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Dissipative solutions to Hamiltonian system, and one conjecture for non autonomous viscous conservation laws and one in measure theory

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Chapter 1

Introduction

In this thesis, we treat two problems related to evolutionary PDE and one problem on metric measure spaces theory.

1.0.1 Dissipative solutions to Hamiltonian systems

The first problem concerns the Lagrangian solutions for an infinite dimensional Hamiltonian system. Since in this class of solutions the energy is dissipated, we will call as dissipative solutions: they are the closure of sticky particle solutions with respect to the weak topology of probability measure. Due to the results of nonexistence and nonuniqueness of sticky particle solution in multidimension [1], the dissipative solutions are a suitable setting for studying problems regarding sticky solutions. The dissipative solutions solve a wide range of systems of equations, among which the pressureless Euler solutions, they gradient flow solutions for Hamiltonian ODE's in the Wasserstein space [5] (see conservative solutions in Section 2.3), they extend the work [7] on dissipative solutions for multi-dimensional pressureless Euler and the work of Hynd [20] on sticky particle solutions with semiconvex potential.

We consider the family of Hamiltonians $H : \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ defined as

$$H(\mu) = \int V(q,p)\mu(dqdp) + \frac{1}{2}\int\int W(q,p,q',p')\mu(dq',dp') \times \mu(dqdp),$$

where $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $W : (\mathbb{R}^d \times \mathbb{R}^d)^2 \to \mathbb{R}$ are functions such that with the assumptions:

1. $V(0) = W(0) = 0$, $\nabla V(0) = 0$, $\nabla W(0) = 0$ and $W(q,p,q',p') = W(q',p',q,p)$;
2. V, W are convex $C^{2,1}$-function
3. V is strictly convex with respect to the first variable.

Note that the first condition comes from the third law of Newton.

We define the projection operator as follows. Denote $\Gamma = L^2((0,T),\mathbb{R}^d)$ and let

$$\mathscr{M}(\Gamma) = \left\{ \eta \in \mathscr{P}_2(L^2((0,T),\mathbb{R}^d)) : \frac{1}{2}\int |\gamma(0)|^2 \eta(d\gamma) \leq 1,\ \frac{1}{2}\int\int_0^T |\dot\gamma(t)|^2 dt\eta(d\gamma) \leq 1 \right\}$$

be a subset of probabilities with a finite second moment. Let $\mathbb{T}_t : \Gamma \to \Gamma$ be the restriction map $\mathbb{T}_t(\gamma) = \gamma\llcorner_{[t,T]}$ and $\eta \in \mathcal{M}$ a fixed probability. Then $\mathbb{P}_t : [L^2(\Gamma,\eta)]^d \to [L^2(\Gamma,\eta,\Omega_t)]^d$ is the *projection* with respect to the algebra $\Omega_t = \mathbb{T}_t^{-1}\big(\mathcal{B}(\Gamma_t)\big)$. This projection can be explicitly computed, it corresponds to the conditional expectation on the curves with the same trajectories on time $[t,T]$.

We say that $\eta \in \mathscr{M}(\Gamma)$ is a *dissipative solution* with initial speed $v_0 \in [L^2(\Gamma,\eta)]^d$ if there is a function $v \in [L^2(\Gamma,\eta,\Omega_t)]^d$, playing the role of the *momentum*, such that for $\mathscr{L}^1$-a.e. t

$$\dot\gamma(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t),\gamma'(t),v(t,\gamma),v(t,\gamma'))\eta(d\gamma'),$$

$$v(t,\gamma) = \mathbb{P}_t\Big(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s),v(s,\gamma))ds - \int_0^t\int \nabla_q W(\gamma(s),\gamma'(s),v(s,\gamma),v(s,\gamma'))\eta(d\gamma')ds\Big).$$

The second equation implies that two particles meeting at time t may either pass through each other or merge, according to the projection $\mathbb{P}_t$. If the projection is the identity, the energy is conserved and we are obtaining the unique conservative solution. The opposite case is when the projection is related to the evaluation map $e_t(\gamma) = \gamma(t)$: the particles are now forced to merge whenever they occupy the same position, the total energy is now dissipated and we recover the sticky particle solutions (if they exist). In the general case $\mathbb{P}_t$ specifies the fraction of the particles merging at a given time; also in this case the total energy is dissipating. See Figure 1.1 for an example of conservative, sticky, and dissipative solutions.

The main result can be summarized as follows:

Theorem 1.0.1. *The following holds*

1. *For every initial datum there exists a dissipative solution. In particular, there is always a unique conservative solution. Also, the Wasserstein distance between a dissipative solution η and the only conservative solution with the same initial datum η_{cons} is bounded by the square root of the energy dissipation, up to a constant:*

$$W_2(\eta,\eta_{cons}) \leq C\sqrt{E(0)-E(T)}, \qquad \text{where } E(t) = H((\gamma(t),v(t))_\sharp\eta).$$

2. *The set of dissipative solutions is weakly compact, and it coincides with the weak closure of the set of sticky particle solutions made of finitely many particles.*

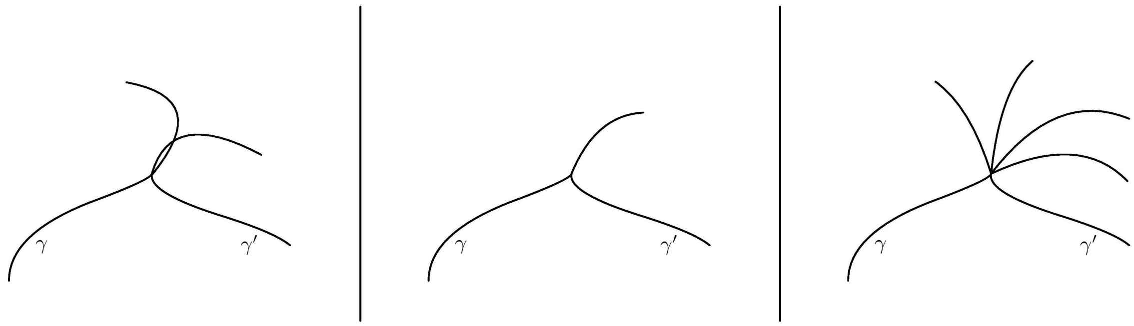


Figure 1.1: from left to right, examples of a conservative solution, a sticky solution and a dissipative solution with the same initial data.

3. *There is a G_δ dense set of initial data such that the unique dissipative solution is the conservative one.*

The uniqueness of the conservative solution is a classical result for convex potential. Here the technical condition of convexity of the potential is crucial, as shown in the following theorem.

Theorem. *For $V = \frac{v^2}{2}$, there exists $W(x - x')$ non-convex defined on $\mathbb{R}$ and discrete initial datum such that there are two distinct conservative solutions.*

In the case of a potential is a quadratic form, that is when force is linear, if an initial datum admits a unique dissipative solution, then such a solution is concentrated on a family of pairwise disjoint trajectories. If the potential is growing locally more than quadratic, then such a condition is not valid anymore, as shown in the following theorem.

Theorem. *There is an initial datum with a unique dissipative solution, i.e. the conservative one, which is not a sticky particle solution: the solution is not concentrated on a set of non-intersecting curves.*

The results involve convex potentials, while the counterexample of non uniqueness considers an internal potential with fast oscillating second derivative. The semi-convex potential case is under investigation. In particular, a result of existence/uniqueness/stability is lacking.

1.0.2 Scalar conservation laws with viscosity

The second research addresses a blowup conjecture of [19], concerning nonlinear non-autonomous conservation laws with viscosity. We consider the problem

$$\begin{cases} u_t + (b(t,x)u^{k+1})_x = u_{xx}, & x \in \mathbb{R},\, t \in (0,\infty), \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

This is a scalar conservation law with viscosity, where the dissipation term is spreading the mass on the whole space, while the drift can try to collect all the mass in one point. The finite time blow-up/global existence will depend on the choice of b and k. In [19] the authors studied the previous problem in the case b Lipschitz. Their idea is to study the L^p-norm of the solution. By using the Gagliardo–Nirenberg–Sobolev inequality they estimate $\frac{d}{dt}\|u(t)\|_p$ in terms of $\|u(t)\|_{\frac{p}{2}}$ and by an iterative method, they proved estimate

$$\|u(T)\|_\infty \le \bar{q}^{\frac{1}{2\bar{q}-k}} \max\left\{\|u_0\|_\infty, \left(\sup_{t\in[0,T]} \mathrm{Lip}(b(t))\right)^{\frac{1}{2\bar{q}-k}} \left(\sup_{t\in[0,T]} \|u(t)\|_{\bar{q}}\right)^{\frac{2\bar{q}}{2\bar{q}-k}}\right\}, \tag{1.2}$$

for any $\bar{q} \ge 1$. This proves the global existence, taking $\bar{q} = 1$, in the case $k < 2$. In the same paper, the authors show a numerical simulation where the L^∞-norm is not controlled. They ended the paper with three open questions. In particular, the last one is the following:

"Is it possible to guarantee global existence for solutions of the problem (1) when $k \ge 2$."

A method used in [26] shows that for $k > 2$, there is a blow-up in finite time for certain solutions. The idea is considering the "energy" $E(u) = \int x^2 u$ and, by considering a specific field b, estimating

$$\frac{d}{dt}E \le 2m - C\frac{m^\alpha}{E^\gamma}, \quad m = \|u\|_1 \text{ mass},$$

for certain $\alpha, \gamma > 0$. For m big enough the energy goes to zero at a finite time, which means that all the mass is concentrated in $\{x = 0\}$ and the solution blows up. In this thesis, the study is extended to the two following distinct cases:

1. b is only integrable:

$$b \in L^\infty_{\text{loc}}((0,\infty), L^{p,\infty}(\mathbb{R})),\ p \in (2,\infty], \qquad \begin{cases} k > 0, \\ u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}); \end{cases}$$

2. b has a weak derivative in x:

$$\begin{cases} b \in L^\infty_{\text{loc}}((0,\infty) \times \mathbb{R}), \\ b_x \in L^\infty_{\text{loc}}((0,\infty), L^{p,\infty}(\mathbb{R})),\ p \in (1,\infty], \end{cases} \qquad \begin{cases} k > \frac{1}{2}, \\ u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ \int |x|^2 |u_0(x)|\, dx < +\infty. \end{cases}$$

The critical values for the blow-up become

$$\mathbf{crit}(p) = \begin{cases} 1 - 1/p & b \in L^{p,\infty}, \\ 2 - 1/p & b_x \in L^{p,\infty}. \end{cases}$$

Note that $\mathbf{crit}(p) = 2$ for $b_x \in L^\infty$. The result of the blow-up/global existence can be summarized as follows.

Theorem. *Assume p is critical ($p = \mathbf{crit}(p)$) or subcritical ($p < \mathbf{crit}(p)$). Every solution is globally defined. In the supercritical case ($p > \mathbf{crit}(p)$) there are initial solutions with a finite time blow-up, that is*

$$\lim_{t\to T} \|u(t)\| = +\infty, \qquad T \in (0, +\infty).$$

This theorem answers the question of [19], showing a different result with respect to what authors conjectured and their numerical simulation showed.

The main idea for the critical case is to study a rescaled solution about the blow-up point at time T. Writing $t = T(1 - e^{-\tau})$, the rescaled functions

$$\begin{aligned} v(\tau, y) &= \sqrt{T}e^{-\tau/2}u\Big(T(1-e^{-\tau}), \sqrt{Te^{-\tau}}y\Big),\\ \tilde{b}(\tau, y) &= (Te^{-\tau})^{\frac{1-k}{2}}b\Big(T(1-e^{-\tau}), \sqrt{Te^{-\tau}}y\Big), \end{aligned}$$

solve the equation

$$v_\tau + \frac{1}{2}(yv)_y + (\tilde{b}v^{1+k})_y = v_{yy}.$$

The equation involves an additional term that is crucial for the following estimates. By studying the derivative of the "entropy"

$$\eta_a(v) = \begin{cases} v^2/2 & 0 \le v \le a,\\ a\,(v - a/2) & a < v < \infty, \end{cases}$$

for a small enough, using Gagliardo–Nirenberg–Sobolev estimate, it can be proved that

$$\liminf_{t\to\infty} \|v\|_\infty = 0.$$

Again by Gagliardo–Nirenberg–Sobolev, it can be achieved the decay

$$\|v(\tau)\|_2^2 \le Ce^{-\frac{\tau}{2}}$$

that leads to the estimate

$$\|u(T(1-e^{-\tau}))\|_2 \le \frac{C}{\sqrt{T}}.$$

Finally, by adapting the estimate (1.2) of [19], the global existence holds.

In the same article, the long behaviour of the solution is studied. In the subcritical case, the L^∞-norm decay of the solution is not guaranteed. In the critical case, the dissipative term dominates the drift and the solution decades as the heat equation. In the supercritical case, the solution requires enough mass in order to blow up in a finite time. If it does not happen and the solution is uniformly bounded, again the solution decades as the heat equation. This can be summarized as follows.

Theorem. *In the critical case*

$$\|u\|_\infty \leq \frac{C}{\sqrt{T}},$$

with C *depending on* $u_0, \|b\|_p$ *(resp.* $\|b_x\|_p$*). In the supercritical, every uniformly bounded solution* $u \leq A$ *satisfies*

$$\|u\|_\infty \leq \frac{\hat{C}}{\sqrt{T}}$$

with $\hat{C}$ *depending on* $u_0, A, k, \|b\|_p$ *(resp.* $\|b_x\|_p$*). In the subcritical case, there are time-independent drifts* $b(x)$ *such that* (1) *admits time independent solutions.*

The multidimensional case is now under investigation, in some particular cases. The difficulty is that, in the critical case, we use the Gagliardo–Nirenberg–Sobolev inequality to estimate the L^∞-norm of the rescaled solution in terms of the mass and the L^2-norm of the derivative. Such an estimate is no longer valid in higher dimensions.

1.0.3 On the Hausdorff Measure of $\mathbb{R}^n$

The last part of the thesis answers a question raised by David H. Fremlin about the Hausdorff measure of $\mathbb{R}^2$ with respect to a distance inducing the Euclidean topology. More specifically the question is the following:

let us consider a metric ρ *on* $\mathbb{R}^2$ *inducing the Euclidean topology,*
is it possible that $\mathcal{H}^2_\rho\left(\mathbb{R}^2\right) = 0$*?*

This problem is taken from a list of problems in Measure Theory proposed by Fremlin, see https://www1.essex.ac.uk/maths/people/fremlin/answer.pdf. In particular, we prove that the Hausdorff n-dimensional measure of $\mathbb{R}^n$ is never 0 when considering a distance inducing the Euclidean topology. This result is achieved by the use of Brouwer degree theory and technical tools of measure theory.

1.1 Structure of the thesis

Part I

The second chapter is organized as follows.

In Section 2.2 we list the notation we will use throughout the chapter: it is in general standard notation in analysis. Section 2.2.1 collects the properties of the Hamiltonian function we require in this thesis, while Section 2.2.2 lists some elementary properties of the space of curves used here.

The conservative flow is studied in Section 2.3: it is mainly a collection of known results, or results which are fairly easy to prove, some of their proofs are collected in Appendix 2.A.

Less standard (but still elementary) is the analysis of initial data for which the trajectories are not crossing: this is done in Section 2.3.1, where it is shown that we can perturb an initial data (in case splitting particles) and get that for a fixed small interval the trajectories are not intersecting (Proposition 2.3.8.

The key part of the chapter begins in Section 2.4. Here the definition of the dissipative solution is given, Definition 2.4.1, and it is shown that it enjoys some properties: it has finite energy (Lemma 2.4.5), it enjoys a concatenation property (Lemma 2.4.6), and $v(t)$ belongs to $\mathrm{BV}_t^{1/2} L^2_\eta$ (Lemma 2.4.7). The latter property gives that the incremental ratio $\frac{\gamma(t+s)-\gamma(t)}{s}$ converges as $s \searrow 0$ to $\dot\gamma$ uniformly in $L^2_{\mathscr{L}^1 \times \eta}$ (Lemma 2.4.8).
These estimates are necessary to construct a forward piecewise conservative approximation to a given dissipative solution, Proposition 2.4.13. There are two key estimates here: the choice of the time interval where the dissipation is small (Lemma 2.4.10), and the comparison between the projection of the conservative solution and the dissipative solution (Lemma 2.4.11 and Corollary 2.4.12).
The nice estimate showing that the distance between the dissipative solution and the conservative one is proportional to the square root of the dissipation of energy is in Proposition 2.4.16, Section 2.4.2, while the analysis of the case where H is purely quadratic (and then the ODEs (2.1) are linear) is in Section 2.4.3, where it is shown that the conservative evolution and the projection $\mathbb{P}_t$ commute. In the same section it is shown that, in this case, the fact that the unique dissipative solution is the conservative one implies that the conservative trajectories are disjoint, Proposition 2.4.19.

Section 2.5 uses standard arguments to deduce that if $\eta_n, v_n(t,\gamma)$ is a family of dissipative solutions converging to $\eta, v(t,\gamma)$ weakly, then the measures $(\gamma(t), \dot\gamma, v_n(t,\gamma))_\sharp \eta_n$ converges weakly to $(\gamma(t), \dot\gamma, v(t,\gamma))_\sharp \eta$, Proposition 2.5.1. This gives a proof of the compactness of dissipative solution, Theorem 2.5.2, which is simpler w.r.t. the proof of the analogous result contained in [7].

Section 2.6 concerns the construction of approximations to a dissipative solution made of finitely many sticky particles. The approach is standard and follows the ideas of [7]. First, if we give final data (position and speed of the particles) and finite set of times t_i and functions $\Upsilon_i(\gamma)$ with $\mathbb{P}_{t_i}(\Upsilon_i) = 0$, one can construct a backward dissipative solution by alternating the backward conservative flow in $[t_i, t_{i+1})$ and requiring that at t_i the projection $\mathbb{P}_{t_i}$ is acting as

$$v(t_i, \gamma) - v(t_i-, \gamma) = \Upsilon_i.$$

In other words, we are specifying the times and the action of the projection. Lemma 2.6.2 shows that this construction is coherent and the result is a dissipative solution.
In the rest of the section one finds suitable times t_i and functions $\Upsilon_i(\gamma_i)$ to obtain the desired approximation to a dissipative solution: first requiring that $\mathbb{P}_t$ acts only at the time t_i (Proposition 2.6.4), then requiring that the approximating dissipative is made of finitely many particles (Proposition 2.6.6), and finally that the backward trajectories of these particles are not intersecting (Proposition 2.6.8).

The result of this section, together with the weak closure of the family of dissipative solutions, implies that the weak closure of the sticky particle solutions is the set of dissipative solutions, Theorem 2.6.9: this shows that it is the natural set for studying this kind of problems.

The last section, Section 2.7, concerns the fact that the set of initial data for which the only dissipative solution is the conservative one is a residual set, Theorem 2.7.1. Its proof uses a quite standard argument, once it is known that $\mu \mapsto H(\mu)$ is l.s.c. (in this section V, W are assumed to be convex), the set of finite particle solutions such that the trajectories are not intersecting is dense (Proposition 2.3.6), and that if the trajectories are non intersecting then the unique solution is the dissipative one (Lemma 2.4.17).

The appendix contains the proof of some elementary facts about the conservative solutions (Appendix 2.A), an example of non-uniqueness for the conservative flow if the Hamiltonian does not satisfy the assumptions of Section 2.2.1 (Appendix 2.B), and an example where the trajectories of the conservative solution are intersecting, but nevertheless the unique solution is the conservative one (giving a counterexample to the converse of Proposition 2.3.8, Appendix 2.C).

Part II

The plan of the third Chapter is the following.

In Section 3.2 we introduce some definitions, notations, and well known results on Lorentz spaces (Section 3.2.1): comparisons, embeddings, interpolations, Hölder's inequalities and convolutions estimates. Since we are using multiplication/convolutions operators and embedding, the Lorentz space setting more or less gives the same estimates as for L^p. We also recall a special case of Gagliardo-Niremberg inequality and prove a useful estimate on the heat kernel.

In Section 3.3 we recall the local existence and uniqueness of the solutions via Duhamel's principle. The assumption $k \geq \frac{1}{2}$ and that $E(u_0) < \infty$ enters only in this section, and are needed if we let b be unbounded. The results in this section are standard, and independent of the main theorems of the thesis.

The main idea of the second part of the thesis is contained in Section 3.4, where we deal with the global existence of the solutions. Differently from the approach of [19], we use a standard rescaling the solutions about the blow-up point at time T, Section 3.4.1. By means of energy estimates for the truncated solution (Lemma 3.4.1) and Gagliardo-Niremberg inequality (Lemma 3.4.2), we show that in the new variables $(\tau, y) \in \mathbb{R}^+ \times \mathbb{R}$ that the rescaled solution decays the L^2-norm as the Heat kernel $\tau^{-\frac{1}{4}}$ (Lemma 3.4.3). This fact will lead to a uniform estimate on the L^2-norm of the original solution u (Corollary 3.4.4), and by adapting the estimate [19, Theorem 3.8] to our case we deduce that $\|u\|_\infty$ is bounded, Theorem 3.4.5 of Section 3.4.4. This concludes the proof of the first part of Theorem 3.1.1. In Section 3.4.5 we study the critical cases, and show that the fact that the norm of b remains constant under rescaling leads to a uniform decay rate, Theorem 3.4.7.

In Section 3.5 we provide examples of solutions blowing up in finite time for k above the critical value. The ideas are taken from [26] and adapted to our situation. Theorem 3.5.2 corresponds to the second part of Theorem 3.1.1 for $k > 1 - \frac{1}{p}$ and $b(t) \in L^{p,\infty}$, while the other case $k > 2 - \frac{1}{p}$ and $b_x(t) \in L^{p,\infty}$ is in the statement of Theorem 3.5.4. As observed already in [26], we notice in Remark 3.5.5 that the L^1-norm of the initial data cannot be too small, otherwise blow-up is not possible.

In Section 3.6 we discuss the long behavior of solutions, proving Theorem 3.1.2. The proof of the main result of this section, Theorem 3.6.1, gives examples of bounded solutions in the subcritical case, and by adapting the analysis of the decay for the critical case we obtain that the solution decay in the critical, or in the supercritical case if we assume that $u \in L^\infty_{t,x}$.

Part III

In Section 4.1 we recall the definition of the Hausdorff measure, and the definition and some properties of the Brouwer Degree.

The last section contains the main idea of the chapter: given a distance ρ on $\mathbb{R}^n$ topologically equivalent to the Euclidean topology, we build a homotopy between the map $\mathrm{id} : (B(0,1), \rho) \to (B(0,1), d_{eucl})$ and a Lipschitz map. Using such homotopy and the Brouwer Degree theory, we prove the existence of a Lipschitz map with the image containing an open set. This allow to conclude that $\mathcal{H}^n_\rho(\mathbb{R}^n) > 0$.

Part I

Chapter 2

Dissipative solutions to Hamiltonian systems

We extend the notion of dissipative particle solutions [7] to the case of Hamiltonian flow in the space of probability measures $\mu \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ in the sense of [5], see also [8]. The Hamiltonian is of the form

$$H(\mu) = \int V(q,p)\mu(dqdp) + \frac{1}{2}\int\int W(q,p,q',p')\mu(dqdp)\mu(dq'dp'),$$

with at most quadratic growth, so that a conservative flow

$$\dot{q} = \nabla_p V + \int \nabla_p W \mu, \quad \dot{p} = -\nabla_q V - \int \nabla_q W \mu$$

is uniquely defined.

The dissipative solution is defined by requiring that the equation of p is replaced by

$$p(t) = \mathbb{P}_t\bigg(p(0) + \int_0^t \bigg[-\nabla_q V - \int \nabla_q W \mu\bigg] ds\bigg).$$

where $\mathbb{P}_t$ is the projection on the space of functions corresponding to the restriction map

$$\mathbb{T}_t \gamma = \gamma \mathbb{1}_{s>t}.$$

Equivalently the particles merge preserving the average momentum p.

We obtain several results on the structure of dissipative solutions; among them, regularity, dissipation of energy, approximations with finite particles solutions, density of conservative solutions. The proofs require additional technical difficulties, not present in the analysis of [7] where $H(q,p) = p^2/2$.

2.1 Introduction

We consider the Hamiltonian function

$$H(\mu) = \int V(q,p)\mu(dqdp) + \frac{1}{2}\int\int W(q,p,q',p')\mu(dqdp)\mu(dq'dp'),$$

where V, W are smooth semiconvex functions with quadratic growth, W symmetric, and $\mu \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is a probability measure with finite quadratic moments. It is known [5] that for every $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ there is a unique solution $t \mapsto \mu(t)$ to the Hamiltonian flow

$$\partial_t \mu(t) + \mathrm{div}_{p,q}\big(\mathrm{J}\nabla H(\mu(t))\mu(t)\big) = 0, \quad \mu(0) = \mu_0.$$

where J is the symplectic matrix

$$\mathrm{J} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \qquad \mathbb{I} \;\; d \times d\text{-identity matrix},$$

and

$$\nabla H(\mu) = \nabla H(q,p;\mu) = \nabla V(q,p) + \int \nabla W(q,p,q',p')\mu(dq'dp').$$

At the level of trajectories (Q, P) in the phase space, these are solutions to

$$\begin{aligned} Q(t,q,p;\mu_0) = q &+ \int_0^t \nabla_v V(Q(s,q,p;\mu_0), P(s,q,p;\mu_0))ds \\ &+ \int_0^t \int \nabla_v W\big(Q(s,q,p;\mu_0), P(s,q,p;\mu_0), Q(s,q',p';\mu_0), P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds, \end{aligned} \tag{2.1a}$$

$$\begin{aligned} P(t,q,p;\mu_0) = p &- \int_0^t \nabla_q V(Q(s,q,p;\mu_0), P(s,q,p;\mu_0))ds \\ &- \int_0^t \int \nabla_q W\big(Q(s,q,p;\mu_0), P(s,q,p;\mu_0), Q(s,q',p';\mu_0), P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds. \end{aligned} \tag{2.1b}$$

In this thesis, we study the existence and stability of solutions when the particles are "sticky", i.e. they are allowed to merge (thus dissipating energy but preserving momentum) if they occupy the same position q: this leads to the notion of dissipative solution.

The prototype example is the sticky particle system, where the Hamiltonian is simply

$$H(\mu) = \int \frac{p^2}{2}\mu(dqdp). \tag{2.2}$$

In this case, the conservative solution is made of straight lines

$$\dot{q}(t) = p(t) + q(0), \quad p(t) = p(0).$$

The condition for sticky particle solution is that the momentum p is conserved when particles merge, i.e. in the case of finitely many particles with mass m_i colliding at time $\bar{t}$

$$q_i(\bar{t}) = q, \; i = i_1, \dots, i_n \quad \Rightarrow \quad \sum_{j=i_1}^{i_n} m_j p_i(\bar{t}-) = p_i(\bar{t}+) \sum_{j=i_1}^{i_n} m_j, \; i = i_1, \dots, i_n.$$

In one space dimension, the above condition is suitable to single out a unique sticky particle solution, see [12, 11, 14, 18, 23] and the references therein for an overview of the results.
In [13], it is shown that when the space dimension d is strictly greater than 1 the existence and uniqueness of a sticky particle solution is in general false: the correct notion that preserves the compactness of solutions is the notion of *dissipative solutions* introduced in [7]. The main results of [7] are that dissipative solutions form a weakly compact set w.r.t. the narrow convergence, and the study of the generality of dissipative solutions.
The fundamental difference between a sticky particle solution and a dissipative solution is that in the first case, if particles occupy the same point in space-time, then they are forced to merge into a single particle; for dissipative solutions, instead, particles do not need to merge, but if they do the conservation of momentum is required. Figure 2.1 shows the different behavior to the three families of solutions (conservative, sticky and dissipative) in the case of initial data made of 2 particles: observe that dissipative solutions allow interactions of fraction of the initial particles.
A simple example of dissipative solutions in 1d is

$$\mu(t, dqdp) = \begin{cases} \frac{\delta_{(t,1)}(dqdp) + \delta_{(-t,-1)}(dqdp)}{2} & t < 0, \\ \frac{1-\alpha}{2}\delta_{(-t,-1)}(dqdp) + \frac{\alpha+\beta}{2}\delta_{(\frac{\beta-\alpha}{\beta+\alpha}t, \frac{\beta-\alpha}{\beta+\alpha})}(dqdp) + \frac{1-\beta}{2}\delta_{(t,1)}(dqdp) & t \geq 0, \end{cases} \tag{2.3}$$

with $\alpha, \beta \in [0, 1]$. One can think that only a fraction α of the first particle decides to merge with a fraction β of the second, resulting in a particle traveling with speed $\frac{\beta-\alpha}{\alpha+\beta}$: the sticky particle solution is obtained when $\alpha = \beta = 1$, and the conservative for $\alpha = \beta = 0$.

The main result of this chapter is the extension of the results of [7] to a general semiconvex Hamiltonian case with quadratic growth.

Theorem 2.1.1. *Assume that $V(q,p)$ is semiconvex and uniformly convex in p, $W(q,p,q',p')$ is convex and symmetric, i.e. $W(q,p,q',p') = W(q',p',q,p)$. Then the following holds.*

1. *The set of dissipative solutions is not empty, and contains all conservative solutions.*

2. *The set of dissipative solutions is weakly compact and coincides with the weak closure of the set of sticky particle solutions made of finitely many particles (Theorem 2.5.2).*

3. *There is a G_δ-set of initial data such that the unique dissipative solution is the conservative one (Theorem 2.7.1).*

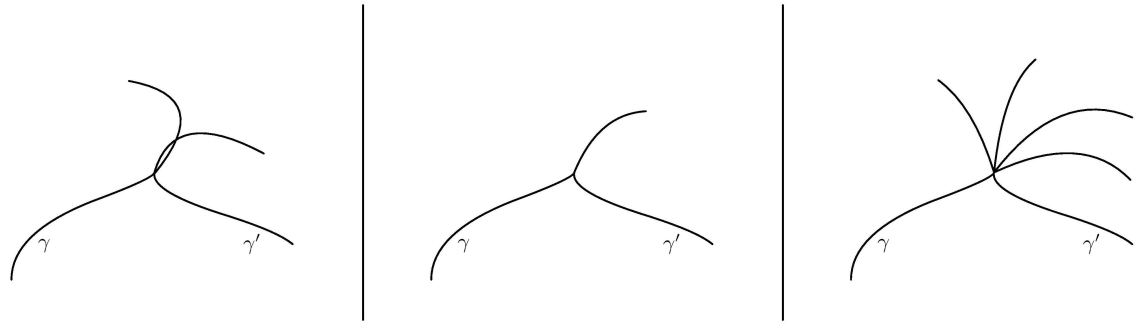


Figure 2.1: from left to right, examples of a conservative solution, a sticky solution and a dissipative solution with the same initial data.

We remark that the first point follows immediately from the very definition of dissipative solutions, Definition 2.4.1.

These results are the exact extension of the results for the sticky particle case. Nevertheless, their proof requires much more effort for the following reason.
For the Hamiltonian (2.2) (and more in general for purely quadratic Hamiltonian, i.e. when the ODEs (2.1) are linear), the dissipative solution at time t can be computed directly from the conservative solution as follows: the position and momentum of merged particles can be obtained by taking the average position and average momentum (i.e. applying the projection $\mathbb{P}_t$ defined below at the conservative flow, Proposition 2.4.18).
For the nonlinear case, the above property is clearly false, and then one has to rely on the integral formulation of the dissipative solutions (Definition 2.4.1): the correct definition of the dissipative solution is actually the central point of the chapter, and we give it here below.

Consider the space of $W^{1,2}$-curves in $\mathbb{R}^d$, and define the family of projections

$$W^{1,2}([0,T],\mathbb{R}^d) \ni \gamma \mapsto \hat{\mathbb{P}}_t\gamma(\tau) = \gamma(t)\mathbf{1}_{\tau<t} + \gamma(\tau)\mathbf{1}_{\tau\geq t}.$$

Let Ω_t be the smallest σ-algebra such that $\hat{\mathbb{P}}_t$ is measurable, and let $\mathbb{P}_t$ be the projection acting on L^2_η. Then $\eta \in \mathbb{P}_2(\Gamma)$ is a dissipative solutions if there is a function $v \in L^2_{\mathscr{L}^1\times\eta}((0,T)\times\Gamma,\mathbb{R}^d)$ such that for $\mathscr{L}^1$-a.e. t

$$\dot{\gamma}(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t),\gamma'(t),v(t,\gamma),v(t,\gamma'))\eta(d\gamma'), \tag{2.4a}$$

$$v(t,\gamma) = \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s),v(s,\gamma))ds - \int_0^t\int \nabla_q W(\gamma(s),\gamma'(s),v(s,\gamma),v(s,\gamma'))\eta(d\gamma')ds\bigg). \tag{2.4b}$$

Following the Hamiltonian nomenclature, the function $v(\cdot,\gamma)$ will be called the momentum of the particle with trajectory γ at time t. The uniform convexity of V in p implies that the first equation, Equation (2.4a), is the graph of Lipschitz function, relating $\dot{\gamma}(t)$ uniquely with $v(t,\gamma)$. The evolution is then described by the second equation, Equation (2.4b), and the projection acts only on this one: it describes the conservation of momentum.

We end this introduction by listing some additional technical results, which have an interest on their own.

- The map $t \mapsto v(t,\gamma)$ belongs to $\mathrm{BV}^{1/2}_t L^2_\eta$ (Lemma 2.4.7): note that the conservative solution has $t \mapsto p(t)$ in $W^{1,2}_\mu$, i.e. the projection reduces the regularity but it still preserves some regularity w.r.t. t. In particular, the incremental ratio $\frac{\gamma(t+s)-\gamma(t)}{s}$ converges to $\dot{\gamma}(t)$ in $L^2_{\mathscr{L}^1 \times \eta}$ (Lemma 2.4.8).

- It is possible to approximate a dissipative solution by alternating the conservative flow and the projection $\mathbb{P}_t$ (Proposition 2.4.13 of Section 2.4.1). This is in some sense the natural idea of a dissipative solution: the particles travel by the conservative flow, and then some of them interact and merge. This approximation collects the interaction at a finite number of times t_i. It is important to remark here that by just throwing in a family of projections, one is not going to construct an approximation of a dissipative solution: indeed the limit is not in general conservative, since the approximation we construct needs to project also the positions $\gamma(t)$. Another approximation we present below will actually construct dissipative solutions.

- In the sticky particle model (or in linear case i.e. H purely quadratic), the dissipation of energy $E(t) = H((\gamma(t), v(t))_\sharp \eta)$ is immediate since the projection commutes with the conservative flow. Here the same result holds, together with the fact that the distance between the conservative solution and the dissipative solution is controlled by $\sqrt{E(s) - E(t)}$; the analysis is however more complicated and requires some preliminary estimates (Section 2.4.2 and Proposition 2.4.16).

- As for the sticky particle systems, in the linear case requiring that the trajectories of the conservative solution are disjoint is a necessary condition in order to have that the only dissipative solution for a given initial data is the conservative one (Proposition 2.4.19). This is however false for the general case, and in Appendix 2.C an explicit example is worked out. We observe here that the first example in [13] shows that the non-crossing of trajectories is not sufficient, if the number of particles is not finite (in the latter case the analysis becomes trivial).

- The compactness of dissipative solutions is exactly the same result as for the sticky particle case [7]. Here (Section 2.5), the proof is slightly simplified: it is based on the use of the $\mathrm{BV}^{1/2}_t L^2_\eta$ compactness of $v(t)$ to prove the weak convergence of $v_n(t)_\sharp \eta_n$, where $\eta_n, v_n(t)$ is a sequence of solutions converging to η (Proposition 2.5.1).

- The construction of a backward dissipative approximation (made only of finitely many particles and also being a sticky particle solution, not just a dissipative solution) to any given dissipative solution is the same as in [7]: the only variation is that since the conservative flow is nonlinear, some additional estimates are needed to assure that there is an arbitrarily small perturbation such that the trajectories are now disjoint in some time interval. The proof of the main result here, Proposition 2.6.8, is indeed based on some elementary properties of the conservative flow, which we present in Section 2.3.1.

- The last result on the genericity of initial data such that the only dissipative solution is the conservative one (Theorem 2.7.1) is exactly as in [7], and follows easily from the analysis of the previous sections.

Finally, it is not clear to us how much of this theory can be extended to convex Hamiltonian with super-quadratic growth: at least we would like to have that the conservative flow is unique and conserves energy. A counterexample to the uniqueness of the dissipative flow is presented in Appendix 2.B, but the Hamiltonian is not even semiconvex.

2.2 Definitions, assumptions, and notations

Some general notation.

- We will work in the space $(t,x) \in [0,T] \times \mathbb{R}^d$ or $(t,q,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.
- In a metric space $(X,\hat{d})$ the ball or radius r about $x \in X$ is written as $B_r^X(x)$. Sometimes $B_r(x)$ if the space is clear from the context.
- The symplectic matrix J is

$$\mathrm{J} = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix} \in \mathbb{R}^{2d\times 2d}. \tag{2.5}$$

- The norm in $\mathbb{R}^d$ is $|\cdot|$, and its scalar product by $(\cdot,\cdot)$. More generally, $\|\cdot\|$ and $(\cdot,\cdot)$ will denote respectively the norm and scalar product on a Hilbert space (here most of the time L^2_ν for some measure ν), and which space is under consideration will be usually clear from the context.
- The letters s,t,τ are reserved for time variable, x,y,z for the space variable in $\mathbb{R}^d$, and for the Hamiltonian variables we will use $(q,p) \in \mathbb{R}^d \times \mathbb{R}^d$. The capital letters X,Y,Z denote the coordinates of N-particles in $\mathbb{R}^d$, and (Q,P) the Hamiltonian coordinates for N-particles; we will also use Q,P in case of a countable or a continuum family of particles. The time interval in which we consider the solution is $[0,T]$. To differentiate variables we will use $x',\tilde{x},\hat{x},\dots$, and the same for $y,z,q,p,X,Q,P,\dots$.
- We write x_i for the i-th component of $x = (x_1,x_2,\dots)$.

- A generic constant is C (if supposedly large), or c (if supposedly small). We use them if their value depends only on some parameters of the problem, if we do not care about this dependence we will use the symbol $\mathcal{O}(1), o(1)$.

- The letter $v = v(t,\gamma)$ is reserved for the Hamiltonian coordinate P on the space of path $t \mapsto \gamma(t) \in \mathbb{R}^d$. The evaluation map $t, \gamma \mapsto \gamma(t)$ will be denoted with $\gamma(t)$ (this slightly differs from the standard notation $e(t)$).

- Since we are in $\mathbb{R}^d$, we will not distinguish between gradient and differential, noting both with ∇_x, ∂_x. For Lipschitz curves $t \mapsto \gamma(t) \in \mathbb{R}^d$ we will use the special notation $d\gamma/dt = \dot\gamma$, and more generally

$$\frac{d}{dt}\phi(t,\gamma(t)) = \dot\phi(t,\gamma(t)).$$

- We will use the Greek letter μ for a measure on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, η for a measure of the space of path $\Gamma = L^2((0,T),\mathbb{R}^d)$, $\mathscr{L}$ for the Lebesgue measure. Sometimes we will parameterize curves as $t \mapsto x(\alpha,t), Q(\alpha,t), P(\alpha,t), \dots$ with $\alpha \in [0,1]$, and we will use the measure $\varpi(d\alpha)$ on the space of parameters: in general, we can take $\varpi = \mathscr{L}^1 \llcorner_{[0,1]}$. A generic measure will be denoted by ν, π, the product of measures will be denoted by $\cdot \times \cdot$.

- To avoid confusion, we underline that when the momentum p of a trajectory γ if considered as a function of the trajectory γ itself, we use the notation $v(t,\gamma)$. When the curve γ is parametrized by $\alpha \in [0,1]$, we use $Q(t,\alpha), P(t,\alpha)$. For special solutions (e.g. discrete in-time dissipative solution), we use the notation X, Y. The choice of using different symbols for equivalent quantities is in order to stress the different domains of definition.

- The push-forward of a measure ν according to a map $\mathbb{T} : X \to Y$ is denoted with $\mathbb{T}_\sharp \nu$.

- The disintegration of a measure ν according to a Borel map $\mathbb{T} : A \to B$, A, B Polish space, is written as

$$\nu = \int \nu_y m(db), \quad m = \mathbb{T}_\sharp \nu.$$

  We often interpret m as the restriction of ν to the σ-algebra $\mathbb{T}^{-1}(\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra on Y, and we will also write

$$\nu = \int \nu_{\mathbb{T}(a)} \nu(da) = \int \nu_a \nu(da).$$

- We will use the notation $L^2_\nu(X,Y)$, Y Hilbert space, for the space of functions $f : X \to Y$ such that

$$\int \|f(x)\|^2_Y \nu(dx) < \infty.$$

- $\mathscr{P}_2(X)$, X Banach space, is the set of probability measures with finite quadratic moments in X. The topology of $\mathscr{P}_2(X)$ is either the narrow topology or the Wasserstein-2 distance W_2.

- The space $\mathrm{BV}^{1/2}([0,T],X)$, X Banach, is defined as the functions $f:[0,T]\mapsto X$ such that
$$\sup\left\{\sum_{i=1}^{N}\|f(t_i)-f(t_{i-1})\|_X^2, 0\le t_0\le t_1\le\cdots\le t_N\le T\right\}<\infty.$$
We will often shorten the notation to $\mathrm{BV}^{1/2}_{\mathscr{L}^1}X$ or $\mathrm{BV}^{1/2}_t X$, e.g. $\mathrm{BV}^{1/2}_t L^2_\eta$.

- A projection operator is denoted by $\mathbb{P}$, with some index in case of dependence from a parameter or to denote the target space.

2.2.1 Hamiltonian

The assumptions in this section are standard, and give the well-posedness of the conservative solution (see Section 2.3, and for more general Hamiltonians see [5, 20]).

We consider the family of Hamiltonians $H:\mathscr{P}_2(\mathbb{R}^d\times\mathbb{R}^d)\to\mathbb{R}$ defined as

$$H(\mu)=\int V(q,p)\mu(dqdp)+\frac{1}{2}\int\int W(q,p,q',p')\mu(dq'dp')\times\mu(dqdp),$$

where $V:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $W:(\mathbb{R}^d\times\mathbb{R}^d)^2\to\mathbb{R}$ are functions such that with the assumptions:

1. $V(0)=W(0)=0$, $\nabla V(0)=0$ and $\nabla W(0)=0$;

2. $W(q,p,q',p')=W(q',p',q,p)$;

3. V,W are functions of class $C^{2,1}$ with first derivatives L-Lipschitz:
$$\left|\nabla V(q,p)-\nabla V(q',p')\right|\le L|(q,p)-(q',p')|, \tag{2.6a}$$
$$\left|\nabla W(x,v,x',v')-\nabla W(y,w,y',w')\right|\le L\left|(x,v,y,w)-(y,w,y',w')\right|; \tag{2.6b}$$

4. there exists $\lambda\ge 0$ such that $(x,v)\mapsto V(x,v)+\lambda|x|^2/2$ is convex, and by (2.6a) it has at most quadratic growth;

5. $(x,v)\mapsto W(x,v,x',v')$ is convex;

6. for all (x,v) it holds
$$\Lambda\mathbb{I}\le\nabla_{pp}V(x,v)\le L\mathbb{I},\qquad \mathbb{I}\text{ identity matrix in }\mathbb{R}^d,$$
i.e. V is uniformly convex w.r.t. v.

Here above and in the following we will denote the differentials of V, W w.r.t. the q, p, q', p' components as $\nabla_q, \nabla_p, \nabla_{q'}, \nabla_{p'}, \nabla_{qq}, \nabla_{pp}, \dots$.

For the measure $\varpi(d\alpha)$ and the map $(\alpha, t) \mapsto (Q(\alpha,t), P(\alpha,t))$, we will use the notation

$$H(t,\varpi) = H(\mu(t)), \quad \mu(t) = (q(t), p(t))_\sharp \varpi,$$

i.e. more explicitly

$$\begin{aligned} H(t,\varpi) = & \int V(Q(\alpha,t), P(\alpha,t))\varpi(d\alpha) \\ & + \frac{1}{2}\int\int W(Q(\alpha,t), P(\alpha,t), q(t,\alpha'), p(t,\alpha'))\varpi(d\alpha)\times\varpi(d\alpha'). \end{aligned}$$

A similar notation will be used for the gradient of H in $\mathscr{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ w.r.t. the Wasserstein distance: we recall that the gradient of a semiconvex functional $H : P_2(\mathbb{R}^d\times\mathbb{R}^d) \to \mathbb{R}$ is the element of minimal L^2_μ-norm in the set of sub-differentials $\partial H(\mu)$ (a reference to the definition of sub-differential can be found in Chapter 10 of [4]). In our case, being the functions V, W of class $C^{2,1}$, it is fairly easy to verify that the gradient of H in $\mathscr{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ is given by

$$\nabla H(\mu) = \nabla V(q,p) + \int \nabla W(q,p,q',p')\mu(dq'dp'),$$

and that $\nabla H(t,\varpi) = \nabla H(\mu(t))$, $\mu(t) = (q(t),p(t))_\sharp\varpi$: we shorten the notation for $\nabla W(q,p,q',p') = (\nabla_q W(q,p,q',p'), \nabla_p W(q,p,q',p'))$.

2.2.2 Probability space of curves

To shorten the notation we will write

$$\Gamma := L^2((0,T),\mathbb{R}^d), \qquad \Gamma_t := L^2((t,T),\mathbb{R}^d),$$

and denote by $\mathbb{T}_t : \Gamma \to \Gamma_t$ the *restriction map*

$$\mathbb{T}_t(\gamma) = \gamma \llcorner_{[t,T]}.$$

The measures η which we are going to consider will be supported on the set

$$\mathscr{M}(\Gamma) = \left\{\eta \in \mathscr{P}(\Gamma) : \frac{1}{2}\int |\gamma(0)|^2\eta(d\gamma) \le \tilde{C}_1, \ \frac{1}{2}\int\int_0^T |\dot\gamma(t)|^2 dt\eta(d\gamma) \le \tilde{C}_2\right\},$$

for some constant $\tilde{C}_1$, $\tilde{C}_2$ which will be estimated explicitly for the solutions (conservative or dissipative) we consider below. It is standard to verify that $\mathscr{M}(\Gamma)$ is a compact subset of $\mathscr{P}_2(\Gamma)$ w.r.t. the narrow convergence. Notice that the bound on $\int\int_0^T |\dot\gamma(t)|^2 dt\eta(d\gamma)$ implies that γ is η-a.e. absolutely continuous (a.c. in the following), so that the value $\gamma(0)$ is well defined η-a.e. giving sense to the first condition in the definition of $\mathscr{M}(\Gamma)$.

Alternatively, observe that $\eta \in \mathscr{M}(\Gamma)$ is concentrated on the space $\gamma \in W^{1,2}((0,T),\mathbb{R}^d)$: hence one can also consider the map

$$W^{1,2}((0,T),\mathbb{R}^d) \ni \gamma \mapsto \hat{\mathbb{P}}_t\gamma(\tau) = \gamma(t)\mathbf{1}_{[0,t)}(\tau) + \gamma(\tau)\mathbf{1}_{[t,T]}(\tau) \in W^{1,2}((0,T),\mathbb{R}^d).$$

One can switch from $\mathbb{T}_t$ to $\hat{\mathbb{P}}_t$ via the bijection

$$W^{1,2}((t,T),\mathbb{R}^d) \ni \gamma \mapsto \gamma(t)\mathbf{1}_{[0,t)}(\tau)+\gamma(\tau)\mathbf{1}_{[t,T]}(\tau) \in \Big\{\gamma \in W^{1,2}((0,T),\mathbb{R}^d), \dot{\gamma}(s) = 0 \; s \in (0,t)\Big\}.$$

Let Ω_t be the descending Borel fibration generated by $\mathbb{T}_t$,

$$\Omega_t = \mathbb{T}_t^{-1}\big(\mathcal{B}(\Gamma_t)\big),$$

i.e. the smallest σ-algebra such that $\mathbb{T}_t$ is Borel, and let $\mathbb{P}_t : L^2_\eta \to L^2_\eta$ be the corresponding projection. The latter functional can be represented by means of the disintegration of $\eta \in \mathscr{P}(\Gamma)$ according to the map $\mathbb{T}_t$: indeed if

$$\eta = \int \omega^t_{\gamma'}(d\gamma)\mathbb{T}_{t\sharp}\eta(d\gamma') = \int \omega_{\mathbb{T}_t(\gamma')}\eta(d\gamma'),$$

where $\Gamma \ni \gamma' \mapsto \omega_{\mathbb{T}_t(\gamma')} \in \mathscr{P}(\Gamma)$ can be taken to be Borel, then [10, Proposition 10.4.18]

$$(\mathbb{P}_t f)(\gamma) = \int f(\gamma')\omega^t_{\mathbb{T}_t(\gamma)}(d\gamma'). \tag{2.7}$$

The $\mathbb{P}_t f$ corresponds to the conditional expectation of f given all the future positions after t.

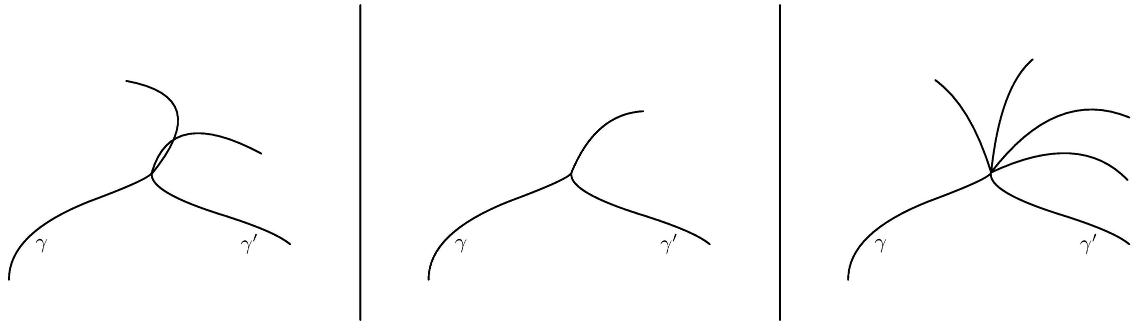
As an example, consider the situation in Figure 2.1, page 20, which for convenience we reproduce above, and let t be the first time of collision. In the conservative case (the one on the left), the measures $\omega^t_{\mathbb{T}_t(\gamma)}$ are Dirac deltas, since the conservative solutions are time-reversible and no information is lost. For the sticky solution, $\omega^t_{\mathbb{T}_t(\gamma)}$ is equal to η (all information is lost), and for the dissipative solution the conditional probabilities describe how the masses of the two entering trajectories γ, γ' is distributed across the exiting trajectories (see formula (2.3)).

The following concatenation property holds: for $s < t$, let $\mathbb{T}_{s\to t} : \Gamma_s \to \Gamma_t$ be the restriction map such that $\mathbb{T}_{s\to t} \circ \mathbb{T}_s = \mathbb{T}_t$. Then by disintegrating according to $\mathbb{T}_{s\to t}$

$$(\mathbb{T}_s)_\sharp\eta(d\gamma') = \int \omega^{s\to t}_{\gamma''}(d\gamma')(\mathbb{T}_t)_\sharp\eta(d\gamma'').$$

Hence

$$\begin{aligned}
\eta &= \int \omega^s_{\gamma'}(\mathbb{T}_s)_\sharp\eta \\
&= \int \bigg[\int \omega^s_{\gamma'}\omega^{s\to t}_{\gamma''}(d\gamma')\bigg](T_{s\to t})_\sharp((\mathbb{T}_s)_\sharp\eta)(d\gamma'') \\
&= \int \bigg[\int \omega^s_{\gamma'}\omega^{s\to t}_{\gamma''}(d\gamma')\bigg](\mathbb{T}_t)_\sharp\eta(d\gamma''),
\end{aligned}$$



and from the uniqueness of disintegration

$$\omega^t_{\gamma''}(d\gamma) = \int \omega^s_{\gamma'}(d\gamma)\omega^{s\to t}_{\gamma''}(d\gamma') \qquad (\mathbb{T}_t)_\sharp\eta\text{-a.e. } \gamma''.$$

The above formula corresponds to the composition property

$$\mathbb{P}_{s\to t} \circ \mathbb{P}_s = \mathbb{P}_t, \qquad (\mathbb{P}_{s\to t} f)(\gamma) = \int f \omega^{s\to t}_\gamma. \tag{2.8}$$

Lemma 2.2.1 ([10, Theorem 10.2.1])**.** *It holds*

$$\lim_{t\searrow s} \mathbb{P}_{s\to t} = \mathbb{P}_s$$

strongly in L^2_η.

2.3 Conservative solutions

Let $H : \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ be the Hamiltonian considered in Section 2.2.1. Here we construct the Hamiltonian flow: the results are classical, we adapt them to a suitable form that will be useful for the study of dissipative solutions. Since this flow preserves the energy, we will call it *conservative flow/solution*, as opposed to the *dissipative flow/solution* of Section 2.4.

The results in this section are pretty much standard: we give for simplicity the proofs in the appendix.

Definition 2.3.1. The measure $\eta \in \mathscr{P}_2(\Gamma)$ is a *conservative solution* if there is an $L^2(\eta)$-function $v : \Gamma \to W^{1,2}((0,T), \mathbb{R}^d)$ such that for η-a.e. γ it holds

$$\gamma(t) = \gamma(0) + \int_0^t \left[\nabla_v V(\gamma(s), v(s,\gamma)) + \int \nabla_v W(\gamma(s), v(s,\gamma), \gamma'(s), v(s,\gamma'(s))\eta(d\gamma')\right] ds, \tag{2.9a}$$

$$v(t,\gamma) = v_0(\gamma) - \int_0^t \left[\nabla_q V(\gamma(s), v(s,\gamma)) + \int \nabla_q W(\gamma(s), v(s,\gamma), \gamma'(s), v(s,\gamma'(s))\eta(d\gamma')\right] ds, \tag{2.9b}$$

where $v_0 \in L^2_\eta$.

Let $\mu(t) \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be the measure

$$\mu(t) = (\gamma(t), v(t))_\sharp \eta.$$

The conservative solution η can be interpreted as the Lagrangian representation of the measure-valued solution $\mu(t)$ to the PDE

$$\partial_t \mu(t) + \operatorname{div}(\mathrm{J}\nabla H(\mu(t))\mu(t)) = 0, \tag{2.10}$$

where J is the $2d \times 2d$ symplectic matrix (2.5). We thus are in the setting of [5] for existence and actually uniqueness of a Hamiltonian flow $t \to \mu_t \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ solving (2.10) and preserving H. The existence and uniqueness of the solution $\mu(t)$ is strictly related to the existence and uniqueness of the Lagrangian representation η: in the linear transport case this is well established [2], while in our case the fact that the measure η is a probability on trajectories $\gamma : [0,T] \to \mathbb{R}^d$ instead of $\gamma : [0,d] \to \mathbb{R}^{2d}$ follows from the fact that the time derivative of (2.9a),

$$\dot\gamma(t) = \nabla_v V(\gamma(s), v(s,\gamma)) + \int \nabla_v W(\gamma(s), v(s,\gamma), \gamma'(s), v(s,\gamma'(s))\eta(d\gamma'),$$

is a bijection between $\dot\gamma$ and $v(\gamma)$.

We give a self-contained proof of these facts in Appendix 2.A.

Define

$$\begin{array}{rcll} F(t) & : & L^2(\eta) & \to \quad L^2(\eta) \\ & & v & \mapsto \quad F(t,v)(\gamma) = \nabla_v V(\gamma(t), v(\gamma)) + \int \nabla_v W(\gamma(t), v(\gamma), \gamma'(t), v(\gamma'))\eta(d\gamma') \end{array} \tag{2.11}$$

Recalling that V is uniformly convex in v and W is convex in v, it is easy to verify that F is well defined for all t, and, moreover, the next proposition holds.

Proposition 2.3.2. *The operator* (2.11) *is uniformly monotone, namely*

$$\Lambda\|v_1 - v_2\|_2^2 \le \big(v_1 - v_2, F(t,v_1) - F(t,v_2)\big) \le 3L\|v_1 - v_2\|_2^2.$$

In particular $F(t)$ is a bi-Lipschitz map of $L^2(\eta)$ into itself. The proof is in Appendix 2.A, page 59.

The next result gives the existence, uniqueness, and continuous dependence. Let $(0,1) \ni \alpha \mapsto Q_0(\alpha), P_0(\alpha)$ be given functions in $L^2(0,1)$.

Proposition 2.3.3. *There exist unique functions* $Q(\alpha,t), P(\alpha,t) \in C_0([0,T], L^2(0,1))$ *satisfying*

$$Q(\alpha,t) = Q_0(\alpha)+\int_0^t \left[\nabla_p V(Q(\alpha,s),P(\alpha,s))+\int \nabla_p W(Q(\alpha,s),P(\alpha,s),Q(\alpha',s),P(\alpha',s)d\alpha'\right]ds,$$

$$P(\alpha,t) = P_0(\alpha)-\int_0^t \left[\nabla_q V(Q(\alpha,s),P(\alpha,s))+\int \nabla_q W(Q(\alpha,s),P(\alpha,s),Q(\alpha',s),P(\alpha',s)d\alpha'\right]ds.$$

Moreover

$$t \mapsto H(t,\mathcal{L}^1\llcorner_{(0,1)}) = \int V(Q(\alpha,t),P(\alpha,t))d\alpha+\int\int W(Q(\alpha,t),P(\alpha,t),Q(\alpha',t),P(\alpha',t))d\alpha d\alpha'$$

is constant, $(\partial_t Q(t), P(t)) \in L^2((0,1), W^{1,2}(0,T))$ *with*

$$\|\partial_t Q(t)\|_{L^2(0,1)}, \|\partial_t P(t)\|_{L^2(0,1)}, \frac{1}{3L}\|\partial_t^2 Q(t)\|_{L^2(0,1)} \leq 3Le^{3Lt}\|(Q_0,P_0)\|_{L^2(0,1)}. \tag{2.12}$$

Finally, if $(Q_0,P_0),(Q_0',P_0')$ *are two different initial data and* $(Q(t),P(t)),(Q'(t),P'(t))$ *the corresponding solutions, then it holds*

$$\big\|(Q(t),P(t))-(Q'(t),P'(t))\big\|_{L^2(0,1)} \leq e^{3Lt}\big\|(Q_0,P_0)-(Q_0',P_0')\big\|_{L^2(0,1)}.$$

Corollary 2.3.4. *The measure* $\eta = Q(\alpha,t)_\sharp \mathscr{L}^1(d\alpha)$ *is a conservative solution concentrated on* $\mathscr{M}(\Gamma)$ *with* $\tilde{C}_1, \tilde{C}_2 = 3LTe^{3LT}\|(x_0,v_0)\|^2_{L^2(0,1)}$.

The measure μ *defined as*

$$\int \phi(x,v)\mu(t;dxdv) = \int_0^1 \phi(Q(\alpha,t),P(\alpha,t))d\alpha$$

is the unique solution to the transport equation (2.10) *with initial data* $\mu(t=0)$.

Proof. The first statement follows from the definition of the conservative solution. The fact that μ is unique and solves the transport equation follows by observing that $t \mapsto (Q(\alpha,t),P(\alpha,t))$ is a characteristic for μ-a.e. α, and the uniqueness result above. □

For the flow at the level of the ODE in the phase space $\mathbb{R}^d \times \mathbb{R}^d$, we will use the following notation: for a given initial data $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ let $(Q(t,q,p;\mu_0), P(t,q,p;\mu_0))$ be the unique flow in $L^\infty((0,T), L^2_{\mu_0}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d))$ such that

$$\begin{aligned} Q(t,q,p;\mu_0) = q &+ \int_0^t \nabla_v V(Q(s,q,p;\mu_0),P(s,q,p;\mu_0))ds \\ &+ \int_0^t \int \nabla_v W\big(Q(s,q,p;\mu_0),P(s,q,p;\mu_0),Q(s,q',p';\mu_0),P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds, \end{aligned} \tag{2.13a}$$

$$P(t,q,p;\mu_0) = p - \int_0^t \nabla_q V(Q(s,q,p;\mu_0), P(s,q,p;\mu_0))ds$$
$$- \int_0^t \int \nabla_q W\big(Q(s,q,p;\mu_0), P(s,q,p;\mu_0), Q(s,q',p';\mu_0), P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds. \tag{2.13b}$$

These equations correspond to the projection on $\mathbb{R}^d \times \mathbb{R}^d$ of (2.9), as it can be easily seen because

$$(\gamma(t), v(t))_\sharp \eta = \mu_t = (Q(t), P(t))_\sharp \mu_0.$$

Using the semigroup property and the fact that (2.13) are time-independent, we have also

$$Q(t-s, Q(s,q,p;\mu_0), P(s,q,p;\mu_0); \mu_s) = Q(t,q,p;\mu_0),$$
$$P(t-s, Q(s,q,p;\mu_0), P(s,q,p;\mu_0); \mu_s) = P(t,q,p;\mu_0).$$

Remark 2.3.5. The conservative flow (Q,P) can be defined to all $\mathbb{R}^d \times \mathbb{R}^d$: indeed the solution μ_t is uniquely defined, and the vector field

$$(q,p) \mapsto \mathrm{J}\nabla H(q,p;\mu_t) = \mathrm{J}\bigg(\nabla V(q,p) + \int \nabla W(q,p,q',p')\mu_t(dq'dp')\bigg)$$

is uniformly Lipschitz. We will use the same notation $(Q,P)(t,q,p;\mu_0)$ as above for the flow extended to the whole $\mathbb{R}^d \times \mathbb{R}^d$.

2.3.1 Non-crossing of trajectories

In the following, we will need to study how generic is the crossing of trajectories of N particles satisfying the Hamiltonian ODE, i.e. solving (2.13) with μ_0 finite sum of Dirac deltas. We will use the notation

$$q_i(t, Q_0, P_0) = Q(t, q_i, p_i; \mu_0),\ \ p_i(t, Q_0, P_0) = P(t, q_i, p_i; \mu_0), \quad \mu_0 = \frac{1}{N}\sum_i \delta_{(q_i,p_i)}.$$

Proposition 2.3.6. *For conservative solutions made of N particles, the set of initial data such that at least two trajectories cross is of codimention $(d-1)$ in $(\mathbb{R}^d \times \mathbb{R}^d)^N$.*

Proof. The condition of the intersection of the particles w.l.o.g. labeled $1,2$ is

$$\big\{(Q_0, P_0) \in \mathbb{R}^d \times \mathbb{R}^d : \exists t \in \mathbb{R}\big(q_1(t,Q_0,P_0) = q_2(t,Q_0,P_0)\big)\big\}.$$

By the implicit function theorem, the condition above defines a $(d-1)$-codimensional surface if

$$\mathrm{rank}\big(\nabla_{Q_0,P_0}(q_1 - q_2)\big) = d.$$

This is implied by the divergence-free property of Hamiltonian flows

$$\det\left(\begin{bmatrix} \nabla_{Q_0,P_0}(q_1-q_2) \\ \vdots \\ \nabla_{Q_0,P_0}q_N \\ \nabla_{Q_0,P_0}p_1 \\ \vdots \\ \nabla_{Q_0,P_0}p_N \end{bmatrix}\right) = \det\left(\begin{bmatrix} \nabla_{Q_0,P_0}q_1 \\ \vdots \\ \nabla_{Q_0,P_0}q_N \\ \nabla_{Q_0,P_0}p_1 \\ \vdots \\ \nabla_{Q_0,P_0}p_N \end{bmatrix}\right) = 1. \qquad \square$$

In the following we will need to perturb a finite particle conservative solution preserving the initial position of the particles and the average speed: more precisely, the initial data are $\{q_{i,j}, p_{i,j}\}_{i,j}$ with

$$q_{i,j} = \bar{q}_i, \quad \sum_j m_{i,j}\bar{p}_{i,j} = \Big(\sum_j m_{i,j}\Big)\bar{p}_i, \tag{2.14}$$

for some constants $m_{i,j} > 0$. In other words, given $\{\bar{q}_i, \bar{p}_i\}_i$, we are allowed to split each particle $\bar{q}_i, \bar{p}_i$ into the particles $\{q_{i,j}, p_{i,j}\}_j$, assigning new initial speeds but preserving the average speed of the particles starting in the same point. The goal of this splitting is again to avoid crossing of trajectories, at least for an interval of time independent of the data $\{\bar{q}_i, \bar{p}_i\}_i, \{q_{i,j}, p_{i,j}\}_{i,j}$.

Remark 2.3.7. A simple example with two distinct particles shows that at least we have to split one trajectory.

Moreover, it is not possible to perturb the trajectories as in (2.14) requiring them not to join forever in the future (or in the past): this can be easily seen in the harmonic case $V = p^2/2$, $W = (q-q')^2/2$. We observe that the situation is different in the case of free motion, i.e. $V = v^2/2$, $W = 0$: indeed one can actually require that the particles do not meet for every $t \neq 0$.

Consider now a conservative solution made of finitely many Dirac deltas $\mu_0 = \sum_i^I m_i \delta_{\bar{q}_i,\bar{p}_i}$.

Proposition 2.3.8. *There exists a time interval $(0,\bar{t})$, independent of μ_0, such that for all $\epsilon > 0$ there is a finite particle solution*

$$\mu_0' = \sum_i \frac{m_i}{2}\big(\delta_{\bar{q}_i,p_{i,1}} + \delta_{\bar{q}_i,p_{i,2}}\big), \quad \frac{p_{i,1}+p_{i,2}}{2} = \bar{p}_i,$$

such that the trajectories $\{q_{i,j}(t)\}$ are not intersecting for $t \in (0,\bar{t})$ and

$$\big|p_{i,j} - \bar{p}_i\big| < \epsilon.$$

Hence it is sufficient to split each particle $(\bar{q}_i, \bar{p}_i)$ in half, see Figure 2.2.

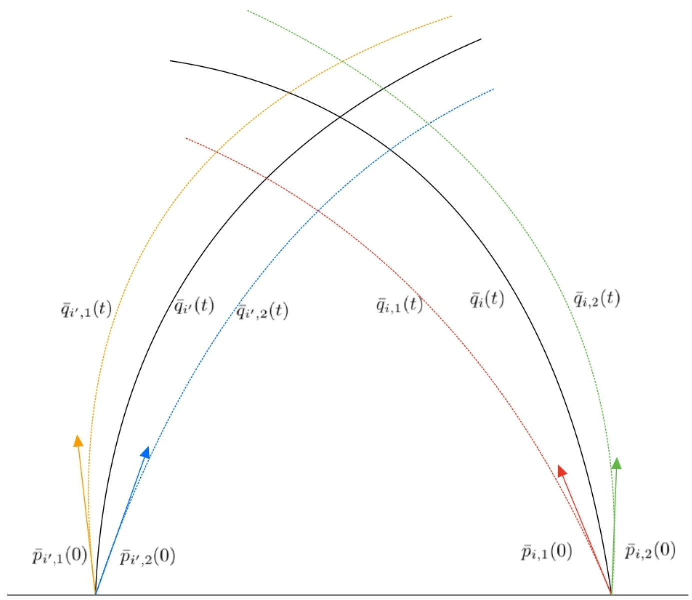


Figure 2.2: two crossing particles are perturbed as in Proposition 2.3.8 to avoid the intersection of the new trajectories.

Proof. Being the conservative flow differentiable w.r.t. the initial data, we compute the derivative of the flow $\bar{Q}(t), \bar{P}(t)$ w.r.t. a perturbation of the form

$$\delta Q_1(0) = (\delta q_{i,1}(0))_i = 0,\ \delta Q_2(0) = (\delta q_{i,2}(0))_i = 0,$$
$$\delta P_1(0) = (\delta p_{i,1}(0))_i = -(\delta p_{i,2}(0))_i = -\delta P_2(0).$$

It is easy to see that the solution satisfies $(\delta Q_1, \delta P_1) = (\delta Q_2, \delta P_2)$ and that the ODE reduces to

$$\frac{d}{dt}\begin{pmatrix} \delta Q_1 \\ \delta P_1 \\ \delta Q_2 \\ \delta P_2 \end{pmatrix} = \mathrm{diag}\big(\nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar{q}_i(t), \bar{p}_i(t); \mu_t)\big) \begin{pmatrix} \delta Q_1 \\ \delta P_1 \\ \delta Q_2 \\ \delta P_2 \end{pmatrix},$$

$$\nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar{q}_i(t), \bar{p}_i(t); \mu_t) = \begin{bmatrix} \nabla_{q_i p_i} V & \nabla_{p_i p_i} V \\ -\nabla_{q_i q_i} V & -\nabla_{q_i p_i} V \end{bmatrix} (\bar{q}_i(t), \bar{p}_i(t))$$
$$+ \int \begin{bmatrix} \nabla_{q_i p_i} W & \nabla_{p_i p_i} W \\ -\nabla_{q_i q_i} W & -\nabla_{q_i p_i} W \end{bmatrix} (\bar{q}_i(t), \bar{p}_i(t), \bar{q}_j, \bar{p}_j) \mu(t; d\bar{q}_j d\bar{p}_j),$$

so that the ODEs are decoupled: $\mu(t) = \sum_i m_i \delta_{\bar{q}_i(t), \bar{p}_i(t)}$ is the solution of (2.10) with initial data $\mu_0 = \sum_i m_i \delta_{\bar{q}_i, \bar{p}_i}$.

To prove the same codimension estimate as in the proof of the previous proposition, it is sufficient to study the ODE

$$\dot{A}_i(t) = \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar{q}_i(t), \bar{p}_i(t); \mu_t) A_i(t), \quad A_i(0) = \mathbb{I}.$$

The assumptions on V, W gives that

$$\left| \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar{q}_i(t), \bar{p}_i(t); \mu_t) \right| \leq 3L,$$

so that one obtains

$$|B_i(t)| = \left| A_i(t) - \mathbb{I} - \int_0^t \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar{q}_i(s), \bar{p}_i(s); \mu_s) ds \right| \leq \big(e^{3Lt} - 1 - 3Lt \big) < \frac{\Lambda}{2} t$$

for $t < \bar{t}$, with $\bar{t}$ depending only on L.

Hence

$$\begin{aligned} \begin{pmatrix} \delta q_{i,j}(t) \\ \delta p_{i,j}(t) \end{pmatrix} &= \left[\mathbb{I} + \int_0^t \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar{q}_i(s), \bar{p}_i(s); \mu_s) ds + B_i(t) \right] \begin{pmatrix} 0 \\ \delta P_{i,j}(0) \end{pmatrix} \\ &= \begin{pmatrix} \big[\int_0^t \nabla^2_{p_i,p_i} H(\bar{q}_i(s), \bar{p}_i(s); \mu_s) ds \big] \delta P_{i,j}(0) \\ \big[\mathbb{I} - \int_0^t \nabla_{q_i,p_i} H(\bar{q}_i(s), \bar{p}_i(s); \mu_s) ds \big] \delta P_{i,j}(0) \end{pmatrix} + B_i(t) \begin{pmatrix} 0 \\ \delta P_{i,j}(0) \end{pmatrix}. \end{aligned}$$

Since, from the uniform convexity of V,

$$\int_0^t \nabla^2_{p_i,p_i} H(\bar{q}_i(s), \bar{p}_i(s); \mu_s) ds \geq \Lambda t \mathbb{I},$$

and $|B_i| \leq \Lambda t/2$ for $t \in [0, \bar{t})$, then

$$\int_0^t \nabla^2_{p_i,p_i} H(\bar{q}_i(s), \bar{p}_i(s); \mu_s) ds + (B_i)_{1,2}$$

is invertible for $t \in (0, \bar{t})$.

We thus conclude that the crossing condition

$$q_{i,j}(t) = q_{i',j'}(t), \quad \text{for some } t \in (0, \bar{t}), j \neq k,$$

gives a $(d-1)$-codimensional manifold in a neighborhood of $q_{i,j} = \bar{q}_i$. Hence there are perturbations $p_{i,j}(0) - \bar{p}_i(0)$ arbitrarily small so that the trajectories do not cross for $t \in (0, \bar{t})$.

□

2.4 Dissipative solution

In this section, we define the dissipative solutions for the Hamiltonian system (2.9), and we show some basic properties.

Definition 2.4.1. We say that $\eta \in \mathscr{M}(\Gamma)$ is a *dissipative solution* with initial speed $v_0 \in L^2_\eta(\Gamma, \mathbb{R}^d)$ if there is a function $v \in L^2_{\mathscr{L}^1 \times \eta}((0,T) \times \Gamma, \mathbb{R}^d)$ such that for $\mathscr{L}^1$-a.e. t

$$\dot\gamma(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t), \gamma'(t), v(t,\gamma), v(t,\gamma'))\eta(d\gamma'), \tag{2.15a}$$

$$v(t,\gamma) = \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s), v(s,\gamma))ds - \int_0^t \int \nabla_q W(\gamma(s), \gamma'(s), v(s,\gamma), v(s,\gamma'))\eta(d\gamma')ds\bigg), \tag{2.15b}$$

where $\mathbb{P}_t$ is the projection (2.7).

One of the reasons for the introduction of the notion of dissipative solution is to obtain a compactness result for solutions as stated in Theorem 2.5.2. As it is known, this property is false for sticky particle solutions: an easy example is obtained by considering two particles in the plane, with the simplest Hamiltonian $H = \int \frac{p^2}{2}$. There is a 1-codimensional cone of initial velocities such that the particles are colliding, and the sticky particle solution is not in the closure of all other solutions (the closure of all other solutions is the free/conservative flow). A more complicated situation is in [13], where an infinite family of dissipative solutions can be constructed but there is no sticky particle solution.

In the next proposition, we show that the notion of dissipative solutions includes the conservative ones.

Proposition 2.4.2. *Any conservative solution is a dissipative solution.*

In particular, every conservative solution is concentrated on a set of trajectories where $\mathbb{P}_t$ coincides with the identity.

Proof. There are two observations to be used here.

1. If $\eta \in \mathscr{M}(\Gamma)$ is a conservative solution, then η is concentrated on trajectories $\gamma \in W^{2,2}((0,T), \mathbb{R}^d)$: this follows from (2.12) of Proposition 2.3.3.

2. The map $v \mapsto F(t,v)$ defined equation (2.11) satisfies

$$F(t, \mathbb{P}_t v) = \mathbb{P}_t(F(t, \mathbb{P}_t v)), \tag{2.16}$$

a property that can be easily deduced from the fact that F is a function of $\gamma(t), v(t)$ only. In particular, from Proposition 2.3.2 we deduce that if $\mathbb{P}_t F(t,v) = F(t,v)$ then

$$\begin{aligned} 0 &= \big(v(t) - \mathbb{P}_t v(t), \mathbb{P}_t(F(t,v) - F(t,\mathbb{P}_t v))\big) \\ &= \big(v(t) - \mathbb{P}_t v(t), F(t,v) - F(t,\mathbb{P}_t v)\big) \geq \Lambda \|v(t) - \mathbb{P}_t v(t)\|_2^2, \end{aligned}$$

i.e. $v(t) = \mathbb{P}_t v(t)$.

From the first point we deduce that if $\mathbb{P}_t\gamma = \mathbb{P}_t\gamma'$, $\gamma, \gamma' \in W^{2,2}((0,T))$, then $\dot\gamma(t) = \dot\gamma'(t)$, i.e. the derivative of γ exists at t and it is the same on the whole level set of $\mathbb{P}_t$: hence

$$\dot\gamma(t) = F(t,v) = \mathbb{P}_t F(t,v).$$

As a consequence, the second point above gives $v(t) = \mathbb{P}_t v(t)$, and using the definition of conservative solution we obtain

$$\begin{aligned} v(t) &= v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s), v(s,\gamma))ds - \int_0^t \int \nabla_q W(\gamma(s), \gamma'(s), v(s,\gamma), v(s,\gamma'))\eta(d\gamma')ds \\ &= \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s), v(s,\gamma))ds - \int_0^t \int \nabla_q W(\gamma(s), \gamma'(s), v(s,\gamma), v(s,\gamma'))\eta(d\gamma')ds \bigg). \end{aligned}$$

Hence for conservative solutions, $\mathbb{P}_t$ is the identity, and a conservative solution is in particular a dissipative solution. □

We observe also that differently from the conservative case where the initial data is encoded into η, here it is not: for example, considering two particles starting at the same point but with different speeds, we can just merge them at $t = 0$, so that their initial speed is different from the initial one. We will see later (Lemma 2.4.7) that, since one can take the dissipative solutions to be right continuous, we can specify the initial v_0 as the limit of $v(t)$ as $t \searrow 0$, and that in some sense the solution is characterized by the initial data *and* the family of projections $\mathbb{P}_t$ (by constructing an approximating sequence depending on the initial data and the projections, Section 2.4.1).

Remark 2.4.3. Equation (2.15a) is exactly the same as Equation (2.9a). The second equation (2.15b) expresses the requirement that when trajectories merge, the function v of the exiting trajectory will be the average of the function v of the incoming trajectories. On the case $V = v^2/2$, $W = W(x,x')$, then $\dot\gamma = v$, so that v coincides with the speed of the trajectory.

It is interesting to note that Equation (2.15a) is compatible with the projection $\mathbb{P}_t$ used in (2.15b): indeed, define the measure $\eta_{\bar t} = (\mathbb{T}_{\bar t})_\sharp \eta$, and by Proposition 2.3.2 find $\tilde v(t) \in L^2_{\eta_{\bar t}}(\Gamma_{\bar t}, \mathbb{R}^d)$ such that

$$\dot\gamma(t) = \nabla_v V(\gamma(t), \tilde v(t,\gamma)) + \int \nabla_v W(\gamma(t), \tilde v(t,\gamma), \gamma'(t), \tilde v(t,\gamma')))\eta_{\bar t}(d\gamma')$$

for $\mathscr{L}^1$-a.e. $t > \bar t$. This function $\tilde v$ is defined on the σ-algebra $\mathcal{B}(\bar t, T) \times \Omega_{\bar t}$, and writing

$$v(t,\gamma) = \tilde v(t, \mathbb{T}_{\bar t}(\gamma)),$$

one deduces immediately that

$$\dot\gamma(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t), v(t,\gamma), \gamma'(t), v(t,\gamma')))\eta(d\gamma'),$$

and Proposition 2.3.2 yields that this is the only solution for $t > \bar t$. In particular, by letting $\bar t \nearrow t$, one deduces that $v(t,\gamma(t))$ is measurable in Ω_t for $\mathscr{L}^1$-a.e. t.

Since (2.15a) gives a one-to-one relation between $\dot\gamma$ and v, we can state alternatively that

Definition 2.4.4. We say that $\eta \in \mathscr{M}(\Gamma)$ is a *dissipative solution* with initial speed $v_0 \in L^2_\eta(\Gamma, \mathbb{R}^d)$ if the function $v \in L^2_{\mathscr{L}^1\times\eta}((0,T)\times\Gamma, \mathbb{R}^d)$ given by the relation

$$\dot\gamma(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t), \gamma'(t), v(t,\gamma), v(t,\gamma'))\eta(d\gamma'),$$

satisfies

$$v(t,\gamma) = \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s), v(s,\gamma))ds - \int_0^t \int \nabla_q W(\gamma(s), \gamma'(s), v(s,\gamma), v(s,\gamma'))\eta(d\gamma')ds\bigg),$$

where $\mathbb{P}_t$ is the projection (2.7).

We begin with a rough energy estimate.

Lemma 2.4.5. *For every dissipative solution*

$$\|(\gamma(t), v(t))\|_{L^2_\eta} \le e^{3Lt}\|(\gamma(0), v_0)\|_{L^2_\eta} \quad \mathscr{L}^1\text{-a.e. } t.$$

The energy $E(t) = H((\gamma(t), v(t)_\sharp\eta)$ is actually decreasing, see Proposition 2.4.16 below. The above lemma shows that the requirement of $\eta \in \mathscr{M}(\Gamma)$ is compatible with the definition of dissipative solution as in Corollary 2.3.4, because of the relation between $\dot\gamma, v$ given by Proposition 2.3.2.

Proof. This is a standard Gronwall estimate.

Being $\mathbb{P}_t$ a contraction, we have by the Lipschitz estimates on $\nabla V, \nabla W$, Points (3), (1) of Page 24, applied to (2.15)

$$\|(\gamma(t), v(t))\|_{L^2_\eta} \le \|(\gamma(0), v_0)\|_{L^2_\eta} + 3L\int_0^t \|(\gamma(s), v(s))\|_{L^2_\eta} dr,$$

for $\mathscr{L}^1$-a.e. t, which gives the statement. □

This next lemma is a concatenation property for dissipative solutions.

Lemma 2.4.6. *It holds*

$$v(t,\gamma) = \mathbb{P}_{s\to t}\bigg(v(s,\gamma) - \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg),$$

where $\mathbb{P}_{s\to t}$ is the projection (2.8).

Proof. Using Remark 2.4.3, for $\mathscr{L}^1$-a.e. $r \geq s$

$$\begin{aligned}&\mathbb{P}_s\Big(\nabla_q V(\gamma(r), v(r,\gamma))dr + \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')\Big)\\&= \nabla_q V(\gamma(r), v(r,\gamma))dr + \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma'),\end{aligned}$$

so that

$$\begin{aligned}&\int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr + \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\\&= \mathbb{P}_s\Big(\int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr + \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big).\end{aligned} \tag{2.17}$$

Hence, directly from the definition of $v(t,\gamma)$ and $\mathbb{P}_t, \mathbb{P}_{s\to t}$

$$\begin{aligned}&\mathbb{P}_{s\to t}\Big(v(s,\gamma) - \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big)\\&= \mathbb{P}_{s\to t}\Big(\mathbb{P}_s\Big(v_0(\gamma) - \int_0^s \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_0^s \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big)\Big)\\&\quad + \mathbb{P}_{s\to t}\Big(- \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big)\\&= \mathbb{P}_t\Big(v_0(\gamma) - \int_0^s \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_0^s \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big)\\&\quad + \mathbb{P}_{s\to t}\Big(\mathbb{P}_s\Big(- \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big)\\&= \mathbb{P}_t\Big(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_0^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\Big),\end{aligned}$$

where we have used (2.17) in the third equality. □

The next estimate plays a key role in the following.

Lemma 2.4.7. *Let $\eta \in \mathscr{M}(\Gamma)$ be a dissipative solution. Then the map $t \to v(t,\gamma)$ is right continuous and belongs to* $\mathrm{BV}^{\frac{1}{2}}([0,T], L^2_\eta(\Gamma, \mathbb{R}^d)))$ *with norm*

$$\|v\|_{\mathrm{BV}_t^{1/2} L^2_\eta} \leq (1 + 6LT)e^{3LT}\|(\gamma(0), v_0)\|^2_{L^2_\eta}. \tag{2.18}$$

Let us denote

$$G(t, v(t))(\gamma) = \nabla_q H(\gamma(t), v(t); \eta) = -\nabla_q V(\gamma(t), v(\gamma)) - \int \nabla_q W(\gamma(t), \gamma'(t), v(\gamma), v(\gamma'))\eta(d\gamma').$$

The proof stems from the fact that $v(t,\gamma)$ is the integral of the L^2_η-function $\nabla_q H(\gamma(t), v(t);\eta)$ w.r.t. time (which would give the a.c. continuity), composed with projection $\mathbb{P}_t$ (which is responsible for energy dissipation and hence for the $\mathrm{BV}^{1/2}$-norm).

Proof. We compute by Lemma 2.4.5

$$\begin{aligned}
\sum_i \|v(t_i) - v(t_{i-1})\|_2^2 &= \sum_i \left[\|v(t_i)\|_2^2 + \|v(t_{i-1})\|^2_{L^2_\eta} - 2\int (v(t_i), v(t_{i-1}))\eta\right] \\
&= \sum_i \left[\|v(t_i)\|^2_{L^2_\eta} + \|v(t_{i-1})\|^2_{L^2_\eta} - 2\int (v(t_i), \mathbb{P}_{t_{i-1}\to t_i} v(t_{i-1}))\eta\right] \\
&= \sum_i \Bigg[\|v(t_i)\|^2_{L^2_\eta} + \|v(t_{i-1})\|^2_{L^2_\eta} \\
&\qquad - 2\int \left(v(t_i), \mathbb{P}_{t_{i-1}\to t_i}\left(v(t_{i-1}) + \int_{t_{i-1}}^{t_i} G(r, v(r))dr\right)\right)\eta \\
&\qquad + 2\int \left(v(t_i), \mathbb{P}_{t_{i-1}\to t_i}\left(\int_{t_{i-1}}^{t_i} G(r, v(r))dr\right)\right)\eta\Bigg] \\
&= \sum_i \left[\|v(t_{i-1})\|^2_{L^2_\eta} - \|v(t_i)\|^2_{L^2_\eta} + 2\int \left(v(t_i), \mathbb{P}_{t_{i-1}\to t_i}\left(\int_{t_{i-1}}^{t_i} G(r, v(r))dr\right)\right)\eta\right] \\
&= \|v(0)\|^2_{L^2_\eta} - \|v(T)\|^2_{L^2_\eta} + 2\sum_i \int \left(v(t_i), \mathbb{P}_{t_{i-1}\to t_i}\left(\int_{t_{i-1}}^{t_i} G(r, v(r))dr\right)\right)\eta \\
&\le \|v(0)\|^2_{L^2_\eta} - \|v(T)\|^2_{L^2_\eta} + 6LT \sup_t \|v(t)\|^2_{L^2_\eta} \\
&\le (6LT + 1)\sup_t \|v(t)\|^2_{L^2_\eta} \le (1 + 6LT)e^{3LT}\|(\gamma(0), v_0)\|^2_{L^2_\eta},
\end{aligned}$$

where in the first inequality we have used the Lipschitz estimate

$$\big\|\nabla_q H(\gamma(t), v_1(t);\eta) - \nabla_q H(\gamma(t), v_2(t);\eta)\big\|_{L^2_\eta} \le 3L\|v_1 - v_2\|_{L^2_\eta}$$

analogous to the second inequality of Proposition 2.3.2.

The $\mathrm{BV}^{1/2}_t L^2_\eta$ regularity gives immediately that the function $t \mapsto v(t)$ is strongly continuous in L^2_η outside countably many times. Moreover, as $t \searrow s$, Lemma 2.2.1 gives

$$\begin{aligned}
&\lim_{t\searrow s} \mathbb{P}_t\left(v(s,\gamma) - \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\right) \\
&= v(s,\gamma)
\end{aligned}$$

in L^2_η, which is the right continuity property. ∎

In particular, by Proposition 2.3.2 we obtain $\dot\gamma \in \mathrm{BV}^{1/2}_t L^2_\eta$ and one can take as initial data

$$v_0(\gamma) = \lim_{t \searrow 0} v(t,\gamma).$$

This will be our choice in the following.

The next results use the $\mathrm{BV}^{1/2}_t L^2_\eta$ estimate on $\dot\gamma(t), v(t)$ to deduce some useful approximation properties: these results are actually valid for generic $\mathrm{BV}^{1/2}X$ functions, we state them in the particular form we will use.

Lemma 2.4.8. *For all η dissipative, it holds*

$$\int_0^{T-s} \int \left| \frac{\gamma(t+s) - \gamma(t)}{s} - \dot\gamma(t) \right|^2 \eta(d\gamma) dt \leq \Big(2(1 + 6LT) e^{3LT} \|(\gamma(0), v_0)\|^2_{L^2_\eta} \Big) s = \mathcal{O}(1) s.$$

Proof. Write

$$\begin{aligned}
\int_0^{T-s} \int \left| \frac{\gamma(t+s) - \gamma(t)}{s} - \dot\gamma(t) \right|^2 \eta(d\gamma) dt &= \int_0^{T-s} \int \left| \frac{1}{s} \int_0^s (\dot\gamma(t+\sigma) - \dot\gamma(t)) d\sigma \right|^2 \eta(d\gamma) dt \\
&\leq \frac{1}{s} \int_0^s \int_0^{T-s} \int |\dot\gamma(t+\sigma) - \dot\gamma(t)|^2 \eta(d\gamma) dt d\sigma \\
&= \frac{1}{s} \int_0^s \left\{ \sum_{k=0}^{[T/s]-2} \int_{ks}^{(k+1)s} + \int_{([T/s]-1)s}^{T-s} \right\} \left[\int |\dot\gamma(t+\sigma) - \dot\gamma(t)|^2 \eta(d\gamma) \right] dt d\sigma \\
&= \frac{1}{s} \int_0^s \int_0^s \left[\sum_{k=0}^{[T/s]-2} \int |\dot\gamma(ks+\tau+\sigma) - \dot\gamma(ks+\tau)|^2 \eta(d\gamma) \right] d\tau d\sigma \\
&\quad + \frac{1}{s} \int_0^s \int_{([T/s]-1)s}^{T-s} \left[\int |\dot\gamma(t+\sigma) - \dot\gamma(t)|^2 \eta(d\gamma) \right] dt d\sigma.
\end{aligned}$$

The first integral is estimated as

$$\begin{aligned}
\frac{1}{s} \int_0^s \int_0^s \left[\sum_{k=0}^{[T/s]-2} \int |\dot\gamma(ks+\tau+\sigma) - \dot\gamma(ks+\tau)|^2 \eta(d\gamma) \right] d\tau d\sigma &\leq \frac{1}{s} \int_0^s \int_0^s \|\dot\gamma\|_{\mathrm{BV}^{1/2}_t(L^2_\eta)} d\tau d\sigma \\
&\leq (1 + 6LT) e^{3LT} \|(\gamma(0), v_0)\|^2_{L^2_\eta} s,
\end{aligned}$$

where we have used the estimate (2.18). Similarly to the last integral

$$\begin{aligned}
\frac{1}{s} \int_0^s \int_{([T/s]-1)s}^{T-s} \left[\int |\dot\gamma(t+\sigma) - \dot\gamma(t)|^2 \eta(d\gamma) \right] dt d\sigma &\leq \frac{1}{s} \int_0^s \int_{([T/s]-1)s}^{T-s} \|\dot\gamma\|_{\mathrm{BV}^{1/2}_t(L^2_\eta)} dt d\sigma \\
&\leq (1 + 6LT) e^{3LT} \|(\gamma(0), v_0)\|^2_{L^2_\eta} s.
\end{aligned}$$

Adding the two estimates we obtain the statement. □

Lemma 2.4.9. *For every $\epsilon > 0$ we can find finitely many times $0 = t_0 < t_1 < \cdots < t_N = T$ such that*

$$\sup_{i, t_{i-1} \leq s < t_i} \|v(s) - v(t_{i-1})\|_{L^2_\eta} \leq \epsilon.$$

Proof. Since $t \mapsto v(t)$ is right continuous, for every t there is δ_t such that

$$\sup_{0 \leq \tau < \delta_t} \|v(t+\tau) - v(t)\|_{L^2_\eta} < \epsilon, \quad \|v(t+\delta_t) - v(t)\|_{L^2_\eta} \geq \epsilon.$$

Starting from $t = 0$, define the sequence of times

$$t_{i+1} = t_{i-1} + \delta_{t_i}, \quad t_0 = 0.$$

The $\mathrm{BV}^{1/2}_t L^2_\eta$-regularity gives that

$$\epsilon \sharp\{t_i\} \leq \sum_i \|v(t_i) - v(t_{i-1})\|^2_{L^2_\eta} \leq (1 + 6LT) e^{3LT} \|(\gamma(0), v_0)\|^2_{L^2_\eta},$$

so that there are at most $\mathcal{O}(\epsilon^{-1})$-many times t_i. □

The source of the discontinuities of the map $t \mapsto v(t)$ is due to the projection $\mathbb{P}_t$: in order to get rid of it, define the function

$$\begin{aligned}
\tilde{v}(t,s,\gamma) &= v(s,\gamma) - \int_s^t \nabla_q H(\gamma(\tau), v(\tau,\gamma); \eta) d\tau \\
&= v(s,\gamma) - \int_s^t \bigg[\nabla_q V(\gamma(\tau), v(\tau,\gamma)) ds + \int \nabla_q W(\gamma(\tau), v(\tau,\gamma), \gamma'(\tau), v(\tau,\gamma')) \eta(d\gamma') \bigg] d\tau,
\end{aligned}$$

so that it holds for all $s \in [0,t]$

$$v(t,\gamma) = \mathbb{P}_t(\tilde{v}(t,s))(\gamma).$$

Lemma 2.4.10. *It holds for $s \leq t$*

$$\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_{L^2_\eta} \leq \|v(t) - v(s)\|_{L^2_\eta} + C(t-s).$$

Proof. Indeed

$$\begin{aligned}
\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_{L^2_\eta} &\leq \|v(t) - v(s)\|_{L^2_\eta} + \int_s^t \|\nabla_q H(\gamma(\tau), v(\tau))\|_{L^2_\eta} d\tau \\
&\leq \|v(t) - v(s)\|_{L^2_\eta} + C(t-s)\|(\gamma(s), v(s))\|_{L^2_\eta}.
\end{aligned}$$

□

2.4.1 Piecewise conservative approximations to dissipative solutions

The next statements aim to compare a dissipative solution and the conservative solution with the same initial data and to construct a piecewise conservative approximation. To shorten the notation, we will write

$$Q(t,\gamma;s,\eta) = Q\big(t-s,\gamma(s),v(s,\gamma),(\gamma(s),v(s))_\sharp\eta\big)$$
$$P(t,\gamma;s,\eta) = P\big(t-s,\gamma(s),v(s,\gamma),(\gamma(s),v(s))_\sharp\eta\big)$$

for the lifting of the conservative solution starting at time s with initial measure $(\gamma(s),v(s))_\sharp\eta$.

The first result is that $(Q(t,\gamma;s,\eta),P(t,\gamma;s,\eta)$ well approximates $(\gamma(t),\tilde{v}(t,s))$.

Lemma 2.4.11. *It holds*

$$\begin{aligned}&\big\|\mathbb{P}_t(Q(t,\gamma;s,\eta),P(t,\gamma;s,\eta))-(\gamma(t),v(t))\big\|_{L^2_\eta}\\&\quad\le\big\|(Q(t,\gamma;s,\eta),P(t,\gamma;s,\eta))-(\gamma(t),\tilde{v}(t,s))\big\|_{L^2_\eta}\le C\int_s^t\|v(\tau)-\tilde{v}(\tau)\|_{L^2_\eta}d\tau.\end{aligned}$$

Note that instead

$$\big\|(Q(t,\gamma;s,\eta),P(t,\gamma,s,\eta))-(\gamma(t),v(t))\big\|_{L^2_\eta}=\mathcal{O}(1)\big(\|v(t)-v(s)\|_{L^2_\eta}+t\big), \tag{2.19}$$

hence by comparing the projection $\mathbb{P}_t(Q,P)$ we gain an estimate which is regular in time.

Proof. It is enough to set $s=0$. We can now compute for $0\le t\le\bar{t}$

$$\begin{aligned}\big\|\mathbb{P}_t(Q(t,\gamma;0,\eta),P(t,\gamma;0,\eta))-(\gamma(t),v(t))\big\|_{L^2_\eta}&\le\big\|(Q(t,\gamma;0,\eta),P(t,\gamma;0,\eta))-(\gamma(t),\tilde{v}(t))\big\|_{L^2_\eta}\\&\le\int_0^t\big\|\nabla H(Q(s,0),P(s,0))-\nabla H(\gamma(s),v(s))\big\|_{L^2_\eta}ds\\&\le 3L\int_0^t\big\|(Q(s,0),P(s,0))-(\gamma(s),v(s))\big\|_{L^2_\eta}ds\\&\le 3L\int_0^t\big\|(Q(s,0),P(s,0))-(\gamma(s),\tilde{v}(s))\big\|_{L^2_\eta}ds\\&\quad+3L\int_0^t\|v(s)-\tilde{v}(s)\|_{L^2_\eta}ds.\end{aligned}$$

Hence by Gronwall's estimate, we obtain

$$\big\|(Q(t,\gamma;0,\eta),P(t,\gamma;0,\eta))-(\gamma(t),\tilde{v}(t))\big\|_{L^2_\eta}\le\int_0^t 3Le^{3L(t-s)}\|v(s)-\tilde{v}(s)\|_{L^2_\eta}ds,$$

which is the second inequality in the statement. The first inequality is deduced from the fact that $\mathbb{P}_t$ is a contraction. □

In Section 2.6 we need a similar estimate as in the above lemma, but for the backward flow: the trivial estimate one obtains from Lemmas 2.4.10, 2.4.11 would give

$$\big\|(Q(s,\gamma;t,\eta),P(s,\gamma;t,\eta))-(\gamma(s),v(s))\big\|_{L^2_\eta}\le C\big((t-s)+\|v(t)-\tilde v(t,s)\|_{L^2_\eta}\big).$$

The next corollary, instead, shows that we can get rid of the second term in the r.h.s. above by considering the conservative solution starting from $(\gamma(t),\tilde v(t,\gamma))$ instead of $(\gamma(t),v(t,\gamma))$.

Corollary 2.4.12. *It holds*

$$\begin{aligned}\bigg\|\Big(Q\big(s-t,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big),P\big(s-t,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big)\Big)-\big(\gamma(s),v(s)\big)\bigg\|_{L^2_\eta}\\ \le C\int_s^t\|v(\tau)-\tilde v(\tau)\|_{L^2_\eta}d\tau.\end{aligned}$$

Recall that

$$s\mapsto Q\big(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big),P\big(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big)$$

is the solution to the conservative flow starting at time t with measure $(\gamma(t),\tilde v(t))_\sharp\eta$.

Proof. The statement is a consequence of the backward stability estimate for the conservative flow and Lemma 2.4.11:

$$\begin{aligned}\Big\|\big(Q(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),v(t))_\sharp\eta),P(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),v(t))_\sharp\eta)\big)-(\gamma(s),v(s))\Big\|_{L^2_\eta}\\ \le C\Big\|(\gamma(t),\tilde v(t))-\big(Q(t,\gamma;s,\eta),P(t,\gamma;s,\eta)\big)\Big\|_{L^2_\eta}\\ \le C\int_s^t\|v(\tau)-\tilde v(\tau)\|_{L^2_\eta}d\tau.\end{aligned}$$

□

We now define the piecewise conservative solutions $\tilde Q(t,\gamma;\eta),\tilde P(t,\gamma;\eta)$ by alternating the conservative flow $Q(t,q,p;\mu_0),P(t,q,p;\mu_0)$ with the projection operator $\mathbb{P}_t$: if $0=t_0<t_1<\cdots<t_N=T$, define for $t\in[0,t_1)$

$$\tilde Q(t,\gamma;\eta)=Q(t,\gamma(0),v_0(\gamma);\mu_0),\ \tilde P(t,\gamma;\eta)=P(t,\gamma(0),v_0(\gamma);\mu_0),\quad \mu_0=(\gamma(0),v_0(\gamma))_\sharp\eta,$$
$$q_{t_1}(\gamma)=\mathbb{P}_{t_1}(\tilde Q(t_1,\gamma;\eta)),\quad p_{t_1}(\gamma)=\mathbb{P}_{t_1}(\tilde P(t_1,\gamma;\eta)),$$

and if $q_{t_i}(\gamma),p_{t_i}(\gamma)$ have been constructed, set for $t\in[t_i,t_{i+1})$

$$\mu_{t_i}=(q_{t_i}(\gamma),p_{t_i}(\gamma))_\sharp\eta,$$
$$\tilde Q(t,\gamma;\eta)=Q(t-t_i,q_{t_i}(\gamma),p_{t_i}(\gamma);\mu_{t_i}),\ \tilde P(t,\gamma;\eta)=P(t-t_i,q_{t_i}(\gamma),p_{t_i}(\gamma);\mu_{t_i}),$$
$$q_{t_{i+1}}(\gamma)=\mathbb{P}_{t_{i+1}}(\tilde Q(t_{i+1},\gamma;\eta)),\quad p_{t_{i+1}}(\gamma)=\mathbb{P}_{t_{i+1}}(\tilde P(t_{i+1},\gamma;\eta)),$$

Proposition 2.4.13. *For every $\epsilon > 0$ there exists a piecewise conservative approximation $(\tilde{Q}, \tilde{P})$ of the dissipative solution η such that*

$$\big\|(\tilde{Q}(t,\gamma;\eta), \tilde{P}(t,\gamma;\eta)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \leq C\epsilon T.$$

Proof. Let $\{t_i\}_i$ be the partition of Lemma 2.4.9: w.l.o.g. we can assume that

$$|t_{i+1} - t_i| \leq \epsilon, \quad i = 0, \dots, N-1. \tag{2.20}$$

Lemma 2.4.10 applied to Lemma 2.4.11 gives that

$$\big\|\mathbb{P}_{t_{i+1}}(Q(t_{i+1},\gamma;t_i,\eta), P(t_{i+1},\gamma;t_i,\eta)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \leq C\epsilon(t_{i+1} - t_i).$$

In each interval $[t_i, t_{i+1})$, we thus estimate by the continuous dependence of the conservative flow

$$\begin{aligned}
&\big\|(\tilde{Q}(t,\gamma;\eta), \tilde{P}(t,\gamma;\eta)) - (Q(t,\gamma;t_i,\eta), P(t,\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&\leq e^{3L(t-t_i)}\big\|(\tilde{Q}(t_i,\gamma;\eta), \tilde{P}(t_i,\gamma;\eta)) - (Q(t_i,\gamma;t_i,\eta), P(t_i,\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&= e^{3L(t-t_i)}\big\|(q_{t_i}(\gamma), p_{t_i}(\gamma)) - (\gamma(t_i), v(t_i))\big\|_{L^2_\eta}.
\end{aligned} \tag{2.21}$$

In particular, we obtain

$$\begin{aligned}
&\big\|(q_{t_{i+1}}(\gamma), p_{t_{i+1}}(\gamma)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq \big\|(q_{t_{i+1}}(\gamma), p_{t_{i+1}}(\gamma)) - \mathbb{P}_{t_{i+1}}((Q(t_{i+1},\gamma;t_i,\eta), P(t_{i+1},\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&\quad + \big\|\mathbb{P}_{t_{i+1}}(Q(t,\gamma;t_i,\eta), P(t,\gamma;t_i,\eta)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq \big\|(\tilde{Q}(t_{i+1},\gamma;\eta), \tilde{P}(t_{i+1},\gamma;\eta)) - (Q(t_{i+1},\gamma;t_i,\eta), P(t_{i+1},\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&\quad + \big\|\mathbb{P}_{t_{i+1}}(Q(t,\gamma;t_i,\eta), P(t,\gamma;t_i,\eta)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq e^{3L(t_{i+1}-t_i)}\big\|(q_{t_i}(\gamma), p_{t_i}(\gamma) - (\gamma(t_i), v(t_i))\big\|_{L^2_\eta} + C\epsilon(t_{i+1} - t_i).
\end{aligned}$$

From the explicit solution to the difference equation

$$a_i = \lambda_i a_{i-1} + b_i, \quad a_i = \bigg(\prod_{j=1}^{i} \lambda_j\bigg) a_0 + \sum_{j=1}^{i} \bigg(\prod_{k=j+1}^{i} \lambda_k\bigg) b_j, \tag{2.22}$$

with the convention $\prod_\emptyset \lambda = 1$, we conclude that

$$\begin{aligned}
&\big\|(q_{t_{i+1}}(\gamma), p_{t_{i+1}}(\gamma)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq \sum_{j=1}^{i} \bigg(\prod_{k=j+1}^{i} e^{3L(t_{i+1}-t_i)}\bigg) C\epsilon(t_{i+1} - t_i) \leq CT\epsilon.
\end{aligned} \tag{2.23}$$

By (2.21), (2.19) and the choice of the intervals as in Lemma 2.4.9 and (2.20), we obtain the statement. □

Remark 2.4.14. The proof above shows that a dissipative solution is uniquely characterized by the initial data and the family of projections $\{\mathbb{P}_t\}_t$. However, these projections are such that $\mathbb{P}_t\gamma(t) = \gamma(t)$: one must check that in the limit the trajectories $t \mapsto \gamma(t)$ are a.c., which is not the case for $t \mapsto \tilde{Q}(t,\gamma)$.

2.4.2 Some useful estimates for dissipative solutions

We now show that the energy

$$\begin{aligned}t \mapsto E(t) &= H((\gamma(t), v(t))_\sharp \eta) \\ &= \int V(\gamma(t), v(t,\gamma))\eta(d\gamma) + \int\int W(\gamma(t), v(t,\gamma), \gamma'(t), v(t,\gamma'))\eta(d\gamma)\eta(d\gamma')\end{aligned}$$

is decreasing, and relate its decrease with the distance of the dissipative solution to the conservative one. The first step is the following estimate.

Lemma 2.4.15. *It holds for $s \leq t$*

$$\Lambda\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_2^2 \leq C\int_s^t \|(\mathbb{I} - \mathbb{P}_\tau)\tilde{v}(\tau,s)\|_2^2 ds + E(s) - E(t). \tag{2.24}$$

Proof. We compute

$$\begin{aligned}E(t) - E(s) &= H((\gamma(t), \mathbb{P}_t\tilde{v}(t,s))_\sharp\eta) - H((\gamma(s), \tilde{v}(s,s))_\sharp\eta) \\ &\leq -\Lambda\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_2^2 + H((\gamma(t), \tilde{v}(t,s))_\sharp\eta) - H((\gamma(s), \tilde{v}(s,s))_\sharp\eta) \\ &= -\Lambda\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_2^2 + \int_s^t \int \nabla H(\gamma(\tau), \tilde{v}(\tau,s))\mathrm{J}\nabla H(\gamma(\tau), v(\tau))\eta(d\gamma)d\tau.\end{aligned}$$

The latter integrand is for $V, W \in C^{2,1}$

$$\begin{aligned}&\int \nabla H(\gamma(\tau), \tilde{v}(\tau,s))\mathrm{J}\nabla H(\gamma(\tau), v(\tau))\eta(d\gamma) \\ &= \int \Big[\nabla H(\gamma(\tau), \mathbb{P}_\tau\tilde{v}(\tau,s)) + \nabla_v\nabla H(\gamma(\tau), \mathbb{P}_\tau\tilde{v}(\tau,s))(\tilde{v}(\tau,s) - \mathbb{P}_\tau\tilde{v}(\tau,s)) \\ &\qquad + \mathcal{O}((\mathbb{I} - \mathbb{P}_\tau)\tilde{v}(\tau,s))^2\Big]\mathrm{J}\nabla H(\gamma(\tau), v(\tau))\eta(d\gamma) \\ &\leq C\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t)\|_2^2,\end{aligned}$$

where we used that for every $g \in L^2_\eta$ by the very definition of projection

$$\int \big(\tilde{v}(\tau,s) - \mathbb{P}_\tau\tilde{v}(\tau,s)\big)\mathbb{P}_\tau g\eta = 0.$$

This is the desired estimate. ☐

Proposition 2.4.16. *The energy $E(t)$ is decreasing in time and it holds*

$$\big\|(Q(t,\gamma;s,\eta),P(t,\gamma;s,\gamma))-(\gamma(t),\tilde{v}(t,\gamma))\big\|_{L^2_\eta}\leq C(t-s)\sqrt{E(s)-E(t)},$$

$$\big\|(Q(t,\gamma;s,\eta),P(t,\gamma;s,\gamma))-(\gamma(t),v(t,\gamma))\big\|_{L^2_\eta}\leq C\sqrt{E(s)-E(t)},$$

Proof. From Lemma 2.4.15 and the right continuity of $t\mapsto v(t)$ we deduce that

$$\begin{aligned}\limsup_{t\searrow s}\frac{E(t)-E(s)}{t-s}&\leq\limsup_{t\searrow s}\frac{1}{t-s}\int_s^t\|(\mathbb{I}-\mathbb{P}_\tau)\tilde{v}(\tau,s)\|_2^2d\tau\\&\leq\limsup_{t\searrow s}\frac{1}{2}\|v(t)-v(s)\|_2^2=0.\end{aligned}$$

Hence $t\mapsto E(t)$ is decreasing.

A Gronwall estimate for (2.24) gives

$$\|(\mathbb{I}-\mathbb{P}_t)\tilde{v}(t)\|_2^2\leq-\int_0^t e^{C(t-s)}DE(ds)\leq C\big(E(0)-E(t)\big),$$

where DE is the measure derivative of the decreasing function $E(t)$, and then by Lemma 2.4.11

$$\begin{aligned}\big\|(Q(t,\gamma;s,\eta),P(t,\gamma;s,\gamma))-(\gamma(t),\tilde{v}(t,\gamma))\big\|_{L^2_\eta}&\leq C\int_s^t\big\|(\mathbb{I}-\mathbb{P}_\tau)\tilde{v}(\tau)\big\|_{L^2_\eta}d\tau\\&\leq C(t-s)\sqrt{E(s)-E(t)},\end{aligned}$$

$$\begin{aligned}&\big\|(Q(t,\gamma;0,\eta),P(t,\gamma;0,\gamma))-(\gamma(t),v(t,\gamma))\big\|_{L^2_\eta}\\&\leq\big\|(Q(t,\gamma;0,\eta),P(t,\gamma;0,\gamma))-(\gamma(t),\tilde{v}(t,\gamma))\big\|_{L^2_\eta}+\big\|(\mathbb{I}-\mathbb{P}_t)\tilde{v}(t)\big\|_{L^2_\eta}\\&\leq C(t-s)\sqrt{E(s)-E(t)}+\big\|(\mathbb{I}-\mathbb{P}_t)\tilde{v}(t)\big\|_{L^2_\eta}\\&\leq C\sqrt{E(s)-E(t)}.\end{aligned}$$

This concludes the proof. □

2.4.3 Some special cases for dissipative solutions

We conclude this section with some special cases, namely when the data are a finite number of Dirac deltas and when the Hamiltonian is purely quadratic.

Lemma 2.4.17. *Assume that*

$$\mu_0=\sum_{n=1}^N m_n\delta_{(q_n,q_n)}$$

and let $Q(t,q_n,p_n;\mu_0),P(t,q_n,p_n;\mu_0)$ be the conservative solution with initial condition μ_0. If the trajectories $\{Q(t,q_n,p_n;\mu_0)\}_n$ do not intersect, then there is a unique dissipative solution with initial data μ_0: in particular it coincides with the conservative one.

Proof. The proof is immediate by observing that since the trajectories never meet then $\mathbb{P}_t = \mathbb{I}$ for all $t \geq 0$. □

In the case of

$$V(x, v) = \frac{1}{2}(q, p)^T A(q, p), \quad W(x, v, x', v') = \frac{1}{2}(x - x', v - v')^T B(q - q', p - p'), \quad (2.25)$$

i.e. V, W are quadratic, with

$$A^T = A, \ A_{22} \geq \Lambda \mathbb{I}, \quad B^T = B \geq 0,$$

then the trajectories of the dissipative solution can be computed by projecting the solution to the conservative one, as in the standard pressureless dynamics. Indeed, the ODEs for the trajectories are

$$\frac{d}{dt}\begin{pmatrix} Q(t, q, p; \mu_0) \\ P(t, q, p; \mu_0) \end{pmatrix} = \mathrm{J}(A + B)\begin{pmatrix} Q(t, q, p; \mu_0) \\ P(t, q, p; \mu_0) \end{pmatrix} + \int \mathrm{J}B\begin{pmatrix} Q(t, q', p'; \mu_0) \\ P(t, q', p'; \mu_0) \end{pmatrix}\mu_0(dq'dp')$$

Hence assuming

$$\int (q, p)\mu_0(dqdp) = 0 \quad \Rightarrow \quad \int \begin{pmatrix} Q(t, q, p; \mu_0) \\ P(t, q, p; \mu_0) \end{pmatrix}\mu_0(dqdp) = 0$$

(i.e. it is preserved in time), we obtain

$$\begin{pmatrix} Q(t, q, p; \mu_0) \\ P(t, q, p : \mu_0) \end{pmatrix} = e^{\mathrm{J}(A+B)t}\begin{pmatrix} q \\ p \end{pmatrix},$$

i.e. the conservative flow is independent from μ_0.

Next, consider the piecewise conservative solution constructed in Proposition 2.4.13: being the projection operator linear, it follows that

$$(\tilde{Q}(t, \gamma; 0, \eta), \tilde{P}(t, \gamma; 0, \eta)) = e^{\mathrm{J}(A+B)t}\mathbb{P}_t(\gamma(0), v_0(\gamma)).$$

Being the above formula independent of the approximation parameter ϵ present in the statement of Proposition 2.4.13, we conclude that

Proposition 2.4.18. *If V, W are quadratic, and η is a dissipative solution with associated descending fibration $\{\Omega_t\}_t$ and projections $\mathbb{P}_t$, then*

$$(\gamma(t), v(t, \gamma)) = e^{\mathrm{J}(A+B)t}\mathbb{P}_t(\gamma(0), v_0).$$

For the quadratic case (2.25), a converse of Lemma 2.4.17 holds: if there is only the conservative solution, then the particle trajectories do not intersect. Note that the examples in [1] show that the existence of a conservative solution with non-intersecting trajectories does not imply that all solutions are conservative (hence there is only one).

Proposition 2.4.19. *If V, W are quadratic and the only dissipative solution is the conservative one, then η is concentrated on a family of non-intersecting curves.*

Proof. We need the following duality result [22]: if $\nu \in \mathscr{P}(X)$, $\nu' \in \mathscr{P}(X')$, X, X' Polish, $\mathcal{Z} \subset X \times X'$ Borel, and

$$\Pi^{\leq}(\nu,\nu') = \left\{ \pi \text{ Borel measure}, \int \phi(x_1)\pi(dx_1dx_2) \leq \int \phi\nu, \int \phi(x_2)\pi(dx_1dx_2) \leq \int \phi\nu' \right\},$$

then

$$\sup\big\{\pi(\mathcal{Z}), \pi \in \Pi^{\leq}(\nu,\nu')\big\} = \min\big\{\nu(A)+\nu'(A'), \mathcal{Z} \subset A \times X' \cup X \times A', A, A' \text{ Borel}\big\}. \quad (2.26)$$

Let η be a dissipative solution such that $(\gamma(0), v_0)_\sharp\eta = \mu_0$. Consider the set

$$\mathcal{Z} = \Big\{(\gamma,\gamma') : \Gamma \times \Gamma : \gamma \neq \gamma' \text{ and } \exists t \in [0,T] \text{ such that } \gamma(t) = \gamma'(t)\Big\},$$

and assume that there is $\pi \in \Pi^{\leq}(\eta,\eta)$ such that $\pi(\mathcal{Z}) > 0$: we can require π to be symmetric, i.e.

$$\int \phi(\gamma,\gamma')\pi(d\gamma d\gamma') = \int \phi(\gamma',\gamma)\pi(d\gamma d\gamma'),$$

because $\mathcal{Z}$ is symmetric. Let

$$\pi = \int \pi_\gamma \eta(d\gamma)$$

be the (not normalized) disintegration.

Define the map

$$(\gamma,\gamma') \mapsto \tau_{\gamma,\gamma'} = \arg\min\big\{\gamma(t) = \gamma'(t)\big\}.$$

and define

$$\varpi(d\gamma d\gamma') = \pi + (\mathbb{I},\mathbb{I})_\sharp(\eta - (P_1)_\sharp\pi).$$

This does not correspond to a measure on Γ, and indeed the same curve γ intersects many others γ': however $(P_1)_\sharp\varpi$ is. Let

$$\mathcal{E}_t = \tau^{-1}_{\gamma,\gamma'}([0,t]),$$

i.e. the couples of curves which cross before t, let $\tilde{\mathbb{P}}_t$ be the corresponding projection, and set

$$t \mapsto (\tilde{\gamma}(t), \tilde{\gamma}'(t)) = \tilde{\mathbb{P}}_t e^{\mathrm{J}(A+B)t}(\gamma(0),\gamma'(0)).$$

It is fairly easy to see that $t \mapsto \tilde{\gamma}(t)$ is continuous, and then that $(\tilde{\gamma})_\sharp\eta$ is a dissipative solution verifying Proposition 2.4.18. Hence from the assumption that there are no dissipative solutions, we deduce that

$$\sup\big\{\pi(\mathcal{Z}), \pi \in \Pi^{\leq}(\eta,\eta)\big\} = 0$$

The duality (2.26) implies that there is an η-negligible set $N = A \cup A'$ such that $\mathcal{Z} \subset N \times N$, and then the proposition is proved. ☐

In general, when the Hamiltonian is not purely quadratic, it may happen that

$$\sup \left\{ \pi(\mathscr{Z}), \pi \in \Pi^{\leq}(\eta, \eta) \right\} > 0$$

even if the unique solution is the conservative one: we will give an explicit example in Appendix 2.C.

2.5 Compactness of Dissipative solutions

It is well known that by Prokhorov's theorem $\mathscr{M}(\Gamma)$ is compact w.r.t. the Wasserstein distance W_p, $p < 2$, being η concentrated on curves in $W^{1,2}$ with uniformly bounded energy. Since the set $\mathscr{M}(\Gamma)$ is tight w.r.t. the cost $\|\cdot\|^2_{L^2}$, the Wasserstein distance W_p, $p < 2$, is equivalent to the narrow convergence.

Proposition 2.5.1. *Let $\{\eta_n\}_{n \in \mathbb{N}}$,$\eta$ dissipative solutions in $\mathscr{M}(\Gamma)$ and suppose $W_p(\eta_n, \eta) \to 0$, $p > 1$. For every continuous bounded function $\phi : (L^2(0,T))^3 \to \mathbb{R}$, it holds*

$$\int \phi(\gamma, \dot{\gamma}, v_n(\gamma)) \eta_n(d\gamma) \to \int \phi(\gamma, \dot{\gamma}, v(\gamma)) \eta(d\gamma).$$

Recall that $v(\gamma)$ is computed by (2.15a).

Proof. The proof is divided into two steps.

Step 1. First of all, we show that there is a family of maps $R_n, R : [0,1] \to \Gamma$ such that

$$\eta_n = (R_n)_\sharp \mathscr{L}^1, \ \eta = R_\sharp \mathscr{L}^1, \quad \lim_n \|R_n - R\|_{L^2(0,1)} = 0.$$

The construction is standard, we repeat it for the reader's convenience.

Let $B_i = B_{r_i}(\gamma_i)$, $i \in \mathbb{N}$, be a family of open balls generating the topology of Γ, and such that

$$\eta(\partial B_i) = 0. \tag{2.27}$$

Define the map $S : \Gamma \to [0,1]$ such that

$$\gamma \mapsto \alpha = S(\gamma) = \sum_i 3^{-i} \chi_{B_i}(\gamma) \in [0,1].$$

The map S is clearly injective.

Define the measures $\mu_n = S_\sharp \eta_n$, $\mu = S_\sharp \eta$, and let $\phi \in C([0,1])$. Then the function

$$\gamma \mapsto \phi\bigg(\sum_i 3^{-i} \chi_{B_i}(\gamma) \bigg)$$

is bounded and continuous outside the set $\cup_i \partial B_i$, so that by [3, Prop.1.62 b] and (2.27) it follows

$$\lim_n \int \phi(\alpha) S_\sharp \eta_n(d\alpha) = \lim_n \int \phi(S(\gamma))\eta_n(d\gamma) = \int \phi(S(\gamma))\eta(d\gamma) = \int \phi(\alpha) S_\sharp \eta(d\alpha),$$

so that

$$\mu_n = S_\sharp \eta_n \rightharpoonup S_\sharp \eta = \mu,$$

i.e. the measures μ_n converges weakly to μ.

Next, consider the unique monotone transport maps $G^n, G : [0,1] \to [0,1]$ such that

$$\mu_n = (G_n)_\sharp \mathscr{L}^1, \quad \mu = (G)_\sharp \mathscr{L}^1.$$

It is elementary to see that

$$\lim_n \|G_n - G\|_{L^p(0,1)} = 0.$$

Finally, if $S^{-1} : [0,1] \to \Gamma$ is a left inverse of S, define the maps

$$R_n = S^{-1} \circ G_n, \quad R = S^{-1} \circ G. \tag{2.28}$$

If $x_n = S(\gamma_n)$, $x = S(\gamma)$ and $x_n \to x$, then every $B_{r_i}(\gamma_i) \ni \gamma$ contains definitely γ_n, hence $\gamma_n \to \gamma$. This shows that $S^{-1} \llcorner_{S(\Gamma)}$ is continuous. Observing now that $G_n(\alpha) \to G(\alpha)$ for $\mathscr{L}^1$-a.e. $\alpha \in [0,1]$, we obtain that $R_n = S^{-1} \circ G_n : [0,1] \to \Gamma$ converges $\mathcal{L}^1$-almost everywhere to $R = S^{-1} \circ G$. Using the estimates

$$\int_0^1 \|R_n(\alpha)\|_{L^2}^2 \mathscr{L}^1(d\alpha) = \int_{[0,1]} \|S^{-1}(\alpha)\|_{L^2}^2 \mu_n(d\alpha) = \int_{L^2(0,T)} \|\gamma\|_{L^2}^2 \eta_n(d\gamma),$$

$$\int_0^1 \|S^{-1} \circ G(\alpha)\|_{L^2}^2 \mathcal{L}^1(d\alpha) = \int_{L^2(0,T)} \|\gamma\|_{L^2}^2 \eta(d\gamma),$$

we deduce that $\|R_n\|_{L^p((0,1),\Gamma)}$ converges to $\|R\|_{L^p((0,1),\Gamma)}$: this together with the $\mathscr{L}^1$-a.e. pointwise convergence implies that $R_n \to R$ in $L^p((0,1),\Gamma)$.

Step 2. The statement thus reduces to

$$\lim_n \int_0^1 \phi\big(R_n(\alpha), \dot{R}_n(\alpha), v_n(R_n(\alpha))\big) d\alpha = \int_0^1 \phi\big(R(\alpha), \dot{R}(\alpha), v(R(\alpha))\big) d\alpha.$$

We claim that

$$\dot{R}_n(\alpha) = \frac{\partial}{\partial t}(R_n(\alpha)) \underset{L^2((0,1),\Gamma)}{\longrightarrow} \frac{\partial}{\partial t}(R(\alpha)) = \dot{R}(\alpha).$$

By Lemma 2.4.8 we have

$$\int_0^{T-s} \left| \frac{R_n(t+s,\alpha) - R_n(\alpha,t)}{s} - \dot{R}_n(\alpha,t) \right|^2 dt d\alpha \leq C(T) s \|(R_n(0), v_n(0))\|_{L^2(0,1)}^2,$$

$$\int_0^{T-s} \left| \frac{R(t+s,\alpha) - R(\alpha,t)}{s} - \dot{R}(\alpha,t) \right|^2 dt d\alpha \leq C(T) s \|(R(0),v(0))\|^2_{L^2(0,1)},$$

Hence by triangle inequality and the convergence $R_n \to R$

$$\begin{aligned}
\limsup_n \int \bigg(\int_0^{T-s} \big| \dot{R}(\alpha,t) - \dot{R}_n(\alpha,t) \big|^2 dt \bigg)^{p/2} d\alpha \\
&\leq C(p) \limsup_n \int \bigg(\int_0^{T-s} \left| \frac{R(t+s,\alpha) - R(\alpha,t)}{s} - \frac{R_n(t+s,\alpha) - R_n(\alpha,t)}{s} \right|^2 ds \bigg)^{p/2} d\alpha \\
&\quad + C(T,p) s \|(R(0),v(0))\|^2_{L^2(0,1)} \\
&= 6C(T,p) s \|(R(0),v(0))\|^2_{L^2(0,1)}.
\end{aligned}$$

Letting $s \searrow 0$ we obtain the desired convergence $\dot{R}_n \to \dot{R}$. Using again Proposition 2.3.2 we deduce that the function $v_n(t, R_n(\alpha))$ converges to $v(t, R(\alpha))$ in $L^2(0,1)$ for $\mathscr{L}^1$-a.e. α.

Finally for a continuous bounded function $\phi : (L^2(0,1))^3 \to \mathbb{R}$

$$\begin{aligned}
\lim_n \int \phi(\gamma, \dot{\gamma}, v_n) \eta_n(d\gamma) &= \lim_n \int \phi\bigg(R_n(\alpha), \frac{\partial R_n(\alpha)}{\partial t}, v_n(R_n(\alpha)) \bigg) d\alpha \\
&= \int \phi\bigg(R(\alpha), \frac{\partial R(\alpha)}{\partial t}, v(R(\alpha)) \bigg) d\alpha = \int \phi(\gamma, \dot{\gamma}, v(\gamma)) \eta(d\gamma). \quad \square
\end{aligned}$$

Theorem 2.5.2. *Let $\{\eta_n\}_{n \in \mathbb{N}}$ be a sequence of dissipative solutions supported on $\mathscr{M}$ such that $W_p(\eta_n, \eta) \searrow 0$, $p > 1$. Then η is dissipative solution.*

Proof. Since (2.15a) is satisfied because of Proposition 2.5.1, we have to prove that equation (2.15b) passes to the limit: if $R_n, R : [0,1] \to \Gamma$ are the functions (2.28) in the proof of Proposition 2.5.1, then (2.15b) can be rewritten as

$$\begin{aligned}
v_n(t, R_n(\alpha,t)) &= \mathbb{P}_{t,n}\big(F(R_n, v_n(R_n)) \big) \\
&= \mathbb{P}_{t,n}\bigg(v_{0,n}(R_n(\alpha)) - \int_0^t \nabla_q V\big(R_n(\alpha,s), v_n(s, R_n(\alpha)) \big) \\
&\qquad - \int_0^t \nabla_q W\big(R_n(\alpha,s), v_n(s, R_n(\alpha)), R_n(s,\alpha'), v_n(s, R_n(\alpha')) \big) d\alpha' \bigg),
\end{aligned}$$

where $\mathbb{P}_{t,n}$ is the projection in $L^2(0,1)$ corresponding to the descending fibration in $(0,1)$ obtained through the map $\mathbb{T}_t \circ R_n$,

$$(0,1) \ni \alpha \mapsto \mathbb{T}_t(R_n(\alpha))(\tau) = R_n(\alpha,t) \mathbf{1}_{[0,t)} + R_n(\alpha,\tau) \mathbf{1}_{[t,T]} \in \Gamma.$$

If $\psi : \Gamma \to \mathbb{R}$ is continuous, then

$$\begin{aligned}
\int \psi(\mathbb{T}_t \circ R_n(\alpha)) F(R_n(\alpha), v_n(R_n(\alpha)))(t) d\alpha &= \int \psi(\mathbb{T}_t \circ R_n(\alpha)) \mathbb{P}_{t,n}\big(F(R_n(\alpha), v_n(R_n(\alpha)))(t) \big) d\alpha, \\
&= \int \psi(\mathbb{T}_t \circ R_n(\alpha)) v_n(t, R_n(\alpha)) d\alpha,
\end{aligned}$$

so that, passing to the limit and by the pointwise convergence of R_n, v_n we obtain

$$\int \psi(\mathbb{T}_t \circ R(\alpha))F(R(\alpha), v(R(\alpha)))(t)d\alpha = \int \psi(\mathbb{T}_t \circ R(\alpha))v(t, R(\alpha))d\alpha,$$

which, due to the arbitrariness of ψ, reads as

$$\mathbb{P}_t(F(R(\alpha), v(R(\alpha)))(t)) = \mathbb{P}_t(v(t, R(\alpha))),$$

or in the original coordinates

$$\mathbb{P}_t(F(\gamma(t), v(t,\gamma))) = \mathbb{P}_t(v(t,\gamma)).$$

By (2.15a) the functions $v(t,\gamma)$ depends only on $(t, \mathbb{T}_t(\gamma))$: this together with the right continuity gives that $\mathbb{P}_t(v(t,\gamma)) = v(t,\gamma)$, and therefore

$$\mathbb{P}_t(F(\gamma(t), v(t,\gamma))) = \mathbb{P}_t(v(t,\gamma)) = v(t,\gamma).$$

which is the requirement to be a dissipative solution. □

The following statement is elementary, because of the quadratic growth of V, W.

Lemma 2.5.3. *The energy $\eta \mapsto H((\gamma(t), v(t)_\sharp\eta)$ is continuous w.r.t. the Wasserstein-2 convergence.*

Remark 2.5.4. Note that for the Hamiltonian

$$H(\mu) = \int \left(\frac{p^2}{2} - \frac{q^2}{2}\right)\mu(dqdp),$$

with the initial data

$$\mu_n = \frac{1}{n^2}\big(\delta_{(n,0)} + \delta_{(-n,0)}\big) + \left(1 - \frac{2}{n^2}\right)\delta_{0,0},$$

the energy is not l.s.c., being

$$H(\mu_n) = -2 < H(\mu_\infty) = 0.$$

Clearly μ_n is not converging to μ w.r.t. Wasserstein-2, but it converges for all $p < 2$.

2.6 Discretization

The aim of this section is to prove that the set of dissipative solutions is the closure of the set of finite particle dissipative solutions. We will first approximate a given dissipative solution with a dissipative solution with dissipation only at finitely many times, then with a dissipative solution with finitely many particles, and finally with a sticky particle solution.

Definition 2.6.1. A dissipative solution is a *discrete in-time dissipative solution* if there exists a partition $0 = t_0 < t_1 < \cdots < t_N = T$ such that in every interval $[t_i, t_{i+1})$ the solution $v(t)$ coincides with the conservative solution with initial measure $(\gamma(t_i), v(t_i))_\sharp \eta$.

Recall that $(Q(t, q, p; \mu_0), P(t, q, p; \mu_0))$ is the conservative trajectory starting from q, p with initial measure μ_0. Thus the above definition can be rewritten as

$$\gamma(t) = Q\big(t - t_i, \gamma(t_i), v(t_i, \gamma); \mu_{t_i}\big), \quad v(t, \gamma) = P\big(t - t_i, \gamma(t_i), v(t_i, \gamma); \mu_{t_i}\big),$$

for $t \in [t_i, t_{i+1})$, with

$$\mu_{t_i} = (\gamma(t_i), v(t_i))_\sharp \eta.$$

We begin by introducing a general method of constructing discrete in-time dissipative solutions. Let $\eta \in \mathscr{P}(\Gamma)$, and consider L^2_η functions $(y(\gamma), w(\gamma))$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$ and for every $i = 1, \dots, N$ let $\Upsilon_i(\gamma) \in L^2_\eta$ be given functions such that

$$\mathbb{P}_{t_i}(\Upsilon_i) = 0.$$

Define the functions $X(t, \gamma), Y(t, \gamma)$ recursively as follows: for $t \in [t_{N-1}, t_N]$ set

$$\begin{cases} X(t, \gamma) = Q\big(t - t_N, y(\gamma), w(\gamma); \mu_T\big), \\ Y(t, \gamma) = P\big(t - t_N, y(\gamma), w(\gamma); \mu_T\big), \end{cases} \qquad \mu_N = (y, w)_\sharp \eta,$$

and for $t \in [t_{i-1}, t_i)$, $i = 1, \dots, N-1$,

$$\begin{cases} X(t, \gamma) = Q\big(t - t_i, X(t_i, \gamma), Y(t_i, \gamma) + \Upsilon_i(\gamma); \mu_{t_i}\big), \\ Y(t, \gamma) = P\big(t - t_i, X(t_i, \gamma), Y(t_i, \gamma) + \Upsilon_i(\gamma); \mu_{t_i}\big), \end{cases} \qquad \mu_i = (X(t_i), Y(t_i))_\sharp \eta. \tag{2.29}$$

In other words, the trajectories X, Y are constructed by alternating the conservative flow $(Q, P)(t, q, p; \mu)$ with the projection $\mathbb{P}_{t_i}$, $i = 1, \dots, N-1$ (Figure 2.3): instead of assigning the initial data, we assign the projections

$$\Upsilon_i = Y(t_i-) - Y(t_i) = (\mathbb{I} - \mathbb{P}_{t_i}) Y(t_i-).$$

Lemma 2.6.2. *The measure $\tilde{\eta} = X_\sharp \eta$ is a dissipative solution with initial velocity $v_0(\gamma) = Y(0, \gamma)$. Moreover*

$$E(0) \leq E(T) + C \sum_i \|\Upsilon_i\|^2_\eta.$$

Proof. The function $\dot{X}(t, \gamma)$ and $Y(t, \gamma)$ satisfies Equation (2.15a) by construction, and Equation (2.9b) holds in each internal $[t_i, t_{i+1})$ with initial data $v(t_i, \gamma)$. We have thus only to verify that

$$v(t_i, \gamma) = \mathbb{P}_{t_i}(v(t_i-))(\gamma), \quad i = 1, \dots, N-1.$$

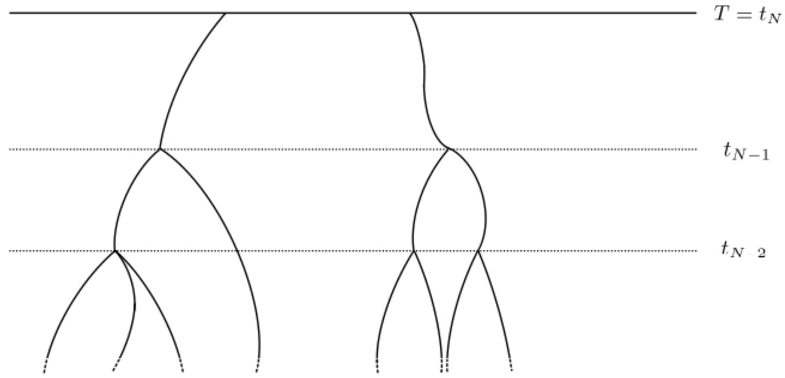


Figure 2.3: the discrete in-time dissipative solution of Lemma 2.6.2.

By construction, $v(t,\gamma)$ depends only on $\gamma\llcorner_{[t,T]}$, so that

$$v(t_i,\gamma) - \mathbb{P}_{t_i}(v(t_i-))(\gamma) = \mathbb{P}_{t_i}\big(v(t_i) - v(t_i-)\big)(\gamma) = \mathbb{P}_{t_i}(-\Upsilon_i)(\gamma) = 0.$$

The energy is jumping only at times t_i of the amount

$$E(t_i-) - E(t_i) = \mathcal{O}(1)\|\Upsilon_i\|^2_\eta,$$

so that the energy estimate holds. □

We next study the stability w.r.t. the data $(y,w), \{\Upsilon_i\}_i$. Let (X,Y) and $(X',\tilde{Y}')$ be discrete in-time dissipative solution with the same time partition and constructed with initial data $(y,w),(y',w')$ and Υ_i,Υ'_i, $i=1,\dots,N-1$.

Lemma 2.6.3. *It holds*

$$\|(X(t),Y(t)) - (X'(t),Y'(t))\|_{L^2_\eta} \leq C\bigg(\|(y,w)-(y',w')\|_{L^2_\eta} + \sum_{i=1}^{N-1}\|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta}\bigg).$$

In particular, by taking $(y',w') = 0, \Upsilon'_i = 0$ we obtain that the solution belongs to $\mathscr{M}(\Gamma)$, with

$$\tilde{C}_1,\tilde{C}_2 = \mathcal{O}(1)\bigg(\|(y,w)\|_{L^2_\eta} + \sum_{i=1}^{N-1}\|\Upsilon_i\|_{L^2_\eta}\bigg).$$

Proof. By stability, for $t \in [t_i,t_{i+1})$ it holds

$$\begin{aligned}
&\|(X(t),Y(t)) - (X'(t),Y'(t))\|_{L^2_\eta} \\
&\quad\leq e^{3L(t_{i+1}-t)}\|(X(t_{i+1}-),Y(t_{i+1}-)) - (X'(t_{i+1}-),Y'(t_{i+1}-))\|_{L^2_\eta} \\
&\quad\leq e^{3L(t_{i+1}-t)}\big(\|\Upsilon_{i+1} - \Upsilon'_{i+1}\|_{L^2_\eta} + \|(X(t_{i+1}),Y(t_{i+1})) - (X'(t_{i+1}),Y'(t_{i+1}))\|_{L^2_\eta}\big).
\end{aligned}$$

The statement is thus a direct application of (2.22) as in the proof of Proposition 2.4.13. □

Proposition 2.6.4. *If η is a dissipative solution, then for every $\epsilon > 0$ there is a discrete in-time dissipative solution $\eta' = X_\sharp \eta$ such that for all $t \in [0, T]$*

$$\|(X(t), Y(t)) - (\gamma(t), v(t))\|_{L^2_\eta} < \epsilon.$$

Proof. The proof is analogous to the proof of Proposition 2.4.13, the only difference being that we will follow the backward solution of Lemma 2.6.2 above, so that, we do not need to apply the projection to the variable Q. In particular, the constructed function is a dissipative solution, as stated in Lemma 2.6.2.

Consider the partition $0 = t_0 < t_1 < \cdots < t_N = T$ of Lemma 2.4.9, and define

$$\Upsilon_i = \tilde{v}(t_i, t_{i-1}, \gamma) - v(t_i, \gamma).$$

Let X, Y be the discrete in-time dissipative solutions constructed in Lemma 2.6.2: at each time step $[t_i, t_{i+1})$ we obtain from Corollary 2.4.12

$$\begin{aligned}
&\big\|(X(t,\gamma), Y(t,\gamma)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \\
&\leq \Big\|(X(t,\gamma), Y(t,\gamma)) \\
&\quad - \Big(Q\big(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta\big), \\
&\qquad\qquad P(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta)\Big)\Big\|_{L^2_\eta} \\
&\quad + \Big\|\Big(Q\big(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta\big), \\
&\qquad\qquad P\big(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta\big)\Big) \\
&\qquad\qquad\quad - (\gamma(t), v(t))\Big\|_{L^2_\eta} \\
&\leq e^{3L(t_i - t)}\big\|(X(t_{i+1}-, \gamma), Y(t_{i+1}-, \gamma)) - (\gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma))\big\|_{L^2_\eta} \\
&\quad + C\int_{t_i}^{t_{i+1}} \|v(s) - \tilde{v}(s)\|_{L^2_\eta} ds \\
&\leq e^{3L(t_i - t)}\big\|(X(t_{i+1}, \gamma), Y(t_{i+1}, \gamma)) - (\gamma(t_{i+1}), v(t_{i+1}, \gamma))\big\|_{L^2_\eta} \\
&\quad + C\epsilon(t_{i+1} - t_i).
\end{aligned}$$

Hence, applying the solution formula (2.22) to the above formula when $t = t_i$ as in (2.23),

$$\big\|(X(t,\gamma), Y(t,\gamma)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \leq C\epsilon.$$

The measure $(X, Y)_\sharp \eta$ satisfies the statement.

□

The next step is to discretize the number of particles.

Definition 2.6.5. A discrete in-time dissipative solution is a *discrete in-time dissipative particle solution* if η is made of Dirac masses. If the number of deltas is finite, it is a *dissipative finite particle solution.*

Proposition 2.6.6. *If η is a discrete in-time dissipative solution and $\epsilon > 0$, then there exists a finite particle solution $\eta' = X_\sharp \eta$ such that for all $t \in [0,T]$*

$$\|(X(t),Y(t)) - (\gamma(t),v(t))\|_{L^2_\eta} \leq \epsilon.$$

Proof. The proof follows immediately from Lemma 2.6.3, if we can find simple functions $(y', w', \{\Upsilon'_i\}_i)$ approximating

$$y(\gamma) = \gamma(T), \quad w(\gamma) = v(T,\gamma), \quad \Upsilon_i(\gamma) = v(t_i-,\gamma) - v(t_i,\gamma)$$

(in the last formula we have used that η is a discrete in-time solution where the projection is applied at times t_i), with the property that

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} + \sum_i \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \epsilon.$$

The existence of such approximations is elementary.

It remains only to prove that the solution (X',Y') of constructed in Lemma 2.6.2 by using $y', w', \{\Upsilon'\}_i$ simple functions is a finite particle solution: this is immediate, since from the explicit form of the solution (2.29) the functions (X',Y') are measurable in the finite algebra generated by $(y', w', \{\Upsilon\}_i)$. □

The last step is to prove that we can construct a sticky particle solution made of finitely many particles.

Definition 2.6.7. A discrete finite particle solution is a *finite sticky particle solution* if for every $t \in [0,T]$ the maps $\mathbb{T}_t$ and e_t induce the same equivalence relation.

Proposition 2.6.8. *If η is a finite dissipative solution and $\epsilon > 0$, there exists $\eta' = X_\sharp \eta$ finite sticky particle solution such that*

$$\|(X(t),Y(t)) - (\gamma(t),v(t))\|_{L^2_\eta} \leq \epsilon.$$

Proof. As in the previous proof, it is enough to find simple functions $y', w', \{\Upsilon'_i\}_i$ approximating

$$y(\gamma) = \gamma(T), \quad w(\gamma) = v(T,\gamma), \quad \Upsilon_i(\gamma) = v(t_i-,\gamma) - v(t_i,\gamma),$$

with the property that

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} + \sum_i \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \epsilon,$$

and, moreover, such that the dissipative solution constructed by Lemma 2.6.2 is actually a sticky particle solution.

Observe that if $\{X_i(t)\}_{i=1}^N$ are the trajectory of a finite dissipative solution η such that

$$X_i(t) \neq X_j(t) \quad \Rightarrow \quad \forall\, 0 \leq s \leq t \; \big(X_i(s) \neq X_j(s) \big),$$

then η is a sticky particle solution: indeed $\{t : X_i(t) = X_j(t)\}$ must be an interval of the form $[t, T]$ or empty, and this means that when two particles collide then they stick together.

W.l.o.g. we can assume that the time steps t_i satisfies

$$t_i - t_{i-1} < \delta_t,$$

being δ_t the time step for which Proposition 2.3.8 holds.

We begin by considering the final data y, w, and let M be the number of particles. If the backward trajectories are not intersecting, then no perturbation is needed. Otherwise, by Proposition 2.3.8 there are arbitrarily small perturbations $y' - y, w' - y$ such that the trajectories are not intersecting in $[t_{N-1}, t_N)$. In particular, we can assume that

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} < \epsilon 2^{-T/\delta_t}.$$

The number of particles is increased by at most $2M$.

Assume to have found perturbations up to time t_{i+1} such that in each time interval $[t_j, t_{j+1})$, $j \geq i+1$, the trajectories are not intersecting, and the number of particles is $2^{N-i-1}M$. The initial data for the backward solution are

$$\gamma(t_{i+1}), \quad v(t_{i+1}, \gamma) + \Upsilon_{i+1}(\gamma).$$

We can then again find perturbations $\Upsilon'_{i+1}(\gamma)$ such that the number of particles is at most $2^{N-i}M$ and

$$\|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \epsilon 2^{-T/\delta_i}.$$

After a finite number of steps we arrive to $t = 0$: the total perturbation is

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} + \sum_i \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \frac{T}{\delta_t} \epsilon 2^{-T/\delta_t} < \epsilon,$$

and the number of particles is at most $2^{T/\delta_t}M$. □

The above result implies directly the following.

Theorem 2.6.9. *The weak closure of the set of finite sticky particle solutions with bounded second-order moments for $\gamma(0), v(0)$ is the set of dissipative solutions.*

2.7 A G_δ dense set of initial data

We have proved that for dissipative solutions the energy $E(t) = H((\gamma(t), v(t))_\sharp \eta)$ is decreasing in time, and actually that the energy dissipation controls the distance from the conservative flow. In this section, we want to prove that the set of initial data for which there is only one dissipative solution (which is then the conservative one) is of second category in the set of initial data.

In this section, we assume that H is convex, so that $\mu \mapsto H(\mu)$ is l.s.c. w.r.t. the narrow convergence. Define for $\mu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ the functional

$$D(\mu) = \max \big\{ H(\mu) - H((\gamma(T), v(T)_\sharp \eta), (\gamma(0), v(0))_\sharp \eta = \mu \big\}.$$

By compactness of $\mathscr{M}(\Gamma)$ and l.s.c. of $\eta \mapsto H((\gamma(t), v(t)_\sharp \eta)$, the maximum is attained. Being the supremum of u.s.c. functionals, $D(\mu)$ is u.s.c., and when $D(\mu) = 0$ then every dissipative solution with the initial data μ has 0 dissipation, i.e. it coincides with the conservative one.

Using Proposition 2.3.6, we deduce the following

Theorem 2.7.1. *The set $D_0 = \{\mu : D(\mu) = 0\} \subset \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is a dense G_δ set w.r.t. narrow convergence.*

Proof. First of all, D_0 is a G_δ-set, being

$$D_0 = \bigcap_n \big\{ \mu : D(\mu) < 2^{-n} \big\}, \quad D(\mu) \text{ u.s.c..}$$

Next, by Proposition 2.6.8, the finite sticky particle solutions are dense and, by Proposition 2.3.6, the set of initial data so that the trajectories are not intersecting is dense in the set of finite sticky particle solutions. Finally, for the non-intersecting trajectories, the unique solution is the conservative one by Lemma 2.4.17.

□

Appendix of Chapter 1

2.A Proofs of Section 2.3

Proposition 3.2, page 28. *The operator* (2.11) *is uniformly monotone, namely*

$$\Lambda\|v_1 - v_2\|_2^2 \le \big(v_1 - v_2, F(t, v_1) - F(t, v_2)\big) \le 3L\|v_1 - v_2\|_2^2. \tag{2.30}$$

Proof of Proposition 2.3.2, page 28. The bound from above follows immediately by the Lipschitz bounds on $\nabla V, \nabla W$, which gives that $F(t)$ is Lipschitz:

$$\|F(t, v_1) - F(t, v_2)\|_2 \le 3L\|v_1 - v_2\|_{L^2_\eta}.$$

For the estimate from below, we observe that by symmetry

$$W(x, v, x', v') = W(x', v', x, v) \quad \Rightarrow \quad \nabla_p W(x, v, x', v') = \nabla_{p'} W(x', v', x, v),$$

so that

$$\begin{aligned}
&\int \Big(v_1(\gamma) - v_2(\gamma), F(t, v_1)(\gamma) - F(t, v_2)(\gamma)\Big)\eta(d\gamma) \\
&= \int \Big(v_1(\gamma) - v_2(\gamma), \nabla_p V(\gamma(t), v_1(\gamma)) - \nabla_p V(\gamma(t), v_2(\gamma))\Big)\eta(d\gamma) \\
&\quad + \int\int \Big(v_1(\gamma) - v_2(\gamma), \nabla_p W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) \\
&\qquad\qquad - \nabla_p W(\gamma(t), v_2(\gamma), \gamma'(t), v_2(\gamma'))\Big)\eta(d\gamma')\eta(d\gamma) \\
&\ge \Lambda\|v_1 - v_2\|^2_{L^2_\eta} \\
&\quad + \frac{1}{2}\int\int \Big(v_1(\gamma) - v_2(\gamma), \nabla_p W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) \\
&\qquad\qquad - \nabla_p W(\gamma(t), v_2(\gamma), \gamma'(t), v_2(\gamma'))\Big)\eta(d\gamma')\eta(d\gamma) \\
&\quad + \frac{1}{2}\int\int \Big(v_1(\gamma') - v_2(\gamma'), \nabla_{p'} W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) \\
&\qquad\qquad - \nabla_{p'} W(\gamma(t), \tilde{v}_2(t, \gamma), \gamma'(t), v_2(\gamma'))\Big)\eta(d\gamma')\eta(d\gamma),
\end{aligned}$$

and then by uniform convexity

$$\left(\begin{pmatrix} v_1(\gamma)-v_2(\gamma) \\ v_1(\gamma')-v_2(\gamma') \end{pmatrix}, \begin{pmatrix} \nabla_p W(\gamma(t),v_1(\gamma),\gamma'(t),v_1(\gamma'))-\nabla_p W(\gamma(t),v_2(\gamma),\gamma'(t),v_2(\gamma')) \\ \nabla_{p'} W(\gamma(t),v_1(\gamma),\gamma'(t),v_1(\gamma'))-\nabla_{p'} W(\gamma(t),v_2(\gamma),\gamma'(t),v_2(\gamma')) \end{pmatrix}\right) \geq 0,$$

and then we conclude

$$\int \Big(v_1(\gamma)-v_2(\gamma), F(t,v_1)(\gamma)-F(t,v_2)(\gamma)\Big)\eta(d\gamma) \geq \Lambda \|v_1-v_2\|^2_{L^2_\eta}.$$

This is the lower bound of (2.30).

□

Proposition 2.3.3, page 29. *There exist unique functions* $Q(\alpha,t), P(\alpha,t) \in C_0([0,T], L^2(0,1))$ *satisfying*

$$Q(\alpha,t) = Q_0(\alpha) + \int_0^t \left[\nabla_p V(Q(\alpha,s),P(\alpha,s)) + \int \nabla_p W(Q(\alpha,s),P(\alpha,s),Q(\alpha',s),P(\alpha',s)d\alpha'\right] ds,$$

$$P(\alpha,t) = P_0(\alpha) - \int_0^t \left[\nabla_q V(Q(\alpha,s),P(\alpha,s)) + \int \nabla_q W(Q(\alpha,s),P(\alpha,s),Q(\alpha',s),P(\alpha',s)d\alpha'\right] ds.$$

Moreover

$$t \mapsto H(t,\mathcal{L}^1 \llcorner_{(0,1)}) = \int V(Q(\alpha,t),P(\alpha,t))d\alpha + \int\int W(Q(\alpha,t),P(\alpha,t),Q(\alpha',t),P(\alpha',t))d\alpha d\alpha'$$

is constant, $(\partial_t Q(t), P(t)) \in L^2((0,1), W^{1,2}(0,T))$ *with*

$$\|\partial_t Q(t)\|_{L^2(0,1)}, \|\partial_t P(t)\|_{L^2(0,1)}, \frac{1}{3L}\|\partial_t^2 Q(t)\|_{L^2(0,1)} \leq 3Le^{3Lt}\|(Q_0,P_0)\|_{L^2(0,1)}.$$

Finally, if $(Q_0,P_0),(Q_0',P_0')$ *are two different initial data and* $(Q(t),P(t)),(Q'(t),P'(t))$ *the corresponding solutions, then it holds*

$$\big\|(Q(t),P(t))-(Q'(t),P'(t))\big\|_{L^2(0,1)} \leq e^{3Lt}\big\|(Q_0,P_0)-(Q_0',P_0')\big\|_{L^2(0,1)}.$$

Proof of Proposition 2.3.3, page 29. The existence, uniqueness, and continuous dependence estimates boil down to the same computation: study the Lipschitz constant of the map

$$(Q(\alpha,t),P(\alpha,t)) \mapsto \left(x_0(\alpha) + \int_0^t \nabla_v H(Q(\alpha,s),P(\alpha,s))ds, v_0(\alpha) - \int_0^t \nabla_q H(Q(\alpha,s),P(\alpha,s))ds\right).$$

We show the continuous dependence: using the Lipschitz estimates for V, W,

$$\begin{aligned}\|Q(t)-Q'(t)\|_2 &= \left\|Q_0 - Q_0' + \int_0^t \big(\nabla_v H(Q(s),P(s))ds - \nabla_v H(Q'(s),P'(s))\big)ds\right\|_2 \\ &\leq \|Q_0-Q_0'\| + 3L\int_0^t \big\|(Q(s),P'(s))-(Q'(s),P'(s))\big\|_2 ds,\end{aligned}$$

Figure 2.4: two different conservative solutions of the Hamiltonian system (2.32), i.e. the stationary solution (red) and the solution $\bar{Q}$ of Proposition 2.B.3 (green).

$$\begin{aligned}\|P(t)-P'(t)\|_2 &= \left\| P_0 - P_0' - \int_0^t \big(\nabla_q H(Q(s),P(s))ds - \nabla_q H(Q'(s),P'(s))\big)ds \right\|_2 \\ &\leq \|P_0 - P_0'\| + 3L\int_0^t \big\|(Q(s),P'(s)) - (Q'(s),P'(s))\big\|_2 ds.\end{aligned}$$

Hence the continuous dependence follows by a Gronwall-type estimate.

A similar estimate gives that for $3Lt < 1$ the above map is a contraction when $(Q_0,P_0) = (Q_0',P_0')$, so that one deduces uniqueness. The convergence to the initial data follows from (2.9).

The estimates on $\ddot{x},\dot{v}$ follow by differentiating the ODE (2.9) and Proposition 2.3.2, and the conservation of energy $H(t,\mathcal{L}^1 \llcorner_{(0,1)})$ directly by differentiating w.r.t. t (which is now allowed since $\dot{x},\dot{v}$ are in $L^2(0,1)$. □

2.B An example of non-uniqueness

We present an example of non-uniqueness for the ODE in dimension 1 with Hamiltonian

$$H(\mu) = \int \frac{v^2}{2}\mu(dxdv) + \int W(x-x')\mu\times\mu(dxdvdx'dv'),$$

where the potential W is not semiconvex. The measure μ will be purely atomic.

The idea of the proof is that with a suitable distribution of masses and a suitable choice of the potential W, the ODE of one particle (here the one located at 0) has an Hölder dependence on the position, allowing for two solutions. The computations below just make this idea precise. Figure 2.4 depicts the two different conservative solutions: one is stationary (red), while in the other the particles are moving toward each other (green).

A natural question is whether this example can be adapted to W semiconvex, where a solution can be constructed [20].

Let

$$\phi(x) = \begin{cases} x & |x| \leq 1, \\ \mathrm{sign}(x) & |x| > 1, \end{cases} \qquad \Phi(x) = \begin{cases} x^2/2 & |x| \leq 1, \\ |x| - 1/2 & |x| > 1. \end{cases}$$

and define

$$\psi(x) = \sum_{n\in\mathbb{N}} \frac{\phi(n^{16}x)}{n^{16}}, \qquad \Psi(x) = \sum_n \frac{\Phi(n^{16}x)}{n^{32}}. \tag{2.31}$$

Let μ_0 be the initial configuration

$$\mu_0(dx) = \delta_0(dx) + \sum_n \frac{1}{n^8}\delta_{n^3}(dx),$$

with speed 0 for all $n = 0, 1, \ldots$.

Define

$$W(x) = \begin{cases} -\Phi(n^{16}(x-n^3))/n^{24} & |x-n^3| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x-n^3| < 1/2, \qquad n \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

which is explicitly

$$W(x) = \begin{cases} -n^8(x-n^3)^2/2 & |x-n^3| \leq n^{-16}, \\ (n^{16}|x| - 1/2)/n^{24} & n^{-16} < |x| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x-n^3| < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Its derivative is

$$W'(x) = \begin{cases} -\phi(n^{16}(x-n^3))/n^8 & |x-n^3| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x-n^3| < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

which is explicitly

$$W'(x) = \begin{cases} -n^8(x-n^3) & |x-n^3| \leq n^{-16}, \\ \text{sign}(x)/n^8 & n^{-16} < |x| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x-n^3| < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Its second derivative is

$$W''(x) = \begin{cases} -n^8 & |x-n^3| \leq n^{-16}, \\ 0 & n^{-16} < |x| \leq 1/3, \\ \text{smooth } \sim n^{-8} & n^{-16} < |x-n^3| < 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

showing that it is not semiconvex.

We have

$$\int x^2 \mu_0(dx) = \sum_n \frac{1}{n^8}(n^3)^2 < \infty,$$

$$\int W(x-y)\mu_0(dx)\mu_0(dy) = \sum_n \frac{1}{n^8}\frac{-\Phi(0)}{n^{12}} = 0.$$

We have observed that the only solutions to

$$n^3 - m^3 = z^3, \quad m, n \in \mathbb{N}, z \in \mathbb{Z}$$

is when $n = m, z = 0$ or $m = 0, n = z$ or $n = 0, z = m$, so that if the positions $x_n(t)$ are such that

$$|x_n - n^3| < 1/4$$

then

$$W(x_n - x_m) \neq 0 \;\Leftrightarrow\; \exists k \in \mathbb{Z}\big(|x_n - x_m - k^3| < 1/2\big) \;\Leftrightarrow\; \exists k \in \mathbb{Z}\big(|n^3 - m^3 - k^3| < 1\big)$$

and the last inequality implies

$$\big(n = m \;\wedge\; k = 0\big) \;\vee \big(n = k \;\wedge\; m = 0\big) \;\vee\; \big(m = -k \;\wedge\; n = 0\big).$$

In particular, if the position remains inside $n^3 + [-1, 1]/4$, then

$$\sum_{n,n'} m_n m_{n'} W(x_n - x_{n'}) = 2\sum_{n\geq 1} m_n W(x_n - x_0) \simeq \sum_n n^{-8-8} < \infty.$$

Let

$$q_n = x_n - n^3.$$

When

$$x_n = n^3 + q_n \in n^3 + [-1/4, 1/4],$$

the equation (2.9) for this case can be rewritten as a system of second order ODEs: using $\dot{q}_n = v_n$, it is immediate to see that

$$\begin{cases} \ddot{q}_0 = \sum_{n=1}^{+\infty} m_i W'(q_n - q_0) \\ \ddot{q}_1 = m_0 W'(q_0 - q_1) \\ \ddot{q}_2 = m_0 W'(q_0 - q_2) \\ \vdots \\ \ddot{q}_n = m_0 W'(q_n - q_0) \\ \vdots \end{cases}$$

which is explicitly (being ϕ antisymmetric)

$$\begin{cases} \ddot{q}_0 = \sum_{n=1}^{+\infty} \frac{\phi(n^{16}(q_0-q_n))}{n^{16}} \\ \ddot{q}_1 = \phi(q_1 - q_0) \\ \ddot{q}_2 = 2^{-8}\phi(2^{16}(q_2 - q_0)) \\ \vdots \\ \ddot{q}_n = n^{-8}\phi(n^{16}(q_n - q_0)) \\ \vdots \end{cases} \tag{2.32}$$

2.B.1 Non-uniqueness for the particle in the origin

Consider the ODE

$$\ddot{u} = \psi(u), \quad u(0) = u'(0) = 0,$$

where ψ is given in (2.31). Multiplying by $\dot{u}$ and integrating

$$\frac{\dot{u}^2}{2} = \Psi(u),$$

which has a solution not identically 0 if and only if

$$\int_0^\delta \frac{1}{\sqrt{\Psi(u)}} du < \infty.$$

We study the Hölder exponent of the function Ψ when $0 \leq u \ll 1$.

Using the explicit formulas for Ψ we have

$$\begin{aligned} \Psi(u) &= \sum_n n^{-32}\Phi(n^{16}u) \\ &= \Bigg(\sum_{n^{16}u\leq 1} n^{-32}\Bigg)\frac{(n^{16}u)^2}{2} + \sum_{n^{16}u>1} n^{-32}\left(n^{16}u - \frac{1}{2}\right) \\ &= \Bigg(\sum_{n^{16}u\leq 1} 1\Bigg)\frac{u^2}{2} + \Bigg(\sum_{n^{16}u>1} n^{-16}\Bigg)u - \frac{1}{2}\sum_{n^{16}u>1} n^{-32}. \end{aligned}$$

We now use the estimates

$$\sum_{n^{16}u\leq 1} \sim \int_1^{u^{-1/16}} d\omega \sim u^{-1/16},$$

$$\sum_{n^{16}u>1} n^{-16} \sim \int_{u^{-1/16}}^{\infty} \omega^{-16} d\omega = \frac{1}{15}u^{15/16},$$

$$\sum_{n^{16}u>1} n^{-32} \sim \int_{u^{-1/16}}^{\infty} \omega^{-32} d\omega = \frac{1}{31} u^{31/16}.$$

Hence for $u \ll 1$ it holds

$$\Psi(u) \sim \left(\frac{1}{2} + \frac{1}{15} - \frac{1}{2}\frac{1}{31} \right) u^{31/16} = \frac{256}{465} u^{31/16}.$$

since

$$\int_0^\delta \frac{1}{\sqrt{\Psi(\omega)}} d\omega \sim \int_0^\delta \omega^{-31/32} d\omega \simeq \delta^{1/32}.$$

Thus there is no uniqueness, and the nonzero solution will behave like t^{32}.

2.B.2 Non-uniqueness - part 2

Before we assumed that the other particles remain in $q_n = 0$ (i.e. $x_n = n^3$). The idea of this section is that they will approach q_0, so that, the actual force in q_0 is larger.

Let $\Theta(t, u)$ be a continuous function such that

$$\frac{3}{2} \geq \Theta(t, u) \geq \Psi'(u) \sim u^{15/16}, \qquad u \in [0, 1/4],$$

and consider the ODE

$$\ddot{u} = \Theta(t, u) \quad \Rightarrow \quad \begin{cases} \dot{t} = 1, \\ \dot{u} = w, \\ \dot{w} = \Theta(t, u). \end{cases}$$

The forward-in-time invariant region $S \subset \mathbb{R}^3$ we consider is the region

$$S = \left\{ 0 \leq u \leq 1/4,\ \sqrt{u}/2 \leq t \leq 2\zeta^{-1}(u),\ \dot{\zeta}(\zeta^{-1}(u)) \leq w \leq 2\sqrt{u} \right\},$$

where $\zeta(t)$ is the graph of the non-zero solution to

$$\dot{u} = \frac{1}{2}\sqrt{\Psi(u)}, \quad u(0) = 0,$$

which we know to behave like

$$\zeta(t) \sim t^{32}, \qquad \dot{\zeta}(\zeta^{-1}(u)) \sim u^{31/32}$$

by the computation at the end of the previous section. The other bound is obtained by solving

$$\dot{u} = w, \quad \dot{w} = 2 \quad \Rightarrow \quad u = t^2, w = t.$$

The region is forward invariant because

$$w = 2\sqrt{u} \quad \Rightarrow \quad \frac{\dot{w}}{\dot{u}} = \frac{\Theta(t,u)}{2\sqrt{u}} \leq \frac{3}{4\sqrt{u}} < \frac{1}{\sqrt{u}} = \partial_u(2\sqrt{u}),$$

$$w = \zeta(u) \quad \Rightarrow \quad \frac{\dot{w}}{\dot{u}} = \frac{\Theta(t,u)}{\zeta(u)} \geq \frac{\Psi'(u)}{\zeta(u)} > \frac{\Psi'(u)/8}{\zeta(u)} = \partial_u(\zeta(u)),$$

$$t = \frac{\sqrt{u}}{2} \quad \Rightarrow \quad \frac{\dot{t}}{\dot{u}} = \frac{1}{w} \geq \frac{1}{2\sqrt{u}} > \frac{1}{4\sqrt{u}} = \partial_u\bigg(\frac{\sqrt{u}}{2}\bigg),$$

$$t = 2\zeta^{-1}(u) \quad \Rightarrow \quad \frac{\dot{t}}{\dot{u}} = \frac{1}{w} \leq \frac{1}{\zeta'(\zeta^{-1}(u))} < \frac{2}{\zeta'(\zeta^{-1}(u))} = \partial_u(2\zeta^{-1}(u)).$$

In all formulas above we are comparing the vector field $(1, w, \Theta(t,u))$ with the tangent to the boundary (whose slope is at the r.h.s. of the previous formulas).

The strict inequalities give that the flow is entering the region S: in particular, every trajectory entering in S at some point $(t, x, w) \in \partial S$ is exiting S in some point in the interior of the region

$$E = \bigg\{u = \frac{1}{4},\ \zeta'(\zeta^{-1}(1/4)) \leq w \leq 1,\ \frac{1}{4} \leq t \leq \zeta^{-1}(1/4)\bigg\}.$$

Lemma 2.B.1. *If $\Psi'(u) \leq \Theta(t,u) \leq 3/2$, there is a trajectory starting from $(0,0)$ at $t = 0$ and reaching $u(\bar{t}) = 1/4$ inside S with $1/2 < \bar{t} < \zeta^{-1}(1/4)$.*

Proof. Consider a sequence of points $(t_n, x_n, w_n) \in \partial S$ converging to $(0,0,0)$, let γ_n be a trajectory starting from (t_n, x_n, w_n) inside S. Then, up to subsequences, the limit trajectory γ satisfies the statement. □

We will denote such a trajectory with $\hat{q}_0(t)$, and we can assume that it is defined for $t \in [0, 1/2]$ and $\hat{q}_0(t) > 0$ for $t > 0$.

We next analyze the other components. The ODE for $|q_n| \leq 1/3$ is

$$\ddot{q}_n = n^{-8}\phi\big(n^{16}(q_n - q_0(t))\big), \quad q_n(0) = \dot{q}_n(0) = 0. \tag{2.33}$$

We assume that the function $q_0(t)$ is given, and it is ≥ 0. Then the ODE above is rewritten as

$$\dot{q}_n = w_n,\ \dot{w}_n = n^{-8}\phi(n^{16}(q_n - q_0(t)),$$

and then the quarter plane $\{q_n, w_n \leq 0\}$ is forward invariant for $t \in [0, 1/3]$: indeed the vector field is of order n^{-8} and Lipschitz, and

$$q_n = 0 \quad \Rightarrow \quad \dot{q}_n \leq 0,$$

$$w_n = 0, \quad \Rightarrow \quad \dot{w}_n \leq 0.$$

We have used that $q_0 \geq 0$ and the uniqueness of the solution: hence

Lemma 2.B.2. *For every $q_0(t) \geq 0$ there is a unique solution $q_n(t)$ to* (2.33) *such that*

$$-\mathcal{O}(n^{-8}) \leq q_n(t), \dot{q}_n(t) \leq 0.$$

Finally, define the compact set

$$K = \mathrm{Lip}([0,1/3],[-1,1]) \times \big(\mathrm{Lip}([0,1/3],[-1,1])\big)^{\mathbb{N}} \subset (C^0([0,1/3],\mathbb{R}))^{\mathbb{N}_0}$$

with the product topology. Given $Q = \{q_n(t)\}_n \in K$, then construct $Q' = \{q'_n(t)\}_n \in K$ as the point whose coordinates are

$$q'_0 = \text{ a solution by Lemma 2.B.1 with } \Theta(t,u) = \sum_n n^{-16}\phi(n^{16}(u - q_n(t))),$$

$$q'_n = \text{ the solution by Lemma 2.B.2.}$$

Since $q_n \leq 0$ for $n \geq 1$, then

$$\Psi'(u) = \sum_n n^{-16}\phi(n^{16}u) \leq \Theta(t,u) = \sum_n n^{-16}\phi(n^{16}(u - q_n(t))) \leq \frac{3}{2}, \qquad 0 \leq u, -q_n(t) \leq \frac{1}{2},$$

so that the assumptions of Lemma 2.B.1 are satisfied for $0 \leq t \leq \frac{1}{3}$.

It is fairly easy to see that $Q \mapsto Q'$ maps K into K.

Repeating the process countably many times, we obtain a family of point $Q_i = \{q_{n,i}\}_n \in K$: assume by compactness that

$$\lim_i q_{n,i}(t) = \bar{q}_n(t)$$

in C^0 up to subsequences. Then

$$\Theta_i(t, q_{0,i}) = \sum_n n^{-16}\phi(n^{16}(q-0,i(t) - q_{n,i}(t))) \to \bar{\Theta}(t,\bar{q}_0) = \sum_n n^{-16}\phi(n^{16}(\bar{q}_0(t) - \bar{q}_n(t)))$$

because the series is uniformly convergent and

$$\phi(n^{16}(q_0(t) - q_n(t)) \to \phi(n^{16}(\bar{q}_0(t) - \bar{q}_n(t)).$$

In particular, since each $q_{0,i}$ is a trajectory in S by Lemma 2.B.1, we deduce

Proposition 2.B.3. *The limit point $\bar{Q} = \{\bar{q}_n(t)\}_n$ is a non constant solution to* (2.32).

2.C An non-trivial example of uniqueness

Consider the space $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$: in this section we will use the notation

$$q = (q_1, q_2, q_3), p = (p_1, p_2, p_3), \quad q, p \in \mathbb{R}^3,$$

i.e. q_i, p_i are the i-th component of the vectors q, p.

Assume

$$H(\mu) = \int \frac{p^2}{2}\mu + \frac{\epsilon}{2}\int\int \frac{q_1^2(q_1')^2}{4}\mu \times \mu = \int \frac{p^2}{2}\mu + \frac{\epsilon}{2}\bigg(\int \frac{q_1^2}{2}\mu\bigg)^2,$$

with $0 < \epsilon \ll 1$. The Hamiltonian has not quadratic growth, but since we will consider solutions for $t \in [-1, 2]$ such that

$$\text{supp}(\mu) \subset \{|q| \leq 12\} \times \mathbb{R}^3,$$

we can alter the function $W(q, q') = \epsilon q_1^2 (q_1')^2/4$ outside $B_{12}^{\mathbb{R}^3}(0)$ arbitrarily.

The Hamiltonian ODE in the second and third coordinates is

$$\dot{q}_i = p_i, \ \dot{p}_i = 0, \quad i = 2, 3,$$

whose solution is

$$q_i(t) = p_i(0)t + q_i(0), \quad i = 2, 3,$$

while the first component satisfies the ODE

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = -\epsilon\bigg(\int \frac{(q_1')^2}{2}\mu(dq')\bigg)q_1. \tag{2.34}$$

For $\alpha \in [-1, 1]$ consider the initial data for $i = 2, 3$

$$(q_2, q_3)(\alpha, 0) = (\alpha, -|\alpha|), \quad (p_2, p_3)(\alpha, 0) = (-\text{sign}(\alpha), 1).$$

The solutions are

$$(q_2, q_3)(\alpha, t) = \big(\alpha - \text{sign}(\alpha)t, t - |\alpha|\big),$$

and then the unique intersection point of the trajectories $q(\alpha), q(\alpha')$ occurs only for

$$\alpha' = -\alpha, \quad t = |\alpha|, \quad (q_2, q_3)(\alpha, |\alpha|) = (0, 0).$$

Claim 1: *there are initial data at $t = -1$ such that the conservative solution $(Q(t), P(t))$ satisfies*

$$q_1(\alpha, |\alpha|) = q_1(-\alpha, |\alpha|) = 0, \quad p_1(\alpha, |\alpha|) = -p_1(-\alpha, |\alpha|) = -1.$$

Proof. It is a standard contraction argument for the map

$$\big\{q_1(\alpha,t),p_1(\alpha,t)\big\}_\alpha \mapsto \bigg\{\int_{|\alpha|}^t p_1(\alpha,\tau)d\tau, -\mathrm{sign}(\alpha) - \epsilon\int_{|\alpha|}^t \bigg(\int \frac{p_1(\alpha',\tau)^2}{2}\mu(d\alpha')\bigg) q_1(\alpha,\tau)d\tau\bigg\},$$

which is a contraction for $\epsilon \ll 1$ and $t \in [-1,2]$.

□

The values $\{Q(\alpha,-1),P(\alpha,-1)\}$ are the initial data for the dissipative solutions we are going to study.

When we consider a possible dissipative solution η, the projection of the motion on the component 2,3 is exactly a sticky particle system in two dimensions, or more precisely it is a dissipative solution to the sticky particle PDE in dimension 2. The third component gives

$$\mathbb{P}_t\bigg(p_3(\alpha,0) - \int_0^t \nabla_x H(q_3(s),p_3(s))ds\bigg) = \mathbb{P}_t(1) = 1,$$

so that the third component of the trajectories of the dissipative solution is again

$$q_3(t) = t - |\alpha|.$$

In particular, only the particles $(|\alpha|,-|\alpha|)$ can interact, and only at time $|\alpha|$, and no additional interactions can occur at a later time.

We can thus write the dissipative solution with the parametrization $q(|\alpha|,\beta),p(|\alpha|,\beta)$, $\beta \in [0,1]$, with the measure $\mathcal{L}^2(d|\alpha|d\beta)$, and the projection $\mathbb{P}_t$ as $\mathbb{P}_{|\alpha|}$ acting on $L^2(d\beta)$. The function $\alpha \mapsto \mathbb{P}_\alpha$ can be assumed Borel, in the sense that for every $f \in L^2(d\beta)$ the function $\alpha \mapsto \mathbb{P}_\alpha(f)$ is Borel.

Let $(\hat{Q},\hat{P})(|\alpha|,\beta)$ be a dissipative solution with initial data $\{Q(\alpha,-1),P(\alpha,-1)\}$, which in the parametrization $(|\alpha|,\beta)$ corresponds to

$$(Q,P)(|\alpha|,\beta,-1) = \begin{cases} (Q,P)(-|\alpha|,-1) & \beta \in [0,1/2], \\ (Q,P)(|\alpha|,-1) & \beta \in (1/2,1]. \end{cases}$$

Let $t_0 \in [1,2]$ be the first time such that

$$\forall t > t_0 \bigg(\int_{t_0}^t \bigg[\int \big|(\mathbb{I} - \mathbb{P}_{|\alpha|})Q(|\alpha|,\beta,\tau)\big|^2 d|\alpha|d\beta\bigg] d\tau > 0\bigg).$$

Here and in the following (Q,P) is the conservative solution, while $(\hat{Q},\hat{P})$ is the dissipative one (both parametrized by $|\alpha|,\beta$).

As an approximation for the dissipative solution, we define

$$(\check{Q},\check{P})(|\alpha|,\beta,t) = \begin{cases} (Q,P)(|\alpha|,\beta) & t \leq |\alpha|, \\ \mathbb{P}_{|\alpha|}(Q,P)(|\alpha|,\beta) & t > |\alpha|, \end{cases}$$

i.e. the trajectories obtained by patching together the conservative solution before the merging time $|\alpha|$, and its projection after the merging time. This is not a solution, since one has for the first component of a trajectory of the approximate dissipative solution above

$$\ddot{\check{q}}_1(|\alpha|,\beta) = -\epsilon\frac{\|q_1\|_2^2}{2}\check{q}_1(|\alpha|,\beta) \neq -\epsilon\frac{\|\check{q}_1\|_2^2}{2}\check{q}_1(|\alpha|,\beta). \tag{2.35}$$

In particular, for $t > t_0$ it holds by Jensen's inequality and strict convexity of $|\cdot|^2$

$$\int_{t_0}^t \frac{\|\check{q}_1\|_2^2}{2} - \frac{\|q_1\|_2^2}{2} d\tau < 0.$$

The contradiction we will arrive is exactly in the inequality above, which implies that the particles $q_1(|\alpha|,\beta \in [0,1/2]), q_1(|\alpha|,\beta \in (1/2,1])$ with arrive late at the merging time $t = |\alpha|$.

Claim 2: *The correction $\delta q_1, \delta p_1$ to $\check{q}_1, \check{p}_1$ satisfies*

$$\dot{\delta q_1} = \delta p_1, \quad \dot{\delta p_1} = -\epsilon\bigg(\int \frac{(\check{q}_1+\delta q_1)^2}{2}\mathcal{L}^2\bigg)\delta q_1 + \epsilon\bigg[\frac{\|q_1\|_2^2}{2} - \int \frac{(\check{q}_1+\delta q_1)^2}{2}\mathcal{L}^2\bigg]\check{q}_1, \tag{2.36}$$

with initial data $(0,0)$.

Proof. Just substitute and use (2.35). □

We next use the following simple estimate: if

$$\dot{x} = v,\ \dot{v} = a(t)x + b(t), \qquad x(0), v(0) = 0,$$

then for every $\delta > 0$ there exists $\bar{t}$ such that for $t \in [0,\bar{t}]$

$$|x(t)| \leq (1+\delta)\int_0^t (t-\tau)|b(\tau)|d\tau, \quad |v(t)| \leq (1+\delta)\int_0^t |b(\tau)|d\tau. \tag{2.37}$$

Moreover $\bar{t} = \frac{\delta}{3(1+\|a\|_\infty)}$ suffices.

Claim 3: *It holds*

$$\int \big|\delta q_1(t)\big|\mathcal{L}^2 \leq 2\epsilon\int_{t_0}^t \bigg(\frac{\|q_1\|_2^2}{2} - \frac{\|\check{q}_1\|_2^2}{2}\bigg)d\tau.$$

Proof. The estimate (2.37) applied to (2.36) yields

$$|\delta q_1(|\alpha|,\beta,t)| \leq (1+\delta)\epsilon \int_{t_0}^{t} \left| \frac{\|q_1\|_2^2}{2} - \int \frac{(\check{q}_1 + \delta q_1)^2}{2} \mathcal{L}^2 \right| |\check{q}_1(|\alpha,\beta,\tau)| d\tau,$$

for

$$0 \leq t - t_0 \leq \bar{t} = \frac{\delta}{3(1 + \epsilon \sup_{\tau \in [t_0,t]} \int \frac{(\check{q}_1(\tau) + \delta q_1(\tau))^2}{2} \mathcal{L}^2)}.$$

It is an easy computation to show that the first equation gives the claim if $\epsilon \ll 1$ and, in particular, the choice $\bar{t} = 1/4$ can be allowed. □

With the above claim, we obtain that

$$\int_{t_0}^{t} \left(\frac{\|q_1(\tau)\|_2^2}{2} - \int \frac{(\check{q}_1(\tau) + \delta q_1(\tau))^2}{2} \mathcal{L}^2 \right) d\tau = \big(1 + \mathcal{O}(\epsilon t)\big) \int_{t_0}^{t} \left(\frac{\|q_1(\tau)\|_2^2}{2} - \frac{\|\check{q}_1(\tau)\|_2^2}{2} \right) d\tau > 0$$

for $0 < t - t_0 < \bar{t}$, and then using again (2.36) we conclude that $\delta q_1 < 0$ in a small positive time interval (t_0, t_1). This implies that for $t_0 < |\alpha| < t_1$ the particles solving the ODE (2.34) (i.e. the ones which have not yet interacted) will have

$$q_1(|\alpha|, \beta \in [0,1/2], |\alpha|) < 0 < q_1(|\alpha|, \beta \in (1/2,1], |\alpha|),$$

contradicting the assumption that they are interacting, i.e. $q_1(|\alpha|, \beta \in [0,1/2], |\alpha|) = q_1(|\alpha|, \beta \in (1/2,1], |\alpha|)$.

Part II

Chapter 3

Existence and blow-up for non-autonomous scalar conservation laws with viscosity

In this chapter we consider a question posed in [19], namely the blow-up of the PDE

$$u_t + (b(t,x)u^{1+k})_x = u_{xx}$$

when b is uniformly bounded, Lipschitz and $k = 2$. We give a complete answer to the behavior of solutions when b belongs to the Lorentz spaces $b \in L^{p,\infty}$, $p \in (2,\infty]$, or $b_x \in L^{p,\infty}$, $p \in (1,\infty]$. See also [9].

3.1 Introduction

We study the global in time existence and long time behavior for the initial value problem

$$\begin{cases} u_t + (b(t,x)u^{k+1})_x = u_{xx}, \quad x \in \mathbb{R},\, t \in (0,\infty), \\ u(0,x) = u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \end{cases} \tag{3.1}$$

where $b(t,x)$ is a non-autonomous drift and u_0 is the initial datum. This question was raised in [19], where the subcritical case was analyzed. It is not restrictive to assume that

$$u(t,x) \geq 0,$$

because the PDE 3.1 is monotone w.r.t. the initial data. Moreover by scaling $u(a^2t, ax)$ we can assume that $\|u_0\|_1 = \|u(t)\|_1 = 1$, where we have used that the PDE is in divergence form.

In this thesis we consider drifts b which are in weak-L^P or with derivative in weak-L^p. More precisely, we make the following assumptions on the initial data u_0 and exponent k:

1. b is only integrable:

$$b \in L^\infty_{\mathrm{loc}}((0,\infty), L^{p,\infty}(\mathbb{R})) \quad \text{with } p \in (2,\infty], \qquad \begin{cases} k > 0, \\ u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}); \end{cases} \tag{3.2}$$

2. b has a weak derivative in x:

$$\begin{cases} b \in L^\infty_{\mathrm{loc}}((0,\infty) \times \mathbb{R}), \\ b_x \in L^\infty_{\mathrm{loc}}((0,\infty), L^{p,\infty}(\mathbb{R})) \quad \text{with } p \in (1,\infty], \end{cases} \qquad \begin{cases} k \geq \frac{1}{2}, \\ u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ E(u_0) := \int x^2 |u_0(x)|\, dx < +\infty. \end{cases} \tag{3.3}$$

The space $L^{p,\infty}$ is the standard Lorentz space, see Subsection 3.2.1 for the precise definition: we just recall here that $L^{p,\infty}$ is also referred to as the weak-L^p space.

The case $k = 0$ corresponds to a linear PDE, which can be studied by means of the Duhamel formula: hence we will restrict ourselves to $k > 0$. The assumption $k \geq \frac{1}{2}$ in (3.3) is due to the fact that the drift b may be unbounded. If b is uniformly bounded that one can remove this assumption. It has actually no influence in the blow-up behaviour, which is a local property.

The aim is to investigate the relation between the exponents k and p so that the solution exists globally in L^∞ and to study the behaviour of $u(t)$ for $t \to \infty$. In the time interval $[0, T)$ where $u(t) \in L^\infty_{\mathrm{loc}}([0,T), L^\infty(\mathbb{R}))$, classical contraction principles show that the solution is unique in the class

$$\begin{aligned} &u \in L^\infty_{\mathrm{loc}}([0,T), L^\infty(\mathbb{R})) \cap C^0([0,T), L^{p'}(\mathbb{R})) \quad \text{under assumptions (3.2)}, \\ &u \in L^\infty_{\mathrm{loc}}([0,T), L^\infty(\mathbb{R})) \cap C^0([0,T), L^2(\mathbb{R})) \quad \text{under assumptions (3.3)}. \end{aligned} \tag{3.4}$$

(See Section 3.3 for details).

The results of this thesis about the existence of a bounded solution can be summarized in the following theorem.

Theorem 3.1.1. *Assume that the drift b and the initial condition u_0 satisfy* (3.2) *with $k \leq 1 - \frac{1}{p}$ or satisfy* (3.3) *with $k \leq 2 - \frac{1}{p}$. Then solution $u(t)$ of* (3.1) *is globally defined* $[0, \infty)$.

Conversely, assume that b, u_0 satisfy (3.2) *with $k > 1 - \frac{1}{p}$ or satisfy* (3.3) *with $k > 2 - \frac{1}{p}$. Then there are bounded initial data such that the corresponding solutions of* (3.1) *blow up in L^∞ in finite time.*

In particular, under the conditions of the first part of the previous statement, the solution is unique in the regularity class (3.4).

The first part of the above statement is contained in Theorem 3.4.5, Section 3.4.4. The second part instead in given in Theorems 3.5.2 and 3.5.4, Section 3.5.

The analysis of the subcritical case $k < 2$ and b bounded, Lipschitz has been done in [19]. The above theorem extends their results to other classes of drift b. The blow-up results follow the analysis in [26], where a specific drift is considered in the multidimensional case with $k = 2$.

Concerning the long time behavior, we assume uniform bounds on b, i.e. we strengthen the conditions on b as follows:

1. under the assumptions (3.2), we also require
$$b \in L^\infty((0,\infty), L^{p,\infty}(\mathbb{R})), \quad p \in (2,\infty]; \tag{3.5}$$
2. under the assumptions (3.3), we also require
$$b \in L^\infty((0,\infty)\times\mathbb{R}),\ b_x \in L^\infty((0,\infty), L^{p,\infty}(\mathbb{R})), \quad p \in (1,\infty]. \tag{3.6}$$

We obtained the following results (Theorem 3.6.1, Section 3.6):

Theorem 3.1.2. *Assume that the solution is bounded for all $t > 0$ and the conditions* (3.2), (3.5), *$k \geq 1 - \frac{1}{p}$ hold, or* (3.3), (3.6), *$k \geq 2 - \frac{1}{p}$ hold. Then $\|u(t)\|_\infty \lesssim t^{-\frac{1}{2}}$ as $t \to \infty$.*

Viceversa, assume (3.2), (3.5) *with $k < 1 - \frac{1}{p}$ or* (3.3), (3.5) *with $k < 2 - \frac{1}{p}$. Then there are drifts b and initial data u_0 such that the corresponding solutions of* (3.1) *do not decay to* 0 *as $t \to \infty$.*

In particular, in the case $b_x \in L^\infty_{\mathrm{loc}}((0,\infty), L^\infty(\mathbb{R}))$ we answer to a question raised in [19], precisely Question 3:

> "Is it possible to guarantee global existence for solutions of the problem (1) when $k \geq 2$, $p = \infty$?" (3.7)

The problem (1) referred above in (3.7) and considered there is the PDE
$$\begin{cases} u_t + (b(x,t)u^{k+1})_x = \mu(t)u_{xx}, \\ u_0 \in L^1(\mathbb{R}) \cap L^\infty,\ u_0 \geq 0 \end{cases} \tag{3.8}$$
with μ strictly positive continuous function and b uniformly bounded and Lipschitz. The PDE (3.8) and (3.1) are equivalent because of the following time transformation:
$$u(t,x) = v(\tau(t),x), \quad \dot\tau(t) = \mu(t), \quad \tau(0) = 0,$$
which leads to the equation (3.1), namely
$$v_t + (\tilde b(\tau,x)v^{k+1})_x = v_{xx}, \quad \tilde b(\tau(t),x) = \frac{b(t,x)}{\mu(t)}.$$

The results of the above theorem can be summarized in the following table: setting
$$\mathbf{critic}(p) = \begin{cases} 1 - 1/p & b \in L^{p,\infty}, \\ 2 - 1/p & b_x \in L^{p,\infty}, \end{cases}$$
we have

	$k < \mathbf{critic}(p)$	$k = \mathbf{critic}(p)$	$k > \mathbf{critic}(p)$
$b \in L^{p,\infty},\ 2 < p \leq \infty$ or $b_x \in L^{p,\infty}, 1 < p \leq \infty$	global existence no decay in general	global existence and decay as $t^{-\frac{1}{2}}$	blow-up in finite time, if bounded then decay as $t^{-\frac{1}{2}}$

3.2 Preliminaries

Given a function $f \in L^1_{\text{loc}}(\mathbb{R})$, and $\alpha \geq 0$ we define the α-moment of f as

$$M_\alpha(f) = \int |x|^\alpha |f|\, dx.$$

In particular we will use the notation

$$m(f) = M_0(f), \text{ as the } \textit{mass} \text{ of } f, \qquad E(f) = M_2(f), \text{ as the } \textit{energy} \text{ of } f.$$

We will write

$$(f \wedge g)(x) := \min\{f(x), g(x)\}, \qquad (f \vee g)(x) := \max\{f(x), g(x)\}.$$

The letter C will be a constant that could change line by line, also we use the symbol $a \lesssim b$ as shorthand for $a \leq Cb$ for some constant C. We will write $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$ hold.

The symbol $*$ will denote the convolution operation:

$$(f * g)(x) = \int f(y)g(x-y)dy.$$

3.2.1 Lorentz spaces

We briefly recall the definition and some results about Lorentz space, see [25].

Let $f^* : (0,\infty) \to \mathbb{R}$ be the symmetric decreasing rearrangement defined by

$$f^*(x) = \inf\Big\{\alpha > 0 : \mathcal{L}^1\big(\{|f| > \alpha\}\big)| \leq x\Big\},$$

and let $f^{**}(x)$ be the function defined by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(y)dy.$$

Define

$$\|f\|_{p,q} = \begin{cases} \left(\displaystyle\int_0^\infty \left[x^{\frac{1}{p}} f^{**}(x)\right]^q \frac{dx}{x}\right)^{\frac{1}{q}} & q \in [1,\infty), p \in (1,\infty) \\ \displaystyle\sup_{x>0} x^{\frac{1}{p}} f^{**}(x) & q = \infty, p \in [1,\infty]. \end{cases} \tag{3.9}$$

Note that

$$\|f\|_{1,\infty} = \|f^*\|_1 = \|f\|_1, \quad \|f\|_{\infty,\infty} = \lim_{x\searrow 0} f^{**}(x) = \|f\|_\infty.$$

By Hardy's inequality [25, Lemma 2.3]

$$\left[\int_0^\infty \left(x^{1/p-1}\int_0^x |f(t)|dt\right)^q \frac{dx}{x}\right]^{1/q} \le p'\left[\int_0^\infty \left(x^{1/p}|f(x)|\right)^q \frac{dx}{x}\right]^{1/q}, \quad \frac{1}{p}+\frac{1}{p'}=1,\ p\in(1,\infty),$$

and some fairly easy computations, one can verify that for $1<p<\infty$ the above definition is equivalent to

$$|||f|||_{p,q} = \begin{cases} \left(\int_0^\infty \left[x^{\frac{1}{p}} f^*(x)\right]^q \frac{dx}{x}\right)^{\frac{1}{q}} & q\in[1,\infty), p\in[1,\infty), \\ \sup_{x>0} x^{\frac{1}{p}} f^*(x) & q=\infty, p\in[1,\infty]. \end{cases} \tag{3.10}$$

For $p=q=\infty$ the equivalence is elementary. For $p=1, q=\infty$ the quantity in (3.10) is the weak-L^1 norm, while (3.9) corresponds to the L^1-space: the L^1-norm in the above definition is realized for $p=q=1$. It is elementary to deduce that for $p\in(1,\infty)$

$$\|f^k\|_{p,q} \le p'|||f^k|||_{p,q} = p'|||f|||^k_{kp,kq} \le p'\|f\|^k_{kp,kq}. \tag{3.11}$$

It is immediate to check that

$$\|f^k\|_{\infty,\infty} = \|f^k\|_\infty = \|f\|^k_\infty = \|f\|^k_{\infty,\infty}.$$

Definition 3.2.1 (Lorentz space)**.** The *Lorentz space* $L^{p,q}$ is the space of (equivalence classes of) measurable functions f such that $\|f\|_{p,q}<\infty$. It is a Banach space with the norm $\|\cdot\|_{p,q}$.

We will use the following results.

Proposition 3.2.2 ([25, Lemma 2.2])**.** *Let* $1<p<\infty$*, then*

$$\|f\|_p \le \|f\|_{p,p} \le p'\|f\|_p, \qquad \textit{where} \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Proposition 3.2.3 ([25, Lemma 2.5])**.** *Suppose* $1<p<\infty$ *and* $1\le q<r\le\infty$*, then*

$$\|f\|_{p,r} \le \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{r}} \|f\|_{p,q}.$$

Another elementary estimate we use is the following.

Lemma 3.2.4. *For $p_1 < p < p_2$ and $q, q_1, q_2 \geq 1$ it holds*

$$\|f\|_{p,q} \leq \left(\frac{1}{\frac{q}{p}-\frac{q}{p_2}} + \frac{1}{\frac{q}{p_1}-\frac{q}{p}}\right)^{\frac{1}{q}} \left(\frac{p}{q_1}\right)^{\frac{1}{q_2}\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}} \left(\frac{p}{q_2}\right)^{\frac{1}{q_1}\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}} \|f\|_{p_1,q_1}^{\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}} \|f\|_{p_2,q_2}^{\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}}.$$

Proof. The fundamental estimate is that, being f^{**} decreasing, then

$$\frac{q}{p}\bar{x}^{\frac{q}{p}}(f^{**}(\bar{x}))^q \leq \int_0^{\bar{x}} \left[x^{\frac{1}{p}} f^{**}(x)\right]^q \frac{dx}{x} \leq \|f\|_{p,q}^q. \tag{3.12}$$

Hence we can write for $q < \infty$

$$\begin{aligned}
\|f\|_{p,q}^q &= \left[\int_0^{\bar{x}} + \int_{\bar{x}}^{\infty}\right] \left[x^{\frac{1}{p}} f^{**}(x)\right]^q \frac{dx}{x} \\
&= \int_0^{\bar{x}} x^{\frac{q}{p}-\frac{q}{p_2}} \left(x^{\frac{1}{p_2}} f^{**}(x)\right)^q \frac{dx}{x} + \int_{\bar{x}}^{\infty} x^{\frac{q}{p}-\frac{q}{p_1}} \left(x^{\frac{1}{p_1}} f^{**}(x)\right)^q \frac{dx}{x} \\
\left[(3.12) \text{ for } p_1, p_2\right] \quad &\leq \int_0^{\bar{x}} x^{\frac{q}{p}-\frac{q}{p_2}} \|f\|_{p_2,q_2}^q \frac{dx}{x} + \int_0^{\bar{x}} x^{\frac{q}{p}-\frac{q}{p_1}} \|f\|_{p_1,q_1}^q \frac{dx}{x} \\
&= \frac{1}{\frac{q}{p}-\frac{q}{p_2}} \left(\frac{p}{q_2}\right)^{\frac{q}{q_2}} \|f\|_{p_2,q_2}^q \bar{x}^{\frac{q}{p}-\frac{q}{p_2}} + \frac{1}{\frac{q}{p_1}-\frac{q}{p}} \left(\frac{p}{q_1}\right)^{\frac{q}{q_1}} \|f\|_{p_1,q_1}^q \bar{x}^{\frac{q}{p}-\frac{q}{p_1}}.
\end{aligned}$$

One can directly check that the same estimate holds for $q = \infty$ as

$$\|f\|_{p,\infty} \leq \left(\frac{p}{q_2}\right)^{\frac{1}{q_2}} \|f\|_{p_2,q_2} \bar{x}^{\frac{1}{p}-\frac{1}{p_2}} + \left(\frac{p}{q_1}\right)^{\frac{1}{q_1}} \|f\|_{p_1,q_1} x^{\frac{1}{p}-\frac{1}{p_1}},$$

and similarly for $q_1 = \infty$ and/or $q_2 = \infty$.

Optimizing w.r.t. to $\bar{x}$,

$$\bar{x}^{\frac{1}{p_1}-\frac{1}{p_2}} = \frac{\left(\frac{p}{q_1}\right)^{\frac{1}{q_1}} \|f\|_{p_1,q_1}}{\left(\frac{p}{q_2}\right)^{\frac{1}{q_2}} \|f\|_{p_2,q_2}},$$

one obtains the statement. □

Theorem 3.2.5 (Hölder's inequality in Lorentz spaces, [25, Theorem 3.4, Theorem 3.5])**.** *If $1 < p, p_1, p_2 < \infty$ and $1 \leq q, q_1, q_2 \leq \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2},$$

then

$$\|fg\|_{p,q} \leq p' \|f\|_{p_1,q_1} \|g\|_{p_2,q_2}.$$

where $1/p + 1/p' = 1$.

If

$$1 = \frac{1}{p_1} + \frac{1}{p_2}, \qquad 1 \le \frac{1}{q_1} + \frac{1}{q_2},$$

then

$$\|fg\|_1 \le \|f\|_{p_1,q_1} \|g\|_{p_2,g_2}.$$

Corollary 3.2.6. *If $f_x \in L^{p,\infty}$, $p > 1$, then*

$$|f(x) - f(x')| \le p' \|f\|_{p,\infty} |x - x'|^{\frac{1}{p'}}.$$

Proof. For $p = \infty$ the estimate is just the Lipschitz regularity of f.

We have by the last formula of Theorem 3.2.5

$$\begin{aligned}|f(x)| &= \left| \int f_x(x') \mathbf{I}_{(0,x)} dx' \right| \\ &\le \|f_x\|_{p,\infty} \|\mathbf{I}_{(0,x)}\|_{p',1} = \|f_x\|_{p,\infty} p' |x|^{\frac{1}{p'}}.\end{aligned}$$

□

Theorem 3.2.7 ([25, Theorem 2.6])**.** *If $1 < p, p_1, p_2 < \infty$ and $1 \le q_1, q_2, q \le \infty$ satisfy*

$$\frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} \le \frac{1}{q_1} + \frac{1}{q_2},$$

then

$$\|f * g\|_{p,q} \le 3p \|f\|_{p_1,q_1} \|g\|_{p_2,q_2}.$$

We recall also Young's theorem on convolution on L^p-spaces:

$$\|f * g\|_p \le \|f\|_{p_1} \|g\|_{p_2}, \quad \frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2},$$

which holds also in the case $p = 1$ (with the constant $3p$ replaced by 1). The case $p = \infty$ gives also [25, Theorem 3.6]

$$\|f * g\|_\infty \le \|f\|_{p,q} \|g\|_{p',q'}, \tag{3.13}$$

with $1/p + 1/p' = 1$, $1/q + 1/q' \ge 1$. We will also use the following variant of the above inequality when $g \in L^\infty$,

$$\|f * g\|_{p,q} \le \|f\|_{p,q} \|g\|_\infty.$$

3.2.2 Gagliardo-Nirenberg inequality

We recall the Gagliardo–Nirenberg interpolation inequality in 1-dimension.

Proposition 3.2.8 (Gagliardo-Nirenberg interpolation inequality,[24])**.** *Let $1 < q < p$, then*

$$\|f\|_p \lesssim \|f_x\|_2^\theta \|f\|_q^{1-\theta}$$

with

$$\frac{1}{p} = -\frac{\theta}{2} + \frac{1-\theta}{q}$$

3.2.3 Heat kernel

Recall that the heat kernel $G : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

In the following we will use the estimate below.

Proposition 3.2.9. *For $1 < p < +\infty$ it holds $\|G_x\|_{p,1} \le Cp' t^{1/2p-1}$.*

Proof. We estimate

$$|G_x(t,x)| \le \frac{C}{t}\mathbf{1}_{[-\sqrt{2t},\sqrt{2t}]}(x) + \mathbf{1}_{\mathbb{R}\setminus[-\sqrt{2t},\sqrt{2t}]}(x)|G_x(t,x)|,$$

thus

$$(G_x(t,x))^* \le \frac{C}{t}\mathbf{1}_{[0,2\sqrt{2t}]}(x) + \mathbf{1}_{[2\sqrt{2t},\infty]}(x) G_x\left(t, \frac{x}{2}\right),$$

therefore

$$\begin{aligned}\|G_x(t)\|_{p,1} \le & \int_0^{2\sqrt{2t}} \left[x^{1/p}\frac{1}{x}\int_0^x \frac{C}{t} dy\right] \frac{dx}{x} \\ & + \int_{2\sqrt{2t}}^{\infty} \left[x^{1/p}\frac{1}{x}\int_{2\sqrt{2t}}^x \left|G_x\left(t, \frac{y}{2}\right)\right| dy\right] \frac{dx}{x} \\ & \simeq p' t^{\frac{1}{2p}-1}.\end{aligned}$$

□

3.3 Local existence and uniqueness

For the sake of completeness, in this section we prove some classical results of local existence and uniqueness for $L^\infty \cap L^1$-solutions of (3.1). Define the integral operator $\Phi[u]$ by Duhamel formula

$$u \mapsto \Phi[u](t) = G(t) * u_0 + \int_0^t G_x(t-s) * (b(s)u^k(s))\, ds.$$

Notice that $\|u(t)\|_1 = \|u_0\|_1$, as required being the PDE in conservation form.

Proposition 3.3.1. *Let $b \in L^\infty_t L^{p,\infty}_x$, $p \in (2,\infty]$, $k \ge 0$, $u_0 \in L^\infty \cap L^1(\mathbb{R})$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $u \mapsto \Phi[u]$ is a contraction in the set*

$$S = \left\{ u \in L^\infty([0,\bar{t}] \times \mathbb{R}) \cap C([0,\bar{t}], L^{p',1}(\mathbb{R})) : \|u\|_\infty \le r,\, u(0) = u_0 \right\},$$

with $\bar{t}$ sufficiently small and $r \ge 2\|u_0\|_\infty$. In particular, there is a unique bounded solution for (3.1), and it belongs to the space $L^\infty((0,\bar{t}) \times \mathbb{R}) \cap C([0,\bar{t}], L^{p'}(\mathbb{R}))$.

We first prove that $t \mapsto \Phi[u](t)$ is continuous in L^∞ when $u \in S$ and $t > 0$: this will be useful later on, and shows also the continuity of $t \mapsto \Phi[u](t)$ in every integral norm $\|\cdot\|_q$, $q \in [1, \infty]$ for $t > 0$.

Lemma 3.3.2. *For every $u \in L^\infty_{t,x}((0, \bar{t}) \times \mathbb{R})$ the function $t \mapsto \Phi[u](t)$ is continuous in $L^\infty(\mathbb{R})$ for $t > 0$.*

Proof. Compute

$$\begin{aligned}
\|u(t+\delta) - u(t)\|_\infty &= \bigg\| (G(t+\delta) - G(t)) * u_0 + \int_0^{t+\delta} G_x(t+\delta-s) * b(s) u^{1+k}(s) ds \\
&\qquad - \int_0^t G_x(t-s) * b(s) u^{1+k}(s) ds \bigg\|_\infty \\
\big[\text{Theorem 3.2.7}\big] \quad &\leq \|G(t+\delta) - G(t)\|_1 \|u_0\|_\infty \\
&\quad + C \int_t^{t+\delta} \|G_x(t+\delta-s)\|_{p',1} \|b(s) u^{1+k}(s)\|_{p,\infty} ds \\
&\quad + C \int_0^t \|G_x(t+\delta-s) - G_x(t-s)\|_{p',1} \|b(s) u^{1+k}(s)\|_{p,\infty} ds \\
\big[\text{Proposition 3.2.9}\big] \quad &\leq \|G(t+\delta) - G(t)\|_1 \|u_0\|_\infty \\
&\quad + C \|b\|_{p,\infty} \|u\|_\infty^{k+1} \left(\delta^{\frac{1}{2p'}} + \int_0^t \|G_x(t+\delta-s) - G_x(t-s)\|_{p',1} ds \right).
\end{aligned}$$

The last terms converge to 0 as $\delta \to 0$ uniformly in every interval of the form $[t_0, t_1]$, $t_0 > 0$. □

Proof of Proposition 3.3.1. We start by deducing the uniform continuity in L^1 as $t \to 0$: this will give the continuity in time for $\|\cdot\|_q$, $q \in [1, \infty)$. From Theorem 3.2.7 in the first inequality we obtain

$$\begin{aligned}
\|u(t) - u_0\|_1 &\leq \|G(t) * u_0 - u_0\|_1 + \int_0^t \|G_x(t-s)\|_1 \|b(s) u^{1+k}(s)\|_1 ds \\
\big[\text{Theorem 3.2.5}\big] \quad &\leq \|G(t) * u_0 - u_0\|_1 + \int_0^t \|G_x(t-s)\|_1 \|b(s)\|_{p,\infty} \|u^{1+k}(s)\|_{p',1} ds \\
\big[\text{Lemma 3.2.4}\big] &\leq \|G(t) * u_0 - u_0\|_1 + C \|b\|_{p,\infty} r^{k+1/p} \int_0^t \frac{1}{\sqrt{t-s}} ds,
\end{aligned}$$

which converges to 0 uniformly once u_0 is fixed. Hence, by Lemma 3.3.2 above, $t \mapsto \Phi[u](t)$ is uniformly continuous in L^1.

We next prove that $\Phi(S) \subset S$: indeed using again Theorem 3.2.5 in the first inequality, it holds

$$\begin{aligned}
\|\Phi[u](t)\|_\infty &\le \|u_0\|_\infty + \int_0^t \|G_x(t-s)\|_{p',1}\|b(s)u^{k+1}(s)\|_{p,\infty}\,ds \\
&\le \|u_0\|_\infty + \int_0^t \|G_x(t-s)\|_{p',1}\|b(s)\|_{p,\infty}\|u(s)\|_\infty^{k+1}\,ds \\
\big[\text{Proposition 3.2.9}\big] \quad &\le \|u_0\|_\infty + Ct^{\frac{1}{2p'}} \operatorname*{ess-sup}_{s\in[0,t]} \|u(s)\|_\infty^{k+1} \\
\big[2\|u_0\|_\infty \le r\big] \quad &\le \frac{r}{2} + C\bar{t}^{\frac{1}{2p'}} r^{k+1},
\end{aligned}$$

where in the first line we have used (3.13). Taking $\bar{t} \ll 1$ it holds that $\Phi(S) \subset S$.

Finally the same computations show that $\Phi[u]$ is a contraction in $C([0,\bar{t}], L^{p',1}(\mathbb{R}))$:

$$\begin{aligned}
\|\Phi[u](t) - \Phi[v](t)\|_{p',1} &\le C\int_0^t \|G_x(t-s)\|_{p',1}\|b(s)\|_{p,\infty}\|u^{1+k}(s) - v^{1+k}(s)\|_{p',1}ds \\
&\le C\|b\|_{p,\infty} t^{\frac{1}{2p'}} r^k \|u-v\|_{C_tL_x^{p',1}},
\end{aligned}$$

so that for $\bar{t} \ll 1$ the statement follows. The case $p = \infty$ follows by Hölder's inequality because $\|b\|_{\infty,\infty} = \|b\|_\infty$, and gives a contraction in $C_tL^1_x$. □

For the second case, i.e. Conditions (3.3), we start by proving that the second moment $M_2(u(t))$ is bounded if the solution u of (3.1) belongs to $L^\infty_{t,x}$.

Lemma 3.3.3. *Let $b \in L^\infty$, $b_x \in L^\infty_t L^{p,\infty}_x$, $p \in (1,\infty]$, $k \ge \frac{1}{2}$, $u_0 \in L^\infty \cap L^1(\mathbb{R})$, $E(u_0) < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $\|u\|_{L^\infty_{t,x}} \le r$, then*

$$\int (1+x^2)|u(t)|dx \le \left(\int (1+x^2)|u_0|dx\right) e^{C(1+r^k)t}.$$

Proof. Using the estimates by Corollary 3.2.6

$$|b(x)| \le \|b_x\|_{p,\infty} x^{1/p'} + |b(0)| \le C\sqrt{1+x^2} \le C(1+|x|),$$

by (3.1) it holds

$$\begin{aligned}\frac{d}{dt}\int(1+x^2)u\,dx &= \int(1+x^2)(u_x-bu^{k+1})_x dx\\ &= -2\int xu_x dx + \int 2xbu^{k+1}dx\\ &\le 2\int u\,dx + 2\int |xb|u^{k+1}\,dx\\ &\le 2\|u\|_1 + C\int(1+x^2)u^{k+1}\,dx\\ &\le C(1+\|u\|_\infty^k)\int(1+x^2)|u|dx.\end{aligned}$$

Since $\|u\|_\infty \le r$, one integrates the differential inequality to obtain the statement. □

Corollary 3.3.4. *Let $b \in L^\infty$, $b_x \in L^\infty_t L^{p,\infty}_x$, $p \in (1,\infty]$, $k \ge \frac{1}{2}$, $u_0 \in L^\infty \cap L^1(\mathbb{R})$, $E(u_0) < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $u(t) \in L^\infty$ and $M_2(u_0) < \infty$, then $t \mapsto \Phi[u](t)$ is uniformly continuous in the interval $t \in [t_0, t_1]$ with $t_0 > 0$,*

Proof. Following the same line as in Lemma 3.3.2 above,

$$\begin{aligned}&\|u(t+\delta)-u(t)\|_\infty\\ &\quad\le \|G(t+\delta)-G(t)\|_1\|u_0\|_\infty + \int_0^\delta \|G_x(\delta-s)\|_2\|b(t+s)u^{1+k}(t+s)\|_2 ds\\ &\qquad + \int_0^t \|G_x(t+\delta-s)-G_x(t-s)\|_2\|b(s)u^{1+k}(s)\|_2 ds\\ &\quad\le \|G(t+\delta)-G(t)\|_1\|u_0\|_\infty + C\Big(\|u^{1+k}\|_{L^2_{t,x}} + \sup_{[t_0,t_1]} E(u(t))^{\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}+k}\Big)\delta^{\frac{1}{4}}\\ &\qquad + C\Big(\|u^{1+k}\|_2 + \sup_{[t_0,t_1]} E(u(t))^{\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}+k}\Big)\int_0^t \|G_x(t+\delta-s)-G_x(t-s)\|_2 ds,\end{aligned}$$

where we recall that $E(u)$ is the second moment of u. As in the previous case, the r.h.s. converges to 0 uniformly for $t \ge t_0$. □

We remind that by Corollary 3.2.6, if $b_x(t) \in L^{p,\infty}$, then $b(t)$ is $\frac{1}{p'}$-Holder and in particular it is defined at every point.

Proposition 3.3.5. *Assume that $b(t, x = 0)$ is uniformly bounded, $b_x \in L^\infty_t L^{p,\infty}_x$, $k \ge 0$ and $u_0 \in L^\infty \cap L^1(\mathbb{R})$ with bounded second order moment $E(u_0)$. Then $u \mapsto \Phi[u]$ is a contraction in the set*

$$S = \Big\{u \in L^\infty([0,\bar{t}]\times\mathbb{R}) \cap C([0,\bar{t}], L^2(\mathbb{R})), \|u\|_\infty \le r, u(0) = u_0\Big\},$$

with $\bar{t}$ sufficiently small and $r > 2\|u_0\|_\infty$. In particular, there is only one bounded solution for (3.1)*, and it belongs to the space $C([0,\bar{t}], L^2(\mathbb{R}))$.*

Proof. Following the same line of the proof of Proposition 3.3.1 we study first the continuity for $t \searrow 0$:

$$\begin{aligned}
\|u(t) - u_0\|_1 &\leq \|G(t) * u_0 - u_0\|_1 + \int_0^t \|G_x(t-s)\|_1 \|b(s)u^{1+k}(s)\|_1 ds \\
&\leq \|G(t) * u_0 - u_0\|_1 \\
&\quad + C\int_0^t \|G_x(t-s)\|_1 \big[(\|b(s,0)\|_\infty + 1)\|u^k(s)\|_\infty \|u^k(s)\|_\infty E(u(s))\big] ds \\
&\leq \|G(t) * u_0 - u_0\|_1 + Cr^k \big[(\|b(x=0)\|_\infty + 1) + \sup_{[0,t]} E(u(t))\big]\sqrt{t},
\end{aligned}$$

which converges to 0 as $t \to 0$. Hence, together with Corollary 3.3.4 we obtain the uniform continuity in every L^p-space.

Next,

$$\begin{aligned}
\|\Phi[u](\bar{t})\|_\infty &\leq \|u_0\|_\infty + \int_0^{\bar{t}} \|G_x(\bar{t}-s)\|_2 \|b(s)u^{k+1}(s)\|_2 \, ds \\
&\leq \|u_0\|_\infty + C\big(\|b(x=0)\|_{L_t^\infty} + 1 + \sup_{[0,\bar{t}]} E(u(t))^{\frac{1}{2}}\big)\|u\|_\infty^{k+\frac{1}{2}} \int_0^{\bar{t}} \|G_x(\bar{t}-s)\|_2 ds \\
&\leq \|u_0\|_\infty + C(1 + e^{C\bar{t}(1+r^k)/2}) r^{k+\frac{1}{2}} \bar{t}^{\frac{1}{4}} \\
&\leq \frac{r}{2} + C(1 + e^{C\bar{t}(1+r^k)/2}) r^{k+\frac{1}{2}} \bar{t}^{\frac{1}{4}},
\end{aligned}$$

Taking $\bar{t} \ll 1$ it holds that $\Phi(S) \subset S$.

Finally, for positive solutions,

$$\begin{aligned}
\int (1+x^2)\big(\Phi[u](\bar{t},x) - \Phi[v](\bar{t},x)\big) &\leq \int_0^{\bar{t}} \|G_x(\bar{t}-s)\|_2 \|b(s)(u^{1+k}(s) - v^{1+k}(s))\|_1 ds \\
&\leq C\Big((\|b(0)\|_{L_t^\infty} + 1)(\|u\|_{L_{t,x}^\infty}^{k-\frac{1}{2}} + \|v\|_{L_{t,x}^\infty}^{k-\frac{1}{2}})\Big)\|u-v\|_{C_t L_x^2} \bar{t}^{\frac{1}{4}} \\
&\quad + \big(E(u)^{1/2}\|u\|_{L_{t,x}^\infty}^{k-\frac{1}{2}} + E(v)^{1/2}\|v\|_{L_{t,x}^\infty}^{k-\frac{1}{2}}\big)\|u-v\|_{C_t L_x^2} \bar{t}^{\frac{1}{4}},
\end{aligned}$$

so that for $\bar{t} \ll 1$ Φ is a contraction. □

Remark 3.3.6. The condition $E(u_0) < +\infty$ that we used in Proposition 3.3.5 will not play a role in the rest of the thesis, in the sense that the estimates obtained are independent on $E(u_0)$. The same can be said for the condition $k \geq \frac{1}{2}$. Clearly for the blow-up it is more interesting to study the PDE for large k.

3.4 Global existence in the subcritical and critical case

In order to prove the global existence we consider a standard rescaling about a possible blow-up at point $(T, \hat{x})$, where w.l.o.g. we assume that $\hat{x} = 0$. We will show that the rescaled solutions decrease to 0 with the appropriate speed in L^2-norm at time T in the critical and subcritical case, so that the original unscaled solution is bounded by using the estimates contained in [19].

3.4.1 Rescaled variables

Set

$$t = T(1 - e^{-\tau}),$$

and define

$$\begin{aligned} v(\tau, y) &= \sqrt{T} e^{-\tau/2} u\Big(T(1-e^{-\tau}), \sqrt{Te^{-\tau}} y\Big), \\ \tilde{b}(\tau, y) &= (Te^{-\tau})^{\frac{1-k}{2}} b\Big(T(1-e^{-\tau}), \sqrt{Te^{-\tau}} y\Big). \end{aligned}$$

The rescaled equation for $v : \mathbb{R}^+ \times \mathbb{R} \to [0, \infty)$ is then

$$v_\tau + \frac{1}{2}(yv)_y + (\tilde{b} v^{1+k})_y = v_{yy}, \quad v(\tau = 0) = \sqrt{T} u_0(\sqrt{T} y).$$

We observe that

$$\|\tilde{b}(\tau)\|_{p,\infty} = (Te^{-\tau})^{\frac{1-k-1/p}{2}} \big\| b(T(1-e^{-\tau}))\big\|_{p,\infty},$$

hence in particular in the critical and subcritical case it holds

$$\sup_{\tau>0} \|\tilde{b}(\tau)\|_{p,\infty} \le T^{\frac{1-1/p-k}{2}} \sup_{t\in(0,T)} \|b(t)\|_{p,\infty} \le CT^{\frac{1-1/p-k}{2}}, \qquad \text{when} \quad k \le 1 - \frac{1}{p} = \frac{1}{p'}. \tag{3.14}$$

In the same way, for $b_x \in L^{p,\infty}(\mathbb{R})$

$$\|\tilde{b}_y(\tau)\|_{p,\infty} = (Te^{-\tau})^{\frac{2-k-1/p}{2}} \big\| b(T(1-e^{-\tau}))\big\|_{p,\infty}, \tag{3.15}$$

and then

$$\sup_{\tau>0} \|\tilde{b}_y(\tau)\|_{p,\infty} \le T^{\frac{2-1/p-k}{2}} \sup_{t\in(0,T)} \|b_x(t)\|_{p,\infty} \le CT^{\frac{2-1/p-k}{2}}, \qquad \text{for} \quad k \le 2 - 1/p. \tag{3.16}$$

Moreover we observe that

$$\|v(\tau)\|_2^2 = \sqrt{T} e^{-\frac{\tau}{2}} \|u(T(1-e^{-\tau}))\|_2^2. \tag{3.17}$$

3.4.2 Entropy dissipation

If $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth $C^{1,1}$-function, then

$$\frac{d}{d\tau}\int \eta(v) = \int \eta' v_\tau = \int \eta'(v) v_{yy} + \int \eta' \Big(-\frac{1}{2} y v - \tilde{b} v^{k+1} \Big)_y. \tag{3.18}$$

Here we consider the entropy η_a given by

$$\eta_a(v) = \begin{cases} v^2/2 & 0 \le v \le a, \\ a\,(v - a/2) & a < v < \infty, \end{cases}$$

and denote

$$v_a = v \wedge a.$$

The parameter a will be chosen later on to be sufficiently small.

Lemma 3.4.1. *Assume the exponent are critical or subcritical, i.e. Conditions (3.2) with $k \le 1 - \frac{1}{p}$ or Conditions (3.3) and $k \le 2 - \frac{1}{p}$. Then for every time $\bar\tau$ there exists a constant $a = a(p, k, \frac{e^{\bar\tau}}{T})$ such that it holds*

$$\frac{d}{d\tau}\int \eta_a(v(\tau, y))\, dy \le -\frac{\|v_{a,y}(\tau)\|_2^2}{2} - \frac{\|v_a(\tau)\|_2^2}{8} \tag{3.19}$$

for $\tau \ge \bar\tau$. If $k = 1 - \frac{1}{p}$ in (3.2) or $k = 2 - \frac{1}{p}$ in (3.3) then $a = a(p, k)$ is independent on $\bar\tau, T$.

Proof. By integration by parts, Equation (3.18) becomes

$$\frac{d}{d\tau}\int \eta_a(v) dy = -\|v_{a,y}\|_2^2 - \frac{1}{4}\|v_a\|_2^2 + \int v_{a,y} \tilde{b} v_a^{k+1}. \tag{3.20}$$

Assume first Conditions (3.2), and let

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}, \quad 2 < p < \infty.$$

Fixed $p > 2$, choose $2 < \tilde{p}' < p'$ such that $(1 + k)\tilde{p}' > 2$ (so the constants below do not depend on $\tilde{p}'$) and compute

$$
\begin{aligned}
\left|\int v_{a,y}\tilde{b}v_a^{k+1}\right| &\leq \|\tilde{b}v_a^{k+1}\|_2\|v_{a,y}\|_2 \\
[\text{Proposition 3.2.2}] \quad &\leq \|\tilde{b}v_a^{k+1}\|_{2,2}\|v_{a,y}\|_2 \\
[\text{Theorem 3.2.5}] \quad &\leq C\|\tilde{b}\|_{p,\infty}\|v_a^{k+1}\|_{p',2}\|v_{a,y}\|_2 \\
[(3.11) \text{ and } (3.14)] \quad &\leq C(p)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{2}}\|v_a\|^{k+1}_{(k+1)p',(k+1)2}\|v_{a,y}\|_2 \\
\left[\begin{array}{c}\text{Proposition 3.2.3}\\ \text{and Lemma 3.2.4, } p' > \tilde{p}' > 2\end{array}\right] \quad &\leq C(p)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{2}}\|v\|_\infty^{(k+1)(1-\frac{\tilde{p}'}{p'})} \\
&\quad \cdot \|v_a\|^{(k+1)\frac{\tilde{p}'}{p'}}_{(k+1)\tilde{p}',(k+1)\tilde{p}'}\|v_{a,y}\|_2 \\
[\text{Proposition 3.2.2}] \quad &\leq C(p)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{2}}\|v\|_\infty^{(k+1)(1-\frac{\tilde{p}'}{p'})}\|v_a\|^{(k+1)\frac{\tilde{p}'}{p'}}_{(k+1)\tilde{p}'}\|v_{a,y}\|_2 \\
\left[\begin{array}{c}\|v_a\|_{(k+1)\tilde{p}'} \leq C(p,k)\|v_{a,y}\|_2^{\frac{1}{2}-\frac{1}{(k+1)\tilde{p}'}}\\ \cdot\|v_a\|_2^{\frac{1}{2}+\frac{1}{(k+1)\tilde{p}'}}\end{array}\right] \quad &\leq C(p,k)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{2}}\|v\|_\infty^{(k+1)(1-\frac{\tilde{p}'}{p'})} \\
&\quad \cdot \|v_{a,y}\|_2^{(\frac{k+1}{2}-\frac{1}{\tilde{p}'})\frac{\tilde{p}'}{p'}+1}\|v_a\|_2^{(\frac{k+1}{2}+\frac{1}{\tilde{p}'})\frac{\tilde{p}'}{p'}} \\
\left[\|v_a\|_\infty \leq \|v_a\|_1^{\frac{1}{3}}\|v_{a,y}\|_2^{\frac{2}{3}}\right] \quad &\leq C(p,k)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{2}}\|v_{a,y}\|_2^{(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})-\frac{1}{p'}+1} \\
&\quad \cdot \|v_a\|_2^{(\frac{k+1}{2}+\frac{1}{\tilde{p}'})\frac{\tilde{p}'}{p'}} \\
\left[\begin{array}{l}\alpha\beta \leq \frac{\alpha^\gamma}{2}+2^{\frac{1}{\gamma-1}}\beta^{\frac{\gamma}{\gamma-1}},\\ \gamma = \frac{2}{(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})-\frac{1}{p'}+1}\end{array}\right] \quad &\leq \frac{1}{2}\|v_{a,y}\|_2^2 + C(p,k)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\
&\quad \cdot \|v_a\|_2^{2\frac{\frac{k+1}{2}\frac{\tilde{p}'}{p'}+\frac{1}{p'}}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\
\left[\|v_a\|_2^2 \leq \|v_a\|_\infty\|v_a\|_1 \leq a\right] \quad &\leq \frac{1}{2}\|v_{a,y}\|_2^2 + C(p,k)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\
&\quad \cdot a^{\frac{\frac{k+1}{3}(2+\frac{\tilde{p}'}{p'})-1}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}}\|v_a\|_2^2 \\
\left[a = \frac{1}{\bar{C}(p,k)}\left(\frac{e^{\bar{\tau}}}{T}\right)^{\frac{1-\frac{1}{p}-k}{\frac{k+1}{3}(2+\frac{\tilde{p}'}{p'})-1}}\right] \quad &\leq \frac{1}{2}\|v_{a,y}\|_2^2 + \frac{1}{8}\|v_a\|_2^2.
\end{aligned}
\tag{3.21}
$$

Substituting (3.21) into (3.20) we get (3.19).

Assume instead (3.3) and $k \leq 2 - \frac{1}{p}$: using here $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$, and a fixed $1 < \tilde{p}' < p'$ such that $(2+k)\tilde{p}' > 2$, we have

$$
\begin{aligned}
\left| \int v_{a,y} \tilde{b} v_a^{k+1} \right| &\leq \frac{1}{k+2} \left| \int \tilde{b}_y v_a^{k+2} \right| \\
\big[\text{Theorem 3.2.5}\big] \quad &\leq C(k) \|\tilde{b}_y\|_{p,\infty} \|v_a^{k+2}\|_{p',1} \\
\big[(3.15),(3.11)\big] \quad &\leq C(p,k)(Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2}} \|v_a\|^{k+2}_{(k+2)p',k+2} \\
\begin{bmatrix} \text{Proposition 3.2.3} \\ \text{and Lemma 3.2.4}, 1 < \tilde{p}' < p' \end{bmatrix} \quad &\leq C(p,k)(Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2}} \|v\|_\infty^{(k+2)(1-\frac{\tilde{p}'}{p'})} \|v_a\|_{(k+2)\tilde{p}'}^{(k+2)\frac{\tilde{p}'}{p'}} \\
\begin{bmatrix} \|v_a\|_{(k+2)\tilde{p}'} \leq C(p,k) \|v_{a,y}\|_2^{\frac{1}{2}-\frac{1}{(k+2)\tilde{p}'}} \\ \cdot \|v_a\|_2^{\frac{1}{2}+\frac{1}{(k+2)\tilde{p}'}} \end{bmatrix} \quad &\leq C(p,k)(Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2}} \|v\|_\infty^{(k+2)(1-\frac{\tilde{p}'}{p'})} \\
&\quad \cdot \|v_{a,y}\|_2^{(\frac{k+2}{2}-\frac{1}{\tilde{p}'})\frac{\tilde{p}'}{p'}} \|v_a\|_2^{(\frac{k+2}{2}+\frac{1}{\tilde{p}'})\frac{\tilde{p}'}{p'}} \\
\big[\|v_a\|_\infty \leq \|v_a\|_1^{\frac{1}{3}} \|v_{a,y}\|_2^{\frac{2}{3}}\big] \quad &\leq C(p,k)(Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2}} \|v_{a,y}\|_2^{(k+2)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})-\frac{1}{p'}} \\
&\quad \cdot \|v_a\|_2^{(\frac{k+2}{2}+\frac{1}{\tilde{p}'})\frac{\tilde{p}'}{p'}} \\
\begin{bmatrix} \alpha\beta \leq \frac{1}{2}\alpha^\gamma + 2^{\frac{1}{\gamma-1}} \beta^{\frac{\gamma}{\gamma-1}}, \\ \gamma = \frac{2}{(k+2)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})-\frac{1}{p'}} \end{bmatrix} \quad &\leq \frac{1}{2}\|v_{a,y}\|_2^2 + C(p,k)(Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2-(k+2)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\
&\quad \cdot \|v_a\|_2^{2\frac{\frac{k+2}{2}\frac{\tilde{p}'}{p'}+\frac{1}{p'}}{2-(k+2)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\
[\|v\|_2^2 \leq \|v\|_\infty \|v\|_1] \quad &\leq \frac{1}{2}\|v_{a,y}\|_2^2 + C(p,k)(Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2-(k+2)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\
&\quad \cdot a^{\frac{\frac{k+2}{3}(2+\frac{\tilde{p}'}{p'})-2}{2-(k+2)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \|v_a\|_2^2 \\
\left[a \leq \frac{1}{\bar{C}(p,k)} \left(\frac{e^\tau}{T}\right)^{\frac{2-\frac{1}{p}-k}{\frac{k+2}{3}(2+\frac{\tilde{p}'}{p'})-2}} \right] \quad &\leq \frac{1}{2}\|v_{a,y}\|_2^2 + \frac{1}{8}\|v_a\|_2^2.
\end{aligned}
\tag{3.22}
$$

so that (3.19) follows as in the other cases.
The case $p = \infty$ for the Condition (3.2)(resp. Condition (3.3)) i.e. $b \in L^\infty_{t,x}$ (resp. $b_x \in L^\infty_{t,x}$), can be treated in a much simpler way, because $\|\tilde{b}\|_{\infty,\infty} = \|b\|_\infty$ (resp. $\|\tilde{b}_y\|_{\infty,\infty} = \|b_x\|_\infty$) :

by following the same lines as above, assuming Condition (3.2), $p = \infty$, it holds

$$
\begin{aligned}
\left|\int v_{a,y}\tilde{b}v_a^{k+1}\right| &\le \|\tilde{b}\|_\infty \|v_a^{k+1}\|_2 \|v_{a,y}\|_2 \\
&\le C(Te^{-\tau})^{\frac{1-k}{2}} \|v_a\|_{(k+1)2}^{k+1} \|v_{a,y}\|_2 \\
\left[\|v_a\|_{2(k+1)} \le C(k)\|v_{a,y}\|_2^{\frac{1}{2}-\frac{1}{2(k+1)}} \|v_a\|_2^{\frac{1}{2}+\frac{1}{2(k+1)}}\right] \quad &\le C(k)(Te^{-\tau})^{\frac{1-k}{2}} \|v_{a,y}\|_2^{1+\frac{k}{2}} \|v_a\|_2^{1+\frac{k}{2}} \\
\left[\alpha\beta \le \frac{1}{2}\alpha^\gamma + 2^{\frac{1}{\gamma-1}}\beta^{\frac{\gamma}{\gamma-1}}, \gamma = \frac{2}{1+\frac{k}{2}}\right] \quad &\le \frac{1}{2}\|v_{a,y}\|_2^2 + C(k)(Te^{-\tau})^{\frac{1-k}{1-\frac{k}{2}}} \|v_a\|_2^{2\frac{1+\frac{k}{2}}{1-\frac{k}{2}}} \\
&\le \frac{1}{2}\|v_{a,y}\|_2^2 + C(p,k)(Te^{-\tau})^{\frac{1-k}{1-\frac{k}{2}}} a^{\frac{2k}{1-\frac{k}{2}}} \|v_a\|_2^2 \\
\left[a = \frac{1}{\bar{C}(k)}\left(\frac{e^{\bar{\tau}}}{T}\right)^{\frac{1-k}{2k}}\right] \quad &\le \frac{1}{2}\|v_{a,y}\|_2^2 + \frac{1}{8}\|v_a\|_2^2,
\end{aligned}
$$

and assuming Condition (3.3), $p = \infty$, it holds

$$
\begin{aligned}
\left|\int v_{a,y}\tilde{b}v_a^{k+1}\right| &\le \frac{1}{k+2}\left|\int \tilde{b}_y v_a^{k+2}\right| \\
&\le C(k)\|\tilde{b}_y\|_\infty \|v_a^{k+2}\|_1 \\
&\le C(k)(Te^{-\tau})^{\frac{2-k}{2}} \|v_a\|_{k+2}^{k+2} \\
\left[\|v_a\|_{k+2} \le C(k)\|v_{a,y}\|_2^{\frac{1}{2}-\frac{1}{k+2}} \|v_a\|_2^{\frac{1}{2}+\frac{1}{k+2}}\right] \quad &\le C(k)(Te^{-\tau})^{\frac{2-k}{2}} \|v_{a,y}\|_2^{\frac{k}{2}} \|v_a\|_2^{2+\frac{k}{2}} \\
\left[\alpha\beta \le \frac{1}{2}\alpha^\gamma + 2^{\frac{1}{\gamma-1}}\beta^{\frac{\gamma}{\gamma-1}}, \gamma = \frac{4}{k}\right] \quad &\le \frac{1}{2}\|v_{a,y}\|_2^2 + C(k)(Te^{-\tau})^{\frac{2-k}{2-\frac{k}{2}}} \|v_a\|_2^{2\frac{2+\frac{k}{2}}{2-\frac{k}{2}}} \\
&\le \frac{1}{2}\|v_{a,y}\|_2^2 + C(k)(Te^{-\tau})^{\frac{2-k}{2-\frac{k}{2}}} a^{\frac{2k}{2-\frac{k}{2}}} \|v_a\|_2^2 \\
\left[a = \frac{1}{\bar{C}(k)}\left(\frac{e^{\bar{\tau}}}{T}\right)^{\frac{2-k}{2k}}\right] \quad &\le \frac{1}{2}\|v_{a,y}\|_2^2 + \frac{1}{8}\|v_a\|_2^2,
\end{aligned}
$$

This concludes the proof. □

Set now

$$
e^{\bar{\tau}} = T, \quad \bar{a} = a\left(p, k, \frac{e^{\bar{\tau}}}{T}\right) = a(p, k, 1), \tag{3.23}
$$

where $a(\bar{\tau}, k, p)$ is given by the previous lemma: notice that it is independent on T.

Lemma 3.4.2. *Assume* (3.2) *and* $0 < k \le 1 - \frac{1}{p}$ *or* (3.3) *and* $0 < k \le 2 - \frac{1}{p}$. *Then it holds*

$$
\int_{\bar{\tau}}^{\infty} \left(\frac{\|v_{\bar{a}}(\tau)\|_\infty^3}{2} + \frac{\|v_{\bar{a}}(\tau)\|_2^2}{8}\right) d\tau \le \frac{3\bar{a}}{2}.
$$

Proof. We consider the case $b(t) \in L^{p,\infty}$, the other case being completely similar. By Lemma 3.4.1 and Gagliardo-Nirenberg inequality we have for $\tau \geq \bar{\tau}$ that

$$\begin{aligned} \frac{d}{d\tau}\int \eta_{\bar{a}} &\leq -\frac{1}{2}\|v_{\bar{a},y}\|_2^2 - \frac{\|v_{\bar{a}}\|_2^2}{8} \\ \left[\|v_{\bar{a}}\|_\infty \leq \|v_{\bar{a},y}\|_2^{2/3}\|v_{\bar{a}}\|_1^{1/3}\right] \quad &\leq -\frac{\|v_{\bar{a}}\|_\infty^3}{2\|v_{\bar{a}}\|_1} - \frac{\|v_{\bar{a}}\|_2^2}{8}. \end{aligned}$$

Integrating and observing that

$$\|v_{\bar{a}}(\tau)\|_1 \leq \|v(\tau)\|_1 = \|u(T(1-e^{-\tau}))\|_1 = \|u_0\|_1 = 1,$$

$$\int \eta_{\bar{a}}(v(\tau,y))dy \leq \frac{1}{2}\|v_{\bar{a}}(\tau)\|_2^2 + \bar{a}\|v(\tau)\|_1 \leq \frac{3}{2}\bar{a}\|u_0\|_1 = \frac{3\bar{a}}{2},$$

we obtain

$$\int_{\bar{\tau}}^\infty \left(\frac{\|v_{\bar{a}}(\tau)\|^3}{2} + \frac{\|v_{\bar{a}}\|_2^2}{8}\right)d\tau \leq \int \eta_{\bar{a}}(v(\bar{\tau},y))dy \leq \frac{3}{2}\bar{a},$$

which is the statement. □

3.4.3 Bounds on the L^2-norm

With the results of the above section, we can estimate the L^2-norms of the solution. The quantities $\bar{a}$ and $\bar{\tau}$ are defined in (3.23).

Lemma 3.4.3. *If b satisfies Conditions* (3.2) *and $k \leq 1 - \frac{1}{p}$ or Conditions* (3.3) *and $k \leq 2 - \frac{1}{p}$, then it holds*

$$\|v(\tau)\|_2^2 \leq \hat{C}(k,p)\bar{a}e^{-\frac{\tau-\bar{\tau}}{2}}.$$

Proof. We consider only the case $b(t) \in L^{p,\infty}$ and $p < \infty$, since the other cases ($b(t) \in L^\infty$ or $b_x(t) \in L^{p,\infty}$) can be obtained with similar computations.

Taking $p' > \tilde{p}' \geq 2$, $(1+k)\tilde{p}' > 2$, with the same computations as in (3.21) we obtain

$$\begin{aligned} \frac{d}{d\tau}\|v\|_2^2 &\leq -\frac{1}{2}\|v_y\|_2^2 + C(p,k)(Te^{-\tau})^{\frac{1-\frac{1}{p}-k}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}}\|v\|_2^{2\frac{\frac{k+1}{2}\frac{\tilde{p}'}{p'}+\frac{1}{p'}}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}} \\ \left[Te^{\bar{\tau}} = 1, \tau \geq \bar{\tau}\right] \quad &= \left(-\frac{1}{2} + C(p,k)\|v\|_2^{2\frac{\frac{k+1}{3}(2+\frac{\tilde{p}'}{p'})-1}{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}}\right)\|v\|_2^2. \end{aligned}$$

The differential inequality

$$\frac{dz}{d\tau} \leq \left(-\frac{1}{2} + C(p,k)z^\alpha\right)z, \qquad \alpha > 0,$$

has a solution bounded solution if for some $\hat{\tau}$

$$C(p,k)z(\hat{\tau})^{\alpha} < \frac{1}{4},$$

and in this case it holds

$$z(\tau) \leq \frac{1}{\left(2C(p,k) + e^{\frac{\alpha}{2}(\tau-\hat{\tau})}(\frac{1}{z^{\alpha}(\hat{\tau})} - 2C(p,k))\right)^{1/\alpha}} \leq 2^{\frac{1}{\alpha}} z(\hat{\tau}) e^{-\frac{\tau-\hat{\tau}}{2}}.$$

Using Lemma 3.4.2, we estimate the first time $\tilde{\tau}$ such that

$$\|v(\tilde{\tau})\|_{\infty} \leq \bar{a}$$

as follows: if $\|v(\tau)\|_{\infty} > a$ for $\tau \in (\bar{\tau}, \tilde{\tau})$, then $\|v_a\|_{\infty} = a$ and then

$$(\tilde{\tau} - \bar{\tau})\bar{a}^3 = \int_{\bar{\tau}}^{\tilde{\tau}} \|v_{\bar{a}}\|_{\infty}^3 d\tau \leq \frac{3\bar{a}}{2}, \quad \text{so that } \tilde{\tau} - \bar{\tau} \leq \frac{3}{2\bar{a}^2}.$$

Also, notice that by (3.21) and the choice of $\bar{a}$

$$C(p,k)\|v(\tilde{\tau})\|_2^{2\alpha} \leq C(p,k)\bar{a}^{\alpha} \leq \frac{1}{8}$$

so that we can take $\hat{\tau} = \tilde{\tau}$. Hence

$$\|v(\tau)\|_2^2 \leq \hat{C}(p,k)\bar{a}e^{-\frac{\tau-\bar{\tau}}{2}}, \quad \tau \geq \bar{\tau} + \frac{3}{2\bar{a}^2}.$$

Hence the constant in the statement can be bounded by $\hat{C}(k,p) \leq 2^{\frac{1-(k+1)(\frac{2}{3}-\frac{1}{6}\frac{\tilde{p}'}{p'})+\frac{1}{p'}}{\frac{k+1}{3}(2+\frac{\tilde{p}'}{p'})-1}} e^{\frac{3}{2\bar{a}^2}}$. □

Corollary 3.4.4. *Assume Conditions* (3.2) *and* $k \leq 1 - \frac{1}{p}$, *or Conditions* (3.3) *and* $k \leq 2 - \frac{1}{p}$*: then it holds*

$$\|u(t)\|_2^2 \leq \hat{C}(p,k)\bar{a}, \qquad \forall t \in [T-1, T). \tag{3.24}$$

Proof. Recalling that by (3.17)

$$\|u(T(1-e^{-\tau}))\|_2^2 = \frac{e^{\tau/2}}{\sqrt{T}}\|v(\tau)\|_2^2,$$

Lemma 3.4.3 and (3.23) give for

$$t = T(1-e^{-\tau}) \geq T(1-e^{-\bar{\tau}}) = T-1$$

that

$$\|u(T(1-e^{-\tau}))\|_2^2 \leq \hat{C}(p,k)\bar{a}\frac{e^{\frac{\bar{\tau}}{2}}}{\sqrt{T}} \leq \hat{C}(p,k)\bar{a}.$$

This proves (3.24). □

3.4.4 Bound on the L^∞-norm

Having proved a uniform bound on the L^2-norm, now we are ready to bound the L^∞-norm. This proves the first part of Theorem 3.1.1

Theorem 3.4.5. *Assume Conditions* (3.2) *with* $k \le 1-\frac{1}{p}$ *or Conditions* (3.3) *with* $k \le 2-\frac{2}{p}$. *Then* u *does not blow up at time* T.

Proof. We do the proof in the first case, the second being completely similar.

We notice first that the same computations leading to [19, Theorem 3.8] gives

$$\|u(T)\|_\infty \le C \max\Big\{\|u(\hat{T})\|_\infty, \sup_{[\hat{T},T]} \|u(t)\|_2^{\frac{1-\frac{1}{p}}{1-\frac{1}{p}-\frac{k}{2}}}\Big\}. \tag{3.25}$$

The statement of the theorem is now a consequence of Corollary 3.4.4 with $\hat{T} = T-1$.

For completeness, we rewrite the proof of [19] for the case $b \in L_t^\infty L_x^{p,\infty}$, since in that paper it is only considered the case $b_x(t) \in L^\infty$.

Step 1 . Write for some $2 < \tilde{p} < p$ (if $p = \infty$ then $\tilde{p} = p$)

$$\begin{aligned}
\frac{d}{dt}\int u^{2n}dx &= -\frac{2(2n-1)}{n}\int (u_x^n)^2 dx - 2n(2n-1)\int u^{2n-2}u_x b u^{1+k} dx \\
&= -\frac{2(2n-1)}{n}\int (u_x^n)^2 dx - 2(2n-1)\int u_x^n b u^{n+k} dx \\
&\le -\frac{2(2n-1)}{n}\|u_x^n\|_2^2 + 2(2n-1)\|u_x^n\|_2 \|bu^{n+k}\|_2 \\
&\le -\frac{2n-1}{n}\|u_x^n\|_2^2 + (2n-1)n\|bu^{n+k}\|_2^2 \\
\left[1 = \frac{2}{p} + \frac{p-2}{p}\right] \quad &\le -\frac{2n-1}{n}\|u_x^n\|_2^2 + Cn(2n-1)\|b\|_{p,\infty}^2 \|u^{n+k}\|_{\frac{2p}{p-2},2}^2 \\
\big[(3.11)\big] \quad &\le -\frac{2n-1}{n}\|u_x^n\|_2^2 + C(p)n(2n-1)\|b\|_{p,\infty}^2 \|u^n\|_{(1+\frac{k}{n})\frac{2p}{p-2},(1+\frac{k}{n})2}^{2(1+\frac{k}{n})}.
\end{aligned} \tag{3.26}$$

For $2 < \tilde{p} < p < \infty$, use now Lemma 3.2.4 with exponents

$$p_1 = \left(1+\frac{k}{n}\right)2 < \bar{p} = \left(1+\frac{k}{n}\right)\frac{2p}{p-2} < p_2 = \left(1+\frac{k}{n}\right)\tilde{p}' = \left(1+\frac{k}{n}\right)\frac{2\tilde{p}}{\tilde{p}-2},$$

$$q_1 = p_1 = \left(1+\frac{k}{n}\right)2, \quad \bar{q} = q_1, \quad q_2 = p_2 = \left(1+\frac{k}{n}\right)\frac{2\tilde{p}}{\tilde{p}-2},$$

so that

$$\|u^n\|_{(1+\frac{k}{n})\frac{2p}{p-2},(1+\frac{k}{n})2} \le C(p,k,n)\|u^n\|_{(1+\frac{k}{n})2,(1+\frac{k}{n})2}^{1-\theta} \|u^n\|_{(1+\frac{k}{n})\frac{2\tilde{p}}{\tilde{p}-2},(1+\frac{k}{n})\frac{2\tilde{p}}{\tilde{p}-2}}^{\theta},$$

with

$$\theta = \frac{\tilde{p}}{p} \in (0,1),$$

$$C(p,k,n) = \left(\frac{p^2}{2(p-\tilde{p})}\right)^{\frac{1}{(1+\frac{k}{n})2}} \left(\frac{p}{p-2}\right)^{\frac{1-\theta}{(1+\frac{k}{n})\frac{2\tilde{p}}{\tilde{p}-2}}} \left(\frac{\frac{p}{p-2}}{\frac{\tilde{p}}{\tilde{p}-2}}\right)^{\frac{\theta}{(1+\frac{k}{n})2}}.$$

By Proposition 3.2.2 we conclude that

$$\|u^n\|_{(1+\frac{k}{n})\frac{2p}{p-2},(1+\frac{k}{n})2} \leq C(p,k,n)\|u^n\|^{1-\theta}_{(1+\frac{k}{n})2}\|u^n\|^{\theta}_{(1+\frac{k}{n})\frac{2\tilde{p}}{\tilde{p}-2}},$$

for another constant $C(p,k,n)$ which is uniformly bounded w.r.t. n once $p, \tilde{p}, k$ are fixed. We can thus continue (3.26) as

$$\frac{d}{dt}\int u^{2n}dx \leq -\frac{2n-1}{n}\|u^n_x\|_2^2 + Cn(2n-1)\|b\|^2_{p,\infty}\|u^n\|^{2(1+\frac{k}{n})(1-\theta)}_{(1+\frac{k}{n})2}\|u^n\|^{2(1+\frac{k}{n})\theta}_{(1+\frac{k}{n})\tilde{p}'}.$$

For $p = \infty$ one obtains the same formula above with $\tilde{p}' = 2$ and $\theta = 1$.

Step 2. Hence $\|u\|_{2n}$ is increasing only if

$$\|u^n_x\|_2 \leq Cn\|u^n\|^{(1+\frac{k}{n})(1-\theta)}_{(1+\frac{k}{n})2}\|u^n\|^{(1+\frac{k}{n})\theta}_{(1+\frac{k}{n})\tilde{p}'}.$$

Then using the embeddings by Gagliardo-Niremberg with exponents

$$\frac{1}{(1+\frac{k}{n})\tilde{p}'} = \frac{\tilde{p}-2}{2\tilde{p}(1+\frac{k}{n})} = -\frac{a}{2} + \frac{1-a}{1}, \quad a = \frac{2}{3}\left(1 - \frac{\tilde{p}-2}{2\tilde{p}(1+\frac{k}{n})}\right),$$

for the function $w = u^n$ we obtain

$$\|w_x\|_2 \leq Cn\|u^n\|^{(1+\frac{k}{n})(1-\theta)}_{(1+\frac{k}{n})2}\big(\|w_x\|_2^a\|w\|_1^{1-a}\big)^{(1+\frac{k}{n})\theta}.$$

Using similarly Gagliardo-Niremberg with

$$\frac{1}{(1+\frac{k}{n})2} = -\frac{b}{2} + 1 - b, \quad b = \frac{2}{3}\left(1 - \frac{1}{(1+\frac{k}{n})2}\right),$$

we can write

$$\begin{aligned}
\|w_x\|_2 &\leq Cn\big(\|w_x\|_2^b\|w\|_1^{1-b}\big)^{(1+\frac{k}{n})(1-\theta)}\big(\|w_x\|_2^a\|w\|_1^{1-a}\big)^{(1+\frac{k}{n})\theta} \\
&= Cn\|w_x\|_2^{(1+\frac{k}{n})[b(1-\theta)+a\theta]}\|w\|_1^{(1+\frac{k}{n})[(1-b)(1-\theta)+(1-a)\theta)]} \\
&= Cn\|w_x\|_2^{(1+\frac{k}{n})\frac{2}{3}(1-\frac{p-2}{(1+\frac{k}{n})2p})}\|w\|_1^{(1+\frac{k}{n})(\frac{1}{3}+\frac{2}{3}\frac{p-2}{(1+\frac{k}{n})2p})}.
\end{aligned}$$

This can be rewritten as

$$\|w_x\|_2^{\frac{2}{3}(1-\frac{1}{p}-\frac{k}{n})} \leq Cn\|w\|_1^{\frac{2}{3}(1-\frac{1}{p}+\frac{k}{2n})}.$$

Using again Gagliardo-Niremberg we get

$$\|w\|_2 \leq C\|w_x\|^{\frac{1}{3}}\|w\|_1^{\frac{2}{3}} \leq Cn^{\frac{\frac{1}{2}}{1-\frac{1}{p}-\frac{k}{n}}}\|w\|_1^{1+\frac{\frac{k}{2n}}{1-\frac{1}{p}-\frac{k}{n}}},$$

which rewritten for u becomes

$$\|u\|_{2n} \leq C^{\frac{1}{n}} n^{\frac{\frac{1}{2n}}{1-\frac{1}{p}-\frac{k}{n}}}\|u\|_n^{1+\frac{\frac{k}{2n}}{1-\frac{1}{p}-\frac{k}{n}}}. \tag{3.27}$$

Step 3. The above estimate implies that

$$\max_{t\in[t_0,t]}\|u(t)\|_{2n} \leq \max\left\{\|u(t_0)\|_{2n}, C^{\frac{1}{n}} n^{\frac{\frac{1}{2n}}{1-\frac{1}{p}-\frac{k}{n}}}\sup_{t\in[t_0,t]}\|u\|_n^{1+\frac{\frac{k}{2n}}{1-\frac{1}{p}-\frac{k}{n}}}\right\},$$

because the solution u is decreasing when (3.27) is not satisfied.

Iterating the procedure for $n = 2^m$, $k = 1, \dots, M$, one obtains

$$\begin{aligned}\max_{t\in[t_0,t]}\|u(t)\|_{2^M} \leq \max\Big\{&\|u(t_0)\|_{2^M},\\ &\dots\\ &C^{\alpha(4,M)}2^{\beta(4,M)}\|u(t_0)\|_4^{\gamma(4,M)},\\ &C^{\alpha(2,M)}2^{\beta(2,M)}\max_{t\in[t_0,t]}\|u\|_2^{\gamma(2,M)}\Big\}.\end{aligned}$$

The constants $\alpha(M',M)$, $\beta(M',M)$, $\gamma(M',M)$ are computed by iterating the exponent of (3.27):

$$\gamma(M',M) = \prod_{n=M'+1}^{M}\frac{1-\frac{1}{p}-\frac{k}{2^n}}{1-\frac{1}{p}-\frac{k}{2^{n-1}}} = \frac{1-\frac{1}{p}-\frac{k}{2^M}}{1-\frac{1}{p}-\frac{k}{2^{M'}}}, \quad \gamma(M,M) = 1,$$

$$\alpha(M',M) = \sum_{n=M'}^{M-1} 2^{-n}\gamma(n+1,M) \leq \bar{\alpha},$$

$$\beta(M',M) = \sum_{n=M'}^{M-1}\frac{n2^{-n-1}}{1-\frac{1}{p}-\frac{k}{2^n}}\gamma(n+1,M) \leq \bar{\beta}.$$

By applying repeatedly Hölder's inequality

$$\|u(t_0)\|_{2^{M'}}^{\gamma(M',M)} \le \left(\|u(t_0)\|_{2^{M'+1}}^{\frac{2}{3}} \|u(t_0)\|_{2^{M'-1}}^{\frac{1}{3}}\right)^{\gamma(M',M)} = \|u(t_0)\|_{2^{M'+1}}^{\gamma(M'+1,M)} \|u(t_0)\|_{2^{M'-1}}^{\gamma(M'-1,M)},$$

we get

$$\max_{t\in[t_0,t]} \|u(t)\|_{2^M} \le C^{\bar{\alpha}} 2^{\bar{\beta}} \max\left\{\|u(t_0)\|_{2^M}, \max_{t\in[t_0,t]} \|u\|_2^{\gamma(2,M)}\right\}.$$

Letting $M \to \infty$ one recovers (3.25). □

We conclude with the following corollary, which follows by using (3.25) in $[0,t]$ because the bound of Corollary 3.4.4 is uniform in t.

Corollary 3.4.6. *Assume that $\|b(t)\|_{p,\infty}$ under Conditions (3.5) with $k \le 1 - \frac{1}{p}$ (or $\|b_x(t)\|_{p,\infty}$ under conditions (3.6) with $k \le 2 - \frac{1}{p}$) is uniformly bounded for $t \in (0,\infty)$. Then $\|u(t)\|_\infty$ is uniformly bounded.*

3.4.5 The critical case $k = 1 - \frac{1}{p}$ or $k = 2 - \frac{1}{p}$

In this section we study the case (3.5) when $\|b(t)\|_{p,\infty}$ is uniform bounded in time and the exponent is critical, i.e. $k = 1 - \frac{1}{p}$, $p > 2$, or Conditions (3.6) with $\|b_x(t)\|_{p,\infty}$ uniformly bounded in time and $k = 2 - \frac{1}{p}$, $p > 1$. We consider only the first case, being the analysis completely similar.

It follows form Lemma 3.4.2 that instead of (3.23) we can just choose $\bar{\tau} = 0$ and $\bar{a} = a(p,k)$ and then Lemma 3.4.3 gives

$$\|v(\tau)\|_2^2 \le \hat{C}(k,p)\bar{a}e^{-\frac{\tau}{2}}.$$

Using the definition of v we obtain for $\tau \ge 0$

$$\|u(T(1-e^{-\tau}))\|_2 \le \frac{C}{\sqrt{T}}. \tag{3.28}$$

Letting $\tau \to \infty$ we obtain the decay of the L^2-norm as for the heat kernel.

Again Chebyshev's inequality applied to Lemma 3.4.2 gives that there exists $\hat{\tau} \in \frac{3}{2\bar{a}^2}[1,2]$ such that $\|v(\hat{\tau})\|_\infty \le \bar{a}$, and then

$$\|u(T(1-e^{\hat{\tau}}))\|_\infty = \frac{\|v(\hat{\tau})\|_\infty}{\sqrt{Te^{-\hat{\tau}}}} \le \frac{\bar{a}e^{\frac{3}{\bar{a}^2}}}{\sqrt{T}}.$$

Using again the estimate (3.25) and noticing that for the critical case $k = 1 - \frac{1}{p}$

$$\frac{\frac{k}{2}}{1 - \frac{1}{p} - \frac{k}{2}} = 2,$$

using (3.28) we get for $\hat{T} = T(1 - e^{-\hat{\tau}}) \geq T(1 - e^{-\frac{3}{2\bar{a}^2}})$

$$\begin{aligned} \|u(T)\|_\infty &\leq C \max\left\{\|u(\hat{T})\|_\infty, \sup_{\hat{T},T} \|u(t)\|_2^2\right\} \\ &\leq C \max\left\{\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{\hat{T}}}\right\} \leq \frac{C}{\sqrt{T}}. \end{aligned}$$

We thus have proved the following

Theorem 3.4.7. *If $b \in L_t^\infty L_x^{p,\infty}$, $p \in (1, \infty]$, and $k = 1 - \frac{1}{2}$, or $b_x \in L_t^\infty L_x^{p,\infty}$, $p \in (1, \infty]$ and $k = 2 - \frac{1}{p}$, then for a constant C independent on u_0 it holds*

$$\|u(t)\|_\infty \leq \frac{C}{\sqrt{t}}.$$

3.5 Blow-up in finite time

In this section we show that above the critical exponent $k > 1 - \frac{1}{p}$ for the case (3.2) or $k > 2 - \frac{1}{p}$ for the case (3.3), the solutions blow up in general: we will prove this statement for time independent $b \in L^p(\mathbb{R})$ in the first case and $b_x \in L^p(\mathbb{R})$ in the second case. A similar result has already been proved in [26], we repeat it here for completeness.

3.5.1 Case (3.2)

Let $0 < \alpha < 1$, $\beta > \alpha$ and $k > 1 - \alpha$. Consider an integrable function $b \in L^p(\mathbb{R})$ satisfying

$$|x|^{-\alpha} \geq -b(x) \geq |x|^{-\alpha} \mathbb{1}_{|x| \leq \bar{x}} + |x|^{-\beta} \mathbb{1}_{|x| > \bar{x}} \tag{3.29}$$

with $\alpha p < 1, \beta p > 1$, and the constant $\bar{x}$ given by

$$1 \leq \bar{x} \leq \left(\frac{\beta + k - 1}{\alpha + k - 1}\right)^{\frac{k}{\beta - \alpha}}. \tag{3.30}$$

Proposition 3.5.1. *Fix a positive measurable function $f : \mathbb{R} \to [0, \infty)$.*

1. *If*

$$\left(\frac{2k}{k - (1 - \alpha)}\right)^{-\frac{k}{3k+1+\alpha}} \left(\int f^{k+1} |xb|\right)^{-\frac{1}{3k+1+\alpha}} E^{\frac{k+1}{3k+1+\alpha}} \leq \bar{x}, \tag{3.31}$$

then

$$m \leq 2 \left(\frac{2k}{k - (1 - \alpha)}\right)^{\frac{2k}{3k+1+\alpha}} \left(\int f^{k+1} |xb|\right)^{\frac{2}{3k+1+\alpha}} E^{\frac{k - (\alpha+1)}{3k+1+\alpha}}. \tag{3.32}$$

2. If

$$\left(\int f^{k+1}|xb|dx\right)^{-\frac{1}{3k+1-\beta}} \left(\frac{2k}{k-(1-\beta})\right)^{-\frac{k}{3k+1+\beta}} E^{\frac{k+1}{3k+1-\beta}} \geq \bar{x}, \tag{3.33}$$

then

$$m \leq (1+2^{\frac{k-(1-\beta)}{k+1}}) \left(\frac{2k}{k-(1-\beta)}\right)^{\frac{2k}{3k+1+\beta}} \left(\int f^{k+1}|xb|dx\right)^{\frac{2}{3k+1+\beta}} E^{\frac{k-(1-\beta)}{3k+1+\beta}}. \tag{3.34}$$

Proof. Case 1. We compute

$$\begin{aligned} m = \int f dx &= \int_{-R}^{R} f dx + \int_{|x|>r} f dx \\ &\leq \int_{-R}^{R} f|x|^{\frac{1-\alpha}{k+1}} \frac{1}{|x|^{\frac{1-\alpha}{k+1}}} dx + \frac{1}{R^2}\int_{|x|>R} |x|^2 f dx \\ &\leq \left(\int_{-R}^{R} f^{k+1}|x|^{1-\alpha} dx\right)^{\frac{1}{k+1}} \left(\int_{-R}^{R} \left(\frac{1}{|x|}\right)^{\frac{1-\alpha}{k}}\right)^{\frac{k}{k+1}} + \frac{1}{R^2}E \\ \text{[if } R \leq \bar{x}] \quad &\leq \left(\int_{\mathbb{R}} f^{k+1}|xb|\right)^{\frac{1}{k+1}} \left(\frac{2k}{k-(1-\alpha)}\right)^{\frac{k}{k+1}} R^{\frac{k-(1-\alpha)}{k+1}} + \frac{1}{R^2}E. \end{aligned}$$

Choosing

$$R = \left(\frac{2k}{k-(1-\alpha)}\right)^{-\frac{k}{3k+1+\alpha}} \left(\int f^{k+1}|xb|\right)^{-\frac{1}{3k+1+\alpha}} E^{\frac{k+1}{3k+1+\alpha}} \leq \bar{x},$$

we obtain

$$m \leq 2\left(\frac{2k}{k-(1-\alpha)}\right)^{\frac{2k}{3k+1+\alpha}} \left(\int f^{k+1}|xb|\right)^{\frac{2}{3k+1-\alpha}} E^{\frac{k-(1-\alpha)}{3k+1+\alpha}},$$

which is estimate (3.32).

Case 2. Similarly to the previous case, we compute

$$\begin{aligned} m = \int f dx &= \int_{|x|\leq M} f dx + \int_{|x|>M} f dx \\ &\leq \int_{|x|\leq M} f|xb|^{\frac{1}{1+k}} \frac{1}{|xb|^{\frac{1}{1+k}}} dx + \frac{1}{M^2}E \\ \text{[if } \bar{x} \leq M] \quad &\leq \left(\int f^{k+1}|xb|dx\right)^{\frac{1}{k+1}} \left(\int_{|x|\leq\bar{x}} \left(\frac{1}{|x|}\right)^{\frac{1-\alpha}{k}} dx + \int_{\bar{x}<|x|\leq M} \left(\frac{1}{|x|}\right)^{\frac{1-\beta}{k}}\right)^{\frac{k}{k+1}} + \frac{1}{M^2}E \\ [(3.30)] \quad &\leq \left(\int f^{k+1}|xb|dx\right)^{\frac{1}{k+1}} \left(\frac{2k}{k-(1-\beta)}\right)^{\frac{k}{k+1}} (2M)^{\frac{k-(1-\beta)}{k+1}} + \frac{1}{M^2}E. \end{aligned}$$

Choosing

$$M := \left(\frac{2k}{k-(1-\beta)}\right)^{-\frac{k}{3k+1+\beta}} \left(\int f^{k+1}|xb|\right)^{-\frac{1}{3k+1+\beta}} E^{\frac{k+1}{3k+1+\beta}},$$

we conclude that

$$m \leq (1+2^{\frac{k-(1-\beta)}{k+1}}) \left(\frac{2k}{k-(1-\beta)}\right)^{\frac{2k}{3k+1+\beta}} \left(\int f^{k+1}|xb|dx\right)^{\frac{2}{3k+1+\beta}} E^{\frac{k-(1-\beta)}{3k+1+\beta}},$$

which is (3.34). □

Theorem 3.5.2. *Assume* (3.2) *and* $k > 1-\frac{1}{p}$. *If* $E(u_0)$ *is sufficiently small and* b *satisfying* (3.29) *and* (3.30), *then the solution of* (3.1) *blows up in finite time.*

Proof. The choice of the constant $\bar{x}$ covers at least one of the two cases (3.31) or (3.33) : indeed

$$\bar{x}^{3k+1+\beta}\left(\frac{2k}{k-(1-\beta)}\right)^{k} \leq \bar{x}^{3k+1+\alpha}\left(\frac{2k}{k-(1-\alpha)}\right)^{k},$$

if and only if (3.30) holds. Then by Proposition 3.5.1 and computing $\frac{dE}{dt}$ by (3.1), one obtains

$$\begin{aligned}\frac{dE}{dt} &\leq 2m - 2\int xbu^{k+1}dx \\ &\leq 2m - C\min\left\{\frac{m^{\frac{3k+1+\alpha}{2}}}{E^{\frac{k-(1-\alpha)}{2}}}, \frac{m^{\frac{3k+1+\beta}{2}}}{E^{\frac{k-(1-\beta)}{2}}}\right\}.\end{aligned}$$

The exponents

$$\frac{k-(1-\alpha)}{2}, \frac{k-(1-\beta)}{2} > 0$$

by the choice of α, β, which gives the blow-up in finite time because of the ODE

$$\dot{y} = a - \frac{b}{y^\gamma}, \quad \gamma > 0$$

has a solution converging to 0 in finite time like

$$y(t) \simeq (T-t)^{\frac{1}{1+\gamma}}$$

if the initial data is $< (b/a)^{1/\gamma}$. This last condition reads as

$$E(t=0) \lesssim \min\left\{m^{\frac{3k+\alpha-2}{k-(1-\alpha)}}, m^{\frac{3k+\beta-2}{k-(1-\beta)}}\right\}.$$

The final observation is that since $m(u)$ is constant, the worst case for $E(u)$ is reached by $u = \|u\|_\infty \mathbf{1}_{(-\frac{m}{2\|u\|_\infty}, \frac{m}{2\|u\|_\infty})}$ and then

$$E(u) = \int x^2 u dx \geq \int_{-\frac{m}{2\|u\|_\infty}}^{\frac{m}{2\|u\|_\infty}} x^2 \|u\|_\infty dx = \frac{m^3}{12\|u\|_\infty^2},$$

and thus the fact that $E(u) \to 0$ forces u to blow up in L^∞ as the Hölder inequality implies. □

3.5.2 Case (3.3)

Let $\alpha \in (0, 1]$, $k > 1 + \alpha$ and consider a smooth odd bounded function b with $b_x \in L^p$ such that for $x \geq 0$

$$\min\{x^\alpha, (\bar{x})^\alpha\} \leq -b(x) \leq \min\{x^\alpha, (2\bar{x})^\alpha\}, \quad \bar{x} \geq 1, \tag{3.35}$$

and $(1 - \alpha)p < 1$.

Proposition 3.5.3. *For every measurable positive function $f : \mathbb{R} \to [0, \infty)$ the following holds.*

1. *If*

$$\left(\frac{2k}{k - (\alpha + 1)}\right)^{-\frac{k}{3k+1-\alpha}} \left(\int f^{k+1}|xb|dx\right)^{-\frac{1}{3k+1-\alpha}} E^{\frac{k+1}{3k+1-\alpha}} \leq \bar{x}, \tag{3.36}$$

   then

$$m \leq 2\left(\frac{2k}{k - (\alpha + 1)}\right)^{\frac{2k}{3k+1-\alpha}} \left(\int f^{k+1}|xb|dx\right)^{\frac{2}{3k+1-\alpha}} E^{\frac{k-(\alpha+1)}{3k+1-\alpha}}.$$

2. *If*

$$\left(\frac{2k}{k - 1}\right)^{-\frac{k}{3k+1}} \left(\int f^{k+1}|xb|dx\right)^{-\frac{1}{3k+1}} E^{\frac{k+1}{3k+1}} \geq \bar{x}, \tag{3.37}$$

   then

$$m \leq (1 + 2^{\frac{k-1}{k+1}}) \left(\frac{2k}{k - 1}\right)^{\frac{2k}{3k+1}} \left(\int f^{k+1}|xb|dx\right)^{\frac{2}{3k+1}} E^{\frac{k-1}{3k+1}}.$$

Proof. Case 1. Define

$$R' := \left(\frac{2k}{k - (\alpha + 1)}\right)^{-\frac{k}{3k+1-\alpha}} \left(\int f^{k+1}|xb|dx\right)^{-\frac{1}{3k+1-\alpha}} E^{\frac{k+1}{3k+1-\alpha}}, \tag{3.38}$$

and compute by Hölder inequality

$$
\begin{aligned}
m = \int f dx &= \int_{|x|\le R'} f dx + \int_{|x|>R'} f dx \\
&\le \left(\int f^{k+1}|x|^{\alpha+1}dx\right)^{\frac{1}{k+1}} \left(\int_{|x|\le R'} \frac{1}{|x|^{\frac{1+\alpha}{k}}} dx\right)^{\frac{k}{k+1}} + \frac{1}{R'^2}E \\
[(3.36)] \quad &= \left(\int f^{k+1}|xb|dx\right)^{\frac{1}{k+1}} \left(\frac{2k}{k-(\alpha+1)}\right)^{\frac{k}{k+1}} (R')^{\frac{k-(\alpha+1)}{k+1}} + \frac{1}{(R')^2}E \\
[(3.38)] \quad &= 2\left(\frac{2k}{k-(\alpha+1)}\right)^{\frac{2k}{3k+1-\alpha}} \left(\int f^{k+1}|xb|dx\right)^{\frac{2}{3k+1-\alpha}} E^{\frac{k-(\alpha+1)}{3k+1-\alpha}},
\end{aligned}
$$

which is the first estimate.
Case 2. Define

$$
M' := \left(\frac{2k}{k-1}\right)^{-\frac{k}{3k+1}} \left(\int f^{k+1}|xb|dx\right)^{-\frac{1}{3k+1}} E^{\frac{k+1}{3k+1}}, \tag{3.39}
$$

and compute

$$
\begin{aligned}
m = \int f dx &= \int_{|x|\le M'} f dx + \int_{|x|>M'} f dx \\
&\le \left(\int f^{k+1}|xb|dx\right)^{\frac{1}{k+1}} \left(\int_{|x|\le M'} \frac{1}{|xb|^{\frac{1}{k}}} dx\right)^{\frac{k}{k+1}} + \frac{1}{(M')^2}E \\
[(3.35),(3.37)] \quad &\le \left(\int f^{k+1}|xb|dx\right)^{\frac{1}{k+1}} \left(2\int_0^{\bar{x}} \frac{1}{|x|^{\frac{1+\alpha}{k}}} dx + 2\int_{\bar{x}}^{M'} \frac{1}{|x(\bar{x})^\alpha|^{\frac{1}{k}}} dx\right)^{\frac{k}{k+1}} + \frac{1}{(M')^2}E \\
[\bar{x}\ge 1] \quad &\le \left(\int f^{k+1}|xb|dx\right)^{\frac{1}{k+1}} \left(\frac{2k}{k-1}\right)^{\frac{k}{k+1}} (2M')^{\frac{k-1}{k+1}} + \frac{1}{(M')^2}E \\
[(3.39] \quad &= (1+2^{\frac{k-1}{k+1}})\left(\frac{2k}{k-1}\right)^{\frac{2k}{3k+1}} \left(\int f^{k+1}|xb|\right)^{\frac{2}{3k+1}} E^{\frac{k-1}{3k+1}}.
\end{aligned}
$$

This is the second estimate in the statement. □

Theorem 3.5.4. *Assume* (3.3) *and* $k > 2 - \frac{1}{p}$. *If* $E(u_0) \ll 1$ *and* b *satisfying* (3.29) *with*

$$
\bar{x} \ge \left(\frac{k-1}{k-(\alpha+1)}\right)^{\frac{k}{\alpha}}, \tag{3.40}
$$

then the solution of (3.1) *blows up in finite time.*

Proof. The choice of the constant c covers at least one of the cases (3.36) or (3.37): indeed, as in the proof of Theorem 3.5.2, the condition (3.40) implies that

$$\bar{x}^{3k+1-\alpha}\left(\frac{2k}{k-(1+\alpha)}\right)^k \leq \bar{x}^{3k+1}\left(\frac{2k}{k-1}\right)^k.$$

Then by Proposition 3.5.3 and (3.1), one obtains

$$\frac{dE}{dt} \leq 2m - C\min\left\{\frac{m^{\frac{3k+1-\alpha}{2}}}{E^{\frac{k-(\alpha+1)}{2}}}, \frac{m^{\frac{3k+1}{2}}}{E^{\frac{k-1}{2}}}\right\},$$

which leads to $E(t) \to 0$ if

$$E(t=0) \lesssim \min\left\{m^{\frac{3k-1-\alpha}{k-(1-\alpha)}}, m^{\frac{3k-1}{k-1}}\right\}.$$

As in the previous case, $E(t) \searrow$ in finite time implies $\|u(t)\|_\infty$ blows up. □

Remark 3.5.5. Observe that if u is bounded then

$$\frac{\|u\|_1^3}{\|u\|_\infty^2} \lesssim E,$$

so that

$$\frac{m^{\frac{2k-(1-\alpha)}{2}}}{E^{\frac{k-(1-\alpha)}{2}}} \gtrsim m^{1-\alpha-\frac{k}{2}}\|u\|_\infty^{k-(1-\alpha)},$$

$$\frac{m^{\frac{3k+1}{2}}}{E^{\frac{k-1}{2}}} \gtrsim \|u\|_\infty^{k-1}m^2.$$

Hence if $m \ll 1$, the blow up may not be possible.

This is also easily verified directly by the estimate (obtained as in (3.21))

$$\begin{aligned}\frac{d}{dt}\|u\|_2^2 &\leq -\|u_x\|_2^2 + C\|b\|_{p,\infty}\|u_x\|_2\|u\|_{(k+1)\tilde{p}'}^{k+1} \\ &\leq -\|u_x\|_2^2 + C\|u\|_1^{\frac{1}{\tilde{p}'}}\|u_x\|_2\|u\|_\infty^{k+1-\frac{1}{\tilde{p}'}}.\end{aligned}$$

Thus $\|u\|_2^2$ is bounded if

$$\|u\|_1^{\frac{1}{p'}}\|u\|_\infty^{k+1-\frac{1}{\tilde{p}}} \lesssim \frac{\|u\|_\infty^{\frac{3}{2}}}{\|u\|_1^{\frac{1}{2}}} \lesssim \|u_x\|_2, \quad \|u\|_\infty^{k-\frac{1}{2}-\frac{1}{p'}} \lesssim \frac{1}{\|u\|_1^{\frac{1}{2}+\frac{1}{p'}}}.$$

where we have used Gagliando-Niremberg inequality. Hence using again the bound (3.25) we have that this bound can be prolonged up to $+\infty$ if $\|u\|_1 \ll 1$.

A completely similar estimate can be done for the case $b_x \in L^\infty L^{p,\infty}$, $p \in (1, \infty]$.

3.6 Long behavior of solutions

In this last section, we discuss the long behavior of the solutions when there are uniform bounds on the drift b:

1. $b \in L^\infty\big((0,\infty), L^{p,\infty}(\mathbb{R})\big)$ for the case (3.2),
2. $b_x \in L^\infty\big((0,\infty), L^{p,\infty}(\mathbb{R})\big)$, $b \in L^\infty_{\mathrm{loc}}\big((0,\infty)\times\mathbb{R}\big)$ for the case (3.3).

We prove the following theorem.

Theorem 3.6.1. *The following holds.*

1. *If $k < 1 - \frac{1}{p}$, then there are drifts $b = b(x) \in L^{p,\infty}$ admitting a stationary solution; similarly, if $k < 2 - \frac{1}{p}$, there are drift $b_x(x) \in L^{p,\infty}_x$ admitting a stationary solution.*
2. *If $k \geq 1 - \frac{1}{p}$, $b \in L^\infty_t L^{p,\infty}_x$ or $k \geq 2 - \frac{1}{p}$, $b_x \in L^\infty_t L^{p,\infty}_x$, every uniformly bounded solution $u(t)$ decays to 0 as $t^{-\frac{1}{2}}$ in the L^∞-norm.*

Proof. Point (1). Define the function

$$u(x) = \frac{1}{(1+x^2)^{\frac{1-1/p}{2k}}},$$

which is a stationary solution to (3.1) in L^1 if $k < 1 - \frac{1}{p}$ and the drift b is given by

$$b(x) = -\frac{1-\frac{1}{p}}{k}\frac{x}{(1+x^2)^{\frac{1+1/p}{2}}}.$$

Being $|b(x)| \sim |x|^{-1/p}$ for $|x| \gg 1$, we have that $b \in L^{p,\infty}(\mathbb{R})$.

For the case $b_x \in L^{p,\infty}$, one can similarly show that

$$u(x) = \frac{1}{(1+x^2)^{\frac{2-\frac{1}{p}}{2k}}},$$

which gives the drift

$$b(x) = -\frac{2-\frac{1}{p}}{k}\frac{x}{(1+x^2)^{\frac{1}{2p}}}.$$

Point (2). We have only to consider the supercritical case, being $k = 1 - \frac{1}{p}$ studied in Section 3.4.5.

For the case $b \in L^\infty_t L^{p,\infty}_x$, we observe that

$$bu^{1+k} = (bu^{k-1+\frac{1}{p}})u^{1-\frac{1}{p}} = \hat{b}u^{1-\frac{1}{p}},$$

with

$$\|\hat{b}\|_{p,\infty} \leq \|u\|_\infty^{k-1+\frac{1}{p}} \|b\|_{p,\infty}.$$

Hence the analysis of the critical case can be applied here, deducing that $\|u(t)\|_\infty \leq \frac{C}{\sqrt{t}}$.

For the case $b_x \in L^{p,\infty}$, we follow the computations of (3.22):

$$\begin{aligned}
\left| \int v_{a,y} \tilde{b} v_a^{k+1} \right| &\leq \frac{1}{k+2} \left| \int \tilde{b}_y v_a^{k+2} \right| \\
&\leq C \|\tilde{b}_y\|_{p,\infty} \|v_a^{k+2}\|_{p',1} \\
&\leq C \|\tilde{b}_y\|_{p,\infty} \|v_a\|_\infty^{k-2+\frac{1}{p}} \|v_a^{2-\frac{1}{p}+2}\|_{p',1} \\
\left[\|v(\tau)\|_\infty \lesssim (Te^{-\tau})^{\frac{1}{2}}\right] \quad &\leq C (Te^{-\tau})^{\frac{2-\frac{1}{p}-k}{2}} (Te^{-\tau})^{\frac{k-2+\frac{1}{p}}{2}} \|v_a\|_{(2-\frac{1}{p})p',2}^{2+2-\frac{1}{p}} \\
\left[\text{as in (3.22)}\right] \quad &\leq \frac{1}{2} \|v_{a,y}\|_2^2 + \frac{1}{8} \|v_a\|_2^2.
\end{aligned}$$

Hence one can repeat the same analysis as in the critical case $k = 2 - \frac{1}{p}$, replacing $\|b_x\|_{p,\infty}$ with $\|b_x\|_\infty \|u\|_\infty^{k-2+\frac{1}{p}}$. In particular one obtains the decay $\|u(t)\|_\infty \lesssim t^{-\frac{1}{2}}$. □

Remark 3.6.2. By slightly changing the exponents in the subcritical case, it is possible to show that actually the vector field can be taken in L^p (or $b_x \in L^p$) in Point (1) of the above theorem.

Part III

Chapter 4

On the Hausdorff Measure of $\mathbb{R}^n$

In this chapter, we answer a question raised by David H. Fremlin about the Hausdorff measure of $\mathbb{R}^2$ with respect to a distance inducing the Euclidean topology. In particular we prove that the Hausdorff n-dimensional measure of $\mathbb{R}^n$ is never 0 when considering a distance inducing the Euclidean topology. Finally, we show via counterexamples that the previous result does not hold in general if we remove the assumption on the topology. See also [6].

4.1 Introduction

This chapter aims to answer an open question stated by D. H. Fremlin in his famous book [17], which can be found moreover indicated by letter code 'GV' on https://www1.essex.ac.uk/maths/people/fremlin/answer.pdf:

$$\begin{aligned}&\textit{let us consider a metric } \rho \textit{ on } \mathbb{R}^2 \textit{ inducing the Euclidean topology,}\\ &\textit{is it possible that } \mathcal{H}^2_\rho\left(\mathbb{R}^2\right) = 0\,?\end{aligned} \tag{Q}$$

By $\mathcal{H}^n_\rho$ we denote the n-dimensional Hausdorff measure according to Definition 4.1.1 below. Before stating our main theorem in Section 4.2, we recall in this introductory section some classical tools for the convenience of the reader (see [16], [21] for further details).

Definition 4.1.1 (Hausdorff measure)**.** Let (X, d) be a metric space. We define the n-dimensional Hausdorff outer measure of $A \in \mathcal{P}(X)$ as

$$\mathcal{H}^n_{\mathrm{d}}(A) := \sup\nolimits_{\delta>0} \mathcal{H}^n_{\delta,\mathrm{d}}(A), \qquad \text{with} \tag{4.1}$$

$$\mathcal{H}^n_{\delta,\mathrm{d}}(A) := \inf\Big\{\sum_{i\in I} \mathrm{diam}(A_i)^n \ : \ A \subseteq \cup_{i\in I} A_i,\ \mathrm{diam}(A_i) \le \delta\Big\}, \tag{4.2}$$

where $\mathrm{diam}(U) = \sup_{x,y\in U} \mathrm{d}(x, y)$ and I is an at most countable collection of indices.

Remark 4.1.2. The usual definition of Hausdorff measure is given scaling the result by a dimensional constant that, for instance, in the Euclidean case is equal to $2^{-n}\omega_n$, where ω_n is the volume of the unit n-ball. We opted to overlook the constant in order to simplify the notation. Clearly, Theorem 4.2.1 is not affected by this choice.

To prove our result we will exploit the following well-known theorem.

Theorem 4.1.3 (Dini)**.** *Let (K, d) be a compact metric space. Let $f_n : K \to \mathbb{R}$ be continuous functions such that*

$$f_n \leq f_{n+1} \qquad \forall n \in \mathbb{N} \tag{4.3}$$

and assume that

$$f(x) = \lim_{n\to+\infty} f_n(x) \qquad \forall x \in K, \tag{4.4}$$

exists and the function $f : K \to \mathbb{R}$ is also continuous. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on K.

Moreover, we briefly recall the definition and some properties of the Brouwer Degree. See for instance [15] for a complete treatment of this topic.

Theorem 4.1.4 (Brouwer Degree)**.** *There exists a unique function, called Brouwer Degree and denoted by* deg*, from the set of couples (D, f), where $D \subset \mathbb{R}^n$ is open and bounded and $f : \bar{D} \to \mathbb{R}^n$ is continuous with $0 \notin f(\partial D)$, into the set $\mathbb{Z}$, which satisfies the following three properties:*

- *(Normalization)* $\deg[\mathrm{id}, D] = 1$ *if* $0 \in D$.
- *(Additivity)* $\deg[f, D] = \deg[f, D_1] + \deg[f, D_2]$ *if D_1 and D_2 are disjoint open subsets of D such that $0 \notin f(\bar{D} \setminus (D_1 \cup D_2))$.*
- *(Homotopy invariance) If $F \in C([0, 1] \times \bar{D}, \mathbb{R}^n)$ and $0 \notin F([0, 1] \times \partial D)$, then* $\deg[F(t, \cdot), D]$ *is independent of* $t \in [0, 1]$.

For the proof of the Brouwer Degree Theorem see [15, Section 1.2.5.].

Definition 4.1.5. If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C(\bar{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, the Brouwer degree $\deg[f, D, z]$ is defined by $\deg[f, D, z] = \deg[f(\cdot) - z, D]$.

Proposition 4.1.6 ([15, Corollary 1.2.5])**.** *If $z \notin f(\bar{D})$, then* $\deg[f, D, z] = 0$.
Equivalently, if $\deg[f, D, z] \neq 0$*, there exists at least one $x \in D$ such that $f(x) = z$.*

4.2 Main results

We are now in the position to state our main theorem.

Theorem 4.2.1. *Let* $(\mathbb{R}^n, \rho)$ *be a metric space with* ρ *inducing the Euclidean topology, then* $\mathcal{H}^n_\rho(\mathbb{R}^n) > 0$.

Proof. Assume by contradiction that there exists a distance ρ in $\mathbb{R}^n$ such that $\mathcal{H}^n_\rho(\mathbb{R}^n) = 0$. We denote by $\mathbb{B}(0,1)$ the closed unit ball with respect to Euclidean metric and we consider the identity map

$$\mathrm{id} : (\mathbb{B}(0,1), \rho) \longrightarrow (\mathbb{B}(0,1), \mathrm{d}_{\mathrm{eucl}}). \tag{4.5}$$

Such a map is a homeomorphism by assumption, but it carries no metric information a priori. Let us write

$$\mathrm{id}(x) = \big(\pi_1(x), \ldots, \pi_n(x)\big) \tag{4.6}$$

and define

$$\pi_i^\varepsilon(x) := \min_{z\in\mathbb{B}(0,1)} \left[\pi_i(z) + \frac{1}{\varepsilon}\rho(x,z)\right] \qquad \forall i = 1, ..., n \quad \forall x \in \mathbb{B}(0,1), \tag{4.7}$$

where we are using that $\mathbb{B}(0,1)$ is compact also for the metric ρ. The latter functions are Lipschitz, since they are the infimum of a family of equi-Lipschitz functions, more precisely

$$|\pi_i^\varepsilon(x) - \pi_i^\varepsilon(y)| \leq \frac{1}{\varepsilon}\rho(x,y) \quad \forall x, y \in \mathbb{B}(0,1). \tag{4.8}$$

We say that the functions π_i^ε converge uniformly in the compact ball $\mathbb{B}(0,1)$ to the components of the identity as $\varepsilon \to 0$. In order to prove that, for every $\varepsilon_m \to 0$ consider a sequence $(z_{\varepsilon_m})_m \subseteq \mathbb{B}(0,1)$ such that

$$\pi_i^{\varepsilon_m}(x) = \pi_i(z_{\varepsilon_m}) + \frac{1}{\varepsilon_m}\rho(x, z_{\varepsilon_m}). \tag{4.9}$$

Since $(z_{\varepsilon_m})_m$ is bounded, by compactness there exists a convergent subsequence. Due to equation (4.9) and the bound

$$1 \geq \pi_i \geq \pi_i^{\varepsilon_m} \geq -1, \tag{4.10}$$

it follows that $\lim_{m\to+\infty} \rho(z_{\varepsilon_m}, x) = 0$, which means that $(z_{\varepsilon_m})_m$ converges to x, leading to the pointwise convergence. Now, since we have $\pi_i^\varepsilon(x) \geq \pi_i^{\varepsilon+\gamma}(x)$ for every $\gamma, \varepsilon > 0$ and for every $x \in \mathbb{B}(0,1)$, by Dini's theorem $\pi_i^{\varepsilon_m}$ converges uniformly to π_i on $\mathbb{B}(0,1)$ for every $i = 1, ..., n$. Summing up we have obtained a sequence

$$F^\varepsilon = (\pi_1^\varepsilon, ..., \pi_n^\varepsilon) : (\mathbb{B}(0,1), \rho) \longrightarrow (\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}) \tag{4.11}$$

such that

$$\mathrm{d}_{\mathrm{eucl}}\big(F^\varepsilon(x), F^\varepsilon(y)\big) \leq C_\varepsilon \rho(x,y) \quad \forall x, y \in \mathbb{B}(0,1) \tag{4.12}$$

with $C_\varepsilon > 0$ and such that it converges uniformly to the identity in $\mathbb{B}(0,1)$. The following claim is of crucial importance.
Claim: there exists $\varepsilon > 0$ such that $F^\varepsilon(\mathbb{B}(0,1))$ has non-empty interior.
Fix $\hat{\varepsilon} > 0$ such that

$$\sup_{x\in\mathbb{B}(0,1)} \mathrm{d}_{\mathrm{eucl}}(F^\varepsilon(x), x) \leq \frac{1}{2} \tag{4.13}$$

for every $\varepsilon \in [0,\hat{\varepsilon}]$ and consider the function

$$F : [0,\hat{\varepsilon}] \times \mathbb{B}(0,1) \to \mathbb{R}^n \tag{4.14}$$

defined by the relation $F(\varepsilon,\cdot) = F^\varepsilon$ for $\varepsilon > 0$ and $F(0,\cdot) = id$. We prove that the function F is a continuous function, or in other words that F is a homotopy between id and $F^{\hat{\varepsilon}}$. First we observe that for every $\varepsilon_m \nearrow \varepsilon$ in $(0,\hat{\varepsilon}]$, given z_ε such that

$$F_i^\varepsilon(x) = \pi_i(z_\varepsilon) + \frac{1}{\varepsilon}\rho(x, z_\varepsilon), \tag{4.15}$$

then

$$\pi_i(z_\varepsilon) + \frac{1}{\varepsilon_m}\rho(x, z_\varepsilon) \geq F_i^{\varepsilon_m}(x) \geq F_i^\varepsilon(x) \tag{4.16}$$

and taking the limit for $m \to +\infty$, we obtain that $\lim_{m\to+\infty} F_i^{\varepsilon_m}(x) = F_i^\varepsilon(x)$. Also, for every $\varepsilon_m \searrow \varepsilon$ in $(0,\hat{\varepsilon}]$ and every $x \in \mathbb{B}(0,1)$, we have

$$1 \geq \pi_i(x) \geq F_i^\varepsilon(x) \geq F_i^{\varepsilon_m}(x) \geq -1. \tag{4.17}$$

Given z_m such that

$$F_i^{\varepsilon_m}(x) = \pi_i(z_m) + \frac{1}{\varepsilon_m}\rho(z_m, x), \tag{4.18}$$

up to a subsequence, we have that $z_m \to \hat{z}$. By equation (4.17), $\hat{z}$ realizes the minimum for $F_i^\varepsilon(x)$, thus we have $F_i^{\varepsilon_m}(x) \to F_i^\varepsilon(x)$. For a generic sequence $\varepsilon_m \to \varepsilon$ in $(0,\hat{\varepsilon}]$ and for every $x \in \mathbb{B}(0,1)$ fixed, up to subsequences we can assume either $\varepsilon_m \searrow \varepsilon$ or $\varepsilon_m \nearrow \varepsilon$, hence we have that $F_i^{\varepsilon_m}(x) \to F_i^\varepsilon(x)$. In general, given $\varepsilon_m \to \varepsilon$ in $(0,\hat{\varepsilon}]$ and $x_m \to x$ in $\mathbb{B}(0,1)$, consider z_m satisfying equation (4.18) as before and observe that

$$\begin{aligned} |F_i^{\varepsilon_m}(x_m) - F_i^\varepsilon(x)| &\leq |F_i^{\varepsilon_m}(x_m) - F_i^{\varepsilon_m}(x)| + |F_i^{\varepsilon_m}(x) - F_i^\varepsilon(x)| \\ &\leq \frac{\rho(x_m, x)}{\varepsilon_m} + o(1) = o(1) \qquad \text{for } m \to +\infty. \end{aligned} \tag{4.19}$$

Finally, consider the last case when $\varepsilon_m \to 0$ and $x_m \to x$ in $\mathbb{B}(0,1)$, then

$$|F_i^{\varepsilon_m}(x_m) - \pi_i(x)| \leq |F_i^{\varepsilon_m}(x_m) - \pi_i(x_m)| + |\pi_i(x_m) - \pi_i(x)| \leq o(1), \tag{4.20}$$

because of uniform convergence, whence F is an homotopy between id and $F^{\bar{\varepsilon}}$.
Now we consider the topological degree of the function $F^{\hat{\varepsilon}}$ with respect to the set $\mathbb{B}(0,1)$

and any point of $B(0,\frac{1}{2})$, the open ball of radius $\frac{1}{2}$. We recall that the map $F^{\hat{\varepsilon}}$ is homotopy equivalent to the map id, and observe that equation (4.13) implies that for any $y \in B(0,\frac{1}{2})$, we have $y \notin F^{\varepsilon}(\mathbb{B}(0,1) \setminus B(0,1))$. Therefore, we can apply homotopy invariance obtaining that

$$1 = \deg(\mathrm{id}, \mathbb{B}(0,1), y) = \deg(F^{\hat{\varepsilon}}, \mathbb{B}(0,1), y) \tag{4.21}$$

for every $y \in B(0,\frac{1}{2})$, hence, by Proposition 4.1.6, it follows $B(0,\frac{1}{2}) \subseteq F^{\hat{\varepsilon}}(\mathbb{B}(0,1))$, proving the claim.
Since $F^{\hat{\varepsilon}}(\mathbb{B}(0,1))$ contains a non-empty open set and $F^{\hat{\varepsilon}}$ is Lipschitz, we get

$$\mathcal{H}^n_{\mathrm{d_{eucl}}}\big(F^{\hat{\varepsilon}}(\mathbb{B}(0,1))\big) \leq C^n_{\varepsilon}\mathcal{H}^n_{\rho}(\mathbb{B}(0,1)) = 0, \tag{4.22}$$

which is a contradiction since the n-dimensional Hausdorff measure on $\mathbb{R}^n$ with the Euclidean distance gives positive measure to not empty open sets. □

Remark 4.2.2. The same proof of Theorem 4.2.1 can be adapted to prove that any nonempty open set A is such that $\mathcal{H}^n_{\rho}(A) > 0$.

Remark 4.2.3. Removing the assumption that ρ induces the Euclidean topology, counterexamples show that $\mathcal{H}^n_{\rho}(\mathbb{R}^n)$ might vanish. Consider, for instance, the metric space $(\mathcal{C}, \mathrm{d})$, where $\mathcal{C} \subset \mathbb{R}$ is the Cantor set and d denotes the usual one-dimensional Euclidean distance. Having $\mathcal{C}$ the cardinality of the continuum, there exist bijections $g_n : \mathcal{C} \to \mathbb{R}^n$. Then, define on $\mathbb{R}^n$ the metric $\rho(x,y) = \mathrm{d}(g_n^{-1}(x), g_n^{-1}(y))$.
Given any collection $(A_i)_{i\in\mathbb{N}}$ that covers $\mathcal{C}$, follows that $(g_n(A_i))_{i\in\mathbb{N}}$ covers $\mathbb{R}^n$ and $\mathrm{diam}(A_i) = \mathrm{diam}(g_n(A_i))\,\forall i \in \mathbb{N}$. Clearly, also the opposite direction applies. Therefore, we have

$$\mathcal{H}^n_{\rho}(\mathbb{R}^n) = \mathcal{H}^n_{d}(\mathcal{C}) = 0 \tag{4.23}$$

that shows a counterexample.

Remark 4.2.4. Note that, under previous assumptions on ρ, it is not true in general that $\dim^{\rho}_{H}(\mathbb{R}^n) = n$. In fact, choosing $\rho(x,y) = \mathrm{d_{eucl}}(x,y)^{1/2}$, the distance ρ induces the Euclidean topology, but in this case

$$\mathcal{H}^s_{\mathrm{d_{eucl}}}(A) = \mathcal{H}^{2s}_{\rho}(A)$$

for all $A \subseteq \mathbb{R}^n,\ s \geq 0$, see for example [16]. For this reason we get that $\dim^{\rho}_{H}(\mathbb{R}^n) = 2n$.

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