

Physics Area - PhD course in
Theoretical Particle Physics

Q-ball Superradiance & Gravitational Waves from First-Order Phase Transitions

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Abstract

The Standard Model of particle physics and the standard cosmological model describe an impressive range of phenomena, from collider measurements to the thermal history of the universe. At the same time, they leave several questions unanswered, including the microscopic origin of dark matter, the origin of the matter-antimatter asymmetry, the small but nonzero neutrino masses, and the possible structure of scalar sectors beyond the observed Higgs field. These open questions motivate the search for new physics both in laboratory experiments and in theoretical modeling. The early universe is a useful arena for this search because it was hot, dense, and intrinsically time dependent. Fields that are difficult to excite today may have controlled the vacuum structure of the high-temperature universe, and their evolution could have produced relic objects or observable backgrounds accessible to experiments. In many cases, the relevant dynamics is not a small perturbation around equilibrium, but instead involves the formation and evolution of localized field configurations, phase transitions, and several other nonlinear processes.

Lattice simulations provide one of the main tools for studying such questions. In a broader physics context, lattice simulations are used in a wide range of systems to study the evolution of one or many interacting degrees of freedom. In early-universe applications, lattice simulations are used to study reheating and preheating after inflation, the formation of defects such as domain walls and cosmic strings, soliton formation and scattering, phase transitions, and the production of gravitational waves from nonlinear stress-energy sources. Similar ideas are also used in condensed-matter and statistical physics, for example to study statistical properties of complex many-body problems. The common feature in all these examples is that the lattice provides a controlled finite-dimensional representation of a field theory whose nonlinear real-time evolution would otherwise be inaccessible analytically.

In this thesis, lattice simulations of classical systems are used extensively alongside analytical and semi-analytical methods. The simulations test the range of validity of simpler (semi-)analytical descriptions, while the semi-analytical approaches make the numerical results physically interpretable and, where applicable, significantly reduce the computational cost relative to brute-force simulations. We study two problems of this type, for which we develop (semi-)analytical approaches and use lattice simulations to benchmark their validity. These problems concern different physical systems, but they share a common methodological theme: both require understanding how nonlinear scalar-field configurations exchange energy with their environment and how this microscopic dynamics affects macroscopic observables:

- The first part is devoted to studying the interaction between Q-balls and scalar perturbations. A Q-ball is a non-topological soliton, namely a localized configuration of scalar fields stabilized by a conserved global charge. Such objects can appear in many extensions of the Standard Model, and they have been discussed in connection with baryogenesis, dark matter, and phase-transition dynamics. For the purposes of this thesis, the key point is that a Q-ball is not merely a static object with a fixed charge and energy: after formation, Q-balls are generically embedded in a heat bath of particles or classical waves that can be described as perturbations around a Q-ball background. Because of its time-dependent nature, the Q-ball can exchange charge and energy

with the surrounding perturbations in a process referred to here as Q-ball superradiance.

We approach this mechanism in the linear regime, where information about the scattering of incoming modes on the Q-ball is encoded in a scattering matrix. The goal is to connect the linear scattering description with amplification or attenuation in the physical energy and charge balance of the Q-ball. Classical lattice simulations are then used as an independent real-time check of the linear analysis and help identify its domain of validity. We then extend the discussion of the single complex scalar field model to the Friedberg-Lee-Sirlin model, where the presence of an additional real scalar field opens a richer set of scattering channels.

- The second part of the thesis focuses on the computational aspects of the gravitational-wave background sourced by first-order phase transitions in the early universe. In a first-order transition, the universe does not move smoothly from one phase to another. Instead, bubbles of the new phase nucleate, expand, collide, and eventually fill space. If the transition is sufficiently strong, the stress-energy tensor produced by bubble collisions and by the energy injected into the plasma can source a stochastic background of gravitational waves. Such a signal would be a direct probe of high-energy dynamics that may be inaccessible to electromagnetic observations or collider experiments. A first-order phase transition is also a nonlinear field-theory problem, and arguably the most direct approach is to simulate the bubble dynamics on a $3 + 1$ D lattice and numerically compute the resulting tensor radiation. Although this approach is powerful and captures the full physics at all relevant scales, it is computationally expensive, especially when the bubble walls are thin, the simulation volume is large, or one needs to scan over model parameters. For this reason, analytical and semi-analytical approximations such as the envelope approximation and the bulk-flow approximation have historically played an important role in the literature.

The main goal of the second part of this thesis is to develop and test a so-called *generalized bulk-flow approximation* for the scalar-field collision contribution to the gravitational-wave spectrum. This approximation will be shown to improve significantly upon previous approximations by taking into account the relevant dynamics of the bubble-wall surface tension before and after collision. It utilizes the separation between macroscopic scales and microscopic physics to treat the model-independent geometry of many bubbles separately from the model-dependent coefficients entering the gravitational-wave spectrum. These coefficients, which effectively describe the post-collision surface tension dynamics through a calibrated decay law, are extracted from more economical two-bubble field simulations on a $1 + 1$ D lattice. As the geometric contribution is reusable and the $1 + 1$ D simulations are generally inexpensive to perform, this approximation greatly reduces the computational cost compared with $3 + 1$ D field simulations and is especially useful for parameter scans of the underlying scalar-field theory.

The two parts of the thesis are therefore connected by a common physical and computational perspective. Both study nonlinear scalar-field configurations that may arise in the contexts of high-energy physics and cosmology and ask how the energy of these configurations changes over time and how these changes affect potentially observable quantities. In both parts, classical field simulations are also used extensively to validate the analytical descriptions. The first part applies this logic to Q-ball perturbations and superradiance, while the second applies it to bubble collisions and gravitational-wave production.

This thesis includes the results of the scientific research I conducted as a Ph.D. student in the Theoretical Particle Physics group at SISSA. The two parts can be read independently and are structured to be self-contained, each with its own introduction, review, technical framework, analytical and numerical results, and relevant discussion. The main results of the first part are based on the publication [1], while those of the second part remain unpublished.

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Part I

Fundamentals of Q-balls and the superradiance phenomena

Outline of Q-ball part in this thesis

Within the scope of this thesis, we focus on perturbations around a Q-ball background and on the physical results obtained by scattering these perturbations off the Q-ball. Q-ball perturbations have been used to analyze normal modes, relaxation, radiation pressure, and energy exchange with incoming waves [2, 3, 4, 1]. This perturbative question is physically motivated because a Q-ball in the early universe is not isolated; rather, it can be immersed in scalar radiation and in other remnants of the same fragmentation process. In concrete cosmological models, additional couplings may also place it in contact with a thermal plasma. As we shall see, even in vacuum, a perturbed Q-ball can exchange energy and charge with radiation through its time-periodic background. The central question is therefore not only whether a Q-ball solution exists, but also how a formed Q-ball responds to the scattering of incoming perturbations.

The outline of the thesis for the Q-ball part is as follows:

- In Chapter. 1, we start with some common examples of topological solitons in the context of field theory, before specializing to Q-balls as non-topological solitons stabilized by a conserved global charge. The chapter reviews the role of Derrick's theorem, the distinction between topological and non-topological stability, the stationary rotating-field ansatz for Q-balls, and the analogy between the Q-ball radial profile equation and a classical particle rolling down the inverted potential with time-dependent friction, which eventually imposes bounds on the internal frequency of Q-balls.
- In Chapter. 2, we first construct the stationary Q-ball solution for a simple model with one complex scalar field, a generic potential, and generic $D+1$ spacetime dimensions, and then revisit the existence criteria, limiting regimes, and virial relation between the various components of the total Q-ball energy in Sections. 2.1 and 2.2. With this foundation in place, Section. 2.3 studies linear perturbations around a Q-ball background, identifies the asymptotic modes that form the basis for the incoming and outgoing scattering states, and expresses particle number conservation before and after scattering processes. Because the background is time dependent, it opens a channel through which the Q-ball can exchange charge and energy with its surroundings by scattering with perturbations. As we shall see, these scattering processes couple particle-like and antiparticle-like frequency channels, which allow perturbations either to deposit charge and energy into the Q-ball or to extract them from it. This effect can be measured quantitatively in terms of amplification factors. These amplification quantities show at linear order how Q-balls evolve through interactions with a medium filled with scalar particles of the same type. We also show that, by introducing a scattering matrix relating incoming and outgoing waves, we can express the relevant observables (total particle number, rate of conversion between modes, charge/energy amplification factors) in terms of the scattering matrix elements. This S-matrix proves to be a useful tool for identifying the underlying symmetry patterns observed in the spectra of these observables.
- In Chapter. 3, this knowledge of Q-ball scattering with particles is specialized to a sextic potential model $U = |\Phi|^2 - |\Phi|^4 + g|\Phi|^6$. The linear predictions can then be verified using lattice simulations to determine when an incoming perturbation is absorbed, reflected, transmitted, or amplified by extracting energy and charge from the Q-ball. With this knowledge of charge/energy amplification factors, we can make phenomenological comments about early-universe Q-balls interacting with ambient radiation or plasma. Finally, we also show that the linear perturbation approach can be generalized beyond the S-wave, so that we can discuss the concept of a cross section for scattering processes between Q-balls and plane waves.
- In Chapter. 4, we extend the same analysis to a multi-field system in the Friedberg-Lee-Sirlin model, where we have a complex scalar field Φ and an extra real scalar field χ whose dynamics are described by the potential

$$U(|\Phi|, \chi) = g_\chi \chi^2 |\Phi|^2 + g_\chi (\chi^2 - v_\chi^2)^2 + m_\Phi^2 |\Phi|^2 + g_\Phi |\Phi|^4. \quad (0.0.1)$$

Here, the nonlinear interaction responsible for localization is generated through renormalizable interactions between different fields, rather than only through higher-order self-interactions of the complex field.

Introduction to Q-balls

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1.1 Basics of soliton

Before discussing what a *Q-ball* is, it is useful to recall the broader idea of a *soliton*. Historically, the subject can be traced back to solitary water waves, most famously the long-lived wave observed by John Scott Russell in a canal in the nineteenth century. In optical systems, optical solitons arise when dispersion effects (which cause different frequencies to travel at different speeds and consequently change the shape of the pulse over time) balance the nonlinear Kerr effect (where the refractive index of a material at a given frequency depends on the light's amplitude or strength). Similarly, in fluid systems, the standard soliton mechanism is the preservation of a localized profile through a balance between dispersion and nonlinear interactions. The same language appears across many areas of physics: shallow-water and tidal-bore waves in fluid mechanics, optical pulses in nonlinear fiber optics, atmospheric solitary waves such as Morning Glory clouds, vortices in superconductors and superfluids, magnetic textures in condensed matter, and localized configurations in relativistic field theories. Figure. 1.1 gives a few visual examples of such soliton-like phenomena in nature.



(a) Qiantang River tidal bore.



(b) Morning Glory cloud.

Figure 1.1: Examples of soliton-like phenomena and applications. Left: the Qiantang River tidal bore, a large tidal-bore wave in Zhejiang, China; image from Zhejiang Provincial People's Government, http://www.ezhejiang.gov.cn/2024-09/19/c_1023588.htm. Right: a Morning Glory cloud, an atmospheric solitary-wave phenomenon; image from Wikimedia Commons, <https://upload.wikimedia.org/wikipedia/commons/f/f0/MorningGloryCloudBurketownFromPlane.jpg>.

The study of solitons later extended to classical field theory, where the term is used more broadly for stable localized finite-energy classical field configurations that behave as particle-like objects. Unlike elementary point particles, a field-theory soliton is an extended configuration with a spatial profile and, in many cases, collective coordinates or internal excitation modes. At the same time, some of its internal properties such as conserved charges can be tracked through interactions, making solitons useful non-perturbative objects for studying stability, scattering, and quantum effects in field theory. *Derrick's theorem* explains why such objects are nontrivial to obtain from ordinary scalar field theories. Consider a time-independent real scalar configuration in D spatial dimensions with total energy $E = E_K + E_S$, where $E_K = \int d^D x \dot{\phi}^2/2$ is the kinetic energy, which is identically zero for this static field configuration. We then have

$$E[\phi] = E_S[\phi] = E_{\text{grad}} + E_{\text{pot}} = \int d^D x \left[\frac{1}{2} \partial_i \phi \partial_i \phi + U(\phi) \right], \quad (1.1.1)$$

where E_S denotes the static energy of the soliton configuration, and the vacuum energy is chosen such that $U(\phi_{\text{vac}}) = 0$ and $E_{\text{grad}}, E_{\text{pot}} \geq 0$ for finite-energy configurations. Under the spatial rescaling $\phi_\xi(\mathbf{x}) = \phi(\xi \mathbf{x})$, with $\mathbf{y} \equiv \xi \mathbf{x}$, we have $d^D x = \xi^{-D} d^D y$ and $\partial_i \phi_\xi(\mathbf{x}) = \xi \partial_i \phi(\mathbf{y})$. The gradient and potential energies therefore scale as

$$E_{\text{grad}}[\phi_\xi] = \frac{1}{2} \int d^D x \partial_i \phi_\xi \partial_i \phi_\xi = \xi^{2-D} E_{\text{grad}}[\phi], \quad (1.1.2)$$

$$E_{\text{pot}}[\phi_\xi] = \int d^D x U(\phi_\xi) = \xi^{-D} E_{\text{pot}}[\phi], \quad (1.1.3)$$

and the total energy becomes

$$E_S[\phi_\xi] = \xi^{2-D} E_{\text{grad}} + \xi^{-D} E_{\text{pot}}. \quad (1.1.4)$$

A stable static soliton would have to be an extremum of this expression at $\xi = 1$, which gives

$$\left. \frac{dE_S[\phi_\xi]}{d\xi} \right|_{\xi=1} = (2-D)E_{\text{grad}} - DE_{\text{pot}} = 0. \quad (1.1.5)$$

For $D \geq 2$ this condition cannot be satisfied by a nontrivial configuration when both energy contributions are non-negative. Note that the theorem does not imply that solitons cannot be constructed in higher dimensions; rather, it states that static scalar lumps require some way around the simple scaling obstruction, such as reducing the effective dimension of the localized profile, considering extended defects with finite tension instead of finite total energy, adding gauge fields or higher-derivative

terms, or allowing time dependence. Soliton constructions whose existence is ensured by a conserved topological charge (which is conserved off shell) are classified as topological solitons, in contrast with non-topological solitons whose stability is usually due to a conserved global Noether charge. Let us now go through some examples of solitons within the context of relativistic field theory to clarify this.

1.1.1 Topological solitons

Topological solitons are stable because finite-energy boundary conditions force the fields at spatial infinity to define a nontrivial map into the vacuum manifold. The corresponding homotopy class cannot be changed by smooth time evolution with finite energy. In the examples below, the integer topological charge is denoted uniformly by Q_T . Several standard examples are:

- *Kinks in $D = 1$* : A simple model is a real scalar field with a double-well potential,

$$\mathcal{L}_{\text{kink}} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U_{\text{kink}}(\phi), \quad U_{\text{kink}}(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2. \quad (1.1.6)$$

Finite energy requires $\phi(x \rightarrow \pm\infty)$ to approach one of the vacua $\phi = \pm v$. The corresponding static kink and antikink solutions are

$$\phi_{\text{kink}}(x) = v \tanh\left[\sqrt{\frac{\lambda}{2}}v(x - x_0)\right], \quad \phi_{\text{antikink}}(x) = -\phi_{\text{kink}}(x), \quad (1.1.7)$$

where x_0 is the arbitrary center of the kink profile. Its mass is identical to its static energy, that is

$$M_{\text{kink}} = E_S[\phi_{\text{kink}}] = \frac{2\sqrt{2\lambda}}{3}v^3. \quad (1.1.8)$$

For this double-well model the topological charge Q_T labels the difference between the endpoint vacua. This charge can be written as the integral of the identically conserved topological current

$$J_T^\mu = \frac{1}{2v}\epsilon^{\mu\nu}\partial_\nu\phi, \quad \partial_\mu J_T^\mu = 0. \quad (1.1.9)$$

The corresponding integer topological charge is

$$Q_T = \int_{-\infty}^{+\infty} dx J_T^0 = \frac{\phi(+\infty) - \phi(-\infty)}{2v}, \quad (1.1.10)$$

giving $Q_T = +1$ for a kink and $Q_T = -1$ for an antikink. This example does not conflict with Derrick's theorem because it lives in $D = 1$, where the scaling condition is not automatically obstructed. One seemingly relevant concept is that of the *domain wall*, which can be imagined as the extension of the kink along the transverse directions in higher dimensions, producing a wall that interpolates between the two vacua. The previous M_{kink} quantity in this case is interpreted as the wall tension per unit transverse area. Such a wall is a topological defect and widely studied in many other contexts, but note that in infinite space it has finite energy per unit area rather than finite total energy, so it is not a localized particle-like soliton by definition.

- *Vortices in the Abelian Higgs model (also known as the Nielsen-Olesen vortex) in $D = 2$* : In $2 + 1$ dimensions, the Abelian Higgs model has the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{vortex}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D_\mu\Phi|^2 - U_{\text{vortex}}(|\Phi|), \\ U_{\text{vortex}}(|\Phi|) &= \frac{\lambda}{2}(|\Phi|^2 - v^2)^2, \quad D_\mu = \partial_\mu - ieA_\mu. \end{aligned} \quad (1.1.11)$$

In the gauge $A_0 = 0$, finite static energy requires the fields to approach the vacuum conditions

$$B \rightarrow 0, \quad D_i\Phi \rightarrow 0, \quad |\Phi| \rightarrow v, \quad r \rightarrow \infty. \quad (1.1.12)$$

Here $B = F_{12}$ is the magnetic field perpendicular to the plane. These conditions still allow the phase of Φ to wind around the vacuum circle at spatial infinity, because the gauge field can compensate the angular variation. Following the Nielsen-Olesen construction, a cylindrically symmetric representative of a sector with winding number n is ⁽¹⁾

$$\Phi(r, \theta) = v f(r) e^{in\theta}, \quad A = A_\theta(r) d\theta, \quad A_\theta(r) = \frac{n}{e} a(r), \quad (1.1.13)$$

with boundary conditions

$$f(0) = a(0) = 0, \quad f(\infty) = a(\infty) = 1. \quad (1.1.14)$$

For winding n , the corresponding generalization is precisely $A_\theta \rightarrow n/e$ and $\Phi \rightarrow v e^{in\theta}$. The two profile functions $f(r)$ and $a(r)$ are fixed by the coupled radial field equations. Their boundary conditions have a direct physical meaning: $f(0) = 0$ and $a(0) = 0$ make the scalar and gauge fields regular at the origin, while $f(\infty) = a(\infty) = 1$ makes the fields approach a vacuum configuration with nonzero winding at spatial infinity. These nonlinear profile equations are not known in closed analytic form in general, and their solutions are normally obtained numerically. The integer winding number can be written as the topological charge associated with the identically conserved current

$$J_T^\lambda = \frac{1}{2\pi i} \varepsilon^{\lambda\mu\nu} \partial_\mu \hat{\Phi}^* \partial_\nu \hat{\Phi}, \quad \hat{\Phi} \equiv \frac{\Phi}{|\Phi|}. \quad (1.1.15)$$

Its charge is

$$Q_T = \int d^2x J_T^0 = \frac{1}{2\pi} \oint d\theta \partial_\theta \arg \Phi = \frac{e}{2\pi} \int d^2x B = n. \quad (1.1.16)$$

This relation expresses the quantization of the magnetic flux for type-II superconductors: if an external magnetic field exceeds a critical value, it penetrates the superconductor in the form of thin tubes, each carrying a quantized unit of magnetic flux. The quantized magnetic flux prevents the vortex from unwinding smoothly. The vortex evades the simple scalar Derrick argument because the Abelian gauge field contributes additional scaling terms to the energy, while the finite-energy boundary condition fixes a nontrivial winding at spatial infinity and prevents it from decaying to a trivial field configuration. Indeed, for a static vortex in the transverse plane one can schematically decompose the energy per unit length as

$$E_S^{\text{vortex}} = E_B + E_{D\Phi} + E_{\text{pot}}, \quad (1.1.17)$$

where $E_B \sim \int d^2x B^2$ is the magnetic energy, $E_{D\Phi} \sim \int d^2x |D_i \Phi|^2$ is the covariant-gradient energy, and E_{pot} is the scalar potential energy. Under the Q_T -preserving rescaling $\Phi_\xi(\mathbf{x}) = \Phi(\xi \mathbf{x})$ and $A_{i,\xi}(\mathbf{x}) = \xi A_i(\xi \mathbf{x})$, these terms scale as

$$E_S^{\text{vortex}}(\xi) = \xi^2 E_B + E_{D\Phi} + \xi^{-2} E_{\text{pot}}. \quad (1.1.18)$$

A stationary vortex can therefore satisfy the virial relation $E_B = E_{\text{pot}}$, unlike the purely scalar Derrick scaling in $D = 2$. At the critical, or BPS, coupling $\lambda = e^2$, the vortex tension is fixed by the magnetic flux:

$$T_{Q_T}^{\text{vortex}} = E_S^{\text{vortex}} = 2\pi v^2 |Q_T|. \quad (1.1.19)$$

- *'t Hooft-Polyakov monopoles in $D = 3$* : We consider the Georgi-Glashow model, consisting of an $\text{SU}(2)$ gauge field $A_\mu = A_\mu^a T^a$ coupled to a Higgs field $\Phi = \Phi^a T^a$ in the adjoint representation,

$$\mathcal{L}_{\text{mon}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - U_{\text{mon}}(\phi^a), \quad (1.1.20)$$

⁽¹⁾ For the unit vortex $n = 1$, we can write the vortex solution explicitly as $A_0 = 0$, $A_i = -\varepsilon_{ij} \hat{x}_j \mathcal{A}(r)$ and $\Phi = v f(r) e^{i\theta}$.

$$U_{\text{mon}}(\phi^a) = \frac{\lambda}{4} \left(\phi^a \phi^a - v^2 \right)^2,$$

where finite energy requires $\hat{\phi}^a \equiv \phi^a/|\phi|$ at spatial infinity to define a map $S_\infty^2 \rightarrow S_{\text{vac}}^2$. The smooth spherically symmetric monopole is described by the hedgehog ansatz

$$\phi^a(\mathbf{x}) = v f(r) \hat{x}^a, \quad A_i^a(\mathbf{x}) = \frac{a(r)}{gr} \varepsilon_{aij} \hat{x}^j, \quad (1.1.21)$$

where $\hat{x}^a = x^a/r$ and g is the non-Abelian gauge coupling. Regularity and finite energy require

$$f(0) = a(0) = 0, \quad f(\infty) = a(\infty) = 1. \quad (1.1.22)$$

The identically conserved topological current reads

$$J_T^\mu = \frac{1}{8\pi} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c, \quad (1.1.23)$$

corresponding to the topological magnetic charge that can be used to classify different monopoles,

$$Q_T = \int d^3x J_T^0 = \frac{1}{8\pi} \int_{S_\infty^2} dS_i \varepsilon_{ijk} \varepsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c \in \mathbb{Z}. \quad (1.1.24)$$

The monopole avoids Derrick's scalar no-go argument, since its energy is not made only from scalar gradient and potential terms: the non-Abelian gauge field also contributes and changes the scaling behavior of the static energy. Schematically, in $D = 3$ the static energy contains

$$E_S^{\text{monopole}} = E_B + E_{D\phi} + E_{\text{pot}}, \quad (1.1.25)$$

where $E_B \sim \int d^3x F_{ij}^a F_{ij}^a$, $E_{D\phi} \sim \int d^3x D_i \phi^a D_i \phi^a$, and E_{pot} is the Higgs potential contribution. Under the gauge-covariant rescaling $A_{i,\xi}(\mathbf{x}) = \xi A_i(\xi \mathbf{x})$ and $\phi_\xi(\mathbf{x}) = \phi(\xi \mathbf{x})$, the energy scales as

$$E_S^{\text{monopole}}(\xi) = \xi E_B + \xi^{-1} E_{D\phi} + \xi^{-3} E_{\text{pot}}. \quad (1.1.26)$$

The extremum condition therefore contains competing positive and negative powers of ξ and is not the scalar Derrick condition. Furthermore, the finite-energy boundary condition fixes a nontrivial map $\hat{\phi} : S_\infty^2 \rightarrow S_{\text{vac}}^2$, so a configuration with nonzero Q_T cannot be continuously deformed into the vacuum while keeping finite energy, explaining the stability of the monopole solutions. The generic monopole mass is determined by the numerical profile functions, but in the BPS limit $\lambda \rightarrow 0$ it is exactly

$$M_{\text{monopole}} = E_S^{\text{monopole}} = \frac{4\pi v}{g} |Q_T|. \quad (1.1.27)$$

- *Skyrmions in $D = 3$* : One way to get around Derrick's argument is to add higher-order terms to the action, which are often encountered in effective low-energy theories. One such example is the Skyrme model, whose Lagrangian is written in terms of an $SU(2)$ -valued field $\mathcal{U}(\mathbf{x}, t)$,

$$\mathcal{L}_{\text{Sk}} = \frac{f_\pi^2}{16} \text{Tr}(L_\mu L^\mu) + \frac{1}{32g^2} \text{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]), \quad (1.1.28)$$

where $L_\mu = \mathcal{U}^\dagger \partial_\mu \mathcal{U}$. Finite energy compactifies physical space to S^3 , so $\mathcal{U}$ defines a map $S^3 \rightarrow SU(2) \simeq S^3$. The unit-baryon Skyrmion is commonly represented by the hedgehog form

$$\mathcal{U}(\mathbf{x}) = e^{iF(r)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}} = \cos F(r) + i\hat{\mathbf{x}} \cdot \boldsymbol{\sigma} \sin F(r), \quad (1.1.29)$$

with $F(0) = \pi$ and $F(\infty) = 0$, where $\boldsymbol{\sigma}$ are the Pauli matrices. The profile $F(r)$ is obtained by solving the Skyrme equation and the boundary conditions ensure that the map wraps the target

$SU(2)$ once. The topological charge, interpreted as *baryon number* in the original model, follows from the identically conserved current

$$J_T^\mu = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{Tr} (L_\nu L_\rho L_\sigma). \quad (1.1.30)$$

The corresponding charge is

$$Q_T = \int d^3x J_T^0 = -\frac{1}{24\pi^2} \int d^3x \varepsilon^{ijk} \text{Tr} (L_i L_j L_k) \in \mathbb{Z}. \quad (1.1.31)$$

Distinct integer values of Q_T label disconnected sectors of finite-energy field configurations. The Skyrmion evades Derrick's theorem because the Lagrangian contains two spatial-derivative orders with different scaling behavior. For a static configuration in $D = 3$, the energy may be written schematically as

$$E_S^{\text{skyrmion}} = E_2 + E_4, \quad E_2 \sim f_\pi^2 \int d^3x (\partial_i \mathcal{U})^2, \quad E_4 \sim \frac{1}{g^2} \int d^3x (\partial_i \mathcal{U})^4. \quad (1.1.32)$$

Here E_2 denotes the contribution from the first term in $\mathcal{L}_{\text{Sk}}$, while E_4 denotes the contribution from the commutator-squared Skyrme term. Under the spatial rescaling $\mathcal{U}_\xi(\mathbf{x}) = \mathcal{U}(\xi\mathbf{x})$, one derivative gives a factor of ξ and the volume element gives ξ^{-3} . Therefore the two-derivative and four-derivative pieces scale as

$$E_S^{\text{skyrmion}}(\xi) = \xi^{-1} E_2 + \xi E_4. \quad (1.1.33)$$

The stationary condition gives $E_2 = E_4$, showing explicitly how the two derivative orders balance. Equivalently, for a configuration with characteristic size R , the two terms scale parametrically as $E_2 \sim f_\pi^2 R$ and $E_4 \sim 1/(g^2 R)$. Balancing them gives $R_{\text{skyrmion}} \sim (f_\pi g)^{-1}$. The exact profile and mass require a numerical solution, but this same estimate gives the characteristic mass scale

$$M_{\text{skyrmion}} = E_S^{\text{skyrmion}} \sim \frac{f_\pi}{g}. \quad (1.1.34)$$

1.1.2 Non-topological solitons

These localized configurations are not protected by a topological winding number. Their stability is instead tied to an ordinary conserved Noether charge and to the energetic fact that, at fixed charge, the coherent lump can be lighter than a collection of free quanta carrying the same charge. The Friedberg-Lee-Sirlin soliton and Coleman's Q-ball are standard examples of this class [5, 6, 7]. Q-balls evade Derrick's theorem by carrying a time-dependent phase, so the field is stationary in energy density but rotates in the internal $U(1)$ field space. This will become clearer during the construction of the Q-ball, where we show this explicitly. The basic physical idea of a Q-ball is therefore that a macroscopic amount of $U(1)$ charge can sometimes be stored more economically in a coherent classical lump than in a collection of free particles. The Q-ball sector therefore contains solutions to the equations of motion that minimize the total energy at fixed total charge Q_Q . Below, in Section. 1.2, we go through the basic construction of the Q-ball in $D = 3$ spatial dimensions, following [6]. Through this discussion, we will also show more explicitly how Derrick's theorem is evaded for the class of Q-ball solutions.

1.2 Q-balls: General features

1.2.1 Thin-wall Q-ball construction of Coleman

To be more specific, let us start with a simple theory of a single complex scalar field $\Phi(\mathbf{x}, t)$ in $3 + 1$ spacetime dimensions governed by the action

$$S[\Phi] = - \int d^4x \mathcal{L}(\Phi) = - \int d^4x \left[\partial_\mu \Phi^\dagger \partial^\mu \Phi + \mathbf{U}(|\Phi|) \right], \quad (1.2.1)$$

where $\mathbf{U}$ is a generic potential that contains no derivatives of the fields and whose leading-order term is quadratic in the fields. Throughout this part, we use the mostly plus metric convention: $\eta_{\mu\nu} = \text{diag}(-, +, +, \dots)$. Q-ball profiles $\Phi = \Phi_Q$, being non-topological solitons by definition, are scalar-field configurations that solve the Euler-Lagrange equation

$$-\partial_\mu \partial^\mu \Phi_Q + \frac{\partial \mathbf{U}}{\partial \Phi_Q^*} = 0. \quad (1.2.2)$$

The theory is invariant under spacetime translations and under the internal phase rotation $\Phi \rightarrow e^{i\alpha} \Phi$. These symmetries correspond respectively to the following conserved Noether currents:

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \partial_\nu \Phi^* - \delta^\mu_\nu \mathcal{L}, \quad (1.2.3)$$

$$J^\mu_Q = -i(\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*), \quad (1.2.4)$$

They imply the conservation of total energy and $U(1)$ global charge,

$$E = \int d^3x \left[|\dot{\Phi}|^2 + |\nabla \Phi|^2 + U(|\Phi|) \right], \quad (1.2.5)$$

$$Q = i \int d^3x (\Phi^* \dot{\Phi} - \Phi \dot{\Phi}^*). \quad (1.2.6)$$

Note that for convenience, we define $U(\phi) \equiv \mathbf{U}(\Phi = \phi/\sqrt{2})$ where $\phi \in \mathbb{R}$. Coleman made several observations during the construction of the Q-ball solution:

- These Q-balls are time-dependent solutions of the form

$$\Phi_Q(\mathbf{x}, t) = \frac{\phi_Q(r)}{\sqrt{2}} e^{-i\omega_Q t}, \quad (1.2.7)$$

where ω_Q is a constant, $r \equiv \|\mathbf{x}\|$, and $\phi_Q(r)$ is a monotonically decreasing function that approaches 0 as $r \rightarrow \infty$. The fact that ϕ_Q depends only on the distance follows from the argument that the solution with minimal energy likely maximizes the symmetry of the system, which in this simple model is spherical symmetry. Meanwhile, the explicit time dependence is necessary to avoid Derrick's theorem, which states that stationary localized solutions to a nonlinear wave equation or nonlinear Klein-Gordon equation in three or more spatial dimensions are unstable. By substituting the Q-ball ansatz into the Euler-Lagrange equation (1.2.2), we obtain an equation for the radial profile,

$$\left(\partial_r^2 + \frac{2}{r} \partial_r \right) \phi_Q = \frac{d}{d\phi_Q} U_{\text{eff}}. \quad (1.2.8)$$

where the effective potential is defined as $U_{\text{eff}} \equiv U - \omega_Q^2 \phi_Q^2/2$.

This raises the question of why we choose this particular time-dependent ansatz. One simple explanation is to start with the total energy of an arbitrary scalar profile Φ_Q with total charge Q_Q , and to enforce the fixed-charge condition by introducing the Lagrange multiplier ω_Q :

$$\begin{aligned} \mathcal{E}_{\omega_Q} &= E_Q + \omega_Q \left(Q_Q - \int d^3x J_Q^0 \right) \\ &= \omega_Q Q_Q + \int d^3x \left[|\dot{\Phi}_Q + i\omega_Q \Phi_Q|^2 + |\nabla \Phi_Q|^2 + \mathbf{U}_{\text{eff}}(\Phi_Q^*, \Phi_Q) \right] \end{aligned} \quad (1.2.9)$$

with the effective potential $\mathbf{U}_{\text{eff}}(\Phi) \equiv \mathbf{U}(\Phi) - \omega_Q^2 |\Phi|^2$, where we used Eqs. (1.2.5) and (1.2.6) in going from the first to the second equality. The first term is positive semi-definite and is minimized when $\dot{\Phi}_Q + i\omega_Q \Phi_Q = 0$, which is readily solved by Eq. (1.2.7) to give the Q-ball ansatz. Note that throughout this thesis, the potential appearing in the Lagrangian for the complex scalar Φ_Q is denoted by $\mathbf{U}$, while the one corresponding to the radial part ϕ_Q is denoted by U , with $U(\phi) = \mathbf{U}(\Phi = \phi/\sqrt{2})$. The same convention applies to U_{eff} .

- In the limit of a very large scalar-field configuration (i.e., the charge $Q_Q \rightarrow \infty$), the Q-ball profile resembles a smoothed-out step function with radius $\sim R_Q$, where $\phi_Q(r < R_Q) = \phi_0$ and $\phi_Q(r > R_Q) = 0$. The transition zone between the inner and outer parts of Q-balls has a thickness of order m_Φ^{-1} ; hence, many references use the term thin-wall Q-ball to refer to this limiting case.

To sketch the construction of Q-balls, as well as their size and total energy as functions of a fixed conserved charge Q_Q , we seek a configuration that minimizes the total energy,

$$\begin{aligned} E_Q &= \int d^3x \left[\left| \dot{\Phi} \right|^2 + |\nabla \Phi|^2 + \mathbf{U} \right] \approx \frac{1}{2} \omega_Q^2 \phi_0^2 V_Q + U_0 V_Q \\ &\approx \frac{1}{2} \frac{Q_Q^2}{\phi_0^2 V_Q} + U_0 V_Q, \end{aligned} \quad (1.2.10)$$

where $V_Q = (4\pi/3)R_Q^3$ is the volume of this configuration, and $U_0 = U(\phi = \phi_0)$. In obtaining the last expression, we used

$$Q_Q = i \int d^3x \dot{\Phi}_Q \Phi_Q^* + \text{h.c.} \approx \omega_Q \phi_0^2 V_Q \quad (1.2.11)$$

to get rid of the dependence on ω_Q . Here we assume that surface-energy effects (corresponding to the gradient term) are negligible compared to the volume-energy contribution,⁽¹⁾ which implies $Q_Q = \omega_Q \phi_0^2 V_Q$ and hence the second equality. For a constant Q_Q , the minimization of E_Q with respect to V_Q gives

$$E_Q = Q_Q \sqrt{\frac{2U_0}{\phi_0^2}}, \quad V_Q = \frac{Q_Q}{\sqrt{2\phi_0^2 U_0}}, \quad (1.2.12)$$

Note that these expressions still show the dependence of the total energy on the field value in the inner core of the Q-ball. By minimizing E_Q with respect to ϕ_0 , we obtain the relation

$$\phi_0 = \underset{\phi}{\operatorname{argmin}} \left[2U(\phi)/\phi^2 \right]. \quad (1.2.13)$$

Finally, substituting V_Q from (1.2.12) into (1.2.11), we obtain the expression for the internal frequency in terms of the field value in the Q-ball core, thereby completing the construction of the large Q-ball solution with known amplitude ϕ_0 and frequency ω_Q :

$$\omega_Q = \sqrt{\frac{2U_0}{\phi_0^2}}. \quad (1.2.14)$$

This rough estimation shows that for large Q-balls, both the total energy E_Q and the volume of the Q-ball are proportional to the total charge Q . These linearly scaling properties resemble those of a ball of ordinary matter, whose existence and radius depend on the number of particles in it. Since the particle number can be equivalently described by Q_Q , this system is called a “Q-ball”, and the homogeneous state in its interior is called “Q-matter”, following Coleman’s convention. Note that this simple derivation so far does not imply that the Q-ball is a spherical configuration, since all shapes of the same volume V_Q are degenerate in total energy. This is due

⁽¹⁾ This is a reasonable approximation for large Q-balls, given that both the potential and kinetic energies scale linearly with respect to $V_Q \propto R_Q^3$. Meanwhile, the gradient term is only non-negligible around the transition zone of thickness $\sim m_\Phi^{-1}$ at the surface of the Q-ball; as a consequence, the surface energy scales as R_Q^2 and should be dominated by the volume terms for sufficiently large R_Q .

to the step where we ignore the contribution from surface tension energy in Eq. (1.2.10). One therefore expects that the true minimum-energy configuration also minimizes the contribution of surface area to the total energy, thereby lifting the degeneracy between all shapes of the same volume to the one with minimum area, in this case a sphere.

- To see the scaling arguments more clearly, let us define

$$I[\phi_Q] = \int d^3x \phi_Q^2, \quad S_{\text{grad}}[\phi_Q] = \frac{1}{2} \int d^3x \partial_i \phi_Q \partial_i \phi_Q, \quad S_{\text{pot}}[\phi_Q] = \int d^3x U(\phi_Q). \quad (1.2.15)$$

For the Q-ball profile, the conserved charge is then $Q_Q = \omega_Q I[\phi_Q]$. At fixed charge, the energy functional takes the schematic form

$$E_Q \equiv E[\phi_Q] = \frac{Q_Q^2}{2I[\phi_Q]} + S_{\text{grad}}[\phi_Q] + S_{\text{pot}}[\phi_Q], \quad (1.2.16)$$

where the first term is the contribution from the internal rotation. Under $\phi_{Q,\xi}(\mathbf{x}) = \phi_Q(\xi \mathbf{x})$, this becomes

$$E_Q[\phi_{Q,\xi}] = \xi^3 \frac{Q_Q^2}{2I[\phi_Q]} + \xi^{-1} S_{\text{grad}}[\phi_Q] + \xi^{-3} S_{\text{pot}}[\phi_Q]. \quad (1.2.17)$$

The extremum condition at $\xi = 1$ gives

$$3 \frac{Q_Q^2}{2I[\phi_Q]} - S_{\text{grad}}[\phi_Q] - 3S_{\text{pot}}[\phi_Q] = 0. \quad (1.2.18)$$

The extra positive term proportional to ξ^3 is absent in the static real-scalar Derrick argument, and the equation above shows explicitly how it can balance the gradient and potential contributions. This is the variational origin of the statement that Q-balls evade Derrick's theorem through time dependence at fixed Noether charge. This extremum condition gives the virial theorem between different components of the Q-ball energy. A more detailed discussion in general D dimensions is given in Appendix. A.1.

- Having constructed the large Q-ball configuration, we now turn to the question of whether such a solution is stable under some simple decay channels. From the definition that a Q-ball is a long-lived solitonic state, decay channels such as the emission of charged mesons by a Q-ball are forbidden. Eq. (1.2.12) then gives the energy per unit charge E/Q as $\sqrt{2U_0/\phi_0^2}$, and as a consequence the Q-ball is stable under meson emission if this number is less than the meson mass m_Φ . We can then make the following statement: new stable solutions of the classical field equations that are absolute minima of the energy at fixed charge Q (Q-balls) appear in the spectrum if the following constraint is satisfied:

$$(\omega_Q^{(-)})^2 \equiv \min [2U/\phi^2] \equiv 2U_0/\phi_0^2 < m_\Phi^2, \quad \phi_0 \neq 0, \quad (1.2.19)$$

with $\omega_Q^{(-)}$ being the effective energy scale of the internal rotation mode, and $m_\Phi^2 \equiv U''(0) = [2U/\phi^2]_{\phi=0}$. As we shall see later, $\omega_Q \rightarrow \omega_Q^{(-)}$ is exactly the large-Q-ball limit that we have assumed so far.

Let us now turn to another potential decay channel: since Q-ball is a homogeneous state of nonzero ϕ , the core of the Q-ball mimics a false vacuum, which raises the concern that the Q-ball could decay by quantum tunneling. In such a process, quantum fluctuations produce a bubble of true vacuum within the false vacuum, which then grows classically until the false-vacuum phase is converted to true vacuum. For the case of Q-ball, one can imagine a cavity formed inside the Q-ball due to quantum fluctuations, expand and finally engulf the whole soliton. However, quantum tunneling conserves both charge and energy; consequently, the Q-ball can only tunnel to a state with the same $E = E_Q$ and $Q = Q_Q$. By the argument of energy minimization at fixed

charge, such a state with fixed charge and minimal energy is a Q-ball; a Q-ball with a cavity must therefore have larger total energy compared to the Q-ball, and therefore quantum tunneling is not a viable decay channel by construction. There are also other possible decay channels, including Q-ball fission into smaller Q-balls or Q-ball evaporation due to couplings with matter fields. We will not go into these stability conditions; some relevant discussions can be found in Appendix. A.2.

1.2.2 From thin-wall to thick-wall: constraints on internal frequency ω_Q

More generally, beyond the thin-wall Q-ball concept introduced by Coleman, a theory that supports the existence of Q-ball solutions must contain a sufficiently attractive self-interaction, so that the energy per unit charge inside a finite-density condensate can fall below the mass of isolated quanta. Equivalently, the scalar self-interaction must make it favorable for charge to be carried by a coherent extended field configuration rather than by independent quanta around the vacuum. The internal rotation then plays the role of a chemical potential conjugate to the conserved charge: if it is too fast, the small-amplitude tail cannot be localized around the vacuum, while if it is too slow, the attractive interaction is not sufficient to support a finite-density charged core. This is the intuitive origin of the finite band of internal frequencies for which localized Q-ball profiles exist. To find the upper and lower bounds of this band of ω_Q , let us recall the radial equation (1.2.8),

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}\right) \phi_Q = \frac{dU_{\text{eff}}}{d\phi_Q}, \quad U_{\text{eff}}(\phi_Q) = U(\phi_Q) - \frac{1}{2} \omega_Q^2 \phi_Q^2. \quad (1.2.20)$$

Multiplying this equation by $d\phi_Q/dr$ gives

$$\frac{d}{dr} \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 - U_{\text{eff}}(\phi_Q) \right] = -\frac{2}{r} \left(\frac{d\phi_Q}{dr} \right)^2 \leq 0. \quad (1.2.21)$$

An example of a potential that allows for a Q-ball solution, together with the effective potential for different values of ω_Q , is shown in Fig. 1.2.

Analogy with a classical ball rolling down a hill with friction: Equation (1.2.21) has the same structure as the Euclidean bounce equation used to describe tunneling from a metastable vacuum to a true vacuum, and it can be solved by similar techniques (see [8] for a more detailed discussion). One useful mechanical analogy is to regard the field value ϕ_Q as the position of a classical particle, while the radial coordinate r plays the role of time. The particle moves in the inverted potential $-U_{\text{eff}}$ and experiences a time-dependent friction force proportional to $2\phi'_Q/r$. Regularity at the origin then requires $\phi'_Q(0) = 0$; otherwise the friction term would be singular at the initial point and the particle would never move. Next, the finite-energy condition requires the profile to relax to the vacuum sufficiently far away from the center of the Q-ball (or at sufficiently late times in the classical analogy), so both ϕ_Q and ϕ'_Q must vanish as $r \rightarrow \infty$. Collectively, the boundary conditions are therefore

$$\phi'_Q(0) = 0, \quad \phi_Q(r \rightarrow \infty) = 0, \quad \phi'_Q(r \rightarrow \infty) = 0. \quad (1.2.22)$$

The finite-energy Q-ball is thus a special trajectory that starts at rest from a nonzero value $\phi_0 \equiv \phi_Q(0)$ and approaches the vacuum $\phi_Q = 0$ as $r \rightarrow \infty$. This observation leads to the *shooting method*, one of the standard numerical methods for solving Eq. (1.2.21) with the boundary conditions (1.2.22). For fixed ω_Q and consequently fixed U_{eff} , one chooses a trial initial value $\phi_0 > 0$, integrates the radial equation outward from $r = 0$ up to some large value of r , and checks whether the resulting profile approaches $\phi_Q = 0$ with vanishing derivative at this large radius. For a simple potential such as the one shown in Fig. 1.2, an unsuccessful trial typically falls into one of two cases:

- *Undershooting:* The particle in the classical analogy does not have enough potential energy, i.e., the initial “height” ϕ_0 is not sufficiently large. It loses its energy due to friction before reaching $\phi_Q = 0$ at late times and therefore ends up trapped at the local minimum of the mechanical potential $-U_{\text{eff}}$ on the starting side.

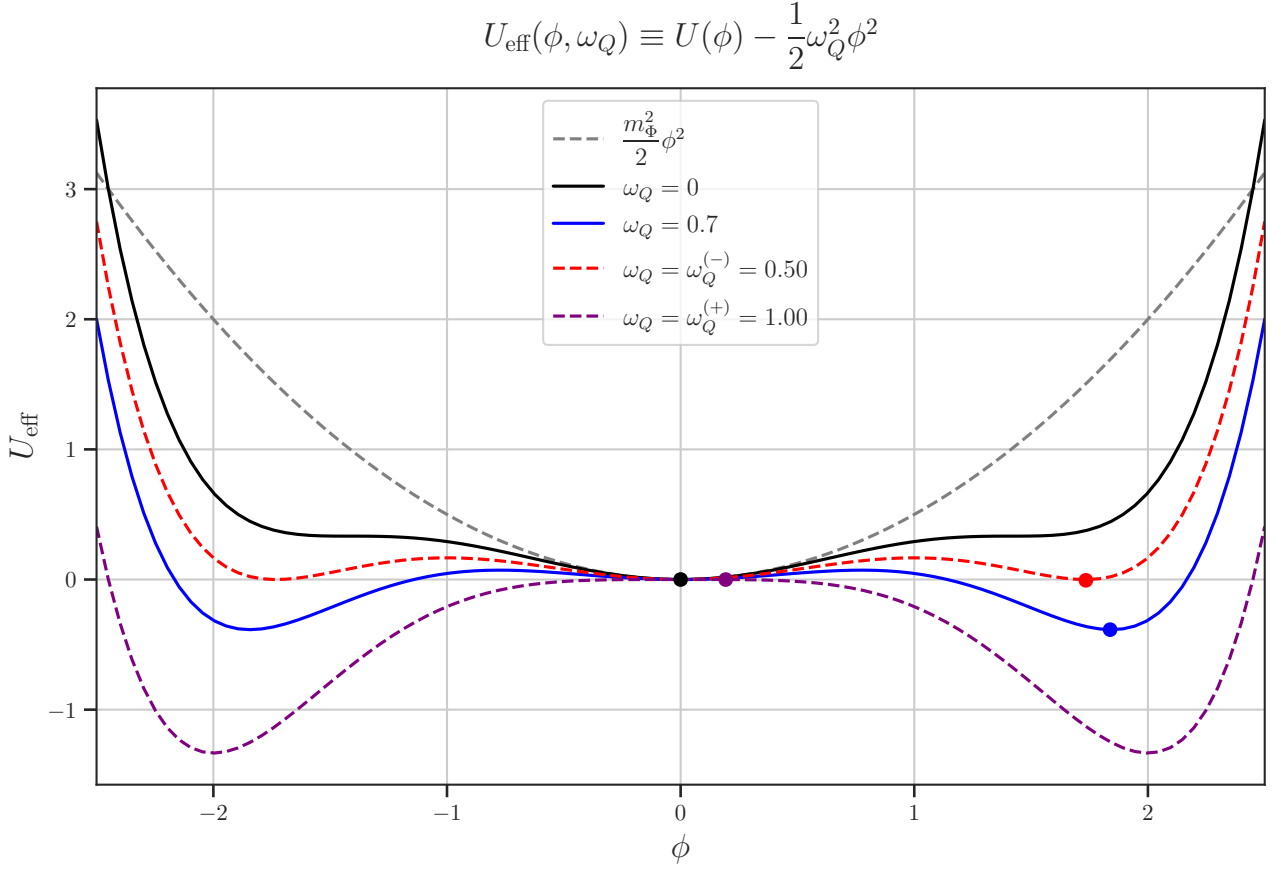


Figure 1.2: An example of a Q-ball effective potential $U_{\text{eff}}(\phi) = \frac{1-\omega_Q^2}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{g}{8}\phi^6$ for several values of ω_Q ; here we choose $g = 1/3$. The black line with $\omega_Q = 0$ corresponds to the original potential $U(\phi)$, and for reference, the dashed gray line corresponds to the mass term of this potential alone. The dashed red and purple lines correspond to the limiting values of ω_Q for which Q-ball solutions exist. The blue line shows an example of an allowed value of ω_Q corresponding to an existing Q-ball solution with this specific potential. The colored markers correspond to the field value at the center of the Q-ball profile, i.e., $\phi_Q(r = 0)$. We will later encounter this potential in the specific numerical study in Chapter. 3.

- *Overshooting:* The initial value ϕ_0 is too large, so the particle has enough energy to cross $\phi_Q = 0$ with nonzero velocity $\phi'_Q(r)$. In the mechanical picture, it can then continue to the other side of the inverted potential instead of asymptotically stopping at the origin, and it may become trapped in the other local minimum. ⁽¹⁾

By classifying each trial as an undershoot or an overshoot, one can iteratively tune ϕ_0 until the large-radius boundary conditions in Eq. (1.2.22) are satisfied to the desired numerical precision.

Existence of Q-ball solutions: This mechanical picture leads to two independent constraints on ω_Q , which may be interpreted as avoiding the two possible failures of the shooting problem.

- The first requirement comes from the undershooting side: suppose that the inverted potential $-U_{\text{eff}}$ is never positive away from the origin, while $U_{\text{eff}}(0) = 0$. Then a particle released from rest at any nonzero ϕ_0 does not have enough mechanical “height” to ever reach $\phi_Q = 0$. The trajectory then turns around before reaching the origin and becomes trapped in oscillations about the local minimum on the starting side of the inverted-potential dynamics, which is the

⁽¹⁾ There can also be trajectories in which the particle crosses the origin, turns around, and only later approaches $\phi_Q = 0$ with the correct large-radius boundary condition. In field theory these correspond to radially excited Q-balls with nodes in the profile. Such excited solutions are generally not the stable ground-state Q-balls at fixed charge, and they will not be discussed further in this thesis.

undershooting obstruction discussed previously. To avoid this, $-U_{\text{eff}}$ must have a positive region away from the origin:

$$\max_{\phi \neq 0} [-U_{\text{eff}}(\phi)] \geq 0 \quad \Longleftrightarrow \quad \boxed{\omega_Q^2 \geq \min_{\phi \neq 0} \left(\frac{2U(\phi)}{\phi^2} \right) \equiv \left(\omega_Q^{(-)} \right)^2.} \quad (1.2.23)$$

This is the lower-frequency condition. Below $\omega_Q^{(-)}$, the attractive part of the scalar self-interaction is not strong enough, at the chosen chemical potential ω_Q , to make a finite-density charged condensate energetically favorable.

- The second requirement for existence of Q-ball solutions comes from the overshooting condition: even if the particle can roll from a finite initial value toward $\phi_Q = 0$, the origin must be a point where it can asymptotically stop as $r \rightarrow \infty$. Thus $\phi_Q = 0$ must be a local maximum of the inverted potential $-U_{\text{eff}}$, or equivalently a local minimum of U_{eff} :

$$U_{\text{eff}}''(0) > 0 \quad \Longleftrightarrow \quad \boxed{\omega_Q^2 < U''(0) \equiv \left(\omega_Q^{(+)} \right)^2.} \quad (1.2.24)$$

If this condition is violated, the origin is not an attracting endpoint of the frictional motion. The shooting trajectory then generically crosses through $\phi_Q = 0$ instead of settling there, and in field-theory language the large-distance tail is oscillatory rather than exponentially suppressed. This is the upper-frequency condition: above $\omega_Q^{(+)}$, the asymptotic vacuum cannot support a localized Q-ball tail.

Figure. 1.3 illustrates the corresponding rolling-particle picture. In combination, the existence of a Q-ball requires

$$\boxed{\omega_Q^{(-)} \leq |\omega_Q| < \omega_Q^{(+)}.} \quad (1.2.25)$$

The two endpoints of this interval have different physical meanings. When U_{eff} is nearly degenerate between the vacuum at $\phi = 0$ and the finite-field region, the particle can spend a long radial “time” near the top of the inverted potential while the friction term decreases, before quickly rolling down the potential hill over a short time interval. The associated Q-ball profile is then large and thin-walled. If $|\omega_Q|$ is below $\omega_Q^{(-)}$, the attractive part of the scalar self-interaction is not strong enough, at that ω_Q , to support a charged core. This contrasts with the case in which $|\omega_Q|$ approaches $\omega_Q^{(+)}$: the Q-ball profile initially stays at ϕ_0 far from the local maximum of $-U_{\text{eff}}$ and starts rolling down under significant friction, leading to the *thick-walled Q-ball* limit with a more diffuse small-amplitude profile. If such a configuration with $\omega_Q > \omega_Q^{(+)}$ is somehow formed, the internal frequency of the background is so large that it would potentially radiate away free quanta and settle into a smaller Q-ball, or even decay into a collection of smaller Q-balls.

For later convenience, define $\phi_+(\omega_Q)$ to be the nonzero value of ϕ that minimizes U_{eff} , and $\phi_-(\omega_Q) \neq 0$ satisfies $U_{\text{eff}}(\phi_-(\omega_Q)) = 0$. Notice that $\phi_-(\omega_Q) \simeq \phi_+(\omega_Q)$ when $\omega_Q \simeq \omega_Q^{(-)}$. The lower limit can be rewritten as $\left(\omega_Q^{(-)} \right)^2 = \frac{2U}{\phi^2} \big|_{\phi_+(\omega_Q^{(-)})}$. For numerical shooting, one therefore tunes the initial value $\phi_0(\omega_Q)$ within the interval

$$\phi_-(\omega_Q) \leq \phi_0(\omega_Q) < \phi_+(\omega_Q). \quad (1.2.26)$$

In conclusion, the two endpoints of the allowed internal frequencies (1.2.25) organize the two standard limiting regimes. The lower bound $|\omega_Q| \simeq \omega_Q^{(-)}$ leads to the large-charge thin-wall regime, while the upper bound $|\omega_Q| \simeq \omega_Q^{(+)}$ leads to the thick-wall regime. The large-charge thin-wall limit can be pictured as a droplet of nearly homogeneous charged matter, while thick-wall Q-balls are diffuse

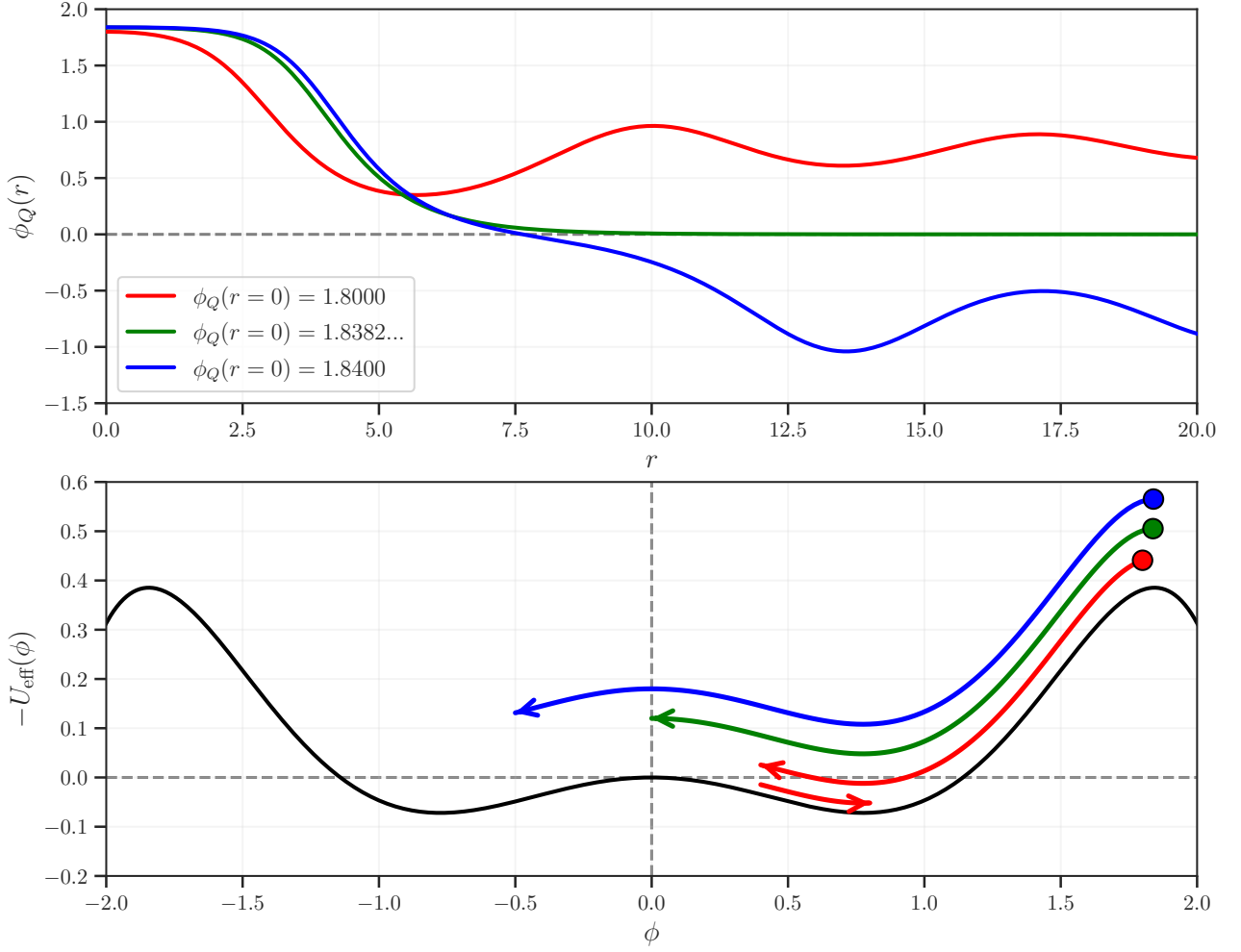


Figure 1.3: Analogy between a Q-ball solution and the classical mechanics of a rolling ball moving down a potential hill with time-dependent friction proportional to the ball’s velocity; recall that the radius r plays the role of time in this classical mechanics analogy. We plot the three cases of overshoot (blue), undershoot (red), and correct shooting (green) that correspond to a Q-ball solution.

objects whose size and energy are controlled by a more delicate balance between gradient energy, mass terms, and nonlinear interactions. Other studies have refined this basic picture beyond the original thin-wall approximation, especially for polynomial potentials where analytic profile approximations and systematic relations between charge, energy, and radius are possible [9, 10, 11].

Q-ball in a single complex scalar field model

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The previous chapter introduced Q-balls as non-topological solitons whose stability is tied to a conserved global charge and whose existence relies on a time-dependent rotation in internal field space. The purpose of this chapter is to turn that qualitative picture into a framework for scattering calculations between a Q-ball and perturbations around it. We consider a single complex scalar field with a generic $U(1)$ -symmetric potential in arbitrary spatial dimensions D , so that the construction is not tied to one special polynomial model. This level of generality is useful because the later questions about energy extraction, charge transfer, and superradiant amplification depend on both the universal structure of the background and the detailed shape of the potential.

The first goal is to construct the stationary Q-ball background as a fixed-charge extremum of the total energy. The internal frequency ω_Q appears as the Lagrange multiplier conjugate to the conserved charge Q_Q , while the radial profile is determined by an effective potential in the usual bounce-like equation. After the basic profile equation is established, we review the limiting regimes of the solution where the internal frequency ω_Q approaches its upper and lower bounds. The thin-wall limit describes a large charged step-like profile with surface corrections, while the thick-wall limit describes a diffuse small-amplitude configuration near the vacuum. The virial relation then gives a useful connection between different energy components, namely the potential energy and surface energy (a.k.a. the gradient energy).

The second goal of the chapter is to formulate the perturbative scattering problem around this background. Because the Q-ball is stationary in energy density but not static as a field configuration, a perturbation with one frequency does not scatter as an isolated mode. Instead, the rotating phase of the background couples two channels with different frequencies $\omega_Q \pm \omega$, and this mode conversion is the basic mechanism by which an incoming wave can exchange charge and energy with the Q-ball. We proceed by introducing the incoming and outgoing amplitude basis, deriving the conserved particle-number norm of the linearized radial system, and defining charge and energy amplification factors in terms of asymptotic fluxes. Finally, the same information is reorganized into an S-matrix, which provides a compact and elegant way to compute observables for arbitrary incoming mixtures and to

expose the symmetry constraints that will be tested in the concrete sextic model in Chapter. 3 and later generalized to the Friedberg-Lee-Sirlin model in Chapter. 4.

2.1 Construction of Q-ball in D spatial dimension

Below, let us generalize the results obtained in the previous introductory chapter, Chapter. 1, to generic $D + 1$ spacetime dimensions. Consider a general complex scalar field with a generic potential in $D + 1$ spacetime dimensions,

$$S[\Phi] = - \int d^{D+1}x \left[\partial_\mu \Phi^\dagger \partial^\mu \Phi + \mathbf{U}(|\Phi|) \right]. \quad (2.1.1)$$

The Q-ball solution belongs to the fixed-charge sector that minimizes the energy content and therefore can be decomposed into a product of a spherically symmetric spatial part and a time-dependent phase,

$$\Phi_Q(\mathbf{x}, t) = \frac{\phi_Q(r)}{\sqrt{2}} e^{-i\omega_Q t} \quad (2.1.2)$$

where the spatial part $\phi_Q(r)$ satisfies the equation ⁽¹⁾

$$\left(\partial_r^2 + \frac{D-1}{r} \partial_r \right) \phi_Q = \frac{d}{d\phi_Q} \left(U - \frac{\omega_Q^2 \phi_Q^2}{2} \right) = \frac{d}{d\phi_Q} U_{\text{eff}}, \quad (2.1.5a)$$

$$\phi_Q'(0) = 0, \quad \phi_Q(r \rightarrow \infty) = 0, \quad \phi_Q'(r \rightarrow \infty) = 0. \quad (2.1.5b)$$

where the boundary conditions ensure regularity of ϕ_Q at the origin and finite energy of the solution. As mentioned, this differential equation is similar to the well-studied bounce equation and similar treatments can be applied. Analytic solutions cannot be obtained in general, although can be found in certain limits. There are several approaches to solving global Q-ball profiles numerically. Arguably the most common approach is the shooting method, which we discussed in Subsection. 1.2.2 and will adopt in our numerical study in Chapter. 3. There are attempts to solve the Q-ball profiles using the relaxation method, which is claimed to have more accurate results with high efficiency. In principle, the coupled ODEs are transformed into a set of matrix equations, which are solved by iteratively updating trial solutions across the grid until the desired precision is reached. A more detailed discussion of the implementation of this method can be found in [14].

Let us now write down explicit formulas for the physical observables corresponding to a Q-ball configuration:

- Conserved $U(1)$ charge

$$Q_Q = i \int d^D x \left(\dot{\Phi}_Q \Phi_Q + \text{h.c.} \right) = \Omega_D \omega_Q \int dr r^{D-1} \phi_Q^2(r). \quad (2.1.6)$$

⁽¹⁾ Note that for simplicity, we almost always consider the zero-angular-momentum solution (i.e., non-rotating Q-balls). A more general solution with nonzero angular momentum in $D = 3$ spatial dimensions in spherical coordinates is $\Phi_Q = \phi_Q(r, \theta) e^{i\omega_Q t + im_Q \varphi}$, where ϕ_Q satisfies

$$\left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta - \frac{m_Q^2}{r^2 \sin^2 \theta} + \omega_Q^2 \right) \phi_Q = \frac{dU}{d\phi}. \quad (2.1.3)$$

This solution has a nonzero J_z given by

$$J_z = - \int d^3 x \left(\partial_t \Phi \partial_\varphi \Phi^* + \partial_t \Phi^* \partial_\varphi \Phi \right) = m_Q Q_Q. \quad (2.1.4)$$

This relation implies that any field configuration having large Q_Q and nonzero m_Q also has large angular momentum. More on the construction and properties of a generic rotating Q-ball can be found, for example, in Refs. [12, 13, 14].

- Energy

$$\begin{aligned}
E_Q &= \int d^D x \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 + U(\phi_Q) + \frac{1}{2} \omega_Q^2 \phi_Q^2 \right] \\
&= \omega_Q Q + \Omega_D \int dr r^{D-1} \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 + U_{\text{eff}}(\phi_Q) \right].
\end{aligned} \tag{2.1.7}$$

Here Ω_D is the solid angle of a D -dimensional sphere.

Legendre relations: recall that the Euclidean action S_{ω_Q} and the effective potential U_{eff} in the Q-ball configuration are given by

$$S_{\omega_Q} = \int d^D x \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 + U_{\text{eff}} \right], \quad U_{\text{eff}} \equiv U - \frac{1}{2} \omega_Q^2 \phi_Q^2. \tag{2.1.8}$$

It is useful to define the functional

$$G_I \equiv \int d^D x \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 + U \right] = E_Q - \left(\frac{1}{2} \omega_Q^2 I \right) = S_{\omega_Q} + \left(\frac{1}{2} \omega_Q^2 \right) I, \tag{2.1.9}$$

where $\frac{1}{2} \omega_Q^2$ is the corresponding Lagrange multiplier and $I \equiv \int d^D x \phi_Q^2$. Since Q-ball solutions give the extrema of E_Q , S_{ω_Q} , and G_I with respect to Q_Q , ω_Q , and I , respectively, one can prove that

$$\left. \frac{\partial E_Q}{\partial Q_Q} \right|_{S_{\omega_Q}} = \omega_Q, \quad \left. \frac{\partial S_{\omega_Q}}{\partial \omega_Q} \right|_{E_Q} = -Q_Q, \quad \left. \frac{\partial G_I}{\partial I} \right|_{S_{\omega_Q}} = \frac{1}{2} \omega_Q^2. \tag{2.1.10}$$

Therefore, one can map E_Q , ω_Q , Q_Q , and G_I to the corresponding thermodynamic variables, specifically the internal energy, chemical potential, particle number, and thermodynamic potentials. For example, the first relation of Eq. (2.1.10) can be rewritten in the form

$$dE_Q/d\omega_Q = \omega_Q dQ_Q/d\omega_Q, \tag{2.1.11}$$

and shows that ω_Q determines how the energy changes when charge is added to or removed from the Q-ball. Another interesting characteristic of the Q-ball is the relation between gradient energy, potential energy, and charge via Derrick's scaling argument. A simple version of this relation was discussed in Section. 1.2; a more general version for arbitrary dimension, together with some comments, can be found in Appendix. A.1. Several different stability criteria are briefly discussed in Appendix. A.2.

2.2 Limiting cases of Q-ball profiles

The stationary ansatz and fixed-charge variational principle derived above are necessary ingredients for constructing a Q-ball, but they do not by themselves guarantee that a physically relevant soliton survives. The first question is whether the scalar potential admits a localized profile with the correct boundary conditions for some value of the internal frequency ω_Q . This is the existence question, and it is most transparently answered by interpreting the radial equation as a frictional motion problem in the effective potential [6, 7]. This interpretation also clarifies whether the solution approaches a thin-wall or thick-wall limit. The derivation of the upper and lower bounds of the Q-ball internal frequency ω_Q in $D = 3$ was already given in Subsection. 1.2.2; note that the arguments used to derive those results are D independent. For reference, let us recall the main existing criteria for a Q-ball governed by a potential U :

- There exists a value $\phi_0 \neq 0$ such that

$$(\omega_Q^{(-)})^2 \equiv \min [2U/\phi^2] \equiv 2U_0/\phi_0^2 < m_\Phi^2, \tag{2.2.1}$$

- The range of internal frequencies for the existence of the solution is bounded from above and below:

$$\omega_Q^{(-)} \leq |\omega_Q| < \omega_Q^{(+)}, \quad (2.2.2)$$

where $\omega_Q^{(-)}$, defined previously, corresponds to the thin-wall limit, and $(\omega_Q^{(+)})^2 \equiv U''(0) = m_\Phi^2$ corresponds to the thick-wall limit.

We discuss these limits separately below in Subsections 2.2.1 and 2.2.2, since their approximations emphasize different parts of the solution: a nearly homogeneous charged core in the first case and a small-amplitude diffuse profile in the second.

Once such a solution exists, one can ask more refined questions: whether the lump is energetically stable against decay into free quanta, whether it is stable against small classical deformations or fission, and how global scaling relations constrain its energy. The purpose of this section is therefore to keep these distinct questions separate: we first derive the allowed range of ω_Q and interpret its endpoints, and then review the two limiting profile regimes, namely the thin-wall Q-ball and the thick-wall Q-ball.

2.2.1 Thin-wall Q-ball

The thin-wall regime is the large-charge endpoint of the existence interval, reached when $|\omega_Q|$ approaches the lower bound $\omega_Q^{(-)}$. In the mechanical picture, the starting point of the trajectory approaches the finite-field maximum of the inverted potential, $\phi_0(\omega_Q) \simeq \phi_+(\omega_Q)$, and in the limiting configuration $\phi_0(\omega_Q) \rightarrow \phi_+^{(-)}$. Here and below we use the shorthand $\phi_\pm^{(\pm)} \equiv \phi_\pm(\omega_Q^{(\pm)})$. Equivalently, $\phi_+^{(-)}$ is the field value at which $2U(\phi)/\phi^2$ is minimized. If we denote the effective potential evaluated at the lower endpoint by

$$U_{\text{eff}}^{(-)}(\phi) \equiv U(\phi) - \frac{1}{2} (\omega_Q^{(-)})^2 \phi^2, \quad (2.2.3)$$

then this endpoint condition can be written as

$$(\omega_Q^{(-)})^2 = \frac{2U(\phi)}{\phi^2} \Big|_{\phi=\phi_+^{(-)}}, \quad U_{\text{eff}}^{(-)}(\phi_+^{(-)}) = U_{\text{eff}}^{(-)}(0) = 0. \quad (2.2.4)$$

The physical picture is then that of a droplet of nearly homogeneous charged matter. The profile stays close to $\phi_+^{(-)}$ over a large interior region and changes rapidly only across a surface layer whose thickness is small compared with the radius. This is why Coleman's original thin-wall approximation captures the leading large- Q_Q behavior, while more refined analyses treat the finite surface region as the first correction [6, 15, 10, 11].

At leading order one may write the real profile as

$$\Phi_Q(x) = \frac{\phi_Q(r)}{\sqrt{2}} e^{-i\omega_Q t} = \frac{\phi_+^{(-)}}{\sqrt{2}} f_Q(r) e^{-i\omega_Q t}, \quad (2.2.5)$$

where $f_Q(r)$ is a dimensionless shape function. In the idealized step-function approximation,

$$f_Q(r) \simeq \Theta(R_Q - r), \quad (2.2.6)$$

and therefore the charge is dominated by the homogeneous core,

$$Q_Q = \Omega_D \omega_Q \int_0^\infty dr r^{D-1} \phi_Q^2(r) \simeq V_Q^{(D)} \omega_Q (\phi_+^{(-)})^2, \quad V_Q^{(D)} \equiv \frac{\Omega_D R_Q^D}{D}. \quad (2.2.7)$$

Using the energy decomposition in Eq. (2.1.7), or equivalently the Legendre relation $E_Q = \omega_Q Q_Q + S_{\omega_Q}$, one obtains

$$E_Q = \omega_Q Q_Q + S_{\omega_Q}[\phi_Q], \quad S_{\omega_Q}[\phi_Q] = \Omega_D \int_0^\infty dr r^{D-1} \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 + U_{\text{eff}}(\phi_Q) \right]. \quad (2.2.8)$$

In the strict endpoint limit the volume contribution to S_{ω_Q} vanishes because $U_{\text{eff}}^{(-)}(\phi_+^{(-)}) = 0$, while the remaining contribution comes from the wall where the profile interpolates between $\phi_+^{(-)}$ and the vacuum. For a thin wall, the wall thickness is controlled by the local shape of the effective potential and stays finite as R_Q becomes large. The surface part of S_{ω_Q} therefore scales like the area, $\Omega_D R_Q^{D-1}$, whereas the charge term $\omega_Q Q_Q$ scales like the core volume, $V_Q^{(D)} \propto R_Q^D$. The ratio of the surface correction to the leading charge energy is consequently of order $1/R_Q$, and becomes negligible in the large-charge limit. Thus the leading thin-wall relation is

$$E_Q \simeq \omega_Q^{(-)} Q_Q, \quad (2.2.9)$$

with corrections controlled by the surface term, scaling schematically like $\Omega_D R_Q^{D-1}$ rather than $V_Q^{(D)}$. This separation between the volume charge and the surface correction is the sense in which the wall is thin: the wall is important for the profile and for finite- Q_Q corrections, but not for the leading large-volume energy per charge.

2.2.2 Thick-wall Q-ball

The opposite limiting regime is reached when $|\omega_Q|$ approaches the upper edge of the allowed interval,

$$|\omega_Q| \rightarrow \omega_Q^{(+)} = m_\Phi, \quad m_\Phi^2 \equiv U''(0). \quad (2.2.10)$$

In this limit the profile probes only the neighborhood of the vacuum. The effective potential can be expanded as

$$U_{\text{eff}}(\phi_Q) = \frac{1}{2} (m_\Phi^2 - \omega_Q^2) \phi_Q^2 + \mathcal{O}(\phi_Q^4), \quad (2.2.11)$$

so the quadratic restoring force becomes weak as $|\omega_Q| \rightarrow m_\Phi$. The terminology “thick wall” should therefore not be read as a literal statement about a sharply defined wall, but rather as the absence of a large homogeneous core separated from the vacuum by a parametrically thin surface. The gradient energy, the small effective mass term, and the leading attractive nonlinear term all compete over the full extent of the object.

Let us investigate the scaling of the Q-ball charge toward the thick-wall limit. Following the thick-wall discussion for polynomial Q-ball potentials in Refs. [15, 10, 11], we assume that the first attractive term near the vacuum has the schematic form

$$U(\phi_Q) = \frac{1}{2} m_\Phi^2 \phi_Q^2 - A \phi_Q^n + \dots, \quad A > 0, \quad n > 2. \quad (2.2.12)$$

For a fixed ω_Q , the corresponding effective potential is

$$U_{\text{eff}}(\phi_Q) = \frac{1}{2} m_{\omega_Q}^2 \phi_Q^2 - A \phi_Q^n + \dots, \quad m_{\omega_Q}^2 \equiv m_\Phi^2 - \omega_Q^2. \quad (2.2.13)$$

The thick-wall limit is the regime $m_{\omega_Q} \ll m_\Phi$, so the profile is broad and small enough that the leading nonlinear term controls the departure from the linear tail. To make the parametric balances explicit, write the profile as

$$\phi_Q(r) = \phi_0 F(\rho), \quad \rho \equiv \frac{r}{R_Q}, \quad (2.2.14)$$

where $F(0) = \mathcal{O}(1)$ and F varies over $\Delta\rho = \mathcal{O}(1)$. Substitution into the radial equation (2.1.5) gives, up to order-one factors depending on the detailed shape of F and on D ,

$$\frac{1}{R_Q^2} \left(F'' + \frac{D-1}{\rho} F' \right) \sim m_{\omega_Q}^2 F - n A \phi_0^{n-2} F^{n-1} + \dots. \quad (2.2.15)$$

The first comparison is between the left-hand side and the linear effective-mass term. Since F changes by an order-one amount across the Q-ball, the gradient contribution scales as ϕ_0/R_Q^2 , while the linear term scales as $m_{\omega_Q}^2 \phi_0$. A localized thick-wall solution therefore has

$$\frac{1}{R_Q^2} \sim m_{\omega_Q}^2 \quad \Rightarrow \quad R_Q \sim m_{\omega_Q}^{-1} = (m_\Phi^2 - \omega_Q^2)^{-1/2}. \quad (2.2.16)$$

The second comparison fixes how the central amplitude approaches zero. In the core, the attractive interaction must compete with the positive quadratic term; otherwise the small-amplitude equation would be purely linear and would not produce a regular nonlinear lump. Equating the two terms in $dU_{\text{eff}}/d\phi_Q$ gives

$$m_{\omega_Q}^2 \phi_0 \sim nA\phi_0^{n-1} \quad \Rightarrow \quad \phi_0(\omega_Q) \sim \left(\frac{m_{\omega_Q}^2}{nA} \right)^{1/(n-2)} \propto m_{\omega_Q}^{2/(n-2)}, \quad (2.2.17)$$

where the final proportionality keeps only the ω_Q dependence for fixed microscopic coupling A . Thus the profile becomes small in amplitude and diffuse in radius as $|\omega_Q|$ approaches m_Φ . The charge scaling is

$$Q_Q \sim \omega_Q \phi_0^2 R_Q^D \sim \omega_Q m_{\omega_Q}^{4/(n-2)-D}, \quad (2.2.18)$$

where $\omega_Q \simeq m_\Phi$ only gives an order-one prefactor in the endpoint scaling. The important point is that, depending on the power $p \equiv 4/(n-2) - D$, the Q-ball can behave differently as it moves toward the thick-wall limit. Since m_{ω_Q} decreases as $|\omega_Q|$ approaches m_Φ from below, the charge behaves as $Q_Q \propto m_{\omega_Q}^p$ near the endpoint. For $p > 0$, the charge decreases to zero in the formal thick-wall limit; for $p < 0$, it grows instead; and for $p = 0$, the leading scaling is marginal and subleading terms decide the behavior. Therefore the same qualitative statement that the profile is broad and small-amplitude does not by itself determine the monotonicity of $Q_Q(\omega_Q)$.

2.3 Q-ball perturbations

Having characterized the stationary Q-ball background, we now ask how this object responds to small disturbances. This is the natural next question, since a Q-ball is not only a static lump of charge; it is a coherent time-dependent condensate whose internal rotation can exchange charge and energy with perturbations. The goal of this section is to investigate scattering processes at linear order, where the background profile is held fixed while the perturbation is treated as a small spherical wave propagating from infinity towards the Q-ball background and bouncing back. The rotating phase of the background makes this scattering problem different from ordinary potential scattering: a single incoming frequency is coupled to two shifted channels, $\omega_Q + \omega$ and $\omega_Q - \omega$, so an outgoing wave need not carry the same physical charge or energy weights as the incoming one. We first derive the coupled radial equations and clarify the incoming/outgoing basis to expand the perturbative solutions. We then explain how the amplitudes are obtained from regular solutions at the origin, derive the conserved particle-number norm of the linearized system, define charge and energy amplification factors that measure the amount of exchanged charge/energy, and finally package the same information into an S-matrix whose unitarity and symmetry produce symmetry patterns in the measured physical spectrum.

2.3.1 Equation of linear perturbations and solutions in limiting cases

The stationary solution constructed above is now treated as a coherent, time-periodic background for small fluctuations. The central structural point is that the background internal phase rotation prevents a single monochromatic perturbation from being an eigenmode by itself. Instead, the perturbation problem closes on a pair of modes with frequencies shifted by the Q-ball chemical potential, and

this two-channel structure is what makes charge and energy exchange possible at linear order. Let us start by considering the perturbation dynamics around the stable Q-ball solution $\Phi = \Phi_Q + \Phi_1(t, r)$, assuming without loss of generality that $\omega_Q > 0$. By varying the Euler-Lagrange equation of the scalar field, Eq. (1.2.2), one obtains the equation of motion for linear perturbations ⁽¹⁾

$$\partial_\mu \partial^\mu \Phi_1 - V(r) \Phi_1 - e^{-2i\omega_Q t} W(r) \Phi_1^* = 0, \quad (2.3.4)$$

where V and W are defined as

$$V = \frac{1}{2} \left(\frac{d^2 U}{d\phi^2} + \frac{1}{\phi} \frac{dU}{d\phi} \right) \Big|_{\phi=\phi_Q}, \quad W = \frac{1}{2} \left(\frac{d^2 U}{d\phi^2} - \frac{1}{\phi} \frac{dU}{d\phi} \right) \Big|_{\phi=\phi_Q}. \quad (2.3.5)$$

Note that the last term of (2.3.4) introduces mode-mixing; consequently no monochromatic solutions exist. We stick to spherically symmetric perturbations, so the spacetime perturbation is decomposed into two radial modes as

$$\Phi_1(t, r) = \eta_+(r) e^{-i\omega_+ t} + \eta_-(r) e^{-i\omega_- t}, \quad (2.3.6)$$

where $\omega_\pm = \omega_Q \pm \omega$. The equation of motion in the frequency domain reads

$$\frac{1}{r^{D-1}} \frac{d}{dr} \left(r^{D-1} \frac{d}{dr} \right) \eta_\pm + (\omega_\pm^2 - V) \eta_\pm - W \eta_\mp^* = 0. \quad (2.3.7)$$

Already at the level of the linear perturbation equations, we see mixing weighted by W between modes $\eta_\pm$ with different energies $\omega_\pm$. One can therefore anticipate that scattering between the perturbations and the Q-ball can lead to energy extraction from the background by converting a perturbation of one frequency into another frequency. Depending on the value of the parameter ω , we can classify these perturbations as follows [4]: ⁽²⁾

- zero modes $\omega = 0$
- bound modes $|\omega_\pm| < m_\Phi$
- half-propagating modes $|\omega_+| > m_\Phi, |\omega_-| < m_\Phi$ or vice versa
- propagating modes $|\omega_\pm| > m_\Phi$

Within the scope of this thesis, we are interested in understanding the interaction of Q-balls with the plasma, so we focus only on the propagating and half-propagating modes. There are two extreme limits where analytical solutions exist:

- **Asymptotic region $r \rightarrow \infty$:** as $\phi_Q(r \rightarrow \infty) = 0$, we have $V(r \rightarrow \infty) = m_\Phi^2$ and $W(r \rightarrow \infty) = 0$, which means the term that mixes the η_+ and η_- perturbations vanishes and the solution is made of linear combinations of two modes. Define the signed momenta

$$k_\pm \equiv \text{sgn}(\omega_\pm) \sqrt{\omega_\pm^2 - m_\Phi^2}. \quad (2.3.8)$$

⁽¹⁾ Specifically, we plug the Q-ball ansatz into the linear perturbation equation

$$\partial_\mu \partial^\mu \Phi_1 - \frac{\partial^2 \mathbf{U}}{\partial \Phi^* \partial \Phi} \Big|_{\Phi=\Phi_Q} \Phi_1 - \frac{\partial^2 \mathbf{U}}{\partial \Phi^* \partial \Phi^*} \Big|_{\Phi=\Phi_Q} \Phi_1^* = 0. \quad (2.3.1)$$

The corresponding derivatives are rewritten in polar coordinates $(|\Phi|, \frac{\Phi}{|\Phi|})$ as

$$\frac{\partial^2 \mathbf{U}}{\partial \Phi^* \partial \Phi} = \frac{1}{4|\Phi|} \frac{\partial}{\partial |\Phi|} \left(|\Phi| \frac{\partial \mathbf{U}}{\partial |\Phi|} \right), \quad (2.3.2)$$

$$\frac{\partial^2 \mathbf{U}}{\partial \Phi^* \partial \Phi^*} = \frac{1}{4} \left(\frac{\Phi}{|\Phi|} \right)^2 \left[\frac{\partial^2 \mathbf{U}}{\partial |\Phi|^2} - \frac{1}{|\Phi|} \frac{\partial \mathbf{U}}{\partial |\Phi|} \right] \quad (2.3.3)$$

and use the conversion $U(\phi_Q) = \mathbf{U}(\Phi_Q)$ stated previously to obtain the expressions for V and W .

⁽²⁾ The only exceptions to this parametrization are the modes corresponding to the Lorentz boost and the Q-ball charge change [16, 17, 4].

The equation of motion for the background perturbation reduces to the simpler Klein-Gordon form for free scalar fields,

$$\eta_{\pm}'' + \frac{D-1}{r} \eta_{\pm}' + k_{\pm}^2 \eta_{\pm} = 0$$

$$\Rightarrow \quad \eta_{\pm}(\omega, r) = \frac{\tilde{A}_{\pm}}{(|k_{\pm}|r)^{\frac{D-2}{2}}} J_{\frac{D-2}{2}}(|k_{\pm}|r) + \frac{\tilde{B}_{\pm}}{(|k_{\pm}|r)^{\frac{D-2}{2}}} Y_{\frac{D-2}{2}}(|k_{\pm}|r) \quad (2.3.9)$$

$$\xrightarrow{r \rightarrow \infty} \frac{A_{\pm}}{(|k_{\pm}|r)^{\frac{D-1}{2}}} e^{i|k_{\pm}|r} + \frac{B_{\pm}}{(|k_{\pm}|r)^{\frac{D-1}{2}}} e^{-i|k_{\pm}|r}, \quad (2.3.10)$$

with $J_{\nu}(z)$ and $Y_{\nu}(z)$ being the Bessel functions of the first and second kind, respectively. ⁽¹⁾ Assuming both modes are propagating spherical waves imposes the reality condition on $k_{\pm}$: $|\omega_Q \pm \omega| > m_{\Phi}$. Since one can choose $\omega_Q > 0$ without loss of generality, this implies that the physical boundary of ω is

$$|\omega| > \omega_Q + m_{\Phi}. \quad (2.3.14)$$

The meaning of the amplitudes $A_{\pm}$ and $B_{\pm}$ is depicted in the figure below.

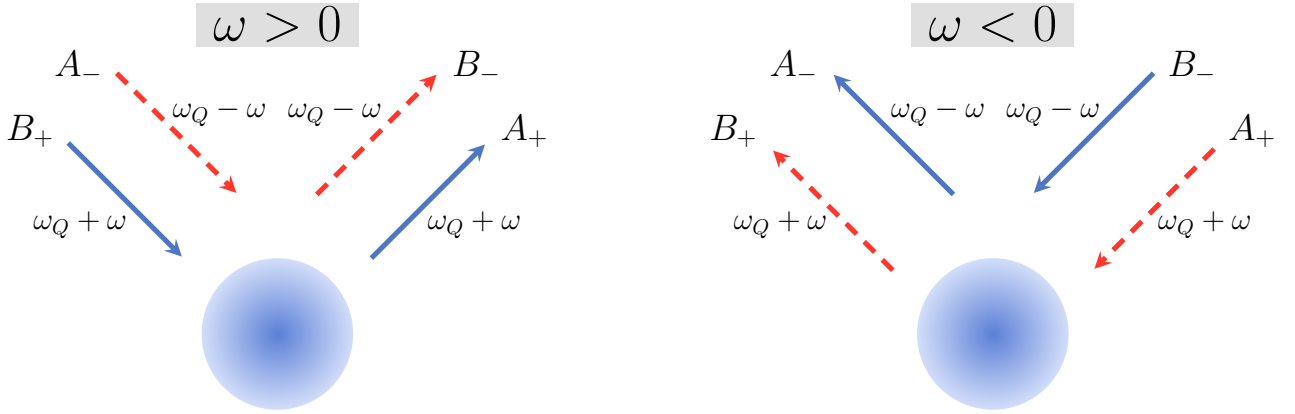


Figure 2.1: Incoming and outgoing waves scattering off a Q-ball. Solid (dashed) lines represent positive (negative) charge, which corresponds to the subscript of A and B when $\omega > 0$ and is flipped when $\omega < 0$.

- **Near-origin region $r \rightarrow 0$:** the spherical radial equation admits a regular and a singular local solution,

$$\eta_{\pm}'' + \frac{D-1}{r} \eta_{\pm}' = 0 \quad \Rightarrow \quad \eta_{\pm}(r \rightarrow 0) = \begin{cases} C_{1,\pm} + C_{2,\pm} r^{2-D}, & D \neq 2, \\ C_{1,\pm} + C_{2,\pm} \log r, & D = 2. \end{cases} \quad (2.3.15)$$

⁽¹⁾ Recall that the solutions can be expressed in the basis of the Hankel functions $H_{\nu}^{(1)} = J_{\nu} + iY_{\nu}$ and $H_{\nu}^{(2)} = J_{\nu} - iY_{\nu}$, and that the asymptotic expansion ($z \rightarrow \infty$) of the Hankel functions takes the following form (see, e.g., [18], section 10.2):

$$H_{\nu}^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \quad H_{\nu}^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}. \quad (2.3.11)$$

From this, we obtain the relation between the coefficients,

$$A_{\pm} = \frac{e^{-i\pi(D-1)/4}}{\sqrt{2\pi}} \tilde{A}_{\pm} - i\tilde{B}_{\pm}, \quad B_{\pm} = \frac{e^{i\pi(D-1)/4}}{\sqrt{2\pi}} \tilde{A}_{\pm} + i\tilde{B}_{\pm}, \quad (2.3.12)$$

$$\tilde{A}_{\pm} = \sqrt{\frac{\pi}{2}} (e^{i\pi(D-1)/4} A_{\pm} + e^{-i\pi(D-1)/4} B_{\pm}), \quad \tilde{B}_{\pm} = -i\sqrt{\frac{\pi}{2}} (e^{i\pi(D-1)/4} A_{\pm} - e^{-i\pi(D-1)/4} B_{\pm}). \quad (2.3.13)$$

Regularity at the origin therefore requires $C_{2,\pm} = 0$, or equivalently

$$\eta_{\pm}(r \rightarrow 0) = C_{\pm}, \quad \eta'_{\pm}(r \rightarrow 0) = 0. \quad (2.3.16)$$

Incoming and outgoing basis: Before defining the amplification factors, let us make explicit the relation between the two asymptotic notations used in this discussion. In the derivation above, the A, B basis uses the positive radial momentum $|k_{\pm}|$, so that the large-radius solution is written as

$$\eta_{\pm}(r \rightarrow \infty) \sim \frac{1}{(|k_{\pm}|r)^{(D-1)/2}} \left[A_{\pm} e^{i|k_{\pm}|r} + B_{\pm} e^{-i|k_{\pm}|r} \right]. \quad (2.3.17)$$

This convention is algebraically convenient because the radial equation depends only on $k_{\pm}^2$ and the Bessel-function identities can be written with positive arguments. However, the labels $A_{\pm}$ and $B_{\pm}$ do not by themselves say whether a wave is incoming or outgoing, because that assignment depends on the sign of the physical frequency $\omega_{\pm}$. For the scattering interpretation it is therefore useful to keep the signed momentum

$$k_{\pm} \equiv \text{sgn}(\omega_{\pm}) \sqrt{\omega_{\pm}^2 - m_{\Phi}^2}, \quad (2.3.18)$$

and write the same asymptotic solution as

$$\eta_{\pm}(r \rightarrow \infty) \sim \frac{1}{(|k_{\pm}|r)^{(D-1)/2}} \left[A_{\pm}^{\text{out}} e^{ik_{\pm}r} + A_{\pm}^{\text{in}} e^{-ik_{\pm}r} \right]. \quad (2.3.19)$$

The conversion between the two bases is then

$$\begin{cases} \omega_{\pm} > 0 : & A_{\pm}^{\text{out}} = A_{\pm}, & A_{\pm}^{\text{in}} = B_{\pm}, \\ \omega_{\pm} < 0 : & A_{\pm}^{\text{out}} = B_{\pm}, & A_{\pm}^{\text{in}} = A_{\pm}. \end{cases} \quad (2.3.20)$$

Equivalently, for $\omega_Q > 0$ and in the two-propagating-channel regime, the assignments are

$$\begin{cases} \omega > 0 : & A_+^{\text{out}} = A_+, & A_+^{\text{in}} = B_+, & A_-^{\text{out}} = B_-, & A_-^{\text{in}} = A_-, \\ \omega < 0 : & A_+^{\text{out}} = B_+, & A_+^{\text{in}} = A_+, & A_-^{\text{out}} = A_-, & A_-^{\text{in}} = B_-. \end{cases} \quad (2.3.21)$$

The A, B notation will be useful for solving the radial boundary-value problem, while the $A^{\text{in,out}}$ notation is better suited for conservation laws, S-matrix relations, and the physical interpretation of incoming and outgoing wave packets. Note that in both amplitude bases, $\{A, B\}$ and $\{A^{\text{in,out}}\}$, we normalize the amplitudes so that they have the same dimension as the underlying scalar field.

2.3.2 Solving outgoing amplitudes based on incoming amplitudes

The observables corresponding to the perturbations discussed later are defined in terms of the asymptotic incoming and outgoing amplitudes. The practical scattering problem is therefore to start from regular perturbations near the Q-ball center, evolve them through the background profile, and read off the map between incoming and outgoing wave amplitudes at large radius. Since Eq. (2.3.7) is linear, this map can be constructed by directly integrating the coupled ODEs for a set of different initial conditions. To be explicit, recall that regularity at the origin fixes the local behavior of the perturbations to the form in Eq. (2.3.16). For each choice of the regular constants $C_{\pm}$, one can integrate the coupled radial system in Eq. (2.3.7) from a small radius to a large matching radius $r_{\infty} \gg R_Q$, where R_Q denotes an effective Q-ball radius. In the asymptotic region the two channels decouple, so each numerical solution can be fitted to the form

$$\eta_{\pm}(r) \simeq \frac{A_{\pm} e^{i|k_{\pm}|r} + B_{\pm} e^{-i|k_{\pm}|r}}{(|k_{\pm}|r)^{(D-1)/2}}. \quad (2.3.22)$$

One convenient way to extract $A_\pm$ and $B_\pm$ is to evaluate the numerical solution at two nearby asymptotic radii, for example r_∞ and αr_∞ with $\alpha = \mathcal{O}(1)$. This gives

$$\begin{aligned} \begin{pmatrix} \eta_\pm(C_\pm, r_\infty) \\ \eta_\pm(C_\pm, \alpha r_\infty) \end{pmatrix} &= \frac{1}{(|k_\pm| r_\infty)^{\frac{D-1}{2}}} \begin{pmatrix} e^{i|k_\pm| r_\infty} & e^{-i|k_\pm| r_\infty} \\ \alpha^{-\frac{D-1}{2}} e^{i\alpha|k_\pm| r_\infty} & \alpha^{-\frac{D-1}{2}} e^{-i\alpha|k_\pm| r_\infty} \end{pmatrix} \begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix} = M_\pm \begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix} \\ \Rightarrow \begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix} &= M_\pm^{-1} \begin{pmatrix} \eta_\pm(C_\pm, r_\infty) \\ \eta_\pm(C_\pm, \alpha r_\infty) \end{pmatrix} \end{aligned} \quad (2.3.23)$$

for each channel. After this extraction, the conversion rules defined previously turn the $A_\pm, B_\pm$ basis into the physical incoming/outgoing basis.

To construct a scattering solution with a prescribed incoming wave, one uses linearity. If (η_+, η_-) is a solution, then $(\xi\eta_+, \xi^*\eta_-)$ is also a solution for any complex constant ξ . We may therefore integrate Eq. (2.3.7) twice, obtaining two independent sets of asymptotic and central data,

$$(A_\pm^{(1)}, B_\pm^{(1)}, C_\pm^{(1)}), \quad (A_\pm^{(2)}, B_\pm^{(2)}, C_\pm^{(2)}), \quad (2.3.24)$$

and then take an appropriate linear combination that removes the undesired incoming component.

2.3.3 Particle number conservation

Recall that the equation of motion for perturbations, Eq. (2.3.7), is similar to a system of coupled wave equations in quantum mechanics. One therefore expects to find a similar description of *particle number conservation* before and after scattering processes. Indeed, it can be shown that these equations of motion can be obtained from the following Lagrangian:

$$\mathcal{L}_\eta = r^{D-1} \sum_{s=\pm} \left[|\partial_r \eta_s|^2 - (\omega_s^2 - V) |\eta_s|^2 \right] + r^{D-1} W(\eta_+^* \eta_-^* + \eta_+ \eta_-). \quad (2.3.25)$$

The Euler-Lagrange equation for η is found by setting $\delta \mathcal{L}_\eta = 0$ and treating r as the time-like variable. Let us note that $\mathcal{L}_\eta$ is invariant under the $U(1)$ symmetry $\eta_\pm \rightarrow e^{\pm i\alpha} \eta_\pm$, with a real constant α . This symmetry is different from the global $U(1)$ used to define Q , as it acts only on the perturbations while keeping the background field fixed. The associated Noether charge is

$$\begin{aligned} J_N^\eta &= ir^{D-1} (\eta_+ \partial_r \eta_+^* - \eta_+^* \partial_r \eta_+) - ir^{D-1} (\eta_- \partial_r \eta_-^* - \eta_-^* \partial_r \eta_-) \\ &= 2r^{D-1} \text{Im} (\eta_+^* \partial_r \eta_+ - \eta_-^* \partial_r \eta_-), \end{aligned} \quad (2.3.26)$$

which satisfies $\partial_r J_N^\eta = 0$, meaning J_N^η is independent of r .⁽¹⁾ At large r , the Noether charge is expressed in terms of the asymptotic amplitudes as $J_N^\eta \propto \sum_{s=\pm} (N_s^{\text{out}} - N_s^{\text{in}})$. At $r \rightarrow 0$, instead, we have $J_N^\eta = 0$. This implies the **conservation of particle number** for generic scattering between incoming and outgoing waves,

$$N_{\text{tot}} = \sum_{s=\pm} N_s^{\text{in}} = \sum_{s=\pm} N_s^{\text{out}}, \quad (2.3.28)$$

where we define the particle numbers

$$N_\pm^{\text{in}} = \frac{|A_\pm^{\text{in}}|^2}{|k_\pm|^{D-2}}, \quad N_\pm^{\text{out}} = \frac{|A_\pm^{\text{out}}|^2}{|k_\pm|^{D-2}}. \quad (2.3.29)$$

This relation is useful for simplifying the expressions for observables and for deriving bounds on amplification factors, as we will see later. It also makes explicit that energy and charge exchange during a scattering process is not due to a change in the total number of scattering particles, but instead to a change in the fraction of particles in different modes, which carry different amounts of charge/energy.

⁽¹⁾ A somewhat more general formulation of this conservation law is to write the Noether flux as

$$\mathbf{J}_N^\eta \equiv 2 \text{Im} (\eta_+^* \nabla \eta_+ - \eta_-^* \nabla \eta_-), \quad (2.3.27)$$

with divergence $\nabla \cdot \mathbf{J}_N^\eta = 0$, i.e., the corresponding flux through any closed surface must be equal to zero. As we will later see, this is an analogue of the conservation of particle number.

2.3.4 Amplification factors

To systematically measure the amount of charge/energy exchanged between the scattering modes and the background, we now turn to the concept of *amplification factors*. It is important first to discuss how we *measure* the change in the physical quantities, and then to show how such a change can be quantified in terms of the incoming/outgoing amplitudes.

Denote the average over a sufficiently long time and over all directions in space of a quantity J as $\langle J \rangle_{T\Omega} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \frac{1}{\Omega_D} \int_{S^D} d\Omega J$. The fluxes are then

- The asymptotic form of the averaged flux of $U(1)$ charge Q is

$$\mathcal{F}_Q^\infty = \lim_{r \rightarrow \infty} r^{D-1} \langle J_Q^r \rangle_{T\Omega} \approx 2s_\omega \sum_{s=\pm} \frac{s}{|k_s|^{D-2}} (|A_s|^2 - |B_s|^2). \quad (2.3.30)$$

- The asymptotic form of the averaged flux of energy is

$$\mathcal{F}_E^\infty = \lim_{r \rightarrow \infty} r^{D-1} \langle T_{rt} \rangle_{T\Omega} \approx 2s_\omega \sum_{s=\pm} \frac{s\omega_s}{|k_s|^{D-2}} (|A_s|^2 - |B_s|^2) \quad (2.3.31)$$

Another way to consider the amount of conserved charge being carried away to infinity is to account for all the scalar charge in an asymptotic spherical shell with interior radius r_1 and exterior radius r_2 (see, e.g., [2]):

- $U(1)$ charge

$$Q_\odot = \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} dr r^{D-1} \langle J_Q^t \rangle_{T\Omega} \xrightarrow{r_{1,2} \rightarrow \infty} -2 \sum_{s=\pm} \frac{\omega_s}{|k_s|^{D-2}} (|A_s|^2 - |B_s|^2). \quad (2.3.32)$$

- Energy

$$E_\odot = \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} dr r^{D-1} \langle T_{tt} \rangle_{T\Omega} \xrightarrow{r_{1,2} \rightarrow \infty} 2 \sum_{s=\pm} \frac{\omega_s^2}{|k_s|^{D-1}} (|A_s|^2 + |B_s|^2). \quad (2.3.33)$$

Based on these two ways to measure emitted $U(1)$ charge and energy, there are two ways to define the amplification factor. Note that flipping the sign of ω also flips the role of the amplitudes corresponding to incoming and outgoing waves, as depicted in Fig. 2.1; $s_\omega \equiv \text{sgn } \omega$ enters the following definitions of amplification factors through this identification.

- Amplification factor as the ratio of the fluxes of incoming and outgoing fields far away ([3])

$$\mathcal{A}_Q^\mathcal{F} = \mathcal{A}_Q^\rho \equiv \mathcal{A}_Q, \quad (2.3.34)$$

$$\mathcal{A}_E^\mathcal{F} \equiv 1 + Z_E^\mathcal{F} = \left| \frac{\sum_{s=\pm} P_s N_s^{\text{out}}}{\sum_{s=\pm} P_s N_s^{\text{in}}} \right| = \left| \frac{\sum_{s=\pm} \frac{s\omega_s |A_s^{\text{out}}|^2}{|k_s|^{D-2}}}{\sum_{s=\pm} \frac{s\omega_s |A_s^{\text{in}}|^2}{|k_s|^{D-2}}} \right|, \quad (2.3.35)$$

where we interpret $P_\pm \equiv \pm\omega_\pm$ as the “energy flux per particle”.

- Amplification factor as the ratio of the charge contained within a spherical shell far away ([2])

$$\mathcal{A}_Q^\rho \equiv 1 + Z_Q^\rho = \left| \frac{\sum_{s=\pm} s N_s^{\text{out}}}{\sum_{s=\pm} s N_s^{\text{in}}} \right| = \left| \frac{\sum_{s=\pm} \frac{s |A_s^{\text{out}}|^2}{|k_s|^{D-2}}}{\sum_{s=\pm} \frac{s |A_s^{\text{in}}|^2}{|k_s|^{D-2}}} \right|, \quad (2.3.36)$$

$$\mathcal{A}_E^\rho \equiv 1 + Z_E^\rho = \frac{\left| \sum_{s=\pm} E_s N_s^{\text{out}} \right|}{\left| \sum_{s=\pm} E_s N_s^{\text{in}} \right|} = \frac{\left| \sum_{s=\pm} \frac{\omega_s^2 |A_s^{\text{out}}|^2}{|k_s|^{D-1}} \right|}{\left| \sum_{s=\pm} \frac{\omega_s^2 |A_s^{\text{in}}|^2}{|k_s|^{D-1}} \right|}, \quad (2.3.37)$$

where we interpret $E_\pm \equiv \frac{\omega_\pm^2}{|k_\pm|}$ as the “energy of each particle”.

The amplification factors defined above are appropriate for the spherically symmetric backgrounds considered in this chapter. If the background itself carries angular momentum, as in spinning Q-balls, one can also define angular-momentum amplification factors by comparing the angular-momentum fluxes of incoming and outgoing modes. This is a separate generalization, discussed for rotating Q-ball backgrounds for example in Refs. [12, 14], and will not be pursued in this work.

Before proceeding, it is useful to clarify why nonzero charge or energy amplification does not contradict the particle-number conservation in Eq. (2.3.28). The conserved quantity N_{tot} is the norm associated with the linearized radial system on a fixed Q-ball background. It is a Wronskian-like conservation law following from the perturbation Lagrangian (2.3.25), not the full Noether charge of the original scalar field. Therefore the equality

$$\sum_{s=\pm} N_s^{\text{in}} = \sum_{s=\pm} N_s^{\text{out}} \quad (2.3.38)$$

only states that the total number of linear scattering quanta, with the normalization in Eq. (2.3.28), is conserved. It does not require the outgoing state to carry the same $U(1)$ charge or the same energy as the incoming state, because the two channels have different weights in the physical fluxes, as shown explicitly in Eqs. (2.3.36)–(2.3.35).

2.3.5 Scattering at linear order via S-matrix formalism

The previous subsections described the scattering problem directly in terms of incoming and outgoing amplitudes. This is the most transparent language for defining physical fluxes and amplification factors, but it becomes cumbersome once one wants to compare different choices of incoming perturbations at the same frequency. For example, a pure incoming η_+ wave, a pure incoming η_-^* wave, and a coherent superposition of the two are all different scattering experiments, even though they are governed by the same linear radial equations on the same Q-ball background. Solving each case separately would obscure the fact that the background response at fixed ω is already encoded in one linear map between the full incoming-amplitude vector and the full outgoing-amplitude vector. The purpose of the S-matrix formulation is to make this linear map explicit. Once the S-matrix is known for a given Q-ball profile and frequency, all outgoing amplitudes follow from any chosen incoming state by matrix multiplication. This gives a compact way to rewrite the charge and energy amplification factors introduced above, including cases with arbitrary mixtures and relative phases of the two incoming modes.

This scattering-matrix language is also useful conceptually. As we will see, the particle-number conservation derived in Eq. (2.3.28) becomes a unitarity condition on the S-matrix. Furthermore, reality and discrete symmetry properties of the linearized equations become algebraic constraints on its entries. These constraints explain several symmetry patterns seen later in the amplification spectra, and they provide useful checks on numerical solutions. Below, we first construct the S-matrix in a general multi-field notation, then specialize it to the single complex scalar model where the two relevant modes are η_+ and η_-^* , showing how to write the particle number and amplification factors in terms of S-matrix elements.

Construction and general properties of the scattering matrix

We first note that the linearized solution around a one-field Q-ball can be generalized to many-field Q-balls, where the system of linear equations of motion describing the scattering of quanta off a

Q-ball has the following form:

$$\left[\nabla^2 + O(r) \right] \vec{\eta}(r) = 0, \quad (2.3.39)$$

where we have introduced the notation $\vec{\eta} \equiv (\eta_+, \eta_-^*)$, and O is a matrix that becomes diagonal at large distances:

$$\lim_{r \rightarrow \infty} O(r) \rightarrow K^2, \quad K \equiv \text{diag}(k_1, k_2, \dots) \quad (2.3.40)$$

where k_i are the momenta of the incoming and outgoing modes. At large distances, the solutions are

$$\vec{\eta}(r \rightarrow \infty) = \frac{1}{(|K|r)^{(D-1)/2}} \left[e^{iKr} \vec{A}^{\text{out}} + e^{-iKr} \vec{A}^{\text{in}} \right]. \quad (2.3.41)$$

Since Eq. (2.3.39) is linear in the perturbations, the incoming and outgoing amplitudes $\vec{A}$ are related by a matrix S , the analogue of the scattering S-matrix.

$$|K|^{(2-D)/2} \vec{A}^{\text{out}} = S |K|^{(2-D)/2} \vec{A}^{\text{in}}. \quad (2.3.42)$$

In general, this matrix S depends on the Q-ball potential, its charge, and the frequency of the incoming mode. It must also satisfy the following properties:

- *The S-matrix is unitary:* this follows from the particle number conservation discussed in Subsection. 2.3.3: $\left\| |K|^{(2-D)/2} \vec{A}^{\text{out}} \right\|^2 = \left\| |K|^{(2-D)/2} \vec{A}^{\text{in}} \right\|^2$, which implies that

$$S^\dagger S - \mathbb{1} = 0 \implies S^{-1} = S^\dagger \quad (2.3.43)$$

- *The S-matrix is symmetric:* this follows from the reality of the function $O(r)$. We immediately see that if $\vec{\eta}(r)$ is a solution of the equations (2.3.39) then $\vec{\eta}^*(r)$ will be a solution as well.

$$\begin{aligned} \vec{\eta}^*(r \rightarrow \infty) &= \frac{1}{(|K|r)^{(D-1)/2}} \left[e^{-i|K|r} \vec{A}^{\text{out}*} + e^{i|K|r} \vec{A}^{\text{in}*} \right] \\ \implies |K|^{(2-D)/2} \vec{A}^{\text{in}*} &= S |K|^{(2-D)/2} \vec{A}^{\text{out}*}, \end{aligned} \quad (2.3.44)$$

where in the second line we used the fact that the S matrix connects amplitudes with negative and positive exponents. Using Eq. (2.3.42), we obtain

$$S S^* = \mathbb{1} \implies S^T = S \quad (2.3.45)$$

The unitarity and symmetry of the S-matrix constrain it to have only $n(n+1)/2$ independent real parameters when $\vec{\eta}$ is an n -dimensional vector.

- *S-matrix in the limit of bound modes:* Let ω_j^{bound} be the frequency where the corresponding wavenumber $k_j = 0$, i.e., the mode j becomes bound. Assuming the solution of the bound mode near the Q-ball is still sufficiently small for linear perturbation theory to remain valid, this solution decays exponentially in r and becomes negligible sufficiently far away. Then particle number conservation implies that there is no conversion between the propagating modes and the bound modes. Consequently, we should obtain the following limits:

$$\left| S^{fi} \left(\omega \rightarrow \omega_i^{\text{bound}} \right) \right| = \left| S^{fi} \left(\omega \rightarrow \omega_f^{\text{bound}} \right) \right| = \delta_{fi}. \quad (2.3.46)$$

In particular, in the one-field model with $\vec{\eta} = (\eta_+, \eta_-^*)$, $\omega_-^{\text{bound}} = m_\Phi + \omega_Q$ and $\omega_+^{\text{bound}} = -m_\Phi - \omega_Q$, we have

$$\left| S \left(\omega \rightarrow \omega_-^{\text{bound}} \right) \right| = \left| S \left(\omega \rightarrow \omega_+^{\text{bound}} \right) \right| = \mathbb{1}_{2 \times 2}, \quad (2.3.47)$$

or equivalently, close to the bound modes, the S-matrix becomes a diagonal matrix of phases. This property is verified numerically.

- *S-matrix in the high-frequency limit:* In the linearized equations, if $|\omega|$ is sufficiently large, the matrix $O(r)$ is approximately diagonal for all r , meaning that all modes decouple and we restore a $U(1)^n$ global symmetry in which each $U(1)$ corresponds to a phase rotation of one mode. We then have n approximately conserved currents, one for each mode, implying $|A_i^{\text{out}}| = |A_i^{\text{in}}|$ for $i = 1, \dots, n$, and hence the S-matrix reduces to a diagonal matrix of phases, or equivalently

$$\lim_{\omega \rightarrow \pm\infty} |S(\omega)| = \mathbb{1}. \quad (2.3.48)$$

This fact has been observed numerically for both one-field and two-field models, which will be discussed later in Chapter. 3 and Chapter. 4.

Application of the S-matrix in single complex scalar field model

In the case of a single complex scalar field model, $\vec{A}^{\text{in,out}} \equiv (A_+, A_-^*)^{\text{in,out}}$. Apart from the properties inherited from the generic construction of the S-matrix for n -field scattering problems discussed above, there are a number of additional observations about the S-matrix in this case:

- *Symmetry of S* implies

$$S^{+-}(\omega) = S^{-+}(\omega). \quad (2.3.49)$$

- *Transformation of the S-matrix under $\mathbb{Z}_2$:* under $\omega \rightarrow -\omega$ we have $\eta_{\pm} \rightarrow \eta_{\mp}^*$ up to a complex scaling factor and a complex conjugation. This imposes the following constraints that are independent of unitarity and symmetry:

$$|S^{++}(-\omega)| = |S^{--}(\omega)|, \quad |S^{+-}(-\omega)| = |S^{-+}(\omega)|, \quad (2.3.50)$$

All of these properties of the S-matrix can be translated into constraints on the incoming-outgoing amplitudes of different modes, for example

$$\left| \frac{k_-}{k_{\pm}} \right|^{D-2} \left| \frac{A_{\pm}^{\text{out}}(\omega)}{A_{\mp}^{\text{out}}(\omega)} \right|_{A_{\mp}^{\text{in}}=0}^2 = \left| \frac{k_+}{k_{\mp}} \right|^{D-2} \left| \frac{A_{\mp}^{\text{out}}(\omega)}{A_{+}^{\text{out}}(\omega)} \right|_{A_{+}^{\text{in}}=0}^2, \quad (2.3.51)$$

or equivalently, in terms of particle number:

$$\boxed{N_{\pm}^{\text{out}}(\omega) \Big|_{N_{\mp}^{\text{in}}=0} = N_{\mp}^{\text{out}}(\omega) \Big|_{N_{+}^{\text{in}}=0}}. \quad (2.3.52)$$

Using the S-matrix language, one can express the outgoing particle number in terms of the S-matrix elements

$$N_i^{\text{in}} = \frac{|A_i^{\text{in}}|^2}{|k_i|^{D-2}} = |\alpha_i|^2 \left| \sum_j \frac{A_j^{\text{in}}}{|k_j|^{(D-2)/2}} \right|^2, \quad (2.3.53)$$

$$N_i^{\text{out}} = \frac{|A_i^{\text{out}}|^2}{|k_i|^{D-2}} = \left| \sum_j S^{ij} \frac{A_j^{\text{in}}}{|k_j|^{(D-2)/2}} \right|^2 = \left| \sum_j S^{ij} \alpha_j \right|^2 \left| \sum_j \frac{A_j^{\text{in}}}{|k_j|^{(D-2)/2}} \right|^2. \quad (2.3.54)$$

Here we define $\alpha_i \equiv \frac{|k_i|^{(2-D)/2} A_i^{\text{in}}}{\sum_j |k_j|^{(2-D)/2} A_j^{\text{in}}}$ as the “amplitude fraction” of mode i , assuming $\sum_j |k_j|^{(2-D)/2} A_j^{\text{in}} \neq 0$. The special case of sending a monochromatic incoming perturbation of mode i corresponds to $\alpha_i = 1$ and $\alpha_{j \neq i} = 0$. Hence, one can interpret $|S^{fi}|^2$ as the fraction of outgoing particle number in mode f given a pure monochromatic incoming mode i .

Note that in computing amplification factors, we always divide a linear combination of N^{out} by another combination of N^{in} , so the factor $\sum_j |k_j|^{(2-D)/2} A_j^{\text{in}}$ disappears. Consequently, the amplification factors defined in Eqs. (2.3.36), (2.3.37), and (2.3.35) can be rewritten in terms of α_i and S^{ij} as follows:

$$\mathcal{A}_Q = \left| \frac{\sum_{s=\pm} s |\sum_{i=\pm} \alpha_i S^{si}|^2}{\sum_{s=\pm} s |\alpha_s|^2} \right| = \left| \frac{|\sum_{i=\pm} \alpha_i S^{+i}|^2 - |\sum_{i=\pm} \alpha_i S^{-i}|^2}{|\alpha_+|^2 - |\alpha_-|^2} \right|, \quad (2.3.55a)$$

$$\mathcal{A}_E^\rho = \left| \frac{\sum_{s=\pm} E_s |\sum_{i=\pm} \alpha_i S^{si}|^2}{\sum_{s=\pm} E_s |\alpha_s|^2} \right| = \left| \frac{\frac{\omega_+^2}{|k_+|} |\sum_{i=\pm} \alpha_i S^{+i}|^2 + \frac{\omega_-^2}{|k_-|} |\sum_{i=\pm} \alpha_i S^{-i}|^2}{\frac{\omega_+^2}{|k_+|} |\alpha_+|^2 + \frac{\omega_-^2}{|k_-|} |\alpha_-|^2} \right|, \quad (2.3.55b)$$

$$\mathcal{A}_E^{\mathcal{F}} = \left| \frac{\sum_{s=\pm} P_s |\sum_{i=\pm} \alpha_i S^{si}|^2}{\sum_{s=\pm} P_s |\alpha_s|^2} \right| = \left| \frac{\omega_+ |\sum_{i=\pm} \alpha_i S^{+i}|^2 - \omega_- |\sum_{i=\pm} \alpha_i S^{-i}|^2}{\omega_+ |\alpha_+|^2 - \omega_- |\alpha_-|^2} \right|, \quad (2.3.55c)$$

where again $E_\pm \equiv \omega_\pm^2 / |k_\pm|$ and $P_\pm \equiv \pm \omega_\pm$ are the energy and energy flux per particle.

In the case of exactly one monochromatic incoming wave, $i = +$ or $i = -$, we have $\alpha_j = \delta_{ji}$, and the expression above reduces to

$$\mathcal{A}_Q = \left| |S^{+i}|^2 - |S^{-i}|^2 \right|, \quad (2.3.56a)$$

$$\mathcal{A}_E^\rho = \left| \frac{E_+}{E_i} |S^{+i}|^2 + \frac{E_-}{E_i} |S^{-i}|^2 \right|, \quad (2.3.56b)$$

$$\mathcal{A}_E^{\mathcal{F}} = \left| \frac{P_+}{P_i} |S^{+i}|^2 + \frac{P_-}{P_i} |S^{-i}|^2 \right|. \quad (2.3.56c)$$

Interestingly, for the one-complex-field case, S is a 2×2 matrix with only 3 independent components. It can be fixed completely by solving Eq. (2.3.7) for only one set of initial conditions at the origin ($r \rightarrow 0$). Indeed in this case we will need to solve the system of equations

$$|K|^{(2-D)/2} \vec{A}^{\text{out}} = S_{2 \times 2} |K|^{(2-D)/2} \vec{A}^{\text{in}} \quad (2.3.57)$$

to find all of the entries of S . Naively, it looks like there are four independent equations since $\vec{A}^{\text{in,out}}$ are complex vectors. However, one equation is redundant due to the unitarity of the S-matrix, which reduces the degrees of freedom of the S-matrix to 3.

In practice, for a one-field complex scalar model, using the S-matrix approach is more flexible for computing the amplification factors than tuning the initial condition to get the desired incoming modes. This is because once the S-matrix is found for a given ω , we can find the amplification factors for all combinations of incoming modes without having to tune the initial condition again. Instead, we simply compute the amplitude fraction α_i and plug it directly into the expression for $\mathcal{A}$ in terms of the S-matrix elements (see Eq. (2.3.55)).

Q-ball superradiance in a concrete model of a single scalar field

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3.1 Sextic potential and benchmark Q-ball backgrounds

After the general construction in the previous chapter, we now specialize to the sextic one-field model used for the explicit background profiles and scattering calculations in this chapter. The dynamics of the complex scalar field is governed by the action

$$S[\tilde{\Phi}] = - \int d^{D+1}\tilde{x} \left[|\tilde{\partial}_\mu \tilde{\Phi}|^2 + \tilde{\mathbf{U}} \right], \quad \tilde{\mathbf{U}} = \tilde{\mu}^2 |\tilde{\Phi}|^2 - \tilde{\lambda} |\tilde{\Phi}|^4 + \tilde{g} |\tilde{\Phi}|^6. \quad (3.1.1)$$

The dimensionless variables used below are obtained by measuring distances in units of $\tilde{\mu}^{-1}$ and field amplitudes in units of $\tilde{\mu}/\sqrt{\tilde{\lambda}}$. Equivalently, we define

$$x^\mu \equiv \tilde{\mu} \tilde{x}^\mu, \quad \Phi \equiv \frac{\sqrt{\tilde{\lambda}}}{\tilde{\mu}} \tilde{\Phi}, \quad g \equiv \frac{\tilde{g} \tilde{\mu}^2}{\tilde{\lambda}^2}, \quad (3.1.2)$$

where g without a tilde denotes the dimensionless sextic coupling from now on. In these variables the action takes the form

$$S[\Phi] = -\mathcal{N}_D \int d^{D+1}x \left[|\partial_\mu \Phi|^2 + \mathbf{U} \right], \quad \mathcal{N}_D \equiv \frac{\tilde{\mu}^{3-D}}{\tilde{\lambda}}, \quad \mathbf{U} = |\Phi|^2 - |\Phi|^4 + g |\Phi|^6. \quad (3.1.3)$$

In the dimensionless convention the elementary scalar mass is normalized to unity, $m_\Phi^2 = U''(0) = 1$. The general existence criteria derived in Eqs. (1.2.23) and (1.2.24) then give the upper bound $|\omega_Q| < \omega_Q^{(+)} = 1$. The lower bound follows from

$$\left(\omega_Q^{(-)}\right)^2 = \min_{\phi \neq 0} \frac{2U(\phi)}{\phi^2}. \quad (3.1.4)$$

Using $U(\phi) = \mathbf{U}(\Phi = \phi/\sqrt{2}) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{g}{8}\phi^6$, we find

$$\frac{2U(\phi)}{\phi^2} = 1 - \frac{1}{2}\phi^2 + \frac{g}{4}\phi^4, \quad \phi^2 = \frac{1}{g} \Rightarrow \left(\omega_Q^{(-)}\right)^2 = 1 - \frac{1}{4g}. \quad (3.1.5)$$

Therefore, this model admits Q-ball solutions within the frequency interval

$$\sqrt{1 - \frac{1}{4g}} \lesssim |\omega_Q| < 1, \quad g > \frac{1}{4}. \quad (3.1.6)$$

Here the lower limit is the ideal thin-wall limit; finite-charge solutions lie slightly above it. The condition $g > 1/4$ is also the condition that the minimum of $2U(\phi)/\phi^2$ be positive. As $g \rightarrow 1/4$ from above, the lower limit tends to zero and the two minima of the limit effective potential become increasingly degenerate in the thin-wall sense. The corresponding Q-ball branch can then extend toward very small internal frequency, and the thin-wall profiles become increasingly broad, with the charge and radius growing as ω_Q approaches 0. Most of the time we will present results corresponding to $g = 1/3$, unless stated otherwise. The effective potential is visualized in Fig. 3.1.

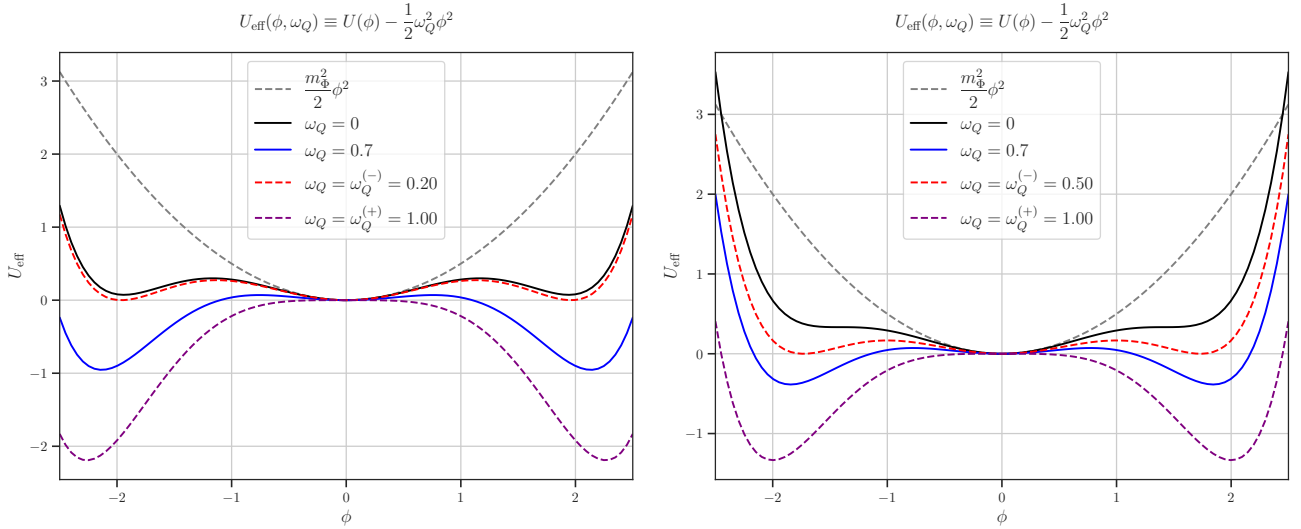


Figure 3.1: Visualization of the effective potential in solving the profile, with $g = 0.26$ (left plot) and $g = 1/3$ (right plot). The interval $\omega_Q \in [\omega_Q^{(-)}, \omega_Q^{(+)})$ corresponds to the black dashed lines. For a Q-ball solution to exist, the potential with varying ω_Q must lie between the dashed red and dashed purple lines, which correspond to $\omega_Q = \omega_Q^{(-)}$ in the thin-wall limit and $\omega_Q = \omega_Q^{(+)}$ in the thick-wall limit.

3.1.1 Solution to Q-ball profiles

Matching analytical and numerical solutions at large radius

We solve Eq. (2.1.5) for the sextic potential using the shooting method described in Subsection. 1.2.2. Note that in setting up the Q-ball background for scattering with perturbations, we need to know the profile at sufficiently large distances from the Q-ball core, and outside the core, the field

value drops exponentially. Therefore, it is more efficient to numerically solve the profile within a sufficiently large radial range $r \in [0, R]$, and then match the tail of the numerical solution with the analytical solution to obtain the field profile of the far-zone Q-ball. Recall the equation of motion, Eq. (2.1.5); in the asymptotic region the field profile is small, so the effective potential can be linearized around the vacuum,

$$\frac{dU_{\text{eff}}}{d\phi_Q} = U_{\text{eff}}''(0)\phi_Q + \mathcal{O}(\phi_Q^2), \quad \kappa_Q^2 \equiv U_{\text{eff}}''(0) = U''(0) - \omega_Q^2. \quad (3.1.7)$$

The condition $\kappa_Q^2 > 0$ is the usual requirement that the Q-ball frequency lies below the elementary scalar mass in the vacuum. Defining $\delta \equiv (D-2)/2$, the linearized equation following from Eq. (2.1.5) is then

$$\left(\partial_r^2 + \frac{D-1}{r} \partial_r - \kappa_Q^2 \right) \phi_Q = 0. \quad (3.1.8)$$

Its two independent radial solutions are the modified Bessel functions $I_\delta(\kappa_Q r)$ and $K_\delta(\kappa_Q r)$. Since $I_\delta(\kappa_Q r)$ grows exponentially at infinity, finite energy selects the K_δ branch,

$$\phi_Q(r \rightarrow \infty) \sim r^{-\delta} K_\delta(\kappa_Q r) \sim \sqrt{\frac{\pi}{2\kappa_Q}} r^{(1-D)/2} e^{-\kappa_Q r}. \quad (3.1.9)$$

We then obtain the Yukawa tail that is used to match with the numerical profile of ϕ_Q at $r = R$ for some sufficiently large R .

Numerical Q-ball profiles

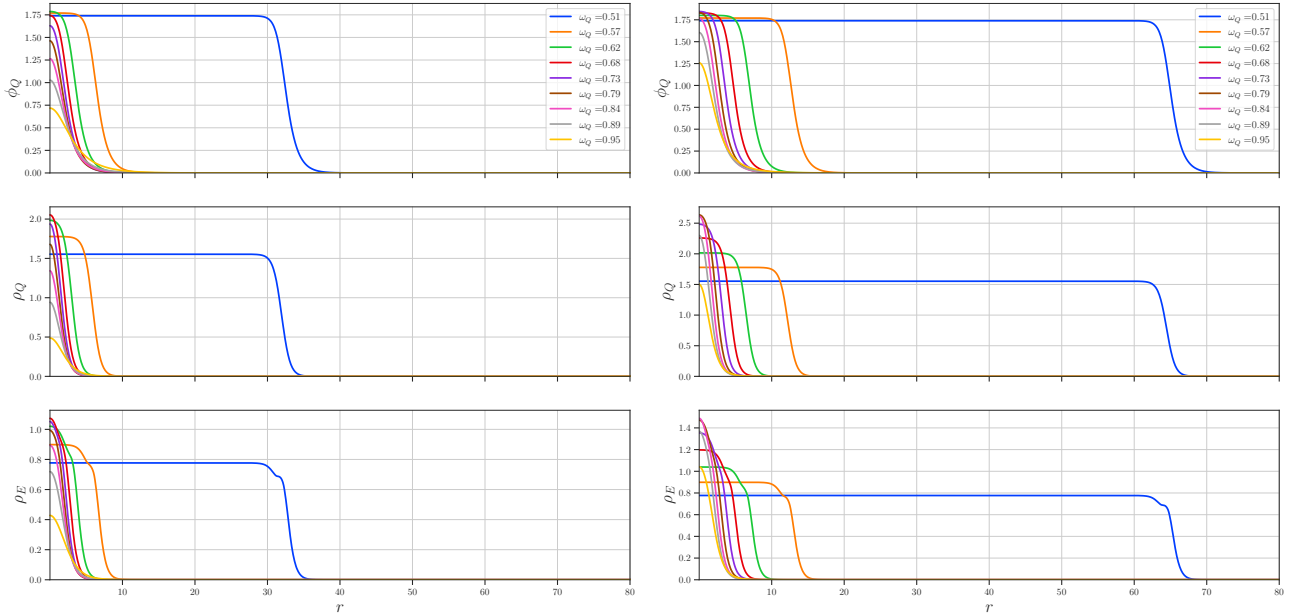


Figure 3.2: The Q-ball profiles with varying internal frequency ω_Q in $D = 2$ (left plots) and $D = 3$ (right plots) spatial dimensions, with self-coupling $g = 1/3$. We show the profiles in the top panel, the $U(1)$ charge density in the middle panel, and the energy density in the bottom panel.

The resulting profiles for $g = 1/3$ are shown in Fig. 3.2 for both $D = 2$ and $D = 3$. The qualitative behavior follows directly from the existence interval derived above. When ω_Q approaches the lower bound, the effective potential develops a nearly degenerate finite-field region, so the solution spends a long radial interval close to a homogeneous interior value before dropping to the vacuum. This is the thin-wall limit: the profile has a broad plateau, the charge density is almost constant inside the core, and the energy density is dominated by an extended volume contribution with an additional

localized surface feature near the wall. As ω_Q is increased toward the upper bound, the central field value tends to decrease and the profile no longer contains a large homogeneous core. The solution then becomes thick-wall-like, with charge and energy densities concentrated near the origin and decaying over a radius of order the inverse asymptotic mass scale.

It is insightful to have a comparison between Q-ball profiles in different spatial dimensions. For the same value of ω_Q , the $D = 3$ profiles are generally wider than the corresponding $D = 2$ profiles. In the mechanical analogy this happens because the radial equation contains the friction term $(D-1)\phi'_Q/r$. A larger spatial dimension increases the energy loss of the mechanical trajectory, so the correct shooting solution starts closer to the finite-field maximum of the inverted effective potential and remains near the core value for a longer radial interval before rolling to the vacuum. The density plots therefore make the dimensional dependence of the thin-wall approximation explicit: the same potential and internal frequency can correspond to a much larger total charge and energy in $D = 3$ than in $D = 2$ because of the larger volume factor. Regarding the stability tests for this specific sextic potential, we present some relevant numerical results in Appendix. A.2.

3.1.2 Q-ball perturbation and amplification factors

For the rest of the scattering analysis, we use the three different $D = 3$ backgrounds shown in Fig. 3.3, with $\omega_Q = 0.65, 0.70, 0.75$. The reported charges and energies also show that the profiles used below satisfy the absolute stability check against decay into free quanta.

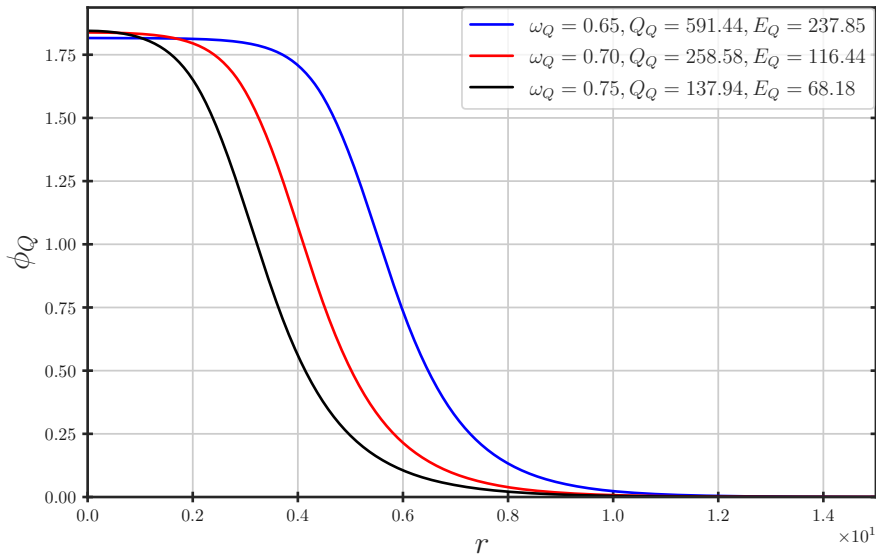


Figure 3.3: The Q-ball profiles with varying internal frequency in $D = 3$ spatial dimensions, with self-coupling $g = 1/3$. The total charge Q_Q and energy E_Q are reported to verify the stability of the Q-ball against decay into free particles, $E_Q < Q_Q m_\Phi$.

Let us proceed by considering the equation of motion of the linear perturbations around a Q-ball background, Eq. (2.3.4),

$$\partial_\mu \partial^\mu \Phi_1 - V(r)\Phi_1 - W(r)\Phi_1^* e^{-2i\omega_Q t} = 0, \quad (3.1.10)$$

where V and W for this specific sextic potential read

$$V = 1 - 2\phi_Q^2 + \frac{9g}{4}\phi_Q^4, \quad W = -\phi_Q^2 + \frac{3g}{2}\phi_Q^4. \quad (3.1.11)$$

Depending on the value of the ω parameter, we can classify these perturbations as follows ⁽¹⁾

⁽¹⁾ The only exceptions to this parametrization are the modes corresponding to the Lorentz boost and the Q-ball charge change [16, 17, 4].

- zero modes $\omega = 0$
- bound modes $|\omega_{\pm}| < 1$
- half-propagating modes $|\omega_{+}| > 1, |\omega_{-}| < 1$ or vice versa
- propagating modes $|\omega_{\pm}| > 1$

After this preliminary discussion, we can proceed to the discussion of charge and energy exchange between the Q-ball and its perturbations in this sextic model. Recall the expressions for the charge-flux and energy-flux amplification factors in Eq. (2.3.55) in $D = 3$ spatial dimensions:

$$\mathcal{A}_Q^{\mathcal{F}} \equiv 1 + Z_Q^{\mathcal{F}} = \frac{\left| \sum_{s=\pm} \frac{s |A_s^{\text{out}}|^2}{|k_s|} \right|}{\left| \sum_{s=\pm} \frac{s |A_s^{\text{in}}|^2}{|k_s|} \right|} = \left| \frac{\sum_{s=\pm} s \left| \sum_{i=\pm} \alpha_i S^{si} \right|^2}{\sum_{s=\pm} s |\alpha_s|^2} \right|, \quad (3.1.12a)$$

$$\mathcal{A}_E^{\mathcal{F}} \equiv 1 + Z_E^{\mathcal{F}} = \frac{\left| \sum_{s=\pm} \frac{s \omega_s |A_s^{\text{out}}|^2}{|k_s|} \right|}{\left| \sum_{s=\pm} \frac{s \omega_s |A_s^{\text{in}}|^2}{|k_s|} \right|} = \left| \frac{\sum_{s=\pm} s \omega_s \left| \sum_{i=\pm} \alpha_i S^{si} \right|^2}{\sum_{s=\pm} s \omega_s |\alpha_s|^2} \right|, \quad (3.1.12b)$$

with $\alpha_i \equiv \frac{|k_i|^{-1/2} A_i^{\text{in}}}{\sum_j |k_j|^{-1/2} A_j^{\text{in}}}$. Note that the amplification factor Z_E^{ρ} used in Ref. [2] is instead based on the charge or energy contained within an asymptotic spherical shell, rather than on the flux through that shell. In the later investigation of the nonlinear regime using lattice simulations, it is more straightforward to measure the flux, and hence to compute the flux amplification factors $Z^{\mathcal{F}}$ instead of Z^{ρ} . Therefore we will not present the results of Z^{ρ} here. Since Eq. (3.1.12) is the only definition used below, the superscript $\mathcal{F}$ will be dropped from now on when no confusion can arise. We can then consider the initial state where only one frequency mode is present (either $N_+^{\text{in}} = 0$ or $N_-^{\text{in}} = 0$). Then, using Eq. (2.3.28), we know that in the final states we will generically have both $\pm$ modes with $N_{\pm}^{\text{out}} \neq 0$, implying in principle that there is energy and charge exchange between the Q-ball and the perturbations.

The results for these quantities are reported in Fig. 3.4. Similarly to what was reported in the 2D case, energy and charge exchange can occur between the Q-ball and its perturbations.

The multiple peaks in Fig. 3.4 are not accidental numerical features. They reflect the fact that the perturbations scatter from an extended, frequency-dependent potential well generated by the Q-ball background. As in the quantum-mechanics problem of scattering from a finite potential well, the superposition between incoming and reflected waves yields interference patterns. At specific frequencies, the energy, and consequently the charge, is amplified, while at other nodes with destructive interference, the amplification factors approach zero. Equivalently, at some frequencies the interference enhances conversion between the η_+ and η_- channels, while at other frequencies it suppresses this conversion. Thinner and larger Q-balls have a larger effective scattering region, so more oscillatory structures can appear within the same frequency range. At sufficiently high frequency, the wave resolves the background only weakly, the off-diagonal mixing becomes perturbatively small, and the S-matrix approaches the identity. This finite-size interference interpretation is supported by the analytical thin-wall treatment of Ref. [19], where a single incoming wave scattering off a large Q-ball gives an amplification factor whose high-frequency dependence reduces to a base part plus a damped oscillatory correction. Schematically,

$$\mathcal{A} \sim 1 + \frac{S_2}{\omega^3} + \frac{S_1}{\omega^4} \cos[R_Q \omega + \text{phase}]. \quad (3.1.13)$$

This formula explains why the amplification spectra contain repeated peaks and troughs, with a characteristic spacing controlled by the inverse Q-ball size. The same analytical approach also gives a way to estimate physical upper bounds on the amplification factors in the large, thin-wall regime.

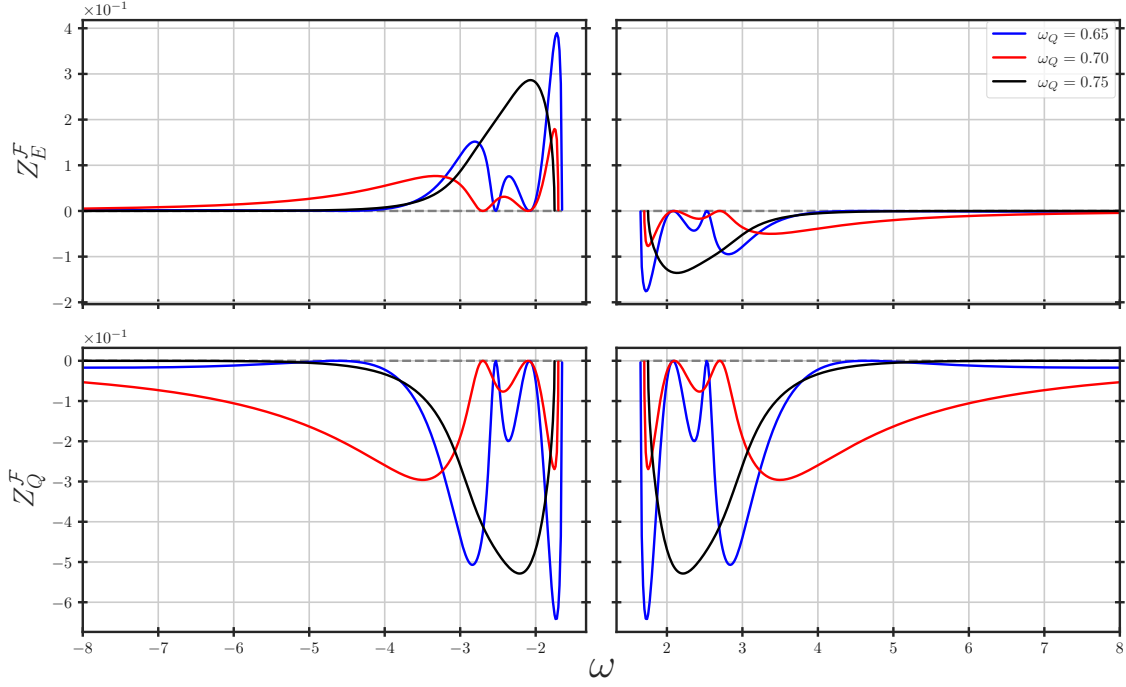


Figure 3.4: The relative amplification factors of the incoming mode η_+ in terms of energy (top figure) and charge (bottom figure) for the various Q-ball profiles shown in Fig. 3.3. We see that both $Z_{E,Q}$ tend to have more peaks for scattering off a thinner Q-ball, i.e., smaller ω_Q . The results for the incoming mode η_- are obtained by exploiting the symmetries in Eqs. (2.3.49) and (2.3.50).

3.1.3 Amplification factors in scattering of quanta

The interpretation becomes more intuitive once we quantize the perturbations and the quantities $N_{\pm}^{\text{in,out}}$ become the probabilities of finding the corresponding particles and antiparticles in the initial and final states. Then the current conservation in Eq. (2.3.28) means that the total number of particles and antiparticles must be conserved, and we will call $\mathbf{J}_\eta$ the particle-number current. Energy exchange with the Q-ball at the linear level can occur only through the following processes, where we use total $U(1)$ charge conservation:

$$\begin{aligned}
Q + \phi &\rightarrow Q + \phi \\
Q + \phi^\dagger &\rightarrow Q + \phi^\dagger \\
Q + \phi &\rightarrow (Q + 2) + \phi^\dagger \\
Q + \phi^\dagger &\rightarrow (Q - 2) + \phi,
\end{aligned} \tag{3.1.14}$$

where $(Q \pm 2)$ denotes Q-balls with the corresponding charges, and $\phi, \phi^\dagger$ are the quanta of the field Φ . The first two processes are just elastic scatterings, so they cannot extract or absorb charge or energy. To analyze the other two, let us assume that the Q-ball has positive charge $Q > 0$. In the process $Q + \phi \rightarrow (Q + 2) + \phi^\dagger$, its charge is increased, so its energy is increased as well; thus, energy is absorbed from the perturbation. Similarly, the process $Q + \phi^\dagger \rightarrow (Q - 2) + \phi$ leads to a decrease in the Q-ball charge and to energy extraction. From this, we immediately see that energy extraction happens only if the incoming particle has charge opposite to the Q-ball charge. As a sanity check, we can calculate the energy amplification from these considerations and compare it with Eq. (3.1.12):

$$\begin{aligned}
\tilde{Z}_E^{\mathcal{F}}(N_+^{\text{in}} = 0, N_-^{\text{in}} = 1) &= \frac{N_+^{\text{out}}}{|\omega_-|} [E(Q) - E(Q - 2)], \\
\tilde{Z}_E^{\mathcal{F}}(N_+^{\text{in}} = 1, N_-^{\text{in}} = 0) &= \frac{N_-^{\text{out}}}{\omega_+} [E(Q) - E(Q + 2)],
\end{aligned} \tag{3.1.15}$$

where with the normalization $N^{\text{in}} = 1$, the factor N_+^{out} takes into account the probability of the transition $Q + \phi^\dagger \rightarrow (Q - 2) + \phi$. Comparing with the expression in Eq. (3.1.12), we see that the

quantities indeed match approximately,

$$\begin{aligned} Z_E^{\mathcal{F}}(N_+^{\text{in}} = 0, N_-^{\text{in}} = 1) &= \frac{\omega_+ N_+^{\text{out}} - \omega_- N_-^{\text{out}}}{|\omega_-|} - 1 = \frac{2\omega_Q}{|\omega_-|} N_+^{\text{out}} \\ &= \frac{2(dE_Q/dQ)N_+^{\text{out}}}{|\omega_-|} \approx \frac{N_+^{\text{out}}}{|\omega_-|} [E(Q) - E(Q-2)], \end{aligned} \quad (3.1.16)$$

where we have used the relations (see, for example, [7, 10]):

$$\frac{dE(Q)}{dQ} = \omega_Q, \quad E(Q+2) - E(Q) = 2\omega_Q + 2\frac{d^2 E_Q}{dQ^2} + \dots \quad (3.1.17)$$

It is clear that the two expressions for energy extraction are equal up to the second and higher derivative terms,

$$Z_E^{\mathcal{F}}(N_+^{\text{in}} = 0, N_-^{\text{in}} = 1) \simeq \tilde{Z}_E^{\mathcal{F}}(N_+^{\text{in}} = 0, N_-^{\text{in}} = 1) \left(1 - \frac{2\frac{d^2 E}{dQ^2}}{E(Q) - E(Q-2)} + \dots \right). \quad (3.1.18)$$

For large values of the Q-ball charge, the second term scales as Q^{-1} and is strongly suppressed. Interestingly, its sign is always positive, since the classical stability of Q-balls dictates [20] that

$$\frac{d^2 E_Q}{dQ^2} = \frac{d\omega_Q}{dQ} < 0. \quad (3.1.19)$$

We illustrate these relations in Fig. 3.5, where we see clearly that the two results match and that the difference is proportional to $Z_E^{\mathcal{F}}$, as expected from Eq. (3.1.17).

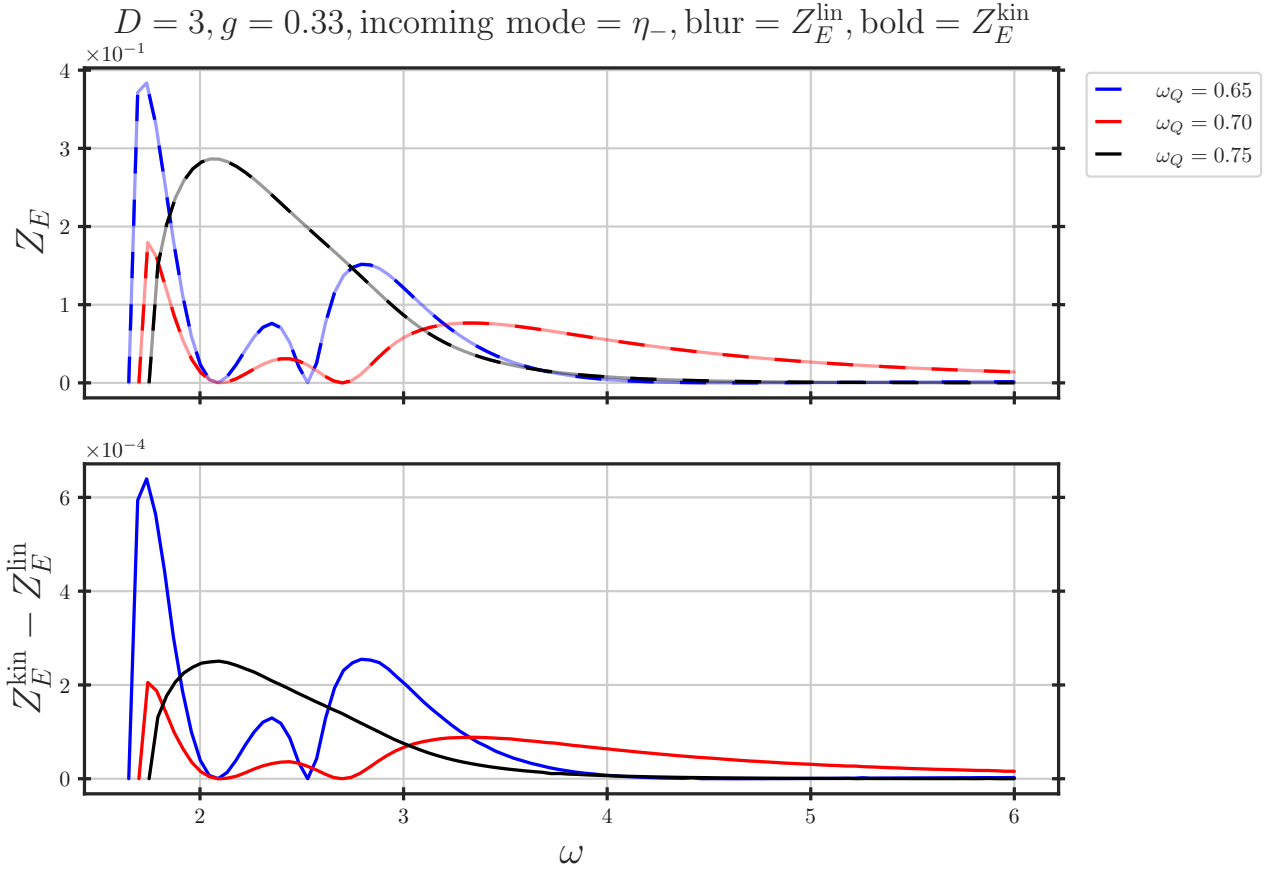


Figure 3.5: The kinematic and linear relative energy amplifications for an incoming $-$ mode (top figure) and their difference (bottom figure); here $Z_E^{\text{kin}} \equiv \tilde{Z}_E^{\mathcal{F}}$ and Z_E^{lin} is obtained using the linear perturbation regime. The shape of $Z_E^{\text{kin}} - Z_E^{\text{lin}}$ matches the shape of Z_E^{kin} for each value of ω_Q , as predicted by Eq. (3.1.18).

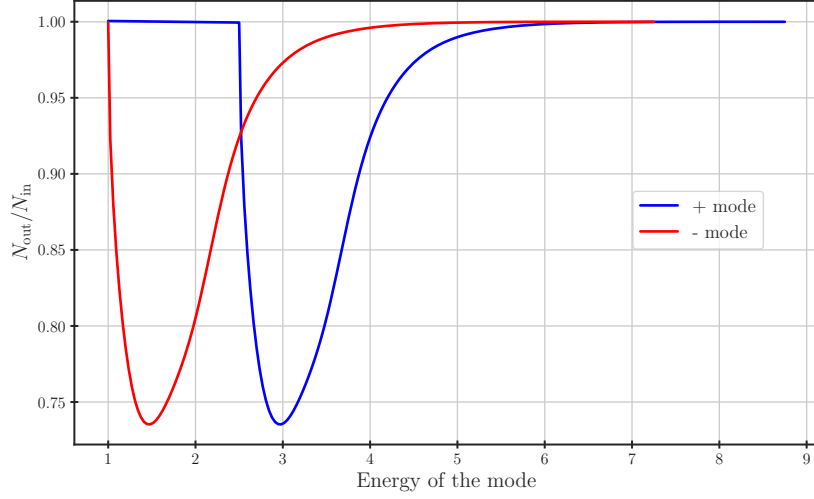


Figure 3.6: Reflection probabilities for the $\pm$ modes $N_{\pm}^{\text{out}}/N_{\pm}^{\text{in}}$, with $g = 1/3$ and $\omega_Q = 0.75$. There is always only one incoming mode, either $+$ or $-$.

On the wave-equation side, the energy extraction can be seen from the following. Let us assume that $\omega_Q > 0$ and $\omega > 0$; taking $\omega < 0$ would simply interchange $\eta_- \leftrightarrow \eta_+$. Then we always have

$$|\omega_+| > |\omega_-|. \quad (3.1.20)$$

Thus, the energy of the η_+ mode is always larger than the energy of the η_- mode. In agreement with Eq. (3.1.15), we therefore have energy extraction only when the η_- (antiparticle) is in the initial state.

Similarly, we can consider the scattering of half-bound modes. In this case, only η_+ can propagate, and from the conservation of the current $\mathbf{J}_\eta$ we immediately obtain

$$N_+^{\text{in}} = N_+^{\text{out}}, \quad (3.1.21)$$

so that the collision is always elastic and no energy or charge exchange with the Q-ball is possible.

3.2 Q-ball evolution in the early universe

The discussion of the linear perturbations and the selection rules for the interactions between the Q-ball and its perturbations leads to a very interesting conclusion. Suppose a Q-ball is formed in the early universe, and we are looking at its evolution due to interactions with the surrounding plasma. If the plasma particles are energetic enough, both of the reactions

$$Q + \phi \rightarrow (Q + 2) + \phi^\dagger \quad Q + \phi^\dagger \rightarrow (Q - 2) + \phi \quad (3.2.1)$$

are open, even though the rates are different (see Fig. 3.6). Note that the blue curve is obtained by shifting the red one by the factor $2\omega_Q$, which follows from the symmetry in Eq. (2.3.51). We can see that, at the classical level, once the energy of the η_+ mode becomes less than $\omega_+ < 1 + 2\omega_Q$, the corresponding η_- becomes bound, and classically we can have only the following processes:

$$\begin{aligned} E &< 1 + 2\omega_Q \\ Q + \phi^\dagger &\rightarrow (Q - 2) + \phi \quad \text{open} \\ Q + \phi &\rightarrow (Q + 2) + \phi^\dagger \quad \text{forbidden.} \end{aligned} \quad (3.2.2)$$

Therefore, in this energy range, Q-balls can only lose their charge due to interactions with the plasma. Note that a very similar conclusion can be obtained just by considering the energy-conservation conditions. Let us suppose that the ϕ particle has energy E_* . The condition is then

$$Q + \phi \rightarrow (Q + 2) + \phi^\dagger \text{ is forbidden if}$$

$$m_\Phi > E(Q) + E_* - E(Q + 2). \quad (3.2.3)$$

The minimum energy at which the inelastic reaction can occur is then

$$E_* = E(Q + 2) - E(Q) + m_\Phi = 1 + 2\omega_Q + 2\frac{d^2 E}{dQ^2} + \dots \quad (3.2.4)$$

Thus, up to higher derivative terms, this expression matches the expression in Eq. (3.2.2).

We remind the reader that these results have been obtained in the linear approximation and ignoring the quantum effects. The validity of the linear approximation will be discussed in the next section; however, for sufficiently small perturbations we expect these results, and in particular the conservation of the current J_N^η defined in Eq. (2.3.26), to be exact. On the other hand, quantum effects are absent in this treatment of perturbations. For example, if we add an interaction

$$\delta\mathcal{L} = \frac{1}{\Lambda_*^2} F_{\mu\nu}^2 |\phi|^2 \quad (3.2.5)$$

to our Lagrangian, the reaction

$$Q + \phi \rightarrow (Q + 1) + \gamma\gamma \quad (3.2.6)$$

will be opened; however, it will be suppressed compared to the “classical transition” by the coupling size and phase-space factors.

3.3 Nonlinear regime

From a theoretical point of view, we have explained that it is safe to use the linear perturbation framework and derive the physically relevant quantities studied here, namely the charge/energy amplification factors, in terms of the perturbation amplitudes. Since we completely ignore the backreaction during the scattering, the violation of particle-number conservation is of higher order and can be safely ignored within this regime. In practice, given an incoming wavepacket, it is useful to justify when one can correctly use the linear regime to study the superradiance effect. Therefore, we proceed here with the analysis of the nonlinearities by solving the system of differential equations on a $1+1$ dimensional lattice, and we verify when these nonlinear effects are subdominant.

3.3.1 Analysis in spacetime domain

Methodology: we discretize the spatial coordinate using a 4th-order finite difference scheme and evolve in time with the 4th-order Runge-Kutta method. We implement an absorbing boundary condition at the boundary corresponding to spatial infinity, i.e once the outgoing waves meet the lattice boundary, it escapes the lattice. Implementing other boundary conditions, such as Dirichlet conditions, is also possible, as long as the boundary is sufficiently far away that reflected waves do not spoil the flux measurements. The initial conditions of the system, including the Q-ball background and a monochromatic Gaussian wavepacket scattering off the Q-ball, read

$$\Phi(0, r) = \Phi_Q(0, r) + \delta \left(\frac{r_0}{r} \right) e^{-\frac{(r-r_0)^2}{2\sigma_r^2}} e^{-is\omega_0 \sqrt{\omega_0^2 - 1}r}, \quad (3.3.1)$$

$$\partial_t \Phi(0, r) = -i[\omega_Q \Phi_Q(0, r) + \omega_0(\Phi(0, r) - \Phi_Q(0, r))]. \quad (3.3.2)$$

The factor $\left(\frac{r_0}{r}\right)$ is negligible when the packet width σ_r is relatively smaller than the distance from the wavepacket to the Q-ball r_0 . This is not the case when ω_0 is close to the threshold of having two propagating modes, i.e., when one of the wavenumbers k_+ or k_- approaches 0, and consequently the wavelength becomes significantly large. In such a case, to capture enough oscillations and obtain good resolution of the mode frequencies, we need sufficiently large σ_r , and hence the asymptotic damping behavior becomes relevant. More technical details on the numerical framework for lattice simulation are discussed in Appendix. B.

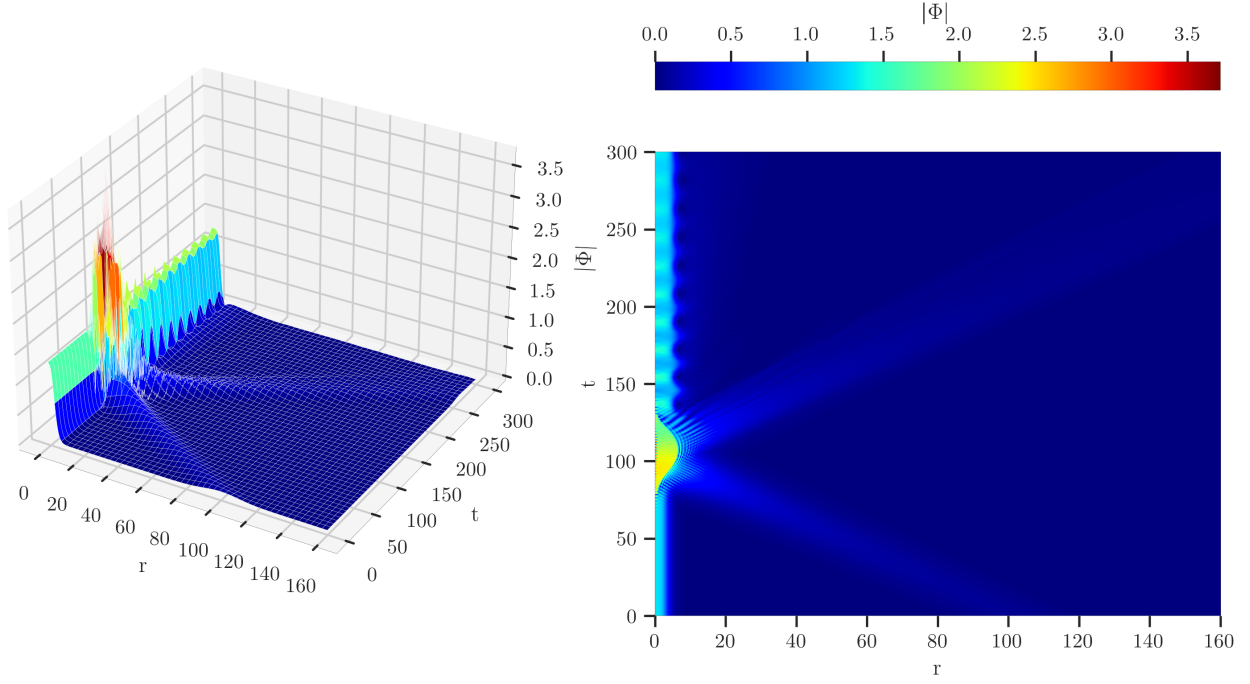


Figure 3.7: The 3D plot and its corresponding heatmap visualize the solution $|\Phi|$ for Q-ball scattering with spherical waves. Here, we use an artificially large wavepacket to visualize the propagation of the incoming and outgoing waves. As a consequence, the scattering is highly nonlinear, significantly deforming the Q-ball and making it oscillate after the collision.

In what follows, we choose the coupling $g = 1/3$ and study only spherically symmetric perturbations.

3.3.2 Analysis in frequency domain

Before extracting amplification factors from fluxes, it is useful to check directly that the lattice evolution contains the same frequency structure as the linear perturbation analysis. For this purpose we take the time series of the scattered field at a fixed spatial slice and Fourier transform it over a time window. In the example shown in Fig. 3.8, the background has $\omega_Q = 0.76$ and the incoming wavepacket amplitude is $\delta = 3 \times 10^{-4}$. We compare two simulations with incoming carrier frequencies

$$\omega_0 = \{3.76, -2.24\} = \{\omega_Q + \omega, \omega_Q - \omega\}, \quad \omega = 3. \quad (3.3.3)$$

Thus the first simulation excites the $+$ channel as the incoming wave, while the second excites the $-$ channel.

The Fourier transform is evaluated at $x_{\text{slice}} = r_0 - 3\sigma_r$ over the interval $t \in [x_{\text{slice}} + \sigma_r, t_{\text{end}}]$, so that the plotted spectrum captures both incoming and outgoing wavepackets after the interaction with the Q-ball. The dominant peaks appear at

$$\omega \simeq \{3.770, -2.226\}, \quad (3.3.4)$$

which agree with the expected shifted frequencies $\omega_+ = 3.76$ and $\omega_- = -2.24$ up to the finite resolution of the wavepacket and Fourier window. More importantly, each simulation contains both frequency components after scattering. The solid blue curve, corresponding to an incoming $+$ mode, has a dominant peak near ω_+ and a smaller peak near ω_- . Conversely, the dashed red curve, corresponding to an incoming $-$ mode, has a dominant peak near ω_- and a smaller peak near ω_+ . This is the frequency-domain signature of the mode mixing already present in Eq. (2.3.7).

There is also a smaller peak around $\omega \simeq 1.1$. We interpret this feature cautiously. It is not one of the two linear asymptotic channels for the chosen incoming frequency, and we tested that its height decreases when the wavepacket amplitude δ is reduced. This suggests that it is associated

with a weak excitation of the Q-ball background or another nonlinear transient produced during the collision, rather than with the leading linear scattering process. For sufficiently small perturbations, the spectrum is dominated by the two expected outgoing frequencies.

This Fourier analysis is therefore a useful qualitative check: it confirms that the lattice simulation produces the correct spectral content and that the linear perturbation equations correctly predict the location of the peaks. However, it is not the best way to obtain quantitative amplification factors. The extracted peak heights are sensitive to the finite width of the wavepacket, the chosen spatial slice, the time window used in the Fourier transform, and the residual overlap with transient radiation from the Q-ball. For this reason, in the next subsection we use integrated charge and energy fluxes as more precise and reliable measurements, since they are more stable numerically and can be compared directly with the amplification factors of the charge/energy fluxes.

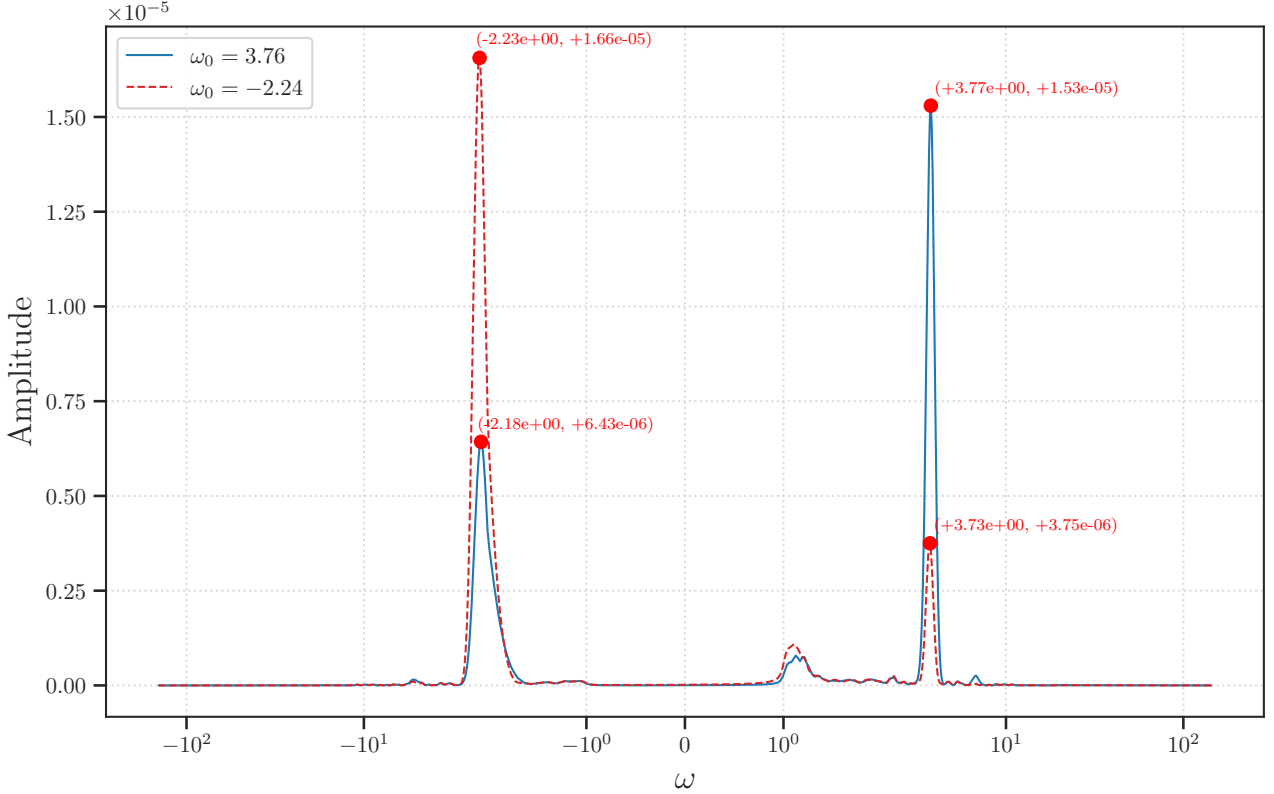


Figure 3.8: Fourier analysis of the field evolution at a fixed spatial slice $x_{\text{slice}} = r_0 - 3\sigma_r$ for a $D = 3$ Q-ball with $g = 1/3$, $\omega_Q = 0.76$, $\delta = 3 \times 10^{-4}$ and $\sigma_r = 5$. The solid blue curve corresponds to an incoming $+$ mode with carrier frequency $\omega_0 = \omega_Q + \omega = 3.76$, while the dashed red curve corresponds to an incoming $-$ mode with $\omega_0 = \omega_Q - \omega = -2.24$, with $\omega = 3$. Both spectra contain the two expected outgoing frequencies, showing the conversion between the η_+ and η_- channels.

3.3.3 Flux amplification factors in lattice simulation

We now explain how the flux amplification factors $Z_E^{\mathcal{F}}$ and $Z_Q^{\mathcal{F}}$ are extracted from the lattice evolution and compared with the linear prediction in Eq. (3.1.12). In the linear analysis, the phase factors of $|k_{\pm}|$ in Eq. (3.1.12) come from the asymptotic normalization of the incoming and outgoing spherical waves. On the lattice, we do not need to fit to obtain the individual amplitudes $A_{\pm}^{\text{in,out}}$. Instead, we compute the physical energy and charge fluxes through a fixed spherical surface. These integrated fluxes provide the lattice analogue of the same flux ratios appearing in Eq. (3.1.12).

The energy current J_E^t and the corresponding time-integrated flux are given by

$$J_E^t(t, r) = -2 \operatorname{Re}(\partial_r \Phi^* \partial_t \Phi) = -2 (\partial_r \Phi_R \partial_t \Phi_R + \partial_r \Phi_I \partial_t \Phi_I), \quad (3.3.5)$$

$$E^{\text{flux}}(t, r) = \int_0^t dt \oint d\Sigma \cdot \mathbf{J}_E(\mathbf{x}, t) = A(r) \int_0^t dt J_E^t(t, r), \quad (3.3.6)$$

where we employed the spherical symmetry to factor out the area of the sphere $A(r)$. Similarly, for the flux of the global $U(1)$ charge,

$$J_Q^t(t, r) = -2 \operatorname{Im}(\Phi^* \partial_t \Phi) = -2 (\Phi_R \partial_t \Phi_I - \Phi_I \partial_t \Phi_R), \quad (3.3.7)$$

$$Q^{\text{flux}}(r) = \int_0^t dt \oint d\Sigma \cdot \mathbf{J}_Q(\mathbf{x}, t) = A(r) \int_0^t dt J_Q^t(t, r). \quad (3.3.8)$$

A visualization of the energy-current signal and the corresponding integrated energy flux at the spatial slice $x = r_{\text{slice}}$ is shown in Fig. 3.9. The upper panel shows the instantaneous flux through the measurement surface. At early times the incoming wavepacket crosses this surface while moving toward the Q-ball, so the outward-oriented flux is negative. After the collision, the scattered wavepacket crosses the same surface in the outward direction, producing a positive flux signal. The lower panel integrates this signed current over time. It first settles to a negative plateau after the incoming packet has crossed the surface, then changes when the outgoing packet arrives, and finally approaches a late-time plateau once the scattered radiation has passed. The inset zooms in on this final plateau, which shows a small residual difference between the outgoing and incoming contributions and is the quantity used to determine the net amplification.

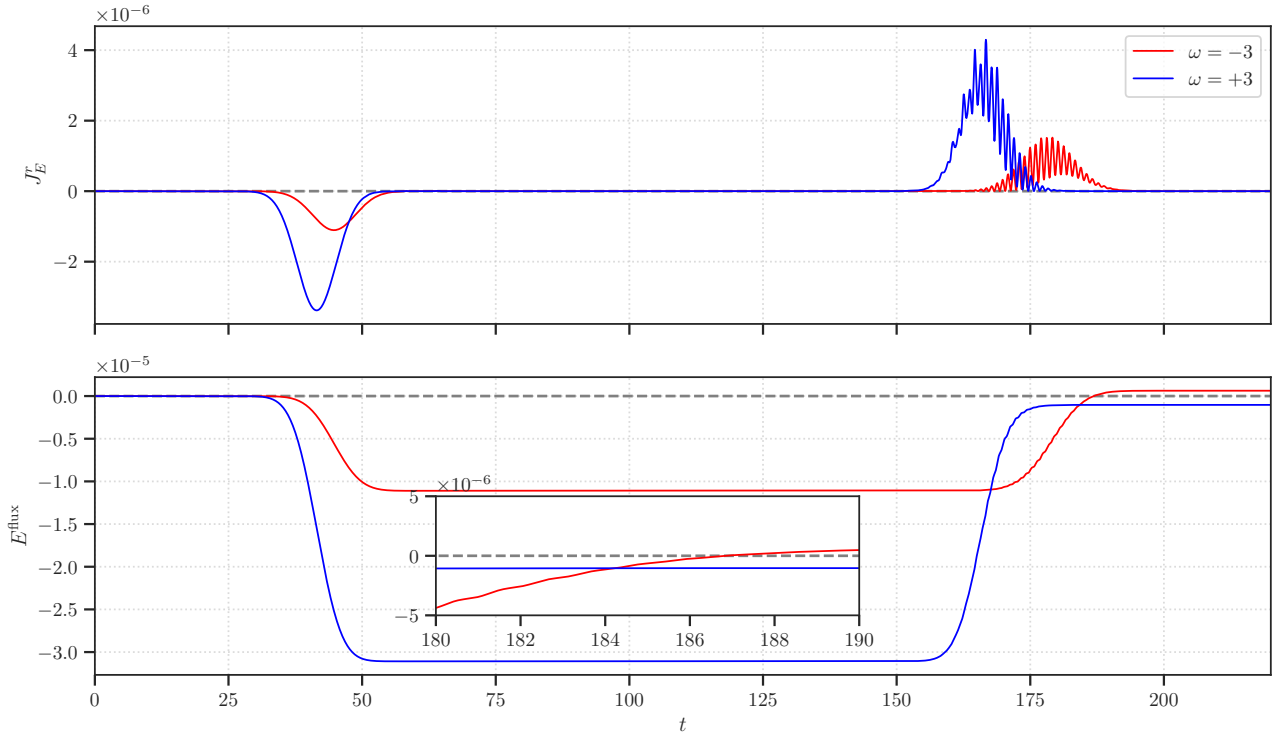


Figure 3.9: Energy flux J_E^t and integrated energy flux E^{flux} for a wavepacket scattering off a non-spinning Q-ball with $\omega_Q = 0.76$ and sextic coupling $g = 1/3$ in $D = 3$. The incoming beam has width $\sigma_r = 5$, center around $r_0 = \{100, 99\}$, and amplitude $\delta = 3 \times 10^{-4}$, corresponding respectively to the different frequencies $\omega_0 = \{-2.24, 3.76\}$, or equivalently $\omega = 3$. With the same parameters given here but with $D = 2$ instead, we reproduced the amplification factors reported in Ref. [3].

With this sign convention, $E^{\text{flux}}(t_*, r_{\text{slice}})$ and $Q^{\text{flux}}(t_*, r_{\text{slice}})$ measure the signed incoming fluxes after the initial packet has crossed the measurement surface. For an incoming wavepacket they are negative because the flux is inward relative to the outward surface normal. The late-time values $E^{\text{flux}}(t = \infty, r_{\text{slice}})$ and $Q^{\text{flux}}(t = \infty, r_{\text{slice}})$ contain the incoming negative contribution plus the outgoing positive contribution. Therefore the ratio below subtracts the incident flux and returns the fractional excess of outgoing over incoming flux, matching the definition $Z^{\mathcal{F}} = \mathcal{A}^{\mathcal{F}} - 1$ used in Eq. (3.1.12).

$$Z_E^{\mathcal{F}} = -\frac{E^{\text{flux}}(t = \infty, r_{\text{slice}})}{E^{\text{flux}}(t = t_*, r_{\text{slice}})}, \quad Z_Q^{\mathcal{F}} = -\frac{Q^{\text{flux}}(t = \infty, r_{\text{slice}})}{Q^{\text{flux}}(t = t_*, r_{\text{slice}})}. \quad (3.3.9)$$

We take $t_* \simeq r_0 - r_{\text{slice}} + \mathcal{O}(10)\sigma_r$ to measure the incoming plateau after the initial wavepacket has passed through the radius r_{slice} , but before the outgoing radiation returns to the same surface. The comparison with the linear calculation is then made by evaluating Eq. (3.1.12) for the corresponding incoming channel and comparing it with the lattice ratios above.

In our analysis, we focus primarily on the validity of the linear perturbation discussion and report it as a function of the perturbation size. Since the amplitude of the perturbation grows as $1/r$ as the wavepacket approaches the Q-ball, it is better to use a rescaled version,

$$\delta_r \equiv \frac{\delta \times r_0}{R_Q \times \phi_Q(0)}, \quad (3.3.10)$$

where R_Q is the radius where the Q-ball field becomes equal to half of its value at the center. The results are shown in Fig. 3.10, where we indicate the discrepancies between the linear and nonlinear approaches. These effects, as expected, grow with the size of the perturbations. Additionally, there are

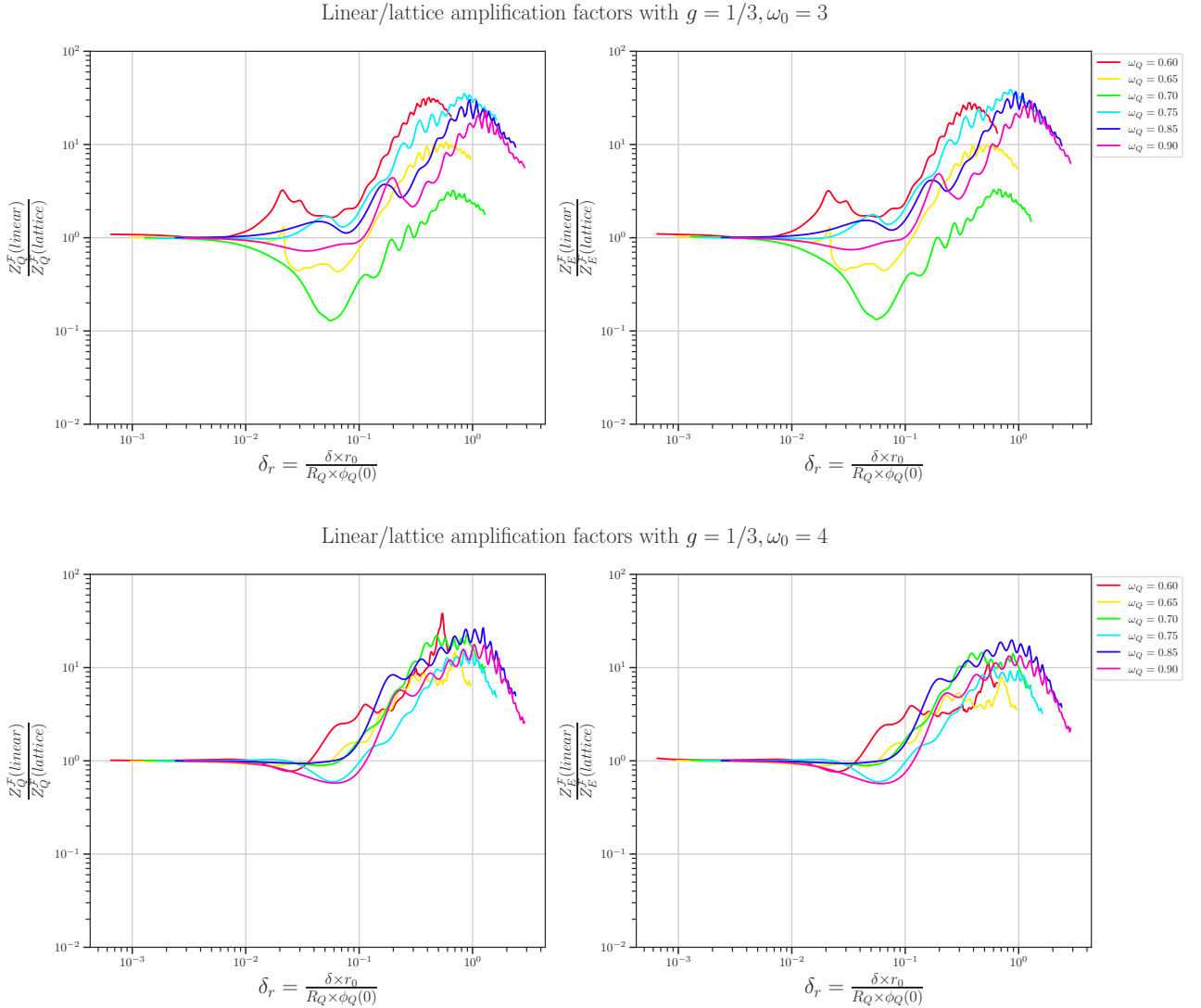


Figure 3.10: Comparison of amplification factors between the linear regime and full lattice simulation results. Here R_Q is defined as the radius where the field becomes half of its value at the center. Recall that the height of the Q-ball profile with given parameters is $\mathcal{O}(1)$, hence δ_r can also be thought of as the relative height between the incoming wavepacket and the Q-ball background. We observe that the convergence rate toward small δ_r is slightly faster with $\omega_0 = 4$ than with $\omega_0 = 3$ due to threshold effects.

some subleading technical effects that feed the discrepancies between the linear and lattice analyses, which we list below:

- The wavepacket width is not sufficiently large, and hence we lose resolution of the incoming wave momenta: This effect becomes particularly important if ω is close to the threshold of having two propagating modes $\pm$, since one of the $k_{\pm}$ becomes very small and we need a broad enough wavepacket σ_r to capture enough oscillations in space. For example, in Fig. 3.10, with $\omega_0 = 4$ we only need to use $\sigma_r = 10$ and still obtain good agreement between the linear and lattice results. Meanwhile, with $\omega_0 = 3$, we are closer to the limit $\omega = 1 + \omega_Q$ (recall that in our set-up, $\omega_0 = \omega_+ = \omega_Q + \omega$). To obtain precision somewhat comparable to the case $\omega_0 \geq 4$, we need to tune $\sigma_r = 30$, and consequently increase the lattice size to correctly capture the full incoming-outgoing wavepackets.
- Precision of the initial Q-ball profile: the profile of the initial Q-ball must be exact; otherwise, it will radiate energy and charge, and this becomes a background for the “measurement” of the charge and energy extraction during the interaction with the perturbations.
- Discretization of time and space, leading to generic systematic errors in the lattice results.

3.3.4 Physics implications

We find that the linear-analysis results remain valid up to perturbations of order $\delta_r \lesssim 10^{-2}$. This leads to the natural question of whether such a regime is realized in the early universe. To effectively describe particle interactions with Q-balls, we must set the charge equal to one. Consequently, the constraint on the amplitude becomes a constraint on the wavepacket size, or coherence length, L_ω :

$$Q = 1 \sim \frac{\mu^3}{\lambda} \delta_{\min}^2 \phi_Q(0)^2 L_\omega^3 \omega \Rightarrow L_\omega \gtrsim \left(\frac{\mu^3}{\lambda} \delta_{\min}^2 \phi_Q^2(0) \omega \right)^{-1/3}, \quad (3.3.11)$$

where $\phi_Q(0)$ and ω are rescaled dimensionless quantities (see Eq. (3.1.3)). Assuming the coherence length is of the order of the inverse plasma temperature, since we expect the wavepacket length to be roughly of order the mean free path, we arrive at the condition

$$\frac{\lambda^{1/2} T^{3/2}}{\mu^{3/2}} \times [\phi_Q(0) \omega^{1/2}]^{-1} < 10^{-2}. \quad (3.3.12)$$

Thus, once the temperature drops by one order of magnitude below the typical energy scale of the Q-ball field (μ), we expect the linear approximation to lead to reliable results. For such temperatures the particle corresponding to the Q-ball perturbation is nonrelativistic, unless we have a very small coupling λ . Then, from Fig. 3.6, we expect that most particle–Q-ball interactions will be either elastic (reflection without gaining or losing energy), or will lead to Q-ball charge reduction.

3.4 Scattering beyond S-wave

The S -wave analysis contains the essential mechanism of charge and energy exchange, but an incident particle in three spatial dimensions is more naturally described by a plane wave. To connect the mode-conversion amplitudes with particle–Q-ball scattering rates, we therefore extend the discussion to higher partial waves. This section explores a technical extension of the same linear scattering problem: the background Q-ball is still non-spinning and spherically symmetric, but the perturbation is allowed to carry orbital angular momentum. Due to the non-spinning background, the scattering problem can be decomposed under a partial-wave expansion, where waves with different angular momentum ℓ decouple, and we return to the problem of solving the linear scattering matrix S_ℓ for each angular momentum. In the end, the S -matrix elements can be converted into total cross sections for elastic and charge-converting channels, showing a more physical picture of the scattering between a Q-ball background and an incoming plane wave.

3.4.1 Partial wave expansion of the S-matrix

In this section, we consider only the case $D = 3$. For a non-spinning Q-ball, the background depends only on r , so different values of ℓ decouple, and we find that the results for $\ell \neq 0$ are qualitatively similar to the $\ell = 0$ case. Indeed, performing the decomposition in terms of angular momentum,

$$\eta_{\pm}(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \eta_{\pm}^{m\ell}(r) e^{im\varphi} P_{\ell}^m(\cos \theta) \quad (3.4.1)$$

where P_{ℓ}^m are associated Legendre polynomials. Substituting this expansion into the linear perturbation equation gives the coupled radial problem

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} \right] \eta_{\pm}^{m\ell} + [\omega_{\pm}^2 - V] \eta_{\pm}^{m\ell} - W(\eta_{\mp}^{-m\ell})^* = 0. \quad (3.4.2)$$

Thus each partial-wave sector is the pair $\eta_{+}^{m\ell}$ and $(\eta_{-}^{-m\ell})^*$, with the global $U(1)$ symmetry associated with each choice of ℓ and m . This $U(1)$ symmetry corresponds to a similar conservation law for the current $\mathbf{J}_{\eta}^{\ell m}$ for all possible values of ℓ, m . The asymptotic solutions for $\eta^{m\ell}$ have the same form as in the previous discussions of S -wave scattering,

$$\eta_{\pm}^{\ell m} = \frac{1}{|k_{\pm}| r} \left[A_{\pm,(\ell,m)}^{\text{in}} e^{-ik_{\pm} r} + A_{\pm,(\ell,m)}^{\text{out}} e^{ik_{\pm} r} \right]. \quad (3.4.3)$$

The relation between the amplitudes becomes

$$\boxed{\frac{|A_{+,(\ell,m)}^{\text{out}}|^2}{|k_{+}|} + \frac{|A_{-,(\ell,-m)}^{\text{out}}|^2}{|k_{-}|} = \frac{|A_{+,(\ell,m)}^{\text{in}}|^2}{|k_{+}|} + \frac{|A_{-,(\ell,-m)}^{\text{in}}|^2}{|k_{-}|}}. \quad (3.4.4)$$

We see that Eq. (3.4.4) imposes number conservation for particles with given angular momentum (ℓ, m) . Recently, Ref. [14] studied perturbations around rotating Q-balls. In this case, states with different values of ℓ are mixed, and the value m of the particle in the $\eta_{\pm}$ fields is related by $m_{\pm} = m_Q \pm m$. However, even in this case, the total number of particles is conserved, and the particle and antiparticle energies are fixed to be $\omega_{\pm}$. Defining $|K| \equiv \text{diag}(|k_{+}|, |k_{-}|)$, we can define the S matrix in this case as well:

$$\boxed{|K|^{-1/2} \vec{A}_{\ell,m}^{\text{out}} = S_{\ell,m} |K|^{-1/2} \vec{A}_{\ell,m}^{\text{in}}}, \quad (3.4.5)$$

so Eq. (3.4.4) is simply the statement that $S_{\ell,m}$ is unitary in this basis. For the non-spinning backgrounds considered here the result is independent of m , and we write the scattering matrix as S_{ℓ} . The same formalism is more involved for spinning Q-balls, where the background carries angular momentum and different angular sectors can mix; see Ref. [14] for a recent analysis of perturbations for spinning Q-balls.

3.4.2 Total cross section

We now convert the partial-wave amplitudes into total cross sections. Consider first an incoming $+$ mode with physical energy $E_{+} = \omega_{+} = \omega + \omega_Q$ and momentum $|k_{+}| = \sqrt{E_{+}^2 - 1}$. Far from the Q-ball, the solution can be written as the sum of an incident plane wave and outgoing spherical waves,

$$\eta_{+}(r, \theta) = \frac{1}{|k_{+}|} \left[e^{ik_{+}z} + f_{++}(\theta) \frac{e^{ik_{+}r}}{r} \right], \quad \eta_{-}(r, \theta) = \frac{1}{|k_{-}|} f_{+-}(\theta) \frac{e^{ik_{-}r}}{r}. \quad (3.4.6)$$

Note that here we use f_{if} , where the first subscript labels the incoming channel and the second labels the outgoing channel, which is the opposite order compared to the S-matrix convention. Expanding the plane wave and the angular scattering amplitudes gives

$$e^{ik_+z} = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(k_+r) P_\ell(\cos\theta), \quad f_{+s}(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_{+,s,\ell} P_\ell(\cos\theta), \quad (3.4.7)$$

where $s = \pm$. Using $j_\ell(z) \sim (2iz)^{-1} (i^{-\ell} e^{iz} - i^\ell e^{-iz})$, the large-radius form of Eq. (3.4.6) becomes

$$\begin{aligned} \eta_+(r, \theta) &= \frac{1}{|k_+|} \sum_{\ell=0}^{\infty} (2\ell+1) \left[\left(f_{++,\ell} + \frac{1}{2ik_+} \right) \frac{e^{ik_+r}}{r} + \frac{(-1)^{\ell+1}}{2ik_+} \frac{e^{-ik_+r}}{r} \right] P_\ell(\cos\theta), \\ \eta_-(r, \theta) &= \frac{1}{|k_-|} \sum_{\ell=0}^{\infty} (2\ell+1) f_{+-, \ell} \frac{e^{ik_-r}}{r} P_\ell(\cos\theta). \end{aligned} \quad (3.4.8)$$

Matching this expression to Eq. (3.4.3) and using the flux-normalized S_ℓ matrix gives

$$f_{+-, \ell} = \frac{(-1)^{\ell+1}}{2ik_+} \sqrt{\frac{|k_+|}{|k_-|}} \left| \frac{A_{-\ell}^{\text{out}}}{A_{+, \ell}^{\text{in}}} \right|_{A_{-, \ell}^{\text{in}}=0} = \frac{(-1)^{\ell+1}}{2ik_+} |S_\ell^{-+}|, \quad (3.4.9a)$$

$$f_{++, \ell} = \frac{i}{2k_+} \left[1 + (-1)^\ell \frac{A_{+, \ell}^{\text{out}}}{A_{+, \ell}^{\text{in}}} \right]_{A_{-, \ell}^{\text{in}}=0} = \frac{i}{2k_+} [1 + (-1)^\ell S_\ell^{++}]. \quad (3.4.9b)$$

Following the usual quantum-mechanics procedure, we arrive at the expression for the cross sections σ_{+f} for producing $f = \pm$ states:

$$\sigma_{++} = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) |f_{++, \ell}|^2, \quad \sigma_{+-} = 4\pi \frac{|k_+|}{|k_-|} \sum_{\ell=0}^{\infty} (2\ell+1) |f_{+-, \ell}|^2. \quad (3.4.10)$$

One can similarly derive the cross sections with the other incoming mode. In terms of S-matrices, the cross sections σ_{if} for various initial states i and final states f are

$$\sigma_{++}(E) = \frac{\pi}{(E^2 - 1)} \sum_{\ell=0}^{\infty} (2\ell+1) \left| 1 + (-1)^\ell S_\ell^{++}(\omega = E - \omega_Q) \right|^2, \quad (3.4.11a)$$

$$\sigma_{+-}(E) = \frac{\pi}{(E^2 - 1)} \sum_{\ell=0}^{\infty} (2\ell+1) \left| S_\ell^{-+}(\omega = E - \omega_Q) \right|^2, \quad (3.4.11b)$$

$$\sigma_{--}(E) = \frac{\pi}{(E^2 - 1)} \sum_{\ell=0}^{\infty} (2\ell+1) \left| 1 + (-1)^\ell S_\ell^{--}(\omega = E + \omega_Q) \right|^2, \quad (3.4.11c)$$

$$\sigma_{-+}(E) = \frac{\pi}{(E^2 - 1)} \sum_{\ell=0}^{\infty} (2\ell+1) \left| S_\ell^{+-}(\omega = E + \omega_Q) \right|^2, \quad (3.4.11d)$$

Here E denotes the positive physical energy of the incoming excitation in dimensionless units. For an incoming $+$ mode, $E = \omega_+ = \omega + \omega_Q$ so the corresponding perturbation parameter is $\omega = E - \omega_Q$. For an incoming $-$ mode, $E = |\omega_-| = \omega - \omega_Q$ so $\omega = E + \omega_Q$. In both cases $E^2 = 1 + |\mathbf{k}|^2$, consistently with the dimensionless rescaling in Eq. (3.1.3). Although the two-channel basis contains $(A_{-, \ell}^{\text{in}, \text{out}})^*$, the total cross sections depend only on the appropriate absolute squares and interference phases shown explicitly above.

For the charge-converting channels, reciprocity gives $S_\ell^{-+}(\omega) = S_\ell^{+-}(\omega)$ in the flux-normalized basis. Comparing the two processes at the same value of the perturbation parameter ω , with $E_+ = \omega + \omega_Q$ and $E_- = \omega - \omega_Q$, one obtains

$$\sigma_{+-}(E_+) = \left[\frac{E_-^2 - 1}{E_+^2 - 1} \right] \sigma_{-+}(E_-). \quad (3.4.12)$$

No analogous relation is implied for the elastic channels, because σ_{++} and σ_{--} contain interference between the incident plane wave and the outgoing wave. Their values are therefore sensitive to the phases of S_ℓ^{++} and S_ℓ^{--} , not only to unitarity and symmetry.

The numerical cross sections are shown in Figs. 3.11, 3.12, and 3.13. Figure 3.11 is a convergence test of the cumulative partial-wave sum. For an incoming $-$ mode scattering from the $\omega_Q = 0.75$ Q-ball, the sum saturates once ℓ exceeds the impact-parameter estimate $\ell_* \sim R_Q |k_-|$, where R_Q is a typical Q-ball radius. This gives $\ell_* \sim \{3, 11\}$ for the two plotted values $\omega = \{2, 4\}$, in agreement with the observed plateaus. The higher-energy case requires more partial waves, but the final cross section is smaller in the conversion channel because the mode-conversion probability itself is energy dependent.

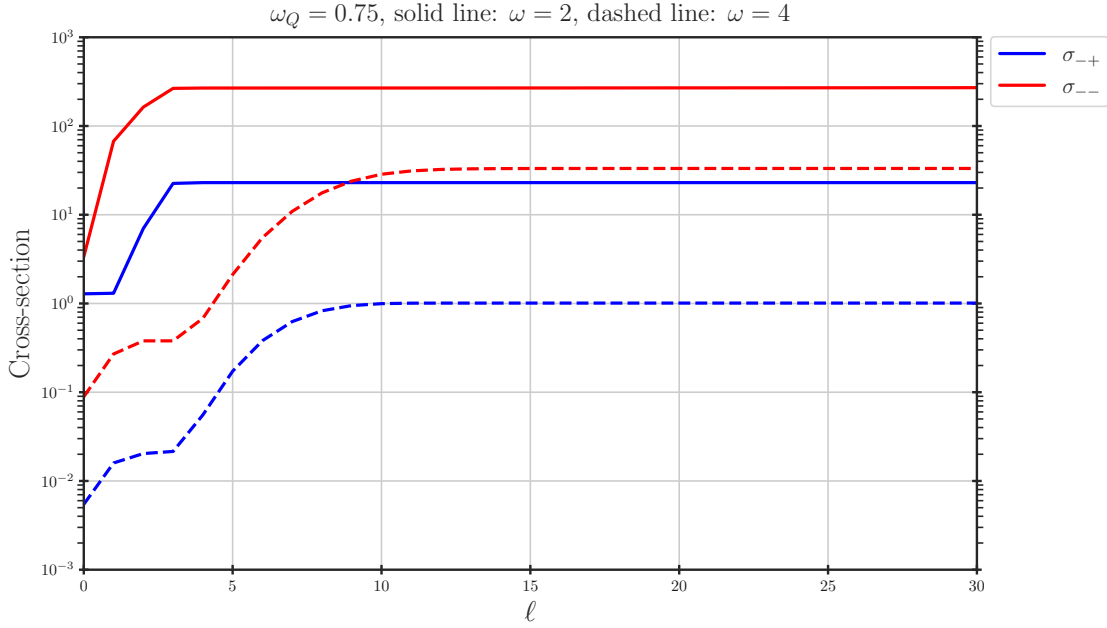


Figure 3.11: The saturation of the total cross section due to the scattering of an incoming particle ($-$ mode) against a Q-ball of frequency $\omega_Q = 0.75$ in the two channels $\sigma(\phi \rightarrow \phi) = \sigma_{-+}$ and $\sigma(\phi \rightarrow \phi^*) = \sigma_{--}$. Given the corresponding effective size of this Q-ball, $R_Q \approx 3.3$, the cross section reaches saturation for the angular momentum values $\ell_* = R_Q |k_-| \sim \{3, 11\}$ for $\omega = \{2, 4\}$, respectively. This cross section is in dimensionless units due to the variable redefinition in Eq. (3.1.3). The correct scaling can be recovered by multiplying it by μ^{-2} .

Figures 3.12 and 3.13 show the energy dependence for incoming $-$ modes. In both backgrounds the elastic channel σ_{--} is larger than the charge-converting channel σ_{-+} over most of the plotted range. The cross sections are enhanced near threshold, where the incoming momentum is small and the partial-wave prefactor $\pi/(E^2 - 1)$ is large. For $\omega_Q = 0.6$ the Q-ball is larger and the effective perturbation potentials are wider, leading to substantially larger cross sections and more visible oscillatory structure than for $\omega_Q = 0.75$. The local peaks and dips can be understood qualitatively as phase-shift effects in a coupled-channel potential well: the functions $V(r)$ and $W(r)$ form a finite radial scattering region, and at special energies the partial waves can interfere destructively or constructively after summing over ℓ . This interpretation is analogous to transmission resonances and reflection zeros in an ordinary one-dimensional quantum-mechanical problem of plane waves scattering from a potential barrier.

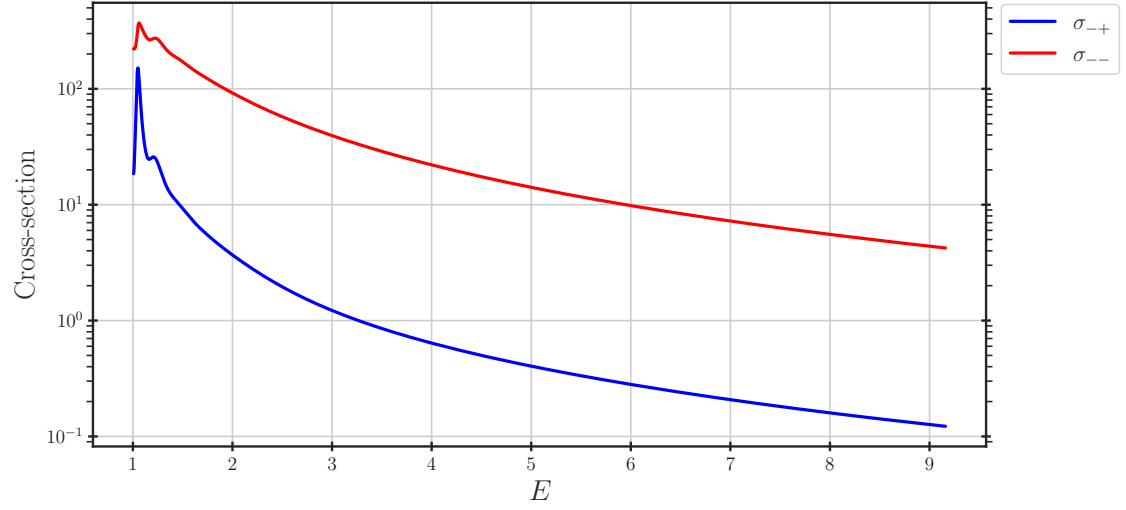


Figure 3.12: The total cross sections of the incoming $-$ mode with energy E scattering off a Q-ball of frequency $\omega_Q = 0.75$. The red line represents the result for the elastic channel, with outgoing mode $-$, while the blue line represents the inelastic one.

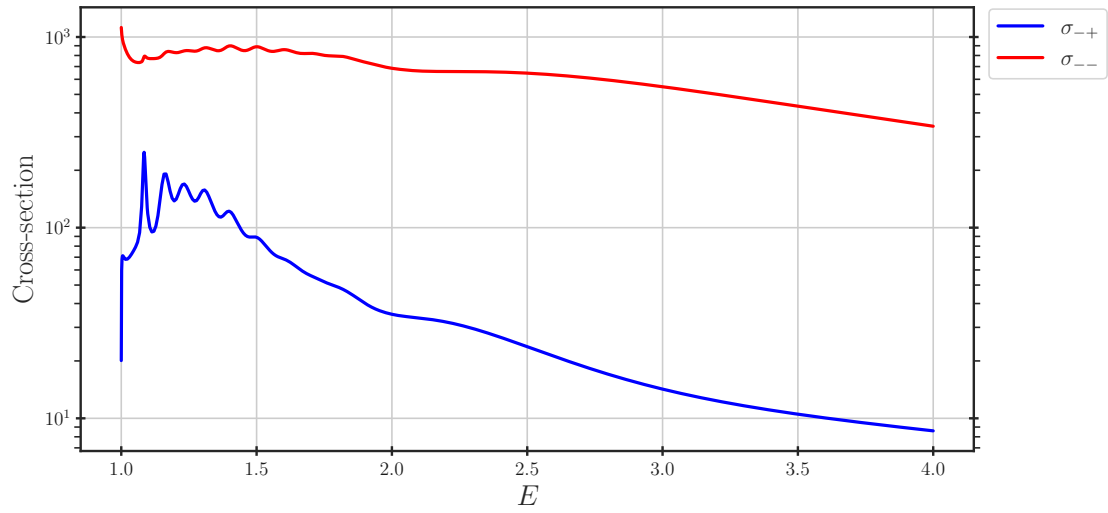


Figure 3.13: Same as Fig. 3.12 but with $\omega_Q = 0.6$. The larger Q-ball background gives larger cross sections and richer oscillatory structure in the energy dependence.

Q-ball superradiance in a concrete model of two scalar fields

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Having studied Q-ball properties and their scattering with perturbations in the context of a single complex scalar field model in the previous chapter, we now extend our investigation to a model containing more scalar fields. Here, the mixing between monochromatic modes is governed not only by terms higher than quadratic order, but also by the nonlinear coupling between different scalars. This introduces richer and more complex patterns of energy and charge amplification, with additional symmetry properties that we observe numerically and that can be derived as consequences of the symmetry of the scattering matrix.

4.1 Introduction

The single-field model discussed in the previous chapter realizes Q-balls through nonlinear self-interactions of the charged complex field. In its simplest polynomial form, the potential contains terms schematically of the form

$$U_{\text{one-field}}(|\Phi|) = m_\Phi^2 |\Phi|^2 - \lambda |\Phi|^4 + g |\Phi|^6, \quad (4.1.1)$$

so that the attractive interaction needed for soliton formation is encoded directly in the self-potential of Φ . Equivalently, the function $U_{\text{one-field}}(|\Phi|)/|\Phi|^2$ can develop a minimum away from the origin, allowing a finite-density lump of $U(1)$ charge to have lower energy per charge than free quanta. This is a minimal and useful description, but it also hides the possible microscopic origin of the attractive interaction inside higher-order terms of the effective potential.

The Friedberg-Lee-Sirlin (FLS) model provides a different and historically important mechanism. Instead of relying only on nonlinear self-interactions of the charged field, the model introduces a neutral real scalar field that couples to the charged complex scalar [20, 21, 22, 7]. To be specific, the Lagrangian description of the FLS model is

$$\mathcal{L}(\Phi, \chi) = -|\partial_\mu \Phi|^2 - \frac{1}{2}(\partial_\mu \chi)^2 - U(|\Phi|, \chi),$$

$$U(|\Phi|, \chi) = g_{\chi\Phi} \chi^2 |\Phi|^2 + g_\chi (\chi^2 - v_\chi^2)^2 + m_\Phi^2 |\Phi|^2 + g_\Phi |\Phi|^4. \quad (4.1.2)$$

The theory is invariant under the global $U(1)$ transformation $\Phi \rightarrow e^{i\alpha}\Phi$, $\chi \rightarrow \chi$, and also under the discrete symmetry $\chi \rightarrow -\chi$, $\Phi \rightarrow \Phi$. The charged field is still protected by this global $U(1)$ symmetry, but the nonlinear dynamics responsible for localization now arise from the coupling between Φ and the neutral field χ . From the viewpoint of the charged quanta, the local effective mass is

$$m_{\Phi,\text{eff}}^2(\chi) = m_\Phi^2 + g_{\chi\Phi} \chi^2. \quad (4.1.3)$$

Outside the soliton, χ approaches its vacuum value v_χ , and the charged field is relatively heavy. Inside the soliton, the charged condensate can push χ away from its vacuum value, thereby reducing the effective mass of Φ and creating a region where it is energetically favorable to store charge. Thus the FLS soliton may be viewed as a charged lump sustained by a rearrangement of the neutral scalar background.

This distinction is conceptually useful for two reasons. First, the FLS construction is renormalizable in four dimensions and therefore gives a simple ultraviolet-complete description for non-topological solitons with more than one scalar degree of freedom. In this sense it differs from the one-field polynomial example, where the attractive interaction is often represented by higher-order effective operators in the charged-field potential. Second, it separates the field carrying the Noether charge from the field mediating the attractive interaction. This separation makes it possible to ask questions that are absent in the one-field model: how the neutral scalar participates in the soliton profile, how it affects stability, how a neutral perturbation can scatter into charged modes or vice versa, and how the appearance of neutral-field perturbation modes impacts the charge/energy extraction mechanism. Because such structures can arise naturally in beyond-the-Standard-Model sectors, FLS-like Q-balls and related multi-field solitons have been studied in connection with dark matter, baryogenesis, and hidden-sector model building [23, 24, 25, 26, 27, 28, 29, 30]. Recent work has revisited the FLS soliton model precisely from this perspective, emphasizing both its similarities to Coleman-type Q-balls and its differences, including the absence of an ordinary Coleman thin-wall limit in part of parameter space [31]. More general multi-field Q-balls with real scalars have also been analyzed as a broader class of theories in which uncharged fields contribute classically to the soliton structure [27].

The model has also remained active in several modern directions. Gauged versions of the FLS model admit electrically and magnetically nontrivial soliton configurations, where a new branch of solutions represents the soliton configuration coupled to a circular magnetic flux [32]. Effective-field-theory descriptions have been used to understand how simplified single-field Q-ball potentials can reproduce full FLS solitons in suitable limits [33]. Most directly connected to this thesis, recent studies of FLS soliton superradiance show that internal rotation can lead to energy extraction in the two-field model, and that the amplification spectra are sensitive to the scalar mass hierarchy and to the treatment of boundary conditions [1, 34].

The purpose of this chapter is to use the FLS model as the next step beyond the one-field analysis. In the previous chapter, perturbations around the Q-ball background involved two charged channels, conventionally denoted η_+ and η_- , whose mixing encoded particle-antiparticle conversion. In the FLS model, the neutral scalar introduces an additional perturbation channel η_χ . The scattering problem is therefore promoted from a two-channel system to a three-channel system,

$$(\eta_+, \eta_-^*) \longrightarrow (\eta_+, \eta_-^*, \eta_\chi). \quad (4.1.4)$$

As a result, the S-matrix has a richer structure: charged perturbations can convert into neutral excitations, neutral perturbations can produce charged outgoing modes, and the energy and charge amplification factors need no longer follow the same simple symmetry pattern as in the one-field model. The discussion below constructs the FLS background, derives the coupled linear perturbation equations, identifies the conserved particle-number current and S-matrix constraints, and then compares the resulting amplification factors with real-time lattice simulations.

4.2 Q-ball in Friedberg-Lee-Sirlin (FLS) model

Following the same approach as in the single complex scalar field model, we first construct the Q-ball profile, which is the time-dependent solution of the field equations of motion obtained by varying the Lagrangian (4.1.2):

$$\begin{cases} \square\chi - \frac{\partial U}{\partial\chi} = 0 \\ \square\Phi - \frac{\partial U}{\partial\Phi^*} = 0 \end{cases} \Rightarrow \begin{cases} \left[\square - 2g_{\chi\Phi}|\Phi|^2 - 4g_{\chi}(\chi^2 - v_{\chi}^2) \right] \chi = 0 \\ \left[\square - g_{\chi\Phi}\chi^2 - m_{\Phi}^2 - 2g_{\Phi}|\Phi|^2 \right] \Phi = 0 \end{cases}. \quad (4.2.1)$$

As is well known [5], this system admits a spherically symmetric Q-ball solution that has a simple time dependence to evade Derrick's theorem,

$$\Phi_Q(\mathbf{x}, t) = \frac{\phi_Q(r)}{\sqrt{2}} e^{-i\omega_Q t}, \quad \chi_Q(\mathbf{x}, t) = \chi_Q(r), \quad (4.2.2)$$

with $\phi_Q(r) \in \mathbb{R}$, where $\chi_Q \rightarrow v_{\chi}$ and $\phi_Q \rightarrow 0$ at $r \rightarrow \infty$. Several public packages implement different algorithms to solve the bounce equations; a recent overview is given in [35], with examples including

- **CosmoTransitions** [36] and **BSMPT** [37, 38, 35], which use path-deformation-based approaches.
- **AnyBubble** [39], based on multiple shooting.
- **BubbleProfiler** [40], based on a semi-analytic tunneling algorithm [41].
- **SimpleBounce** [42], based on gradient flow methods [43].
- **FindBounce** [44], based on the polygonal bounce approach [45].

We choose to implement the path-deformation method to solve the FLS soliton profiles, which is the main topic of Subsection. 4.2.1 below.

4.2.1 Path deformation and the FLS soliton profiles

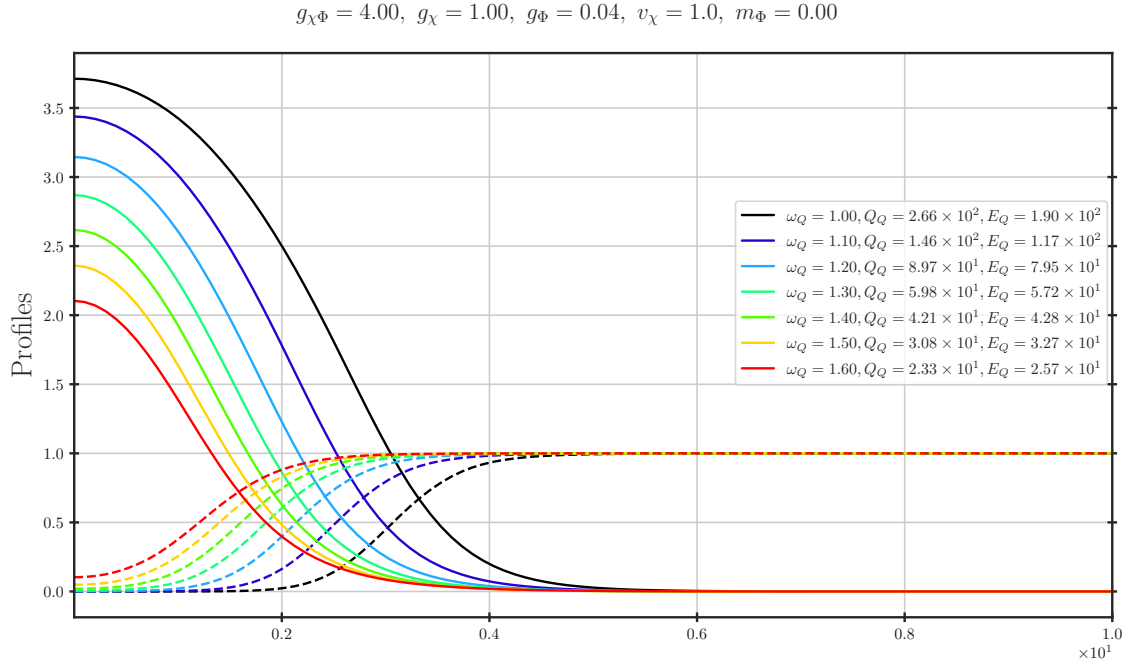


Figure 4.1: The profiles of FLS solitons (ϕ_Q : solid line, χ_Q : dashed line) with various internal frequencies ω_Q , with reported total charge Q_Q and total energy E_Q . All these profiles pass the sanity check of stability against decay into free particles, i.e., $E_Q < Q_Q \sqrt{m_{\Phi}^2 + g_{\chi\Phi} v_{\chi}^2}$.

The numerical determination of the FLS profile is a two-field boundary-value problem. After inserting the ansatz for Φ_Q and χ_Q , the radial equations take the same form as a multi-field bounce equation,

$$\frac{d^2\varphi}{dr^2} + \frac{D-1}{r} \frac{d\varphi}{dr} = \nabla U_{\text{eff}}(\varphi), \quad \varphi(r) \equiv \begin{pmatrix} \phi_Q(r) \\ \chi_Q(r) \end{pmatrix}, \quad (4.2.3)$$

where U_{eff} is the frequency-dependent effective potential for the profile. The profile is regular at the origin and must approach the asymptotic vacuum $(\phi_Q, \chi_Q) = (0, v_\chi)$ as $r \rightarrow \infty$. For a single field, the overshoot/undershoot method works because the ordering of initial conditions determines whether the trajectory overshoots or undershoots the target vacuum. In several fields this one-dimensional topology is lost, so tuning the initial point becomes tuning a whole trajectory in field space.

The path-deformation method addresses this by first assuming that the solution lies on a fixed path in field space (see, e.g., [36]). Let this trial path be written as $\varphi_{\text{path}}(x)$, where x is chosen to be the distance along the path so that $|d\varphi/dx| = 1$. Substituting $\varphi(r) = \varphi_{\text{path}}(x(r))$ into the multi-field equation and decomposing the result into tangent and normal components gives

$$\frac{d^2x}{dr^2} + \frac{D-1}{r} \frac{dx}{dr} = \frac{\partial}{\partial x} U_{\text{eff}}[\varphi_{\text{path}}(x)], \quad (4.2.4)$$

$$\frac{d^2\varphi_{\text{path}}}{dx^2} \left(\frac{dx}{dr} \right)^2 = \nabla_{\perp} U_{\text{eff}}(\varphi_{\text{path}}). \quad (4.2.5)$$

The first equation is an ordinary one-dimensional bounce equation along the assumed path and can be solved by overshoot/undershoot. The second equation is the consistency condition that the assumed path must satisfy in the directions perpendicular to itself. Equivalently, defining the normal force

$$\mathbf{N} = \frac{d^2\varphi_{\text{path}}}{dx^2} \left(\frac{dx}{dr} \right)^2 - \nabla_{\perp} U_{\text{eff}}(\varphi_{\text{path}}), \quad (4.2.6)$$

the correct multi-field solution is reached when $\mathbf{N}$ vanishes, up to numerical tolerance. In the mechanical analogy, the trial path is a track constraining the fictitious particle; $\mathbf{N}$ is the normal force that the track must exert to keep the particle on it. If the track coincides with the true solution, no such external normal force is required.

The practical algorithm is therefore iterative. One starts from a guessed path connecting the approximate interior point $\varphi_0 \simeq \varphi(0)$ to the asymptotic point $\varphi_F = (0, v_\chi)$. For a given path, one solves Eq. (4.2.4) to obtain the one-dimensional profile $x(r)$. Using this profile, one evaluates the normal force (4.2.6) at many points along the path. The path points are then shifted in the direction of this force with a controlled step size. In the implementation described in [36], the normal force is evaluated on a relatively dense set of points along the path, rescaled by the characteristic path length and the largest force scale, and multiplied by an adaptive small step size.⁽¹⁾ Several such deformation steps can be performed while keeping the one-dimensional solution $x(r)$ and dx/dr fixed; after the perpendicular force has been reduced, the one-dimensional bounce along the new path is recomputed and the deformation cycle is repeated. For thick-wall solutions, the starting point φ_0 is also allowed to move, while in thin-wall situations it is kept close to the interior minimum. The step size must be chosen carefully: sufficiently small deformations tend to smooth local path errors, while too large a step can reverse and amplify spurious wiggles. The iteration stops once the perpendicular force is small compared with the force along the path, meaning that solving the one-dimensional problem along the deformed path also approximately solves the full multi-field equations.

We use the method to obtain the two-field trajectory $(\phi_Q(r), \chi_Q(r))$ and cross-check the implementation against **CosmoTransitions**. Some numerical FLS profiles are presented in Fig. 4.1.

⁽¹⁾ One example of parameterizing this path implemented by **CosmoTransitions** is to use splines plus a linear component connecting its endpoints,

$$\varphi_{\text{path}}(y) = \sum_i \beta_i \mathbf{B}_i(y) + (\varphi_0 - \varphi_F)y + \varphi_F, \quad 0 \leq y \leq 1, \quad (4.2.7)$$

where the spline basis functions vanish at the endpoints so that the endpoint values remain fixed during a deformation step. After moving the points, the spline coefficients are refitted to the deformed path.

4.3 Perturbations around a FLS Q-ball background

4.3.1 Equation of linear perturbations and solutions in limiting cases

We then consider small perturbations around the background of such a Q-ball configuration, i.e., $\Phi = \Phi_Q + \Phi_1$ and $\chi = \chi_Q + \chi_1$. The corresponding equations of motion up to first order in the perturbation fields are

$$\begin{cases} \left[\square - \frac{\partial^2 U}{\partial \chi^2} \Big|_{(\chi, \Phi) \rightarrow (\chi_Q, \Phi_Q)} \right] \chi_1 - \left[\frac{\partial^2 U}{\partial \chi \partial \Phi} \Big|_{(\chi, \Phi) \rightarrow (\chi_Q, \Phi_Q)} \Phi_1 + h.c \right] = 0 \\ \left[\square - \frac{\partial^2 U}{\partial \Phi^* \partial \Phi} \Big|_{(\chi, \Phi) \rightarrow (\chi_Q, \Phi_Q)} \right] \Phi_1 - \frac{\partial^2 U}{\partial (\Phi^*)^2} \Big|_{(\chi, \Phi) \rightarrow (\chi_Q, \Phi_Q)} \Phi_1^* - \frac{\partial^2 U}{\partial \Phi^* \partial \chi} \Big|_{(\chi, \Phi) \rightarrow (\chi_Q, \Phi_Q)} \chi_1 = 0 \end{cases} \quad (4.3.1)$$

$$\Rightarrow \begin{cases} \left[\square - g_{\chi\Phi} \phi_Q^2 - 4g_{\chi} (3\chi_Q^2 - v_{\chi}^2) \right] \chi_1 - \sqrt{2} g_{\chi\Phi} \chi_Q \phi_Q (\Phi_1 e^{+i\omega_Q t} + \Phi_1^* e^{-i\omega_Q t}) = 0 \\ \left(\square - g_{\chi\Phi} \chi_Q^2 - m_{\Phi}^2 - 2g_{\Phi} \phi_Q^2 \right) \Phi_1 - g_{\Phi} \phi_Q^2 e^{-2i\omega_Q t} \Phi_1^* - \sqrt{2} g_{\chi\Phi} \chi_Q \phi_Q e^{-i\omega_Q t} \chi_1 = 0 \end{cases} \quad (4.3.2)$$

Due to linearity, this system admits plane-wave solutions of the following form:

$$\chi_1 = \eta_{\chi} e^{-i\omega t} + \eta_{\chi}^* e^{+i\omega t}, \quad \Phi_1 = \eta_{+} e^{-i\omega_{+} t} + \eta_{-} e^{-i\omega_{-} t}, \quad (4.3.3)$$

with $\omega_{\pm} \equiv \omega_Q \pm \omega$. By substituting this ansatz into (4.3.2), we obtain the equations for the different modes:

$$\left[\frac{1}{r^{D-1}} \partial_r (r^{D-1} \partial_r) + \omega^2 - g_{\chi\Phi} \phi_Q^2 - 4g_{\chi} (3\chi_Q^2 - v_{\chi}^2) \right] \eta_{\chi} - \sqrt{2} g_{\chi\Phi} \chi_Q \phi_Q (\eta_{+} + \eta_{-}^*) = 0 \quad (4.3.4a)$$

$$\left[\frac{1}{r^{D-1}} \partial_r (r^{D-1} \partial_r) + \omega_{+}^2 - g_{\chi\Phi} \chi_Q^2 - m_{\Phi}^2 - 2g_{\Phi} \phi_Q^2 \right] \eta_{+} - g_{\Phi} \phi_Q^2 \eta_{-}^* - \sqrt{2} g_{\chi\Phi} \chi_Q \phi_Q \eta_{\chi} = 0 \quad (4.3.4b)$$

$$\left[\frac{1}{r^{D-1}} \partial_r (r^{D-1} \partial_r) + \omega_{-}^2 - g_{\chi\Phi} \chi_Q^2 - m_{\Phi}^2 - 2g_{\Phi} \phi_Q^2 \right] \eta_{-} - g_{\Phi} \phi_Q^2 \eta_{+}^* - \sqrt{2} g_{\chi\Phi} \chi_Q \phi_Q \eta_{\chi}^* = 0 \quad (4.3.4c)$$

From these equations, we can explicitly obtain the linearity property: if $(\eta_{\chi}^{(1)}, \eta_{+}^{(1)}, \eta_{-}^{(1)})$ and $(\eta_{\chi}^{(2)}, \eta_{+}^{(2)}, \eta_{-}^{(2)})$ are solutions of the system, then $(\eta_{\chi}^{(1)} + \xi \eta_{\chi}^{(2)}, \eta_{+}^{(1)} + \xi \eta_{+}^{(2)}, \eta_{-}^{(1)} + \xi^* \eta_{-}^{(2)})$ is also a solution of the system, with ξ a complex constant. Analytical solutions of (4.3.4) exist in two extreme limits:

- **Asymptotic region $r \rightarrow \infty$:** as $\phi_Q(r \rightarrow \infty) = 0$ and $\chi_Q(r \rightarrow \infty) = v_{\chi}$, the system of linearized equations reduces to

$$\begin{aligned} & \left[\frac{1}{r^{D-1}} \partial_r (r^{D-1} \partial_r) + k_{\chi}^2 \right] \eta_{\chi} = 0 \\ & \Rightarrow \eta_{\chi}(r \rightarrow \infty) = \frac{\tilde{A}_{\chi}}{(k_{\chi} r)^{\frac{D-2}{2}}} J_{\frac{D-2}{2}}(k_{\chi} r) + \frac{\tilde{B}_{\chi}}{(k_{\chi} r)^{\frac{D-2}{2}}} Y_{\frac{D-2}{2}}(k_{\chi} r) \\ & \xrightarrow{r \rightarrow \infty} \frac{A_{\chi}}{(k_{\chi} r)^{\frac{D-2}{2}}} e^{ik_{\chi} r} + \frac{B_{\chi}}{(k_{\chi} r)^{\frac{D-2}{2}}} e^{-ik_{\chi} r}, \end{aligned} \quad (4.3.5)$$

$$\begin{aligned} & \left[\frac{1}{r^{D-1}} \partial_r (r^{D-1} \partial_r) + k_{\pm}^2 \right] \eta_{\pm} = 0 \\ & \eta_{\pm}(r \rightarrow \infty) = \frac{\tilde{A}_{\pm}}{(k_{\pm} r)^{\frac{D-2}{2}}} J_{\frac{D-2}{2}}(k_{\pm} r) + \frac{\tilde{B}_{\pm}}{(k_{\pm} r)^{\frac{D-2}{2}}} Y_{\frac{D-2}{2}}(k_{\pm} r) \\ & \xrightarrow{r \rightarrow \infty} \frac{A_{\pm}}{(k_{\pm} r)^{\frac{D-2}{2}}} e^{ik_{\pm} r} + \frac{B_{\pm}}{(k_{\pm} r)^{\frac{D-2}{2}}} e^{-ik_{\pm} r}, \end{aligned} \quad (4.3.6)$$

where the momenta of the perturbations

$$k_\chi^2 \equiv \omega^2 - 8g_\chi v_\chi^2, \quad k_\pm^2 \equiv \omega_\pm^2 - g_\chi \Phi v_\chi^2 - m_\Phi^2. \quad (4.3.7)$$

Here $8g_\chi v_\chi^2$ and $g_\chi \Phi v_\chi^2 + m_\Phi^2$ are respectively the squared masses of the neutral and charged fields far away from the soliton's core. If we assume all modes correspond to propagating spherical waves, we need to impose the reality condition on the wavenumber, i.e.,

$$\begin{cases} k_\chi^2 > 0 \\ k_\pm^2 > 0 \end{cases} \Leftrightarrow \begin{cases} |\omega| > \sqrt{8g_\chi} v_\chi \\ |\omega| > \omega_Q + \sqrt{g_\chi \Phi v_\chi^2 + m_\Phi^2} \end{cases}. \quad (4.3.8)$$

- **Near-origin region $r \rightarrow 0$:** similarly to the case of the one-field Q-ball, we have

$$\eta_\chi(\mathbf{C}_\chi, r \rightarrow 0) = C_{1,\chi} + \frac{C_{2,\chi}}{(|k_\chi| r)^{D-2}}, \quad (4.3.9)$$

$$\eta_\pm(\mathbf{C}_\pm, r \rightarrow 0) = C_{1,\pm} + \frac{C_{2,\pm}}{(|k_\pm| r)^{D-2}}. \quad (4.3.10)$$

For every choice of $(\mathbf{C}_\chi, \mathbf{C}_\pm)$, the initial-value problem gives a set of $(A_\chi, A_\pm, B_\chi, B_\pm)$ by simply inverting the values of the perturbations at two points far away, e.g., at αr_∞ and r_∞ with $r_\infty \gg R_Q$ and $\alpha \sim \mathcal{O}(1)$, where R_Q is the effective radius of the Q-ball. This is the same approach to solving the incoming-outgoing amplitudes that we used in the discussion of the one-field model.

Incoming-outgoing basis for amplitudes: the tuning of initial conditions to have only one incoming mode can be done by using the linearity property of the perturbations in the same manner as in the one-field model discussed previously. Similarly, it is also useful to express the perturbations in the incoming-outgoing basis:

$$\begin{cases} \eta_\chi(r \rightarrow \infty) = \frac{1}{(|k_\chi| r)^{(D-1)/2}} (A_\chi^{\text{out}} e^{ik_\chi r} + A_\chi^{\text{in}} e^{-ik_\chi r}) \\ \eta_\pm(r \rightarrow \infty) = \frac{1}{(|k_\pm| r)^{(D-1)/2}} (A_\pm^{\text{out}} e^{ik_\pm r} + A_\pm^{\text{in}} e^{-ik_\pm r}) \end{cases}$$

$$k_\chi \equiv s_\omega \sqrt{\omega^2 - 8g_\chi v_\chi^2}, \quad k_\pm \equiv \pm s_\omega \sqrt{\omega_\pm^2 - g_\chi \Phi v_\chi^2 - m_\Phi^2}. \quad (4.3.11)$$

The conversion between the two conventions, given $\omega_Q > 0$ without loss of generality, is

$$\begin{cases} \omega > 0 : & A_+^{\text{out}} = A_+, & A_+^{\text{in}} = B_+, & A_-^{\text{out}} = B_-, & A_-^{\text{in}} = A_-, & A_\chi^{\text{out}} = A_\chi, & A_\chi^{\text{in}} = B_\chi, \\ \omega < 0 : & A_+^{\text{out}} = B_+, & A_+^{\text{in}} = A_+, & A_-^{\text{out}} = A_-, & A_-^{\text{in}} = B_-, & A_\chi^{\text{out}} = B_\chi, & A_\chi^{\text{in}} = A_\chi. \end{cases} \quad (4.3.12)$$

4.3.2 Particle number conservation

Note that the system of equations of motion (4.3.4) has a $U(1)$ symmetry $\eta_\chi \rightarrow e^{+i\alpha} \eta_\chi$ and $\eta_\pm \rightarrow e^{\pm i\alpha} \eta_\pm$, as well as a discrete symmetry $\eta_\chi \rightarrow \eta_\chi^*$ and $\eta_\pm \rightarrow \eta_\mp$ inherited from the global $U(1)$ symmetry and the $\mathbb{Z}_2$ symmetry of the original FLS Lagrangian. Due to this $U(1)$ symmetry, there is a conserved quantity that we later interpret as particle number. To see this more clearly, let us write down a Lagrangian corresponding to the above Euler-Lagrange equations:

$$\begin{aligned} \frac{\mathcal{L}_\eta}{r^{D-1}} = & \sum_{s=\pm, \chi} |\partial_r \eta_s|^2 + \sum_{\eta \in \{\eta_+, \eta_-, \eta_\chi\}} m_\eta^2 \eta^* \eta \\ & + \left[\left(\sqrt{2} g_\chi \Phi \chi_Q \phi_Q \right) (\eta_+ + \eta_-^*) \eta_\chi + g_\Phi \phi_Q^2 \eta_+^* \eta_- + h.c. \right], \end{aligned} \quad (4.3.13)$$

where the “effective mass” m_η^2 hides the terms inside the brackets in Eq. (4.3.4) following the r derivative term. From the $U(1)$ symmetry transformation of $\mathcal{L}_\eta$, we obtain the conserved current

$$\begin{aligned} J_N^\eta(r) &= \sum_{s=\pm,\chi} \frac{\partial \mathcal{L}_\eta}{\partial (\partial_r \eta_s)} \frac{\delta \eta_s}{\delta \alpha} + h.c \\ &= i r^{D-1} \sum_{s=\pm,\chi} s (\eta_s \partial_r \eta_s^*) + h.c, \end{aligned} \quad (4.3.14)$$

with $\partial_r J_N^\eta(r) = 0$. Similarly to the one-field Q-ball case in the linear perturbation analysis, this current $J_N^\eta(r)$ is a conserved current corresponding to total particle-number conservation. ⁽¹⁾ At large r , the Noether charge is expressed in terms of the asymptotic amplitudes as

$$J_N^\eta = 2 \left(\frac{|A_\chi|^2 - |B_\chi|^2}{k_\chi^{D-2}} + \frac{|A_+|^2 - |B_+|^2}{k_+^{D-2}} - \frac{|A_-|^2 - |B_-|^2}{k_-^{D-2}} \right). \quad (4.3.16)$$

At $r \rightarrow 0$, instead, we have $J_N^\eta = 0$. This implies the **conservation of particle number** for generic scattering between incoming and outgoing waves:

$$N_{\text{tot}} = \frac{|A_\chi|^2}{k_\chi^{D-2}} + \frac{|A_+|^2}{k_+^{D-2}} + \frac{|B_-|^2}{k_-^{D-2}} = \frac{|B_\chi|^2}{k_\chi^{D-2}} + \frac{|B_+|^2}{k_+^{D-2}} + \frac{|A_-|^2}{k_-^{D-2}} \quad (4.3.17)$$

or, equivalently, in the incoming-outgoing basis:

$$N_{\text{tot}} = \sum_{s=\pm,\chi} N_s^{\text{in}} = \sum_{s=\pm,\chi} N_s^{\text{out}}, \quad (4.3.18)$$

where we define the particle numbers

$$N_s^{\text{in}} = \frac{|A_s^{\text{in}}|^2}{|k_s|^{D-2}}, \quad N_s^{\text{out}} = \frac{|A_s^{\text{out}}|^2}{|k_s|^{D-2}}. \quad (4.3.19)$$

4.3.3 Amplification factors and the scattering matrix in FLS model

Similarly to the one-field case, the quantities of interest are the amplification factors of charge and energy, which basically tell us under which circumstances these physical quantities are enhanced or reduced after a wavepacket scatters off an FLS soliton. The derivation is similar to that in Subsection. 2.3.4, so we only write down the results here:

- Amplification factor as the ratio of the energy contained within a spherical shell far away

$$\mathcal{A}_E^\rho = \left| \frac{\frac{\omega^2 |A_\chi|^2}{(k_\chi)^{D-1}} + \frac{\omega_+^2 |A_+|^2}{k_+^{D-1}} + \frac{\omega_-^2 |B_-|^2}{k_-^{D-1}}}{\frac{\omega^2 |B_\chi|^2}{(k_\chi)^{D-1}} + \frac{\omega_+^2 |B_+|^2}{k_+^{D-1}} + \frac{\omega_-^2 |A_-|^2}{k_-^{D-1}}} \right|^{s_\omega} \equiv 1 + Z_E^\rho. \quad (4.3.20)$$

Note that since only Φ is charged under $U(1)_Q$, there are no contributions from A_χ and B_χ in $\mathcal{A}_Q^\rho$.

⁽¹⁾ Another equivalent way to express this particle-number conservation law is

$$\mathbf{J}_N^\eta = 2 \text{Im} (\eta_\chi \nabla \eta_\chi^* + \eta_+ \nabla \eta_+^* - \eta_- \nabla \eta_-^*), \quad \nabla \cdot \mathbf{J}_N^\eta = 0 \quad (4.3.15)$$

- Amplification factor as the ratio of the fluxes of incoming and outgoing fields far away

$$\mathcal{A}_Q^{\mathcal{F}} = \left(\frac{\frac{|A_+|^2}{k_+^{D-2}} - \frac{|B_-|^2}{k_-^{D-2}}}{\frac{|B_+|^2}{k_+^{D-2}} - \frac{|A_-|^2}{k_-^{D-2}}} \right)^{s_\omega} \equiv 1 + Z_Q^{\mathcal{F}}, \quad (4.3.21)$$

$$\mathcal{A}_E^{\mathcal{F}} = \left| \frac{\frac{\omega |A_\chi|^2}{(k_\chi)^{D-2}} + \frac{\omega_+ |A_+|^2}{k_+^{D-2}} - \frac{\omega_- |B_-|^2}{k_-^{D-2}}}{\frac{\omega |B_\chi|^2}{(k_\chi)^{D-2}} + \frac{\omega_+ |B_+|^2}{k_+^{D-2}} - \frac{\omega_- |A_-|^2}{k_-^{D-2}}} \right|^{s_\omega} \equiv 1 + Z_E^{\mathcal{F}}. \quad (4.3.22)$$

Following the discussion of scattering in the S-matrix formalism in Subsection. 2.3.5, we now consider the extension of the S-matrix for the amplitude basis of the FLS model, $\vec{A}^{\text{in,out}} \equiv (A_+, A_*, A_\chi)^{\text{in,out}}$, where

$$|K|^{(2-D)/2} \vec{A}^{\text{out}} = S |K|^{(2-D)/2} \vec{A}^{\text{in}}, \quad (4.3.23)$$

with $K = \text{diag}(k_+, k_-, k_\chi)$. All properties of the S-matrix in the one-field model propagate to the FLS model, and can generically be generalized straightforwardly to more complex models with more fields. These include unitarity, $S^\dagger S = \mathbb{1}$, and symmetry, $S^T = S$, which constrain the number of degrees of freedom of the scattering matrix to 6. However, solving (4.3.4) with one initial condition leads to only $(2n - 1) = 5$ equations, which are not sufficient to fix the S-matrix. Furthermore, due to the richer field content, more complex symmetry patterns arise. Specifically, under the $\mathbb{Z}_2$ transformation $\omega \rightarrow -\omega$, we have $(\eta_\pm, \eta_\chi) \rightarrow (\eta_\mp^*, \eta_\chi^*)$ up to a complex scaling factor and a complex conjugation. This observation follows directly from the equations of motion. It imposes the following constraints, which are independent of unitarity and symmetry:

$$|S^{++}(-\omega)| = |S^{--}(\omega)|, \quad |S^{+\chi}(-\omega)| = |S^{-\chi}(\omega)|, \quad (4.3.24)$$

$$|S^{+-}(-\omega)| = |S^{+-}(\omega)|, \quad |S^{\chi\chi}(-\omega)| = |S^{\chi\chi}(\omega)|. \quad (4.3.25)$$

Together with the constraints from symmetry, $S^{ij}(\omega) = S^{ji}(\omega)$, and unitarity, these imply constraints between the incoming-outgoing amplitudes, or particle numbers, of the perturbations, for example

$$\boxed{\begin{aligned} \frac{N_+^{\text{out}}(\omega)}{N_-^{\text{in}}(\omega)} \Big|_{N_{+, \chi}^{\text{in}}=0} &= \frac{N_-^{\text{out}}(\omega)}{N_+^{\text{in}}(\omega)} \Big|_{N_{\chi, -}^{\text{in}}=0}, & \frac{N_-^{\text{out}}(-\omega)}{N_+^{\text{in}}(-\omega)} \Big|_{N_{-, \chi}^{\text{in}}=0} &= \frac{N_+^{\text{out}}(\omega)}{N_-^{\text{in}}(\omega)} \Big|_{N_{+, \chi}^{\text{in}}=0}, & (4.3.26a) \\ \frac{N_+^{\text{out}}(\omega)}{N_\chi^{\text{in}}(\omega)} \Big|_{N_{+, -}^{\text{in}}=0} &= \frac{N_\chi^{\text{out}}(\omega)}{N_+^{\text{in}}(\omega)} \Big|_{N_{\chi, -}^{\text{in}}=0}, & \frac{N_-^{\text{out}}(\omega)}{N_\chi^{\text{in}}(\omega)} \Big|_{N_{-, -}^{\text{in}}=0} &= \frac{N_\chi^{\text{out}}(\omega)}{N_-^{\text{in}}(\omega)} \Big|_{N_{\chi, +}^{\text{in}}=0}. & (4.3.26b) \end{aligned}}$$

Next, recall that the amplification factors of a spherical incoming wavepacket can be expressed in terms of the S-matrix elements

$$\mathcal{A}_Q = \left| \frac{|\sum_i \alpha_i S^{+i}|^2 - |\sum_i \alpha_i S^{-i}|^2}{|\alpha_+|^2 - |\alpha_-|^2} \right|, \quad (4.3.27)$$

$$\mathcal{A}_E^{\mathcal{F}} = \left| \frac{\omega_+ |\sum_i \alpha_i S^{+i}|^2 - \omega_- |\sum_i \alpha_i S^{-i}|^2 + \omega |\sum_i \alpha_i S^{\chi i}|^2}{\omega_+ |\alpha_+|^2 - \omega_- |\alpha_-|^2 + \omega |\alpha_\chi|^2} \right|, \quad (4.3.28)$$

where, similarly to the one-field discussion in Subsection. 2.3.5, $\alpha_i \equiv \frac{|k_i|^{(2-D)/2} A_i^{\text{in}}}{\sum_j |k_j|^{(2-D)/2} A_j^{\text{in}}}$ is the “amplitude fraction” of mode i , assuming $\sum_j |k_j|^{(2-D)/2} A_j^{\text{in}} \neq 0$. There is no change in the expression

for $\mathcal{A}_Q$ because the additional mode in the FLS model is neutral, while $\mathcal{A}_E$ indeed has an extra contribution from this neutral mode.

Let us look at the energy exchange between the Q-ball and its perturbations. We set $\omega_Q > 0$ and assume that $\omega > 0$; the case $\omega < 0$ corresponds to an exchange of the $\pm$ modes. Then the energies of the coupled perturbations satisfy

$$\omega_+ > \omega > |\omega_-| \Rightarrow E_+ > E_\chi > E_- . \quad (4.3.29)$$

Let us assume that the initial state is η_- . Then the only possible reactions are

$$\begin{aligned} Q + \phi^\dagger &\rightarrow (Q - 2) + \phi \\ Q + \phi^\dagger &\rightarrow (Q - 1) + \chi . \end{aligned} \quad (4.3.30)$$

The relations in Eq. (4.3.29) predict energy extraction in both cases with ϕ and χ final states, and the total energy release is given by

$$\begin{aligned} 1 + Z_E &= \left| \frac{\omega}{\omega_+} + \frac{\omega_Q}{\omega_+} (1 + Z_Q) \right| = \left| 1 + \frac{\omega_Q}{\omega_+} Z_Q \right| , \\ \text{where: } 1 + Z_Q &= N_+^{\text{out}} - N_-^{\text{out}} , \quad N_+^{\text{in}} = 1, N_-^{\text{in}} = 0 . \end{aligned} \quad (4.3.31)$$

Following the same logic, one can see that if the initial state is η_+ , both possible reactions

$$\begin{aligned} Q + \phi &\rightarrow (Q + 1) + \chi \\ Q + \phi &\rightarrow (Q + 2) + \phi^\dagger \end{aligned} \quad (4.3.32)$$

proceed with energy absorption. If the initial state is χ , then the energy enhancement is equal to

$$\begin{aligned} 1 + Z_E &= \mathcal{A}_E^F = \left| 1 + \frac{\omega_Q}{\omega} \frac{N_+^{\text{out}} - N_-^{\text{out}}}{N_\chi^{\text{in}}} \right| \\ &= \left| 1 + \frac{\omega_Q}{\omega} \left[\left| \frac{k_\chi}{k_+} \right| \left| \frac{A_+^{\text{out}}}{A_\chi^{\text{in}}} \right|^2 - \left| \frac{k_\chi}{k_-} \right| \left| \frac{A_-^{\text{out}}}{A_\chi^{\text{in}}} \right|^2 \right] \right| . \end{aligned} \quad (4.3.33)$$

One can see that, depending on the values of N_+^{out} or N_-^{out} , there can be energy release or absorption by the Q-ball. In particle language, there are two possible reactions:

$$\begin{aligned} Q + \chi &\rightarrow (Q + 1) + \phi^\dagger \text{ energy absorption,} \\ Q + \chi &\rightarrow (Q - 1) + \phi \text{ energy release.} \end{aligned} \quad (4.3.34)$$

Depending on their rates, the energy will be either absorbed or released. The squared S-matrix elements $|S^{fi}|^2$ corresponding to $N_f^{\text{out}}/N_i^{\text{in}}$ for processes with a monochromatic incoming mode can be calculated numerically, and we report our results for a particular set of couplings in Fig. 4.2. Once the components S^{fi} are known, it is straightforward to compute the amplification factors for energy and charge. Numerical results for specific scattering off a Q-ball are shown in Figs. 4.3 and 4.4. As expected from the previous discussion, the energy is amplified with $\omega < 0$ and attenuated with $\omega > 0$, while the charge of the incoming mode is always attenuated. Typically, the amplification effects are stronger than the attenuation effects, and these energy/charge exchange mechanisms are much weaker for high-frequency incoming modes. At frequencies ω very close to the threshold of having two propagating charged modes, both Z_E and Z_Q drop to zero. At frequencies lower than this threshold, we have elastic scattering where no energy or charge transfer occurs.

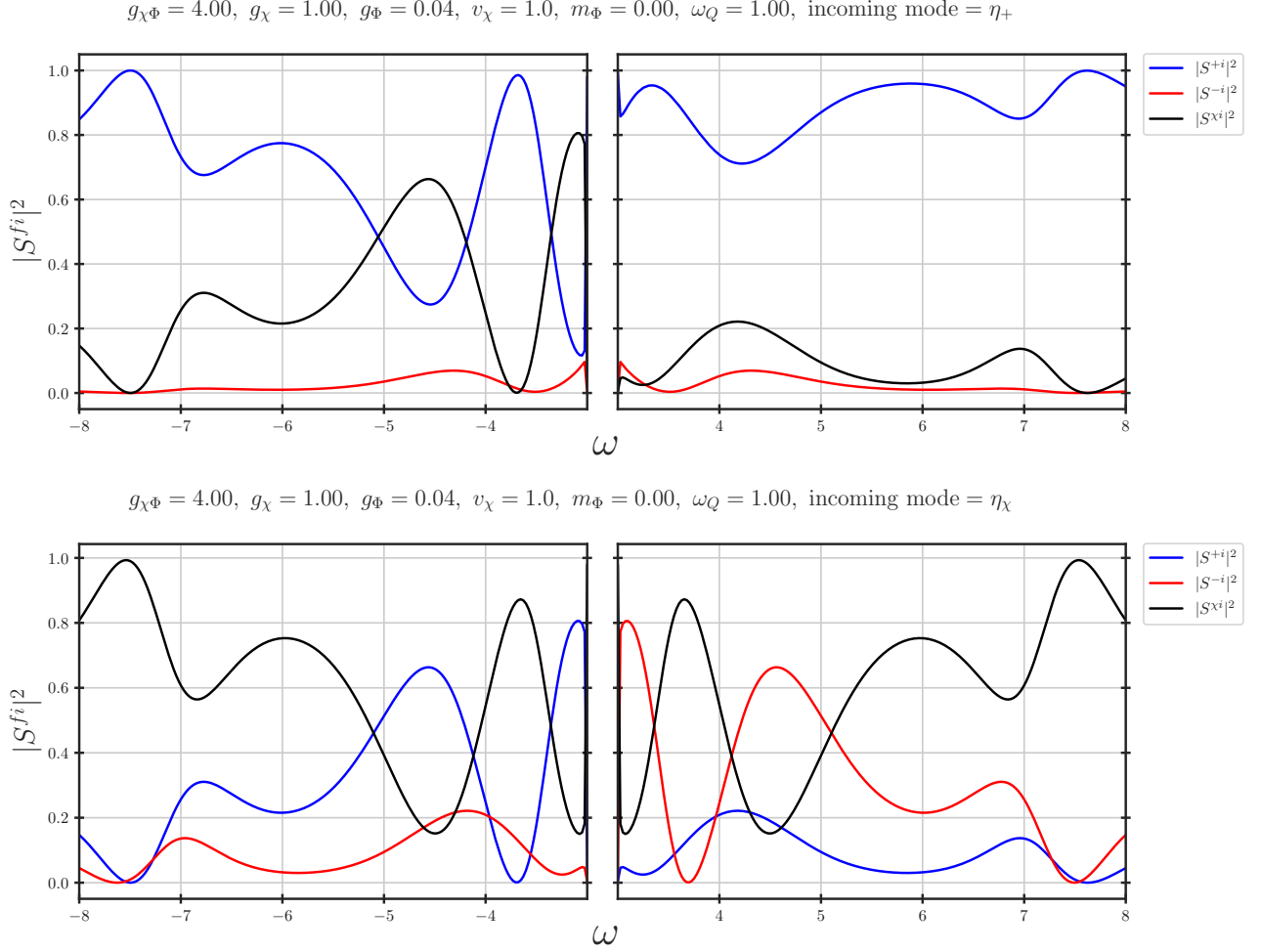


Figure 4.2: Squared S-matrix elements, with incoming mode η_+ (top plot) and η_χ (bottom plot). Different colors represent different outgoing modes.

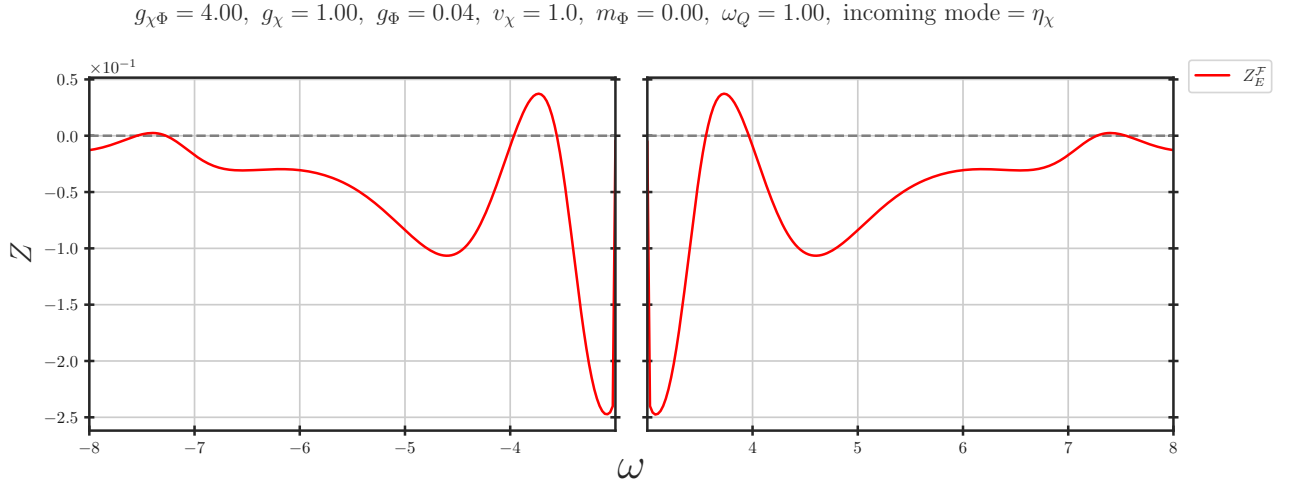


Figure 4.4: Energy amplification factors with incoming mode η_χ .

In Fig. 4.4, we present the energy extraction for the η_χ incoming mode. The symmetry of Z under $\omega \rightarrow -\omega$ follows the symmetry properties of the S matrix. Note that the sign of Z is not fixed, and we can have both energy absorption and extraction depending on the energy of the initial mode. Since the incoming charge is 0, it is not well-defined to talk about Z_Q here.

More numerical results for the S-matrix elements and amplification factors in the FLS model are

$g_{\chi\Phi} = 4.00$, $g_{\chi} = 1.00$, $g_{\Phi} = 0.04$, $v_{\chi} = 1.0$, $m_{\Phi} = 0.00$, $\omega_Q = 1.00$, incoming mode = η_+

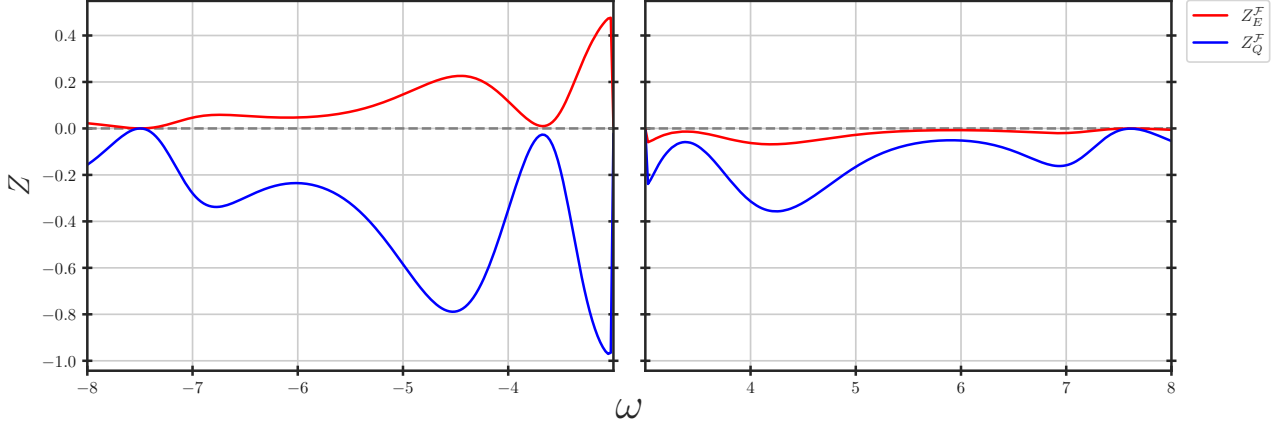


Figure 4.3: Energy and charge amplification factors with incoming mode η_+ .

presented in Appendix. C. There we observe more clearly the symmetry patterns for various scattering channels, which are consequences of the symmetry properties of the scattering matrix. At the end of this appendix, we also discuss the setup for the lattice simulation of the two-field model, which serves as a test of the circumstances under which we can trust the linear regime.

4.3.4 Propagating and bounded modes

Depending on the model parameters, some of the three modes $(\eta_{\pm}, \eta_{\chi})$, which can transform into each other, will become bounded. Based on this, we can identify the various parameter-space regions according to the propagating field content. To be explicit, let us choose $\omega_Q > 0$; and based on the sign of k_{χ}^2 and $k_{\pm}^2$ defined in Eq. 4.3.7, the conditions for modes being propagating/bounded are the followings:

- $\omega^2 > 8g_{\chi}v_{\chi}^2$ & $(\omega \pm \omega_Q)^2 > g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$: all three modes $\eta_{\pm}, \eta_{\chi}$ are propagating.
- $\omega^2 < 8g_{\chi}v_{\chi}^2$ & $(\omega \pm \omega_Q)^2 > g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$: $\eta_{\pm}$ are propagating, and η_{χ} is bounded.
- $\omega^2 > 8g_{\chi}v_{\chi}^2$ & $(\omega \pm \omega_Q)^2 < g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$: $\eta_{\pm}, \eta_{\chi}$ are propagating, and $\eta_{\mp}$ is bounded.
- $\omega^2 < 8g_{\chi}v_{\chi}^2$ & $(\omega \mp \omega_Q)^2 < g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$ & $(\omega \pm \omega_Q)^2 > g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$: $\eta_{\pm}$ is propagating, and $\eta_{\mp}, \eta_{\chi}$ are bounded.
- $\omega^2 > 8g_{\chi}v_{\chi}^2$ & $(\omega \pm \omega_Q)^2 < g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$: only η_{χ} is propagating.
- $\omega^2 < 8g_{\chi}v_{\chi}^2$ & $(\omega \pm \omega_Q)^2 < g_{\chi\Phi}v_{\chi}^2 + m_{\Phi}^2$: all of the modes are bounded.

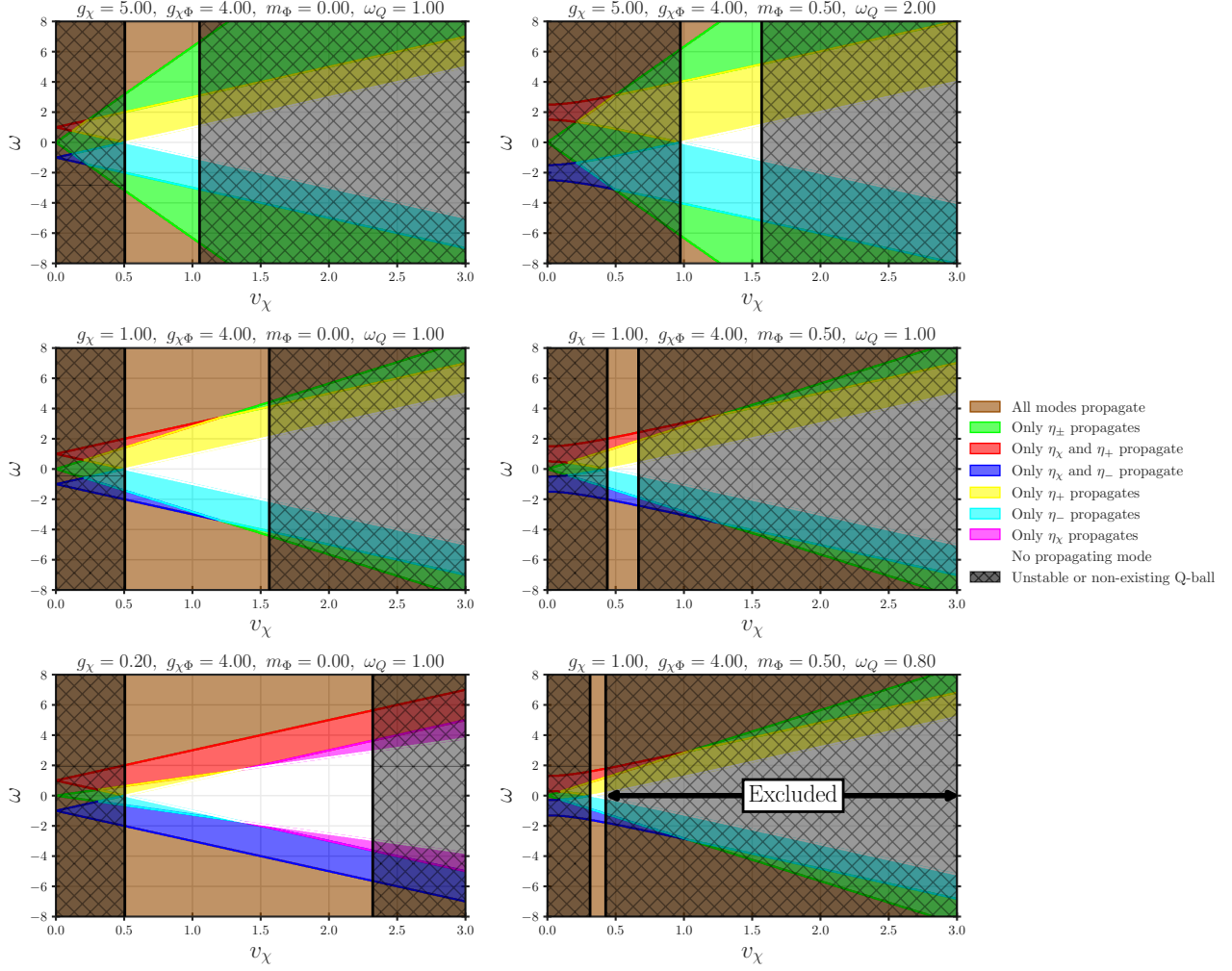


Figure 4.5: Constraints on (v_χ, ω) for different propagating modes as a “color mixer” plot. The first column corresponds to $m_\Phi = 0$ with varying g_χ , while the second column corresponds to $m_\Phi = 0.5$ with varying ω_Q . The gray shaded region indicates parameter space where a Q-ball profile is either nonexistent or not stable under decay into quanta for the fixed g_ϕ coupling value $g_\phi = 0.04$. Note that modifying only this coupling will alter the gray area, while leaving everything else unchanged. The parameter set that we consistently used in the previous calculations corresponds to the middle left figure.

These regions are visualized in Fig. 4.5, where the symmetry $(\eta_\pm, \eta_\chi) \rightarrow (\eta_\mp^*, \eta_\chi^*)$ under $\omega \rightarrow -\omega$ is reflected by the symmetry of the color patterns around the axis $\omega = 0$. Furthermore, for arbitrary potential parameters, at sufficiently high ω one can always find that all modes propagate. Conversely, with fixed ω and a sufficiently large symmetry-breaking scale v_χ , we observe that all modes become bounded, provided that Q-ball solutions exist for such a potential.

Summary of Q-ball superradiance

In this work, we expanded upon the investigation of Q-ball perturbations, drawing on previous works [2, 3], with particular emphasis on non-rotating Q-balls in 3+1 dimensions. Our findings enhance the understanding of Q-ball formation mechanisms in early-universe cosmology, particularly within models involving solitosynthesis.

Compared to the original works, the novelty of our results lies in three distinct yet interconnected directions.

- First, we provide an intuitive explanation of the energy extraction process, linking it to differences in the binding energies of Q-balls with varying charges, analogous to energy release in nuclear reactions.
- Second, we analyzed in detail the physics of linear perturbations in the context of a single complex scalar field model. We identified the symmetry properties of various transition rates and carefully determined the validity range of linear perturbation analysis for Q-balls using the 1 + 1D lattice simulation for the sextic potential

$$U = |\Phi|^2 - |\Phi|^4 + g|\Phi|^6$$

This enables us to estimate when, in the early universe, linear analysis leads to reliable predictions for Q-ball interactions with plasma particles. Interestingly, we find that the scattering of nonrelativistic particles off the Q-ball most likely leads to its charge and energy reduction at the classical level. This is the case where the Q-balls, after formation, are surrounded by a heat bath of sufficiently low temperature.

- Third, we analyzed, for the first time, the perturbations of Q-ball solution of the FLS model, whose potential is given by

$$U(|\Phi|, \chi) = g_\chi \Phi \chi^2 |\Phi|^2 + g_\chi (\chi^2 - v_\chi^2)^2 + m_\Phi^2 |\Phi|^2 + g_\Phi |\Phi|^4.$$

We demonstrated that energy extraction occurs in this case via transformations among modes with three distinct energies. We also illustrated that all relevant physical quantities such as particle number and amplification factors can be expressed through the language of a scattering matrix connecting incoming and outgoing modes. We then use the properties of this scattering matrix to show symmetry patterns observed within the spectrum of these observables, both in the context of single field model and the FLS model.

Part II

A Surface-Tension-Based Approximation for Gravitational Waves from Vacuum Bubble Collisions

Outline of Gravitational Wave part in this thesis

Within the scope of this thesis, we focus on stochastic gravitational waves sourced by first-order phase transitions, with particular emphasis on the regime where the scalar-field bubble-collision contribution dominates. Gravitational waves are physically interesting because they can carry information about violent processes in the early universe that may be inaccessible to collider experiments or electromagnetic observations. One of the theoretical difficulties is that the source of interest is intrinsically nonlinear: bubbles nucleate, expand, collide, and then undergo post-collision field evolution whose details depend on the underlying scalar theory. Full real-time lattice simulations can follow this dynamics directly, but they are in general expensive to perform. The central question in this part is therefore how to retain the important microscopic information about post-collision scalar-field dynamics while keeping the many-bubble gravitational-wave calculation computationally tractable. This motivates the generalized bulk-flow method, an approximation scheme based on tracing the evolution of the bubble-wall surface tension before and after collision.

This part of the thesis is organized as follows:

- In Chapter. 6, we review the basics of gravitational waves. Section. 6.1 reviews the weak-field description of metric perturbations and the effective energy density carried by the gravitational waves. Section. 6.2 then shifts to discuss stochastic backgrounds and the corresponding broad landscape of sources, either of cosmological or astrophysical origins. The chapter closes by summarizing detector frequency bands relevant for interpreting a stochastic signal.
- In Chapter. 7, we specialize the general SGWB discussion to first-order phase transitions in the early universe. Section. 7.1 reviews the finite-temperature effective potential, bubble nucleation and percolation, and the macroscopic transition parameters that enter gravitational-wave predictions. The chapter then briefly discusses stochastic gravitational waves from first-order phase transitions, including the main three source classes: scalar-field bubble collisions, acoustic waves and turbulence. After this general discussion, Section. 7.2 focuses on derivation of approximations used specifically in the regime where the scalar-field bubble-collision contribution is dominant. Starting from Weinberg’s formula and the separation between macroscopic scales of phase transition versus the microphysics scale, the envelope and bulk-flow approximations are then reviewed as two limiting prescriptions for the post-collision evolution of the scalar-wall surface tension. This chapter ends by analyzing some drawbacks of these previously developed approximation, which motivates the development of the generalized bulk-flow approximation in the next chapter.
- Chapter. 8 focuses on the construction, validation and application of the generalized bulk-flow approximation for various systems. We start by first constructing the generalized bulk-flow approximation in Section. 8.1. The construction keeps the reusable many-bubble geometry of the envelope and bulk-flow calculations, but replaces their fixed post-collision rules by a calibrated expansion of the surface-tension decay. We then explain how this approximation is extremely efficient in computing the GW spectrum with good precision by separating the geometry-dependent functions $C_{+,x}^{(\xi)}$ from the model-dependent coefficients a_ξ . Section. 8.2 validates the method using two-bubble lattice simulations, where the directly computed gravitational-wave spectrum provides a benchmark for various approximation schemes. This validation is performed for both a real scalar quartic potential and a complex scalar logarithmic potential, including the role of relative phase in the two-field system. Finally, Section. 8.3 applies the resulting method to many-bubble configurations with different nucleation histories. Here we analyze general features of the resulting spectra of GBF approximation including their peak positions, amplitudes, infrared and ultraviolet slopes, and their dependence on numerical and physical inputs such as box size, lattice geometry and boundary conditions.

Introduction to Gravitational Waves

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General Relativity, formulated by Albert Einstein and published in its comprehensive form in 1916, is the modern classical description of gravitation for macroscopic systems. Its central idea is that gravity is a manifestation of spacetime geometry: matter and radiation determine curvature, while freely falling bodies follow the geometry encoded in the metric. One of the most remarkable predictions of this theory is the existence of gravitational waves. Einstein first obtained them in the weak-field approximation as propagating solutions of the linearized gravitational field equations. They can be described, in the usual physical language, as propagating ripples in spacetime curvature. The idea that a disturbance of the gravitational field should propagate as a wave is natural once gravity is understood as a relativistic field. In electrodynamics, replacing electrostatics by the full dynamical Maxwell theory reveals electromagnetic waves, which carry information about the time evolution of charged systems. Similarly, when a mass-energy distribution changes in time, the corresponding change in the gravitational field propagates outward from the source as gravitational waves with the speed of light. Gravitational waves have, however, a distinctive feature: the field that propagates is the metric itself. As the metric is the object that defines spacetime, a gravitational wave changes the geometry through which it passes and therefore changes the proper distance between freely falling test masses. This fundamental phenomenon is the core principle upon which modern gravitational wave detection experiments are built.

For many decades gravitational waves were primarily a theoretical prediction and a target for increasingly sensitive experiments. The first decisive observational evidence was indirect: the binary pulsar PSR B1913+16, discovered by Russel Hulse and Joseph Taylor, showed over time that the pulse frequency shortened, revealing that the stars were spiraling inward at a rate of energy loss nearly identical to the predicted emission of gravitational waves [46, 47]. This agreement established that compact relativistic binaries behave as radiating systems in the way required by General Relativity.

The first direct detection came from GW150914. On September 14, 2015 at 09:50:45 UTC, the two Advanced LIGO detectors observed a coherent transient signal from the merger of two stellar-mass black holes [48]. The chirp signal swept upward in frequency from about 35 Hz to 250 Hz and lasted over 0.2 seconds. Figure 6.1 shows the LIGO measurements of the gravitational waves at the Hanford and Livingston detectors, together with the waveform reconstruction in comparison with the theoretical prediction. This observation of GW150914 thus confirmed the prediction of General Relativity and

opened a new approach to extracting direct information about the source dynamics through the phase and amplitude of the waveform. This makes gravitational-wave astronomy complementary to electromagnetic and particle observations, and especially powerful for compact objects, strong-field gravity, and early-universe phenomena.

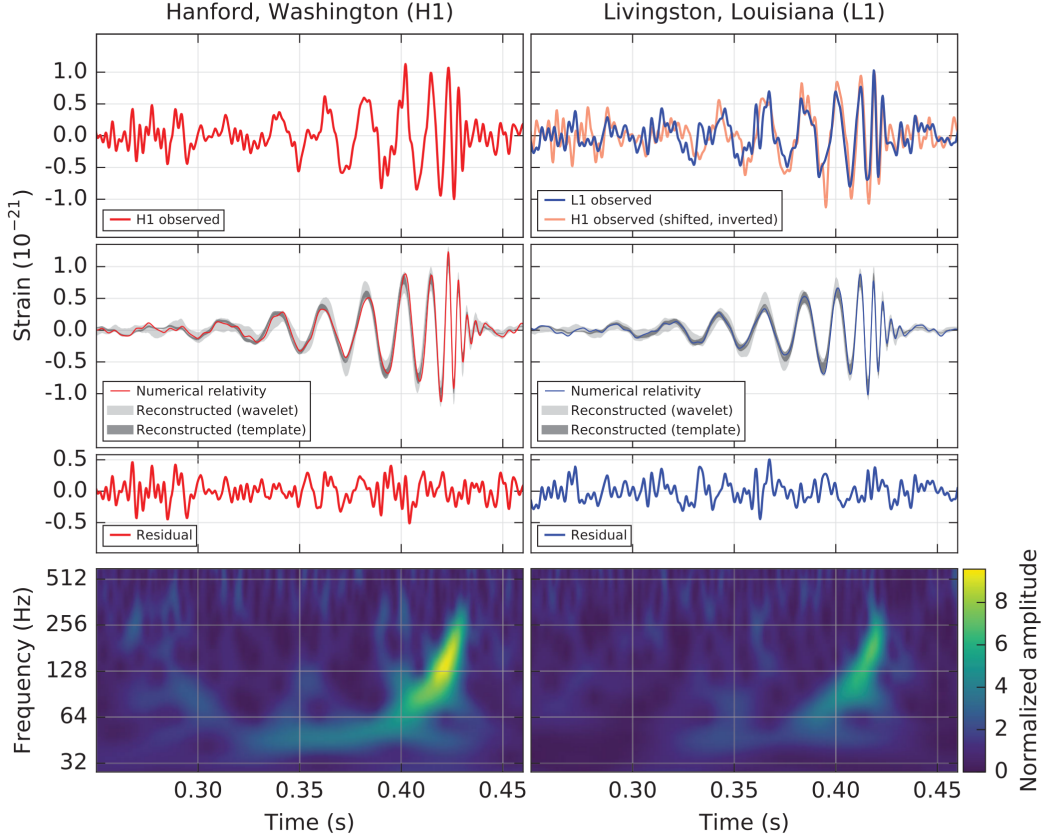


Figure 6.1: The first direct gravitational-wave detection, GW150914, observed by the LIGO Hanford and Livingston detectors. The panels show the measured strain, comparison with numerical-relativity and reconstructed waveforms, residuals after subtracting the best-fit waveform, and the time-frequency representation of the chirp. The image is from Fig. 1 of Ref. [48].

The present chapter uses this observational and theoretical framework as the starting point for the gravitational-wave part of the thesis. We first review the weak-field description of tensor perturbations, their physical polarizations in vacuum gravitational-wave solutions, their interaction with test masses, and the energy carried by gravitational radiation. We then turn to discuss various sources for stochastic gravitational wave background, before focusing in the following chapters on the source of primary interest for this thesis, namely first-order phase transitions.

6.1 Foundations of Gravitational Waves

6.1.1 Linearized gravity and tensor perturbations

Gravitational waves are most conveniently introduced in the weak-field regime of General Relativity, where the spacetime metric is written as a small perturbation around Minkowski space,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \quad (6.1.1)$$

with indices raised and lowered by the mostly-plus background metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The inverse metric and connection are then expanded as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2), \quad \Gamma_{\mu\nu}^{\rho} = \frac{1}{2}\eta^{\rho\sigma}(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}) + \mathcal{O}(h^2). \quad (6.1.2)$$

Starting from the full Einstein equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}, \quad (6.1.3)$$

one expands the Einstein tensor as $G_{\mu\nu} = G_{\mu\nu}^{(1)} + \mathcal{O}(h^2)$. The linearized Ricci tensor and Ricci scalar are

$$R_{\mu\nu}^{(1)} = \frac{1}{2}(\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \square h_{\mu\nu} - \partial_\mu\partial_\nu h), \quad (6.1.4)$$

$$R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \square h, \quad (6.1.5)$$

where $\square = -\partial_t^2 + \nabla^2$ with the mostly-plus signature. Keeping only these linear terms turns gravity into a relativistic field theory for a symmetric rank-two field propagating on a fixed background. This linearized description is the appropriate starting point for the gravitational radiation produced by cosmological phase transitions, since the final gravitational-wave amplitude is extremely small even when the source dynamics is highly nonlinear.

The perturbation $h_{\mu\nu}$ itself contains gauge redundancy inherited from diffeomorphism invariance. Under an infinitesimal coordinate transformation, one may shift

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad (6.1.6)$$

without changing the physical geometry at linear order. It is useful to introduce the trace-reversed metric perturbation

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad h \equiv \eta^{\rho\sigma}h_{\rho\sigma}. \quad (6.1.7)$$

By choosing the Lorenz gauge (a.k.a harmonic gauge or de Donder gauge, which is the analog of the Lorenz gauge for gauge fields)

$$\partial^\mu \bar{h}_{\mu\nu} = 0, \quad (6.1.8)$$

the derivative-mixing terms in $G_{\mu\nu}^{(1)}$ are removed and one obtains

$$G_{\mu\nu}^{(1)} = -\frac{1}{2}\square\bar{h}_{\mu\nu}. \quad (6.1.9)$$

Substituting this expression into Eq. (6.1.3) gives the sourced linearized Einstein equation in the Lorenz gauge

$$\square\bar{h}_{\mu\nu} = (-\partial_t^2 + \nabla^2)\bar{h}_{\mu\nu} = -16\pi GT_{\mu\nu}. \quad (6.1.10)$$

Equation (6.1.10) is one of the main results of the weak-field expansion. It shows that the perturbation of flat spacetime propagates as a wave with the speed of light, as derived from Einstein's theory of General Relativity in the weak-field regime. After the gauge condition has removed the derivative mixing between components of $h_{\mu\nu}$, each component of the trace-reversed perturbation obeys a flat-space sourced wave equation. The equation is linear in $\bar{h}_{\mu\nu}$, but note that it is not merely a collection of decoupled scalar wave equations, since the source is the stress-energy tensor. Also note that the field still carries the gauge redundancy inherited from diffeomorphism invariance. The gauge condition and the field equation are also mutually consistent only when the source is conserved at this order. Indeed, taking ∂^μ of Eq. (6.1.10) and using $\partial^\mu \bar{h}_{\mu\nu} = 0$ gives

$$\partial^\mu T_{\mu\nu} = 0, \quad (6.1.11)$$

which is the flat-background conservation law appropriate to linearized gravity. This is the linearized remnant of the covariant conservation law in the full theory.

As in electrodynamics, the causal solution to Eq. (6.1.10) can be written in terms of a retarded potential,

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3\mathbf{x}' \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \quad (6.1.12)$$

in units where the speed of light is one, and the integral extends over the past light-cone of the event $(t, \mathbf{x})$. Thus the metric perturbation observed far from a compact source is determined by the source stress-energy evaluated at the retarded time. The homogeneous equation, obtained by setting $T_{\mu\nu} = 0$, describes the propagation of gravitational waves after they have left the source region. The next subsection analyzes this vacuum propagation problem and shows how the gauge freedom reduces the apparent ten components of $h_{\mu\nu}$ down to the two physical tensor polarizations.

6.1.2 Gravitational waves in the linear approximation

Far from the source we may treat the wave as a weak perturbation propagating through vacuum. In the Lorenz gauge, Eq. (6.1.10) then reduces to the homogeneous flat-space wave equation

$$\square \bar{h}_{\mu\nu} = (-\partial_t^2 + \nabla^2) \bar{h}_{\mu\nu} = 0. \quad (6.1.13)$$

This is the basic result for solving gravitational waves within the linearized regime. A common approach is to build a general solution as a superposition of plane-wave solutions, so let us look for a plane-wave solution of the form

$$\bar{h}_{\mu\nu}(x) = \text{Re}(H_{\mu\nu} e^{ik_\rho x^\rho}), \quad (6.1.14)$$

where $H_{\mu\nu}$ is a constant symmetric, potentially complex, matrix describing the polarization of the wave, and k^μ is a real wavevector. Substitution into Eq. (6.1.13) gives the null condition $k_\mu k^\mu = 0$, which expresses the fact that gravitational waves propagate at the speed of light relative to the Minkowski background. The Lorenz gauge condition gives a second algebraic constraint,

$$k^\mu H_{\mu\nu} = 0, \quad (6.1.15)$$

i.e., the gravitational waves are transverse. For a plane wave traveling in the z direction one may take

$$k_\mu = \omega(-1, 0, 0, 1), \quad (6.1.16)$$

so that Eq. (6.1.15) implies $H_{0\mu} + H_{3\mu} = 0$. At this stage the wave is transverse in the Lorenz-gauge sense, but the gauge is not yet fixed completely. To achieve that, let us now consider the residual gauge freedom, which is crucial for identifying the correct physical degrees of freedom of a propagating tensor field. A residual coordinate transformation $\tilde{x}^\mu = x^\mu - \xi^\mu$ generated by $\xi_\mu = X_\mu e^{ik_\rho x^\rho}$ with $k_\mu k^\mu = 0$ preserves the Lorenz gauge, because it obeys $\square \xi_\mu = 0$. Under this residual transformation, the trace-reversed perturbation changes as

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho. \quad (6.1.17)$$

Substituting $\xi_\mu = X_\mu e^{ik_\rho x^\rho}$ into Eq. (6.1.17), the plane-wave amplitude of the trace-reversed perturbation changes as

$$H_{\mu\nu} \rightarrow H'_{\mu\nu} = H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k_\rho X^\rho). \quad (6.1.18)$$

The four residual gauge freedoms X_μ are then chosen to achieve the “longitudinal” and “trace-free” gauge,

$$H'_{0\mu} = 0, \quad \eta^{\mu\nu} H'_{\mu\nu} = 0. \quad (6.1.19)$$

With these constraints, the trace of the perturbation vanishes, so $h'_{\mu\nu} = \bar{h}'_{\mu\nu}$, and the remaining nonzero components are purely spatial and transverse. This residual gauge fixing is known as the transverse-traceless (TT) gauge, in which

$$h_{0\mu}^{\text{TT}} = 0, \quad k^i h_{ij}^{\text{TT}} = 0, \quad \delta^{ij} h_{ij}^{\text{TT}} = 0. \quad (6.1.20)$$

In this gauge the perturbation is traceless, so $\bar{h}_{\mu\nu} = h_{\mu\nu}$. For a wave propagating along z , the additional conditions also imply $h_{3\mu}^{\text{TT}} = 0$. Thus the plane-wave amplitude can be written as

$$H_{\mu\nu}^{\text{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}, \quad (6.1.21)$$

where the two constants H_+ and $H_\times$ represent the two independent polarization modes. Equivalently, the spatial part of the metric perturbation at fixed phase can be written as

$$h_{ij}^{\text{TT}}(t, z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} = \begin{pmatrix} H_+ & H_\times & 0 \\ H_\times & -H_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \times \cos[\omega(t - z) + \text{phase}], \quad (6.1.22)$$

where h_+ and $h_\times$ denote the two time-dependent linear polarization modes. This is one way to show that gravitational waves, at each spacetime point, have two degrees of freedom. The tensor nature of these modes is what distinguishes gravitational radiation from scalar or vector perturbations.

Polarization of gravitational waves

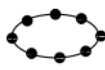
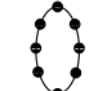
ωt	h_+	$h_\times$
0		
$\pi/2$		
π		
$3\pi/2$		

Figure 6.2: Deformation of a ring of test masses under the $+$ and $\times$ polarizations of a gravitational wave, from Ref. [49].

The two polarization components h_+ and $h_\times$ become observable through their effect on the relative motion of freely falling test masses. This statement is slightly subtle, because the answer depends on which coordinate frame is used to describe the same invariant physics. The cleanest strategy is to keep track of both descriptions: in TT coordinates one sees directly how the proper distance changes, while in a local inertial detector frame one sees the same effect as geodesic deviation [49].

Start with the geodesic equation for a freely falling point particle,

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (6.1.23)$$

where τ is the proper time along the worldline. To first order in the metric perturbation, the Christoffel symbols are

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} \eta^{\alpha\beta} (\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}) + \mathcal{O}(h^2). \quad (6.1.24)$$

Consider a particle initially at rest in TT coordinates, with four-velocity $u^\mu = dx^\mu/d\tau = (1, 0, 0, 0)$ at zeroth order. The spatial part of Eq. (6.1.23) is then controlled by

$$\Gamma_{00}^i = \frac{1}{2} \eta^{ij} (2\partial_0 h_{j0} - \partial_j h_{00}) = 0, \quad (6.1.25)$$

because TT gauge has $h_{00} = h_{0i} = 0$. Thus a test mass initially at rest remains at fixed spatial coordinate position x^μ in TT coordinates as the gravitational waves pass through. Physical experiments, however, measure proper distance: based on the metric solution for gravitational waves, Eq. (6.1.22), we have

$$ds^2 = (1 + h_+) dx^2 + (1 - h_+) dy^2 + 2h_\times dx dy. \quad (6.1.26)$$

First, assume $H_\times = 0$ and $H_+ \neq 0$, so that h_+ oscillates over spacetime. For two particles separated along the x axis by coordinate distance 2δ , the proper distance is

$$ds^2 = 2\delta \sqrt{1 + h_+} = 2\delta \left(1 + \frac{1}{2} h_+ \right) + \mathcal{O}(h^2). \quad (6.1.27)$$

For two particles separated along the y axis by the same coordinate distance,

$$ds^2 = 2\delta \sqrt{1 - h_+} = 2\delta \left(1 - \frac{1}{2} h_+ \right) + \mathcal{O}(h^2). \quad (6.1.28)$$

A positive h_+ therefore stretches one transverse direction while compressing the orthogonal direction, and the pattern reverses after half a period. If $h_+ = 0$ and $h_\times \neq 0$, the same stretch-compress pattern is obtained along axes rotated by 45° . This is the origin of the names “plus” and “cross” polarizations. Figure. 6.2 illustrates these deformations for a ring of initially circularly arranged test masses.

The discussion above is the measurement principle behind gravitational-wave experiments such as laser interferometric detectors. Such a detector measures the relative tidal deformation of approximately freely suspended test masses [49]. In an interferometer, a laser beam is split into two nearly orthogonal arms, reflected by mirrors acting as test masses, and recombined at the output. A passing gravitational wave changes the relative proper lengths of the arms, so the two returning light beams acquire a time-dependent phase difference. By inspecting the phase difference, one can determine whether there has been any change in the length difference of the two arms, and hence detect whether a gravitational wave has passed through the detector. The LIGO detectors, for example, have test-mass mirrors separated by $L = 4$ km [48]. They are sensitive enough to detect a relative change in arm length of order $\delta L/L \sim 10^{-21}$ and have operated for more than a decade and detected roughly ~ 400 gravitational-wave signals.

TT projection tensor

For later use, it is helpful to construct a projection operator that maps a symmetric tensor to its TT part. Given a unit vector $\hat{\mathbf{k}}$ in the propagation direction, define the spatial projector

$$P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j. \quad (6.1.29)$$

The projector P_{ij} is symmetric, transverse, and idempotent:

$$P_{ij} = P_{ji}, \quad P_{ij}\hat{k}_j = 0, \quad P_{il}P_{lj} = P_{ij}, \quad P_{ii} = 2. \quad (6.1.30)$$

The final identity is specific to three spatial dimensions and says that P_{ij} projects onto the two-dimensional plane orthogonal to $\hat{\mathbf{k}}$. The TT projection operator can be built from the P operator as follows:

$$\Lambda_{ij,lm} = P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm}. \quad (6.1.31)$$

The TT part of a symmetric spatial tensor S_{ij} is then extracted by $S_{ij}^{\text{TT}} = \Lambda_{ij,lm}(\hat{\mathbf{k}})S_{lm}$. The TT projector inherits the corresponding symmetry, transversality, tracelessness, and idempotence properties by construction:

$$\Lambda_{ij,lm} = \Lambda_{ji,lm} = \Lambda_{ij,ml} = \Lambda_{lm,ij}, \quad (6.1.32)$$

$$\hat{k}_i\Lambda_{ij,lm} = 0, \quad \delta_{ij}\Lambda_{ij,lm} = 0, \quad (6.1.33)$$

$$\Lambda_{ij,ab}\Lambda_{ab,lm} = \Lambda_{ij,lm}. \quad (6.1.34)$$

These identities ensure that S_{ij}^{TT} is transverse, traceless, symmetric, and unchanged by applying the TT projection multiple times.

6.1.3 Energy carried by gravitational waves

Gravitational waves carry energy and momentum, as is already suggested by the fact that they can exchange energy with freely falling test masses. However, there is no local, gauge-invariant stress tensor for the gravitational field in the same sense as for matter. Instead, an effective stress-energy tensor can be obtained by asking how a rapidly varying gravitational wave curves a slowly varying background geometry [49, 50, 51]. This, however, requires going beyond the strictly linear theory on Minkowski spacetime: if the background were fixed to be $\eta_{\mu\nu}$ from the start, the wave could not backreact on the geometry. We therefore decompose the metric into a slowly varying background, a first-order wave perturbation, and a second-order correction,

$$g_{\mu\nu} = g_{\mu\nu}^B + \epsilon h_{\mu\nu} + \epsilon^2 j_{\mu\nu} + \mathcal{O}(\epsilon^3), \quad (6.1.35)$$

where ϵ is a bookkeeping parameter set to unity at the end of the calculation; its presence is equivalent to saying that $j_{\mu\nu}$ is of order $\mathcal{O}(|h_{\mu\nu}^2|)$. The field $h_{\mu\nu}$ is the rapidly varying first-order gravitational wave, while $j_{\mu\nu}$ is the second-order metric perturbation sourced by the wave self-interaction. The split is meaningful when there is a clear separation between the gravitational-wave wavelength λ and the characteristic length scale λ_B on which $g_{\mu\nu}^B$ changes, $\lambda \ll \lambda_B$, which is also known as the geometric-optics regime. Within this regime, one can quantify the impact of gravitational waves on spacetime curvature by averaging over an intermediate scale ℓ satisfying $\lambda \ll \ell \ll \lambda_B$, thereby removing the oscillatory short-scale part while leaving the slow background geometry. Expanding the Einstein tensor around $g_{\mu\nu}^B$ gives schematically

$$G_{\mu\nu}[g] = G_{\mu\nu}[g^B] + \epsilon G_{\mu\nu}^{(1)}[h; g^B] + \epsilon^2 G_{\mu\nu}^{(1)}[j; g^B] + \epsilon^2 G_{\mu\nu}^{(2)}[h; g^B] + \mathcal{O}(\epsilon^3). \quad (6.1.36)$$

Here $G_{\mu\nu}^{(1)}[\dots; g^B]$ is the linearized Einstein operator expanded over the perturbation around the background $g_{\mu\nu}^B$, while $G_{\mu\nu}^{(2)}[h; g^B]$ is the part of the Einstein tensor quadratic in $h_{\mu\nu}$. Recall that we are considering the Einstein equation in vacuum, in a sufficiently far-field zone with respect to the localized source, and hence $G_{\mu\nu}[g] = 0$. We then equate Eq. (6.1.36) order by order in ϵ . The zeroth-order equation determines the background in the absence of gravitational waves, $G_{\mu\nu}[g^B] = 0$, while the first-order equation $G_{\mu\nu}^{(1)}[h; g^B] = 0$ describes the propagation of gravitational waves on that background. The energy carried by the gravitational waves first appears at second order, because the

average of the oscillatory first-order field vanishes, whereas products of two first-order fields can have a nonzero slowly varying component. The averaged second-order equation can then be written as an effective Einstein equation for the slowly varying part of the metric, i.e., terms that vary on length scales $\sim \lambda_B$,

$$G_{\mu\nu} [g_{\alpha\beta}^B + \epsilon^2 \langle j_{\alpha\beta} \rangle] = 8\pi G \langle t_{\mu\nu}^{\text{GW}} \rangle + \mathcal{O}(\epsilon^3), \quad t_{\mu\nu}^{\text{GW}} \equiv -\frac{1}{8\pi G} G_{\mu\nu}^{(2)}[h; g^B], \quad (6.1.37)$$

Thus the gravitational wave affects the long-wavelength geometry in the same way as a matter source with the averaged stress tensor $\langle t_{\mu\nu}^{\text{GW}} \rangle$, where the average $\langle \dots \rangle$ is taken over the intermediate scale ℓ satisfying $\lambda \ll \ell \ll \lambda_B$ because of the separation of scales discussed above. It can be shown that, up to corrections of order λ^2/λ_B^2 that can be neglected, and by choosing a gauge in which the perturbation is transverse and traceless with respect to the background, the effective stress-energy sourced by linear metric perturbations simplifies to [50]

$$\langle t_{\mu\nu}^{\text{GW}} \rangle = \frac{1}{32\pi G} \langle \nabla_\mu^B h_{\alpha\beta} \nabla_\nu^B h^{\alpha\beta} \rangle, \quad (6.1.38)$$

where indices are raised with the inverse background metric $(g^B)^{\mu\nu}$. The gravitational-wave energy density is the time-time component of the averaged effective tensor. In the far zone with a local flat-spacetime limit, the same expression can be written only in terms of the physical transverse-traceless components. In particular, the gauge-invariant energy density is

$$\rho_{\text{GW}} = \langle t_{00}^{\text{GW}} \rangle = \frac{1}{32\pi G} \langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \rangle = \frac{1}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle, \quad (6.1.39)$$

where the dot denotes a derivative with respect to physical time. Physically, ρ_{GW} is the averaged energy per unit physical volume stored in the radiative tensor perturbations after the source has produced the waves. The associated spectral density is defined by decomposing this energy density into logarithmic frequency intervals,

$$\rho_{\text{GW}} = \int_0^\infty d\ln \omega \frac{d\rho_{\text{GW}}}{d\ln \omega}. \quad (6.1.40)$$

For later calculations, it is useful to turn Eq. (6.1.39) into a formula for the energy flux of gravitational waves. Consider an outgoing gravitational wave observed at a large distance r from a localized source. In the far zone, the transverse-traceless waveform has the form $h_{ij}^{\text{TT}}(t, r\hat{\mathbf{k}}) = f_{ij}(t - r, \hat{\mathbf{k}})/r + \mathcal{O}(r^{-2})$ in units $c = 1$, as will be shown shortly. Therefore $\partial_r h_{ij}^{\text{TT}} = -\dot{h}_{ij}^{\text{TT}} + \mathcal{O}(r^{-2})$. Equivalently, the radial energy flux equals the local energy density of the outgoing radiation, giving

$$\frac{dE_{\text{GW}}}{dA dt} = \langle t_{00}^{\text{GW}} \rangle = \frac{1}{32\pi G} \langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \rangle. \quad (6.1.41)$$

Recall that in the weak-field approximation on a flat background, we have

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \quad \partial^\mu \bar{h}_{\mu\nu} = 0, \quad (6.1.42)$$

where $T_{\mu\nu}$ is the matter stress-energy tensor of the source. Outside the source, the radiative spatial part can be projected onto the transverse-traceless sector. Using the TT projector defined in Eq. (6.1.31), the sourced wave equation may be written schematically as

$$\ddot{h}_{ij}^{\text{TT}} - \nabla^2 h_{ij}^{\text{TT}} = 16\pi G T_{ij}^{\text{TT}}, \quad T_{ij}^{\text{TT}}(\mathbf{k}, t) = \Lambda_{ij,lm}(\hat{\mathbf{k}}) T_{lm}(\mathbf{k}, t). \quad (6.1.43)$$

The projection removes the trace and longitudinal parts of the spatial stress. Thus only the non-spherical, time-dependent part of T_{ij} contributes to radiation; homogeneous vacuum energy, isotropic

pressure, and exactly spherical expansion do not radiate directly. The retarded solution of Eq. (6.1.43) then gives, at leading order in $1/r$,

$$h_{ij}^{\text{TT}}(t, r\hat{\mathbf{k}}) = \frac{4G}{r} \Lambda_{ij,lm}(\hat{\mathbf{k}}) \int d^3x' T_{lm}(t - r + \hat{\mathbf{k}} \cdot \mathbf{x}', \mathbf{x}') + \mathcal{O}(r^{-2}). \quad (6.1.44)$$

With Weinberg's Fourier-transform convention, ⁽¹⁾

$$\tilde{T}_{ij}(\hat{\mathbf{k}}, \omega) = \frac{1}{2\pi} \int dt d^3x e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} T_{ij}(\mathbf{x}, t), \quad (6.1.45)$$

Eq. (6.1.44) becomes

$$h_{ij}^{\text{TT}}(t, r\hat{\mathbf{k}}) = \frac{4G}{r} \Lambda_{ij,lm}(\hat{\mathbf{k}}) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-r)} \tilde{T}_{lm}(\hat{\mathbf{k}}, \omega). \quad (6.1.46)$$

This expression describes the solution of the metric perturbation in TT gauge as a function of Fourier modes of the energy-momentum tensor of the localized source. Substituting this waveform into Eq. (6.1.41), integrating over time, and using the projector identity $\Lambda_{ij,ab}\Lambda_{ij,lm} = \Lambda_{ab,lm}$ gives the total gravitational radiation energy over a solid angle $\Omega_{\mathbf{k}}$:

$$\frac{dE_{\text{GW}}}{d\Omega_{\mathbf{k}}} = 2G \int_0^{\infty} d\omega \omega^2 \Lambda_{ij,lm}(\hat{\mathbf{k}}) \tilde{T}_{ij}^*(\hat{\mathbf{k}}, \omega) \tilde{T}_{lm}(\hat{\mathbf{k}}, \omega). \quad (6.1.47)$$

Equivalently, the energy emitted per logarithmic frequency interval and per solid angle is

$$\boxed{\frac{dE_{\text{GW}}}{d \ln \omega d\Omega_{\mathbf{k}}} = 2G\omega^3 \Lambda_{ij,lm}(\hat{\mathbf{k}}) \tilde{T}_{ij}^*(\hat{\mathbf{k}}, \omega) \tilde{T}_{lm}(\hat{\mathbf{k}}, \omega).} \quad (6.1.48)$$

This is Weinberg's formula, which will be used later in the derivation of our generalized bulk-flow approximation in Chapter. 7 to quantify the gravitational spectrum sourced by bubble collisions in a first-order phase transition.

6.2 Stochastic Backgrounds and Their Sources

6.2.1 Stochastic gravitational wave background (SGWB)

For a stochastic background, $h_{ij}(\mathbf{x}, t)$ is not a deterministic waveform but a random tensor field. The statistical description is especially natural for cosmological sources. For many sources of cosmological origin (i.e those associated with events or processes in the early Universe, which take place well before the formation of stars and galaxies), the statistical description is among one of their natures: a causal process in the early universe can correlate the emitted radiation only over scales no larger than the causal horizon at the time of production, while today a detector receives the superposition of radiation from many such regions redshifted into our present horizon [51]. For other sources of astrophysical origin which typically consist of populations of compact objects like white dwarfs, neutron stars, or black holes, the emitted gravitational waves could be too weak or too numerous to be differentiated by the detector. Stochastic gravitational-wave backgrounds (SGWB) are therefore formed from the superposition of gravitational-wave signals where each source cannot be individually resolvable. Probing the SGWB therefore provides information regarding the statistical and/or population properties of the sources.

The baseline stochastic background considered in detector forecasts is usually taken to be statistically homogeneous, isotropic, unpolarized, and Gaussian [51, 52, 53]. Homogeneity and isotropy follow from the FLRW background in the absence of a preferred direction. The absence of net polarization

⁽¹⁾ Several modern textbooks, such as Maggiore's [49], use instead $\tilde{T}_{ij}^{\text{M}}(\hat{\mathbf{k}}, \omega) = \int dt d^3x e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} T_{ij}(\mathbf{x}, t)$, so that $\tilde{T}_{ij}^{\text{M}} = 2\pi \tilde{T}_{ij}$ in the convention of Eq. (6.1.45). In this convention the same result reads $dE_{\text{GW}}/(d \ln \omega d\Omega_{\mathbf{k}}) = G\omega^3 \Lambda_{ij,lm} \tilde{T}_{ij}^{\text{M}*} \tilde{T}_{lm}^{\text{M}}/(2\pi^2)$.

is expected when the source does not violate parity, so that the two tensor helicities are produced with equal statistical weight. Gaussianity is often justified by the central-limit behavior of a signal built from many uncorrelated emitting regions. These assumptions can fail for anisotropic, parity-violating, or strongly non-Gaussian sources, but they give the standard reference case for first-order phase-transition spectra unless additional model ingredients are specified. A common dimensionless quantity used to characterize a stochastic background is the gravitational-wave energy density per logarithmic frequency interval,

$$\Omega_{\text{GW}}(\omega) \equiv \frac{1}{\rho_c} \frac{d\rho_{\text{GW}}}{d\ln\omega}, \quad (6.2.1)$$

where ρ_c is the critical energy density at the epoch at which the spectrum is being quoted. For present-day observational constraints we take $\rho_c = \rho_{c,0} = 3H_0^2/(8\pi G)$ and write $\Omega_{\text{GW},0}$. In theoretical calculations performed at the time of production, it is instead often more convenient to normalize to the total energy density of the system or to the background energy density at the given time. This distinction is only a matter of convention, provided the normalization and the redshifting of the spectrum are stated explicitly. The use of $d\ln\omega$ rather than $d\omega$ is useful because gravitational-wave spectra typically extend over many orders of magnitude in frequency. Detector-oriented literature often quotes the same GW spectrum as a function of the ordinary frequency $f = \omega/(2\pi)$. This is only a change of variable, since $d\ln f = d\ln\omega$. In this thesis I will use the frequency ω for gravitational-wave spectra, as well as for the Fourier transform of the stress-energy tensor. When experimental frequency bands are quoted in hertz, I will write them explicitly in terms of $\omega/(2\pi)$.

The same information can be described in terms of the total energy emitted into gravitational waves. Suppose that the source calculation spans a representative physical volume V_* at the production time, outside of which the source is switched off and the radiation can be treated as freely propagating. The emitted gravitational-wave energy is then related to the energy density by

$$E_{\text{GW}} = V_* \rho_{\text{GW}}, \quad (6.2.2)$$

or, differentially in frequency and propagation direction,

$$\frac{d\rho_{\text{GW}}}{d\ln\omega} = \frac{1}{V_*} \int d\Omega_{\mathbf{k}} \frac{dE_{\text{GW}}}{d\ln\omega d\Omega_{\mathbf{k}}}. \quad (6.2.3)$$

Here $d\Omega_{\mathbf{k}}$ is the solid-angle element of the emitted wave vector. For an exactly isotropic stochastic background the angular integral simply gives 4π times the emission per unit solid angle, but keeping the angular dependence is useful before averaging over bubble configurations. Combining Eq. (6.2.3) with Eq. (6.2.1) gives

$$\Omega_{\text{GW}}(\omega) = \frac{1}{\rho_c V_*} \int d\Omega_{\mathbf{k}} \frac{dE_{\text{GW}}}{d\ln\omega d\Omega_{\mathbf{k}}}, \quad (6.2.4)$$

where ρ_c denotes the reference critical density at the epoch at which the spectrum is quoted. E_{GW} is not a new observable independent of ρ_{GW} and Ω_{GW} ; it is sometimes convenient to compute this quantity in lattice simulations. One can then use Weinberg's formula, derived previously in Eq. (6.1.48), to compute the gravitational-wave spectrum $\Omega_{\text{GW}}(\omega)$ from the energy-momentum tensor of the stochastic sources.

6.2.2 Sources, foregrounds, and detector frequency bands

The discussion above characterizes a stochastic gravitational-wave spectrum by its statistical properties and by its spectral energy density. As a consequence, a measured $\Omega_{\text{GW}}(\omega)$ reveals information about the sources that produce it through its amplitude and structure in different frequency bands. The sources are usually divided into two distinct categories: cosmological sources and astrophysical sources. Here we briefly describe properties of the spectra produced by different sources in each category before moving on to the detector observational bands that can potentially probe them.

Cosmological sources

Cosmological backgrounds are generated in the early Universe, before the formation of ordinary astrophysical objects. Their stochastic character can be intrinsic, as for quantum or thermal fluctuations amplified by an early-Universe process, or it can arise from the incoherent superposition of many unresolved events or defects. The spectrum is therefore a probe of high-energy physics: the characteristic frequencies are tied to physical time or length scales at production, while the amplitude depends on the fraction of the total energy density converted into tensor perturbations, and the shape can reveal further source characteristics, including, e.g., its dynamics and lifetime. Broad reviews of cosmological SGWB sources, their detector targets, and their role in fundamental physics can be found, e.g., in Refs. [51, 54, 55].

Cosmic inflation is one important source of a cosmological SGWB. In a simple single-field slow-roll picture, tensor fluctuations of the metric are stretched beyond the horizon and later re-enter as a nearly scale-invariant stochastic tensor background. Consequently the resulting SGWB prediction is broad and nearly scale invariant. The amplitude of this spectrum can be described through a parameter r that represents the ratio between the amplitudes of the power spectra of tensor and scalar perturbations of the metric. CMB studies using BICEP/Keck data and Planck [56] put a bound on $r < 0.032$, evaluated at a pivot scale of 0.05Mpc^{-1} at 95% confidence, which implies a present-day amplitude of $\Omega_{\text{GW}} \lesssim 10^{-16}$ at frequency $\sim 10^{-17}$ Hz, far below the reach of planned GW detectors. The value of r is itself model dependent, so a measurement of primordial tensors helps discriminate among inflationary scenarios. Beyond single-field inflation, additional degrees of freedom such as spectator fields or particle production during inflation can source tensor modes classically and generate an extra GW component distinct from the vacuum fluctuations of the metric [57].

Reheating and preheating can also source a primordial background after inflation [58, 59]. In these scenarios the homogeneous inflaton condensate transfers energy to other fields, sometimes through highly non-perturbative and inhomogeneous dynamics. During parametric resonance, particle species coupled to the inflaton are created in energetic bursts; the transferred energy can grow exponentially within a few inflaton oscillations, and field modes in selected momentum bands build large spatial gradients. The TT part of the stress tensor of these gradients then sources GWs while the system is far from equilibrium. The resulting background is therefore qualitatively different from the nearly scale-invariant vacuum signal of slow-roll inflation. The source is active only during the finite interval in which resonance, backreaction, and nonlinear relaxation generate large anisotropic stresses, so the spectrum is usually peaked at several scales, in contrast with the scale-invariant spectrum produced by cosmic inflation. For example, through a lattice simulation study, Ref. [59] found that the resulting present-day SGWB can reach amplitudes of $h^2\Omega_{\text{GW}} \approx \mathcal{O}(10^{-11}) - \mathcal{O}(10^{-13})$, with peaks around $f_p \approx \mathcal{O}(10^7) - \mathcal{O}(10^8)$ Hz.

First-order phase transitions are the cosmological source class central to this thesis. They proceed through nucleation and expansion of bubbles of the new phase, and gravitational radiation is sourced by the anisotropic stresses associated with scalar-field collisions, acoustic motion in the plasma, and magnetohydrodynamic turbulence [60, 61, 62, 63, 64]. The corresponding spectrum is typically peaked at the scale set by the nucleation rate of the new phase, usually parameterized by β^{-1} , or equivalently by the interbubble distance at collision R_* , while the amplitude depends on the released energy and on how that energy is partitioned among scalar gradients, bulk fluid motion, and turbulent plasma motion. Short-lived collision sources have an amplitude scaling set by the square of the source duration in Hubble units, schematically $(H_*/\beta)^2$, whereas long-lived acoustic sources can receive an enhancement as the shear stress survives after bubble collisions. The infrared side is constrained by causality for transient sources, while the ultraviolet falloff and the peak amplitude are generally source-dependent features. This is why the later phase-transition section separates scalar-field bubble collisions, acoustic waves, and MHD turbulence rather than using a single universal template. First-order phase transitions form a collection of one of the main cosmological sources and are central to the scope of this thesis; hence, they are discussed in more detail later in Subsection. 7.1.3.

Cosmic strings and other topological defects form a further class of cosmological sources. Cosmic strings are line-like defects that can form when the vacuum manifold of a symmetry-breaking transition has a nontrivial first homotopy group. In the Nambu–Goto description the string is treated

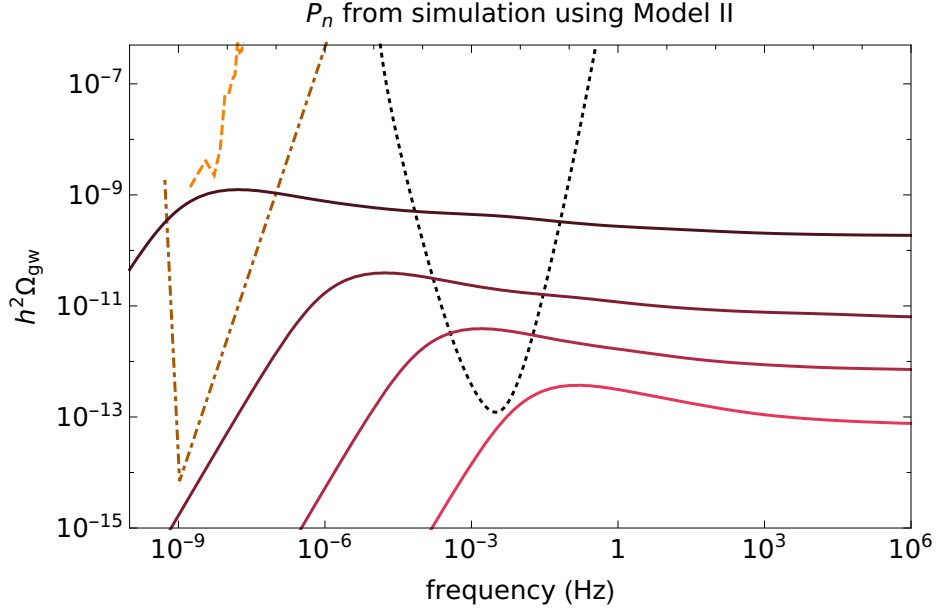


Figure 6.3: Cosmic-string SGWB for different string tensions $G\mu$ (solid curves), compared with the sensitivity of the European Pulsar Timing Array (EPTA, dashed orange curve), Square Kilometer Array (SKA) (dash-dotted dark orange), and LISA (dotted black curve) from Ref. [67]. The solid curves correspond to $G\mu \in \{10^{-10}, 10^{-13}, 10^{-15}, 10^{-17}\}$ from the darkest (top) curve to the lightest (bottom) one. The curve with $G\mu = 10^{-17}$ corresponds to the lowest bound that will be set by LISA.

as an effectively infinitesimal relativistic object with tension, or energy per unit length, μ , and its gravitational strength is measured by the dimensionless combination $G\mu$. After formation, the network is expected to approach a scaling regime, in which long strings continually self-intersect and intercommute to produce subhorizon loops. These loops oscillate under their own tension and, in the standard Nambu–Goto treatment, lose energy predominantly into GWs. The total power emitted by a loop is usually parametrized as $P = \Gamma G\mu^2$, where numerical loop simulations and phenomenological modeling commonly use $\Gamma \simeq 50$ [65, 66, 67]. Because the network radiates over a long cosmological history, the resulting SGWB is much broader than the signal of a short-lived phase transition. It can contain an extended radiation-era plateau, changes associated with the radiation-to-matter transition, and features caused by variations in the relativistic degrees of freedom or by non-standard expansion histories. Its amplitude and shape depend on $G\mu$, the loop-size distribution, the intercommutation probability, the loop power spectrum P_n ⁽¹⁾, and the cosmological history. As illustrated in Fig. 6.3, varying $G\mu$ moves the predicted background through different observational windows; for sufficiently small tensions the peak leaves the Pulsar Timing Array (PTA) band and the millihertz band probed by Laser Interferometer Space Antenna (LISA) becomes especially relevant.

Astrophysical sources

Astrophysical backgrounds are produced by the incoherent superposition of gravitational waves emitted by late-Universe astrophysical systems that are not individually resolved by a given experiment. This includes extragalactic populations, whose redshift and sky distributions are often treated

⁽¹⁾ For a loop of length l , the oscillation period is $T = l/2$, so the angular emission frequencies are $\omega_n = 2\pi n/T = 4\pi n/l$, equivalently $f_n = \omega_n/(2\pi) = 2n/l$ in the ordinary-frequency convention used in the cosmic-string literature. An averaged loop power spectrum expresses the distribution of the total power among these modes. For example, Ref. [67] uses the normalized model

$$P_n = \frac{\Gamma}{\zeta(q)} n^{-q}, \quad (6.2.5)$$

Here P_n is the dimensionless modal power coefficient and $\zeta(q)$ is the Riemann zeta function; the physical power in the n th harmonic is $G\mu^2 P_n$, with $\sum_n P_n = \Gamma$. The spectral index is taken as $q = 4/3, 5/3$, or 2 for emission dominated by cusps, kinks, or kink–kink collisions, respectively.

statistically on cosmological scales, and Galactic foregrounds such as unresolved white-dwarf binaries, for which the anisotropic sky distribution and the time-dependent response of the detector to the sky can be essential. The stochastic character is therefore partly source dependent and partly detector dependent: a deterministic signal from a single binary, supernova, or neutron star becomes part of the SGWB only when it is below threshold, overlaps with many similar events, or cannot be separated from the data stream. Broad reviews of SGWB astrophysical source statistics and observational prospects can be found in Refs. [68, 55]. For discrete populations, the duty cycle, namely the mean ratio between the event duration and the time interval between consecutive events, is a useful diagnostic. Rare events produce a burst-like or shot-noise signal, partially overlapping events produce a “popcorn” background, while high-duty-cycle populations result in a continuous background [68, 69].

For an isotropic population of transient or finite-duration sources, a useful starting point is the practical theorem of Ref. [70]. If $\dot{N}(z)$ is the comoving event rate density, defined per comoving volume and per source-frame time, the present spectrum can be written in units $c = 1$ as

$$\Omega_{\text{GW}}(\omega) = \frac{1}{\rho_c} \int dz \frac{\dot{N}(z)}{(1+z)H(z)} \left[\omega_r \frac{dE_{\text{GW}}}{d\omega_r} \right]_{\omega_r=(1+z)\omega}, \quad (6.2.6)$$

where ρ_c is the present critical density, $H(z)$ is the Hubble rate, ω_r is the source-frame angular frequency, and $dE_{\text{GW}}/d\omega_r$ is the emitted spectral energy of one event, possibly averaged over masses, spins, environments, and other population parameters. Equation (6.2.6) shows that an astrophysical background is not only a measure of GW emission from an individual source, but also of the cosmic or Galactic population that supplies such sources. This relation illustrates why astrophysical sources can both obscure primordial backgrounds and carry information about stellar evolution, compact-object formation, galaxy assembly, and source environments.

Compact-binary backgrounds constitute one of the most important astrophysical source classes. The relevant sub-classes differ mainly by the masses, environments, and spatial distributions of the binaries, and therefore by the frequency band in which their unresolved emission is observed. For a circular binary whose evolution is driven only by GW emission, the inspiral part of the single-source spectrum gives the familiar population scaling $\Omega_{\text{GW}} \propto \omega^{2/3}$ over the frequency range where the sources are in the adiabatic inspiral regime and where factors such as merger-ringdown physics and environmental effects do not dominate [70, 71]. Some examples of the compact-binary sub-classes relevant for SGWB searches are:

- *Stellar-origin compact-binary coalescences.* These are the relevant compact-binary sub-class for ground-based interferometers and include binary black holes, binary neutron stars, and neutron-star–black-hole binaries. The low-frequency part of their unresolved background is controlled by the inspiral scaling above, while the turnover and high-frequency falloff depend on the mass distribution and on merger-ringdown physics. The amplitude and redshift dependence are determined by the merger-rate history. For example, for stellar-origin binary black holes, Ref. [72] computes the SGWB using three different methods and confirms that the spectral shape of the SGWB in the LISA band agrees with the analytical prediction of a single power law, $h^2\Omega_{\text{GW}}(f) = h^2\Omega_{\text{GW}}(f_*)(f/f_*)^{2/3}$, where f_* is an arbitrary pivot frequency. The estimated SGWB amplitude at the reference frequency lies within the range $h^2\Omega_{\text{GW}}(f = 3 \times 10^{-3}\text{Hz}) \in [5.65, 11.5] \times 10^{-13}$, in agreement with most previous estimates.
- *Massive and supermassive black-hole binaries.* These form the same broad compact-binary category, but with much larger central objects. The cosmic population of supermassive black-hole binaries is the standard astrophysical interpretation of a nanohertz SGWB in Pulsar Timing Array data [73]. Further departures from the pure GW-driven power law can arise, e.g., from environmental interactions with gas and stars, as these accelerate the merger at large distances and hence deplete the GW stochastic background at low frequencies [74].

Detector frequency bands

The observational bands help identify which source populations and which time scales are relevant. Figure 6.4 is an overview of the SGWB produced by different sources, as well as the observational

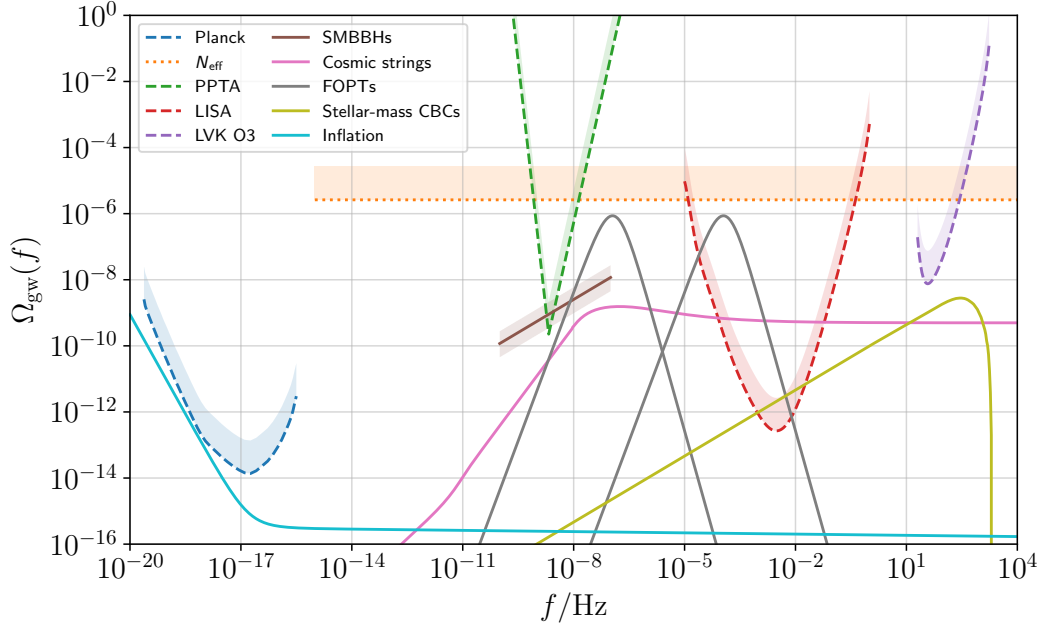


Figure 6.4: Overview of possible stochastic gravitational-wave backgrounds and observational constraints across ordinary frequency $f = \omega/(2\pi)$, from Ref. [55]. The curves include representative cosmological sources, such as inflation, cosmic strings, and first-order phase transitions, as well as astrophysical backgrounds from supermassive black-hole binaries and stellar-mass compact binaries. The dashed curves indicate observational constraints or sensitivities, while the dotted N_{eff} line, where N_{eff} is the effective number of relativistic species, is an integrated bound and should not be read as a frequency-local sensitivity curve.

constraints. Overall, this plot illustrates that different sources populate different frequency ranges and therefore can be probed by different types of experiments. It also shows why the same spectral variable Ω_{GW} contains information about very different physics, from nearly scale-invariant inflationary backgrounds to peaked phase-transition spectra and compact-binary backgrounds.

Pulsar Timing Array (PTA) constraints probe $\omega/(2\pi)$ in the nanohertz range, corresponding to periods of years to decades. In 2023 several PTA collaborations reported evidence for a common stochastic signal in this band, including NANOGrav, EPTA with InPTA, PPTA, and CPTA [73, 75, 76, 77]. The leading astrophysical interpretation is a background from supermassive black hole binaries, but cosmic strings and low-temperature phase transitions are also studied as possible cosmological contributions.

The millihertz band is the natural target for space-based laser interferometers. LISA is designed to probe $\omega/(2\pi)$ over the broad low-frequency range roughly spanning $10^{-4} - 10^{-1}$ Hz, with sensitivity extending outside this interval depending on the adopted sensitivity criterion and with best sensitivity around the millihertz range [78]. This band is central for first-order phase transitions near the electroweak to multi-TeV scale, where β/H_* is roughly $10^2 - 10^3$ [64, 79]. This range is also potentially populated by astrophysical sources, e.g., Extreme Mass Ratio Inspiral (EMRI) systems describing long-lasting inspirals from months to a few years. The Galactic double-white-dwarf population is also an important source that produces a confusion-limited noise background at low frequencies ($\log f \lesssim -2.8$), within the reach of LISA sensitivity, and might be detectable [80]. Related proposed missions such as Taiji and TianQin target similar low-frequency gravitational waves and provide complementary millihertz sensitivity, from 10^{-4} Hz to 1 Hz [81, 82].

At higher frequencies, decihertz experiments such as the Deci-hertz Interferometer Gravitational wave Observatory (DECIGO) and Big Bang Observer (BBO) have been proposed as probes of primordial stochastic backgrounds [83, 84]. They occupy the gap between LISA-like interferometers and ground-based detectors. For first-order phase transitions, this band can correspond to higher transition temperatures than the millihertz band. Ground-based interferometers, including the current

LIGO-Virgo-KAGRA network and future third-generation detectors such as the Einstein Telescope and Cosmic Explorer, probe a frequency band ranging from tens of hertz to kilohertz [85, 86, 87]. In this band the leading astrophysical stochastic contribution for current ground-based searches is expected from unresolved stellar-origin compact-binary coalescences, especially binary black holes and neutron-star binaries [55, 88].

Stochastic Gravitational-Wave Background Sourced by a First-Order Phase Transition

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The previous chapters introduced gravitational waves as propagating tensor perturbations and emphasized that a stochastic gravitational-wave background can preserve information about violent processes in the early universe. First-order phase transitions provide one of the most concrete realizations of this idea. When the universe cools down and ends up in a metastable phase, the transition proceeds through the nucleation, expansion, and collision of true-vacuum bubbles. The resulting system is intrinsically out of equilibrium, contains large anisotropic stresses, and can source a stochastic gravitational-wave background whose peak frequency and amplitude are tied to the descriptions of the phase transition.

The purpose of this chapter is twofold. We begin the first part by reviewing the finite-temperature field-theory picture behind cosmological phase transitions. We start from the effective potential, explain symmetry restoration at high temperature, and introduce the first-order phase transition with bubble nucleation, percolation, completion, as well as the macroscopic parameters used to connect a particle-physics model to a gravitational-wave source. The second part turns from this general review to the regime where bubble-collision is the dominant source, where we go through the construction of the generalized bulk-flow approximation. This approximation keeps the geometric efficiency of the envelope and bulk-flow descriptions, but replaces their limiting post-collision assumptions by a parametrized decay of the wall surface tension. This construction isolates the dependence on microscopic collision dynamics into a small set of coefficients a_ξ while keeping the dependence on the bubble nucleation history in reusable geometric functions. Therefore, once we perform simulations over nucleation histories to capture the average geometric functions, changing the underlying model amounts to recomputing the coefficients a_ξ , which can be done in a relatively inexpensive $1 + 1\text{D}$ lattice simulation, compared with a much more expensive $3 + 1\text{D}$ simulation of bubbles plus metric dynamics.

7.1 Review: Cosmological phase transitions in the early universe

7.1.1 Finite temperature effective potential

The basic idea behind a cosmological phase transition is that the state preferred by a quantum field theory can change as the temperature of the universe changes. At zero temperature one often describes the phases of a scalar theory by minimizing a potential energy, but due to the screening effect from finite temperature (or finite density) upon the propagation of a particle and its interaction with other particles, the potential describing the underlying field also receive corrections from the thermal bath. As we will see, these corrections have a significant impact on the dynamics of the theory and can determine whether a phase transition is possible. Let us follow [89] in this review, where basic physics can be illustrated with a single real scalar field with a discrete $\mathbb{Z}_2$ symmetry $\phi \leftrightarrow -\phi$. The zero-temperature tree-level potential can be written as

$$V_0(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4, \quad m^2 > 0, \quad \lambda > 0, \quad (7.1.1)$$

with extrema obtained from solving $dV_0/d\phi = 0$, which yields $\phi = 0$ and $\phi = \pm v$ with $v = m/\sqrt{\lambda}$. The symmetric point $\phi = 0$ is an unstable local maximum as $d^2V_0/d\phi^2 = -m^2 < 0$ there, while the other two are degenerate stable minima. Choosing one of these minima as the vacuum breaks the reflection symmetry even though the Lagrangian still has it, and the physical scalar mass in the broken phase is $m_{\text{phys}}^2 = d^2V_0/d\phi^2|_{\phi=v} = 2m^2$. After shifting the additive constant in the potential so that the broken minimum has zero energy, the same potential takes the form

$$V_0(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2. \quad (7.1.2)$$

This constant shift does not change the scalar equation of motion in nongravitational field theory. It does matter for cosmology because an absolute vacuum-energy density contributes to the expansion rate. The smallness of the observed vacuum energy is therefore a separate cosmological issue, not something explained by this toy model.

The intuitive finite-temperature effect is that particles in the thermal bath modify the propagation of the scalar field. If perturbation theory is constructed around the unstable point $\phi = 0$ at zero temperature, one obtains a particle with an imaginary mass term, which results in exponentially growing perturbations. However, due to the interaction between the scalar field and the thermal bath, the particle mass receives a finite “plasma” mass correction, which helps damp this exponential growth. Schematically, the effective mass of the particle around $\phi = 0$ is $m_{\text{eff}}^2(T) = -m^2 + c_T T^2/4$, and the sign of this effective mass determines whether the perturbation expansion around the symmetric vacuum is justified. The solution of $m_{\text{eff}}^2(T = T_c) = 0$ then determines a critical temperature scale T_c : if $T > T_c$, the symmetric point $\phi = 0$ is stable and the symmetry is restored; otherwise, the broken minima are the preferred vacua of the theory. A more rigorous treatment of symmetry restoration accounts for the effects of the thermal background on the classical potential: $V_{\text{eff}}(\phi, T) = V_0(\phi, T) + V_{\text{CW}}(\phi) + V_T(\phi)$, where V_{CW} is the zero-temperature one-loop Coleman-Weinberg quantum correction, and V_T is the one-loop thermal correction:

$$V_T(\phi, T) = \frac{T^4}{2\pi^2} J_B\left(\frac{M^2(\phi)}{T^2}\right), \quad J_B(y) = \int_0^\infty dx x^2 \log\left[1 - \exp\left(-\sqrt{x^2 + y}\right)\right], \quad (7.1.3)$$

where $M^2(\phi)$ is the field-dependent mass squared of the fluctuation being integrated out.⁽¹⁾ For the simple quartic scalar potential in Eq. (7.1.1), this mass is $M^2(\phi) = -m^2 + 3\lambda\phi^2$. To see what the

⁽¹⁾ In a general theory, all light species that couple to the scalar background contribute to the one-loop thermal free energy. With $m_i^2(\phi)$ the field-dependent mass squared and n_i a positive degeneracy factor, one may write

$$V_T(\phi, T) = T^4/(2\pi^2) \left\{ \sum_{i \in B} n_i J_B(m_i^2(\phi)/T^2) - \sum_{i \in F} n_i J_F(m_i^2(\phi)/T^2) \right\}$$

Equivalently, the fermionic minus sign can be absorbed into a signed degeneracy factor. The fermionic function differs by the Fermi-Dirac sign, $J_F(y) = \int_0^\infty dx x^2 \log[1 + \exp(-\sqrt{x^2 + y})]$.

bosonic thermal function does, it is useful to expand it in the high-temperature limit $y = m_i^2(\phi)/T^2 \ll 1$ (or equivalently $T \gg T_c$) [90]:

$$J_B(y)|_{y \ll 1} = -\frac{\pi^4}{45} + \frac{\pi^2}{12}y - \frac{\pi}{6}y^{3/2} - \frac{1}{32}y^2 \log \frac{y}{a_B} + \mathcal{O}(y^3), \quad (7.1.4)$$

where $a_B = 16\pi^2 \exp(3/2 - 2\gamma_E)$ and γ_E is the Euler-Mascheroni constant. This leads to

$$V_T(\phi, T) = \frac{\lambda}{8}T^2\phi^2 - \frac{\pi^2}{90}T^4 + \dots, \quad (7.1.5)$$

and we see clearly that $V''(\phi=0)$ changes sign at the temperature $T_c \simeq 2m/\sqrt{\lambda}$. For $T < T_c$, $\phi=0$ is a local maximum, while for $T > T_c$ it is the global minimum and therefore becomes the true ground state of the model. Hence the symmetry $\mathbb{Z}_2$ is restored in a sufficiently high-temperature background.

7.1.2 First-order phase transition

A first-order phase transition (FOPT) is a transition in which two locally stable phases coexist over some range of control parameters and are separated by a free-energy barrier. In thermodynamic context, the transition is associated with a discontinuity in the first derivative of the free energy with respect to some thermal dynamic variables. Familiar examples include boiling or freezing water, where the two phases coexist at the transition temperature and the conversion proceeds through nucleation and growth of bubbles or droplets. The cosmological examples relevant for this thesis follow the same pattern. As the universe expands and cools, finite-temperature corrections change the effective potential, and a new lower-free-energy minimum can appear while the field remains trapped in the older high-temperature minimum. This metastable phase is the false vacuum, and the transition proceeds when true-vacuum bubbles nucleate, expand, and eventually collide. The collisions, together with the accompanying scalar-field and plasma dynamics, provide the out-of-equilibrium anisotropic stress that can source a stochastic gravitational-wave background which is the main topic of study later in this thesis.

An example of an FOPT potential

The Z_2 scalar model reviewed above is mainly a symmetry-restoration example. In the leading high-temperature treatment, the curvature at $\phi=0$ changes sign at T_c , and the order parameter moves continuously from the symmetric point to the broken minimum as the universe cools. There is no barrier between degenerate minima at the critical temperature in this example; the transition from the symmetric phase to the broken phase is therefore second order in this mean-field description. To discuss tunneling, we consider adding a symmetry-breaking cubic term to the potential:

$$V(\phi) = \frac{\lambda}{4}\phi^2(\phi - v)^2 - \frac{\lambda}{2}\epsilon v \phi^3, \quad \lambda > 0, \quad v > 0, \quad \epsilon \geq 0. \quad (7.1.6)$$

Here v sets the field scale of the second minimum and ϵ is a dimensionless parameter that tilts the potential. At $\phi=0$ one has $V(0)=0$ and $V''(0)=\lambda v^2/2 > 0$, so this point is a local minimum. The two nonzero stationary points are

$$\phi_{\text{top}} = \frac{v}{4} \left[3(1 + \epsilon) - \sqrt{9(1 + \epsilon)^2 - 8} \right], \quad (7.1.7)$$

$$\phi_{\sigma} = \frac{v}{4} \left[3(1 + \epsilon) + \sqrt{9(1 + \epsilon)^2 - 8} \right], \quad (7.1.8)$$

-
- For $y \ll 1$

$$J_F(y) = 7\pi^4/360 - (\pi^2/24)y - (1/32)y^2 \log(y/a_F) + \mathcal{O}(y^3), \quad a_F = \pi^2 \exp(3/2 - 2\gamma_E)$$

- For $y \gg 1$,

$$J_F(y) \approx +\sqrt{\pi/2}y^{3/4}e^{-\sqrt{y}}(1 + 15y^{-1/2}/8).$$

which follow from solving $dV/d\phi = 0$. The smaller stationary point ϕ_{top} is the top of the barrier, while the larger one ϕ_{σ} is the second local minimum. The subscript σ in ϕ_{σ} is only a label for this stationary point and should not be confused with the wall tension σ_{wall} introduced below. For $\epsilon = 0$, the two minima at $\phi = 0$ and $\phi = v$ are degenerate. For $\epsilon > 0$, the second minimum has lower potential energy, so $\phi = 0$ is a metastable false vacuum and ϕ_{σ} is the true vacuum. The barrier between them is the feature absent in the minimal Z_2 symmetry-restoration example.

A phase transition occurs when the preferred minimum changes as the universe cools. We denote the high-temperature metastable minimum by $\phi_{\text{FV}}(T)$ and the low-temperature minimum by $\phi_{\text{TV}}(T)$. At the critical temperature the two phases have equal free energy,

$$V_{\text{eff}}(\phi_{\text{FV}}(T_c), T_c) = V_{\text{eff}}(\phi_{\text{TV}}(T_c), T_c). \quad (7.1.9)$$

Below T_c , the true-vacuum phase has lower free energy, and it is useful to define

$$\rho_{\text{vac}}(T) = \Delta V(T) = V_{\text{eff}}(\phi_{\text{FV}}(T), T) - V_{\text{eff}}(\phi_{\text{TV}}(T), T) > 0. \quad (7.1.10)$$

The name “false vacuum” should be read with care at finite temperature: these are phases of the thermal free energy, not necessarily zero-temperature vacua ϕ_{σ} . The essential feature of a first-order transition is the barrier between ϕ_{FV} and ϕ_{TV} . Because of this barrier, the field cannot pass homogeneously and classically to the lower-free-energy phase immediately after T_c . For a sketch of the finite-temperature effective potential for a first-order phase transition, see Fig. 7.1.

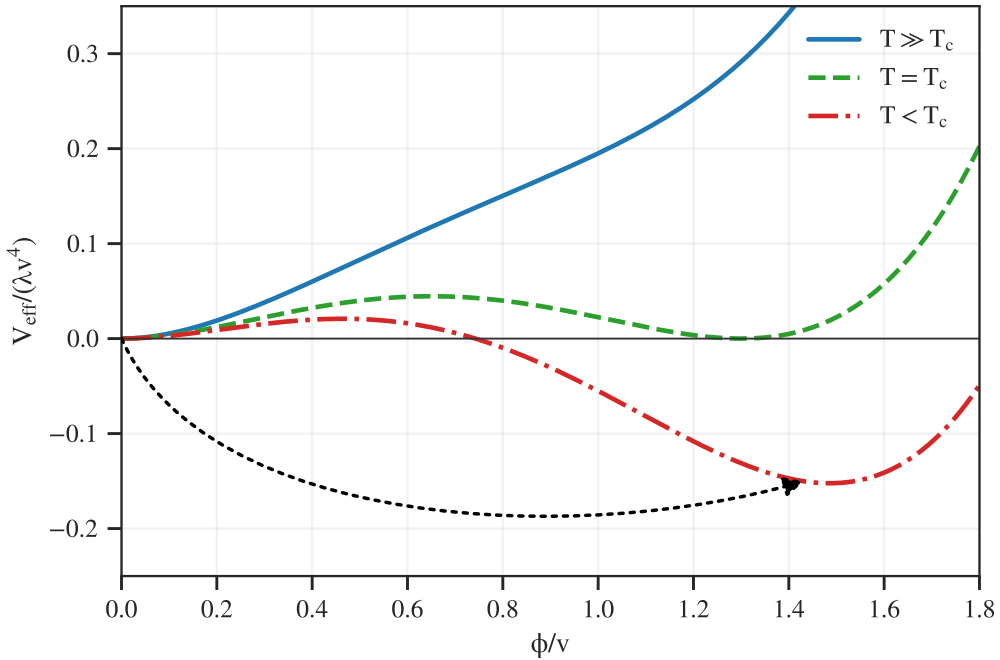


Figure 7.1: Schematic finite-temperature effective potential for a first-order transition corresponding to the potential (7.1.6). As the temperature falls below T_c , the false minimum ϕ_{FV} remains locally stable while the true minimum ϕ_{TV} has lower free energy and triggers a first-order phase transition. The black dotted arrow is a schematic projection of the tunneling path in field space; its endpoint is the tunneling value $\phi_b(0)$ of the finite-temperature $O(3)$ bounce for the plotted $T < T_c$ potential. The barrier prevents a higher-order phase transition, and the free energy difference $\rho_{\text{vac}}(T)$ then drives the subsequent bubble expansion once it is nucleated.

Bounce actions and critical bubbles

The transition from the metastable phase to the stable phase proceeds by bubble nucleation. A bubble is a localized field configuration in which ϕ is near ϕ_{TV} in its interior and approaches ϕ_{FV} far away. Small bubbles shrink because the wall costs gradient and potential energy. Large bubbles

expand because the volume gain from converting false-vacuum free energy into true-vacuum free energy dominates the wall cost. The critical bubble is the saddle point separating these two behaviors. At zero temperature, the leading semiclassical calculation is the Coleman bounce [91]. The Euclidean action for a scalar profile in flat spacetime is

$$S_E[\phi] = \int d\tau d^3x \left[\frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + V_{\text{eff}}(\phi, 0) - V_{\text{eff}}(\phi_{\text{FV}}, 0) \right], \quad (7.1.11)$$

where τ is Euclidean time. The subtraction of $V_{\text{eff}}(\phi_{\text{FV}}, 0)$ sets the false-vacuum action density to zero. In flat space and at zero temperature, the lowest-action solution has $O(4)$ symmetry and depends only on $r = \sqrt{\tau^2 + \mathbf{x}^2}$. It satisfies

$$\frac{d^2 \phi_b}{dr^2} + \frac{3}{r} \frac{d\phi_b}{dr} = \frac{\partial V_{\text{eff}}(\phi_b, 0)}{\partial \phi_b}, \quad \left. \frac{d\phi_b}{dr} \right|_{r=0} = 0, \quad \phi_b(r \rightarrow \infty) = \phi_{\text{FV}}. \quad (7.1.12)$$

The solution $\phi_b(r)$ is the bounce profile, and its value at $r = 0$ is the escape point. This bounce equation can be solved conceptually by the overshoot/undershoot method, as previously discussed in Chapter. 1, and will not be discussed again here. The corresponding decay rate of the false vacuum per physical volume per physical time has the schematic form

$$\Gamma_4 \simeq A_4 e^{-S_4}, \quad S_4 = 2\pi^2 \int_0^\infty dr r^3 \left[\frac{1}{2} \left(\frac{d\phi_b}{dr} \right)^2 + V_{\text{eff}}(\phi_b, 0) - V_{\text{eff}}(\phi_{\text{FV}}, 0) \right], \quad (7.1.13)$$

where A_4 is a fluctuation prefactor, for which a dimensional estimate is sufficient in many applications.

The Euclidean bounce also has a direct Lorentzian interpretation. If the field profile changes rapidly near a radius R , the analytic continuation from Euclidean to real time describes a spherical bubble nucleated with radius R . In the ideal vacuum problem the bubble wall then expands with a trajectory approaching the speed of light. This lightlike expansion is not assumed in the later plasma discussion, where the macroscopic wall speed is denoted by v_w and can be subluminal due to the friction between the wall and the surrounding plasma.

The thin-wall approximation gives a useful analytic limit of the $O(4)$ result. It applies when the free-energy difference ρ_{vac} between the two minima is small compared with the barrier height, so that the bubble radius is much larger than the wall thickness. Let σ_{wall} denote the surface tension, i.e. the energy per unit area stored in the wall profile. Approximating the Euclidean bubble as a four-dimensional ball with a thin three-dimensional surface gives

$$S_4(R) = 2\pi^2 R^3 \sigma_{\text{wall}} - \frac{\pi^2}{2} R^4 \rho_{\text{vac}}. \quad (7.1.14)$$

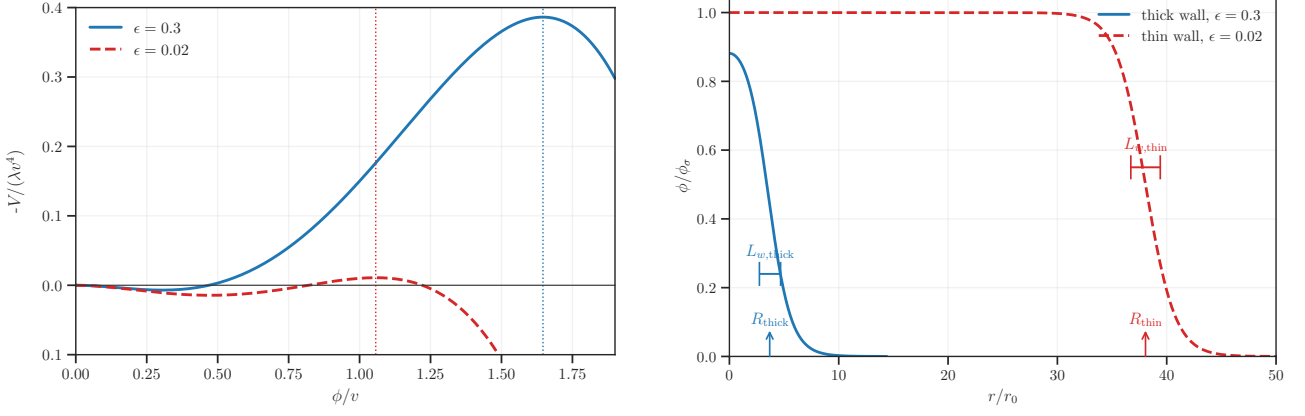
The first term is the wall contribution, since the area of a three-sphere of radius R is $2\pi^2 R^3$. The second term is the volume gain, since the volume of a three-ball is $\pi^2 R^4/2$. Minimizing Eq. (7.1.14) gives

$$R_c = \frac{3\sigma_{\text{wall}}}{\rho_{\text{vac}}}, \quad S_4^{\text{tw}} = \frac{27\pi^2 \sigma_{\text{wall}}^4}{2\rho_{\text{vac}}^3}. \quad (7.1.15)$$

This formula is the four-dimensional Euclidean version of the surface-versus-volume competition. Figure. 7.2 shows the inverted tunneling potential and the corresponding thick-wall and thin-wall bounce profiles.

The finite-temperature calculation modifies this picture because Euclidean time is periodic with period $1/T$. The bounce must satisfy $\phi(\tau + 1/T, \mathbf{x}) = \phi(\tau, \mathbf{x})$. When the temperature is low compared with the inverse size of the zero-temperature bounce, this periodicity is unimportant and the $O(4)$ solution gives the leading exponent. At sufficiently high temperature the saddle point of the Euclidean action is approximately independent of τ . Between these two limits there can also be genuinely time-dependent periodic bounces, so comparing only S_4 with S_3/T should be understood as a limiting or approximate criterion. Therefore, in the high-temperature limit, the Euclidean action factors as $S_E = S_3/T$, where the three-dimensional action is

$$S_3(T) = 4\pi \int_0^\infty dr r^2 \left[\frac{1}{2} \left(\frac{d\phi_b}{dr} \right)^2 + V_{\text{eff}}(\phi_b, T) - V_{\text{eff}}(\phi_{\text{FV}}, T) \right]. \quad (7.1.16)$$



(a) Inverted tunneling potential for the thick-wall and thin-wall parameter choices.

(b) Numerical $O(4)$ bounce profiles with radius and wall-width markers.

Figure 7.2: Inverted potential and corresponding $O(4)$ critical bubble profiles based on the potential in Eq. (7.1.6) with $\epsilon = 0.3$ for the thick-wall profile and $\epsilon = 0.02$ for the thin-wall profile. The potential panel uses the dimensionless variables ϕ/v and $-V/(\lambda v^4)$, which remove the overall field and energy scales from the polynomial potential and leave the shape controlled solely by ϵ . The profile panel uses r/r_0 , with $r_0 = (\sqrt{\lambda}v)^{-1}$, and ϕ/ϕ_σ , where ϕ_σ is the true-vacuum field value for the chosen ϵ . The critical-bubble radius is marked at R , defined numerically by $\phi(R) = \phi(0)/2$, and the wall width is marked as $L_w = R_{\text{out}} - R_{\text{in}}$, where $\phi(R_{\text{in}}) = \phi(0)[1 - \tanh(-0.5)]/2$ and $\phi(R_{\text{out}}) = \phi(0)[1 - \tanh(0.5)]/2$.

The corresponding $O(3)$ bounce equation for the critical bubble profile is

$$\frac{d^2\phi_b}{dr^2} + \frac{2}{r} \frac{d\phi_b}{dr} = \frac{\partial V_{\text{eff}}(\phi_b, T)}{\partial \phi_b}, \quad \left. \frac{d\phi_b}{dr} \right|_{r=0} = 0, \quad \phi_b(r \rightarrow \infty) = \phi_{\text{FV}}(T). \quad (7.1.17)$$

The thermal nucleation rate is therefore controlled by the exponent $S_3(T)/T$,

$$\Gamma_3(T) \simeq A_3(T) e^{-S_3(T)/T}, \quad (7.1.18)$$

with a prefactor $A_3(T)$ of order the fourth power of the relevant temperature or microscopic scale.

As for the $O(4)$ bubble, we can also apply the thin-wall approximation to gain insight into the relevant properties of the critical bubble in this $O(3)$ case. For a spherical thermal bubble of radius R , the free-energy cost is approximately

$$S_3^{\text{tw}}(T) = 4\pi R^2 \sigma_{\text{wall}}(T) - \frac{4\pi}{3} R^3 \rho_{\text{vac}}(T). \quad (7.1.19)$$

The first term is the positive surface cost, and the second term is the negative volume gain. The stationary point of $S_3^{\text{tw}}(T)$ gives the critical radius

$$R_c = \frac{2\sigma_{\text{wall}}(T)}{\rho_{\text{vac}}(T)}. \quad (7.1.20)$$

The corresponding thin-wall three-dimensional action is

$$S_3^{\text{tw}}(T) = \frac{16\pi\sigma_{\text{wall}}^3(T)}{3\rho_{\text{vac}}^2(T)}. \quad (7.1.21)$$

The thermal tunneling exponent is then $S_3^{\text{tw}}(T)/T$. We see a similarity in both $O(3)$ and $O(4)$ bubbles: the critical radius is linearly proportional to the ratio between the surface tension and the vacuum energy difference.

Nucleation, percolation, and completion

After nucleation, the true-vacuum bubbles expand, collide, and fill the physical space of the false vacuum until completion. It is important to distinguish several temperatures that characterize different scales during the phase transition. The critical temperature T_c is the temperature at which the two minima of the thermal potential are degenerate, as introduced above. The nucleation temperature T_n is often defined by asking when the expected number of bubbles nucleated inside one Hubble volume becomes of order unity. These two temperature scales correspond to the time scales denoted as t_c and t_n . Under adiabatic cooling, where the time-temperature relation is $dt/dT = -(TH(T))^{-1}$, this may be written schematically as [92, 93]

$$N(T_n) = \int_{t_c}^{t_n} dt \frac{\Gamma(t)}{H^3(t)} = \int_{T_n}^{T_c} \frac{dT}{T} \frac{\Gamma(T)}{H^4(T)} \simeq 1. \quad (7.1.22)$$

A more systematic measure of the progress of the transition is the probability that a fixed comoving point $\mathbf{x}$ is still in the false phase at time t . Assume that bubble nucleations are local and uncorrelated on the spacetime scales used in the transition calculation, and that after nucleation the wall trajectory is determined by the macroscopic wall speed v_w . Under these assumptions $\Gamma(t)$ gives the mean number of bubble nucleations per physical volume per physical time. A nucleation at $\mathbf{x}'$ and time t' converts the chosen point $\mathbf{x}$ by time t to the true vacuum phase if $|\mathbf{x}' - \mathbf{x}| < r(t, t')$, where the comoving radius reached by the wall is

$$r(t, t') = \int_{t'}^t d\bar{t} \frac{v_w}{a(\bar{t})}. \quad (7.1.23)$$

The probability that $\mathbf{x}$ has not yet been reached by any bubble is then ⁽¹⁾

$$P_{\text{FV}}(t) = e^{-I(t)}, \quad I(t) = \frac{4\pi}{3} \int_{t_c}^t dt' \Gamma(t') a^3(t') r^3(t, t'). \quad (7.1.24)$$

The quantity $I(t)$ counts the volume fraction that would be converted if all bubbles were allowed to expand through each other. The exponential relation in Eq. (7.1.24) is what converts this extended volume into the actual probability for a point to remain in the false phase and therefore accounts for bubble overlap. The ensemble-averaged converted volume fraction is then $P_{\text{TV}} = 1 - P_{\text{FV}} = 1 - e^{-I}$. We can rewrite the result in terms of temperature by using $dT/dt \simeq -H(T)T$ and $a(T) \propto T^{-1}$. The product $a(T')r(T, T')$ may then be written, up to the irrelevant normalization of a , as

$$a(T')r(T, T') = \frac{1}{T'} \int_T^{T'} \frac{v_w d\tilde{T}}{H(\tilde{T})}. \quad (7.1.25)$$

Substituting this into the time-domain result gives

$$I(T) = \frac{4\pi}{3} \int_T^{T_c} \frac{dT'}{T'^4} \frac{\Gamma(T')}{H(T')} \left[\int_T^{T'} \frac{v_w d\tilde{T}}{H(\tilde{T})} \right]^3. \quad (7.1.26)$$

⁽¹⁾ Define the spacetime region of nucleation events whose bubbles would have reached $\mathbf{x}$ by time t ,

$$\mathcal{D}_{\mathbf{x}}(t) = \{(t', \mathbf{x}') \mid t_c \leq t' \leq t, |\mathbf{x}' - \mathbf{x}| < r(t, t')\}.$$

In a small cell $A \subset \mathcal{D}_{\mathbf{x}}(t)$, the probability for such a nucleation event is

$$\Delta p_A = \Gamma(t_A) a^3(t_A) \Delta t_A \Delta^3 x_A + \mathcal{O}(\Delta p_A^2).$$

The probability of no such nucleation event in all cells is therefore

$$P_{\text{FV}}(t) = \lim_{\Delta p_A \rightarrow 0} \prod_{A \subset \mathcal{D}_{\mathbf{x}}(t)} (1 - \Delta p_A) = \exp \left[- \int_{\mathcal{D}_{\mathbf{x}}(t)} dt' d^3 x' \Gamma(t') a^3(t') \right].$$

Statistical homogeneity makes the result independent of the chosen point $\mathbf{x}$, and the spatial integral over the comoving ball of radius $r(t, t')$ gives the expression for $I(t)$ in Eq. (7.1.24).

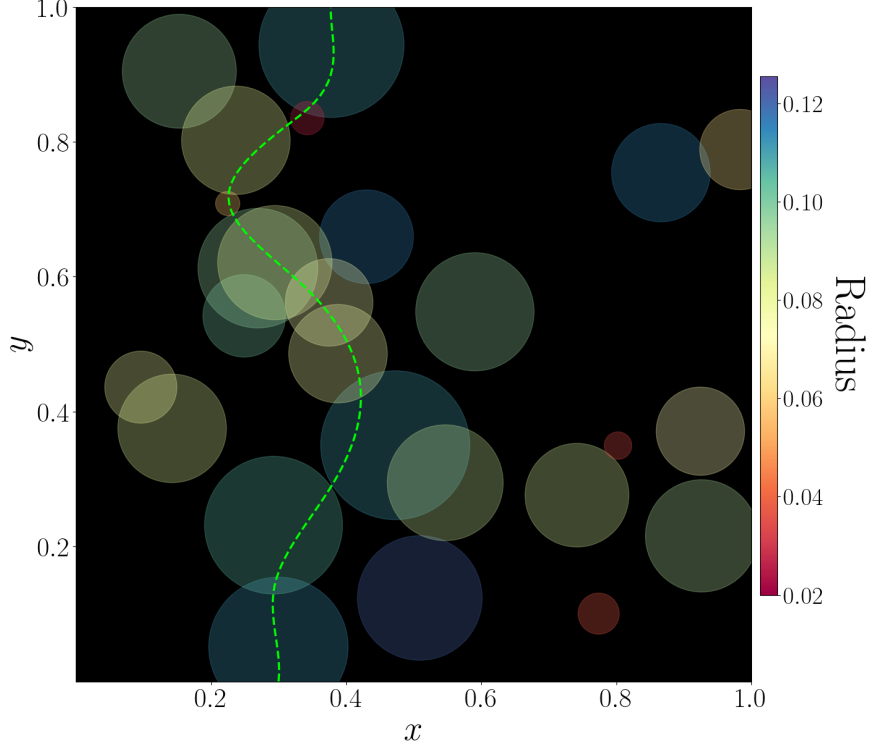


Figure 7.3: Visualization of the onset of percolation in a simple three-dimensional bubble-nucleation simulation, shown as a two-dimensional slice, from Ref. [96]. At the plotted time the false-vacuum fraction is approximately $P_{\text{FV}} \simeq 0.7$, close to the percolation criterion used in that work, and the dashed curve marks a connected path of bubbles spanning the simulation volume.

This temperature-space form assumes adiabatic cooling with slowly varying relativistic degrees of freedom, so that $dT/dt \simeq -H(T)T$ and $a(T) \propto T^{-1}$ are good approximations during the nucleation and percolation interval. ⁽¹⁾

Another relevant scale is the percolation temperature T_p , corresponding to the temperature near bubble overlap and collision. It can be defined as $P_{\text{FV}}(T_p) = 1/e$, equivalently $I(T_p) = 1$, following Ref. [94], or by $P_{\text{FV}}(T_p) \approx 0.71$, as used in Ref. [95] and references therein. The corresponding cosmic time is denoted by t_p , and can be used to mark the characteristic time for GW production. Percolation begins when there is a connected cluster of bubbles that spans the entire Universe, as illustrated in Fig. 7.3.

The time that marks successful completion of the phase transition, t_{end} , may require a stronger condition than the percolation scale, especially in vacuum-dominated or strongly supercooled cases. To derive it, take a fixed comoving reference volume V_{com} . Its expected physical false-vacuum volume is

$$\mathcal{V}_{\text{FV}}(t) = a^3(t)V_{\text{com}}P_{\text{FV}}(t). \quad (7.1.28)$$

A necessary completion condition is therefore $d\mathcal{V}_{\text{FV}}/dt < 0$, i.e., the physical false-vacuum volume starts decreasing. Dividing by $\mathcal{V}_{\text{FV}}$ and using $P_{\text{FV}} = e^{-I}$ gives

$$\frac{1}{\mathcal{V}_{\text{FV}}} \frac{d\mathcal{V}_{\text{FV}}}{dt} \Big|_{t=t_{\text{end}}} = \left(3H(t) - \frac{dI}{dt} \right) \Big|_{t=t_{\text{end}}}. \quad (7.1.29)$$

⁽¹⁾ In this expression $H(T)$ should include all components relevant for the background expansion, including the radiation energy density ρ_{rad} and the false-vacuum energy in a strongly supercooled transition ρ_{vac} , i.e.,

$$H(T)^2 = \frac{1}{3M_{\text{pl}}^2}(\rho_{\text{rad}} + \rho_{\text{vac}}) = \frac{1}{3M_{\text{pl}}^2} \left(\frac{T^4}{\xi_g^2} + \rho_{\text{vac}} \right), \quad (7.1.27)$$

with $\xi_g = \sqrt{30/(\pi^2 g_*)}$, where $g_* \approx 106.75$ is the number of degrees of freedom.

Finally, using $dT/dt \simeq -H(T)T$ gives the temperature-dependent form used in supercooled-transition analyses, e.g., Refs. [92, 93, 96],

$$\frac{1}{\mathcal{V}_{\text{FV}}} \frac{d\mathcal{V}_{\text{FV}}}{dt} \Big|_{t=t_{\text{end}}} = H(T) \left(3 + T \frac{dI}{dT} \right) \Big|_{T=T_{\text{end}}} < 0. \quad (7.1.30)$$

This condition expresses the fact that the probability of remaining in the false phase must decrease fast enough to compensate for the growth of physical volume if we want to use the percolation scale to mark the end of the phase transition. For transitions where Eq. (7.1.30) is satisfied only at temperatures below T_p , the concept of percolation is questionable, as two nearby bubbles may never meet when their growth in size is slower than the expansion of their separation.

Macroscopic parameters for gravitational-wave sources

For gravitational-wave calculations, the detailed microphysics of a first-order phase transition is usually summarized by a small set of macroscopic parameters [64, 97, 51]. These parameters do not replace the underlying finite-temperature field theory, but they provide the bridge between a particle-physics model and the shape, amplitude, and frequency scale of the gravitational-wave signal. The most commonly used quantities are the transition temperature T_* , the strength parameter α , the transition rate β , and the wall velocity v_w . Efficiency factors such as κ_ϕ and κ_{sw} then specify how the released energy is distributed among the relevant sources.

The temperature T_* denotes the epoch at which the gravitational-wave source is evaluated. In the literature this may refer to the nucleation temperature, the percolation temperature, or a reheating temperature after the transition, which for weak or moderately strong transitions are often numerically close. This temperature scale also fixes the Hubble rate at collision

$$H_*^2 = \frac{8\pi G}{3} \rho_{\text{tot}}(T_*), \quad (7.1.31)$$

and consequently it fixes the redshift between the production epoch and today. For example, during radiation domination one has

$$\rho_{\text{tot}}(T_*) = \rho_{\text{rad}}(T_*) = \frac{\pi^2}{30} g_* T_*^4, \quad (7.1.32)$$

with g_* being the effective number of relativistic degrees of freedom [64, 51].

The strength parameter α measures the released energy relative to the radiation bath. In the vacuum-dominated limit, where thermal corrections to the released energy are negligible, we have

$$\alpha \simeq \frac{\rho_{\text{vac}}}{\rho_{\text{rad}}(T_*)}, \quad (7.1.33)$$

where ρ_{vac} denotes the zero-temperature vacuum-energy difference.

The parameter β measures how rapidly the transition proceeds. It was introduced above through the local exponential approximation to the nucleation rate,

$$\Gamma(t) \simeq \Gamma_* e^{\beta(t-t_*)}. \quad (7.1.34)$$

The dimensionless ratio β/H_* compares the duration of the transition with the Hubble time. A large value of β/H_* corresponds to a short transition, many bubbles per Hubble volume, and a smaller characteristic length scale. For an approximately exponential nucleation history with slowly varying v_w , the mean bubble separation is often estimated as [64, 79]

$$R_* \simeq (8\pi)^{1/3} \frac{v_w}{\beta}, \quad (7.1.35)$$

up to convention-dependent definitions of the average radius or separation. This scale typically sets the peak wavelength of the gravitational-wave spectrum.

The wall velocity v_w controls both the geometry of the source and the partition of the released energy. It enters directly in the bubble size at collision, in the hydrodynamic solution around each wall,

and in the efficiency factors used in gravitational-wave templates. In a plasma, v_w is determined by a balance between the driving pressure and microscopic friction whenever a stationary solution exists [98, 97]. If the friction saturates below the available driving pressure, the wall can enter a runaway-like regime in which a large fraction of the released energy remains in the scalar field. For fluid sources, the value of v_w also distinguishes subsonic deflagrations, supersonic detonations, and hybrid solutions. The detailed hydrodynamic matching conditions are not needed for the work developed within the scope of this thesis; what matters for the later gravitational-wave discussion is that different wall regimes deposit energy into different stress-energy channels.

It is therefore useful to separate the released energy into scalar-field energy near the wall, bulk kinetic energy of the plasma, and later nonlinear or turbulent plasma motion, which are three different channels that contribute to the generation of gravitational waves. We denote by κ_ϕ the fraction of released energy stored in the scalar-field configuration, by κ_{sw} the fraction converted into coherent sound waves in the fluid, and by κ_{turb} the fraction associated with nonlinear vortical or magnetohydrodynamic motion. The combinations that enter gravitational-wave amplitudes are often written schematically as

$$K_X \equiv \frac{\kappa_X \alpha}{1 + \alpha}, \quad (7.1.36)$$

where X labels the source channel. We will briefly comment on the spectrum generated by these channels in Subsection. 7.1.3.

7.1.3 Stochastic Gravitational Waves from First-Order Phase Transitions

Recall that a first-order phase transition occurs through many bubbles nucleated randomly at different positions and times. The gravitational-wave radiation arriving today is not a deterministic waveform of a single event, but is instead a stochastic background built from the incoherent superposition of many causally independent regions and characterized by a spectral energy density Ω_{GW} .

The classification of gravitational-wave sources in a first-order phase transition follows directly from the stress-energy tensor. Tensor perturbations are sourced only by the transverse-traceless part of the spatial stress, so the relevant question is where the transition stores anisotropic stress and for how long. The standard decomposition separates three contributions: scalar-field bubble collisions, acoustic waves in the plasma, and magnetohydrodynamic turbulence. The scalar-field contribution is associated with gradients and kinetic energy in the bubble walls. It is most important when the released energy remains in the scalar sector, as can happen in vacuum-dominated or runaway-like transitions, e.g., supercooled first-order phase transitions. For transitions with an appreciable thermal plasma, however, most of the released energy is transferred to bulk fluid motion of the plasma. The resulting sound waves can act as a long-lasting source compared with the duration of bubble collisions, although this lifetime can be shortened once shocks or nonlinear fluid motion develop. At later times, nonlinear fluid motion and magnetic fields can source magnetohydrodynamic turbulence, which is usually treated as a separate contribution with larger theoretical uncertainties.

General features of the sources and the SGWB spectra

Given that each contribution has different source dynamics, a different lifetime, a different efficiency factor, and consequently a different spectral shape, we expect different sources to have distinct features that can be observed in the output spectrum. The distinctions among the spectral shapes of different source classes are generally reflected in physical features of the spectra, including scaling laws in different frequency ranges, the width of the transition region between scaling laws, and the spectral amplitudes at these transition frequencies.

The characteristic spectral scale is set by the typical size and duration of the source. For an exponentially growing nucleation rate, we use the mean bubble separation defined in Eq. (7.1.35) and commonly adopted in FOPT gravitational-wave studies

$$R_* \simeq (8\pi)^{1/3} \frac{v_w}{\beta}, \quad (7.1.37)$$

In different studies, this definition may vary slightly. For example, in sound-wave studies v_w is replaced by $\max(v_w, c_s)$ in this estimate, because the relevant acoustic shell can extend with the sound speed rather than purely with the wall speed [79]. When comparing formulae below, R_* should therefore be read as the macroscopic interbubble scale, with the precise convention stated by the template being used. A source varying on this scale emits most efficiently at angular frequencies of order $\omega_* \sim R_*^{-1}$ at the time of production. ⁽¹⁾ The subscript $*$ denotes quantities evaluated at production, and frequencies redshift as $\omega_0 = (a_*/a_0)\omega_*$. When translating ordinary-frequency f from the literature, we use $\omega = 2\pi f$ while leaving β and H_* as inverse-time rates. Thus the peak angular frequency observed today is controlled primarily by a_* , or equivalently the temperature at the production epoch T_* , by β/H_* , and by v_w .

The amplitude follows a complementary logic: a larger released energy fraction increases the available anisotropic stress, while a slower transition gives the source more time and a larger coherence scale relative to the Hubble radius. Both factors can have a significant impact on the amplitude of the SGWB. The low-frequency behavior also has a general interpretation. For causal sources active for a finite time, the spectrum is constrained by the finite correlation length of the stress tensor, leading to a rising infrared spectrum whose precise power depends on the source model. The ultraviolet behavior is instead sensitive to the microphysics, including, e.g., the detailed wall profile at collision, the fluid shell structure, or the turbulent cascade. Consequently, phenomenological templates are useful, but their domain of validity must be tied to the assumptions about the underlying source.

Essential features of different sources

More detailed discussions of these three source classes are given in Appendix. D.1. Below, we summarize the essential features of the three source classes as follows:

- *Scalar-field bubble collisions* are sourced by the TT part of the gradient stress of the order-parameter field. For a canonical scalar field the spatial stress is

$$T_{ij}^\phi = \partial_i \phi \partial_j \phi + \delta_{ij} \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right].$$

The isotropic δ_{ij} part is removed by the TT projection, leaving $(\partial_i \phi \partial_j \phi)^{\text{TT}}$ as the radiating source. This channel is most relevant when a significant fraction of the released energy remains in the scalar walls, as in vacuum-dominated, sufficiently supercooled, or runaway-like transitions [99, 61, 100]. In the standard envelope approximation the wall stress is removed after first collision, so the source is short-lived, with duration set by the transition time scale β^{-1} . This gives the characteristic collision scaling $\Omega_{\text{GW}} \propto (H_*/\beta)^2 K_\phi^2$, with K_ϕ the scalar-wall energy fraction [61, 64]. The Caprini et al. envelope-template fit is [64]

$$\begin{aligned} h^2 \Omega_{\text{GW},0}^{\text{env}}(\omega) &= 1.67 \times 10^{-5} \left(\frac{H_*}{\beta} \right)^2 K_\phi^2 \left(\frac{100}{g_*} \right)^{1/3} \left(\frac{0.11 v_w^3}{0.42 + v_w^2} \right) S_{\text{env}}(\omega), \\ S_{\text{env}}(\omega) &= \frac{3.8(\omega/\hat{\omega}_{\text{env},0})^{2.8}}{1 + 2.8(\omega/\hat{\omega}_{\text{env},0})^{3.8}}, \\ \frac{\hat{\omega}_{\text{env},0}}{2\pi} &= 16.5 \times 10^{-3} \text{ mHz} \left(\frac{0.62}{1.8 - 0.1 v_w + v_w^2} \right) \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \end{aligned}$$

Its spectral shape depends strongly on the post-collision prescription: envelope spectra have an infrared rise close to ω^3 and a shallow ultraviolet falloff close to ω^{-1} . An improvement over the envelope approximation is the bulk-flow approximation developed in [101], which keeps persistent shell stress and can give a much shallower infrared rise and a steeper ultraviolet falloff. In Section. 7.2, we will discuss the construction of both the envelope and bulk-flow approximations in more detail, including some drawbacks of these approaches before developing a more flexible approach called the generalized bulk-flow approximation in the next chapter.

⁽¹⁾ This angular frequency ω is also called the physical wave number k in other literature. We keep the notation ω throughout the thesis for consistency.

- *Acoustic waves in the plasma* are sourced by coherent bulk fluid motion generated when the expanding bubble walls transfer energy and momentum to a thermal plasma. For an ideal relativistic fluid, the spatial stress is

$$T_{ij}^{\text{fluid}} = \gamma^2(e + p)v_i v_j + p\delta_{ij}.$$

The pressure term is projected out and the TT source is the shear part $[\gamma^2(e + p)v_i v_j]^{\text{TT}}$. After the scalar walls collide and disappear, the longitudinal velocity field can continue to carry anisotropic stress as sound waves, which is why this source can dominate in non-runaway thermal transitions [62, 63]. If the acoustic stage remains linear and weakly damped for a Hubble time, its amplitude scales parametrically as $\Omega_{\text{GW}}^{\text{sw}} \propto (H_* R_*) K_{\text{sw}}^2$, or equivalently as $(H_*/\beta) K_{\text{sw}}^2$ up to wall-velocity and convention-dependent factors, with K_{sw} the fluid kinetic fraction. The Caprini et al. sound-wave template used for forecasts is [64]

$$\begin{aligned} h^2 \Omega_{\text{GW},0}^{\text{sw}}(\omega) &= 2.65 \times 10^{-6} \left(\frac{H_*}{\beta} \right) K_{\text{sw}}^2 \left(\frac{100}{g_*} \right)^{1/3} v_w S_{\text{sw}}(\omega), \\ S_{\text{sw}}(\omega) &= \left(\frac{\omega}{\hat{\omega}_{\text{sw},0}} \right)^3 \left[\frac{7}{4 + 3(\omega/\hat{\omega}_{\text{sw},0})^2} \right]^{7/2}, \\ \frac{\hat{\omega}_{\text{sw},0}}{2\pi} &= 1.9 \times 10^{-2} \text{ mHz} \frac{1}{v_w} \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \end{aligned}$$

This longer source lifetime is the main reason sound waves usually dominate over scalar-field collisions in non-runaway thermal transitions. The commonly used sound-wave template has a cubic infrared rise and an ultraviolet falloff $\Omega_{\text{GW}}^{\text{sw}} \propto \omega^{-4}$, while the more detailed sound-shell model can contain two scales where power law changes, which are controlled by the mean bubble separation and by the shell thickness [102, 103, 64].

- *Magnetohydrodynamic turbulence* describes the later nonlinear plasma stage, in which vortical fluid motion and magnetic anisotropic stress contribute to the TT source. Schematically, the TT source is

$$T_{ij}^{\text{MHD}} = [w\gamma^2 v_i v_j + T_{ij}^B]^{\text{TT}}, \quad T_{ij}^B = \frac{1}{2} B^2 \delta_{ij} - B_i B_j,$$

where $w = e + p$ is the enthalpy density and the second expression gives the magnetic stress tensor in physical Lorentz-Heaviside conventions. The isotropic magnetic pressure is again removed by the TT projection. This source is usually modeled as being fed by a fraction of the kinetic energy already deposited in the plasma, often written phenomenologically as $\kappa_{\text{turb}} = \epsilon \kappa_{\text{sw}}$ with $\epsilon < 1$ [64, 79]. Different from the bubble collisions or sound waves, the gravitational spectrum scales as $\Omega_{\text{GW}}^{\text{turb}} \propto (H_* R_*) K_{\text{turb}}^{3/2}$, where K_{turb} is the turbulent kinetic-plus-magnetic source fraction. The Caprini et al. turbulence template, assuming non-helical Kolmogorov-type turbulence, is [64]

$$\begin{aligned} h^2 \Omega_{\text{GW},0}^{\text{turb}}(\omega) &= 3.35 \times 10^{-4} \left(\frac{H_*}{\beta} \right) K_{\text{turb}}^{3/2} \left(\frac{100}{g_*} \right)^{1/3} v_w S_{\text{turb}}(\omega), \\ S_{\text{turb}}(\omega) &= \frac{(\omega/\hat{\omega}_{\text{turb},0})^3}{[1 + (\omega/\hat{\omega}_{\text{turb},0})]^{11/3} (1 + 8\pi\omega/\omega_{H,0})}, \\ \frac{\hat{\omega}_{\text{turb},0}}{2\pi} &= 2.7 \times 10^{-2} \text{ mHz} \frac{1}{v_w} \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}, \\ \frac{\omega_{H,0}}{2\pi} &= 16.5 \times 10^{-3} \text{ mHz} \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \end{aligned}$$

In this template, the spectrum rises causally in the far infrared, behaves effectively as ω^2 over the intermediate sub-Hubble range due to the finite-lifetime factor, and falls as $\omega^{-5/3}$ at high frequency [104, 64]. This high-frequency falloff is shallower than the simple acoustic ω^{-4} tail, so turbulence may be comparatively visible in the ultraviolet tail even when it is subdominant near the main plasma peak.

7.2 Approximate methods for GW calculations

Constructing approximate methods for computing GWs generated by bubble collisions during a first-order phase transition is important because a direct $3 + 1$ D field-theory simulation of bubble expansion, collision, and post-collision evolution is computationally expensive. Such approximate approaches are possible because our physical system contains several well-separated length and time scales. A bubble is nucleated with a microscopic critical size R_c , its wall has a microscopic thickness L_w at collision, and the anisotropic GW source is instead controlled by the macroscopic bubble radius at collision, denoted by R_* . For a transition that completes in less than a Hubble time, this macroscopic collision scale is smaller than the Hubble radius, so that the regime relevant for the thin-wall treatments is schematically

$$R_c, L_w \ll R_* \lesssim H_*^{-1}. \quad (7.2.1)$$

The first hierarchy is the basis of the thin-wall description at collision: the detailed radial profile of the scalar wall can be integrated out and replaced by an effective surface stress, whose angular geometry and collision history determine the anisotropic source. The second hierarchy allows the GW calculation during the active collision period to be treated locally in flat spacetime, with the cosmological expansion entering only through the final normalization and redshifting of the spectrum. This hierarchy of scales also explains why a direct $3 + 1$ D field-theory simulation is expensive. The simulation box must contain many bubbles with an interbubble distance of order R_* and must be evolved long enough to capture the relevant source history. Meanwhile, the lattice spacing must remain sufficiently small to resolve the much smaller wall thickness L_w , and for highly relativistic walls this resolution requirement becomes even more severe. In practice, this typically requires access to high-performance computing resources, and a systematic scan over transition parameters quickly becomes costly. The difficulty is especially sharp in the regime most relevant for potentially observable scalar-collision signals: bubbles that have grown to a larger R_* can produce stronger GW radiation and are therefore more likely to be probed by GW experiments, but increasing R_*/L_w is precisely what makes the full lattice problem harder.

The aim of this section is to construct a hierarchy of thin-wall prescriptions for the anisotropic stress generated predominantly by expanding and colliding bubbles, based on the separation of scales mentioned earlier. We first write the expression for computing the GW spectrum in a form adapted to a many-bubble configuration, following Ref. [61]. The envelope approximation and the bulk-flow approximation then appear as two limiting choices for the time dependence of each wall element after its first collision. We begin with Weinberg's formula, Eq. (6.1.48), which relates the spatial components of the stress-energy tensor $T_{ij}(\mathbf{x}, t)$ to the energy radiated in the direction $\hat{\mathbf{k}}$:

$$\begin{aligned} \frac{dE_{\text{GW}}}{d \ln \omega d\Omega_{\hat{\mathbf{k}}}} &= 2G\omega^3 \Lambda_{ij,lm}(\hat{\mathbf{k}}) \tilde{T}_{ij}^*(\hat{\mathbf{k}}, \omega) \tilde{T}_{lm}(\hat{\mathbf{k}}, \omega) \\ &= 2G\omega^3 \left(\text{Tr} [P_{\hat{\mathbf{k}}} \tilde{T}^* P_{\hat{\mathbf{k}}} \tilde{T}] - \frac{1}{2} |\text{Tr} [P_{\hat{\mathbf{k}}} \tilde{T}]|^2 \right), \end{aligned} \quad (7.2.2)$$

Here the Λ tensor is defined via the projector $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{\mathbf{k}}_i \hat{\mathbf{k}}_j$ along the unit vector $\hat{\mathbf{k}}$ as

$$\begin{aligned} \Lambda_{ij,lm}(\hat{\mathbf{k}}) &= P_{il}(\hat{\mathbf{k}}) P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2} P_{ij}(\hat{\mathbf{k}}) P_{lm}(\hat{\mathbf{k}}) \\ &= \delta_{il} \delta_{jm} - 2\delta_{il} \hat{\mathbf{k}}_j \hat{\mathbf{k}}_m + \frac{1}{2} \hat{\mathbf{k}}_i \hat{\mathbf{k}}_j \hat{\mathbf{k}}_l \hat{\mathbf{k}}_m - \frac{1}{2} \delta_{ij} \delta_{lm} + \frac{1}{2} \delta_{ij} \hat{\mathbf{k}}_l \hat{\mathbf{k}}_m + \frac{1}{2} \delta_{lm} \hat{\mathbf{k}}_i \hat{\mathbf{k}}_j. \end{aligned}$$

The stress-energy tensor is

$$\begin{aligned} \tilde{T}_{ij}(\hat{\mathbf{k}}, \omega) &= \frac{1}{2\pi} \int dt d^3x e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} T_{ij}(\mathbf{x}, t) \\ &= \frac{1}{2\pi} \int dt e^{i\omega t} \sum_{n=1}^N \int d\Omega_{\mathbf{x}} dr e^{-i\omega \hat{\mathbf{k}} \cdot (\mathbf{x}_n + r \hat{\mathbf{x}})} r^2 T_{ij,n}(r, t), \end{aligned} \quad (7.2.3)$$

where we have split the spatial integral into regions around the nucleation centers of N bubbles. Here, $\mathbf{x}_n$ refers to the nucleation coordinates of the n -th bubble, and $\hat{\mathbf{x}}$ is a unit vector pointing from the center of the bubble toward the solid-angle element $d\Omega_{\mathbf{x}}$. Assuming that the energy is concentrated in a thin shell along the light cone compared with the bubble radius, the anisotropic part of the scalar stress that contributes to the transverse-traceless source can be written locally as

$$T_{ij,n}(r, t) \simeq \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \rho_{\text{grad},n}(r, t), \quad (7.2.4)$$

up to terms proportional to δ_{ij} that do not contribute after the TT projection. Here $\rho_{\text{grad},n}$ denotes the gradient contribution localized near the wall of the n -th bubble in the angular direction $\hat{\mathbf{x}}$. The corresponding surface tension carried by this wall element is defined by the radial integral

$$\sigma_n(t - t_n, t_{n,c} - t_n; \hat{\mathbf{x}}) \equiv \int dr \rho_{\text{grad},n}(r, t), \quad (7.2.5)$$

where $t_{n,c} = t_{n,c}(\hat{\mathbf{x}})$ is the first-collision time of that angular wall element in the direction $\hat{\mathbf{x}}$, and the integration is performed within a small window around the wall location that captures the dominant contribution of $\rho_{\text{grad},n}$. Since the support of $\rho_{\text{grad},n}$ is narrow compared with the bubble radius R_n , the factor r^2 can be evaluated at R_n inside the radial integral.

$$\int dr r^2 e^{-i\omega \hat{\mathbf{k}} \cdot (\mathbf{x}_n + r \hat{\mathbf{x}})} T_{ij,n}(r, t) \simeq e^{-i\omega \hat{\mathbf{k}} \cdot (\mathbf{x}_n + R_n \hat{\mathbf{x}})} \int dr r^2 T_{ij,n}(r, t), \quad (7.2.6)$$

$$\simeq e^{-i\omega \hat{\mathbf{k}} \cdot (\mathbf{x}_n + R_n \hat{\mathbf{x}})} \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j R_n^2 \int dr \rho_{\text{grad},n}(r, t), \quad (7.2.7)$$

$$\simeq \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j e^{-i\omega \hat{\mathbf{k}} \cdot (\mathbf{x}_n + R_n \hat{\mathbf{x}})} R_n^2 \sigma_n(t - t_n, t_{n,c} - t_n; \hat{\mathbf{x}}), \quad (7.2.8)$$

$$\equiv \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j e^{-i\omega \hat{\mathbf{k}} \cdot (\mathbf{x}_n + R_n \hat{\mathbf{x}})} \frac{R_n^3}{3} f_\sigma(t - t_n, t_{n,c} - t_n) \rho_{\text{vac}}. \quad (7.2.9)$$

In the last line we introduced the dimensionless shape function f_σ of the surface tension:

$$\sigma_n(t - t_n, t_{n,c} - t_n; \hat{\mathbf{x}}) \equiv \frac{\rho_{\text{vac}}}{3} R_n f_\sigma(t - t_n, t_{n,c} - t_n). \quad (7.2.10)$$

This convention is chosen so that an uncollided wall element has $f_\sigma = 1$, corresponding to the usual energy-conservation result $\sigma_n = \rho_{\text{vac}} R_n / 3$ for the surface tension of a relativistic thin shell. In the above $R_n \equiv t - t_n$ is the radius of the n -th bubble, $t_{n,c}(\hat{\mathbf{x}})$ is the first collision time of a wall element with another bubble, ρ_{vac} is the difference in vacuum energy between the metastable and true vacua, while $f_\sigma(t - t_n, t_{n,c} - t_n)$ accounts for the time dependence of the effective surface tension before and after collision. The notation emphasizes that, after neglecting cosmological expansion, taking the critical bubble size $R_c = 0$, and setting the wall velocity to the speed of light, $v_w = 1$, the wall-element dynamics depends on the age of the bubble and on the time since nucleation at which the first collision occurs. The remaining angular dependence of a wall element is carried by $t_{n,c} = t_{n,c}(\hat{\mathbf{x}})$. All in all, we have

$$\tilde{T}_{ij}(\hat{\mathbf{k}}, \omega) = \frac{1}{2\pi} \int dt d^3x e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} T_{ij}(\mathbf{x}, t) = \rho_{\text{vac}} C_{ij}(\hat{\mathbf{k}}, \omega), \quad (7.2.11a)$$

$$C_{ij}(\hat{\mathbf{k}}, \omega) = \frac{1}{6\pi} \sum_n \int dt e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x}_n)} A_{n,ij}(\hat{\mathbf{k}}, \omega), \quad (7.2.11b)$$

$$A_{n,ij}(\hat{\mathbf{k}}, \omega, t) = \int_{S_n} d\Omega_{\mathbf{x}} e^{-i\omega(t - t_n) \hat{\mathbf{k}} \cdot \hat{\mathbf{x}}} \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \left[(t - t_n)^3 f_\sigma(t - t_n, t_{n,c} - t_n) \right] \quad (7.2.11c)$$

Note that here and in what follows we shall assume that the initial bubble radius $R_c = 0$ and that all bubbles are moving at the speed of light $v_w = 1$. ⁽¹⁾ Note that the phase factors in Eq. (7.2.11) should not be replaced by the quadrupole limit $\hat{\mathbf{k}} \cdot \mathbf{x} \rightarrow 0$ except in the strict low-frequency regime.

⁽¹⁾ For finite wall velocity, Ref. [101] shows that one can often obtain the spectrum by rescaling a $v_w = 1$ calculation. In particular, the bubble radius becomes $R_n = v_w(t - t_n)$, the phase associated with the bubble center changes from $e^{i\omega(t_n - z_n)}$ to $e^{i\omega(t_n - v_w z_n)}$ in the frame where $\hat{\mathbf{k}}$ is along the z -axis, and the remaining dependence enters through the product $v_w \omega$. In the present subsection we keep the simpler $v_w = 1$ setup because the later GBF calibration is performed in the ultra-relativistic scalar-wall regime.

As emphasized in Ref. [61], the bubble-collision source is not a compact localized source in the usual quadrupole sense, and this approximation can overestimate the emitted radiation and shift the peak frequency at frequencies comparable to, or above, the inverse bubble scale. Keeping the full phases $e^{-i\omega\hat{\mathbf{k}}\cdot\mathbf{x}}$ is therefore part of the physical content of the envelope and bulk-flow calculations.

Let $\mathbf{x}_n = (x_n, y_n, z_n)$ denote the location of the n -th bubble in the Cartesian coordinate system attached to the reference bubble. Following Ref. [101], we can choose a coordinate system such that the unit vector $\hat{\mathbf{k}} \rightarrow \hat{\mathbf{k}}' = (0, 0, 1)$ and so that the projections simplify to

$$\begin{aligned} \frac{dE_{\text{GW}}}{d\ln\omega d\Omega_{\mathbf{k}}} &= G\omega^3 \left(|\tilde{T}_{xx} - \tilde{T}_{yy}|^2 + 2|\tilde{T}_{xy}|^2 + 2|\tilde{T}_{yx}|^2 \right), \\ &= 4G\rho_{\text{vac}}^2\omega^3(|C_+|^2 + |C_\times|^2). \end{aligned} \quad (7.2.12)$$

The functions $C_{+,\times}$ are explicitly given by

$$C_{+,\times}(\omega) = \frac{1}{6\pi} \sum_{n=1}^N \int dt e^{i\omega(t-z_n)} A_{n,+,\times}(\omega, t), \quad (7.2.13a)$$

$$A_{n,+,\times}(\omega, t) = \int_{-1}^1 d\cos\theta_x e^{-i\omega v_w(t-t_n)\cos\theta_x} B_{n,+,\times}(\cos\theta_x, t), \quad (7.2.13b)$$

$$B_{n,+,\times}(\cos\theta_x, t) = \frac{\sin^2\theta_x}{2} \int_0^{2\pi} d\phi_x X_{+,\times}(\phi_x) \left[(t-t_n)^3 f_\sigma(t-t_n, t_{n,c}-t_n) \right]. \quad (7.2.13c)$$

In the final line, we define $X_+ \equiv \cos 2\phi_x$ and $X_\times \equiv \sin 2\phi_x$. Here, z_n and t_n denote the z -coordinate and the nucleation time of the n -th bubble, respectively, and $t_{n,c}(\cos\theta_x, \phi_x)$ denotes the time at which the wall element of the n -th bubble undergoes the first collision. ⁽¹⁾ In this way, we can compute $B_{n,+,\times}$ once and reuse it for all input frequencies ω , reducing the computational cost by a factor of $\mathcal{O}(n_\omega)$ compared with using Eq. (7.2.11) directly.

So far, we have not made any strong assumption about the surface tension σ , or equivalently the shape function f_σ . There is, however, one important scale that is not fixed by geometry arguments alone and helps determine f_σ : the relaxation time t_{relax} of the stress carried by a wall element after its first collision. This scale depends on the underlying scalar-field theory and consequently on the post-collision dynamics, so it must be modeled based on a physically sound assumption or calibrated rather than assumed universally:

- We start with the first approach, which elaborates assumptions about the scaling behavior of the wall surface tension before and after collision and leads to two previously developed approximations: the *envelope approximation* and the *bulk-flow approximation* in Subsections. 7.2.1 and 7.2.2. Such assumptions are supported by observations from certain lattice simulation and hence are well-motivated. The envelope approximation corresponds to the limiting case $t_{\text{relax}} \ll R_*$, in which the anisotropic stress in a collided wall segment disappears on microscopic time scales, possibly through dissipation of energy into particle waves. The bulk-flow approximation corresponds to the opposite idealization in which the shell-like stress remains coherent on scales comparable to, or longer than, R_* , and then dilutes only because the shell area continues to grow while there is no more latent heat released to the bubble to drive the wall motion.
- The second approach, based on calibrating the wall surface tension, plays the main role in constructing a new approximation called *generalized bulk-flow*, which keeps the geometric simplification of the thin-wall treatment while allowing the post-collision relaxation to be inferred from microscopic dynamics via a lower-dimensional simulation. The development, verification,

⁽¹⁾ In the two-bubble case, one can show that t_n can be set to zero via an appropriate boost, while z_n can be set to zero by a spatial shift.

and application of this approximation are the main goals of the second part of this thesis and will be presented in more detail in Chapter. 8.

Below we review the envelope and bulk-flow prescriptions as two limiting assumptions about t_{relax} before moving on to the generalized bulk-flow approximation in the next chapter.

7.2.1 Envelope approximation

Let us assume that we are in a vacuum-dominated phase transition and can ignore plasma effects; hence the bubble motion accelerates toward the speed of light until the instant of collision. The envelope approximation assumes that the energy-momentum tensor corresponding to the expanding bubble instantaneously vanishes in the angular segments where the bubble surfaces have collided with neighboring bubbles. Then the surface energy density of the bubble, assuming energy conservation, must scale as

$$\sigma^{\text{env}}(t - t_n, t_{n,c} - t_n) = \frac{\rho_{\text{vac}}}{3}(t - t_n)\Theta[(t_{n,c} - t_n) - (t - t_n)], \quad (7.2.14)$$

which corresponds to the following choice of the f_σ -function in Eq. (7.2.13)

$$f_\sigma^{\text{env}}(t - t_n, t_{n,c} - t_n) = \Theta[(t_{n,c} - t_n) - (t - t_n)] = \Theta(t_{n,c} - t) \quad (7.2.15)$$

where $t_{n,c} \equiv t_{n,c}(\theta, \phi)$ is the first-collision time of a surface element of the reference bubble indexed by n , with explicit dependence on the angular coordinates (θ, ϕ) in the spherical coordinate associated with the reference bubble. Since $v_w = 1$, the quantity $t_{n,c} - t_n$ also gives the collision radius of that wall element.

What is the physics behind this approximation? Why can we treat the evolution of the energy-momentum tensor as a step function? The motivation is the following: after the phase transition, the system should equilibrate to the true vacuum. The envelope approximation assumes that this happens on time scales much smaller than the typical bubble time at collision, $t_{n,c} - t_n \sim \mathcal{O}(R_*)$,

$$t_{\text{relax}} \ll R_*. \quad (7.2.16)$$

Indeed, if the relaxation time is controlled fully by the field microphysics set by the wall width at collision L_w , i.e., $t_{\text{relax}} \sim L_w \sim m^{-1}$, the relaxation period is much shorter than the macroscopic bubble radius at collision R_* . As is well known and will be shown in the numerical results in the next chapter, the peak frequency of the envelope spectrum is of order R_*^{-1} , and within the IR window of frequencies $\omega \lesssim R_*^{-1}$, the GW spectrum scales as $\Omega_{\text{GW}} \sim \omega^3$ by causality arguments for a localized, finite-duration stochastic source [105]. In the UV window $R_*^{-1} \lesssim \omega \lesssim L_w^{-1}$, the GW tail decays as $\Omega_{\text{GW}} \sim \omega^{-1}$, which is a known result from many-bubble simulations [61] and is later derived analytically in [106]. For cosmological transitions, $R_*/L_w \sim H^{-1}m$ can be parametrically large and the envelope window spans many orders of magnitude in frequency, covering the entire observationally relevant band.

Given the assumption of the envelope approximation, it is clear this ansatz fails to capture post-collision wall dynamics. These include, e.g., trapped-vacuum re-emission and bulk motion of the scalar field along the wall after collision, which can potentially source the generation of GWs on a longer time scale than L_w .

7.2.2 Bulk-flow approximation

The bulk-flow ansatz keeps the same pre-collision behavior as the envelope approximation but assigns a different evolution to wall elements after their first collision. The guiding principle is energy conservation in a thin shell [101]. Before collision, the latent heat converted into wall or shell energy grows with the volume swept out by the bubble, while the area of the shell grows as R_n^2 . The surface energy density therefore grows linearly with the bubble radius. After a given wall element first collides, no further latent heat is injected into that angular direction, and the wall is assumed

to continue propagating as a free-streaming kinetic shell. The energy previously stored in that wall element is then spread over an area that continues to grow as R_n^2 , which implies a post-collision dilution of surface tension proportional to R_n^{-2} . More rigorously, in terms of the first-collision time $t_{n,c}(\theta, \phi)$, this gives

$$\begin{aligned} \sigma^{\text{BF}}(t - t_n, t_{n,c} - t_n) &= \frac{\rho_{\text{vac}}}{3}(t - t_n)\Theta[t_{n,c} - t] \\ &\quad + \frac{\rho_{\text{vac}}}{3}(t_{n,c} - t_n) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^2 \Theta[t - t_{n,c}], \end{aligned} \quad (7.2.17)$$

The first collision time of the wall element determines when this transition from growth to dilution begins. Subsequent collisions are not treated as independent dynamical events in this ansatz. In this sense the envelope and bulk-flow prescriptions describe two limiting assumptions about post-collision relaxation:

Envelope: instantaneous relaxation,

Bulk-flow: continued propagation with only area dilution.

The latter is still an idealized model rather than a full solution of the post-collision field or fluid dynamics. It can be a useful limiting description when the relaxation time is comparable to the collision scale,

$$t_{\text{relax}} \sim R_*. \quad (7.2.18)$$

When can such a scenario happen? One example model of a complex scalar field investigated in Subsection 8.2.5 illustrates that with collision between two vacuum bubbles of different phases, the peculiar post-collision dynamics give rise to an almost perfect elastic reflection of the walls.

The calculation of the GW spectrum proceeds in exactly the same way as for the envelope case. The only difference is that, with the scaling ansatz for the surface tension in Eq. 7.2.17, the corresponding surface-tension shape function becomes

$$f_{\sigma}^{\text{BF}}(t - t_n, t_{n,c} - t_n) = \Theta(t_{n,c} - t) + \Theta(t - t_{n,c}) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^3 \quad (7.2.19)$$

The bulk-flow model should therefore be read as a controlled phenomenological prescription for a long-lived shell source. It assumes zero wall thickness, preserves the spherical propagation of each wall element, keeps the post-collision propagation speed unchanged, and treats the total source as a linear superposition of the individual bubble shells. These assumptions are precisely what make the calculation economical, but they also delimit the model: effects such as shell fragmentation, changes of propagation speed after collision, nonlinear overlap of fluid shells, etc. are not included.

7.2.3 Drawbacks of these approximations

The envelope and bulk-flow approximations are useful because they reduce the complicated post-collision field dynamics to a simple rule for the surface tension of each wall element and significantly reduce the computational resources needed for GW calculations. The same reduction also makes clear what physical situations they cannot describe.

The envelope approximation assumes that, once a wall element has collided, the anisotropic stress carried by that element is removed from the source. This is appropriate only when the relaxation time after collision is microscopic compared with the bubble size, $t_{\text{relax}} \ll R_*$. It therefore misses any situation in which a significant fraction of the released energy remains in a coherent post-collision configuration. Examples include scalar-field configurations in which the collision excites reflected walls, trapped false-vacuum regions, or long-lived scalar radiation along the original wall direction, as well as thermal transitions in which the plasma continues to carry anisotropic stress in the form of sound waves, and later turbulence, after the wall tension has disappeared. In such cases the envelope

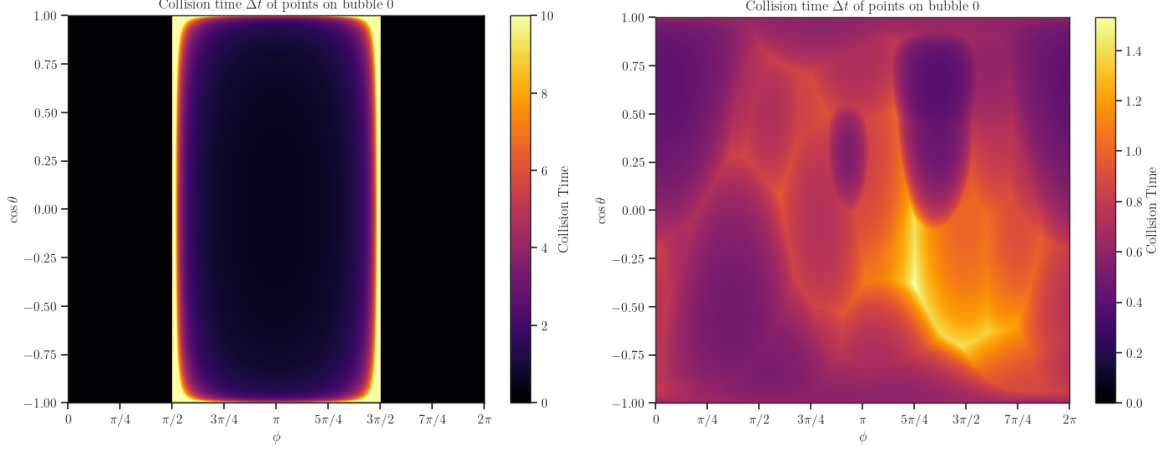


Figure 7.4: Visualization of the first-collision time $t_{n,c}(\phi, \cos \theta)$ for surface elements on a reference bubble colliding with other bubbles in a two-bubble system (left) and a many-bubble system (right).

approximation removes the source too early and cannot encode the model-dependent time profile of the stress after first collision. Its sharp removal of collided regions also produces kinks and cusps in the source energy distribution, which are part of the reason why the high-frequency GW spectrum shows a slow descent as $\sim k^{-1}$, significantly different from smoother long-lived source models [101].

The bulk-flow approximation addresses one of these problems by allowing a collided shell element to persist after first collision. Its physical assumption is that

- After collision, no new latent heat is injected into that angular direction, and the energy shell continues to propagate. The surface energy density of the walls after collision dilutes only because the shell area grows, not because of emission of matter waves.
- Bubble collisions contains surface elements that collide at different times and with different local orientations. It is assumed that the dilution of the surface energy is independent of the angle between the surface elements of the wall at collision.

This makes it a useful limiting model for a long-lived source with $t_{\text{relax}} \sim R_*$ or larger. However, it is still a fixed phenomenological prescription, as it assumes an infinitesimally thin shell, unchanged post-collision propagation speed, preserved spherical propagation of each bubble element, and a linear superposition of independent bubble shells. Later collisions of already-collided shells do not introduce new dynamics. Consequently, the bulk-flow model cannot describe cases where the shell loses energy by changing speed, fragments, broadens, develops a model-dependent relaxation law, or interacts non-linearly with overlapping shells. In comparisons with hydrodynamic simulations, Ref. [101] emphasizes a more specific limitation of this picture. Both the bulk-flow model and hydrodynamic simulations contain anisotropic stress that persists after the phase transition, but in the bulk-flow model the shells keep expanding and the characteristic stress scale drifts toward smaller wave numbers. In hydrodynamic simulations the correlation length instead appears to remain approximately constant after the transition, and the peak scale is associated with the thickness of the sound shells rather than with the average bubble size. Possible reasons discussed in Ref. [101] are that sound shells may lose energy by changing propagation speed, fragmenting the initially spherical shell picture, and that later overlap of sound shells can matter because the anisotropic stress is quadratic in the fluid velocity.

These drawbacks motivate the generalized bulk-flow approximation developed in Chapter. 8. This approximation keeps the many-bubble geometry encoded in the collision-time surface $t_{n,c}(\cos \theta_x, \phi_x)$ and parameterizes the fixed post-collision rule for σ using a basis of allowed late-time relaxations, with the envelope and bulk-flow prescriptions appearing as limiting cases. The coefficients corresponding to this basis are then calibrated from lower-dimensional field-theory simulations of two-bubble collisions, so the model dependence of the microscopic post-collision dynamics enters through the fitted surface-tension evolution rather than through an assumed relaxation law. Thus GBF directly addresses one main weakness of the envelope and bulk-flow prescriptions by allowing flexibility in t_{relax} depending

on the field dynamics governed by, e.g., the potential for the phase transition. It also preserves the computational advantage of the previous geometric methods, since the expensive many-bubble part can still be organized into reusable $C_{+,\times}^{(\xi)}$ functions. The remaining limitations should also be kept explicit. GBF is still a thin-wall, geometric approximation to the scalar-field collision source, and it does not by itself replace full hydrodynamic simulations when the dominant source is a nonlinear plasma velocity field.

Generalized Bulk-Flow Approximation: Method Validation and Results

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The previous chapter identified scalar-field bubble collisions as one possible source of a stochastic gravitational-wave background from a first-order phase transition, and reviewed the envelope and bulk-flow approximations as two limiting descriptions of the post-collision source. Building on that discussion, the present chapter first constructs the generalized bulk-flow approximation in Section. 8.1, whose central result is the master equation (8.1.9). The remaining purpose of the chapter is to test the validity of the Generalized Bulk-Flow (GBF) approximation against two-bubble scalar-field simulations and apply it to many-bubble systems. The method reduces the computational cost of predicting GWs and incorporates model-dependent post-collision wall dynamics through fitted coefficients. The comparisons below assess its accuracy in the tested two-bubble configurations; its application to many-bubble systems relies on the transferability assumptions stated in Section. 8.1. Specifically:

- In Section. 8.2, we use two-bubble collisions as a framework to test the accuracy of the GBF approximation constructed at the beginning of this chapter; such tests are missing in previous literature. In this configuration the scalar field evolution has an $O(2, 1)$ symmetry, so the problem can be reduced to a $1 + 1$ D lattice calculation in the coordinates (s, z) . The output of this simulation is twofold. First, the simulations provide the surface-tension evolution along the wall trajectory, from which the GBF coefficients a_ξ are fitted. Second, the same field profiles are used to compute the gravitational-wave spectrum directly, giving a first-principles benchmark for the GBF prediction. This comparison is performed for a real scalar quartic potential and for a complex scalar model with a logarithmic potential, including the dependence on the relative phase in the complex field.

- The second part of the chapter applies the resulting approximation to realistic many-bubble configurations. As the GBF framework reduces the source to geometric functions and fitted post-collision coefficients, it makes it possible to study larger boxes, larger bubble counts, and longer effective source durations than would be practical in a brute-force scalar-field $3 + 1$ D evolution. The many-bubble analysis therefore focuses on understanding general features of a GBF spectrum for various scaling powers ξ of the surface tension, including the peak position and amplitude of the spectrum, the infrared and ultraviolet scaling laws, finite-box effects, and the dependence on lattice geometry and boundary conditions.

8.1 Foundation of the Generalized Bulk-Flow Approximation (GBF)

8.1.1 Construction of the Generalized Bulk-Flow Approximation

In the previous chapter, we introduced two approximations for the stress-energy evolution during bubble collisions. These correspond to opposite limits of the field relaxation time, which is not sufficient to capture the full post-collision physics. What we would like to propose is the assumption that a more realistic evolution of the stress-energy density lies somewhere between these two limits. This translates to the following ansatz for the surface tension: ⁽¹⁾

$$\sigma^{\text{GBF}}(t - t_n, t_{n,c} - t_n) = \frac{\rho_{\text{vac}}}{3} \left[\Theta(t_{n,c} - t) (t - t_n) + \Theta(t - t_{n,c}) (t_{n,c} - t_n) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^2 \sum_{\xi} a_{\xi} \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^{\xi} \right]. \quad (8.1.1)$$

where the envelope corresponds to $a_{\xi} = 0, \forall \xi$, and the bulk-flow is the case with $a_0 = 1, a_{\xi \neq 0} = 0$. In practice, as will be shown in Section. 8.2, the coefficients a_{ξ} are computed by computing the post-collision surface tension in a two-bubble simulation and fitting the result with Eq. (8.1.1). This makes the GBF ansatz more sensitive to the microscopic wall dynamics of the scalar theory and can take care of models where the energy lost after the collision is not complete, which is one of the main improvement in comparison with envelope and bulk-flow approximations. However, there are still some assumptions that need to be justified for this approach

- The fitted quantity is the surface tension carried by the outgoing energy shell. Technically, we follow the light cone propagation associated with the post-collision wall trajectory and integrate the gradient energy density in a narrow window around it. Energy radiated outside the chosen window is therefore treated as energy that no longer contributes to the long-lived shell source and is assumed as part of the envelope contribution. This is not always the case. One such example is when the false vacuum is trapped within the collided space and oscillates in a time period of order $\mathcal{O}(R_*)$ before relaxing. This oscillating period also generates GWs which should not be captured using the envelope description.
- We assume that coefficients calibrated in the two-bubble setup can be applied to wall elements in a many-bubble configuration, without an explicit dependence on collision angle or subse-

⁽¹⁾ Note that in this ansatz (8.1.1), we choose to expand the post-collision surface tension along the light cone tracing the location of the bubble in the basis of rational functions $\left\{ \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^{\xi} \mid \xi \in \mathbb{N} \right\}$. This is motivated by the following points:

- The special limits $\xi = 0$ and $\xi \rightarrow \infty$ correspond respectively to totally elastic and totally inelastic wall collisions, which, as we will show shortly, are the limits of the envelope and pure bulk-flow approximations. Still, there is no argument against choosing another sensible basis, e.g., by multiplying the rational functions by an exponential damping factor at late times. It is simply natural to choose rational functions as a basis of expansion, because we expect the surface tension to drop with some power of time as the bubble expands in the post-collision regime.
- All intermediate values of ξ then correspond to subleading corrections, and in practice we observe that keeping one such subleading term gives significantly better predictions compared with both the envelope and bulk-flow approximations.

quent collisions. This universality assumption can be partly tested in the two-bubble system numerically by varying the initial bubble separation, perform the 1 + 1D simulation and relevant measurements, and test the stability of the fitted coefficients over this separation. We however will not explicitly establish independence from arbitrary bubble positions and nucleation times in our numerical results and instead keep it as a generic assumption to proceed.

This ansatz (8.1.1) for the surface tension then leads to the following expression for the surface-tension shape function f_σ :

$$\begin{aligned} f_\sigma^{\text{GBF}}(t - t_n, t_{n,c} - t_n) &= \Theta(t_{n,c} - t) + \Theta(t - t_{n,c}) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^3 \sum_{\xi} a_{\xi} \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^{\xi} \\ &= a_{\text{env}} f_\sigma^{\text{env}} + \sum_{\xi} a_{\xi} f_\sigma^{(\xi)}, \end{aligned} \quad (8.1.2)$$

with $a_{\text{env}} \equiv (1 - \sum_{\xi} a_{\xi})$ is the envelope contribution coming from instantaneous relaxation of part of the stress-energy tensor after wall collision. Here the envelope template is defined as previously in Eq. (7.2.15)

$$f_\sigma^{\text{env}}(t - t_n, t_{n,c} - t_n) = \Theta(t_{n,c} - t), \quad (8.1.3)$$

and the ξ -th template is

$$f_\sigma^{(\xi)}(t - t_n, t_{n,c} - t_n) = \Theta(t_{n,c} - t) + \Theta(t - t_{n,c}) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^{3+\xi}. \quad (8.1.4)$$

The decomposition of Eq. (7.2.13) then follows from the fact that the functions $C_{+, \times}$ are linear in the surface-tension shape function f_σ . Let $C_i^{\text{env}}(\omega)$, with $i = +, \times$, denote the result of evaluating Eq. (7.2.13) using f_σ^{env} , and let $C_i^{(\xi)}(\omega)$ denote the result obtained with $f_\sigma^{(\xi)}$. Explicitly, this means

$$\begin{aligned} C_i^{\text{GBF}}(\omega) &= C_i^{\text{env}}(\omega) + \sum_{\xi} a_{\xi} [C_i^{(\xi)}(\omega) - C_i^{\text{env}}(\omega)] \\ &= a_{\text{env}} C_i^{\text{env}}(\omega) + \sum_{\xi} a_{\xi} C_i^{(\xi)}(\omega), \quad a_{\text{env}} \equiv 1 - \sum_{\xi} a_{\xi} \end{aligned} \quad (8.1.5a)$$

$$C_i^{(\xi)}(\omega) = \frac{1}{6\pi} \sum_{n=1}^N \int dt e^{i\omega(t-z_n)} A_{n,i}^{(\xi)}(\omega, t), \quad (8.1.5b)$$

$$A_{n,i}^{(\xi)}(\omega, t) = \int_{-1}^1 d\cos\theta_x e^{-i\omega v_w(t-t_n)\cos\theta_x} B_{n,i}^{(\xi)}(\cos\theta_x, t), \quad (8.1.5c)$$

$$B_{n,i}^{(\xi)}(\cos\theta_x, t) = \frac{\sin^2\theta_x}{2} \int_0^{2\pi} d\phi_x X_i(\phi_x) [(t - t_n)^3 f_\sigma^{(\xi)}(t - t_n, t_{n,c} - t_n)], \quad (8.1.5d)$$

where $i = +, \times$ labels the polarization degrees of freedom, $X_+(\phi_x) = \cos 2\phi_x$, and $X_\times(\phi_x) = \sin 2\phi_x$. The resulting gravitational-wave spectrum is

$$\frac{dE_{\text{GW}}}{d\ln\omega d\Omega_{\mathbf{k}}} = 4G\rho_{\text{vac}}^2\omega^3 \sum_{i=+, \times} \left| a_{\text{env}} C_i^{\text{env}} + \sum_{\xi} a_{\xi} C_i^{(\xi)} \right|^2. \quad (8.1.6)$$

So far, we have not said anything about the coefficients a_{ξ} or how to calculate them. However, the whole procedure makes sense only if the coefficients a_{ξ} are fixed by the fundamental Lagrangian of the system and are, at least approximately, independent of the bubble positions and nucleation times. This is one of the verification steps that we study numerically in Section. 8.2.

On general grounds, the GBF ansatz (8.1.2) captures both types of effects characterizing the stress-energy evolution discussed previously in the motivation of the envelope and bulk-flow approximations:

- **Short-distance effects:** On time scales set by the wall width, $t_{\text{relax}}^{(\text{short})} = t - t_{n,c} \sim m^{-1}$, the field quickly relaxes toward the true vacuum, dissipating the localized stress-energy. This effect is captured by the envelope-like approximation given the scale separation $R_* \gg m^{-1}$. In other words, it is captured by the factor

$$a_\infty = a_{\text{env}} = 1 - \sum_{\xi} a_{\xi} \quad (8.1.7)$$

- **Long-distance effects** capture the physics on time scales of the same order as, or larger than, the bubble size at collision, $t_{\text{relax}}^{(\text{long})} = t - t_{n,c} \gtrsim R_*$. Similar to the construction of the pure bulk-flow approximation, we assume an elastic collision where the energy stored in the “shell” is assumed to drop like R_n^2 , which is encoded by a_0 , plus a subleading correction a_{ξ} for $\xi \geq 1$ that parameterizes model-dependent deviations from exact elastic propagation. Assuming $\xi = 1$ is the dominant subleading correction at late times or large distances, and ignoring the remaining contributions from $\xi > 1$ terms, the surface-tension ansatz becomes

$$\sigma_{\xi=0,1}^{\text{GBF}}(t - t_n, t_{n,c} - t_n) = \frac{\rho_{\text{vac}}}{3} \left\{ \Theta(t_{n,c} - t)(t - t_n) + \Theta(t - t_{n,c})(t_{n,c} - t_n) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^2 \left[a_0 + a_1 \left(\frac{t_{n,c} - t_n}{t - t_n} \right) \right] \right\} \quad (8.1.8)$$

The coefficients a_0 , a_1 , and $a_{\text{env}} \equiv 1 - a_0 - a_1$ can be extracted using first-principles two-bubble simulations, which are relatively inexpensive computationally.

The calculation of the gravitational-wave spectrum then becomes strikingly fast. All we need to calculate the $C_{+,\times}$ functions *only once* for various stress-energy evolution scenarios, i.e., $\xi = \infty, 0, 1$, and store the obtained results. For the reduced ansatz in Eq. (8.1.8), the bulk-flow template is $C_i^{\text{BF}} \equiv C_i^{(0)}$. Within the reduced ansatz and its transferability assumptions, the gravitational-wave spectrum is given by

$$\frac{dE_{\text{GW}}}{d \ln \omega d\Omega_{\mathbf{k}}} = 4G\rho_{\text{vac}}^2 \omega^3 \sum_{i=+,\times} \left| a_{\text{env}} C_i^{\text{env}} + a_0 C_i^{\text{BF}} + a_1 C_i^{(1)} \right|^2. \quad (8.1.9)$$

Within this approximation, the coefficients a_{ξ} encode the modeled post-collision surface-tension evolution, and the C functions encode the chosen bubble positions and nucleation times for fixed wall kinematics and basis functions. This separation follows algebraically once the same coefficients are assigned to all wall elements.

With the theoretical framework for the GBF approximation established, the next step is to elaborate the procedure for extracting the GBF coefficient a_{ξ} using a 1+1D lattice simulation of two-bubble collision. We then demonstrate its utility in reliably predicting the GW spectrum generated by collisions in a two-bubble system before extending this approximation to many-bubble systems. In order to go from the simple two-bubble case to many-bubble scenario, we also elaborate the test for stability of the fitted coefficients a_{ξ} over bubble separation d in the 1+1D to simulate the fact that in many-bubble system, bubbles are randomly nucleated and hence we should have a range over inter-bubble distance.

Before closing the construction of the GBF approximation, let us note that a similar idea was used in Ref. [107] to compute the GW spectrum of a complex scalar system, which is one of the models we investigate in Section. 8.2. One difference is that while the approach of Ref. [107] uses another scaling function f_{ρ} that tracks the evolution of the peak amplitude of the gradient energy density ρ_{grad} , we choose the dimensionless function f_{σ} , which captures the shape of the surface tension over

time. This has the slight advantage of smoothing out potential oscillations of the peak ρ_{grad} along the wall in the post-collision regime, and hence potentially yields smaller errors in fitting the coefficients a_ξ . Another difference is in the fitting approach for the shape function: while Ref. [107] fits f_ρ with a somewhat arbitrary choice of scaling powers, our approach mainly focuses on the dominant terms in f_σ that survive long after collision, corresponding to $\xi = 0$, $\xi = 1$, and the envelope term $\xi = \infty$. This is because we are interested in capturing long-distance effects where bubble collisions take place at relatively large bubble sizes, as this regime typically generates larger GW signals.

8.1.2 Sketch of the implementation of GBF

Complexity of the GBF approximation compared to 3 + 1D simulation

It is useful to make the computational advantage of the GBF approximation over 3+1D simulation more explicit in terms of scaling with the numerical parameters. Let N_{bubbles} be the number of bubbles entering one nucleation history, $n_{\cos\theta}$ and n_ϕ the angular resolutions in $\cos\theta_x$ and ϕ_x , n_t the number of time samples used in the integrals of Eq. (7.2.13), n_ω the number of frequencies at which the spectrum is evaluated, and n_ξ the number of retained GBF basis functions. The total number of operations in the GBF workflow can be split into three main contributions. The first two contributions are model-independent for a fixed nucleation history: the construction of the first-collision time of each wall element, denoted $N_{\text{op}}^{\text{GBF}, t_{n,c}}$, and the computation of the table of functions $C_{+, \times}^{(\xi)}(\omega)$, denoted $N_{\text{op}}^{\text{GBF}, C_{+, \times}}$, once the first-collision table has been cached. The third contribution $N_{\text{op}}^{\text{GBF}, \text{model}}$ is model-dependent: the 1 + 1D two-bubble simulation used to extract the coefficients a_ξ . Here N_{op} is the cost function that measures the number of operations required to compute the corresponding quantity. The current implementation first constructs the first-collision surface $t_{n,c}(\cos\theta_x, \phi_x)$ on the angular grid. For each reference bubble and angular direction, the code loops over candidate colliding bubbles and checks whether the corresponding collision point is blocked by any other bubble. Thus the worst-case scaling of this geometric preprocessing step is

$$N_{\text{op}}^{\text{GBF}, t_{n,c}} = \mathcal{O}\left(N_{\text{bubbles}}^3 n_{\cos\theta} n_\phi\right). \quad (8.1.10)$$

Once this collision table is available, either because it has just been computed or because it has been cached, the computation of the table of functions $C_{+, \times}^{(\xi)}(\omega)$ can be decomposed into two separate costs. The cost of the C -table computation is again a sum of two separate costs. First, for each bubble, time sample, $\cos\theta_x$ point, and retained basis template, the code computes the ϕ_x integral entering $B_{n, +, \times}^{(\xi)}(\cos\theta_x, t)$. In combination with the computation of first collision time, this part scales as

$$N_{\text{op}}^{\text{GBF}, B} = N_{\text{op}}^{\text{GBF}, t_{n,c}} + \mathcal{O}(N_{\text{bubbles}} n_{\cos\theta} n_t n_\xi n_\phi). \quad (8.1.11)$$

Second, once the B table is available, the remaining operations are the frequency-dependent sums over $\cos\theta_x$ and t that build $C_{+, \times}^{(\xi)}(\omega)$. These scale as

$$N_{\text{op}}^{\text{GBF}, C|B} = \mathcal{O}(N_{\text{bubbles}} n_{\cos\theta} n_t n_\xi n_\omega). \quad (8.1.12)$$

Therefore the full cost of building the reusable $C_{+, \times}^{(\xi)}(\omega)$ table is

$$\begin{aligned} N_{\text{op}}^{\text{GBF}, C_{+, \times}} &= N_{\text{op}}^{\text{GBF}, B} + N_{\text{op}}^{\text{GBF}, C|B} \\ &= \mathcal{O}\left(N_{\text{bubbles}}^3 n_{\cos\theta} n_\phi\right) + \mathcal{O}(N_{\text{bubbles}} n_{\cos\theta} n_t n_\xi n_\phi) + \mathcal{O}(N_{\text{bubbles}} n_{\cos\theta} n_t n_\xi n_\omega). \end{aligned} \quad (8.1.13)$$

One important point is that there is no multiplicative factor $n_\phi n_\omega$, because the ϕ_x integral is independent of ω and is not repeated for every frequency. For an ensemble of N_{hist} statistically independent nucleation histories, Eqs. (8.1.10) and (8.1.13) are multiplied by N_{hist} . For a fixed underlying scalar field theory, the remaining model-dependent cost is the extraction of the coefficients a_ξ from a 1 + 1D two-bubble simulation. If a two-bubble calibration uses a one-dimensional box of size L_{box} , lattice spacing Δx , end time t_{end} , and time step $\Delta t = \mathcal{O}(\Delta x)$, this calibration scales as

$$N_{\text{op}}^{\text{GBF}, \text{model}} = \mathcal{O}\left[\left(\frac{L_{\text{box}}}{\Delta x}\right) \left(\frac{t_{\text{end}}}{\Delta t}\right)\right], \quad (8.1.14)$$

or equivalently $\mathcal{O}(L_{\text{box}} t_{\text{end}} / \Delta x^2)$ after using $\Delta t = \mathcal{O}(\Delta x)$, up to constant factors associated with the number of scalar fields and the fitting procedure. If several bubble separations, phases, or model parameter points are used in the calibration, this expression should be multiplied by the number of such calibration runs. This estimate still contains the microscopic wall-resolution requirement. If L_w is the bubble-wall width in the wall rest frame and γ_{max} is the largest wall Lorentz factor that the calibration must resolve, then the lattice spacing should satisfy roughly $\Delta x \lesssim L_w / \gamma_{\text{max}}$. Taking the largest useful lattice spacing, $\Delta x \sim L_w / \gamma_{\text{max}}$, gives the resolved calibration cost

$$N_{\text{op}}^{\text{GBF,model}} \Big|_{\Delta x \sim L_w / \gamma_{\text{max}}} = \mathcal{O} \left[L_{\text{box}} t_{\text{end}} \frac{\gamma_{\text{max}}^2}{L_w^2} \right]. \quad (8.1.15)$$

This scaling is the practical reason for separating the microscopic dynamics into the coefficients a_ξ and the many-bubble geometry into the reusable C functions.

The scaling of a direct 3 + 1D scalar-field lattice simulation is qualitatively different. In the approach of Ref. [100], one evolves the scalar field and an auxiliary metric perturbation on a three-dimensional grid, uses a finite-difference stencil for the field gradients, takes time steps constrained by the lattice spacing, and periodically Fourier-transforms the metric perturbations to perform the transverse-traceless projection. Let L_{box} be the spatial box size, t_{end} the end time, Δx the lattice spacing, and Δt the time step. With a time step satisfying the Courant-Friedrichs-Lewy condition $\Delta t = \mathcal{O}(\Delta x)$ ⁽¹⁾, the local evolution cost scales as

$$N_{\text{op}}^{3+1\text{D}} = \mathcal{O} \left[\left(\frac{L_{\text{box}}}{\Delta x} \right)^3 \left(\frac{t_{\text{end}}}{\Delta t} \right) \right] = \mathcal{O} \left(\frac{L_{\text{box}}^3 t_{\text{end}}}{\Delta x^4} \right) \quad (8.1.16)$$

This quantity shows the scaling of the number of operations required for the local field and metric evolution on the lattice. The wall-resolution requirement discussed above also applies to the direct 3 + 1D scalar-field simulation. Again taking $\Delta x \sim L_w / \gamma_{\text{max}}$ gives the resolved direct-simulation cost

$$N_{\text{op}}^{3+1\text{D}} \Big|_{\Delta x \sim L_w / \gamma_{\text{max}}} = \mathcal{O} \left[L_{\text{box}}^3 t_{\text{end}} \frac{\gamma_{\text{max}}^4}{L_w^4} \right], \quad (8.1.17)$$

which exhibits the severe quartic sensitivity to the microscopic wall scale. For fixed physical L_{box} and t_{end} , refining the lattice gives the familiar Δx^{-4} cost of an explicit three-dimensional time evolution. Equivalently, at fixed dimensionless runtime $t_{\text{end}} / L_{\text{box}}$ the cost scales as $\mathcal{O}[(L_{\text{box}} / \Delta x)^4]$. The first two GBF contributions, Eqs. (8.1.10) and (8.1.13), produce a model-independent $C_{+,x}^{(\xi)}$ table for the chosen nucleation histories. Once this table is available, the fair comparison with a direct scalar-field computation is between the remaining model-dependent 1 + 1D calibration cost in Eq. (8.1.15) and the full 3 + 1D cost in Eq. (8.1.17). The difference is then parametrically clear: resolving the same Lorentz-contracted wall costs $\mathcal{O}(\gamma_{\text{max}}^2 / L_w^2)$ in the 1 + 1D calibration, but $\mathcal{O}(\gamma_{\text{max}}^4 / L_w^4)$ in the direct 3 + 1D simulation, before accounting for the physical volume factors $L_{\text{box}} t_{\text{end}}$ and $L_{\text{box}}^3 t_{\text{end}}$ that appear in the one-dimensional and three-dimensional evolutions, respectively. The memory requirement follows an argument similar to the runtime estimate. This is why the GBF implementation is particularly advantageous for the cases where the goal is to compare many nucleation histories of large (hence thin-walled) bubbles at collision while keeping the expensive microscopic dynamics confined to a lower-dimensional simulation.

Flowchart for implementing the GBF approximation

The computational framework of the generalized bulk-flow approximation is summarized in Fig. 8.1, which makes explicit the separation achieved in this chapter between the universal geometric part of the GBF construction and the model-dependent information carried by the wall dynamics.

⁽¹⁾ The Courant-Friedrichs-Lewy condition is the stability condition for explicit finite-difference evolution of hyperbolic equations, such as the evolution of the scalar field in the context of phase transition. It states that the numerical domain of dependence must contain the physical domain of dependence, so information cannot propagate across more lattice sites in one time step than the finite-difference stencil can communicate. In units where the maximal signal speed is one, this requires the Courant factor $\Delta t / \Delta x$ to be smaller than an order-one stability bound set by the discretization and spatial dimension. In practice, throughout the 1 + 1D simulations discussed in this thesis, we choose $\Delta t / \Delta x$ in the range 0.2–0.9.

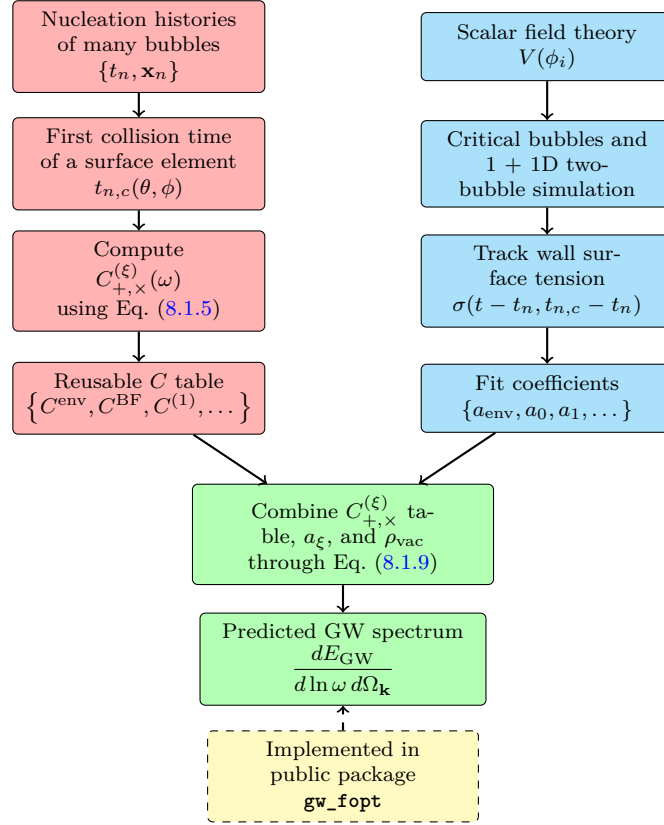


Figure 8.1: Flow chart for computing the gravitational-wave spectrum with the generalized bulk-flow approximation. The left branch depends on the bubble nucleation history and produces a table of geometric functions $C_{+,x}^{(\xi)}(\omega)$ for a chosen frequency grid, which is model-independent and hence reusable for arbitrary underlying scalar-field theories of first-order phase-transition processes. The right branch is model-dependent and extracts the coefficients a_{ξ} from two-bubble simulations by fitting the post-collision surface tension along the bubble wall. These two ingredients are combined to efficiently obtain the GW spectrum. An implementation of this procedure can be found in the package **gw_fopt**, which is publicly available at https://github.com/hoquoctrung98/gw_fopt. Most of the numerical results shown in Chapter. 8 are produced using this code.

8.2 Validation of the method

In Section. 8.1, we introduced a generalized bulk-flow approximation for the stochastic GW signal. As shown in Eq. 8.1.9, the results depend on the coefficients a_{ξ} , which are unknown and must be obtained from numerical simulations. The simplest bubble configuration consists of two bubbles nucleated at the same time, which we use to fit these coefficients. According to Ref. [108], in the absence of an observer, the evolution of two vacuum bubbles has $O(2,1)$ symmetry. ⁽¹⁾ We make use of this symmetry by utilizing the hyperbolic coordinates (s, ψ, φ, z) in place of the Cartesian coordinates (t, x, y, z) . The patches $t^2 \geq x^2 + y^2$ and $t^2 \leq x^2 + y^2$ are labeled as $+$ and $-$, respectively. In region $+$, the coordinates and metric dl^2 are given by

$$t = s \cosh \psi, \quad x = s \sinh \psi \cos \varphi, \quad y = s \sinh \psi \sin \varphi, \quad (8.2.1)$$

$$dl^2 = ds^2 - s^2 d\psi^2 - s^2 \sinh^2 \psi d\varphi^2 - dz^2, \quad (8.2.2)$$

and in the complementary region $-$ we have

$$t = s \sinh \psi, \quad x = s \cosh \psi \cos \varphi, \quad y = s \cosh \psi \sin \varphi, \quad (8.2.3)$$

$$dl^2 = -ds^2 + s^2 d\psi^2 - s^2 \cosh^2 \psi d\varphi^2 - dz^2, \quad (8.2.4)$$

⁽¹⁾ At zero temperature, the critical bubbles are formed at rest. Therefore, any pair of bubbles is nucleated at the same time with an appropriate choice of Lorentz frame.

where we use the mostly minus metric signature. The field equations describing the two-bubble system in the patches $\pm$ are

$$\pm \frac{\partial^2 \Phi_{\pm}}{\partial s^2} \pm \frac{2}{s} \frac{\partial \Phi_{\pm}}{\partial s} - \frac{\partial^2 \Phi_{\pm}}{\partial z^2} + \frac{dV}{d\Phi_{\pm}} = 0. \quad (8.2.5)$$

Note that the equation for Φ_+ is a hyperbolic partial differential equation (PDE) that can be put on the lattice and solved numerically, while the evolution of Φ_- is described by an elliptic PDE whose solution is given by the analytical continuation of the initial bubble configuration:

$$\Phi_-(s, z) = \Phi_0 \left(\sqrt{s^2 + (z - d/2)^2} \right) + \Phi_0 \left(\sqrt{s^2 + (z + d/2)^2} \right), \quad (8.2.6)$$

with d being the bubble separation at the nucleation epoch. The equation above has just one spatial coordinate and is much easier to solve than the full three-dimensional simulation. We implement a simple leapfrog integration to evolve the initial two-bubble profile from $s = 0$ to $s = s_{\max}$, manually tuning the spatial step to correctly capture the bubble wall at $s_{\max}$.

8.2.1 Gravitational waves generated by two-bubble collision

Let us now discuss how to obtain the GW spectrum from the two-bubble evolution. Recall the expression for the stress-energy tensor with the artificial time cutoff factor $C(t)$:

$$\begin{aligned} T_{ij}(\hat{\mathbf{k}}, \omega) &= \frac{1}{2\pi} \int dt d^3x C(t) e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} T_{ij}(\mathbf{x}, t) \\ &= \frac{1}{2\pi} \int dt d^3x C(t) e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} \partial_i \Phi \partial_j \Phi \\ &= \frac{1}{2\pi} \int dt d^3x C(t) e^{i\omega(t - \hat{\mathbf{k}} \cdot \mathbf{x})} \sum_f \left(\partial_i \phi^{(f)} \partial_j \phi^{(f)} \right), \end{aligned} \quad (8.2.7)$$

where there is a summation over different fields denoted by f with canonical kinetic term. By choosing $\hat{\mathbf{k}} = (\sin \theta_k, 0, \cos \theta_k)$ without loss of generality ⁽¹⁾, one can then perform a coordinate transformation to cylindrical coordinates (ρ, φ, z) to simplify the computations, and applying the following identity for the integral over the azimuthal angle φ ,

$$\int_0^{2\pi} d\varphi e^{i\beta \cos \varphi} \cos(n\varphi) = 2\pi i^n J_n(\beta), \quad (8.2.8)$$

where J_i denotes Bessel functions of the first kind, the only non-zero elements are T_{xx} , T_{yy} , T_{zz} , and $T_{xz} = T_{zx}$:

$$\begin{aligned} T_{xx} &= \frac{1}{2\pi} \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_{\rho} \Phi)^2 \int_0^{2\pi} d\varphi e^{-i \sin \theta_k \omega \rho \cos \varphi} \cos^2 \varphi \\ &= \frac{1}{2} \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_{\rho} \Phi)^2 [J_0(\sin \theta_k \omega \rho) - J_2(\sin \theta_k \omega \rho)] \end{aligned} \quad (8.2.9)$$

$$\begin{aligned} T_{yy} &= \frac{1}{2\pi} \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_{\rho} \Phi)^2 \int_0^{2\pi} d\varphi e^{-i \sin \theta_k \omega \rho \cos \varphi} \sin^2 \varphi, \\ &= \frac{1}{2} \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_{\rho} \Phi)^2 [J_0(\sin \theta_k \omega \rho) + J_2(\sin \theta_k \omega \rho)] \end{aligned} \quad (8.2.10)$$

$$\begin{aligned} T_{zz} &= \frac{1}{2\pi} \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_z \Phi)^2 \int_0^{2\pi} d\varphi e^{-i \sin \theta_k \omega \rho \cos \varphi}, \\ &= \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_z \Phi)^2 J_0(\sin \theta_k \omega \rho) \end{aligned} \quad (8.2.11)$$

⁽¹⁾ Our system of two identical bubbles at $z = \pm d/2$ possesses a rotational symmetry along the z -axis; therefore any vector $\hat{\mathbf{k}}$ can be rotated to lie in the Oxz plane.

$$\begin{aligned}
T_{xz} &= \frac{1}{2\pi} \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_\rho \Phi \partial_z \Phi) \int_0^{2\pi} d\varphi e^{-i \sin \theta_k \omega \rho \cos \varphi} \cos \varphi, \\
&= -i \int dt e^{i\omega t} C(t) \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \int_0^{\infty} d\rho \rho (\partial_\rho \Phi \partial_z \Phi) J_1(\sin \theta_k \omega \rho),
\end{aligned} \tag{8.2.12}$$

As $\Phi = \Phi(s, z)$, it is more convenient to define $u \equiv \cosh \psi$ for the patch $t^2 \geq \rho^2$ and $u \equiv \sinh \psi$ for the patch $t^2 < \rho^2$. The map from cylindrical coordinates to hyperbolic ones is then

$$t = su, \quad \rho = s(u^2 - 1)^{1/2}, \quad u \in [1, \infty), \quad \text{for } t^2 \geq \rho^2, \tag{8.2.13}$$

$$t = su, \quad \rho = s(u^2 + 1)^{1/2}, \quad u \in [0, \infty), \quad \text{for } t^2 < \rho^2, \tag{8.2.14}$$

with $s \in [0, \infty)$ and $z \in (-\infty, \infty)$. In practice, due to the finite size of the simulation, we have $s \in [0, s_{\max}]$ and $z \in [-L, L]$ for sufficiently large lattice size L . The corresponding Jacobian and radial derivative are

$$\int dt d\rho \rightarrow \int ds du \frac{s}{(u^2 \mp 1)^{1/2}}, \quad \frac{\partial \phi}{\partial \rho} \rightarrow \mp (u^2 \mp 1)^{1/2} \frac{\partial \phi}{\partial s}, \quad (t^2 \gtrless \rho^2). \tag{8.2.15}$$

The stress-energy tensor elements integrated over the coordinates (s, u, φ, z) can then be rewritten as follows:

$$\begin{aligned}
T_{xx} &= \frac{1}{2} \int ds s^2 \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \\
&\times \left[(\partial_s \Phi_+)^2 \int_1^{\infty} du C(su) (u^2 - 1) e^{i\omega su} \left[J_0(\sin \theta_k \omega s \sqrt{u^2 - 1}) - J_2(\sin \theta_k \omega s \sqrt{u^2 - 1}) \right] \right. \\
&\quad \left. + (\partial_s \Phi_-)^2 \int_0^{\infty} du C(su) (u^2 + 1) e^{i\omega su} \left[J_0(\sin \theta_k \omega s \sqrt{u^2 + 1}) - J_2(\sin \theta_k \omega s \sqrt{u^2 + 1}) \right] \right]
\end{aligned} \tag{8.2.16}$$

$$\begin{aligned}
T_{yy} &= \frac{1}{2} \int ds s^2 \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \\
&\times \left[(\partial_s \Phi_+)^2 \int_1^{\infty} du C(su) (u^2 - 1) e^{i\omega su} \left[J_0(\sin \theta_k \omega s \sqrt{u^2 - 1}) + J_2(\sin \theta_k \omega s \sqrt{u^2 - 1}) \right] \right. \\
&\quad \left. + (\partial_s \Phi_-)^2 \int_0^{\infty} du C(su) (u^2 + 1) e^{i\omega su} \left[J_0(\sin \theta_k \omega s \sqrt{u^2 + 1}) + J_2(\sin \theta_k \omega s \sqrt{u^2 + 1}) \right] \right]
\end{aligned} \tag{8.2.17}$$

$$\begin{aligned}
T_{zz} &= \int ds s^2 \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \\
&\times \left[(\partial_z \Phi_+)^2 \int_1^{\infty} du C(su) e^{i\omega su} J_0(\sin \theta_k \omega s \sqrt{u^2 - 1}) \right. \\
&\quad \left. + (\partial_z \Phi_-)^2 \int_0^{\infty} du C(su) e^{i\omega su} J_0(\sin \theta_k \omega s \sqrt{u^2 + 1}) \right]
\end{aligned} \tag{8.2.18}$$

$$\begin{aligned}
T_{xz} &= i \int ds s^2 \int_{-\infty}^{\infty} dz e^{-i \cos \theta_k \omega z} \\
&\times \left[(\partial_s \Phi_+ \partial_z \Phi_+) \int_1^{\infty} du C(su) \sqrt{u^2 - 1} e^{i\omega su} J_1(\sin \theta_k \omega s \sqrt{u^2 - 1}) \right. \\
&\quad \left. - (\partial_s \Phi_- \partial_z \Phi_-) \int_0^{\infty} du C(su) \sqrt{u^2 + 1} e^{i\omega su} J_1(\sin \theta_k \omega s \sqrt{u^2 + 1}) \right].
\end{aligned} \tag{8.2.19}$$

These expressions are consistent with Eq. (21) in Ref. [109] and are more general because they also work for the case $\Phi_+(-z) \neq \Phi_+(z)$. Furthermore, with the choice $\hat{\mathbf{k}} = (\sin \theta_k, 0, \cos \theta_k)$, the projector reduces to

$$P(\hat{\mathbf{k}}) = \begin{pmatrix} \cos^2 \theta_k & 0 & -\sin \theta_k \cos \theta_k \\ 0 & 1 & 0 \\ -\sin \theta_k \cos \theta_k & 0 & \sin^2 \theta_k \end{pmatrix}, \tag{8.2.20}$$

and finally, the energy radiated in a solid angle around direction $\hat{\mathbf{k}}$ is

$$\begin{aligned} \frac{dE_{\text{GW}}(\hat{\mathbf{k}}, \omega)}{d\omega d\Omega} &= 2G\omega^2 \left(\text{Tr}[PT^*PT] - \frac{1}{2} |\text{Tr}[PT]|^2 \right) \\ &= G\omega^2 \left| \cos^2 \theta_k T_{xx} - T_{yy} + \sin^2 \theta_k T_{zz} - \sin(2\theta_k) T_{xz} \right|^2. \end{aligned} \quad (8.2.21)$$

Note that, for consistency, when computing the energy-stress tensor components, we need to impose the same time-integral cutoff $C(t)$ in both the 1 + 1D lattice simulation and the computation of the coefficients $C_{+, \times}$ in the various approximations.

8.2.2 Validating the GBF approximation against the SGWB computation from two-bubble lattice simulations

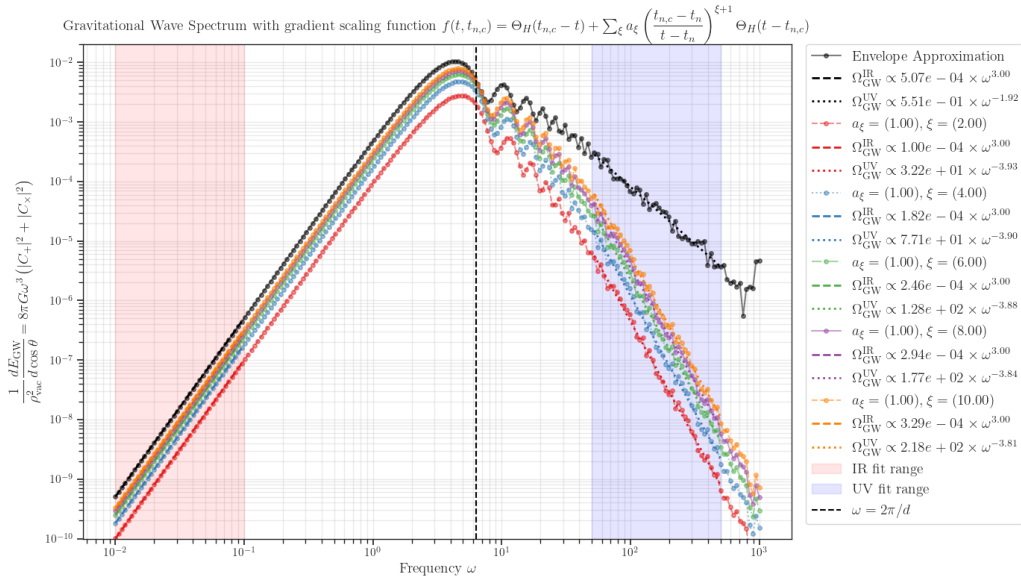


Figure 8.2: GW spectrum using GBF approximation for different power ξ in a two-bubble system. The black curve corresponds to the envelope approximation. Note that the IR scaling powers for various ξ are approximately the same as in the envelope case. This is an artifact of the two-bubble system, which has no time cutoff set by the completion of the phase transition as in the many-bubble case: as $t \rightarrow \infty$, half of each bubble is still expanding without ever collapsing. As we will show later in Section. 8.3, for a many-bubble system with an appropriate boundary condition, such a time cutoff arises naturally and hence has a significant impact on the shape of the GBF spectra.

In what follows, we validate the GBF approximation by applying it to concrete models and comparing the output with that of the two-bubble simulation on a (s, z) lattice. A sketch of the procedure is as follows:

- *Critical bubbles as initial condition for lattice field evolution:* the critical bubble profile is obtained by solving the bounce equation together with appropriate boundary conditions

$$\begin{cases} \frac{d^2\phi_c}{dr^2} + \frac{3}{r} \frac{d\phi_c}{dr} - \frac{dV}{d\phi_c} = 0 \\ \phi_c(r \rightarrow \infty) = \phi_{\text{FV}}, \quad \frac{d\phi_c}{dr}(r=0) = 0 \end{cases}. \quad (8.2.22)$$

Given either a boost factor γ at collision or the bubble separation d at nucleation, we can construct a two-bubble profile as the initial condition at $s = 0$. ⁽¹⁾ In practice, we use

⁽¹⁾ Given the boost factor γ at collision time $s_{\text{col}} \approx d/2$, one obtains $d \approx \gamma R_c$, with R_c being the critical bubble radius.

CosmoTransitions [36] to obtain the numerical profile for the critical bubble. Some useful scales that will be used in the simulations include the wall-width parameter L_w , defined as the difference of inner and outer radii of the initial profile

$$L_w = R_{\text{out}} - R_{\text{in}}, \quad \text{with} \quad \begin{cases} R_{\text{in}} = \text{argmin}_r \left[\phi_c(r) - \frac{\phi_c(r=0)}{2} (1 - \tanh(-0.5)) \right] \\ R_{\text{out}} = \text{argmin}_r \left[\phi_c(r) - \frac{\phi_c(r=0)}{2} (1 - \tanh(0.5)) \right] \end{cases}, \quad (8.2.23)$$

and the critical radius R_c , defined by $\phi_c(r = R_c) = \frac{1}{2}\phi_c(r = 0)$. The initial field is evolved from $s = 0$ to some s_{end} using a simple leap-frog integration.

- *Extracting the surface tension along the bubble wall:* We then compute the surface tension numerically from the field evolution using

$$\sigma(t - t_n, t_{n,c} - t_n) = \int_{z_{\text{wall}} - \Delta}^{z_{\text{wall}} + \Delta} dz \, \rho_z = \int_{z_{\text{wall}} - \Delta}^{z_{\text{wall}} + \Delta} dz \, \frac{(\partial_z \phi)^2}{2}, \quad (8.2.24)$$

where $z_{\text{wall}} \equiv \text{argmax}_z \rho_z$ is the value of z that maximizes ρ_z , which traces the position of the expanding wall. In this way, we make sure that we capture the surface tension around the wall of one of the bubbles before and after collision.

- *Extracting the GBF coefficients a_ξ by fitting the surface tension $\sigma(t - t_n, t_{n,c} - t_n)$ according to Eq. (8.1.1):* since the envelope contribution corresponds to $\xi = \infty$, we cannot directly obtain the coefficient a_∞ from fitting directly. There are two approaches that we consider later in Section. 8.2:

- Choose the fitting function containing three powers $\xi \in [0, 1, 18]$, where the quickly decaying part $\xi = 18$ mimics the envelope contribution. Using this approach, we also need to impose the continuity condition for the surface tension before and after collision, which implies $\sum_{\xi=0,1,18} a_\xi = 1$, and furthermore we require $a_\xi > 0$. This avoids letting points around the nonlinear oscillation period close to the collision spoil the fitting. Later, we denote this fitting strategy as **GBF $\xi \in [0, 1, 18]$** .
- Choose the fitting function with only two powers $\xi \in [0, 1]$ at sufficiently late time to capture only the large-distance effects, and impose the condition $a_\infty = 1 - \sum_\xi a_\xi \geq 0$. This approach is expected to work well when the microphysics at collision time is negligible compared with the bubble size at collision. Later, we denote this fitting strategy as **GBF $\xi \in [0, 1, \infty]$** .

As we will validate later in the two-bubble system, both of these fitting approaches give quite similar GW spectra. More details on the fitting strategies are discussed in Appendix. D.2.

- With the input of the field evolution on the lattice, the stress-energy components are computed and plugged into Eq. (8.2.21) to obtain the “exact” GW spectrum. We then plot this exact spectrum together with GBF and the previously developed envelope and bulk-flow approximations as a comparison.

8.2.3 Real scalar field with quartic potential

We start the investigation with a single-component scalar field, whose dynamics are governed by the action

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + \frac{\delta}{3} \phi^3 - \frac{\lambda}{4} \phi^4 \right). \quad (8.2.25)$$

To simplify the parametric dependence of the dynamics, we perform the following reparameterization:

$$x^\mu \rightarrow \frac{\sqrt{\lambda}}{\delta} x^\mu, \quad \phi \rightarrow \frac{\delta}{\lambda} \phi. \quad (8.2.26)$$

Under this transformation, the potential reduces to

$$V(\phi) = \frac{\bar{\lambda}}{9}\phi^2 - \frac{1}{3}\phi^3 + \frac{1}{4}\phi^4, \quad (8.2.27)$$

with the new coupling $\bar{\lambda} \equiv \frac{m^2}{m_c^2} = \frac{9\lambda m^2}{2\delta^2}$, where m_c^2 is the critical mass. The classical dynamics of the phase transition is now characterized by only one fundamental coupling, $\bar{\lambda}$. For the purpose of having a first-order phase transition, we impose the following constraints on the coupling:

$$\bar{\lambda} \in (0, 1) \Rightarrow \begin{cases} \phi_{\text{FV}} = 0, & \phi_{\text{TV}} = \frac{1}{2} \left(1 + \sqrt{1 - \frac{8}{9}\bar{\lambda}} \right), \\ V(\phi_{\text{TV}}) = 0, & V(\phi_{\text{FV}}) = -\frac{\left(3 + \sqrt{9 - 8\bar{\lambda}} \right)^2 \left(3 + \sqrt{9 - 8\bar{\lambda}} - 4\bar{\lambda} \right)}{2592} \end{cases} \quad (8.2.28)$$

For $\bar{\lambda} \leq 0$, there is no barrier separating the two vacua, and hence no first-order phase transition can take place. With $\bar{\lambda} \geq 1$, ϕ_{FV} becomes the global minimum, and at $\bar{\lambda} = 1$, we have a degenerate potential with $V(\phi_{\text{FV}}) = V(\phi_{\text{TV}})$. The limits of thick-wall and thin-wall critical bubbles therefore correspond to $\bar{\lambda} \rightarrow 0_+$ and $\bar{\lambda} \rightarrow 1_-$, respectively.

In order to verify the GBF approximation in a broad picture, let us look at two reference points: $\bar{\lambda} = 0.84$, corresponding to bubbles with a larger critical radius, and $\bar{\lambda} = 0.1$, corresponding to a smaller-radius profile. The value $\bar{\lambda} = 0.84$ has been frequently used in the literature by other groups, e.g., Refs. [108, 110]. A visualization of the potential shape and the corresponding critical profiles are shown in Fig. 8.3. The evolution of the energy density ρ_z obtained by solving (8.2.5) is shown in Fig. 8.4. We observe that for the smaller critical profile, $\bar{\lambda} = 0.1$, most of the energy stored within the wall after collision still moves as a bulk with speed close to c . In contrast, for the larger critical bubbles with $\bar{\lambda} = 0.84$, part of the energy after collision is trapped close to the collision surface. Since the GBF assumes that the bubble surface continues to propagate with constant velocity after collision, it misses the trapped energy, and we expect the prediction using the approximation to be poorer for this coupling, as later shown to be the case.

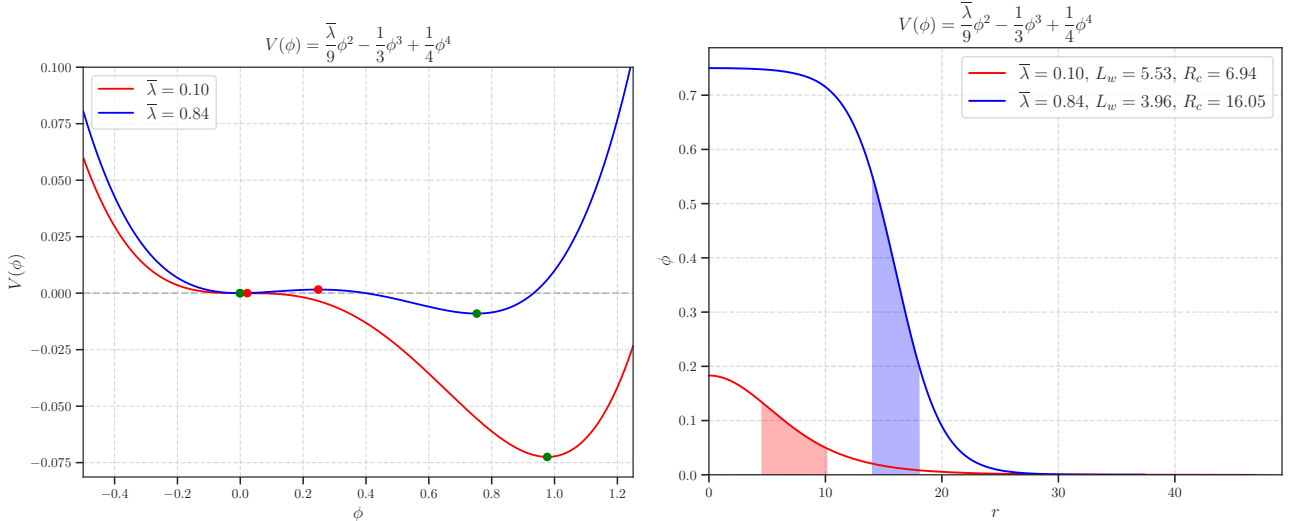


Figure 8.3: The potential (8.2.27) and the corresponding critical bubble profiles with $\bar{\lambda} = 0.1$ and 0.84 respectively. The profiles are obtained by solving the bounce equation (8.2.22) using the public package **CosmoTransitions**.

Stability of the coefficients

We note that the energy conversion from latent heat to gravitational waves is efficient when bubbles start to collide at large sizes. A natural observation arises: when fitting the surface tension

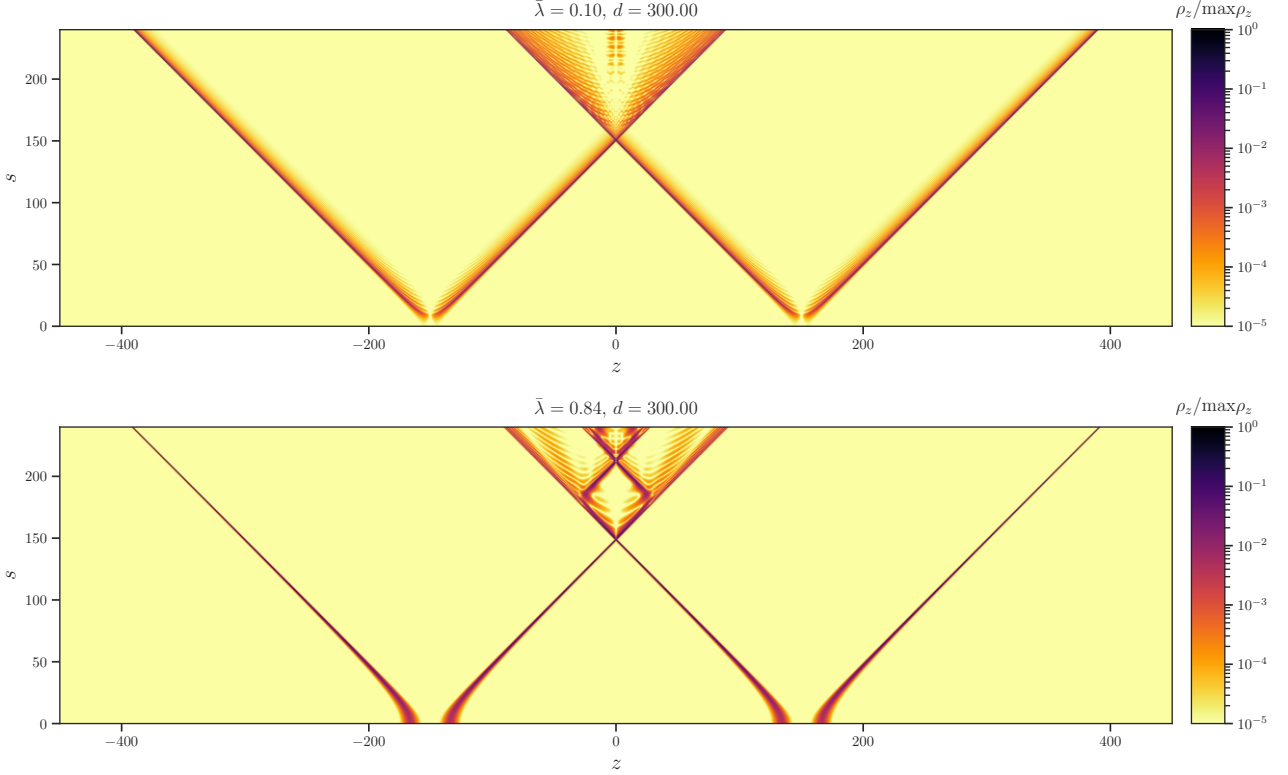


Figure 8.4: The evolution of energy density ρ_z for quartic potential with $\bar{\lambda} = 0.1$ (top) and 0.84 (bottom). Note that here we normalize ρ_z by its maximum, and the color bar is on a logarithmic scale, which forces us to introduce a cutoff at the minimum value of ρ_z .

to extract the coefficients a_ξ used in our GBF approximation, if these fitting coefficients stabilize at sufficiently large intra-bubbles scales, we can perform one comparison between the GBF predictions and the exact results computed using the $(1+1)D$ simulation in Eq. (8.2.21) at one large bubble separation d and expect the conclusion to hold for a broad range of d . This then enables us to apply these fitting parameters to the many-bubble context where bubbles are nucleated at different times and hence have different distances from each other. Therefore, we study the stability of fitting parameters against different bubble separations in this subsection.

Note that, in the case of two bubbles nucleated at time zero and separated by the distance $2s_{\text{col}}$, Eq. 8.1.8 takes the following form:

$$\frac{\sigma(t)}{(\rho_{\text{vac}}/3)s_{\text{col}}} = \Theta(s_{\text{col}} - t) \frac{t}{s_{\text{col}}} + \Theta(t - s_{\text{col}}) \left(\frac{s_{\text{col}}}{t} \right)^2 \left(a_0 + a_1 \frac{s_{\text{col}}}{t} \right) \quad (8.2.29)$$

i.e., the plot must remain invariant if we plot it in terms of the variables $(t/s_{\text{col}}, \sigma/s_{\text{col}})$, and indeed we see that this is the case; see the results in Figs. 8.5 and 8.6. Fitting the coefficients a_i further confirms that at sufficiently large separation between the bubbles, stability of the fit is achieved.

Comparison with full calculation

We start with the comparison of the full calculation with the two-bubble configuration. For this purpose one can use the public code provided by [108], (which we have modified...). In this case one needs to introduce an artificial time cutoff in the simulation, since the two-bubble system has a quadrupole moment that keeps growing in time. The results are shown in Figs. 8.7 and 8.8.

We see that our approximate method gives reasonable agreement with the first-principles calculation compared with previously developed methods, namely the envelope approximation and the $\xi = 0$ bulk-flow approximation. This is especially true when the surface element moves as a bulk after the bubbles collide, where the energy content is well captured by the surface tension along the bubble wall position, i.e., the case $\bar{\lambda} = 0.1$ in Fig. 8.7.

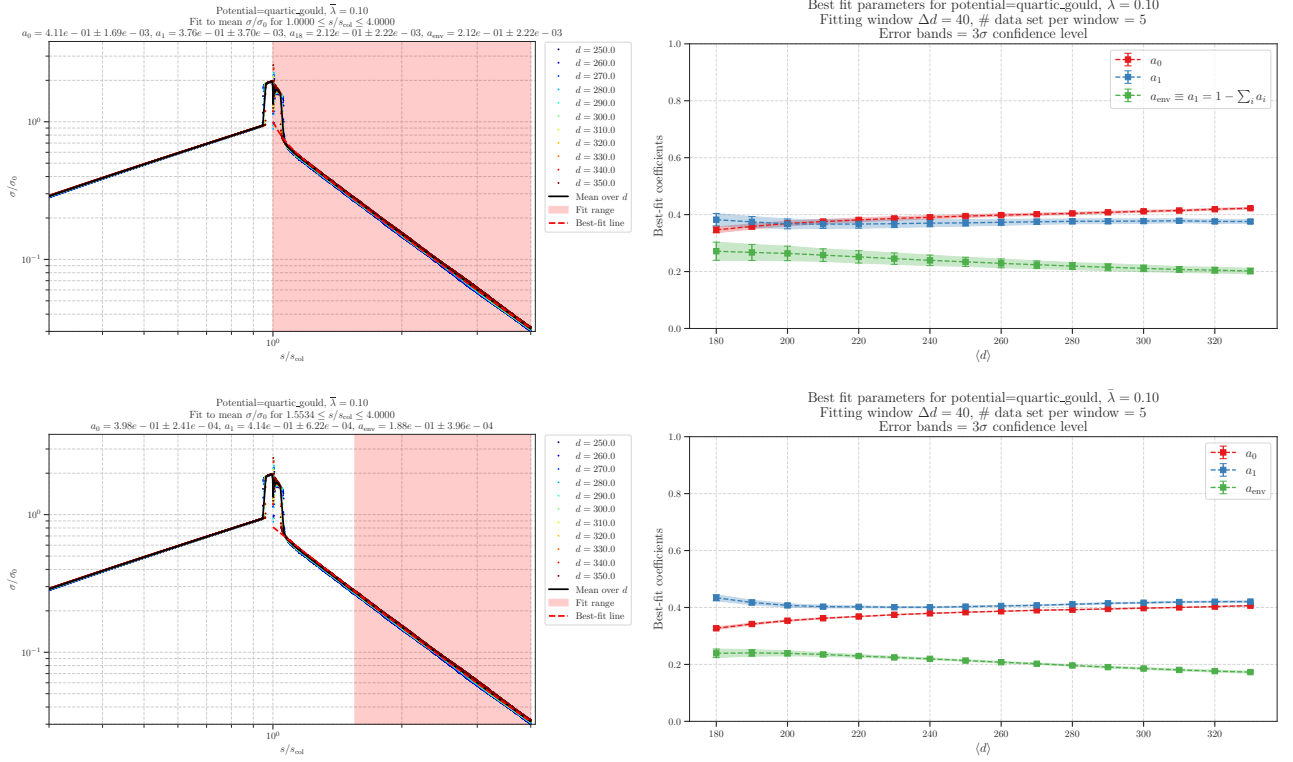


Figure 8.5: The fitting coefficients a_{env}, a_0, a_1 (left plots) and the corresponding stability of the fitting coefficients over a range of several distances (right plots), with $\bar{\lambda} = 0.1$. For the upper plots, the fitting is performed after the collision ($s/s_{\text{col}} \in [1, 4]$) with three powers $\xi \in [0, 1, 18]$; here $\xi = 18$ mimics the envelope contribution, and we impose the continuity condition for the surface tension before and after $s = s_{\text{col}}$, i.e., $\sum_{\xi=0,1,18} a_{\xi} = 1$, and furthermore we require $a_{\xi} > 0$. In the lower plots, the fitting is performed at sufficiently late time to capture only the large-distance effect without being spoiled by the nonlinear oscillation period. Here we fit only with two powers $\xi \in [0, 1]$ and impose $a_0, a_1 > 0$ and $a_{\text{env}} = 1 - a_0 - a_1 > 0$. We see that both fitting strategies work well, with small error bands, and are relatively stable over large bubble separations. In particular, the value of a_{env} is almost the same in both cases.

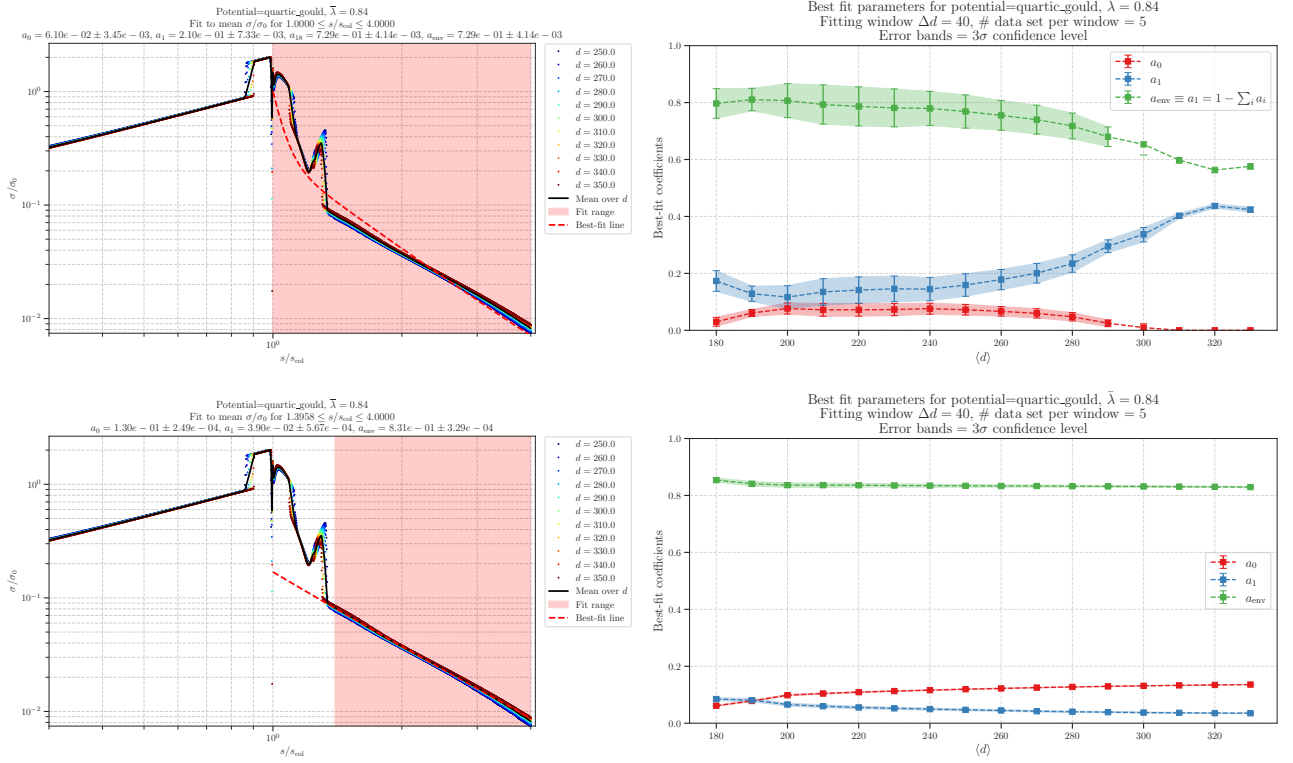


Figure 8.6: A similar plot to Fig. 8.5, here with $\bar{\lambda} = 0.84$. In this case, due to the long oscillation time after collision, the first fitting strategy with $\xi = [0, 1, 18]$ (top right plot) is not stable over distances, while the second strategy with $\xi = [0, 1, \infty]$ (bottom right plot) avoids this by cutting off the nonlinear interaction period, and the fitting is much more stable over large distances.

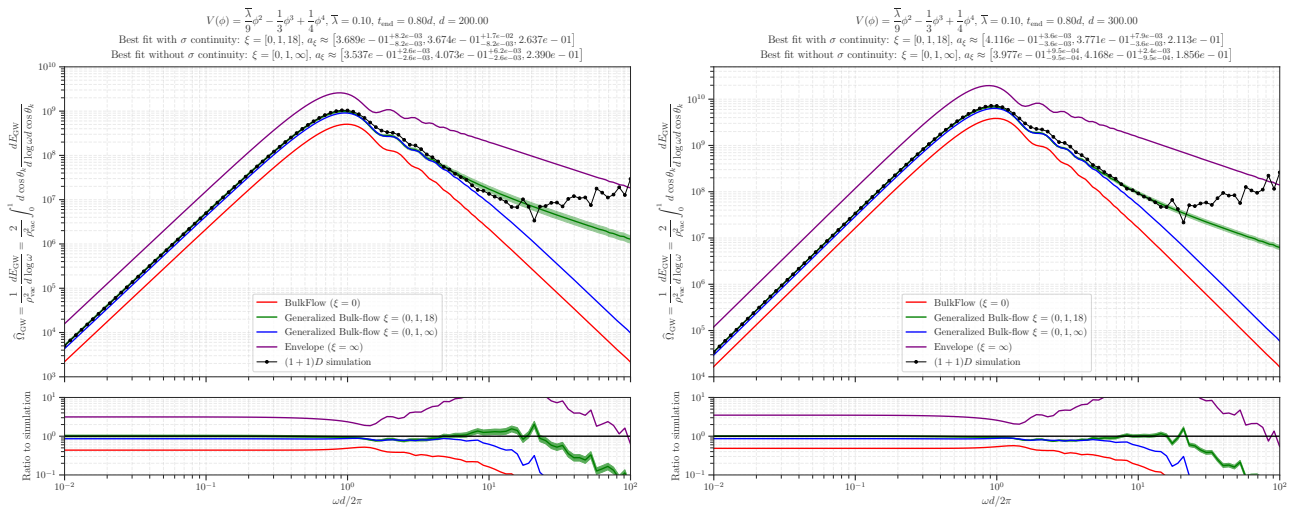


Figure 8.7: Comparison between the GW spectrum from the 1+1D simulation (black points) and the generalized bulk-flow approximations (green and blue lines) with $\bar{\lambda} = 0.1$, where $d = 200$ (left) and $d = 300$ (right).

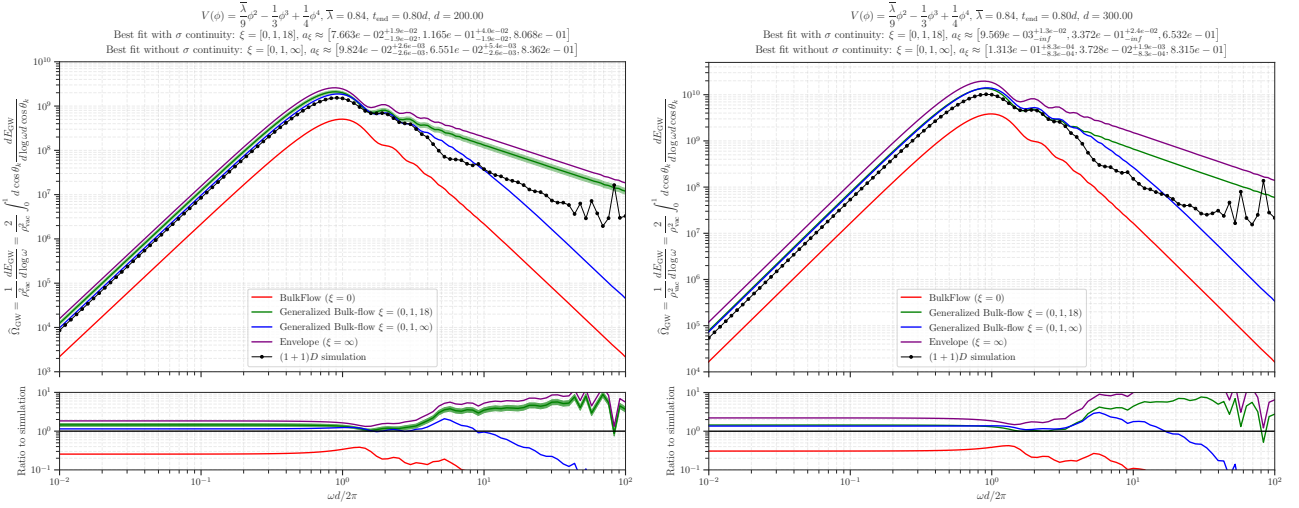


Figure 8.8: A similar plot to Fig. 8.7, here with $\bar{\lambda} = 0.84$. Here the generalized bulk-flow approximation is less accurate, potentially because, with the chosen time cutoff $s_{\text{end}} = 0.8d = 1.6s_{\text{col}}$, the contribution of the oscillation phase around collision is rather large for $\bar{\lambda} = 0.84$ compared with $\bar{\lambda} = 0.1$, as can be observed from Figs. 8.5 and 8.6.

8.2.4 Validation with multi-bubble system

At this point we can start comparing the GBF with the full 3+1D lattice calculation for the multi-bubble systems. An honest comparison requires both calculations to have exactly the same bubble nucleation histories; otherwise, the results will be biased by differences in the bubble configurations. For this purpose we perform a comparison with a first-principles lattice simulation by Toby Opferkuch and Malte Buschmann using package **Sledgehamr** [111], where 64 bubbles were nucleated at the same time in a box of size 200. This calculation was performed for the coupling $\bar{\lambda} = 0.84$. The results for different end times are reported in Fig. 8.9.

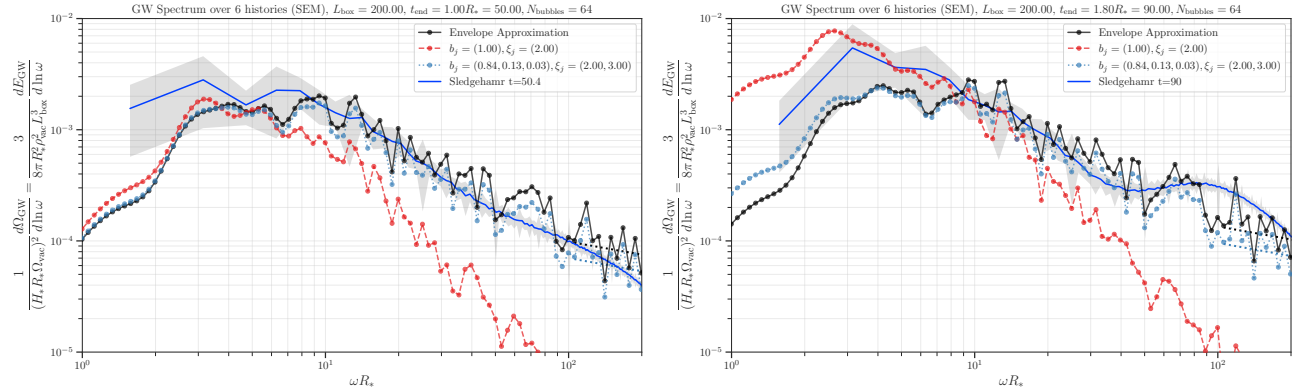


Figure 8.9: Comparison between the first-principles calculations and the GBF for the coupling $\bar{\lambda} = 0.84$ at different time cutoffs $t_{\text{end}} = R_*$ (top panel) and $t_{\text{end}} = 1.8R_*$ (bottom panel). The dotted red, blue and black lines correspond to bulk-flow, generalized bulk-flow and envelope predictions respectively. The blue solid line is the result from lattice simulation, with the shaded region is the corresponding error band.

Let us make a few comments about these results. First, the full simulation results are located between the predictions for the envelope and BF ansatzes. This is expected, since by definition these approximations correspond to opposite limits of the field relaxation time. We can also see that the UV tail is very well reproduced by the GBF, which shows that it is indeed controlled mainly by the drop of the energy-momentum tensor at the microphysics scale. The small peak on the full simulation curve comes from an effect in the microphysical dynamics that is not captured by our approximation.

However, note that in the simulation this peak is roughly at $\omega R_* \sim 100$, where the average interbubble distance was $R_* \sim 200/N_b^{1/3} \sim 50$. Thus we are discussing frequencies of order the mass scale of the fundamental theory, $\omega \sim m$, or distances $L \sim m^{-1}$. At such short scales there are field oscillations; see, for example, Fig. 8.6, which obviously cannot be captured by the GBF approximation. On the other hand, in more realistic configurations $m^{-1} \ll R_*$, so the peak associated with the microphysical dynamics will be located in the far UV tail. In the IR, the agreement becomes worse, but even in this case we are talking at most about a factor of 2–3 discrepancy. We are able to reproduce the peak position reasonably.

To summarize, GBF definitely presents a clear improvement compared with the BF or envelope approximation. Scanning through the model parameter space becomes much faster and more efficient. From the examples shown, we see that it can reasonably describe the GW spectrum.

Time cutoff in realistic simulation The BF approximation has the feature that it must be cut at some finite time; otherwise, its IR tail has a k^{-1} scaling, contradicting the requirements from causality. This anomalous behavior follows from the fact that BF assumes that the spherical wave of the stress-energy tensor propagates for an infinite amount of time after the collision without losing total energy (the local energy density drops as r^{-2}). To deal with this property, one has to introduce some artificial cutoff in the simulation. For example, there will be secondary collisions between the wave fronts, which will necessarily modify the stress-energy-momentum evolution. This time can be used as an estimate for a time cutoff in the BF or GBF approximations.

8.2.5 Complex scalar field with log potential

Let us now consider another example to illustrate the applicability of the GBF approximation. We consider a potential describing the dynamics of a conformal model composed of n scalar fields ϕ_j , $j \in \{1, 2, \dots, n\}$, along a direction $\phi = \sum_{j=1}^n \alpha_j \phi_j$ with $\sum_{j=1}^n \alpha_j^2 = 1$; see, e.g., Ref. [112]. In the regime $T \gg M_k$, where M_k is the tree-level mass of the scalar field ϕ , the one-loop finite-temperature effective potential along the ϕ direction is given by

$$V_{\text{eff}} = B\phi^4 \left[\log \left(\frac{\phi}{v} \right) - \frac{1}{4} \right] + CT^2\phi^2, \quad (8.2.30)$$

where the first part is the Coleman-Weinberg potential at zero temperature, and the second part describes the one-loop finite-temperature corrections. Under the transformation

$$B = \frac{(\kappa + 2)\rho_{\text{vac}}}{v^4}, \quad C = \frac{\kappa\rho_{\text{vac}}}{T^2v^2}, \quad (8.2.31)$$

we obtain the following potential expression that will be used throughout our computations:

$$\frac{V(\Phi)}{\rho_{\text{vac}}} = 1 + \kappa \frac{|\Phi|^2}{v^2} + \frac{|\Phi|^4}{v^4} \left[(\kappa + 2) \log \left(\frac{|\Phi|^2}{v^2} \right) - (\kappa + 1) \right] \quad (8.2.32)$$

$$\kappa > 0 \quad \Rightarrow \quad \begin{cases} \Phi_{\text{TV}} = v, & \Phi_{\text{FV}} = 0 \\ V(\Phi_{\text{TV}}) = 0, & V(\Phi_{\text{FV}}) = \rho_{\text{vac}}. \end{cases} \quad (8.2.33)$$

We can normalize the field and the potential such that $\rho_{\text{vac}} = v = 1$ for simplicity. Therefore, similar to the quartic-potential case discussed previously, the dynamics of bubble nucleation and collision are captured solely by the coupling parameter κ . However, unlike the previous single-component scalar-field model, here we have a complex scalar field, and therefore the phase of the true-vacuum bubbles also enters the dynamics and affects the generated gravitational-wave spectrum. Since we also use the two-bubble system to verify the GBF, we need to pay attention to the relative phase between two nucleated bubbles, which will be denoted $\Delta\varphi$ for the rest of the discussion.

Let us also note that this potential has been studied in [107] for GWs generated during a strongly supercooled phase transition, where the symmetry breaking originates from radiative corrections and

finite temperature corrections. As previously mentioned, in that paper, the authors use an approach similar to the generalized bulk-flow approximation, but instead of fitting the surface tension, they fit the maximum of the kinetic energy density for manually chosen powers ξ . Furthermore, they imposed the continuity of the energy density before and after collision.

We start by considering two values of the coupling, $\kappa = 0.2$ and $\kappa = 1$, whose potential and critical profiles are depicted in Fig. 8.10.

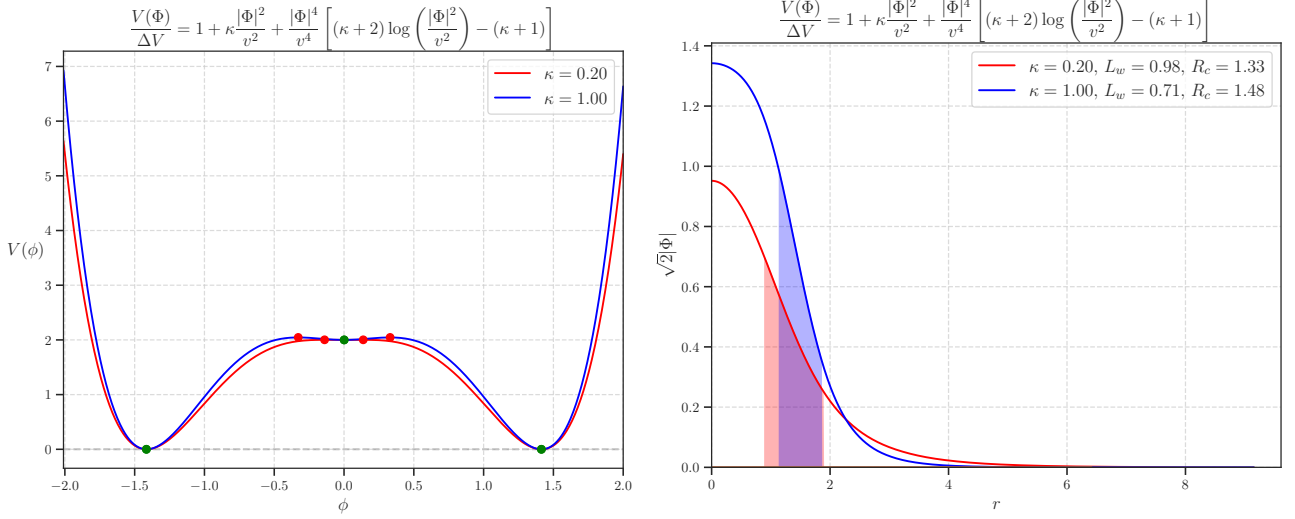


Figure 8.10: The potential (8.2.32) and the corresponding critical bubble profiles with $\kappa = 0.1$ and 1.0.

As shown in Fig. 8.11, the dynamics of bubble profiles after collision have different characteristics depending on their relative phase $\Delta\varphi$. We therefore expect different fitting coefficients for the surface tension, resulting in different predictions for the final spectrum. Let us investigate the case $\Delta\varphi = 0$, i.e., only the real part of the underlying complex field is “activated” and evolves over time. Later on, we also show the results for a generic $\Delta\varphi$ and the final GW spectrum obtained by averaging over these phases.

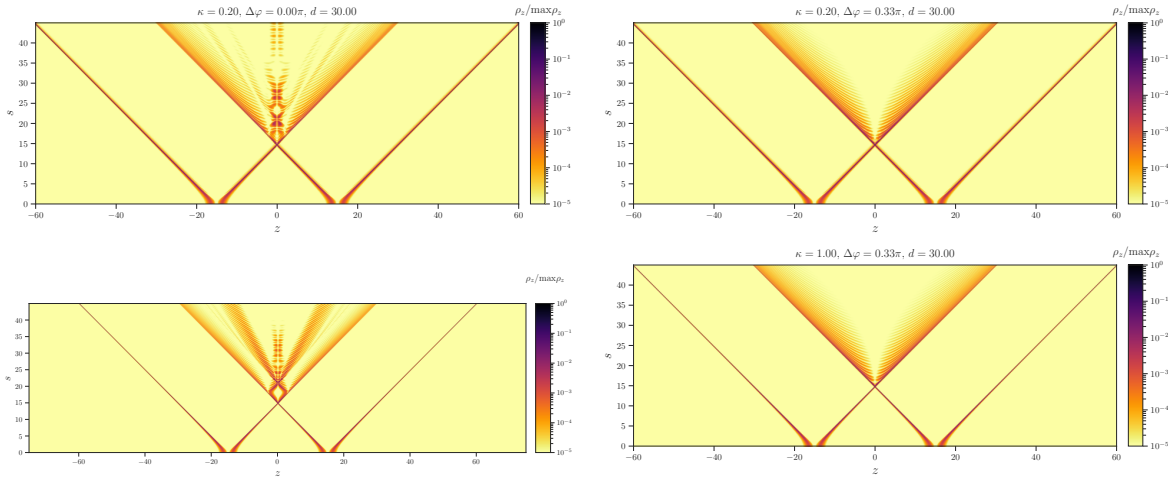


Figure 8.11: The evolution of energy density ρ_z for potential (8.2.32) with $\kappa = 0.2$ (top) and 1.0 (bottom). The left plots correspond to the relative phase $\Delta\varphi = 0$, and the right plots correspond to $\Delta\varphi = \pi/3$. In the case with $\Delta\varphi = 0$, only the real part of the complex scalar field evolves while the imaginary part remains constant, and there are short time intervals where the walls after collision bounce between the false vacuum and true vacuum; consequently, the energy is somewhat trapped between the colliding surfaces. Once we turn on the relative phase, both the real and imaginary parts evolve dynamically, and there is no visible trapping effect in the right plots.

Results for spectrum averaged over $\Delta\varphi$

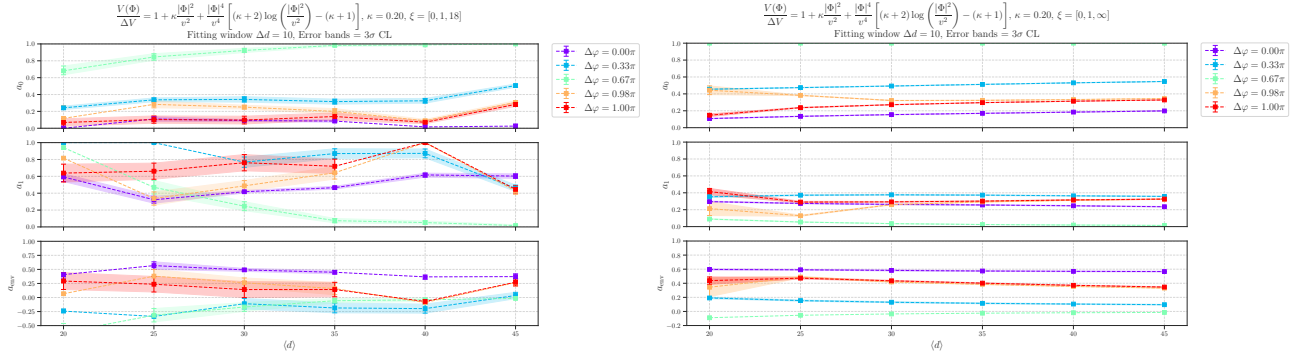


Figure 8.12: The fitting coefficients with different relative phases of colliding bubbles, with fitting powers $\xi = [0, 1, 18]$ (left) and $\xi = [0, 1, \infty]$ (right). In this figure, we consider the inter-bubble distance $d = 30$.

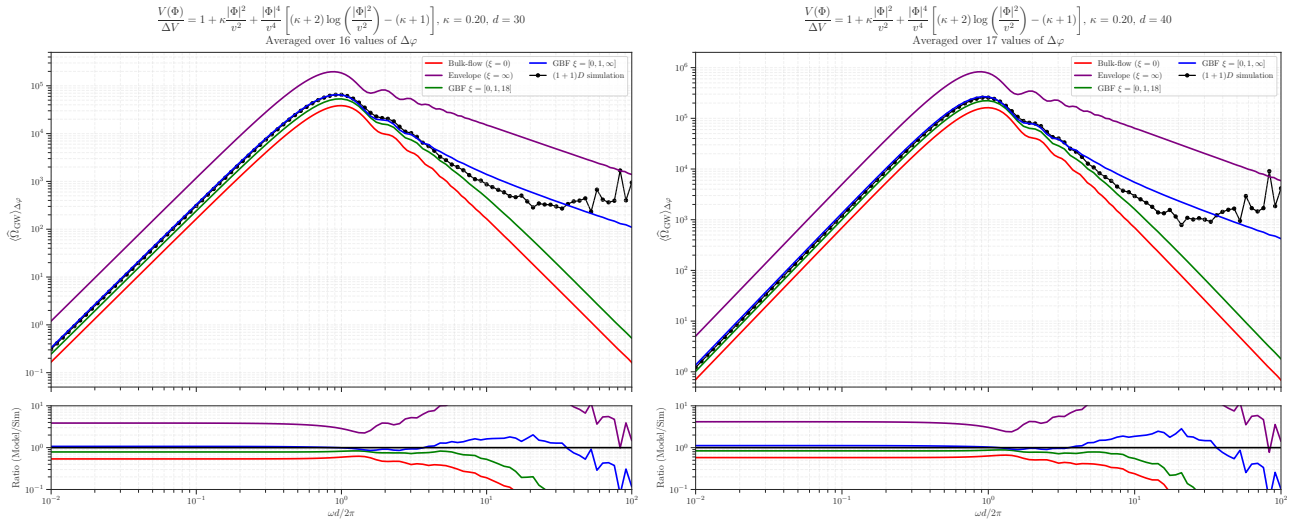


Figure 8.13: The final spectrum obtained by averaging over different relative phases within the range $\Delta\varphi \in [0, \pi]$, with inter-bubble distance $d = 30$ (left) and $d = 40$ (right). The GBF results on both fitting strategies agree well with the $(1+1)D$ simulation results.

In a more realistic situation, bubbles with random phases are nucleated with the same rate. It therefore requires a generalization of the GW spectrum computation where two bubbles are nucleated with a nonzero relative phase, after which we average the results over all these $\Delta\varphi$. We start by finding the GBF coefficients a_ξ by fitting the surface tension as a function of time s/s_{col} for various d , as shown in Fig. 8.12. We observe a peculiar trend here: as $\Delta\varphi$ increases from 0 up to $\sim 0.6\pi$, the envelope coefficient decreases; for $0.6\pi \lesssim \Delta\varphi \lesssim 0.87\pi$, there is almost no envelope contribution; and for $0.87\pi \lesssim \Delta\varphi \lesssim \pi$, a_{env} increases again. Note that due to numerical errors in the fitting, in several cases $a_{\text{env}} = 1 - a_0 - a_1$ is negative but still quite close to zero, while a_0 is larger than, but still close to, 1. This indicates that, for these relative phase values, almost no energy is dissipated during the collision, and the bulk-flow approximation with $\xi = 0$ is a good approximation for capturing the GW spectrum.

8.3 Application of the method

This section presents a more realistic study of GWs generated by a system of many bubbles. The phase transition is modeled by different approaches to introducing true-vacuum bubbles within the false-vacuum region of the lattice, and we aim to answer the following question: are the general

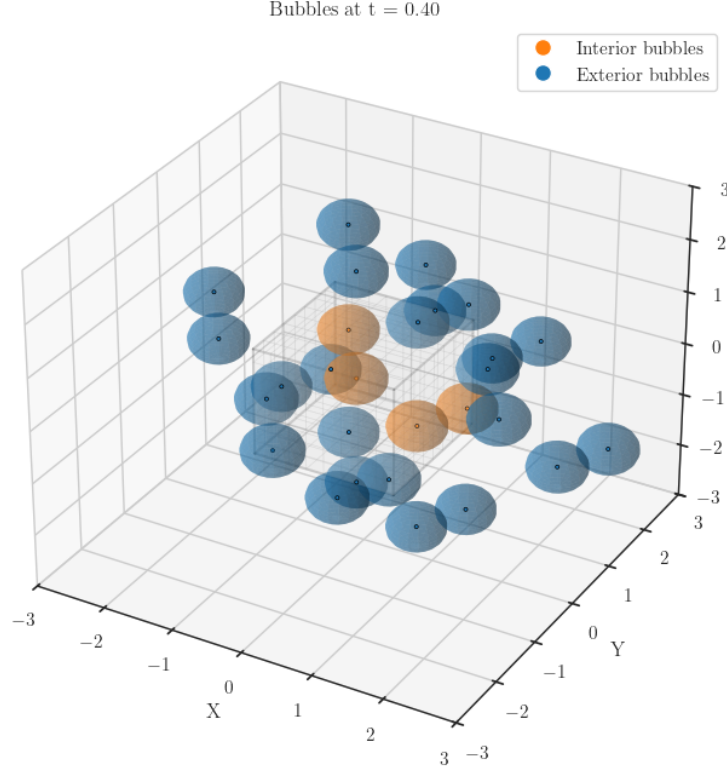


Figure 8.14: Bubble configuration at a fixed time slice in a Cartesian lattice, with exterior bubbles (blue) are generated using periodic boundary condition.

features of the GWs generated during a FOPT sensitive to the different ways of nucleating bubbles, or do they depend only on the average bubble distance at percolation? The different nucleation types under investigation include

- **Exponential nucleation:** Bubbles are introduced with an exponentially increasing rate per unit volume.
- **Constant nucleation:** Somewhat similar to exponential nucleation, but now bubbles are introduced with a constant rate per unit volume.
- **Simultaneous nucleation:** A fixed number of bubbles are nucleated at the start of the simulation and distributed uniformly inside the lattice.

Several papers have performed $3 + 1$ D numerical simulations to study the difference between nucleation types, e.g., Ref. [100]. In our study, we use the developed GBF approximation to take a more detailed look at the effects of nucleation strategies with even larger lattices and more bubbles. We also quantify potential sources of uncertainty in setting up simulations, including the lattice shape, size, and boundary conditions imposed to introduce a cutoff on the envelope contribution. In short:

- **Finite lattice size:** Several scales are involved in the simulation of GWs on the lattice using the GBF approximation, including the mean bubble radius at collision R_* , the lattice size L_{box} , and the end time of the simulation t_{end} . Compared with a full $3 + 1$ D dynamical simulation, the GBF has the advantage of avoiding the bubble-wall thickness scale by effectively setting it to zero, thereby reducing the computational cost associated with the dense grid needed for better wall resolution.

The first issue arises immediately: a finite box only allows Fourier modes $k_i = 2\pi n_i/L$, so any GW power at wavelengths $\lambda_{\text{GW}} \gtrsim L_{\text{box}}$ is absent. We therefore require the peak frequency to be far from the IR limit, i.e., $\hat{\omega} \gtrsim 2\pi/L_{\text{box}}$, and since $\hat{\omega} \sim R_*^{-1}$, we want a sufficiently large lattice $L_{\text{box}} \gtrsim 2\pi R_*$. This issue can be partially tamed by applying periodic boundary conditions, where one can imagine continuously stacking the boxes to build an even larger lattice and hence,

in principle, go deeper into the IR.

Another issue is that, with exponential or constant nucleation, a small box implies that only a few bubbles are nucleated, leading to a poor realization of the homogeneous stochastic GW source and potentially changing the shape or amplitude of the spectrum, e.g., shifting the peak position or creating bumps at random frequencies. One way to solve this issue, apart from simply increasing the box size, is to average the spectrum over multiple nucleation histories. An advantage of this approach over using a larger box is that we can easily distribute the workloads on a cluster, thereby reducing the waiting time.

- **Lattice shape and corresponding boundary conditions:** A historical option used in the early development of the envelope approximation is the *spherical lattice* [99], which pairs with reflecting boundary conditions, i.e., by creating a mirror of the bubble configuration on the lattice boundary ⁽¹⁾. This has the advantage of computing the GW spectrum along arbitrary directions.

Another common choice is a Cartesian lattice, most often a *cubic lattice*. This choice is common in 3 + 1D numerical simulations of the full field dynamics, and has been implemented in Ref. [101] to study the $\xi = 0$ bulk-flow approximation in order to obtain results compatible with hydrodynamic simulations. However, a cubic lattice introduces a source of spurious anisotropy in the stress-energy tensor, because usually only directions perpendicular to the six faces are considered. Similar to the arguments on finite lattice size, better statistics requires running more simulations or increasing the lattice volume to have a larger bubble count.

Finally, note that even though the bubbles inside the lattice are nucleated by enforcing causality, i.e., no bubble is formed inside existing bubbles, the exterior bubbles generated by the boundary condition generally do not respect causality. However, we expect such a violation of causality to have a negligible effect on the average spectrum given sufficiently large bubble counts per simulation, partly because only the contributions of the interior bubbles are summed over.

In all results derived below, we use a cubic lattice with periodic boundary conditions, unless stated otherwise. For each nucleation history, we average the gravitational-wave spectrum over six directions, and later we average the spectrum over all nucleation histories.

For exponential nucleation, we have two relevant length scales: the average radius at collision $R_* \sim \beta^{-1}$, and the end time of the simulation t_{end} , which is manually imposed in the simulations but can be physically set by the time scale at which the phase transition stops, e.g., $t_{\text{end}} \sim H^{-1}$. It is therefore convenient to define the energy fraction in GW radiation per octave (Refs. [99, 101, 61])

$$\Omega_{\text{GW}} = \frac{1}{E_{\text{tot}}} \frac{dE_{\text{GW}}}{d \ln \omega} = \left(\frac{H}{\beta} \right)^2 \left(\frac{\alpha}{\alpha + 1} \right)^2 \Delta(\omega/\beta, v_w), \quad (8.3.1)$$

where the Hubble parameter is defined as

$$H^2 = \frac{8\pi G \rho_{\text{tot}}}{3} = \frac{8\pi G (\rho_{\text{vac}} + \rho_{\text{rad}})}{3}, \quad (8.3.2)$$

and $\alpha \equiv \rho_{\text{vac}}/\rho_{\text{rad}}$ denotes the latent heat fraction. According to Eq. (7.2.12), in the generalized bulk-flow framework we have

$$\Delta(\omega/\beta, v_w) = \left(\frac{\omega^3}{\beta^3} \right) \frac{3v_w^6 \beta^5}{2\pi V} \int d\hat{\mathbf{k}} \Lambda_{ij,lm}(\hat{\mathbf{k}}) C_{ij}^* C_{lm}, \quad (8.3.3)$$

where C_{ij} captures the geometry of the nucleation process. This function Δ is what will be computed and fitted in the many-bubble context below.

As we will see later, the GW spectrum sourced by bubble collision has two frequencies corresponding to R_*^{-1} and t_{end}^{-1} that mark the places where the scaling power law changes. To capture

⁽¹⁾ In the case of the envelope approximation, reflecting and absorbing boundary conditions are equivalent.

various features of the GW spectrum, namely amplitude, scaling powers, and the frequencies at which these scaling powers change, we fit the numerical results with the following function: ⁽¹⁾

$$\Delta(\omega) = \hat{\Delta} \left(\frac{\omega}{\hat{\omega}_2} \right)^{p_{\text{IR}1}} \left(\frac{1 + (\omega/\hat{\omega}_1)^{1/\varepsilon_1}}{1 + (\hat{\omega}_2/\hat{\omega}_1)^{1/\varepsilon_1}} \right)^{\varepsilon_1(p_{\text{IR}2} - p_{\text{IR}1})} \left(\frac{1 + (\omega/\hat{\omega}_2)^{1/\varepsilon_2}}{2} \right)^{\varepsilon_2(p_{\text{UV}} - p_{\text{IR}2})} \quad (8.3.7)$$

where $\hat{\Delta}$ and $\hat{\omega}$ are the peak amplitude and frequency, respectively, $p_{\text{IR}1}$, $p_{\text{IR}2}$, and p_{UV} are the scaling powers within the ranges $\omega \lesssim \hat{\omega}_1$, $\hat{\omega}_1 \lesssim \omega \lesssim \hat{\omega}_2$, and $\omega \gtrsim \hat{\omega}_2$, and $\varepsilon_{1,2}$ parameterize the width of the peak. The fitting is done using the Levenberg-Marquardt method implemented in the package LMFIT [113].

8.3.1 Exponential nucleation

Here we describe step by step how to reproduce the envelope and bulk-flow gravitational-wave spectrum in Ref. [101] with an exponential nucleation history, i.e., $\Gamma(t) = \Gamma_* e^{\beta(t-t_*)}$. We start with the procedure to generate various nucleation histories given a fixed nucleation rate β . In what follows, we use the following parameters: the nucleation rate $\beta = 0.05$ and a cubic lattice of size $L_{\text{box}} = 20/\beta$. The fraction $V_{\text{FV}}(t)/V_{\text{lattice}}$ is obtained by a brute-force Monte Carlo approach, where we randomly sample points inside the lattice and count the fraction of points outside all bubbles, including bubbles nucleated inside the lattice and bubbles outside the lattice generated using periodic boundary conditions.

The distribution of nucleated bubbles over time $N_{\text{bubbles}}(t)$ can be determined by solving the integro-differential equation below

$$\frac{dN_{\text{bubbles}}(t)}{dt} \approx \Gamma_0 V_{\text{lattice}} e^{\beta t} \exp \left[-\frac{4\pi}{3} \int_0^t dt' \frac{dN_{\text{bubbles}}(t')}{dt'} \frac{(t-t')^3}{V_{\text{lattice}}} \right]. \quad (8.3.8)$$

For $v_w \neq 1$, the factor $(t-t')^3$ should be replaced by $v_w^3(t-t')^3$, and in an expanding cosmological background by the corresponding physical bubble volume constructed from the comoving radius in

⁽¹⁾ This is a special case of a more general fitting function with n scales $\hat{\omega}_{1\dots n}$, the same number of parameters $\varepsilon_{1\dots n} \geq 0$ to control the width of the broken power law at the corresponding scale, and an amplitude parameter at a given scale $\hat{\Delta} = \Delta(\hat{\omega}_m)$

$$\Delta(\omega) = \hat{\Delta} \left(\frac{\omega}{\hat{\omega}_m} \right)^{p_0} \prod_{k=1}^n \left(\frac{1 + (\omega/\hat{\omega}_k)^{1/\varepsilon_k}}{1 + (\hat{\omega}_m/\hat{\omega}_k)^{1/\varepsilon_k}} \right)^{\varepsilon_k(p_k - p_{k-1})} \quad (8.3.4)$$

Assuming the separation of scales $\hat{\omega}_1 \ll \hat{\omega}_2 \ll \dots \ll \hat{\omega}_n$ and sufficiently small width ε_k , this function has the following asymptotic behaviors

$$\begin{aligned} \Delta(\hat{\omega}_k \ll \omega \ll \hat{\omega}_{k+1}) &\approx \hat{\Delta} \left(\frac{\omega}{\hat{\omega}_m} \right)^{p_k} \prod_{j=1}^n \frac{1}{\left(1 + (\hat{\omega}_m/\hat{\omega}_j)^{1/\varepsilon_j} \right)^{\varepsilon_j(p_j - p_{j-1})}} \cdot \prod_{j=1}^k \left(\frac{\hat{\omega}_m}{\hat{\omega}_j} \right)^{p_j - p_{j-1}} \\ &\equiv \hat{\Delta} \left(\frac{\omega}{\hat{\omega}_m} \right)^{p_k} D_k. \end{aligned} \quad (8.3.5)$$

It is sometimes useful to talk about the integrated spectrum, which involves the quantity $I_{\Delta}(\omega_{\text{max}}) = \int_0^{\omega_{\text{max}}} d \ln \omega \Delta(\omega)$, with $\omega_{k_{\text{max}}} \leq \omega_{\text{max}} \leq \omega_{k_{\text{max}}+1}$. We can approximate this integral using asymptotic function

$$I_{\Delta}(\omega_{\text{max}}) = \hat{\Delta} \left[\sum_{k=0}^{k_{\text{max}}} D_k \mathcal{I}_k(x_k, x_{k+1}) + D_k \mathcal{I}_k(x_{k_{\text{max}}}, x_{\text{max}}) \right], \quad (8.3.6)$$

where

$$x_0 = 0, \quad x_k \equiv \frac{\hat{\omega}_k}{\hat{\omega}_m}, \quad x_{\text{max}} \equiv \frac{\hat{\omega}_k}{\hat{\omega}_m}, \quad \mathcal{I}_k(x_a, x_b) = \begin{cases} \frac{x_b^{p_k} - x_a^{p_k}}{p_k}, & p_k \neq 0 \\ \ln \left(\frac{x_b}{x_a} \right), & p_k = 0 \end{cases}.$$

The form of one scale and two power laws in Eq. (8.3.7) is the case $n = 2$, with $p_{\text{IR},1} \equiv p_0$, $p_{\text{IR},2} \equiv p_2$, $p_{\text{UV}} \equiv p_2$.

Eq. (7.1.24). Defining the dimensionless parameters $n_{\text{bubbles}} \equiv N_{\text{bubbles}}/(\beta^3 V_{\text{lattice}})$, $\bar{\Gamma}_0 \equiv \Gamma_0 \beta^{-4}$, and $\tau = \beta t$, we can rewrite this equation as

$$\frac{dn_{\text{bubbles}}(\tau)}{d\tau} \approx \bar{\Gamma}_0 e^\tau \exp \left[-\frac{4\pi}{3} \int_0^\tau d\tau' \frac{dn_{\text{bubbles}}(\tau')}{d\tau'} (\tau - \tau')^3 \right]. \quad (8.3.9)$$

There are several approaches to solve this equation. One such approach is by introducing auxiliary moments:

$$M_k(\tau) = \int_0^\tau d\tau' \frac{dn_{\text{bubbles}}(\tau')}{d\tau'} (\tau - \tau')^k, \quad k = 0, 1, 2, 3, \quad (8.3.10)$$

we can rewrite Eq. (8.3.9) in terms of a system of ODEs

$$\frac{dn_{\text{bubbles}}(\tau)}{d\tau} = \frac{dM_0(\tau)}{d\tau} = \bar{\Gamma}_0 e^\tau \exp \left[-\frac{4\pi}{3} M_3(\tau) \right] \quad (8.3.11a)$$

$$\frac{dM_1(\tau)}{d\tau} = M_0(\tau) \quad (8.3.11b)$$

$$\frac{dM_2(\tau)}{d\tau} = 2M_1(\tau) \quad (8.3.11c)$$

$$\frac{dM_3(\tau)}{d\tau} = 3M_2(\tau) \quad (8.3.11d)$$

This system can be solved numerically with the initial condition $[M_0, M_1, M_2, M_3] \Big|_{\tau=0} = [0, 0, 0, 0]$.

Note that $\bar{\Gamma}_0$ must be sufficiently small for the ODE system and the initial condition to be self-consistent. A visualization of bubble density and its derivative as a function of time is shown in Fig. 8.15.

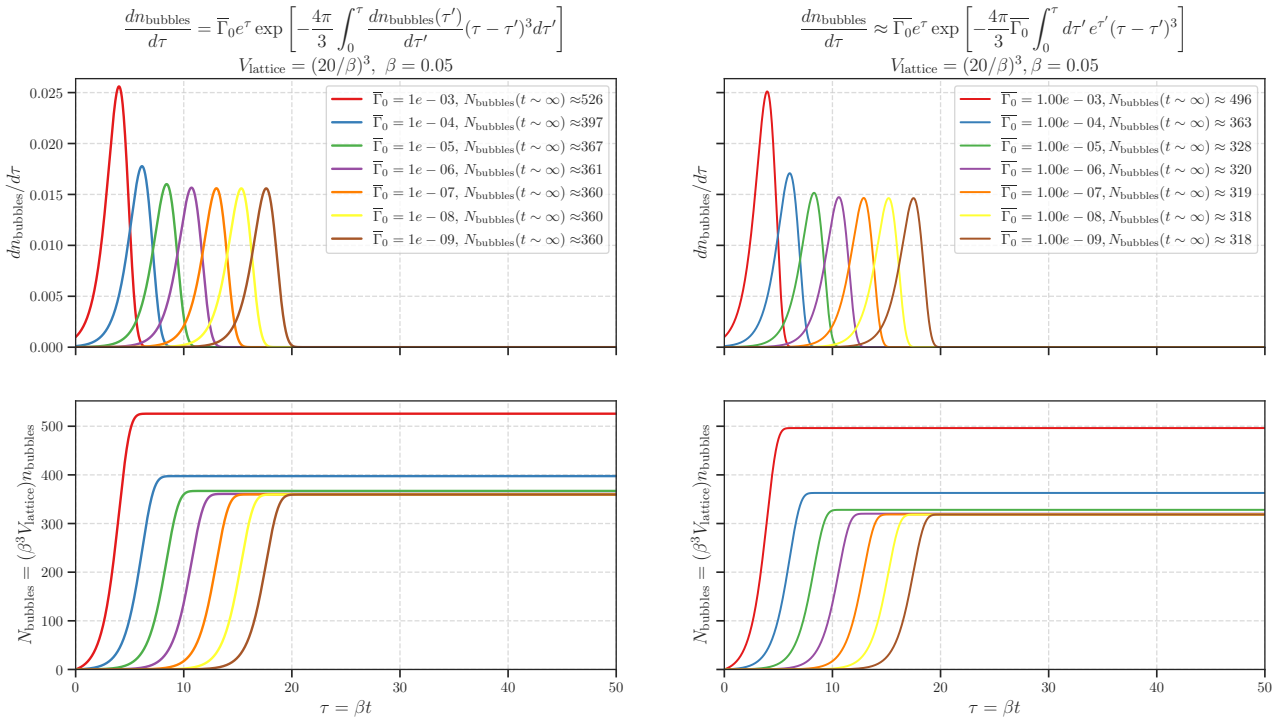


Figure 8.15: Plot of the total number of bubbles inside a box lattice of size $20/\beta$, obtained by solving Eq. (8.3.9) numerically (left) or using the approximation (right). Different lines correspond to different choices of Γ_0 . Here we observe that with sufficiently small Γ_0 , the total number of bubbles nucleated inside the lattice approaches a constant value.

Having obtained the nucleation histories, we then proceed to compute the GW spectrum using Eq. (7.2.13). Note that the summation over bubbles, indexed by n in the given formula, is performed

only over bubbles nucleated within the lattice, ignoring the bubbles generated by the periodic boundary conditions. For a cross-check between our numerical results for the envelope and $\xi = 0$ bulk-flow approximations developed in Ref. [101], see Appendix. D.3.1.

Next, we investigate the impact of box sizes on the spectrum. The expectation is that by increasing the volume of the lattice, the GW spectra should be less sensitive to the nucleation histories and hence should converge. The numerical results are shown in Fig. 8.16, with details of the fitting parameters, including best-fit values and $1\text{-}\sigma$ confidence intervals, are given in Fig. 8.17. For numerical values of the fitting parameters of the GW spectrum in Fig. 8.17, see Tables. D.2, D.6, and D.4 in the appendix.

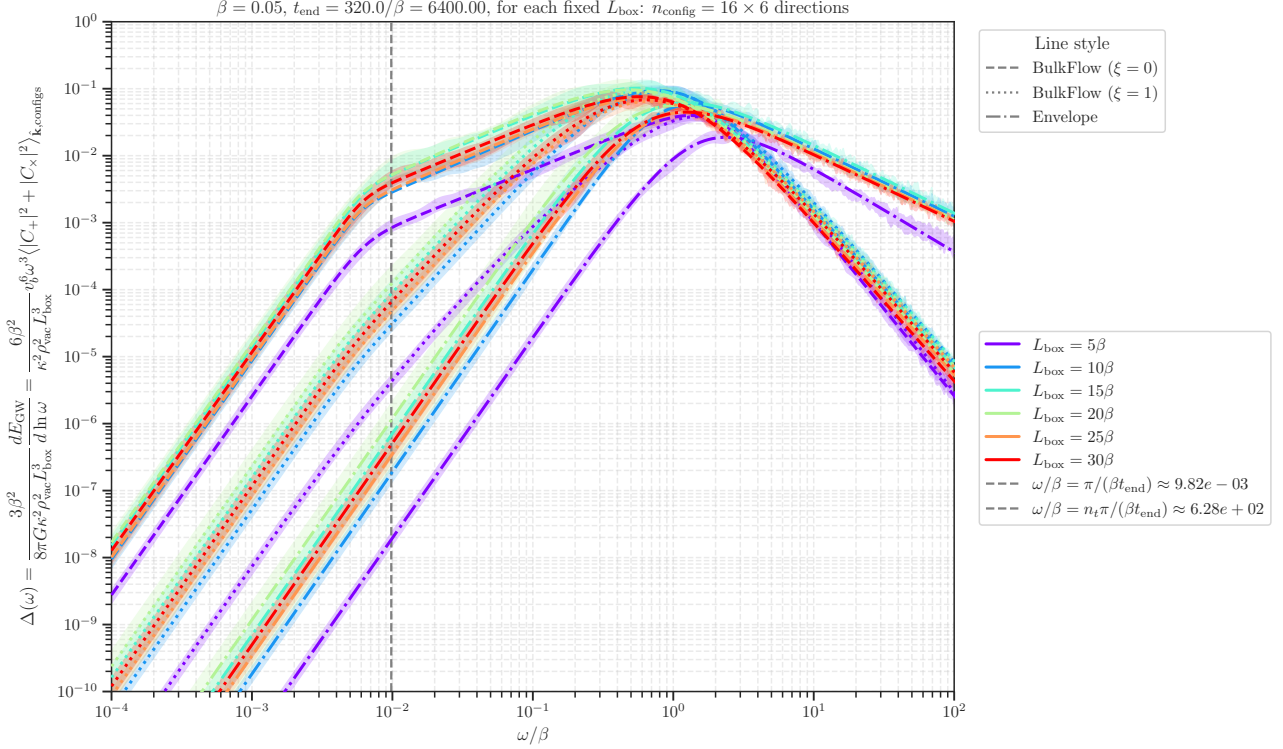


Figure 8.16: Effects of changing lattice sizes with $L_{\text{box}} = \{10\beta, 15\beta, 20\beta, 25\beta, 30\beta\}$, with fixed end time of integration $t_{\text{end}} = 320/\beta$. For each given L_{box} , we generate 16 configurations and take the average of the spectrum over both configurations and 6 directions. The fitting of spectrum with Eq. (8.3.7) is performed within the frequency range $\omega \in [10^{-4}, 10^2]$.

Some observations regarding the results of fitting the GW spectra for different approximations, namely envelope with $\xi = 0$, and $\xi = 1$ GBF, are:

- We observe a sudden change in the IR power law of the $\xi = 0$ and $\xi = 1$ bulk-flow spectra: for $\omega \lesssim \pi/t_{\text{end}}$, the spectra scale similarly to the envelope spectrum, i.e., following the usual scaling set by causality, while for $\omega \gtrsim \pi/t_{\text{end}}$, the spectra of the GBF approximation scale with a smaller a_{IR} compared with the envelope. This change in IR power law is more obvious for $\xi = 0$ than for $\xi = 1$.
- Overall, we confirm that for sufficiently large boxes, $\beta L_{\text{box}} \gtrsim 10$, the resulting GW spectrum and consequently the fitting coefficients, including the amplitude, peak frequencies, and scaling powers, stabilize. Note that the width is very sensitive to the number of points around the frequency where the power law changes; therefore, it is not as stable as the other fitting parameters.
- The frequency of the first change in power law from p_{IR1} to p_{IR2} in bulk-flow spectra is approximately $\hat{\omega}_1 \approx \pi/t_{\text{end}}$.
- In the deep IR, we obtain $p_{\text{IR1}} \approx 3$, which is a known result following from causality. Meanwhile, in the deep UV, the envelope scales with power $p_{\text{UV}} \approx -1$, while the GBF with $\xi = [0, 1]$ scales

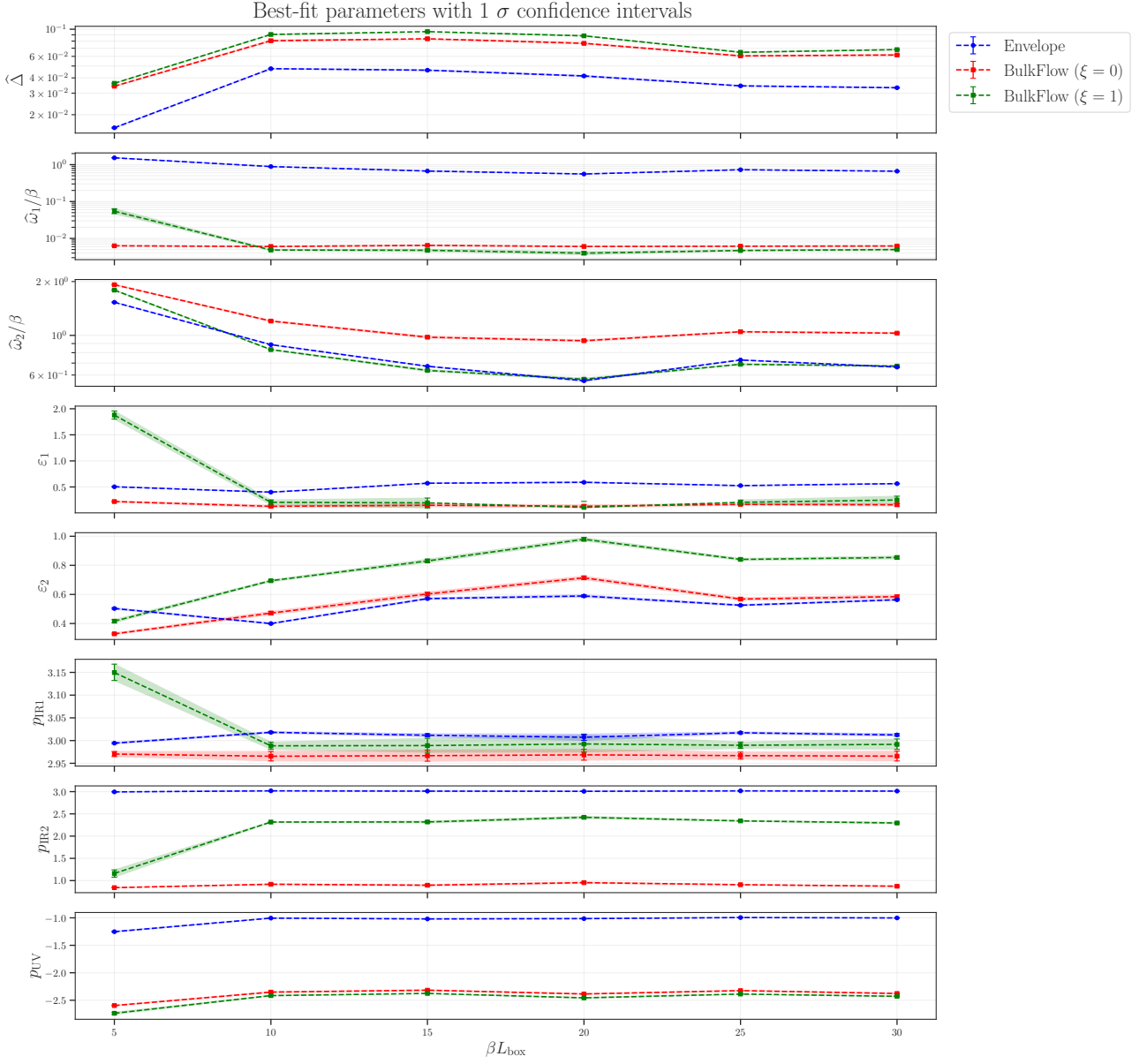


Figure 8.17: Changes of the best-fit parameters as a function of lattice size for simultaneous nucleation and exponential nucleation. This is simply a visualization of the results shown in Tables D.2, D.3, and D.4.

with $p_{UV} \approx -2.5$. We check with larger values of ξ and see that they all share the same behavior in the UV, confirming that this is a consequence of the long-lasting stress-energy source.

8.3.2 Simultaneous nucleation

Here we consider simultaneous nucleation with fixed box size and varying total number of bubbles per nucleation history N_{bubbles} . We study how the GW spectrum changes with different numbers of bubbles at fixed box size. The visualization of the resulting GW spectra with varying n_{bubbles} is shown in Fig. 8.18. We also fit these GW spectra to extract common features of the shape and amplitude, with the fitting parameters corresponding to the envelope approximation, $\xi = 0$ bulk-flow approximation, and $\xi = 1$ GBF approximation are visualized in Fig. 8.19. The numerical results of these fittings can be found in the Appendix, see Tables. D.5, D.6, and D.7.

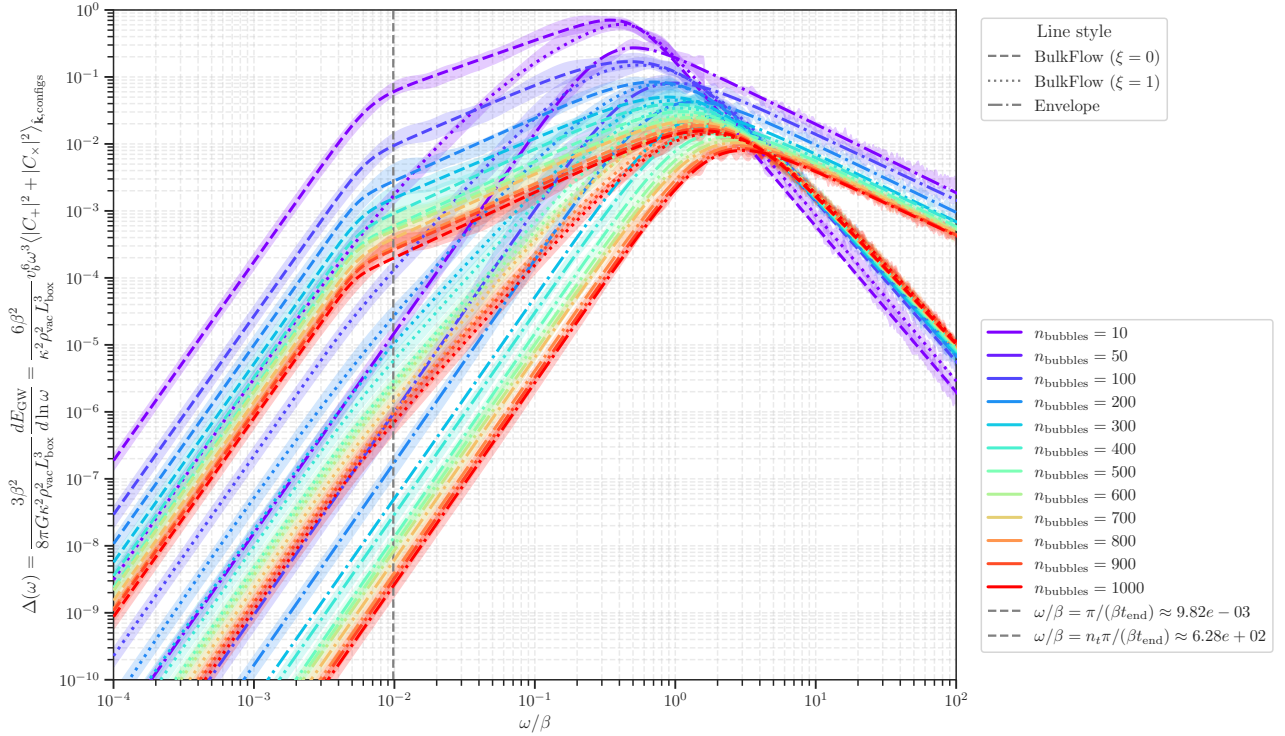


Figure 8.18: The spectrum obtained by simultaneously nucleating a fixed number of bubbles at $t = 0$. Note that the parameter β in this plot is merely a scale, chosen so that we plot the same quantity on the y-axis throughout the paper, but it does not play any role in the nucleation process. The error bands of the spectrum over nucleation histories for simultaneous nucleation are narrower compared with exponential nucleation, potentially because the latter introduces randomness not only in nucleation position but also in nucleation time.

Several observations can be drawn from the fitting results for simultaneous nucleation histories:

- Overall, the best-fit parameters of the generalized bulk-flow approximation ($\xi = 1$) have larger uncertainties compared with those of the envelope and bulk-flow ($\xi = 0$) approximations.
- The best-fit values of the three power laws p_{IR1} , p_{IR2} , and p_{UV} are relatively stable over N_{bubbles} . This means that with an appropriate choice of N_{bubbles} for simultaneous nucleation, we can generate the same spectrum as in exponential nucleation.
- As expected, $\hat{\omega}_1 \approx \pi/t_{\text{end}}$ and is insensitive to the number of bubbles. This is not the case for $\hat{\omega}_2$, the peak of the spectrum, which is of order $\mathcal{O}(1)$ and tends to shift toward the UV for larger N_{bubbles} .
- On the other hand, the peak amplitude decreases as N_{bubbles} increases, implying that larger bubbles at collision, and hence smaller bubble density, generate larger GW signals.

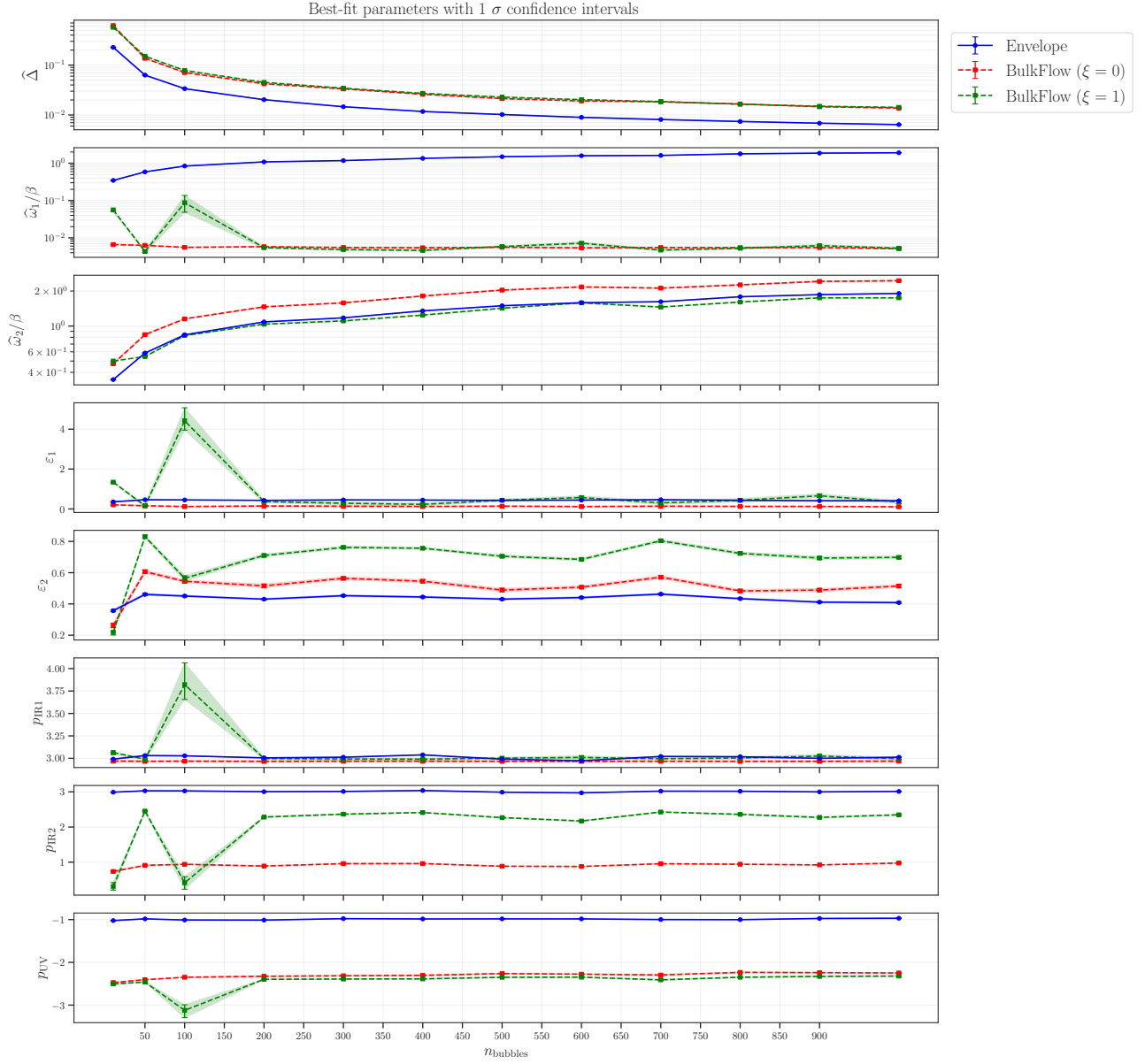


Figure 8.19: Best-fit parameters and the 1σ error band as a function of the total number of bubbles in each set of nucleation histories. This is a visualization of the results shown in Tables. [D.5](#), [D.6](#), and [D.7](#).

Summary of gravitational-wave part

In this part of the thesis, we constructed, verified, and applied a semi-analytic approximation called the *generalized bulk-flow* (GBF) approximation for the scalar-field bubble-collision contribution to a stochastic gravitational-wave background from first-order phase transitions. Assuming the separation of scales $R_c, L_w \ll R_* \lesssim H_*^{-1}$, we emphasized that the relaxation time t_{relax} is generally model-dependent rather than universal, unlike in the previously developed envelope and bulk-flow approximations. This motivates our study of the GBF approximation, which include the followings:

- First, we construct the GBF approximation which generalizes the geometric structure of envelope and bulk-flow calculations by allowing the post-collision scalar stress to be calibrated from the microscopic field dynamics of the bubble wall, resulting in more reliable predictions of GW spectra. Keeping only the leading large-distance corrections of the bulk-flow part corresponding to $\xi = 0$ and $\xi = 1$ gives the reduced surface-tension fit

$$\sigma(t - t_n, t_{n,c} - t_n) = \frac{\rho_{\text{vac}}}{3} \left[\Theta(t_{n,c} - t) (t - t_n) + \Theta(t - t_{n,c}) (t_{n,c} - t_n) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^2 \left(a_0 + a_1 \frac{t_{n,c} - t_n}{t - t_n} \right) \right],$$

with the envelope contribution quantified as $a_{\text{env}} = 1 - a_0 - a_1$. These equations isolate two pieces of physics: the model-dependent coefficients a_ξ describe microscopic post-collision relaxation, while the model-independent functions $C_{+, \times}^{(\xi)}$ capture the many-bubble geometry. This leads to the practical master formula

$$\frac{dE_{\text{GW}}}{d \ln \omega d\Omega_{\mathbf{k}}} = 4G\rho_{\text{vac}}^2 \omega^3 \sum_{i=+, \times} \left| a_{\text{env}} C_{\text{env}}^i + a_0 C_{\text{BF}}^i + a_1 C_1^i \right|^2.$$

- Second, we explicitly validated the GBF approximation for the two-bubble system, a step missing from the previous literature (see Section. 8.2). For this system, the exact GW spectrum is directly computed using field-evolution data and used as a benchmark for the various approximations, with the following specific models:

- Quartic potential for a real field:

$$V(\phi) = \frac{\bar{\lambda}}{9} \phi^2 - \frac{1}{3} \phi^3 + \frac{1}{4} \phi^4.$$

- Log potential for a complex scalar field:

$$\frac{V(\Phi)}{\rho_{\text{vac}}} = 1 + \kappa \frac{|\Phi|^2}{v^2} + \frac{|\Phi|^4}{v^4} \left[(\kappa + 2) \log \left(\frac{|\Phi|^2}{v^2} \right) - (\kappa + 1) \right].$$

We illustrated that, for both models, the GBF approximation provides excellent accuracy in comparison with the envelope and bulk-flow approximations.

- Finally, we applied the GBF to many-bubble systems with either exponential or simultaneous nucleation, and illustrated the extraction of general features of the resulting GW spectra for different choices of ξ by fitting with the template

$$\Delta(\omega) = \hat{\Delta} \left(\frac{\omega}{\hat{\omega}_2} \right)^{p_{\text{IR1}}} \left(\frac{1 + (\omega/\hat{\omega}_1)^{1/\varepsilon_1}}{1 + (\hat{\omega}_2/\hat{\omega}_1)^{1/\varepsilon_1}} \right)^{\varepsilon_1(p_{\text{IR2}} - p_{\text{IR1}})} \left(\frac{1 + (\omega/\hat{\omega}_2)^{1/\varepsilon_2}}{2} \right)^{\varepsilon_2(p_{\text{UV}} - p_{\text{IR2}})}$$

These features include the amplitude, scaling powers, and characteristic frequencies. The effects of the macroscopic box size on the predictions of the GBF were also studied numerically during this application, showing that with a sufficiently large box, $L_{\text{box}}\beta \gtrsim 10$, the resulting GW spectrum stabilizes.

The usefulness of the GBF is twofold. First, given that we need to compute the functions $C_{+, \times}^{(\xi)}$ only once, the cost of changing the underlying model amounts to recomputing the coefficients a_ξ by performing a relatively inexpensive 1 + 1D lattice simulation of a two-bubble system and fitting the surface tension along the bubble wall as a function of time. As a consequence, once the geometric contributions are computed and tabulated, computing the resulting GW spectrum is extremely fast in comparison with direct 3+1D field simulations. Second, it replaces the fixed post-collision prescriptions of those approximations by a small set of coefficients a_ξ that are calibrated from lower-dimensional two-bubble simulations and therefore carry information about the microscopic scalar-field dynamics. The comparisons carried out in this thesis show that this strategy gives a clear improvement over the envelope and pure bulk-flow limits in the examples studied, while also discussing the regimes where the GBF ansatz is not sufficient, such as when having trapping vacuum post-collision which oscillates and generates a significant contribution to the final GW spectrum.

Part III

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Extras on Q-ball in a single complex scalar field

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A.1 Virial theorem and characteristic slope

The virial theorem gives a useful global constraint between gradient and potential functionals of a stationary profile. Recall that the charge and gradient energy of a Q-ball solution are

$$Q_Q = \omega_Q \int d^D x \phi_Q^2, \quad E_{\text{grad}} = \int d^D x \frac{1}{2} (\nabla \phi_Q)^2. \quad (\text{A.1.1})$$

Let the stationary profile be rescaled as $\phi_{Q,\xi}(r) = \phi_Q(\xi r)$ while keeping ω_Q fixed. The reduced action at fixed ω_Q ,

$$S_{\omega_Q}[\phi_Q] = \Omega_D \int_0^\infty dr r^{D-1} \left[\frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 + U_{\text{eff}}(\phi_Q) \right] \quad (\text{A.1.2})$$

then transforms as

$$S_{\omega_Q}[\phi_{Q,\xi}] = \xi^{2-D} S_{\text{grad}}[\phi_Q] + \xi^{-D} S_{\text{pot}}^{\text{eff}}[\phi_Q],$$

$$\text{where we define } \begin{cases} S_{\text{grad}}[\phi_Q] \equiv \Omega_D \int_0^\infty dr r^{D-1} \frac{1}{2} \left(\frac{d\phi_Q}{dr} \right)^2 \\ S_{\text{pot}}^{\text{eff}}[\phi_Q] \equiv \Omega_D \int_0^\infty dr r^{D-1} U_{\text{eff}} \\ S_{\text{pot}}[\phi_Q] \equiv \Omega_D \int_0^\infty dr r^{D-1} U \end{cases} \quad (\text{A.1.3})$$

Extremizing at $\xi = 1$ gives

$$(D-2)S_{\text{grad}}[\phi_Q] + D S_{\text{pot}}^{\text{eff}}[\phi_Q] = 0. \quad (\text{A.1.4})$$

This is the fixed- ω_Q virial relation. Equivalently, using $S_{\text{pot}}^{\text{eff}}[\phi_Q] = S_{\text{pot}}[\phi_Q] - \omega_Q Q_Q/2$, it can be written in the form

$$D S_{\text{pot}}[\phi_Q] = (2-D)S_{\text{grad}}[\phi_Q] + \frac{D}{2} \omega_Q Q_Q. \quad (\text{A.1.5})$$

The charge term is what allows a time-dependent Q-ball to evade Derrick's theorem. In the static limit $Q_Q \rightarrow 0$, Eq. (A.1.5) becomes

$$DS_{\text{pot}}[\phi_Q] = (2 - D)S_{\text{grad}}[\phi_Q]. \quad (\text{A.1.6})$$

For $D \geq 2$ and a potential normalized so that $U(\phi_Q) \geq 0$ relative to the vacuum, the two sides cannot both be positive unless the profile is trivial. Thus an ordinary static scalar lump with finite energy is excluded in $D \geq 2$ under these assumptions. The Q-ball avoids this conclusion because the internal rotation contributes the positive term $(D/2)\omega_Q Q_Q$ to the virial balance.

The same relation gives a compact expression for the Q-ball energy:

$$E_Q = \omega_Q Q_Q + S_{\omega_Q}[\phi_Q] = \omega_Q Q_Q + \frac{2}{D}S_{\text{grad}}[\phi_Q]. \quad (\text{A.1.7})$$

This energy identity motivates the characteristic slope used in the stationary-property literature. Following the discussion in Ref. [9], we define

$$\gamma_Q(\omega_Q) \equiv \frac{E_Q}{\omega_Q Q_Q}. \quad (\text{A.1.8})$$

Using Eq. (A.1.7), this can be written as

$$\gamma_Q = 1 + \frac{2S_{\text{grad}}[\phi_Q]}{D\omega_Q Q_Q} = 1 + \left[D - 2 + D \frac{S_{\text{pot}}}{S_{\text{grad}}} \right]^{-1}. \quad (\text{A.1.9})$$

As both S_{grad} and S_{pot} are non-negative quantities, for $D \geq 2$ we always have $\gamma_Q \geq 1$. With $D = 1$, this is not the case: $\gamma_Q \geq 1$ only when $S_{\text{pot}} \geq S_{\text{grad}}$. An example where the equality $\gamma_Q = 1$ holds is the ideal large-charge thin-wall limit, where the gradient energy is localized near the wall and is subleading compared with the rotational charge energy $\omega_Q Q_Q$. Whenever the wall size is not negligible, the energy contains an additional positive contribution that is not accompanied by a proportional increase in the Noether charge. The departure of γ_Q from unity therefore directly measures the relative size of the gradient, or surface, energy compared with $\omega_Q Q_Q$. Finally, if γ_Q is approximately constant, the chemical-potential relation $dE_Q/dQ_Q = \omega_Q$ implies the power-law scaling

$$E_Q \propto Q_Q^{1/\gamma_Q}. \quad (\text{A.1.10})$$

As we will examine later in Chapter. 3, there are many examples where γ_Q is indeed insensitive to ω_Q , so (A.1.10) is a reliable way to estimate Q-ball energy once the total charge is known. For a more detailed discussion of the virial relation as a tool for understanding the energy landscape of Q-balls, see Ref. [114]. There the generalized virial identity is used not only as a consistency condition, but also as a way to organize approximate analytic estimates of Q-ball energies in arbitrary spatial dimensions. These results are useful background for interpreting γ_Q and the virial balance as more than formal identities: they provide diagnostics for how the Q-ball energy changes with profile size, charge, and dimensionality.

A.2 Four kinds of stability conditions

The existence conditions derived above answer only whether the radial profile equation admits a localized stationary solution. They do not guarantee that this solution is the physically relevant state at fixed charge, nor that it is stable against perturbations or decay channels. It is therefore useful to separate several notions of stability. Absolute stability asks whether the lump is energetically favored over free scalar quanta carrying the same charge. Classical stability is about whether the branch has an unstable fluctuation mode under small deformations at fixed charge. Fission stability then considers whether one Q-ball can lower its energy by splitting into smaller Q-balls. Finally, if the scalar sector

is coupled to additional light fermions, surface evaporation gives a separate decay channel that is not captured by the purely bosonic profile equation [115, 116, 117]. In the numerical results presented in Chapter. 3, we mainly illustrate tests for absolute stability and classical stability; the fermion criterion is quoted only as a standard limitation of the purely scalar analysis.

Criterion	Physical question	Diagnostic
Existence	Does a localized solution exist?	$\omega_Q^{(-)} \leq \omega_Q < \omega_Q^{(+)}$
Absolute stability	Can it decay into free quanta?	$E_Q/Q_Q < m_\Phi$
Classical stability	Is the profile stable to small deformations?	$dQ_Q/d\omega_Q < 0$ for $\omega_Q > 0$
Fission stability	Can one Q-ball split into smaller Q-balls?	$d^2 E_Q/dQ_Q^2 < 0$
Fermion evaporation	Can a Q-ball evaporate by emitting fermions?	$dQ_Q/(dt dA) \leq \omega_Q^3/(192\pi^2)$

Table A.1: Summary of the stability notions used in the Q-ball literature and their role in this chapter.

- **Absolute stability:** this is the requirement that the Q-ball energy be smaller than the energy of Q_Q free scalar quanta at rest [6, 15, 10, 11, 118],

$$E_Q(\omega_Q) < E_{\text{free}} \quad \Longleftrightarrow \quad \frac{E_Q}{Q_Q} < m_\Phi. \quad (\text{A.2.1})$$

This criterion states that it is energetically favorable for the charge to be stored in a coherent Q-ball rather than in isolated quanta. Failing this inequality means that decay into free particles is energetically allowed, so the object is at best metastable unless an additional obstruction suppresses the decay.

- **Classical stability:** a Q-ball can be classically stable if small fluctuations around the solution oscillate and therefore remain small over time. The criterion for ordinary global Q-balls can be written as

$$\frac{\omega_Q}{Q_Q} \frac{dQ_Q}{d\omega_Q} \leq 0 \quad \Longleftrightarrow \quad \frac{d^2 S_{\omega_Q}}{d\omega_Q^2} \geq 0, \quad (\text{A.2.2})$$

see, e.g., [5, 7, 15, 9, 118]. For the positive ω_Q branch used in this chapter, this reduces to $dQ_Q/d\omega_Q \leq 0$. Using the Legendre relations, the same condition can also be expressed as

$$\frac{d}{d\omega_Q} \left(\frac{E_Q}{Q_Q} \right) = -\frac{S_{\omega_Q}}{Q_Q^2} \frac{dQ_Q}{d\omega_Q} \geq 0, \quad (\text{A.2.3})$$

provided $S_{\omega_Q} > 0$ on the branch under consideration. Derivations based on the fixed-charge energy functional are given in Refs. [5, 7]. Another related analysis for one- and two-field Q-balls is discussed in Refs. [119, 120]; those works also emphasize that the ordinary global-Q-ball criterion should not be applied blindly to gauged Q-balls.

- **Stability against fission:** a Q-ball is stable against fission if it is energetically unfavorable for one soliton of charge $Q_1 + Q_2$ to split into two separated solitons of charges Q_1 and Q_2 . The discussion of ground-state stability in Ref. [118] makes explicit why, for an ordinary global Q-ball branch, this condition follows from the same sign that appears in the classical stability criterion. Using the chemical-potential relation in Eq. (2.1.11),

$$\frac{dE_Q}{dQ_Q} = \omega_Q, \quad (\text{A.2.4})$$

one obtains

$$\frac{d^2 E_Q}{dQ_Q^2} = \frac{d\omega_Q}{dQ_Q}. \quad (\text{A.2.5})$$

Hence, on the positive-frequency branch, the classical stability condition $dQ_Q/d\omega_Q < 0$ implies

$$\frac{d^2 E_Q}{dQ_Q^2} < 0. \quad (\text{A.2.6})$$

The energy is then a concave function of the charge along the stable branch. Integrating $d^2 E_Q/dQ_Q^2 < 0$ from an intermediate charge Q_i to $Q_i + Q_2$ gives

$$\left. \frac{dE_Q}{dQ_Q} \right|_{Q_i+Q_2} < \left. \frac{dE_Q}{dQ_Q} \right|_{Q_i}. \quad (\text{A.2.7})$$

A second integration over Q_i from 0 to Q_1 , together with $E_Q(0) = 0$, yields

$$E_Q(Q_1 + Q_2) < E_Q(Q_1) + E_Q(Q_2). \quad (\text{A.2.8})$$

Therefore the combined soliton has lower energy than two separated solitons with the same total charge, and fission into smaller Q-balls is energetically forbidden. This fission statement is conceptually distinct from classical stability: classical stability concerns infinitesimal field perturbations around one solution, while fission stability compares the energy of different charge partitions. If the final state also contains free quanta, the absolute-stability condition discussed above must also be imposed, since it is more general than stability against fission.

- **Stability against fermion evaporation:** if the Q-ball field has Yukawa couplings to light fermions, the conserved scalar charge can be depleted by particle production at the surface. This is a genuinely different issue from the scalar-only stability tests above. The profile may be stable as a solution of the bosonic field equations, but once an interaction such as $y\Phi\psi\psi + \text{h.c.}$ is present, the time-dependent Q-ball background can create fermions. Here y denotes a Yukawa coupling and ψ denotes a light fermion field; the symbol is chosen to avoid confusion with the sextic self-coupling used later. Cohen, Coleman, Georgi and Manohar showed that for large thin-wall Q-matter decaying into effectively massless fermions the outgoing flux saturates because the final-state fermion phase space near the surface is filled by Pauli blocking [115]. This then gives the standard surface evaporation bound

$$\frac{dQ_Q}{dt dA} \leq \frac{\omega_Q^3}{192\pi^2}. \quad (\text{A.2.9})$$

Here dA denotes an infinitesimal surface-area element of the Q-ball, and dQ_Q/dt is the charge-loss rate through fermion emission. The role of this criterion is therefore to flag an additional model-dependent lifetime constraint that must be imposed when, e.g., the Q-ball is embedded in a theory with light fermions.

As an example illustrating the stability criteria for Q-balls, we consider the sextic potential studied in Chapter. 3,

$$\mathbf{U} = |\Phi|^2 - |\Phi|^4 + g|\Phi|^6. \quad (\text{A.2.10})$$

We perform a scan over different internal frequencies ω_Q . Fig. A.1 summarizes which of these numerical profiles are useful as stable backgrounds for the scattering problem. The upper two rows show that E_Q and Q_Q grow rapidly as ω_Q is lowered toward the thin-wall limit, as expected from the large-radius behavior. The green points indicate profiles that pass the convergence, absolute-stability, and classical-stability checks. ⁽¹⁾ The yellow points fail the absolute-stability condition $E_Q/Q_Q < m_\Phi$, while the red markers identify the region where the classical branch criterion $dQ_Q/d\omega_Q < 0$ is violated. In $D = 2$, the acceptable branch extends close to the upper bound and the failures are concentrated near $\omega_Q \simeq 1$. In $D = 3$, the absolute-stability failure sets in over a larger portion of the high-frequency side,

⁽¹⁾ Here, by convergence test, we check the value of the field profile at the maximum radius set by the solver, which ensures that this radius is sufficiently large to cover a significant part of the tail.

and the classical-instability markers appear inside this endpoint region. This difference is important for the rest of the chapter: the scattering backgrounds must be chosen from the green part of the branch, not merely from the formal existence interval.

The third row plots the characteristic slope $\gamma_Q = E_Q/(\omega_Q Q_Q)$. Near the thin-wall side, γ_Q is close to unity, reflecting that the relation between total charge and energy is approximately $E_Q \approx \omega_Q Q_Q$, with the wall contribution subleading. Moving away from the thin-wall limit, γ_Q departs from one because gradient and finite-size effects contribute a larger fraction of the total energy. Finally, the bottom row checks the chemical-potential relation derived in Eq. (2.1.11). The numerical derivative $dE_Q/d\omega_Q$ tracks $\omega_Q dQ_Q/d\omega_Q$ along the smooth part of the branch with high precision for all values of ω_Q .

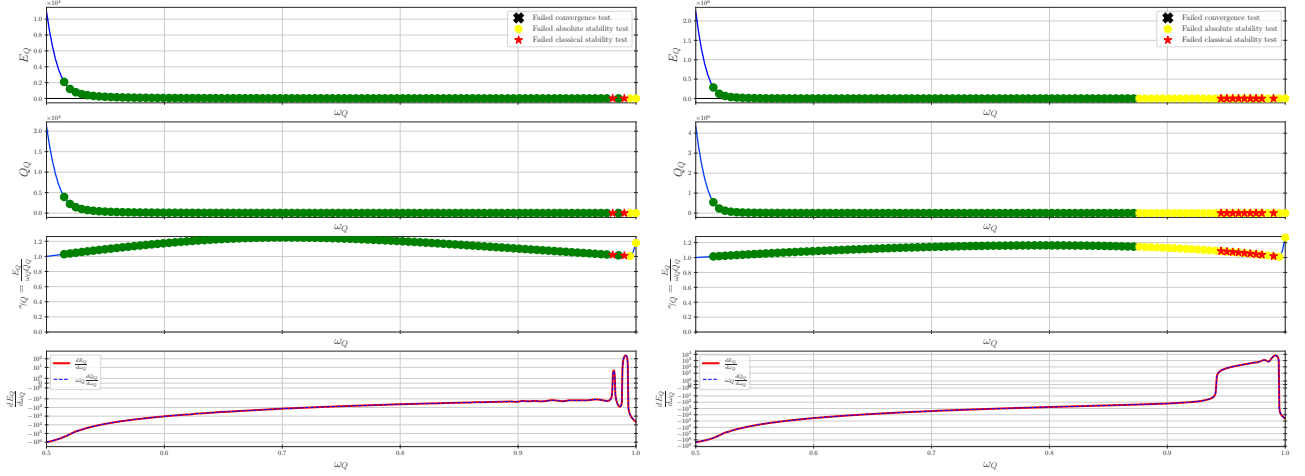


Figure A.1: Here we show the changes of total energy, total $U(1)$ charge and γ_Q as a function of ω_Q on the top three rows, with different colors indicating different tests failed, and green means all tests are passed. In the bottom plot, we also show numerically the chemical potential relation in Eq. (2.1.11).

Numerical framework for Q-ball scattering simulation

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B.1 Evolution of a scalar profile in time domain

B.1.1 The equations and boundary conditions

Equation of motion: Here we aim to solve the full dynamics of the scalar field in D spatial dimensions, i.e., the Euler–Lagrange equation (1.2.2)

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi + \frac{\partial U}{\partial \Phi^*} = 0. \quad (\text{B.1.1})$$

Given $U(\Phi) = |\Phi|^2 - |\Phi|^4 + g|\Phi|^6$ and assuming spherical symmetry, the above equation of motion can be decomposed into the following system of ODEs

$$\begin{cases} \frac{\partial \Phi_R}{\partial t} = \Pi_R \end{cases} \quad (\text{B.1.2a})$$

$$\begin{cases} \frac{\partial \Phi_I}{\partial t} = \Pi_I \end{cases} \quad (\text{B.1.2b})$$

$$\begin{cases} \frac{\partial \Pi_R}{\partial t} = \left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right) \Phi_R - \Phi_R (1 - 2|\Phi|^2 + 3g|\Phi|^4) \end{cases} \quad (\text{B.1.2c})$$

$$\begin{cases} \frac{\partial \Pi_I}{\partial t} = \left(\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right) \Phi_I - \Phi_I (1 - 2|\Phi|^2 + 3g|\Phi|^4) \end{cases} \quad (\text{B.1.2d})$$

where $\Phi = \Phi_R + i\Phi_I$.

Initial conditions: In order to obtain a solution to the wave equation, it is required to supplement the two initial conditions $\Phi(r, t = 0) = \Phi_0(r)$ and $\frac{\partial \Phi}{\partial t}(r, t = 0)$. For example, to illustrate the

scattering of a monochromatic wavepacket off a Q-ball configuration, we impose the following initial conditions

$$\begin{cases} \Phi(0, r) = \Phi_Q(0, r) + \delta e^{-\frac{(r-r_0)^2}{2\sigma_r^2}} e^{-is\omega_0\sqrt{\omega_0^2-1}r} \\ \partial_t \Phi(0, r) = -i[\omega_Q \Phi_Q(0, r) + \omega_0(\Phi(0, r) - \Phi_Q(0, r))] \end{cases} \quad (\text{B.1.3a})$$

$$\quad (\text{B.1.3b})$$

Boundary conditions: Because we want to simulate the system in an infinitely large spatial domain, it is reasonable to implement the so-called absorbing boundary conditions (a.k.a. radiation boundary conditions, Sommerfeld boundary conditions, or open boundary conditions). This BC enables a wave to leave the computational domain and travel undisturbed through the computational boundary. Since the current problem is one-dimensional with respect to the radial coordinate, we impose the following condition at the right boundary $r = r_\infty$

$$\lim_{r \rightarrow \infty} r^{\frac{D-1}{2}} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right)^\# \Phi(t, r) \Big|_{r=r_\infty} = 0, \quad (\text{B.1.4})$$

where $\#$ is an arbitrary positive integer which is chosen depending on the finite difference scheme. In the current code, we implement $\# = 2$. For the left boundary, we have two options:

- **Positive r domain:** with $D \geq 2$, regularity at the origin requires the Neumann boundary condition

$$\frac{\partial \Phi}{\partial r} \Big|_{r=0^+} = 0. \quad (\text{B.1.5})$$

In practice, we implement the condition at $r = r_\epsilon$ such that $0 < r_\epsilon \ll 1$ to avoid the division by zero.

- **Real r domain:** this obviously makes sense in $D = 1$. Since the equation of motion (1.2.2) is invariant under reflection $r \rightarrow -r$, it is possible to extend the computational domain from $r \in [r_\epsilon, r_\infty]$ to $r \in [-r_\infty, r_\infty]$ and enforce one more absorbing boundary condition on the left,

$$\lim_{r \rightarrow \infty} r^{\frac{D-1}{2}} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r} \right)^\# \Phi(t, r) \Big|_{r=-r_\infty} = 0. \quad (\text{B.1.6})$$

Note that for $D \geq 2$, the Neumann BC at the origin is still required. Nonetheless, if the initial conditions are well-defined (i.e., satisfy the Neumann BC), the evolution of such a profile would also satisfy the Neumann BC automatically. For the current problem, we require spherical symmetry, so this second method calculates twice the number of “meaningful points” because of the reflection symmetry. However, if we do not have spherical symmetry (e.g., scattering of multiple Q-balls, or the perturbation itself is not spherically symmetric), this method can be easily generalized to arbitrary dimension $D \geq 2$.

B.1.2 Finite difference discretization

Here we describe how to use fourth-order finite differences to discretize space and a fourth-order Runge–Kutta method to evolve the system numerically in time. We start by discretizing the space and time domain:

$$\begin{cases} r \in [r_i, r_f] \rightarrow r_j, & \text{with } j \in [0, N_r] \\ t \in [0, t_f] \rightarrow t_n, & \text{with } n \in [0, N_t] \end{cases} \quad (\text{B.1.7})$$

Because we need to solve a second-order PDE with respect to r and t and want to use a Runge–Kutta method to integrate over t , we let $\pi = \Phi_t(r, t)$ and derive a system of equations for each spatial point that is first-order in time by discretizing the following PDEs:

$$\Phi_t(r, t) = \Pi(r, t), \quad \Pi_t(r, t) = \Phi_{rr}(r, t) + \frac{D-1}{r} \Phi_r(r, t) - \frac{\partial U}{\partial \Phi^*}(r, t). \quad (\text{B.1.8})$$

- **Finite difference of the PDE:** First, we use central finite difference to discretize the spatial derivatives

$$\Phi_{rr}(r, t) = -\frac{\Phi(r + 2\Delta r, t) - 16\Phi(r + \Delta r, t) + 30\Phi(r, t) - 16\Phi(r - \Delta r, t) + \Phi(r - 2\Delta r, t)}{12(\Delta r)^2} + \mathcal{O}(\Delta r^4), \quad (\text{B.1.9})$$

$$\frac{1}{r}\Phi_r(r, t) = -\frac{1}{r}\frac{\Phi(r + 2\Delta r, t) - 8\Phi(r + \Delta r, t) + 8\Phi(r - \Delta r, t) - \Phi(r - 2\Delta r, t)}{12(\Delta r)^2} + \mathcal{O}(\Delta r^4). \quad (\text{B.1.10})$$

Substituting this back to (B.1.8) gives the ODE for the collection of “middle points”

$$\Pi_t(r, t) = \sum_{s=-2}^2 C_s^M \Phi(r + s\Delta r, t) - \frac{\partial U}{\partial \Phi^*}(r, t), \quad (\text{B.1.11})$$

or for simplicity, let us denote the time step by n and spatial step by i

$$(\Pi_t)_j^n = \sum_{s=-2}^2 C_s^M \Phi_{j+s}^n - \frac{\partial U}{\partial (\Phi^*)_j^n}, \quad (\text{B.1.12})$$

where Φ_i^n is the approximation of $\Phi(r, t)$ at $r = r_i + j\Delta r$ and $t = n\Delta t$, and

$$\left\{ \begin{array}{l} C_{-2}^M = \frac{(D-1)\Delta r - r}{12(\Delta r)^2} \\ C_{-1}^M = -\frac{8((D-1)\Delta r - 2r)}{12(\Delta r)^2} \\ C_0^M = -\frac{5}{2\Delta r^2} \\ C_1^M = \frac{8((D-1)\Delta r + 2r)}{12(\Delta r)^2} \\ C_2^M = -\frac{(D-1)\Delta r + r}{12(\Delta r)^2} \end{array} \right. \quad (\text{B.1.13})$$

- **Finite difference equations of points near the boundary** Since the fourth-order finite-difference stencil described above requires two neighboring points on both the left and right, we need another way to write the equation for points near the spatial boundaries. Since the boundary points are treated differently because of the Neumann and/or absorbing boundary conditions, we now consider the points next to the left or right boundaries, i.e., Φ_1^n and $\Phi_{N_r}^n$. Let us begin with Φ_1^n ; it requires 8 points to achieve the correction up to fourth order,

$$\begin{aligned} \Phi_{rr}(r, t) = \frac{1}{12(\Delta r)^2} & \left[10\Phi(r - \Delta r, t) - 15\Phi(r, t) - 4\Phi(r + \Delta r, t) \right. \\ & \left. + 14\Phi(r + 2\Delta r, t) - 6\Phi(r + 3\Delta r, t) + \Phi(r + 4\Delta r, t) \right] \\ & + \mathcal{O}(\Delta r^4), \end{aligned} \quad (\text{B.1.14})$$

$$\begin{aligned} \frac{1}{r}\Phi_r(r, t) = -\frac{1}{12r\Delta r} & \left[3\Phi(r - \Delta r, t) + 10\Phi(r, t) - 18\Phi(r + \Delta r, t) \right. \\ & \left. + 6\Phi(r + 2\Delta r, t) - \Phi(r + 3\Delta r, t) \right] + \mathcal{O}(\Delta r^4), \end{aligned} \quad (\text{B.1.15})$$

together with (B.1.8) gives

$$\Pi_t(r, t) = \sum_{s=-1}^4 C_s^L \Phi(r - s\Delta r, t) - \frac{\partial U}{\partial \Phi^*}(r, t). \quad (\text{B.1.16})$$

Applying this to the next-to-left boundary point $j = 1$ yields

$$(\Pi_t)_1^n = \sum_{s=-1}^4 C_s^L \Phi_{1+s}^n - \frac{\partial U}{\partial(\Phi^*)_1^n}, \quad (\text{B.1.17})$$

with

$$\left\{ \begin{array}{l} C_{-1}^L = \frac{-3(D-1)\Delta r + 10r}{12r\Delta r^2} \\ C_0^L = \frac{-5(2(D-1)\Delta r + 3r)}{12r\Delta r^2} \\ C_1^L = \frac{2(9(D-1)\Delta r - 2r)}{12r\Delta r^2} \\ C_2^L = \frac{-2(3(D-1)\Delta r - 7r)}{12r\Delta r^2} \\ C_3^L = \frac{(D-1)\Delta r - 6r}{12r\Delta r^2} \\ C_4^L = \frac{1}{12\Delta r^2} \end{array} \right. . \quad (\text{B.1.18})$$

Similarly, we can derive the following finite difference relation

$$\Phi_{rr}(r, t) = \frac{\Phi(r-4\Delta r, t) - 6\Phi(r-3\Delta r, t) + 14\Phi(r-2\Delta r, t) - 4\Phi(r-\Delta r, t) - 15\Phi(r, t) + 10\Phi(r+\Delta r, t)}{12(\Delta r)^2} + \mathcal{O}(\Delta r^4), \quad (\text{B.1.19})$$

$$\frac{1}{r}\Phi_r(r, t) = -\frac{\Phi(r-3\Delta r, t) - 6\Phi(r-2\Delta r, t) + 18\Phi(r-\Delta r, t) - 10\Phi(r, t) - \Phi(r+\Delta r, t)}{r\Delta r} + \mathcal{O}(\Delta r^4). \quad (\text{B.1.20})$$

Substituting these expansions back to the Klein-Gordon equation gives

$$\Pi_t(r, t) = \sum_{s=-4}^1 C_s^R \Phi(r+s\Delta r, t) - \frac{\partial U}{\partial(\Phi^*)_1^n}(r, t). \quad (\text{B.1.21})$$

By applying this to the next-to-right boundary point $i = N_r - 1$ we have

$$(\Pi_t)_{N_r-1}^n = \sum_{s=-4}^1 C_s^R \Phi_{N_r-1+s}^n - \frac{\partial U}{\partial(\Phi^*)_1^n}, \quad (\text{B.1.22})$$

with

$$\left\{ \begin{array}{l} C_{-4}^R = \frac{1}{12\Delta r^2} \\ C_{-3}^R = -\frac{(D-1)\Delta r + 6r}{12r\Delta r^2} \\ C_{-2}^R = \frac{2(3(D-1)\Delta r + 7r)}{12r\Delta r^2} \\ C_{-1}^R = -\frac{2(9(D-1)\Delta r + 2r)}{12r\Delta r^2} \\ C_0^R = \frac{5(2(D-1)\Delta r - 3r)}{12r\Delta r^2} \\ C_1^R = \frac{3(D-1)\Delta r + 10r}{12r\Delta r^2} \end{array} \right. . \quad (\text{B.1.23})$$

- **Finite difference equations for boundary points:** Let us first consider applying the absorbing boundary conditions for both left and right boundary points. Recall from (B.1.4) and (B.1.6), we need to impose

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r}\right)^2 \Phi(t, r) = \Pi_t(t, r) \pm 2\Pi_r(t, r) + \Phi_{rr}(t, r) = 0, \quad (\text{B.1.24})$$

where the plus (minus) sign corresponds to the right (left) boundary. We start with the finite difference for the left boundary

$$\Pi_r(r, t) = \frac{1}{2} \sum_{s=0}^4 B_s^\Pi \Pi(r + s\Delta r, t) + \mathcal{O}(\Delta r^4), \quad (\text{B.1.25})$$

$$\Phi_{rr}(r, t) = - \sum_{s=0}^5 B_s^\Phi \Phi(r + s\Delta r, t) + \mathcal{O}(\Delta r^4). \quad (\text{B.1.26})$$

with

$$B_{s=1\dots 4}^\Pi = -\frac{1}{6\Delta r} (25, -48, 36, -16, 3), \quad (\text{B.1.27})$$

$$B_{s=1\dots 5}^\Phi = -\frac{1}{12\Delta r^2} (45, -154, 214, -156, 61, -10) \quad (\text{B.1.28})$$

Plugging these back to $(\partial_t - \partial_r)^2 \Phi = 0$ calculated at $j = 0$ yields

$$(\Pi_t)_0^n = \sum_{s=0}^4 B_s^\Pi \Pi_s^n + \sum_{s=0}^5 B_s^\Phi \Phi_s^n. \quad (\text{B.1.29})$$

Notice that we may consider the “positive r -domain,” with the Neumann boundary conditions constraining the behavior of $\Phi(r, t)$ near the origin, instead of extending the calculation domain to the negative side and applying the absorbing boundary condition there. This implementation has unresolved issues and only works for $D = 1$ (which is trivial, as this is the only case where the term leading to the divergence $\propto r^{-1}\Phi_r$ does not contribute); for dimensions larger than 1, we obtain immediate divergence of the field profile. Nevertheless, as noted above, the equation of motion has reflection symmetry: once the initial condition is correctly imposed, the Neumann boundary condition is automatically simulated during the evolution of the scalar field. We can therefore use a trick by extending the “positive r -domain” with ghost points at negative r , where the value of Φ is taken from the calculated points at positive r by reflection. In the end, we succeeded in reducing the number of calculated points by half while maintaining the correctness of the numerical simulation.

One can similarly obtain the finite difference equation for the right absorbing boundary condition, whose coefficients can be recycled from B_s^Π and B_s^Φ

$$(\Pi_t)_0^n = \sum_{s=0}^4 B_s^\Pi \Pi_{N_r-s}^n + \sum_{s=0}^5 B_s^\Phi \Phi_s^n. \quad (\text{B.1.30})$$

Up to this point, we have derived the ODE that is first-order in time for all points at a fixed time step. Collectively, from (B.1.8), (B.1.12), (B.1.17), (B.1.22), (B.1.29), and (B.1.30), we can write

$$\frac{d}{dt} \begin{pmatrix} \Phi_i^n \\ \Pi_i^n \end{pmatrix} = F((\Phi^n, \Pi^n)). \quad (\text{B.1.31})$$

This system can be easily integrated over time using the fourth-order Runge–Kutta method. Specifically,

$$\begin{pmatrix} \Phi_i^{n+1} \\ \Pi_i^{n+1} \end{pmatrix} = \begin{pmatrix} \Phi_i^n \\ \Pi_i^n \end{pmatrix} + \frac{\Delta t}{6} (K_1 + 2K_2 + 2K_3 + K_4), \quad (\text{B.1.32})$$

with

$$\begin{cases} K_1 &= F((\Phi^n, \Pi^n)) \\ K_2 &= F\left((\Phi^n, \Pi^n) + \frac{\Delta t}{2} K_1\right) \\ K_3 &= F\left((\Phi^n, \Pi^n) + \frac{\Delta t}{2} K_2\right) \\ K_4 &= F((\Phi^n, \Pi^n) + \Delta t K_3) \end{cases} \quad (\text{B.1.33})$$

B.2 Validation of the numerical framework

B.2.1 Check code validity by solving linear wave equation $\Phi_{tt} - \Phi_{xx} = 0$

As a simple check, we consider $D = 1$ and turn off the potential U , and hence the nonlinearity, leaving a 1D wave equation. We consider the time evolution with a Gaussian initial condition

$$\phi(x, t = 0) = \phi_0 e^{-\frac{(x-x_0)^2}{2\sigma^2}}, \quad \frac{d\phi}{dt}(x, t = 0) = \frac{v_0 \phi_0}{\sigma^2} e^{-\frac{(x-x_0)^2}{2\sigma^2}},$$

with $v_0 = c = 1$. In the following examples, there are two solutions based on the boundary conditions:

- Solution with Dirichlet boundary condition where the boundary points are kept constant. The waves propagate toward the boundary and bounce back.
- Solution with absorbing (a.k.a. open or Sommerfeld) boundary condition. The waves keep propagating outside the boundary.

The solution is found using either the forward-time central-space (FTCS) method or fourth-order Runge–Kutta, which gives a sensible result depicting a right-moving Gaussian wave packet.

B.2.2 Initial Q-ball profile with $U(\Phi) = |\Phi|^2 - |\Phi|^4 + g|\Phi|^6$

Next, we test the time evolution with an initial Q-ball configuration,

$$\phi(x, t = 0) = \phi_Q(x), \quad \frac{d\phi}{dt}(x, t = 0) = i\omega_Q \phi_Q(x). \quad (\text{B.2.1})$$

We expect that a “good” Q-ball profile solution is stable over time.

To test the stability of the Q-ball configuration, we consider the following scaled initial conditions, which have the same charge Q as the given Q-ball,

$$\Phi(0, r) = \alpha^{-D/2} \Phi_Q(0, \alpha r), \quad \partial_t \Phi(0, r) = -i\alpha^{-D/2} \omega_Q \Phi_Q(0, \alpha r). \quad (\text{B.2.2})$$

We expect that with $\alpha = 1$ (i.e., the exact Q-ball initial conditions), the configuration is stable. For other values, a conclusion similar to the initial Gaussian case can be drawn: if α is close to but not exactly 1, we observe oscillatory behavior. If α is close to 0 or much larger than 1, the initial configuration decays to plane waves and/or smaller Q-balls.

B.2.3 Energy and charge conservation

Since the underlying theory conserves both energy and $U(1)_Q$ charge, we expect the numerical results also reflect these symmetries. Indeed, Fig. B.4 shows that the numerical implementation satisfies the conservation laws.

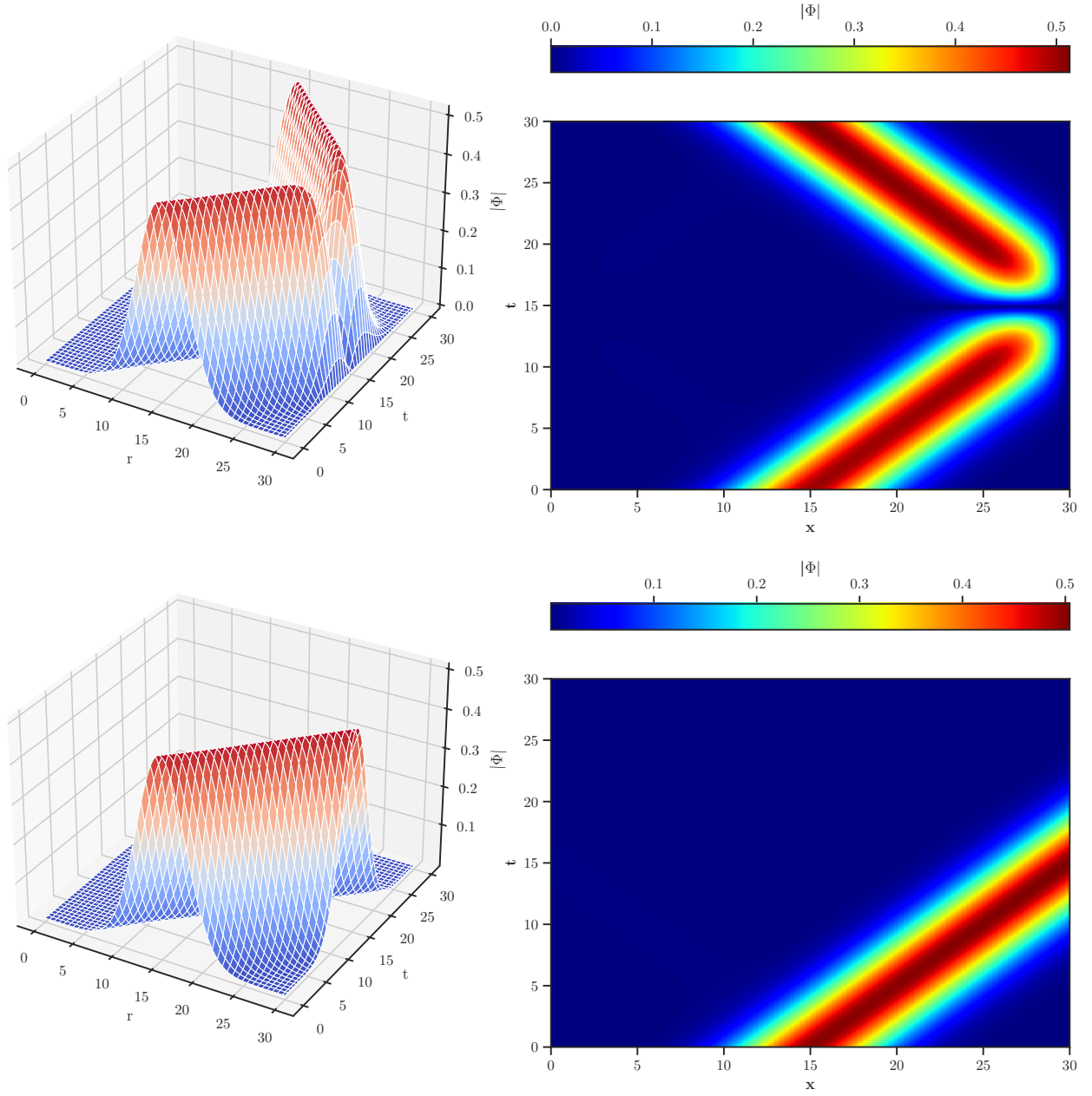


Figure B.1: At the top is the solution with Dirichlet boundary conditions, where the incoming wave bounces off the right wall. At the bottom, with the absorbing boundary condition implemented, the wave propagates outside the calculation domain.

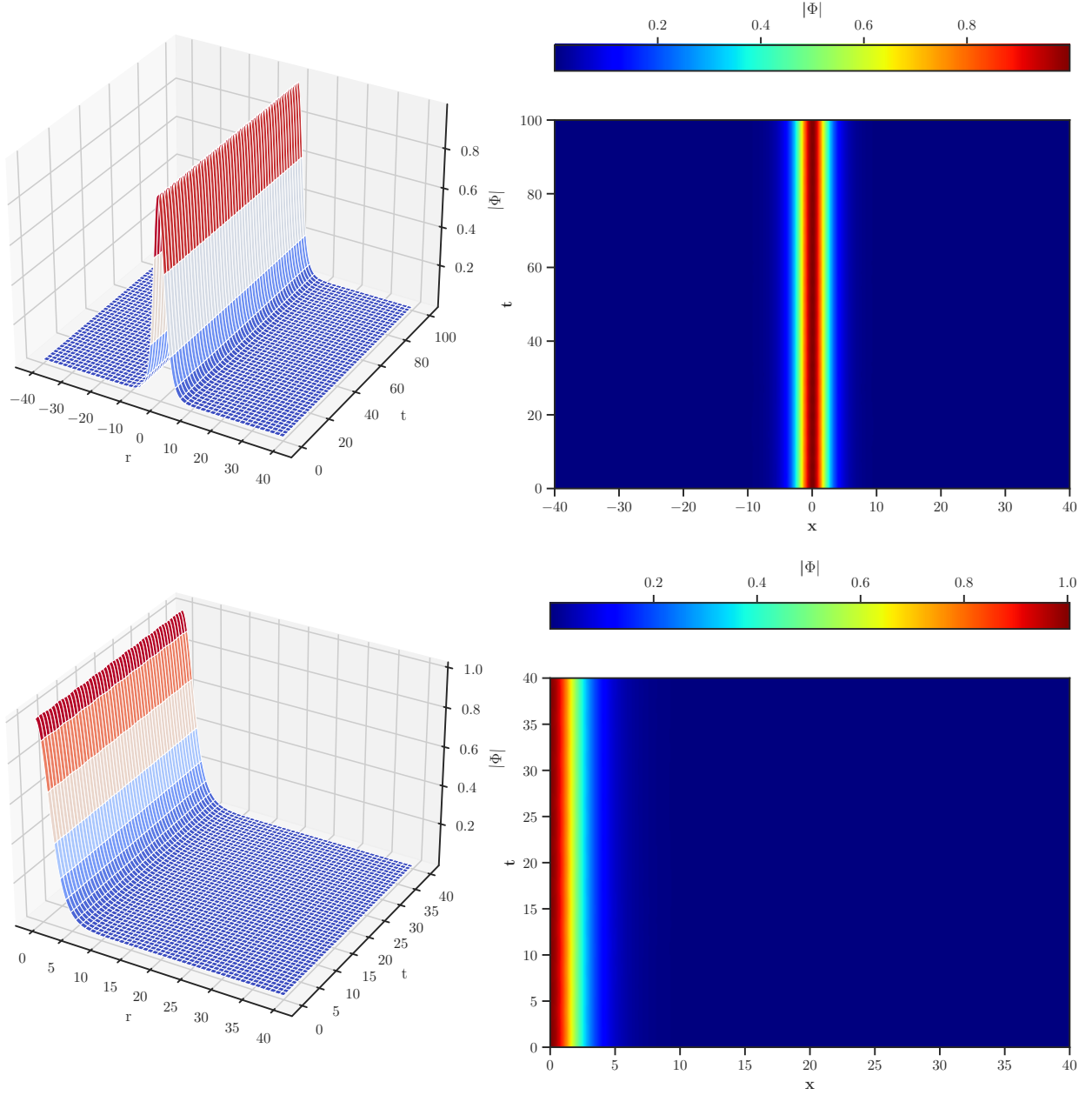


Figure B.2: The evolution of a Q-ball profile in $D = 2$, $g = 1/3$, and $\omega_Q = 0.8$. We solve using two methods: either by analytically expanding the solution to the negative- r domain, or by considering only positive r and applying the Neumann boundary condition using reflection symmetry. We can see that it is stable over time.

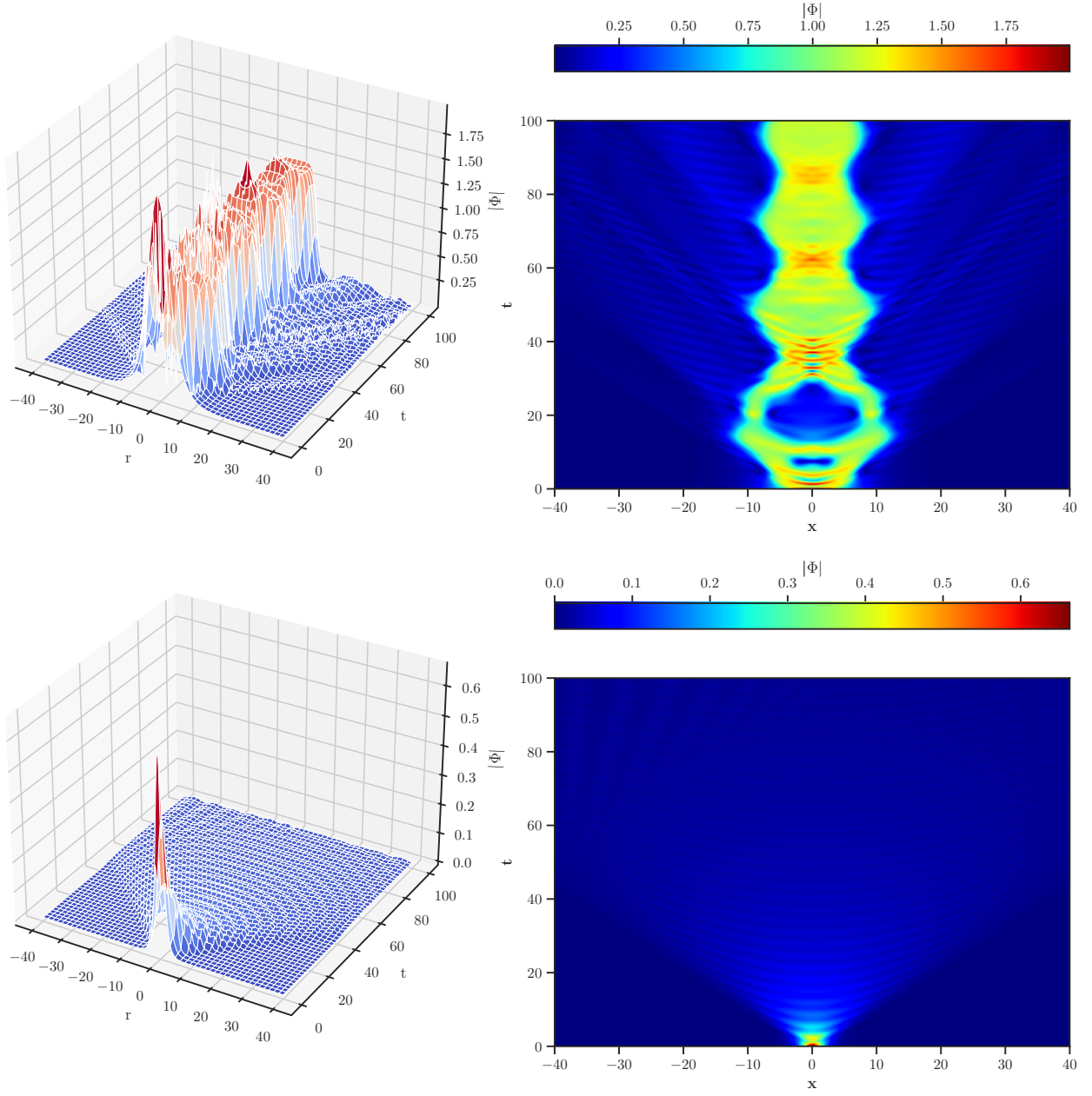


Figure B.3: The scaled Q-ball with $\alpha = 0.5$ (top figure) and $\alpha = 1.5$ (bottom figure).

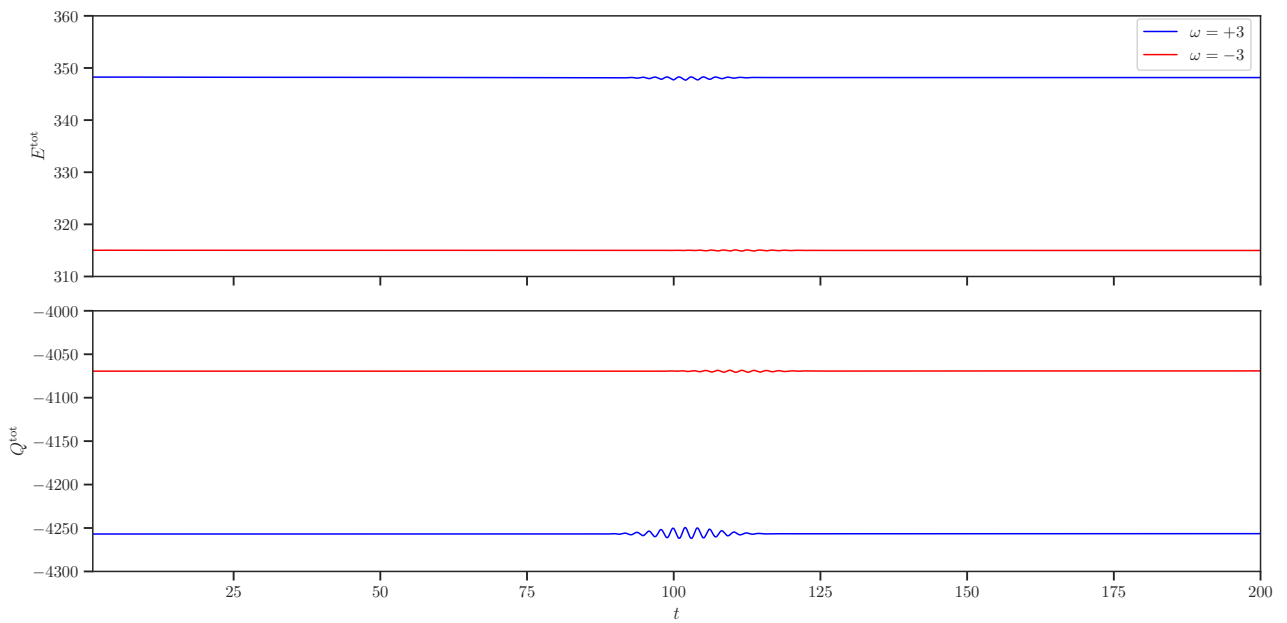


Figure B.4: Total energy E^{tot} and total charge Q^{tot} of the whole field configuration, including both the Q-ball and the scattered wavepacket. We observe a small fluctuation of E^{tot} and Q^{tot} centered around $t \sim 100$, i.e., the time around which the interaction between the Q-ball and the wavepacket is significant. This might be due to numerical error. For the rest of the time domain, the total charges remain constant.

Extra numerical results for FLS model

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C.1 S-matrix elements and amplification factors

In this section we report the absolute squares of the scattering matrix elements $|S_{fi}|^2$, as well as the amplification factors $Z_{E,Q}$ for Q-balls with various charges. Again, the quantity $|S^{fi}|^2$ is interpreted as the fraction of outgoing particle number in mode f for a pure monochromatic incoming mode i . The results are presented in Figs. C.1, C.2, C.3, C.4, and C.5. For all of these plots, we fix the FLS model couplings to $g_{\chi\Phi} = 4, g_{\chi} = 1, g_{\Phi} = 0.04, v_{\chi} = 1, m_{\Phi} = 0$ and vary the Q-ball charge (ω_Q) and the incoming mode. All of the plots assume S-wave scattering. The symmetry patterns follow as a consequence of the S-matrix properties discussed previously in Subsection. 4.3.3.

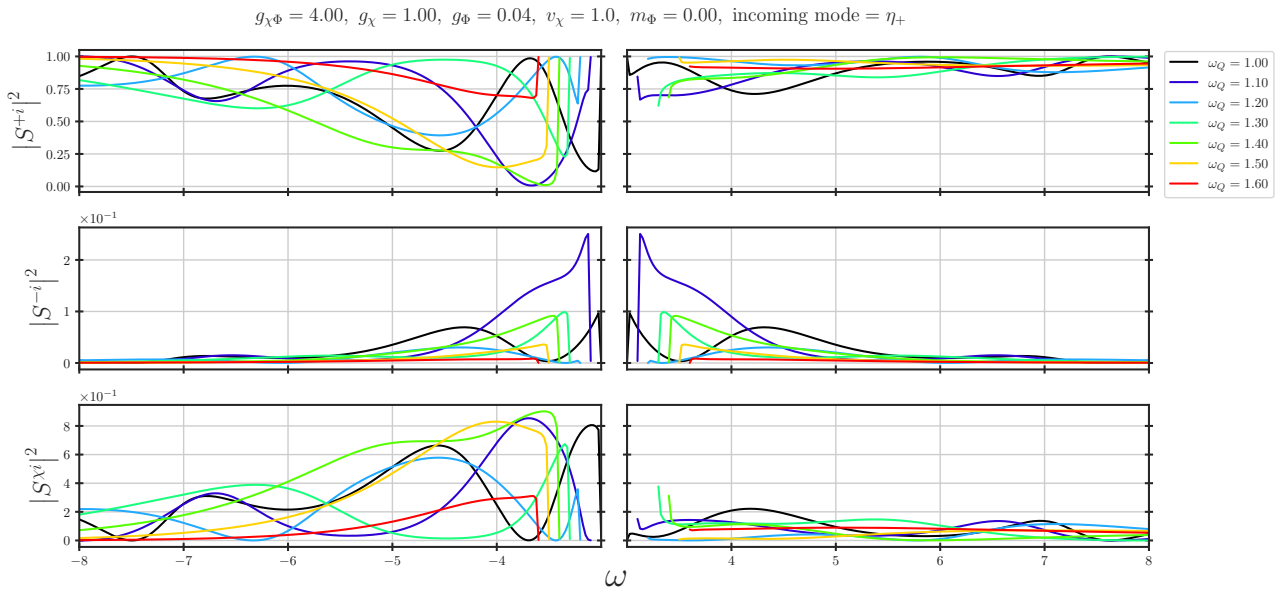


Figure C.1: $|S^{f+}(\omega)|^2$ for various ω_Q . The enhancement/attenuation of squared amplitude for $+$ and χ modes exhibited greater fluctuations in the negative ω domain compared to the positive one. Meanwhile, the amplification effect on the $-$ mode is even with respect to ω .

$g_{\chi\Phi} = 4.00$, $g_{\chi} = 1.00$, $g_{\Phi} = 0.04$, $v_{\chi} = 1.0$, $m_{\Phi} = 0.00$, incoming mode = η_-

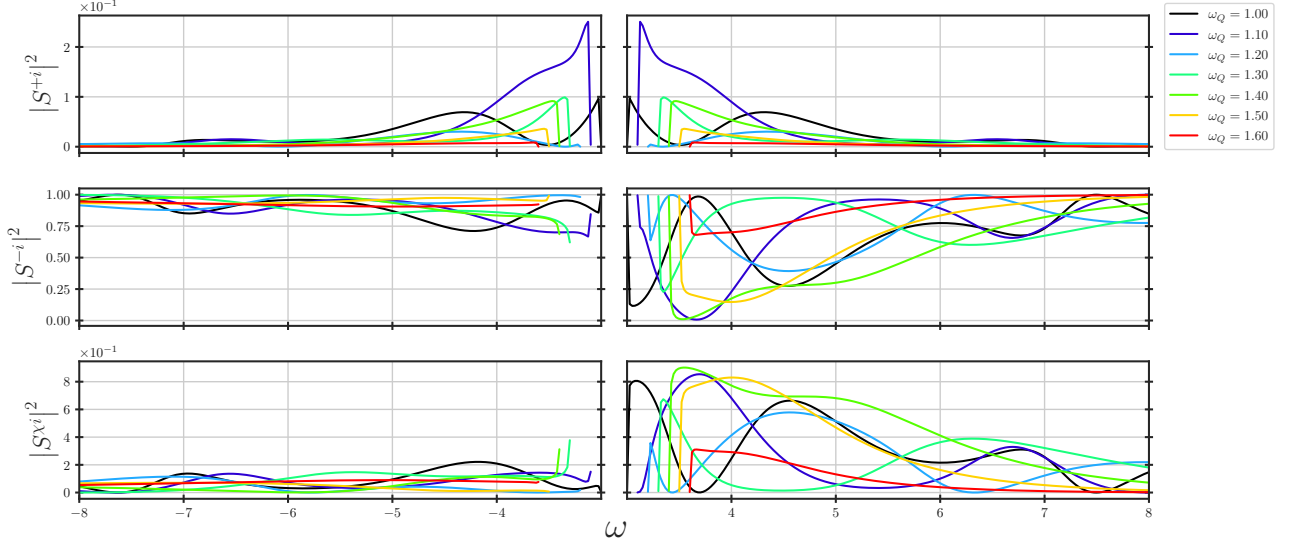


Figure C.2: $|S^{f-}(\omega)|^2$ for various ω_Q . In comparison with Fig. C.1, one can see the symmetries under $(\eta_{\pm}, \eta_{\chi}) \leftrightarrow (\eta_{\mp}^*, \eta_{\chi}^*)$.

$g_{\chi\Phi} = 4.00$, $g_{\chi} = 1.00$, $g_{\Phi} = 0.04$, $v_{\chi} = 1.0$, $m_{\Phi} = 0.00$, incoming mode = η_{χ}

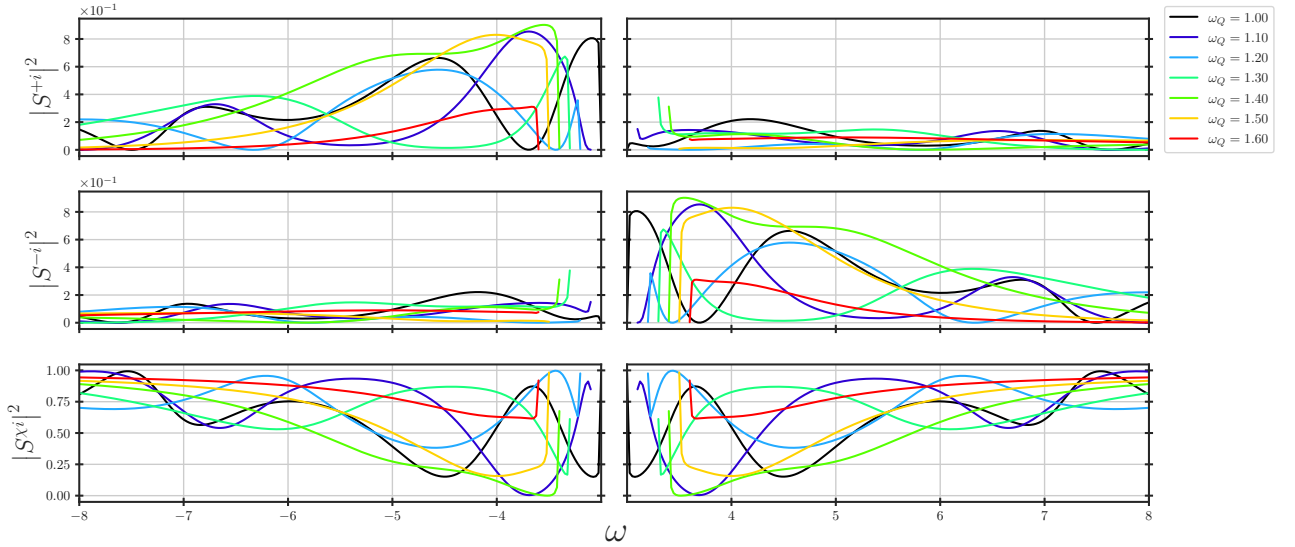


Figure C.3: $|S^{f\chi}(\omega)|^2$ for various ω_Q . While the result for $f = \chi$ is an even function of ω , we also have the symmetries between $f = +$ and $f = -$ under $\omega \rightarrow -\omega$.

$$g_{\chi\Phi} = 4.00, \quad g_{\chi} = 1.00, \quad g_{\Phi} = 0.04, \quad v_{\chi} = 1.0, \quad m_{\Phi} = 0.00, \quad \text{incoming mode} = \eta_+$$

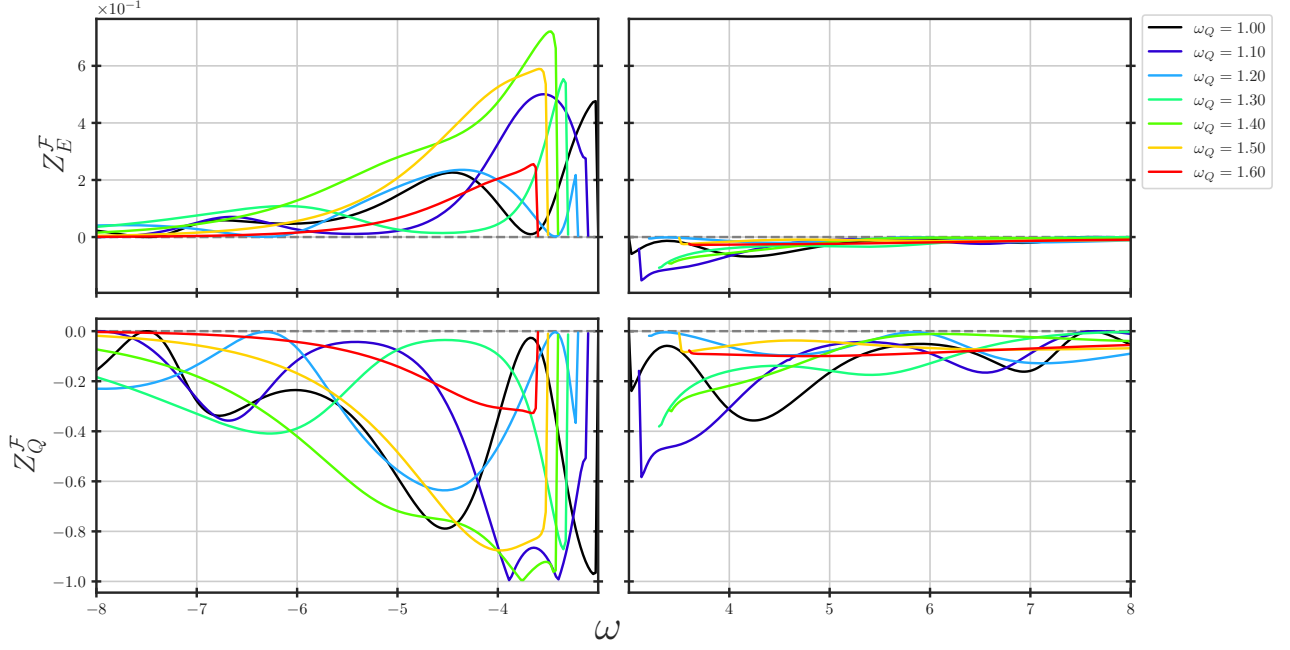


Figure C.4: Energy and charge amplification factors as a function of incoming mode frequency with positively charged incoming mode. The amplification/attenuation effects tend to be stronger for negative ω than for positive ω . The Z_Q^F is no longer symmetric under $\omega \rightarrow -\omega$ as in the one-field case due to the opening of a new scattering channel $+\rightarrow\chi$.

$$g_{\chi\Phi} = 4.00, \quad g_{\chi} = 1.00, \quad g_{\Phi} = 0.04, \quad v_{\chi} = 1.0, \quad m_{\Phi} = 0.00, \quad \text{incoming mode} = \eta_{\chi}$$

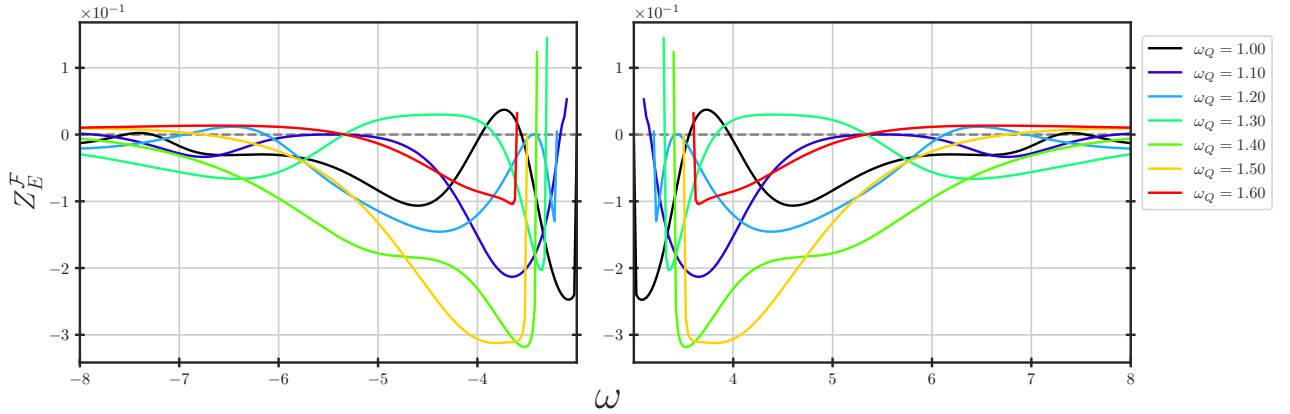


Figure C.5: Energy amplification factors as a function of incoming mode frequency with neutral incoming mode. We observe the symmetry pattern under $\omega \rightarrow -\omega$ in this case, which is a consequence of a property of the S-matrix discussed in previous appendix. Since there is no net charge for the initial states, it is not well-defined to talk about the charge amplification.

One common pattern observed from these plots is that the quantities under consideration tend to be larger at small $|\omega|$ and decay to their asymptotic values (0 or 1, depending on the specific case) as $|\omega|$ increases. This implies the total elastic scattering of the incoming mode, or equivalently in the language of the S-matrix: $\lim_{\omega \rightarrow \pm\infty} S(\omega) = \mathbb{1}$.

C.2 Lattice simulation for two-field model

In this appendix we briefly discuss the numerical setup for the lattice calculations in the case of the two-field Q-ball. The numerical framework is the same as in the one-field simulation, with fourth-order finite-difference discretization in space and a fourth-order Runge–Kutta integrator for the numerical time evolution. The perturbed Q-ball initial conditions in $D = 3$ with internal frequency ω_Q in the FLS potential are

$$\Phi(0, r) = \Phi_Q(0, r) + \delta_\Phi \left(\frac{r_0}{r} \right) e^{-\frac{(r-r_0)^2}{2(\sigma_r^\Phi)^2}} e^{-ik_\Phi r}, \quad (\text{C.2.1a})$$

$$\partial_t \Phi(0, r) = -i\omega_Q \Phi_Q(0, r) - i\omega_\Phi \delta_\Phi \left(\frac{r_0}{r} \right) e^{-\frac{(r-r_0)^2}{2(\sigma_r^\Phi)^2}} e^{-ik_\Phi r}, \quad (\text{C.2.1b})$$

$$\chi(0, r) = \chi_Q(0, r) + \delta_\chi \left(\frac{r_0}{r} \right) e^{-\frac{(r-r_0)^2}{2(\sigma_r^\chi)^2}} \cos(k_\chi r), \quad (\text{C.2.1c})$$

$$\partial_t \chi(0, r) = \delta_\chi \left(\frac{r_0}{r} \right) \omega_\chi e^{-\frac{(r-r_0)^2}{2(\sigma_r^\chi)^2}} \sin(k_\chi r), \quad (\text{C.2.1d})$$

where $k_\chi = \sqrt{\omega^2 - 8g_\chi v_\chi^2}$ and k_Φ is either k_+ or k_- depending on what incoming mode one wants to study, $k_\pm = \sqrt{\omega_\pm^2 - g_{\chi\Phi}^2 v_\chi^2 - m_\Phi^2}$. For simplicity, we only consider one type of incoming mode for each scattering, so we turn on either δ_Φ or δ_χ , but not both in the same simulation. We perform a scan of amplification factors over various Q-balls and different values of incoming mode frequency ω . We use the same set of ω_Q values as in the previous results (see, e.g., the profiles for these Q-balls in Fig. 4.1). The final results are visualized in Figs. C.6 and C.7 for $\omega < 0$ and $\omega > 0$, respectively. For these calculations we have kept the size of the perturbation fixed with the parameters $\sigma_r^\Phi = 10$ and $\delta_\Phi = 5 \times 10^{-4}$. We can see that for small values of ω the agreement between the lattice and linear description becomes worse. This is related to the larger wavelength of the perturbations, which at some point exceeds the width of the wave packet. Obviously the linear description cannot provide reliable results in this limit.

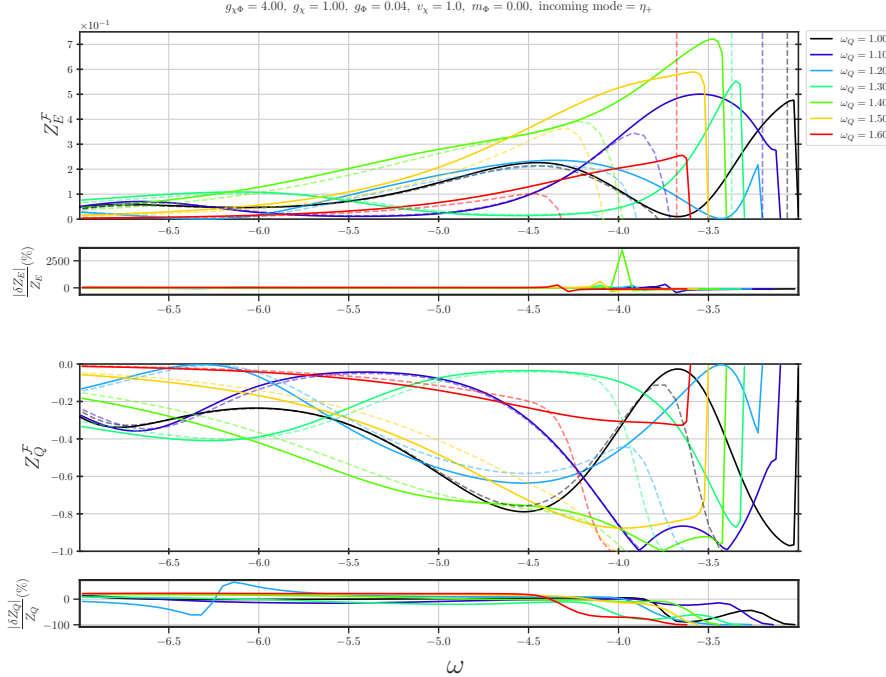


Figure C.6: Comparison between amplification factors in the linear regime (solid lines) and lattice results (dashed lines) for $\omega < 0$ and various Q-balls, with different colors corresponding to different ω_Q . Here we have kept the perturbation size fixed with $\sigma_r^\Phi = 10$ and $\delta_\Phi = 5 \times 10^{-4}$.

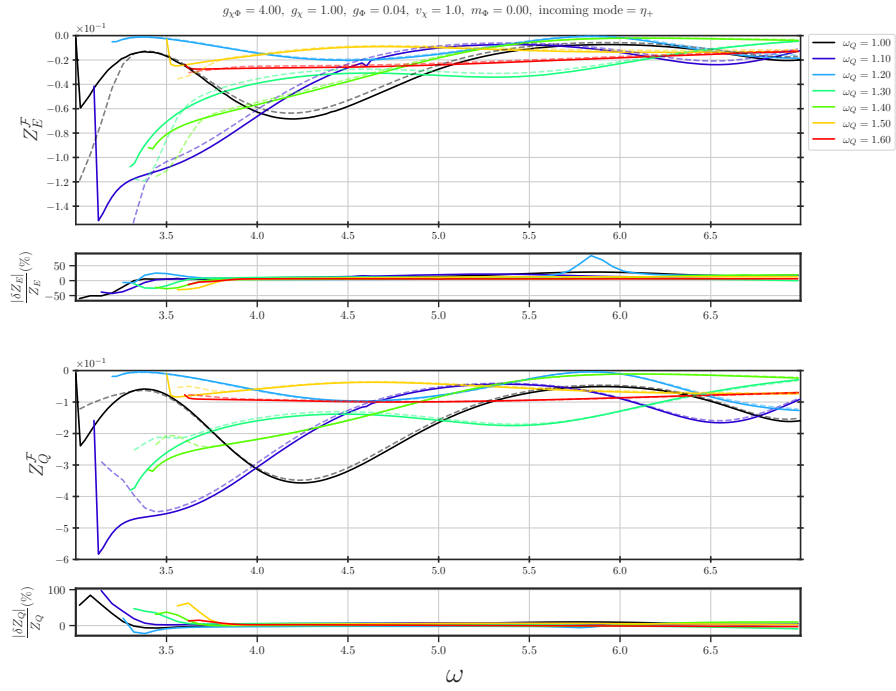


Figure C.7: Comparison between amplification factors in the linear regime (solid lines) and lattice results (dashed lines) for $\omega > 0$ and various Q-balls, with different colors corresponding to different ω_Q . Here we have kept the perturbation size fixed with $\sigma_r^\Phi = 10$ and $\delta_\Phi = 5 \times 10^{-4}$.

Extras on SGWB sourced by FOPT

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D.1 Classification of SGWB sources from FOPT

D.1.1 Scalar-field bubble collisions

The scalar-field bubble-collision contribution is among the most direct gravitational-wave sources associated with the order-parameter field itself. It is relevant when the released vacuum energy is not efficiently transferred to a long-lived thermal plasma, as can happen in vacuum-dominated, sufficiently supercooled transitions. In this regime the dominant TT source is the anisotropic part of the scalar stress-energy tensor rather than the fluid shear stress. For a canonical scalar field, this tensor component reads

$$T_{ij}^\phi = \partial_i \phi \partial_j \phi + \delta_{ij} \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right]. \quad (\text{D.1.1})$$

The second term is isotropic in the spatial indices and is removed by the TT projection introduced in Eq. (6.1.31). Therefore the radiating part of the scalar source is controlled by the spatial-gradient structure $(\partial_i \phi \partial_j \phi)^{\text{TT}}$. An isolated bubble is approximately spherical, so its angular stress integrates to no net transverse-traceless radiation. The gravitational-wave signal appears when many bubbles collide: the wall network loses spherical symmetry, the gradient energy is redistributed in space and time, and the TT-projected stress develops a non-vanishing quadrupolar component.

In the relativistic thin-wall limit this physical picture becomes especially transparent [61, 121, 101]. Consider a bubble nucleated at time t_n , with center $\mathbf{x}_n$ and radius $R_n(t) \simeq v_w(t - t_n)$. The scalar gradients are localized on the wall and point approximately along the outward normal $\hat{\mathbf{x}}$, so the thin-wall stress of the n -th bubble may be represented schematically as

$$T_{ij}^{(n)}(\mathbf{x}, t) \propto \hat{x}_i \hat{x}_j \sigma_n(t, \hat{\mathbf{x}}) \delta[r - R_n(t)], \quad (\text{D.1.2})$$

where σ_n is the surface energy density carried by that angular wall element that depends on the energy difference ρ_{vac} between true and false vacua as well as the radius R_n of the given bubble. As

Model	$\{a, b\}$ for $v_w \simeq 1$	$\{a, b\}$ for $v_w \ll 1$	$\hat{\Delta}(v_w)$	$\hat{\omega}/\beta$
Envelope	$\{2.9, 0.9\}$	$\{2.95, 1\}$	$\frac{0.44v_w^3}{1 + 8.28v_w^3}$	$\frac{1.96}{1 - 0.051v_w + 0.88v_w^2}$
Bulk flow	$\{0.9, 2.1\}$	$\{0.95, 2.9\}$	$\frac{0.0866v_w^3}{1 + 0.354v_w^3}$	$\frac{1.24}{1 - 0.047v_w + 0.58v_w^2}$

Table D.1: Spectral-shape and peak-quantity fits from Tables 1 and 2 of Ref. [101]. The first entry in $\{a, b\}$ is the infrared power and the second entry is the absolute value of the ultraviolet power. The dimensionless peak amplitude $\hat{\Delta}$ enters Eq. (D.1.3).

the SGWB spectrum is a function of the energy-momentum of the source, Eq. (D.1.2) implies that the wall velocity and the nucleation history determine which angular wall elements contribute coherently to the spectrum at a given frequency.

One of the first approximations for scalar-field bubble collisions is the envelope approximation [99, 122, 61]. It keeps only the uncollided parts of the bubble walls and removes the stress of a wall element immediately after its first collision. Equivalently, the post-collision relaxation time of the scalar gradients is assumed to be much shorter than the macroscopic bubble radius at collision. Under this assumption the problem is reduced to the geometry of the uncollided wall envelope: the stress tensor is built from the remaining wall segments, projected onto its TT part, and inserted into the GW formula of Eq. (6.1.48). In this envelope approximation, the uncollided surface areas of the bubbles measure the level of spherical symmetry breaking, hence the name. One distinct feature of this approximation is that, in a realistic bubble nucleation history, there is a natural time cutoff at which essentially no uncollided surface elements are left, and hence no gravitational waves are produced after this time cutoff. This, in turn, gives rise to a unique power spectrum with different scaling laws in both the IR and UV compared with other scenarios with long-lasting sources.

Konstandin’s bulk-flow model implements a different post-collision assumption [101]. It was introduced as a simplified model for persistent fluid-shell motion, where, by assuming that the surface tension scales as R_n^{-2} after bubble collision, the bulk motion of the wall element contributes as a long-lasting source. More specifically, in the bulk-flow case the energy per surface element is fixed after first collision and is diluted by the subsequent growth of the shell area. This dynamical treatment of the bubble wall after collision has significant impacts on the lifetime, smoothness, and coherence of the TT source, which consequently change the shape of the GW spectrum.

For an exponentially increasing nucleation rate, the transition duration is characterized by β^{-1} . The collision spectrum peaks at a frequency of order the inverse bubble size at collision, rather than at a microscopic wall-thickness scale. Konstandin parameterizes the production-time GW fraction per logarithmic frequency interval as

$$\Omega_{\text{GW},*}(\omega) = \omega \frac{dE_{\text{GW}}}{d\omega} \frac{1}{E_{\text{tot}}} = \left(\frac{H_*}{\beta} \right)^2 K_\phi^2 \Delta(\omega/\beta, v_w), \quad (\text{D.1.3})$$

where Δ is a dimensionless spectral function, $K_\phi \equiv \kappa_\phi \alpha / (1 + \alpha)$, the efficiency factor κ_ϕ is defined as the fraction of latent heat deposited in a thin shell close to the phase transition front, and v_w is the bubble wall speed. The factor $(H_*/\beta)^2$ reflects how short the collision stage is compared with a Hubble time, while $\kappa\alpha/(1 + \alpha)$ measures the fraction of the total energy budget carried by the source. This dimensionless spectrum is fitted by the following broken power law:

$$\Delta(\omega/\beta, v_w) = \hat{\Delta} \frac{(a + b)\hat{\omega}^b \omega^a}{b\hat{\omega}^{a+b} + a\omega^{a+b}}, \quad (\text{D.1.4})$$

where $\hat{\omega}$ and $\hat{\Delta}$ denote the peak angular frequency and the peak value of the dimensionless spectral function for the approximation under consideration. The low-frequency side scales as $\Delta \propto \omega^a$, while the high-frequency side scales as $\Delta \propto \omega^{-b}$. The fit coefficients below are numerical results of Ref. [101], obtained by averaging over 48 nucleation histories in a box of size $(20/\beta)^3$ with roughly 300 bubbles per simulation.

The comparison between the two approximations is given in Table. D.1. The envelope spectrum has an infrared rise close to ω^3 and a relatively shallow ultraviolet falloff close to ω^{-1} , which reflects the assumption that the source is short-lived after the time cutoff set by bubble collisions. The bulk-flow spectrum instead rises approximately as ω in the infrared and falls faster in the ultraviolet, with a power between roughly $\omega^{-2.1}$ and $\omega^{-2.9}$ over the fitted velocity range. The infrared enhancement is a consequence of the long-lived post-collision source: expanding shells continue to shift power toward larger length scales. This apparent ω^1 infrared behavior should not be interpreted as a universal causal theorem; it is instead tied to the idealized long-lived source in a non-expanding background, while on scales below the Hubble frequency one expects the usual causal ω^3 behavior to be restored. The stronger ultraviolet suppression reflects the smoother stress history in the bulk-flow model, and in Ref. [101] it is emphasized that a cancellation between the finite transition-time contribution and the analytically integrated late-time contribution leads to a high-frequency scaling of the source functions consistent with this behavior.

The peak fits show how the wall velocity changes the observable features. The small-velocity behavior $\hat{\Delta} \propto v_w^3$ can be understood directly from the geometric normalization of the collision source. ⁽¹⁾ The bulk-flow peak frequency is generally lower than the envelope peak frequency for the same v_w , reflecting the longer source lifetime and the continued propagation of stress after first collision. In both cases, increasing β/H_* shifts the observed peak to higher frequency and suppresses the amplitude through $(H_*/\beta)^2$. Increasing the transition strength α and the relevant efficiency factor κ enhances the amplitude through the squared energy fraction in Eq. (D.1.3), but does not by itself fix the spectral shape. The shape is controlled primarily by the lifetime and post-collision evolution of the stress: prompt disappearance gives the envelope spectrum while persistent shell motion gives the bulk-flow spectrum.

For phenomenological applications, the same fit can be written directly for the observed spectrum today using the template of Ref. [101]:

$$h^2\Omega_{\text{GW},0}(\omega_0) = h^2\hat{\Omega}_{\text{GW},0} \frac{(a+b)\hat{\omega}_0^b\omega_0^a}{b\hat{\omega}_0^{a+b} + a\omega_0^{a+b}}, \quad (\text{D.1.6})$$

where the coefficients a , b , $\hat{\Delta}$, and $\hat{\omega}/\beta$ are chosen from Table. D.1. Here ω_0 denotes the angular frequency observed today, while $\hat{\omega}$ without the subscript denotes the production-time peak angular frequency appearing in Table. D.1. The peak position today obtained from many-bubble simulation is

$$\hat{\omega}_0 = 2\pi \left(2.62 \times 10^{-3} \text{ mHz} \right) \left(\frac{\hat{\omega}}{\beta} \right) \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \quad (\text{D.1.7})$$

The corresponding peak amplitude is

$$h^2\hat{\Omega}_{\text{GW},0} = 1.67 \times 10^{-5} \hat{\Delta} \kappa^2 \left(\frac{H_*}{\beta} \right)^2 \left(\frac{\alpha}{1+\alpha} \right)^2 \left(\frac{100}{g_*} \right)^{1/3}. \quad (\text{D.1.8})$$

Here T_* and g_* are the temperature and effective number of relativistic degrees of freedom at the transition, and κ is the efficiency factor appropriate to the source model.

The envelope approximation is therefore useful because it gives a concrete and predictive collision spectrum, but it is also dynamically restrictive, in the sense that it assumes the scalar stress disappears at first contact, whereas the true field evolution can contain reflected walls, scalar radiation,

⁽¹⁾ In the thin-wall context, the dimensionless spectrum Δ contains an explicit factor v_w^6/V multiplying the TT-projected geometric correlator of the source. For a large simulation volume, only bubbles within a correlation volume contribute coherently, so the double sum over bubbles effectively scales linearly with the number of independent bubbles. At fixed nucleation time scale β^{-1} , a slow wall has a characteristic bubble radius $R_* \sim v_w/\beta$, and hence a characteristic bubble volume $R_*^3 \sim v_w^3/\beta^3$. The number of independent bubbles in a volume V therefore scales as $N \sim V\beta^3/v_w^3$. Equivalently, the geometric correlator scales as

$$\int d\mathbf{k} \Lambda_{ij,lm} C_{ij}^* C_{lm} \propto N\beta^{-8} \propto \frac{V}{v_w^3\beta^5}, \quad (\text{D.1.5})$$

where the powers of β follow from the dimensions of the Fourier-space source functions C_{ij} . Combining this with the explicit prefactor $v_w^6\beta^5/V$ in $\Delta(\omega/\beta, v_w)$ gives the net small-velocity scaling $\Delta \propto v_w^3$, and hence $\hat{\Delta} \propto v_w^3$.

oscillons, or long-lived shells depending on the potential and on the available dissipation channels. Lattice simulations of vacuum transitions, which evolve the scalar field directly, show that the envelope approximation can reproduce the order of magnitude and broad peak structure when scalar-field gradients dominate, but post-collision dynamics can change both the intermediate-frequency behavior and the ultraviolet tail [100]. This sensitivity to post-collision evolution is part of the reason why the scalar collision problem is not exhausted by the envelope model.

D.1.2 Acoustic waves in the plasma

The acoustic contribution is the gravitational radiation sourced by the coherent fluid motion left behind by expanding bubbles in a thermal plasma. It is qualitatively different from the scalar-field collision signal discussed above. In the envelope approximation the source is localized on uncollided bubble-wall surfaces and disappears at first contact. In a coupled field-fluid system, instead, the expanding wall deposits energy and momentum into the plasma, producing compression and rarefaction waves which remain after the scalar wall network has disappeared [62, 63].

For an ideal relativistic fluid with rest-frame energy density e , pressure p , enthalpy density $e + p$, and four-velocity $u^\mu = \gamma(1, \mathbf{v})$, the spatial part of the stress tensor is $T_{ij} = \gamma^2(e + p)v_i v_j + p\delta_{ij}$. Since the isotropic pressure term is removed by the TT projection, the fluid shear source may be represented by

$$T_{ij}^f = \gamma^2(e + p)v_i v_j. \quad (\text{D.1.9})$$

The simulations of Refs. [62, 63] evolve an auxiliary tensor sourced by the unprojected fluid stress T_{ij}^f and by the scalar gradient stress, and apply the TT projection in momentum space to recover the physical metric perturbation.

A useful diagnostic of the source strength is the enthalpy-weighted RMS fluid-velocity parameter $\bar{U}_f$, defined in the simulations by

$$(\bar{e} + \bar{p})\bar{U}_f^2 = \frac{1}{V} \int_V d^3x T_{ii}^f, \quad (\text{D.1.10})$$

where the bars denote spatial averages over the simulation volume V . If the flow is nonrelativistic and the enthalpy density is approximately homogeneous, Eq. (D.1.10) reduces to

$$\bar{U}_f^2 \simeq \frac{1}{V} \int_V d^3x \mathbf{v}^2, \quad (\text{D.1.11})$$

which is the usual volume RMS three-velocity squared. The fluid kinetic-energy fraction is then ([93])

$$K_{\text{sw}} \equiv \frac{1}{\bar{e}V} \int_V d^3x T_{ii}^f = (1 + w_{\text{eos}})\bar{U}_f^2 \simeq \frac{\kappa_{\text{sw}}\alpha}{1 + \alpha}, \quad (\text{D.1.12})$$

where $w_{\text{eos}} = \bar{p}/\bar{e}$ is the equation-of-state parameter. The first equality is the definition of K_{sw} as the fluid shear/kinetic contribution normalized to the mean energy density, while the last relation is the relation between $\bar{U}_f$ and the hydrodynamic efficiency factor derived in more detail in Ref. [63]. For a relativistic radiation fluid $w_{\text{eos}} \simeq 1/3$. The quantity κ_{sw} is the efficiency for converting the released vacuum energy into bulk fluid motion that sources acoustic waves.

Let us dive into more technical details of the theoretical background of the computations, following the unequal-time-correlator (UETC) formulation. Denote $\Pi^2(k, t_1, t_2)$ as the angularly averaged correlator of the TT-projected fluid shear stress, including the contractions with $\Lambda_{ij,lm}(\mathbf{k})$ and the Fourier delta function enforcing momentum conservation. In the flat-space simulation normalization used in Refs. [62, 63], this correlator gives

$$\frac{d\rho_{\text{GW}}(k)}{d\ln k} = \frac{2Gk^3}{\pi} \int_0^t dt_1 dt_2 \cos[k(t_1 - t_2)] \Pi^2(k, t_1, t_2). \quad (\text{D.1.13})$$

In this subsection we follow the convention of Refs. [62, 63] and use k as the wave-number variable in the source calculation, rather than the angular frequency ω used elsewhere in this chapter. For freely

propagating GWs in units with $c = 1$, the physical wave number is equal to the production-frame angular frequency. For a statistically stationary acoustic source active over a time τ_{sw} , Hindmarsh et al. model the UETC between two source times t_1 and t_2 as

$$\Pi^2(k, t_1, t_2) \simeq \left[(\bar{e} + \bar{p}) \bar{U}_f^2 \right]^2 L_f^3 \tilde{\Pi}^2(k L_f, z), \quad z = k(t_1 - t_2), \quad (\text{D.1.14})$$

where t_1 and t_2 are the two times at which the fluid shear stress is evaluated, L_f is a characteristic length scale of the velocity field, and $\tilde{\Pi}^2$ is a dimensionless UETC shape function. Define the dimensionless spectral-density function by

$$\tilde{P}_{\text{GW}}(k L_f) = \frac{1}{k L_f} \int dz \frac{\cos z}{2} \tilde{\Pi}^2(k L_f, z). \quad (\text{D.1.15})$$

Combining Eqs. (D.1.13)–(D.1.15), converting ρ_{GW} to $\Omega_{\text{GW}} = \rho_{\text{GW}}/\rho_c$ at production, and using $\rho_c = 3H_*^2/(8\pi G)$ gives

$$\frac{d\Omega_{\text{GW},*}^{\text{sw}}}{d \ln k} = 3(1 + w_{\text{eos}})^2 \bar{U}_f^4 (H_* \tau_{\text{sw}}) (H_* L_f) \frac{(k L_f)^3}{2\pi^2} \tilde{P}_{\text{GW}}(k L_f), \quad (\text{D.1.16})$$

where, as usual, the asterisk denotes evaluation at the time of GW production. Define the integrated dimensionless efficiency parameter

$$\tilde{\Omega}_{\text{GW}} \equiv \int \frac{dk}{k} \frac{(k L_f)^3}{2\pi^2} \tilde{P}_{\text{GW}}(k L_f), \quad (\text{D.1.17})$$

Integrating Eq. (D.1.16) over k then gives

$$\Omega_{\text{GW},*}^{\text{sw}} = 3(1 + w_{\text{eos}})^2 \bar{U}_f^4 (H_* \tau_{\text{sw}}) (H_* L_f) \tilde{\Omega}_{\text{GW}}. \quad (\text{D.1.18})$$

This equation is the central acoustic result of Ref. [63]. It says that the model dependence enters through the fluid kinetic energy, the source lifetime, and the macroscopic fluid length scale. In particular, this expression says that the gravitational-wave power spectrum grows linearly with time for as long as the velocity perturbations stay active. The simulations in Ref. [63] find an approximately universal normalization $8\pi \tilde{\Omega}_{\text{GW}} = 0.8 \pm 0.1$ for the weak and intermediate transitions studied, where the uncertainty is the root mean square fluctuation between simulations.

Using Eq. (D.1.12), the integrated result can equivalently be written as

$$\Omega_{\text{GW},*}^{\text{sw}} = 3K_{\text{sw}}^2 (H_* \tau_{\text{sw}}) (H_* L_f) \tilde{\Omega}_{\text{GW}}, \quad (\text{D.1.19})$$

For a long-lived acoustic source, the effective source lifetime is set by the Hubble time, $\tau_{\text{sw}} \simeq H_*^{-1}$, and Eq. (D.1.19) reduces to $\Omega_{\text{GW},*}^{\text{sw}} = 3K_{\text{sw}}^2 (H_* L_f) \tilde{\Omega}_{\text{GW}}$. Let us compare GWs sourced by sound waves with GWs sourced by scalar-field bubble collisions in the envelope approximation. For relativistic walls the envelope result can be written as [61]

$$\Omega_{\text{GW},*}^{\text{env}} \simeq C_{\text{env}}(v_w) \left(\frac{H_*}{\beta} \right)^2 K_\phi^2, \quad C_{\text{env}}(v_w) = \frac{0.11 v_w^3}{0.42 + v_w^2}, \quad K_\phi = \frac{\kappa_\phi \alpha}{1 + \alpha}. \quad (\text{D.1.20})$$

Here κ_ϕ denotes the fraction of released energy stored in scalar-field wall or gradient energy. The acoustic source instead depends on the fluid kinetic fraction K_{sw} in Eq. (D.1.12). For the following parametric comparison we assume K_{sw} and K_ϕ are of the same order of magnitude, so that the ratio isolates the dependence on source lifetimes and length scales rather than the model-dependent energy partition between scalar walls and plasma.

Recall that the length scale in Eq. (D.1.19) is the correlation scale of the velocity field. In the simulations of Ref. [63], identifying this scale with the mean bubble separation changes the gravitational-wave spectrum by no more than a factor of 2. Thus, for the parametric comparison below we identify

(1)

$$L_f \simeq R_* \simeq (8\pi)^{1/3} \frac{\max(v_w, c_s)}{\beta}, \quad (\text{D.1.22})$$

Since the envelope expression in Eq. (D.1.20) was quoted for relativistic walls, the simple ratio below takes $v_w > c_s$, so that $\max(v_w, c_s) = v_w$. Taking the ratio between Eqs. (D.1.19) and (D.1.20) then gives

$$\frac{\Omega_{\text{GW},*}^{\text{sw}}}{\Omega_{\text{GW},*}^{\text{env}}} \simeq \frac{3(8\pi)^{1/3} v_w}{C_{\text{env}}(v_w)} \tilde{\Omega}_{\text{GW}} \beta \tau_{\text{sw}} \left(\frac{K_{\text{sw}}}{K_\phi} \right)^2 = \frac{3(8\pi)^{1/3} (0.42 + v_w^2)}{0.11 v_w^2} \tilde{\Omega}_{\text{GW}} \beta \tau_{\text{sw}} \left(\frac{K_{\text{sw}}}{K_\phi} \right)^2. \quad (\text{D.1.23})$$

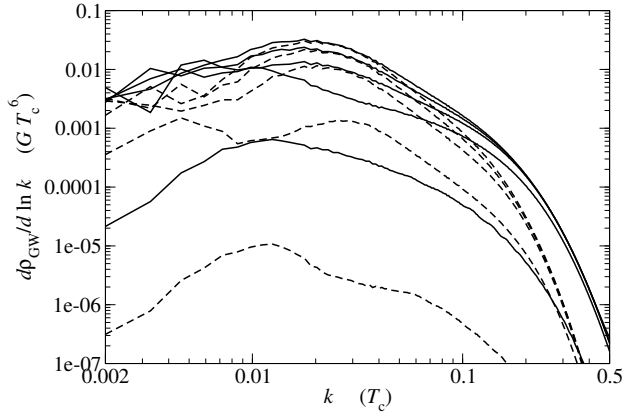
For $\tau_{\text{sw}} \simeq H_*^{-1}$ and $K_{\text{sw}} \sim K_\phi$, the enhancement is controlled by β/H_* . Thermal electroweak-scale transitions typically have $\beta/H_* \gtrsim \mathcal{O}(10^2)$, so the acoustic gravitational-wave signal can exceed the envelope signal by two or more orders of magnitude [63]. As expected, the gravitational-wave signal is enhanced when it is sourced by sound waves, because the scalar wall source is active for a time of order β^{-1} , while the sound-wave shear stress can remain active for a Hubble time. This dominance is expected when the wall reaches a small terminal velocity and consequently most of the released energy goes into the plasma. This rough estimate of the sound-wave contribution relative to the envelope approximation need not hold for vacuum-dominated or runaway transitions, where a significant scalar-field wall energy fraction survives, or for deflagrations where one should take into account the suppression of the tunneling rate ahead of the bubble wall, where the fluid is heated by the release of latent heat.

To sum up, one of the main numerical observations is that the scalar-field gradient measure grows and decays with the total bubble-wall area, whereas $\bar{U}_f$ grows with the bubble volume and remains approximately constant after bubble wall collision. Another important observation is that the resulting post-collision fluid velocity power spectrum is dominantly longitudinal (corresponding to compressional components) rather than transverse (rotational components), explaining why the source is called acoustic: the dominant fluid perturbations are sound waves rather than fully developed incompressible turbulence. This comparison is illustrated by the simulation diagnostics in Fig. D.1.

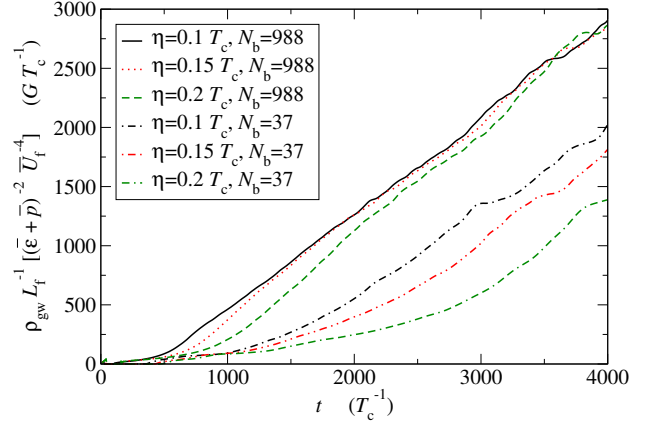
- Fig. a shows a simulation in which the total GW power contains a scalar-field contribution during the collision stage, while the dashed curves keep only the fluid source. The power-law behavior visible in the total spectrum during the early stage is attributed in Ref. [63] to scalar gradient energy localized near the bubble walls, and can be approximated by the envelope contribution of the bubble walls. However, this scalar contribution is short-lived: after the scalar domain walls disappear, the remaining source is the fluid velocity field, which plays the dominant role in the total spectrum at late times in the simulation. As the fluid contribution becomes increasingly relevant, both the amplitude and the scaling power law change over time. For the weak deflagration simulations, the GW spectrum at sufficiently large time falls approximately as $k^{-p_{\text{UV}}}$ with $p_{\text{UV}} \gtrsim 3$; for faster walls, the high- k falloff is steeper or harder to extract from the available dynamic range [63]. This contrasts with the k^{-1} high-frequency behavior associated with the envelope approximation and gives a possible discriminator between thermal plasma transitions and vacuum-like scalar collisions.
- Fig. b tests the scaling of ρ_{GW} over time t , which is equivalent to Eq. (D.1.18) if we replace τ_{sw} with t . At late times the curves grow approximately linearly with similar slopes, even across simulations with different bubble separations, which agrees with the formulation of Ω_{GW} sourced by sound waves in the fluid.

⁽¹⁾ For an exponential nucleation rate, the usual detonation estimate is $R_* \simeq (8\pi)^{1/3} v_w / \beta$. However, for deflagrations the wall is subsonic and the plasma disturbance or reheating/shock front propagates ahead of the wall at roughly the sound speed, so the region in which further nucleations are affected is controlled by c_s rather than by v_w [79, 93]. Therefore, a more correct bubble separation scale is

$$R_* \simeq (8\pi)^{1/3} \frac{\max(v_w, c_s)}{\beta}, \quad (\text{D.1.21})$$



(a) Fluid-only and total GW power during the collision stage.



(b) Scaled fluid-sourced GW energy density during the acoustic stage.

Figure D.1: Simulation diagnostics from Ref. [63]. Panel a compares the GW power spectra sourced by the fluid alone (dashed curves) with the total field-plus-fluid power (solid curves) over a time interval of $500/T_c$. Panel b shows the time series of $\rho_{\text{GW}} L_f^{-1} [(\bar{\epsilon} + \bar{p})^{-2} \bar{U}_f^{-4}]$, which describes the evolution of the gravitational-wave energy density relative to an estimate of the square of the final fluid shear stresses. Here η is an adjustable friction parameter, which enters the evolution equation as $[\partial_\mu T^{\mu\nu}]_{\text{fluid}} = -\eta U^\mu \partial_\mu \phi \partial^\nu \phi$.

The present-day fit quoted in Ref. [64] is

$$h^2 \Omega_{\text{GW},0}^{\text{sw}}(\omega) = 2.65 \times 10^{-6} \left(\frac{H_*}{\beta} \right) K_{\text{sw}}^2 \left(\frac{100}{g_*} \right)^{1/3} v_w S_{\text{sw}}(\omega), \quad (\text{D.1.24})$$

where $K_{\text{sw}} = \kappa_{\text{sw}} \alpha / (1 + \alpha)$ is the fluid kinetic fraction introduced in Eq. (D.1.12). Here S_{sw} is the dimensionless sound-wave spectral shape, with fitting expression

$$S_{\text{sw}}(\omega) = \left(\frac{\omega}{\hat{\omega}_{\text{sw},0}} \right)^3 \left[\frac{7}{4 + 3(\omega/\hat{\omega}_{\text{sw},0})^2} \right]^{7/2}, \quad (\text{D.1.25})$$

and $\hat{\omega}_{\text{sw},0}$ is the corresponding present-day peak angular frequency

$$\frac{\hat{\omega}_{\text{sw},0}}{2\pi} = 1.9 \times 10^{-2} \text{ mHz} \frac{1}{v_w} \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \quad (\text{D.1.26})$$

Eq. (D.1.25) imposes a cubic infrared rise and gives an ω^{-4} ultraviolet tail. It was introduced as a practical forecast template for generic wall speeds away from the sound speed and from the speed of light, and it was based on simulations with $\alpha \lesssim 0.1$ [64].

Note that the Hubble-time substitution $\tau_{\text{sw}} \simeq H_*^{-1}$ is a lifetime assumption about the linear acoustic stage. The simulations and analytic arguments behind Eqs. (D.1.18) and (D.1.24) support this substitution when the acoustic source remains linear and weakly damped for at least one Hubble time, with a flow correlation scale well below the Hubble radius. A slightly more careful approach is to consider the timescale set by the nonlinear effects (including the generation of shocks, turbulence and the damping of the fluid flow) [103]

$$\tau_{\text{sh}} \sim \frac{L_f}{\bar{U}_\parallel} \simeq \frac{R_*}{\bar{U}_f}, \quad (\text{D.1.27})$$

where $\bar{U}_\parallel$ is the RMS longitudinal, or compressional, velocity of the sound waves. In the mostly acoustic simulations discussed above, the flow is dominantly longitudinal, so the last estimate uses $L_f \simeq R_*$ and $\bar{U}_\parallel \simeq \bar{U}_f$. The condition for a Hubble-limited acoustic source is then

$$H_* \tau_{\text{sh}} \sim \frac{H_* R_*}{\bar{U}_f} \gtrsim 1. \quad (\text{D.1.28})$$

If instead $H_*\tau_{\text{sh}} < 1$, nonlinear bulk motion occurs before Hubble expansion shuts off the source, and Eq. (D.1.18) should be evaluated with a shorter effective lifetime, parametrically $\tau_{\text{sw}} \sim \tau_{\text{sh}}$. Recall from Eq. (D.1.12) that $\bar{U}_f$ is a function of α and v_w . Thus, for a given terminal wall velocity, $H_*\tau_{\text{sh}}$ can be expressed as a function of α and β/H , making it possible to identify the boundary between long-lasting ($H_*\tau_{\text{sh}} \gtrsim 1$) and short-lasting ($H_*\tau_{\text{sh}} \lesssim 1$) plasma sound-wave sources in the $(\alpha, \beta/H)$ plane. Runaway or nearly vacuum transitions require additional caution because the assumptions of a terminal wall velocity and a plasma-dominated acoustic source may themselves fail.

There are many follow-up works that aim to extend the sound-wave picture, motivated by the results of the simulations. One example is the so-called sound shell model, which refines the acoustic picture by replacing the one-scale spectral template with a hydrodynamic construction of the velocity power spectrum in Ref. [103]. It treats the post-collision velocity field as a linear superposition of sound shells sourced by many bubbles, with each shell initialized by the self-similar fluid profile around an expanding wall and then propagated as an acoustic perturbation. This construction introduces two physical length scales: the mean bubble separation R_* , which controls correlations between different bubbles, and the shell thickness ΔR_* , parametrically of order $R_*|v_w - c_s|/v_w$ up to convention-dependent factors of order unity. When v_w is close to the sound speed, $\Delta R_* \ll R_*$ and the two scales can be resolved separately. It is useful to write the corresponding gravitational-wave spectrum as the double-broken power law [102, 103]

$$\frac{d\Omega_{\text{GW},*}^{\text{sw}}}{d\ln k} \propto \begin{cases} (kR_*)^{p_{\text{IR}}}, & k\Delta R_*, kR_* \ll 1, \\ (kR_*)^1, & k\Delta R_* \ll 1 \ll kR_*, \\ (kR_*)^{-3}, & 1 \ll k\Delta R_*, kR_*, \end{cases} \quad (\text{D.1.29})$$

for a sufficiently thin shell and while the linear sound shell assumptions hold. In the original analysis of Ref. [102], the infrared exponent was $p_{\text{IR}} = 5$. The later refined analysis of Ref. [103] identified an error in this low-wavenumber part of the argument: by enforcing the causal analyticity constraint on the velocity correlator, the idealized acoustic infrared scaling is updated to be approximately $p_{\text{IR}} = 9$. It should also be noted that this statement concerns the idealized acoustic contribution; at sufficiently small k , bubble-collision contributions with a different causal scaling can obscure the acoustic infrared behavior in simulations. The intermediate k^1 regime is absent or very narrow when $\Delta R_* \sim R_*$, which is why a single broken-power-law fit can be an adequate phenomenological summary for wall speeds not close to c_s or for simulations and detector measurements that do not resolve both breaks. Conversely, near $v_w \simeq c_s$ the simple shape in Eq. (D.1.25) should not be interpreted as the full sound shell prediction, because it misses the extra break and the shell-thickness dependence.

To sum up, the acoustic source explains why scalar-field bubble collisions are not generically the dominant GW source in ordinary thermal transitions. The reason is that the lifetime of the TT source is as important as the efficiency with which latent heat is converted into that source. We proceed with the following discussion of turbulence, which describes what can happen after the acoustic stage becomes nonlinear.

D.1.3 Magnetohydrodynamic turbulence in the plasma

Apart from sound waves, the percolation of bubbles during a phase transition could also generate swirling and vortical motions within the surrounding fluid called turbulence. Stated another way, turbulence is the nonlinear fluid source that can follow the acoustic stage of a first-order phase transition. Because the primordial plasma in the early universe is fully ionized and therefore has high electrical conductivity, these nonlinear plasma motions couple to the magnetic field and induce magnetohydrodynamic (MHD) turbulence. Recall that the acoustic signal discussed above is carried mainly by longitudinal, or compressional, fluid motion, whereas turbulence corresponds to the vortical part of the velocity field, which the simulations discussed previously find to be subdominant compared with the compressional component. However, after the characteristic time $\tau_{\text{sh}} \sim (v_w/\sqrt{\kappa_v\alpha})\beta^{-1}$, we expect shocks to form and eventually convert the energy budget stored in sound waves into a turbulent source, provided $H_*\tau_{\text{sh}} < 1$. It is therefore not a third independent reservoir of released vacuum energy in the same sense as scalar walls and bulk sound waves; rather, it is usually modeled as a late-time channel fed by some fraction of the kinetic energy already deposited in the plasma [64, 79, 123].

An MHD description is appropriate because the early-universe plasma has very large kinetic Reynolds numbers ($\sim 10^{13}$ at the EW epoch, which makes turbulent motion set in rapidly in the fluid) and magnetic Reynolds numbers (which lead to the amplification of magnetic fields generated during the phase transition). For a flow stirred on a physical length scale L_f with characteristic velocity U_f , the kinetic Reynolds number is [104]

$$\text{Re}(L_f) = \frac{U_f L_f}{\nu}, \quad (\text{D.1.30})$$

where ν is the kinematic viscosity. Similarly, the magnetic Reynolds number is $\text{R}_m(L_f) = U_f L_f / \mu$, with μ the magnetic diffusivity. When both numbers are large, the stirred fluid develops a turbulent cascade and magnetic fields are advected and amplified rather than being immediately dissipated. The largest eddies are seeded on the phase-transition stirring scale, commonly taken to be of order the mean bubble separation or fluid-flow scale, $L_f \sim R_*$. The eddy turnover time on this scale is

$$\tau_{\text{tu}} \sim \frac{L_f}{\bar{U}_\perp}, \quad (\text{D.1.31})$$

where $\bar{U}_\perp$ is the RMS vortical, or transverse, component of the fluid velocity. This is the analog of the shock time in Eq. (D.1.27), but with the transverse turbulent velocity replacing the longitudinal acoustic velocity.

At the level of the GW source, the relevant object is again the TT part of the spatial stress tensor. The MHD source contains both the vortical fluid shear and the magnetic anisotropic stress. Schematically,

$$\Pi_{ij}^{\text{MHD}} = \left[w \gamma^2 v_i v_j + T_{ij}^B \right]^{\text{TT}}, \quad (\text{D.1.32})$$

where $w = e + p$ is the enthalpy density and T_{ij}^B denotes the magnetic stress. For example, in Lorentz-Heaviside physical-field conventions $T_{ij}^B = B^2 \delta_{ij} / 2 - B_i B_j$ up to the overall sign convention for the spatial stress, while in comoving-field conventions the appropriate powers of the scale factor are absorbed into the definition of B_i [104, 124]. The isotropic magnetic pressure term is removed by the TT projection, just as the isotropic fluid pressure was removed in Eq. (D.1.9). The sign convention of the standalone magnetic term does not affect an autocorrelation-only estimate, but it would matter if one retained fluid-magnetic cross-correlations.

The strength of the turbulent source is usually parametrized by the fraction of released energy ending up in the turbulent source sector. Adapting the notation of the standard template, define

$$K_{\text{turb}} \equiv \frac{\kappa_{\text{turb}} \alpha}{1 + \alpha}, \quad (\text{D.1.33})$$

where κ_{turb} is the efficiency factor used in the turbulence template [64]. In this phenomenological notation K_{turb} is the template-level analog of the vortical kinetic fraction $K_\perp$ used in the scaling discussion of Ref. [79], with possible magnetic stresses included as part of the effective MHD source. In many phenomenological studies one further writes $\kappa_{\text{turb}} = \epsilon \kappa_{\text{sw}}$, with ϵ a small phenomenological fraction, because the turbulent reservoir cannot exceed the total kinetic energy deposited in the plasma [64, 79]. The LISA update emphasizes that this fraction is not yet a reliably predicted microscopic quantity: simulations of weak and intermediate transitions found only a small vortical component, while stronger deflagrations and shock-dominated evolution may generate more vorticity [63, 125, 79].

The characteristic amplitude scaling differs from the acoustic result because turbulence decorrelates on an eddy turnover time. For a turbulent source that persists longer than a Hubble time but decorrelates on τ_{tu} , the estimate reviewed in Ref. [79] gives

$$\Omega_{\text{GW},*}^{\text{turb}} \propto K_{\text{turb}}^{3/2} H_* R_*, \quad (\text{D.1.34})$$

up to a dimensionless spectral factor and order-one constants. Note the different dependence on K_{turb} compared to the sound-wave source (Eq. (D.1.24)) and the bubble collision source (Eq. (D.1.3)). The

explanation is that, for a purely vortical kinetic source, the power $K_{\text{turb}}^{3/2}$ follows parametrically from $\Omega_{\text{GW},*}^{\text{turb}} \propto K_{\text{turb}}^2 H_* \tau_{\text{tu}}$ together with $\tau_{\text{tu}} \sim R_*/\bar{U}_\perp$ and $\bar{U}_\perp^2 \sim K_{\text{turb}}$. In the MHD template this scaling should be read as an effective parametrization of the turbulent kinetic and magnetic source energy, not as a first-principles prediction of the magnetic-field efficiency. If instead the turbulent source is assumed to be short-lived, with $\tau_{\text{on}} \sim \tau_{\text{corr}} \sim \tau_{\text{tu}} \ll H_*^{-1}$, the same scaling arguments lead to a different dependence, $\Omega_{\text{GW},*}^{\text{turb}} \propto K_{\text{turb}}(H_* R_*)^2$ [79]. This sensitivity to the source lifetime and UETC model is one of the main reasons why turbulence is less settled than the acoustic contribution.

The present-day turbulence template used for eLISA [64], assuming Kolmogorov-type turbulence as proposed in Ref. [122], is modeled as

$$h^2 \Omega_{\text{GW},0}^{\text{turb}}(\omega) = 3.35 \times 10^{-4} \left(\frac{H_*}{\beta} \right) K_{\text{turb}}^{3/2} \left(\frac{100}{g_*} \right)^{1/3} v_w S_{\text{turb}}(\omega), \quad (\text{D.1.35})$$

where the dimensionless spectral shape is found analytically and given by

$$S_{\text{turb}}(\omega) = \frac{(\omega/\hat{\omega}_{\text{turb},0})^3}{[1 + (\omega/\hat{\omega}_{\text{turb},0})]^{11/3} (1 + 8\pi\omega/\omega_{H,0})}. \quad (\text{D.1.36})$$

Here ω is the observed angular frequency today, and $\hat{\omega}_{\text{turb},0}$ is the angular-frequency version of the stirring-scale characteristic frequency. Ref. [64], following the redshift convention of Ref. [122], defines $\omega_{H,0}$ as

$$\frac{\omega_{H,0}}{2\pi} = \left(\frac{a_*}{a_0} \right) H_* = 16.5 \times 10^{-3} \text{ mHz} \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \quad (\text{D.1.37})$$

Similar to the sound-wave source, the peak frequency is connected with the bubble size R_* (and equivalently with β) toward the end of the phase transition, specifically $\omega_{\text{turb}} \simeq (2\pi)(3.5/2)(\beta/v_w)$, which becomes, after redshifting,

$$\frac{\hat{\omega}_{\text{turb},0}}{2\pi} = 2.7 \times 10^{-2} \text{ mHz} \frac{1}{v_w} \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \quad (\text{D.1.38})$$

The notation $\hat{\omega}_{\text{turb},0}$ should be interpreted with a small caveat: in the standard turbulence template this parameter is inherited from the stirring scale and is not tuned so that S_{turb} in (D.1.36) reaches its exact maximum at $\omega = \hat{\omega}_{\text{turb},0}$. As in the sound-wave spectrum (D.1.24), the GW spectrum (D.1.34) is enhanced by a factor of β/H_* compared with the bubble-collision source (D.1.8), which is a characteristic feature of sources that last longer than the average duration of the phase transition $1/\beta$.

The shape in Eq. (D.1.36) encodes several physical assumptions. For frequencies below the redshifted Hubble scale it has the causal cubic rise. For the intermediate sub-Hubble range $\omega_{H,0} \ll \omega \ll \hat{\omega}_{\text{turb},0}$, the extra finite-lifetime factor changes the rise effectively to ω^2 . For $\omega \gg \hat{\omega}_{\text{turb},0}$ it falls as $\omega^{-5/3}$, reflecting the Kolmogorov turbulence model assumed in the template [104, 64]. This ultraviolet falloff is much shallower than the ω^{-4} tail in the simple acoustic fit of Eq. (D.1.25). Consequently, even if turbulence is subdominant near the main plasma peak, it can become comparatively more visible in the high-frequency tail when ϵ is not too small.

Figure. D.2 gives an illustrative example of this comparison between the acoustic and turbulent plasma contributions. It should be interpreted only as a historical visualization of the ansatz-based decomposition, because the erratum to Ref. [97] states that the plotted spectrum used an incorrect normalization for the sound-wave contribution and that no updated plot was provided.

Helical magnetic fields provide another possible modification. If a helical magnetic component is generated, approximate magnetic-helicity conservation can drive an inverse cascade, transferring magnetic power from smaller to larger scales. See [126] for a detailed numerical simulation study of models with helical inverse cascade MHD turbulence. In short, the resulting GW spectrum can develop a lower-frequency MHD feature and can be strongly polarized because magnetic helicity is parity odd. It is also found that the gravitational-wave signal sourced by helical MHD turbulent motion exceeds that of bubble collisions as well as that from hydrodynamical turbulence alone. Such

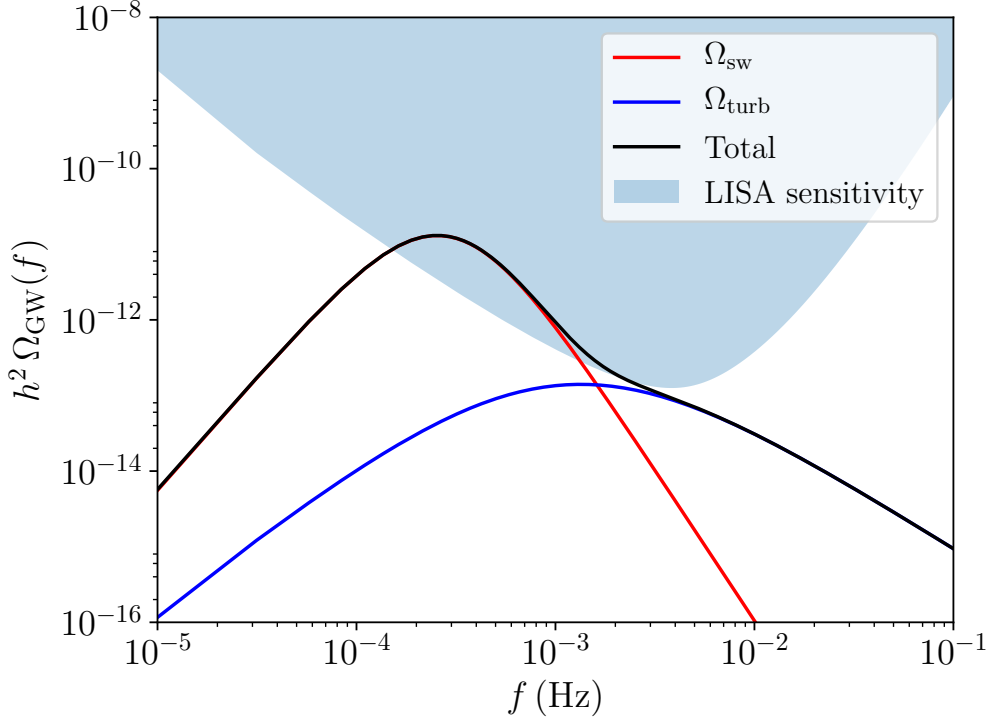


Figure D.2: Example phase-transition gravitational-wave spectrum from Fig. 6 of Ref. [97], showing the sound-wave and turbulence contributions together with the total signal and a LISA sensitivity curve.

effects are not included in the non-helical template of Eqs. (D.1.35)–(D.1.38), so these expressions should not be regarded as a universal fit for MHD turbulence, but rather as a fit for Kolmogorov-type turbulence.

Putting these results together, the most conservative interpretation is that turbulence is a physically motivated but comparatively uncertain plasma contribution. It is expected to be subdominant in weak or intermediate thermal transitions for which the acoustic source remains long-lived and $\kappa_{\text{turb}} \ll \kappa_{\text{sw}}$. It can become more relevant if shocks shorten the acoustic lifetime ($H_* \tau_{\text{sh}} \ll 1$) and leave bulk kinetic energy for nonlinear plasma motion, assuming a sizeable vortical fraction $K_\perp$ is generated from the fluid flow [79, 93]. In non-runaway plasma spectra the turbulence template can be comparatively more important in the high-frequency tail because it falls more slowly than the sound-wave template [64, 124]. This completes the classification of first-order-transition GW sources before returning to scalar-field collision approximations in the next chapter, where the generalized bulk-flow approximation is developed to give more accurate predictions for the GW spectra compared with previous studies of the envelope and bulk-flow approximations.

D.2 Extracting and fitting the surface tension

In solving the evolution of two bubbles on a (s, z) lattice (Eq. (8.2.5)), we implement a simple leap-frog integration. The spatial step is chosen to be $dz = 0.1 L_w^{\text{col}}$, where L_w^{col} is the wall width at the collision time, so that we have a good resolution of the wall throughout the simulation.

The fitting procedure includes the following steps:

- We perform 1 + 1D simulations over various distances and extract the surface tension σ along the bubble wall. The surface density is computed numerically from the field evolution

$$\sigma(t - t_n, t_{n,c} - t_n) = \int_{z_{\text{wall}} - \Delta}^{z_{\text{wall}} + \Delta} dz \rho_z = \int_{z_{\text{wall}} - \Delta}^{z_{\text{wall}} + \Delta} dz \frac{(\partial_z \phi)^2}{2}, \quad (\text{D.2.1})$$

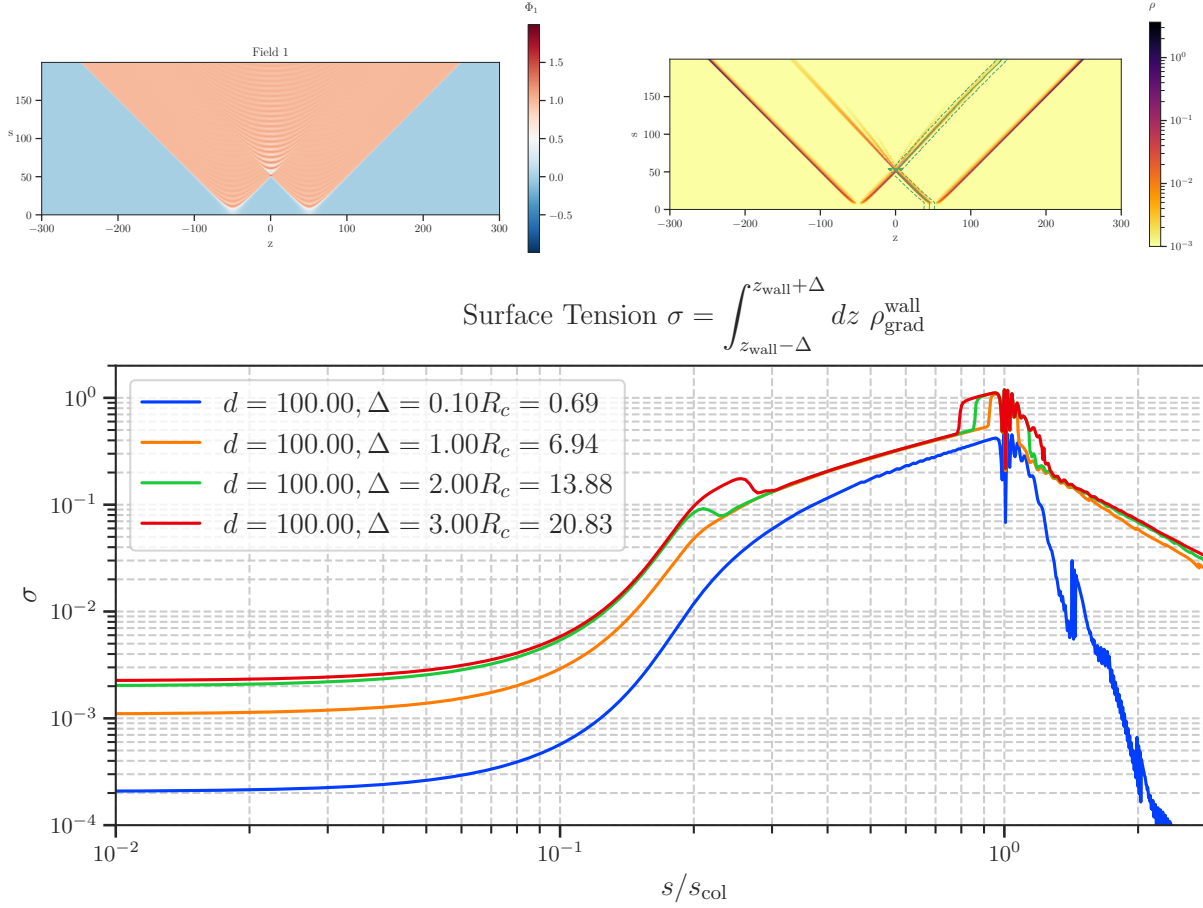


Figure D.3: Evolution of field profiles for a two-bubble collision (top left) and gradient energy density (top right). The green lines on the top right plot capture the region $(z_{\text{wall}} - \Delta, z_{\text{wall}} + \Delta)$ where we compute the surface tension σ . With sufficiently large Δ , the scaling of surface tension right before and after collision time converges, which is shown on the bottom plot. Note that close to $s = s_{\text{col}}$, the integration window captures both walls and hence the jump in σ . The larger Δ is, the broader this nonlinear oscillation window becomes.

where $z_{\text{wall}} \equiv \text{argmax}_z \rho_z$ is the value of z that maximizes ρ_z . In this way, we make sure that we are capturing the surface tension around the wall of one of the bubbles before and after collision. See Fig. D.3 for a visualization of the evolution of the field and energy density over time.

- The surface tension can then be fitted with Eq. (8.1.1)

$$\begin{aligned} \sigma(t - t_n, t_{n,c} - t_n) = & \Theta[(t_{n,c} - t_n) - (t - t_n)] \sigma_0(t - t_n) \\ & + \Theta[(t - t_n) - (t_{n,c} - t_n)] \sigma_0(t_{n,c} - t_n) \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^2 \sum_{\xi} a_{\xi} \left(\frac{t_{n,c} - t_n}{t - t_n} \right)^{\xi}, \end{aligned}$$

in order to extract σ_0 and the dimensionless fitting coefficients a_{ξ} . Specifically, we fit the surface tension before collision (e.g., $s/s_{\text{col}} \in (0.5, 0.8)$) to extract σ_0 , where $\sigma \sim \sigma_0(s/s_{\text{col}})$. The fitting range should be chosen not too close to the nucleation time (as the bubble walls are just starting to expand) and not too close to the collision time. The value of σ_0 is checked to be almost the same for all distances, and will be used to normalize σ for the fitting of a_{ξ} . Next, we perform the fitting of the average of σ/σ_0 over all distances, treating the variation of this quantity across d as errors.

- Finally, having the best-fit coefficients a_{ξ} from the fitting of the surface tension, the GW spectrum under the generalized bulk-flow approximation is directly obtained as discussed in the main text.

For simplicity, throughout the application of the generalized bulk-flow approximation we focus on large-distance effects, where there are three main contributions to the GW spectrum: an envelope contribution a_{env} , a ‘Konstandin’ bulk-flow contribution a_0 , and a mid-power correction term a_1 . Let us define $\hat{s} = s/s_{\text{col}} = 2s/d_i$, $\hat{\sigma}_{d_i}(\hat{s}) \equiv \sigma_{d_i}(\hat{s})/\sigma_0$, where $\sigma_{d_i}(\hat{s})$ is the surface tension along the bubble wall with bubble separation d_i . Recall that before collision, we have the scaling behavior $\hat{\sigma}_{d_i}(\hat{s}) \sim \hat{s}$, and we need to fit the behavior of $\hat{\sigma}_{d_i}(\hat{s})$ after collision, i.e., for $\hat{s} > 1$. In order to obtain the best fit parameters a_{ξ} , we minimize the following χ^2 function

$$\chi^2(a_{\xi}) = \sum_{\hat{s}} \left(\frac{\hat{\sigma}(a_{\xi}; \hat{s}) - \bar{\hat{\sigma}}(\hat{s})}{\delta_{\hat{\sigma}}(\hat{s})} \right)^2, \quad (\text{D.2.2})$$

where

$$\begin{aligned} \hat{\sigma}(a_{\xi}; \hat{s}) &= \sum_{\xi} \frac{a_{\xi}}{\hat{s}^{\xi+2}}, \\ \bar{\hat{\sigma}}(\hat{s}) &= \frac{1}{N_d} \sum_{i=1}^{N_d} \frac{\hat{\sigma}_{d_i}(\hat{s})}{\sigma_0}, \\ \delta_{\hat{\sigma}}(\hat{s}) &= \max_i \sigma_{d_i}(\hat{s}) - \min_i \sigma_{d_i}(\hat{s}), \end{aligned}$$

Next, recall the visualization of the fitting below. Before collision, after a short period after the nucleation, the surface tension approaches a linear dependence on time (hence bubble radius), then the bubble walls collide at $s/s_{\text{col}} \sim 1$, oscillate and decay in the end with seemingly fixed power law. We mainly focus on the scenario where bubbles collide when their sizes are sufficiently large such that the oscillation period at collision (which scales as the size of the wall L_w) is negligible. The reason is twofold. First, the generalized bulk-flow approximation cannot capture the microphysics at the wall collision time due to its nonlinear nature, and we do not expect this approximation to work well in these cases. Second, the larger the bubble walls at collision, the stronger the emitted GWs, which is the realistic case we want to consider.

D.3 Extra results of GW spectra for many-bubble systems

D.3.1 Cross-check with Konstandin’s results

The cross-check between our implementation and Konstandin’s results for envelope and bulk-flow approximation for a system of many bubbles are summarized in Fig. D.5. It shows a good agreement

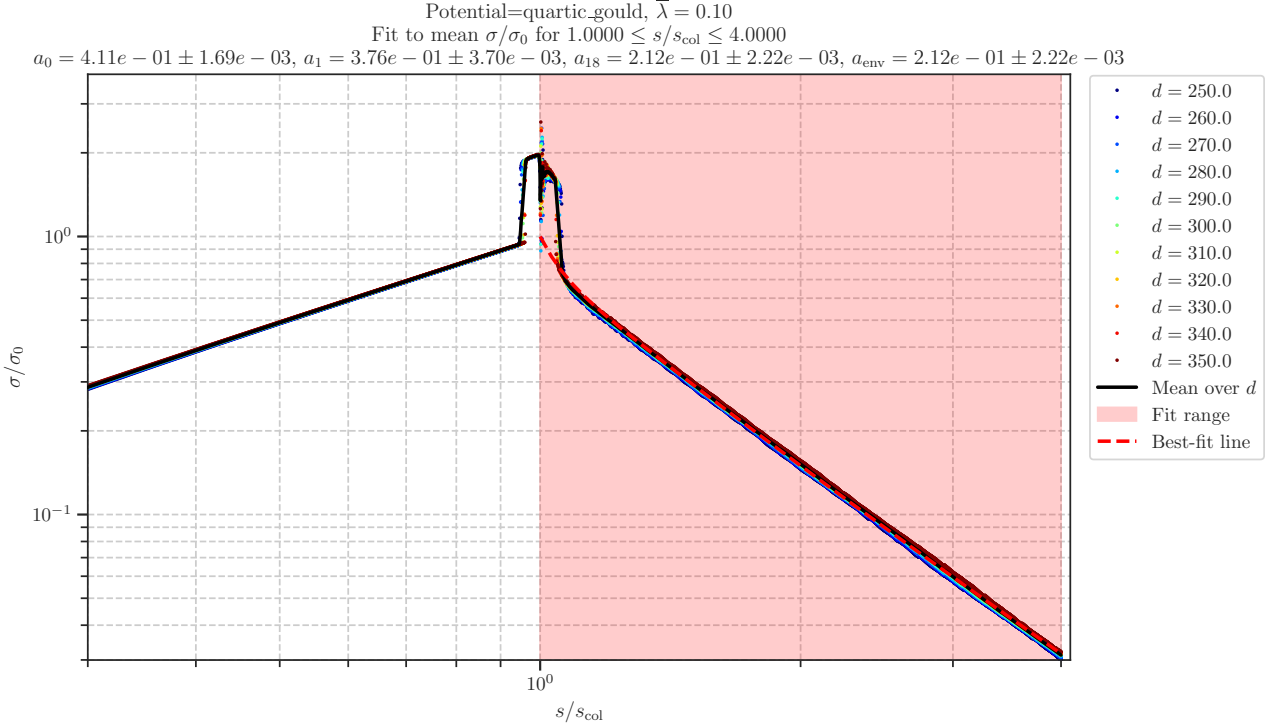


Figure D.4: In 8.2.3, we choose to calculate the GW spectrum at $d = 300$. Therefore, the choices of distances in the fitting should form a window centered at this value.

within the error bands. Small differences is potentially due to the statistical factors from nucleation histories.

D.3.2 Exponential nucleation

Here we put the tables of fitting parameters corresponding to exponential nucleation histories for different box sizes, basically the data corresponding to Fig. 8.17. These results are summarized in Tables. D.2, D.3, and D.4 for the fitting parameters corresponding to envelope, bulk-flow ($\xi = 0$) and generalized bulk-flow ($\xi = 1$) respectively.

Table D.2: Best-fit parameters with asymmetric uncertainties for envelope approximation.

βL_{box}	$\hat{\Delta}$	$\hat{\omega}_2/\beta$	ε_2	p_{IR2}	p_{UV}
5	$0.016^{+0.000}_{-0.000}$	$1.532^{+0.007}_{-0.007}$	$0.503^{+0.005}_{-0.005}$	$2.994^{+0.002}_{-0.002}$	$-1.252^{+0.008}_{-0.008}$
10	$0.048^{+0.000}_{-0.000}$	$0.887^{+0.003}_{-0.003}$	$0.400^{+0.004}_{-0.004}$	$3.018^{+0.002}_{-0.002}$	$-1.005^{+0.004}_{-0.004}$
15	$0.046^{+0.000}_{-0.000}$	$0.672^{+0.003}_{-0.003}$	$0.571^{+0.006}_{-0.006}$	$3.012^{+0.003}_{-0.003}$	$-1.020^{+0.004}_{-0.004}$
20	$0.041^{+0.000}_{-0.000}$	$0.558^{+0.005}_{-0.005}$	$0.589^{+0.008}_{-0.008}$	$3.007^{+0.007}_{-0.007}$	$-1.014^{+0.004}_{-0.004}$
25	$0.034^{+0.000}_{-0.000}$	$0.728^{+0.003}_{-0.003}$	$0.526^{+0.005}_{-0.005}$	$3.017^{+0.002}_{-0.002}$	$-0.995^{+0.004}_{-0.004}$
30	$0.033^{+0.000}_{-0.000}$	$0.665^{+0.003}_{-0.003}$	$0.563^{+0.005}_{-0.005}$	$3.012^{+0.003}_{-0.003}$	$-1.002^{+0.004}_{-0.004}$

D.3.3 Simultaneous nucleation

Here we put the tables of fitting parameters corresponding to simultaneous nucleation histories for different number of bubbles, basically the data corresponding to Fig. 8.17. These results are summarized in Tables. D.5, D.6, and D.7 for the fitting parameters corresponding to envelope, bulk-flow ($\xi = 0$) and generalized bulk-flow ($\xi = 1$) respectively.

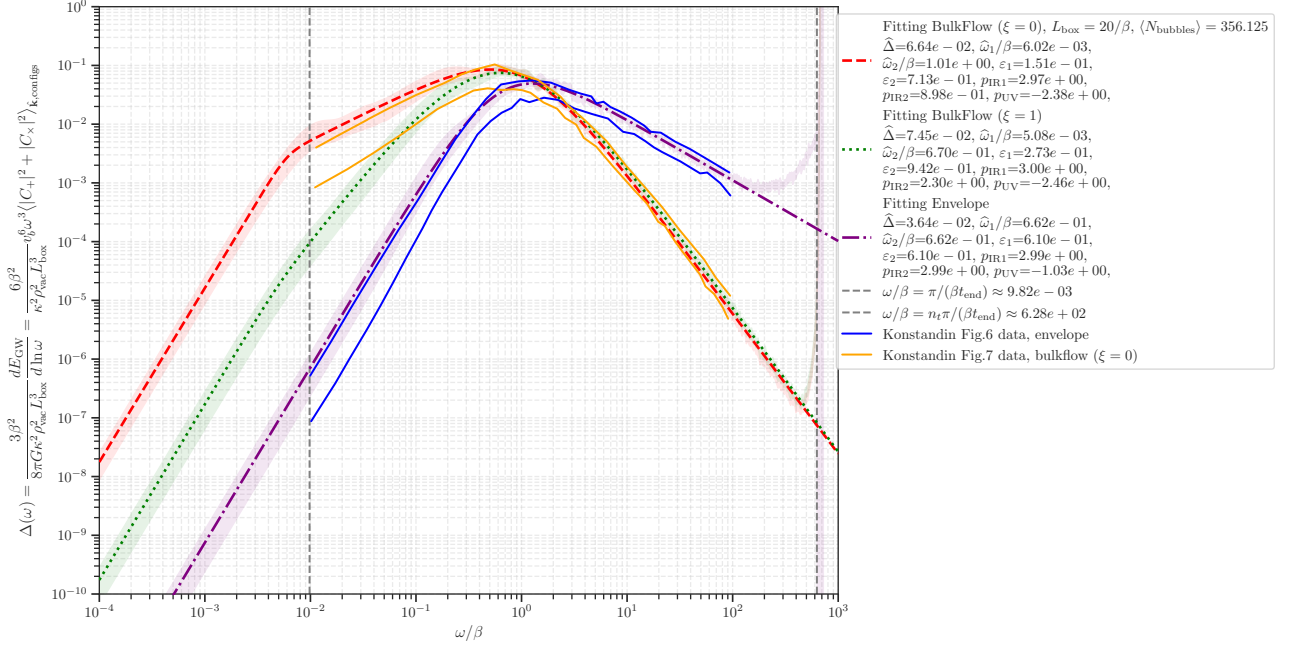


Figure D.5: The fitting results of envelope approximation (purple curve), $\xi = 0$ GBF approximation (red curve) and $\xi = 1$ GBF (green curve), in comparison with the envelope and $\xi = 0$ GBF (solid lines) in [101]. The error bands correspond to mean $\pm 1.96 \times$ standard error of the mean, where we already averaged the spectrum over six directions for each nucleation history before using this to compute the error over histories. The fitting is performed from $\omega = 10^{-3}$ up to the largest frequency before we have divergence in the spectrum (i.e., up to $\sim \omega = 10^2$). The errors of the fitting parameters correspond to the 1σ confidence intervals.

Table D.3: Best-fit parameters with asymmetric uncertainties for bulk-flow approximation ($\xi = 0$).

βL_{box}	$\hat{\Delta}$	$\hat{\omega}_1/\beta$	$\hat{\omega}_2/\beta$	ε_1	ε_2	p_{IR1}	p_{IR2}	p_{UV}
5	$0.034^{+0.000}_{-0.000}$	$0.006^{+0.000}_{-0.000}$	$1.919^{+0.010}_{-0.009}$	$0.220^{+0.011}_{-0.011}$	$0.329^{+0.006}_{-0.006}$	$2.970^{+0.006}_{-0.006}$	$0.841^{+0.004}_{-0.004}$	$-2.597^{+0.007}_{-0.007}$
10	$0.081^{+0.001}_{-0.001}$	$0.006^{+0.000}_{-0.000}$	$1.202^{+0.011}_{-0.011}$	$0.129^{+0.021}_{-0.022}$	$0.472^{+0.012}_{-0.012}$	$2.966^{+0.011}_{-0.010}$	$0.918^{+0.009}_{-0.009}$	$-2.352^{+0.009}_{-0.009}$
15	$0.084^{+0.001}_{-0.001}$	$0.006^{+0.000}_{-0.000}$	$0.977^{+0.011}_{-0.011}$	$0.149^{+0.026}_{-0.027}$	$0.603^{+0.013}_{-0.013}$	$2.967^{+0.012}_{-0.012}$	$0.896^{+0.014}_{-0.014}$	$-2.317^{+0.007}_{-0.007}$
20	$0.077^{+0.001}_{-0.001}$	$0.006^{+0.000}_{-0.000}$	$0.934^{+0.011}_{-0.011}$	$0.130^{+0.026}_{-0.028}$	$0.714^{+0.012}_{-0.012}$	$2.969^{+0.012}_{-0.012}$	$0.954^{+0.015}_{-0.015}$	$-2.386^{+0.007}_{-0.007}$
25	$0.061^{+0.001}_{-0.001}$	$0.006^{+0.000}_{-0.000}$	$1.048^{+0.008}_{-0.008}$	$0.166^{+0.016}_{-0.016}$	$0.568^{+0.009}_{-0.009}$	$2.967^{+0.008}_{-0.008}$	$0.908^{+0.007}_{-0.007}$	$-2.325^{+0.006}_{-0.006}$
30	$0.062^{+0.001}_{-0.001}$	$0.006^{+0.000}_{-0.000}$	$1.029^{+0.009}_{-0.009}$	$0.158^{+0.022}_{-0.022}$	$0.585^{+0.011}_{-0.010}$	$2.966^{+0.011}_{-0.011}$	$0.874^{+0.010}_{-0.010}$	$-2.377^{+0.007}_{-0.007}$

Table D.4: Best-fit parameters with asymmetric uncertainties for bulk-flow approximation ($\xi = 1$).

βL_{box}	$\hat{\Delta}$	$\hat{\omega}_1/\beta$	$\hat{\omega}_2/\beta$	ε_1	ε_2	p_{IR1}	p_{IR2}	p_{UV}
5	$0.036^{+0.000}_{-0.000}$	$0.055^{+0.009}_{-0.008}$	$1.791^{+0.012}_{-0.012}$	$1.880^{+0.078}_{-0.078}$	$0.416^{+0.011}_{-0.011}$	$3.150^{+0.018}_{-0.017}$	$1.161^{+0.080}_{-0.084}$	$-2.737^{+0.018}_{-0.021}$
10	$0.091^{+0.001}_{-0.001}$	$0.005^{+0.000}_{-0.000}$	$0.832^{+0.004}_{-0.004}$	$0.205^{+0.047}_{-0.048}$	$0.694^{+0.006}_{-0.006}$	$2.988^{+0.008}_{-0.008}$	$2.316^{+0.009}_{-0.009}$	$-2.416^{+0.006}_{-0.006}$
15	$0.096^{+0.001}_{-0.001}$	$0.005^{+0.000}_{-0.000}$	$0.637^{+0.008}_{-0.007}$	$0.195^{+0.092}_{-0.099}$	$0.830^{+0.012}_{-0.012}$	$2.989^{+0.016}_{-0.015}$	$2.319^{+0.021}_{-0.022}$	$-2.375^{+0.008}_{-0.008}$
20	$0.088^{+0.001}_{-0.001}$	$0.004^{+0.000}_{-0.000}$	$0.569^{+0.008}_{-0.008}$	$0.108^{+0.115}_{-nan}$	$0.979^{+0.012}_{-0.012}$	$2.993^{+0.019}_{-0.018}$	$2.423^{+0.024}_{-0.024}$	$-2.457^{+0.008}_{-0.008}$
25	$0.065^{+0.000}_{-0.000}$	$0.005^{+0.000}_{-0.000}$	$0.689^{+0.004}_{-0.004}$	$0.205^{+0.044}_{-0.045}$	$0.841^{+0.007}_{-0.007}$	$2.990^{+0.007}_{-0.007}$	$2.342^{+0.010}_{-0.010}$	$-2.387^{+0.006}_{-0.006}$
30	$0.068^{+0.000}_{-0.000}$	$0.005^{+0.000}_{-0.000}$	$0.673^{+0.006}_{-0.006}$	$0.249^{+0.073}_{-0.072}$	$0.854^{+0.009}_{-0.009}$	$2.992^{+0.012}_{-0.011}$	$2.295^{+0.017}_{-0.018}$	$-2.428^{+0.007}_{-0.007}$

Table D.5: Best-fit parameters with asymmetric uncertainties for envelope approximation. Note that in order for the fitting to work with envelope spectrum, we put the constraints $\hat{\omega}_1 = \hat{\omega}_2$, $\varepsilon_1 = \varepsilon_2$ and $p_{IR1} = p_{IR2}$.

n_{bubbles}	$\hat{\Delta}$	$\hat{\omega}_2/\beta$	ε_2	p_{IR2}	p_{UV}
50	$0.057^{+0.003}_{-0.003}$	$0.576^{+0.014}_{-0.013}$	$0.372^{+0.028}_{-0.028}$	$3.015^{+0.020}_{-0.020}$	$-0.795^{+0.020}_{-0.020}$
100	$0.030^{+0.001}_{-0.001}$	$0.772^{+0.018}_{-0.017}$	$0.379^{+0.025}_{-0.025}$	$3.080^{+0.024}_{-0.023}$	$-0.837^{+0.022}_{-0.023}$
200	$0.019^{+0.001}_{-0.001}$	$1.002^{+0.016}_{-0.016}$	$0.319^{+0.020}_{-0.020}$	$2.996^{+0.012}_{-0.012}$	$-0.794^{+0.018}_{-0.018}$
300	$0.013^{+0.001}_{-0.001}$	$1.084^{+0.021}_{-0.020}$	$0.377^{+0.023}_{-0.023}$	$3.023^{+0.015}_{-0.014}$	$-0.787^{+0.021}_{-0.022}$
400	$0.010^{+0.000}_{-0.000}$	$1.201^{+0.022}_{-0.021}$	$0.375^{+0.020}_{-0.020}$	$3.089^{+0.015}_{-0.015}$	$-0.779^{+0.022}_{-0.023}$
500	$0.010^{+0.000}_{-0.000}$	$1.376^{+0.023}_{-0.023}$	$0.336^{+0.021}_{-0.021}$	$2.985^{+0.011}_{-0.011}$	$-0.781^{+0.021}_{-0.022}$
600	$0.008^{+0.000}_{-0.000}$	$1.418^{+0.026}_{-0.025}$	$0.360^{+0.021}_{-0.021}$	$3.022^{+0.017}_{-0.016}$	$-0.751^{+0.022}_{-0.022}$
700	$0.007^{+0.000}_{-0.000}$	$1.480^{+0.024}_{-0.024}$	$0.393^{+0.021}_{-0.021}$	$3.027^{+0.011}_{-0.011}$	$-0.791^{+0.022}_{-0.023}$
800	$0.007^{+0.000}_{-0.000}$	$1.601^{+0.026}_{-0.026}$	$0.358^{+0.019}_{-0.018}$	$3.044^{+0.012}_{-0.011}$	$-0.776^{+0.022}_{-0.023}$
900	$0.006^{+0.000}_{-0.000}$	$1.684^{+0.024}_{-0.024}$	$0.320^{+0.017}_{-0.017}$	$3.009^{+0.009}_{-0.009}$	$-0.742^{+0.019}_{-0.019}$

Table D.6: Best-fit parameters with asymmetric uncertainties for bulk-flow approximation ($\xi = 0$).

n_{bubbles}	$\hat{\Delta}$	$\hat{\omega}_1/\beta$	$\hat{\omega}_2/\beta$	ε_1	ε_2	p_{IR1}	p_{IR2}	p_{UV}
50	$0.123^{+0.004}_{-0.004}$	$0.014^{+0.001}_{-0.001}$	$0.891^{+0.027}_{-0.026}$	$0.139^{+0.054}_{-0.057}$	$0.649^{+0.034}_{-0.033}$	$2.930^{+0.047}_{-0.043}$	$0.902^{+0.043}_{-0.042}$	$-2.424^{+0.023}_{-0.025}$
100	$0.070^{+0.002}_{-0.002}$	$0.013^{+0.001}_{-0.001}$	$1.133^{+0.024}_{-0.024}$	$0.107^{+0.064}_{-0.086}$	$0.542^{+0.025}_{-0.024}$	$2.920^{+0.063}_{-0.057}$	$0.939^{+0.029}_{-0.028}$	$-2.307^{+0.017}_{-0.018}$
200	$0.042^{+0.001}_{-0.001}$	$0.012^{+0.001}_{-0.001}$	$1.432^{+0.032}_{-0.031}$	$0.113^{+0.056}_{-0.067}$	$0.511^{+0.027}_{-0.027}$	$2.917^{+0.054}_{-0.049}$	$0.903^{+0.024}_{-0.024}$	$-2.285^{+0.023}_{-0.024}$
300	$0.033^{+0.001}_{-0.001}$	$0.012^{+0.000}_{-0.000}$	$1.560^{+0.028}_{-0.027}$	$0.128^{+0.048}_{-0.052}$	$0.554^{+0.022}_{-0.021}$	$2.919^{+0.044}_{-0.041}$	$0.959^{+0.021}_{-0.020}$	$-2.281^{+0.019}_{-0.019}$
400	$0.026^{+0.001}_{-0.001}$	$0.012^{+0.001}_{-0.001}$	$1.740^{+0.038}_{-0.036}$	$0.108^{+0.061}_{-0.080}$	$0.550^{+0.025}_{-0.024}$	$2.919^{+0.059}_{-0.054}$	$0.979^{+0.022}_{-0.022}$	$-2.254^{+0.023}_{-0.024}$
500	$0.022^{+0.000}_{-0.000}$	$0.012^{+0.000}_{-0.000}$	$1.983^{+0.033}_{-0.032}$	$0.123^{+0.040}_{-0.043}$	$0.450^{+0.018}_{-0.018}$	$2.914^{+0.039}_{-0.036}$	$0.867^{+0.014}_{-0.014}$	$-2.203^{+0.017}_{-0.018}$
600	$0.020^{+0.000}_{-0.000}$	$0.011^{+0.001}_{-0.001}$	$1.993^{+0.031}_{-0.030}$	$0.087^{+0.064}_{-nan}$	$0.498^{+0.017}_{-0.017}$	$2.921^{+0.063}_{-0.057}$	$0.945^{+0.018}_{-0.018}$	$-2.197^{+0.017}_{-0.017}$
700	$0.018^{+0.000}_{-0.000}$	$0.012^{+0.000}_{-0.000}$	$2.052^{+0.031}_{-0.030}$	$0.129^{+0.043}_{-0.046}$	$0.529^{+0.020}_{-0.019}$	$2.917^{+0.041}_{-0.038}$	$0.937^{+0.015}_{-0.015}$	$-2.214^{+0.018}_{-0.019}$
800	$0.017^{+0.000}_{-0.000}$	$0.012^{+0.000}_{-0.000}$	$2.127^{+0.036}_{-0.035}$	$0.104^{+0.043}_{-0.050}$	$0.447^{+0.019}_{-0.018}$	$2.917^{+0.041}_{-0.039}$	$0.941^{+0.015}_{-0.014}$	$-2.141^{+0.018}_{-0.019}$
900	$0.015^{+0.000}_{-0.000}$	$0.012^{+0.000}_{-0.000}$	$2.302^{+0.035}_{-0.034}$	$0.095^{+0.046}_{-0.058}$	$0.453^{+0.018}_{-0.017}$	$2.917^{+0.046}_{-0.043}$	$0.929^{+0.014}_{-0.013}$	$-2.160^{+0.018}_{-0.018}$

Table D.7: Best-fit parameters with asymmetric uncertainties for generalized bulk-flow approximation ($\xi = 1$).

n_{bubbles}	$\hat{\Delta}$	$\hat{\omega}_1/\beta$	$\hat{\omega}_2/\beta$	ε_1	ε_2	p_{IR1}	p_{IR2}	p_{UV}
50	$0.139^{+0.002}_{-0.002}$	$0.011^{+0.001}_{-0.001}$	$0.598^{+0.011}_{-0.010}$	$0.183^{+0.123}_{-0.118}$	$0.769^{+0.016}_{-0.016}$	$2.976^{+0.037}_{-0.029}$	$2.324^{+0.034}_{-0.038}$	$-2.422^{+0.012}_{-0.013}$
100	$0.075^{+0.001}_{-0.001}$	$0.010^{+0.002}_{-0.002}$	$0.783^{+0.011}_{-0.010}$	$0.158^{+0.213}_{-0.158}$	$0.699^{+0.013}_{-0.014}$	$2.971^{+0.075}_{-0.049}$	$2.363^{+0.027}_{-0.030}$	$-2.339^{+0.011}_{-0.011}$
200	$0.045^{+0.001}_{-0.001}$	$0.011^{+0.001}_{-0.001}$	$1.044^{+0.012}_{-0.011}$	$0.213^{+0.102}_{-0.103}$	$0.639^{+0.014}_{-0.014}$	$2.969^{+0.042}_{-0.034}$	$2.220^{+0.021}_{-0.022}$	$-2.311^{+0.014}_{-0.014}$
300	$0.034^{+0.000}_{-0.000}$	$0.010^{+0.001}_{-0.001}$	$1.112^{+0.015}_{-0.014}$	$0.219^{+0.180}_{-0.152}$	$0.732^{+0.015}_{-0.016}$	$2.975^{+0.062}_{-0.040}$	$2.346^{+0.024}_{-0.029}$	$-2.354^{+0.016}_{-0.016}$
400	$0.027^{+0.000}_{-0.000}$	$0.010^{+0.001}_{-0.001}$	$1.212^{+0.013}_{-0.012}$	$0.154^{+0.169}_{-nan}$	$0.729^{+0.013}_{-0.013}$	$2.972^{+0.059}_{-0.042}$	$2.423^{+0.019}_{-0.020}$	$-2.334^{+0.015}_{-0.015}$
500	$0.023^{+0.000}_{-0.000}$	$0.012^{+0.001}_{-0.001}$	$1.436^{+0.019}_{-0.018}$	$0.372^{+0.121}_{-0.111}$	$0.641^{+0.015}_{-0.015}$	$2.996^{+0.056}_{-0.042}$	$2.195^{+0.024}_{-0.028}$	$-2.278^{+0.015}_{-0.015}$
600	$0.020^{+0.000}_{-0.000}$	$0.011^{+0.004}_{-0.005}$	$1.499^{+0.040}_{-0.031}$	$0.664^{+0.467}_{-0.277}$	$0.651^{+0.020}_{-0.021}$	$3.079^{+0.324}_{-0.124}$	$2.206^{+0.055}_{-0.092}$	$-2.257^{+0.019}_{-0.020}$
700	$0.018^{+0.000}_{-0.000}$	$0.073^{+0.202}_{-0.056}$	$1.688^{+0.063}_{-0.054}$	$2.226^{+0.553}_{-0.439}$	$0.568^{+0.056}_{-0.078}$	$3.309^{+0.349}_{-0.187}$	$1.240^{+0.468}_{-0.637}$	$-2.369^{+0.089}_{-0.155}$
800	$0.017^{+0.000}_{-0.000}$	$0.007^{+0.004}_{-0.005}$	$1.623^{+0.054}_{-0.056}$	$1.054^{+1.029}_{-0.589}$	$0.624^{+0.030}_{-0.030}$	$3.288^{+nan}_{-0.276}$	$2.199^{+0.110}_{-0.220}$	$-2.221^{+0.019}_{-0.023}$
900	$0.015^{+0.000}_{-0.000}$	$0.009^{+0.003}_{-0.004}$	$1.731^{+0.047}_{-0.035}$	$0.818^{+0.573}_{-0.301}$	$0.631^{+0.020}_{-0.023}$	$3.163^{+0.408}_{-0.138}$	$2.207^{+0.059}_{-0.113}$	$-2.234^{+0.018}_{-0.019}$

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