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Quantum Fields and the Cosmological Constant

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Abstract: It has been shown that if one solves self-consistently the semiclassical Einstein equations in the presence of a quantum scalar field, with a cutoff on the number of modes, spacetime become flatter when the cutoff increases. Here, we extend the result to include the effect of fields with spin 0, 1/2, 1 and 2. With minor adjustments, the main result persists. Remarkably, one can have positive curvature even if the cosmological constant in the bare action is negative.

Keywords: QFT in curved spacetime; renormalization; cosmological constant; backreaction

1. Introduction

The cosmological constant problem consists of various questions [1]. The first and most striking one concerns the effect of vacuum fluctuations on the curvature of spacetime. If one thinks of a quantum field as a large number of oscillators, the vacuum energy is the sum of their ground state energies. Since a curved manifold looks flat at short distances, the divergences of a quantum field in curved spacetime are the same as in flat spacetime. Based on this observation, it is a generally accepted practice to calculate the expectation value of the energy–momentum (EM) tensor in a fixed (typically flat) metric and then to insert the result in the semiclassical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle \,, \quad (1)$$

where Λ is a cosmological constant. Since the expectation value of the EM tensor grows quartically with the ultraviolet cutoff, even a very low cutoff leads to an unacceptably large curvature.[1] If quantum field theory is valid up to the Planck scale, the whole universe should be of Planck size. This is often presented as the greatest discrepancy between theory and observation in the history of science. Clearly, there must be something wrong on the side of the theory.[2]

Becker and Reuter have argued that it is inconsistent to use the EM tensor computed in one metric as the source of another metric: the metric has a nontrivial effect on the expectation value of the EM tensor that cannot be ignored and changes the result drastically. Instead, one has to calculate the EM tensor self-consistently, in the metric that solves the semiclassical Einstein Equation (1) *in the presence of that EM tensor* [2,3]. This is, in general, a very hard task. As an illustration of how it could work, Becker and Reuter have considered a maximally symmetric background, where the only free parameter of the metric is the radius, or equivalently the scalar curvature R. The concrete calculation (that we shall review in Section 2) is performed in Euclidean space, i.e., on a sphere. Since the spectrum is

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discrete, it is natural to put a cutoff N on the principal quantum number. For fixed N, one can then compute the curvature self-consistently, and surprisingly it decreases like $1/N^2$. The result has been extended also to the noncompact, negative curvature case [4].

There are some aspects of this calculation that require some comments. The cutoff N is dimensionless, instead of having the standard dimensions of mass. Furthermore, notice that calling it a cutoff is slightly misleading: it suggests that it is a number that has to be sent to infinity in order to recover the continuum. Then, according to the standard quantum field theoretic interpretation, quantities computed at fixed N would not be physical: only renormalized quantities, from which the infinities have been subtracted, would be physical. As emphasized by Becker and Reuter, here one must adopt a different interpretation: the cutoff has to be taken seriously as a physical quantity, with each N defining a different world that could in principle exist as a physical system.[3] Also, note that the equation of motion has to be solved for finite cutoff. If one wants to do so, the cutoff can be sent to infinity, but only after the equation of motion has been solved.

Conceptually, this calculation bears some similarity to the ADM calculation of the self-energy of a massive particle [6], also reviewed in [7] and more recently revisited in [8]. Normally, if we consider a point particle of mass m_0 to be the limit $\epsilon \to 0$ of a shell of radius ϵ, the energy of its gravitational field is $m_0 - \frac{m_0^2}{2\epsilon}$. Sending the cutoff to zero and solving the equation of motion generates a singular solution with infinite self-energy. This is analogous to the standard way of computing the cosmological constant. Instead, ADM finds that, using the general relativistic equation of motion at finite ϵ, the energy is proportional to $\sqrt{\epsilon}$ [6]. Thus, taking the limit $\epsilon \to 0$, an uncharged particle with finite "bare" mass has zero physical gravitational mass, and the solution it generates is flat space. In Becker and Reuter's calculation, the "bare" cosmological constant is the analog of the bare mass. Also, in this case, the equation of motion has to be solved for finite cutoff N, and only then can the cutoff be removed, leaving flat space, irrespective of the bare cosmological constant. The fact that the ADM calculation is purely classical, whereas the Becker–Reuter one is quantum, does not detract very much from the analogy: in both cases, the essential point is the use of the gravitational equation of motion before removing the regulating parameter.

The paper [2] dealt with a single scalar field, while [3] dealt with quantum fluctuations of the metric itself, where some undesirable technical features appeared. In this paper, we consider the more general case of scalar, fermion, vector and symmetric tensor quantum field theories (corresponding, on flat space, to spin zero, one half, one and two) on a four-dimensional sphere. The fields are first taken in isolation and then in arbitrary combinations so that the results can be applied to semi-realistic models consisting of various combinations of free fields. For more physical models, one has to include the effects of interactions. In the scalar case, this has been discussed in [9], where it was found that putting the classical metric on shell (i.e., solving the semiclassical Einstein Equation (1)) actually removes the divergences in the mass and quartic self-couplings. This adds to the similarities with the ADM calculation, where it was found that if the particle also has a charge, its electromagnetic self-energy is made finite by the interaction with gravity.

This paper is structured as follows. In Section 2, we set the stage, we review the computation of the one loop effective action of fields in Euclidean de Sitter and we introduce the use of the N-cutoffs. Sections 3–6 are devoted to the treatment of scalars, fermions, Maxwell fields and gravitons, respectively. New is the inclusion of the fermions and the Maxwell fields. In Section 7, we consider all the kind of fields and study different combinations and how they contribute to the effective curvature. Section 8 contains a discussion about a recently published paper of Branchina et al. [10], where a similar computation is carried out with a different measure. Finally, we conclude in Section 9.

2. Quantum Fields on Euclidean de Sitter Space

In this section, we review the general logic of the calculation of [2], following the treatment in [9]. We assume that we are in a semiclassical regime where the metric $g_{\mu\nu}$ can be treated as a classical field with the (Euclidean) Hilbert action (and cosmological term)

$$S_H(g) = \frac{1}{16\pi G} \int d^4x \sqrt{g}[2\Lambda - R] \,, \tag{2}$$

interacting with a quantum field ϕ with action $S_m(\phi; g)$. The one loop effective action (EA), which includes the effect of the backreaction of the quantum field, is given by the somewhat abstract formula

$$\Gamma(g, \phi) = S_H(g) + S_m(g, \phi) + \frac{1}{2}\mathrm{Tr}\log(\Delta/\mu^2) \tag{3}$$

where

$$\Delta = -\nabla^2 + E \,, \tag{4}$$

is a Laplace-type operator, with E given by the mass squared of the field and possibly other terms proportional to the curvature. The mass μ a physically unimportant reference scale. Variation in the EA with respect to the metric yields the semiclassical Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle \tag{5}$$

where the l.h.s. comes from the classical Hilbert action (with a bare cosmological term) and the r.h.s is the VEV of the EM tensor of the quantum field.

For our purposes, it will be sufficient to study the EA on a Euclidean de Sitter space, i.e., a sphere S^4. Then, the only metric degree of freedom is the radius r or equivalently the (constant) scalar curvature $R = 12/r^2$. Equation (5) reduces to the single trace equation

$$-R + \Lambda = 8\pi G\langle T^\mu_\mu\rangle \,. \tag{6}$$

We can also obtain this equation directly by deriving the EA with respect to R. For example, at classical level, recalling that the volume of the four-sphere is

$$V_4 = \frac{384\pi^2}{R^2} \,, \tag{7}$$

the Hilbert action for a spherical metric can be written

$$S_H(R) = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} \tag{8}$$

and deriving with respect to R we obtain the classical equation

$$R = 4\Lambda \,. \tag{9}$$

Aside from eliminating all gravitational degrees of freedom except for the overall scale, the advantage of working on a sphere is that the spectrum of Laplacians is well known, and therefore it is possible to calculate the EA without resorting to heat kernel asymptotics.

The operation Tr in the definition (3) of the EA is a functional trace that on the sphere can be written explicitly as a sum over all eigenstates of the Laplacian. We regulate the sum by putting an upper bound N on the principal quantum number ℓ:

$$\frac{1}{2}\mathrm{Tr}_N \log(\Delta/\mu^2) = \frac{1}{2}\sum_{\ell=1}^{N} m_\ell \log(\lambda_\ell/\mu^2) \,, \tag{10}$$

We can compare this procedure to the more standard one of cutting off the sum at some cutoff C with dimension of mass, such that

$$\lambda_\ell < C^2 \,. \tag{11}$$

At this stage, there is no significant difference between the two procedures because the dimensionless and the dimensionful cutoffs are simply related by

$$C^2 = \lambda_N \,. \tag{12}$$

In fact, the divergences for $N \to \infty$ match those for $C \to \infty$. However, we shall see in the following that when we demand that the background metric satisfies Einstein's equations, the two procedures lead to very different conclusions.

In the next sections, we specify the calculation to fields with different spins: we consider scalar fields, fermions, vector fields and the graviton.

3. Scalar Field

For a scalar field, the eigenvalues and their multiplicities are

$$\lambda_\ell = \frac{R}{12}\ell(\ell+3) + E\,, \qquad m_\ell = \frac{1}{6}(\ell+1)(\ell+2)(2\ell+3) \tag{13}$$

where E is the squared mass plus possibly a nonminimal term proportional to R, and $\ell = 0, 1 \ldots$. In the massless, minimally coupled case, the EA is

$$\Gamma_N(R) = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} + \frac{1}{2}\mathbf{f}_S(N)\log\left(\frac{R}{12\mu^2}\right) + \frac{1}{2}\mathbf{T}_S(N,0)\,, \tag{14}$$

where

$$\mathbf{f}_S(N) = \sum_{\ell=1}^{N} m_\ell = \frac{1}{12}N(N+4)(N^2+4N+7) \tag{15}$$

is the total number of scalar modes included in the trace and

$$\mathbf{T}_S(N,z) = \sum_{\ell=1}^{N} m_\ell \log(\ell(\ell+3)+z)\,, \tag{16}$$

is the quantum trace for a sphere of radius $R = 12\mu^2$ and $z = 12E/R$. The functions $\mathbf{f}_S$ and $\mathbf{T}_S$ diverge quartically for $N \to \infty$, In the following, we shall also need the function

$$\mathbf{S}_S(N,z) \equiv \frac{\partial \mathbf{T}_S(N,z)}{\partial z} = \sum_{\ell=1}^{N} m_\ell \frac{1}{\ell(\ell+3)+z} \tag{17}$$

that only diverges quadratically.

The equation of motion for the metric is[4]

$$\frac{24\pi}{GR^3}(-R+4\Lambda) = \frac{1}{2R}\mathbf{f}_S(N)\,, \tag{18}$$

whose solution is a sphere of curvature

$$\begin{aligned} R &= \frac{24\pi}{G\mathbf{f}_S(N)}\left(-1 \pm \sqrt{1+\frac{G\Lambda\mathbf{f}_S(N)}{3\pi}}\right) \\ &= 48\sqrt{\frac{\pi\Lambda}{G}}\left(\frac{1}{N^2}-\frac{4}{N^3}\right) + \left(\frac{600\sqrt{\pi\Lambda}}{\sqrt{G}} - \frac{288\pi}{G}\right)\frac{1}{N^4} + \ldots \end{aligned} \tag{19}$$

where in the expansion we have selected the solution with the upper sign because the other gives negative curvature. This is Becker and Reuter's main result. We see that, opposite to standard lore, the curvature of spacetime *decreases* in units of $\sqrt{\Lambda/G}$, when more quantum modes are included in the calculation.

In the massive case, the EA for the metric becomes

$$\Gamma_N(R) = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} + \frac{1}{2}\mathbf{f}_S(N)\log\left(\frac{R}{12\mu^2}\right) + \frac{1}{2}\mathbf{T}_S\left(N, \frac{12m^2}{R}\right). \tag{20}$$

The quantum trace is independent of ϕ, so the scalar potential does not receive any quantum corrections. Since the sum is finite, we can take the derivative under the sum so that the equation of motion for the metric can be written in the form

$$\frac{24\pi}{GR^3}(-R+4\Lambda) = \frac{1}{2R}\left[\mathbf{f}_S(N) - \frac{12m^2}{R}\mathbf{S}_S\left(N, \frac{12m^2}{R}\right)\right]. \tag{21}$$

This equation cannot be solved analytically, but one can make an ansatz

$$R = \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4} + \ldots \tag{22}$$

and determine the coefficients K_n iteratively. Expanding for small masses (and assuming $\Lambda > 0$),

$$K_2 = 48\sqrt{\frac{\pi\Lambda}{G}} + 12m^2 + O(m^4) \tag{23}$$

$$K_3 = -192\sqrt{\frac{\pi\Lambda}{G}} - 48m^2 + O(m^4) \tag{24}$$

$$K_4 = \left(\frac{600\sqrt{\pi\Lambda}}{\sqrt{G}} - \frac{288\pi}{G}\right) + 128m^2 + \ldots \tag{25}$$

We refer to [9] for more coefficients. Note that the mass-independent terms are the same as in (19).

A few comments on orders of magnitude are in order: in these calculations, it seems natural to choose Λ of order one in Planck units. Then, the coefficient K_2 is of the order of the square Planck mass. If we were to choose $\Lambda = 0$, there would still be a N^{-2} suppression with a much smaller coefficient K_2 of the order of the square of the scalar mass. If $\Lambda = 0$ and the scalar field is also massless, the solution with the upper sign in (19) is identically zero. The one with the lower sign has $K_2 = 0$ and $K_3 = 0$, and R decreases like N^{-4}, with a coefficient K_4 that is again of order one in Planck units, but negative, so not compatible with the assumed positive curvature. This problem is specific to the scalar field, and we shall return to it later when we discuss other matter fields.

4. Fermion

We consider at first the case of a Dirac field minimally coupled to gravity. The effective action of the system is

$$\Gamma_N(g,\psi) = S_H(g) + S_m(g,\psi) - \mathrm{Tr}_N \log(\mathcal{D}/\mu) \tag{26}$$

where $\mathcal{D}$ is the Dirac operator. Its eigenvalues and multiplicities are given by [11]

$$\lambda_\ell^\pm = \pm\sqrt{\frac{R}{12}}(\ell+2), \qquad m_\ell = 4\binom{\ell+3}{\ell}, \qquad \ell = 0,1,2\ldots \tag{27}$$

The trace sum is over the eigenvalues with $\ell \leq N$. We follow the standard method of evaluating $\mathrm{Tr}_N \log(\mathcal{D}/\mu) = \frac{1}{2}\mathrm{Tr}\log(\mathcal{D}^2/\mu^2)$ [12].

On a spherical background, the effective action (26) becomes

$$\Gamma_N(R) = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} - \frac{1}{2}\mathbf{f}_D(N)\log R/\mu^2 + C(N)\,, \tag{28}$$

where

$$\mathbf{f}_D(N) = 2\sum_{\ell=0}^{N} m_\ell = \frac{1}{2}(N+1)(N+2)(N+3)(N+4) \tag{29}$$

$$= \frac{1}{3}(N^4 + 10N^3 + 35N^2 + 50N + 24) \tag{30}$$

is the total number of modes with $\ell < N$ (the factor two is due to the existence of eigenvalues with both signs). The coefficient of the N^4 term is four times that of $\mathbf{f}_S$, reflecting that a Dirac spinor has four degrees of freedom. Furthermore, extremizing the action with respect to R amounts to solving

$$\frac{24\pi}{GR^3}(-R + 4\Lambda) = -\frac{1}{2R}\mathbf{f}_D\,. \tag{31}$$

The solutions of this equation are

$$R = \frac{24\pi}{G\mathbf{f}_D}\left[1 \pm \sqrt{1 - \frac{G\Lambda\mathbf{f}_D}{3\pi}}\right] \tag{32}$$

In order for the solution to be real, we must have $\frac{G\Lambda\mathbf{f}_D}{3\pi} < 1$. If $\Lambda > 0$, this only happens for a finite range of values of N. For negative Λ, the solutions exist for all N, and the one with the upper sign is as follows:

$$R = 24\sqrt{\frac{\pi|\Lambda|}{G}}\frac{1}{N^2} + O\left(\frac{1}{N^3}\right) \tag{33}$$

Remarkably, this is the same solution that we found for the scalar, aside from a factor 2.

Let us now generalize to the case of a fermion with mass m_D. In this case, the effective action reads

$$\Gamma_N = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} - \frac{1}{2}\mathbf{f}_D(N)\log\left(\frac{R}{12\mu^2}\right) - \frac{1}{2}\mathbf{T}_D(N, z_D)\,, \tag{34}$$

where from (16) we have introduced the function

$$\mathbf{T}_D(N,z) = 2\sum_{n=0}^{N} m_n \log\left((2+n)^2 + z\right) \tag{35}$$

with the parameter $z = z_D$

$$z_D = \frac{12m_D^2}{R}\,. \tag{36}$$

The equation of motion is analogous to (21) and is given by:

$$\frac{24\pi}{GR^3}(-R + 4\Lambda) = \frac{1}{2R}(-\mathbf{f}_D + z_D\mathbf{S}_D(N, z_D))\,, \tag{37}$$

with the function $\mathbf{S}_D$ defined from (17), namely,

$$\mathbf{S}_D(N,z) = \frac{\partial \mathbf{T}_D(N,z_D)}{\partial z} = 2\sum_{\ell=0}^{N} m_\ell \frac{1}{(2+\ell)^2+z}\,. \tag{38}$$

We can search solutions with the form

$$R = \frac{K_2}{N^2} + \frac{K_3}{N^3} + \frac{K_4}{N^4} + \dots \tag{39}$$

Inserting into the equation of motion and asking for the divergences N^6, N^5 and N^4 to cancel, we obtain the equation

$$K_2^2 - 24K_2 m_D^2 + 576\pi G\Lambda + 288 m_D^4 \log\left(1 + \frac{K_2}{12 m_D^2}\right) = 0 \tag{40}$$

for K_2 and similar ones for K_3 and K_4, which can be solved in the limit $m_D \to 0$. Assuming $\Lambda < 0$, we obtain

$$K_2 = 24\sqrt{\frac{-\pi\Lambda}{G}} + 12 m_D^2 + O(m_D^3) \tag{41}$$

$$K_3 = -120\sqrt{\frac{-\pi\Lambda}{G}} - 60 m_D^2 + O(m_D^3) \tag{42}$$

$$K_4 = -480\sqrt{\frac{-\pi\Lambda}{G}} + 72\frac{\pi}{G} + 12 m_D^2\left[\frac{127}{6} + 3\sqrt{-\frac{\pi}{G\Lambda}} + \log\left(\frac{G m_D^2}{2\sqrt{-\pi G\Lambda}}\right)\right] + O(m_D^3) \tag{43}$$

As in the case of the scalar, if we choose $\Lambda = 0$, the mass of the fermion gives the dimension to the coefficient K_2. However, if the fermion is massless, $K_2 = 0$, and the first nonzero coefficient is K_4, which this time is positive. Thus, the choice $\Lambda = 0$ is possible for purely fermionic massless matter but not for massless scalars.[5] We shall return to this point later.

5. Maxwell Field

Now, we want to consider an abelian gauge field A_μ. To fix the gauge, we have to include in the matter action a gauge fixing term for the gauge boson and the associated ghost contribution. The Euclidean gauge fixed action is

$$S_M = \int d^4x\sqrt{g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\nabla_\mu A^\mu)^2 + \nabla_\mu \bar{c}\nabla^\mu c\right], \tag{44}$$

where α is a gauge parameter. Commuting derivatives, the bosonic part of this action is

$$\frac{1}{2}\int d^4x\sqrt{g}A_\mu\left[-\nabla^2 g^{\mu\nu} + \left(1 - \frac{1}{\alpha}\right)\nabla^\mu\nabla^\nu + \frac{1}{4}R g^{\mu\nu}\right]A_\nu\,. \tag{45}$$

The standard way to dispose of the nonminimal terms $\nabla^\mu\nabla^\nu$ in the operator is to choose the Feynman gauge $\alpha = 1$. We can avoid fixing the gauge parameter by decomposing A^μ in its transverse and longitudinal components

$$A^\mu = a^\mu + \nabla^\mu\phi \tag{46}$$

with

$$\nabla_\mu a^\mu = 0\,. \tag{47}$$

In this way, the action becomes

$$S = \int d^4x\sqrt{g}\left[\frac{1}{2}a_\mu\left(-\nabla^2 + \frac{R}{4}\right)a^\mu + \frac{1}{\alpha}\phi(-\nabla^2)^2\phi + \nabla_\mu\bar{c}\nabla^\mu c\right] . \tag{48}$$

The decomposition (46) is a change in variables that has a Jacobian $J = \det'(-\nabla^2/\mu^2)$, where the prime means that the zero mode has to be left out.

The effective action for the system is therefore given by

$$\begin{aligned}\Gamma_N(g, a, \phi, \bar{c}, c) = S_{EH}(g) + S_m(g, a, \phi, \bar{c}, c) \\ + \frac{1}{2}\mathrm{Tr}_{aN}\log\left(\left(-\nabla^2 + \frac{R}{d}\right)\frac{1}{\mu^2}\right) + \frac{1}{2}\mathrm{Tr}_{\phi N}\log\left((-\nabla^2)^2\frac{1}{\mu^4}\right) \\ - \mathrm{Tr}_{\bar{c}cN}\log\left((-\nabla^2)\frac{1}{\mu^2}\right) - \frac{1}{2}\mathrm{Tr}'_{JN}\log\left((-\nabla^2)\frac{1}{\mu^2}\right) .\end{aligned} \tag{49}$$

Importantly, the contributions of the longitudinal photon and of the ghosts cancel exactly. For the transverse field, the eigenvalues and multiplicities are

$$\lambda_\ell = \frac{\ell(\ell+3) - 1}{12}R , \qquad m_\ell = \frac{1}{2}\ell(\ell+3)(2\ell+3) . \tag{50}$$

The Jacobian is a scalar determinant, so its calculation is based on the eigenvalues and multiplicities given previously.

The effective action can then be expressed as

$$\Gamma_N = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} + \frac{1}{2}\mathbf{f}_M\log\frac{R}{12\mu^2} + \frac{1}{2}\mathbf{T}_a(N, -1) - \frac{1}{2}\mathbf{T}_S(N, 0) . \tag{51}$$

Here, the total number of physical degrees of freedom is

$$\mathbf{f}_M = \mathbf{f}_a - \mathbf{f}_S = \frac{1}{6}N(N^3 + 8N^2 + 17N + 4) , \tag{52}$$

with $\mathbf{f}_a$ and $\mathbf{f}_S$ the total number of modes of the transverse vector and the Jacobian, respectively. The function $\mathbf{T}_a$ is defined similarly to (16) but with the eigenvalues (50). We observe that the coefficients of the N^4 and N^3 terms are exactly twice those of a scalar, reflecting the fact that a Maxwell field has two degrees of freedom.

Extremizing the effective action with respect to R yields

$$0 = \frac{24\pi}{GR^3}(R - 4\Lambda) + \frac{1}{2R}\mathbf{f}_M . \tag{53}$$

This is exactly the same equation of the scalar case, except for the replacement of $\mathbf{f}_S$ by $\mathbf{f}_M$. Again, we have to choose the solution with the positive sign and assume $\Lambda > 0$, leading to

$$R \approx 16\sqrt{\frac{3\pi\Lambda}{G}}\left(\frac{1}{N^2} - \frac{4}{N^3}\right) + \left(\frac{-232\sqrt{3\pi\Lambda}}{\sqrt{G}} + \frac{96\pi}{G}\right)\frac{1}{N^4} + \ldots \tag{54}$$

We emphasize that the result is fully independent of the gauge parameter α. We interpret this as being due to the fact that the background Maxwell field $A_\mu = 0$ is a solution of the equations of motion. Moreover, we observe that the coefficient K_4 is positive, analogously to the case of Dirac fields.

6. Graviton

Next, we consider the case of metric fluctuations. Also, in this case, we need to include a gauge fixing term and the corresponding ghost action. Furthermore, we choose to use the exponential parametrization of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu \,, \tag{55}$$

where h in the exponent is a mixed tensor $h^\rho{}_\nu$ such that $h_{\mu\nu} = \bar{g}_{\mu\rho}h^\rho{}_\nu$ is symmetric. For the graviton field, we use the York decomposition

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\nabla}^2\sigma + \frac{1}{4}\bar{g}_{\mu\nu} h \,, \tag{56}$$

where $h^{TT}_{\mu\nu}$ is transverse and traceless, the vector ξ is transverse, σ is a spin-0 field, h is the trace of $h_{\mu\nu}$ and all covariant derivatives are calculated with the background metric. Note that h is the infinitesimal variation in the conformal factor of the metric. Under an infinitesimal diffeomorphism ϵ^μ, it changes by $\delta h = 2\bar{\nabla}_\mu \epsilon^\mu$. The field σ is also gauge-variant in such a way that the combination $h - \Box\sigma$ is gauge-invariant. It is very convenient to choose the "unimodular" gauge $h = 0$ that removes one of the three gauge degrees of freedom.[6] The remaining invariance under volume-preserving diffeomorphisms can be fixed by the condition $F_\mu = 0$, where

$$F_\mu = \left(\delta^\nu_\mu - \bar{\nabla}_\mu \frac{1}{\bar{\nabla}^2}\bar{\nabla}^\nu\right)\bar{\nabla}_\rho h^\rho{}_\nu = \left(\bar{\nabla}^2 + \frac{R}{4}\right)\xi_\mu \,. \tag{57}$$

As discussed in Section 5.4.6 of [13], this leads to a very simple form for the (off-shell) effective action

$$\Gamma(\bar{g}) = S(\bar{g}) + \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{(2)}/\mu^2\right) - \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{(1)}/\mu^2\right) , \tag{58}$$

where

$$\Delta^{(2)} = -\bar{\nabla}^2 + \frac{1}{6}\bar{R} \tag{59}$$

acts on $h^{TT}_{\mu\nu}$ and

$$\Delta^{(1)} = -\bar{\nabla}^2 - \frac{1}{4}\bar{R} \tag{60}$$

acts on ξ_μ.

The spectra of these operators are

$$\lambda^{(2)}_\ell = \frac{\ell(\ell+3)-2}{12}R \,, \qquad m^{(2)}_\ell = \frac{5}{6}(\ell+4)(\ell-1)(2\ell+3) \tag{61}$$

with $\ell = 2, 3\ldots$ and

$$\lambda^{(1)}_\ell = \frac{\ell(\ell+3)-6}{12}R \,, \qquad m^{(1)}_\ell = \frac{1}{2}\ell(\ell+3)(2\ell+3) \tag{62}$$

with $\ell = 2, 3\ldots$

Altogether, the effective action can be expressed in terms of the **T**-functions and the number of degrees of freedom

$$\Gamma_N = \frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} + \frac{1}{2}(\mathbf{f}^{(2)} - \mathbf{f}^{(1)})\log\frac{R}{12\mu^2} + \frac{1}{2}\mathbf{T}^{(2)}(N,-1) - \frac{1}{2}\mathbf{T}^{(1)}(N,-6) \,, \tag{63}$$

where $\mathbf{f}_G = \mathbf{f}^{(2)} - \mathbf{f}^{(1)}$,

$$\mathbf{f}^{(2)}(N) = \sum_{n=2}^{N} m^{(2)}(n) = \frac{5}{12}N(N-1)(N+4)(N+5) \tag{64}$$

$$\mathbf{f}^{(1)}(N) = \sum_{n=2}^{N} m^{(1)}(n) = \frac{1}{4}N(N+1)(N+3)(N+4)\,, \tag{65}$$

and similar definitions for the $\mathbf{T}$-functions in (16). Since the arguments of these functions are constant, they do not affect the equations of motion, and we do not need to give them in detailed form.

The total $\mathbf{f}$-function is

$$\mathbf{f}_G = \mathbf{f}^{(2)} - \mathbf{f}^{(1)} = \frac{1}{6}(N^4 - 8N^3 - N^2 - 68N)$$

and we observe that, like in the Maxwell case, the first two terms in the large-N expansion of the function $\mathbf{f}_G$ are the same of a free scalar, in accordance with the count of degrees of freedom. Thus, in pure gravity, assuming $\Lambda > 0$,

$$R = 24\sqrt{\frac{2\pi\Lambda}{G}}\left(\frac{1}{N^2} - \frac{4}{N^3}\right) + \left(\frac{588\sqrt{2\pi\Lambda}}{\sqrt{G}} - \frac{144\pi}{G}\right)\frac{1}{N^4} + \dots \tag{66}$$

Let us comment on the differences between this treatment and the one in [3], where the linear split of the metric was used,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,. \tag{67}$$

With this split, all the components of the metric in the York decomposition are sensitive to the cosmological constant term. In particular, the transverse traceless component is governed by the operator [7]

$$-\bar{\nabla}^2 + \frac{2}{3}\bar{R} - 2\Lambda\,. \tag{68}$$

The cosmological term has the same effect as a mass and appears in the effective action through the functions $\mathbf{T}(N, -24\Lambda/R)$. This greatly complicates the analysis because it introduces a new dependence of the effective action on curvature. Moreover, if $\Lambda > 0$, as one would naively expect, the cosmological term has the same effect as a negative squared mass, with the result that the poles of the function $\mathbf{T}(N, -24\Lambda/R)$ appear at positive rather than negative R. There are two ways of avoiding this problem. One is to choose $\Lambda < 0$, in which case the poles occur for negative R, as in the case of a normal massive field. Another way that we have followed is to use the exponential parametrization of the metric (see also [14,15] for a discussion on the parametrization), in which case the cosmological constant only affects the propagation of the trace mode h and only off-shell. We have eliminated this effect by the gauge choice $h = 0$. We note that if we put a gauge parameter α in the gauge fixing term $\int F_\mu F^\mu$, the result turns out to be independent of α [13]. This is because the sphere is "almost on shell", in the sense that its metric satisfies nine out of ten Einstein's equations. Only the overall scale factor is in general off-shell. On the other hand, if we were to repeat this calculation using the linear splitting of the metric and another gauge, the result would depend on the gauge parameters, which is to be expected insofar as our background is not fully on shell.

Having separately analyzed all the contributions from the different quantum fields, we will consider them together, in order to understand their interplay and study physically realistic systems.

7. All Together

When there is a graviton, N_S scalars with mass m_S, N_D Dirac fields with mass m_D and N_M massless gauge fields, the effective action reads

$$\begin{aligned}\Gamma_N = &\frac{48\pi\Lambda}{GR^2} - \frac{24\pi}{GR} + \\ &+ \frac{1}{2}(\mathbf{f}_G(N) + N_S\mathbf{f}_S(N) - N_D\mathbf{f}_D(N) + N_M\mathbf{f}_M(N))\log\left(\frac{R}{12\mu^2}\right) \\ &+ \frac{1}{2}(N_S\mathbf{T}_S(N,z_S) - N_D\mathbf{T}_D(N,z_D)) \end{aligned} \tag{69}$$

where $z_S = 12m_S^2/R$, $z_D = 12m_D^2/R$ and we omit constants.

We simplify the discussion by assuming that at fundamental level all fields are massless, which could be the case if all masses are generated dynamically or by a Higgs mechanism. Then, all the **T**-terms in the effective action can be ignored, and the equation of motion for R is very simple

$$\frac{24\pi}{GR^3}(-R + 4\Lambda) = \frac{1}{2R}\mathbf{f}_{TOT}(N) \tag{70}$$

with

$$\begin{aligned}\mathbf{f}_{TOT} &= \mathbf{f}_G + N_S\mathbf{f}_S - N_D\mathbf{f}_D + N_M\mathbf{f}_N \\ &= \frac{1}{12}\Delta N^4 + \frac{2}{3}\Delta' N^3 + \frac{1}{12}\Delta'' N^2 + \frac{1}{3}\Delta''' N - 8N_D\,,\end{aligned} \tag{71}$$

where

$$\begin{aligned}\Delta &= N_S - 4N_D + 2N_M + 2 \\ \Delta' &= -N_S + 5N_D - 2N_M - 2 \\ \Delta'' &= 23N_S - 140N_D + 34N_M - 2 \\ \Delta''' &= 7N_S - 50N_D + 2N_M - 34\end{aligned} \tag{72}$$

In particular, note that Δ is the number of bosonic minus fermionic degrees of freedom. The solution for the Ricci scalar is given by

$$\begin{aligned}R^{\pm} &= \frac{24\pi}{G\mathbf{f}_{TOT}(N)}\left(-1 \pm \sqrt{1 + \frac{G\Lambda\mathbf{f}_{TOT}(N)}{3\pi}}\right) \\ &= \frac{K_2^{\pm}}{N^2} + \frac{K_3^{\pm}}{N^3} + \frac{K_4^{\pm}}{N^4} + O(N^{-5})\,.\end{aligned} \tag{73}$$

Let us now comment on the possible cases. In previous sections, we have always assumed that Λ is positive for bosonic fields or negative for fermionic ones. Now that both types of fields are present, we do not make a priori any such assumption. Then, we find in general for the coefficients

$$\begin{aligned}K_2^{\pm} &= \pm 48\sqrt{\frac{\pi}{G}}\frac{\Lambda}{\sqrt{\Lambda}}\frac{1}{\sqrt{\Delta}} \\ K_3^{\pm} &= \pm 192\sqrt{\frac{\pi\Lambda}{G}}\frac{\Delta'\sqrt{\Delta}}{\Delta^2} \\ K_4^{\pm} &= \pm 24\sqrt{\frac{\pi}{G}}\frac{\Lambda}{\sqrt{\Lambda}}\frac{48\Delta'^2 - \Delta\Delta''}{\Delta^{5/2}} - \frac{288\pi}{G\Delta}\,.\end{aligned} \tag{74}$$

The reality of the solution requires that the product $\Delta\Lambda$ be positive. Thus, if bosons dominate, we must choose $\Lambda > 0$, and then in order to have positive curvature, we have to select the solution R^+. If fermions dominate, we must choose $\Lambda < 0$ and the solution R^-. This is in agreement with previous assumptions. In any case, one can achieve positive curvature, independently of the sign of the cosmological constant in the action.

8. Questions About the Measure

Recently, there appeared the paper [10], that also makes use of a dimensionless cutoff N but in a more traditional field-theoretic context. In order to understand the differences, let us first briefly describe their approach in the case of a scalar field.

The first difference is the functional measure. Whereas in our approach the functional integration measure for the scalar field has been implicitly assumed to be the Fujikawa measure [16–18]

$$\Pi_x d\phi(x)(\det g(x))^{1/4}\mu \tag{75}$$

as carefully discussed in [2], they use the measure [19] [8]

$$\Pi_x d\phi(x)(\det g(x))^{1/4}(g^{00}(x))^{1/2} \tag{76}$$

with the result that the effective action (3) is replaced by

$$\Gamma'(g,\phi) = S_H(g) + S_m(g,\phi) + \frac{1}{2}\mathrm{Tr}\log\left(\frac{12}{R}\Delta\right). \tag{77}$$

In our approach, the measure must involve an arbitrary dimensionful factor μ that also appears in the trace log formula, whereas in [10] the measure and the operator appearing in the trace log are automatically dimensionless, with the result that no external scale μ is needed. We refer the reader to [21] for a discussion about the physical implications of the choice of measure, particularly in relation to diffeomorphism invariance.

Regardless of those considerations, if we compare the two formulas for the effective action, we find that

$$\Gamma'(g,\phi) = \Gamma(g,\phi) - \frac{1}{2}\log\left(\frac{R}{12\mu^2}\Delta\right)\mathbf{f}_S(N) \tag{78}$$

and we observe that the second term exactly removes the third term in the r.h.s. of (14). Thus, for a massless scalar field, the equation of motion does not receive any correction at one loop: the solution is $R = 4\Lambda$ independently of N.

This result holds for any massless field. For a spinor ψ, a Maxwell field A_μ (including the ghosts c and $\bar{c}$) and a graviton $h_{\mu\nu}$ (including ghosts $\bar{C}^\mu$ and C_ν) the measures we implicitly used are the Fujikawa measures (all in four dimensions)

$$\begin{aligned}
&\Pi_x\left(dA_\mu(x)(\det g(x))^{1/8}\right)\left(d\bar{c}(x)(\det g(x))^{1/4}\right)\left(dc(x)(\det g(x))^{1/4}\right)\\
&\Pi_x\left(d\bar{\psi}(x)(\det g(x))^{1/4}d\psi(x)(\det g(x))^{1/4}\right)\\
&\Pi_x\left(dg_{\mu\nu}(x)d\bar{C}^\mu(x)(\det g(x))^{3/8}dC_\nu(x)(\det g(x))^{1/8}\right)
\end{aligned} \tag{79}$$

whereas the measures advocated in [10] are [9]

$$\begin{aligned}
&\Pi_x \left(dA_\mu(x) d\bar{c}(x) dc(x) g^{00} (\det g(x))^{1/4} \right) \\
&\Pi_x \left(d\bar{\psi}(x) d\psi(x) (\det g(x))^{3/8} \right) \\
&\Pi_x \left(dg_{\mu\nu}(x) d\bar{C}^\mu(x) dC_\nu(x) g^{00} (\det g(x))^{-1} \right)
\end{aligned} \tag{80}$$

As in the case of the scalar, the effect of the additional powers of the determinant is to make the kinetic operator appearing in the Tr log's dimensionless and hence to remove from the effective action the terms containing the function **f** so that the equations of motion do not undergo any quantum correction.

On the other hand, for massive fields (with the bare cosmological constant playing the role of mass term for the gravitational field), the effective action Γ' receives contributions proportional to $\log N^2 m^4/R^2$ and $(N^2 + \log N^2) m^2/R$ that can be interpreted as quantum corrections of Λ/G and $1/G$. (This is in sharp contrast to our measure, with which there are no terms proportional just to $1/R$ or $1/R^2$ in the one loop contribution.) In the presence of scalars, fermions and the graviton, the effective cosmological constant and Newton constant are given by [10]

$$\frac{\Lambda_{eff}}{G_{eff}} = \frac{\Lambda_B}{G_B} - \frac{N_S m_S^4}{8\pi} \log N^2 + \frac{N_D m_D^4}{4\pi} \log N^2 - \frac{3\Lambda_B^2}{\pi} \log N^2 \tag{81}$$

$$\frac{1}{G_{eff}} = \frac{1}{G_B} - N_S \frac{m_S^2}{24\pi} (N^2 + 2\log N^2) + N_D \frac{m_D^2}{12\pi} (N^2 - \log N^2) + \frac{\Lambda_B}{2\pi} (N^2 - 8\log N^2) \, . \tag{82}$$

These formulae assume that all the particles of the same type have the same mass. If not, the main contribution to each term comes from the heaviest particles of each type.

In the normal QFT logic, the N-dependence, which gives rise to infinities when $N \to \infty$, can be compensated by the N-dependence of the bare parameters Λ_B and G_B. Then, Λ_{eff} and G_{eff} can be interpreted as renormalized couplings, and they must be independent of N.

9. Discussion

The treatment of the cosmological constant we described in Sections 2–7 differs from the standard one in two main ways: the most evident one is that the ultraviolet cutoff is the dimensionless number N, rather than the usual C with dimensions of mass. This could be just a minor difference because one can always convert N to C, as in (12). It becomes more significant when combined with the second difference, which is the use of the semiclassical Einstein Equation (1) *before* taking the limit $N \to \infty$. This is equivalent to using the EM tensor regulated with the N-cutoff in the semiclassical Einstein equations, rather than a renormalized EM tensor. This implies a different physical interpretation of the cutoff. One normally does not think of the ultraviolet regulator as having physical meaning: it is merely an artificial mathematical device used in intermediate steps of the calculations, ultimately leading to renormalized quantities. It is only the latter that are given physical meaning, and in particular it is the renormalized EM tensor that is used in the semiclassical Einstein Equation (1). Our use of the N-regulated EM tensor looks weird, if not outright wrong, from this point of view. As already emphasized in [2], in the approach we follow here, the cutoff must be regarded as having a physical meaning: it is related to the total number of degrees of freedom in the universe. The systems with finite N are described in [2] as "gravity-coupled approximants", systems that are physically realizable and can give a good description of the real universe.

It is noteworthy that the quantum corrections do not contain any term of the same form as the ones that are present in the classical Hilbert action, so that there is no renormalization of the cosmological and Newton constant. The quantum corrections are nonlocal and can be interpreted as the terms that generate the trace anomaly in the field equations. Another consequence is that the quantum-corrected field equations are very different from the classical ones: the quantum effects are so strong that one can have a positive curvature even if the "bare" cosmological constant is negative. This fact could have interesting applications, which we shall not enter into.

This approach is close in spirit to the old idea that gravity may act as a universal regulator [23–25]. In the case of the self-energy of a charged particle, this has been discussed in [6]; see also the recent discussion in [8]. It has indeed been shown in [9] that the use of the N-cutoff, in conjunction with Einstein's equations, leads to finiteness of the quantum corrections to the mass and quartic self-interaction of a scalar theory. The full implications of this result and the generalization to other interacting theories are left for future work.

This work can be extended in several directions. First of all, the computation in a Lorentzian setting would be compelling in order to address the cosmological constant problem in a cosmological spacetime. Here, some complications arise due to the more involved eigenvalues and eigenstates of the Laplace operator. These should, however, reflect some physical IR properties related to the Lorentzian signature on the one side and give universal results in the UV limit on the other side. Secondly, it would be interesting to study related self-consistent flow equations, by taking finite differences in N as iterative differential equations. Finally, this work gives a different perspective on the role of the cutoff in QFT, especially within a gravitational setting. Having a dimensionless cutoff at hand allows a regularized system to be studied without the need to introduce any (energy) scale, opening new avenues for the interpretation of the cutoff as a physical quantity. This could be related to a quantitative notion of information of the system, encoded in the number of degrees of freedom.

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Notes

1. Alternatively, if one tries to cancel the divergences with Λ, one is faced with a severe fine tuning issue.
2. One simple solution is to say that one out of the infinitely many degrees of freedom of the gravitational field, namely, the total volume of spacetime, is not dynamical. If the total volume is fixed, it is natural to further fix the whole volume form by a gauge choice, leading to unimodular gravity. While this has the desired effect, it is hard to imagine how to test unimodular vs. ordinary gravity. In this paper, we assume that the total volume is dynamical and focus on the calculation of the vacuum energy.

3 See [5] for a similar discussion on the role of the cutoff from a different point of view.
4 This is just the trace of (1).
5 One has to take the limit $m_D \to 0$ before the limit $\Lambda \to 0$.
6 We stress that in this paper we deal exclusively with the dynamics of the global scale factor of the universe. Its fluctuations are the constant mode of the trace h and are physical, but they only give a finite contribution to Γ that we neglect here. In perturbation theory, the non-constant fluctuations of the trace, whose high frequency modes make the action unbounded from below, can be eliminated by a gauge choice.
7 Note that this operator coincides with $\Delta^{(2)}$ on shell.
8 Equivalent results would be obtained by using the measure $\Pi_x d\phi(x)(\det g(x))^{1/8}$ [20].
9 Since reference [10] does not treat vector fields, for this case, we use the measure given in [22].

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