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Integrable operators, $\overline{\partial}$-problems, KP and NLS hierarchy

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Abstract
We develop the theory of integrable operators $\mathcal{K}$ acting on a domain of the complex plane with smooth boundary in analogy with the theory of integrable operators acting on contours of the complex plane. We show how the resolvent operator is obtained from the solution of a $\overline{\partial}$-problem in the complex plane. When such a $\overline{\partial}$-problem depends on auxiliary parameters we define its Malgrange one form in analogy with the theory of isomonodromic problems. We show that the Malgrange one form is closed and coincides with the exterior logarithmic differential of the Hilbert–Carleman determinant of the operator $\mathcal{K}$. With suitable choices of the setup we show that the Hilbert–Carleman

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determinant is a τ-function of the Kadomtsev–Petviashvili (KP) or nonlinear Schrödinger hierarchies.

Keywords: integrable systems, d-bar problems, integrable operators, regularized determinants

1. Introduction

A key notion in the theory of solvable integrable systems is that of τ–function, (see [29] for a comprehensive historical perspective) and in many instances such τ–function coincides with the Fredholm determinant of an appropriate integral operator.

A distinguished class of integral operators are the so called *integrable operators*: the theory of such operators has its roots in the work of Jimbo *et al* [37] that ultimately led to the construction by Its *et al* [33, 34] of a Riemann–Hilbert problem to express the kernel of their resolvent operators. Their original motivation for studying these operators comes from the theory of quantum integrable models. The theory of integrable operators later found applications in many fields of mathematics such as random matrices and integrable partial differential equations: for example the *gap probabilities* in determinantal random point processes (and more generally the generating function of occupation numbers) are expressible as a Fredholm determinant [17, 48] and this is at the core of the celebrated Tracy–Widom distribution for fluctuations of the largest eigenvalue of a random matrix in a Gaussian Unitary Ensemble [50]. An integrable operator is an integral operator acting on $L^2(\Sigma, |\mathrm{d}w|)\otimes\mathbb{C}^n$ of the form

$$\mathcal{K}[v](z) = \int_\Sigma K(z,w)\, v(w)\, \mathrm{d}w, \quad z\in\Sigma,$$

where Σ is some oriented contour in the complex plane and the kernel $K(z,w)\in \mathrm{Mat}(n\times n,\mathbb{C})$ has a special form

$$K(z,w) := \frac{f^T(z)\, g(w)}{z-w}, \quad f(z), g(z)\in \mathrm{Mat}(r\times n,\mathbb{C}), \tag{1.1}$$

where f and g are rectangular $r\times n$ matrices and for the time being we only assume that f and g are smooth along the connected components of Σ. The condition for K to be nonsingular requires

$$f^T(z)\, g(z) \equiv 0.$$

In the most relevant applications, the operators of the form (1.1) are trace class operators. An important observation in [33] is that the resolvent operator

$$\mathcal{R} = \mathcal{K}(\mathrm{Id}-\mathcal{K})^{-1} = (\mathrm{Id}-\mathcal{K})^{-1} - \mathrm{Id}, \tag{1.2}$$

where Id is the identity operator, is in the same class, namely

$$\mathcal{R}[v](z) = \int_\Sigma R(z,w)\, v(w)\, \mathrm{d}w, \quad z\in\Sigma,$$

where the resolvent kernel has also the form of an integrable operator:

$$R(z,w) := \frac{F^T(z)\, G(w)}{z-w}, \quad F(z), G(z)\in \mathrm{Mat}(r\times n,\mathbb{C}). \tag{1.3}$$

Here $F^T(z) = (\mathrm{Id} - \mathcal{K})^{-1}[f^T]$ and $G = [g](\mathrm{Id} - \mathcal{K})^{-1}$, where $(\mathrm{Id} - \mathcal{K})^{-1}$ in the first relation is acting to the right while in the second relation its action is to the left. Another crucial observation of [33] (see also the introduction of [31]) is that the determination of $\mathcal{R}$ is equivalent to the solution of an associated Riemann–Hilbert (RH) problem for a $r \times r$ matrix $\Gamma(z)$ analytic in $\mathbb{C}\backslash\Sigma$ that satisfies the boundary value relation (sometimes referred to as 'jump relation')

$$\begin{aligned} &\Gamma_+(z) = \Gamma_-(z) J(z), \quad z \in \Sigma, \quad J(z) = \mathbf{1} + 2\pi i f(z) g^T(z) \\ &\Gamma(z) \to \mathbf{1}, \quad \text{as } |z| \to \infty. \end{aligned} \tag{1.4}$$

Here $\Gamma_\pm(z)$ denote the boundary values of the matrix $\Gamma(z)$ as z approaches from the left and right the oriented contour Σ and $\mathbf{1}$ is the identity matrix in $\mathrm{Mat}(r \times r, \mathbb{C})$. The matrices F and G that define the resolvent kernel (1.3) are related to the solution Γ of the RH problem (1.4) by the relation

$$F(z) = \Gamma(z) f(z), \quad G(z) = \left(\Gamma(z)^T\right)^{-1} g(z). \tag{1.5}$$

This connection between the Fredholm determinant and the RH problem has been exploited in several contexts where the kernel depends on parameters and the study of the asymptotic behaviour of the Fredholm determinant for large values of the parameters is obtained via the Deift–Zhou nonlinear steepest descent method of the corresponding RH problem [19]. This analysis has been successfully implemented for $n = 1$ and $r = 2$ in a large class of kernels originating in random matrices, orthogonal polynomials, probability and partial differential equations (see e.g. [9, 18, 19, 29, 32]).

The knowledge of the resolvent operator allows to write variational formulae for the Fredholm determinant of the operator $\mathrm{Id} - \mathcal{K}$ as

$$\delta \log\det(\mathrm{Id} - \mathcal{K}) = -\mathrm{Tr}\left((\mathrm{Id} - \mathcal{K})^{-1} \circ \delta\mathcal{K}\right) = -\mathrm{Tr}\left((\mathrm{Id} + \mathcal{R}) \circ \delta\mathcal{K}\right), \tag{1.6}$$

where here and below δ stands for exterior total differentiation in the space of parameters. Clearly the one-form $\omega := \delta \log\det(\mathrm{Id} - \mathcal{K})$ is closed in the space of parameters.

In the theory of isomonodromic deformations an analogous close one form is called the Malgrange one form and it is the logarithmic derivate of the isomonodromic τ-function introduced by the Kyoto school headed by Jimbo *et al* [38] in the context of the inverse monodromy problem for a linear system of first order ODEs in the complex plane with regular or irregular singularities. An open problem in the theory of isomonodromic deformations is whether its τ-function can be identified with a Fredholm determinant. When the isomonodromic problem is related to some of the Painlevé equations, this problem has a positive answer see e.g. [14, 20, 21, 26], and also the applications in [7, 8].

Another important class of integral equations whose solution can be reduced to a Fredholm determinant is the class of Hankel composition operators that first appeared in the theory of inverse scattering on the line (see e.g. [22, 23] for initial data vanishing at infinity and [25] for step-like initial data). In a more general setting such operators can be reduced to integrable operators via Fourier transform, see for example the works [5, 10, 39]. Applications of this class of operators to the theory of random matrices, integrable probability and integro-differential Painlevé equations and non commutative Painlevé equations are obtained in [2, 11–13, 15, 16, 49].

The common feature of all these works is the appearance, in one way or another, of a Riemann–Hilbert problem, namely, a boundary value problem of a matrix with discontinuities across a contour (or union thereof) with boundary values related multiplicatively by a group-like element J (the 'jump matrix') as in (1.4).

The goal of the present manuscript is to enlarge the class of integrable operators by considering operators $\mathcal{K}$ acting on $L^2(\mathscr{D}, \mathrm{d}^2w)\otimes\mathbb{C}^n$ where $\mathscr{D}$ is a bounded domain of the complex plane with a matrix kernel $K(z,\overline{z},w,\overline{w})\in Mat(n\times n,\mathbb{C})$, namely

$$\mathcal{K}[v](z,\overline{z}) = \iint_{\mathscr{D}} K(z,\overline{z},w,\overline{w})\, v(w)\,\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2i},\quad z,\overline{z}\in\mathscr{D}, \tag{1.7}$$

$$K(z,\overline{z},w,\overline{w}) := \frac{f^T(z,\overline{z})\, g(w,\overline{w})}{z-w},$$

$$f^T(z,\overline{z})\, g(z,\overline{z}) \equiv 0 \equiv \left(\partial_{\overline{z}} f(z,\overline{z})\right)^T g(z,\overline{z}),\quad f,g\in\mathcal{C}^1\left(\overline{\mathscr{D}},\mathrm{Mat}(r\times n,\mathbb{C})\right), \tag{1.8}$$

where $\overline{\mathscr{D}}$ is the closure of $\mathscr{D}$ in $\mathbb{C}$. Here and below, we use the symbol $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ to specify the derivative with respect to $\overline{z}$ (known as Wirtinger antiholomorphic derivative). The dependence of f and g on z and $\overline{z}$ is to remind the reader that f and g are in general smooth matrix functions on the complex plane. We remark that the condition (1.8) guarantees that the operator $\mathcal{K}$ is a Hilbert–Schmidt operator. Our results are the following.

- In section 2 we show that the resolvent of the integral operator $\mathrm{Id}-\mathcal{K}$ is obtained through the solution of a $\overline{\partial}$-Problem (instead of a RH problem) for a matrix–valued function Γ:

$$\begin{aligned} &\partial_{\overline{z}}\Gamma(z,\overline{z}) = \Gamma(z,\overline{z})\, M(z,\overline{z});\qquad \Gamma(z,\overline{z}) \underset{z\to\infty}{\to} \mathbf{1},\\ &M(z,\overline{z}) = \pi f(z,\overline{z})\, g^T(z,\overline{z})\,\chi_{\mathscr{D}}(z), \end{aligned} \tag{1.9}$$

  where $\chi_{\mathscr{D}}(z)$ is the characteristic function of the domain $\mathscr{D}$. Note that the matrix $M(z,\overline{z})$ is nilpotent because of (1.8). We show that the $\overline{\partial}$-Problem is solvable if and only if the operator $\mathrm{Id}-\mathcal{K}$ is invertible. Furthermore we show, in analogy with integrable operators defined on contours, that the kernel of the resolvent is

$$R(z,\overline{z},w,\overline{w}) = \frac{F(z,\overline{z})^T\, G(w,\overline{w})}{z-w},\quad F(z,\overline{z}) = \Gamma(z,\overline{z}) f(z,\overline{z}),\;\; G(z,\overline{z}) = \left(\Gamma^T(z,\overline{z})\right)^{-1} g(z,\overline{z})$$

  where Γ solves the $\overline{\partial}$-problem (1.9).
- In section 3 we consider the regularized determinant (Hilbert–Carleman determinant, equation 7.8 [27]) of the operator $\mathcal{K}$. This is defined as the Fredholm determinant of the trace class operator $\mathcal{T}_{\mathcal{K}} := \mathrm{Id}-(\mathrm{Id}-\mathcal{K})e^{\mathcal{K}}$, namely

$$\det\nolimits_2(\mathrm{Id}-\mathcal{K}) := \det(\mathrm{Id}-\mathcal{T}_{\mathcal{K}}) = \det\left((\mathrm{Id}-\mathcal{K})\, e^{\mathcal{K}}\right). \tag{1.10}$$

  Using the Jacobi variational formula (1.6) we show that

$$\delta \log\det\nolimits_2(\mathrm{Id}-\mathcal{K}) = \omega, \tag{1.11}$$

$$\omega := -\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,\delta M(z)\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}. \tag{1.12}$$

The one form ω is shown to be closed. In analogy with the literature on Riemann–Hilbert problems on contours [3, 4] we call ω the Malgrange one form of the $\overline{\partial}$-Problem. The corresponding τ function of the $\overline{\partial}$-Problem is henceforth defined by

$$\delta \log \tau := \omega. \tag{1.13}$$

Therefore we have that

$$\tau = \det{}_2 (\mathrm{Id} - \mathcal{K}). \tag{1.14}$$

If the operator $\mathcal{K}$ is of trace class, then the τ-function can be expressed as a Fredholm determinant by the relation $\tau = \det(\mathrm{Id} - \mathcal{K}) e^{Tr(\mathcal{K})}$. We also show (section 3.1) that the formula (1.12) defines a closed one–form under the less restrictive assumption that $M(z,\bar{z})$ is traceless but not nilpotent, and this allows us to define a τ–function (up to multiplicative constants) of the $\overline{\partial}$-problem

$$\partial_{\bar{z}} \Gamma(z,\bar{z}) = \Gamma(z,\bar{z}) M(z,\bar{z}); \qquad \Gamma(z,\bar{z}) \underset{z\to\infty}{\to} \mathbf{1}, \tag{1.15}$$

by (1.13). Note that in this more general case, where $M(z,\bar{z})$ is traceless but not nilpotent, the τ-function of the $\overline{\partial}$-problem is well defined but it is not in general related to a Hilbert–Carleman determinant of some integral operator.

- Finally in section 4 we use the results of the previous points and we consider the $\overline{\partial}$-problem (1.15) where M is a 2×2 matrix of the form

$$M(z,\bar{z},\boldsymbol{t}) = e^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} M_0(z,\bar{z}) e^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\xi(z,\boldsymbol{t}) = \sum_{j=1}^{+\infty} z^j t_j$ and $M_0(z,\bar{z})$ is a traceless matrix supported on a compact set $\mathscr{D}$; we show that the corresponding τ-function of the $\overline{\partial}$-problem (1.15) is a Kadomtsev–Petviashvili (KP) τ-function, namely it satisfies Hirota's bilinear relations for the KP hierarchy (see e.g. [29]). We remark that the $\overline{\partial}$-problem (1.15) has already appeared in the study of the Cauchy problem for the KP equation, in [1, 44].

We then specialize the matrix M of the $\overline{\partial}$-problem in (1.15) to the nilpotent and traceless form

$$M(z,\bar{z};x,t) = \pi f(z,\bar{z};x,t) g^T(z,\bar{z};x,t), \quad x \in \mathbb{R}, \; t \geqslant 0$$

with

$$f(z,\bar{z};x,t) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \beta(z,\bar{z}) e^{-i(zx+z^2t)} \chi_{\mathscr{D}}(z) \\ -\beta^*(z,\bar{z}) e^{i(zx+z^2t)} \chi_{\mathscr{D}^*}(\bar{z}) \end{bmatrix}$$
$$g(z,\bar{z};x,t) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \beta^*(z,\bar{z}) e^{i(zx+z^2t)} \chi_{\mathscr{D}^*}(\bar{z}) \\ \beta(z,\bar{z}) e^{-i(zx+z^2t)} \chi_{\mathscr{D}}(z) \end{bmatrix}, \tag{1.16}$$

where $\beta^*(z,\bar{z}) = \overline{\beta(\bar{z},z)}$ is a smooth function and $\chi_{\mathscr{D}}$, $\chi_{\mathscr{D}^*}$ are respectively the characteristic functions of a simply connected domain $\mathscr{D} \subset \mathbb{C}^+$ and its conjugate $\mathscr{D}^*$. Here $\mathbb{C}^+$ is the upper half space. We show that the τ-function of the $\overline{\partial}$-problem (1.15) is the τ-function for the focusing Nonlinear Schrödinger (NLS) equation and coincides with

Hilbert–Carleman determinant of the operator $\mathcal{K}$ on $L^2(\mathscr{D}\cup\overline{\mathscr{D}})$ with integrable kernel $K(z,\overline{z},w,\overline{w}) = \frac{f^T(z,\overline{z})\,g(w,\overline{w})}{z-w}$, namely[7]

$$\partial_x^2 \log\tau(x,t) = \partial_x^2 \log\det_2(\mathrm{Id}-\mathcal{K}) = |\psi(x,t)|^2, \tag{1.17}$$

where the complex function $\psi(x,t)$ solves the focusing NLS equation

$$i\,\partial_t\psi + \frac{1}{2}\partial_x^2\psi + |\psi|^2\psi = 0. \tag{1.18}$$

While we can write the solution to the NLS equation in the form (1.17), the analytical properties of such family of initial data and solutions (e.g. the long-time behaviour) have still to be explored. It is shown in [6] that such family of initial data naturally emerge in the limit of an infinite number of solitons. We also illustrate how, for a specific choice of the domain $\mathscr{D}$ and of the function β, the $\overline{\partial}$-problem (1.9) can be reduced to a standard RH problem.

2. Integrable operators and $\overline{\partial}$-problems

Let $\mathscr{D}\subset\mathbb{C}$ be a compact union of domains with smooth boundary and denote by $\mathcal{K}$ the integral operator acting on the space $L^2(\mathscr{D},\mathrm{d}^2z)\otimes\mathbb{C}^n$ with a kernel $K(z,w)$ of the form

$$K(z,\overline{z},w,\overline{w}) := \frac{f^T(z,\overline{z})\,g(w,\overline{w})}{z-w},\quad f(z,\overline{z}),g(z,\overline{z})\in Mat(r\times n,\mathbb{C}), \tag{2.1}$$

$$f^T(z,\overline{z})\,g(z,\overline{z})\equiv 0 \quad\text{and}\quad (\partial_{\overline{z}}f(z,\overline{z}))^T g(z,\overline{z})\equiv 0,\quad z,\overline{z}\in\mathscr{D}. \tag{2.2}$$

Here the matrix-valued functions $f,g\in\mathcal{C}^1(\overline{\mathscr{D}},\mathrm{Mat}(r\times n,\mathbb{C}))$, namely no analyticity is required and for this reason we indicate the dependence on both variables z and $\overline{z}$.

The vanishing requirements along the locus $z=w$ are sufficient to guarantee that the kernel K admits a well-defined value on the diagonal and it is continuous on $\mathscr{D}\times\mathscr{D}$

$$\lim_{w\to z} K(z,\overline{z},w,\overline{w}) = K(z,\overline{z},z,\overline{z}) = \partial_z f^T(z,\overline{z})\,g(z,\overline{z}). \tag{2.3}$$

We have emphasized that the kernel and the functions are not holomorphically dependent on the variables; that said, from now on we omit the explicit dependence on $\overline{z}$, trusting that the class of functions we are dealing with will be clear by the context each time. The operator $\mathcal{K}$ acts as follows on functions

$$\mathcal{K}[\varphi](z) := \iint_{\mathscr{D}} K(z,w)\,\varphi(w)\,\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2i},\quad \varphi\in L^2\left(\mathscr{D},\mathrm{d}^2z\right)\otimes\mathbb{C}^n. \tag{2.4}$$

We introduce the following $\overline{\partial}$-problem for an $r\times r$ matrix-valued function $\Gamma(z,\overline{z})$.

Problem 2.1. Find a matrix-valued function $\Gamma(z,\overline{z})\in GL_r(\mathbb{C})$ such that

$$\partial_{\overline{z}}\Gamma(z) = \Gamma(z)M(z);\qquad \Gamma(z)\underset{z\to\infty}{\to}\mathbf{1} \tag{2.5}$$

[7] We omit explicit notation of the x,t dependence from the kernel.

where $\mathbf{1}$ is the identity in $GL_r(\mathbb{C})$ and

$$M(z) := \begin{cases} \pi f(z) g^T(z), & \text{for } z \in \mathscr{D}, \\ 0 & \text{for } z \in \mathbb{C} \backslash \mathscr{D}. \end{cases} \tag{2.6}$$

We first show that

Lemma 2.2. *If a solution of the* $\overline{\partial}$*-problem 2.1 exists, it is unique. Furthermore* $\det \Gamma(z) \equiv 1$.

Proof. If Γ is a solution of the $\overline{\partial}$-problem 2.1 then

$$\partial_{\bar{z}} \det \Gamma = Tr(adj(\Gamma) \partial_{\bar{z}} \Gamma) = Tr(adj(\Gamma) \Gamma M) \tag{2.7}$$

where $\operatorname{adj}(\Gamma)$ denotes the adjugate matrix (the transposed of the co-factor matrix). Here Tr denotes the matrix trace. Now the product in the last formula yields $\operatorname{adj}(\Gamma)\Gamma = (\det \Gamma)\mathbf{1}$, so that

$$\partial_{\bar{z}} \det \Gamma = \det(\Gamma) Tr(M) = 0 \tag{2.8}$$

where the last identity follows from the fact that M is traceless because $\operatorname{Tr}(M) = \operatorname{Tr}(M^T) = \operatorname{Tr}(f^T(z) g(z)) = 0$ thanks to the assumption (2.2). Thus $\det \Gamma$ is an entire function which tends to $\mathbf{1}$ at infinity, and hence it is identically equal to 1 by Liouville's theorem.

Now, if Γ_1, Γ_2 are two solutions, it follows easily that $R(z) := \Gamma_1 \Gamma_2^{-1}$ is an entire matrix-valued function which tends to the identity matrix $\mathbf{1}$ at infinity and hence, by Liouville's theorem $R(z) \equiv \mathbf{1}$, thus proving the uniqueness. □

Theorem 2.3. *The operator* $\operatorname{Id} - \mathcal{K}$*, with* $\mathcal{K}$ *as in (2.4) and kernel* $K(z,w)$ *of the form (2.1) (2.2), is invertible in* $L^2(\mathscr{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n$ *if and only if the* $\overline{\partial}$*-problem 2.1 admits a solution. The resolvent* $\mathcal{R}$ *of* $\mathcal{K}$ *has kernel given by:*

$$R(z,w) := \frac{f^T(z) \Gamma^T(z) \left(\Gamma^T(w)\right)^{-1} g(w)}{z - w}, \quad (z,w) \in \mathscr{D} \times \mathscr{D} \tag{2.9}$$

where $\Gamma(z)$ *is a* $r \times r$ *matrix that solves the* $\overline{\partial}$*-problem 2.1.*

Proof. Suppose that the $\overline{\partial}$-problem 2.1 is solved by $\Gamma(z)$; we now show that the operator $(\operatorname{Id} - \mathcal{K})$ is invertible. Let us define the operator

$$\mathcal{R} : L^2\left(\mathscr{D}, \mathrm{d}^2 z\right) \otimes \mathbb{C}^n \to L^2\left(\mathscr{D}, \mathrm{d}^2 z\right) \otimes \mathbb{C}^n$$

with kernel $R(z,w)$ given by (2.9). To verify that $\mathcal{R}$ is the resolvent of the operator $\mathcal{K}$ we need to check the following condition

$$\begin{gathered} (\operatorname{Id} + \mathcal{R}) \circ (\operatorname{Id} - \mathcal{K}) = \operatorname{Id} \\ \Downarrow \\ \mathcal{R} \circ \mathcal{K} = \mathcal{R} - \mathcal{K}. \end{gathered} \tag{2.10}$$

To this end we compute the kernel of $\mathcal{R}\circ\mathcal{K}$ namely

$$(R\circ K)(z,w) := \iint_{\mathscr{D}} R(z,\zeta)\,K(\zeta,w)\,\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i}$$

$$= \iint_{\mathscr{D}} \frac{f^T(z)\,\Gamma^T(z)\,\overbrace{\left(\Gamma^T(\zeta)\right)^{-1} g(\zeta) f^T(\zeta)}^{=-\frac{1}{\pi}\partial_{\overline{\zeta}}\left(\Gamma^T(\zeta)\right)^{-1}}\,g(w)}{(z-\zeta)(\zeta-w)}\,\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i}$$

$$= -\frac{f^T(z)\,\Gamma^T(z)}{z-w}\iint_{\mathscr{D}} \partial_{\overline{\zeta}}\left(\Gamma^T(\zeta)\right)^{-1}\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-w}\right)\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i\pi}\,g(w). \tag{2.11}$$

If we consider the generalized Cauchy-Pompeiu formula for the matrix $(\Gamma^T(z))^{-1}$ we can express it in integral form as

$$\left(\Gamma^T(z)\right)^{-1} = \mathbf{1} - \iint_{\mathscr{D}} \frac{\partial_{\overline{\zeta}}\left(\Gamma^T(\zeta)\right)^{-1}}{\zeta - z}\,\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2\pi i},\quad z\in\mathbb{C}. \tag{2.12}$$

We substitute (2.12) into (2.11):

$$\begin{aligned}(R\circ K)(z,w) &= -\frac{f^T(z)\,\Gamma^T(z)}{z-w}\left(\left(\left(\Gamma^T(z)\right)^{-1}-\mathbf{1}\right)-\left(\left(\Gamma^T(w)\right)^{-1}-\mathbf{1}\right)\right)g(w)\\ &= -\frac{f^T(z)\,\Gamma^T(z)}{z-w}\left(\left(\Gamma^T(z)\right)^{-1}-\left(\Gamma^T(w)\right)^{-1}\right)g(w)\\ &= \frac{f^T(z)\,\Gamma^T(z)\left(\Gamma^T(w)\right)^{-1}g(w)}{z-w}-\frac{f^T(z)g(w)}{z-w} = R(z,w)-K(z,w).\end{aligned} \tag{2.13}$$

This shows that indeed $\mathcal{R}$ satisfies the resolvent equation (2.10) and hence the operator $\mathrm{Id}-\mathcal{K}$ is invertible.

Viceversa, let us now suppose that the operator $\mathrm{Id}-\mathcal{K}$ is invertible and denote

$$\mathcal{R} = (\mathrm{Id}-\mathcal{K})^{-1}-\mathrm{Id}.$$

We now verify that $\mathcal{R}$ has kernel

$$R(z,w) = \frac{F(z)^T\,G(w)}{z-w} \tag{2.14}$$

where the matrices $F(z)$ and $G(z)$ are defined as

$$\begin{aligned}F^T &:= (\mathrm{Id}-\mathcal{K})^{-1}\left[f^T\right]\\ G &:= [g]\,(\mathrm{Id}-\mathcal{K})^{-1},\end{aligned} \tag{2.15}$$

where in the second relation the operator is acting from the left.

Indeed we verify the condition (2.10) with R given by (2.14):

$$
\begin{aligned}
(R\circ K)(z,w) &= \iint_{\mathscr{D}} \frac{F^T(z)\,G(\zeta)f^T(\zeta)\,g(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i} \\
&= \frac{1}{z-w}\left(\iint_{\mathscr{D}} \frac{F^T(z)\,G(\zeta)f^T(\zeta)\,g(w)}{(z-\zeta)}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\right. \\
&\quad \left. + \iint_{\mathscr{D}} \frac{F^T(z)\,G(\zeta)f^T(\zeta)\,g(w)}{(\zeta-w)}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\right) \\
&= \frac{1}{z-w}\left(\mathcal{R}\left[f^T\right](z)\,g(w) + F^T(z)\left([G]\,\mathcal{K}\right)(w)\right),
\end{aligned}
$$

where we use the notation $([G]\mathcal{K})(w)$ to denote the operator that is acting on the left. Adding and subtracting the kernels $K(z,w)$ and $R(z,w)$, we obtain

$$
\begin{aligned}
(R\circ K)(z,w) = &\frac{1}{z-w}\left((\mathrm{Id}+\mathcal{R})\left[f^T\right](z)\,g(w) - F^T(z)\left([G]\,(\mathrm{Id}-\mathcal{K})\right)(w)\right) \\
&+ R(z,w) - K(z,w)\,,
\end{aligned} \tag{2.16}
$$

we see that we must have

$$
(\mathrm{Id}+\mathcal{R})\left[f^T\right](z) = F^T(z)\,,\quad g(w) = \left([G]\,(\mathrm{Id}-\mathcal{K})\right)(w)
$$

which is equivalent to (2.15). With the definitions (2.15) the contributions in the first line of (2.16) cancel out and the condition (2.10) is satisfied. To conclude the proof we need to verify that

$$
F(z) = \Gamma(z)f(z)\,,\;\; G(z) = \left(\Gamma^T(z)\right)^{-1} g(z) \tag{2.17}
$$

where the matrix Γ solves the $\bar{\partial}$-problem 2.1. To this end, let us define the matrix $\widetilde{\Gamma}(z)$

$$
\widetilde{\Gamma}(z) := \mathbf{1} - \iint_{\mathscr{D}} \frac{F(\zeta)\,g^T(\zeta)}{\zeta - z}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}, \quad z\in\mathbb{C}. \tag{2.18}
$$

From this definition it follows that

$$
\begin{aligned}
f^T(z)\,\widetilde{\Gamma}^T(z) &= f^T(z) - \iint_{\mathscr{D}} \frac{f^T(z)\,g(\zeta)\,F^T(\zeta)}{\zeta - z}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i} \\
&= f^T(z) + \mathcal{K}\left[F^T\right](z) \\
&= f^T(z) + F^T(z) - (\mathrm{Id}-\mathcal{K})\left[F^T\right](z) \\
&= F^T(z)
\end{aligned} \tag{2.19}
$$

which implies

$$
F(z) = \widetilde{\Gamma}(z)f(z)\,. \tag{2.20}
$$

We now substitute (2.20) in the definition (2.18):

$$
\widetilde{\Gamma}(z) = \mathbf{1} - \iint_{\mathscr{D}} \frac{\widetilde{\Gamma}(\zeta)f(\zeta)\,g^T(\zeta)}{\zeta - z}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}. \tag{2.21}
$$

Then, following the general Cauchy formula (2.12), we find that the matrix $\widetilde{\Gamma}(z)$ satisfies

$$\partial_{\bar{z}}\widetilde{\Gamma}(z) = \pi\widetilde{\Gamma}(z)f(z)\,g^T(z)\,. \tag{2.22}$$

Finally, since the support $\mathscr{D}$ of M is compact, the equation (2.18) implies that $\widetilde{\Gamma}$ is analytic outside of $\mathscr{D}$ and tends to $\mathbf{1}$ as $|z|\to\infty$. Thus $\widetilde{\Gamma}$ solves the same $\overline{\partial}$-problem 2.1 and since the solution is unique, it must coincide with Γ.

In the same way, we can prove the other result $G(z) = (\Gamma^T(z))^{-1}g(z)$. We define another matrix $\hat{\Gamma}(z)$ as

$$\hat{\Gamma}(z) := \mathbf{1} + \iint_{\mathscr{D}} \frac{f(\zeta)\,G^T(\zeta)}{\zeta - z}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}, \quad z\in\mathbb{C}. \tag{2.23}$$

We multiply both side of (2.23) by $g^T(z)$ and we get

$$\begin{aligned} g^T(z)\,\hat{\Gamma}(z) &= g^T(z) + \mathcal{K}^{\mathcal{T}}\left[G^T\right](z) \\ &= g^T(z) + G^T(z) - \left(\mathrm{Id} - \mathcal{K}^T\right)\left[G^T\right](z) \\ &= G^T(z) \end{aligned} \tag{2.24}$$

which implies

$$G(z) = \hat{\Gamma}^T(z)\,g(z)\,. \tag{2.25}$$

In the above relations we have used the operator $\mathcal{K}^T$ that has kernel $K^T(\zeta,z)$ (namely, the transposition is only in the matrix indices). We then substitute (2.25) in the definition (2.23) and we obtain the integral equation

$$\hat{\Gamma}(z) := \mathbf{1} + \iint_{\mathscr{D}} \frac{f(\zeta)\,g^T(\zeta)\,\hat{\Gamma}(\zeta)}{\zeta - z}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}, \quad z\in\mathbb{C}. \tag{2.26}$$

From the general Cauchy formula (2.12), we have that $\hat{\Gamma}(z)$ satisfies the $\overline{\partial}$-problem

$$\partial_{\bar{z}}\hat{\Gamma}(z) = -\pi f(\zeta)\,g^T(\zeta)\,\hat{\Gamma}(\zeta)\,,\text{for } z\in\mathscr{D}, \tag{2.27}$$

which coincides with the $\overline{\partial}$-problem for the matrix $(\Gamma(z))^{-1}$. Since the solution is unique, then $\hat{\Gamma}(z)$ coincides with $(\Gamma(z))^{-1}$. □

3. The Fredholm determinant

In section 2 we have linked the solution of the $\overline{\partial}$-problem 2.1 to the existence of the inverse of $\mathrm{Id}-\mathcal{K}$. From the conditions (2.1) we conclude that $\mathcal{K}$ is a Hilbert–Schmidt operator with a well-defined and continuous diagonal in $\mathscr{D}\times\mathscr{D}$: according to [47] this is sufficient to define the Fredholm determinant for the operator $\mathrm{Id}-\mathcal{K}$, as explained in the following remark.

Remark 3.1. In general, for a Hilbert-Schmidt operator $\mathcal{A}$, the Fredholm determinant is not defined but we can still define a regularization of it, called the *Hilbert–Carleman determinant*

$$\det_2(\mathrm{Id}-\mathcal{A}) := \det\left((\mathrm{Id}-\mathcal{A})\, e^{\mathcal{A}}\right). \tag{3.1}$$

We observe that $\det\left((\mathrm{Id}-\mathcal{A})\mathrm{e}^{\mathcal{A}}\right) = \det(\mathrm{Id}-\mathcal{T}_{\mathcal{A}})$ with $\mathcal{T}_{\mathcal{A}} := \mathrm{Id}-(\mathrm{Id}-\mathcal{A})\mathrm{e}^{\mathcal{A}}$ and the operator $\mathcal{T}_{\mathcal{A}}$ is trace class because it has the representation

$$\mathcal{T}_{\mathcal{A}} = -\sum_{n=2}^{\infty}\frac{n-1}{n!}\mathcal{A}^n.$$

If $\mathcal{A}$ is of trace class we can rewrite the Hilbert–Carleman determinant as

$$\det_2(\mathrm{Id}-\mathcal{A}) = \det(\mathrm{Id}-\mathcal{A})\, e^{Tr(\mathcal{A})}. \tag{3.2}$$

Moreover, as for the Fredholm determinant, the Hilbert–Carleman determinant can be represented by a series

$$\det_2(\mathrm{Id}-\mathcal{A}) = 1+\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\Psi_n(\mathcal{A}) \tag{3.3}$$

where $\Psi_n(\mathcal{A})$ is given by the *Plemelj-Smithies formula*

$$\Psi_n(\mathcal{A}) = \det\begin{bmatrix} 0 & n-1 & 0 & \dots & 0 & 0\\ Tr\left(\mathcal{A}^2\right) & 0 & n-2 & \dots & 0 & 0\\ Tr\left(\mathcal{A}^3\right) & Tr\left(\mathcal{A}^2\right) & 0 & \dots & 0 & 0\\ . & . & . & . & . & .\\ Tr\left(\mathcal{A}^n\right) & Tr\left(\mathcal{A}^{n-1}\right) & Tr\left(\mathcal{A}^{n-2}\right) & \dots & Tr\left(\mathcal{A}^2\right) & 0\end{bmatrix}.$$

It is shown that if $\mathcal{A}$ is Hilbert–Schmidt then (3.3) converges ([27], chapter 10, theorem 3.1).

Let us now assume that $\mathcal{K}$ depends smoothly on parameters $\boldsymbol{t} = (t_1, t_2, \dots, t_j, \dots)$with $t_j \in \mathbb{C}, \forall j \geqslant 1$: we want to relate solutions of the $\overline{\partial}$-problem 2.1 with the variational equations for the determinant.

Proposition 3.2. *Let us suppose that the matrix $M(z,\bar{z})$ in the $\overline{\partial}$-problem 2.1, depends smoothly on some parameters* $\boldsymbol{t}$*, while remaining identically nilpotent. Then the solution $\Gamma(z)$ of the $\overline{\partial}$-problem 2.1 is related to the logarithmic derivative of the Hilbert–Carleman determinant of* $\mathrm{Id}-\mathcal{K}$ *as follows:*

$$\delta\log\left[det_2(\mathrm{Id}-\mathcal{K})\right] = -\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,\delta M(z)\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}, \tag{3.4}$$

where δ stands for the total differential in the space of parameters $\boldsymbol{t}$*.*

Proof. Using the Jacobi variational formula (1.6), we can rewrite the LHS of (3.4) as

$$\delta\log\left[\det_2(\mathrm{Id}-\mathcal{K})\right] = \delta\log\left[\det\left((\mathrm{Id}-\mathcal{K})\, e^{\mathcal{K}}\right)\right] = -Tr(\mathcal{R}\circ\delta\mathcal{K}), \tag{3.5}$$

where $\mathcal{R}\circ\delta\mathcal{K}$ is a trace class operator, since it is the composition of two Hilbert–Schmidt operators. Here Tr denotes the trace on the Hilbert space $L^2(\mathscr{D},\mathrm{d}^2z)\otimes\mathbb{C}^n$. The composition of the two operators $\mathcal{R}\circ\delta\mathcal{K}$ produces the kernel

$$
\begin{aligned}
(R\circ\delta K)(z,w) &= \iint_{\mathscr{D}} \frac{f^T(z)\Gamma^T(z)\left(\Gamma^T(\zeta)\right)^{-1} g(\zeta)\,\delta\left(f^T(x)\,g(w)\right)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar\zeta\wedge\mathrm{d}\zeta}{2i}\\
&= \iint_{\mathscr{D}} \frac{f^T(z)\Gamma^T(z)\left(\Gamma^T(\zeta)\right)^{-1} g(\zeta) f^T(\zeta)\,\delta g(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar\zeta\wedge\mathrm{d}\zeta}{2i} &(3.6)\\
&+ \iint_{\mathscr{D}} \frac{f^T(z)\Gamma^T(z)\left(\Gamma^T(\zeta)\right)^{-1} g(\zeta)\,\delta f^T(\zeta)\,g(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar\zeta\wedge\mathrm{d}\zeta}{2i} &(3.7)
\end{aligned}
$$

where we have omitted explicit notation of the dependence on $\boldsymbol{t}$ of the functions f,g,F,G,Γ.

We focus on the term in (3.6). Using the identity $\frac{1}{(z-\zeta)(\zeta-w)}=\frac{1}{z-w}\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-w}\right)$, we obtain

$$
(3.6)=\frac{f^T(z)\Gamma^T(z)}{z-w}\left(\iint_{\mathscr{D}}\left(\Gamma^T(\zeta)\right)^{-1} g(\zeta)f^T(\zeta)\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-w}\right)\frac{\mathrm{d}\bar\zeta\wedge\mathrm{d}\zeta}{2i}\right)\delta g(w)\,. \quad (3.8)
$$

In order to compute the trace we need to compute the kernel (3.8) along the diagonal $z=w$ and hence we consider $\lim_{w\to z}(3.8)$. Observe that $(\Gamma^T(\zeta))^{-1}g(\zeta)f^T(\zeta)=-\frac{1}{\pi}\partial_{\bar\zeta}(\Gamma^T(\zeta))^{-1}$, and hence we can apply the formula (2.12) to eliminate the integral and rewrite (3.8) as follows

$$
\begin{aligned}
(3.8) &= -\frac{f^T(z)\Gamma^T(z)\left(\left(\Gamma^T(z)\right)^{-1}-\mathbf{1}\right)\delta g(w)}{z-w}+\frac{f^T(z)\Gamma^T(z)\left(\left(\Gamma^T(w)\right)^{-1}-\mathbf{1}\right)\delta g(w)}{z-w} &(3.9)\\
&= \frac{f^T(z)\left(\Gamma^T(z)\left(\Gamma^T(w)\right)^{-1}-\mathbf{1}\right)\delta g(w)}{z-w}. &(3.10)
\end{aligned}
$$

We can now easily compute the expansion of (3.10) along the diagonal $w\to z$ by Taylor's formula, keeping in mind that Γ is not a holomorphic function inside $\mathscr{D}$:

$$
\begin{aligned}
(3.10) &= f^T(z)\,\partial_z\Gamma^T(z)\left(\Gamma^T(z)\right)^{-1}\delta g(z)+\frac{\bar z-\bar w}{z-w}\overbrace{f^T(z)\,g(z)}^{\equiv 0}f^T(z)\,\delta g(w)+\mathcal{O}(|z-w|) &(3.11)\\
&= f^T(z)\,\partial_z\Gamma^T(z)\left(\Gamma^T(z)\right)^{-1}\delta g(z)+\mathcal{O}(|z-w|)\,. &(3.12)
\end{aligned}
$$

Using the above expression we conclude that the trace in $L^2(\mathscr{D},\mathrm{d}^2z)\otimes\mathbb{C}^n$ of (3.6) is

$$
Tr((3.6))=\iint_{\mathscr{D}} Tr\left(f^T(z)\,\partial_z\Gamma(z)^T\left(\Gamma^T(z)\right)^{-1}\delta g(z)\right)\frac{\mathrm{d}\bar z\wedge\mathrm{d}z}{2i}. \quad (3.13)
$$

Using the cyclicity of the trace and its invariance under transposition of the arguments, we reorder the terms (3.13) to the form

$$
Tr((3.6))=\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)f(z)\,\delta g^T(z)\right)\frac{\mathrm{d}\bar z\wedge\mathrm{d}z}{2i}. \quad (3.14)
$$

We now consider the term (3.7). Taking its trace yields:

$$
\begin{aligned}
Tr((3.7)) &= -\iint_{\mathscr{D}}\iint_{\mathscr{D}} \frac{Tr\left(f^T(z)\Gamma^T(z)\left(\Gamma^T(\zeta)\right)^{-1}g(\zeta)\delta f^T(\zeta)g(z)\right)}{(z-\zeta)^2}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2i}\\
&= -\iint_{\mathscr{D}}\iint_{\mathscr{D}} \frac{Tr\left(g(z)f^T(z)\Gamma^T(z)\left(\Gamma^T(\zeta)\right)^{-1}g(\zeta)\delta f^T(\zeta)\right)}{(z-\zeta)^2}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2i}.
\end{aligned}
\tag{3.15}
$$

We observe that the integrand is in L^2_{loc} because the numerator vanishes to order $\mathcal{O}(|z-\zeta|)$ along the diagonal

$$
Tr\left(g(z)f^T(z)\Gamma^T(z)\left(\Gamma^T(\zeta)\right)^{-1}g(\zeta)\delta f^T(\zeta)\right) = Tr\left(g(\zeta)\overbrace{f^T(\zeta)g(\zeta)}^{=0}\delta f^T(\zeta)\right) + \mathcal{O}(|z-\zeta|),
\tag{3.16}
$$

and hence the integrand is $\mathcal{O}(|z-\zeta|^{-1})$ which is locally integrable with respect to the area measure. We can now relate this integral to $\partial_z\Gamma$ as follows. Using the formula (2.12) and the $\overline{\partial}$-problem 2.1 we can rewrite $\Gamma^T(\zeta)$ as

$$
\Gamma^T(\zeta) = \mathbf{1} - \iint_{\mathscr{D}} \frac{\partial_{\bar{z}}\left(\Gamma^T(z)\right)}{z-\zeta}\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i} = \mathbf{1} - \iint_{\mathscr{D}} \frac{M^T(z)\Gamma^T(z)}{z-\zeta}\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}.
\tag{3.17}
$$

Taking the holomorphic derivative with respect to ζ we get

$$
\partial_\zeta\Gamma^T(\zeta) = -\iint_{\mathscr{D}} \frac{g(z)f^T(z)\Gamma^T(z)}{(z-\zeta)^2}\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2i}.
$$

Plugging the result into (3.15) we obtain

$$
\begin{aligned}
(3.15) &= -\iint_{\mathscr{D}} Tr\left(\left(\Gamma^T(\zeta)\right)^{-1}g(\zeta)\delta f(\zeta)\left(\iint_{\mathscr{D}}\frac{g(z)f^T(z)\Gamma^T(z)}{(z-\zeta)^2}\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2i}\right)\right)\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\\
&= \iint_{\mathscr{D}} Tr\left(\left(\Gamma^T(\zeta)\right)^{-1}g(\zeta)\delta f^T(\zeta)\partial_\zeta\left(\Gamma^T(\zeta)\right)\right)\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\\
&= \iint_{\mathscr{D}} Tr\left(g(\zeta)\delta f^T(\zeta)\partial_\zeta\Gamma^T(\zeta)\left(\Gamma^{-1}(\zeta)\right)^T\right)\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\\
&= \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(\zeta)\partial_\zeta\Gamma(\zeta)\delta f(\zeta)g^T(\zeta)\right)\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i},
\end{aligned}
\tag{3.18}
$$

so that

$$
Tr((3.7)) = \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(\zeta)\partial_\zeta\Gamma(\zeta)\delta f(\zeta)g^T(\zeta)\right)\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}.
\tag{3.19}
$$

Combining (3.14) and (3.19) we have obtained that

$$\begin{aligned} -Tr(\mathcal{R}\circ\delta\mathcal{K}) &= -Tr((3.6)+(3.7)) \\ &= -\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,\delta\left(f(z)\,g^T(z)\right)\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2i} \\ &= -\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,\delta M(z)\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}. \end{aligned} \tag{3.20}$$

This concludes the proof of proposition 3.2. □

3.1. Malgrange one form and τ-function

From proposition 3.2 we define the following one form on the space of deformations, which we call *Malgrange one form* following the terminology in [3]:

$$\omega := -\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,\delta M(z)\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}, \tag{3.21}$$

where $\Gamma(z)$ is the solution of the $\overline{\partial}$-problem 2.1 and $M(z)$ is defined in (2.6). For the operator $\mathcal{K}$ defined in (2.4), the proposition 3.2 implies that

$$\omega = \delta\log\det{}_2\left(\mathrm{Id}-\mathcal{K}\right), \tag{3.22}$$

and hence ω is an exact (and hence closed) one form in the space of deformation parameters the operator $\mathcal{K}$ may depend upon. The form ω can be shown to be closed under weaker assumptions on the matrix M than the ones that appears in the $\overline{\partial}$-problem 2.1 as the following theorem shows.

Theorem 3.3. *Suppose that the $r\times r$ matrix $M = M(z,\bar{z};\boldsymbol{t})$ is smooth and compactly supported in $\mathscr{D}$ (uniformly with respect to the parameters $\boldsymbol{t}$), depends smoothly on $\boldsymbol{t}$ and the matrix trace $Tr(M)\equiv 0$. Let $\Gamma(z,\bar{z};\boldsymbol{t})$ be the solution of the $\overline{\partial}$-problem 1.15. Then the exterior differential of the one-form ω defined in (3.21) vanishes:*

$$\delta\omega(\boldsymbol{t}) = 0. \tag{3.23}$$

Proof. From the $\overline{\partial}$-problem we obtain

$$\delta\left(\partial_{\bar{z}}\Gamma\right) = \Gamma\delta M + \delta\Gamma M \quad\Rightarrow\quad \delta\Gamma(z) = \iint_{\mathscr{D}} \frac{\Gamma(w)\,\delta M(w)\,\Gamma^{-1}(w)}{(w-z)^2}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}\Gamma(z). \tag{3.24}$$

Using (3.24) we can compute

$$\begin{aligned} \delta\omega &= -\iint_{\mathscr{D}} Tr\left(\delta\left(\Gamma^{-1}\partial_z\Gamma\wedge\delta M\right)\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i} \\ &= \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}\delta\Gamma\Gamma^{-1}\partial_z\Gamma\wedge\delta M\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i} - \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}\delta\partial_z\Gamma\wedge\delta M\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}. \end{aligned} \tag{3.25}$$

From (3.24) we deduce

$$\delta\partial_z\Gamma(z) = -\iint_{\mathscr{D}} \frac{\Gamma(w)\,\delta M(w)\,\Gamma^{-1}(w)}{(w-z)^2}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}\Gamma(z) + \delta\Gamma(z)\,\Gamma(z)^{-1}\,\partial_z\Gamma(z). \tag{3.26}$$

Substituting (3.26) in the equation (3.25) we obtain:

$$\delta\omega = \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\left(\iint_{\mathscr{D}} \frac{\Gamma(w)\,\delta M(w)\,\Gamma^{-1}(w)}{(w-z)^2}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}\right)\Gamma(z)\wedge\delta M(z)\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}. \tag{3.27}$$

The crux of the proof is now the correct evaluation of the iterated integral:

$$\delta\omega = \iint_{\mathscr{D}} \frac{\mathrm{d}^2 z}{\pi}\iint_{\mathscr{D}} \frac{\mathrm{d}^2 w}{\pi}\frac{F(z,w)}{(z-w)^2},$$
$$F(z,w) := Tr\left(\Gamma(w)\,\delta M(w)\,\Gamma^{-1}(w)\wedge\Gamma(z)\,\delta M(z)\,\Gamma^{-1}(z)\right). \tag{3.28}$$

By applying Fubini's theorem, since the integrand is antisymmetric in the exchange of the variables $z\leftrightarrow w$, we quickly conclude that the integral is zero. However the integrand is singular along the diagonal $\Delta := \{z=w\}\subset\mathscr{D}\times\mathscr{D}$ and we need to make sure that the integrand is absolutely summable.

Recalling that $F(z,w) = -F(w,z)$, so that $F(z,z)\equiv 0$, we now compute the Taylor expansion of $F(z,w)$ with respect to w near z;

$$F(z,w) = 0 + \partial_w F(z,z)(w-z) + \partial_{\overline{w}} F(z,z)(\overline{w}-\overline{z}) + \mathcal{O}\left(|z-w|^2\right). \tag{3.29}$$

Thus $\frac{|F(z,w)|}{|z-w|^2} = \mathcal{O}(|z-w|^{-1})$ which is integrable with respect to the area measure. Hence application of Fubini's theorem is justified. □

From this theorem, we can define a τ-function associated to the the $\overline{\partial}$-problem 1.15 by

$$\tau(\boldsymbol{t}) = \exp\left(\int \omega(\boldsymbol{t})\right). \tag{3.30}$$

In general the above τ-function is defined only up to scalar multiplication and hence should be rather thought of as a section of an appropriate line bundle over the space of deformation parameters, depending on the context. However, for M in the form specified in (2.6) we know from proposition 3.2 that we can *identify* the τ-function with the regularized Hilbert–Carleman determinant:

$$\tau(\boldsymbol{t}) = \exp\left(\int \omega(\boldsymbol{t})\right) = \det_2\left(\mathrm{Id} - \mathcal{K}(\boldsymbol{t})\right). \tag{3.31}$$

In the next section, by choosing a specific dependence on the parameters $\boldsymbol{t}$ in the more general setting of M as in theorem 3.3 we are going to show that $\tau(\boldsymbol{t})$ is a KP τ-function in the sense that it satisfies Hirota bilinear relations [30].

4. $\tau(\boldsymbol{t})$ as a KP τ-function

The celebrated KP equation is a PDE for a scalar function $u = u(t_1,t_2,t_3)$ of the form

$$3\partial_{t_2}^2 u = \partial_{t_1}\left(4\partial_{t_3} u - \partial_{t_1}^3 u - 6u\partial_{t_1} u\right), \tag{4.1}$$

where t_3 is identified with the time and $t_1,t_2\in\mathbb{R}$ are spatial variables. The above equation is called KPII equation. The change of variable $t_2\to i t_2$ transform the KPII equation into the KPI equation. When the function $u(t_1,t_2,t_3)$ is t_2-independent, one recovers the celebrated Korteweg de Vries equation. The function u is related to the τ-function of the KP equation as

$$\partial_{t_1}^2 \log\tau(t_1,t_2,t_3) = \frac{1}{2}u(t_1,t_2,t_3)$$

and τ satisfies the Hirota bilinear equation

$$\left(3\mathcal{D}_2^2 - 4\mathcal{D}_1\mathcal{D}_3 + \mathcal{D}_1^4\right)\tau^2 = 0, \tag{4.2}$$

where $\mathcal{D}_j$ is the Hirota derivative with respect to t_j, defined as

$$\mathcal{D}_j p(\boldsymbol{t}) q(\boldsymbol{t}) := \left(\partial_{t_j} - \partial_{t'_j}\right)(p(\boldsymbol{t}) q(\boldsymbol{t}'))|_{\boldsymbol{t}=\boldsymbol{t}'} \tag{4.3}$$

for any function $p(\boldsymbol{t})$ and $q(\boldsymbol{t})$ depending smoothly on an infinite number of 'times' $\boldsymbol{t} = (t_1,t_2,t_3,\dots)$. The concept of τ-function can be generalized to the infinite set of times $\boldsymbol{t}$. A τ-function of the KP hierarchy, $\tau(\boldsymbol{t})$, can be characterized as a function of (formally) an infinite number of variable which satisfies the Hirota Bilinear relation

$$\mathrm{Res}_{z=\infty}(\tau(\boldsymbol{t} - [z^{-1}])\tau(\boldsymbol{s} + [z^{-1}])e^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} = 0 \tag{4.4}$$

where $\boldsymbol{t} \pm [z^{-1}]$ is the *Miwa Shift*, defined as:

$$\boldsymbol{t} \pm [z^{-1}] := \left(t_1 \pm \frac{1}{z}, t_2 \pm \frac{1}{2z^2}, \dots, t_j \pm \frac{1}{jz^j}, \dots\right). \tag{4.5}$$

The residue in (4.4) is meant in the formal sense, namely by considering the coefficient of z^{-1} in the expansion at infinity and it can be thought of as the limit of $\oint_{|z|=R}$ as $R \to +\infty$. If the functions of z intervening in (4.4) can be written as analytic functions in a deleted neighbourhood of ∞, then the residue is a genuine integral; this is the case of interest below.

As described in [36], the equation (4.4) implies that the tau function satisfy an equation of the Hirota type

$$P(\mathcal{D}_1,\mathcal{D}_2,\dots)\tau^2 = 0 \tag{4.6}$$

and $P(\mathcal{D}_1,\mathcal{D}_2,\dots)$ is a polynomial in $(\mathcal{D}_1,\mathcal{D}_2,\dots)$. In particular, if we consider the first three times t_1,t_2 and t_3, and $t_k = 0$ for $k \geqslant 3$ the equation (4.4) is equivalent to the KP equation in Hirota's form (4.2).

4.1. Hirota bilinear relation for the KP hierarchy

In this section we consider a specific type of dependence of the matrix M in (2.6) of the $\overline{\partial}$-problem 2.1 with respect to the 'times':

$$M(z,\boldsymbol{t}) = e^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} M_0(z) e^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}, \tag{4.7}$$

with

$$\xi(z,\boldsymbol{t}) = \sum_{j=1}^{+\infty} z^j t_j \tag{4.8}$$

and with $M_0(z)$ a 2×2 traceless matrix compactly supported on $\mathscr{D}$. The rest of this section is devoted to the verification of the Hirota bilinear relation (4.4) for the KP tau function constructed from the $\overline{\partial}$- problem.

The main result is the following.

Theorem 4.1. *Let $\Gamma(z,\boldsymbol{t})$ be the 2×2 matrix solution of the $\overline{\partial}$-problem*

$$\partial_{\bar z}\Gamma(z,\boldsymbol{t}) = \Gamma(z,\boldsymbol{t})M(z,\boldsymbol{t}), \quad M(z,\boldsymbol{t}) = e^{\frac{\xi(z,t)}{2}\sigma_3}M_0(z)e^{-\frac{\xi(z,t)}{2}\sigma_3} \tag{4.9}$$

$$\Gamma(z,\boldsymbol{t}) \underset{z\to\infty}{\to} \mathbf{1} \tag{4.10}$$

where $M_0(z)$ is a traceless 2×2 matrix whose entries are $\mathcal{C}^1$-functions on a compact domain $\mathscr{D}\subset\mathbb{C}$ and the function ξ is given by the formal sum $\xi(z,\boldsymbol{t}) = \sum_{j=1}^{+\infty} z^j t_j$. Then the function

$$\tau(\boldsymbol{t}) = \exp\left(\int \omega(\boldsymbol{t})\right), \tag{4.11}$$

with ω defined in (3.21) is a KP τ-function, i.e. it satisfies the Hirota Bilinear relation (4.4).

Remark 4.2. In this setting the KP τ-function is in general complex–valued. Under appropriate additional symmetry constraints for the matrix M_0 and the domain $\mathscr{D}$ we can obtain a real–valued τ-function. The class of solutions obtained from (4.11) is different from the class studied in [44–46] and represented via a Fredholm determinant.

We prove the theorem in several steps. We first analyze the effect of the Miwa shifts on the τ-function. For this purpose we need to determine how the Miwa shift acts on the matrices $\Gamma(z,\boldsymbol{t})$ and $M(z,\boldsymbol{t})$. We consider $M(z,\boldsymbol{t}\pm[\zeta^{-1}])$ first.

$$M\left(z,\boldsymbol{t}\pm\left[\zeta^{-1}\right]\right) = e^{\frac{1}{2}\xi\left(z,\boldsymbol{t}\pm\left[\zeta^{-1}\right]\right)\sigma_3}M_0(z)e^{-\frac{1}{2}\xi\left(z,\boldsymbol{t}\pm\left[\zeta^{-1}\right]\right)\sigma_3}$$

from the definition of $\xi(z,t)$ (4.8)

$$\xi\left(z,\boldsymbol{t}\pm\left[\zeta^{-1}\right]\right) = \sum_{j=1}^{+\infty} z^j\left(t_j \pm \frac{1}{j\zeta^j}\right) = \sum_{j=1}^{+\infty} z^j t_j \pm \sum_{j=1}^{+\infty}\frac{z^j}{j\zeta^j} = \xi(z,\boldsymbol{t}) \mp \ln\left(1-\frac{z}{\zeta}\right)$$

and we have that

$$M\left(z,\boldsymbol{t}\pm\left[\zeta^{-1}\right]\right) = \left(1-\frac{z}{\zeta}\right)^{\mp\frac{\sigma_3}{2}} M(z,\boldsymbol{t})\left(1-\frac{z}{\zeta}\right)^{\pm\frac{\sigma_3}{2}}. \tag{4.12}$$

For the matrices $\Gamma(z,\boldsymbol{t}\pm[\zeta^{-1}])$ we need to consider the two cases separately. Let us start with the negative shift $\Gamma(z,\boldsymbol{t}-[\zeta^{-1}])$.

$$\begin{aligned}\partial_{\bar z}\Gamma\left(z,\boldsymbol{t}-\left[\zeta^{-1}\right]\right) &= \Gamma\left(z,\boldsymbol{t}-\left[\zeta^{-1}\right]\right)M\left(z,\boldsymbol{t}-\left[\zeta^{-1}\right]\right)\\ &= \Gamma\left(z,\boldsymbol{t}-\left[\zeta^{-1}\right]\right)\left(1-\frac{z}{\zeta}\right)^{+\frac{\sigma_3}{2}} M(z,\boldsymbol{t})\left(1-\frac{z}{\zeta}\right)^{-\frac{\sigma_3}{2}}\\ &= \Gamma\left(z,\boldsymbol{t}-\left[\zeta^{-1}\right]\right)D(z,\zeta)M(z,\boldsymbol{t})D^{-1}(z,\zeta)\end{aligned} \tag{4.13}$$

where

$$D(z,\zeta) = \begin{bmatrix}1-\frac{z}{\zeta} & 0\\ 0 & 1\end{bmatrix}.$$

From (4.13), we notice that the matrix $\Gamma(z, t-[\zeta^{-1}])D(z,\zeta)$ satisfies the $\overline{\partial}$-problem 2.1, i.e. there exists a connection matrix $C(z)$ such that

$$\Gamma\left(z,\boldsymbol{t}-\left[\zeta^{-1}\right]\right)=C(z)\,\Gamma(z,\boldsymbol{t})\,D(z,\zeta)^{-1}, \tag{4.14}$$

where obviously $C(z)$ depends also on ζ and $\boldsymbol{t}$.

The matrix $C(z)$ is determined by the conditions that both $\Gamma(z,\boldsymbol{t})$ and $\Gamma(z,\boldsymbol{t}-[\zeta^{-1}])$ must tend to $\mathbf{1}$ for $z\to\infty$ and are regular at $z=\zeta$

$$\begin{gathered}\lim_{z\to\infty}\left(1-\frac{z}{\zeta}\right)^{-1}C(z)\begin{bmatrix}\Gamma_{11}(z,\boldsymbol{t})\\ \Gamma_{12}(z,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\quad \lim_{z\to\infty}C(z)\begin{bmatrix}\Gamma_{21}(z,\boldsymbol{t})\\ \Gamma_{22}(z,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix},\\ \lim_{z\to\zeta}\left(1-\frac{z}{\zeta}\right)^{-1}C(z)\begin{bmatrix}\Gamma_{11}(z,\boldsymbol{t})\\ \Gamma_{12}(z,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}\Gamma_{11}(\zeta,\boldsymbol{t})\\0\end{bmatrix}.\end{gathered} \tag{4.15}$$

Solving the system (4.15), we obtain that the matrix $C(z)$ has the following form [38]

$$C(z)=\begin{bmatrix}\left(1-\frac{z}{\zeta}\right)+\frac{\partial_z\Gamma_{12}(\infty)\Gamma_{21}(\zeta)}{\zeta\Gamma_{11}(\zeta)} & -\frac{\partial_z\Gamma_{12}(\infty)}{\zeta}\\ -\frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)} & 1\end{bmatrix}. \tag{4.16}$$

Following the same ideas, we can find a similar formula for $\Gamma(z,t+[\zeta^{-1}])$

$$\Gamma\left(z,\boldsymbol{t}+\left[\zeta^{-1}\right]\right)=\tilde{C}(z)\,\Gamma(z,\boldsymbol{t})\,\tilde{D}(z,\zeta)^{-1} \tag{4.17}$$

with

$$\tilde{D}(z,\zeta)=\begin{bmatrix}1 & 0\\ 0 & 1-\frac{z}{\zeta}\end{bmatrix}.$$

Also in this case, we have three conditions similar to (4.15):

$$\begin{gathered}\lim_{z\to\infty}\tilde{C}(z)\begin{bmatrix}\Gamma_{11}(x,\boldsymbol{t})\\ \Gamma_{12}(x,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\quad \lim_{z\to\infty}\left(1-\frac{z}{\zeta}\right)^{-1}\tilde{R}(z,\zeta)\begin{bmatrix}\Gamma_{21}(x,\boldsymbol{t})\\ \Gamma_{22}(x,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\\ \lim_{z\to x}\left(1-\frac{z}{\zeta}\right)^{-1}\tilde{C}(z)\begin{bmatrix}\Gamma_{21}(x,\boldsymbol{t})\\ \Gamma_{22}(x,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}0\\ \Gamma_{22}(\zeta,\boldsymbol{t})\end{bmatrix}\end{gathered} \tag{4.18}$$

and we find out that $\tilde{C}(z)$ has the following form:

$$\tilde{C}(z)=\begin{bmatrix}1 & -\frac{\Gamma_{12}(\zeta)}{\Gamma_{22}(\zeta)}\\ -\frac{\partial_z\Gamma_{21}(\infty)}{\zeta} & \left(1-\frac{z}{\zeta}\right)+\frac{\partial_z\Gamma_{21}(\infty)\Gamma_{12}(\zeta)}{\zeta\Gamma_{22}(\zeta)}\end{bmatrix}. \tag{4.19}$$

We need to show how the Miwa shift acts on the Malgrange one form. We define $\delta_{[\zeta]}$ the differential deformed including the external parameter ζ

$$\delta_{[\zeta]}:=\sum_{j=1}^{+\infty}\mathrm{d}t_j\partial_{t_j}+\mathrm{d}\zeta\partial_\zeta=\delta+\delta_\zeta. \tag{4.20}$$

Lemma 4.3. *When $\zeta \notin \mathscr{D}$ the Miwa shift (4.5) acts on the Malgrange one form (3.21) in the following way:*

$$\omega\left(\boldsymbol{t} \pm \left[\zeta^{-1}\right]\right) = \omega(\boldsymbol{t}) + \delta_{[\zeta]} \ln\left(\left(\Gamma^{\mp 1}(\zeta)\right)_{11}\right) \mp \delta_{[\zeta]}\gamma(\zeta)\,, \tag{4.21}$$

where $\Gamma(z)$ solves the $\overline{\partial}$-problem 2.1 and $\gamma(\zeta)$ is a $\boldsymbol{t}$ independent function defined as

$$\gamma(\zeta) := \iint_{\mathscr{D}} \log\left(\frac{\zeta}{\zeta - z}\right)(\partial_z M_0(z))_{11} \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i}\,,\ \zeta \in \mathbb{C}\backslash\mathscr{D}, \tag{4.22}$$

that is is analytic (for $\zeta \notin \mathscr{D}$) and goes to zero as $\zeta \to \infty$, and $M_0(z)$ is defined as in Theorem 4.1.

Observe that since $\mathrm{Tr}M_0 = 0$ we may express the formula in terms of the $(2,2)$ entry instead. The proof of this lemma is presented in the appendix B. Now we can state the following proposition:

Proposition 4.4. *For $\zeta \notin \mathscr{D}$ the following relations holds:*

$$\frac{\tau\left(\boldsymbol{t} - \left[\zeta^{-1}\right]\right)}{\tau(\boldsymbol{t})} = \Gamma_{11}(\zeta,\boldsymbol{t})\, e^{\gamma(\zeta)} \qquad \frac{\tau\left(\boldsymbol{t} + \left[\zeta^{-1}\right]\right)}{\tau(\boldsymbol{t})} = \Gamma_{11}^{-1}(\zeta,\boldsymbol{t})\, e^{-\gamma(\zeta)}, \tag{4.23}$$

where $\tau(t)$ is defined in (3.30), $\Gamma(z)$ solves the $\overline{\partial}$-problem 2.1 and $\gamma(\zeta)$ is defined in (4.22)

Proof. From lemma 4.3 and the equation (3.30), we rewrite (4.21) as

$$\delta_{[\zeta]} \ln\tau\left(\boldsymbol{t} \pm \left[\zeta^{-1}\right]\right) = \delta_{[\zeta]} \ln\tau(\boldsymbol{t}) + \delta_{[\zeta]} \ln\left(\left(\Gamma^{\mp 1}(\zeta)\right)_{11}\right) \mp \delta_{[\zeta]}\gamma(\zeta) \tag{4.24}$$

an then, from the properties of the logarithm the statement (4.23) is proved. □

Remark 4.5. The exponential term $\mathrm{e}^{\gamma(\zeta)}$ could be absorbed by a gauge transformation in the formalism of the infinite dimensional Grassmannian manifold of Segal-Wilson ([29], chapter 4). Such gauge transformations have no effect on the Hirota bilinear relation (4.4).

Let us now define the matrix $H(z)$ as

$$H(z) := H(z;\boldsymbol{t},\boldsymbol{s}) := \Gamma(z,\boldsymbol{t})\, e^{(\xi(z,\boldsymbol{t}) - \xi(z,\boldsymbol{s}))E_{11}}\,\Gamma^{-1}(z,\boldsymbol{s}) \tag{4.25}$$

where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma(z,\bar{z},\boldsymbol{t})$ solves the $\overline{\partial}$-problem 2.1 and $\boldsymbol{s} = (s_1, s_2, \ldots, s_j, \ldots)$ denotes another set of values for the deformation parameters.

Lemma 4.6. *The matrix H(z) defined in (4.25) is analytic for all $z \in \mathbb{C}$.*

Proof. For $z \notin \mathscr{D}$ the statement is trivial, so we consider the case of $z \in \mathscr{D}$.

We apply the operator $\partial_{\bar{z}}$ to the matrix (4.25)

$$\begin{aligned}
\partial_{\bar{z}} H(z) &= \partial_{\bar{z}} \Gamma(z,\boldsymbol{t})\, e^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} \Gamma^{-1}(z,\boldsymbol{s}) + \Gamma(z,\boldsymbol{t})\, e^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} \partial_{\bar{z}} \Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t})\, M(z,\boldsymbol{t})\, e^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} \Gamma^{-1}(z,\boldsymbol{s}) + \\
&\quad - \Gamma(z,\boldsymbol{t})\, e^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} M(z,\boldsymbol{s})\, \Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t}) \left(e^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} M_0(z) e^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} e^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} + \right. \\
&\quad \left. - e^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} e^{\frac{\xi(z,\boldsymbol{s})}{2}\sigma_3} M_0(z) e^{-\frac{\xi(z,\boldsymbol{s})}{2}\sigma_3} \right) \Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t}) \left(e^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} e^{\frac{\xi(z,\boldsymbol{t})}{2}\mathbb{I}} M_0(z) e^{-\xi(z,\boldsymbol{s})E_{11}} + \right. \\
&\quad \left. - e^{\xi(z,\boldsymbol{t})E_{11}} M_0(z) e^{\frac{\xi(z,\boldsymbol{s})}{2}\mathbb{I}} e^{-\frac{\xi(z,\boldsymbol{s})}{2}\sigma_3} \right) \Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t}) \left(e^{\xi(z,\boldsymbol{t})E_{11}} M_0(z) e^{-\xi(z,\boldsymbol{s})E_{11}} - e^{\xi(z,\boldsymbol{t})E_{11}} M_0(z) e^{-\xi(z,\boldsymbol{s})E_{11}} \right) \Gamma^{-1}(z,\boldsymbol{s}) \\
&= 0
\end{aligned}$$

and this proves the statement. □

We are now ready to prove the main result of the section, namely theorem 4.1.

Proof of theorem 4.1. Let us compute the residue

$$\begin{aligned}
&\mathrm{Res}_{z=\infty} \left(\tau\left(\boldsymbol{t} - \left[z^{-1}\right]\right) \tau\left(\boldsymbol{s} + \left[z^{-1}\right]\right) e^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} \right) \\
&= \tau(\boldsymbol{t})\,\tau(\boldsymbol{s})\, \mathrm{Res}_{z=\infty} \left(\frac{\tau\left(\boldsymbol{t}-\left[z^{-1}\right]\right)}{\tau(\boldsymbol{t})} \frac{\tau\left(\boldsymbol{s}+\left[z^{-1}\right]\right)}{\tau(\boldsymbol{s})} e^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} \right) \\
&= \tau(\boldsymbol{t})\,\tau(\boldsymbol{s})\, \mathrm{Res}_{z=\infty} \left(\Gamma_{11}(z,\boldsymbol{t}) \left(\Gamma^{-1}(z,\boldsymbol{s})\right)_{11} e^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} \right) \\
&= \tau(\boldsymbol{t})\,\tau(\boldsymbol{s}) \lim_{R\to\infty} \oint_{|z|=R} \Gamma_{11}(z,\boldsymbol{t}) e^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} (\Gamma^{-1}(z,\boldsymbol{s}))_{11} \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{4.26}$$

Consider the first diagonal element of the matrix $H(z)$. From the analyticity of $H(z)$ proved in lemma 4.6 we get

$$\begin{aligned}
0 = \left(\oint_{|z|=R} H(z) \frac{\mathrm{d}z}{2\pi i} \right)_{11} &= \oint_{|z|=R} \Gamma_{11}(z,\boldsymbol{t})\, e^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} \left(\Gamma^{-1}(z,\boldsymbol{s})\right)_{11} \frac{\mathrm{d}z}{2\pi i} + \\
&\quad - \oint_{|z|=R} \Gamma_{12}(z,\boldsymbol{t})\, \Gamma_{21}(z,\boldsymbol{s}) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{4.27}$$

So, we can rewite (4.26) as

$$(4.26) = \tau(\boldsymbol{t})\,\tau(\boldsymbol{s}) \lim_{R\to\infty} \oint_{|z|=R} \Gamma_{12}(z,\boldsymbol{t})\, \Gamma_{21}(z,\boldsymbol{s}) \frac{\mathrm{d}z}{2\pi i}. \tag{4.28}$$

Since both $\Gamma_{12}(z,\boldsymbol{t})$ and $\Gamma_{21}(z,\boldsymbol{s})$ are analytic for $|z|$ sufficiently large (given that $\mathscr{D}$ is compact) and

$$\Gamma(z,\boldsymbol{t}) \sim \mathbf{1} + \mathcal{O}\left(z^{-1}\right) \quad \text{for}\, z \to \infty,$$

it follows that (4.28) is zero because the integrand is $\mathcal{O}(z^{-2})$, and the statement is proved. □

4.2. Fredholm determinant and reduction to focusing NLS equation

In this subsection we make a specific choice of the matrix M_0 of the form

$$M_0(z) = \begin{bmatrix} 0 & \beta(z)^2 \chi_{\mathscr{D}} \\ -(\beta^*(z))^2 \chi_{\mathscr{D}^*} & 0 \end{bmatrix}, \tag{M_0}$$

where $\beta(z) = \beta(z,\bar{z})$ is a smooth function on $\mathscr{D} \subset \mathbb{C}_+$ and $\chi_{\mathscr{D}}$ ($\chi_{\mathscr{D}^*}$) is the characteristic function of $\mathscr{D}$ ($\mathscr{D}^*$). We observe that M_0 satisfies the Schwarz symmetry

$$\overline{M_0(\bar{z})} = \sigma_2 M_0(z)\sigma_2, \quad \text{where } \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{4.29}$$

For this choice of the matrix M_0, the matrix M becomes

$$M(z;x,t) = \pi f(z;x,\boldsymbol{t})\, g^T(z;x,\boldsymbol{t}), \quad x \in \mathbb{R},\ t \geqslant 0$$

with

$$f(z;x,t) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \beta(z) e^{-i\xi(z,\boldsymbol{t})} \chi_{\mathscr{D}}(z) \\ -\beta^*(z) e^{i\xi(z,\boldsymbol{t})} \chi_{\mathscr{D}^*}(z) \end{bmatrix} \quad g(z;x,t) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \beta^*(z) e^{i\xi(z,\boldsymbol{t})} \chi_{\mathscr{D}^*}(z) \\ \beta(z) e^{-i\xi(z,\boldsymbol{t})} \chi_{\mathscr{D}}(z) \end{bmatrix}. \tag{4.30}$$

Observe that we have made a re-scaling of the times $t_j \to -2i t_j$ for the purpose of matching with the most common form of the coefficients in the resulting NLS equation later on. Combining proposition 3.4 with theorem 4.1 we have the following lemma.

Lemma 4.7. *Let us consider the operator $\mathcal{K}$ acting on $L^2(\mathscr{D} \cup \mathscr{D}^*)$ with kernel*

$$K(z,w;x,\boldsymbol{t}) = \frac{f^T(z;x,\boldsymbol{t})\, g(w;x,\boldsymbol{t})}{z-w}, \text{ with } f \text{ and } g \text{ as above}$$

Then the Hilbert–Carleman determinant

$$\log det_2(\mathrm{Id} - \mathcal{K}) = \tau(\boldsymbol{t}), \tag{4.31}$$

is a τ-function of the KP II hierarchy, up to the aforementioned re-scaling of times.

We remark again that this class of solutions is different from the one derived in [44–46]. Let us consider now the $\bar{\partial}$-problem

$$\begin{aligned} &\partial_{\bar{z}}\Gamma(z,\boldsymbol{t}) = \Gamma(z,\boldsymbol{t})\, e^{-i\xi(z,\boldsymbol{t})\sigma_3} M_0(z)\, e^{i\xi(z,\boldsymbol{t})\sigma_3} && \text{for } z \in \mathscr{D} \cup \mathscr{D}^* \\ &\Gamma(z,\boldsymbol{t}) \underset{z\to\infty}{\to} \mathbf{1} \end{aligned} \tag{4.32}$$

with, M_0 as in (M_0), $\xi(z,\boldsymbol{t})$ as in (4.8) and notice the introduction of i in the exponents of the exponentials.

Theorem 4.8. *Let $\Gamma(z,\boldsymbol{t})$ be the solution of the $\bar{\partial}$-problem (4.32) and let*

$$\psi(\boldsymbol{t}) := 2i \lim_{z\to\infty} z\,(\Gamma(z,\boldsymbol{t}) - \mathbf{1})_{12}\,.$$

Then the function $\psi = \psi(\boldsymbol{t})$ satisfies the NLS hierarchy [23, 41] written in the recursive form

$$i\partial_{t_m}\psi_1 = 2\psi_{m+1}, \quad \psi_1 := \psi,\ m \geqslant 1, \tag{4.33}$$

$$\psi_m = \frac{i}{2}\partial_{t_1}\psi_{m-1} + \psi_1 h_{m-1}, \quad \partial_{t_1} h_m = 2\,Im\left(\psi_1\overline{\psi}_m\right), \tag{4.34}$$

where ψ_m and h_m are functions of $\boldsymbol{t}$ and $h_1 := 0$.

The proof of this theorem is classical and is deferred to appendix A. In particular the second flow gives the focusing NLS equation

$$i\partial_{t_2}\psi + \frac{1}{2}\partial_{t_1}^2\psi + |\psi|^2\psi = 0,$$

where comparing with the notation in the introduction $t_2 = t$ and $t_1 = x$. The third flows gives the so called complex modified KdV equation

$$\partial_{t_3}\psi + \frac{\partial_{t_1}^3\psi}{4} + \frac{3}{2}|\psi|^2\partial_{t_1}\psi = 0.$$

Setting $t_k = 0$ for $k \geqslant 4$ one obtains that $v(t_1,t_2,t_3) := 2|\psi_1(t_1,t_2,t_3)|^2$ satisfies the KP equation (4.1) after the rescalings $v = -4u$ and $t_j \to \frac{i}{2}t_j$. The solution of the focusing NLS equation in terms of Fredholm determinant as in (1.17) follows in a straighforward way from (4.31).

4.2.1. Reduction to a RH problem. In this section we show that for particular choices of the function $\beta(z)$ and of the domain $\mathscr{D}$ one can reduce the $\overline{\partial}$-problem (4.32) to a standard RH problem when z is outside $\mathscr{D}$ and $\mathscr{D}^*$. The easiest way to solve the $\overline{\partial}$-problem (4.32) is to split it in components

$$\Gamma(z,\boldsymbol{t}) = \left[\vec{A}(z,\boldsymbol{t})\,\vec{B}(z,\boldsymbol{t})\right]$$

so that

$$\begin{aligned}&\partial_{\bar z}\vec{A}(z,\boldsymbol{t}) = 0\\ &\partial_{\bar z}\vec{B}(z,\boldsymbol{t}) = \beta(z)^2 e^{-2i\xi(z,\boldsymbol{t})}\vec{A}(z,\boldsymbol{t})\end{aligned} \quad \text{for } z \in \mathscr{D} \tag{4.35}$$

$$\begin{aligned}&\partial_{\bar z}\vec{A}(z,\boldsymbol{t}) = -\beta^*(z)^2 e^{2i\xi(z;\boldsymbol{t})}\vec{B}(z,\boldsymbol{t})\\ &\partial_{\bar z}\vec{B}(z,\boldsymbol{t}) = 0\end{aligned} \quad \text{for } z \in \mathscr{D}^* \tag{4.36}$$

with the boundary condition

$$\vec{A} \sim \begin{pmatrix}1\\0\end{pmatrix} \quad \vec{B} \sim \begin{pmatrix}0\\1\end{pmatrix} \quad \text{as } z \to \infty. \tag{4.37}$$

From the equations (4.35) and (4.36) one deduces that

- $\vec{A}(z,\boldsymbol{t})$ is analytic for z in the set $\mathscr{D}$;
- $\vec{B}(z,\boldsymbol{t})$ is analytic for z in the set $\mathscr{D}^*$.

From the Cauchy–Pompeiu formula, we can rewrite the equations (4.35) and (4.36) as a system of two integral equations

$$\begin{aligned}\vec{A}(z,\boldsymbol{t}) &= \begin{bmatrix}1\\0\end{bmatrix} + \iint_{\mathscr{D}^*} \frac{\vec{B}(w,\boldsymbol{t})\,\beta^*(w)^2\, e^{2i\xi(z,\boldsymbol{t})}}{w-z}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}\\ \vec{B}(z,\boldsymbol{t}) &= \begin{bmatrix}0\\1\end{bmatrix} - \iint_{\mathscr{D}} \frac{\vec{A}(w,\boldsymbol{t})\,\beta(w)^2\, e^{-2i\xi(z,\boldsymbol{t})}}{w-z}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}.\end{aligned} \tag{4.38}$$

Let us assume now that $\beta(z)$ is analytic in $\mathscr{D}$ simply connected and the boundary of $\mathscr{D}$ is sufficiently smooth so that it can be described by the so–called Schwarz function $S(z)$ [28] of the domain $\mathscr{D}$ through the equation

$$\overline{z} = S(z)\,.$$

The Schwarz function admits an analytic extension to a maximal domain $\mathscr{D}^0 \subset \mathscr{D}$. Here we assume that $\mathscr{L} := \mathscr{D}\setminus\mathscr{D}^0$ consist of a *mother-body*, i.e. a collection of smooth arcs. An example of this is the ellipse. Using Stokes theorem and the Schwarz function of the domain, we can reduce the area integral in (4.38) to a contour integral, namley

$$\begin{aligned}\vec{A}(z,\boldsymbol{t}) &= \begin{bmatrix}1\\0\end{bmatrix} + \oint_{\partial\mathscr{D}^*} \frac{\vec{B}(w,\boldsymbol{t})\,S^*(w)\,\beta^*(w)^2\, e^{2i\xi(z,\boldsymbol{t})}}{w-z}\frac{\mathrm{d}w}{2\pi i}\\ \vec{B}(z,\boldsymbol{t}) &= \begin{bmatrix}0\\1\end{bmatrix} - \oint_{\partial\mathscr{D}} \frac{\vec{A}(w,\boldsymbol{t})\,S(w)\,\beta(w)^2\, e^{-2i\xi(z,\boldsymbol{t})}}{w-z}\frac{\mathrm{d}w}{2\pi i},\ z\notin\mathscr{D},\end{aligned} \tag{4.39}$$

where the boundary $\partial\mathscr{D}$ is oriented anticlockwise. By analiticity, we can shrink the contour integral to the mother-body $\mathscr{L}$, namely

$$\begin{aligned}\vec{A}(z,\boldsymbol{t}) &= \begin{bmatrix}1\\0\end{bmatrix} + \oint_{\mathscr{L}^*} \frac{\vec{B}(w,\boldsymbol{t})\,\Delta S^*(w)\,\beta^*(w)^2\, e^{2i\xi(z,\boldsymbol{t})}}{w-z}\frac{\mathrm{d}w}{2\pi i}\\ \vec{B}(z,\boldsymbol{t}) &= \begin{bmatrix}0\\1\end{bmatrix} - \oint_{\mathscr{L}} \frac{\vec{A}(w,\boldsymbol{t})\,\Delta S(w)\,\beta(w)^2\, e^{-2i\xi(z,\boldsymbol{t})}}{w-z}\frac{\mathrm{d}w}{2\pi i},\ z\notin\mathscr{D},\end{aligned} \tag{4.40}$$

where $\Delta S(w) = S_-(w) - S_+(w)$, with $S_\pm(z)$ the boundary values of S on the oriented contour $\mathscr{L}$. The orientation of $\mathscr{L}$ is inherited by the orientation of $\partial\mathscr{D}$. We can express the system (4.40) in matrix form

$$\widetilde{\Gamma}(z,\boldsymbol{t}) = \mathbf{1} + \int_{\mathscr{L}\cup\mathscr{L}^*} \frac{\widetilde{\Gamma}(w,\boldsymbol{t})\,e^{-i\xi(z,\boldsymbol{t})\sigma_3}\widetilde{M}(w)\,e^{i\xi(z,\boldsymbol{t})\sigma_3}}{w-z}\frac{\mathrm{d}w}{2\pi i} \tag{4.41}$$

where

$$\widetilde{M}(z) = \begin{bmatrix} 0 & \Delta S(z)\,\beta(z)^2\,\chi_{\mathscr{L}}(z)\\ -\Delta S^*(z)\,\beta^*(z)^2\,\chi_{\mathscr{L}^*}(z) & 0\end{bmatrix}. \tag{4.42}$$

Using then the Sokhotski–Plemelj formula we can rewrite the above integral equation as a RH problem for a matrix function $\widetilde{\Gamma}(z,\boldsymbol{t})$ analytic in $\mathbb{C}\setminus\{\mathscr{L}\cup\mathscr{L}^*\}$ such that

$$\begin{aligned}\widetilde{\Gamma}_+(z,\boldsymbol{t}) &= \widetilde{\Gamma}_-(z)\,e^{-i\xi(z,\boldsymbol{t})\sigma_3}\left(\mathbf{1}+\widetilde{M}(z)\right)e^{i\xi(z,\boldsymbol{t})\sigma_3},\quad z\in\mathscr{L}\cup\mathscr{L}^*,\\ \widetilde{\Gamma}(z,\boldsymbol{t}) &= \mathbf{1}+\mathcal{O}\left(z^{-1}\right),\quad \text{as}\, z\to\infty.\end{aligned} \tag{4.43}$$

We remark that $\Gamma(z,\boldsymbol{t})$ and $\widetilde{\Gamma}(z,\boldsymbol{t})$ coincides only for $z\in\mathbb{C}\backslash\{\mathscr{D}\cup\mathscr{D}^*\}$. However for our purpose, namely the solution of the nonlinear Scrödinger equation, only the terms of $\widetilde{\Gamma}(z,\boldsymbol{t})$ for $z\to\infty$ are needed. When the domain $\mathscr{D}$ is an ellipse we show in [6] that the initial data for the nonlinear Scrödinger equation is step-like oscillatory.

5. Conclusions

The $\overline{\partial}$-problems treated in this manuscript differ from the $\overline{\partial}$-problem introduced in [42, 43] to study asymptotic behaviour of orthogonal polynomials or PDEs with non analytic initial data respectively. In those cases the $\overline{\partial}$-problem is a by-product of the steepest descent Deift-Zhou method extended to the case where the jump-matrix is not analytic but otherwise the initial problem is an ordinary RHP; in our case, the initial data is defined from the solution of the $\overline{\partial}$-problem and is encoded in the domain $\mathscr{D}$ and in the matrix M of the $\overline{\partial}$-problem (1.9). An equation similar to (4.32) was also studied by Zhu *et al* [52], with the aim to find solutions for the defocusing/focusing NLS with nonzero boundary conditions. Another class of $\overline{\partial}$-problems different from the one considered in the present manuscript has emerged in the study of normal matrix models [35] and the KP equation [1] as well as other integrable equation [24]. In the former case the relevant $\overline{\partial}$-problem is

$$\partial_{\bar{z}}Y(z)=\overline{Y(z)}M(z)$$

for a 2×2 matrix $Y(z)$ with some normalization conditions at infinity; in the latter case the $\overline{\partial}$-problem corresponds to an equation of the form $\partial_{\bar{z}}\mu(z;x,t)=J(z;x,t)\mu(-\bar{z};x,t)$ for a scalar function μ

A generalization that could be considered is one where instead of the 'pure' $\overline{\partial}$-problem (2.1) one has a mixed $\overline{\partial}$ and Riemann–Hilbert problem; this would correspond to an operator for example acting on $L^2(\mathscr{D},\mathrm{d}^2z)\oplus L^2(\Sigma,|\mathrm{d}z|)$ (typically with $\partial\mathscr{D}\subseteq\Sigma$); this type of problems would use, in the computation of the exterior derivative of the Malgrange form, the full Cauchy–Pompeiu formula. We defer this investigation to future efforts.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Connection between the $\overline{\partial}$-Problem and Lax-pair formalism

In this section we prove theorem 4.8 by deriving the corresponding Zakharov–Shabat Lax pair [51] for the solution of the $\overline{\partial}$-problem (4.32). To simplify the presentation, we restrict only to the first flow, namely we set $t_1 = x$, $t_2 = t$ and $t_j = 0$ for $j \geqslant 3$. The general case can be treated in a similar way. Let us consider the matrix

$$\Psi(z;x,t) = \Gamma(z;x,t)\,\mathrm{e}^{-\mathrm{i}\left(zx+z^2t\right)\sigma_3}, \tag{A.1}$$

where Γ is a solution of the $\overline{\partial}$-problem 4.32 so that we obtain the $\overline{\partial}$-problem

$$\begin{cases} \partial_{\bar z}\Psi(z) = \Psi(z)\,M_0(z) \\ \Psi(z) = \left(\mathbf{1} + \mathcal{O}\left(z^{-1}\right)\right)\mathrm{e}^{-\mathrm{i}\left(zx+z^2t\right)\sigma_3} \quad \text{as } z \to \infty\,. \end{cases} \tag{A.2}$$

We denote the terms of the expansion of Ψ near $z = \infty$ as follows:

$$\Psi(z;x,t) = \left(\mathbf{1} + \sum_{\ell=1}^{\infty} \frac{\Gamma_\ell(x,t)}{z^\ell}\right) e^{-i\left(zx+z^2t\right)\sigma_3}. \tag{A.3}$$

The first observation is that Ψ satisfies the Schwartz-like symmetry

$$\Psi(z;x,t)^\dagger\,\Psi(\bar z;x,t) \equiv \mathbf{1}, \tag{A.4}$$

which follows from the uniqueness of the solution after observing that the matrix $\Phi(z;x,t) := \Psi(z;x,t)^\dagger$ solves the same $\overline{\partial}$-problem, thanks to the property $M(z;x,t) = -M(\bar z;x,t)^\dagger$. Given that $\det\Psi \equiv 1$ we can rewrite the symmetry as

$$\Psi(z;x,t) = \sigma_2\,\Psi(\bar z;x,t)^\dagger\,\sigma_2. \tag{A.5}$$

This translates to the following symmetry for the matrices $\Gamma_\ell(x,t)$:

$$\Gamma_\ell(x,t) = \sigma_2\overline{\Gamma_\ell(x,t)}\sigma_2. \tag{A.6}$$

Since the operators ∂_x and $\partial_{\bar z}$ commute, we can see that $\partial_x\Psi$ satisfies the problem (A.2)

$$\partial_{\bar z}\partial_x\Psi = \partial_x\Psi\,M_0(z)\,. \tag{A.7}$$

It now follows that the matrix $U(z;x,t) := \partial_x\Psi(\Psi^{-1})$ is an entire function in z. Indeed

$$\begin{aligned} \partial_{\bar z}\left(\partial_x\Psi\Psi^{-1}\right) &= \left(\partial_{\bar z}\partial_x\Psi\right)\Psi^{-1} + \partial_x\Psi\left(\partial_{\bar z}\Psi^{-1}\right) \\ &= \left(\partial_x\Psi\right)M_0\Psi^{-1} - \left(\partial_x\Psi\right)\Psi^{-1}\partial_{\bar z}\Psi\left(\Psi^{-1}\right) \\ &= \left(\partial_x\Psi\right)M_0\Psi^{-1} - \left(\partial_x\Psi\right)M_0\Psi^{-1} = 0. \end{aligned}$$

Thus we obtain the following equation:

$$\partial_x\Psi(z;x,t) = U(z;x,t)\,\Psi(z;x,t)\,. \tag{A.8}$$

In order to determine the z–dependence of $U(z;x,t)$ we consider the asymptotic of $\Psi(z;x,t)$ for $z \to \infty$ by differentiation of the asymptotic behaviour specified in (A.2)

$$\partial_x\Psi \sim -\left(\mathbf{1} + \frac{\Gamma_1(x,t)}{z} + \mathcal{O}\left(z^{-2}\right)\right) iz\sigma_3 e^{-\mathrm{i}\left(zx+z^2t\right)\sigma_3} + \left(\frac{\partial_x\Gamma_1(x,t)}{z} + \mathcal{O}\left(z^{-2}\right)\right) e^{-\mathrm{i}\left(zx+z^2t\right)\sigma_3}.$$

Upon substitution in (A.8) we get

$$\begin{aligned} U(z;x,t) &= \partial_x \Psi \left(\Psi^{-1}\right) \\ &\sim -iz\left(\mathbb{I} + \frac{\Gamma_1(x,t)}{z}\right)\sigma_3\left(\mathbf{1} - \frac{\Gamma_1(x,t)}{z}\right) + \mathcal{O}\left(z^{-1}\right) \\ &= -iz\sigma_3 - i\left[\Gamma_1(x,t),\sigma_3\right] + \mathcal{O}\left(z^{-1}\right), \end{aligned} \tag{A.9}$$

and since we know that $U(z;x,t)$ is entire, we conclude that U is the polynomial in z of first degree obtained by dropping the $\mathcal{O}(z^{-1})$ in (A.9). Due to the symmetry (A.6) the matrix $\Gamma_1(x,t)$ has the form

$$\Gamma_1(x,t) = \begin{bmatrix} a(x,t) & b(x,t) \\ -\overline{b}(x,t) & \overline{a}(x,t) \end{bmatrix}, \tag{A.10}$$

from which we find

$$[\Gamma_1(x,t),\sigma_3] = \begin{bmatrix} 0 & -2b(x,t) \\ -2\overline{b}(x,t) & 0 \end{bmatrix}.$$

We thus conclude that the matrix $U(z;x,t)$ has the form

$$U(z;x,t) = \begin{bmatrix} -iz & 2ib(x,t) \\ 2i\overline{b}(x,t) & iz \end{bmatrix}. \tag{A.11}$$

The same arguments can be applied for the parameter t. In that case, $\Psi(z;x,t)$ satisfy the ODE

$$\partial_t \Psi(z;x,t) = V(z;x,t)\Psi(z;x,t) \tag{A.12}$$

where $V(z;x,t)$ is an entire function in z. Following the same idea as before, we expand $\partial_t\Psi(z;x,t)$ for $z\to\infty$

$$\partial_t\Psi \sim -iz^2\left(\mathbb{I} + \frac{\Gamma_1(x,t)}{z} + \frac{\Gamma_2(x,t)}{z^2} + \mathcal{O}\left(z^{-3}\right)\right)\sigma_3 e^{-\mathrm{i}\left(zx+z^2t\right)\sigma_3} + \mathcal{O}\left(z^{-1}\right)e^{-\mathrm{i}\left(zx+z^2t\right)\sigma_3} \tag{A.13}$$

and we have

$$\begin{aligned} \partial_t\Psi\left(\Psi^{-1}\right) = V(z;x,t) &\sim -\left(\mathbb{I} + \frac{\Gamma_1(x,t)}{z} + \frac{\Gamma_2(x,t)}{z^2} + \mathcal{O}\left(z^{-3}\right)\right) \\ &\quad \times iz^2\sigma_3\left(\mathbb{I} - \frac{\Gamma_1(x,t)}{z} - \frac{\Gamma_2(x,t)}{z^2} + \frac{(\Gamma_1(x,t))^2}{z^2} + \mathcal{O}\left(z^{-3}\right)\right) \\ &= -iz^2\sigma_3 - iz\left[\Gamma_1(x,t),\sigma_3\right] - i\left[\Gamma_2(x,t),\sigma_3\right] + i\left[\Gamma_1(x,t),\sigma_3\right]\Gamma_1(x,t) + \mathcal{O}\left(z^{-1}\right). \end{aligned} \tag{A.14}$$

We similarly conclude that $V(z;x,t)$ is the polynomial part of the above expression, a quadratic polynomial in z. To complete the calculation we need to relate the matrix $\Gamma_2(x,t)$ to the ∂_x derivative of Γ_1 by taking the expansion of both sides of the Lax equation (A.8) as $z\to\infty$, and using the explicit expression of U given in (A.9). The term $\mathcal{O}(z^{-1})$ in (A.8) provides the equation:

$$\partial_x\Gamma_1(x,t) = i\left[\Gamma_2(x,t),\sigma_3\right] - i\left[\Gamma_1(x,t),\sigma_3\right]\Gamma_1(x,t). \tag{A.15}$$

The (1, 1) entry of (A.15) yields the relation

$$\partial_x a(x,t) = -2i|b(x,t)|^2 \tag{A.16}$$

while the off diagonal give

$$([\Gamma_2(x,t),\sigma_3])_{12} = \left(\overline{[\Gamma_2(x,t),\sigma_3]}\right)_{21} = -2b\bar{a} - i\partial_x b. \tag{A.17}$$

In conclusion, the matrix $V(z;x,t)$ is

$$\begin{aligned} V(z;x,t) &= -iz^2\sigma_3 - iz[\Gamma_1(x,t),\sigma_3] - i([\Gamma_2(x,t),\sigma_3] - [\Gamma_1(x,t),\sigma_3]\Gamma_1(x,t)) \\ &= -iz^2\sigma_3 - iz[\Gamma_1(x,t),\sigma_3] - \partial_x\Gamma_1(x,t) \\ &= \begin{bmatrix} -iz^2 + 2i|b|^2 & 2zb - \partial_x b \\ 2z\bar{b} + \partial_x\bar{b} & iz^2 - 2i|b|^2 \end{bmatrix}. \end{aligned} \tag{A.18}$$

Summarizing, the matrix $\Psi(z;x,t)$ solves the $\bar{\partial}$-problem A.2 as well as the two linear PDEs

$$\begin{aligned} \partial_x\Psi &= U(z;x,t)\Psi = \begin{bmatrix} -iz & \psi \\ -\bar{\psi} & iz \end{bmatrix}\Psi \\ \partial_t\Psi &= V(z;x,t)\Psi = \begin{bmatrix} -iz^2 + \frac{i}{2}|\psi|^2 & z\psi + \frac{i}{2}\partial_x\psi \\ -z\bar{\psi} + \frac{i}{2}\partial_x\bar{\psi} & iz^2 - \frac{i}{2}|\psi|^2 \end{bmatrix}\Psi \end{aligned} \tag{A.19}$$

where we have set $\psi(x,t) := 2ib(x,t)$. We can see that the matrices $U(z;x,t)$ and $V(z;x,t)$ are in the form of the Lax pair of the NLS (1.17), namely, the zero curvature equations [51]

$$\partial_x\partial_t\Psi \equiv \partial_t\partial_x\Psi \quad \Leftrightarrow \quad \partial_t U - \partial_x V + [U,V] \equiv 0 \tag{A.20}$$

and the latter is equivalent to the NLS equation (1.17).

Appendix B. Proof of lemma 4.3

In this section we give the proof of lemma 4.3. Since the computations of $\omega(\boldsymbol{t} \pm [\zeta^{-1}])$ are the same, we give the proof only for $\omega(\boldsymbol{t} - [\zeta^{-1}])$.

From (3.21), (4.12) and (4.14), we get

$$\begin{aligned} &\omega\left(\boldsymbol{t} - \left[\zeta^{-1}\right]\right) \\ &= -\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}\left(z,\boldsymbol{t} - \left[\zeta^{-1}\right]\right)\partial_z\Gamma\left(z,\boldsymbol{t} - \left[\zeta^{-1}\right]\right)\delta_{[\zeta]}M\left(z,\boldsymbol{t} - \left[\zeta^{-1}\right]\right)\right)\frac{d\bar{z}\wedge dz}{2\pi i} \\ &= -\iint_{\mathscr{D}} Tr\left(D(z)\Gamma^{-1}(z)C^{-1}(z)\partial_z\left(C(z)\Gamma(z)D^{-1}(z)\right)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{d\bar{z}\wedge dz}{2\pi i} \\ &= -\iint_{\mathscr{D}} Tr\left(D(z)\Gamma^{-1}(z)\partial_z\Gamma(z)D^{-1}(z)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{d\bar{z}\wedge dz}{2\pi i} + \qquad \text{(B.1)} \\ &\quad -\iint_{\mathscr{D}} Tr\left(D(z)\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{d\bar{z}\wedge dz}{2\pi i} + \qquad \text{(B.2)} \\ &\quad -\iint_{\mathscr{D}} Tr\left(D(z)\partial_z D^{-1}(z)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{d\bar{z}\wedge dz}{2\pi i}. \qquad \text{(B.3)} \end{aligned}$$

We now consider the three parts (B.1)–(B.3), separately.

B.1. Computation of (B.1)

We find:

$$
\begin{aligned}
(B.1) = &-\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i}+\\
&-\iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\left[D^{-1}(z)\,\delta_\zeta D(z),M(z,\boldsymbol{t})\right]\right)\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i}\\
&=\omega(\boldsymbol{t})+\iint_{\mathscr{D}} Tr\left(D^{-1}(z)\,\delta_\zeta D(z)\left[\Gamma^{-1}(z)\,\partial_z\Gamma(z),M(z,\boldsymbol{t})\right]\right)\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i}.
\end{aligned}
$$

Since $\zeta\notin\mathscr{D}$, the matrix $D^{-1}(z)$ in (4.13) is analytic in $\mathscr{D}$ and using the $\overline{\partial}$-problem for Γ we can rewrite the two integrals as

$$
\begin{aligned}
(B.1) = \omega(\boldsymbol{t}) &+ \iint_{\mathscr{D}} \partial_{\bar z} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,D^{-1}(z)\,\delta_\zeta D(z)\right)\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i}+\\
&-\iint_{\mathscr{D}} Tr\left(\partial_z M(z,\boldsymbol{t})\,D^{-1}(z)\,\delta_\zeta D(z)\right)\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i}.
\end{aligned}
\tag{B.4}
$$

We now observe that the last integral is independent of $\boldsymbol{t}$, due to the fact that $D(z)$ is diagonal. Moreover, using

$$
D^{-1}(z)\,\delta_\zeta D(z) = -\frac{z}{\zeta(z-\zeta)}E_{11}\mathrm{d}\zeta,
$$

where $E_{11}=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, we find

$$
-\iint_{\mathscr{D}} Tr\left(\partial_z M(z,\boldsymbol{t})\,D^{-1}(z)\,\partial_\zeta D(z)\right)\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i} = \iint_{\mathscr{D}}\frac{z}{\zeta(z-\zeta)}\left(\partial_z M_0(z)\right)_{11}\frac{\mathrm{d}\bar z\wedge \mathrm{d}z}{2\pi i}.
\tag{B.5}
$$

The RHS of (B.5) equals $\partial_\zeta\gamma(\zeta)$. Now, the integrand of the remaining integral in (B.4) does not have a pole in $\mathscr{D}$ and we can use Stokes' Theorem

$$
\oint_{\partial\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\,D^{-1}(z)\,\partial_\zeta D(z)\right)\frac{\mathrm{d}z}{2\pi i} = \oint_{-\partial\mathscr{D}}\frac{z}{\zeta(z-\zeta)}\left(\Gamma^{-1}(z)\,\partial_z\Gamma(z)\right)_{11}\frac{\mathrm{d}z}{2\pi i}
$$

where $-\partial\mathscr{D}$ is the border of $\mathscr{D}$ oriented clockwise. Since $\Gamma(z)$ is analytic outside $\mathscr{D}$, we can apply Cauchy's residue Theorem and pick up the residues at $z=\zeta$ (there is no residue at $z=\infty$ because the integrand is $\mathcal{O}(z^{-2})$):

$$
\begin{aligned}
&\oint_{-\partial\mathscr{D}}\frac{z}{\zeta(z-\zeta)}\left(\Gamma_{22}(z)\,\partial_z\Gamma_{11}(z)-\partial_z\Gamma_{21}(z)\,\Gamma_{12}(z)\right)\frac{\mathrm{d}z}{2\pi i}\\
&\quad=\Gamma_{22}(\zeta)\,\partial_\zeta\Gamma_{11}(\zeta)-\partial_\zeta\Gamma_{21}(\zeta)\,\Gamma_{12}(\zeta)
\end{aligned}
$$

so that

$$
(B.1) = \omega(\boldsymbol{t}) + \left(\Gamma_{22}(\zeta)\,\partial_\zeta\Gamma_{11}(\zeta)-\partial_\zeta\Gamma_{21}(\zeta)\,\Gamma_{12}(\zeta)\right)\mathrm{d}\zeta+\delta_\zeta\gamma(\zeta).
\tag{B.6}
$$

B.2. Computation of (B.2)

Let us consider (B.2):

$$
\begin{aligned}
(B.2) = &- \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, \delta M(z,\boldsymbol{t})\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} + \\
&- \iint_{\mathscr{D}} Tr\left(M(z,\boldsymbol{t})\, \Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, D^{-1}(z)\, \delta_\zeta D(z)\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
&+ \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, M(z,\boldsymbol{t})\, D^{-1}(z)\, \delta_\zeta D(z)\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
= &- \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, \delta M(z,\boldsymbol{t})\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
&+ \iint_{\mathscr{D}} Tr\left(\partial_{\bar{z}}\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, D^{-1}(z)\, \delta_\zeta D(z)\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
&+ \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \overline{\partial}\Gamma(z)\, D^{-1}(z)\, \delta_\zeta D(z)\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i}.
\end{aligned} \tag{B.7}
$$

Since the only singularity is at $z=\zeta$, which is outside the domain $\mathscr{D}$, we can apply Stokes' Theorem to the integration and we get

$$
\begin{aligned}
(B.2) = &- \iint_{\mathscr{D}} Tr\left(C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, \delta M(z,\boldsymbol{t})\, \Gamma^{-1}(z)\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
&+ \oint_{\partial\mathscr{D}} Tr\left(\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, D^{-1}(z)\, \delta_\zeta D(z)\right) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{B.8}
$$

Now observe that

$$
\Gamma(z,\boldsymbol{t})\, \delta M(z,\boldsymbol{t})\, \Gamma^{-1}(z,\boldsymbol{t}) = \partial_{\bar{z}}\left[(\delta\Gamma(z,\boldsymbol{t}))\, \Gamma^{-1}(z,\boldsymbol{t})\right]. \tag{B.9}
$$

Using (B.9) in the first integral of (B.8), we can rewrite it as a contour integral

$$
\begin{aligned}
&- \iint_{\mathscr{D}} Tr\left(\Gamma^{-1}(z)\, C^{-1}(z)\, \partial_z C(z)\, \Gamma(z)\, \delta M(z,\boldsymbol{t})\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
&= - \iint_{\mathscr{D}} \partial_{\bar{z}} Tr\left(C^{-1}(z)\, \partial_z C(z)\, \delta\Gamma(z)\, \Gamma^{-1}(z)\right) \frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} \\
&= \oint_{-\partial\mathscr{D}} Tr\left(C^{-1}(z)\, \partial_z C(z)\, \delta\Gamma(z)\, \Gamma^{-1}(z)\right) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned}
$$

From the explicit expression of C in (4.16) we obtain

$$
C^{-1}(z) = \frac{1}{\det C(z)} adj(C(z)) = \frac{1}{\left(1-\frac{z}{\zeta}\right)} \begin{bmatrix} 1 & -\frac{\partial_z\Gamma_{12}(\infty)}{\zeta} \\ \frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)} & \left(1-\frac{z}{\zeta}\right) - \frac{\partial_z\Gamma_{12}(\infty)\Gamma_{12}(\zeta)}{\zeta\Gamma_{11}(\zeta)} \end{bmatrix}
$$

$$
\partial_z C(z) = -\frac{1}{\zeta} E_{11}
$$

and

$$Tr\left(C^{-1}(z)\partial_z C(z)\delta\Gamma(z)\Gamma^{-1}(z)\right) = \frac{1}{(z-\zeta)}\left((\delta\Gamma(z)\Gamma^{-1}(z))_{11} + \frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}(\delta\Gamma(z)\Gamma^{-1}(z))_{12}\right)$$
$$= \frac{\delta\Gamma_{11}(z)\Gamma_{22}(z) - \delta\Gamma_{12}(z)\Gamma_{21}(z)}{(z-\zeta)} + \frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}\left(\frac{\delta\Gamma_{12}(z)\Gamma_{11}(z) - \delta\Gamma_{11}(z)\Gamma_{12}(z)}{(z-\zeta)}\right).$$

We thus conclude that the first integral in (B.8) is given by

$$\oint_{-\partial\mathscr{D}} Tr\left(C^{-1}(z)\partial_z C(z)\delta\Gamma(z)\Gamma^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi i} = \delta\Gamma_{11}(\zeta)\Gamma_{22}(\zeta) - \frac{\delta\Gamma_{11}(\zeta)}{\Gamma_{11}(\zeta)}\Gamma_{12}(\zeta)\Gamma_{21}(\zeta)$$
$$= \frac{\delta\Gamma_{11}(\zeta)}{\Gamma_{11}(\zeta)} = \delta\ln\Gamma_{11}(\zeta). \tag{B.10}$$

To compute the second integral in (B.8) we expand the trace and obtain

$$Tr\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\partial_\zeta D(z)\right)$$
$$= -\frac{z}{\zeta(z-\zeta)^2}\Gamma_{11}(z)\left(\Gamma_{22}(z) - \frac{\Gamma_{12}(z)\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}\right).$$

So we are left with a contour integral with a double pole at $z=\zeta$ and a simple pole at $z=\infty$. Using the explicit expression (4.16) for the matrix C we obtain:

$$\oint_{\partial\mathscr{D}} Tr\left(\Gamma^{-1}(z)\,C^{-1}(z)\,\partial_z C(z)\,\Gamma(z)\,D^{-1}(z)\,\partial_\zeta D(z)\right)\frac{\mathrm{d}z}{2\pi i}$$
$$= \oint_{-\partial\mathscr{D}} \frac{z}{\zeta(z-\zeta)^2}\Gamma_{11}(z)\left(\Gamma_{22}(z) - \frac{\Gamma_{12}(z)\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}\right)\frac{\mathrm{d}z}{2\pi i}$$
$$= -\frac{1}{\zeta} + \frac{\det\Gamma(\zeta)}{\zeta} + \partial_\zeta(\Gamma_{11}(\zeta)\Gamma_{22}(\zeta)) - \frac{\partial_\zeta\left(\Gamma_{11}(\zeta)\Gamma_{12}(\zeta)\right)\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}$$
$$= \partial_\zeta\ln\Gamma_{11}(\zeta) + \Gamma_{11}(\zeta)\,\partial_\zeta\Gamma_{22}(\zeta) - \partial_\zeta\Gamma_{12}(\zeta)\,\Gamma_{21}(\zeta). \tag{B.11}$$

Combining (B.10) with (B.11) we have

$$(B.2) = \delta_{[\zeta]}\left(\ln\left(\Gamma_{11}(\zeta)\right)\right) + \left(\Gamma_{11}(\zeta)\,\partial_\zeta\Gamma_{22}(\zeta) - \Gamma_{21}(\zeta)\,\partial_\zeta\Gamma_{12}(\zeta)\right)\mathrm{d}\zeta. \tag{B.12}$$

B.3. Computation of (B.3)

This term turns out to vanish; indeed

$$(B.3) = \iint_{\mathscr{D}} Tr\left(D^{-1}(z)\partial_z D(z)\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i} +$$
$$- \iint_{\mathscr{D}} Tr\left(D^{-1}(z)\partial_z D(z)\left[M(z,\boldsymbol{t}), D^{-1}(z)\delta_\zeta D(z)\right]\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}$$
$$= -\iint_{\mathscr{D}} Tr\left(\delta\xi(z,t)D^{-1}(z)\partial_z D(z)\left[M(z,\boldsymbol{t}),\sigma_3\right]\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i} +$$
$$- \iint_{\mathscr{D}} Tr\left(D^{-1}(z)\partial_z D(z)\left[M(z,\boldsymbol{t}), D^{-1}(z)\delta_\zeta D(z)\right]\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\pi i}$$

and the integrand vanishes identically because of the cyclicity of the trace and the fact that D is a diagonal matrix. In conclusion, adding the equations (B.6) and (B.12), we obtain

$$\omega\left(\boldsymbol{t}-\left[\zeta^{-1}\right]\right)=\omega\left(\boldsymbol{t}\right)+\delta_{[\zeta]}\ln\left(\Gamma_{11}\left(\zeta\right)\right)+\delta_{[\zeta]}\gamma\left(\zeta\right). \tag{B.13}$$

Substituting $C(z)$ and $D(z)$ with $\tilde{C}(z)$ and $\tilde{D}(z)$ respectively and using the nonsingular condition for K (2.1), we find $\omega(\boldsymbol{t}+[\zeta^{-1}])$ with similar calculations and we get the following result

$$\omega\left(\boldsymbol{t}+\left[\zeta^{-1}\right]\right)=\omega\left(\boldsymbol{t}\right)+\delta_{[\zeta]}\ln\left(\Gamma_{11}^{-1}\left(\zeta\right)\right)-\delta_{[\zeta]}\gamma\left(\zeta\right). \tag{B.14}$$

and this proves the lemma 4.3. □

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