$ has a complex structure, which we denote by $\mathbb{J}$. Moreover, it has a Hermitian inner product

$$\langle A, B\rangle_J := \int_M \langle A, B\rangle_{g_J} \frac{\omega^n}{n!}$$

and the two combine to give a Kähler form, given at the point J by

$$\mathbf{\Omega}_J(A, B) = \langle JA, B\rangle_J.$$

So $\mathscr{J}$ is an infinite-dimensional Kähler manifold. Inside $\mathscr{J}$, we consider the complex subspace $\mathscr{J}^{\mathrm{int}}$ of *integrable* almost complex structures of $\mathscr{J}$. Its tangent space is given by those $\alpha \in T^{0,1}_J\mathscr{J}$ such that $\bar\partial\alpha = 0$.

Consider the group of Hamiltonian symplectomorphisms of (M, ω), denoted by $\mathscr{G}$. This is the infinite-dimensional Lie group of time-one flows of Hamiltonian vector fields on M, and it acts on $\mathscr{J}$ by pull-back:

$$J \in \mathscr{J}, \phi \in \mathscr{G} \qquad \phi^*J := \mathrm{d}\phi^{-1} \circ J \circ \mathrm{d}\phi.$$

Lemma 1.13. *The Lie algebra of $\mathscr{G}$ can be identified with the space $C^\infty_0(M)$ of the smooth functions on M with ω-average zero.*

Proof. Consider a 1-parameter subgroup $\{\phi_t\}$ of $\mathscr{G}$. Then $\phi_t^*\omega = \omega$. So, denoting by η_t the vector field which time-one flow is the symplectomorphism ϕ_t, we have

$$\mathcal{L}_{\eta_t}\omega = 0.$$

By the Cartan magic formula, $\mathrm{d}\omega(\eta_t, \cdot) = 0$. Therefore, η_t is Hamiltonian vector field with some Hamiltonian function h. We can assume that h has mean-value zero in M, as the Hamiltonian function is unique up to a constant. So $\mathrm{Lie}(\mathscr{G}) = C^\infty_0(M)$. □

The following theorem is due to Fujiki [28] and Donaldson [19].

Theorem 1.14. *The action of $\mathscr{G}$ on $\mathscr{J}$ is Hamiltonian with moment map*

$$\begin{aligned} \mu : \mathscr{J} &\longrightarrow \mathrm{Lie}(\mathscr{G})^* \\ J &\longmapsto \mathrm{Scal}(\omega, J) - \widehat{S}. \end{aligned} \tag{1.3}$$

If J is integrable, $\mathrm{Scal}(\omega, J)$ is the scalar curvature of the metric g_J. Otherwise, it is the Hermitian scalar curvature of the Chern connection on TM, which is not the same as the Levi-Civita connection in general. In particular cscK metrics on M correspond to $J \in \mathscr{J}^{\mathrm{int}}$ such that $\mu(J) = 0$. The function $\mathrm{Scal}(J) - \widehat{S}$ is viewed as an element of $C_0^\infty(M)^*$ by identifying $\mathrm{Lie}(\mathscr{G})^*$ with its dual via the $L^2(\omega)$-product on M, i.e.

$$\phi \mapsto \langle \mathrm{Scal}(\omega, J) - \widehat{S}, \phi \rangle_{L^2}.$$

We next introduce two operators: the infinitesimal action of $\mathscr{G}$, denoted P, and the differential of the scalar curvature, denoted Q. Consider the scalar curvature map with respect to the complex structure:

$$\begin{aligned} \mathcal{S} : \mathscr{J} &\longrightarrow C_0^\infty(X) \\ J &\mapsto \mathrm{Scal}(\omega, J) - \widehat{S}. \end{aligned} \tag{1.4}$$

For fixed $J \in \mathscr{J}$, the infinitesimal action of $\mathscr{G}$ on $\mathscr{J}$ is given by the operator

$$\begin{aligned} P : C_0^\infty(M) &\longrightarrow T_J \mathscr{J} \\ h &\longmapsto \mathcal{L}_{\eta_h} J, \end{aligned}$$

where η_h is the Hamiltonian vector field with Hamiltonian function h. Let Q be the derivative at J of the map (1.4), i.e.

$$\begin{aligned} Q : T_J \mathscr{J} &\longrightarrow C_0^\infty(M) \\ A &\longmapsto \mathrm{d}_J \mathcal{S}(A). \end{aligned} \tag{1.5}$$

We next show that the operators P and Q are adjoint and how they are used to prove that the map (1.4) is a moment map. These operators and their properties play a central role throughout this work. In particular, the operator P is used also in §1.3.1 to define a notion of complexified orbits of the action of the group $\mathscr{G}$ on $\mathscr{J}$, even if $\mathscr{G}$ does not admit a genuine complexification. In §1.4 we use again the operator P to describe deformations of cscK manifolds and a version of Luna's slice theorem. We will introduce a relative version of P in §2.3.2 and we will use it to describe deformations of holomorphic submersions with a fixed base.

We begin by giving an alternative definition of P.

Lemma 1.15. *The operator P can be written as*

$$P(h) = 2J\bar{\partial}\eta_h^{1,0} + 2J\overline{(\bar{\partial}\eta_h^{1,0})}, \tag{1.6}$$

where η_h is the Hamiltonian vector field with Hamiltonian function h.

Proof. Recall that, by definition of the Lie derivative, for any vector field ξ

$$(\mathcal{L}_{\eta_h} J)(\xi) = [\eta_h, J\xi] - J[\eta_h, \xi].$$

In coordinates, we write ξ as $\xi^c \partial_c + \xi^{\bar{d}} \partial_{\bar{d}}$, and we have

$$\mathcal{L}_\xi J(\partial_a) = \underbrace{\xi(i\partial_a)}_{=0} - i\partial_a(\xi) - J\underbrace{(\xi(\partial_a))}_{=0} + J(\partial_a(\xi)) = -2i\left(\partial_a \xi^{\bar{d}}\right)\partial_{\bar{d}},$$

and analogously $\mathcal{L}_\xi J(\partial_{\bar{b}}) = 2i\left(\partial_{\bar{b}}\xi^c\right)\partial_c$. Plugging in the expression for the Hamiltonian vector field $\eta_h = \omega^{c\bar{d}}\partial_{\bar{d}} h\, \partial_c + \omega^{c\bar{d}}\partial_c h\, \partial_{\bar{d}}$ we obtain

$$\begin{aligned} \mathcal{L}_{\eta_h} J(\partial_a) &= -2i\partial_a(\omega^{c\bar{d}}\partial_c h)\partial_{\bar{d}}, \\ \mathcal{L}_{\eta_h} J(\partial_{\bar{b}}) &= 2i\partial_{\bar{b}}(\omega^{c\bar{d}}\partial_{\bar{d}} h)\partial_c. \end{aligned}$$

Thus
$$\mathcal{L}_{\eta_h} J = 2J\partial_{\bar{b}}(\omega^{c\bar{d}}\partial_{\bar{d}}h)\partial_c \otimes \mathrm{d}\bar{z}^b + 2J\partial_a(\omega^{c\bar{d}}\partial_c h)\partial_{\bar{d}} \otimes \mathrm{d}z^a.$$
On the other hand $\bar{\partial}\eta_h^{1,0} = \partial_{\bar{b}}(\omega^{c\bar{d}}\partial_{\bar{d}}h)\partial_c \otimes \mathrm{d}\bar{z}^b$, so
$$\mathcal{L}_{\eta_h} J = 2J\bar{\partial}\eta_h^{1,0} + \overline{2J\bar{\partial}\eta_h^{1,0}},$$
which proves the thesis. □

In light of Lemma 1.15, we can and will modify our operator P and consider as its definition the following:
$$\begin{aligned} P : C_0^\infty(M) &\longrightarrow T_J^{1,0}\mathscr{J} \\ h &\longmapsto \bar{\partial}\eta_h^{1,0}. \end{aligned} \tag{1.7}$$
Moreover, if J is integrable, $J\mathcal{L}_{\eta_h}J = \mathcal{L}_{J\eta_h}J$, so
$$\bar{\partial}\eta_h^{1,0} + \overline{(\bar{\partial}\eta_h^{1,0})} = -2J\mathcal{L}_{\eta_h}J = -2\mathcal{L}_{J\eta_h}J.$$
Remark 1.16. Fix $J \in \mathscr{J}$ integrable complex structure. Let $h \in C_0^\infty(M)$ and η_h be the associated Hamiltonian vector field. Using the definition of a Hamiltonian vector field, the Cartan magic formula and the relation $\omega(Ju, v) + \omega(u, Jv) = 0$, we have
$$\mathcal{L}_{J\eta_h}\omega = \mathrm{d}(\omega(J\eta_h, \cdot)) = -\mathrm{d}(\omega(\eta_h, J\cdot)) = -\mathrm{d}(-\mathrm{d}h(J\cdot)) = \mathrm{d}(J\mathrm{d}h).$$
If J is integrable, by writing the expression in local coordinates, we obtain that
$$\mathcal{L}_{J\eta_h}\omega = \mathrm{d}(J\mathrm{d}h) = -2i\partial\bar{\partial}h. \tag{1.8}$$

The fact that (1.3) is a moment map means by definition that the L^2-inner product $\langle\mu(J), \phi\rangle$ is a Hamiltonian function for the infinitesimal vector field $\mathcal{L}_{\eta_\phi}J$, i.e.
$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle\mu, \phi\rangle(J_t) = \mathbf{\Omega}_{J_0}(\dot{J}_0, \mathcal{L}_{\eta_\phi}J_0).$$
By writing the moment map condition in terms of P, Q, for $A \in T_J\mathscr{J}$, we obtain the following relation:
$$\langle Q(A), \phi\rangle_{L^2} = -\mathbf{\Omega}_J\left(\frac{1}{2}JP(\phi), A\right) = \mathbf{\Omega}_J\left(A, \frac{1}{2}JP(\phi)\right) = \langle JA, \frac{1}{2}JP(\phi)\rangle_J = \frac{1}{2}\langle A, P(\phi)\rangle_J.$$
Therefore $Q^* = \frac{1}{2}P$, and conversely $P^* = 2Q$. The operators Q and P are thus adjoint.

Remark 1.17. Let J_1, J_2 be two almost complex structures compatible with ω, and assume that $J_2 = f^*J_1$ for some $f \in \mathrm{Diff}(M)$. Denoting by $g(J_i, \omega)$, $i = 1, 2$ the corresponding Riemannian metrics, we have
$$g(J_2, \omega) = f^*g(J_1, (f^{-1})^*\omega).$$
If moreover $f \in \mathscr{G}$, we have that $g(J_1, \omega)$ and $g(J_2, \omega)$ are isometric.

Let $\mathcal{D}^*\mathcal{D}$ be the Lichnerowicz operator with respect to the metric g_J. Its coordinates expression is [75, Theorem 4.2]
$$\mathcal{D}^*\mathcal{D}(\phi) = g_J^{d\bar{b}}g_J^{a\bar{c}}\nabla_d\nabla_a\nabla_{\bar{c}}\nabla_{\bar{b}}\phi = g_J^{d\bar{b}}g_J^{a\bar{c}}\nabla_a\nabla_d\nabla_{\bar{c}}\nabla_{\bar{b}}\phi.$$
We use this expression to relate the operators P and Q. With the following result we prove that along the direction given by the infinitesimal action of $\mathscr{G}$ on $\mathscr{J}$, the derivative of the scalar curvature $\mathcal{S}$ is the real part of the Lichenrowicz operator.

Lemma 1.18 ([20]). *The maps P and Q satisfy the following property:*

$$Q(P(\phi)) = \mathrm{Re}(\mathcal{D}^*\mathcal{D}\phi). \tag{1.9}$$

Proof. Start by writing Q as the composition of two operators,

$$Q_1 : T_J\mathscr{J} \subset \Omega^{0,1}(T^{1,0}M) \longrightarrow \Omega^{1,0}(M)$$

and

$$Q_2 : \Omega^{1,0}(M) \longrightarrow C^\infty(M).$$

The operator Q_1 is defined as

$$Q_1(\alpha) = \mathrm{Re}(\nabla_a \alpha^a_{\ \bar{b}} d\bar{z}^b),$$

for $\alpha \in T^{0,1}_J\mathscr{J}$. As for Q_2, let $\theta = \theta_a dz^a \in \Omega^{1,0}(M)$. Then

$$Q_2(\theta) = -\mathrm{div}(J\theta^\sharp) = \mathrm{div}(ig^{a\bar{b}}\theta_a\partial_{\bar{b}})$$

where $\theta^\sharp$ is the vector field obtained by θ by raising the index with the metric g. Moreover, Q_2 satisfies the equation

$$Q_2(\theta)\omega^n = n d\theta \wedge \omega^{n-1}.$$

We have that

$$\begin{aligned} Q(A) = Q_2(Q_1(A)) = Q_2\left(\mathrm{Re}(\nabla_a A^a_{\ \bar{b}} d\bar{z}^b)\right) &= -\mathrm{div}\left(J\mathrm{Re}(g^{c\bar{b}}\nabla_a A^a_{\ \bar{b}}\partial_c)\right) \\ &= -\mathrm{div}\left(\mathrm{Re}(ig^{c\bar{b}}\nabla_a A^a_{\ \bar{b}}\partial_c)\right) = \mathrm{div}\left(\mathrm{Im}(g^{c\bar{b}}\nabla_a A^a_{\ \bar{b}}\partial_c)\right) \\ &= \mathrm{Im}\left(\nabla_c(g^{c\bar{b}}\nabla_a A^a_{\ \bar{b}})\right). \end{aligned} \tag{1.10}$$

Let us compose this with the operator

$$P(\phi)^{1,0} = \bar{\partial}\left(\xi_\phi\right)^{1,0} = \partial_{\bar{c}}\left(\omega^{a\bar{b}}\partial_{\bar{b}}\phi\right) d\bar{z}^c \otimes \partial_a.$$

Hence we obtain

$$\begin{aligned} Q(P(\phi)) &= Q\left(\partial_{\bar{c}}\left(\omega^{a\bar{b}}\partial_{\bar{b}}\phi\right) d\bar{z}^c \otimes \partial_a\right) \\ &= Q\left(ig^{a\bar{b}}\ \nabla_{\bar{c}}\nabla_{\bar{b}}\phi\ d\bar{z}^c \otimes \partial_a\right) \\ &= \mathrm{Im}\left(i\nabla_d\left(g^{d\bar{b}}\nabla_a g^{a\bar{c}}\nabla_{\bar{b}}\nabla_{\bar{c}}\phi\right)\right) \\ &= \mathrm{Re}(\mathcal{D}^*\mathcal{D}\phi), \end{aligned}$$

as claimed. ◻

We can compare the linearisation (1.9) of the scalar curvature map S, where ω is fixed, to the linearisation of the scalar curvature map described in §1.1, where the complex structure is fixed and the Kähler potential varies. When (ω, J) has constant scalar curvature, the linearisation of the scalar curvature coincides with the Lichenrowich operator, which is real. Thus it coincides with the expression (1.9). However, in the proof of (1.9) we do not use any hypothesis on the scalar curvature being constant, so when the reference metric $g(\omega, J)$ has non-constant scalar curvature the two linearisations may be different. In the next section, we will expand on the interplay between fixing the complex structure and varying the Kähler potential and fixing the symplectic form and varying the complex structure.

1.3.1 Complexified orbits and Kähler potentials

The group $\mathscr{G}$ does not admit a formal complexification. Nonetheless, there is a notion of complexified orbits for the action of $\mathscr{G}$ on $\mathscr{J}$, at the infinitesimal level. These orbits play a role in the interaction between the complex structure and its deformations and the changes in the Kähler metric within its Kähler class.

Lemma 1.19. *Let $J \in \mathscr{J}$ and let O_J be the orbit of J for the action of $\mathscr{G}$. Then*

$$T_J O_J = \left\{ JP(h) \mid h \in C_0^\infty(M) \right\}.$$

Lemma 1.19 is a consequence of Remark 1.15, where P is related to the infinitesimal action of $\mathscr{G}$. In order to define a formal complexification of $\mathscr{G}$, we proceed by complexifing the tangent space to the orbits. Indeed, we can consider the complexification of the Lie algebra of $\mathscr{G}$, i.e. $C_0^\infty(M, \mathbb{C})$. Thus we can complexify the infinitesimal action P to the operator

$$P^{\mathbb{C}} : C_0^\infty(M, \mathbb{C}) \longrightarrow T_J \mathscr{J}$$

defined as follows: if $h = u + iv \in C_0^\infty(M, \mathbb{C})$, then

$$P^{\mathbb{C}}(h) = P(u) + JP(v).$$

The complexified infinitesimal action defines for every $J \in \mathscr{J}$ the set

$$\mathcal{D}_J = \{(P(h), JP(h)) \mid h \in \mathrm{Lie}(\mathscr{G})\}.$$

Lemma 1.20. *1. For every almost complex structure J, $\mathcal{D}_J$ is a subset of $T_J \mathscr{J}$, hence $\mathcal{D}$ is a distribution;*

2. The distribution $\mathcal{D}$ is integrable.

For a proof of this result in the case of J integrable complex structure, see [76, Chapter 4].

Definition 1.21. The leaves of the foliation $\mathcal{D}$ are defined to be the tangent spaces to the *complexified orbits* of $\mathscr{G}^c$ or $\mathscr{G}^c$-orbits.

We next explain that the complexified orbits can be interpreted in terms of Kähler metrics in a fixed Kähler class. Let J be an integrable compatible complex structure on (M, ω), and consider the space $\mathcal{K}_J(\omega)$ of Kähler potentials in the class $[\omega]$ (1.1). The following rather technical proposition of Donaldson [20, p.17] is the key property in the interpretation of the relation between $\mathscr{G}^c$-orbits and $K_J(\omega)$. We will extend it to the setting of fibrations in Proposition 4.5.

Proposition 1.22. *For every $\omega_\phi \in \mathcal{K}_J(\omega)$ there exist $f \in \mathrm{Diff}_0(M)$ such that $f^*\omega_\phi = \omega$ and (M, ω_ϕ, J) is isomorphic to (M, ω, f^*J).*

Proof. Consider J fixed, and pick a Kähler metric in the class $[\omega]$:

$$\omega_\phi = \omega + 2i\partial\bar{\partial}\phi.$$

Using the relation (1.8), a path ω_t between ω and ω_ϕ for $t \in [0,1]$ can be written as

$$\omega_t = \omega + 2it\partial\bar{\partial}\phi = \omega - t\mathrm{d}(J\mathrm{d}\phi).$$

For each $t \in [0,1]$ let η_t be the Hamiltonian vector field

$$\eta_t = \mathrm{grad}^{\omega_t} \dot{\phi}_t.$$

Consider the vector field $\xi_t = J\eta_t$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = -\mathrm{d}(J\mathrm{d}\phi) = -\mathcal{L}_{\xi_t}\omega_t, \tag{1.11}$$

where the second equality is given again by (1.8). Define $\{f_t, t \in [0,1]\}$ to be the *isotopy* of the time-dependente vector field ξ_t, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \xi_t(f_t), \quad f_0 = \mathrm{id}.$$

The following result is a standard application of Moser's trick in symplectic geometry [8, 6.4]: for a smooth family η_t of p-forms

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t^*\eta_t = f_t^*\left(\mathcal{L}_{\xi_t}\eta_t + \frac{\mathrm{d}\eta_t}{\mathrm{d}t}\right). \tag{1.12}$$

Applying the relations (1.12) and (1.11) to ω_t lead

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t^*\omega_t = 0,$$

which implies that $f_t^*\omega_t = f_0^*\omega = \omega$. Let J_t be the pull-back f_t^*J. Then, for $t = 1$, the two Riemannian metrics $g(\omega, f_1^*J)$ and $g(\omega_\phi, J)$ are isometric, hence the Kähler manifolds (M, ω, f_1^*J) and (M, ω_ϕ, J) are isomorphic. □

In particular, if we fix $J \in \mathscr{J}$ integrable, from Proposition 1.22 we have a map

$$\begin{aligned} F : \left\{\phi \in C^\infty(M, \mathbb{R}) \mid \omega_\phi \in \mathcal{K}_J(\omega)\right\} &\longrightarrow \mathscr{J} \\ \phi &\longmapsto F_\phi J := f_1^* J. \end{aligned} \tag{1.13}$$

Using again equation (1.12), which can be generalised to all tensors, we see that the differential at 0 of F is given by

$$\mathrm{d}_0F(\dot\phi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_t = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^*J = \mathcal{L}_{\xi_t}J = \mathcal{L}_{JX_{\dot\phi}(\omega)}J = J\mathcal{L}_{\xi_{\dot\phi}(\omega)}J = -\frac{1}{2}P(\dot\phi).$$

In particular, we have obtained that the differential $\mathrm{d}_0F(\dot\phi)$ lies in the leaf $\mathcal{D}_J$. This means that a variation of the Kähler form in a given Kähler class for J fixed corresponds to a variation of the complex structure J in the same $\mathscr{G}^c$-orbit, for ω fixed. In other terms, the $\mathscr{G}^c$-orbits of J integrable are in bijection with the space $\mathcal{K}_J(\omega)$ of Kähler metrics in the class $[\omega]$.

1.4 Deformation theory of Kähler metrics with constant scalar curvature

In this section we follow Székelyhidi [74], although similar results were obtained also by Brönnle in his PhD thesis [6]. Let (M, L) be a polarised Kähler manifold and let ω be a fixed Kähler form in $c_1(L)$. We fix $J \in \mathscr{J}$ an integrable complex structure on (M, ω) such that $\mathrm{Scal}(\omega, J)$ is constant. In this section, we describe the deformations of the complex structure J and we explain which deformations still define a cscK metric. We will use these results in §2.3, where we describe the deformations of *families* of cscK complex structures.

The deformations of the complex structure are encoded in a complex

$$C_0^\infty(M,\mathbb{C}) \xrightarrow{P^{\mathbb{C}}} T_J\mathscr{J} \xrightarrow{\bar\partial} \Omega^{0,1}(T^{1,0}M).$$

Let $\widetilde{H}^1$ be the cohomology of the complex. Then $\widetilde{H}^1$ can be described as

$$\widetilde{H}^1 = \left\{\alpha \in T_J\mathscr{J} \mid P^*\alpha = \bar\partial\alpha = 0\right\}. \tag{1.14}$$

This is a finite-dimensional vector space since it is the kernel of the elliptic operator

$$\square = PP^* + (\bar\partial^*\bar\partial)^2 \tag{1.15}$$

on $T_J\mathscr{J}$. Consider the group of Hamiltonian isometries of (M,ω,J), denoted by K: it is the group of functions $\varphi \in \mathscr{G}$ such that

$$\mathrm{d}\varphi^{-1}J\mathrm{d}\varphi = J.$$

In particular, the group K is the stabiliser of the complex structure J for the action of $\mathscr{G}$ which means that, by definition, it is the intersection of $\mathscr{G}$ with $\mathrm{Aut}(M,J)$. The Lie algebra of K, denoted by $\mathfrak{k}$, consists of smooth functions over M such that their Hamiltonian vector field is also holomorphic, thus it can be identified with the kernel of P. The group K can be complexified and from Theorem 1.4 its complexification is $\mathrm{Aut}(M,L)$.

The map (1.13) can be generalised to a map between Sobolev spaces

$$\begin{aligned} F : L^2_k &\to \mathscr{J}^2_{k-2} \\ \phi &\mapsto F_\phi(J) \end{aligned}$$

where $\mathscr{J}^2_{k-2}$ is the set of almost complex structures on (M,ω) with coefficients in L^2_{k-2}. This map is K-equivariant, with respect to the natural pull-back action on L^2_k[1].

From Kuranishi [48], we may construct a holomorphic embedding

$$\Phi_1 : V_1 \to \mathscr{J} \tag{1.16}$$

where $V_1 \subset \widetilde{H}^1$ is a ball around the origin. The map Φ_1 maps the origin 0 to the reference complex structure J. The group K acts naturally on $\widetilde{H}^1$ by pull-back, and hence on V_1, and the map Φ_1 is K-equivariant. Moreover, V_1 parametrises $\mathscr{G}^c$-orbits of integrable complex structures near J_0, in the sense that the $\mathscr{G}^c$-orbit of every integrable complex structure near J_0 intersects the image of Φ_1. The following theorem is mainly due to Kuranishi [48]. A proof of items 1. and 2. adapted to take into account the compatibility with the symplectic form can be found in [12, §6]. The third claim is due to Székelyhidi [74].

Theorem 1.23. *There exists a ball around the origin $V \subset \widetilde{H}^1$ and a K-equivariant map*

$$\Phi : V \to \mathscr{J} \tag{1.17}$$

such that $\Phi(0) = J$ and

1. *the $\mathscr{G}^c$-orbit of every integrable complex structure near J_0 intersects the image of Φ;*
2. *if two points x and x' of V are in the same orbit for the complexified action of K, and $\Phi(x)$ is integrable, then their images $\Phi(x)$ and $\Phi(x')$ are in the same $\mathscr{G}^c$-orbit;*

[1] One can see this by using the following relation on the pull-back of a Hamiltonian vector field: $F^*(\xi_h(\omega)) = \xi_h(F^*\omega)$.

3. $\mathrm{Scal}(\omega, \Phi(x)) - \widehat{S}$ *is an element of the Lie algebra of K.*

The open ball V is a local slice of the $\mathscr{G}^c$-action near the reference complex structure J. We will refer to it as the *Kuranishi space* and to Φ as the *Kuranishi map*. Since we allow also non integrable almost complex structure, the slice is an actual ball. Instead, in the original work by Kuranishi, the set V parametrises only *integrable* complex structures, hence it is a complex analytic subspace of our V. Moreover, while the map Φ_1 (1.16) is holomorphic, our Kuranishi map Φ (1.17) is not holomorphic in general: this is due to the fact that Φ is perturbed from Φ_1 using the implicit function theorem in order to meet the third requirement of Theorem 1.23.

Consider the symplectic form on V pulled-back via Φ from the Kähler form $\mathbf{\Omega}$ on $\mathscr{J}$. We will denote this symplectic form again by Ω, since it is essentially the same. Then we obtain a moment map for the K-action on V:

$$\begin{aligned} \mu : V &\longrightarrow \mathfrak{k} \\ x &\longmapsto S(\omega, \Phi(x)) - \widehat{S}, \end{aligned} \tag{1.18}$$

where $\mathfrak{k}$ is identified with its dual via the L^2-product of functions. So the question about which deformations of the complex structure J admit a cscK metric can be restated in terms of finding a $K^{\mathbb{C}}$-orbit which contains a zero of the moment map μ. We will expand on this in the next section.

Until this point, we have considered deformations of the complex structure of a fixed symplectic manifold and we have treated them as $(0,1)$-forms with values in the $(1,0)$-tangent bundle. Another point of view consists in considering deformations of the manifold M as a family $\pi : \mathcal{U} \longrightarrow T$, where π is a smooth proper morphism and T is a complex space. The simplest example is when T is the double point $\mathrm{Spec}\mathbb{C}[\varepsilon]/\varepsilon^2$. However, if we consider also non integrable deformations, we can consider T to be an open disk.

Definition 1.24. We say that the deformation family is *complete* if, for any other deformation family $\mathcal{U}' \longrightarrow T'$, there exists a map
$$\tau : T' \longrightarrow T$$
such that $\mathcal{U}' = \tau^*\mathcal{U}$. Moreover, if the differential of τ at 0 is unique we say that the deformation family is *versal*, and *universal* if τ itself is unique.

Ehresmann's fibration theorem [44, Theorem 6.2.2] guarantees that the two points of view are interchangeable.

Theorem 1.25 (Ehresmann)**.** *Let $\pi : \mathcal{U} \longrightarrow T$ be a proper family of differentiable manifolds. If T is connected, then all the fibres are diffeomorphic.*

Throughout this work, we will make extensive use of Ehresmann's theorem to view any family of deformations of a given complex manifold as a family of deformations of its complex structure. In particular Kuranishi's Theorem [48] gives the existence of a versal deformation family centred at M with the complex structure J and base the Kuranishi space V. Moreover, the Kuranishi deformation family is complete for nearby complex structures.

1.4.1 Reduction to the finite dimensional problem

In this section we follow Inoue [45, §3] to compute a moment map for the linearised action of K on the tangent space to the Kuranishi space (see also [62, §3.5]). Such a computation will be used in the definition of an optimal symplectic connection and its linearisation. We then report the proof of a theorem of Székelyhidi [74] on finding a $K^{\mathbb{C}}$-orbit in V which contains a zero of the moment map (1.18).

On V we have the symplectic form Ω pulled back from the one of $\mathscr{J}$ and by definition the moment map (1.18) satisfies

$$\mathrm{d}_{v_0}\langle \mu, f\rangle(v) = \Omega_{v_0}(v, \mathcal{L}_{\eta_f} v_0),$$

where η_f is the Hamiltonian vector field of f, also denoted by $\mathrm{grad}^{\omega} f$. We will also use the notation $\sigma_{v_0}(f)$ for the infinitesimal action given by the Lie derivative of v_0 along the Hamiltonian vector field associated with f. The origin of V is a fixed point of the action. By identifying T_0V with $\widetilde{H}^1$, we consider on $\widetilde{H}^1$ the linear symplectic form

$$\Omega_0(\cdot,\cdot) = \boldsymbol{\Omega}_{J_0}(\mathrm{d}_0\Phi\cdot, \mathrm{d}_0\Phi\cdot), \tag{1.19}$$

and the linear action of K induced by the one on V. For any $f \in \mathfrak{k}$, consider the endomorphism of $\widetilde{H}^1$

$$A_f(t) = \mathrm{d}_0\left(y \mapsto \exp(tf)\cdot y\right),$$

where by $\exp(tf)$ we denote the 1-parameter subgroup of K defined by the element $f \in \mathfrak{k}$. It corresponds via Φ to the flow of the Hamiltonian vector field η_f on M, which we denote by ρ_t^f. The operator $A_f(t)$ is a unitary operator, since it is linear and symplectic, because the group K acts by symplectomorphisms on V. Let

$$A_f := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A_f(t). \tag{1.20}$$

We have the following properties:

1. A_f is a skew-hermitian endomorphism of $(\widetilde{H}^1, J_0)$ and $A_f(t) = \exp(tA_f)$;
2. For $v \in \widetilde{H}^1$, denote by $\mathbf{v}$ a vector field on V such that $\mathbf{v}\mid_0 = v$. Then

$$A_f(v) = \partial_t \mid_{t=0} A_f(t)v = \partial_t \mid_{t=0} \left((\rho_t^f)_*(\mathbf{v})\right)_0 = -(\mathcal{L}_{\eta_f}\mathbf{v})_0 = [\mathbf{v}, \eta_f]_0. \tag{1.21}$$

Definition 1.26. We define a map $\nu : \widetilde{H}^1 \longrightarrow \mathfrak{k}$ by

$$\langle \nu(v), f\rangle = \frac{1}{2}\Omega_0(A_f v, v).$$

The map ν can be characterised as a moment map by relating it to the scalar curvature (1.18) as follows [45, §3]. We begin by computing the second derivative of the moment map μ, as follows:

$$\begin{aligned}
\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \langle \mu(tv), f\rangle &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \frac{\mathrm{d}}{\mathrm{d}t}\mu(tv), f\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \mathrm{d}_{tv}\mu(v), f\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Omega_{tv}(v, \mathcal{L}_{\eta_f}(tv)) \\
&= \Omega_0(v, -A_f v) \\
&= \langle \nu(v), f\rangle.
\end{aligned}$$

In particular, written as a matrix product, the above relation becomes

$$v^\intercal \cdot \mathrm{Hess}_0(\langle \mu, f\rangle) \cdot v = -v^\intercal \Omega_0 A_f v,$$

hence the matrix $\Omega_0 A_f$ is symmetric. Therefore

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \nu(v_t), f\rangle \overset{\text{def}}{=} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{1}{2}\Omega_0(A_f v_t, v_t) \\
&= \frac{1}{2}\Omega_0(A_f v_0, \dot{v}_0) + \frac{1}{2}\Omega_0(A_f \dot{v}_0, v_0) \\
&= \Omega_0(A_f v_0, \dot{v}_0) \\
&= \Omega_0(\dot{v}, (\mathcal{L}_{\eta_f}\mathbf{v})_0),
\end{aligned}$$

i.e. ν satisfies the moment map condition. The following theorem is due to Székelyhidi [74]. We report here a proof, which differs slightly from the original and is more in line with the techniques used in the following chapters.

Theorem 1.27. *Let $v \in V$ be a GIT-polystable point for the $K^{\mathbb{C}}$-action on $\widetilde{H}^1$. Then there is a $v_0 \in V$ in the same $K^{\mathbb{C}}$-orbit as v such that $\mu(v_0) = 0$.*

Proof. Since v is GIT-polystable for the linearised action, by the Kempf-Ness theorem 1.12 there is a zero of the moment map ν in the same $K^{\mathbb{C}}$-orbit of v. Without loss of generality, we can assume that $\nu(v) = 0$. Hence we have the following expansion of μ for small t:

$$\mu(tv) = \mu(0) + t\mathrm{d}_0\mu(v) + \frac{t^2}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mu(tv) + O(t^3). \tag{1.22}$$

By our hypothesis (M, ω, J) is cscK, so $\mu(0) = 0$. Moreover, the differential $\mathrm{d}_0\mu(v)$ vanishes, since the origin is a fixed point of the K-action, so

$$\mu(tv) = \frac{t^2}{2}\nu(v) + O(t^3). \tag{1.23}$$

By the above computation and the polystability of v, the second derivative $\nu(v)$ vanishes. Hence $\mu(tv) = O(t^3)$.

Consider now the map

$$\begin{aligned}
\mu_t : \mathfrak{k}_\ell &\longrightarrow \mathfrak{k}_{\ell-4} \\
f &\longrightarrow \mu(\exp(if)\cdot tv).
\end{aligned}$$

where $\mathfrak{k}_\ell$ is the $W^{2,\ell}$-completion of $\mathfrak{k}$. For $x \in V$, denote by K_x the stabiliser of x with respect to the K-action, and by $\mathfrak{k}_x$ its Lie algebra. Then for all f in $\mathfrak{k}_x$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu(tx), f\rangle = \Omega_{tx}(x, \sigma_{tx}(f)) = 0,$$

where $\sigma_x : \mathfrak{k} \longrightarrow T_xV$ is the infinitesimal action. Hence for all x in V, $\mu(x)$ belongs to $\mathfrak{k}_x^\perp$. We next wish to apply the implicit function theorem to the map

$$\mu_t : (\mathfrak{k}_{tv}^\perp)_\ell \longrightarrow (\mathfrak{k}_{tv}^\perp)_{\ell-4}.$$

Let $D\mu_t$ be the linearisation of μ_t at the origin. If we show that

1. $D\mu_t$ is an isomorphism;

2. there exists ε such that $\|\mu_t(0)\| < \frac{\varepsilon}{\|D\mu_t^{-1}\|}$;

then we can conclude that there exists $f \in (\mathfrak{k}_{tv}^{\perp})_\ell$ such that $\mu_t(f) = 0$, i.e. there exists a zero of the moment map μ in the same $K^{\mathbb{C}}$-orbit as tv. Since μ_t is elliptic, by standard elliptic regularity f is smooth.

To prove the first requirement, let us compute $D\mu_t$:

$$D\mu_t(f) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mu_t(sf) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mu\left(\exp(isf)\cdot(tv)\right) = \mathrm{d}_{tv}\mu \circ \sigma_{tv}(f).$$

For $x \in V$ define $Q_x = \sigma_x^*\sigma_x$, where σ_x^* denotes the adjoint with respect to the L^2-norm induced by Ω. Then

$$Q_x = \mathrm{d}_x\mu \circ \sigma_x.$$

Indeed, for $f, h \in \mathfrak{k}$,

$$\langle Q_x(f), h\rangle_{L^2(\Omega)} = \Omega\left(\sigma_x(f), \sigma_x(h)\right) = \mathrm{d}_x\langle \mu, h\rangle(\sigma_x(f)).$$

Now $Q_x : \mathfrak{k}_x^{\perp} \to \mathfrak{k}_x^{\perp}$ is an isomorphism, thus we have proved the first requirement.

To prove the second requirement, we show that

$$\|Q_x^{-1}\| < ct^{-2}, \tag{1.24}$$

where $x = \exp(if)\cdot(tv)$ for $f \in \mathfrak{k}$ such that $\|f\| < \delta$ for some $\delta > 0$. Indeed, since $\mu_t(0) = O\left(t^3\right)$, there exists a constant ε' sufficiently small such that $\|\mu_t(0)\| < \varepsilon' t^2$. If we prove that (1.24) holds, then

$$\|\mu_t(0)\|\|Q_x^{-1}\| < \left(\varepsilon' t^2\right)\left(ct^{-2}\right) < \varepsilon' c.$$

In particular, since Q_x^{-1} coincides with $D\mu_t^{-1}$ at $x = \exp(if)\cdot(tv)$, it follows that

$$\|\mu_t(0)\| < \frac{\varepsilon}{\|D\mu_t^{-1}\|},$$

where $\varepsilon = \varepsilon' c$. The proof of the estimate (1.24) follows exactly as in [74, Prop 8] and we report it here for completeness. There is a constant c such that for all $h \in \mathfrak{k}_x^{\perp}$ and for all $x = \exp(if)\cdot v$ with $\|f\| < \delta$, we have

$$\|\sigma_x(h)\|_{\Omega_0}^2 \geq c\|h\|^2,$$

where Ω_0 is the metric (1.19) on $\widetilde{H}^1$. If V is sufficiently small, then the metric Ω can be bounded below by $\frac{1}{2}\Omega_0$. Then, since $\sigma_{tx}(h) = t\sigma_x(h)$,

$$\|\sigma_{tx}(h)\|_{\Omega} \geq \frac{1}{2}t\|\sigma_x(h)\|_{\Omega_0}.$$

It follows that

$$\langle \sigma_{tx}^*\sigma_{tx}(h), h\rangle = \|\sigma_{tx}(h)\|_{\Omega}^2 \geq \frac{1}{4}t^2 c\|h\|,$$

so $\|Q_x^{-1}\| < ct^{-2}$. $\square$

1.5 The moduli space of cscK manifolds

Fujiki and Schumacher [31] have constructed a moduli space of polarised cscK manifolds with a discrete group of automorphisms, using the Kuranishi theory we described in §1.4. Dervan, Neumann [13] and Inoue [45] extended their result to the case when the group of automorphisms is not discrete. We conclude this chapter with a brief discussion of their construction and results, which we will use in various instances in the following chapters.

Theorem 1.28. *There is a Hausdorff complex space $\mathcal{M}^{cscK}$ that parametrises Kähler manifolds with constant scalar curvature. The space $\mathcal{M}^{cscK}$ admits a Weil-Petersson type Kähler metric α_{WP} that represents the first Chern class of a line bundle over $\mathcal{M}^{cscK}$.*

In the case of discrete automorphism group, the moduli space is defined locally around a cscK manifold (M, L) as the Kuranishi space V defined in Theorem 1.23 quotient by the action of the group of automorphisms. More precisely, we consider the complex analytic subspace V^{int} of V of integrable deformations of the complex structure of M. Fujiki and Schumacher prove that there exists an open neighbourhood of the origin of V^{int} where the cscK equation has a solution, using the implicit function theorem and the fact the automorphism group is discrete. The local structure of the moduli space is then given as the quotient of V^{int} by the action of $\mathrm{Aut}(M, L)$, which is a *finite* discrete group. In particular, the moduli space has locally the structure of a complex orbifold space. When (M, L) has continuous automorphisms, the local structure of the moduli space is instead that of a GIT quotient $V^{int} /\!\!/ \mathrm{Aut}(M, L)$.

We next describe the definition of the Weil-Petersson type Kähler metric on the moduli space. We start by recalling that the notions of a smooth Hermitian metric and of a Kähler metric are well posed on a complex space [31, Definitions 1.1, 1.2]. Let (M, L) be a cscK manifold of complex dimension m and let $\mathcal{U} \to V^{int}$ be the Kuranishi family with central fibre (M, L) described with the discussion following Definition 1.24. In particular, $\mathcal{U} \to V^{int}$ is a family of cscK manifolds. From Theorem 1.23 we get an injective map

$$\mathrm{d}_0\Phi : T_0V^{int} \to H^0(M, T^{1,0}M). \tag{1.25}$$

We use this map to pull-back the standard L^2-product on $H^0(M, T^{1,0}M)$. More precisely, let $\alpha, \beta \in T_0V$. Then we can consider α and β as *harmonic* $(0, 1)$-forms with values in $T^{1,0}M$, where harmonicity is defined with respect to the Laplacian type operator (1.15). Then for each $t \in V^{int}$,

$$\alpha_t := \langle \alpha, \beta \rangle_t = \int_{M_t} \Lambda_{\omega_t} \mathrm{Tr}_{\omega_t}(\alpha\bar{\beta})\omega_t^m.$$

Definition 1.29. The *Weil-Petersson metric* on V^{int} is the two-form given by the collection $\{\alpha_t\}$.

The Weil-Petersson metric satisfies the following fibre-integral formula.

Theorem 1.30 ([13, 31])**.** *The Weil-Petersson metric on V^{int} coincides with the $(1, 1)$-form:*

$$\alpha_{WP} = \frac{\widehat{S}}{m+1} \int_{\mathcal{U}/V} \omega^{m+1} - \int_{\mathcal{U}/V} \rho \wedge \omega^m, \tag{1.26}$$

where $\widehat{S}$ is the average scalar curvature of M and ρ is the relative Ricci form induced by ω on the fibres of $\mathcal{U} \to V$:

$$\rho = i\partial\bar{\partial} \log(\omega^m).$$

To produce an actual Kähler metric on the moduli space, the two-form (3.7) is then glued on the local charts $V^{int} /\!\!/ \mathrm{Aut}(M, L)$. Moreover, for any family $\pi_X : X \to B$ of cscK manifolds, a Weil-Petersson type form can be defined by the expression (3.7) as a fibre integral for the fibration π_X. More precisely, π_X induces a map

$$q : B \to \mathcal{M}^{cscK},$$

such that the pull-back of the Weil-Petersson metric on $\mathcal{M}^{cscK}$ via q is a semipositive closed two-form on B and it can be written as the fibre integral (3.7) over X/B. Moreover, α_{WP} is positive definite if and only if the map (1.25) for the family $X \to B$ is an immersion.

Chapter 2

Holomorphic submersions

Let $\pi_X : X \longrightarrow B$ be a holomorphic submersion between compact Kähler manifolds. By Ehresmann's fibration theorem 1.25 all fibres are diffeomorphic. Let n be the dimension of B and m be the dimension of the fibres, so that $\dim(X) = n + m$.

Definition 2.1. A line bundle H_X on the total space X of the submersion π_X is said to be *relatively ample* if its restrictions to the fibres of π_X are ample. Similarly, a representative $\omega \in c_1(H_X)$ is called a *relatively Kähler metric* if its restrictions to the fibres of π_X are Kähler metrics.

In the following, we always assume that X admits a relatively ample line bundle H_X and B admits an ample line bundle L, and we consider the line bundles as fixed. Thus when we write $\pi_X : X \longrightarrow B$ we always mean $\pi_X : (X, H_X) \longrightarrow (B, L)$.

Let ω be a relatively Kähler metric on X in $c_1(H_X)$ and ω_B a Kähler metric on B in $c_1(L)$. We can define a Kähler metric on X by taking the relative Kähler metric ω and adding a large multiple of the base metric, pulled-back on X

$$\omega_k = \omega + k\pi_X^*\omega_B \qquad k \gg 0.$$

We often omit the pull-back and write $\omega + k\omega_B$. The relatively Kähler form ω determines a splitting of the tangent space

$$TX = \mathcal{V} \oplus \mathcal{H}^\omega \tag{2.1}$$

where $\mathcal{V}_x = T_x X_{\pi_X(x)}$ is the tangent space to the fibre, and

$$\mathcal{H}^\omega_x = \{u \in T_x X \mid \omega(u, v) = 0 \ \forall v \in \mathcal{V}_x\}$$

In the context of symplectic fibrations ω is called a *symplectic connection* [54, Chapter 6] and the splitting (2.1) induces a splitting on all tensor bundles. We denote by ω_F the purely vertical part of ω and by $\omega_{\mathcal{H}}$ the purely horizontal part of ω.

In what follows will also need the groups of automorphisms of the projections π.

Definition 2.2. For $\pi_X : X \longrightarrow B$ the group of *relative automorphisms* of the fibration, or the group of *automorphisms of* π, is

$$\mathrm{Aut}(\pi_X) := \left\{ f \in \mathrm{Aut}(X, H_X) \mid \pi_X \circ f = \pi_X \right\}.$$

2.1 Splitting of the function space

In this section, we assume that the fibres of X each admits a constant scalar curvature Kähler metric. We also assume that the spaces $H^0(X_b, T^{1,0}X_b)$ of holomorphic vector fields on the fibre X_b are isomorphic as Lie algebras for all b. The following lemma explains that one can define a relatively Kähler metric on the total space which is relatively cscK.

Lemma 2.3 ([14, Lemma 3.8]). *For any $b \in B$, let ω_b be a cscK metric on the fibre X_b in the class $c_1(H_X|_b)$. Then there exists a $\omega \in c_1(H_X)$ which is relatively cscK.*

Denoting by I the complex structure of X, we next explain how the relative Kähler metric (ω, I) induces a splitting of the space of smooth functions on X. Let

$$\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} : C^\infty(X, \mathbb{R}) \to C^\infty(X, \mathbb{R})$$

be the vertical Lichnerowicz operator, defined fibrewise as $\left(\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}}\varphi\right)|_{X_b} = \mathcal{D}^*_b\mathcal{D}_b\ \varphi\big|_{X_b}$. It is a real operator since the fibrewise metric is cscK. By integrating a function $\varphi \in C^\infty(X, \mathbb{R})$ over the fibres of π_X, we define a projection

$$\begin{aligned} C^\infty(X, \mathbb{R}) &\longrightarrow C^\infty(B, \mathbb{R}) \\ \varphi &\longmapsto \int_{X/B} \varphi\omega^m. \end{aligned}$$

Its kernel is given by the space $C^\infty_0(X, \mathbb{R})$ of functions that have fibrewise mean value zero. A key step in the study of optimal symplectic connections is that we can further split this space as follows.

Consider a real vector bundle $E \to B$ [14, §3.1], whose fibre over $b \in B$ is the real finite-dimensional vector space $\ker_0(\mathcal{D}^*_b\mathcal{D}_b)$ of holomorphy potentials on the fibre X_b with mean-value zero with respect to ω_b. E is well defined as a vector bundle since we are assuming that the complex dimension of the Lie algebra $\mathfrak{h}(X_b)$ of holomorphic vector fields on X_b is independent of b. Its smooth global sections are

$$C^\infty(E) = \ker_0\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}}.$$

In [34, Lemma 2.7], Hallam showed - using the Cartan decomposition for the space $\mathfrak{h}(X_b)$ of holomorphic vector fields of the fibre - that E_b can be also viewed as the vector space of all Kähler potentials φ_b on X_b of mean-value zero for which $\omega_b + i\partial\bar{\partial}\varphi_b$ is still cscK. We can split $C^\infty_0(X)$ as

$$C^\infty_0(X, \mathbb{R}) = C^\infty(E) \oplus C^\infty(R),$$

where $C^\infty(R)$ is the fibrewise L^2-orthogonal complement with respect to the fibre metric ω_b, i.e. for all $\varphi \in \ker_0\mathcal{D}^*_b\mathcal{D}_b$, $\psi \in C^\infty(R)$

$$\langle\varphi, \psi\rangle_b := \int_{X_b} \varphi\psi\omega_b^m = 0.$$

In the end, we obtain

$$C^\infty(X, \mathbb{R}) = C^\infty(B) \oplus C^\infty(E) \oplus C^\infty(R). \tag{2.2}$$

We denote by $p_E : C^\infty(X) \to C^\infty(E)$ the projection.

Since we are interested in deformations of the complex structure of X, sometimes we will denote the vector bundle E as $E(\omega, I)$ to underline its dependence on the Kähler metric. Notice that if we change just the relatively Kähler metric ω to $\omega + i\partial\bar{\partial}\varphi$, for $\varphi \in C^\infty(X)$, the vector bundles $E(\omega, I)$ and $E(\omega + i\partial\bar{\partial}\varphi, I)$ are isomorphic. We give the following definition.

Definition 2.4. We denote by $\mathcal{K}_E$ the space of all smooth functions $\varphi \in C^\infty(X)$ such that $\omega + i\partial\bar{\partial}\varphi$ is still a *fibrewise cscK metric*.

The following proposition [17, Lemma 4.20] relates the space $\mathcal{K}_E$ to the vector bundle $E \to B$, justifying the notation.

Proposition 2.5. *Let $\varphi_t : [0,1] \to \mathcal{K}_E$ be a smooth path. Then for all t*

$$\dot{\varphi}_t \in C^\infty(B) \oplus C^\infty(E(\omega + i\partial\bar{\partial}\varphi_t, I))$$

that is to say that, up to a function on the base, $\dot{\varphi}_t$ is a fibrewise holomorphy potential with respect to $\omega + i\partial\bar{\partial}\varphi_t$.

Again, if we want to underline the dependence on the complex structure, we write $\mathcal{K}_E(I)$.

2.2 Optimal symplectic connections

In this section, we give the definition of optimal symplectic connection. We begin in §2.2.1 by describing optimal symplectic connections on fibrations with cscK fibres, following Dervan-Sektnan [14]. In §2.2.2 we describe a generalisation of optimal symplectic connection to fibrations with K-semistable fibres.

2.2.1 The relatively cscK case

Let $\omega \in c_1(X)$ be a relatively Kähler metric. Then ω defines two curvature quantities on X, the symplectic curvature and the relative Ricci curvature.

Definition 2.6. The *symplectic curvature* of ω is a two-form on B with values in the fibrewise Hamiltonian vector fields given as follows: if $v_1, v_2 \in \mathfrak{X}(B)$,

$$F_{\mathcal{H}}(v_1, v_2) = [v_1^\sharp, v_2^\sharp]^{\text{vert}},$$

where $v_j^\sharp$ denotes the horizontal lift.

Let γ^* be the map which associates to a fibrewise Hamiltonian vector field its fibrewise Hamiltonian function with fibrewise mean value zero. Thus we consider $\gamma^*(F_{\mathcal{H}})$, which is a two-form on B with values in $C_0^\infty(X, \mathbb{R})$, and we pull it back on X. Notice that the two-form $\gamma^*(F_{\mathcal{H}})$ depends only on the relatively symplectic form and not on the complex structure. Moreover, the symplectic curvature is related to the symplectic connection as follows [14, §3.2]:

$$\omega_{\mathcal{H}} = \gamma^*(F_{\mathcal{H}}) + \pi_X^*\beta,$$

where β is a two-form on B.

Definition 2.7. The relatively Kähler form ω induces a Hermitian metric $\wedge^m g_\omega$ on the top-wedge power of the vertical tangent bundle

$$\bigwedge^m \mathcal{V}^{1,0} = -K_{X/B}.$$

Explicitly, if $A, B \in \bigwedge^m \mathcal{V}^{1,0}$, we can write locally $A = f_A \partial_{w^1} \wedge \cdots \wedge \partial_{w^m}$ and $B = f_B \partial_{w^1} \wedge \cdots \wedge \partial_{w^m}$. Then

$$\langle A, B \rangle_{\wedge^m g_\omega} = \det(g_\omega) f_A \overline{f_B}.$$

Its curvature, denoted by ρ, represents the first Chern class of the relative anticanonical bundle $-K_{X/B}$. We call ρ the *relative Ricci curvature* of ω and we denote by $\rho_{\mathcal{H}}$ its purely horizontal part.

We can now give the definition of an optimal symplectic connection according to [14].

Definition 2.8. Let ω be a relatively cscK metric. Then ω is an *optimal symplectic connection* if

$$p_E\left(\Delta_{\mathcal{V}}(\Lambda_{\omega_B}\gamma^*(F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}\right) = 0. \tag{2.3}$$

This is a second-order elliptic equation on the vector bundle $E \to B$. In the following, we will use the notation $\Theta(\omega, I) = \Delta_{\mathcal{V}}(\Lambda_{\omega_B}\gamma^*(F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}$.

The optimal symplectic connection equation arises from expanding the scalar curvature of ω_k in negative powers of k. Indeed, to leading order the expansion reads

$$\mathrm{Scal}(\omega_k) = \widehat{S}_b + O\left(k^{-1}\right),$$

where the leading order term is the scalar curvature of the fibres, which we assume to be constant and is independent of b. The left-hand side of equation (2.3) is then the projection onto $C^\infty(E)$ of the k^{-1}-term of the expansion. x The linearisation of the equation at a solution is given by the operator $\mathcal{R}^*\mathcal{R}$ [14, §4.3], where

$$\mathcal{R}(\varphi_E) = \bar{\partial}_B \nabla^{1,0}_{\mathcal{V}}\varphi_E \tag{2.4}$$

and the adjoint is computed with respect to the Hermitian metric $\omega_F + \omega_B$. Here $\nabla^{1,0}_{\mathcal{V}}\varphi_E$ is a section of the holomorphic tangent bundle; the vertical part of $\bar{\partial}\nabla^{1,0}_{\mathcal{V}}\varphi_E$ vanishes since $\varphi \in C^\infty(E)$ and the horizontal part is denoted by the expression (2.4). The operator (2.4) can be described as follows [14, §4.3]: let $\mathcal{D}^*_k\mathcal{D}_k$ be the Lichnerowicz operator with respect to the Kähler metric ω_k. It can be written as a power series expansion in negative powers of k:

$$\mathcal{D}^*_k\mathcal{D}_k = \mathcal{L}_0 + k^{-1}\mathcal{L}_1 + O\left(k^{-2}\right),$$

where $\mathcal{L}_0$ is the *vertical* Lichnerowicz operator $\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}}$. Then for φ, ψ fibrewise holomorphy potentials

$$\int_X \varphi\mathcal{L}_1(\psi)\omega^m \wedge \omega_B^n = \int_X \langle\mathcal{R}\varphi, \mathcal{R}\psi\rangle_{\omega_F+\omega_B}\omega^m \wedge \omega_B^n.$$

This means that the operator $\mathcal{R}^*\mathcal{R}$ can actually be seen as $p_E \circ \mathcal{L}_1$ restricted to $C^\infty_E(X)$. The kernel of $\mathcal{R}$, thus of $\mathcal{R}^*\mathcal{R}$, consists of fibrewise holomorphy potentials which are global holomorphy potentials on X with respect to ω_k.

2.2.2 Optimal symplectic connections in general

Let now $(Y, H_Y) \to (B, L)$ be a polarised holomorphic submersion with $\omega \in c_1(H_Y)$ a relatively Kähler metric.The following assumptions restrict the class of admissible fibrations to those whose fibres satisfy a stability property defined in terms of K-stability. More precisely we assume that:

1. the fibres Y_b are analytically K-semistable, which means by definition that they each admit a degeneration to a cscK manifold X_b. We also assume that the deformation is compatible with the fibration structure in the following sense: there exists a holomorphic map $\widehat{\pi} : (\mathcal{X}, \mathcal{H}) \to (B, L) \times S$, parametrised by a disk S, such that for $s \neq 0$, the family $(\mathcal{X}_s, \mathcal{H}_s) \to B$ is isomorphic to the original fibration $\pi_Y : (Y, H_Y) \to B$ and the central fibration at $s = 0$ is a family $\pi_X : (X, H_X) \to B$ whose fibres are cscK;

2. the automorphism groups $\mathrm{Aut}_0(X_b, H_b)$ of the fibres are all isomorphic.

The first hypothesis is a stability assumption. We will refer to the submersion $X \to B$ as the *relatively cscK degeneration* of $Y \to B$. The second hypothesis holds if and only if the spaces $H^0(X_b, T^{1,0}X_b)$ are isomorphic as Lie algebras, which we assumed in §2.1 to define the vector bundle $E \to B$. It is needed to define optimal symplectic connections and it is a key hypothesis for the construction of the moduli space of fibrations in §4.1.

A relative version of Ehresmann's theorem, which will be proved as Proposition 2.19, implies that we can locally trivialise the family in such a way that all $\mathcal{X}_s$ are diffeomorphic. So we can take the perspective of fixing ω and seeing $\mathcal{X} \to B \times S$ as a family of compatible complex structures $\{J_s\}$ which keep π_X a holomorphic submersion and are all biholomorphic except for J_0.

Let

$$\mathscr{J}_\pi = \left\{J \text{ almost complex structure compatible with } \omega \text{ and s.t. } \mathrm{d}\pi \circ J = J_B \circ \mathrm{d}\pi\right\}. \tag{2.5}$$

Compatibility with ω means that $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$ and that $\omega_b \circ J \mid_{X_b}$ is non degenerate and positive definite. The tangent space at I to $\mathscr{J}_\pi$ can be identified with

$$T_I^{0,1}\mathscr{J}_\pi = \left\{A \in \Omega^{0,1}(\mathcal{V}^{1,0}) \mid \omega_F(A\cdot,\cdot) + \omega_F(\cdot, A\cdot) = 0\right\}. \tag{2.6}$$

As in §1.4, for any fibre X_b let V_b be the Kuranishi space, K_b the group of Hamiltonian isometries and Ψ_b the Kuranishi map (1.17) of the fibre. Let $x_{s,b} \in V_b$ be such that $\Psi_b(x_{s,b}) = J_s \mid_{X_b}$. Let μ_b be the moment map (1.18). Then we can define a section of $C^\infty(E)$ as

$$\mu_s(b) = p_E\left(\mathrm{Scal}_{X_b}\left(\omega_b, \Psi_b(x_{s,b})\right)\right). \tag{2.7}$$

Note that Ψ_b may not vary smoothly with b, but when applied to $x_{s,b}$ it gives the complex structure $J_s \mid_{X_b}$. Since J_s is a complex structure defined on the whole X, it varies smoothly with the base, so μ_s is a smooth section. For each fibre X_b we can linearise the action to the tangent space to V_b at 0 as in §1.4, so we can define another section ν of E by

$$\nu(b) = \nu_b(v_b). \tag{2.8}$$

Here ν_b is the moment map defined in Definition 1.26 for the linear action of K_b on $\widetilde{H}^1(X_b)$, and $v_b \in \widetilde{H}^1(X_b)$ is tangent at $0 \in V_b$. Even if ν_b does not necessarily vary smoothly with b, ν is a smooth section because there is an expansion

$$\mu_s(b) = \frac{s^2}{2}\nu + O(s^3) \tag{2.9}$$

as explained in Proposition 1.23, and μ_s is smooth.

Definition 2.9. We say that the relative Kähler form ω is an *optimal symplectic connection* on $(Y, H_Y) \to B$ if it satisfies the equation

$$p_E\left(\Delta_{\mathcal{V}}(\Lambda_{\omega_B}\gamma^*(F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}\right) + \frac{\lambda}{2}\nu = 0 \tag{2.10}$$

for some constant $\lambda > 0$.

The first term is the left-hand side of the optimal symplectic connection equation (2.3) for fibrewise cscK metrics, and it involves only the complex structure I of the relatively cscK degeneration. The second term represents the deformation of the complex structure, in terms of the first-order deformation of the fibres.

Remark 2.10. Similarly to the relatively cscK case, the equation arises by expanding the scalar curvature of the Kähler metric (ω_k, J_s) in inverse powers of k and powers of s and then relating the two parameters to obtain a single expansion. The left-hand side of (2.10) is then the projection onto $C^\infty(E(\omega, I))$ of the sub-leading order term of such an expansion. We explain the expansion of the scalar curvature in detail in §3.1.

In §3.3, we will prove that the linearisation of the equation at a solution is given by an operator

$$\widehat{\mathcal{L}} = \mathcal{R}^*\mathcal{R} + \mathcal{A}^*\mathcal{A},$$

where $\mathcal{R}$ is the operator (2.4) and $\mathcal{A}$ is obtained by extending the map (1.20) to a fibrewise map. Its kernel is given by the fibrewise I-holomorphy potentials which are global holomorphy potentials with respect to J_s.

The definition of an optimal symplectic connection can be generalised to that of an extremal symplectic connection as follows.

Definition 2.11. The relative Kähler form ω is an *extremal symplectic connection* on Y if

$$\widehat{\mathcal{L}}\left(p_E(\Theta(\omega, I)) + \frac{\lambda}{2}\nu\right) = 0.$$

In particular, if ω is an extremal symplectic connection, the fibrewise I-holomorphy potential

$$h_1 := p_E\left(\Theta(\omega, I)\right) + \frac{\lambda}{2}\nu \tag{2.11}$$

is a holomorphy potential for the complex structure of Y. The definition of an extremal symplectic connection extends the one given by Dervan-Sektnan on relatively cscK fibrations [14, §3.4].

2.3 Deformations of fibrations

In this section, we study in more detail the deformations of a holomorphic fibrewise cscK fibration. In particular, we prove a relative version of Ehresmann's fibration theorem in Proposition 2.19, which allows us to view families of fibrations as families of complex structures in $\mathscr{J}_\pi$ on the same underlying smooth fibration. The main result of this section is a relative version of Kuranishi's Theorem 1.23, which will be needed to describe the linearisation of the optimal symplectic connection equation (2.10) and to study the moduli space of its solutions.

We start by giving a description in local coordinates of a first-order deformation $A \in T_I\mathscr{J}_\pi$. In local computations, we make the following notational conventions:

$\{w^1, \dots, w^m\}$ denote vertical holomorphic coordinates; indices are denoted with the letters $a, b, c, \dots$;

$\{z^1, \dots, z^n\}$ denote holomorphic coordinates on the base; indices are denoted with the letters $i, j, k, \dots$;

$\{\zeta^1, \dots, \zeta^{n+m}\}$ denote holomorphic coordinates on the total space; indices are denoted with the letters $p, q, r, \dots$.

We write $A \in T_J\mathscr{J}_\pi$ locally as

$$A = A^a_{\bar{b}} d\bar{w}^b \otimes \partial_{w^a} + A^a_{\bar{j}} d\bar{z}^j \otimes \partial_{w^a},$$

since A takes values in the vertical vector fields. The following lemma explains the relation between $A^a{}_{\bar{b}}$ and $A^a{}_{\bar{j}}$.

Lemma 2.12. *For $A \in T_I \mathscr{J}_\pi$ we have that:*

1. *A vanishes on horizontal vector fields;*
2. *$A^a{}_{\bar{j}} = A^a{}_{\bar{c}} (\omega_F)^{d\bar{c}} (\omega)_{d\bar{j}}$.*

Proof. As for the first claim, if $u \in \mathcal{V}$, $v \in \mathcal{H}^\omega$, then

$$\omega(u, Av) = \omega_F(u, Av) = -\omega_F(Au, v) = 0. \tag{2.12}$$

Indeed, the first equality comes from the fact that Av is vertical and ω coincides with ω_F on a pair of vertical vector fields. The middle equality follows from the compatibility of the deformation with the fibrewise symplectic structure (2.6). The last equality follows from the fact that v is horizontal. So Av is horizontal, since the relation (2.12) holds for any $u \in \mathcal{V}$; but Av is also vertical. Thus $Av = 0$. This proves the first claim.

We prove the second claim. While ∂_{w^a}, $\partial_{\bar{w}^a}$ are vertical vector fields on X, ∂_{z^j}, $\partial_{\bar{z}^j}$ are not horizontal in general. So we have a splitting

$$\partial_{\bar{z}^j} = \varepsilon_{\bar{j}} + \eta_{\bar{j}} \qquad \text{with} \qquad \varepsilon_{\bar{j}} \in \mathcal{H}^\omega,\ \eta_{\bar{j}} \in \mathcal{V}.$$

Then from $\omega(\partial_{w^a}, \varepsilon_{\bar{j}}) = 0$ it follows that

$$(\omega)_{a\bar{b}}\, \eta^b_{\bar{j}} = (\omega)_{a\bar{j}}.$$

So $\eta_{\bar{j}} = (\omega_F)^{a\bar{c}} (\omega_X)_{a\bar{j}} \partial_{\bar{w}^c}$. Thus we can write the horizontal part of $\partial_{\bar{z}^j}$ as

$$\varepsilon_{\bar{j}} = \partial_{\bar{z}^j} - (\omega_F)^{a\bar{c}} (\omega)_{a\bar{j}} \partial_{\bar{w}^c}.$$

Since A takes value in the vertical vector fields, $A(\varepsilon_{\bar{j}}) = 0$, so

$$0 = -A^a{}_{\bar{c}} (\omega_F)^{d\bar{c}} (\omega)_{d\bar{j}} \partial_{w^a} + A^a{}_{\bar{j}}\, \partial_{w^a},$$

hence the claim. □

The following lemma explains the relation between a $J \in \mathscr{J}_\pi$ and the splitting (2.1) of the tangent bundle of X induced by ω, by showing that the elements of $\mathscr{J}_\pi$ in a neighborhood of I differ from I only on the vertical vector bundle.

Lemma 2.13. *Any $J \in \mathscr{J}_\pi$ preserves the splitting of the tangent space $TX = \mathcal{V} \oplus \mathcal{H}^\omega$. Moreover, $J(u) = I(u)$ for all $u \in \mathcal{H}^\omega$.*

Proof. For the first fact to hold, we have to prove that $J(\mathcal{V}) \subseteq \mathcal{V}$ and $J(\mathcal{H}^\omega) \subseteq \mathcal{H}^\omega$.

1. Let $v \in \mathcal{V}$. Then
$$\mathrm{d}\pi(Jv) = J_B(\underbrace{\mathrm{d}\pi(v)}_{=0}) = 0,$$
so $Jv \in \mathcal{V}$.

2. Let $u \in \mathcal{H}^\omega$. Then $\omega(u, v) = 0$ for every $v \in \mathcal{V}$. So

$$\omega(Ju, v) = -\omega(u, Jv) = 0,$$

since Jv is vertical by the previous step.

We now prove that indeed $J(u) = I(u)$ for all $J \in \mathscr{J}_\pi$ and all $u \in \mathcal{H}^\omega$. Consider for instance a first order deformation $I + \varepsilon A$. Since A vanishes on horizontal vector fields by Lemma 2.12, if $u \in \mathcal{H}^\omega$, $(I + \varepsilon A)(u) = I(u)$. Let now J_s be a path in $\mathscr{J}_\pi$ which joins I to J. Then

$$\partial_s J_s(\mathcal{H}^\omega) = A_s(\mathcal{H}^\omega) = 0,$$

so for all s we have $J_s(u) = I(u)$, for all $u \in \mathcal{H}^\omega$, from which the claim follows. □

In particular, the last part of the proof shows that the horizontal parts of the operators $\partial, \bar\partial$ with respect to I remain the same for any J in $\mathscr{J}_\pi$.

Remark 2.14. Let $k \gg 0$ be such that $\omega + k\omega_B$ is a Kähler form on X. Then $\mathscr{J}_\pi$ embeds into $\mathscr{J}(\omega + k\omega_B)$. Indeed for $J \in \mathscr{J}_\pi$

$$\omega_k(J\cdot, J\cdot) = \omega(J\cdot, J\cdot) + k\pi^*\omega_B(J\cdot, J\cdot)$$

and $\pi^*\omega_B(J\cdot, J\cdot) = \omega_B(\mathrm{d}\pi J\cdot, \mathrm{d}\pi J\cdot) = \omega_B(J_B\mathrm{d}\pi\cdot, J_B\mathrm{d}\pi\cdot) = \pi^*\omega_B(\cdot, \cdot)$.

2.3.1 Families of holomorphic submersions

In this section, we give a more rigorous definition of a family of fibrations and we prove a relative version of Ehresmann's fibration theorem. Families of fibrations are in particular families of holomorphic maps, for which a deformation theory has been developed by Horikawa [41, 42], under the assumption that $H^2(X, TX)$ embeds into $H^2(X, \pi^*TB)$.

Definition 2.15. A family of holomorphic submersions onto B with central fibre $X \to B$ is the data of $(\mathcal{X}, \widehat{\pi}, p, S)$, where:

1. $\mathcal{X}, S$ are complex manifolds;
2. $p : \mathcal{X} \to S$ and $\widehat{\pi} : \mathcal{X} \to B \times S$ are proper holomorphic submersions;
3. there is a distinguished point $0 \in S$ such that $\widehat{\pi}_0$ induces $\pi : X \to B$ and $p = \mathrm{pr}_2 \circ \widehat{\pi}$.

A family of holomorphic maps $(\mathcal{X}, \widehat{\pi}, p, S)$ onto B is *complete* if for any other family $(\mathcal{X}', \widehat{\pi}', p', S')$ with the same central fibre, there exists a map $\tau : S' \to S$, defined locally on neighbourhoods of the distinguished points, such that the family $(\mathcal{X}', \widehat{\pi}', p', S')$ can be obtained by $(\mathcal{X}, \widehat{\pi}, p, S)$ via pull-back using τ.

In order to parametrise compatible almost complex structures, we need to take into account that all the fibres admit a relative Kähler metric in the same cohomology class as the central fibre (X, ω). Following Schumacher [67], we introduce the definition of a polarised family.

Definition 2.16. A polarised family $(p : \mathcal{M} \to S, \gamma)$ is a family of compact complex manifolds with a section $\gamma \in \Gamma(S, R^2p_*\mathbb{R})$ such that $\gamma|_{M_t} \in H^2(M_t, \mathbb{R})$ is a Kähler class. Analogously, a polarised family of maps onto B is a family of maps $(\mathcal{X}, \widehat{\pi}, p, S)$ with a section $\gamma \in \Gamma(S, R^2p_*\mathbb{R})$ such that $\gamma|_{X_s} \in H^2(X_s, \mathbb{R})$ is a relative Kähler class with respect to the projection $\widehat{\pi}_s : X_s \to B$.

The following theorem guarantees that a polarised family exists for a Kähler manifold (M, γ_0). In the next section, we will prove an analogous result for holomorphic submersions.

Theorem 2.17 (Schumacher [67])**.** *Let (X, γ_0) be a Kähler manifold, with $\lambda_0 \in H^2(X, \mathbb{R})$ a Kähler class. Let $p : \mathcal{X} \longrightarrow S$ be a versal family with central fibre (X, λ_0) (which exists by Kuranishi's Theorem). Then there exists $S' \subset S$ such that λ_0 can be extended to a global section $\gamma \in \Gamma(S', R^2 p_* \mathbb{R})$, thus $(\mathcal{X} \longrightarrow S', \gamma)$ is a polarised family.*

We conclude this section with a proof of a relative version of Ehresmann's fibration theorem [44, Proposition 6.2.2]. For a smooth proper morphism $p : M \longrightarrow N$ with N connected, Ehresmann's fibration theorem says that the fibres of p are all diffeomorphic, and more precisely that M is locally diffeomorphic to a product. To generalise it to fibrations, we need the following definition.

Definition 2.18. Let $F : M \longrightarrow N$ be a smooth map, u a vector field on M, u' a vector field on N. We say that u and u' are *F-related* if

$$F \circ \phi^t_{u'} = \phi^t_u \circ F.$$

Then we can extend Ehresmann's theorem 1.25 to our setting as follows.

Proposition 2.19 (Relative Ehresmann's Theorem)**.** *Let $(\mathcal{X}, \widehat{\pi}, p, S)$ be a family of holomorphic maps onto B with S connected. Then there exists a diffeomorphism $\tau : \mathcal{X} \xrightarrow{\sim} X \times S$ which commutes with the projections to B, i.e.*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\tau", "\sim"'] \arrow[d, "\widehat{\pi}"'] & X \times S \arrow[dl, "\pi \times \mathrm{id}"] \\ B \times S & \end{tikzcd}$$

Proof. Up to restricting to a segment, we can assume that S is a small open neighbourhood of the origin in $\mathbb{R}$. We can then consider the vector field $u = \partial_s$, and view it as a vector field in $B \times S$, denoted by u'. This means that, denoting by $\phi^t_{u'}$ its flow and $\mathrm{pr}_2 : B \times S \longrightarrow S$ the second projection, u' is pr_2-related to u. It is a consequence of the implicit function theorem that if $F : M \longrightarrow N$ is a smooth *submersion* of manifolds, then for any vector field on N there exists a vector field on M which is F-related to it. Let then v be a vector field on $\mathcal{X}$ which is $\widehat{\pi}$-related to u', i.e.

$$\widehat{\pi} \circ \phi^t_v = \phi^t_{u'} \circ \widehat{\pi}.$$

Then, using $p = \mathrm{pr}_2 \circ \widehat{\pi}$, we obtain

$$p \circ \phi^t_v = \phi^t_u \circ p,$$

thus v is p-related to u. Hence we can define a map

$$\begin{aligned} \tau : \mathcal{X} &\longrightarrow X \times S \\ z &\longmapsto (\phi^{-t}_v(z), p(z)), \end{aligned}$$

which is a diffeomorphism with inverse

$$(x, s) \longmapsto \phi^s_v(x).$$

Since v is $\widehat{\pi}$-related to u', this diffeomorphism commutes with the projections to B, as required. □

We can then formalise the degeneration family of fibrations introduced in §2.2.2. Let $(X, H_X) \to (B, L)$ be a polarised holomorphic submersion with $\omega \in c_1(H_X)$ a relatively Kähler metric. Then we consider the following setting: $(\mathcal{X}, \mathcal{H}, \widehat{\pi}, p, S)$ is a smooth *polarised family* of maps onto B with central fibration $(X, H_X) \to B$, where we can assume for simplicity that S is a disk in $\mathbb{C}$. In particular, the line bundle $\mathcal{H}$ on $\mathcal{X}$ restricts to a relatively ample line bundle $\mathcal{H}_s$ on each fibration $\mathcal{X}_s \to B$. Ehresmann's theorem implies that we can locally trivialise the family in such a way that all $\mathcal{X}_s$ are diffeomorphic, so we can interpret the family as a family of almost complex structures $\{J_s\}$ on X which preserve the projection onto B.

Moreover, since (X_s, H_s) is a small deformation of (X, H_X), we have that $c_1(H_Y) = c_1(H_X)$. Then by Moser's theorem [8, Theorem 7.2] we can assume that ω is relatively Kähler with respect to the complex structures J_s, up to modifying (X_s, H_s) by a small diffeomorphism. So we can view a family $\mathcal{X} \to B \times S$ as a family of complex structures on X which keep π_X a holomorphic submersion and ω a relatively Kähler metric.

Remark 2.20. A possible example of a degeneration family $(\mathcal{X}, \mathcal{H}, \widehat{\pi}, p, S)$ is obtained by assuming that S is one-dimensional and there is an action of $\mathbb{C}^*$ on $S \times B$ which lifts to $(\mathcal{X}, \mathcal{H})$ such that $\widehat{\pi}$ is $\mathbb{C}^*$-equivariant. It follows that for $s \neq 0$ the fibrations $(\mathcal{X}_s, \mathcal{H}_s) \to B$ are all biholomorphic. In this context, one can think of the family $(\mathcal{X}, \mathcal{H}, \widehat{\pi}, p, S)$ as a family of test configurations varying holomorphically over B.

2.3.2 Relative Kuranishi's Theorem

As in the previous section, we consider a holomorphic submersion $\pi_X : (X, H_X) \to (B, L)$ with a relative Kähler metric ω and a complex structure I which is fibrewise cscK. We require a relative version of Székelyhidi's and Brönnle's deformation theory described in §1.4. Let $\mathscr{J}_\pi$ the space of compatible almost complex structures defined in (2.5) and $T_J \mathscr{J}_\pi$ be its tangent space at a point J. If $A \in T_J \mathscr{J}_\pi$ also $JA \in T_J \mathscr{J}_\pi$, so $\mathscr{J}_\pi$ admits an almost complex structure. Consider the map

$$\begin{aligned} P_{\mathcal{V}} : C_0^\infty(X, \mathbb{R}) &\longrightarrow T_I^{0,1} \mathscr{J}_\pi \\ \varphi &\longmapsto \bar\partial (\mathrm{grad}^{\omega_F} \varphi)^{1,0}, \end{aligned}$$

which is the relative version of the map defined in (1.7). Let $\widetilde{H}^1_{\mathcal{V}}$ be the kernel in $T_I^{0,1} \mathscr{J}_\pi$ of the operator

$$\Box_{\mathcal{V}} = P_{\mathcal{V}} P_{\mathcal{V}}^* + (\bar\partial^* \bar\partial)^2$$

inside $T_I^{0,1} \mathscr{J}_\pi$. This is an elliptic operator because $P_{\mathcal{V}} P_{\mathcal{V}}^*$ is trivial in horizontal directions, where the adjoint is computed with respect to any Kähler metric on X which restricts to ω_F vertically. So its kernel is a finite dimensional vector space and it can be described as

$$\widetilde{H}^1_{\mathcal{V}} = \left\{ \alpha \in T_I^{0,1} \mathscr{J}_\pi \mid P_{\mathcal{V}}^* \alpha = 0 = \bar\partial \alpha \right\}.$$

Fibrewise, $\Box_{\mathcal{V}}$ restricts to the operator (1.15) and $\widetilde{H}^1_{\mathcal{V}}$ restricts to the vector space described in (1.14). In particular, $\widetilde{H}^1_{\mathcal{V}}$ depends only on the vertical part of the metric, ω_F.

Consider the smooth fibre bundle $\mathcal{K} \to B$ whose fibre K_b is the stabiliser of $I|_{X_b}$ for the $\mathscr{G}_b$-action. in particular, K_b coincides with the group $\mathrm{Isom}(X_b, \omega_b)$ of biholomorphic isometries of the fibre. Thanks to our hypothesis 2, the groups K_b are finite-dimensional with dimension independent of b. The group of global sections of $\mathcal{K}$ is

$$K_\pi := \mathrm{Isom}(\pi_X, \omega) = \left\{ f \in \mathrm{Aut}(X) \mid f^* \omega = \omega \text{ and } \pi_X \circ f = \pi_X \right\}. \tag{2.13}$$

We next prove a relative version of Kuranishi Theorem, adapted from Chen-Sun [12, §6].

Theorem 2.21 (Relative Kuranishi Theorem)**.** *There exists a neighborhood of the origin $V_\pi \subset \widetilde{H}^1_{\mathcal{V}}$ and a K_π-equivariant holomorphic map*

$$\Psi : V_\pi \to \mathscr{J}_\pi$$

such that:

1. *$\Psi(0) = I$;*
2. *If $v_1, v_2 \in V$ and $v_1|_b \in K_b^{\mathbb{C}} \cdot v_2|_b$ for all b, and if $\Psi(v_1)$ is integrable, then $\Psi(v_1)|_{X_b}$ is in the same $\mathscr{G}_b^c$-orbit as $\Psi(v_2)|_{X_b}$;*
3. *For any $J \in \mathscr{J}_\pi$ integrable close to I, there exists J' in the image of Ψ such that, for all b, J'_b is in the same $\mathscr{G}_b^c$-orbit as J_b.*

Proof. We can identify any J close to I with an element $\alpha \in T_I^{0,1}\mathscr{J}_\pi$, i.e. with a $(0,1)$-form with values in the vertical holomorphic tangent bundle compatible with ω_F. So we have an embedding from an open subset in $T_I^{0,1}\mathscr{J}_\pi$ into $\mathscr{J}_\pi$:

$$f : \mathcal{U}(T_I^{0,1}\mathscr{J}_\pi) \hookrightarrow \mathscr{J}_\pi.$$

Given $b \in B$, we denote by ρ_b the restriction $\mathscr{J}_\pi \to \mathscr{J}(X_b)$. Then $f_b(\alpha\mid_{X_b}) = \rho_b \circ f(\alpha)$. We define now a new embedding $\hat{f} : \mathcal{U}(T_I^{0,1}\mathscr{J}_\pi) \hookrightarrow \mathscr{J}_\pi$ as follows:

$$\hat{f}(\alpha) = \int_{\mathcal{K}/B} g^{-1} f(g \cdot \alpha) \mathrm{d}\mu_{\mathcal{K}/B}(g),$$

where $\mathrm{d}\mu_{\mathcal{K}/B}$ is the fibrewise Haar measure on $\mathcal{K} \to B$. Then $\hat{f}$ is such that

$$\hat{f}(k \cdot \alpha)\mid_b = k\mid_b \cdot \hat{f}(\alpha)\mid_b .$$

Now, α is an integrable deformation if and only if it satisfies [44, Lemma 6.1.2]

$$N(\alpha) = \bar{\partial}\alpha + [\alpha, \alpha] = 0. \tag{2.14}$$

Note that if α is integrable its restriction to each fibre is also integrable, so equation (2.14) holds also fibrewise. For any $b \in B$, let $H_b : T_I^{0,1}\mathscr{J}_\pi\mid_{X_b} \to \widetilde{H}^1_b$ be the L^2_k-orthogonal projection and let G_b be the Green operator of $\Box_b$, defined by the condition

$$\mathbb{1} = G_b \Box_b + H_b = \Box_b G_b + H_b.$$

Let α be an integrable deformation. A simple computation starting from (2.14) leads to the identity

$$\alpha\mid_b + G_b \bar{\partial}_b^* \bar{\partial}_b \bar{\partial}_b^* [\alpha\mid_b, \alpha\mid_b] = H_b \alpha\mid_b .$$

Then we can define a map

$$\begin{aligned} F : B \times T_I^{0,1}\mathscr{J}_\pi &\to T_I^{0,1}\mathscr{J}_\pi \\ (b, \alpha) &\mapsto \alpha\mid_b + G_b \bar{\partial}_b^* \bar{\partial}_b \bar{\partial}_b^* [\alpha\mid_b, \alpha\mid_b], \end{aligned}$$

where $T_I^{0,1}\mathscr{J}_\pi$ is endowed with the Sobolev L^2_k-norm. The differential of F in the second component at the origin is the identity, since $G_b \bar{\partial}_b^* \bar{\partial}_b \bar{\partial}_b^* [\alpha\mid_b, \alpha\mid_b]$ is quadratic in α. Hence by the implicit function theorem we can locally invert F and the inverse varies smoothly with b. We consider the inverse restricted to an open ball in $\widetilde{H}^1_{\mathcal{V}}$, which we define to be V_π, and for $x \in V_\pi$ we denote it by $\alpha(x)$. Thus we have a family

$$U := \{\alpha(x) \mid x \in V_\pi\} \subset T^{0,1}_J \mathscr{J}_\pi, \tag{2.15}$$

and we can define

$$\begin{aligned} \Psi : V_\pi &\to \mathscr{J}_\pi \\ x &\mapsto \hat{f}(\alpha(x)). \end{aligned}$$

We begin by proving that this map satisfies the required properties. Denoting

$$U^{\mathrm{int}} = \{\alpha(x) \mid N(\alpha(x)) = 0\}$$

and

$$U^{\mathrm{int}}_{\mathcal{V}} = \{\alpha(x) \mid N_b(\alpha(x)\mid_b) = 0 \ \forall b \in B\},$$

we want to prove that U^{int} is an analytic subset of U. We begin by showing that $U^{\mathrm{int}}_{\mathcal{V}}$ is an analytic subset of U. On each fibre X_b, $\alpha(x)\mid_b$ is integrable if and only if $H_b[\alpha(x)\mid_b, \alpha(x)\mid_b] = 0$. Indeed

$$\begin{aligned} N_b\left(\alpha(x)\mid_b\right) &= \bar{\partial}_b \alpha(x)\mid_b + \left[\alpha(x)\mid_b, \alpha(x)\mid_b\right] \\ &= 2G_b \bar{\partial}^*_b \bar{\partial}_b \bar{\partial}^*_b \left[N(\alpha(x)\mid_b), \alpha(x)\mid_b\right] + H\left[\alpha(x)\mid_b, \alpha(x)\mid_b\right]. \end{aligned} \tag{2.16}$$

The map

$$\begin{aligned} B \times V_\pi &\to \widetilde{H}_{\mathcal{V}} \\ (b, v) &\mapsto H_b[\alpha(x)\mid_b, \alpha(x)\mid_b] \end{aligned}$$

is holomorphic, so $U^{\mathrm{int}}_{\mathcal{V}}$ is an analytic subset of U. Then, denoting $\overline{U}^{\mathrm{int}}$ the analytic family given by the Kuranishi Theorem [12, Lemma 6.1] applied to X, we see that U^{int} is the intersection of $\overline{U}^{\mathrm{int}}$ and $U^{\mathrm{int}}_{\mathcal{V}}$, so it is itself an analytic family. Moreover, when restricted to the fibre X_b, both maps f and F are K_b-equivariant and holomorphic, so (2) is also proved.

We prove (3). Let $J_x = \Psi(x) \in \mathscr{J}_\pi$, and fix $b \in B$. Given $\xi \in \Gamma(X, \mathcal{V})$ a vertical vector field, define

$$\begin{aligned} F_\xi : X &\to X \\ p &\mapsto \exp_p(\xi_p, g_{\pi(p)}), \end{aligned}$$

where $g_{\pi(p)}$ is the Riemannian metric on the fibre $X_{\pi(p)}$ with respect to J_x. Following [12, Lemma 6.1], we fix $v \in V_\sigma \subset V_\pi$, thought of as a tangent vector at x in V_π, and we define a map

$$\begin{aligned} R_b : \mathcal{U}(L^2_{k+2}(X_b, \mathbb{C})) &\to L^2_{k+2}(X_b, TX_b) \\ \varphi_b &\mapsto \xi_b(\varphi_b, v\mid_b) \end{aligned}$$

such that $R_b(0) = 0$ and

1. $\mathrm{d}_0 R_b(\varphi_b) = \mathrm{grad}^{\omega_b}(\mathrm{Re}(\varphi_b)) + J_v\mid_{X_b} \mathrm{grad}^{\omega_b}(\mathrm{Im}(\varphi_b))$;
2. $F^*_{\xi_b(\varphi_b, v|_b)} J_x \mid_{X_b} \in \mathscr{G}^c_b \cdot J_x \mid_{X_b}$.

Since this map is defined via the implicit function theorem, the vector field $\xi_b(\varphi_b, v\mid_b)$ varies smoothly with b, thus defining a global vertical vector field on X (more details about this technique of using the implicit function theorem to prove smooth dependence on b are given in the proof of Proposition 2.23 below). So we can define a global map

$$\begin{aligned} R : \mathcal{U}(L^2_{k+2}(X,\mathbb{C})) &\to L^2_{k+2}(X,\mathcal{V}) \\ \varphi &\mapsto \xi(\varphi,v) \quad \text{s.t.} \quad \xi(\varphi,v)\,|_{X_b} = \xi(\varphi\,|_b, v\,|_b). \end{aligned}$$

The complex structure $F^*_{\xi(\varphi,v)}J_x$ on X satisfies the following properties:

1. it is compatible with ω. Indeed

$$\begin{aligned} &\omega(F^*_{\xi(\varphi,v)}J_x\cdot,\cdot) + \omega(\cdot, F^*_{\xi(\varphi,v)}J_x\cdot) = \\ &\omega_F(F^*_{\xi(\varphi,v)}J_x\cdot,\cdot) + \omega_F(\cdot, F^*_{\xi(\varphi,v)}J_x\cdot) + \omega_{X,\mathcal{H}}(J_x\cdot,\cdot) + \omega_{X,\mathcal{H}}(\cdot, J_x\cdot). \end{aligned}$$

   The first two terms sum to zero because the complex structure $F^*_{\xi(\varphi,v)}J_x$ is fibrewise compatible with the fibrewise Kähler form. The last two terms sum to zero since $J_x \in \mathscr{J}_\pi$;

2. it preserves π, since the differential commutes with pull-back;

3. it satisfies property 2 above for every $b \in B$.

Let now $\alpha(\varphi,v)$ be the $(0,1)$-form with values in the holomorphic vertical tangent space which is the pre-image of $F^*_{\xi(\varphi,v)}J_x$ via Ψ. Then from (2.16) it follows that $\alpha(\varphi,v)$ fibrewise satisfies an elliptic equation of the form

$$\Box_{\mathcal{V}} T(\varphi,v,N(\alpha)) = 2\bar\partial^*_{\mathcal{V}}\bar\partial_{\mathcal{V}}\bar\partial^*_{\mathcal{V}}\left[T(\varphi,v,N(\alpha)), S(\varphi,v,\alpha)\right], \tag{2.17}$$

where $T(0,v,N) = N$ and $S(0,v,\alpha) = \alpha$.

Let now $J \in \mathscr{J}^{\mathrm{int}}_\pi$ be close to I in L^2_k. The proof now goes exactly as in [12, Lemma 6.1], and we report it here for completeness. Since J is integrable, the corresponding vector-valued $(0,1)_I$-form α_J satisfies (2.17) for all (φ,v). Consider the L^2_k-projections

$$\Pi_1 : L^2_k\left(T^{0,1}_I\mathscr{J}_\pi\right) \to \mathrm{Im}(P_{\mathcal{V}}), \qquad \Pi_2 : L^2_k\left(T^{0,1}_I\mathscr{J}_\pi\right) \to \widetilde{H}^1_{\mathcal{V}}$$

and consider the map $\chi : \mathcal{U}(L^2_{k+2}(X,\mathbb{C})) \times V_\sigma \to \mathrm{Im}(P_{\mathcal{V}}) \times \widetilde{H}^1_{\mathcal{V}}$ defined by

$$(\varphi,v) \mapsto \left(\Pi_1\left(F^*_{\xi(\varphi,v)}J_v\right), \Pi_2\left(F^*_{\xi(\varphi,v)}J_v\right)\right).$$

Remark that if $\alpha,\beta \in T^{0,1}_I\mathscr{J}_\pi$ satisfy (2.17) and they are such that $(\Pi_1\alpha,\Pi_2\alpha) = (\Pi_1\beta,\Pi_2\beta)$, then by ellipticity it follows that $\alpha = \beta$. The differential of the map χ is $\mathrm{d}_{(0,0)}\chi(\varphi,v) = (P_{\mathcal{V}}(\varphi),v)$: it is surjective and the kernel corresponds to fibrewise holomorphy potentials, so it is finite dimensional fibrewise. Thus again by the implicit function theorem, there exist (φ,v) such that $(\Pi_1(\alpha_J),\Pi_2(\alpha_J)) = \chi(\varphi,v)$, hence by the ellipticity argument $\alpha_J = F^*_{Y(\varphi,v)}J_x$. □

Remark 2.22 (Versal deformations). The proof of the relative Kuranishi Theorem guarantees the existence of *versal* deformations. Recall from Definition 1.24 that deformation $\mathcal{X} \to B \times V_\pi$ with central fibre (X,I) is called versal if any other family $\mathcal{X}' \to B \times V'_\pi$ (centred at I) is obtained by pullback via a map $f : V'_\pi \to V_\pi$, which might not be unique but whose differential is uniquely determined. This is proven in the third step of Theorem 2.21, where a single complex structure J is considered instead of a second family $\{J_{t'}\}$. The pullback is given by the exponential map F_ξ, where $\xi = \xi(\varphi,v)$ is uniquely determined by the vector v tangent to the complex structure J.

Proposition 2.23. *Possibly after shrinking V_π, we can perturb the map Ψ to*

$$\Phi : V_\pi \longrightarrow \mathscr{J}_\pi \tag{2.18}$$

such that

$$\mathrm{Scal}_{\mathcal{V}}\,(\omega, \Phi(x)) - \widehat{S}_b \in C^\infty(E).$$

Remark that the claim holds fibrewise as a consequence of Theorem 1.23, so we just need to check that the complex structure we find on each fibre X_b varies smoothly with b. This relies on the fact that the proof involves the implicit function theorem.

Proof of Proposition 2.23. For every $b \in B$, the Lie algebra of K_b is $\mathfrak{k}_b = \ker \mathcal{D}_b^* \mathcal{D}_b$, which is exactly the fibre E_b of the vector bundle E defined in §2.1. Let U_l be a small ball around the origin of $L^{2,l}(R)$. Define a map

$$\begin{aligned} G : B \times V_\pi \times U_l &\longrightarrow L^{2,l-4}(R) \\ (b, x, \varphi) &\longmapsto \pi_{L^{2,l-4}(R)} \mathrm{Scal}\left(\omega_b, F_{\varphi_b}(\Psi(x)\mid_b)\right), \end{aligned}$$

where $\Psi(x)\mid_b$ defines an element in $\mathscr{J}_\pi$ from Theorem 1.23, and the map F is the one defined in (1.13). The derivative along the third component of G at 0 of a function φ is given by $P^*_{\mathcal{V}} P_{\mathcal{V}}(\varphi)$, which is an isomorphism $L^{2,l}(R) \to L^{2,l-4}(R)$. By the implicit function theorem, for every b, Ψ can be perturbed to $\Phi_b : V_b \to \mathscr{J}(X_b)$ in such a way that $\mathrm{Scal}(\omega_b, \Phi_b(x)) \in \mathfrak{k}_b$, and Φ_b varies smoothly with b. Thus we find a map

$$\Phi : V_\pi \longrightarrow \mathscr{J}_\pi$$

such that $\mathrm{Scal}_{\mathcal{V}}(\omega, \Phi(x)) - \widehat{S}_b \in C^\infty(E)$. □

Let us return now to considering a holomorphic submersion $\pi_X : X \to B$ with a relatively cscK metric (ω, I). By viewing ω as fixed and varying the complex structure, we consider a family $\{J_s\}$ such that (ω, J_s) are relatively Kähler metrics on $X \to B$. Theorem 2.21, together with Proposition 2.23, allow us to extend definition of the sections μ_s (2.7) and ν (2.8) of $C^\infty(E)$ to the following maps.

Definition 2.24. Let μ_π be the map

$$\begin{aligned} \mu_\pi : V_\pi &\longrightarrow C^\infty(E, I) \\ x &\longmapsto \mathrm{Scal}_{\mathcal{V}}(\omega, \Phi(x)) - \widehat{S}_b \end{aligned}$$

and let ν_π be the map

$$\begin{aligned} \nu_\pi : \widetilde{H}^1 &\longrightarrow C^\infty(E) \\ v &\longmapsto \nu_\pi(v), \end{aligned}$$

where $\nu_\pi(v)\mid_b = \nu_b(v\mid_{X_b})$ and ν_b is the map defined in Definition 1.26.

Remark 2.25. If $x_s \in V_\pi$ corresponds to J_s via the relative Kuranishi map (2.18), we have that $\mu_\pi(x_s) = \mu_s$, where $\mu_s \in C^\infty(E)$ is the section defined in (2.7). Similarly, if $v \in \widetilde{H}^1_{\mathcal{V}}$ is the deformation of the family $\{J_s\}$, then $\nu_\pi(v)$ is the section $\nu \in C^\infty(E)$ defined in (2.8). By applying Proposition 2.23 we can perturb μ_π to end up in $C^\infty(E)$, so we do not see the projection as in (2.7).

From the definition (2.7), the perturbation given by Proposition 2.23 and the expansion (2.9) of $\mu_s \in C^\infty(E)$ it follows that, if $v \in \widetilde{H}^1_{\mathcal{V}}$ is the deformation of the family $\{J_s\}$,

$$\mathrm{Scal}_{\mathcal{V}}(\omega, J_s) - \widehat{S}_b = \mu_\pi(x_s) = \frac{s^2}{2}\nu_\pi(v) + O\left(s^3\right). \tag{2.19}$$

This expansion will be used in §3.1, where we derive the optimal symplectic connection equation from an expansion of the scalar curvature.

Chapter 3

Extremal metrics on the total space

As before, let $\widehat{\pi} : (\mathcal{X}, \mathcal{H}) \to (B, L) \times S$ be a degeneration of a fibration $\pi_Y : (Y, H_Y) \to B$ with central fibration $\pi_X : (X, H_X) \to B$, endowed with a $\mathbb{C}^*$-action on $B \times S$ which lifts to $(\mathcal{X}, \mathcal{H})$. Let ω be a relatively cscK metric on X. As in §2.2.2, we can assume that ω is relatively Kähler also on Y. It follows that the general fibrations $X_s \to B \times \{s\}$ are all biholomorphic to $Y \to B$. For $k \gg 0$, consider the Kähler form

$$\omega_k = \omega + k\omega_B,$$

where ω_B is a fixed Kähler metric on B. In this chapter, we obtain the optimal symplectic connection from the expansion of the scalar curvature in powers of k. We then compute the linearisation of the optimal symplectic operator and we use optimal symplectic connections to construct constant scalar curvature and extremal Kähler metrics in the class $c_1(H_Y) + kc_1(L)$.

3.1 Expansion of the scalar curvature

In this subsection, we derive an expansion of the scalar curvature $\mathrm{Scal}(\omega_k, J_s)$, in powers of s and inverse powers of k, from which we deduce the optimal symplectic connection equation (2.10). Recall from [14, §4.1] that

$$\mathrm{Scal}(\omega_k, J_s) = \mathrm{Scal}_{\mathcal{V}}(\omega, J_s) + k^{-1}\left(\mathrm{Scal}(\omega_B) + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}\omega_{\mathcal{H}}) + \Lambda_{\omega_B}\rho_{\mathcal{H}}\right) + O\left(k^{-2}\right).$$

Clearly, the k^{-1} term - denoted $T_{k^{-1}}$ - depends on s, so we can write

$$T_{k^{-1}}(\omega_B, \omega, J_s) = T_{k^{-1}}(\omega_B, \omega, I) + O(s).$$

Proposition 3.1. *By choosing $s^2 = \lambda k^{-1}$ for $\lambda > 0$ and using the expansion (2.19) for the vertical scalar curvature we obtain*

$$\mathrm{Scal}(\omega_k, J_s) = \widehat{S}_b + k^{-1}\left(\psi_B + p_E(\Delta_{\mathcal{V}}(\Lambda_{\omega_B}\omega_{\mathcal{H}}) + \Lambda_{\omega_B}\rho_{\mathcal{H}}) + \frac{\lambda}{2}\nu(v) + \psi_R\right) + O\left(k^{-3/2}\right)$$

where:

1. *ψ_B is a function on the base given by*

$$\psi_B = \mathrm{Scal}(\omega_B) + \int_{X/B}\left(\Lambda_{\omega_B}\rho_{\mathcal{H}}\right)\omega^m.$$

2. $\psi_R \in C^\infty(R, I)$.

Proof. For the first item, in [14, §4.1] based on [26, Lemma 2.3] it is shown that

$$\int_{X/B} (\Lambda_{\omega_B} \rho_{\mathcal{H}}) \omega^m = -\Lambda_{\omega_B} \alpha_{\mathrm{WP}},$$

where α_{WP} is the Weil-Petersson metric defined in (3.7). Moreover, the k^{-1}-term depends only on I because the $O(s)$-part ends up in $O\left(k^{-3/2}\right)$. Its expression is obtained following [14, §4.2]. □

Thus the optimal symplectic connection equation implies that the $C^\infty(E)$-part of the k^{-1}-term of the expansion of the scalar curvature vanishes. Note that $\widehat{S}_b$ is a topological constant independent of b, since all the fibres are diffeomorphic.

Remark 3.2. For simplicity, we have assumed that all the manifolds involved are projective varieties. This is not a strictly necessary assumption, as everything could be carried out in the Kähler case: instead of a relative polarisation on X_s, we fix a Kähler class α_s, and on the base a Kähler class β. We remark that the constant $\widehat{S}_b$, scalar curvature of the fibres of $X = X_{s=0}$, is still independent of b: indeed locally on $U \subseteq B$, X is diffeomorphic to $F \times U$, where (F, α_F) is a model fibre. Then $\alpha \simeq p_1^* \alpha_F$, where p_1 is the projection onto the first factor $F \times U \to F$. Now,

$$\int_{X_b} \alpha_b^m = \int_F \alpha_F^{m-1} \cdot c_1(X_b).$$

So

$$\widehat{S}_b = \frac{c_1(X_b) \cdot \alpha_b^{m-1}}{\alpha_b^m}$$

is independent of b.

3.2 Linearisation of the fibrewise map ν

We restrict our attention to a single fibre, so we consider a manifold (M, ω), where I is a cscK complex structure and $v \in \widetilde{H}^1$ is a deformation of I. We wish to linearise the map ν defined in 1.26.

Let $\varphi_E \in \ker_{\mathbb{R}}(\mathcal{D}_0^* \mathcal{D}_0)$. Then $\frac{1}{2}\nabla^g \varphi_E$ is a real holomorphic vector field, where g is the Riemannian metric induced by ω and I. Let $\rho(t)$ be the flow of the vector field

$$\xi_{\varphi_E} := \nabla^g \varphi_E = -I \mathrm{grad}^\omega \varphi_E.$$

We wish to study how $\nu(v)$ changes when changing ω to $\rho(t)^*\omega$, so we must compute

$$\partial_t \left.\right|_{t=0} \nu_t(v_t) = \partial_t \left.\right|_{t=0} \nu_t(v) + \mathrm{d}_v \nu \left(\partial_t \left.\right|_{t=0} v_t\right). \tag{3.1}$$

In this expression, ν_t is the moment map for the action of $K^{\mathbb{C}}$ defined in 1.26 computed with respect to the Kähler form $\rho(t)^*\omega$, and $v_t = \rho(t)^* v$. Remark that $\rho(t)$ is a 1-parameter group of diffeomorphisms in $K^{\mathbb{C}}$ because it is the flow of a holomorphic vector field that admits a holomorphy potential.

As in §1.4, let $\Phi : V \to \mathscr{J}$ be the Kuranishi map (2.18) which maps 0 to I, and $\widetilde{H}^1$ the deformation space, which we identify with the tangent space $T_0 V$. The map Φ is K-equivariant, hence locally $K^{\mathbb{C}}$-equivariant. In particular

$$\Phi\left(\rho(t)^* x\right) = \rho(t)^* \Phi(x) \qquad \text{for } x \in V.$$

Hence the pair $(\omega, \rho(t)^* x)$ corresponds via Φ to a compatible pair $(\omega, \rho(t)^*\Phi(x))$ and this also holds for our $v \in \widetilde{H}^1$, which is itself an element of V_π. We have that

$$\partial_t \;|_{t=0}\; \rho(t)^* v = \left(\mathcal{L}_{\xi_{\varphi_E}} \mathbf{v}\right)\Big|_0 , \tag{3.2}$$

where $\mathbf{v}$ is a vector field on V_π such that $\mathbf{v}\;|_0 = v$. By abuse of notation, we will often denote this derivative by $\mathcal{L}_{\xi_{\varphi_E}} v$.

Lemma 3.3. *For $v \in \widetilde{H}^1$, $\nu_t(v) = \nu(v_t)$.*

Proof. Again, this follows from equivariance. Consider again the moment map $\mu(x) = S(\omega, \Phi(x))$ and denote by μ_t the map

$$\begin{aligned} \mu_t : V &\longrightarrow \mathfrak{k} \\ x &\mapsto S(\omega, \rho(t)^*\Phi(x)). \end{aligned}$$

Because Φ and μ_t are (locally) $K^{\mathbb{C}}$-invariant, and in light of the above computation, we obtain

$$S(\omega, \rho(t)^*\Phi(x)) = S(\omega, \Phi(\rho(t)^* x)) = \mu(\rho(t)^* x) = \rho(t)^*\mu(x).$$

Now let $v \in T_0 V = \widetilde{H}^1$. Then

$$\mu_t(sv) = \rho(t)^*\mu(sv) = \rho(t)^*\left[\frac{s^2}{2}\nu(v) + O(s^3)\right].$$

But also

$$\mu_t(sv) = \frac{s^2}{2}\nu_t(v) + O(s^3).$$

Thus $\nu_t(v) = \rho(t)^*\nu(v) = \nu(\rho(t)^* v)$, as claimed. ∎

Using this lemma and equation (3.2), the derivative (3.1) becomes

$$\partial_t \;|_{t=0}\; \nu_t(v_t) = 2\, \mathrm{d}_v \nu \left(\mathcal{L}_{\xi_{\varphi_E}} v\right).$$

Thus using the definition of moment map we can compute the linearisation of the map ν as follows. Letting $\psi \in \mathfrak{k}$,

$$\mathrm{d}_v\langle \nu, \psi\rangle(\mathcal{L}_{\xi_{\varphi_E}} v) = \Omega_0\left(\mathcal{L}_{\xi_{\varphi_E}} v, \mathcal{L}_{\eta_\psi} v\right),$$

where $\eta_\psi = \mathrm{grad}^\omega \psi$. Recall the linearised infinitesimal action induced by $\psi \in \mathfrak{k}$ defined in (1.20), and denoted A_ψ. We showed in (1.21) that

$$A_\psi v = -\left(\mathcal{L}_{\eta_\psi} \mathbf{v}\right)\;|_0 .$$

Thus using the definition of Ω_0,

$$\begin{aligned} \mathrm{d}_v\langle \nu, \psi\rangle\left(\mathcal{L}_{\xi_{\varphi_E}} v\right) &= \int_M \langle I \mathrm{d}_0\Phi\left(\mathcal{L}_{\xi_{\varphi_E}} v\right), \mathrm{d}_0\Phi\left(\mathcal{L}_{\eta_\psi} v\right)\rangle_\omega\; \omega^m \\ &= \int_M \langle \mathrm{d}_0\Phi\left(\mathcal{L}_{\eta_{\varphi_E}} v\right), \mathrm{d}_0\Phi\left(\mathcal{L}_{\eta_\psi} v\right)\rangle_\omega\; \omega^m \\ &= \int_M \langle \mathrm{d}_0\Phi\left(A_{\varphi_E} v\right), \mathrm{d}_0\Phi\left(A_\psi v\right)\rangle_\omega\; \omega^m, \end{aligned} \tag{3.3}$$

where $\langle \cdot, \cdot \rangle_\omega$ is the inner product induced by the Riemannian metric $g(\omega, I)$.

3.3 Linearisation of the optimal symplectic connection equation

Let us now return to the fibration setting. Letting $\varphi, \psi \in C^\infty(E)$, by applying (3.3) and the fact that the map ν_π defined in 2.24 is defined fibrewise, we obtain

$$\langle \mathrm{d}_v \nu_\pi(\mathcal{L}_{\xi_\varphi} v), \psi \rangle = \int_X \langle \mathrm{d}_0 \Phi \left(A_\varphi v \right), \mathrm{d}_0 \Phi \left(A_\psi v \right) \rangle_{\omega_F} \omega_F^m \wedge \omega_B^n. \tag{3.4}$$

Here, the map A_ψ acts vertically, because it is induced by the infinitesimal action of the group of holomorphic isometries of every fibre. By using equation (3.4), we obtain the following result.

Lemma 3.4. *Let $\widehat{\mathcal{L}}$ be the linearisation of the equation (2.10) at a solution, composed with the projection p_E. Then*

$$\langle \widehat{\mathcal{L}}(\varphi), \psi \rangle = \int_X \langle \mathcal{R}\varphi, \mathcal{R}\psi \rangle_{\omega_F + \omega_B} \omega_F^m \wedge \omega_B^n + \lambda \int_X \langle \mathrm{d}_0 \Phi \left(A_\varphi v \right), \mathrm{d}_0 \Phi \left(A_\psi v \right) \rangle_{\omega_F} \omega_F^m \wedge \omega_B^n,$$

where $\mathcal{R}$ is the operator (2.4), which gives the linearisation of the optimal symplectic connection equation at a solution.

From this expression it follows that $\widehat{\mathcal{L}}$ is self adjoint.

We now study the kernel of $\widehat{\mathcal{L}}$. Since v is fixed in our setting, we can define the maps

$$\begin{aligned} A : C^\infty(E) &\to \widetilde{H}^1_{\mathcal{V}} \qquad \text{and} \qquad & \mathcal{A} : C^\infty(E) &\to T_I^{0,1} \mathscr{J}_\pi \\ \psi &\mapsto A_\psi v & \psi &\mapsto \mathrm{d}_0 \Phi \left(A_\psi v \right). \end{aligned} \tag{3.5}$$

Lemma 3.5. *A function $\psi \in C^\infty(E)$ is in the kernel of A if and only if ψ is a fibrewise holomorphy potential with respect to all J_s, i.e. $\psi \in C^\infty(E, J_s)$.*

Proof. Let $\psi \in \ker A$ and take $x_s \in V_\pi$ is such that $x_0 = 0$ and $\dot{x}_0 = v$. Then

$$\begin{aligned} 0 = A_\psi(v) &= \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \mathrm{d}_0 (x \mapsto \exp(t\psi) \cdot x)(v) \\ &= \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \exp(t\psi) \cdot x_s \\ &= \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left(\rho^{\eta_\psi}(t) \right)^* x_s \\ &= \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} \mathcal{L}_{\eta_\psi} J_{x_s}. \end{aligned}$$

where by $\rho^{\eta_\psi}(t)$ we denote the flow of the vertical vector field $\eta_\psi = \mathrm{grad}^{\omega_F} \psi$, and all the equalities hold fibrewise since the Hamiltonian action we consider is a fibrewise action. So $\mathcal{L}_{\eta_\psi} J_{x_s}$ is fibrewise constant, i.e.

$$\left(\mathcal{L}_{\eta_\psi} J_{x_s} \right)_{\mathcal{V}} = \left(\mathcal{L}_{\eta_\psi} J_0 \right)_{\mathcal{V}} = 0$$

for all s. This can be rephrased as

$$\bar{\partial}_{s,\mathcal{V}} \eta_\psi = 0$$

for all s, where $\bar{\partial}_{s,\mathcal{V}}$ is the vertical $\bar{\partial}$-operator computed with respect to J_s. Notice that η_ψ is a real vector field which corresponds (under the isomorphism between real vector fields and $(1,0)_s$ vector fields) to $J_s\nabla^{1,0}_{s,\mathcal{V}}\psi$, where $\nabla_{s,\mathcal{V}}\psi$ denotes the vertical vector field which on each fibre is the Riemannian gradient with respect to the fibrewise metric induced by (ω_F, J_s). Since J_s is integrable,

$$\left(\mathcal{L}_{\eta_\psi}J_s\right)_{\mathcal{V}} = \left(\mathcal{L}_{J_s\nabla^{1,0}_{s,\mathcal{V}}\psi}J_s\right)_{\mathcal{V}} = \left(J_s\mathcal{L}_{\nabla^{1,0}_{s,\mathcal{V}}\psi}J_s\right)_{\mathcal{V}} = 0,$$

so ψ is a fibrewise holomorphic potential for J_s. □

Proposition 3.6. *The kernel of $\widehat{\mathcal{L}}$ is given by*

$$\ker\widehat{\mathcal{L}} = \left\{\psi \in C^\infty(E,I) \mid \bar{\partial}_s(\nabla^{1,0}_{s,\mathcal{V}}\psi) = 0\ \forall s\right\},$$

i.e. the functions in the kernel are those fibrewise I-holomorphy potentials which are global holomorphy potentials with respect to all J_s.

Proof. Since Φ is an embedding, $\mathrm{d}_0\Phi$ is injective, so for $\psi \in C^\infty(E)$, $\psi \in \ker\widehat{\mathcal{L}}$ if and only if $\psi \in \ker\mathcal{R}$ and $\psi \in \ker A$.

As seen in §2.2.1, the kernel of $\mathcal{R}$ consists of fibrewise holomorphy potentials which are also global holomorphy potentials. Thus $\psi \in C^\infty(E)$ lies in $\ker\widehat{\mathcal{L}}$ if and only if

$$\bar{\partial}_B\nabla^{1,0}_{\mathcal{V}}\psi = 0 \quad \text{and} \quad \bar{\partial}_{s,\mathcal{V}}\nabla^{1,0}_{s,\mathcal{V}}\psi = 0.$$

as shown in Lemma 3.5. From these two conditions, and in light of Lemma 2.13, which implies that $\bar{\partial}_B$ does not depend on s, we have

$$\bar{\partial}_s\mathrm{grad}^{\omega_F}\psi = \bar{\partial}_{s,\mathcal{V}}\mathrm{grad}^{\omega_F}\psi + \bar{\partial}_B\mathrm{grad}^{\omega_F}\psi = 0,$$

as claimed. □

Remark 3.7. In [17, §4.1] it is explained that the kernel of the operator $\mathcal{R}$ is given by the Lie algebra of the group $\mathrm{Aut}(\pi_X)$ of automorphisms of the projection, described in Definition 2.2. In our case, the kernel of the linearisation $\widehat{\mathcal{L}}$ is the intersection

$$\ker\widehat{\mathcal{L}} = \mathrm{Lie}(\mathrm{Aut}(\pi_s)) \cap \mathrm{Lie}(\mathrm{Aut}(\pi_X)),$$

where we denote by $\pi_X : X \to B$ the central fibration and we view $\{J_s\}$ as a family of complex structures on the same underlying smooth manifolds, compatible with the projection and with ω.

We wish to see that $\widehat{\mathcal{L}}$ is elliptic as a differential operator on the global sections of $E \to B$. Let us split $\mathcal{A}$ in (3.5) as the composition of the two operators

$$\begin{aligned} A_1 : C^\infty(E) &\to \Gamma(\mathcal{V}) \\ \varphi &\mapsto \mathrm{grad}^{\omega_F}\varphi \end{aligned}$$

and

$$\begin{aligned} A_2 : \Gamma(\mathcal{V}) &\to T_I\mathscr{J}_\pi \\ \eta &\mapsto -(\mathcal{L}_\eta v). \end{aligned}$$

To give a local expression, we make use of *Riemannian* coordinates, and we denote again the vertical coordinates with the letters $a, b, c, \ldots$ and the horizontal coordinates with the letters $i, j, k, \ldots$, as in §2.3. We have:

$$(A_2(\eta))^a{}_b = -\left(\mathcal{L}_\eta v\right)^a{}_b = -\eta^c \partial_c v^a{}_b - v^a{}_c \partial_b \eta^c + v^c{}_b \partial_c \eta^a$$
$$(A_2(\eta))^a{}_j = (A_2(\eta))^a{}_c (\omega_F)^{dc} (\omega)_{dj}$$

where the second expression follows from Lemma 2.12. Thus when $\varphi \in C^\infty(E)$ and $\eta = \omega_F^{de} \partial_e \varphi \partial_d$,

$$(\mathcal{A}(\varphi))^a{}_b = -v^a{}_c \partial_b (\omega_F^{cd} \partial_d \varphi) + v^c{}_b \partial_c (\omega_F^{ad} \partial_d \varphi) + T_1(\varphi)$$
$$(\mathcal{A}(\varphi))^a{}_j = -v^a{}_c \partial_b (\omega_F^{cd} \partial_d \varphi)(\omega_F)^{eb}(\omega)_{ej} + v^c{}_b \partial_c (\omega_F^{ad} \partial_d \varphi)(\omega_F)^{eb}(\omega)_{ej} + T_1'(\varphi),$$

where $T_1(\varphi)$ and $T_1'(\varphi)$ are terms involving first order vertical derivatives of φ. Thus we see that $\mathcal{A}$ is a second order differential operator, and all the derivatives of φ involved are vertical. The adjoint of A_1 is given by

$$A_1^*(\eta) = \mathrm{div}(I\eta).$$

Indeed, we can compute the divergence with respect to any Kähler metric g whose Kähler form restricts vertically to ω_F, and the result depends only on the vertical part:

$$\begin{aligned}\langle A_1 \varphi, \eta \rangle_{L^2} &= \int_X g_{ac}\, \omega_F^{ab}\, \partial_b \varphi\, \eta^c \, \mathrm{d}Vol_g = \int_X -i g_{ac} g^{ab}\, (\nabla_b \varphi)\, \eta^c \, \mathrm{d}Vol_g \\ &= \int_X -i (\nabla_c \varphi)\, \eta^c \, \mathrm{d}Vol_g = \int_X \varphi\, \nabla_c (i\eta^c)\, \mathrm{d}Vol_g = \langle \varphi, A_1^* \eta \rangle_{L^2}.\end{aligned}$$

To compute the adjoint of A_2 we make use of the following lemma.

Lemma 3.8. *Let $w \in \widetilde{H}^1_{\mathcal{V}}$ and let g_F be the vertical Riemannian metric induced by (ω_F, I). Then*

$$g_F(\mathcal{L}_\eta v, w) = g_F(w, \nabla v(\eta)) + g_F(vw - wv, \nabla \eta).$$

The proof of the lemma is obtained by computing the different quantities in Riemannian coordinates [66, §4.2].

In light of the lemma, the adjoint to A_2 can be formally written as

$$A_2^*(w) = -(\nabla v)^* w - \nabla^*([v, w]).$$

If $w = \mathcal{A}(\varphi)$, the first term is of order 3. So we have:

$$\mathcal{A}^* \mathcal{A}(\varphi) = -\mathrm{div}\left(I \nabla^*_{\mathcal{V}} (v (\mathcal{L}_{\eta_\varphi} v)_{\mathcal{V}} - (\mathcal{L}_{\eta_\varphi} v)_{\mathcal{V}} v) \right) + \text{lower order terms}.$$

From this expression, we see that all the quantities involved are vertical. This means that, as an operator on the global sections of the vector bundle E, the operator

$$\mathcal{A}^* \mathcal{A} : C^\infty(E) \to C^\infty(E)$$

is of order 0. Indeed, let us denote by r the rank of E and consider a local frame $h_1, \ldots, h_r$ of E. Then we can write a local section $h = \sum_i f_i h_i$, with $f_i \in C^\infty(B)$. Then

$$\mathcal{A}^* \mathcal{A}(h) = \sum_i f_i \mathcal{A}^* \mathcal{A}(h_i).$$

Thus, as an operator on the global sections $C^\infty(E)$, the operator $\widehat{\mathcal{L}}$ is elliptic, since $\mathcal{R}^* \mathcal{R}$ is from [14, §4] and $\mathcal{A}^* \mathcal{A}$ is of lower order. We have established the following:

Theorem 3.9. *Let $\widehat{\mathcal{L}}$ be the linearisation of the optimal symplectic connection equation (2.10). Then $\widehat{\mathcal{L}}$ is an elliptic operator of order two on the global sections of E which is self-adjoint and whose kernel consists of fibrewise I-holomorphy potentials which are also global J_s-holomorphy potentials for all s.*

3.4 Automorphisms of the optimal symplectic connection equation

Let $(X, H_X) \to B$ be a relatively cscK fibration. Consider the complex group $\mathrm{Aut}(X, H_X)$ of automorphisms of X lifting to H_X. Its Lie algebra is given by the holomorphic vector fields which vanish somewhere, and we denote it by $\mathfrak{h}_0$. Recall from (2.13) the group of relative Hamiltonian isometries

$$K_\pi := \mathrm{Isom}(\pi_x, \omega) = \left\{ f \in \mathrm{Diffeo}(X) \mid f^*\omega = \omega \text{ and } \pi_X \circ f = \pi_X \right\}$$

and from Definition 2.2 the group of relative automorphisms

$$\mathrm{Aut}(\pi_X) = \left\{ f \in \mathrm{Aut}(X, H_X) \mid \pi_X \circ f = \pi_X \right\}.$$

We denote by $\mathfrak{h}_\pi$ the Lie algebra of $\mathrm{Aut}(\pi_X)$ and $\mathfrak{k}_\pi$ the Lie algebra of K_π. An element in $\mathfrak{h}_\pi$ is a holomorphic vector field which vanishes somewhere and whose flow lies in $\mathrm{Aut}(\pi_X)$, while an element of $\mathfrak{k}_\pi$ is a holomorphic vector field which corresponds to a Killing vector field under the identification of the real tangent bundle $T_{\mathbb{R}}X$ with the holomorphic tangent bundle $T^{1,0}X$. The following fibration version of Theorem 1.4 is a result of Dervan and Sektnan [14], [17].

Lemma 3.10.
1. *Let ω be an optimal symplectic connection and $f \in \mathrm{Aut}(\pi_X)$. Then $f^*\omega$ is an optimal symplectic connection.*
2. *Let ω be an optimal symplectic connection. Then*
$$\mathfrak{h}_\pi = \mathfrak{k}_\pi \oplus I\mathfrak{k}_\pi.$$

In particular, the lemma implies that $K_\pi^{\mathbb{C}}$ is contained in $\mathrm{Aut}(\pi)$ with equality holding if (ω, I) is an optimal symplectic connection.

We next prove an analogous result for the optimal symplectic connection equation (2.10) on a fibration with K-semistable fibres. Let $(Y, H_Y) \to (B, L)$ be such a fibration admitting a degeneration to $(X, H_X) \to (B, L)$ and let V_π be the Kuranishi space of π_X. Let $(\mathcal{X}, \mathcal{H}) \to (B, L) \times S$ be the degeneration family. The family of complex structures $\{J_s\}$ with $J_0 = I$ corresponds to a family $\{y_s\}$ of points in V_π such that y_0 is the origin of V_π. Let v be the tangent vector at the origin of V_π that represents the degeneration family, i.e.

$$v = \partial_s|_{s=0} y_s.$$

Consider the stabiliser of v for the action of K_π, denoted $K_{\pi,v}$. Then

$$K_{\pi,v} = \left\{ f \in K_\pi \mid f^* v = v \right\},$$

and

$$G_{\pi,v} = (K_\pi^{\mathbb{C}})_v. \tag{3.6}$$

For $f \in G_{\pi,v}$

$$\partial_s|_{s=0} y_s = v = f^* v = f^* \left(\partial_s|_{s=0} y_s \right) = \partial_s|_{s=0} \left(f^* y_s \right).$$

Therefore
$$\partial_s|_{s=0}(y_s - f^* y_s) = 0,$$
so $v = f^* v$. So the elements of $G_{\pi,v}$ are automorphisms of the complex structure I of the relatively cscK degeneration $X \longrightarrow B$ that preserve the projection π_X and are also automorphisms of the complex structures J_s. Moreover, the pull-back of the optimal symplectic connection operator via $f \in G_{\pi,v}$ satisfies

$$f^*\left(\frac{1}{2}\nu(v) + p_E(\Theta(\omega, I))\right) = \frac{1}{2}\nu(v) + p_E(\Theta(f^*\omega, I)).$$

Indeed, since ν is $K_\pi^{\mathbb{C}}$-equivariant,

$$f^*\nu(v) = \nu(f^*v) = \nu(v),$$

and by Lemma 3.10,
$$f^*(p_E(\Theta(\omega, I))) = p_E(\Theta(f^*\omega, I)).$$

We have proven the following.

Lemma 3.11. *Let ω be an optimal symplectic connection and $f \in G_{\pi,v}$. Then $f^*\omega$ is an optimal symplectic connection. Moreover, if φ is a fibrewise I-holomorphy potential whose flow of the gradient lies in $G_{\pi,v}$, φ is in the kernel of the linearisation $\widehat{\mathcal{L}}$.*

Let $\mathfrak{g}_{\pi,v}$ be the Lie algebra of $G_{\pi,v}$, consisting on those holomorphic vector fields whose flow lies in $K_\pi^{\mathbb{C}}$ and which preserve v. In particular, preserving v means that they extend to holomorphic vector fields with respect to all J_s. Let $\mathfrak{k}_{\pi,v}$ be the Lie algebra of $K_{\pi,v}$, of Killing holomorphic vector fields whose flow preserves v. We can then prove a version of Theorem 1.4 for our setting.

Theorem 3.12. *Let ω be an optimal symplectic connection. Then*

$$\mathfrak{g}_{\pi,v} = \mathfrak{k}_{\pi,v} \oplus I\mathfrak{k}_{\pi,v}.$$

In particular $K_{\pi,v}$ is a reductive subgroup of $G_{\pi,v}$.

Proof. From Theorem 3.9, the kernel $\widehat{\mathcal{L}}$ of the linearisation of the optimal symplectic connection equation consists of fibrewise I-holomorphy potentials which are also global J_s-holomorphy potentials for all s. From the discussion above, this is in bijection with the Lie algebra $\mathfrak{g}_{\pi,v}$, and $\mathfrak{k}_{\pi,v}$ corresponds to the real vector fields in $\mathfrak{g}_{\pi,v}$. Since $\widehat{\mathcal{L}}$ is a real operator, $\widehat{\mathcal{L}}(u + iv) = 0$ if and only if $\widehat{\mathcal{L}}(u) = 0$ and $\widehat{\mathcal{L}}(v) = 0$. □

3.5 Special Kähler metrics on the total space: the adiabatic limit

Let $(\mathcal{X}, \mathcal{H}) \to B \times S$ be a family of submersions with central fibre the fibration $(X, H_X) \to (B, L)$ as before. In this section we construct approximate constant scalar curvature and extremal metrics on the total space of $\pi_Y : (Y, H_Y) \to (B, L)$, assuming the optimal symplectic connection and the extremal symplectic connection, respectively. We first construct approximate solutions in the case of a discrete automorphism group and in the presence of automorphisms, and then we perturb the approximate solutions by applying the implicit function theorem. We do so by using an *adiabatic limit*, such as in [24, 14].

We will later need to choose ω_B appropriately, to produce cscK and extremal metrics on Y. To do so, we use the moduli theory of cscK manifolds explained in §1.5. Let $\mathcal{M}^{cscK}$ be the moduli space of polarised cscK manifolds. Since our central fibration $\pi : X \to B$ has cscK fibres, it induces a map $q : B \to \mathcal{M}^{cscK}$. The pull-back via q of the Weil-Petersson metric, denoted α_{WP}, is a closed smooth $(1,1)$-form on B, and it has the expression (3.7):

$$\alpha_{WP} = \frac{\widehat{S}_b}{m+1}\int_{X/B}\omega^{m+1} - \int_{X/B}\rho\wedge\omega^m, \tag{3.7}$$

where ρ is the relative Ricci form defined in Section 2.2.1 and m is the dimension of the fibres. Recall that α_{WP} is positive semi-definite in general.

Definition 3.13 ([72, 24])**.** The Kähler metric $\omega_B \in c_1(L)$ is

1. *twisted cscK* with respect to α if there exists a constant c_B such that $\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha = c_B$;
2. *twisted extremal* with respect to α if $\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha \in \ker \mathcal{D}_B$, where $\mathcal{D}_B$ is the Lichnerowicz operator on B.

Definition 3.14. The group of automorphisms of the moduli map is

$$\mathrm{Aut}(q) = \left\{ f \in \mathrm{Aut}(B, L) | q \circ f = f \right\}.$$

If we denote by h_B the twisted extremal holomorphy potential, then the linearisation of the twisted extremal operator at a solution is given by the map [35, §2], [15, §3.2]

$$\mathcal{L}_\alpha(\varphi) = -\mathcal{D}_B^*\mathcal{D}_B\varphi + \frac{1}{2}\langle\nabla\Lambda_{\omega_B}\alpha, \nabla\varphi\rangle + \langle i\partial\bar{\partial}\varphi, \alpha\rangle. \tag{3.8}$$

The kernel of this operator is given by the holomorphy potentials of those vector fields whose flow lies in $\mathrm{Aut}(q)$ [15, Proposition 3.5]. We prove the following results.

Theorem 3.15. *Assume that the group* $\mathrm{Aut}(q)$ *and the group* $\mathrm{Aut}(Y, H_Y)$ *are discrete. Let* ω_B *be twisted cscK with respect to the pull-back via* q *of the Weil-Petersson metric on the moduli space of cscK manifolds. Let* ω *be an optimal symplectic connection on* $(Y, H_Y) \to (B, L)$*. Then for all* $k \gg 0$ *there exists a constant scalar curvature Kähler metric on* Y*, in the class* $[\omega] + k[\omega_B]$.

When automorphisms are present, we use extremal symplectic connections and twisted extremal metrics on the base to prove the existence of extremal metrics on the total space. Recall from Definition 2.11 that ω is an extremal symplectic connection on Y if

$$\widehat{\mathcal{L}}\left(p_E(\Theta(\omega, I)) + \frac{\lambda}{2}\nu\right) = 0,$$

so that the function

$$h_1 := p_E(\Theta(\omega, I)) + \frac{\lambda}{2}\nu$$

is a holomorphy potential for the complex structure of Y.

Theorem 3.16. *Assume that there is an action of* $\mathrm{Aut}(\pi_Y)$ *on* $(\mathcal{X}, \mathcal{H})$ *which is equivariant with respect to the projection onto* S *and that all automorphisms of the moduli map* q *lift to* (Y, H_Y)*. Let* ω_B *be a twisted extremal metric on* B *with respect to the pull-back via* q *of the Weil-Petersson metric on the moduli space of cscK manifolds. Let* ω *be an extremal symplectic connection for on* $(Y, H_Y) \to (B, L)$*. Then for all* $k \gg 0$ *there exists an extremal Kähler metric on* Y*, in the class* $[\omega] + k[\omega_B]$.

3.5.1 Approximate solutions in the case of discrete automorphism group

In this section we construct approximate constant scalar curvature Kähler metrics on the total space of $\pi_s : (X_s, H_s) \to (B, L)$, where (X_s, H_s) is a deformation of a fibration $\pi_X : (X, H_X) \to (B, L)$ whose fibres are cscK. We make the assumptions of Theorem 3.15:

1. $\mathrm{Aut}(X_s, H_s)$ is discrete and (X_s, H_s) admits an optimal symplectic connection. Thanks to Proposition 3.6, this guarantees that the operator $\widehat{\mathcal{L}}$ is invertible and also that the global Lichnerowicz operator on X_s with respect to ω_k is invertible.

2. The base form $\omega_B \in c_1(L)$ is twisted cscK with respect to the pull-back via q of the Weil-Petersson metric, as in Definition 3.13, and the group $\mathrm{Aut}(q)$ is discrete. As recalled in the discussion following Definition 3.14, this implies that the linearisation at a solution of the twisted cscK equation on the base is invertible;

Let $k \gg 0$ be such that

$$\omega_k = \omega + k\omega_B$$

is a Kähler metric on X, and let $s^2 = \lambda k^{-1}$ for $\lambda > 0$. We relate s and k as above, namely $s^2 = \lambda k^{-1}$, so we will sometimes denote also the corresponding complex structure by J_k. Since all the J_s are isomorphic, Theorem 3.15 still gives the existence of a cscK metric in each adiabatic class for all J_s. The adiabatic limit technique consists in constructing inductively approximated solutions, which have constant scalar curvature up to a certain order in $k^{-1/2}$, then using the implicit function theorem to perturb an approximate solution to a genuine solution. The following result establishes the approximate solution.

Proposition 3.17. *With the assumptions listed above, for all $k \gg 0$ and for each r there exist functions*

$$f_{B,1}, \dots, f_{B,r} \in C^\infty(B) \qquad f_{E,1}, \dots, f_{E,r} \in C^\infty(E) \qquad f_{R,1}, \dots, f_{R,r} \in C^\infty(R)$$

and constants

$$\widehat{S}_1, \dots, \widehat{S}_r$$

such that the Kähler potentials

$$h^B_{k,r} = \sum_{j=2}^{r} \frac{f_{B,j}}{k^{j-2}} \qquad h^E_{k,r} = \sum_{j=2}^{r} \frac{f_{E,j}}{k^{(j-1)/2}} \qquad h^R_{k,r} = \sum_{j=2}^{r} \frac{f_{R,j}}{k^{j/2}}$$

satisfy

$$\mathrm{Scal}\left(\omega_k + i\partial\bar\partial\left(h^B_{k,r} + h^E_{k,r} + h^R_{k,r}\right), J_k\right) = \widehat{S}_b + \sum_{j=1}^{r} \frac{\widehat{S}_j}{k^{j/2}} + O\left(k^{(-r-1)/2}\right).$$

Proof. With the hypotheses of ω being an optimal symplectic connection and ω_B being a twisted cscK metric on the base, we have

$$\mathrm{Scal}(\omega_k) = \widehat{S}_b + k^{-1}\left(c_B + \psi_{R,1}\right) + O\left(k^{-3/2}\right), \tag{3.9}$$

where $\psi_{R,1} \in C^\infty(R)$. In order to make the k^{-1}-term constant we add a potential $k^{-1}f \in C^\infty(R)$ to ω_k. Then

$$\mathrm{Scal}(\omega_k + k^{-1}i\partial\bar\partial f) = \widehat{S}_b + k^{-1}\left(c_B + \psi_{R,1} - \mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} f\right) + O\left(k^{-3/2}\right),$$

where the linearisation of the scalar curvature to order 0 in k coincides with (minus) the Lichnerowicz operator with respect to the complex structure I, since the scalar curvature is constant in order 0, and the higher order terms fall into $O\left(k^{-3/2}\right)$. Since $\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}}$ is a fibrewise elliptic differential operator and $C^\infty(R)$ is orthogonal to its kernel, we can find a solution $f_{R,1}$ of

$$\psi_{R,1} - \mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} f = constant. \tag{3.10}$$

Summing up, we have proved step $n = 1$ of Proposition 3.17, with $f_{B,1} = 0 = f_{E,1}$. We define

$$\omega_{k,1} = \omega_k + k^{-1} i\partial\bar{\partial} f_{R,1}$$

such that

$$\mathrm{Scal}(\omega_{k,1}) = \widehat{S}_b + k^{-1}\widehat{S}_1 + O\left(k^{-3/2}\right).$$

To proceed with the approximate solutions, we need the linearisation of the scalar curvature at a metric $(\omega_{k,r}, J_k)$.

Lemma 3.18. *The linearisation of the scalar curvature of $\omega_{k,r}$ satisfies*

$$\mathcal{L}_{k,r} = -\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} + k^{-1} D_1 + k^{-3/2} D_{3/2} + k^{-2} D_2 + O\left(k^{-5/2}\right),$$

where

1. *$D^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}}$ is the vertical Lichnerowicz operator with respect to the complex structure I;*
2. *If $f \in C^\infty(B)$, $D_{j+1/2}(f) = 0$ for all j;*
3. *If $f \in C^\infty(B)$, $D_1(f) = 0$ and*

$$\int_{X/B} D_2(f)\omega^m \wedge \omega_B^n = -\mathcal{L}_\alpha(f),$$

   where $\mathcal{L}_\alpha$ is the linearisation of the twisted cscK equation on the base, with twisting the Weil-Petersson form α_{WP}, at a solution, defined in (3.8).
4. *If $f \in C^\infty(E)$, then*

$$p_E \circ D_1(f) = -p_E \circ \widehat{\mathcal{L}}(f).$$

Proof of the Lemma. Let us distinguish the parameter s of the deformation of the complex structure from the parameter k of the polarisation. Consider the case $n = 0$, so that we compute the scalar curvature of the metric (ω_k, J_s). Then

$$\mathcal{L}_k = \mathcal{L}_{k,0} + O(s), \tag{3.11}$$

where $\mathcal{L}_{k,0}$ is the linearisation of the scalar curvature of (ω_k, I). In [14, Proposition 4.11] it is proven that

$$\mathcal{L}_{k,0} = -D^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} + k^{-1}D'_1 + k^{-2}D'_2 + O\left(k^{-3}\right),$$

from which we see that the term of order zero is indeed the vertical I-Lichnerowicz operator. This proves claim (1). By imposing the relation $s^2 = \lambda k^{-1/2}$ we see that the $O(s)$-term in (3.11) admits an expansion in powers of $k^{-1/2}$:

$$k^{-1}D''_1 + k^{-3/2}D''_{3/2} + k^{-2}D''_2 + O\left(k^{-5/2}\right).$$

Claim (2) follows from the fact that the deformation of the complex structure is vertical, thus all the terms involved in the expansion of the scalar curvature coming from the deformation do not have a $C^\infty(B)$-component.

Claims (3) and (4) follow as in [14, Proposition 4.11]. □

The proof of Proposition 3.17 now goes by induction, using Lemma 3.18. We explain in detail steps $r = \frac{3}{2}$ and $r = 2$. We start from the expansion

$$\mathrm{Scal}(\omega_{k,1}) = \widehat{S}_b + k^{-1}\widehat{S}_1 + k^{-3/2}(\psi_{E,3/2} + \psi_{R,3/2}) + O\left(k^{-2}\right).$$

We add a potential $k^{-1/2} f_E$ to $\omega_{k,1}$. Thus we have

$$\mathrm{Scal}\left(\omega_{k,1} + k^{-1/2} i\partial\bar{\partial} f_E\right) = \widehat{S}_b + k^{-1}\widehat{S}_1 + k^{-2/3}\left(\psi_{E,3/2} + D_1(f) + \psi_{R,3/2}\right) + O\left(k^{-2}\right).$$

Using Lemma 3.18, our hypothesis on the automorphism group of (X_s, H_s) and the fact that the linearisation $\widehat{\mathcal{L}}$ of the optimal symplectic connection equation at a solution is elliptic, as proved in Theorem 3.9, we can find $f_{E,2}$ such that

$$\psi_{E,2/3} + p_E \circ D_1(f_{E,2}) = constant.$$

This makes the $C^\infty(E)$-term constant to order $k^{-3/2}$. We next add a potential $k^{-3/2} f_R \in C^\infty(R)$ and we obtain

$$\begin{aligned}\mathrm{Scal}\left(\omega_{k,1} + i\partial\bar{\partial}\left(k^{-1/2} f_{E,2} + k^{-3/2} f_R\right)\right) &= \widehat{S}_b + k^{-1}\widehat{S}_1 + \\ &+ k^{-3/2}\left(c_{E,3/2} + \psi'_{R,3/2} - \mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} f_R\right) + O\left(k^{-2}\right).\end{aligned}$$

Once again, using the fibrewise ellipticity of $\mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}}$ and the fact that $C^\infty(R)$ is orthogonal to its kernel, we obtain a solution $f_{R,2}$ of the equation

$$\psi'_{R,2} - \mathcal{D}^*_{\mathcal{V}}\mathcal{D}_{\mathcal{V}} f_R = constant.$$

Thus we have constructed a Kähler metric on X_s constant up to order $k^{-3/2}$:

$$\omega_{k,3/2} = \omega_{k,1} + i\partial\bar{\partial}\left(k^{-1/2} f_{E,2} + k^{-3/2} f_{R,2}\right).$$

As for the step $r = 2$, we explain how to deal with the $C^\infty(B)$-term. We add a potential f_B to $\omega_{k,3/2}$, which amounts to adding a potential $k^{-1} f_B$ to ω_B. Since the scalar curvature of the base affects the order k^{-1}-term and not the order zero term, the combined effect on the linearisation is of order k^{-2}. This allows us to write

$$\begin{aligned}\mathrm{Scal}(\omega_{k,3/2} + i\partial\bar{\partial} f_B) &= \widehat{S}_b + k^{-1}\widehat{S}_1 + k^{-3/2}\widehat{S}_{3/2} + \\ &+ k^{-2}\left(\psi_{B,2} - D_2(f_B) + \psi_{E,2} + \psi_{R,2}\right) + O\left(k^{-5/2}\right).\end{aligned}$$

Thanks to Lemma 3.18 and to our hypothesis on the automorphism group of the moduli map,

$$\psi_{B,2} - p_B \circ D_2(f_B) = constant$$

admits a solution, which we denote $f_{B,2}$. This makes the $C^\infty(B)$-term constant to order k^{-2}.

The corrections to the $C^\infty(E)$-term and to the $C^\infty(B)$-term now work exactly as in the case $r = 3/2$. □

Notice that the order is important: one can make the $C^\infty(E)$-term constant without affecting the $C^\infty(B)$-term, but it cannot work the other way around, and similarly for the $C^\infty(R)$-term.

Remark 3.19. The very first step of the approximate solution procedure, which the expansion (3.9), comes from the fact that in Proposition 2.23 we have modified the Kuranishi map Φ in order to meet the requirement that $\mathrm{Scal}_{\mathcal{V}}(\omega, \Phi(x))$ is a section of E, for $x \in V_\pi$. If we do not deform the Kuranishi map in this way, we can write the vertical scalar curvature as the sum of the projection onto $C^\infty(E)$ and the projection onto $C^\infty(R)$. The $C^\infty(E)$-part is the map μ_π defined in 2.24, while the $C^\infty(R)$-part introduces a term of order $k^{-1/2}$ in the expansion (3.9), which then becomes

$$\mathrm{Scal}(\omega_k) = \widehat{S}_b + k^{-1/2}\,\psi_{R,0} + k^{-1}\left(c_B + \psi_{R,1}\right) + O\left(k^{-2}\right).$$

We can get rid of this term by adding a potential $k^{-1/2} i\partial\bar{\partial}\varphi_{R,0}$ to ω_k, as in equation (3.10). Indeed, the linearisation given by Lemma 3.18 of the scalar curvature acquires an extra term $\sqrt{k}D_{1/2}$, which is non-zero only on $C^\infty(R)$, so it does not affect the $C^\infty(E)$ and $C^\infty(B)$ parts in the k^{-1}-term.

3.5.2 Approximate solutions in the presence of automorphisms

In this section, we allow the base and the total space to have automorphisms. As before let $\widehat{\pi} : (\mathcal{X}, \mathcal{H}) \to (B, L) \times S$ be a degeneration of the fibration $\pi_Y : (Y, H_Y) \to B$ to $\pi_X : (X, H_X) \to B$. Let $\omega \in c_1(H)$ be a relatively cscK metric on X; since Y is a small deformation of X, $c_1(H) = c_1(H_Y)$, so we can assume that ω is relatively Kähler on Y, as explained in §2.2.2.

We make the hypotheses of Theorem 3.16 concerning the groups of automorphisms $\mathrm{Aut}(\pi_Y)$ and $\mathrm{Aut}(q)$ defined in 2.2 and 3.14:

1. There is an action of $\mathrm{Aut}(\pi_Y)$ on $(\mathcal{X}, \mathcal{H})$ which is equivariant with respect to the projection onto S. This means that $\mathrm{Aut}(\pi_Y)$ acts on each X_s as a subgroup of automorphisms of (X_s, H_s). Since the action extends to the central fibration, this assumption allows us to view $\mathrm{Aut}(\pi_Y)$ as a subgroup of $\mathrm{Aut}(\pi)$. In particular, recall from Remark 3.7 that $\ker\widehat{\mathcal{L}} = \mathrm{Lie}(\mathrm{Aut}(\pi_Y)) \cap \mathrm{Lie}(\mathrm{Aut}(\pi))$. With this assumption, we obtain

$$\mathrm{Ker}\ \widehat{\mathcal{L}} = \mathrm{Lie}(\mathrm{Aut}(\pi_Y)),$$

   and h_1 is a holomorphy potential also on X.

2. All automorphisms of the moduli map q lift to (Y, H_Y).

The first hypothesis is motivated by the analogous definition of test configurations which are equivariant with respect to the automorphisms of the fibres, which are used to test K-polystability of polarised manifolds.

Recall from Definition 3.13 that a twisted extremal metric on B, with twisting form the Weil-Petersson form α_{WP} (3.7), satisfies the condition

$$\mathrm{Scal}(\omega_B) - \Lambda_\omega \alpha_{WP} = b_1 \in \ker\mathcal{D}_B,$$

where $\mathcal{D}_B$ is the Lichnerowicz operator on the base.

Remark 3.20. Let $\widehat{g}$ be a lift of an automorphism of q to (Y, H_Y). We claim that $\widehat{g}$ lies in $\mathrm{Aut}(X, H_X)$. Indeed, denoting by J the complex structure of Y and I the complex structure of X, we have

$$\mathrm{d}\widehat{g} \circ J = J \circ \widehat{g}.$$

But since $\widehat{g}$ is an automorphism in the base direction, it is equivalent to say that

$$\mathrm{d}\widehat{g} \circ J_{\mathcal{H}} = J_{\mathcal{H}} \circ \widehat{g},$$

where $J_{\mathcal{H}}$ is the horizontal part of J. Now, $J_{\mathcal{H}} = I_{\mathcal{H}}$, since the deformation of the complex structure which we are considering is only in the vertical direction. Thus $\widehat{g}$ is a lift of an automorphism of B to X as well.

Definition 3.21. We denote the group of automorphisms of (Y, H_Y) which are also automorphisms of (X, H_X) as $\mathrm{Aut}(Y/X, H_Y)$.

In light of this definition we have the inclusion $\mathrm{Aut}(\pi_Y) \subseteq \mathrm{Aut}(Y/X, H_Y)$ and, if $\widehat{\mathrm{Aut}}(q)$ is a lift of $\mathrm{Aut}(q)$ to (Y), then $\widehat{\mathrm{Aut}}(q) \subseteq \mathrm{Aut}(Y/X, H_Y)$. Thus we can recover the following result from [14, Proposition 3.14].

Lemma 3.22. *Suppose that all automorphisms of q lift to Y. Then there is a short exact sequence*

$$0 \longrightarrow \mathrm{Lie}(\mathrm{Aut}(\pi_Y)) \longrightarrow \mathrm{Lie}(\mathrm{Aut}(Y, H_Y)) \longrightarrow \mathrm{Lie}(\mathrm{Aut}(q)) \longrightarrow 0.$$

Remark 3.23. Let us denote by ξ_E the holomorphic vector field on Y which arises from the extremal symplectic connection condition:

$$\xi_E = J_s \nabla_{\mathcal{V}} \left(p_E(\Theta(\omega, I)) + \frac{\lambda}{2}\nu \right),$$

and ξ_q the holomorphic vector field on B which arises from the twisted extremal condition:

$$\xi_q = J_B \nabla_B (\mathrm{Scal}(\omega_B) - \Lambda_\omega \alpha_{WP}).$$

By our assumptions, ξ_E is a holomorphy potential on X, and ξ_q lifts to a holomorphic vector field on Y (and on X). Nonetheless, the holomorphy potential of ξ_q on Y is a function $\widetilde{b}_1$ such that

$$\widetilde{b}_1 = k\pi^* b_1 + O(1).$$

Again from Remark 3.20, $\widetilde{b}_1$ is holomorphic potential for a lift of ξ_q also on X. As in [14], we need to assume the following invariance properties: ω is invariant under the flow of ξ_E and of the pull-back of ξ_q. In order to make this assumptions reasonable to work with, we consider a maximal torus T_E in $\mathrm{Aut}(\pi_Y)$ which contains the flow of ξ_E, and a maximal torus T_q in $\mathrm{Aut}(B, L)$ which contains the flow of ξ_q. The pull back $\widehat{T_q}$ lies in $\mathrm{Aut}(Y/X, H_Y)$. Then we fix a maximal torus T in $\mathrm{Aut}(Y, H_Y)$ which contains T_E and T_q, and we require that ω is invariant with respect to T. From Lemma 3.22, we obtain a splitting $\mathrm{Lie}(T) = \mathrm{Lie}(T_E) + \mathrm{Lie}(T_q)$, so indeed we have $T \subset \mathrm{Aut}(Y/X, H_Y)$ as well.

Moreover, an analogous splitting holds also for the complexification $T^{\mathbb{C}}$, so we can write every vector field $\xi \in \mathrm{Lie}(T^{\mathbb{C}})$ as $\xi_E + \xi_q$. If h_E is the holomorphy potential of ξ_E with respect to ω and h_B is the holomorphy potential of ξ_q on the base with respect to ω_B, then $h_E + k\pi_Y^* h_B$ is a holomorphy potential of ξ on Y (and on X).

Define the *extremal symplectic connection operator*

$$\mathcal{P} : C^\infty(Y, \mathbb{R}) \times C^\infty(E) \longrightarrow C^\infty(Y, \mathbb{R})$$

by

$$\mathcal{P}(\varphi, h_1) = p_E\left(\Theta(\omega + i\partial\bar{\partial}\varphi, J_s)\right) + \frac{\lambda}{2}\nu_\varphi - h_1 - \frac{1}{2}\langle \nabla h_1, \nabla\varphi \rangle_{\omega_F}.$$

The linearisation at $(h_1, 0)$ applied to (h_1, ψ) is obtained, as for the extremal operator described in (1.2), as follows:

$$\widehat{\mathcal{L}}(\psi) - h_1 - \frac{1}{2}\langle \nabla h_1, \nabla \psi \rangle_{\omega_F},$$

where $\widehat{\mathcal{L}}$ is the real operator of the linearisation of the optimal symplectic connection equation described in Lemma 3.4 and the map sending φ to $\langle \nabla h_1, \nabla \varphi \rangle_{\omega_F}$ is linear. We can write

$$\langle \nabla h_1, \nabla \psi \rangle_{\omega_F} = \frac{1}{2}\nabla h_1(\psi) + \frac{1}{2} iJ\nabla h_1(\psi),$$

so if we assume that ψ is invariant under the torus T, the second term vanishes and linearisation is a real operator.

With all of these assumptions in place, we can obtain approximate solutions to the extremal equation much as in §3.5.1.

Proposition 3.24. *Let $(\mathcal{X}, \mathcal{H}) \to B \times S$ be a degeneration of a smooth fibration $\pi_Y : (Y, H_Y) \to B$ to a smooth relatively cscK fibration $\pi_X : (X, H_X) \to B$, equivariant with respect to $\mathrm{Aut}(\pi_Y)$. Let ω be an extremal symplectic connection on X_s, invariant under the torus T described in Remark 3.23. Let ω_B a twisted extremal metric on the base, and assume that all automorphisms of q lift to Y. Then for each $r > 1$ there exist functions*

$$f_{B,1}, \dots, f_{B,r} \in C^\infty(B)^T, \qquad f_{E,1}, \dots, f_{E,r} \in C^\infty(E)^T, \qquad f_{R,1}, \dots, f_{R,r} \in C^\infty(R)^T,$$

base holomorphy potentials

$$b_1, \dots, b_r \in C^\infty(B)^T,$$

fibre holomorphy potentials

$$h_1, \dots, h_r \in C^\infty(E)^T$$

and a constant c such that, letting

$$h^B_{k,r} = \sum_{j=2}^{r} \frac{f_{B,j}}{k^{j-2}}, \qquad h^E_{k,r} = \sum_{j=2}^{r} \frac{f_{E,j}}{k^{(j-1)/2}}, \qquad h^R_{k,r} = \sum_{j=2}^{r} \frac{f_{R,j}}{k^{j/2}}$$

and

$$\eta_{k,r} = c + \sum_{j=1}^{r} \left(b_j k^{(-j-1)/2} + h_j k^{-j/2} \right),$$

the Kähler metric

$$\omega_{k,r} = \omega_k + i\partial\bar{\partial}\left(h^B_{k,r} + h^E_{k,r} + h^R_{k,r} \right)$$

satisfies

$$\mathrm{Scal}\left(\omega_{k,r}, J_k\right) = \eta_{k,r} + \frac{1}{2}\langle \nabla \eta_{k,r}, \nabla\left(h^B_{k,r} + h^E_{k,r} + h^R_{k,r} \right)\rangle_{\omega_k} + O\left(k^{-(r+1)/2}\right).$$

3.5.3 Solution to the non-linear equation

In order to have genuine solutions, we perturb $\omega_{k,r}$ to a genuine extremal metric by using a quantitative version of the implicit function theorem, as in [24, 7, 15, 14]. In particular, all the cited works rely on Fine's paper [24], though the difference with Fine's setting is that we are considering the base and the total space to have automorphisms, so the linearised operators will have a non-trivial kernel to deal with.

Theorem 3.25 ([7, Theorem 25]). *Let $F : B_1 \to B_2$ be a differentiable map of Banach spaces such that D_0F is surjective with right-inverse P. Let*

1. *$\delta' > 0$ be such that the non-linear operator $(F - D_0F)$ is Lipschitz in $B_{\delta'}(0)$ with constant $\frac{1}{2\|P\|}$, i.e. for $x_1, x_2 \in B_{\delta'}(0) \subseteq B_1$, we have*
$$\|(F - D_0F)(x_1) - (F - D_0F)(x_2)\|_{B_2} \leq \frac{1}{2\|P\|}\|x_1 - x_2\|_{B_1};$$
2. *$\delta = \frac{\delta'}{2\|P\|}$.*

Then for all $y \in B_2$ such that $\|y - F(0)\| < \delta$, there exists $x \in B_1$ such that $F(x) = y$.

To apply the theorem to the extremal operator, one should bound both the right inverse of the linearisation and the non-linear operator. Denote by $L^2_{0,p}$ the Sobolev spaces of functions on Y computed with respect to $\omega_{k,r}$, and remark that these do not depend on k, since the Sobolev norms are equivalent for different values of k [7, Remark 30].

Let $\mathfrak{t}$ be the Lie algebra of T, where T is the torus of automorphisms described in Remark 3.23. Let $\bar{\mathfrak{t}}$ be the set of holomorphy potentials whose flow lies in T. We denote by $(L^2_{0,p})^T$ the space of T-invariant functions in $L^2_{0,p}$.

For each k, r denote by $\gamma_{k,r}$ the Kähler potential defined in Proposition 3.24, such that the approximately extremal metric $\omega_{k,r}$ is given by $\omega_k + i\partial\bar{\partial}\gamma_{k,r}$. For each k, r we define the map

$$\begin{aligned} \tau_{k,r} : \mathfrak{t} &\to C^\infty(X, \mathbb{R}) \\ \xi &\mapsto k\pi_Y^* h_B + h_q + \frac{1}{2}\langle \nabla\gamma_{k,r}, \nabla(k\pi_Y^* h_B + h_q)\rangle_{\omega_k}, \end{aligned}$$

where h_B and h_q are the holomorphy potentials defined in Remark 3.23. The map $\tau_{k,r}$ associates to a T-invariant holomorphic vector field the correspondent holomorphy potential with respect to $\omega_{k,r}$.

We apply the theorem to the operators

$$\begin{aligned} &F_{k,r} : (L^2_{0,p+4})^T \times \bar{\mathfrak{t}} \to (L^2_{0,p})^T \\ &F_{k,r}(\varphi, h) = \mathrm{Scal}(\omega_{k,r} + i\partial\bar{\partial}\varphi) - \frac{1}{2}\langle \nabla\eta_{k,r}, \nabla\gamma_{k,r}\rangle - \eta_{k,r} - \frac{1}{2}\langle \nabla(\tau_{k,r}(h)), \nabla\varphi\rangle - \tau_{k,r}(h), \end{aligned}$$

where $\eta_{k,r}$ is the Kähler potential which makes $\omega_{k,r}$ approximately extremal. The linearisation of $F_{k,r}$ is the operator

$$\begin{aligned} &G_{k,r} : (L^2_{0,p+4})^T \times \bar{\mathfrak{t}} \to (L^2_p)^T \\ &(\varphi, h) \mapsto -\mathcal{D}^*_{k,r}\mathcal{D}_{k,r}(\varphi) + \frac{1}{2}\langle \nabla(\mathrm{Scal}(\omega_{k,r}) - \tau_{k,r}(h)), \nabla\varphi\rangle - \tau_{k,r}(h). \end{aligned}$$

The proof requires two steps: the first one is to ensure that the linearisation is an isomorphism with bounded inverse $P_{k,r}$. Theorem 3.25 then gives δ_k such that if $\|F_{k,r}(0)\| < \delta_k$, a zero of $F_{k,r}$ exists. Since we want to find a zero for all k, the second step is to find a value of r for which the norm $\|F_{k,r}(0)\|$ converges to zero quicker than δ_k. The first step is contained in the following lemma [15, Lemma 6.6], based on [24, Lemmas 6.5,6.6,6.7].

Lemma 3.26. *There exists a constant C independent of k such that $G_{k,r}$ has a right inverse $P_{k,r}$ such that*
$$\|P_{k,r}\| \leq Ck^{5/2}.$$

The second step relies on the following result [15, Lemma 6.6], which is a consequence of the mean value theorem.

Lemma 3.27. *Let $\mathcal{N}_{k,r} = F_{k,r} - \mathrm{d}_0 F_{k,r}$ be the nonlinear part of the extremal operator. Then there are constant c, C such that for all r sufficiently large, if $f_i \in (L^2_{p+4})^T \times \bar{\mathfrak{t}}$ for $i = 1, 2$ satisfy $\|f_i\| \leq c$, then*

$$\left\|\mathcal{N}_{k,r}(f_1) - \mathcal{N}_{k,r}(f_2)\right\|_{L^2_p} \leq C \left(\|f_1\|_{L^2_{p+4}(\omega_{k,r})} + \|f_2\|_{L^2_{p+4}(\omega_{k,r})} \right) \left\|f_1 - f_2\right\|_{L^2_{p+4}(\omega_{k,r})}.$$

By applying the implicit function Theorem 3.25, we can now complete the proof of Theorem 3.16 as follows. Lemma 3.27 implies that $\mathcal{N}_{k,r}$ is Lipschitz on any ball of radius ρ sufficiently small, with Lipschitz constant ρC. Thus the radius δ' on which $\mathcal{N}_{k,r}$ is Lipschitz with constant $(2\|P\|_{k,r})^{-1}$ is bounded below by some multiple of $k^{-5/2}$. Hence $\delta = \delta'(2\|P\|)^{-1}$ is bounded below by a multiple of k^{-5}. In order to apply the implicit function theorem, it remains to bound $F_{k,r}(0,0)$. The point-wise bound $F_{k,r} = O(k^{(-r-1)/2})$ is provided by Proposition 3.24. Results of Fine [24, Lemma 5.6, 5.7] can be applied directly to our situation in order to have a $L^2_p(\omega_{k,r})$-bound on $F_{k,r}(0)$ of order $k^{5-\frac{1}{2}}$, when $r > 5$. Thus the hypotheses of the implicit function theorem are satisfied and $\|F_{k,r}(0)\|$ converges to zero quicker than δ_k.

Chapter 4

The moduli space of holomorphic submersions

We construct the moduli space of fibrations admitting optimal symplectic connections, with discrete relative automorphism group. Our main reference in the construction is Fujiki-Schumacher [31], where the moduli space of cscK manifolds in the case of discrete automorphism group is defined. Let $Y \to B$ be a fibration that degenerates to a relatively cscK holomorphic submersion $X \to B$, as described in §2.3.1. We first prove that the set of deformations of $Y \to B$ that still degenerate to a relatively cscK fibration (possibly different from $X \to B$) forms a locally closed analytic subset of the relative Kuranishi space. We then prove that the solutions to the optimal symplectic connection equation (2.10) form an open set inside the locally closed subset of admissible deformations. This allows us to define a local moduli space of optimal symplectic connections. Finally, we glue the local moduli spaces and we prove that we obtain a global Hausdorff complex space which parametrises optimal symplectic connections.

4.1 Openness of the setting

Given a fibration $\pi_Y : (Y, H_Y) \to (B, L)$ with analytically K-semistable fibres, we assume as in §2.2.2 that there exists a degeneration of π_Y to a fibration $\pi_X : (X, H_X) \to (B, L)$ such that the fibres of π_X are cscK. In particular, we can consider $Y \to B$ and $X \to B$ as the same symplectic fibration $\pi : (M, \omega) \to B$, and the degeneration as a deformation of a complex structure J to I, where (ω, I) has fibrewise constant scalar curvature. The goal of this section is to understand for which deformations J' of J we can still find a relatively cscK degeneration X', and to construct such a degeneration.

We begin by working locally in B. Let $\mathcal{U} \subseteq B$ be a coordinate open subset of B. The fibre over the origin of $\mathcal{U}$, denoted M_0, has a constant scalar curvature metric (ω_0, I_0). Let K be the group of Hamiltonian isometries of (ω_0, I_0) and let $\widetilde{H}^1_0$ be the vector space (1.14) parametrising first-order deformations of (M_0, ω_0, I_0). Since (ω_0, I_0) has constant scalar curvature, the complexification of K is the group

$$G = \mathrm{Aut}_0(M_0, H_0).$$

We will employ Definition 1.9 of GIT-stability for the action of a reductive group on an affine space, applied the definition of stability to the vector space $\widetilde{H}^1_0$. The complex structure I_0 corresponds to the origin in the Kuranishi space V_0, which is fixed by the action of G. Therefore its orbit is closed and its stabiliser is the group G itself, so it is a polystable point. A key result for our construction is the fact that the closure of the orbit of every point in $\widetilde{H}^1_0$ contains a unique polystable orbit (Lemma 1.8).

Let V_0 be the subspace of the Kuranishi space which parametrises integrable almost complex structures; it is a locally closed analytic subspace of $\widetilde{H}^1_0$ because it is defined by the vanishing of the Nijenhuis tensor. Thus the family $X \to B$ can be described locally over $\mathcal{U}$ as a family $\{x_b\}$ in V_0. By our hypothesis 2, the automorphism group of (M_b, H_b) is isomorphic to G for all $b \in B$. Therefore the points $\{x_b \mid b \in \mathcal{U}\}$ are all fixed by the action of G and are hence polystable.

The family $Y \to B$ can be described locally over $\mathcal{U}$ as a family $\{y_b \mid b \in \mathcal{U}\}$ of points such that for each b the closure of the G-orbit of y_b contains the polystable point x_b [12, Theorem 1.3]. By Lemma 1.8, x_b is the only polystable orbit in the closure of the orbit of y_b. Let V_0^+ be the set of all semistable points that have a fixed point in the closure of their orbit. Then the map

$$F : V_0^+ \to V_0 \tag{4.1}$$

that maps a semistable point to the corresponding fixed point is well-defined.

Lemma 4.1. *The set V_0^+ is an analytic subvariety of V_0 and the map (4.1) is holomorphic.*

Proof. The space V_0 is an open subset of the vector space $\widetilde{H}^1$. Let d be the dimension of $\widetilde{H}^1$, so each point $z \in V_0$ has coordinates

$$(z_1, \ldots, z_d)\,.$$

Let us begin with the case when G is isomorphic to a complex torus $(\mathbb{C}^*)^r$. The fixed points of the action can be described by the vanishing of the coordinates

$$z_{i_1} = \cdots = z_{i_h} = 0 \quad \text{for} \quad i_1, \ldots, i_h \in \{1, \ldots, d\},$$

thus they form an analytic subspace of V_0.

By the Hilbert-Mumford criterion 1.11, a point y_j is semistable if it is semistable for the action of every 1-parameter subgroup of the group G. Let $\rho(t) : \mathbb{C}^* \hookrightarrow G$ be a 1-parameter subgroup. The action of $\rho(t)$ can be written as $(t^{a_1}, \ldots, t^{a_d})$, where the numbers a_j are the *weights* of the action. Then V_0 splits into a sum of weight spaces $V_0^{\text{pos}} \oplus V_0^{\text{fix}} \oplus V_0^{\text{neg}}$, where $\rho(t)$ acts on V_0^{pos} with positive weights, on V_0^{neg} with negative weights and fixes the subspace V_0^{fix}. A semistable point that has a fixed point in the closure of its orbit is described by the following condition: if a coordinate in V_0^{pos} is nonzero, then all coordinates in V_0^{neg} vanish. Applying this condition to all 1-parameter subgroups yields a set of polynomial equations that define the semistable points in V_0^+. Thus, the semistable points correspond to an analytic subset of V_0. The map F is the projection onto the set described by $\{z_{i_1} = \ldots = z_{i_h} = 0\}$, thus it is holomorphic.

Let now G be any reductive group, and let y be a semistable point and x be a polystable and fixed point in the closure of its orbit. The fixed points of G form a vector subspace V_0^{fix} also in this case. Moreover there exists a 1-parameter subgroup $\lambda_x : \mathbb{C}^* \hookrightarrow G$ such that

$$\lim_{t \to 0} \lambda_x(t) \cdot y = x.$$

It follows that the map $y \mapsto x$ is the projection onto the vector subspace of fixed points for λ_x, hence it is holomorphic. Consider the composite map

$$V_0 \to V_0 \xrightarrow{pr} V_0^{\text{fix}} \tag{4.2}$$

where the first map is the projection onto the vector subspace of fixed points for λ_x and the second map is the projection onto the subspace of fixed points for the whole group G. The map is holomorphic because it is a composition of holomorphic projections. We prove that it coincides with the map (4.1). Let $\widetilde{x}' = \lim_{t\to 0} \lambda_x(t)\cdot y'$. Then

$$\overline{G\cdot \widetilde{x}'} \subseteq \overline{G\cdot y'}.$$

The unique polystable orbit contained in $\overline{G\cdot \widetilde{x}'}$ is also a polystable orbit in $\overline{G\cdot y'}$, so it must coincide with the fixed-point orbit $\{x'\}$. Thus flowing along the orbit of $\widetilde{x}'$ amounts to projecting onto the subspace of fixed points, and so the map (4.2) maps any point y' to the fixed point x' in the closure of its orbit. □

Remark 4.2. The map (4.1) is analogous to the one given by the Byałinicki-Birula decomposition [4, 46].

Let V_π be the Kuranishi space of the fibration $X \to B$ defined in Theorem 2.21. Consider the subspace

$$V_\pi^+ := \left\{ y \in V_\pi \mid y|_{X_b} \in V_b^+ \right\}.$$

We remark that V_π^+ depends on the complex structure of the reference fibration $X \to B$ and its deformation $Y \to B$. We denote by $\mathscr{J}_\pi^+$ the image of V_π^+ via the relative Kuranishi map (2.18).

Lemma 4.3. *V_π^+ is a locally closed subvariety of V_π.*

Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two open coordinate subsets of B with non-empty intersection and let 0_1 be the origin of $\mathcal{U}_1$ and 0_2 be the origin of $\mathcal{U}_2$. Denote by I_1 and I_2 the complex structures of the fibres X_{0_1} and X_{0_2}, and G_1, G_2 their groups of automorphisms. By assumption, G_1 and G_2 are isomorphic, and we will denote them by G. Recall that the Kuranishi space is *versal* and more specifically that it is a complete deformation space for the nearby fibres. We use the versality of the Kuranishi map to glue the spaces V_1^+ and V_2^+ constructed in Lemma 4.1 to a variety V^+ and prove that V_π^+ is obtained as the intersection of said variety with the relative Kuranishi space V_π. More precisely, versality of the Kuranishi space means that there is a map

$$\tau_{21} : V_2 \longrightarrow V_1,$$

not necessarily unique.

The map τ_{21} can be taken to be G-equivariant. In fact, the G-equivariance can be traced back to the proof of Kuranishi's Theorem 1.23. The map τ_{21} is defined using the implicit function theorem, which can be applied to a K-equivariant map to provide an implicit inverse function which is K-equivariant. Since G is the complexification of K, we obtain that τ_{21} is G-equivariant. The equivariance also implies that the image of V_2^+ is V_1^+, so we can restrict τ_{21} to

$$\widetilde{\tau}_{21} : V_2^+ \longrightarrow V_1^+.$$

Moreover the map $\widetilde{\tau}_{21}$ has an inverse that is constructed reversing the roles of V_1 and V_2, so it is an isomorphism. In fact, although the map τ_{21} is not canonical, the restriction to $\widetilde{\tau}_{21}$ is fixed by the reference K-semistable fibration $Y \to B$. Each Kuranishi space V_b is a complex subspace of the vector space $\widetilde{H}_b^1$, described as the kernel of the elliptic operator $P_b P_b^* + (\bar{\partial}_b^* \bar{\partial}_b)^2$ (1.14). So we can use the isomorphism $\widetilde{\tau}_{21}$ to glue the spaces V_b^+ to a subvariety V^+ of the kernel of the fibrewise elliptic operator

$$P_{\mathcal{V}} P_{\mathcal{V}}^* + (\bar{\partial}_{\mathcal{V}}^* \bar{\partial}_{\mathcal{V}})^2.$$

Therefore the intersection

$$V_\pi^+ = V^+ \cap V_\pi$$

is a locally closed subvariety of V_π. □

The following lemma shows that we can glue the local fibration constructed in Lemma 4.1 to a global fibration over B.

Lemma 4.4. *Let $Y' = (M, \omega, J') \to B$ be a fibration with complex structure J' represented by $y' \in V_\pi^+$. Then Y' degenerates to $X' = (M, \omega, I') \to B$ such that*

1. *(ω, I') is relatively cscK;*
2. *the groups $\mathrm{Aut}(X'_b, H'_b)$ are isomorphic for all $b \in B$.*

Proof. Let $\mathcal{U} \subseteq B$ be an open coordinate subset. By the relative Kuranishi Theorem 1.23, for each $b \in \mathcal{U}$ there exists a point $y'_b \in V_0$ such that $\Phi_0(y'_b)$ is in the same $\mathcal{G}^c$-orbit of J'_b. The fact that the map (4.1) is holomorphic implies that we can find polystable points $\{x'_b\}$ such that $\{\Phi_0(x'_b)\}$ are a holomorphic family of cscK complex structures over $\mathcal{U}$ that are deformations of $\{J'_b\}$. Then we can construct a local relatively cscK fibration from the pullback diagram

$$\begin{tikzcd} X'_{\mathcal{U}} \arrow[r] \arrow[d] & \mathcal{M}_{\mathcal{U}} \arrow[d, "p_{\mathcal{U}}"] \\ \mathcal{U} \arrow[r, "i"'] & V_0 \end{tikzcd} \tag{4.3}$$

where $p_{\mathcal{U}} : \mathcal{M}_{\mathcal{U}} \to V_0$ is Kuranishi's versal family and $i(b) = x'_b$.

Now we glue the local fibrations to a fibration $X' \to B$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ and G as in Lemma 4.3. The associated Kuranishi maps are denoted respectively by $\Phi_1 : V_1 \to \mathscr{J}$ and $\Phi_2 : V_2 \to \mathscr{J}$.

For $b \in \mathcal{U}_1 \cap \mathcal{U}_2$, consider the complex structure J'_b. Since J'_b can be regarded as a deformation of both I_1 and I_2, there exist points $y'_{b,1} \in V_1$ and $y'_{b,2} \in V_2$ such that

$$\Phi_1(y'_{b,1}) = J'_b = \Phi_2(y'_{b,2}).$$

The diagram (4.3) produces two fibrations $X'_1 \to \mathcal{U}_1$ and $X'_2 \to \mathcal{U}_2$. In order to glue the local fibrations we need to prove that there is an isomorphism

$$\phi_{12} : X'_1|_{\mathcal{U}_1 \cap \mathcal{U}_2} \xrightarrow{\sim} X'_2|_{\mathcal{U}_1 \cap \mathcal{U}_2}$$

and that it satisfies the cocycle condition

$$\phi_{23} \circ \phi_{12} = \phi_{13}$$

on a triple intersection $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ of open subsets of B. In particular, the first condition produces a global compact complex manifold X', while the cocycle condition implies that X' admits a submersion onto B.

To prove the existence of the isomorphism ϕ_{12} we use again the fact that the Kuranishi space induces complete deformations on the nearby fibres. The map

$$\tau_{21} : V_2 \to V_1$$

is such that $\mathcal{M}_2 = \tau_{21}^* \mathcal{M}_1$. In particular, we have the following diagram

$$\begin{tikzcd} \mathcal{M}_2 \arrow[d] & & \mathcal{M}_1 \arrow[d] \\ V_2 \arrow[rr, "\tau_{21}"] & & V_1 \\ X'_2 \arrow[r] & \mathcal{U}_1 \cap \mathcal{U}_2 \arrow[ul, hook, "i_2"] \arrow[ur, hook, "i_1"'] & X'_1. \arrow[l] \end{tikzcd}$$

If we show that the diagram is commutative, i.e. $\tau_{21} \circ i_2 = i_1$, then

$$X'_2|_{\mathcal{U}_1 \cap \mathcal{U}_2} = i_2^* \mathcal{M}_2 \simeq (\tau_{21} \circ i_2)^* \mathcal{M}_1 = i_1^* \mathcal{M}_1 = X'_1|_{\mathcal{U}_1 \cap \mathcal{U}_2}.$$

The commutativity follows from the fact that the map τ_{21} is G-equivariant. Indeed the points $i_1(b)$ and $i_2(b)$ are defined as the fixed-point limits of semistable orbits. Moreover, an equivariant map between two spaces on which there is an action of the same group G sends fixed points to fixed points and the closure of the orbit of $i_1(b)$ to the closure of the orbit of $i_2(b)$.

We now show the cocycle condition. Let $\mathcal{U}$ be a triple intersection $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ and consider the following diagram

$$\begin{tikzcd}
V_1 & V_2 \arrow[l, "\tau_{21}"'] & V_3 \arrow[l, "\tau_{32}"'] \arrow[ll, bend right, "\tau_{31}"'] \\
& \mathcal{U} \arrow[ul, hook, "i_1"] \arrow[u, hook, "i_2"] \arrow[ur, hook, "i_3"'] &
\end{tikzcd}.$$

On V_3 we have two families pulled-back from V_1, namely $\tau_{31}^* \mathcal{M}_1$ and $(\tau_{21} \circ \tau_{32})^* \mathcal{M}_1$. They induce two distinct families on $\mathcal{U}$, pulled-back using i_3. Although in general it is *not* true that τ_{31} is equal to the composition $\tau_{21} \circ \tau_{32}$, the commutativity of the arrows proved above implies that

$$\tau_{21} \circ \tau_{32} \circ i_3 = \tau_{31} \circ i_3.$$

Therefore the two families $\tau_{31}^* \mathcal{M}_1$ and $(\tau_{21} \circ \tau_{32})^* \mathcal{M}_1$ coincide. □

4.2 Openness of the space of optimal symplectic connections

Let $Y \to B$ be a fibration with K-semistable fibres, and assume that it degenerates to a fibration $X \to B$ with cscK fibres, in the sense of §2.2.2. Let $Y' \to B$ be a deformation of $Y \to B$ in $\mathscr{J}_\pi^+$. Then $Y' \to B$ admits a degeneration to $X' \to B$, whose fibres are cscK, as explained in §4.1. The goal of this section is to show that if Y admits an optimal symplectic connection then Y' also does.

We denote by I the complex structure of X, by J the complex structure of Y and we assume that $Y \to B$ is generated by $v_0 \in V_\pi$. We also assume that (ω, J) is an optimal symplectic connection. Let V_π^+ the subvariety of V_π which describes the family of complex structures $\mathscr{J}_\pi^+$. The following is a relative version of Proposition 1.22.

Proposition 4.5. *For every $\varphi \in \mathcal{K}_E(I)$ there exists $f \in \mathrm{Diff}_0(M)$ such that $f^* \omega_\varphi = \omega$ and $(M, \omega_\varphi, I) \to B$ is isomorphic to $(M, \omega, f^* I) \to B$.*

Proof. Let us consider a potential $\varphi \in \mathcal{K}_E(I)$ and a path $\{\varphi_t\}$ in $\mathcal{K}_E(I)$ from 0 to φ. A result of Hallam [34, Theorem 3.3] guarantees that this path exists and that it is smooth. We define the relatively cscK metrics

$$\omega_t = \omega + 2i\partial\bar{\partial}\varphi_t$$

and the Kähler metrics

$$\omega_{k,t} = \omega_t + k\omega_B,$$

where the $\partial, \bar{\partial}$ operators are taken with respect to the relatively cscK complex structure I. From Proposition 2.5, we have that $\dot{\varphi}_t \in C^\infty(E(\omega_t, I)) \oplus C^\infty(B)$, for all t. Thus the fibrewise Hamiltonian vector fields

$$\eta_t := \mathrm{grad}^{\omega_t} \dot{\varphi}_t$$

are well-defined. Consider the vertical vector fields

$$\xi_t := (\nabla^{g_t}\dot{\varphi}_t)_{\mathcal{V}} = (I\mathrm{grad}^{\omega_t}\dot{\varphi}_t)_{\mathcal{V}}.$$

Then fibrewise

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = -\mathcal{L}_{I\eta_t}\omega_t = \mathcal{L}_{\xi_t}\omega_t. \tag{4.4}$$

Let $\{f_t, t \in [0,1]\}$ be the *isotopy* of the time-dependent vector field ξ_t, i.e. the collection of diffeomorphisms of M such that

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t = \xi_t(f_t), \quad f_0 = \mathrm{id}.$$

Since ξ_t is vertical, $f_t \in \mathrm{Diffeo}(M, \pi)$. As in the proof of Proposition 1.22, we apply the following property (1.12):

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t^*\eta_t = f_t^*\left(\mathcal{L}_{\xi_t}\eta_t + \frac{\mathrm{d}\eta_t}{\mathrm{d}t}\right),$$

where ξ_t and f_t are a time-dependent vector field and its isotopy respectively. Applying it to ω_t gives the fibrewise relation

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t^*\omega_t = 0,$$

which implies

$$f_t^*\omega_t = f_0^*\omega = \omega.$$

Define $J_t = f_t^*I$. Then, for $t = 1$, the two metrics $g(\omega, f_1^*I)$ and $g(\omega + 2i\partial\bar{\partial}\varphi, I)$ are fibrewise isometric, i.e.

$$(M, \omega, f_1^*I) \simeq (M, \omega + 2i\partial\bar{\partial}\varphi, I)$$

as relative Kähler manifold with fibrewise constant scalar curvature. □

Thus, given an integrable relatively cscK complex structure I in $\mathscr{J}_\pi$, we have a map

$$\begin{aligned} F_\pi : \mathcal{K}_E(I) &\longrightarrow \mathscr{J}_\pi \\ \varphi &\longmapsto f_1^*I =: F_\pi(\varphi, I) \end{aligned} \tag{4.5}$$

which locally parametrises all integrable complex structures in the same diffeomorphism class of I that are fibrewise cscK with respect to the fixed ω. Its differential at the origin is

$$\mathrm{d}_0F_\pi(\dot{\varphi}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_t = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^*I = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{L}_{\xi_t}I = \mathcal{L}_{(I\mathrm{grad}^{\omega}\dot{\varphi})_{\mathcal{V}}}I = -\frac{1}{2}\bar{\partial}(\mathrm{grad}^{\omega}\dot{\varphi})_{\mathcal{V}},$$

where, again by Proposition 2.5, $\dot{\varphi} \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$.

Now let $v_0 \in V_\pi$ be the deformation of the complex structure which generates the family $Y \to B$ and let Φ be the relative Kuranishi map (2.18). Then we can define the map

$$\begin{aligned} F'_\pi : \mathcal{K}_E(I) &\to \widetilde{H}^1_{\mathcal{V}} \\ \varphi &\mapsto f_1^*v_0 =: F'_\pi(\varphi, I, v_0). \end{aligned}$$

Its differential at the origin computed at $\dot{\varphi} \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$ is

$$\mathrm{d}_0F'_\pi(\dot{\varphi}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^*v_0 = \mathcal{L}_{(I\mathrm{grad}^{\omega}\dot{\varphi})_{\mathcal{V}}}v_0. \tag{4.6}$$

Definition 4.6. We denote with $\mathcal{P}$ the set of triples $(\varphi, x, v) \in C^\infty(M) \times TV_\pi$ such that $\varphi \in \mathcal{K}_E(\Phi(x))$ and x is $K_\pi^{\mathbb{C}}$-polystable. The optimal symplectic connection operator is the map

$$\begin{aligned} &\mathcal{G} : \mathcal{P} \to C^\infty(X) \\ &(\varphi, x, v) \mapsto p_{E(\varphi,x)}\left(\Theta(\omega, F_\pi(\varphi, \Phi(x)))\right) + \frac{\lambda}{2}\nu_{\varphi,x}\left(F'_\pi(\varphi, \Phi(x), v)\right). \end{aligned}$$

In this expression $E(\varphi, x)$ is the vector bundle of fibre holomorphy potentials with respect to the Kähler structure $(\omega, F_\pi(\varphi, \Phi(x)))$ and $\nu_{\varphi,x}$ is the map (2.8) computed with respect to the complex structure $F_\pi(\varphi, \Phi(x))$.

We now compute the differential of $\mathcal{G}$ along the φ-variable computed at $(0, 0, v_0)$. To do so, we need the following technical result on the contraction with ω_k.

Lemma 4.7. *Let α be a covariant 2-tensor of type $(1,1)$. Then*

$$\Lambda_{\omega_k}\alpha = \Lambda_{\mathcal{V}}\alpha + \frac{1}{k}\Lambda_{\omega_B}\alpha + O\left(k^{-2}\right),$$

where Λ_{ω_k} denotes the contraction with the Kähler metric ω_k, $\Lambda_{\mathcal{V}}$ denotes the contraction in the vertical direction and Λ_{ω_B} denotes the contraction with respect to the base metric.

Proof. At each point $x \in X_b \subset X$, the matrix $[\omega_k]$ is block-diagonal:

$$\begin{pmatrix} [\omega_F] & 0 \\ 0 & [\omega_{k,\mathcal{H}}] \end{pmatrix}.$$

In local coordinates around the point x

$$(\omega_k)^{p\bar{q}}\alpha_{p\bar{q}} = (\omega_F)^{a\bar{b}}\alpha_{a\bar{b}} + (\omega_{k,\mathcal{H}})^{i\bar{j}}\alpha_{i\bar{j}}.$$

The horizontal part of ω_k, denoted $\omega_{k,\mathcal{H}}$ splits as $\omega_{\mathcal{H}} + k\omega_B$, where $\omega_{\mathcal{H}}$ is the horizontal part of ω. Let $[\omega_{k,\mathcal{H}}]$, $[\omega_{\mathcal{H}}]$ and $[\omega_B]$ be the matrices of coefficients of the two-forms $\omega_{k,\mathcal{H}}$, $\omega_{\mathcal{H}}$ and ω_B respectively. We can write

$$[\omega_{k,\mathcal{H}}] = k\left(\frac{[\omega_{\mathcal{H}}][\omega_B]^{-1}}{k} + \mathbb{1}\right)[\omega_B],$$

where $\mathbb{1}$ is the identity matrix and the base form ω_B induces a Riemannian metric on the horizontal tangent bundle, so its inverse is well defined. The inverse of the matrix $[\omega_{k,\mathcal{H}}]$ can be expanded in inverse powers of k as

$$\begin{aligned} [\omega_{k,\mathcal{H}}]^{-1} &= k^{-1}[\omega_B]^{-1}\left(\frac{[\omega_{\mathcal{H}}][\omega_B]^{-1}}{k} + \mathbb{1}\right)^{-1} \\ &= k^{-1}[\omega_B]^{-1}\sum_{i=0}^{\infty}\left(-\frac{[\omega_{\mathcal{H}}][\omega_B]^{-1}}{k}\right)^i \\ &= \frac{1}{k}[\omega_B]^{-1} + O\left(k^{-2}\right). \end{aligned}$$

This implies the claim. ◻

We then write

$$\mathcal{G}(\varphi, x, v) = \mathcal{G}_1(\varphi, x) + \mathcal{G}_2(\varphi, x, v),$$

and we split the computation into two separate lemmas.

Lemma 4.8. *Let $\varphi \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$. The differential along the first variable of $\mathcal{G}_1$ is*

$$D_1\mathcal{G}_1|_{(0,0,v_0)}(\varphi) = -\frac{1}{2}\mathcal{R}^*\mathcal{R}(\varphi).$$

Proof. Let $\{J_t\}$ be a family of relatively cscK complex structures compatible with ω, such that $J_0 = I$ and consider the scalar curvature of (ω_k, J_t),

$$\mathrm{Scal}(\omega_k, J_t) = \mathrm{Scal}(\omega_k, I) + t\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{Scal}(\omega_k, J_t) + O(t^2). \tag{4.7}$$

Let $\alpha = \partial_{t=0}J_t$, and define

$$Q_k(\alpha) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\mathrm{Scal}(\omega_k, J_t).$$

From Lemma 1.18 we obtain

$$Q_k(\alpha) = \mathrm{Im}\left((g_k)^{p\bar{q}}\,\nabla_p\nabla_a\alpha^a_{\ \bar{q}}\right) = \mathrm{Re}\left((\omega_k)^{p\bar{q}}\,\nabla_p\nabla_a\alpha^a_{\ \bar{q}}\right).$$

We need to compute the sub-leading order term of Q_k along the differential of the map F_π (4.5)

Let us consider $\alpha = \bar{\partial}\left(\mathrm{grad}^\omega\varphi_E\right)^{1,0}_{\mathcal{V}}$, where $\varphi_E \in C^\infty_E(X)$ is a fibrewise holomorphy potential. Then

$$Q_k(\alpha) = k^{-1}\mathrm{Re}\left(\mathcal{R}^*\mathcal{R}(\varphi_E)\right) + O\left(k^{-2}\right), \tag{4.8}$$

where $\mathcal{R}(\varphi_E) = \bar{\partial}_B\mathrm{grad}_{\mathcal{V}}\varphi_E$ and the adjoint is computed with respect to $\omega_F + \omega_B$. As explained in §2.2.1, the operator $\mathcal{R}^*\mathcal{R}$ can actually be seen as $p_E \circ \mathcal{L}_1$ restricted to $C^\infty(E(\omega, I))$. Its kernel consists of fibrewise holomorphy potentials which are global holomorphy potentials on X with respect to ω_k.

A local coordinate expression for α is

$$\alpha = \bar{\partial}\left(Y_{\varphi_E}\right)^{1,0}_{\mathcal{V}} = \partial_{\bar{z}^j}\left(\omega_F^{a\bar{b}}\partial_{\bar{w}^b}\varphi_E\right)\partial_{w^a}\otimes\mathrm{d}\bar{z}^j = \nabla_{\bar{j}}\left(\omega_F^{a\bar{b}}\nabla_{\bar{b}}\varphi_E\right)\partial_{w^a}\otimes\mathrm{d}\bar{z}^j,$$

where we have used that the component of α with the covariant index of vertical type vanishes, since the potential φ_E is a fibrewise holomorphy potential. Therefore, using Lemma 4.7,

$$\begin{aligned}
Q_k(\alpha) &= k^{-1}\mathrm{Re}\left(\omega_B^{i\bar{j}}\,\nabla_i\nabla_a\nabla_{\bar{j}}\left(\omega_F^{a\bar{b}}\nabla_{\bar{b}}\varphi_E\right)\right) + O\left(k^{-2}\right) = \\
&= k^{-1}\mathrm{Re}\left(\omega_B^{i\bar{j}}\,\omega_F^{a\bar{b}}\,\nabla_i\nabla_a\nabla_{\bar{j}}\nabla_{\bar{b}}\varphi_E\right) + O\left(k^{-2}\right) = \\
&= k^{-1}\mathrm{Re}\left(\mathcal{R}^*\mathcal{R}(\varphi_E)\right) + O\left(k^{-2}\right).
\end{aligned}$$

A similar computation also holds if we consider, instead of a potential in $C^\infty(E(\omega, I))$, a map $\varphi \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$, so that $\alpha = \bar{\partial}(Y_\varphi)^{1,0}_{\mathcal{V}}$. This choice amounts to considering an element in the image of the differential of the map F_π (4.5). In this case

$$Q_k(\alpha) = \mathrm{Re}(\mathcal{L}_0\varphi) + \frac{1}{k}\mathrm{Re}(\mathcal{L}_1\varphi) + O\left(k^{-2}\right),$$

where $\mathcal{L}_0(\varphi) = \mathcal{L}_0(\varphi_B) = 0$, since φ_B is constant when restricted to a fibre, so we obtain the same equation as (4.8). To obtain the correct coefficient in the claimed expression note that

$$\mathrm{d}_0 F_\pi(\varphi) = -\frac{1}{2}\alpha. \qquad \square$$

Lemma 4.9. *Let $\varphi \in C^\infty(E(\omega, I)) \oplus C^\infty(B)$. The differential along the first variable of $\mathcal{G}_2$ is*

$$D_1\mathcal{G}_2|_{(0,0,v_0)}(\varphi) = -\mathcal{A}^*\mathcal{A}(\varphi).$$

Proof. We compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nu_t\left(f_t^* v_0\right),$$

where ν_t is the map ν computed with respect to the complex structure $f_t^*\Phi(0)$. Now, f_t is the isotopy of the vector field $\xi_t = (I\mathrm{grad}^{\omega_t}\dot\varphi_t)_{\mathcal{V}}$, where $\dot\varphi_t$ is in $C^\infty(E_t) \oplus C^\infty(B)$ and $\dot\varphi_0 = \varphi$. In the expression of ξ_t we are fixing the complex structure I and varying the Kähler form ω_t. In particular, ξ_t is a fibrewise holomorphic vector field with respect to I. This implies that $f_t^* I = I$, so $\nu_t = \nu$. Therefore

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nu_t\left(f_t^* v_0\right) = \mathrm{d}_{v_0}\nu\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^* v_0\right).$$

Using the expression (4.6) we obtain

$$\mathrm{d}_{v_0}\nu\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^* v_0\right) = \mathrm{d}_{v_0}\nu\left(\mathcal{L}_{(I\mathrm{grad}^\omega\varphi)_{\mathcal{V}}} v_0\right).$$

The right-hand side can be written as $-\mathcal{A}^*\mathcal{A}(\varphi)$ following the description (3.4). The minus sign follows from the relation

$$(I\mathrm{grad}^\omega\varphi)_{\mathcal{V}} = -\nabla_{\mathcal{V}}\varphi,$$

where $\nabla_{\mathcal{V}}$ is the vertical Riemannian gradient. $\square$

We define the operator

$$\begin{aligned} \mathcal{G} : \mathcal{P}^{2,\ell} &\longrightarrow W^{2,\ell-2}(X) \\ (\varphi, x, v) &\longmapsto p_{E(x,\varphi)}\left(\Theta(\omega, F_\pi(\varphi, \Phi(x)))\right) + \frac{\lambda}{2}\nu_{\varphi,x}\left(F'_\pi(\varphi, \Phi(x), v)\right), \end{aligned} \tag{4.9}$$

where $\mathcal{P}^{2,\ell}$ is the space defined in Definition 4.6, but the functions are considered to be in the Sobolev space $W^{2,\ell}(X)$ instead of smooth.

Proposition 4.10. *Let $\pi_X : X \to B$ be a holomorphic submersion with a fibrewise cscK structure (ω, I) and let $\pi_Y : Y \to B$ be a deformation of $X \to B$ with complex structure J. Let V_π be the Kuranishi space based at I and let $v_0 \in V_\pi$ represent the complex structure J. Assume that*

1. *(ω, J) is an optimal symplectic connection;*
2. *the relative automorphism group*

   $$\mathrm{Aut}(\pi_Y) := \left\{f \in \mathrm{Aut}(Y, H_Y) \mid f \circ \pi = \pi\right\}$$

   is discrete.

3. *the group* $\mathrm{Aut}(X_b, H_b)$ *is independent of* B*, and will be denoted* G*.*

Then for any small deformation v *of* v_0 *in* V_π^+ *there exists a pair* $(x, v) \in TV_\pi$ *such that* $(\omega, \Phi(x))$ *is relatively cscK and* v *generates a complex structure* J'*, and a Kähler potential* φ *such that*

$$\omega + i\partial\bar{\partial}\varphi$$

is an optimal symplectic connection with respect to J'*, where the* $\partial, \bar{\partial}$ *operators are with respect to* $\Phi(x)$*.*

Proof. The proof consists of proving that the operator 4.9 is an elliptic operator with a trivial kernel so that we can apply the implicit function theorem. We note that this can be done even though V_π^+ may be a singular complex space. Indeed, let $\mathcal{X} \to B \times V_\pi^+$ be a family of holomorphic submersions such that the fibre over $0 \in V_\pi^+$ is $X \to B$. By the Kuranishi theorem [48, §1] we can locally consider a smooth trivialisation of the family over V_π^+ such that the complex structures of the fibrations $\mathcal{X}_x \to B \times \{x\}$ form a smooth family $\{J_s\}$.

As before, let I denote the complex structure of X. As we assume that (ω, I, v_0) is an optimal symplectic connection, $\mathcal{G}(0, 0, v_0) = 0$. The derivative with respect to the first component, given by Lemma 4.8 and 4.9, is

$$\mathrm{d}_1\mathcal{G}_{(0,0,v_0)}(\varphi) = -\mathcal{R}^*\mathcal{R}(\varphi) - \lambda\mathcal{A}^*\mathcal{A}(\varphi) = -\widehat{\mathcal{L}}(\varphi).$$

The hypothesis on the automorphism group implies that the kernel of the linearisation is empty, so the implicit function theorem guarantees that there exists a path

$$(\varphi(x, v), x, v) \in \mathcal{P}^{2,\ell}$$

such that locally around $(0, 0, v_0)$

$$\mathcal{G}(\varphi(x, v), x, v) = 0. \tag{4.10}$$

The function $\varphi(x, v)$ is smooth by the standard theory of regularity of solutions to elliptic partial differential equations, applied to the operator $\mathcal{G}$ [3, Theorem 41]. Therefore solutions to (4.10) produce optimal symplectic connections. □

4.3 The moduli space of optimal symplectic connections

Let $(Y, H_Y) \to B$ be a relatively K-semistable holomorphic submersion with a degeneration to a relatively cscK fibration $(X, H_X) \to B$. Assume that we have a relatively Kähler metric (ω, J) on Y that is an optimal symplectic connection. Let W be the subset of the Kuranishi space V_π that corresponds to fibrations satisfying the hypotheses of Proposition 4.10. Then W is an open subset of the locally closed subvariety V_π^+ described in §4.1 by Proposition 4.10.

Lemma 4.11 ([27, Corollary to Proposition 2])**.** *The group* $\mathrm{Aut}(\pi_Y)$ *is a subgroup of* $\mathrm{Aut}(Y, H_Y)$ *with finitely many connected components.*

In particular, under our assumption $\mathrm{Aut}(\pi_Y)$ is a finite discrete group. Let V_{π_Y} be the Kuranishi space of the fibration π_Y and let

$$\tau : V_{\pi_Y} \to V_\pi$$

be the map given by completeness of the Kuranishi space. If we denote $\tau^{-1}W =: W_Y$, then W_Y is a locally closed subvariety of V_{π_Y}.

Let $\mathcal{Y} \to B \times W_Y$ be the Kuranishi family of fibrations which admit an optimal symplectic connection, with central fibration $Y \to B$. The quotient

$$W_Y/\mathrm{Aut}(\pi_Y) \tag{4.11}$$

is a local complex space and it is Hausdorff since it is the quotient of a variety by a finite group. We now explain that we can glue the quotients (4.11) to obtain a global Hausdorff moduli space $\mathcal{M}$ of fibrations that admit an optimal symplectic connection.

Remark 4.12. The moduli space $\mathcal{M}$ depends on the group $G = \mathrm{Aut}_0(X_b, H_b)$. In other words, $\mathcal{M}$ parametrises all fibrations $\pi_Y : Y \to B$ such that they have discrete relative automorphism group and such that they degenerate to a relatively cscK fibration whose fibres have G as their automorphism group.

The following result builds on [31, Proposition 6.5] and [30, Lemma 3.8].

Lemma 4.13. *Let $\mathcal{Y}$ and $\mathcal{Y}'$ over W_Y be two families of fibrations that admit an optimal symplectic connection. The group of isomorphisms between $\mathcal{Y}$ and $\mathcal{Y}'$ that preserve the fibration structure, denoted* $\mathrm{Isom}_{W_Y}(\mathcal{Y}, \mathcal{Y}', B)$*, is proper over W_Y.*

Proof. Let $y_t \to \bar{y}$ be a convergent sequence in W_Y and consider a family of isomorphisms $f_t : \mathcal{Y}_{y_t} \to \mathcal{Y}'_{y_t}$ preserving the projection onto B. Such isomorphisms are fibrewise isometries with respect to the underlying fibrewise Riemannian metrics. Therefore there exists a subsequence $\{f_{t_k}\}$ which converges in the C^m-topology to a fibrewise C^m-isometry $f : \mathcal{Y}_{\bar{y}} \to \mathcal{Y}'_{\bar{y}}$ [30, Lemma 3.8]. Moreover, f is a biholomorphism because it is the limit of biholomorphic maps. Therefore the convergence takes place in $\mathrm{Isom}_{W_Y}(\mathcal{Y}, \mathcal{Y}', B)$, which is then proper over W_Y. □

The global definition of $\mathcal{M}$ relies on the following lemma, whose proof follows from [1, XI.6].

Lemma 4.14. *Let $\mathcal{Y} \to B \times W_Y$ be a family of fibrations that admit an optimal symplectic connection. For any points $y, w \in W_Y$, the fibrations $Y_y \to B$ and $Y_w \to B$ are isomorphic if and only if there exists an element $g \in \mathrm{Aut}(\pi_Y)$ such that $g(w) = y$.*

Proof. We can prove the lemma for the entire relative Kuranishi space V_{π_Y}. The statement follows from:

1. For any $y \in V_{\pi_Y}$, the automorphism group $\mathrm{Aut}(\pi_{Y_y})$ is contained in $\mathrm{Aut}(\pi_Y)$;
2. For any $y \in V_{\pi_Y}$ there exists an $\mathrm{Aut}(\pi_{Y_y})$-invariant open neighbourhood $\mathcal{U}_y$ such that any isomorphism between fibres of $\mathcal{Y}|_{\mathcal{U}_y} \to \mathcal{U}_y$ is induced by an element of $\mathrm{Aut}(\pi_{Y_y})$.

We begin by proving the second statement. Assume by contradiction that there exist two sequences $\{y_n\}$ and $\{w_n\}$ both converging to y and that there exist $\{g_n\}$ in $\mathrm{Aut}(\pi_{Y_y}) \setminus \mathrm{Aut}(\pi_Y)$ such that $g_n \cdot w_n = y_n$. Then by Lemma 4.13 there exists $g \in \mathrm{Aut}(\pi_{Y_y})$ such that $g_n \to g$. Up to replacing g_n with $g_n g^{-1}$ and w_n with $g^{-1} w_n$ we may assume that g is the identity of $\mathrm{Aut}(\pi_{Y_y})$. Consider the map

$$\begin{aligned} \mathrm{Aut}(\pi_Y) \times V_{\pi_Y} &\to \widetilde{\mathrm{H}}^1_{\mathcal{V}} \\ (\eta, y) &\mapsto \eta \cdot y. \end{aligned}$$

By Theorem 2.21, this is a local biholomorphism at (id, y). So we have $F(\mathrm{id}, y_n) = F(g_n, w_n)$. Hence $g_n = \mathrm{id}$, a contradiction.

The first claim then follows exactly as in [1, p.204]. We report the proof for completeness. Set

$$I = \left\{(y, g) \in W_Y \times \mathrm{Aut}(\pi_{y_y})\right\}.$$

Then I is equal to $\mathrm{Isom}(\mathcal{Y}, \mathcal{Y}, B)$. Up to shrinking W_Y we can assume that any connected component I' of I intersects $\{0\} \times \mathrm{Aut}(\pi_Y)$. Let $\widehat{I} = \{(y, g) \in I' \mid g \in \mathrm{Aut}(\pi_Y)\}$. Then $\widehat{I}$ is non-empty and Zariski-closed. It follows from 2 that $\widehat{I}$ contains an open set. Then $\widehat{I} = I'$, which concludes the proof. □

To glue the charts (4.11), we use the completeness of the Kuranishi space. Let $Y_1 \to B$ be another relatively K-semistable fibration which admits an optimal symplectic connection and is close to $Y \to B$. Then the Kuranishi theorem gives a map

$$\tau : V_{\pi_{Y_1}} \to V_{\pi_Y}$$

such that a family $\mathcal{Y}_1 \to B \times V_{\pi_{Y_1}}$ is isomorphic to the pull-back via τ of $\mathcal{Y} \to B \times V_{\pi_Y}$. Consider the composition of this map with the inclusion $i : W_{Y_1} \to V_{\pi_{Y_1}}$:

$$\tau \circ i : W_{Y_1} \to V_{\pi_Y}.$$

Let $y_1 \in W_{Y_1}$. Then y_1 represents a fibration with an optimal symplectic connection, i.e. the image of y_1 via the Kuranishi map is a complex structure J_{y_1} such that (ω, J_{y_1}) is an optimal symplectic connection. Therefore, there is a representative y of J_{y_1} in V_{π_Y} which belongs to W_Y, so

$$\alpha := \tau \circ i : W_{Y_1} \to W_Y.$$

Lemma 4.14 allows us to pass to the quotient and obtain an isomorphism

$$\widetilde{\alpha} : W_{Y_1}/\mathrm{Aut}(\pi_{Y_1}) \to W_Y/\mathrm{Aut}(\pi_Y), \tag{4.12}$$

which is uniquely determined (while α itself might not be). Indeed, the inverse is constructed by reverting the roles of Y and Y_1. Therefore, we can use it to glue the local charts to give $\mathcal{M}$ the structure of a complex space.

Proposition 4.15. *The space $\mathcal{M}$ is a Hausdorff complex space with at most countably many connected components.*

Proof. The countability follows from [30, Theorem 7.3]. We prove the Hausdorff property. Let $\mathcal{Y} \to B \times W_Y$ and $\mathcal{Y}_1 \to B \times W_{Y_1}$ be two families of fibrations which admit an optimal symplectic connection. Let $y_t \to \bar{y}$ be a sequence in W_Y and $y_{1t} \to \bar{y}_1$ be a sequence in W_{Y_1} and assume that Y_{y_t} is isomorphic to $Y_{1,y_{1t}}$ as fibrations over B. Following the proof of [27, Proposition 10], we show that $Y_{\bar{y}} \to B$ is isomorphic to $Y_{1,\bar{y}_1}$. Let

$$\widehat{W} = W_Y \times W_{Y_1},$$

and let $\widehat{\mathcal{Y}} \to B \times \widehat{W}$ and $\widehat{\mathcal{Y}}_1 \to B \times \widehat{W}$ be the pull-back of $\mathcal{Y}$ and $\mathcal{Y}_1$ using the projections of $\widehat{W}$ onto the first and second component respectively. Consider

$$\Sigma = \{(y, y_1) \in \widehat{W} \mid Y_y \to B \text{ is isomorphic to } Y_{1,y_1} \to B\}.$$

It follows from the properness of $\mathrm{Isom}(\widehat{\mathcal{Y}}, \widehat{\mathcal{Y}}_1, B)$ that Σ is a locally closed analytic subvariety of $\widehat{W}$. Therefore $(\bar{y}, \bar{y}_1) \in \Sigma$, which concludes the proof. □

We have proven the following.

Corollary 4.16. *There exists a Hausdorff complex space $\mathcal{M}$ which parametrises holomorphic submersions over a fixed base admitting an optimal symplectic connection, with fixed relative automorphism group.*

4.4 A Weil-Petersson type Kähler metric

In this section, we define a Kähler metric on the moduli space of fibrations admitting an optimal symplectic connection. We do so by describing a relative version of the theory of Weil-Petersson type metrics developed by Fujiki and Schumacher for cscK manifolds [31, Sections 8, 9]. In particular, we first define the Weil-Petersson metric locally on the sets W_Y and subsequently we extend the definition to the charts $W_Y/\mathrm{Aut}(\pi_Y)$ and then to the moduli space $\mathcal{M}$.

Consider a family of holomorphic submersions that admit an optimal symplectic connection, denoted by $\mathcal{Y} \to B \times W_Y$, and let $Y \to B$ be the central fibration. Let ω_t be also the optimal symplectic connection on $Y_t \to B$. For each k sufficiently large, $\omega_{t,k} = \omega_t + k\omega_B$ is a Kähler form on Y_t, and we denote by $\omega_{t,F}$ its purely vertical part. Let $\mathcal{Y} \to W_Y$ be the composition with the second projection.

From Theorem 2.21 together with Proposition 2.23, for each $t \in W_Y$ there is an injective map

$$\mathrm{d}_t\Phi : T_t W_Y \hookrightarrow \widetilde{H}^1_{\mathcal{V}}(Y_t) \subseteq \Omega^{0,1}(\mathcal{V}^{1,0}_{Y_t}) \tag{4.13}$$

that identifies a vector α in the Zariski tangent space $T_t W_Y$ with a $(0,1)$-form valued in the $(1,0)$-tangent bundle of Y_t, which we will also denote by α. The map (4.13) is equivariant with respect to the action of $\mathrm{Aut}(\pi_Y)$. Therefore we can define an inner product on $T_t W_Y$ by pulling back the L^2-product on $\Omega^{0,1}(\mathcal{V}^{1,0}_{Y_t})$ induced by the Hermitian metric associated to $\omega_{t,F} + \omega_B$. For any $\alpha, \beta \in T_t W_Y$, its imaginary part is given by

$$\Omega_t(\alpha,\beta) := \langle \alpha, \beta\rangle_{\omega_{t,F}+\omega_B} = \int_{Y_t} \Lambda_{\omega_{t,F}+\omega_B} \mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta})\omega_t^m \wedge \omega_B^n, \tag{4.14}$$

where we denote by Λ the contraction of the covariant part and by Tr the trace of the contravariant part. We give the following definition.

Definition 4.17. The *relative Weil-Petersson metric* on W_Y, denoted by Ω_{WP}, is the two-form $\{\Omega_t\}_{t\in W_Y}$.

Using the compatibility of the deformations with the Kähler form, we write the trace in coordinates as

$$\Lambda_{\omega_{t,F}+\omega_B}\mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta}) = \alpha^a{}_{\bar{q}}\overline{\beta^b{}_{\bar{p}}}(\omega_{t,F})_{a\bar{b}}(\omega_{t,F}+\omega_B)^{\bar{q}p} = \alpha^a{}_{\bar{b}}\overline{\beta^b{}_{\bar{a}}} + \Lambda_{\omega_B}\mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta}).$$

Therefore the integral (4.14) can be written as the sum

$$\int_{Y_t} \alpha^a{}_{\bar{b}}\overline{\beta^b{}_{\bar{a}}}\omega_t^m \wedge \omega_B^n + \int_{Y_t} \Lambda_{\omega_B}\mathrm{Tr}_{\omega_{t,F}}(\alpha\overline{\beta})\omega_t^m \wedge \omega_B^n.$$

Remark 4.18. The first term can be split over Y_t as

$$\int_B \left(\int_{Y_{t,b}} \alpha^a{}_{\bar{b}}\overline{\beta^b{}_{\bar{a}}}\omega_t^m \right) \omega_B^n. \tag{4.15}$$

In particular, it vanishes when α and β restrict to the trivial deformation on the fibres. This is the case when $\mathcal{Y} \to B \times W_Y$ is a family of holomorphic submersions with rigid fibres, for example.

We next prove that (4.14) is closed and positive definite. Consider for each $t \in W_Y$ a k-dependent inner product on $\widetilde{H}^1_{\mathcal{V}}(Y_t)$, defined using the Kähler form $\omega_{t,k}$ of Y_t:

$$\Omega_{k,t}(\alpha,\beta) := \int_{Y_t} \langle \alpha,\beta\rangle_{\omega_{t,k}}\omega_{t,k}^{n+m} = \int_{Y_t} \Lambda_{\omega_{t,k}}\mathrm{Tr}_{\omega_{t,k}}(\alpha\overline{\beta})\omega_{t,k}^{n+m}.$$

The collection $\{\Omega_{t,k}\} =: \Omega_k$ is the Weil-Petersson type Hermitian metric defined in Definition 1.29 for any family of smooth polarised varieties. Using Lemma 4.7, we can write in local holomorphic coordinates

$$\langle \alpha, \beta \rangle_{\omega_{t,k}} = \alpha^a{}_{\bar{q}} \overline{\beta^b{}_{\bar{p}}} (\omega_{t,k})_{a\bar{b}} (\omega_{t,k})^{\bar{q}p} = \alpha^a{}_{\bar{b}} \overline{\beta^b{}_{\bar{a}}} + k^{-1} \Lambda_{\omega_B} \mathrm{Tr}_{\omega_{t,F}}(\alpha \overline{\beta}) + O\left(k^{-2}\right).$$

The expansion in powers of k of $\omega_{t,k}$ reads

$$\omega_{t,k}^{n+m} = k^n \omega_t^m \wedge \omega_B^n + \frac{k^{n-1}}{nm} \omega_t^{m+1} \wedge \omega_B^{n-1} + O\left(k^{n-2}\right).$$

Then

$$\begin{aligned}\Omega_{t,k}(\alpha, \beta) = k^n \int_{Y_t} \alpha^a{}_{\bar{b}} \overline{\beta^b{}_{\bar{a}}} \omega_t^m \wedge \omega_B^n + k^{n-1} \Bigg[\int_{Y_t} \alpha^a{}_{\bar{b}} \overline{\beta^b{}_{\bar{a}}} \omega_t^{m+1} \wedge \omega_B^{n-1} \\ + \int_{Y_t} \Lambda_{\omega_B} \mathrm{Tr}_{\omega_{t,F}}(\alpha \overline{\beta}) \omega_t^m \wedge \omega_B^n \Bigg] + O\left(k^{n-2}\right).\end{aligned} \tag{4.16}$$

Then the two terms in the sum (4.14) are the first and third coefficients of this expansion.

We next describe a fibre integral formula for the Weil-Petersson metric on W_Y. Let $\omega_{\mathcal{Y},k}$ be the relatively Kähler metric on W_Y such that its restriction to each Y_t is the Kähler metric $\omega_t + k\omega_B$, where ω_t is an optimal symplectic connection on $Y_t \to B$. Let also $\rho_{\mathcal{Y},k}$ be the curvature of the Hermitian structure induced by $\omega_{\mathcal{Y},k}$ on the relative anticanonical bundle

$$-K_{\mathcal{Y}/W_Y} = \bigwedge^{n+m} \mathcal{V}_{\mathcal{Y}/W_Y},$$

where $\mathcal{V}_{\mathcal{Y}/W_Y}$ denotes the vertical tangent bundle of the fibration $\mathcal{Y} \to W_Y$. From the fibre integral formula of Theorem (1.30), the k-dependent $(1,1)$-form Ω_k can be written as a fibre integral over the map $\mathcal{Y} \to W_Y$:

$$\Omega_k(\omega_{\mathcal{Y},k}) = -\int_{\mathcal{Y}/W_Y} \rho_{\mathcal{Y},k} \wedge \omega_{\mathcal{Y},k}^{n+m} + \frac{1}{n+m+1} \int_{\mathcal{Y}/W_Y} \mathrm{Scal}_{\mathcal{V}}(\omega_{\mathcal{Y},k}) \omega_{\mathcal{Y},k}^{n+m+1}. \tag{4.17}$$

By expanding $\Omega_k(\omega_{\mathcal{Y},k})$ in powers of k, we can find a fibre integral formula for the Weil-Petersson metric (4.14). Since the base B is fixed, the relative metric $\omega_{\mathcal{Y},k}$ can be written as

$$\omega_{\mathcal{Y},k} = \widehat{\omega} + k\omega_B,$$

were the restriction of $\widehat{\omega}$ to each Y_t is the optimal symplectic connection ω_t. Then $\rho_{\mathcal{Y},k}$ can be expanded in powers of k as

$$\begin{aligned}\rho_{\mathcal{Y},k} &= i\partial\bar{\partial} \log\left(k^n \widehat{\omega}^m \wedge \omega_B^n + O\left(k^{n-1}\right)\right) \\ &= i\partial\bar{\partial} \log\left(\widehat{\omega}^m \wedge \omega_B^n\right) + O\left(k^{-1}\right) \\ &= i\partial\bar{\partial} \log \det(\widehat{\omega}) + i\partial\bar{\partial} \log \det(\omega_B) + O\left(k^{-1}\right),\end{aligned}$$

where the second line follows from the fact that $\widehat{\omega}^m \wedge \omega_B^n$ is a volume form. The k^{-1}-term is exact, because the two volume forms $\omega_{\mathcal{Y},k}^{m+n}$ and $\widehat{\omega}^m \wedge \omega_B^n$ both induce the class $c_1(-K_{\mathcal{Y}/W_Y})$. Moreover, $\det(\widehat{\omega})$ is the relative determinant of $\widehat{\omega}$ and $i\partial\bar{\partial} \log \det(\widehat{\omega})$ is the curvature of the Hermitian metric induced by $\widehat{\omega}$ on the relative anticanonical bundle $-K_{\mathcal{Y}/B\times W_Y}$. To compute the expansion of the vertical scalar curvature, Proposition 3.1 gives the expansion

$$\mathrm{Scal}(\omega_k) = \widehat{S}_b + k^{-1}\left(\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha_{\mathrm{WP}} + p_E(\Theta(\omega))\right) + O\left(k^{-3/2}\right),$$

where $\widehat{S}_b$ is the average scalar curvature of the fibres and α_{WP} is the Weil-Petersson metric on B induced by the relatively cscK degeneration $X \to B$. Then, since $\widehat{\omega}$ is an optimal symplectic connection when restricted to each Y_t, the vertical scalar curvature of $\omega_{\mathcal{Y},k}$ admits an expansion as

$$\mathrm{Scal}_{\mathcal{V}}(\omega_{\mathcal{Y},k}) = \widehat{S}_b + k^{-1}\left(\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\widehat{\alpha}\right) + O\left(k^{-2}\right),$$

where $\widehat{\alpha}$ is a closed two-form on B. Then the leading order term of (4.17) is

$$I_0 = -\int_{\mathcal{Y}/W_Y} i\partial\bar{\partial}\log\det(\widehat{\omega}) \wedge \widehat{\omega}^m \wedge \omega_B^n + \frac{1}{n+m+1}\int_{\mathcal{Y}/W_Y} \widehat{S}_b\widehat{\omega}^{m+1} \wedge \omega_B^n$$

The sub-leading order term is the sum of the four integrals

$$\begin{aligned} I_1 &= -\int_{\mathcal{Y}/W_Y} i\partial\bar{\partial}\log\det(\widehat{\omega}) \wedge \widehat{\omega}^{m+1} \wedge \omega_B^{n-1}, \\ I_2 &= \frac{1}{n+m+1}\int_{\mathcal{Y}/W_Y} \widehat{S}_b\widehat{\omega}^{m+2} \wedge \omega_B^n, \\ I_3 &= -\frac{1}{n}\int_{\mathcal{Y}/W_Y} \mathrm{Scal}(\omega_B)\widehat{\omega}^{m+1} \wedge \omega_B^n, \\ I_4 &= \frac{1}{n+m+1}\int_{\mathcal{Y}/W_Y} \left(\mathrm{Scal}(\omega_B) - \Lambda_{\omega_B}\widehat{\alpha}\right)\widehat{\omega}^{m+1} \wedge \omega_B^n. \end{aligned} \tag{4.18}$$

We can use this expansion to prove a fibre integral formula in our setting.

Lemma 4.19. *The Weil-Petersson metric $\Omega_{WP}(\omega_{\mathcal{Y}})$ (4.14) can be written as the fibre integral*

$$I_0 + I_2 + I_3 + I_4. \tag{4.19}$$

Proof. It follows from [31, Lemma 8.5] applied to the family $\mathcal{Y} \to B \times W_Y$ that the two-form defined as the collection of the integrals

$$\int_{Y_t} \alpha^a{}_{\bar{b}}\overline{\beta^b{}_{\bar{a}}}\omega_t^{m+1} \wedge \omega_B^{n-1}$$

is equal to I_1. Indeed, working locally in $D \subset B \times W_Y$, Fujiki and Schumacher prove that one can trivialise the family $\mathcal{Y}$ over D as $Y_{0,0} \times D$ such that the horizontal distribution induced by $\widehat{\omega}$ is preserved. Then, given a family $\{\beta_t\}$ of vertical deformations of the complex structure of $Y_{0,0}$ which represent the family $Y_{0,0} \times D \to D$, [31, Lemma 8.5] gives the equality

$$i\partial\bar{\partial}\log\det(\widehat{\omega}) = \mathrm{Tr}(\partial_t\beta_t|_{t=0}\overline{\partial_t\beta_t}|_{t=0}),$$

where $\partial_t\beta_t$ is the map (4.13). $\square$

Lemma 4.20. *The two-form Ω_{WP} is closed and positive-definite on W_Y.*

Proof. Since the map (4.13) is injective, the integral (4.14) is positive. To prove closedness, we show that the terms I_0, I_2, I_3 and I_4 in Lemma 4.19 are closed. The terms I_0 and I_2 are closed because they are the fibre integrals of a closed form. The term I_4 can be written as

$$\int_{\mathcal{Y}/W_Y} (\rho_B - \widehat{\alpha}) \wedge \widehat{\omega}^{m+1} \wedge \omega_B^{n-1},$$

where ρ_B is the Ricci form of ω_B. In particular $(\rho_B - \widehat{\alpha}) \wedge \omega_B^{n-1}$ is a top degree form on B, hence it is closed. So I_4 is closed. Analogously, I_3 is closed. □

Let $h_{WP}(W_Y)$ be the Hermitian metric on the tangent bundle to W_Y induced by the two-form Ω_{WP}.

Theorem 4.21. *The Hermitian metric $h_{WP}(W_Y)$ induces a global Kähler metric on the moduli space $\mathcal{M}$.*

Proof. The theorem follows from the fact that the action of the finite group $\mathrm{Aut}(\pi_Y)$ is induced by an automorphism of the Kuranishi family, so the Weil-Petersson metric $h_{WP}(W_Y)$ is invariant for the action of $\mathrm{Aut}(\pi_Y)$. Therefore it defines a metric on the quotient $W_Y/\mathrm{Aut}(\pi_Y)$. □

Remark 4.22. We have defined a Weil-Petersson metric on W_Y that is independent of the adiabatic parameter k. This is reasonable from the point of view of describing the moduli space of fibrations using the optimal symplectic connection alone, which is only relatively Kähler. However, it is possible that a different kind of a Weil-Petersson type metric could be defined by taking a sequence of Kähler metrics that depend on k, where the adiabatic construction plays a bigger role.

4.4.1 The determinant line bundle for the Weil-Petersson metric

We construct a line bundle on W_Y, and hence on the moduli space, such that the Weil-Petersson metric represents its first Chern class. To do so, we appeal to the theory of Deligne pairings [22], [23], [5, §1]. Let $M \to B$ be a flat, projective morphism between complex algebraic varieties of relative dimension d and consider $d+1$ line bundles $L_0, ..., L_d$ on M. The push-forward of the intersection product of $L_0, ..., L_d$ is an isomorphism class of line bundles on B, represented by the cohomology class

$$\int_{M/B} c_1(L_0) \wedge \cdots \wedge c_1(L_d). \tag{4.20}$$

The *Deligne pairing* of $L_0, \ldots, L_d$, denoted by $\langle L_0, \ldots, L_d \rangle_{M/B}$, is a canonical choice of a line bundle on B such that (4.20) is its first Chern class. The construction is symmetric, multilinear and functorial. Moreover, if $h_0, \ldots, h_d$ are Hermitian metrics on $L_0, \ldots, L_d$ respectively, the theory provides a metric $\langle h_0, \ldots, h_d \rangle_{M/B}$ on $\langle L_0, \ldots, L_d \rangle_{M/B}$. Denoting by $\omega_0, \ldots, \omega_d$ the curvature forms of $h_0, \ldots, h_d$ respectively, the curvature of $\langle h_0, \ldots, h_d \rangle_{M/B}$ is given by the fibre integral

$$\int_{M/B} \omega_0 \wedge \cdots \wedge \omega_d. \tag{4.21}$$

The fibre integral formula for the Weil-Petersson metric on W_Y of Lemma 4.19 is a special case of the expression (4.21). To describe it as the curvature form of a line bundle on W_Y, we first recall the following result of Fujiki and Schumacher for the k-dependent Weil-Petersson metric.

Proposition 4.23 ([31, §9]). *The k-dependent Weil-Petersson type Kähler metric $\Omega_k(\omega_{\mathcal{Y},k})$ represents the first Chern class of the line bundle*

$$-\langle -K_{\mathcal{Y}/W_Y}, \mathcal{L}_k^{n+m}\rangle_{\mathcal{Y}/W_Y} + \frac{1}{n+m+1}\frac{-K_Y\cdot(H_Y+kL)^{n+m-1}}{(H_Y+kL)^{n+m}}\langle \mathcal{L}_k^{n+m+1}\rangle_{\mathcal{Y}/W_Y}, \tag{4.22}$$

where the constant

$$\widehat{S}_Y := \frac{-K_Y\cdot(H_Y+kL)^{n+m-1}}{(H_Y+kL)^{n+m}}$$

is the average scalar curvature of Y, and hence of each Y_t, with respect to the metric $\omega + k\omega_B$.

We use Proposition 4.23 to prove the following.

Proposition 4.24. *There exists a line bundle $\mathcal{D}(Y)$ on W_Y whose first Chern class is represented by the Weil-Petersson metric* (4.14).

Proof. Let $\widehat{\mathcal{H}} \to \mathcal{Y}$ be the relatively ample line bundle induced by each relative polarisation $H_t \to Y_t$. More precisely, we consider the fibration $\mathcal{Y} \to W_Y$ as the composition of

$$\mathcal{Y} \to B\times W_Y \to W_Y$$

so $\widehat{\mathcal{H}}$ is relatively ample with respect to the first projection. Moreover, since the base B is fixed, we can pull-back L to $\mathcal{Y}$ and consider it as a line bundle on $\mathcal{Y}$. So we can define a relatively ample line bundle $\mathcal{L}_k$ over $\mathcal{Y}$ as

$$\mathcal{L}_k = \widehat{\mathcal{H}}\otimes kL, \tag{4.23}$$

whose fibre over Y_t is $\mathcal{L}_k|_t = H_t\otimes kL$. Its first Chern class contains the relative metric $\omega_{\mathcal{Y},k}$, where $\widehat{\omega}$ is in $c_1(\widehat{\mathcal{H}})$. We define a line bundle $\mathcal{D}(Y)\to W_Y$ by using the fibre integral formula (4.19) and the expansion in powers of k of the line bundle (4.22). Expanding in k the intersection product $\mathcal{L}_k^{n+m}$, $\mathcal{L}_k^{n+m+1}$ and the expression (4.22) we obtain that the Weil-Petersson metric $\Omega_{WP}(\omega_{\mathcal{Y},k})$ represents the first Chern class of the line bundle $\mathcal{D}(Y)$ given as the tensor product of the line bundles

$$\begin{aligned}
\mathcal{D}_0(Y) &= -\langle -K_{\mathcal{Y}/B\times W_Y}, \widehat{\mathcal{H}}^m, L^n\rangle_{\mathcal{Y}/W_Y},\\
\mathcal{D}_2(Y) &= \frac{1}{n+m+1}\frac{-K_{Y/B}\cdot L^n\cdot H_Y^{m-1}}{L^n\cdot H_Y^m}\left(\langle\widehat{\mathcal{H}}^{m+1}, L^n\rangle_{\mathcal{Y}/W_Y} + \langle\widehat{\mathcal{H}}^{m+2}, L^{n-1}\rangle_{\mathcal{Y}/W_Y}\right),\\
\mathcal{D}_3(Y) &= -\langle -K_B, \widehat{\mathcal{H}}^{m+1}, L^{n-1}\rangle_{\mathcal{Y}/W_Y},\\
\mathcal{D}_4(Y) &= \frac{1}{n+m+1}\frac{\left(-K_{Y/B}\cdot H_Y^{m-1}\cdot L^n\right)\left(H_Y^{m+1}\cdot L^{n-1}\right)}{H_Y^m\cdot L^n}\langle\widehat{\mathcal{H}}^{m+1}, L^n\rangle_{\mathcal{Y}/W_Y},
\end{aligned}$$

defined using the Deligne pairing. Indeed, by expanding the expression (4.22) in powers of k we see that the sum of the leading order term and the sub-leading order term is given by

$$\mathcal{D}_0(Y)+\mathcal{D}_1(Y)+\mathcal{D}_2(Y)+\mathcal{D}_3(Y)+\mathcal{D}_4(Y),$$

where $\mathcal{D}_1(Y)$ is given by

$$\mathcal{D}_1(Y) = -\langle -K_{\mathcal{Y}/B\times W_Y}, \widehat{\mathcal{H}}^{m+1}, L^{n-1}\rangle_{\mathcal{Y}/W_Y}.$$

However, its first Chern class is represented by the term (4.18), which does not appear in the fibre integral formula for the relative Weil-Petersson metric of Lemma 4.19. This concludes the proof. □

We have constructed a line bundle $\mathcal{D}_Y$ on W_Y whose first Chern class is the Weil-Petersson metric $\Omega_{WP}(\omega_{\mathcal{Y},k})$. Let now $p : W_Y \to \mathcal{M}$ be the composition of the maps

$$W_Y \to W_Y/\mathrm{Aut}(\pi_Y) \hookrightarrow \mathcal{M},$$

where the first map is the quotient by the group action and the second map is the inclusion of a local chart in the moduli space. It follows from Lemma 4.14 that the orbits of the $\mathrm{Aut}(\pi_Y)$-action on W_Y correspond to isomorphic manifolds in the family $\mathcal{Y} \to W_Y$. Therefore the line bundle $\mathcal{D}(Y)$ is invariant for the action of $\mathrm{Aut}(\pi_Y)$ and thus descends to a line bundle $\widetilde{\mathcal{D}}(Y)$ on the quotient $W_Y/\mathrm{Aut}(\pi_Y)$.

Theorem 4.25. *There exists a line bundle $\mathcal{F}$ on the moduli space $\mathcal{M}$ such that its restriction to each chart $W_Y/\mathrm{Aut}(\pi_Y)$ it is isomorphic to $\widetilde{\mathcal{D}}(Y)$.*

Proof. Let $W_{Y_1}/\mathrm{Aut}(\pi_{Y_1})$ and $W_Y/\mathrm{Aut}(\pi_Y)$ be two local charts of $\mathcal{M}$ with non empty intersection. Then using completeness of the Kuranishi space there exists an isomorphism $\widetilde{\alpha}$: $W_{Y_1}/\mathrm{Aut}(\pi_{Y_1}) \to W_Y/\mathrm{Aut}(\pi_Y)$ (4.12) which preserves the relative polarisation (4.23) and the submersions onto the base B. By functoriality of the Deligne pairings, the pull-back $\widetilde{\alpha}^*\widetilde{\mathcal{D}}(Y_1)$ is then isomorphic to $\widetilde{\mathcal{D}}(Y)$. Therefore, on the intersection of $W_Y/\mathrm{Aut}(\pi_Y)$ and $W_{Y_1}/\mathrm{Aut}(\pi_{Y_1})$ there is a morphism of line bundles $\chi : \widetilde{\mathcal{D}}(Y_1) \xrightarrow{\sim} \widetilde{\mathcal{D}}(Y)$. Let $\varphi_Y : \widetilde{\mathcal{D}}(Y) \to W_Y/\mathrm{Aut}(\pi_Y) \times \mathbb{C}$ and $\varphi_{Y_1} : \widetilde{\mathcal{D}}(Y_1) \to W_{Y_1}/\mathrm{Aut}(\pi_{Y_1}) \times \mathbb{C}$ be local trivialisations and, on the intersection,

$$\psi_{Y_1 Y} := \varphi_Y \circ \chi \circ \varphi_{Y_1}^{-1}.$$

The map $\psi_{Y_1 Y}$, viewed as a function on $\mathbb{C}$ is invertible, with inverse $\psi_{Y Y_1}$. Indeed, the map χ is an isomorphism because $\widetilde{\alpha}$ is. The same argument proves that the cocycle condition holds. □

The following corollary is a consequence of Theorem 4.25 and of [31, Proposition 1.7].

Corollary 4.26. *Any compact analytic subspace of $\mathcal{M}$ is projective.*

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