

Modular flavour symmetries and modulus stabilisation

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ABSTRACT: We study the problem of modulus stabilisation in the framework of the modular symmetry approach to the flavour problem. By analysing simple UV-motivated CP-invariant potentials for the modulus τ we find that a class of these potentials has (non-fine-tuned) CP-breaking minima in the vicinity of the point of \mathbb{Z}_3^{ST} residual symmetry, $\tau \simeq e^{2\pi i/3}$. Stabilising the modulus at these novel minima breaks spontaneously the CP symmetry and can naturally explain the mass hierarchies of charged leptons and possibly of quarks.

KEYWORDS: CP Violation, Theories of Flavour, Supergravity Models

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Contents

1	Introduction	1
2	Framework	5
2.1	Modular symmetry as a flavour symmetry	5
2.2	Scalar potential	6
2.3	q - and u -expansions	9
3	Results	11
3.1	Numerical analysis of minima for various m, n	11
3.2	CP-violating minima of $V_{m,0}$	12
4	Summary and conclusions	16
A	Modular forms	19
B	Rigid SUSY limit	20
C	u-expansions	21

1 Introduction

Understanding the origins of flavour in both the quark and lepton sectors, i.e., of the patterns of quark masses and mixing, of charged-lepton and neutrino masses and of neutrino mixing and CP violation in the quark and lepton sectors, is one of the most challenging fundamental problems in contemporary particle physics [1]. Although this problem arose more than 20 years ago, all efforts to find a satisfactory solution have essentially failed. A universal, elegant, natural and viable theory of flavour is still lacking. Constructing such a theory would be a major achievement and a breakthrough in particle physics.

The unsatisfactory status of the flavour problem together with the remarkable progress made in the studies of neutrino oscillations (see, e.g., [2]), which began 23 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by the SuperKamiokande experiment [3] and led to the determination of the pattern of 3-neutrino mixing consisting of two large and one small mixing angles, stimulated renewed attempts to seek new approaches to the lepton as well as to the quark flavour problems. A step forward in this direction was made in 2017 in ref. [4], where the idea of using modular invariance as a flavour symmetry was put forward. The first phenomenologically viable lepton flavour models based on modular symmetry appeared in the first half of 2018 [5–7] and, since then, the modular-invariance approach to the flavour problem has been and continues to be intensively investigated and developed with encouraging results.

In the modular-invariance approach, the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian are expressed in terms of modular forms of a certain level N and a limited number of coupling constants. The modular forms are functions of a single complex scalar field τ — the modulus. Both the modulus τ and the modular forms have specific transformation properties under the action of the modular group $\Gamma \equiv \text{SL}(2, \mathbb{Z})$. The matter fields are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$, while the modular forms furnish irreducible representations of the same group. For $N \leq 5$, the finite modular groups Γ_N are isomorphic to the permutation groups S_3 , A_4 , S_4 and A_5 (see, e.g., [8]) and the groups Γ'_N are isomorphic to their double covers. These groups are quite extensively used in flavour model building (see, e.g., [9–13]). The modular symmetry described by the finite modular group $\Gamma_N^{(\prime)}$ plays the role of a flavour symmetry and the theory is assumed to be invariant under the whole modular group Γ .

A very appealing feature of this approach is that the vacuum expectation value (VEV) of the modulus τ can be the only source of flavour symmetry breaking, such that flavons are not needed.¹ Another appealing feature of the discussed framework is that the VEV of τ can also be the only source of breaking of CP symmetry [14].

There is no VEV of τ which preserves the full modular symmetry. However, as pointed out in [15] and exploited in [16–18], there exist three values in the modular group fundamental domain, which do not break the modular symmetry completely. These so-called “fixed points” are $\tau_{\text{sym}} = i, \omega, i\infty$, with $\omega \equiv \exp(2\pi i/3) = -1/2 + \sqrt{3}/2 i$ (the ‘left cusp’), and, for theories based on Γ_N invariance, preserve $\mathbb{Z}_2^S, \mathbb{Z}_3^{ST}$, and \mathbb{Z}_N^T residual symmetries, respectively.² After the flavour symmetry is fully or partially broken, the modular forms and thus the elements of the Yukawa coupling and fermion mass matrices get fixed. Correspondingly, the fermion mass matrices exhibit a certain symmetry-constrained flavour structure.

The approach to the flavour problem based on modular invariance has been widely explored so far primarily in the framework of supersymmetric (SUSY) theories. Within the framework of rigid ($\mathcal{N} = 1$) SUSY, modular invariance is assumed to be a property of the superpotential, whose holomorphicity restricts the number of allowed terms. Following a bottom-up approach, phenomenologically viable and “minimal” lepton flavour models based on modular symmetry, which do not include flavons, have been constructed first using the groups $\Gamma_4 \simeq S_4$ [6] and $\Gamma_3 \simeq A_4$ [7]. A “non-minimal” model with flavons based on $\Gamma_2 \simeq S_3$ has been proposed in [5]. After these studies, the interest in the approach grew significantly and a large variety of models has been constructed and extensively studied. This includes:³

- i) lepton flavour models based on the groups $\Gamma_4 \simeq S_4$ [15, 21–23], $\Gamma_5 \simeq A_5$ [17, 24], $\Gamma_3 \simeq A_4$ [16, 23, 25–27], $\Gamma_2 \simeq S_3$ [28] and $\Gamma_7 \simeq \text{PSL}(2, \mathbb{Z}_7)$ [29],

¹The first modular-invariant “minimal” lepton flavour model without flavons was constructed in [6].

²In the case of the double cover groups Γ'_N , these residual symmetries are augmented by \mathbb{Z}_2^R [19].

³A rather complete list of the articles on modular-invariant models of lepton and/or quark flavour, which appeared by March of 2021, can be found in [20]. We cite here only a representative sample.

- ii) models of quark flavour [30] and of quark-lepton unification [31–36],
- iii) models with multiple moduli, considered first phenomenologically in [15, 16] and further studied, e.g., in [37–39],
- iv) models in which the formalism of the interplay of modular and generalised CP (gCP) symmetries, developed and applied first to the lepton flavour problem in [14], is explored [18, 40–43].

Also the formalism of the double cover finite modular groups Γ'_N , to which top-down constructions typically lead (see, e.g., [44, 45] and references therein), has been developed and viable flavour models have been constructed for the cases of $\Gamma'_3 \simeq T'$ [46], $\Gamma'_4 \simeq S'_4$ [19, 47] and $\Gamma'_5 \simeq A'_5$ [48, 49]. Recently, the framework has been further generalised to arbitrary finite modular groups (i.e., those not described by series Γ'_N) in ref. [50]. It is hoped that the results obtained in the bottom-up modular-invariant approach to the lepton and quark flavour problems will eventually connect with top-down results (see, e.g., [51–61]), based on UV-complete theories.

In practically all phenomenologically viable lepton and/or quark flavour models based on modular invariance, the VEV of the modulus is treated as a free parameter which is determined by confronting model predictions with experimental data. Its value is critical for phenomenological viability and can vary significantly, depending on the model. For instance, viable $\Gamma_4 \simeq S_4$ lepton flavour models consistent with the available data on lepton masses and mixing have been obtained in [15] for values of τ relatively close to the symmetric point $\tau_{\text{sym}} = i$, very close to the boundary of the fundamental domain, at $\text{Re}\tau \simeq \pm 0.5$, as well as for $\tau \simeq \pm 0.143 + 1.523i$ and $\tau \simeq \pm 0.179 + 1.397i$. In [41], where modular A_4 lepton and quark flavour models have been considered, the authors find viable models for different values of τ close to $\tau_{\text{sym}} = i$, and values of τ close to the imaginary axis ($\text{Re}\tau = 0$) with rather large $\text{Im}\tau \simeq 2.67$. In [62], viable lepton flavour models with A_4 modular symmetry have been presented for values of τ close to each of the three symmetric points, $\tau_{\text{sym}} = i, \omega, i\infty$. It should be clear from this discussion that determining the VEV of τ from first principles and not from fits to the data could be used as a powerful selection criteria for the proposed flavour models.

An additional unique feature of the modular approach to the flavour problem, as shown recently in ref. [20], is that one can obtain fermion (charged-lepton and quark) mass hierarchies from the properties of the modular forms — without fine-tuned constants — provided the VEV of the modulus τ takes a value close to the one of the symmetric points $\tau_{\text{sym}} = \omega$ (the left cusp) or $\tau_{\text{sym}} = i\infty$.⁴ In practically all modular-invariant flavour models without flavons considered before the appearance of [20], the charged-lepton and quark mass hierarchies were successfully reproduced with the help of severe fine-tuning of the limited number of coupling constants present in the Yukawa couplings. In the viable fine-tuning-free model constructed in [20], data requires τ to have a value near the cusp

⁴Given the exponential dependence of modular forms on $\text{Im}\tau$, “ ∞ ” effectively means a number sufficiently bigger than one, e.g., a number $\sim (2-3)$.

$\tau_{\text{sym}} = \omega$, selecting a best-fit point with

$$\tau \simeq -0.496 + 0.877i. \tag{1.1}$$

The viable region for τ actually corresponds to a small ring around the cusp, of radius $|u| \simeq 0.007$, with $u \equiv (\tau - \omega)/(\tau - \omega^2)$. The smallness of this quantity is at the basis of the mechanism giving rise to fermion mass hierarchies in this context. The question to address is whether such data-driven values of τ can be naturally justified by a dynamical principle, e.g., from a top-down perspective.

Attempts to determine the value of τ on the basis of dynamical considerations were made in, e.g., [22, 40, 58]. In [22], the authors consider lepton flavour models with $\Gamma_3 \simeq A_4$ modular symmetry arising from the breaking of $\Gamma_4 \simeq S_4$ symmetry by anomalies. Fitting the available data on charged-lepton masses, neutrino mixing angles and neutrino mass-squared differences, they determine the values of the modulus for which the models are phenomenologically viable. They further attempt to obtain these values within the supergravity framework, constructing relatively simple superpotentials. The latter are assumed to be generated non-perturbatively by hidden sector dynamics and involve singlet modular forms of weights 4 or 6. Only the linear combination of potentials, each involving one of the two singlet modular forms is shown to have absolute minima at some of the requisite CP-nonconserving values of τ . However, this combination effectively contains one additional complex parameter which violates CP symmetry explicitly. In a follow-up study [40], the possibility of spontaneously breaking the CP symmetry in theories of flavour based on $\Gamma_3 \simeq A_4$, $\Gamma_2 \simeq S_3$ and $\Gamma_4 \simeq S_4$ was analysed. Superpotentials analogous to those used in [22] were constructed, leading, however, to CP-invariant potentials for the modulus τ . The authors of [40] have found these potentials to have absolute minima at different CP-conserving values of τ , related by the T transformation ($\tau \xrightarrow{T} \tau + 1$). The same result was found in [40] to hold also in theories with global supersymmetry and essentially the same superpotentials. In [58] the authors have considered three-form fluxes in Type IIB string theory and derived the preferred values of τ by investigating the possible configurations of flux compactifications on a $T^6/(Z_2 \times Z'_2)$ orbifold (exploring the so-called “string landscape”). The number of stable vacua depends on a certain positive integer $N_{\text{flux}}^{\text{max}}$ and reads 312, 2918 and 2886221, for $N_{\text{flux}}^{\text{max}} = 10, 100$ and 1000, respectively. These vacua correspond to stabilised values of the modulus τ in the fundamental domain of the modular group. The most probable of these are found to lie on the border of the fundamental domain $\text{Re } \tau = -1/2$, on the imaginary axis $\text{Re } \tau = 0$, and on the arc, $\tau = \exp(i\alpha)$ with, e.g., $\cos \alpha = -1/4$. All these values are CP-conserving [14]. Actually, the so-derived stabilising values of τ are shown to cluster at the CP-conserving symmetry point $\tau = \omega$.⁵

The possibility of spontaneous breaking of the CP symmetry by the modulus VEV was investigated in [63] in a supergravity framework in which the dilaton is also present. The authors showed that if the dilaton is stabilised by non-perturbative corrections to the Kahler potential, by varying the four parameters present in the relevant effective potential

⁵See also [53], where τ is found to be stabilised at the CP-conserving right cusp, $\tau = \exp(i\pi/3)$.

it is possible to find minima of the potential at CP-violating VEVs of the modulus τ inside the fundamental domain of the modular group close to the symmetric point $\tau = i$.

In the present article we address the problem of modulus stabilisation by analysing known and relatively simple supergravity-motivated modular- and CP-invariant potentials for the modulus τ . In section 2 we describe the framework we employ, giving some details on how modular symmetry can act as a flavour symmetry (section 2.1). We introduce the modular- and CP-invariant potentials for τ in section 2.2 and derive in section 2.3 their q - (and u -) expansions which prove useful for the analyses of the potentials. We present the main results of our study in section 3, including the found novel global CP-breaking minima of the considered potentials. Section 4 contains a summary of our results. Some technical details are given in appendices A to C.

2 Framework

2.1 Modular symmetry as a flavour symmetry

The modular approach to flavour is based on invariance under the action of the modular group $\Gamma \equiv \text{SL}(2, \mathbb{Z})$. While in this section we summarise the defining features of this (bottom-up) framework, the reader is referred to [19, 20] for a more detailed description.

The modular group is generated by three elements S, T and R obeying $(ST)^3 = R^2 = \mathbb{1}$, $S^2 = R$ and $RT = TR$. A generic element γ of this group acts on the modulus chiral superfield τ as a fractional linear transformation,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \quad \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}. \quad (2.1)$$

Its action on matter superfields instead reads [4, 64, 65]

$$\gamma \in \Gamma : \quad \psi_i \rightarrow (c\tau + d)^{-k} \rho_{ij}(\gamma) \psi_j, \quad (2.2)$$

where k is the modular weight of ψ and ρ is a unitary representation of Γ . Modular symmetry plays the role of a discrete flavour symmetry when $\rho(\gamma) = \mathbb{1}$ for $\gamma \equiv \mathbb{1} \pmod{N}$. In this case ρ is effectively a (unitary) representation of the finite quotient group $\Gamma'_N \simeq \text{SL}(2, \mathbb{Z}_N)$ characterised by the integer level $N \geq 2$. For $N \leq 5$, $\Gamma'_{2,3,4,5}$ are isomorphic to the double covers S'_3, A'_4, S'_4, A'_5 of the well-known permutation groups S_3, A_4, S_4, A_5 , with $S'_3 \equiv S_3$. If the R generator acts trivially on fields, one is instead dealing with the inhomogeneous finite modular group Γ_N , a quotient of $\bar{\Gamma} \simeq \text{PSL}(2, \mathbb{Z}) \simeq \text{SL}(2, \mathbb{Z})/\mathbb{Z}_2^R$. For small N , the latter finite groups are isomorphic to the already quoted permutation groups $\Gamma_{2,3,4,5} \simeq S_3, A_4, S_4, A_5$.

Modular symmetry may determine the structure of fermion mass matrices, as it severely constrains the form of the superpotential $W(\tau, \psi)$, thanks also to its holomorphicity. To ensure the modular transformation properties of W , Yukawa couplings and fermion mass terms must generically depend on τ and transform in a way similar to fields — they are multiplets $Y(\tau)$ of modular forms of level N , characterised by their own integer weights $k_Y > 0$ and representations of the flavour symmetry group $\Gamma_N^{(l)}$. Since the number of

available independent modular forms is finite (and small for small k_Y), only a limited set may contribute to W . Thus, the number of superpotential parameters is restricted, and mass and coupling matrices are determined once the lowest (scalar) component of the modulus τ acquires a VEV. This setup can be remarkably predictive.⁶

The breakdown of modular symmetry is parameterised by the VEV of the modulus ($\text{Im } \tau > 0$).⁷ This VEV can always be restricted to the fundamental domain \mathcal{D} of the modular group, defined by the union

$$\mathcal{D} \equiv \left\{ \tau \in \mathcal{H} : -\frac{1}{2} \leq \text{Re } \tau < \frac{1}{2}, |\tau| > 1 \right\} \cup \left\{ \tau \in \mathcal{H} : -\frac{1}{2} < \text{Re } \tau \leq 0, |\tau| = 1 \right\}, \quad (2.3)$$

see also figure 1. Any choice of the VEV of the modulus in the upper-half plane can be related to a single $\tau \in \mathcal{D}$ via a modular transformation, and it is therefore physically equivalent to it (see also section 4 of [15]). Instead, two elements in the fundamental domain cannot be related by a modular transformation and are thus physically inequivalent. By convention, the right half of the boundary of \mathcal{D} — including the right half of the unit arc — is excluded from the above definition, since it is equivalent to the left half.⁸

Even though there is no value of τ which preserves the full modular symmetry, as we have already noted, specific residual symmetries remain at certain symmetric points $\tau = \tau_{\text{sym}}$. There are only three (inequivalent) symmetric points [15]:⁹

- $\tau_{\text{sym}} = i$, invariant under S , preserving \mathbb{Z}_4^S (note that $S^2 = R$);
- $\tau_{\text{sym}} = i\infty$, invariant under T , preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$; and
- $\tau_{\text{sym}} = \omega = \exp(2\pi i/3)$ (the “left cusp”), invariant under ST , preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

In models where τ deviates slightly from one of these values τ_{sym} , fermion mass hierarchies may be generated as powers of the small deviation $|\tau - \tau_{\text{sym}}|$ (or as powers of $|q| = e^{-2\pi \text{Im } \tau}$ in the case $\tau_{\text{sym}} = i\infty$) [20].¹⁰ Finally, these symmetric values preserve the CP symmetry of a CP- and modular-invariant theory [14, 19]. In such a theory, the \mathbb{Z}_2^{CP} symmetry is preserved for $\text{Re } \tau = 0$ or for τ lying on the border of \mathcal{D} , but is broken for other generic values of the modulus.

2.2 Scalar potential

While in bottom-up approaches the VEV of τ is scanned over the fundamental domain in a fit to the data, it is clearly desirable to have a dynamical reason for its specific,

⁶One needs to control the form of the (joint) Kähler potential of the modulus and matter fields $K(\tau, \psi)$ [66], typically taken to have a minimal form in a bottom-up approach. This problem is the subject of ongoing research (see, e.g., [44, 67]).

⁷Flavon fields are not required in this approach, and we do not consider them here.

⁸Any point on the right boundary $\text{Re } \tau = 1/2$ can be obtained from a point with the same $\text{Im } \tau$ on the left boundary $\text{Re } \tau = -1/2$ by a T -transformation, while any point on the right half of the arc with given $\text{Re } \tau > 0$ and $\text{Im } \tau$ can be obtained by an S -transformation from the point on the left half of the arc with the same $\text{Im } \tau$.

⁹The R generator is unbroken for any value of τ and a \mathbb{Z}_2^R symmetry is always preserved [19].

¹⁰The residual symmetries are at play at the level of the whole action, and corrections to the Kähler are not expected to qualitatively affect the hierarchies in the fermion mass spectrum.

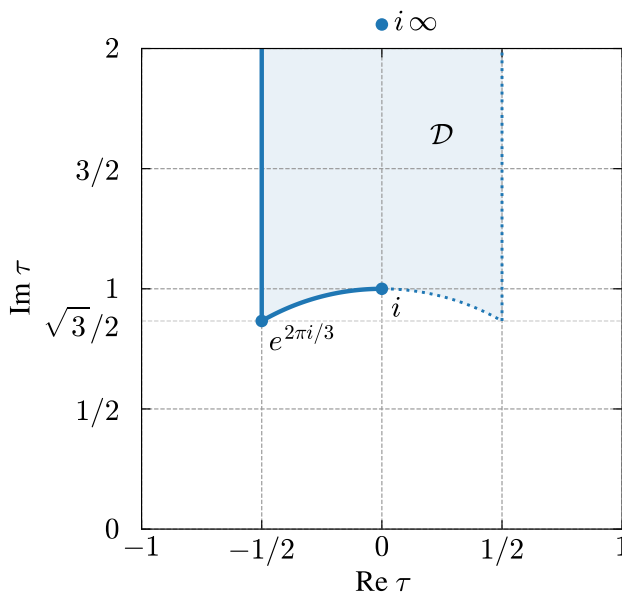


Figure 1. The fundamental domain \mathcal{D} of the modular group Γ and its three symmetric points $\tau_{\text{sym}} = i\infty, i, \omega$. The value of τ can be restricted to \mathcal{D} by a suitable modular transformation. (figure from ref. [19].)

phenomenologically viable value(s). This is the issue we address here. In this work we analyse a known class of simple modular-invariant potentials which are functions of τ alone. Since we are concerned only with the contribution involving the modulus, these turn out to be simplified models, which nevertheless are explicit examples of $\mathcal{N} = 1$ supergravity models.¹¹ We thus focus on the simple Kähler potential [69],

$$K(\tau, \bar{\tau}) = -\Lambda_K^2 \log(2 \text{Im } \tau), \tag{2.4}$$

where Λ_K is a scale (mass dimension one).

We discuss next the form of simple superpotentials $W(\tau)$, following refs. [68, 69]. Keeping in mind the single-modulus case, the relevant $\mathcal{N} = 1$ supergravity action depends on the Kähler-invariant function

$$G(\tau, \bar{\tau}) = \kappa^2 K(\tau, \bar{\tau}) + \log \left| \kappa^3 W(\tau) \right|^2, \tag{2.5}$$

where $\kappa^2 = 8\pi/M_P^2$, M_P being the Planck mass. Given the choice of eq. (2.4) for the Kähler potential, modular invariance of G implies that the superpotential W carries modular weight $-\mathbf{n}$, where $\mathbf{n} = \kappa^2 \Lambda_K^2$. We consider integer values of \mathbf{n} , in line with [68, 69]. The superpotential can then be parameterised in terms of the Dedekind η function (see ap-

¹¹A fully realistic string compactification is expected to involve other moduli, as well as gauge bosons and matter fields [68]. Therefore, the investigated potentials correspond to a subsector of the full theory. In particular, we do not identify their minimum values with the cosmological constant, as the latter receives contributions from other subsectors as well.

pendix A) and a modular-invariant function H , as

$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2n}}, \quad (2.6)$$

where Λ_W is a mass scale so that $H(\tau)$ is dimensionless. The most general H (without singularities in the fundamental domain) can be cast in the following form [69]:

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)), \quad (2.7)$$

making use of the Klein j function, which is invariant under the action of the modular group $\text{SL}(2, \mathbb{Z})$ (see appendix A). Here, m and n are non-negative integers and \mathcal{P} is a polynomial in $j(\tau)$.

The scalar potential in $\mathcal{N} = 1$ supergravity is given by (see, e.g., [70])

$$V = e^{\kappa^2 K} \left(K^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3\kappa^2 |W|^2 \right), \quad (2.8)$$

where $D_i \equiv \partial_i + \kappa^2 (\partial_i K)$, $K^{i\bar{j}}$ is the inverse of the Kähler metric $K_{i\bar{j}} \equiv \partial_i \partial_{\bar{j}} K$, and ∂_i ($\partial_{\bar{j}}$) is the derivative with respect to the corresponding field (its conjugate). In our setup, the only field is the modulus τ , and the scalar potential follows from the explicit forms of $K(\tau, \bar{\tau})$ and $W(\tau)$ given above,

$$V(\tau, \bar{\tau}) = \frac{\Lambda_V^4}{(2 \text{Im } \tau)^n |\eta(\tau)|^{4n}} \left[\left| iH'(\tau) + \frac{n}{2\pi} H(\tau) \hat{G}_2(\tau, \bar{\tau}) \right|^2 \frac{(2 \text{Im } \tau)^2}{n} - 3|H(\tau)|^2 \right], \quad (2.9)$$

where we have defined $\Lambda_V = (\kappa^2 \Lambda_W^6)^{1/4}$ as the mass scale of the potential, and \hat{G}_2 is the non-holomorphic Eisenstein function of weight 2 (see, e.g., [69]). It is given by

$$\hat{G}_2(\tau, \bar{\tau}) = G_2(\tau) - \frac{\pi}{\text{Im } \tau}, \quad (2.10)$$

where G_2 is its holomorphic counterpart (see appendix A). G_2 can be related to the Dedekind function via

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi} G_2(\tau). \quad (2.11)$$

It is not difficult to show that the potential $V(\tau, \bar{\tau})$ is modular-invariant. We briefly discuss its global SUSY limit and its minima in this limit in appendix B.

We are interested in the simple cases investigated in refs. [68, 69], for which $n = 3$ corresponds to the number of compactified complex dimensions.¹² With this choice, the scalar potential reads:

$$V(\tau, \bar{\tau}) = \frac{\Lambda_V^4}{8(\text{Im } \tau)^3 |\eta|^{12}} \left[\frac{4}{3} \left| iH' + \frac{3}{2\pi} H \hat{G}_2 \right|^2 (\text{Im } \tau)^2 - 3|H|^2 \right]. \quad (2.12)$$

¹²The compactification of 6 dimensions may bring about three moduli τ_i ($i = 1, 2, 3$), corresponding to the radii of three two-tori. For simple potentials symmetric under the exchange of the τ_i , the preferred minimum is found to occur at $\tau_1 = \tau_2 = \tau_3 = \tau$ [69]. This result gives support to studying the case of only one modulus, as is done here.

In what follows, we analyse the global minima of this potential. We consider the form of $H(\tau)$ given in eq. (2.7) for different values of m and n . Following again [69] (see also [68]), we take the simplest choice $\mathcal{P}(j) = 1$, which nevertheless yields non-trivial results. In this case, the potential V can be shown to be CP-symmetric, i.e., to be invariant under a reflection with respect to the imaginary axis [14], $\tau \rightarrow -\bar{\tau}$. This follows from the fact that under the reflection $\tau \rightarrow -\bar{\tau}$, the functions H , η , j and \hat{G}_2 are transformed to their conjugates (while $H' \rightarrow -H'^*$, $\eta' \rightarrow -\eta'^*$). This \mathbb{Z}_2^{CP} symmetry is present for a more general choice of $\mathcal{P}(j)$, provided the polynomial coefficients are real or share a common complex phase.

Despite the modular- and CP- invariance of V , the vacuum, which breaks the modular symmetry for any value of the modulus τ , may also spontaneously break the CP symmetry. Extrema not lying at CP-conserving points make up an inequivalent (degenerate) pair, at some τ and $-\bar{\tau}$. In ref. [69], it was conjectured that all extrema of V would lie at CP-conserving values of τ , i.e., either on the boundary of the fundamental domain \mathcal{D} or on the imaginary axis. Therein, the cases $(m, n) = (0, 0), (1, 1), (0, 3)$ were explicitly examined. The global minima of the corresponding potentials were indeed found to lie at $\tau \simeq 1.2i$ (imaginary axis), $\tau \simeq \pm 0.24 + 0.97i$ (equivalent minima on the unit arc) and $\tau = i$, respectively. While we have verified these particular results, we have further found that potentials with $n = 0$ but $m > 0$ do allow for CP-breaking global minima. Moreover, these minima are found to be located in the vicinity of the left cusp $\tau = \omega$, at values of $|\tau - \omega|$ favoured by the mechanism put forward in [20] to explain fermion (charged-lepton and quark) mass hierarchies, as we will see in the following sections.

2.3 q - and u -expansions

As a first step in the analysis of the potential given in eq. (2.12), we express the functions j (and therefore H , H') and \hat{G}_2 in terms of the Dedekind η and its derivatives. Rewriting the latter is immediate via eq. (2.11), while the former can be expressed as

$$j = \left[\frac{72}{\pi^2 \eta^6} \left(\frac{\eta'}{\eta^3} \right)' \right]^3, \tag{2.13}$$

see appendix A. For a broad numerical analysis, it then suffices to know the q -expansion of η , i.e., its expansion in powers of $q = e^{2\pi i \tau}$, up to a certain order. This expansion has the well-known form

$$\eta = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}} = q^{1/24} \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22}) \right) \tag{2.14}$$

and converges rapidly within the fundamental domain, where $|q| \leq e^{-\sqrt{3}\pi} \simeq 0.004$.

A preliminary q -expansion analysis of the potential for different (m, n) with $n \neq 0$ reveals CP-conserving global minima on the imaginary axis and on the unit arc, as expected. For $(m, 0)$ with $m \neq 0$, it further reveals CP-breaking minima in the vicinity of the left cusp $\tau = \omega$ (paired with inequivalent minima in the vicinity of the right cusp $-\bar{\tau} = \omega + 1$). To

guarantee a robust numerical analysis as well as an analytical understanding of these CP-breaking minima, we now develop the expansion of the potential in terms of a parameter $u \equiv (\tau - \omega)/(\tau - \omega^2)$, which quantifies the deviation of τ from the left cusp.

This effort is also warranted in light of the results of ref. [20], where the same convenient parameterisation of deviations from the left cusp was motivated and used. In particular, stabilising τ in the vicinity of ω or at a point with large $\text{Im } \tau$ can provide a non-fine-tuned, i.e., natural explanation of the three generation charged-lepton and quark mass hierarchies, based on the smallness of $|u|$ or $|q|$. In contrast, the stabilisation of τ in the vicinity of i , even if possible, cannot offer an explanation of these mass hierarchies in terms of small deviations from this symmetry point without severe fine-tuning [20] or some additional non-minimal input [71]. Finally, it is known that the potential under analysis diverges for large $\text{Im } \tau$ [69], leaving $\tau \simeq \omega$ as the most interesting case to investigate.

We have seen that the potential can be fully expressed in terms of η and its derivatives. To obtain its u -expansion, it is enough to determine the u -expansion of η . Using $\tau = \omega^2(\omega^2 - u)/(1 - u)$, one can write η as a function of u . It further proves useful to define¹³ $\tilde{\eta}(u) = (1 - u)^{-1/2}\eta(u)$, since symmetry dictates $\tilde{\eta}$ to be a power series in u^3 . Indeed, it is easy to show using $\tau \xrightarrow{T} \tau + 1$ and $\tau \xrightarrow{S} -1/\tau$ that under the modular transformation $\gamma = ST$ the variable u transforms as $u \xrightarrow{ST} \omega^2 u$. Taking further into account that $\eta(T\tau) = \exp(i\pi/12)\eta(\tau)$ and $\eta(S\tau) = \sqrt{-i\tau}\eta(\tau)$, one obtains $\eta(ST\tau) = \sqrt{-\omega(\tau + 1)}\eta(\tau)$. Thus, we have

$$\tilde{\eta}(u) \xrightarrow{ST} \tilde{\eta}(\omega^2 u) = \frac{\sqrt{-\omega(\tau + 1)}}{\sqrt{1 - \omega^2 u}}\eta(u) = \frac{1}{\sqrt{1 - u}}\eta(u) = \tilde{\eta}(u), \quad (2.15)$$

meaning $\tilde{\eta}(u)$ is invariant under $\gamma = ST$. Given the transformation property of u , it follows that only powers of u^3 survive in the u -expansion of $\tilde{\eta}$. Each coefficient in this expansion is given by its own power series in $q(\omega) = -e^{-\sqrt{3}\pi}$, which can be determined by expressing q in terms of u in the known q -expansion for η , eq. (2.14). One can prove analytically that these coefficients are real up to a common phase. Numerically, we obtain

$$\begin{aligned} \tilde{\eta}(u) &\simeq e^{-i\pi/24} \left(0.800579 - 0.573569u^3 - 0.780766u^6 - 0.150007u^9 \right) + \mathcal{O}(u^{12}) \\ &\equiv e^{-i\pi/24} \left(\tilde{\eta}_0 + \tilde{\eta}_3 u^3 + \tilde{\eta}_6 u^6 + \tilde{\eta}_9 u^9 \right) + \mathcal{O}(u^{12}), \end{aligned} \quad (2.16)$$

which is an expansion in powers of u^3 , as anticipated. As a final step, recall that $\eta(u) = \sqrt{1 - u}\tilde{\eta}(u)$, so its u -expansion is trivially related to that of $\tilde{\eta}$. The u -expansions of other modular forms are collected in appendix C.

Let us restate the relevance of these results. Knowing the u -expansion of η allows one to implement the u -expansion of the potential V . The use of such an expansion allows for a clear analysis of the shape of V and of its extrema in the vicinity of the left cusp $\tau = \omega$, converging faster than the usual q -expansion. The corresponding results are shown in the next section.

¹³Unless explicitly stated, we always take the principal branch of the roots appearing in our discussion.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$m = 0$	$0.000 + 1.235i$	$0.000 + 1.000i$	$0.000 + 1.000i$	$0.000 + 1.000i$
$m = 1$	$\mp 0.484 + 0.884i$	$-0.238 + 0.971i$	$-0.190 + 0.982i$	$-0.163 + 0.987i$
$m = 2$	$\mp 0.492 + 0.875i$	$-0.286 + 0.958i$	$-0.239 + 0.971i$	$-0.211 + 0.978i$
$m = 3$	$\mp 0.495 + 0.872i$	$-0.312 + 0.950i$	$-0.267 + 0.964i$	$-0.239 + 0.971i$

Table 1. Values of the modulus τ at the global minima of the potential $V(\tau, \bar{\tau})$, eq. (2.12), obtained numerically for various m and n .

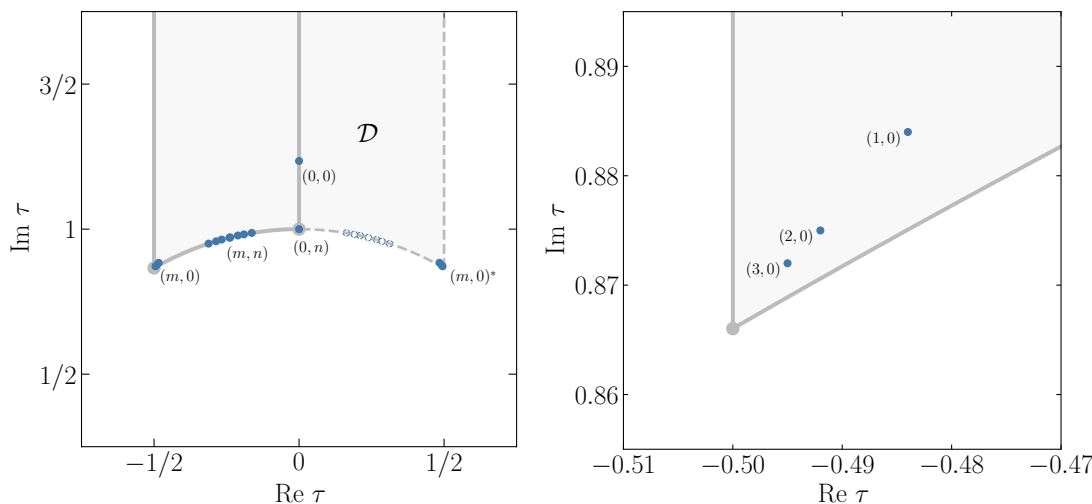


Figure 2. Global minima of the potentials $V(\tau, \bar{\tau})$, eq. (2.12), see text for details. Note that points on the right half of the unit arc, which are CP-conjugates of the (m, n) minima, are excluded as they lie outside the fundamental domain. The right panel shows the series $(m, 0)$ in the vicinity of the left cusp in more detail.

3 Results

3.1 Numerical analysis of minima for various m, n

As discussed in section 2.3, the q -expansions of η and its derivatives allow to compute the potential $V(\tau, \bar{\tau})$, eq. (2.12), to arbitrary precision at any point within the fundamental domain (2.3).¹⁴ Making use of this fact, we find global minima of the potential numerically for $0 \leq m, n \leq 3$ ($\mathcal{P}(j) = 1$), see table 1 and figure 2. As a cross-check, we note that for the special cases of $(m, n) = (0, 0), (1, 1), (0, 3)$ considered in ref. [69] our results are consistent with the values reported therein. This numerical analysis suggests that the minima fall into several classes depending on values of m and n :

(0, 0) is a single minimum at $\tau \simeq 1.2i$ on the imaginary axis, corresponding to the case $m = n = 0$;

(0, n) is a single minimum at the symmetric point $\tau = i$ attained when $m = 0, n \neq 0$;

¹⁴In practice, we numerically implement a q -expansion for the potential V . We have checked that the same, stable q -expansions are obtained independently of expressing H as a function of j or as a function of η .

$(m, 0)$ and $(m, 0)^*$ are a pair of degenerate minima for each $m \neq 0$ and $n = 0$: $(m, 0)$ is located in the vicinity of the left cusp $\tau = \omega$, approaching this symmetric point as m increases, while $(m, 0)^*$ is its CP-conjugate;

(m, n) is a series of minima on the unit arc, corresponding to $m \neq 0, n \neq 0$; these minima shift towards $\tau = \omega$ ($\tau = i$) along the arc as m (n) grows.

An important observation is that the minima belonging to classes $(0, 0), (0, n), (m, n)$ lie either on the boundary of the fundamental domain \mathcal{D} or the imaginary axis, in line with the conjecture of ref. [69]; these minima are CP-conserving. However, the $(m, 0)^{(*)}$ minima slightly depart from the left (right) cusp symmetric point and the boundary. This property makes such minima an interesting possibility from the phenomenological viewpoint, as they can naturally explain both CP violation [14] and hierarchical mass patterns [20] in an economical way. Therefore, we now turn to a discussion of the $(m, 0)^{(*)}$ minima and the corresponding $V_{m,0}$ potentials.

3.2 CP-violating minima of $V_{m,0}$

Since the $(m, 0)^*$ minima are trivially related to the $(m, 0)$ minima via CP reflection, we concentrate here on the latter series only, i.e., we study the behaviour of $V_{m,0}$ in the vicinity of the left cusp. The minima of interest deviate only slightly from the boundary; to make sure these deviations are not a numerical artefact of our q -expansions, we re-expand the potential in terms of u , as described in section 2.3.

Note that $V_{m,0}$ is a real-valued non-holomorphic function of u , therefore it expands in powers of $|u|$ rather than u itself, with coefficients possibly depending on the phase of u . Denoting this phase as ϕ , i.e., $u = |u|e^{i\phi}$, $\phi \in [-\pi/3, 0]$ (see appendix C), we find:

$$V_{m,0} = \Lambda_V^4 \frac{1728^m}{\sqrt{3} \tilde{\eta}_0^{12}} \left\{ -1 - 2|u|^2 + (A_m^2 - 3)|u|^4 \right\} + \mathcal{O}(|u|^6), \quad (3.1)$$

where

$$A_m \equiv \frac{864 |\tilde{\eta}_3|^3}{\pi^6 \tilde{\eta}_0^{27}} m + \frac{6 |\tilde{\eta}_3|}{\tilde{\eta}_0} \simeq 68.78 m + 4.30 \quad (3.2)$$

and $\tilde{\eta}_i$ are coefficients of the u -expansion of $\tilde{\eta}(u)$ defined in eq. (2.16) (in particular, $\tilde{\eta}_0 = |\eta(\omega)|$).

Apart from the overall scale, the potential $V_{m,0}$ in eq. (3.1) depends on only one parameter — m , which takes positive integer values. One can see from eq. (3.2) that the quartic term coefficient $(A_m^2 - 3)$ is positive for any $m \geq 1$, so up to $\mathcal{O}(|u|^6)$ the potential has the well-known Mexican-hat profile, similar to the Higgs potential in the Standard Model (see figure 4). This clearly indicates that the cusp $\tau = \omega \leftrightarrow |u| = 0$ is not the minimum. Instead, this point is a local maximum, while the true minimum is attained at

$$|u|_{\min} \simeq (A_m^2 - 3)^{-1/2} \simeq A_m^{-1} = \frac{0.0145}{m + 0.0625}. \quad (3.3)$$

Comparing this approximation with the minima obtained numerically from the q -expansions for $m \leq 7$, we find excellent agreement, as shown in figure 3.

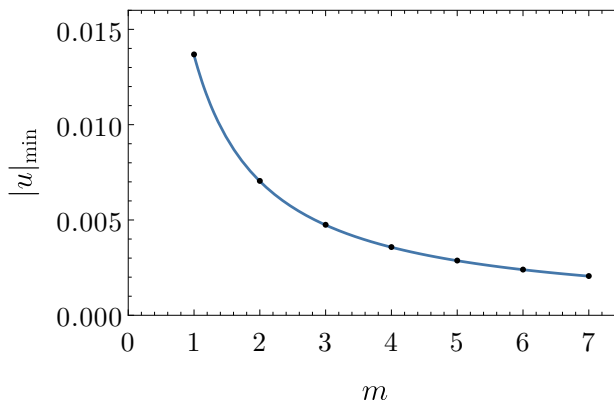


Figure 3. Deviation of the minimum of $V_{m,0}(\tau, \bar{\tau})$ from the left cusp $\tau = \omega$ measured by $|u| = |(\tau - \omega)/(\tau - \omega^2)|$. Values obtained numerically (black dots) match the analytical approximation of eq. (3.3) (blue line).

Eq. (3.3) has an important phenomenological implication. In the vicinity of the left cusp, fermion mass matrix entries in modular-invariant theories are proportional to powers of the small parameter $\epsilon \sim |u|$ [20]. With a suitable choice of fermion field representations under the modular group, this leads to a hierarchical mass pattern of the form $(1, \epsilon, \epsilon^2)$ for three generations of fermions. Hence eq. (3.3) describes possible values of the small parameter responsible for the hierarchy of fermion masses. In particular, we see that $\epsilon \sim 0.01$ for small m which is consistent with the observed mass hierarchy of charged leptons and quarks.

A model in which hierarchical charged lepton masses and the observed lepton mixing pattern of two large and one small angles are generated naturally without fine-tuning in the vicinity of the left cusp was constructed in section 4.2 of ref. [20]. Statistical analysis showed that this S'_4 model is phenomenologically viable at 3σ confidence level for $\epsilon \in [0.0163, 0.0214]$, $\epsilon \simeq 2.8|u|$, independently of the phase of u , with the best fit value of $|u| \simeq 0.00664$. On the other hand, eq. (3.3) yields a series $\epsilon \simeq 0.0383, 0.0197, 0.0133, \dots$ for $m = 1, 2, 3, \dots$. Quite remarkably, choosing $m = 2 \leftrightarrow \epsilon \simeq 0.0197$ one gets a value of ϵ within the phenomenologically allowed range of the model.¹⁵ In the original construction the small values of $|u|$ and correspondingly of ϵ , for which the model is viable, are unexplained. Here we find a natural explanation for these small values, which is general and does not rely on the discussed specific model. In other words, the potential $V_{m,0}$ completes the non-fine-tuned model presented in ref. [20] by providing a model-independent universal dynamical origin of the smallness of the deviation of τ from its symmetric value.¹⁶

So far we have not discussed the phase of u at the minimum, ϕ_{\min} . It may seem from eq. (3.1) that the potential is independent of ϕ , thus having a flat direction. However, expanding $V_{m,0}$ to higher orders in $|u|$ reveals a mild dependence on ϕ : up to an overall

¹⁵A more careful analysis of the chi-squared function shows that this value of ϵ lies in the 1.1σ range.

¹⁶Although this model was considered in the context of global SUSY, it can be trivially modified to fit into the supergravity framework by shifting modular weights of the fields so that the superpotential carries weight -3 rather than 0 .

factor, we have

$$V_{m,0} \propto -1 - 2|u|^2 + (A_m^2 - 3)|u|^4 + (-4 + 2A_m^2 + B_m^2 \cos 6\phi)|u|^6 + 2A_mB_m^2 \cos 3\phi |u|^7 + (-5 + 3A_m^2 + 2B_m^2 \cos 6\phi)|u|^8 + \mathcal{O}(|u|^9), \quad (3.4)$$

where

$$B_m^2 \equiv \frac{864 |\tilde{\eta}_3|^3}{\pi^6 \tilde{\eta}_0^{27}} m \left[\frac{864 |\tilde{\eta}_3|^3}{\pi^6 \tilde{\eta}_0^{27}} (m-2) + \frac{3(31 \tilde{\eta}_3^2 - 10 \tilde{\eta}_0 \tilde{\eta}_6)}{\tilde{\eta}_0 |\tilde{\eta}_3|} \right] + \frac{6(7 \tilde{\eta}_3^2 - 2 \tilde{\eta}_0 \tilde{\eta}_6)}{\tilde{\eta}_0^2} \quad (3.5)$$

$$\simeq 4730.60 m^2 - 2069.73 m + 33.26.$$

Comparing the last expression with $A_m^2 \simeq 4730.60 m^2 + 591.32 m + 18.48$, we notice that

$$B_m \sim A_m \simeq |u|_{\min}^{-1}. \quad (3.6)$$

This means that in the vicinity of the minimum:

- terms of order 6 and higher in $|u|$ are indeed negligible compared to the quadratic and quartic term, which further justifies the validity of approximation (3.1) for the estimation of $|u|_{\min}$;
- the ϕ -dependent parts of $\mathcal{O}(|u|^6)$ and $\mathcal{O}(|u|^7)$ terms are comparable, so they are equally important for the estimation of ϕ_{\min} ;
- the ϕ -dependent part of $\mathcal{O}(|u|^8)$ term is negligible compared to the corresponding parts of the two previous terms.

We expect that the last condition holds also for higher-order terms, so that the ϕ -dependent contribution to the potential is dominated by

$$B_m^2 \cos 6\phi |u|^6 + 2A_mB_m^2 \cos 3\phi |u|^7 \propto \cos 6\phi + 2A_m |u| \cos 3\phi \simeq \cos 6\phi + 2 \cos 3\phi \quad (3.7)$$

at $|u| = |u|_{\min}$. Expression (3.7) is minimised in the region of interest $[-\pi/3, 0]$ at the following unique value of ϕ :

$$\phi_{\min} \simeq -\frac{2\pi}{9}, \quad (3.8)$$

independently of m , in excellent agreement with the minima obtained numerically.

In the case $m = 2$ relevant for the non-fine-tuned model of ref. [20], one gets

$$u_{\min} \simeq \frac{0.0145}{2 + 0.0625} e^{-2\pi i/9} \leftrightarrow \tau_{\min} \simeq -0.492 + 0.875i \quad (3.9)$$

(cf. table 1), which is again consistent with the allowed range of τ reported in ref. [20].

While eq. (3.3) shows that the minimum deviates from the *symmetric point*, which may be responsible for mass hierarchies, eq. (3.8) indicates that the minimum deviates also from the *boundary* of the fundamental domain, providing an origin of CP breaking. Indeed, $\phi=0$ corresponds to the left vertical boundary, while $\phi=-\pi/3$ corresponds to the arc (see appendix C), so that the minimum lies in between. This can be seen in the left panel of figure 4, which shows the potentials $V_{m,0}$, $m = 1, 2, 3$, in the vicinity of the cusp.

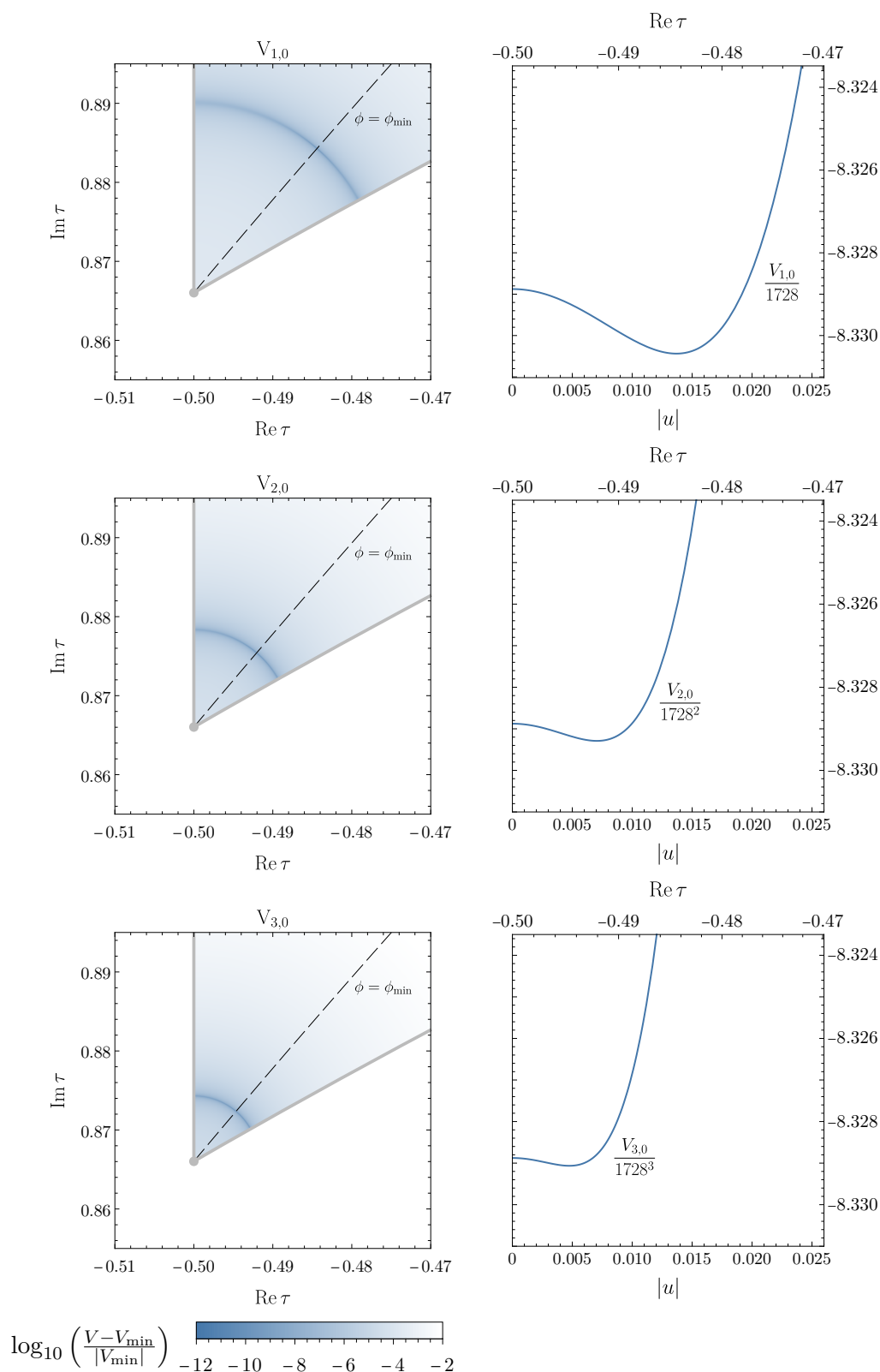


Figure 4. Potentials $V_{m,0}(\tau, \bar{\tau})$, $m = 1, 2, 3$, in the vicinity of the cusp (left panel) and their 1-dimensional projections onto the curve $\phi = \phi_{\min}$ (right panel), in units of Λ_V^4 (see text for details).

To make the potential shapes clearly visible, we use the logarithmic scale $\log_{10} \left(\frac{V - V_{\min}}{|V_{\min}|} \right)$, where V_{\min} is the minimum value of the corresponding potential. As anticipated, the potentials have a deep narrow “trench” around $|u| = |u|_{\min}$, while the dependence on the phase ϕ is almost unnoticeable. To illustrate further the Mexican-hat shape of the potentials, in the right panel we report their 1-dimensional profiles as one varies $|u|$ while keeping $\phi = \phi_{\min}$ fixed (black dashed line in the left panel).

We have seen in this section that the $V_{m,0}$ potentials have special properties, which are important from the phenomenological viewpoint. It may seem however that the choice

$$H(\tau) = (j(\tau) - 1728)^{m/2} , \tag{3.10}$$

which leads to such potentials, is not distinguished within the more general class (2.7) from the outset, and thus could be considered as a form of tuning: indeed, one could expect to have generically a non-trivial polynomial $\mathcal{P}(j)$ as well as $n \neq 0$, which could change the behaviour of V dramatically. We would like to point out that the series (3.10) *does* actually play a special role in the full set of $H(\tau)$ given by eq. (2.7): in fact, eq. (3.10) describes a subset of all possible $H(\tau)$ which vanish only at the symmetric point $\tau = i$ (which is itself distinguished by modular symmetry).¹⁷ In view of this special property, we expect that $V_{m,0}$ potentials should arise naturally in certain top-down completions without any need for special tuning to avoid non-trivial \mathcal{P} and non-zero n .

4 Summary and conclusions

In the present article we have investigated the problem of modulus stabilisation in theories of flavour based on modular symmetry. The modulus τ — a complex scalar field — plays a fundamental role in the modular-invariance approach to the lepton and quark flavour problems. It has specific transformation properties under the action of the modular group $\Gamma \equiv \text{SL}(2, \mathbb{Z})$. The VEV of the modulus τ can be the only source of breaking of both the modular symmetry and the flavour symmetry, described in the approach by a finite inhomogeneous (homogeneous) modular group $\Gamma_N^{(l)}$. Thus, flavons are not needed. For $N \leq 5$, the finite modular groups Γ_N are isomorphic to the permutation groups S_3, A_4, S_4 and A_5 , while $\Gamma_N^{(l)}$ are isomorphic to their double covers. In the “minimal” models without flavons, the VEV of τ can also be the only source of breaking of CP symmetry when it does not lie on the imaginary axis $\text{Re } \tau = 0$ or on the border of the fundamental domain \mathcal{D} of the modular group, where it has CP-conserving values. In the discussed approach to the flavour problem, the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian are expressed in terms of modular forms of a certain level N and a limited number of coupling constants. The modular forms are functions of the modulus τ , have specific

¹⁷To prove this, note that $\mathcal{P}(j(\tau))$ factorises into monomials of the form $(j(\tau) - z)$, $z \in \mathbb{C}$, and without loss of generality $z \neq 1728$ (otherwise the monomial can be absorbed into $(j(\tau) - 1728)^{m/2}$ by redefining m). These monomials vanish for some $\tau \neq i$ in the fundamental domain \mathcal{D} , since $j : \mathcal{D} \rightarrow \mathbb{C}$ is a bijection and $j(i) = 1728$ (see appendix A). Therefore, for $H(\tau)$ to vanish only at $\tau = i$, $\mathcal{P}(j(t))$ has to be trivial. Finally, since $j(\omega) = 0$, n has to be zero, as otherwise $H(\tau)$ vanishes at $\tau = \omega$.

transformation properties under the action of the modular group $SL(2, \mathbb{Z})$ and furnish irreducible representations of the finite modular, i.e., flavour symmetry, group $\Gamma_N^{(\prime)}$. The matter fields are assumed also to transform in representations of $\Gamma_N^{(\prime)}$. After the flavour symmetry is (fully or partially) broken by the VEV of τ , the modular forms and thus the elements of the Yukawa coupling and fermion mass matrices get fixed. Correspondingly, the fermion mass matrices exhibit a certain symmetry-determined flavour structure which depends via the modular forms used on the VEV. Thus, τ 's VEV is critical for the phenomenological viability of a given modular-invariant flavour model.

Although there is no VEV of τ which preserves the full modular symmetry, there exist three values in the modular group fundamental domain, which break the modular symmetry only partially [15]. Only one of these residual symmetry points is relevant for our analysis, namely, $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3) = -1/2 + \sqrt{3}/2 i$ (the “left cusp”), at which the \mathbb{Z}_3^{ST} symmetry is preserved. In models where τ deviates slightly from $\tau_{\text{sym}} = \omega$, charged lepton (and possibly quark) mass hierarchies may arise naturally from the properties of the modular forms as powers of the small deviation $|\tau - \tau_{\text{sym}}|$ without the use of fine-tuned constants [20].

Following a bottom-up approach, a large number of viable “minimal” lepton and quark flavour models based on modular symmetry, which do not include flavons, has been constructed. In the overwhelming majority of these models the VEV of the modulus has been determined by confronting model predictions with experimental data and can vary significantly depending on the model. There have been a few attempts to determine the modulus VEV from a dynamical principle (see section 1). The results of these attempts revealed, in particular, that in the predominant number of cases the specific CP-invariant potentials used for τ lead to CP-conserving VEVs of τ .

In the present study of modulus stabilisation we have considered relatively simple UV-motivated modular- and CP-invariant potentials of the modulus, $V_{m,n}(\tau, \bar{\tau})$ (eq. (2.12)), m, n being non-negative integer numbers, proposed and analysed within the framework of supergravity theories in ref. [69] and further studied in ref. [68]. They can be expressed in terms of the Dedekind eta function, $\eta(\tau)$, and its derivatives. Using the well-known q -expansion of $\eta(\tau)$ (eq. (2.14)), which allows to compute the potential $V_{m,n}(\tau, \bar{\tau})$ to arbitrary precision in any point of the fundamental domain, we have derived the absolute minima of $V_{m,n}$ for $m, n = 0, 1, 2, 3$ (table 1 and figure 2) and any $m > 0$ for $n = 0$ (section 3.2). It was conjectured in ref. [69] that all extrema of $V_{m,n}$ would correspond to CP-conserving values of τ , i.e., would lie either on the boundary of the fundamental domain \mathcal{D} or on the imaginary axis. In [69] the cases $(m, n) = (0, 0), (1, 1), (0, 3)$ were explicitly examined and the global minima of the corresponding potentials were indeed found to lie at $\tau \simeq 1.2 i$ (imaginary axis), $\tau \simeq \pm 0.24 + 0.97 i$ (equivalent minima on the unit arc) and $\tau = i$, respectively. While we have verified these results we showed also that i) the potentials $V_{0,n}$ for $n = 1, 2$ have the same absolute minimum as $V_{0,3}$, ii) the potentials $V_{m,n}$ with $m, n = 1, 2, 3$, have absolute minima at the unit arc, which shift towards $\tau = \omega$ ($\tau = i$) along the arc as m (n) grows. Most importantly, we have further found that potentials with $n = 0$ but given $m > 0$ do allow for a pair of degenerate (CP-conjugate) global minima at

τ_{\min} and $(-\bar{\tau}_{\min})$, which break CP symmetry spontaneously. Moreover, τ_{\min} are found to be located in the vicinity of the left cusp $\tau = \omega$ (figure 4), at values of $|\tau_{\min} - \omega|$ favoured by the mechanism put forward in [20] to explain fermion (charged-lepton and quark) mass hierarchies. As the found CP-breaking minima deviate only slightly from the fundamental domain boundary, to make sure these deviations are not a numerical artefact of the used q -expansions of $\eta(\tau)$, we re-expanded $\eta(\tau)$ and the potential $V_{m,0}$ in terms of the parameter $u = (\tau - \omega)/(\tau - \omega^2)$ which quantifies the deviation of τ from the left cusp (section 2.3). Expressed in terms of u this potential is shown to depend, apart from the overall scale, on just one parameter — m , which takes positive integer values. We found that up to $\mathcal{O}(|u|^6)$ (with $|u|^6$ giving negligible contribution), the potential $V_{m,0}$ has the well-known Mexican-hat profile, similar to the Higgs potential in the Standard Model (eq. (3.1)), with absolute minimum at $|u|_{\min} \simeq 0.0145/(m + 0.0625)$ — in excellent agreement with the minima obtained numerically from the q -expansions for $m \leq 7$ (figure 3). Using further the expansions up to $\mathcal{O}(|u|^8)$ (with $|u|^8$ being negligibly small) we have found also that $\arg(u_{\min}) \simeq -2\pi/9 = -40^\circ$ independently of m .

An S'_4 lepton flavour model constructed in [20], in which the charged lepton mass hierarchies are generated naturally for τ in the vicinity of the left cusp and the observed lepton mixing is reproduced without fine-tuning, was found to be phenomenologically viable at 3σ C.L. for $\epsilon \simeq 2.8|u| \in [0.0163, 0.0214]$. It is quite remarkable that for $m = 2$, the potential $V_{2,0}$ has an absolute minimum at $|u|_{\min} \simeq 0.00705$ corresponding to $\epsilon \simeq 0.0197$ lying in the $\sim 1\sigma$ allowed range of the model. Thus, the potential $V_{2,0}$ completes this non-fine-tuned lepton flavour model by providing a dynamical origin of the smallness of the deviation of τ from its left cusp symmetric value.

We note finally that the results of our study of modulus stabilisation do not depend on the choice of the finite modular group $\Gamma_N^{(\prime)}$ as a group of flavour symmetry, of the modular weights of the matter fields and of the representations of $\Gamma_N^{(\prime)}$ assumed to be furnished by the matter fields, which define a modular-invariant model of flavour. In this sense they are universal. They have a direct impact on the phenomenology of the modular-invariant models of flavour since they lay out preferred regions in the fundamental domain of the modular group for stabilisation of the modulus. Our results may have also implications for the problem of CP violation in supersymmetric extensions of the Standard Model.

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A Modular forms

The *Dedekind eta function* is a modular form of weight $1/2$ defined as

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} \phi(q), \quad (\text{A.1})$$

where $q \equiv e^{2\pi i\tau}$ and $\phi(q)$ is known as the Euler function. The eta function admits the expansions

$$\begin{aligned} \eta &= q^{1/24} \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22}) \right) \\ &= q^{1/24} \left(1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \mathcal{O}(q^7) \right)^{-1}, \end{aligned} \quad (\text{A.2})$$

and satisfies $\eta(T\tau) = \eta(\tau + 1) = e^{i\pi/12} \eta(\tau)$ and $\eta(S\tau) = \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$.

The *Eisenstein series of weight $2k$* is defined for integer $k > 1$ as

$$G_{2k}(\tau) = \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ (n_1, n_2) \neq (0,0)}} (n_1 + n_2\tau)^{-2k}, \quad (\text{A.3})$$

and converges to a holomorphic function in the upper-half plane: a modular form of weight $2k$. While the series does not converge for $k = 1$, one can still define the $G_2(\tau)$ function via a specific prescription on the order of summation (see, e.g., [72], where it is denoted G_2^*). This function is related to G_4 by the identity [69]

$$\frac{5}{2\pi} G_4 = i G_2' + \frac{G_2^2}{2\pi}. \quad (\text{A.4})$$

Using eq. (2.11), one can further show that G_2 is not quite a modular form of weight 2, since under a generic modular transformation $\gamma \in \text{SL}(2, \mathbb{Z})$

$$\frac{\eta'(\tau)}{\eta(\tau)} \xrightarrow{\gamma} (c\tau + d)^2 \frac{\eta'(\tau)}{\eta(\tau)} + \frac{1}{2} c(c\tau + d). \quad (\text{A.5})$$

Noting how $\text{Im } \tau$ transforms under the action of the modular group, it further follows that

$$\frac{1}{4i \text{Im } \tau} \xrightarrow{\gamma} \frac{|c\tau + d|^2}{4i \text{Im } \tau} = \frac{(c\tau + d)^2}{4i \text{Im } \tau} - \frac{1}{2} c(c\tau + d). \quad (\text{A.6})$$

One can then define \hat{G}_2 as given in eq. (2.10), which transforms as a weight 2 form, at the cost of being non-holomorphic.

Finally, the *Klein j function* or *j -invariant* (sometimes called the absolute modular invariant) is a modular form of zero weight. It can be defined in terms of the Dedekind eta and G_4 as

$$j(\tau) \equiv \frac{3^6 5^3 G_4(\tau)^3}{\pi^{12} \eta(\tau)^{24}}. \quad (\text{A.7})$$

Using eqs. (2.11) and (A.4), one can relate the j function to the Dedekind eta and its derivatives, showing that

$$j = \left(\frac{72}{\pi^2} \frac{\eta\eta'' - 3\eta'^2}{\eta^{10}} \right)^3 = \left[\frac{72}{\pi^2\eta^6} \left(\frac{\eta'}{\eta^3} \right)' \right]^3, \quad (\text{A.8})$$

as given in eq. (2.13). This function $j : \mathcal{D} \rightarrow \mathbb{C}$ is a one-to-one map between points in the fundamental domain and the whole complex plane. In particular, $j(\tau)$ takes real values only for CP-conserving values of τ , i.e., when τ is on the border of \mathcal{D} or when $\text{Re } \tau = 0$. The j function is not holomorphic at $\tau = i\infty$, where it diverges. At the remaining symmetric points, one has $j(\omega) = 0$ and $j(i) = 1728 = 12^3$. Thus, it admits as q -expansion the Laurent series

$$j = 744 + \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \mathcal{O}(q^4). \quad (\text{A.9})$$

While here only the first terms in this well-known expansion are reported, in practice many more powers of q were taken into account in our analyses, guaranteeing numerical convergence and stability. It is further known that j has a triple zero at $\tau = \omega$, that $(j - 1728)$ has a double zero at $\tau = i$ and that the derivative j' vanishes only at these values of τ : $j'(\omega) = j'(i) = 0$ (see, e.g., [73]).

B Rigid SUSY limit

In the limit of rigid $\mathcal{N} = 1$ SUSY, one has $M_P \rightarrow \infty$ and $\kappa \rightarrow 0$. Keeping the form of $K(\tau, \bar{\tau})$ given in eq. (2.4), one sees that the scalar potential of eq. (2.8) becomes

$$V = K^{i\bar{j}} \partial_i W \partial_{\bar{j}} W^*. \quad (\text{B.1})$$

Note that, in this limit, $\mathbf{n} = \kappa^2 \Lambda_K^2 \rightarrow 0$. The superpotential of eq. (2.6) reduces to $W(\tau) = \Lambda_W^3 H(\tau)$ and we arrive at the simple result

$$V(\tau, \bar{\tau}) = \frac{4\Lambda_W^6}{\Lambda_K^2} (\text{Im } \tau)^2 |H'(\tau)|^2. \quad (\text{B.2})$$

Taking H of the form given in eq. (2.7), with $\mathcal{P}(j) = 1$, one finds that there is always a value of $\tau \in \mathcal{D}$ for which $H'(\tau) = 0$. Hence, global minima of this potential correspond to the zeros of H' . This function is given by

$$H' = j' (j - 1728)^{m/2} j^{n/3} \left[\frac{m}{2} \frac{1}{j - 1728} + \frac{n}{3} \frac{1}{j} \right]. \quad (\text{B.3})$$

Apart from the trivial case $m = n = 0$, the zeros of H' — and correspondingly the global minima of V — are located at CP-conserving values of τ , namely at $\tau = i$, $\tau = \omega$, or at points for which the factor in square brackets vanishes, which correspond to real $j \in [0, 1728]$ and values of τ on the arc. This result suggests that supergravity effects are important for the presence of CP-violating minima in the discussed class of simple superpotentials. More specifically, the presence of the term $\propto \hat{G}_2$ and the term $\propto 3|H|^2$

in the potential V in eqs. (2.9) and (2.12) seem to be crucial for the spontaneous breaking of the CP symmetry. Let us further note that, in the case of a non-trivial $\mathcal{P}(j)$, this polynomial can be engineered to produce minima at arbitrary points in the fundamental domain.

C u -expansions

Recalling the definition of u given in section 2.3,

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2} \Leftrightarrow \tau = \omega^2 \frac{\omega^2 - u}{1 - u}, \tag{C.1}$$

one finds

$$\operatorname{Re} \tau = -\frac{1}{2} - \frac{\sqrt{3} \operatorname{Im} u}{|1 - u|^2}, \quad \operatorname{Im} \tau = \frac{\sqrt{3}}{2} \frac{1 - |u|^2}{|1 - u|^2}, \tag{C.2}$$

and conversely

$$\operatorname{Re} u = \frac{\operatorname{Re} \tau + |\tau|^2 - 1/2}{|\tau - \omega^2|^2}, \quad \operatorname{Im} u = -\frac{\sqrt{3}}{2} \frac{1 + 2 \operatorname{Re} \tau}{|\tau - \omega^2|^2}. \tag{C.3}$$

Writing $u = |u|e^{i\phi}$, as in section 3.2, one can check that $\operatorname{Re} u > 0$ and

$$\phi = -\arctan \left(\frac{\sqrt{3}}{2} \frac{1 + 2 \operatorname{Re} \tau}{\operatorname{Re} \tau + |\tau|^2 - 1/2} \right) \tag{C.4}$$

within the fundamental domain (excluding $\tau = \omega$, where $u = 0$ and ϕ is indeterminate). By analysing this expression, it follows that the phase of u varies in the interval $[-\pi/3, 0]$. Namely, it reaches its highest value of $\phi = 0$ at the left boundary of the fundamental domain, $\operatorname{Re} \tau = -1/2$. Its lowest value corresponds to the maximum value of the argument of the arctangent, attained at the arc $|\tau|^2 = 1$, for which $\phi = -\arctan \sqrt{3} = -\pi/3$.

Following the procedure described in section 2.3, one can obtain the u -expansions of modular forms, such as that of $\eta = \sqrt{1-u} \tilde{\eta}$, which can be extracted from eq. (2.16). For j , \tilde{G}_2 and \tilde{G}_4 , having defined \tilde{G}_2 and \tilde{G}_4 via

$$G_2(u) \equiv \frac{2\pi}{\sqrt{3}} \left((1-u) + (1-u)^2 \tilde{G}_2 \right), \tag{C.5}$$

$$G_4(u) \equiv (1-u)^4 \tilde{G}_4(u),$$

we find

$$j(u) \simeq -237698 u^3 - 1.17505 \times 10^7 u^6 - 2.78879 \times 10^8 u^9 + \mathcal{O}(u^{12}),$$

$$\tilde{G}_2(u) \simeq 4.29865 u^2 + 14.7827 u^5 + 18.155977 u^8 + \mathcal{O}(u^{11}), \tag{C.6}$$

$$\tilde{G}_4(u) \simeq 22.6272 u + 243.166 u^4 + 716.769 u^7 + \mathcal{O}(u^{10}).$$

Note that, while both j and $\tilde{\eta}$ are invariant under $\gamma = ST$, one can check that $\tilde{G}_2 \xrightarrow{ST} \omega \tilde{G}_2$ and $\tilde{G}_4 \xrightarrow{ST} \omega^2 \tilde{G}_4$. Recalling that $u \xrightarrow{ST} \omega^2 u$ and $j(\omega) = 0$, the peculiar structure in powers of u of eq. (C.6) follows.

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