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Doubly Intermittent Maps with Critical Points and Singularities

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DEDICATION

To my son, Abubakar Muhammad Mubarak, my wife, Hadiza Idris and my loving parents, Hajiya Hauwa Muhammad and Malam Muhammad Salis.

Abstract

The first chapter is a joint work with Douglas Coates and Stefano Luzzatto [CLM22]. We study a class of one-dimensional full branch maps admitting two indifferent fixed points as well as critical points and/or unbounded derivative. Under some mild assumptions we prove the existence of a unique invariant mixing absolutely continuous probability measure, study its rate of decay of correlation and prove a number of limit theorems.

The second chapter is a joint work with Tanja I. Schindler (in preparation). We extend the definition of a subclass of the maps introduced in [CLM22] to a more robust class by replacing some of the parameters they studied with slowly varying functions. In particular, when $\beta = 1$ which is a boundary case in [CLM22] where acip measure cease to exist, introducing the slowly varying functions near ± 1 creates a spectrum of new parameters within $\beta = 1$ with a more subtle boundary base on the slowly varying functions. Under some mild assumptions we prove existence of a unique invariant absolutely continuous probability measure.

The third chapter is a joint work with Marks Ruziboev (in preparation). We study existence of equivariant measure and quenched mixing rates for a class of random interval maps with two indifferent fixed points and singular points. Using a random tower construction we prove the existence of an equivariant absolutely continuous probability measures with polynomial decay of correlation.

Introduction

Ergodic theory is an area of mathematics that uses geometric and analytic tools to study statistical behavior of dynamical systems (DS). Given a DS $f : M \to M$ (say measurable) on a phase space M and an observable $\phi : M \to \mathbb{R}$ (with some regularity), $(\Phi_n := \phi \circ f^n)_n$ is a sequence of real random variable (RV). In the case where these RVs are independent and identically distributed (iid), there are two pertinent statistical characterization: *Strong Law of Large Number (SLLN)*;

$$\frac{1}{n}\sum_{i=1}^{n-1} \phi \circ f^i \to \int \phi d\mu \tag{1}$$

for μ a.e x and *Central Limit Theorem (CLT)*; there exists a $\sigma^2 \geq 0$ and a $\mathcal{N}(0, \sigma^2)$ random variable V such that

$$\lim_{n\to\infty} \mu\left(\frac{\sum_{i=0}^{n-1} \phi \circ g^i}{\sqrt{n}} \leq x\right) = \mu(V \leq x), \tag{2}$$

for every $x \in \mathbb{R}$ for which the function $x \mapsto \mu(V_{\sigma^2} \leq x)$ is continuous. The former (1) asserts that the time average converges to the space average and it (convergence) is guaranteed by *Birkoff Ergodic Theorem* for μ that is $f-$invariant and ergodic whereas the latter (2) asserts that even though the individual RVs may have distributions that are far from normal distribution, the normalized sum of the sequence converges in distribution to a normal RV. We want the measure μ to be physically meaning full (to describe a large number of points), so we want it to be absolutely continuous w.r.t lebesgue measure. A third statistical characterization for dynamical systems (RVs) is the rate of decay of correlation function which measures the speed of convergence in (1); for Hölder continuous functions $\varphi, \psi : [-1, 1] \to \mathbb{R}$ and $n \geq 1$, we define the *correlation function*

$$\mathcal{C}_n(\varphi, \psi) := \left| \int \varphi\psi \circ g^n d\mu - \int \varphi d\mu \int \psi d\mu \right|. \tag{3}$$

In Chapter 1, we introduce a family of one-dimensional full branch maps with generalized parameter $\beta \geq 0$, which admits double indifferent fixed points and allows critical points and/or unbounded derivative. For $\beta \in [0, 1)$, we have the statistical properties (1), (2),(3) (and stable laws) for the family. To obtain these, a Young-Tower is constructed for the family of maps and based on the tail estimates (other conditions on the observable and parameters of the map) the results was obtained by applying general results in [You99; Gou04a]. For $\beta \geq 1$, we have an invariant, ergodic and absolutely continuous measure that is only sigma finite. It is not a probability measure because the tail of the tower is not summable.

In Chapter 2, we consider a subfamily of the maps introduced in Chapter 1 restricting to the transition parameter value $\beta = 1$ where invariant probability measure cease to exist. We replace some parameters with slowly varying functions near ± 1 creating a spectrum of new parameters within $\beta = 1$ with a more subtle boundary base on the slowly varying functions. we have the

statistical properties (1), (3) for the family by building Young-Tower for the family as in Chapter 1 and applying general results in [You99; Hol05].
In Chapter 3, we study an iid random dynamical systems. The fibre maps that constitutes the random dynamics are maps admiting two indifferent fixed points and a singularity known as Pikovsky maps and the base map $\sigma : [\alpha_1, \alpha_2]^{\mathbb{Z}} \to [\alpha_1, \alpha_2]^{\mathbb{Z}}$ is a 2-sided shift map where $[\alpha_1, \alpha_2]$ is an interval of parameter of the fibre maps. We obtain statistical properties for the system by building a Random Young-Tower for this family and a rate of sub-exponential quenched; i.e., for almost every realisation, rate is obtained using the machinery developed in [BBR19].

Possible Future Projects

The work done in this thesis introduces a new family of maps, and the result present in here a first study of some of the properties of this maps. There are many possible future projects which can be investigated. We mention here a few:

1. For $\beta \geq 1$ in Chapter 1, the invariant measure is infinite. In this case more general limit theorems such as Darling-Kac theorem for ergodic sums of integrable functions, the arcsine law for occupation times of sets of infinite measure, and the (Dynkin-Lamperti) arcsine law for waiting times, is expected similar to work in [TZ06].

2. For the family of maps considered in Chapter 2 with $\beta = 1$, under some conditions on the slowly varying function we have an acip measure but there is no result on limit theorem for such boundary case not even with one intermittent fixed point. This is an interesting future direction.

3. It should be possible to obtain versions of Theorems C and D in Chapter 1 in the settings of Chapter 2 with $\beta = 1$ by applying general results from [You99; Hol05; AM21].

4. It would also be interesting to study other statistical limit laws beyond mixing and CLT such as 'quantitative recurrence' which include study of hitting and return time recurrence.

5. A general question is how invariant ergodic measure change under perturbation of the map. Building on work in [BS16; Kor16] for the LSV maps, it expected that the measure vary continuously or even differentiably with respect to parameters.

6. An interesting question relating to ideas in Chapter 3 is statistical stability of invariant density for random non-uniformly expanding dynamical systems similar to the work in [AV02; Alv04].

7. Existence of equivariant measures and quenched mixing rates could be obtained in a more general settings than Pikovsky maps; in particular we could study random dynamics where the fibre maps are the system of maps introduced in Chapter 1 or in the system of intermittent cusp maps studies in [G+10].

8. The qualitative analysis of invariant measure given in [G+10] for the intermittent cusp map should give information on the qualititive behaviour of the systems introduced in Chapter 1 and 2.

Contents

Doubly Intermittent Maps with Critical Points and Singularities

1.1 Introduction and Statement of Results

The purpose of this work is to study the ergodic properties of a large class of full branch interval maps with two branches, including maps with *two* indifferent fixed points (which, as we shall see below, affects both the results and the construction of the induced map which we require). We also allow the derivative to go to *zero* as well as to *infinity* at the boundary between the two branches, and we do not assume any symmetry, even the domains of the branches can be of arbitrary length. Such maps are known to exhibit a wide range of behaviour from an ergodic point of view and many of them have been extensively studied, we give a detailed literature review below.

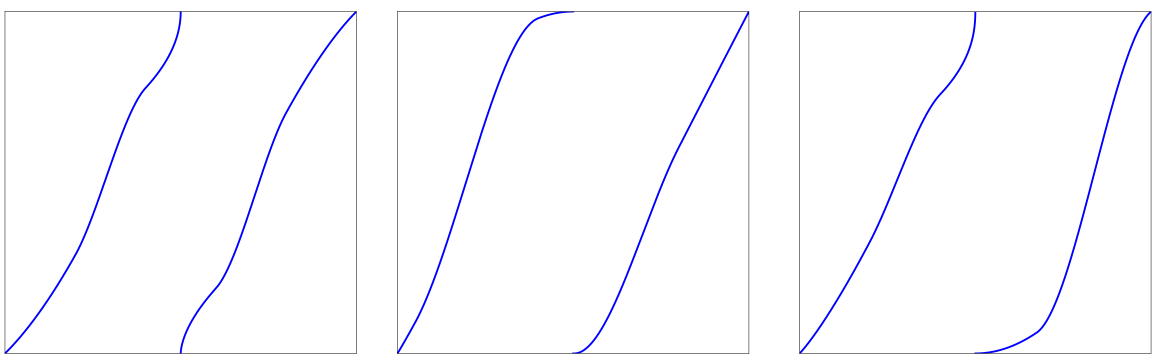


Figure 1.1: Graph of g for various possible values of parameters.

In Section 1.1.1 we give the precise definition of the class of maps we consider, which includes many cases already studied in the literature as well as many cases which have not yet been studied; in section 1.1.2 we give the precise statements of our results; in section 1.1.3 we give a literature review of related results and include specific examples of maps in our family; in Section 1.2 we give a detailed outline of our proof, emphasising several novel aspects of our construction and arguments. Then in Section 1.3 we give the construction and estimates related to our "double-induced" map and in Section 1.4 apply these estimates to complete the proofs of our results

1.1.1 Full Branch Maps

We start by defining the class of maps which we consider in this paper. Let I, I_-, I_+ be compact intervals, let $\mathring{I}, \mathring{I}_-, \mathring{I}_+$ denote their interiors, and suppose that $I = I_- \cup I_+$ and $\mathring{I}_- \cap \mathring{I}_+ = \emptyset$.

(A0) $g : I \to I$ is *full branch*: the restrictions $g_- : \mathring{I}_- \to \mathring{I}$ and $g_+ : \mathring{I}_+ \to \mathring{I}$ are orientation preserving C^2 diffeomorphisms and the only fixed points are the endpoints of I.

To simplify the notation we will assume that

$$I = [-1, 1], \quad I_- = [-1, 0], \quad I_+ = [0, 1]$$

but our results and proofs will be easily seen to hold in the general setting.

(A1) There exists constants $\ell_1, \ell_2 \geq 0$, $\iota, k_1, k_2, a_1, a_2, b_1, b_2 > 0$ such that:

(i) if $\ell_1, \ell_2 \neq 0$ and $k_1, k_2 \neq 1$, then

$$g(x) = \begin{cases} x + b_1(1 + x)^{1+\ell_1} & \text{in } U_{-1}, \\ 1 - a_1|x|^{k_1} & \text{in } U_{0-}, \\ -1 + a_2 x^{k_2} & \text{in } U_{0+}, \\ x - b_2(1 - x)^{1+\ell_2} & \text{in } U_{+1}, \end{cases} \tag{1.1}$$

where

$$U_{0-} := (-\iota, 0], \quad U_{0+} := [0, \iota), \quad U_{-1} := g(U_{0+}), \quad U_{+1} := g(U_{0-}). \tag{1.2}$$

(ii) If $\ell_1 = 0$ and/or $\ell_2 = 0$ we replace the corresponding lines in (1.1) with

$$g|_{U_{\pm 1}}(x) := \pm 1 + (1 + b_1)(x + 1) \mp \xi(x), \tag{1.3}$$

where ξ is C^2, $\xi(\pm 1) = 0, \xi'(\pm 1) = 0$, and $\xi''(x) > 0$ on U_{-1} and $\xi''(x) < 0$ on U_{+1}. If $k_1 = 1$ and/or $k_2 = 1$, then we replace the corresponding lines in (1.1) with the assumption that $g'(0_-) = a_1 > 1$ and/or $g'(0_+) = a_2 > 1$ respectively, and that g is monotone in the corresponding neighbourhood.

Remark 1.1.1. It is easy to see that the definition in (1.1) yields maps with dramatically different derivative behaviour depending on the values of ℓ_1, ℓ_2, k_1, k_2, including having neutral or expanding fixed points and points with zero or infinite derivative, see Remark 1.1.3 for a detailed discussion. For the moment we just remark that the assumptions described in part ii) of condition **(A1)** are consistent with (1.1) but significantly relax the definition given there as in these cases (1.1) would imply that the map is affine in the corresponding neighbourhood, whereas we only need expansivity. In particular this allows us to include uniformly expanding maps in our class of maps. In the calculations below we will explicitly consider the cases $\ell_1 = 0$ and/or $\ell_2 = 0$, which correspond to assuming that one or both the fixed points are expanding instead of neutral, since they yield different estimates (several quantities decay exponentially rather than polynomially in these cases) and different results, and still include some maps which, as far as we know, have not been studied in the literature. For simplicity, on the other hand, we will not consider explicitly the cases $k_1 = 1$ and/or $k_2 = 1$, which just correspond to assuming the derivative at one or both sides of the discontinuity is finite instead of being zero or infinite. These correspond to much simpler special cases and the required estimates follow by arguments which are very similar to arguments and calculation we give here, and which are essentially already considered in the literature, but treating them explicitly would require a significant amount of additional notation and calculations.

Our final assumption can be intuitively thought of as saying that g is uniformly expanding outside the neighbourhoods $U_{0\pm}$ and $U_{\pm 1}$. This is however much stronger than what is needed, and therefore we formulate a weaker and more general assumption for which we need to describe some aspects of the topological structure of maps satisfying condition **(A0)**. First of all we define

$$\Delta_0^- := g^{-1}(0,1) \cap I_- \quad \text{and} \quad \Delta_0^+ := g^{-1}(-1,0) \cap I_+. \tag{1.4}$$

Then we define iteratively, for every $n \geq 1$, the sets

$$\Delta_n^- := g^{-1}(\Delta_{n-1}^-) \cap I_- \quad \text{and} \quad \Delta_n^+ := g^{-1}(\Delta_{n-1}^+) \cap I_+ \tag{1.5}$$

as the n'th preimages of Δ_0^-, Δ_0^+ inside the intervals I_-, I_+. It follows from **(A0)** that $\{\Delta_n^-\}_{n\geq 0}$ and $\{\Delta_n^+\}_{n\geq 0}$ are mod 0 partitions of I_- and I_+ respectively, and that the partition elements depend *monotonically* on the index in the sense that $n > m$ implies that $\Delta_n^\pm$ is closer to ± 1 than $\Delta_m^\pm$, in particular the only accumulation points of these partitions are -1 and 1 respectively. Then, for every $n \geq 1$, we let

$$\delta_n^- := g^{-1}(\Delta_{n-1}^+) \cap \Delta_0^- \quad \text{and} \quad \delta_n^+ := g^{-1}(\Delta_{n-1}^-) \cap \Delta_0^+. \tag{1.6}$$

Notice that $\{\delta_n^-\}_{n\geq 1}$ and $\{\delta_n^+\}_{n\geq 1}$ are mod 0 partitions of Δ_0^- and Δ_0^+ respectively and also in these cases the partition elements depend monotonically on the index in the sense that $n > m$ implies that $\delta_n^\pm$ is closer to 0 than $\delta_m^\pm$, (and in particular the only accumulation point of these partitions is 0). Notice moreover, that

$$g^n(\delta_n^-) = \Delta_0^+ \quad \text{and} \quad g^n(\delta_n^+) = \Delta_0^-.$$

We now define two non-negative integers $n_\pm$ which depend on the positions of the partition elements $\delta_n^\pm$ and on the sizes of the neighbourhoods $U_{0\pm}$ on which the map g is explicitly defined. If $\Delta_0^- \subseteq U_{0-}$ and/or $\Delta_0^+ \subseteq U_{0+}$, we define $n_- = 0$ and/or $n_+ = 0$ respectively, otherwise we let

$$n_+ := \min\{n : \delta_n^+ \subset U_{0+}\} \quad \text{and} \quad n_- := \min\{n : \delta_n^- \subset U_{0-}\}. \tag{1.7}$$

We can now formulate our final assumption as follows.

(A2) There exists a $\lambda > 1$ such that for all $1 \leq n \leq n_\pm$ and for all $x \in \delta_n^\pm$ we have $(g^n)'(x) > \lambda$.

Notice that **(A2)** is an expansivity condition for points outside the neighbourhoods $U_{0\pm}$ and $U_{\pm 1}$ but is much weaker than assuming that the derivative of g is greater than 1 outside these neighbourhoods, which would be unnatural and unnecessarily restrictive in the presence of critical points. This completes the set of conditions which we require, and for convenience we let

$$\widehat{\mathfrak{F}} := \{g : I \to I \text{ which satisfy } \textbf{(A0)}\text{-}\textbf{(A2)}\}$$

The class $\widehat{\mathfrak{F}}$ contains many maps which have been studied in the literature, including uniformly expanding maps and various well known intermittency maps with a single neutral fixed point. We will give a more in-depth literature review in Section 1.1.3. Here we make a few technical remarks concerning these assumptions before proceeding to state our results in the next subsection.

Remark 1.1.2 (Remark on notation). To simplify many statements which will be made through the paper, it will be useful to recall some relatively standard notation as follows. Given sequences (s_n) and (t_n) of non-negative terms, we write $s_n = O(t_n)$, or $s_n \lesssim t_n$, if s_n/t_n is uniformly bounded above; $s_n \approx t_n$ if s_n/t_n is uniformly bounded away from 0 and ∞; $s_n = o(t_n)$ if $s_n/t_n \to 0$ as $n \to \infty$; and $s_n \sim t_n$ if $s_n/t_n = 1 + o(1)$, i.e. if s_n/t_n converges to 1 as $n \to \infty$.

Remark 1.1.3. Changing the parameter values ℓ_1, ℓ_2, k_1, k_2 gives rise to maps with quite different characteristics. For example, if $\ell_1 > 0$, we have

$$g'|_{U_{-1}}(x) = 1 + b_1(1+\ell_1)(1+x)^{\ell_1} \quad \text{and} \quad g''|_{U_{-1}}(x) = b_1(1+\ell_1)\ell_1(1+x)^{\ell_1-1}. \tag{1.8}$$

Then $g'(-1) = 1$ and the fixed point -1 is a *neutral* fixed point. Similarly, when $\ell_2 > 0$ the fixed point 1 is a neutral fixed point. On the other hand, when $\ell_1 = 0$, from (1.3) we have

$$g'|_{U_{-1}}(x) = 1 + b_1 + \xi'(x) \quad \text{and} \quad g''|_{U_{-1}}(x) = \xi''(x) \tag{1.9}$$

and thus the fixed point -1 is *hyperbolic repelling* with $g'(-1) = 1 + b_1$. When $k_1 \neq 1$ we have

$$g'|_{U_{0-}}(x) = a_1 k_1 |x|^{k_1-1} \quad \text{and} \quad g''|_{U_{0-}}(x) = a_1 k_1(k_1-1)|x|^{k_1-2}. \tag{1.10}$$

Then $k_1 \in (0,1)$ implies that $|g'|_{U_{0-}}(x)| \to \infty$ as $x \to 0$, in which case we say that $g|_{U_{0-}}$ has a (one-sided) *singularity* at 0, whereas $k_1 > 1$ implies that $|g'|_{U_{0-}}(x)| \to 0$ as $x \to 0$, and therefore we say that $g|_{U_{0-}}$ has a (one-sided) *critical point* at 0. Analogous observations hold for the various values of ℓ_2 and k_2 and Figure 1.1 shows the graph of g for various combinations of these exponents.

For future reference we mention also some additional properties which follow from **(A1)**. First of all notice that if $\ell_1 \in (0,1)$ we have $g''(x) \to \infty$ but if $\ell_1 > 1$ we have $g''(x) \to 0$, as $x \to -1$ and, as we shall see, this qualitative difference in the higher order derivative plays a crucial role in the ergodic properties of g. Analogous observations apply to $g|_{U_1}$ when $\ell_2 > 0$. Secondly, notice also that for every $x \in U_{-1}$ we have

$$g''(x)/g'(x) \lesssim (1+x)^{\ell_1-1} \tag{1.11}$$

and an analogous bound holds for $x \in U_1$. Similarly, in U_0 we have

$$|g''(x)|/|g'(x)| \lesssim x^{-1}, \tag{1.12}$$

and notice that in this case the bound does not actually depend on the value of k_1 or k_2 and in particular does not depend on whether we have a critical point or a singularity. Finally, we note that when $\ell_1 = 0$, it follows from (1.9) and from the assumption that $\xi''(x) > 0$ that

$$\xi'(x) > \xi(x)/(1+x) \tag{1.13}$$

for every $x \in U_{-1}$. Indeed, notice that $1 + x$ is just the distance between x and -1 and thus $\xi(x)/(1+x)$ is the slope of the straight line joining the point $(-1, 0)$ to $(x, \xi(x))$ in the graph of ξ, which is exactly the *average* derivative of ξ in the interval $[-1, x]$. Since $\xi'' > 0$, the derivative is monotone increasing and thus the derivative ξ' is maximal at the endpoint x, which implies (1.13). The same statement of course holds for $\ell_2 = 0$ and for all $x \in U_{+1}$.

1.1.2 Statement of Results

Our first result is completely general and applies to all maps in $\widehat{\mathfrak{F}}$.

Theorem A. Every $g \in \widehat{\mathfrak{F}}$ admits a unique (up to scaling by a constant) invariant measure which is absolutely continuous with respect to Lebesgue; this measure is σ-finite and equivalent to Lebesgue.

This is perhaps not completely unexpected but also certainly not obvious in the full generality of the maps in $\widehat{\mathfrak{F}}$, especially for maps which admit critical points (which can, moreover, be of arbitrarily high order). Our construction gives some additional information about the measure given in Theorem A, in particular the fact that its density with respect to Lebesgue is locally Lipschitz and unbounded only at the endpoints ± 1. We will show that, depending on the exponents k_1, k_2, ℓ_1, ℓ_2, the density may or may not be integrable and so the measure may or may not be finite. More specifically, let

$$\beta_1 := k_2\ell_1, \quad \beta_2 := k_1\ell_2, \quad \text{ and } \quad \beta := \max\{\beta_1, \beta_2\}.$$

We will show that the density is Lebesgue integrable at -1 or 1 respectively if and only if β_1 and β_2 respectively are < 1. In particular, letting

$$\mathfrak{F} := \{g \in \widehat{\mathfrak{F}} \text{ with } \beta < 1\}$$

we have the following result.

Theorem B. A map $g \in \widehat{\mathfrak{F}}$ admits a unique ergodic invariant *probability* measure μ_g absolutely continuous with respect to (indeed equivalent to) Lebesgue *if and only if* $g \in \mathfrak{F}$.

Notice that the condition $\beta < 1$ is a restriction only on the *relative* values of k_1 with respect to ℓ_2 and of k_2 with respect to ℓ_1. It still allows k_1 and/or k_2 to be *arbitrarily large*, thus allowing arbitrarily "degenerate" critical points, as long as the corresponding exponents ℓ_2 and/or ℓ_1 are sufficiently small, i.e. as long as the corresponding neutral fixed points are not too degenerate.

We now give several non-trivial results about the statistical properties maps $g \in \mathfrak{F}$ with respect to the probability measure μ_g. To state our first result recall that the *measure-theoretic entropy* of g with respect to the measure μ is defined as

$$h_\mu(g) := \sup_{\mathcal{P}} \left\{ \lim_{n\to\infty} \frac{1}{n} \sum_{\omega_n \in \mathcal{P}_n} -\mu(\omega_n) \ln \mu(\omega_n) \right\}$$

where the supremum is taken over all finite measurable partitions $\mathcal{P}$ of the underlying measure space and $\mathcal{P}_n := \mathcal{P} \vee f^{-1}\mathcal{P} \vee \cdots \vee f^{-n}\mathcal{P}$ is the dynamical refinement of $\mathcal{P}$ by f.

Theorem C. Let $g \in \mathfrak{F}$. Then μ_g satisfies the *Pesin entropy formula:* $h_{\mu_g}(g) = \int \log |g'| d\mu_g$.

For Hölder continuous functions $\varphi, \psi : [-1, 1] \to \mathbb{R}$ and $n \geq 1$, we define the *correlation function*

$$\mathcal{C}_n(\varphi, \psi) := \left| \int \varphi\psi \circ g^n d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

It is well known that μ_g is *mixing* if and only if $\mathcal{C}_n(\varphi, \psi) \to 0$ as $n \to \infty$. We say that μ_g is *exponentially mixing*, or satisfies *exponential decay of correlations* if there exists a $\lambda > 0$ such that for all Hölder continuous functions φ, ψ there exists a constant $C_{\varphi,\psi}$ such that $\mathcal{C}_n(\varphi, \psi) \leq C_{\varphi,\psi} e^{-\lambda n}$. We say that μ_g is *polynomially mixing*, or satisfies *polynomial decay of correlations*, with rate $\alpha > 0$ if for all Hölder continuous functions φ, ψ there exists a constant $C_{\varphi,\psi}$ such that $\mathcal{C}_n(\varphi, \psi) \leq C_{\varphi,\psi} n^{-\alpha}$.

Theorem D. Let $g \in \mathfrak{F}$. If $\beta = 0$ then μ_g is exponentially mixing, if $\beta \in (0, 1)$ then μ_g is polynomially mixing with rate $(1 - \beta)/\beta$.

Notice that the polynomial rate of decay of correlations $(1-\beta)/\beta$ itself decays to 0 as β approaches 1, which is the *transition parameter* at which the invariant measure ceases to be finite. Intuitively, as $\beta \to 1$, the measure, while still equivalent to Lebesgue, is increasingly concentrated in neighbourhoods of the neutral fixed points, which *slow down* the decay of correlations.

Our final result concerns a number of *limit theorems* for maps $g \in \mathfrak{F}$, which depend on the parameters of the map and, in some cases, also on some additional regularity conditions. These are arguably some of the most interesting results of the paper, and those in which the existence of two indifferent fixed points, instead of just one, really comes into play, giving rise to quite a complex scenario of possibilities. We start by recalling the relevant definitions. For integrable functions φ with $\int \varphi d\mu = 0$ we define the following limit theorems.

CLT φ satisfies a *central limit theorem* with respect to μ if there exists a $\sigma^2 \geq 0$ and a $\mathcal{N}(0,\sigma^2)$ random variable V such that

$$\lim_{n\to\infty} \mu\left(\frac{\sum_{k=0}^{n-1} \varphi \circ g^k}{\sqrt{n}} \leq x\right) = \mu(V \leq x),$$

for every $x \in \mathbb{R}$ for which the function $x \mapsto \mu(V_{\sigma^2} \leq x)$ is continuous.

CLT$_{\mathbf{ns}}$ φ satisfies a *non-standard central limit theorem* with respect to μ if there exists a $\sigma^2 \geq 0$ and a $\mathcal{N}(0,\sigma^2)$ random variable V such that

$$\lim_{n\to\infty} \mu\left(\frac{\sum_{k=0}^{n-1} \varphi \circ g^k}{\sqrt{n \log n}} \leq x\right) = \mu(V \leq x),$$

for every $x \in \mathbb{R}$ for which the function $x \mapsto \mu(V_{\sigma^2} \leq x)$ is continuous.

SL$_\alpha$ φ satisfies a *stable law* of index $\alpha \in (1,2)$, with respect to a measure μ, if there exists a stable random variable W_α such that

$$\lim_{n\to\infty} \mu\left(\frac{\sum_{k=0}^{n-1} \varphi \circ g^k}{n^{1/\alpha}} \leq x\right) = \mu(W_\alpha \leq x),$$

for every $x \in \mathbb{R}$ for which the function $x \mapsto \mu(W_\alpha \leq x)$ is continuous.

Finally, we say that an observable $\varphi : [-1,1] \to \mathbb{R}$ is a *co-boundary* if there exists a measurable function $\chi : [-1,1] \to \mathbb{R}$ such that $\varphi = \chi \circ g - \chi$. We are now ready to state our result on the various limit theorems which hold under some conditions on the parameters and on the observable φ. In order to state these conditions it is convenient to introduce the following variable:

$$\beta_\varphi := \begin{cases} 0 & \text{if } \varphi(-1) = 0 \text{ and } \varphi(1) = 0 \\ \beta_1 & \text{if } \varphi(-1) \neq 0 \text{ and } \varphi(1) = 0 \\ \beta_2 & \text{if } \varphi(-1) = 0 \text{ and } \varphi(1) \neq 0 \\ \beta & \text{if } \varphi(-1) \neq 0 \text{ and } \varphi(1) \neq 0, \end{cases} \tag{1.14}$$

We can then state our results in all cases in a clear and compact way as follows.

Theorem E. Let $g \in \mathfrak{F}$ and $\varphi : [-1,1] \to \mathbb{R}$ be Hölder continuous with $\int \varphi d\mu = 0$ and satisfying

($\mathcal{H}$) $\nu_1 > (\beta_1 - 1/2)/k_2$ and $\nu_2 > (\beta_2 - 1/2)/k_1$, where ν_1, ν_2 are the Hölder exponents of $\varphi|_{[-1,0]}$ and $\varphi|_{(0,1]}$ respectively. Then

1. if $\beta_\varphi \in [0, 1/2)$ then φ satisfies CLT,
2. if $\beta_\varphi = 1/2$ then φ satisfies CLT_{ns},
3. if $\beta_\varphi \in (1/2, 1)$ then φ satisfies $\text{SL}_{1/\beta_\varphi}$.

In case 3 we can replace the Hölder continuity condition $(\mathcal{H})$ by the weaker (in this case) condition

$(\mathcal{H}')$ $\nu_1 > (\beta_1 - \beta_\varphi)/k_2$ and $\nu_2 > (\beta_2 - \beta_\varphi)/k_1$.

Moreover, in all cases where CLT holds we have that $\sigma^2 = 0$ if and only if φ is a coboundary.

Remark 1.1.4. Our results highlight the fundamental significance of the value of the observable φ at the two fixed points, and how *the fixed point at which φ is non-zero*, in some sense *dominates*, and determines the kind of limit law which the observable satisfies. If φ is non-zero at both fixed points, then it is the larger exponent which dominates.

Remark 1.1.5. Note that $(\mathcal{H})$ and $(\mathcal{H}')$ are automatically satisfied for various ranges of β_1, β_2, for example if $\beta \leq 1/2$ then $(\mathcal{H})$ always holds and if $\beta = \beta_\varphi$ then $(\mathcal{H}')$ always holds. These Hölder continuity conditions arise as technical conditions in the proof and it is not clear to us if they are really necessary and what could be proved without them. It may be the case, for example, that some limit theorems still hold under weaker regularity conditions on φ.

Remark 1.1.6. We remark also that the compact statement of Theorem E somewhat "conceals" quite a large number of cases which express an intricate relationship between the map parameters and the values and regularity of the observable. For example, the case $\beta_\varphi = 0$ allows all possible values $\beta_1, \beta_2 \in [0, 1)$ and the case $\beta_\varphi = \beta_1$ allows all possible values of $\beta_2 \in [0, 1)$. We therefore have a huge number of possible combinations which do not occur in the case of maps with just a single intermittent fixed point.

1.1.3 Examples and Literature Review

There is an extensive literature on the dynamics and statistical properties of full branch maps, which have been studied systematically since the 1950s. Their importance stems partly from the fact that they occur very naturally, for example any smooth non-invertible local diffeomorphism of $\mathbb{S}^1$ is a full branch map, but also, and perhaps most importantly, because many arguments in Smooth Ergodic Theory apply in this setting in a particularly clear and conceptually straightforward way. Indeed, arguably, most existing techniques used to study hyperbolic (including non-uniformly hyperbolic) dynamical systems are essentially (albeit often highly non-trivial) extensions and generalisations of methods first introduced and developed in the setting of one-dimensional full branch maps.

Our class of maps $\widehat{\mathfrak{F}}$ is quite general and includes many one-dimensional full branch maps which have been studied in the literature as well as many maps which have *not* been previously studied. We give below a brief survey of some of these examples and indicate for which choices of parameters these correspond to maps in our family[1].

Arguably one of the very first and simplest general class of maps for which the existence of an invariant ergodic and absolutely continuous probability measure was proved are *uniformly expanding full branch* maps with derivatives uniformly bounded away from 0 and infinity, a result often referred

[1]Recall that we have fixed the domains of the branches of our maps as $[-1, 0)$ and $(0, 1]$ for convenience. In the examples below, when listing parameters, we slightly abuse notation and assume an affine change of coordinates which transforms the given domains into the ones used in our class.

to as the *Folklore Theorem* and generally attributed to Renyi. Some particularly simple examples of uniformly expanding maps are piecewise affine maps such as those given by

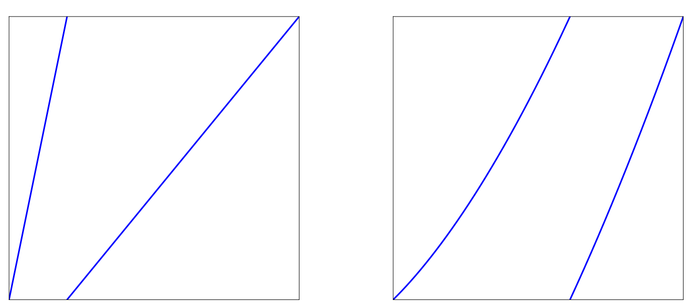
$$g(x) = \begin{cases} ax & \text{for } x \in [0, 1/a] \\ \frac{a}{a-1}\left(x - \frac{1}{a}\right) & \text{for } x \in (1/a, 1] \end{cases} \tag{1.15}$$

for parameters $a > 1$, see Figure 1.2a. These are easily seen to be contained in the class $\widehat{\mathfrak{F}}$ with parameters $(\ell_1, \ell_2, k_1, k_2, a_1, a_2, b_1, b_2) = (0, 0, 1, 1, a, a/(a-1), a-1, a/(a-1)-1)$.

In the late '70s, physicists Maneville and Pomeau [PM80] introduced a simple but extremely interesting generalisation consisting of a class of full branch one-dimensional maps $g : [0, 1] \to [0, 1]$, which they called *intermittency maps*, defined by

$$g(x) = x(1 + x^\alpha) \mod 1 \tag{1.16}$$

for $\alpha > 0$, see Figure 1.2b (notice that for $\alpha = 0$ this just gives the map $g(x) = 2x \bmod 1$, which is just (1.15) with $a = 2$). These maps can be seen to be contained in our class $\widehat{\mathfrak{F}}$ by taking the parameters $(\ell_1, \ell_2, k_1, k_2, a_1, a_2, b_1, b_2) = (\alpha, 0, 1, 1, a, a, 1, 1)$, where $a = g'(x_0)$, and $x_0 \in (0, 1)$ is the boundary of the intervals on which the two branches of the map are defined. The Maneville-Pomeau maps are interesting because the uniform expansivity condition fails at a single fixed point on the boundary of the interval, where we have $g'(0) = 1$ Their motivation was to model fluid flow where long period of stable flow is followed with an intermittent phase of turbulence, and they showed that this simple model indeed seemed to exhibit such dynamical behaviour. It was then shown in [Pia80] that for $\alpha > 2$, the intermittency maps *failed* to have an invariant ergodic and absolutely continuous probability measure and satisfies the extremely remarkable property that *the time averages of Lebesgue almost every point converge to the Dirac-delta measure δ_0 at the neutral fixed point*, even though these orbits are dense in $[0, 1]$ and the fixed point is topologically repelling.



(a) Graph of (1.15) ($a = 5$)   (b) Graph of (1.16) ($\alpha = 9/10$)

Figure 1.2: Graphs of a piecewise affine, Manneville-Pomeau, and Liverani-Saussol-Vaienti maps.

Various variations of intermittency maps have been studied extensively from various points of views and with different techniques yielding quite deep results, see e.g. [LSV99; You99; Sar01; Mel09; PS09; FL01; Gou04a; Gou04b; NTV18; CHT21; Fre+16; Kor16; BS16; Ter16; SS13; Zwe03]. One well known version is the so-called Liverani-Saussol-Vaienti (LSV) map $g : [0, 1] \to [0, 1]$ introduced in [LSV99] and defined by

$$g(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{for } x \in [0, 1/2] \\ 2x - 1 & \text{for } x \in (1/2, 1) \end{cases} \tag{1.17}$$

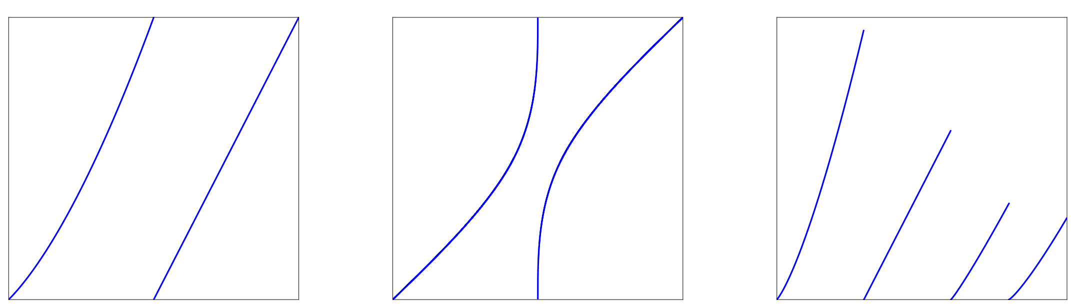
with parameter $\alpha > 0$, see Figure 1.3a. This maintains the essential features of the Maneville-Pomeau maps (1.16), i.e. it is uniformly expanding except at the neutral fixed point at the origin, but in slightly simplified form where the two branches are always defined on the fixed domains $[0, 1/2]$ and $1/2, 1)$ and the second branch is affine, both of which make the map family easier to study, including the effect of varying the parameter. The family of LSV maps (1.17) can be seen to be contained in our class $\widehat{\mathfrak{F}}$ by taking the parameters $(\ell_1, \ell_2, k_1, k_2, a_1, a_2, b_1, b_2) = (\alpha, 0, 1, 1, 2, 2, 2^\alpha, 1)$.

In an earlier paper [Pik91a], Pikovsky had introduced the maps $g : \mathbb{S}^1 \to \mathbb{S}^1$, defined (in a somewhat unwieldy way) by the implicit equation

$$x = \begin{cases} \frac{1}{2\alpha}(1 + g(x))^\alpha & \text{for } x \in [0, 1/2\alpha] \\ g(x) + \frac{1}{2\alpha}(1 - g(x))^\alpha & \text{for } x \in (1/2\alpha, 1) \end{cases} \tag{1.18}$$

for $x \in [0, 1)$, and then by the symmetry $g(x) = g(-x)$ for $x \in (-1, 0]$, see Figure 1.3b. These maps have a neutral fixed point at the left end point, like in (1.16) and (1.17) but with the added complication of having unbounded derivative at the boundary between the domains of the two branches. On the other hand the definition is specifically designed in such a way that the order of intermittency is the inverse of the order of the singularity and, together with the symmetry of the two branches, this implies that Lebesgue measure is invariant for all values of the parameter $\alpha > 0$. Ergodic and statistical properties of these maps were studied in [AA04; Cri+10; BM14] and they can be seen to be contained within our class $\widehat{\mathfrak{F}}$ by taking the parameters $(\ell_1, \ell_2, k_1, k_2, a_1, a_2, b_1, b_2) = (\alpha - 1, \alpha - 1, 1/\alpha, 1/\alpha, (2\alpha)^{1/\alpha}, (2\alpha)^{1/\alpha}, 1/2\alpha, 1/2\alpha)$.

Finally, [Ino92; Cui21] consider a class of maps, see Figure 1.3c for an example, with a single intermittent fixed point and multiple critical points with each critical point mapping to the fixed point. These include some maps which are more general than those we consider here as they are defined near the fixed and critical points through some bounds rather than explicitly as we do here, but are also more restrictive as they only allow for a single neutral fixed point. Under a condition on the product of the orders of the neutral and (the most degenerate) critical point which is exactly analogous to our condition $\beta < 1$, the existence of an invariant ergodic probability measure is proved which exhibits decay of correlations but no bounds are given for the rate of decay and no limit theorems are obtained.



(a) Graph of (1.17) ($\alpha = 9/10$) (b) Graph of (1.18) ($\alpha = 3$) (c) An example from [Cui21]

Figure 1.3: Graphs of previsously studied generalisations of the Manneville-Pomeau map.

1.2 Overview of the proof

We discuss here our overall strategy and prove our Theorems modulo some key technical Propositions which we then prove in the rest of the paper. Our argument can be naturally divided into three main steps which we describe in some detail in the following three subsections.

1.2.1 The induced map

The first step of our arguments is the construction of an *induced* full branch Gibbs-Markov map, also known as a *Young Tower*. This is relatively standard for many systems, including intermittent maps, however, the inducing domain which we are obliged to use here due to the presence of two indifferent fixed points is *different from the usual inducing domains* and requires a more sophisticated *double inducing* procedure, which we outline here and describe and carry out in detail in Section 1.3. Recall the definition of Δ_0^- in (1.4) and, for $x \in \Delta_0^-$, let

$$\tau(x) := \min\{n \geq 1 : g^n(x) \in \Delta_0^-\}$$

be the first return time to Δ_0^-. Then we define the *first-return induced map*

$$G : \Delta_0^- \to \Delta_0^- \quad \text{by} \quad G(x) := g^{\tau(x)}(x). \tag{1.19}$$

We say that a first return map (or, more generally, any induced map), *saturates* the interval I if

$$\bigcup_{n\geq 0} \bigcup_{i=0}^{n-1} g^i(\{\tau = n\}) = \bigcup_{n\geq 0} g^n(\{\tau > n\}) = I \pmod 0. \tag{1.20}$$

Intuitively, saturation means that the return map "reaches" every part of the original domain of the map g, and thus the properties and characteristics of the return map reflect, to some extent, all the relevant characteristics of g.

Remark 1.2.1. If G is a first return induced map, as in our case, then all sets of the form $g^i(\{\tau = n\})$ are pairwise disjoint and therefore form a partition of I mod 0.

The first main result of the paper is the following.

Proposition 1.2.2. *Let $g \in \widehat{\mathfrak{F}}$. Then $G : \Delta_0^- \to \Delta_0^-$ is first return induced Gibbs-Markov map which saturates I.*

We give the precise definition of Gibbs-Markov map, and prove Proposition 1.2.2, in Section 1.3. In Section 1.3.1 we describe the topological structure of G and show that it a full branch map with countably many branches which saturates I (we will define G as a composition of two full branch maps, see (1.37) and (1.40), which is why we call the construction a double inducing procedure); in Section 1.3.2 we obtain key estimates concerning the sizes of the partition elements of the corresponding partition; in Section 1.3.3 we show that G is uniformly expanding; in Section 1.3.4 we show that G has bounded distortion. From these results we get Proposition 1.2.2 from which we can then obtain our first main Theorem.

Proof of Theorem A. By standard results G admits a unique ergodic invariant probability measure $\hat{\mu}_-$, supported on Δ_0^-, which is equivalent to Lebesgue measure m and which has Lipschitz continuous density $\hat{h}_- = d\hat{\mu}_-/dm$ bounded above and below. We then "spread" the measure over the original interval I by defining the measure

$$\tilde{\mu} := \sum_{n=0}^{\infty} g_*^n(\hat{\mu}_-|\{\tau \geq n\}) \tag{1.21}$$

where $g^n_*(\hat\mu_-|\{\tau \ge n\})(E) := \hat\mu_-(g^{-n}(E) \cap \{\tau \ge n\})$. Again by standard arguments, we have that $\tilde\mu$ is a sigma-finite measure which is ergodic and invariant for g and, using the non-singularity of g, it is absolutely continuous with respect to Lebesgue. The fact that G saturates I implies moreover that $\tilde\mu$ is equivalent to Lebesgue, which completes the proof. □

Remark 1.2.3. We emphasize that we are not assuming *any* symmetry in the two branches of the map g. It is not important that the branches are defined on intervals of the same length and, depending on the choice of constants, we might even have a critical point in one branch and a singularity with unbounded derivative on the other. Interestingly, however, there is some symmetry in the construction in the sense that for $x \in \Delta_0^+$, we can define the first return map $G_+ : \Delta_0^+ \to \Delta_0^+$ in a completely analogous way to the definition of G above (see discussion in Section 1.3.1). Moreover, the conclusions of Proposition 1.2.2 hold for G_+ and thus G_+ admits a unique ergodic invariant probability measure $\hat\mu_+$ which is equivalent to Lebesgue measure m and such that the density $\hat h_+ := d\hat\mu_+/dm$ is Lipschitz continuous and bounded above and below. The two maps G and G_+ are clearly distinct, as are the measures $\hat\mu_-$ and $\hat\mu_+$, but exhibit a subtle kind of symmetry in the sense that the corresponding measure $\tilde\mu$ obtained by substituting $\hat\mu_-$ by $\hat\mu_+$ in (1.21) is, up to a constant scaling factor, exactly the same measure.

Corollary 1.2.4. *The density $\tilde h$ of $\tilde\mu|_{\Delta_0^-\cup\Delta_0^+}$ is Lipschitz continuous and bounded and $\tilde\mu|_{\Delta_0^-} = \hat\mu$.*

Proof. Since G is a first return induced map it follows that the measure $\tilde\mu$ defined in (1.21) satisfies $\tilde\mu|_{\Delta_0^-} = \hat\mu$ and so the density $\tilde h$ of $\tilde\mu$ is Lipchitz continuous and bounded away from both 0 and infinity on Δ_0^-. Moreover, as mentioned in Remark 1.2.3, $\tilde\mu|_{\Delta_0^+}$ is equal, up to a constant, to the measure $\hat\mu_+$ and so the density of $\tilde\mu|_{\Delta_0^+}$ is also Lipschitz continuous and bounded away from 0 and infinity. □

Remark 1.2.5. We have used above the notation G rather than G_- for simplicity as this is the map which plays a more central role in our construction, see Remark 1.3.3 below. Similarly, we will from now on simply use the notation $\hat\mu$ to denote the measure $\hat\mu_-$.

1.2.2 Orbit distribution estimates

The second step of the argument is aimed at establishing conditions under which the measure $\tilde\mu$ is finite, and can therefore be renormalized to a probability measure $\mu := \tilde\mu/\tilde\mu(I)$, and aimed at studying the ergodic and statistical properties of μ. Our approach here differs even more significantly from existing approaches in the literature, although it does have some similarities with the argument of [Cri+10]: rather than starting with estimates of the tail of the inducing time (which would themselves anyway be significantly more involved than in the usual examples of intermittency maps with a single critical point due to our double inducing procedure), we carry out *more general estimates* on the *distribution* of iterates of points in I_- and I_+ before they return to Δ_0^-. More precisely, we define the functions $\tau^\pm(x) : \Delta_0^- \to \mathbb{N}$ by

$$\tau^+(x) := \#\{1 \le i \le \tau : g^i(x) \in I_+\}, \quad \text{and} \quad \tau^-(x) := \#\{1 \le j \le \tau : g^j(x) \in I_-\}. \tag{1.22}$$

These functions *count the number of iterates of x in I_- and I_+ respectively before returning to Δ_0^-.* Then for any $a, b \in \mathbb{R}$ we define *weighted combination* $\tau_{a,b} : \Delta_-^0 \to \mathbb{R}$ by

$$\tau_{a,b}(x) = a\tau^+(x) + b\tau^-(x) \tag{1.23}$$

As we shall see as part of our construction of the induced map, both of these functions are *unbounded* and their level sets have a non-trivial structure in Δ^0_- and, moreover, the *inducing time* function $\tau : \Delta_0^- \to \mathbb{N}$ of the induced map G_- corresponds exactly to $\tau_{1,1}$ so that

$$\tau(x) = \tau_{1,1}(x) = \tau^+(x) + \tau^-(x). \tag{1.24}$$

The key results of this part of the proof consists of explicit and sharp asymptotic bounds for the distribution of $\tau_{a,b}$ for different values of a, b, from which we can then obtain as an immediate corollary the rates of decay of the inducing time function τ, and which will also provide the core estimates for the various distributional limit theorems. To state our results, let

$$B_1 := a_1^{-1/k_1}(\ell_2 b_2)^{-1/\beta_2} \quad \text{and} \quad B_2 := a_2^{-1/k_2}(\ell_1 b_1)^{-1/\beta_1}, \tag{1.25}$$

(the expressions defining the constants B_1, B_2 will appear in the proof of Proposition 1.3.5 below).

Recall from Corollary 1.2.4 that the density $\tilde{h}$ of $\tilde{\mu}$ is bounded on $\Delta_0^- \cup \Delta_0^+$ and let $\tilde{h}(0^-)$ and $\tilde{h}(0^+)$ denote the values of this density on either side of 0 . Then, for any $a, b \geq 0$, we let

$$C_a := \tilde{h}(0^-)B_1 a^{1/\beta_2}, \quad \text{and} \quad C_b := \tilde{h}(0^+)B_2 b^{1/\beta_1}. \tag{1.26}$$

Then we have the following distributional estimates.

Proposition 1.2.6. *Let $g \in \widehat{\mathfrak{F}}$. Then for every $a, b \geq 0$ we have the following distribution estimates. For every $\gamma \in [0, 1)$*

$$\tilde{\mu}(a\tau^+ + b\tau^- > t) = \begin{cases} C_b t^{-1/\beta_1} + C_a t^{-1/\beta_2} + o(t^{-\gamma-1/\beta}) & \text{if } \ell_1, \ell_2 > 0 \\ C_b t^{-1/\beta_1} + o(t^{-\gamma-1/\beta_1}) & \text{if } \ell_1 > 0, \ell_2 = 0 \\ C_a t^{-1/\beta_2} + o(t^{-\gamma-1/\beta_2}) & \text{if } \ell_1 = 0, \ell_2 > 0 \\ O\left((1+b_1)^{-t/k_2} + (1+b_2)^{-t/k_1}\right) & \text{if } \ell_1 = 0, \ell_2 = 0 \end{cases} \tag{1.27}$$

$$\tilde{\mu}(a\tau^+ - b\tau^- > t) = \begin{cases} C_a t^{-1/\beta_2} + o(t^{-\gamma-1/\beta}) & \text{if } \ell_2 > 0 \\ O\left((1+b_2)^{-t/k_1}\right), & \text{if } \ell_2 = 0 \end{cases} \tag{1.28}$$

$$\tilde{\mu}(a\tau^+ - b\tau^- < -t) = \begin{cases} C_b t^{-1/\beta_1} + o(t^{-\gamma-1/\beta}) & \text{if } \ell_1 > 0 \\ O\left((1+b_1)^{-t/k_2}\right), & \text{if } \ell_1 = 0. \end{cases} \tag{1.29}$$

Remark 1.2.7. We have assumed in Proposition 1.2.6 that $a, b \geq 0$ to avoid stating explicitly too many cases, but one can easily read off the tails for $\tau_{a,b}$ for arbitrary $a, b \in \mathbb{R}$. For example, if $a < 0$ and $b > 0$ we can write $\tilde{\mu}(-a\tau^+ + b\tau^- > t) = \tilde{\mu}(a\tau^+ - b\tau^- < -t)$ and. get the corresponding estimate from (1.29). Notice moreover, that the estimates for $\ell_1 = 0$ and/or $\ell_2 = 0$ are exponential.

Recall from Corollary 1.2.4 that $\hat{\mu} = \tilde{\mu}$ on the inducing domain Δ_{0^-} and therefore all the above estimates hold for $\hat{\mu}$ with exactly the same constants. In particular by Proposition 1.2.6 and (1.24), we immediately get the corresponding estimates for the tail $\hat{\mu}(\tau > t) = \tilde{\mu}(\tau > t)$.

Corollary 1.2.8. *If $\beta = 0$ then $\hat{\mu}(\tau > t)$ decay exponentially as $t \to +\infty$. If $\beta > 0$ then there exists a positive constant C_τ (which can be computed explicitly) such that*

$$\hat{\mu}(\tau > t) \sim C_\tau t^{-1/\beta}.$$

Proposition 1.2.6 will be proved in Section 1.4.1, here we show how it implies Theorems B, C, D.

Proof of Theorems B, C, and D. From the definition of $\tilde{\mu}$ in (1.21) and since $g^{-n}(I) = I$ we have

$$\tilde{\mu}(I) := \sum_{n=0}^{\infty} \hat{\mu}_-(g^{-n}(I) \cap \{\tau > n\}) = \sum_{n=0}^{\infty} \hat{\mu}_-(I \cap \{\tau > n\}) = \sum_{n=0}^{\infty} \hat{\mu}_-(\tau > n).$$

By Corollary 1.2.8, if $\beta = 0$, the quantities $\hat{\mu}_-(\tau > n)$ decay exponentially and, if $\beta > 0$ we have

$$\tilde{\mu}(I) = C \sum_{n=1}^{\infty} n^{-\frac{1}{\beta}}(1 + o(1)),$$

for some $C > 0$. This implies that $\tilde{\mu}(I) < \infty$ if and only if $\beta \in [0, 1)$, i.e. if and only if $g \in \mathfrak{F}$. Thus, for $g \in \mathfrak{F}$ we can define the measure $\mu_g := \tilde{\mu}/\tilde{\mu}(I)$, which is an invariant ergodic probability measure for g, and is unique because it is equivalent to Lebesgue, thus proving Theorem B. Theorem C follows from Theorem A in [AM21] by noticing that $\mathcal{P} = \{(-1, 0), (0, 1)\}$ is a Lebesgue mod 0 generating partition such that $H_{\mu_g}(\mathcal{P}) < \infty$ and $h_{\mu_g}(g, \mathcal{P}) < \infty$, and therefore $h_{\mu_g}(g) < \infty$. Finally, Theorem D follows by well known results [You99] which show that the decay rate of the tail of the inducing times provides upper bounds for the rates of decay of correlations as stated. □

1.2.3 Distribution of induced observables

The last part of our argument is focused on obtaining the limit theorems stated in Theorem E. When $\beta = 0$ the decay of correlations is exponential and the result follows from [You99]. Similarly, after having established Proposition 1.2.2 and Corollary 1.2.8, the case that only one of ℓ_1, ℓ_2 is positive implies that there is only one intermittent fixed point, and thus essentially reduces to the argument given in [Gou04a, Theorem 1.3] for the LSV map. We only therefore need to consider the case that both $\ell_1, \ell_2 > 0$, which implies in particular that $\beta \in (0, 1)$.

Given an observable $\varphi : [0, 1] \to \mathbb{R}$, we define the *induced observable* $\Phi : \Delta_0^- \to \mathbb{R}$ by

$$\Phi(x) := \sum_{k=0}^{\tau(x)-1} \varphi \circ g^k.$$

Definition 1.2.9. We write $\Phi \in \mathcal{D}_\alpha$ if $\exists c_1, c_2 \geq 0$, with *at least one* of c_1, c_2 non-zero, such that

$$\hat{\mu}(\Phi > t) = c_1 t^{-\alpha} + o(t^{-\alpha}) \quad \text{and} \quad \hat{\mu}(\Phi < -t) = c_2 t^{-\alpha} + o(t^{-\alpha}), \tag{1.30}$$

In certain settings, limit theorems can be deduced from properties of the induced observable Φ. In particular, it is proved in Theorems 1.1 and 1.2 of [Gou04a] that, precisely in our setting [2]:

$$\text{if } \Phi \in L^2(\hat{\mu}) \text{ then } \varphi \text{ satisfies } CLT, \tag{1.31}$$

$$\text{if } \Phi \in \mathcal{D}_2 \text{ then } \varphi \text{ satisfies } CLT_{\text{ns}}, \tag{1.32}$$

$$\text{if } \Phi \in \mathcal{D}_\alpha \text{ with } \alpha \in (1, 2) \text{ then } \varphi \text{ satisfies } SL_\alpha. \tag{1.33}$$

We will argue that in each case of Theorem E, the induced observable Φ satisfies one of the above. To prove this, we first decompose a general observable $\varphi : [-1, 1] \to \mathbb{R}$ by letting $a := \varphi(1)$ and $b := \varphi(-1)$ and writing

$$\varphi = \varphi_{a,b} + \tilde{\varphi} \quad \text{where} \quad \varphi_{a,b} := b\chi_{[-1,0)} + a\chi_{[0,1]} \text{ and } \tilde{\varphi} := \varphi - \varphi_{a,b}, \tag{1.34}$$

[2]The assumptions of [Gou04a, Theorems 1.1 and 1.2] are that φ is Hölder continuous and G is an induced Gibbs-Markov map with invariant absolutely continuous probability measure $\hat{\mu}$ and return time satisfying $\hat{\mu}(\tau > n) = O(n^{-\gamma})$ for some $\gamma > 1$, which holds in our case by Corollary 1.2.8 and the fact that $\beta \in (0, 1)$.

where $\chi_{[-1,0)}, \chi_{[0,1]}$ are the characteristic functions of the intervals $[-1,0)$ and $(0,1]$ respectively. The induced observable of φ is the sum of the induced observables of $\varphi_{a,b}$ and $\tilde{\varphi}$ giving

$$\Phi(x) = \sum_{k=0}^{\tau(x)-1} \varphi_{a,b} \circ g^k(x) + \sum_{k=0}^{\tau(x)-1} \tilde{\varphi} \circ g^k(x) = \tau_{a,b} + \tilde{\Phi}, \tag{1.35}$$

where $\tilde{\Phi}$ denote the induced observable of $\tilde{\varphi}$, and $\tau_{a,b}$ is defined in (1.23), indeed, $\varphi_{a,b} \circ g^k(x)$ takes only two possible values, a or b, depending on whether $g^k(x) \in (0,1]$ or $g^k(x) \in [-1,0)$, and therefore the corresponding induced observable is precisely $\tau_{a,b}$.

To prove Theorem E we obtain regularity and distribution results for the induced observables $\tau_{a,b}$ and $\tilde{\Phi}$ and substitute them into (1.35) to get the various cases (1.31)-(1.33). The motivation for the decomposition (1.34) is given by the observation that $\tilde{\varphi}(-1) = \tilde{\varphi}(1) = 0$, which allows us to prove the following estimate for the corresponding induced observable $\tilde{\Phi}$.

Proposition 1.2.10. *Let $g \in \mathfrak{F}$ with $\beta \in (0,1)$ and let $\tilde{\varphi} : [-1,1] \to \mathbb{R}$ be a Hölder continuous observable such that $\tilde{\varphi}(-1) = \tilde{\varphi}(1) = 0$. Then*

$$(\mathcal{H}) \implies \tilde{\Phi} \in L^2 \quad \text{and} \quad (\mathcal{H}') \implies \hat{\mu}(\pm\tilde{\Phi} > t) = o(t^{-1/\beta_\varphi}). \tag{1.36}$$

Proposition 1.2.6 gives results for $\tau_{a,b}$.

Corollary 1.2.11 (Corollary to Proposition 1.2.6)**.** *If at least one of $a := \varphi(1)$, $b := \varphi(-1)$ is non-zero then:*

$$\beta_\varphi \in [0,1/2) \implies \tau_{a,b} \in L^2(\hat{\mu}), \quad \text{and} \quad \beta_\varphi \in [1/2,1) \implies \tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}.$$

We prove Corollary 1.2.11 and Proposition 1.2.10 in Section 1.4.2. For now we show how they imply Theorem E.

Proof of Theorem E. If $\varphi(-1) = \varphi(1) = 0$ then $\tau_{a,b} \equiv 0$ and so $\Phi = \tilde{\Phi}$, Proposition 1.2.10 implies that $\Phi \in L^2(\hat{\mu})$ and so (1.31) holds. If at least one of $\varphi(-1), \varphi(1)$ is non-zero, we have two cases. If $\beta_\varphi \in (0,1/2)$, Proposition 1.2.10 and Corollary 1.2.11 give that both $\tau_{a,b}, \tilde{\Phi} \in L^2(\hat{\mu})$, which implies that $\Phi \in L^2(\hat{\mu})$ and therefore (1.31) holds. If $\beta_\varphi \in [1/2,1)$ then $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$ by Corollary 1.2.11 and $\hat{\mu}(\pm\tilde{\Phi} > t) = o(t^{-1/\beta_\varphi})$ by Proposition 1.2.10, and therefore $\Phi = \tau_{a,b} + \tilde{\Phi} \in \mathcal{D}_{1/\beta_\varphi}$ since the tail of $\tilde{\Phi}$ is negligible compared to that of $\tau_{a,b}$. Whence, (1.32) holds when $\beta_\varphi = 1/2$ and (1.33) holds otherwise. □

Remark 1.2.12. The relation between Φ and $\tau_{a,b}$ is given formally in (1.35) but it can be useful to have a heuristic idea of this relationships. Given a point $x \in \delta_{i,j}$ with i, j both large we know that most of the first i iterates $x, g(x), \ldots, g^{i-1}(x)$ will lie near the fixed point 1. Similarly, most of the next j iterates $g^i(x), \ldots, g^{i+j-1}$ will lie near the fixed point -1. Thus, if we assume that φ is "sufficiently well behaved" near 1 and -1 (in a sense that is made precise by conditions $(\mathcal{H})$ and $(\mathcal{H}')$), it is reasonable to hope that the induced observable Φ at the point x will behave like $\Phi(x) = \sum_{k=0}^{n-1} \varphi \circ g^k \approx ai + bj = \tau_{a,b}(x)$ when $a = \varphi(1), b = \varphi(-1)$ are not both zero.

1.3 The Induced Map

In this section we prove Proposition 1.2.2. We begin by recalling one of several essentially equivalent definitions of Gibbs-Markov map.

Definition 1.3.1. An interval map $F : I \to I$ is called a (full branch) Gibbs-Markov map if there exists a partition $\mathcal{P}$ of I (mod 0) into open subintervals such that:

1. F is *full branch*: for all $\omega \in \mathcal{P}$ the restriction $F|_\omega : \omega \to int(I)$ is a C^1 diffeomorphism;
2. F is *uniformly expanding*: there exists $\lambda > 1$ such that $|F'(x)| \geq \lambda$ for all $x \in \omega$ for all $\omega \in \mathcal{P}$;
3. F has *bounded distortion*: there exists $C > 0, \theta \in (0,1)$ s.t. for all $\omega \in \mathcal{P}$ and all $x, y \in \omega$,

$$\log \left| \frac{F'(x)}{F'(y)} \right| \leq C\theta^{s(x,y)},$$

where $s(x,y) := \inf\{n \geq 0 : F^n x \text{ and } F^n y \text{ lie in different elements of the partition } \mathscr{P}\}$.

We will show that the first return map G defined in (1.19) satisfies all the conditions above as well as the saturation condition (1.20). In Section 1.3.1 we describe the topological structure of G and show that it is a full branch map with countably many branches which saturates I; this will require only the very basic topological structure of g provided by condition **(A0)**. In Section 1.3.2 we obtain estimates concerning the sizes of the partition elements of the corresponding partition; this will require the explicit form of the map g as given in **(A1)**. In Section 1.3.3 we show that G is uniformly expanding; this will require the final condition **(A2)**. Finally, in Section 1.3.4 we use the estimates and results obtained to show that G has bounded distortion.

1.3.1 Topological Construction

In this section we give an explicit and purely topological construction of the first return maps $G^- : \Delta_0^- \to \Delta_0^-$ and $G^+ : \Delta_0^- \to \Delta_0^-$ which essentially depends only on condition **(A0)**, i.e. the fact that g is a full branch map with two orientation preserving branches. Recall first of all the definitions of the sets $\Delta_n^\pm$ and $\delta_n^\pm$ in (1.5) and (1.6). It follows immediately from the definitions and from the fact that each branch of g is a C^2 diffeomorphism, that for every $n \geq 1$, the maps $g : \delta_n^- \to \Delta_{n-1}^+$ and $g : \delta_n^+ \to \Delta_{n-1}^-$ are C^2 diffeomorphisms, and, for $n \geq 2$, the same is true for the maps $g^{n-1} : \Delta_{n-1}^- \to \Delta_0^-$, and $g^{n-1} : \Delta_{n-1}^+ \to \Delta_0^+$, which implies that for every $n \geq 1$, the maps

$$g^n : \delta_n^- \to \Delta_0^+ \quad \text{and} \quad g^n : \delta_n^+ \to \Delta_0^-$$

are C^2 diffeomorphisms. We can therefore define two maps

$$\widetilde{G}^- : \Delta_0^- \to \Delta_0^+ \quad \text{and} \quad \widetilde{G}^+ : \Delta_0^+ \to \Delta_0^- \quad \text{by} \quad \widetilde{G}^\pm|_{\delta_n^\pm} := g^n. \tag{1.37}$$

Notice that these are *full branch* maps although they have *different domains and ranges*, indeed the domain of one is the range of the other and viceversa. The fact that they are full branch allows us to *pullback* the partition elements $\delta_n^\pm$ *into each other*: for every $m, n \geq 1$ we let

$$\delta_{m,n}^- := g^{-m}(\delta_n^+) \cap \delta_m^- \quad \text{and} \quad \delta_{m,n}^+ := g^{-m}(\delta_n^-) \cap \delta_m^+.$$

Then, for $m \geq 1$, the sets $\{\delta_{m,n}^-\}_{n\geq 1}$ and $\{\delta_{m,n}^+\}_{n\geq 1}$ are partitions of δ_m^- and δ_m^+ respectively and so

$$\mathscr{P}^- := \{\delta_{m,n}^-\}_{m,n\geq 1} \quad \text{and} \quad \mathscr{P}^+ := \{\delta_{m,n}^+\}_{m,n\geq 1} \tag{1.38}$$

are partitions of Δ_0^-, Δ_0^+ respectively, with the property that for every $m, n \geq 1$, the maps

$$g^{m+n} : \delta_{m,n}^- \to \Delta_0^- \quad \text{and} \quad g^{m+n} : \delta_{m,n}^+ \to \Delta_0^+ \tag{1.39}$$

are C^2 diffeomorphisms. Notice that $m+n$ is the *first return time* of points in $\delta^-_{m,n}$ and $\delta^+_{m,n}$ to Δ^-_0 and Δ^+_0 respectively and we have thus constructed two *full branch first return induced maps*

$$G^- := \widetilde{G}^+ \circ \widetilde{G}^- : \Delta^-_0 \to \Delta^-_0 \quad \text{and} \quad G^+ := \widetilde{G}^- \circ \widetilde{G}^+ : \Delta^+_0 \to \Delta^+_0. \tag{1.40}$$

for which we have $G^-|_{\delta^-_{m,n}} = g^{m+n}$ and $G^+|_{\delta^+_{m,n}} = g^{m+n}$.

Lemma 1.3.2. *The maps G^- and G^+ are full branch maps which saturate I*

Proof. The full branch property follows immediately from (1.39). It then also follows from the construction that the families

$$\{g^j(\delta^-_{m,n})\}_{\substack{m,n\geq 1\\ 0\leq j<m+n}} \quad \text{and} \quad \{g^j(\delta^+_{m,n})\}_{\substack{m,n\geq 1\\ 0\leq j<m+n}}$$

of the images of the partition elements (1.38) are each formed by a collection of *pairwise disjoint* intervals which satisfy

$$\bigcup_{\delta^-_{m,n}\in\mathcal{P}^-} \bigcup_{j=0}^{m+n-1} g^j(\delta^-_{m,n}) = \bigcup_{\delta^+_{m,n}\in\mathcal{P}^+} \bigcup_{j=0}^{m+n-1} g^j(\delta^-_{m,n}) = I \mod 0$$

and therefore clearly satisfy (1.20), giving the saturation. □

Remark 1.3.3. Notice that the map G^- is exactly the first return map G defined in (1.19) and therefore Lemma 1.3.2 implies the first part of Proposition 1.2.2.

1.3.2 Partition Estimates

The construction of the full branch induced maps $G^\pm : \Delta^\pm_0 \to \Delta^\pm_0$ in the previous section is purely topological and works for any map g satisfying condition **(A0)**. In this section we proceed to estimate the sizes and positions of the various intervals defined above, and this will require more information about the map, especially the forms of the map as given in **(A1)**. Before stating the estimates we introduce some notation. First of all, we let $(x^-_n)_{n\geq 0}$ and $(x^+_n)_{n\geq 0}$ be the boundary points of the intervals Δ^-_n, Δ^+_n so that $\Delta^-_0 = (x^-_0, 0), \Delta^+_0 = (0, x^+_0)$ and, for every $n \geq 1$ we have

$$\Delta^-_0 = (x^-_0, 0), \qquad \Delta^+_0 = (0, x^+_0), \qquad \Delta^-_n = (x^-_n, x^-_{n-1}), \qquad \Delta^+_n = (x^+_{n-1}, x^+_n). \tag{1.41}$$

The following proposition gives the speed at which the sequences $(x^+_n), (x^-_n)$ converge to the fixed points $1, -1$ respectively and gives estimates for the size of the partition elements $\Delta^\pm_n$ for large n in terms of the values of ℓ_1 and ℓ_2. To state the result we let

$$C_1 = (\ell_1 b_1)^{-1/\ell_1}, \quad C_2 = (\ell_2 b_2)^{-1/\ell_2}, \quad C_3 = \ell_1^{-(1+1/\ell_1)} b_1^{-1/\ell_1} \quad C_4 = \ell_2^{-(1+1/\ell_2)} b_2^{-1/\ell_2}.$$

Proposition 1.3.4. *If $\ell_1 = 0$, then*

$$(1+b_1+\epsilon)^{-n} \lesssim 1 + x^-_n \lesssim (1+b_1)^{-n} \quad \textit{and} \quad |\Delta^-_n| \lesssim (1+b_1)^{-n}. \tag{1.42}$$

If $\ell_1 > 0$, then

$$1 + x^-_n \sim C_1 n^{-1/\ell_1} \quad \textit{and} \quad |\Delta^-_n| \sim C_3 n^{-(1+1/\ell_1)}. \tag{1.43}$$

If $\ell_2 = 0$, then

$$(1+b_2+\epsilon)^{-n} \lesssim 1 - x^+_n \lesssim (1+b_2)^{-n} \quad \textit{and} \quad |\Delta^+_n| \lesssim (1+b_2)^{-n}. \tag{1.44}$$

If $\ell_2 > 0$, then

$$1 - x^+_n \sim C_2 n^{-1/\ell_2} \quad \textit{and} \quad |\Delta^+_n| \sim C_4 n^{-(1+1/\ell_2)}. \tag{1.45}$$

Proof. We will prove (1.42) and (1.43) and then (1.44) and (1.45) follow by exactly the same arguments. Notice first of all that from (1.7) we have $\delta_n^+ \subset U_{0+}$ for all $n > n_+$ and, since from (1.2) we have that $U_{-1} := g(U_{0+})$, this implies that $\Delta_{n-1}^- := g(\delta_n^+) \subset U_{-1}$ for all $n > n_+$, and thus $\Delta_n^- \subset U_{-1}$ for all $n \geq n_+$ which, by the definition of x_n in (1.41), implies that $x_n^- \in U_{-1}$ for $n \geq n_+$.

Now suppose that $\ell_1 > 0$. For $n \geq n_+$, by definition of the x_n^-, we have that $g(x_{n+1}^-) = x_n^-$, and so $1+x_n^- = 1+x_{n+1}^- + b_1(1+x_{n+1}^-)^{1+\ell_1}$. Setting $z_n = 1+x_n^-$ we can write this as $z_n = z_{n+1}(1+b_1 z_{n+1}^{\ell_1})$ and, taking the power $-\ell_1$ and expanding we get

$$\frac{1}{z_n^{\ell_1}} = \frac{1}{z_{n+1}^{\ell_1}}(1+b_1 z_{n+1}^{\ell_1})^{-\ell_1} = \frac{1}{z_{n+1}^{\ell_1}}\left(1 - \ell_1 b_1 z_{n+1}^{\ell_1} + O(z_{n+1}^{2\ell_1})\right) = \frac{1}{z_{n+1}^{\ell_1}} - \ell_1 b_1 + o(1).$$

From the above we know that $z_n^{-\ell_1} = z_{n-1}^{-\ell_1} + b_1\ell_1 + o(1)$ and applying this relation recursively we obtain that $z_n^{-\ell_1} = \ell_1 b_1 n + o(n)$ which yields $x_n^- + 1 = (\ell_1 b_1 n)^{-1/\ell_1}(1+o(1))$, thus giving the first statement in (1.43). Now, by definition $\Delta_n^- = [x_n^-, x_{n-1}^-) = [x_n^-, g(x_n^-))$, so, for all n large enough, $|\Delta_n^-| = g(x_n^-) - x_n^- = b_1(1+x_n^-)^{1+\ell_1}$. Inserting $x_n^- + 1 \sim C_1 n^{-1/\ell_1}$ into this expression for $|\Delta_n^-|$ then yields $|\Delta_n^-| \sim \ell_1^{-(1+1/\ell_1)} b_1 n^{-(1+1/\ell_1)}$, completing the proof of (1.43).

Now, for $\ell_1 = 0$, since $g(x_n^-) = x_{n-1}^-$, mean value theorem implies $(1+b_1) \leq (x_{n-1}^- + 1)/(x_n^- + 1) \leq (1+b_1+o(1))$ which can be written as $(1+b_1+o(1))^{-1}(x_{n-1}^- + 1) \leq x_n^- + 1 \leq (1+b_1)^{-1}(x_{n-1}^- + 1)$. Iterating this relation we obtain the claimed bounds for $x_n^- + 1$. As in the previous case we may calculate using (1.3) that $|\Delta_n^-| = g(x_n^-) - x_n^- = -1 + (1+b_1)(1+x_n^-) + \xi(x_n^-) - x_{n-1} = b_1(1+x_n^-) + o(1) \lesssim (1+b_1)^{-n}$, which concludes the proof.

□

To get analogous estimates for the intervals δ_n^-, δ_n^+, we let $(y_n^-)_{n\geq 0}$ and $(y_n^+)_{n\geq 0}$ be the boundary points of the intervals δ_n^-, δ_n^+ respectively, so that for every $n \geq 1$ we have

$$\delta_n^- = (y_{n-1}^-, y_n^-) \quad \text{and} \quad \delta_n^+ = (y_n^+, y_{n-1}^+).$$

In particular, $y_0^- = x_0^-$, $y_0^+ = x_0^+$, and $g(y_n^-) = x_{n-1}^+$, $g(y_n^+) = x_{n-1}^-$ for $n \geq 1$. Then we let

$$B_1 = a_1^{-1/k_1}(\ell_2 b_2)^{-1/\beta_2}, \quad B_2 = a_2^{-1/k_2}(\ell_1 b_1)^{-1/\beta_1}, \quad B_3 = B_1/\beta_2, \quad B_4 = B_2/\beta_1.$$

Recall that B_1, B_2 have already been defined in (1.25).

Proposition 1.3.5. *If $\ell_1 = 0$, then for every $\varepsilon > 0$*

$$\left(\frac{1}{1+b_2+\varepsilon}\right)^{\frac{n}{k_1}} \lesssim -y_n^- \lesssim \left(\frac{1}{1+b_2}\right)^{\frac{n}{k_1}} \quad \text{and} \quad |\delta_n^-| \lesssim \left(\frac{1}{1+b_2}\right)^n. \tag{1.46}$$

If $\ell_1 > 0$, then

$$y_n^- \sim -B_1 n^{-\frac{1}{\beta_2}}, \quad \text{and} \quad |\delta_n^-| \sim B_3 n^{-\left(1+\frac{1}{\beta_2}\right)}. \tag{1.47}$$

If $\ell_2 = 0$, then for every $\varepsilon > 0$

$$\left(\frac{1}{1+b_1+\varepsilon}\right)^{\frac{n}{k_2}} \lesssim y_n^+ \lesssim \left(\frac{1}{1+b_1}\right)^{\frac{n}{k_2}}, \quad \text{and} \quad |\delta_n^+| \lesssim \left(\frac{1}{1+b_1}\right)^n. \tag{1.48}$$

If $\ell_2 > 0$, then

$$y_n^+ \sim B_2 n^{-\frac{1}{\beta_1}}, \quad \text{and} \quad |\delta_n^+| \sim B_4 n^{-\left(1+\frac{1}{\beta_1}\right)}, \tag{1.49}$$

Proof. We will prove (1.46) and (1.47), as (1.48) and (1.49) follow by analogous arguments. Suppose first that $\ell_1 > 0$. As $x_n^+ \to 1$, and as $g_-(y_n^-) = x_{n-1}^+$ we know that for all n sufficiently large we have $g_-(y_n^-) = 1 - a_1(-y_n^-)^{k_1} = x_{n-1}^+$. Solving for y_n this gives

$$y_n^- = -\left((1 - x_{n-1}^+)/a_1\right)^{1/k_1} = -a_1^{-1/k_1}\,(\ell_2 b_2 n)^{-1/\ell_2 k_1}\,(1 + o(1))$$

which is the first statement in (1.47). Now we turn our attention to the size of the intervals δ_n^-. First let us note that for any $\gamma > 0$ we have that $n^{-\gamma} - (n+1)^{-\gamma} = n^{-\gamma}\left[1 - (1 + 1/n)^{-\gamma}\right] = n^{-\gamma}\left[1 - \left(1 - \gamma/n + O(n^{-2})\right)\right] = \gamma n^{-(1+\gamma)}(1 + O(1/n))$. and therefore

$$|\delta_n| = y_n^- - y_{n+1}^- = B_1(n^{-1/\ell_2 k_1} - (n+1)^{-1/\ell_2 k_1})(1 + o(1)) = \frac{B_1}{\ell_2 k_1} n^{-(1+1/\ell_2 k_1)}(1 + o(1))$$

which completes the proof of (1.47). Now for $\ell_2 = 0$ we proceed as before, and by (1.44) we get

$$(1 + b_2 + \varepsilon)^{-n/k_1} \lesssim -y_n^- = \left((1 - x_{n-1}^+)/a_1\right)^{1/k_1} \lesssim (1 + b_2)^{-n/k_1}.$$

For the size of the interval δ_n^-, we may use the mean value theorem to conclude that

$$g'(u_n) = \frac{x_{n-2}^+ - x_{n-1}^+}{y_{n-1}^- - y_n^-} = \frac{|\Delta_{n-1}^+|}{|\delta_n^-|},$$

for some $u_n \in \delta_n^-$. As g' is monotone on U_0^- we know, from the above and (1.44), that

$$|\delta_n^-| \lesssim |\Delta_{n-1}^+|/g'(y_n^-) \lesssim (1 + b_2 + \varepsilon)^{-n/k_1}(1 + b_2)^{-n} \lesssim (1 + b_2)^{-n}.$$

which concludes the proof. □

1.3.3 Expansion Estimates

Proposition 1.3.6. *For every $g \in \widehat{\mathfrak{F}}$ the first return map $G : \Delta_0^- \to \Delta_0^-$ is uniformly expanding.*

It is enough to prove uniform expansivity for the two maps $\widetilde{G}^-, \widetilde{G}^+$, recall (1.37), since this implies the same property for their composition $G = G^-$, recall (1.40). To simplify the notation we will only prove the statement for $\widetilde{G}^+$, i.e. we will prove that $x \in \delta_n^+ \Rightarrow (g^n)'(x) > \lambda$. The fact that $x \in \delta_n^- \Rightarrow (g^n)'(x) > \lambda$ follows by an identical argument.

Notice first of all that if $k_2 \in (0, 1)$, the derivative in Δ_0^+ is greater than 1 and therefore the uniform expansion of $\widetilde{G}^+$ is immediate. So throughout this section we will assume that $k_2 \geq 1$ which means that g has a critical point and the derivative of g in Δ_0^+ can be arbitrarily small. For points outside the neighbourhood U_{0+} on which the map g has a precise form, more precisely for $1 \leq n \leq n_+$ and for $x \in \delta_n^+$, the expansivity is automatically guaranteed by condition **(A2)**, but for points close to 0 where the derivative can be arbitrarily small the statement is non-trivial. It ultimately depends on writing $\widetilde{G}^+(x) := g^n(x)$ for $x \in \delta_n^+$, so that $(\widetilde{G}^+)'(x) = (g^n)'(x) = (g^{n-1})'(g(x))g'(x)$, and then showing that the small derivative $g'(x)$ near the critical point is compensated by sufficiently large number of iterates where the derivative is > 1. This clearly relies very much on the partition estimates in Section 1.3.2 which provide a relation between the position of points, and therefore their derivatives, and the corresponding values of n. A relatively straightforward computation using those estimates shows that we get expansion for *sufficiently large* $n \geq 1$, which is quite remarkable but not enough for our purposes as it does not give a complete proof of expansivity for $\widetilde{G}^+$ at every point in Δ_0^+. We therefore need to use a somewhat more sophisticated approach that shows that the derivative of $\widetilde{G}^+$ has a kind of *"monotonicity"* property in the following sense. Define the function $\phi : \Delta_0^+ \setminus \delta_1^+ \to \Delta_0^+$ given implicitly by $g^2 = g \circ \phi$ and explicitly by

$$\phi := (g|_{U_{0+}})^{-1} \circ g|_{U_{-1}} \circ g|_{U_{0+}} \tag{1.50}$$

Notice that ϕ is the bijection which makes the diagram in Figure 1.4 commute.

$$\begin{array}{ccc} \delta_{n+1}^+ & \xrightarrow{\phi} & \delta_n^+ \\ \downarrow g & & \downarrow g \\ \Delta_n^- & \xrightarrow{g} & \Delta_{n-1}^- \end{array}$$

Figure 1.4: Definition of the map $\phi : \Delta_0^+ \setminus \delta_1^+ \to \Delta_0^+$.

The key step in the proof of Proposition 1.3.6 is the following lemma.

Lemma 1.3.7. *For all $n \geq n^+$ and $x \in \delta_{n+1}^+$ we have*

$$(g^2)'(x) > g'(\phi(x)).$$

Remark 1.3.8. Lemma 1.3.7 is equivalent to $(g^2)'(x)/g'(\phi(x)) > 1$ which is equivalent to

$$\frac{g'(x)}{g'(\phi(x))} g'(g(x)) > 1. \tag{1.51}$$

Notice that the ratio $g'(x)/g'(\phi(x))$ is < 1 and measures how much derivative is "lost" when choosing the initial condition x instead of the initial condition $\phi(x)$ (since $\phi(x) > x$ and the derivative is monotone increasing), whereas $g'(g(x)) > 1$ measures how much derivative is "gained" from performing an extra iteration of g. The Lemma says that the gain is more than the loss.

Proof. To simplify the notation let us set $a = a_2$, $b = b_1$, $k = k_2$, and $\ell = \ell_1$. Notice first of all that by the form of g in U_{0+} given in **(A1)** we have

$$\frac{g'(x)}{g'(\phi(x))} = \left(\frac{x}{\phi(x)}\right)^{k-1} = \left(\frac{\phi(x)}{x}\right)^{1-k} \tag{1.52}$$

Recall that $k > 1$ and $x < \phi(x)$ and so the ratio above is < 1. To estimate $g'(g(x))$ we consider two cases depending on ℓ. If $\ell > 0$, using the form of g given in **(A1)** and plugging into 1.50 we get

$$\phi(x) = \left[x^k + ba^\ell x^{k(\ell+1)}\right]^{1/k} \qquad \text{and therefore} \qquad \left(\frac{\phi(x)}{x}\right)^k = 1 + ba^\ell x^{k\ell}$$

and, therefore, using the form of G in U_{-1}, this gives

$$g'(g(x)) = g'(-1 + ax^k) = 1 + ba^\ell x^{k\ell} + b\ell a^\ell x^{k\ell} = \left(\frac{\phi(x)}{x}\right)^k + b\ell a^\ell x^{k\ell} \tag{1.53}$$

From (1.52) and (1.53) and the fact that $x < \phi(x)$ we immediately get

$$\frac{g'(x)}{g'(\phi(x))} g'(g(x)) = \left(\frac{\phi(x)}{x}\right)^{1-k} \left[\left(\frac{\phi(x)}{x}\right)^k + b\ell a^\ell x^{k\ell}\right] > \frac{\phi(x)}{x} > 1$$

which establishes (1.51) and completes the case that $\ell > 0$. For $\ell = 0$, proceeding as above we obtain

$$\phi(x) = \Big[(1+b)x^k + \xi(g(x))/a\Big]^{1/k} \quad \text{and therefore} \quad \left(\frac{\phi(x)}{x}\right)^k = (1+b) + \frac{\xi(g(x))}{ax^k}. \tag{1.54}$$

Since $g(x) = -1 + ax^k$, from (1.13) we have $\xi'(g(x)) \geq \xi(g(x))/(1+g(x)) = \xi(g(x))/ax^k)$, and so

$$g'(g(x)) = (1+b) + \xi'(g(x)) \geq (1+b) + \frac{\xi(g(x))}{ax^k} = \left(\frac{\phi(x)}{x}\right)^k.$$

Together with (1.52), as above, we get the statement in this case also. □

As an almost immediate consequence of Lemma 1.3.7 we get the following.

Corollary 1.3.9. *For all $n \geq n^+$ and $x \in \delta^+_{n+1}$ we have*

$$(\widetilde{G}^+)'(x) > (\widetilde{G}^+)'(\phi(x)).$$

Proof. By Lemma 1.3.7 and (1.51), for any $1 \leq m \leq n$ we have

$$(g^{m+1})'(x) = g'(x)g'(g(x)) \cdots g'(g^m(x)) = \frac{g'(x)g'(g(x))}{g'(\phi(x))}(g^m)'(\phi(x)) > (g^m)'(\phi(x)). \tag{1.55}$$

□

Proof of Proposition 1.3.6. Condition **(A2)** implies that $(\widetilde{G}^+)'(x) \geq \lambda$ for all $x \in \delta^+_n$ for $1 \leq n \leq n^+$. Then, for $x \in \delta^+_{n^++1}$ we have $\phi(x) \in \delta^+_{n^+}$ and therefore

$$(\widetilde{G}^+)'(x) > (\widetilde{G}^+)'(\phi(x)) \geq \lambda$$

Proceeding inductively we obtain the result. □

1.3.4 Distortion Estimates

Proposition 1.3.10. *For all $g \in \widehat{\mathfrak{F}}$ there exists a constant $\mathfrak{D} > 0$ such that for all $0 \leq m < n$ and all $x, y \in \delta^\pm_n$,*

$$\log \frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} \leq \mathfrak{D}|g^n(x) - g^n(y)|.$$

As a consequence we get that G is a Gibbs-Markov map with constants $C = \mathfrak{D}\lambda$ and $\theta = \lambda^{-1}$.

Corollary 1.3.11. *For all $x, y \in \delta_{i,j} \in \mathcal{P}$ with $x \neq y$ we have*

$$\log \left|\frac{G'(x)}{G'(y)}\right| \leq \mathfrak{D}\lambda^{-s(x,y)+1}.$$

Proof. Let $n := s(x,y)$. Since G is uniformly expanding, we have $1 \geq |G^n(x) - G^n(y)| = |(G^{n-1})'(u)||G(x) - G(y)| \geq \lambda^{n-1}|G(x) - G(y)|$ and therefore $|G(x) - G(y)| \leq \lambda^{-n+1}$. By Proposition 1.3.10 this gives $\log|G'(x)/G'(y)| \leq \mathfrak{D}|G(x) - G(y)| \leq \mathfrak{D}\lambda^{-n+1} = \mathfrak{D}\lambda^{-s(x,y)+1}$. □

Proof of Proposition 1.3.10. We begin with a couple of simple formal steps. First of all, by the chain rule, we can write

$$\log \frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} = \log \prod_{i=m}^{n-1} \frac{g'(g^i(x))}{g'(g^i(y))} = \sum_{i=m}^{n-1} \log \frac{g'(g^i(x))}{g'(g^i(y))}.$$

Then, since $g^i(x), g^i(y)$ are both in the same smoothness component of g, by the Mean Value Theorem, there exists $u_i \in (g^i(x), g^i(y))$ such that

$$\log \frac{g'(g^i(x))}{g'(g^i(y))} = \log g'(g^i(x)) - \log g'(g^i(y)) = \frac{g''(u_i)}{g'(u_i)} |g^i(x) - g^i(y)|.$$

Substituting this into the expression above, and writing $\mathfrak{D}_i := g''(u_i)/g'(u_i)$ for simplicity, we get

$$\log \frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} = \sum_{i=m}^{n-1} \mathfrak{D}_i |g^i(x) - g^i(y)| \leq \sum_{i=0}^{n-1} \mathfrak{D}_i |g^i(x) - g^i(y)|. \tag{1.56}$$

We will bound the sum above in two steps. First of all we will show that it admits a uniform bound $\widehat{\mathfrak{D}}$ independent of m, n. We will then use this bound to improve our estimates and show that by paying a small price (increasing the uniform bound to a larger bound $\mathfrak{D} := \widehat{\mathfrak{D}}^2/|\Delta_0^-|$) we can include the term $|g^n(x) - g^n(y)|$ as required. Ultimately this gives a stronger result since it takes into account the closeness of the points x, y.

Let us suppose first for simplicity that $x, y \in \delta_n^+$, the estimates for δ_n^- are identical. Then for $1 \leq i < n$ we have that $g^i(x), g^i(y), u_i \in \Delta_{n-i}^-$ and therefore we can bound (1.56) by

$$\sum_{i=0}^{n-1} \mathfrak{D}_i |g^i(x) - g^i(y)| \leq \mathfrak{D}_0 |x - y| + \sum_{i=1}^{n-1} \mathfrak{D}_i |g^i(x) - g^i(y)| \leq \mathfrak{D}_0 \delta_n^+ + \sum_{i=1}^{n-1} \mathfrak{D}_i |\Delta_{n-i}^-|. \tag{1.57}$$

From (1.12) and using the relationship between the y_n^+ and the x_n^- we may bound the first term by

$$\mathfrak{D}_0 |\delta_n^+| \lesssim u_0^{-1} |\delta_n^+| \lesssim \frac{y_n^+ - y_{n+1}^+}{y_{n+1}^+} \lesssim \left(\frac{1 + x_n^-}{1 + x_{n+1}^-} \right)^{1/k} - 1 \to c < \infty \tag{1.58}$$

where we have used the fact that that for some sequence $\xi_n \to -1$ we have $(1 + x_n^-)/(1 + x_{n+1}^-) = g'(\xi_n)$ which converges to 1 if $\ell > 0$ (and therefore $c = 0$) or $1 + b_1$ otherwise (and therefore $c = b_1$). If $\ell_1 = 0$ then $\mathfrak{D}_i$ is uniformly bounded for $i > 0$, if $\ell_1 > 0$, then from (1.11) and (1.43) we know that

$$\mathfrak{D}_i |\Delta_{n-i}^-| \lesssim (1 + u_i)^{\ell_1 - 1} |\Delta_{n-i}^-| \lesssim (n-i)^{-\frac{\ell_1 - 1}{\ell_1}} (n-i)^{-\left(1 + \frac{1}{\ell_1}\right)} = (n-i)^{-2}. \tag{1.59}$$

Then by (1.58) and (1.59) we find that

$$\widehat{\mathfrak{D}} := \mathfrak{D}_0 \delta_n^+ + \sum_{i=1}^{\infty} \mathfrak{D}_i \Delta_{n-i} \leq \mathfrak{D}_0 + \sum_{i=1}^{\infty} \mathfrak{D}_i \Delta_{n-i} < \infty. \tag{1.60}$$

Substituting this back into (1.57) and then into (1.56) we get

$$\log \left| \frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} \right| \leq \widehat{\mathfrak{D}} \tag{1.61}$$

which completes the first step in the proof, as discussed above. We now take advantage of this bound to improve our estimates as follows. By a standard and straightforward application of the Mean Value Theorem, (1.61) implies that the diffemorphisms $g^n : \delta_n^+ \to \Delta_0^-$ and $g^{n-m} : \Delta_{n-m}^- \to \Delta_0^-$ all have uniformly bounded distortion in the sense that for every $x, y \in \delta_n^+$ and $1 \le m < n$ we have

$$\frac{|x-y|}{|\delta_n^+|} \le \exp(\widehat{\mathfrak{D}}) \frac{|g^n(x) - g^n(y)|}{|\Delta_0^-|} \tag{1.62}$$

and

$$\frac{|g^m(x) - g^m(y)|}{|\Delta_{n-m}^+|} \le \widehat{\mathfrak{D}} \frac{|g^{n-m}(g^m(x)) - g^{n-m}(g^m(y))|}{|\Delta_0^-|} = \exp(\widehat{\mathfrak{D}}) \frac{|g^n(x)) - g^n(y)|}{|\Delta_0^-|}. \tag{1.63}$$

Therefore

$$|x-y| \le \frac{\widehat{\mathfrak{D}}}{|\Delta_0^-|} |g^n(x) - g^n(y)||\delta_n^+| \quad \text{and} \quad |g^m(x) - g^m(y)| \le \frac{\widehat{\mathfrak{D}}}{|\Delta_0^-|} |g^n(x) - g^n(y)||\Delta_{n-m}^-|.$$

Substituting these bounds back into (1.56) (with $i = m$), and letting $\mathfrak{D} := \widehat{\mathfrak{D}}^2/|\Delta_0^-|$, we get

$$\begin{aligned}
\frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} &\le \sum_{i=0}^{n-1} \mathfrak{D}_i |g^i(x) - g^i(y)| = \mathfrak{D}_0|x-y| + \sum_{i=1}^{n-1} \mathfrak{D}_i |g^i(x) - g^i(y)| \\
&\le \mathfrak{D}_0 \frac{\widehat{\mathfrak{D}}}{|\Delta_0^-|} |g^n(x) - g^n(y)||\delta_n^+| + \sum_{i=1}^{n-1} \mathfrak{D}_i \frac{\widehat{\mathfrak{D}}}{|\Delta_0^-|} |g^n(x) - g^n(y)||\Delta_{n-i}^-| \\
&= \frac{\widehat{\mathfrak{D}}}{|\Delta_0^-|} \left[\mathfrak{D}_0 |\delta_n^+| + \sum_{i=1}^{n-1} \mathfrak{D}_i |\Delta_{n-i}^-| \right] |g^n(x) - g^n(y)| \\
&\le \frac{\widehat{\mathfrak{D}}^2}{|\Delta_0^-|} |g^n(x) - g^n(y)| = \mathfrak{D}|g^n(x) - g^n(y)|.
\end{aligned}$$

Notice that the last inequality follows from (1.60). This completes the proof. □

We state here also a simple corollary of Propositions 1.3.5 and 1.3.10 which we will use in Section 1.4.

Lemma 1.3.12. *For all $i, j \ge 1$ we have*

$$\tilde{\mu}(\delta_{i,j}) \lesssim \begin{cases} i^{-(1+1/\beta_2)} j^{-(1+1/\beta_1)}, & \text{if } \ell_1, \ell_2 > 0 \\ (1+b_2)^{-i}, & \text{if } \ell_1 = 0, \ell_2 > 0 \\ (1+b_1)^{-j}, & \text{if } \ell_2 = 0, \ell_1 > 0 \\ \min\{(1+b_1), (1+b_2)\}^{-i-j}, & \text{if } \ell_1 = 0, \ell_2 = 0 \end{cases} \tag{1.64}$$

Proof. Proposition 1.3.10 implies that $|\delta_{ij}| \approx |\delta_i^-||\delta_j^+|$ uniformly for all $i, j \ge 1$. Indeed, more precisely, it implies $\mathfrak{D}^{-1}|g^i(\delta_{ij})|/|g^i(\delta_i)| \le |\delta_{ij}|/|\delta_i| \le \mathfrak{D}|g^i(\delta_{ij})|/|g^i(\delta_i)|$ which implies $|\delta_i^-||\delta_j^+|/\mathfrak{D}|\Delta_0^+| \le |\delta_{ij}| \le \mathfrak{D}|\delta_i^-||\delta_j^+|/|\Delta_0^+|$. As $\tilde{\mu}$ is equivalent to Lebesgue on $\Delta_0^- \cup \Delta_0^+$ we obtain the Lemma immediately from Proposition 1.3.5. □

1.4 Statistical Properties

In Section 1.4.1 we prove Proposition 1.2.6 and in Section 1.4.2 we prove Proposition 1.2.10 and Corollary 1.2.11. As discussed in Section 1.2.3 this completes the proof of Theorem E.

1.4.1 Distribution and Tail Estimates

In this section we prove Proposition 1.2.6. We will only explicitly prove (1.27) and (1.28) as the proof of (1.29) is identical to that of (1.28). For $a, b \geq 0$ consider the following decompositions

$$\tilde{\mu}(a\tau^+ + b\tau^- > t) = \tilde{\mu}(a\tau^+ > t) + \tilde{\mu}(b\tau^- > t) \tag{1.65}$$
$$- \tilde{\mu}(a\tau^+ > t, b\tau^- > t) + \tilde{\mu}(a\tau^+ + b\tau^- > t, \max\{a\tau^+, b\tau^-\} \leq t) \tag{1.66}$$

and

$$\tilde{\mu}(a\tau^+ - b\tau^- > t) = \tilde{\mu}(a\tau^+ > t) \tag{1.67}$$
$$- \tilde{\mu}(a\tau^+ > t, a\tau^+ - b\tau^- \leq t). \tag{1.68}$$

We can then reduce the proof to two further Propositions. First of all we give precise asymptotic estimates of the terms $\mu(a\tau^+ > t)$, $\mu(b\tau^- > t)$ which make up (1.65) and (1.67).

Proposition 1.4.1. *For every $a, b \geq 0$ and for every $\gamma \in (0, 1)$*

$$\tilde{\mu}(a\tau^+ > t) = \begin{cases} C_a t^{-1/\beta_2} + o(t^{-\gamma - 1/\beta_2}) & \text{if } \ell_1 > 0 \\ O\left((1 + b_2)^{-t/ak_1}\right) & \text{if } \ell_1 = 0 \end{cases} \tag{1.69}$$

and

$$\tilde{\mu}(b\tau^- > t) = \begin{cases} C_b t^{-1/\beta_1} + o(t^{-\gamma - 1/\beta_1}) & \text{if } \ell_2 > 0 \\ O\left((1 + b_1)^{-t/bk_2}\right) & \text{if } \ell_2 = 0 \end{cases} \tag{1.70}$$

Then, we show that the remaining terms (1.66) and (1.68) in the decompositions above have negligible contribution to the leading order asymptotics of the tail.

Proposition 1.4.2. *If at least one of ℓ_1, ℓ_2 are not zero, then for every $a, b \geq 0$, $\gamma \in (0, 1)$ we have*

$$\tilde{\mu}(a\tau^+ > t, b\tau^- > t) + \tilde{\mu}(a\tau^+ + b\tau^- > t, \max\{\tau^+, b\tau^-\} \leq t) = o(t^{-\gamma - 1/\beta}) \tag{1.71}$$

and

$$\tilde{\mu}(a\tau^+ > t, a\tau^+ - b\tau^- \leq t) = o(t^{-\gamma - 1/\beta}) \tag{1.72}$$

As we shall see, (1.72) actually holds for all $\gamma \in (0, 1/\beta)$ (where $1/\beta > 1$ since $\beta \in (0, 1)$ by assumption) but we will not need this stronger statement. We prove Proposition 1.4.1 in Section 1.4.1 and Proposition 1.4.2 in Section 1.4.1, but first we show how they imply Proposition 1.2.6.

Proof of Proposition 1.2.6. To prove (1.27), first suppose that least one of ℓ_1, ℓ_2 is non-zero. Substituting the corresponding lines of (1.69) and (1.70) into (1.66) and substituting (1.71) into (1.68) we obtain (1.27) in this case. If $\ell_1 = \ell_2 = 0$ we only need to establish upper bound for $\tilde{\mu}(a\tau^+ + b\tau^- > t)$ rather than an asymptotic equality and therefore, instead of the decomposition in (1.65) and (1.66), we can use the fact that

$$\tilde{\mu}(a\tau^+ + b\tau^- > t) \leq \tilde{\mu}(a\tau^+ > t) + \tilde{\mu}(b\tau^- > t) \tag{1.73}$$

The result then follows by inserting the corrsponding lines of (1.69) and (1.70) into (1.73).

To prove (1.28), if $\ell_2 > 0$ the result follows by substituting the corresponding line of (1.69) into (1.67) and substituting (1.72) into (1.68). Again, if $\ell_2 = 0$ we only need to establish upper bound for $\tilde{\mu}(a\tau^+ - b\tau^- > t)$ rather than an asymptotic equality and therefore, instead of the decomposition in (1.67) and (1.68), we can use the fact that

$$\tilde{\mu}(a\tau^+ - b\tau^- > t) \leq \tilde{\mu}(a\tau^+ > t) \tag{1.74}$$

The result then follows by inserting the corresponding line of (1.69) into (1.74). □

Leading order asymptotics

We prove Proposition 1.4.1 via two lemmas which show in particular how the values $\tilde{h}(0^-), \tilde{h}(0^+)$ of the density of the measure $\tilde{\mu}$ turn up in the constants C_a, C_b defined in (1.26). Our first lemma shows that the tails of the distributions $\tilde{\mu}(\tau^+ > t)$ and $\tilde{\mu}(\tau^- > t)$ have a very geometric interpretation.

Lemma 1.4.3. *For every $t > 0$ we have*

$$\tilde{\mu}(\tau^+ > t) = \tilde{\mu}(y^-_{\lceil t \rceil}, 0) \quad \textit{and} \quad \tilde{\mu}(\tau^- > t) = \tilde{\mu}(0, y^+_{\lceil t \rceil}). \tag{1.75}$$

Remark 1.4.4. While the first statement in (1.75) is relatively straightforward, the second statement is not at all obvious since τ^- is defined on Δ_0^- and there is no immediate connection with the interval $(0, y^+_{\lceil t \rceil})$ in Δ_0^+. As we shall see, the proof of Lemma 1.4.3 requires a subtle and interesting argument.

Remark 1.4.5. Since $\tilde{\mu}$ is equivalent to Lebesgue measure on Δ_0^- and Δ_0^+, we immediately have that $\tilde{\mu}(y^-_{\lceil t \rceil}, 0) \approx |y^-_{\lceil t \rceil}|$ and $\tilde{\mu}(0, y^+_{\lceil t \rceil}) \approx y^+_{\lceil t \rceil}$, and we can then use (1.47) and (1.49), and Lemma 1.4.3, to get upper bounds for the distributions $\tilde{\mu}(\tau^+ > t)$ and $\tilde{\mu}(\tau^+ > t)$. This is however not enough for our purposes as we require sharper estimates for the distributions, and we therefore need a more sophisticated argument which yields the statement in the following lemma.

Lemma 1.4.6. *For every $t > 0$ we have*

$$\tilde{\mu}(y^-_{\lceil t \rceil}, 0) = y^-_{\lceil t \rceil} \tilde{h}(0^-) + O((y^-_{\lceil t \rceil})^2) \quad \textit{and} \quad \tilde{\mu}(y^+_{\lceil t \rceil}, 0) = y^+_{\lceil t \rceil} \tilde{h}(0^+) + O((y^+_{\lceil t \rceil})^2).$$

Before proving these two lemmas we show how they imply Proposition 1.4.1.

Proof of Proposition 1.4.1. Let us first show (1.69). Recall from the definition of C_a in (1.26) that $a = 0 \Rightarrow C_a = 0$, so if $a = 0$ there is nothing to prove. Let us suppose then that $a > 0$. By Lemmas 1.4.3 and 1.4.6 we have

$$\tilde{\mu}(a\tau^+ > t) = \tilde{\mu}(\tau^+ > t/a) = y^-_{\lceil t/a \rceil} \tilde{h}(0^-) + O((y^-_{\lceil t/a \rceil})^2).$$

Then, using the asymptotic estimates (1.46) and (1.47) for y_n^- in Proposition 1.3.5; and since $O(t^{-2/\beta_1}) = o(t^{-\gamma - 1/\beta_1})$ for every $\gamma \in (0, 1)$; and by the definition of C_a in (1.26), we obtain

$$\mu(a\tau^+ > t) = \begin{cases} B_1 \tilde{h}(0^-)(t/a)^{-1/\beta_2} + O(t^{-2/\beta_2}) = C_a t^{-1/\beta_2} + o(t^{-\gamma - 1/\beta_2}) & \text{if } \ell_1 > 0 \\ O\left((1 + b_2)^{-t/ak_1}\right) & \text{if } \ell_1 = 0 \end{cases}$$

yielding (1.69). To show (1.70) we can proceed similarly to the above. As before, if $b = 0$ there is nothing to prove so we assume $b > 0$ in which case Lemmas 1.4.3 and 1.4.6 we have $\tilde{\mu}(b\tau^- > t) = \tilde{\mu}(\tau^- > t/b) = y^+_{\lceil t/b \rceil} \tilde{h}(0^-) + O((y^+_{\lceil t/b \rceil})^2)$. Now using (1.48) and (1.49), and arguing as above we find that (1.29) holds for every $\gamma \in (0, 1)$. □

We complete this section with the proofs of Lemmas 1.4.3 and 1.4.6.

Proof of Lemma 1.4.3. By definition, recall (1.22), $\tau^+(x) = i, \tau^-(x) = j$ for all $x \in \delta_{i,j}$, and therefore

$$\tilde{\mu}(\tau^+ > t) = \sum_{i>t}\sum_{j=1}^{\infty}\tilde{\mu}(\delta_{i,j}) \quad \text{and} \quad \tilde{\mu}(\tau^- > t) = \sum_{j>t}\sum_{i=1}^{\infty}\tilde{\mu}(\delta_{i,j}). \tag{1.76}$$

We claim that for every $i, j \geq 1$ we have

$$\sum_{j=1}^{\infty}\tilde{\mu}(\delta_{i,j}) = \tilde{\mu}(\delta_i^-) \quad \text{and} \quad \sum_{i=1}^{\infty}\tilde{\mu}(\delta_{i,j}) = \tilde{\mu}(\delta_j^+). \tag{1.77}$$

Then, substituting (1.77) into (1.76) we get

$$\tilde{\mu}(\tau^+ > t) = \sum_{i>t}\sum_{j=1}^{\infty}\tilde{\mu}(\delta_{i,j}) = \sum_{i>t}\tilde{\mu}(\delta_i^-) = \tilde{\mu}(y^-_{\lceil t\rceil}, 0), \tag{1.78}$$

and

$$\tilde{\mu}(\tau^- > t) = \sum_{j>t}\sum_{i=1}^{\infty}\tilde{\mu}(\delta_{i,j}) = \sum_{j>t}\tilde{\mu}(\delta_i^+) = \tilde{\mu}(0, y^+_{\lceil t\rceil})$$

which is exactly the statement (1.75) in the Lemma.

Thus it only remains to prove (1.77). As already mentioned in Remark 1.4.4, despite the apparent symmetry between the two statements, the situation in the two expressions is actually quite different. Indeed, from the topological construction of the induced map, for each $i \geq 1$ we have

$$\delta_i^- = \bigcup_{j=1}^{\infty}\delta_{i,j} \tag{1.79}$$

which, since the intervals $\delta_{i,j}$ are pairwise disjoint, clearly implies the first equality i (1.77). The second equalityis not immediate since, for each fixed $j \geq 1$, , the intervals $\delta_{i,j}$ are *spread out* in Δ_0^-, with each $\delta_{i,j}$ lying inside the corresponding interval δ_i^-, and indeed the $\delta_{i,j}$ do not even belong to δ_j^+ and therefore we cannot just substitute i and j to get a corresponding version of (1.79). We use instead a simple but clever argument inspired by a similar aergument in [Cri+10, Lemma 8] which takes advantage of the invariance of the meaure $\tilde{\mu}$. Recall first of all from the construction of the induced map, that $g^{-1}(\delta_j^+)$ consists of exactly two connected components, one is exactly the interval $\delta_{1,j}$ and the other one is a subinterval of Δ_1^+. So for any $j \geq 1$ we have

$$g^{-1}(\delta_j^+) = \delta_{1,j} \cup \{x : \Delta_1^+ : g(x) \in \delta_j^+\}.$$

By the invariance of the measure $\tilde{\mu}$, and since these two components are disjoint, this implies

$$\tilde{\mu}(\delta_j^+) = \tilde{\mu}(g^{-1}(\delta_j^+)) = \tilde{\mu}(\delta_{1,j}) + \tilde{\mu}(\{x : \Delta_1^+ : g(x) \in \delta_j^+\}) \tag{1.80}$$

The preimage of the set $\{x : \Delta_1^+ : g(x) \in \delta_j^+\}$ itself also has two disjoint connected components

$$g^{-1}\{x \in \Delta_1^+ : g(x) \in \delta_j^+\} = \delta_{2,j} \cup \{x \in \Delta_2^+ : g^2(x) \in \delta_j^+\}$$

and therefore, again by the invariance of $\tilde{\mu}$, we get

$$\tilde{\mu}(g^{-1}\{x \in \Delta_1^+ : g(x) \in \delta_j^+\}) = \tilde{\mu}(\delta_{2,j}) + \tilde{\mu}(\{x \in \Delta_2^+ : g^2(x) \in \delta_j^+\})$$

and, substituting this into (1.80), we get

$$\tilde{\mu}(\delta_j^+) = \tilde{\mu}(g^{-1}(\delta_j^+)) = \tilde{\mu}(\delta_{1,j}) + \tilde{\mu}(\delta_{2,j}) + \tilde{\mu}(\{x \in \Delta_2^+ : g^2(x) \in \delta_j^+\}).$$

Repeating this procedure n times gives

$$\tilde{\mu}(\delta_j^+) = \sum_{i=1}^{n} \tilde{\mu}(\delta_{i,j}) + \tilde{\mu}(\{x \in \Delta_n^+ : g^n(x) \in \delta_j^+\})$$

and therefore inductively, we obtain (1.77), thus completing the proof. □

Proof of Lemma 1.4.6. From Lemma 1.4.3 we can give precise estimates for $\tilde{\mu}(\tau^{\pm} > t)$ in terms of the $y_{\lceil t \rceil}$ by making use of the fact that $\tilde{h}$ is Lipschitz on $\Delta_0^{\pm}$ (see Corollary 1.2.4). Indeed,

$$\tilde{\mu}(\tau^- > t) = \tilde{\mu}(0, y_{\lceil t \rceil}^+) = \int_0^{y_{\lceil t \rceil}^+} \tilde{h}(x)dx = y_{\lceil t \rceil}^+ \tilde{h}(0^+) + \int_0^{y_{\lceil t \rceil}^+} \tilde{h}(x) - \tilde{h}(0^+)dx.$$

Using the fact that the density is Lipschitz we have

$$\left| \int_0^{y_{\lceil t \rceil}^+} \tilde{h}(x) - \tilde{h}(0^+)dx \right| \lesssim \int_0^{y_{\lceil t \rceil}^+} xdx \lesssim (y_{\lceil t \rceil}^+)^2$$

and so $\tilde{\mu}(\tau^- > t) = y_{\lceil t \rceil}^+ \tilde{h}(0^+) + O((y_{\lceil t \rceil}^+)^2)$. The statement for $\mu(\tau^+ > t)$ follows in the same way. □

Higher order asymptotics

In this subsection we prove Proposition 1.4.2. For clarity we prove (1.71) and (1.72) in two separate lemmas. We will make repeated use of some upper bounds for the measure $\tilde{\mu}(\delta_{i,j})$ of the partition elements which are given in Lemma 1.3.12

Lemma 1.4.7. *If at least one of ℓ_1, ℓ_2 are not zero, then for every $a, b \geq 0$*

$$\tilde{\mu}(a\tau^+ > t, b\tau^- > t) + \tilde{\mu}(a\tau^+ + b\tau^- > t, \max\{a\tau^+, b\tau^-\} \leq t) = o(t^{-\gamma - 1/\beta}) \tag{1.81}$$

for any $\gamma \in (0, 1)$.

Proof. First note that if one of a, b is 0 then (1.81) is automatically satisfied.

Now suppose that $a, b > 0$. For the first term in (1.81), from Lemma 1.3.12 we get

$$\tilde{\mu}(a\tau^+ > t, b\tau^- > t) = \sum_{i=t/a}^{\infty} \sum_{j=t/b}^{\infty} \tilde{\mu}(\delta_{i,j}) \lesssim \sum_{i=t/a}^{\infty} \sum_{j=t/b}^{\infty} (ij)^{-(1+1/\beta)} \lesssim t^{-2/\beta} \tag{1.82}$$

which is $o(t^{-\gamma - 1/\beta})$ for every $\gamma \in (0, 1/\beta)$ and therefore in particular for every $\gamma \in (0, 1)$.

For the second term in (1.81) we obtain from Lemma 1.3.12 that

$$\begin{aligned} \tilde{\mu}(a\tau^+ + b\tau^- > t, \max\{a\tau^+, b\tau^-\} \leq t) &= \sum_{i=1}^{t/a} \sum_{j=\frac{t-ai+1}{b}}^{t/b} \tilde{\mu}(\delta_{i,j}) \lesssim \sum_{i=1}^{t/a} \sum_{j=\frac{t-ai+1}{b}}^{t/b} i^{-1-1/\beta} j^{-1-1/\beta} \qquad (1.83) \\ &\lesssim \sum_{i=1}^{t/a} i^{-1-1/\beta} \left[\left(\frac{t - ai + 1}{b} \right)^{-1/\beta} - \left(\frac{t}{b} \right)^{-1/\beta} \right] \\ &\lesssim t^{-1/\beta} \sum_{i=1}^{t/a} i^{-1-1/\beta} \left[\left(1 - \frac{ai - 1}{t} \right)^{1/\beta} - 1 \right] \end{aligned}$$

Making the change of variables $k = \lceil ai - 1 \rceil$ and using that the first term in the sum is 0 we obtain

$$\tilde{\mu}(a\tau^+ + b\tau^- > t, \max\{a\tau^+, b\tau^-\} \le t) \lesssim t^{-1/\beta} \sum_{k=1}^{t-1} k^{-1-1/\beta} \left[\left(1 + \frac{k}{t}\right)^{1/\beta} - 1 \right].$$

Let us set $a_k(t) := k^{-1-1/\beta} \left[(1 + \frac{k}{t})^{1/\beta} - 1 \right]$ and use the binomial theorem to get

$$a_k(t) = \frac{1}{k^{1+1/\beta}} \sum_{m=1}^{\infty} \binom{-1/\beta}{m} \left(-\frac{k}{t}\right)^m = \frac{1}{tk^{1/\beta}} \sum_{m=1}^{\infty} \binom{-1/\beta}{m-1} \left(\frac{1}{m} \left(\frac{1}{\beta} - 1 \right) + 1 \right) \left(-\frac{k}{t}\right)^{m-1}.$$

As $\left(\frac{1}{\beta} - 1\right)/m$ is uniformly bounded above by some constant depending only on β we obtain

$$a_k(t) \lesssim k^{-1/\beta} t^{-1} \sum_{k=0}^{\infty} \binom{-1/\beta}{m-1} \left(-\frac{k}{t}\right)^{m-1} = k^{-1/\beta} t^{-1} \left(1 - \frac{k}{t}\right)^{-1/\beta}.$$

Using the fact that $n/(n-1) < 2$ and that $1/\beta > 1$ we may conclude

$$\begin{aligned} \tilde{\mu}(a\tau^+ + b\tau^- > t, \max\{a\tau^+, b\tau^-\} \le t) &\lesssim t^{-1-1/\beta} \sum_{k=1}^{t-1} \left(\frac{t}{k(t-k)} \right)^{1/\beta} \lesssim t^{-1-1/\beta} \sum_{k=1}^{t-1} \frac{t}{k(t-k)} \\ &\lesssim t^{-1-1/\beta} \int_1^{t-1} \frac{t}{x(t-x)} dx \lesssim t^{-1-1/\beta} \log(t) = o(t^{-\gamma-1/\beta}) \end{aligned}$$

for any $\gamma \in (0,1)$. □

Lemma 1.4.8. *If ℓ_1, ℓ_2 are not both zero, then for every $a, b \ge 0$, $\gamma \in (0, 1/\beta)$ we have*

$$\tilde{\mu}(a\tau^+ > t, a\tau^+ - b\tau^- \le t) = o(t^{-\gamma-1/\beta}).$$

Proof. By Lemma 1.3.12 we get

$$\begin{aligned} \tilde{\mu}(a\tau^+ > t, a\tau^+ - b\tau^- \le t) &= \sum_{i > t/a} \tilde{\mu}(\tau^+ = i, b\tau^- \ge ai - t) = \sum_{i>t/a} \sum_{j \ge (ai-t)/b} \tilde{\mu}(\delta_{i,j}) \\ &\lesssim \sum_{i>t/a} i^{-(1+1/\beta)} (ai - t)^{-1/\beta} \lesssim \sum_{i=1}^{\infty} (i+t)^{-(1+1/\beta)} i^{-1/\beta}. \end{aligned}$$

We claim that

$$\sum_{i=1} (t+i)^{-1-1/\beta} i^{-1/\beta} \lesssim t^{-\gamma-1/\beta}$$

for every $0 < \gamma < 1/\beta$, which is equivalent to showing that

$$\sum_{i=1} \frac{t^{\gamma+1/\beta}}{(t+i)^{1+1/\beta} i^{1/\beta}} \le C$$

for some $C > 0$ independent of t. Indeed, for every i,

$$\frac{t^{\gamma+1/\beta}}{(t+i)^{1+1/\beta} i^{1/\beta}} \le \frac{(t+i)^{\gamma+1/\beta}}{(t+i)^{1+1/\beta} i^{1/\beta}} = \frac{1}{(t+i)^{1-\gamma} i^{1/\beta}} = \frac{1}{(t/i+1)^{1-\gamma} i^{1-\gamma+1/\beta}} \le \frac{1}{i^{1-\gamma+1/\beta}}$$

which is summable for every $0 < \gamma < 1/\beta$. This implies the claim and thus the lemma. □

1.4.2 Estimates for the induced observables

In this section we prove Corollary 1.2.11 and Proposition 1.2.10. We recall (see paragraph at the beginning of Section 1.2.3) that we will only explicitly treat the case that $\ell_1, \ell_2 > 0$ (and thus in particular $\beta > 0$). Throughout this section we will assume that φ is a Hölder obervable and define $a = \varphi(1)$, $b = \varphi(-1)$.

Proof of Corollary 1.2.11

We first consider the case where $\beta_\varphi = \beta$. Recall from (1.14) that $\beta_\varphi = \beta$ occurs when φ is non-zero at a fixed point corresponding to the maximum of β_1, β_2; and that $\beta_\varphi \neq \beta$ occurs when $\beta_1 \neq \beta_2$ and φ is zero at the fixed point corresponding to the minimum of β_1, β_2.

Lemma 1.4.9. *If $\beta_\varphi = \beta$, then $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$. In particular, if $\beta_\varphi \in (0, 1/2)$ then $\tau_{a,b} \in L^2(\hat{\mu})$.*

Proof. Notice first of all that if $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$ then, in particular, $\tilde{\mu}(\pm\tau_{a,b} > t) \lesssim t^{-1/\beta_\varphi}$ and so, if $\beta_\varphi \in (0, 1/2)$ we obtain that $\tau_{a,b} \in L^2(\hat{\mu})$. Thus we just need to prove that $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$.

Suppose first that a, b do not have opposite signs, i.e. either $a, b \geq 0$ or $a, b \leq 0$, in which case the distribution of $\tau_{a,b}$ is determined by the first case of (1.27). Therefore, taking $\gamma = 0$ we get

$$\tilde{\mu}(\tau_{a,b} > t) = C_a t^{-1/\beta_2} + C_b t^{-1/\beta_1} + o(t^{-1/\beta}) = ct^{-1/\beta} + o(t^{-1/\beta})$$

for some constant $c > 0$, which may be equal to C_a, C_b, or $C_a + C_b$, depending on the relative values of β_1, β_2. If $a, b \geq 0$, and exactly the same tail for $\tilde{\mu}(\tau_{a,b} < -t)$ if $a, b \leq 0$. By (1.30) and the fact that $\beta_\varphi = \beta$ we get that $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$, thus proving the result in this case. If $a \geq 0$, $b \leq 0$ the distribution of $\tau_{a,b}$ is given by (1.28) and (1.29) and so, taking $\gamma = 0$ gives

$$\tilde{\mu}(\tau_{a,b} > t) = C_a t^{-1/\beta_2} + o(t^{-1/\beta}) = c_1 t^{-1/\beta} + o(t^{-1/\beta})$$

and

$$\tilde{\mu}(\tau_{a,b} < -t) = C_{|b|} t^{-1/\beta_1} + o(t^{-1/\beta}) = c_2 t^{-1/\beta} + o(t^{-1/\beta})$$

where $c_1 = C_a$ and $c_2 = C_{|b|}$ if $\beta_2 = \beta$ and $\beta_1 = \beta$ respectively, and equal to 0 otherwise. At least one of the c_1, c_2 has to be non-zero as $\beta_\varphi = \beta$ implies that φ is non-zero at a fixed point corresponding to the largest of β_1, β_2 and so if $\beta_1 = \max\{\beta_1, \beta_2\}$ we know from (1.26) that $c_2 = C_{|b|} > 0$ and if $\beta_2 = \max\{\beta_1, \beta_2\}$ we know from (1.26) that $c_1 = C_a > 0$. Thus, since $\beta = \beta_\varphi$, we get $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$. If $a \leq 0$, $b \geq 0$ the same argument holds exchanging the roles of the positive and negative tails. ☐

Proof of Corollary 1.2.11. We have already proved the result for $\beta_\varphi = \beta$ in Lemma 1.4.9 so we can assume that $\beta_\varphi \neq \beta$. This implies that $\beta_1 \neq \beta_2$ and that φ is only non-zero at the fixed point corresponding to the smallest of the β_1, β_2. This situation can arise in two ways: either (i) $a \neq 0$, $b = 0$ and $\beta_\varphi = \beta_2 < \beta_1$; or (ii) $a = 0$, $b \neq 0$ and $\beta_\varphi = \beta_1 < \beta_2$. We will assume (i) and give an explcit proof of the Lemma. The proof of the Lemma in situation (ii) then follows in the same way.

Under our assumptions we know from Proposition 1.2.6 that the tail of $\tau_{a,b}$ is determined by (1.27), and we recall from (1.26) that $C_b = 0$. If $\beta_\varphi = \beta_2 = 0$ then we know from the second line of (1.27) that

$$\hat{\mu}(\pm\tau_{a,b} > t) \lesssim t^{-\gamma - 1/\beta_1}.$$

Since $\beta = \beta_1 < 1$ by assumption, we may choose $\gamma \in [0, 1)$ such that $\gamma + 1/\beta_1 > 2$ yielding $\tau_{a,b} \in L^2(\hat{\mu})$. If $\beta_\varphi \in (0, 1/2)$ then the first line of (1.27) gives that

$$\hat{\mu}(\pm\tau_{a,b} > t) \lesssim \max\{t^{-1/\beta_\varphi}, t^{-\gamma - 1/\beta}\}$$

for any $\gamma \in [0,1)$. Choosing γ as before so that $\gamma + 1/\beta_1 > 2$ we again obtain that $\tau_{a,b} \in L^2(\hat{\mu})$. If $\beta_\varphi = \beta_2 \in [1/2, 1)$ then, choosing $\gamma \in [0,1)$ so that $\gamma + 1/\beta_1 > 1/\beta_2$ we know from (1.27) that the non-zero tail of $\tau_{a,b}$ is given by

$$\hat{\mu}(\pm\tau_{a,b} > t) \lesssim C_a t^{-1/\beta_2} + o(t^{-\gamma-1/\beta_1}) = C_a t^{-1/\beta_2} + o(t^{-1/\beta_2})$$

yielding $\tau_{a,b} \in \mathcal{D}_{1/\beta_\varphi}$. □

Proof of Proposition 1.2.10

Proof of Proposition 1.2.10. For a point $x \in \delta^-_{i,j}$ we know that $\tau(x) = i + j$ and that

$$g^k(x) \in \Delta^+_{i-k} \quad \forall\, 1 \le k \le i; \qquad \text{and} \qquad g^{i+k}(x) \in \Delta^-_{j-k} \quad \forall\, 1 \le k \le j-1. \tag{1.84}$$

Recall that by Proposition 1.3.4 we have $1 - x^+_n \lesssim n^{-1/\ell_2}$, and $|\Delta^+_n| \lesssim n^{-(1+1/\ell_2)} \ll 1 - x^+_n$, which means that we can use the fact that $\tilde{\varphi}(1) = 0$ and the Hölder continuity of $\tilde{\varphi}_{(0,1]}$ to obtain

$$|\tilde{\varphi} \circ g^k(x)| \lesssim (1 - x^+_{i-k})^{\nu_2} \le (i-k)^{-\nu_2/\ell_2}, \tag{1.85}$$

for all $1 \le k \le i-1$. Similarly, using the fact that $\tilde{\varphi}(-1) = 0$ and the Hölder continuity of $\tilde{\varphi}_{[-1,0)}$,

$$|\tilde{\varphi} \circ g^{i+k}(x)| \lesssim (1 + x^+_{j-k})^{\nu_1} \le (j-k)^{-\nu_1/\ell_1}, \tag{1.86}$$

for all for $1 \le k \le j-1$. For $x \in \delta_{i,j}$ we know from (1.85) and (1.86) that

$$|\tilde{\Phi}(x)| \lesssim \sum_{k=1}^{i-1} (i-k)^{-\frac{\nu_2}{\ell_2}} + \sum_{k=1}^{j-1} (j-k)^{-\frac{\nu_1}{\ell_1}}. \tag{1.87}$$

We now consider two cases. Suppose first that $\ell_1 < \nu_1$ and $\ell_2 < \nu_2$. Then $|\tilde{\Phi}(x)|$ is uniformly bounded in x, as both (1.85) and (1.86) are summable in k and therefore the sums in (1.87) both converge. Therefore $\tilde{\Phi} \in L^q(\hat{\mu})$ for every $q > 0$, in particular $\tilde{\Phi} \in L^2(\hat{\mu})$ giving the first implication in (1.36), and by Chebyshev's inequality, $\hat{\mu}(\pm\tilde{\Phi} > t) = O(t^{-q})$ for every $q > 0$, giving the second implication in (1.36). Notice that we have not required in this case the conditions $(\mathcal{H})$ and $(\mathcal{H}')$.

Now suppose that $\ell_1 \ge \nu_1$ and/or $\ell_2 \ge \nu_2$ and suppose also that

$$\nu_1 > \frac{\beta_1 - 1/q}{k_2} \qquad \text{and} \qquad \nu_2 > \frac{\beta_2 - 1/q}{k_1}. \tag{1.88}$$

Notice that for $q = 2$ this gives exactly condition $(\mathcal{H})$ and for $q = 1/\beta_\varphi$ this gives exactly $(\mathcal{H}')$. We can also suppose without loss of generality that in fact $\ell_1 > \nu_1$ and/or $\ell_2 > \nu_2$ since we can decrease slightly the Hölder exponent while still satisfying (1.88). In this case the sums in (1.87) diverge but admit the following bounds:

$$|\tilde{\Phi}(x)| \lesssim \sum_{k=1}^{i-1} (i-k)^{-\frac{\nu_2}{\ell_2}} + \sum_{k=1}^{j-1} (j-k)^{-\frac{\nu_1}{\ell_1}} \lesssim i^{1-\frac{\nu_2}{\ell_2}} + j^{1-\frac{\nu_1}{\ell_1}}.$$

We can then bound the integral by

$$\int_{\Delta_0^-} |\tilde{\Phi}(x)|^q dm \lesssim \left[\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\delta_{i,j}| \left(i^{1-\frac{\nu_2}{\ell_2}} + j^{1-\frac{\nu_1}{\ell_1}} \right)^q \right]$$

Then, since $|\delta_{i,j}| = O(i^{-(1+1/\beta_2)} j^{-(1+1/\beta_1)})$ we get

$$\int_{\Delta_0^-} |\tilde{\Phi}(x)|^q dm \lesssim \left[\sum_{i,j=1}^{\infty} \frac{i^{q-\frac{q\nu_2}{\ell_2}}}{i^{1+\frac{1}{\beta_2}} j^{1+\frac{1}{\beta_1}}} + \sum_{i,j=1}^{\infty} \frac{j^{q-\frac{q\nu_1}{\ell_1}}}{i^{1+\frac{1}{\beta_2}} j^{1+\frac{1}{\beta_1}}} \right] \lesssim \sum_{i=1}^{\infty} i^{q-\frac{q\nu_2}{\ell_2}-1-\frac{1}{\beta_2}} + \sum_{j=1}^{\infty} j^{q-\frac{q\nu_1}{\ell_1}-1-\frac{1}{\beta_1}}$$

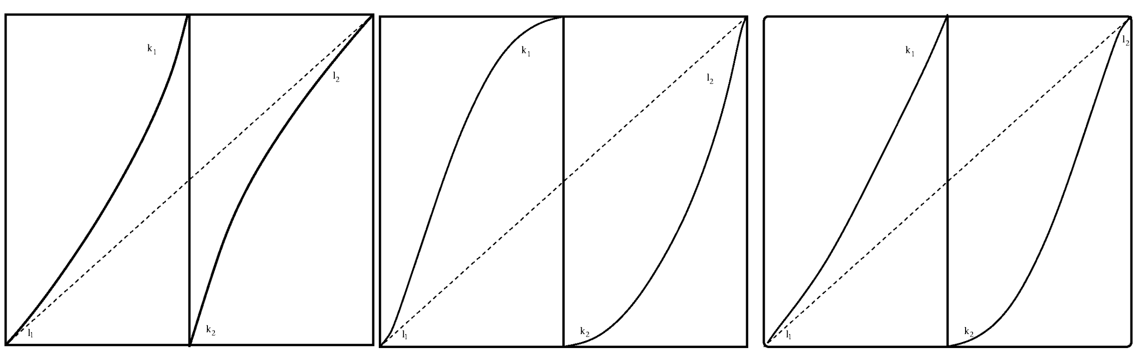
The latter sums are bounded exactly when (1.88) holds. As mentioned above, for $q = 2$ this is exactly condition ($\mathcal{H}$) and therefore we get that $\tilde{\Phi} \in L^2(\hat{\mu})$. For $q = 1/\beta_\varphi$ this is exactly condition ($\mathcal{H}'$) and therefore we get that $\tilde{\Phi} \in L^q(\hat{\mu})$. In fact if (1.88) holds for $q = 1/\beta_\varphi$ then there exists some $\varepsilon > 0$ such that (1.88) holds for all $q \in [1/\beta_\varphi, 1/\beta_\varphi + \varepsilon)$ and therefore $\tilde{\Phi} \in L^q(\hat{\mu})$ for every $q \in [1/\beta_\varphi, 1/\beta_\varphi + \varepsilon)$. From this and Chebyschev's inequality we get $\hat{\mu}(\pm\tilde{\Phi} > t) = o(t^{-1/\beta_\varphi})$. □

Doubly Intermittent Maps with Regularly Varying Tail

2.1 Introduction and Statement of Results

In this chapter, we combined results in [CLM22] and [Hol05]. We extend the definition of a subclass of the maps introduced in [CLM22] to a more robust class by replacing some of the parameters they studied with slowly varying functions. The implication is that we require different technique (Karamata Theory [Bin+89]) to obtain similar estimates used in [CLM22]. In particular, when $\beta = 1$ which is a boundary case in [CLM22] where acip measure cease to exist, introducing the slowly varying functions near ± 1 creates a spectrum of new parameters within $\beta = 1$ with a more subtle boundary base on the slowly varying functions. Under some mild assumptions we prove existence of a unique invariant absolutely continuous probability measure.

[Hol05] studied a similar kind of maps by extending definition of the intermittency maps and prove existence of acip measure and mixing properties for a boundary parameter of the intermittent maps. See [LSV99; You99] for statistical and mixing results of intermittent maps.



(a) (b) (c)

Figure 2.1: Graph of g for various possible values of parameters.

2.1.1 Maps with Regularly Varying Tails

First, we define the class of maps with two intermittent fixed points, singularity and regularly varying tail which is to be studied in this paper. Let I, I_-, I_+ be compact intervals, let $\mathring{I}, \mathring{I}_-, \mathring{I}_+$ be their interiors and suppose $\mathring{I}_- \cap \mathring{I}_+ = \emptyset$ and $I = I_- \cup I_+$. For brevity we assume that $I = [-1, 1]$, $I_- = [-1, 0]$, $I_+ = [0, 1]$.

Furthermore, we assume there exists $g : I \to I$ satisfying the following properties:

(A0) $g : I \to I$ is *full branch* such that $g_- = g|_{\mathring{I}_-} : \mathring{I}_- \to \mathring{I}$ and $g_+ = g|_{\mathring{I}_+} : \mathring{I}_+ \to \mathring{I}$ are orientation preserving C^2 diffeomorphisms with only $\{-1, 1\}$ as fixed points.

(A1) There exist constants $\ell_1, \ell_2, \iota, k_1, k_2 > 0$ and functions

$$\mathcal{L}_{-1} : U_{-1} + 1 \to [0, \infty) \quad \text{and} \quad \mathcal{L}_1 : U_{+1} - 1 \to [0, \infty)$$

are both slowly varying at 0 such that

(i)

$$g(x) = \begin{cases} x + (1+x)^{1+\ell_1} \mathcal{L}_{-1}(1+x) & \text{in } U_{-1}, \\ 1 - a_1 |x|^{k_1} & \text{in } U_{0-}, \\ -1 + a_2 x^{k_2} & \text{in } U_{0+}, \\ x - (1-x)^{1+\ell_1} \mathcal{L}_1(1-x) & \text{in } U_{+1}, \end{cases} \tag{2.1}$$

where

$$U_{0-} := (-\iota, 0], \quad U_{0+} := [0, \iota), \quad U_{-1} := g(U_{0+}), \quad U_{+1} := g(U_{0-}). \tag{2.2}$$

(ii) $\mathcal{L}_{-1}, \mathcal{L}_1$ are positive constants or

$$\lim_{x \to -1} \mathcal{L}_{-1}(1+x) = \infty, \qquad \lim_{x \to +1} \mathcal{L}_1(1-x) = \infty, \tag{2.3}$$

$$\mathcal{L}'_{-1} > 0 \quad \text{and} \quad \mathcal{L}'_1 > 0. \tag{2.4}$$

For some $M > 0$ the function $\mathcal{L}_{-1}(1/x)$ and $\mathcal{L}_1(1/x)$ are choosen to be C^ω (real analytic) on $[M, \infty)$.

Notice that when $\mathcal{L}_{-1}, \mathcal{L}_1$ are positive constants, (2.1) corresponds to maps in [CLM22] with $\mathcal{L}_{-1} = b_1$ and $\mathcal{L}_1 = b_2$. Thus, we will restrict to the case where $\mathcal{L}_{-1}, \mathcal{L}_1$ satisfy (2.3).

Our final assumption can be intuitively thought of as saying that g is uniformly expanding outside the neighbourhoods $U_{0\pm}$ and $U_{\pm 1}$. This is however much stronger than what is needed, and therefore we formulate a weaker and more general assumption for which we need to describe some aspects of the topological structure of maps satisfying condition **(A0)**. First of all we define

$$\Delta_0^- := g^{-1}(0, 1) \cap I_- \quad \text{and} \quad \Delta_0^+ := g^{-1}(-1, 0) \cap I_+. \tag{2.5}$$

Then we define iteratively, for every $n \geq 1$, the sets

$$\Delta_n^- := g^{-1}(\Delta_{n-1}^-) \cap I_- \quad \text{and} \quad \Delta_n^+ := g^{-1}(\Delta_{n-1}^+) \cap I_+ \tag{2.6}$$

as the n'th preimages of Δ_0^-, Δ_0^+ inside the intervals I_-, I_+. It follows from **(A0)** that $\{\Delta_n^-\}_{n \geq 0}$ and $\{\Delta_n^+\}_{n \geq 0}$ are mod 0 partitions of I_- and I_+ respectively, and that the partition elements depend *monotonically* on the index in the sense that $n > m$ implies that $\Delta_n^\pm$ is closer to ± 1 than $\Delta_m^\pm$, in particular the only accumulation points of these partitions are -1 and 1 respectively. Then, for every $n \geq 1$, we let

$$\delta_n^- := g^{-1}(\Delta_{n-1}^+) \cap \Delta_0^- \quad \text{and} \quad \delta_n^+ := g^{-1}(\Delta_{n-1}^-) \cap \Delta_0^+. \tag{2.7}$$

Notice that $\{\delta_n^-\}_{n \geq 1}$ and $\{\delta_n^+\}_{n \geq 1}$ are mod 0 partitions of Δ_0^- and Δ_0^+ respectively and also in these cases the partition elements depend monotonically on the index in the sense that $n > m$

implies that $\delta_n^\pm$ is closer to 0 than $\delta_m^\pm$, (and in particular the only accumulation point of these partitions is 0). Notice moreover, that

$$g^n(\delta_n^-) = \Delta_0^+ \quad \text{and} \quad g^n(\delta_n^+) = \Delta_0^-.$$

We now define two non-negative integers $n_\pm$ which depend on the positions of the partition elements $\delta_n^\pm$ and on the sizes of the neighbourhoods $U_{0\pm}$ on which the map g is explicitly defined. If $\Delta_0^- \subseteq U_{0-}$ and/or $\Delta_0^+ \subseteq U_{0+}$, we define $n_- = 0$ and/or $n_+ = 0$ respectively, otherwise we let

$$n_+ := \min\{n : \delta_n^+ \subset U_{0+}\} \quad \text{and} \quad n_- := \min\{n : \delta_n^- \subset U_{0-}\}. \tag{2.8}$$

We can now formulate our final assumption as follows.

(A2) There exists a $\lambda > 1$ such that for all $1 \leq n \leq n_\pm$ and for all $x \in \delta_n^\pm$ we have $(g^n)'(x) > \lambda$.

Let

$$\widehat{\mathfrak{G}} := \{g : I \to I \text{ satisfying } \textbf{(A0)} - \textbf{(A2)}\}.$$

We remark that $\widehat{\mathfrak{F}} \cap \{\ell_1, \ell_2 > 0\} \subset \widehat{\mathfrak{G}}$ where $\widehat{\mathfrak{F}}$ is the class of maps introduced in [CLM22].

For later use we mention some properties. For $\ell_1 > 0$, we have by Proposition 1.5.8 in [Bin+89] that

$$g'|_{U_{-1}}(x) \sim 1+(1+\ell_1)(1+x)^{\ell_1}\mathcal{L}_{-1}(1+x) \quad \text{and} \quad g''|_{U_{-1}}(x) \sim (1+\ell_1)\ell_1(1+x)^{\ell_1-1}\mathcal{L}_{-1}(1+x), \tag{2.9}$$

for x tending to -1. We remark that although Proposition 1.5.8 in [Bin+89] is stated for regularly varying functions at infinity, we can still use it. See Remark 2.5.8.

By the precise form of (2.1), we have $g'(-1) = 1$ and thus the fixed point -1 is a *neutral* fixed point. Similarly, when $\ell_2 > 0$ the fixed point 1 is a neutral fixed point. Notice that changing the parameter values k_1, k_2 gives rise to maps with quite different characteristics. For all $k_1 > 0$ we have

$$g'|_{U_{0-}}(x) = a_1 k_1 |x|^{k_1-1} \quad \text{and} \quad g''|_{U_{0-}}(x) = a_1 k_1(k_1-1)|x|^{k_1-2}. \tag{2.10}$$

So $k_1 \in (0,1)$ implies that $|g'|_{U_{0-}}(x)| \to \infty$ as $x \to 0$, thus $g|_{U_{0-}}$ has a *singularity* at 0 (one-sided), while $k_1 > 1$ implies that $|g'|_{U_{0-}}(x)| \to 0$ as $x \to 0$, thus we say that $g|_{U_{0-}}$ has a *critical point* at 0 (one-sided). Analogous observations hold for the various values of ℓ_2 and k_2 and Figure 2.1 shows the graph of g for various combinations of these exponents.

For future reference we mention additional properties which follow from **(A1)**. Observe that if $\ell_1 \in (0,1]$ we have $g''(x) \to \infty$ but if $\ell_1 > 1$ we have $g''(x) \to 0$, as $x \to -1$ and, as we shall see, this qualitative difference in the higher order derivative plays a crucial role in the ergodic properties of g. Here, we notice that the behaviour for $\ell_1 = 1$ is determined by the slowly varying function $\mathcal{L}_{-1}$. Similar observations holds for $g|_{U_1}$ when $\ell_2 > 0$. Also observe that for every $x \in U_{-1}$ we have

$$g''(x)/g'(x) \lesssim (1+x)^{\ell_1-1}\mathcal{L}_{-1}(1+x) \tag{2.11}$$

and for every $x \in U_{0+}$

$$|g''(x)|/|g'(x)| \lesssim x^{-1}. \tag{2.12}$$

Here and in the following we denote by $a(x) \lesssim b(x)$ that there exists a constant C such that $a(x) \leq Cb(x)$ for all x in the respective domain.

2.1.2 Statement of Results

Our first result is completely general and applies to all maps in $\widehat{\mathfrak{G}}$.

Theorem A. Every $g \in \widehat{\mathfrak{G}}$ admits a unique (up to scaling by a constant) invariant σ-finite measure which is equivalent to Lebesgue.

Notice that this result extents Theorem A in [CLM22] for $g \in \widehat{\mathfrak{F}} \cap \{\ell_1, \ell_2 > 0\}$. By our construction we have that the density with respect to Lebesgue of the measure given in Theorem A is locally Lipschitz and unbounded only at the endpoints ± 1. Depending on the slowly varying functions $\mathcal{L}_{-1}$ and $\mathcal{L}_1$, we will see that the density may or may not be integrable and thus the measure may or may not be finite. More specifically, let

$$\beta_1 := k_2\ell_1, \quad \beta_2 := k_1\ell_2, \quad \text{and} \quad \beta := \max\{\beta_1, \beta_2\}.$$

For every slowly varying function monotone on $[M, \infty)$ there is another slowly varying function $\mathcal{L}^{\#}$ with properties given in Lemma 2.5.4. We call $(\mathcal{L}, \mathcal{L}^{\#})$ a *de Bruijn conjugate pair.*
Let

$$\chi_n^1 := n^{-\frac{1}{\beta_2}} (\mathcal{L}_1^{\#}(n^{-\frac{1}{\ell_2}}))^{-\frac{1}{k_1}}, \qquad \chi_n^2 := n^{-\frac{1}{\beta_1}} (\mathcal{L}_{-1}^{\#}(n^{-\frac{1}{\ell_1}}))^{-\frac{1}{k_2}}, \tag{2.13}$$

$$\mathfrak{G} := \left\{ g \in \widehat{\mathfrak{G}} : \max\left\{ \sum_{n=1}^{\infty} \chi_n^1, \sum_{n=1}^{\infty} \chi_n^2 \right\} < \infty \right\}.$$

Observe that $\mathfrak{G}$ contains all $g \in \widehat{\mathfrak{G}}$ with $\beta < 1$. In addition it contains also functions g where $\beta_1 = 1$ or $\beta_2 = 1$. In this chase the growth rate of $\mathcal{L}_1^{\#}$ (or $\mathcal{L}_{-1}^{\#}$) decide if $\sum_{n=1}^{\infty} \chi_n^1 < \infty$ (or $\sum_{n=1}^{\infty} \chi_n^2 < \infty$ respectively) or not. Notice that when $\mathcal{L}_{-1}$ is a positive constant, we have $\chi_n^2 \lesssim n^{-\frac{1}{\beta_1}}$ and the condition is the same as in [CLM22].

Theorem B. A map $g \in \widehat{\mathfrak{G}}$ admits a unique ergodic invariant *probability* measure μ_g equivalent to Lebesgue *if and only if* $g \in \mathfrak{G}$.

Notice that of particular interest is the case when $\beta = 1$ which is a boundary case in [CLM22] where acip measure cease to exist. However, introducing the slowly varying functions $\mathcal{L}_{-1}, \mathcal{L}_1$ creates a spectrum of new parameters within $\beta = 1$ with a more suble boundary base on the slowly varying functions. Observe also that the condition $\beta = 1$ is a restriction only on the *relative* values of k_1 with respect to ℓ_2 and of k_2 with respect to ℓ_1. It still allows k_1 and/or k_2 to be *arbitrarily large*, therefore letting more "degenerate" critical points, so far the corresponding exponents ℓ_2 and/or ℓ_1 are significantly small, i.e. so far the corresponding neutral fixed points are not so degenerate. Furhermore, we notice that the condition on $\mathcal{L}_{-1}, \mathcal{L}_1$ changes according to the precise choice of k_1, k_2, ℓ_1, ℓ_2.

2.2 Overview of the proof

We expain our overall strategy and prove our theorems. Some key technical propositions will however be proved in later part of the paper.

2.2.1 The induced map

In this section, we construct the first return map G and show that it is a full branch map with infinitely many branches, prove asymptotic estimates related to the construction, finally show that G is uniformly expanding and has bounded distortion.
Define the first return time

$$\tau : \Delta_0^- \to \mathbb{N} \quad \text{by} \quad \tau(x) := \min\{n > 0 : g^n(x) \in \Delta_0^-\}$$

and the *first-return induced map*

$$G : \Delta_0^- \to \Delta_0^- \quad \text{by} \quad G(x) := g^{\tau(x)}(x). \tag{2.14}$$

An induced map is said to saturate the interval I if

$$\bigcup_{n\geq 0} \bigcup_{i=0}^{n-1} g^i(\{\tau = n\}) = \bigcup_{n\geq 0} g^n(\{\tau > n\}) = I \pmod 0. \tag{2.15}$$

Intuitively, saturation means that the return map "reaches" every part of the original domain of the map g, and thus the properties and characteristics of the return map reflect, to some extent, all the relevant characteristics of g.

Remark 2.2.1. If G is a first return induced map, as in our case, then each pair of sets of the form $g^i(\{\tau = n\})$, $i = 0, \dots, n-1$ either coincides or is disjoint and therefore those sets form a partition of I mod 0.

The following proposition will be a main result leading to the proof of Theorem A and Theorem B.

Proposition 2.2.2. *Let $g \in \widehat{\mathfrak{G}}$. Then $G : \Delta_0^- \to \Delta_0^-$ given as in (2.14) is a first return induced Gibbs-Markov map which saturates I.*

We give the definition of Gibbs-Markov map and prove Proposition 2.2.2 in Section 2.3. In Section 2.3.1 we describe the topological structure of G and show that it is a full branch map with countably many branches which saturates I (we will define G as a composition of two full branch maps, see (2.18) and (2.19), which is why we call the construction a double inducing procedure); in Section 2.3.2 we obtain key estimates concerning the sizes of the partition elements of the corresponding partition; in Section 2.3.4 we show that G is uniformly expanding; in Section 2.3.3 we show that G has bounded distortion. From these results we get Proposition 2.2.2 from which we can then obtain our first main Theorem A.

Proof of Theorem A. Let $g \in \widehat{\mathfrak{G}}$. By proposition 2.2.2, G is Gibbs-Markov map which saturates I. Together with Theorem 3.13 in [Alv20] we have that G admits a unique ergodic invariant probability measure $\hat{\mu}_-$, supported on Δ_0^-, which is equivalent to Lebesgue measure m and which has Lipschitz continuous density $\hat{h}_- = d\hat{\mu}_-/dm$ bounded above and below.
We then "spread" the measure over the original interval I by defining the measure

$$\tilde{\mu} := \sum_{n=0}^{\infty} g_*^n(\hat{\mu}_-|\{\tau \geq n\}) \tag{2.16}$$

where $g_*^n(\hat{\mu}_-|\{\tau \geq n\})(E) := \hat{\mu}_-(g^{-n}(E) \cap \{\tau \geq n\})$. By Theorem 3.18 in [Alv20], we have that $\tilde{\mu}$ is a sigma-finite measure which is ergodic and invariant for g and absolutely continuous with respect to Lebesgue. The fact that G saturates I implies moreover that $\tilde{\mu}$ is equivalent to Lebesgue, which completes the proof. □

Remark 2.2.3. We can define the first return map $G_+ : \Delta_0^+ \to \Delta_0^+$ in a completely analogous way to the definition of G above. Moreover, the conclusions of Proposition 2.2.2 hold for G_+ and thus G_+ admits a unique ergodic invariant probability measure $\hat{\mu}_+$ which is equivalent to Lebesgue measure m and such that the density $\hat{h}_+ := d\hat{\mu}_+/dm$ is Lipschitz continuous and bounded above and below. The two maps G and G_+ are clearly distinct, as are the measures $\hat{\mu}_-$ and $\hat{\mu}_+$, but exhibit a subtle kind of symmetry in the sense that the corresponding measure $\tilde{\mu}$ obtained by substituting $\hat{\mu}_-$ by $\hat{\mu}_+$ in (2.16) is, up to a constant scaling factor, exactly the same measure.

Corollary 2.2.4. *Let $g \in \widehat{\mathfrak{G}}$. The density $\tilde{h}$ of $\tilde{\mu}|_{\Delta_0^- \cup \Delta_0^+}$ is Lipschitz continuous and bounded such that $\tilde{\mu}|_{\Delta_0^-} = \hat{\mu}_-$.*

Proof. The proof follows in exactly the same manner as Corollary 2.4 [CLM22]. □

Remark 2.2.5. We have used above the notation G rather than G_- for simplicity as this is the map which plays a more central role in our construction. Similarly, we will from now on simply use the notation $\hat{\mu}$ to denote the measure $\hat{\mu}_-$.

Proposition 2.2.6. *Let $g \in \widehat{\mathfrak{G}}$. Then there are constants $C_1, C_2 > 0$ such that*

$$\tilde{\mu}(\tau > n) \sim C_1 \chi_n^1 + C_2 \chi_n^2$$

with χ^1, χ^2 defined in (2.13).

We prove Proposition 2.2.6 in Section 2.4. We now use it to prove Theorem B.

Proof of Theorem B. Let $g \in \widehat{\mathfrak{G}}$. As $g^{-n}(I) = I$, we have by definition of $\tilde{\mu}$ (2.16) that

$$\tilde{\mu}(I) := \sum_{n=0}^{\infty} \hat{\mu}_-(g^{-n}(I) \cap \{\tau > n\}) = \sum_{n=0}^{\infty} \hat{\mu}_-(I \cap \{\tau > n\}) = \sum_{n=0}^{\infty} \hat{\mu}_-(\tau > n).$$

Together with Proposition 2.2.6 we have $\tilde{\mu}(I) \sim C_1 \sum_{n=0}^{\infty} \chi_n^1 + C_2 \sum_{n=0}^{\infty} \chi_n^2$ which is bounded if and only if $g \in \mathfrak{G}$. We now define the required measure by $\mu_g := \tilde{\mu}/\tilde{\mu}(I)$. □

2.3 The Induced Map

In this section we prove Proposition 2.2.2. We begin by recalling one of several essentially equivalent definitions of Gibbs-Markov maps.

First, we define the *separation time* for any $x, y \in I$ to be zero if x, y lie in different elements of $\mathscr{P}$ and

$$s(x, y) := \inf\{n \geq 0 : G^n x \text{ and } G^n y \text{ lie in different elements of the partition } \mathscr{P}\}. \tag{2.17}$$

With this we are able to give the definition of Gibbs-Markov maps.

Definition 2.3.1. An interval map $F : I \to I$ is called a (full branch) Gibbs-Markov map if there exists a partition $\mathcal{P}$ of I (mod 0) into open subintervals such that:

1. F is *full branch*: for all $\omega \in \mathcal{P}$ the restriction $F|_\omega : \omega \to int(I)$ is a C^1 diffeomorphism;
2. F is *uniformly expanding*: there exists $\lambda > 1$ such that $|F'(x)| \geq \lambda$ for all $x \in \omega$ for all $\omega \in \mathcal{P}$;

3. F has *bounded distortion*: there exists $C > 0, \theta \in (0,1)$ s.t. for all $\omega \in \mathcal{P}$ and all $x, y \in \omega$,

$$\log \left| \frac{F'(x)}{F'(y)} \right| \le C\theta^{s(x,y)}.$$

We will show that the first return map G defined in (2.14) satisfies all the conditions above as well as the saturation condition (2.15). In Section 2.3.1 we describe the topological structure of G and show that it is a full branch map with countably many branches which saturates I; this will require only the very basic topological structure of g provided by condition **(A0)**. In Section 2.3.2 we obtain estimates concerning the sizes of the partition elements of the corresponding partition; this will require the explicit form of the map g as given in **(A1)**. In Section 2.3.4 we show that G is uniformly expanding; this will require the final condition **(A2)**. Finally, in Section 2.3.3 we use the estimates and results obtained to show that G has bounded distortion.

2.3.1 Topological Construction

In this section we make some topological construction which depends only on condition **(A0)**. The construction is the same as in [CLM22]. However, since we need the notation for our following calculation, we give it in full detail.
Let $\Delta_0^+ := g_+^{-1}(-1,0)$ and $\Delta_0^- := g_-^{-1}(0,1)$. For every $n \ge 1$, we define iteratively the sets

$$\Delta_n^- := g^{-1}(\Delta_{n-1}^-) \cap I_- \quad \text{and} \quad \Delta_n^+ := g^{-1}(\Delta_{n-1}^+) \cap I_+.$$

The collections $\{\Delta_n^-\}_{n\ge0}$ and $\{\Delta_n^+\}_{n\ge0}$ are partitions of I_- and I_+ respectively.
For all $n > 1$ we again define

$$\delta_n^- := g_-^{-1}(\Delta_{n-1}^+) = g^{-1}(\Delta_{n-1}^+) \cap \Delta_0^- \quad \text{and} \quad \delta_n^+ := g_+^{-1}(\Delta_{n-1}^-) = g^{-1}(\Delta_{n-1}^-) \cap \Delta_0^+.$$

Again the collections $\{\delta_n^-\}_{n\ge0}$ and $\{\delta_n^+\}_{n\ge0}$ are partitions of Δ_0^- and Δ_0^+ respectively and for every $n \ge 1$,

$$g : \delta_n^- \to \Delta_{n-1}^+, \quad g : \delta_n^+ \to \Delta_{n-1}^-, \quad g^{n-1} : \Delta_{n-1}^- \to \Delta_0^-, \quad \text{and} \quad g^{n-1} : \Delta_{n-1}^+ \to \Delta_0^+$$

are C^2 diffeomorphisms. Thus, $g^n : \delta_n^- \to \Delta_0^+$ and $g^n : \delta_n^+ \to \Delta_0^-$ are also C^2 diffeomorphisms. These give rise to two *full branch induced maps*

$$\widetilde{G}^- : \Delta_0^- \to \Delta_0^+ \quad \text{and} \quad \widetilde{G}^+ : \Delta_0^+ \to \Delta_0^- \tag{2.18}$$

defined by $\widetilde{G}^-|_{\delta_n^-} = g^n$ and $\widetilde{G}^+|_{\delta_n^+} = g^n$. For all $m, n \ge 1$ we again define

$$\delta_{m,n}^- := (G^-)^{-1}(\delta_n^+) \cap \delta_m^- \quad \text{and} \quad \delta_{m,n}^+ := (G^+)^{-1}(\delta_n^-) \cap \delta_m^+.$$

Then $\{\delta_{m,n}^-\}_{n\ge1}$ and $\{\delta_{m,n}^+\}_{n\ge1}$ are partitions for δ_m^- and δ_m^+ respectively and thus

$$\mathscr{P}^- := \{\delta_{m,n}^-\}_{m,n\ge1} \quad \text{and} \quad \mathscr{P}^+ := \{\delta_{m,n}^+\}_{m,n\ge1}$$

are partitions of Δ_0^-, Δ_0^+ respectively, such that for every $m, n \ge 1$, the maps

$$g^{m+n} : \delta_{m,n}^- \to \Delta_0^- \quad \text{and} \quad g^{m+n} : \delta_{m,n}^+ \to \Delta_0^+$$

are C^2 diffeomorphisms. These further give rise to two *full branch induced maps*

$$G^- := \widetilde{G}^+ \circ \widetilde{G}^- : \Delta_0^- \to \Delta_0^- \quad \text{and} \quad G^+ := \widetilde{G}^- \circ \widetilde{G}^+ : \Delta_0^+ \to \Delta_0^+ \tag{2.19}$$

with return time $R : \Delta^\pm \to \mathbb{Z}$ defined as

$$R|_{\delta_{m,n}^-} = m + n \quad \text{and} \quad R|_{\delta_{m,n}^+} = m + n.$$

2.3.2 Partition Estimates

In this section we give asymptotic estimates of the partition which depend on the form of the map given in **(A1)**. Let $(x_n^-)_{n\geq 0}$ and $(x_n^+)_{n\geq 0}$ be the boundary points of the intervals $(\Delta_n^-)_{n\geq 0}$ and $(\Delta_n^+)_{n\geq 0}$ respectively such that

$$\Delta_0^- = (x_0^-, 0), \qquad \Delta_0^+ = (0, x_0^+), \qquad \Delta_n^- = (x_n^-, x_{n-1}^-), \qquad \Delta_n^+ = (x_{n-1}^+, x_n^+) \tag{2.20}$$

for $n \geq 1$.

Let $\bar{\mathcal{L}}_{-1}(x) = 1/\mathcal{L}_{-1}(1/x)$.
The following result gives the asymptotic rates of convergence of (x_n^-) and (x_n^+) to -1 and 1 respectively.

Proposition 2.3.2.

$$1 + x_n^- \sim \ell_1^{-\frac{1}{\ell_1}} n^{-\frac{1}{\ell_1}} \mathcal{L}_{-1}^{\#}(n^{-\frac{1}{\ell_1}}), \quad \text{and} \quad 1 - x_n^+ \sim \ell_2^{-\frac{1}{\ell_2}} n^{-\frac{1}{\ell_2}} \mathcal{L}_1^{\#}(n^{-\frac{1}{\ell_2}}). \tag{2.21}$$

$$|\Delta_n^-| \sim \ell_1^{-\frac{1}{\ell_1}} n^{-(\frac{1}{\ell_1}+1)} \mathcal{L}_{-1}^{\#}(n^{-\frac{1}{\ell_1}}) \quad \text{and} \quad |\Delta_n^+| \sim \ell_2^{-\frac{1}{\ell_2}} n^{-(\frac{1}{\ell_2}+1)} \mathcal{L}_1^{\#}(n^{-\frac{1}{\ell_2}}).$$

Proof of Proposition 2.3.2. Let $z_n^- := 1 + x_n^-$ and $z_n^+ := 1 - x_n^+$. As

$$z_n^- := 1 + g(x_{n+1}^-) = z_{n+1}^-(1 + \mathcal{R}_{-1,\ell_1}(z_{n+1}^-)),$$

we have by Proposition 1 in [Hol05]
$z_n^- \sim \frac{1}{\ell_1^{\frac{1}{\ell_1}} \bar{\mathcal{R}}_{-1,\ell_1}^{-1}(n)}$, where $\bar{\mathcal{R}}_{-1,\ell_1}(n) := \frac{1}{\mathcal{R}_{-1,\ell_1}(\frac{1}{n})} = n^{\ell_1} \bar{\mathcal{L}}_{-1}(n)$ and $\bar{\mathcal{L}}_{-1}(n) = \left(\mathcal{L}_{-1}(\frac{1}{n})\right)^{-1}$.
Together with Lemma 2.5.5 and the fact that $(L^{\#})^{-1} = (L^{-1})^{\#}$ we get $\bar{\mathcal{R}}_{-1}^{-1}(n) = n^{\frac{1}{\ell_1}} \left(\bar{\mathcal{L}}_{-1}^{\#}\right)(x^{\frac{1}{\ell_1}})$.
By the mean value theorem, we have

$$|\Delta_n^-| = |x_{n-1}^- - x_n^-| = |z_{n-1}^- - z_n^-| \sim \frac{\mathrm{d}z_n^-}{\mathrm{d}n}$$

which together with (2.21) gives the required estimates for $|\Delta_n^-|$. Here, we note that z_n^- is of course a function in $\mathbb{N}$. However, $\mathcal{L}_-$ is defined on the whole interval $[0,1]$ and thus z_n could also be analytically extended to $\mathbb{R}$ making $\mathrm{d}z_n^-/\mathrm{d}n$ a meaningful expression.
We obtain estimates for $1 - x_n^+$ and $|\Delta_n^+|$ by an analogous argument. □

Let $(y_n^-)_{n\geq 0}$ and $(y_n^+)_{n\geq 0}$ be the boundary points of the intervals $(\delta_n^-)_{n\geq 0}$ and $(\delta_n^+)_{n\geq 0}$ respectively such that

$$\delta_n^- = [y_{n-1}^-, y_n^-) \quad \text{and} \quad \delta_n^+ = (y_n^+, y_{n-1}^+] \tag{2.22}$$

for $n \geq 1$.

Corollary 2.3.3. *We have the following asymptotics:*

$$y_n^- \sim -a_1^{-\frac{1}{k_1}} n^{-\frac{1}{\beta_2}} \ell_2^{-1} (\mathcal{L}_1^{\#}(n^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}} \quad \text{and} \quad y_n^+ \sim a_2^{-\frac{1}{k_2}} n^{-\frac{1}{\beta_1}} \ell_1^{-1} (\mathcal{L}_{-1}^{\#}(n^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}, \tag{2.23}$$

$$|\delta_n^-| \sim a_1^{-\frac{1}{k_1}} \beta_2^{-1} \ell_2^{-1} n^{-(1+\frac{1}{\beta_2})} (\mathcal{L}_1^{\#}(n^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}} \quad \text{and} \quad |\delta_n^+| \sim a_2^{-\frac{1}{k_2}} \beta_1^{-1} \ell_1^{-1} n^{-(1+\frac{1}{\beta_1})} (\mathcal{L}_{-1}^{\#}(n^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}.$$

Proof. By our topological construction $g_+(y_n^+) = x_{n-1}^-$ and Definition of g_+ (2.1) we have that $y_n^+ = \left(\frac{1}{a_2}(1+x_{n-1}^-)\right)^{\frac{1}{k_2}}$.
Together with Proposition 2.3.2 we the result. By the mean value theorem we have

$$|\delta_n^+| = |y_{n-1}^+ - y_n^+| \sim \frac{\mathrm{d}y_n^+}{\mathrm{d}n}$$

which together with (2.23) gives the required estimates for $|\delta_n^+|$. We obtain estimates for y_n^- and $|\delta_n^-|$ by analogous argument.

□

2.3.3 Distortion Estimates

Proposition 2.3.4. *For all $g \in \widehat{\mathfrak{G}}$ there exists a constant $\mathfrak{D} > 0$ such that for all $0 \leq m < n$ and all $x, y \in \delta_n^\pm$,*

$$\log \frac{\left(g^{n-m}\right)'(g^m(x))}{\left(g^{n-m}\right)'(g^m(y))} \leq \mathfrak{D}|g^n(x) - g^n(y)|.$$

Proof of Proposition 2.3.7. We assume for simplicity that $x, y \in \delta_n^+$, the estimates for δ_n^- are the same. By the chain rule, we write

$$\log \frac{\left(g^{n-m}\right)'(g^m(x))}{\left(g^{n-m}\right)'(g^m(y))} = \log \prod_{i=m}^{n-1} \frac{g'(g^i(x))}{g'(g^i(y))} = \sum_{i=m}^{n-1} \log \frac{g'(g^i(x))}{g'(g^i(y))}.$$

Since $(g^i(x), g^i(y)) \subset \Delta_{n-i}^-$ (a smooth component of g) for $1 \leq i < n$, by mean value theorem there is $u_i \in \left(g^i(x), g^i(y)\right)$ such that

$$\log \frac{g'(g^i(x))}{g'(g^i(y))} = \log g'(g^i(x)) - \log g'(g^i(y)) = \frac{g''(u_i)}{g'(u_i)}|g^i(x) - g^i(y)|.$$

We substitute this into the above expression taking $\mathfrak{D}_i := g''(u_i)/g'(u_i)$ to get

$$\log \frac{\left(g^{n-m}\right)'(g^m(x))}{\left(g^{n-m}\right)'(g^m(y))} = \sum_{i=m}^{n-1} \mathfrak{D}_i|g^i(x) - g^i(y)| \leq \sum_{i=0}^{n-1} \mathfrak{D}_i|g^i(x) - g^i(y)| \leq \mathfrak{D}_0\delta_n^+ + \sum_{i=1}^{n-1} \mathfrak{D}_i|\Delta_{n-i}^-|. \tag{2.24}$$

By estimates in Proposition 2.3.2, Corollary 2.3.3 and bounds (2.11) (2.12) we have

$$\begin{aligned} \mathfrak{D}_i|\Delta_{n-i}^-| \lesssim & (1+x_{n-i}^-)^{\ell_1 - 1}\mathcal{L}_{-1}(1+x_{n-i}^-)|\Delta_{n-i}^-| \\ \lesssim & (n-i)^{-2}\mathcal{L}_{-1}^{\#}((n-i)^{-\frac{1}{\ell_1}})^{\ell_1}\mathcal{L}_{-1}((n-i)^{-\frac{1}{\ell_1}}\mathcal{L}_{-1}^{\#}((n-i)^{-\frac{1}{\ell_1}})) \end{aligned} \tag{2.25}$$

and using the relationship between the y_n^+ and the x_n^- we have

$$\mathfrak{D}_0|\delta_n^+| \lesssim u_0^{-1}|\delta_n^+| \lesssim \frac{y_n^+ - y_{n+1}^+}{y_{n+1}^+} \lesssim \left(\frac{1+x_n^-}{1+x_{n+1}^-}\right)^{1/k} - 1 \to 0 \tag{2.26}$$

where for all $n \geq 1$ we have applied mean value theorem to the diffeomophism $g : [-1, x_{n+1}^-] \to [-1, x_n^-]$ to get some sequence $\xi_n \to -1$ with $(1+x_n^-)/(1+x_{n+1}^-) = g'(\xi_n) \to 1$.

Substituting (2.26) and (2.26) into (2.25) we get

$$\log \frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} \lesssim \mathfrak{D}_0\delta_n^+ + \sum_{i=1}^{n-1} i^{-2}\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})^{\ell_1}\mathcal{L}_{-1}\left(\frac{\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})}{i^{\frac{1}{\ell_1}}}\right) := \widehat{\mathfrak{D}} < \infty. \tag{2.27}$$

Boundedness for the first summand is immediate. For the second summand there exists $C > 0$ such that

$$\sum_{i=1}^{n-1} i^{-2}\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})^{\ell_1}\mathcal{L}_{-1}\left(\frac{\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})}{i^{\frac{1}{\ell_1}}}\right) < \sum_{i=1}^{\infty} i^{-2}\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})^{\ell_1}\mathcal{L}_{-1}\left(\frac{\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})}{i^{\frac{1}{\ell_1}}}\right) < C\sum_{i=1}^{\infty} i^{-3/2} < \infty,$$

since the slowly varying function fulfills $\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})^{\ell_1}\mathcal{L}_{-1}\left(\frac{\mathcal{L}_{-1}^{\#}(i^{-\frac{1}{\ell_1}})}{i^{\frac{1}{\ell_1}}}\right) \lesssim i^{1/2}$, for i sufficiently large. Together with Mean Value Theorem we have that the diffeomorphisms $g^n : \delta_n^+ \to \Delta_0^-$ and $g^{n-m} : \Delta_{n-m}^- \to \Delta_0^-$ all have uniformly bounded distortion; that is

$$|x-y| \leq \frac{e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-|}|g^n(x)-g^n(y)||\delta_n^+| \quad \text{and} \quad |g^i(x)-g^i(y)| \leq \frac{e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-|}|g^n(x)-g^n(y)||\Delta_{n-m}^-|$$

for all $x, y \in \delta_n^+$ and $1 \leq m < n$. Substituting back into Equation (2.24)

$$\begin{aligned}\log \frac{(g^{n-m})'(g^m(x))}{(g^{n-m})'(g^m(y))} &\leq \frac{e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-|}\left[\mathfrak{D}_0|\delta_n^+| + \sum_{i=1}^{n-1}\mathfrak{D}_i|\Delta_{n-i}^-|\right]|g^n(x)-g^n(y)| \\ &\leq \frac{e^{\widehat{\mathfrak{D}}}\widehat{\mathfrak{D}}}{|\Delta_0^-|}|g^n(x)-g^n(y)| \\ &= \mathfrak{D}|g^n(x)-g^n(y)|,\end{aligned}$$

where $\mathfrak{D} := e^{\widehat{\mathfrak{D}}}\widehat{\mathfrak{D}}/|\Delta_0^-|$. Notice that in the first inequality the term in brackets corresponds to Equation (2.27) which is uniformly bounded. □

As an immediate consequence we get the following two corollaries.

Corollary 2.3.5. *For all $i, j \geq 1$ we have*

$$\tilde{\mu}(\delta_{i,j}^-) \lesssim i^{-(1+\frac{1}{\beta_2})}(\mathcal{L}_1^{\#}(i^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}} j^{-(1+\frac{1}{\beta_1})}(\mathcal{L}_{-1}^{\#}(j^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}. \tag{2.28}$$

Proof. Proposition 2.3.4 implies that

$$\frac{1}{\mathfrak{D}|\Delta_0^+|}|\delta_i^-||\delta_j^+| \leq |\delta_{i,j}| \leq \frac{\mathfrak{D}}{|\Delta_0^+|}|\delta_i^-||\delta_j^+|.$$

Applying estimates in Corollary 2.3.3 gives the result as $\tilde{\mu}$ is equivalent to Lebesgue on $\Delta_0^- \cup \Delta_0^+$. □

Corollary 2.3.6. *There exist $C > 0$ and $\theta \in (0, 1)$ such that*

$$\log\left|\frac{G'(x)}{G'(y)}\right| \leq C\theta^{s(x,y)}$$

for all $x, y \in \delta_{i,j} \in \mathcal{P}$ with $x \neq y$ where s is the separation time defined in (2.17).

2.3.4 Expansion Estimate

Proposition 2.3.7. *For every $g \in \mathfrak{G}$ the first return map $G : \Delta_0^- \to \Delta_0^-$ is uniformly expanding.*

Define $\phi : \Delta_0^+ \setminus \delta_1^+ \to \Delta_0^+$ explicitly by

$$\phi \coloneqq (g|_{U_{0+}})^{-1} \circ g|_{U_{-1}} \circ g|_{U_{0+}} \tag{2.29}$$

which is a bijection given by the commutative diagram in Figure 2.2.

$$\begin{array}{ccc} \delta_{n+1}^+ & \xrightarrow{\phi} & \delta_n^+ \\ \big\downarrow g & & \big\downarrow g \\ \Delta_n^- & \xrightarrow{g} & \Delta_{n-1}^- \end{array}$$

Figure 2.2: ϕ satisfies $g^2 = g \circ \phi$.

We start by the following prelimary lemma.

Lemma 2.3.8. *For all $x \in \delta_{n+1}^+$, $n \geq n^+$ we have*

$$(\widetilde{G}^+)'(x) > (\widetilde{G}^+)'(\phi(x)).$$

Sublemma 2.3.9. *For all $x \in \delta_{n+1}^+$, $n \geq n^+$ we have*

$$(g^2)'(x) > g'(\phi(x)).$$

Proof. We will prove the equivalent statement

$$\frac{g'(x)}{g'(\phi(x))} g'(g(x)) > 1. \tag{2.30}$$

By the definition of g in U_{0+} given in **(A1)** we have that

$$\frac{g'(x)}{g'(\phi(x))} = \left(\frac{x}{\phi(x)}\right)^{k_2 - 1} = \left(\frac{\phi(x)}{x}\right)^{1-k_2}. \tag{2.31}$$

The form of g in U_{0+} given in **(A1)** together with (2.29) give

$$\phi(x) = [x^{k_2} + a_2^{\ell_1} x^{1+k_2} \mathcal{L}_{-1}(a_2 x^{k_2}]^{\frac{1}{k_2}} \quad \text{and thus} \quad \left(\frac{\phi(x)}{x}\right)^{k_2} = 1 + a^{\ell_1} x \mathcal{L}_{-1}(a_2 x^{k_2}). \tag{2.32}$$

Again by the form of the map g given in **(A1)** we have

$$\begin{aligned} g'(g(x)) &= 1 + a_2^{\ell_1} x (1 + \ell_1) \mathcal{L}_{-1}(a_2 x^{k_2}) + a_2^{\ell_1+1} x^{k_2+1} \mathcal{L}'_{-1}(a_2 x^{k_2}) \\ &\geq 1 + a_2^{\ell_1} x \mathcal{L}_{-1}(a_2 x^{k_2}) + a_2^{\ell_1+1} x^{k_2+1} \mathcal{L}'_{-1}(a_2 x^{k_2}) \\ &\geq \left(\frac{\phi(x)}{x}\right)^{k_2} + a_2^{\ell_1+1} x^{k_2+1} \mathcal{L}'_{-1}(a_2 x^{k_2}) \end{aligned} \tag{2.33}$$

where the last inequality is given by (2.32).

Since $x < \phi(x)$ and $k_2 > 1$, (2.31) is less than 1. Inserting (2.31) and (2.33) into (2.30) we have

$$\begin{aligned}\frac{g'(x)}{g'(\phi(x))}g'(g(x)) &\geq \left(\frac{\phi(x)}{x}\right)^{1-k_2}\left(\left(\frac{\phi(x)}{x}\right)^{k_2} + a_2^{\ell_1+1}x^{k_2+1}\mathcal{L}'_{-1}(a_2x^{k_2})\right)\\ &\geq \frac{\phi(x)}{x} + \left(\frac{\phi(x)}{x}\right)^{1-k_2} a_2^{\ell_1+1}x^{k_2+1}\mathcal{L}'_{-1}(a_2x^{k_2})\\ &\geq 1 + \left(\frac{\phi(x)}{x}\right)^{1-k_2} a_2^{\ell_1+1}x^{k_2+1}\mathcal{L}'_{-1}(a_2x^{k_2}).\end{aligned}$$

Observe that the last term is positive as $\mathcal{L}'_{-1} > 0$ is given by **(A1)**(ii). □

Proof of Lemma 2.3.7. It suffices to show uniform expansion for $(\widetilde{G}^+)'$ because it holds for $(\widetilde{G}^-)'$ via identical procedure and $G' := (\widetilde{G}^-)' \circ (\widetilde{G}^+)'$. We will restrict ourselves to the case where $k_2 > 1$ otherwise expansion is trivial.

Let $x \in \delta^+_{n+1}$, $n < n^+$. Then $(\widetilde{G}^+)'$ is uniformly expanding by **(A2)**.

Suppose $n \geq n^+$. As $x < \phi(x)$ and $K_2 > 1$, we have by the geometry of g that for any $1 < k < n$

$$g'(g^k(x)) > g'(g^{k-1}(\phi(x))).$$

This implies that for any $1 \leq m \leq n$

$$\begin{aligned}(g^{m+1})'(x) &= g'(x)g'(g(x))\cdots g'(g^m(x))\\ &> g'(x)g'(g(x))g'(g\phi(x))\cdots g'(g^{m-1}\phi(x)).\end{aligned} \tag{2.34}$$

Applying (1.51) we have the desired result. □

2.4 Tail Estimates

In this section we prove Proposition 2.2.6. We start first with the following lemma

Lemma 2.4.1.

$$\sum_{\substack{i<n,j<n\\ i+j=n}} \tilde{\mu}(\delta^-_{i,j}) = o(|y^\pm_n|).$$

(A1) (ii) implies the following sublemma.

Sublemma 2.4.2.

$$(\mathcal{L}^\#_1(n^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}} \to 0 \quad \textit{and} \quad (\mathcal{L}^\#_{-1}(n^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}} \to 0, \quad \textit{as } n \to \infty. \tag{2.35}$$

Proof of Lemma 2.4.1. By bounded distortion (2.28) and Corollary 2.3.3 we have

$$\begin{aligned}\frac{1}{|y^-_n|}\sum_{\substack{i<n,j<n\\ i+j=n}} \tilde{\mu}(\delta^-_{i,j}) &\lesssim n^{\beta_2}\mathcal{L}^\#_1(n^{-\frac{1}{\ell_2}})^{-\frac{1}{k_1}}\sum_{i=1}^n i^{-(1+\frac{1}{\beta_2})}(\mathcal{L}^\#_1(i^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}}\sum_{j=n-i+1}^n j^{-(1+\frac{1}{\beta_1})}(\mathcal{L}^\#_{-1}(j^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}\\ &\lesssim n\mathcal{L}^\#_1(n^{-\frac{1}{\ell_2}})^{-\frac{1}{k_1}}\sum_{i=1}^n i^{-(1+\frac{1}{\beta_2})}\sum_{j=n-i+1}^n j^{-(1+\frac{1}{\beta_1})}\end{aligned} \tag{2.36}$$

where the last inequality is given by (2.35) and the fact that $\beta \le 1$. For $\beta < 1$ in (2.36), we obtain the desired result by applying Lemma 4.7 in [CLM22]. For $\beta = 1$, we take the difference of the tails for the sum over j in (2.36) to get

$$\frac{1}{|y_n^-|}\sum_{\substack{i<n,j<n\\ i+j=n}} \tilde{\mu}(\delta_{i,j}^-) \lesssim (\mathcal{L}_1^\#(n^{-\frac{1}{\ell_2}}))^{-\frac{1}{k_1}} \sum_{i=1}^{n} \frac{1}{i(n-i+1)}.$$

Resolving to partial fraction we get

$$\frac{1}{|y_n^-|}\sum_{\substack{i<n,j<n\\ i+j=n}} \tilde{\mu}(\delta_{i,j}^-) \lesssim \frac{(\mathcal{L}_1^\#(n^{-\frac{1}{\ell_2}}))^{-\frac{1}{k_1}}}{n+1}\left(\sum_{i=1}^{n}\frac{1}{i} + \sum_{i=1}^{n}\frac{1}{(n-i+1)}\right).$$

This converges to 0 by the fact that

$$\frac{(\mathcal{L}_1^\#(n^{-\frac{1}{\ell_2}}))^{-\frac{1}{k_1}}}{n}\sum_{i=1}^{n}\frac{1}{i} \sim \frac{\log(n)\mathcal{L}_1^\#(n^{-\frac{1}{\ell_2}})^{-\frac{1}{k_1}}}{n} \to 0,$$

as $n \to \infty$. □

We now prove Proposition 2.2.6.

Proof of Proposition 2.2.6. We can split $\tilde{\mu}(\tau > n)$ into the following four sums:

$$\tilde{\mu}(\tau > n) = \sum_{i>n}\sum_{j=1}^{\infty}\tilde{\mu}(\delta_{i,j}^-) + \sum_{j>n}\sum_{i=1}^{\infty}\tilde{\mu}(\delta_{i,j}^-) - \sum_{i>n}\sum_{j>n}\tilde{\mu}(\delta_{i,j}^-) + \sum_{\substack{i<nj<n\\ i+j=n}}\tilde{\mu}(\delta_{i,j}^-). \tag{2.37}$$

Since by the topology of our partition $\delta_i^- = \bigcup_{j=1}^{\infty}\delta_{i,j}$ and $\bigcup_{i=n}^{\infty}\delta_i^- = [y_n^-, 0)$, the first sum in (2.37) is

$$\sum_{i>n}\sum_{j=1}^{\infty}\tilde{\mu}(\delta_{i,j}^-) = \sum_{i>n}\tilde{\mu}(\delta_i^-) = \tilde{\mu}((y_n^-, 0)).$$

Together with Lemma 4.6 in [CLM22] we have $\tilde{\mu}((y_n^-, 0)) \sim \tilde{h}(0^-)y_n^-$.
Since the measure is invariant, we have by Lemma 4.3 in [CLM22] that the second sum

$$\sum_{j>n}\sum_{j=1}^{\infty}\tilde{\mu}(\delta_{i,j}^-) = \tilde{\mu}((0, y_n^+)).$$

Together with Lemma 4.6 in [CLM22] we have $\tilde{\mu}((0, y_n^+)) \sim \tilde{h}(0^+)y_n^+$. Since the fourth sum decays faster than the first two by Lemma 2.4.1, we only have to show

$$\sum_{i>n}\sum_{j>n}\tilde{\mu}(\delta_{i,j}^-) = o(|y_n^\pm|).$$

By bounded distortion (2.28) we have

$$\begin{aligned}
\frac{1}{|y_n^-|}\sum_{i>n}\sum_{j>n}\tilde{\mu}(\delta_{i,j}^-) &\lesssim \frac{1}{|y_n^-|}\left(\sum_{i>n} i^{-(1+\frac{1}{\beta_2})}(\mathcal{L}_1^\#(i^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}}\right)\left(\sum_{j>n} j^{-(1+\frac{1}{\beta_1})}(\mathcal{L}_{-1}^\#(j^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}\right)\\
&\lesssim \frac{1}{|y_n^-|}\left(n^{-\frac{1}{\beta_2}}(\mathcal{L}_1^\#(n^{-\frac{1}{\ell_2}}))^{\frac{1}{k_1}}\right)\left(n^{-\frac{1}{\beta_1}}(\mathcal{L}_{-1}^\#(n^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}\right)\\
&\lesssim \left(n^{-\frac{1}{\beta}}(\mathcal{L}_{-1}^\#(n^{-\frac{1}{\ell_1}}))^{\frac{1}{k_2}}\right)
\end{aligned}$$

which converges to 0 by (2.35). We applied Lemma 2.5.7 to get the second inequality. □

2.5 Appendix

In this section we list some relevant results on regularly varying function we have used. Although the results stated here are for functions slowly varying at ∞, by setting $\widetilde{\mathcal{L}}(x) = \mathcal{L}(1/x)$ we have that $\widetilde{\mathcal{L}}$ is a slowly varying function in zero. See Remark 2.5.8.

Lemma 2.5.1 (Theorem 1.4.1 (Characterization of regularly varying functions)[Bin+89])**.** *Every function $\mathcal{R}$ regularly varying at ∞ with index γ can be written as $\mathcal{R}(x) = x^\gamma \mathcal{L}(x)$ where $\mathcal{L}$ is slowly varying.*

Lemma 2.5.2 (Theorem 1.3.1 (Representation of slowly varying functions)[Bin+89])**.** *Let $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$ be slowly varying at ∞. Then there exist $C_0 > 0$ and measurable functions $\epsilon, \zeta : [C_0, \infty) \to \mathbb{R}^+$ $\zeta(x) \to c \in \mathbb{R}^+, \epsilon(x) \to 0$ as $x \to \infty$ and*

$$\mathcal{L}(x) = \zeta(x) \exp\left\{ \int_{C_0}^{x} \frac{\epsilon(t)}{t} dt \right\} \text{ for all } x \geq C_0.$$

Though the following is a widely used standard result, we will give a proof here.

Lemma 2.5.3 (Proposition 1.5.1 [Bin+89])**.** *Let $\mathcal{L}$ be a slowly varying function in infinity and let $\ell \neq 0$, then*

$$x^\ell \mathcal{L}(x) \to \begin{cases} \infty & \text{if } \ell > 0 \\ 0 & \text{if } \ell < 0, \end{cases}$$

as $x \to \infty$.

Lemma 2.5.4 (Theorem 1.5.13 [Bin+89])**.** *If $\mathcal{L}$ is slowly varying in infinity, there exists a slowly varying function $\mathcal{L}^\#$, unique up to asymptotic equivalence such that*

$$\mathcal{L}(x)\mathcal{L}^\#(x\mathcal{L}(x)) \to 1, \quad \mathcal{L}^\#(x)\mathcal{L}(x\mathcal{L}^\#(x)) \to 1,$$

as $x \to \infty$ and $\mathcal{L}^{\#\#} = \mathcal{L}$.

Lemma 2.5.5 (Theorem 1.5.15 [Bin+89])**.** *Let $a, b > 0$ and $g(x) \sim x^a \mathcal{L}^a(x^b)$ where $\mathcal{L}$ is slowly varying in infinity. Suppose f is an asymptotic inverse of g (i.e. $g(f(x)) \sim f(g(x)) \sim x$). Then*

$$f(x) \sim x^{\frac{1}{ab}} \mathcal{L}^{\#\frac{1}{b}}(x^{\frac{1}{a}}).$$

Lemma 2.5.6 (Proposition 1.5.8 [Bin+89])**.** *If $\mathcal{L}$ is slowly varying, M is so large that $\mathcal{L}(x)$ is locally bounded in $[M, \infty)$, and $\alpha > -1$, then*

$$\int_M^\alpha t^\alpha \mathcal{L}(t) dt \sim x^{\alpha+1}\mathcal{L}(x)/(\alpha+1) \quad \text{as} \quad x \to 0.$$

Lemma 2.5.7 (Proposition 1.5.9b [Bin+89])**.** *If $\mathcal{L}$ is slowly varying and $\alpha < -1$ then $\int^\infty t^\alpha \mathcal{L}(t) dt$ converges and*

$$\frac{x^{\alpha+1}\mathcal{L}(x)}{\int_x^\alpha t^\alpha \mathcal{L}(t) dt} \to -\alpha - 1 \quad \text{as} \quad x \to \infty.$$

Remark 2.5.8. By setting $\widetilde{\mathcal{L}}(x) = \mathcal{L}(1/x)$ we have that $\widetilde{\mathcal{L}}$ is a slowly varying function in zero. For this function analogous results can be stated. In particular the same characterization statement holds and we have for the de Bruijn conjugate that $\widetilde{\mathcal{L}}^\#(x) = \mathcal{L}^\#(1/x)$ is a possible choice and thus analogous statements for Lemma 2.5.4 and Lemma 2.5.5 can be stated for convergence $x \to 0$.

Moreover, the statement of Lemma 2.5.3 changes for $\ell \neq 0$ into

$$x^\ell \widetilde{\mathcal{L}}(x) \to \begin{cases} 0 & \text{if } \ell > 0 \\ \infty & \text{if } \ell < 0, \end{cases}$$

as $x \to 0$.

Random Doubly Intermittent Maps with Singularities

3.1 Introduction and Statement of Results

In recent years quenched statistical properties of random dynamical systems attracted considerable amount of attention of experts in dynamical systems. Even in the uniformly hyperbolic setting, random dynamical systems required non-trivial adjustments to obtain limit theorems [Dra+20]. Also, it showed a richer dynamical phenomena than deterministic systems, due to presence of extra "degree of freedom", for instance the so called critical intermittency phenomenon observed in [AGH18; Hom+22; HPR21; KZ22].

For limit theorems including quenched versions of central limit theorem, almost sure invariance principle in expanding or expanding on average setting and uniformly hyperbolic setting we refer to [DH21; DHS21; Dra+20] and references therein. In the case this system is not expanding on average one can study the statistical properties using random induced maps in this direction we refer to [BBMD02; BBMD03; ABR22; Alv+22; BBR19] for results concerning decay of correlations and [Haf22; Su22; NPT21] for quenched limit theorems.

The aim of this paper is to study quenched statistical properties of a random dynamical system built over a class of maps with two intermittent points with a singularity at the origin introduced in physics literature. This class of maps was introduced by Pikovsky [Pik91b] and further studied in [RG04] by Arthuso and Cristodoro. Ergodic and statistical properties of these class of maps was studied in [G+10; CLM22]. These maps are topologically conjugate to the doubling map on the torus and tent map on the unit interval. The main feature is that both of maps have intermittent fixed points and a singularity. This makes the inducing more complicated than the Liverani-Saussol-Vaienti maps. Here we consider random compositions of maps from the above class and obtain quenched decay of correlations. In contrast to random LSV maps studied in [BBR19], the existence of singularities makes it impossible to obtain sharp estimates on the tail of the return times. Although, the results still show that the fastest mixing system dominates the rates of mixing, quenched mixing rates are strictly slower than that of the fastest mixing system.

3.1.1 Setup

In this section we introduce the class of random dynamical systems that we will study in this work.

Pikovsky Maps

Let $\mathbb{T} = I \backslash \sim$, $I = [-1, 1]$, $I^- = [-1, 0)$ and $I^+ = (0, 1]$. Consider the family of deterministic maps $g_\alpha : \mathbb{T} \to \mathbb{T}$ introduced in [Pik91b] with graph 3.1 and defined implicitly by the equation

$$x = \begin{cases} \frac{1}{2\alpha}(1 + g(x))^\alpha & \text{if } 0 \le x \le \frac{1}{2\alpha}, \\ g(x) + \frac{1}{2\alpha}(1 - g(x))^\alpha & \text{if } \frac{1}{2\alpha} \le x \le 1 \end{cases} \tag{3.1}$$

and for negative values of x by taking $g(-x) = -g(x)$, which are continuous on $\mathbb{T}\backslash\{0\}$ and C^2 on $\mathbb{T}\backslash\{0\} \cup \{1\}$. The map reduces to the doubling map for $\alpha = 1$. For all $\alpha > 1$, the family g_α preserve Lebesgue measure and it is polynomially mixing with rate $\frac{1}{\alpha - 1}$ [Cri+10].

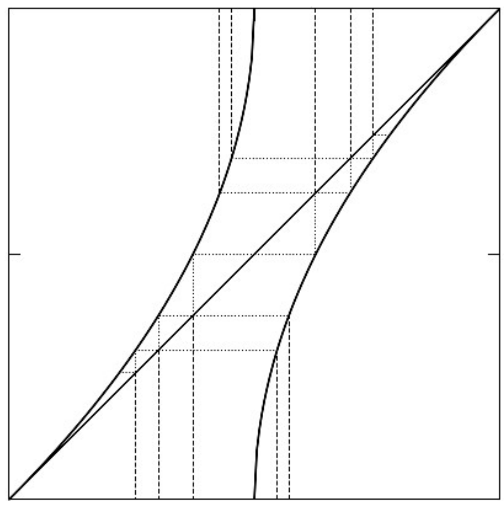


Figure 3.1: Graph of g

By [Lemma 2, [G+10]], in a neighbourhood of 1^-

$$\begin{aligned} g_\alpha(x) &= 1 - (1 - x) - (1 - x)^\alpha\left(\frac{1}{2\alpha} + u_\alpha(x)\right) \\ g'_\alpha(x) &= 1 + (1 - x)^{\alpha - 1}\left(\frac{1}{2} + v_\alpha(x)\right) \\ g''_\alpha(x) &= (1 - x)^{\alpha - 2}\left(-\frac{\alpha - 1}{2} + w_\alpha(x)\right) \end{aligned} \tag{3.2}$$

where $u_\alpha, v_\alpha, w_\alpha : \mathbb{T} \to \mathbb{T}$ such that $\lim_{x \to 1} u(x) = \lim_{x \to 1} v(x) = \lim_{x \to 1} w(x) = 0$.

Fix two real numbers $1 < \alpha_1 < \alpha_2 < \infty$ and consider a class of Pikovsky maps $(g_\alpha)_{\alpha \in [\alpha_1, \alpha_2]}$. By (3.1) and (3.2) we have

$$u_\alpha(x) = \frac{1}{\alpha}\left(1 - \left(\frac{1 - g_\alpha(x)}{1 - x}\right)^\alpha\right)$$

which implies that $\sup_{\alpha \in [\alpha_1, \alpha_2]} u_\alpha \le M < \infty$. By similar analysis we have that v_α, w_α is uniformly bounded. Let ν be a probability measure on $[\alpha_1, \alpha_2]$. Let $\Omega = [\alpha_1, \alpha_2]^{\mathbb{Z}}$ and $P = \nu^{\mathbb{Z}}$. Then the two sided shift map $\sigma : \Omega \to \Omega$ preserves P. Let $\alpha : \Omega \to [\alpha_1, \alpha_2]$ be the projection to the zeroth coordinate. We will write $g_\omega = g_{\omega_0}$. The **Random Pikovsky Maps** is given by the skew product map $S : \Omega \times I \to \Omega \times I$ defined by

$$S(\omega, x) = (\sigma\omega, g_\omega x).$$

Compositions of S is given by

$$S^n(\omega, x) = (\sigma^n\omega, g^n_\omega x), \quad \text{where} \quad g^n_\omega = g_{\sigma^{n-1}\omega} \circ g_{\sigma^{n-2}\omega} \circ \ldots \circ g_{\sigma\omega} \circ g_\omega.$$

Since g_ω preserves the Lebesgue measure m for all $\omega \in \Omega$, S preserves $P \times m$.

3.1.2 Statement of Results

Definition 3.1.1. A family of measures μ_ω on X_ω is said to be equivariant if $(g_\omega)_*\mu_\omega = \mu_{\sigma\omega}$.

Definition 3.1.2. Given observables $\psi, \varphi : \Omega \times I \to \mathbb{R}$ we define the future and past fibre-wise correlations as

$$Cor^{(f)}_{n,\omega} = \int (\varphi_{\sigma^n\omega} \circ g^n_\omega)\psi_\omega d\mu_\omega - \int \varphi_{\sigma^n\omega} d\mu_{\sigma^n\omega} \int \psi_\omega d\mu_\omega,$$

$$Cor^{(p)}_{n,\omega} = \int (\varphi_\omega \circ g^n_{\sigma^{-n}\omega})\psi_\omega d\mu_{\sigma^{-n}\omega} - \int \varphi_\omega d\mu_\omega \int \psi_{\sigma^{-n}\omega} d\mu_{\sigma^{-n}\omega}$$

respectively.

Notice that (3.1) preserve Lebesgue measure automatically implies that for all $\omega \in \Omega$, g_ω preserves the Lebesgue measure. For the measure $\nu : [\beta_1, \beta_2] \to \mathbb{R}$ we define the following distributions.

DD A measure ν has *discrete distribution* if $\nu([\beta_1, t]) = p_1\delta_{\beta_1}([\beta_1, t]) + p_2\delta_{\beta_2}([\beta_1, t])$.

UD A measure ν has *uniform distribution* if $\nu([\beta_1, t]) = \frac{1}{\beta_2 - \beta_1}(t - \beta_1)$.

We first consider the case in which the measure ν has discrete or uniform distribution.

Theorem 3.1.3. *Suppose ν satisfies DD or UD. Then for every $\delta > 0$ there exist a full measure subset $\Omega_0 \subset \Omega$ and a random variable $C_\omega : \Omega_0 \to \mathbb{R}_+$ so that for any $\varphi \in L^\infty(\mathbb{T})$ and $\psi \in \mathcal{C}^\eta(\mathbb{T})$:*

(1) "Future" correlations: $\left|\int(\varphi \circ g^n_\omega)\psi dm - \int \varphi dm \int \psi dm\right| \leq C_\omega C_{\varphi,\psi} n^{-(\frac{1}{\alpha_1 - 1} - \delta)}$;

(2) "Past" correlations: $\left|\int(\varphi \circ g^n_{\sigma^{-n}\omega})\psi dm - \int \varphi dm \int \psi dm\right| \leq C_\omega C_{\varphi,\psi} n^{-(\frac{1}{\alpha_1 - 1} - \delta)}$,

for almost every $\omega \in \Omega$, for some $C_{\varphi,\psi} > 0$ which is uniform for all $\omega \in \Omega$. Further, there exist constants $C, u' > 0$ and $0 < v < 1$ such that $P\{C_\omega > n\} \leq Ce^{-u'n^{v'}}$ for all $n \in \mathbb{N}$.

Theorem 3.1.3 is a special case of a more general result which allows any measures ν that satisfies some general assumption. We will see later that these assumption is natural if ν has discrete or uniform distribution.
We start by stating the assumptions required for the Random Pikovsky Maps. Let

$$\begin{aligned} A_k(\omega) :=& (\alpha_1 - 1)\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)} + u_{\alpha(\sigma^{n-k}\omega)}(x^+_k(\sigma^{n-k}\omega))\right]\left[\frac{c_k(\alpha_1)}{k^{\frac{1}{\alpha_1 - 1}}}\right]^{\alpha(\sigma^{n-k}\omega) - \alpha_1} \\ &- \frac{\alpha_1(\alpha_1 - 1)}{2}\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)} + u_{\alpha(\sigma^{n-k}\omega)}(x^+_k(\sigma^{n-k}\omega))\right]^2\left[\frac{c_k(\alpha_2)}{k^{\frac{1}{\alpha_2 - 1}}}\right]^{2\alpha(\sigma^{n-k}\omega) - \alpha_1 - 1}. \end{aligned} \tag{3.3}$$

We will see later that the sequences $\{c_k(\alpha_1)\}_k$ and $\{u_{\alpha(\sigma^{n-k}\omega)}(x^+_k(\sigma^{n-k}\omega))\}_k$ in 3.3 depend on both the base map and the deterministic map (3.1) with parameter α_1.
The following assumption on the distribution of ν guarantees the convergence of the expectation of 3.3.

(A) For all $\omega \in \Omega$ there are constants $q = q(\nu) \geq 0$ and $c(\nu) > 0$ such that

$$\frac{(\log n)^q}{n} \sum_{k=1}^{n} E_\nu A_k(\omega) \to c(\nu); \tag{3.4}$$

Theorem 3.1.4. *Suppose $\{g_\omega\}_{\omega\in\Omega}$ satisfy* **(A)**. *Then for every $\delta > 0$ there exist a full measure subset $\Omega_0 \subset \Omega$ and a random variable $C_\omega : \Omega_0 \to \mathbb{R}_+$ so that for any $\varphi \in L^\infty(\mathbb{T})$ and $\psi \in \mathcal{C}^\eta(\mathbb{T})$:*

(1) "Future" correlations: $\left|\int(\varphi \circ g_\omega^n)\psi dm - \int \varphi dm \int \psi dm\right| \leq C_\omega C_{\varphi,\psi} n^{-(\frac{1}{\alpha_1 - 1} - \delta)}$;

(2) "Past" correlations: $\left|\int(\varphi \circ g_{\sigma^{-n}\omega}^n)\psi dm - \int \varphi dm \int \psi dm\right| \leq C_\omega C_{\varphi,\psi} n^{-(\frac{1}{\alpha_1 - 1} - \delta)}$,

for almost every $\omega \in \Omega$, for some $C_{\varphi,\psi} > 0$ which is uniform for all $\omega \in \Omega$. Further, there exist constants $C, u' > 0$ and $0 < v < 1$ such that $P\{C_\omega > n\} \leq Ce^{-u'n^{v'}}$ for all $n \in \mathbb{N}$.

Here we prove that if ν is the discrete or uniform distribution then **(A)** holds and thus Theorem 3.1.3 follows as a special case of Theorem 3.1.4. The rest of the paper is then devoted to the proof of Theorem 3.1.4.

Proof of Theorem 3.1.3 assuming Theorem 3.1.4. **Discrete Probability Distribution**

Let $\rho_k(\alpha(\omega)) = \frac{1}{2\alpha(\sigma^{n-k}\omega)} + u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))$. Using the fact that P is σ-invariance we have

$$E_\nu A_k(\omega) = (\alpha_1 - 1)\left(\rho_k(\alpha_1)p_1 + p_2\rho_k(\alpha_2)\left[\frac{c_k(\alpha_1)}{k^{\frac{1}{\alpha_1-1}}}\right]^{\alpha_2-\alpha_1}\right)$$
$$- \frac{\alpha_1(\alpha_1-1)}{2}\left(p_1\rho_k(\alpha_1)\left[\frac{c_k(\alpha_2)}{k^{\frac{1}{\alpha_2-1}}}\right]^{\alpha_1-1} + p_2\rho_k(\alpha_1)\left[\frac{c_k(\alpha_2)}{k^{\frac{1}{\alpha_2-1}}}\right]^{2\alpha_2-\alpha_1-1}\right)$$

Thus noting that the sequences $c_k(\alpha_1)$ and $c_k(\alpha_2)$ are bounded we have

$$\frac{1}{n}\sum_{k=1}^{n} E_\nu A_k(\omega) = \frac{p_1(\alpha_1-1)}{2\alpha_1} + p_1\frac{1}{n}\sum_{k=1}^{n} u_{\alpha_1}(x_k^+(\alpha_1)) + O(n^{-\delta}),$$

where $\delta := \min\{\frac{\alpha_2-\alpha_1}{\alpha_1-1}, \frac{\alpha_1-1}{\alpha_2-1}, \frac{2\alpha_2-\alpha_1-1}{\alpha_2-1}\} > 0$. Therefore Assumption **(A)** is satisfied by taking $q := 0$ and $c(\nu) := \frac{p_1(\alpha_1-1)}{2\alpha_1} > 0$.

Uniform Probability Distribution

Here we assume that ν is the normalized Lebesgue measure on $[\alpha_1, \alpha_2]$. Recall that in this case we have the following asymptotics (Lemma 5.6 [BBR19]) as $u \to \infty$ for every $c \geq 1$

$$E_\nu(e^{-c(\alpha(\omega)-\alpha_1)u}) \sim \frac{e^{-(c-1)\alpha_1 u}}{cu(\alpha_2-\alpha_1)}. \tag{3.5}$$

Using the above asymptotics for the summands in the definition of $A_k(\omega)$ twice: first with $c = 1$ and $u = \log\frac{c_k(\alpha_1)}{k^{1/(\alpha_1-1)}}$, then the second time with $c = 2$ and $u = \log\frac{c_k(\alpha_2)}{k^{1/(\alpha_2-1)}}$ we obtain

$$\frac{\log n}{n}\sum_{k=1}^{n} E_\nu A_k(\omega) \sim \frac{1}{2\alpha_1}\frac{(\alpha_1-1)^2}{\alpha_2-\alpha_1}(1 - e^{(\alpha_2-\alpha_1)\log\frac{c_k(\alpha_1)}{k^{1/(\alpha_2-1)}}}) + O(n^{-\delta}),$$

$\delta := \min\{\frac{\alpha_2-\alpha_1}{\alpha_1-1}, \frac{\alpha_1}{\alpha_2-1}\} > 0$. Thus, Assumption **(A)** holds with $q = 1$, $c(\nu) = \frac{(\alpha_1-1)^2}{\alpha_2-\alpha_1}\frac{1}{2\alpha_1}$. ∎

3.2 Random Induced Map

3.2.1 Construction of Random Tower

In this section we give topological construction of the random tower which solely depends on definition of the random maps. We begin by defining, for every $n \geq 1$, the sets

$$\Delta_n^-(\omega) := g_\omega^{-1}(\Delta_{n-1}^-(\sigma\omega)) \cap I_- \quad \text{and} \quad \Delta_n^+(\omega) := g_\omega^{-1}(\Delta_{n-1}^+(\sigma\omega)) \cap I_+$$

where $\Delta_0^-(\omega) = (g_\omega^{-1}(0), 0) \cap I_-, \Delta_0^+(\omega) = (0, g_\omega^{-1}(0)) \cap I_+$. This implies that $\{\Delta_n^-(\omega)\}_{n\geq 0}$ and $\{\Delta_n^+(\omega)\}_{n\geq 0}$ are mod 0 partitions of I_- and I_+ respectively. We again define recursively, for every $n \geq 1$,

$$\delta_n^-(\omega) := g_\omega^{-1}(\Delta_{n-1}^+(\sigma\omega)) \cap \Delta_0^-(\omega) \quad \text{and} \quad \delta_n^+(\omega) := g_\omega^{-1}(\Delta_{n-1}^-(\sigma\omega)) \cap \Delta_0^+(\omega).$$

We have that the collections $\{\delta_n^-(\omega)\}_{n\geq 1}$ and $\{\delta_n^+(\omega)\}_{n\geq 1}$ are mod 0 partitions of $\Delta_0^-(\omega)$ and $\Delta_0^+(\omega)$ respectively. Further, for every $n \geq 1$, $g_\omega : \delta_n^-(\omega) \to \Delta_{n-1}^+(\sigma\omega)$ and $g_\omega : \delta_n^+(\omega) \to \Delta_{n-1}^-(\sigma\omega)$ are C^2 diffeomorphisms and $g_\omega^n : \delta_n^-(\omega) \to \Delta_0^+(\sigma^n\omega)$ and $g_\omega^n : \delta_n^+(\omega) \to \Delta_0^-(\sigma^n\omega)$ are C^2 diffeomorphisms.
For all $i \geq 1$, define

$$\delta_{ij}^-(\omega) = \delta_i^-(\omega) \cap g_\omega^{-i}\left(\delta_j^+(\sigma^i\omega)\right) \quad \text{and} \quad \delta_{ij}^+(\omega) = \delta_i^+(\omega) \cap g_\omega^{-i}\left(\delta_j^-(\sigma^i\omega)\right).$$

for $j \geq 1$. Again the collection $\{\delta_{ij}^-(\omega)\}_{i,j\geq 1}$ is a partition for $\delta_i^-(\omega)$ and $g_\omega^{i+j} : \delta_{ij}^-(\omega) \to \Delta_0^-(\sigma^{i+j}\omega)$ is a *full branch induced map* which is a C^2 diffeomorphism.
Set $\Lambda_\omega = \Delta_0^-(\omega)$ and let $\mathcal{P}_\omega = \{\delta_{ij}^-(\omega) | i, j \geq 1\}$ be a partition of Λ_ω. Define the return time $R_\omega : \Lambda_\omega \to \mathbb{N}$ as

$$R_\omega|_{\delta_{ij}^-(\omega)} = i + j, \text{ for all } i, j \geq 1. \tag{3.6}$$

For each $\omega \in \Omega$ we define the tower Δ_ω as

$$\Delta_\omega = \bigcup_{\ell=0}^{\infty} \bigcup_{k=2}^{\infty} \bigcup_{i+j=\ell+k} \delta_{ij}^-(\sigma^{-\ell}\omega) \times \{\ell\}.$$

The collection $(\Delta_\omega, F_\omega)$ gives rise to a **fibred system** and the tower map $F_\omega : (\omega, \Delta_\omega) \to (\sigma\omega, \Delta_{\sigma\omega})$ is a **fibred map**. As in [BBR19; ABR22] we define a tower projection

Definition 3.2.1 (Tower Projection Map)**.** For a.e $\omega \in \Omega$ and $(x, \ell) \in \Delta_\omega$, we define projection map as $\pi_\omega : \Delta_\omega \to \Lambda_\omega$ as $\pi_\omega(x, \ell) = g_{\sigma^{-\ell}\omega}^\ell(x)$.

We now state the conditions in [BBR19] which we will verify for the Random Tower.

(P1) **Markov**: for each $\Lambda_j(\omega)$ the map $F_\omega^{R_\omega}|\Lambda_j(\omega) : \Lambda_j(\omega) \to \Lambda$ is a bijection;

(P2) **Bounded distortion**: There are constants $D > 0$ and $0 < \gamma < 1$ such that for all ω and each $\Lambda_j(\omega)$ the map $F_\omega^{R_\omega}|\Lambda_j(\omega)$ and its inverse are non-singular with respect to m with corresponding Jacobian $JF_\omega^{R_\omega}|\Lambda_j(\omega)$ which is positive and for each $x, y \in \Lambda_j(\omega)$ satisfies the following

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)} - 1\right| \leq D\gamma^{s(F_\omega^{R_\omega}(x,0), F_\omega^{R_\omega}(y,0))}; \tag{3.7}$$

(P3) **Weak expansion**: $\mathcal{P}_\omega$ is a generating partition for F_ω i.e. diameters of the partitions $\vee_{j=0}^{n} F_\omega^{-j}\mathcal{P}_{\sigma^j\omega}$ converge to zero as n tends to infinity;

(P4) **Return time asymptotics**: There are constants $C > 0$, $a > 1$, $b \geq 0$, $u > 0$, $v > 0$, a full measure subset $\Omega_1 \subset \Omega$ and a random variable $n_1 : \Omega_1 \to \mathbb{N}$ such that

$$\begin{cases} m\{x \in \Lambda \mid R_\omega(x) > n\} \leq C\frac{(\log n)^b}{n^a}, \text{ whenever } n \geq n_1(\omega), \\ P\{n_1(\omega) > n\} \leq Ce^{-un^v}; \end{cases} \tag{3.8}$$

(P5) **Aperiodicity**: There are $N \in \mathbb{N}$ and $\{t_i \in \mathbb{Z}_+ \mid i = 1, 2, ..., N\}$ such that g.c.d.$\{t_i\} = 1$ and $\epsilon_i > 0$ so that for almost every $\omega \in \Omega$ and $i = 1, 2, \dots N$ we have $m\{x \in \Lambda \mid R_\omega(x) = t_i\} > \epsilon_i$.

(P6) **Finiteness**: There exists an $M > 0$ such that $m(\Delta_\omega) \leq M$ for all $\omega \in \Omega$.

(P7) **Annealed return time asymptotics**: There are constants $C > 0$, $\hat{b} \geq 0$ and $a > 1$ such that $\int_\Omega m\{x \in \Lambda | \, R_\omega = n\} dP \leq C\frac{(\log n)^{\hat{b}}}{n^{a+1}}$.

Proposition 3.2.2. *The fibred system (Δ, F), where $\Delta = \{\Delta_\omega\}_\omega$ and $F = \{F_\omega\}_\omega$ satisfy conditions (P1), (P3) and (P5).*

Proof. $g_\omega^{R_\omega}|_{\delta_{ij}^-(\omega)} : \delta_{ij}^-(\omega) \to \Lambda_\omega$ is a bijection by construction, which implies (P1). (P3) follows because g is uniformly expanding on $\Delta_0^-(\omega)$ and its derivative is always bigger than one. By (3.6) we have all the reutrn times starting from 2 for all $\omega \in \Omega$. Thus aperiodicity assumption (P5) is satisfied. ◻

It remains to prove bounded distortion (P2), return time asymptotic (P4) and finiteness (P6). To do this we need an estimate for $\delta_{ij}^\pm(\omega)$ for almost every $\omega \in \Omega$ and $i, j \geq 1$.

3.2.2 Technical Estimates

In this section, we prove asymptotic estimates for partitions $\{\Delta_n^\pm(\omega)\}_n$ and $\{\delta_n^\pm(\omega)\}_n$ and their endpoints for almost all ω under a natural assumption on the asymptotic of expectation of ν.
We start by defining sequences $\{x_n^\pm(\omega)\}_{n\geq 0}$ of the endpoints of $\{\Delta_n^\pm(\omega)\}_n$ as

$$x_n^-(\omega) = \left(g_\omega^n \mid_{(-1,0)}\right)^{-1}(0) \text{ and } x_n^+(\omega) = \left(g_\omega^n \mid_{(0,1)}\right)^{-1}(0) \tag{3.9}$$

for $n \geq 0$. Thus, for $n \geq 0$ hold that

$$\Delta_n^-(\omega) = (x_{n+1}^-(\omega), x_n^-(\omega)) \quad \text{ and } \quad \Delta_n^+(\omega) = (x_{n-1}^+(\omega), x_n^+(\omega)).$$

Further, define the sequences $\{y_n^\pm(\omega)\}_{n\geq 1}$ for $n \geq 1$ by

$$y_n^-(\omega) = \left(g_\omega \mid_{(-1,0)}\right)^{-1}\left(x_{n-1}^+(\sigma\omega)\right) \text{ and } y_n^+(\omega) = \left(g_\omega \mid_{(0,1)}\right)^{-1}\left(x_{n-1}^-(\sigma\omega)\right). \tag{3.10}$$

Thus, it holds that for $n \geq 1$ the intervals

$$\delta_n^-(\omega) = (y_{n-1}^-(\omega), y_n^-(\omega)) \quad \text{ and } \quad \delta_n^+(\omega) = (y_n^+(\omega), y_{n-1}^+(\omega)).$$

In this section we use the following estimate obtained in [G+10, Lemma 2].

Lemma 3.2.3 (Lemma 2, [G+10])**.** *For all n, $x^{\pm}_{n+1} = x^{\pm}_n \pm \frac{1}{2\alpha}(1 \mp x^{\pm}_n)^{\alpha}$,*

$$1 - x^+_n \sim \left(\frac{2\alpha}{\alpha-1}\right)^{-\frac{1}{\alpha-1}} n^{-\frac{1}{\alpha-1}} \quad \text{and} \quad x^-_n + 1 \sim \left(\frac{2\alpha}{\alpha-1}\right)^{-\frac{1}{\alpha-1}} n^{-\frac{1}{\alpha-1}}.$$

$$\Delta^-_n \sim \frac{1}{2\alpha}\left(\frac{2\alpha}{\alpha-1}\right)^{-\frac{\alpha}{\alpha-1}} n^{-\frac{\alpha}{\alpha-1}} \quad \text{and} \quad y^-_n \sim \left(\frac{2\alpha}{\alpha-1}\right)^{-\frac{\alpha}{\alpha-1}} n^{-\frac{\alpha}{\alpha-1}}.$$

Definition of $x^{\pm}_n(\omega)$'s implies that for all $\omega \in \Omega$ holds:

$$1 \mp x^{\pm}_n(\alpha_1) \le 1 \mp x^{\pm}_n(\omega) \le 1 \mp x^{\pm}_n(\alpha_2). \tag{3.11}$$

Lemma 3.2.4.

$$\begin{aligned} (1 \mp x^{\pm}_{n-1}(\alpha_1))^{\alpha(\omega)} &\lesssim y^{\mp}_n(\omega) \lesssim (1 \mp x^{\pm}_{n-1}(\alpha_2))^{\alpha(\omega)}; \\ (1 \mp x^{\pm}_{n-1}(\alpha_1))^{\alpha(\omega)} &\lesssim m(\delta^{\mp}_n(\omega)) \lesssim (1 \mp x^{\pm}_{n-1}(\alpha_2))^{\alpha(\omega)} \end{aligned} \tag{3.12}$$

Proof. By definition 3.1, $y^+_n(\omega) = \frac{1}{2\alpha(\omega)}(1 + x^-_{n-1}(\sigma\omega))^{\alpha(\omega)}$. Applying (3.11) we get the estimate for $y^+_n(\omega)$. This proves the first set of inequalities. Noting that $m(\delta^+_n(\omega)) = y^+_{n-1}(\omega) - y^+_n(\omega)$ and applying the first set of inequalities in (3.12) we get the desired bound for $m(\delta^+_n(\omega))$. □

Lemma 3.2.5. *For each $t > 0$ we have*

$$P\left\{\frac{(\log n)^q}{n}\left|\sum_{k=1}^n A_k - \sum_{k=1}^n E_\nu A_k\right| > t\right\} \le \exp\left[-\frac{2nt^2}{(\log n)^{2q} s^2}\right] \tag{3.13}$$

where A_k is given in (3.3).

Proof of Sublemma 3.2.5. Apply Hoeffding Inequality [Theorem 1, [Hoe63]] to

$$P\left\{\frac{1}{n}\left|\sum_{k=1}^n A_k - \sum_{k=1}^n E_\nu A_k\right| > \frac{t}{(\log n)^q}\right\}$$

noting that$-(1/2+M)^2 < A_k < (1/2+M)$ and taking $s = (1/2+M) + (1/2+M)^2$. □

Lemma 3.2.6. *Suppose **(A)** hold. Then for every $0 < c < c(\nu)$ there are constants $C > 0$, $u > 0$, $v > 0$ and a random variable $n_1 : \Omega \to \mathbb{N}$ such that $[1 - x^+_n(\omega)]^{\alpha_1 - 1} \le \frac{1}{c} \cdot \frac{(\log n)^q}{n}$ for all $n > n_1(\omega)$ and $P\{n_1(\omega) > n\} \le C\exp[-un^v]$ for all $n \in \mathbb{N}$. Moreover,*

$$\limsup_n \frac{n^{\frac{1}{\alpha_1-1}}(1 - x^+_n(\omega))}{\left[(\log n)^{\frac{q}{\alpha_1-1}}\right]} \le \frac{1}{[c(\nu)]^{\frac{1}{\alpha_1-1}}}.$$

Proof. We have from Lemma 3.2.3 that $1 - x^+_n(\alpha_1) \sim (2\alpha_1/(\alpha_1-1))^{-\frac{1}{\alpha_1-1}} n^{-\frac{1}{\alpha_1-1}}$. Define $c_n(\alpha_1) = (1 - x^+_n(\alpha_1))n^{\frac{1}{\alpha_1-1}}$ and write $(1 - x^+_n(\alpha_1)) = c_n(\alpha_1)n^{-\frac{1}{\alpha_1-1}}$. Then $c(\alpha_1) = \lim_n c_n(\alpha_1) = (2\alpha_1/(\alpha_1-1))^{-\frac{1}{\alpha_1-1}}$. Similarly we have $c_n(\alpha_2)$ and $c(\alpha_2)$.
By the definition of the map in (3.1) and (3.2) we have

$$\begin{aligned} &\left[1 - x^+_{n-1}(\sigma\omega)\right]^{-(\alpha_1-1)} = \left[1 - g_\omega(x^+_n(\omega))\right]^{-(\alpha_1-1)} \\ &= \left[1 - x^+_n(\omega)\right]^{-(\alpha_1-1)}[1 + \left[\frac{1}{2\alpha(\omega)} + u_{\alpha(\omega)}(x^+_n(\omega))\right](1 - x^+_n(\omega))^{\alpha(\omega)-1}]^{-(\alpha_1-1)}. \end{aligned} \tag{3.14}$$

Using the inequality

$$[1+y]^{-\beta} \leq 1-\beta y+\frac{\beta(1+\beta)}{2} y^{2}$$

with $y=\left[\frac{1}{2\alpha(\omega)}+u_{\alpha(\omega)}(x_n^+(\omega))\right](1-x_n^+(\omega))^{\alpha(\omega)-1}$ we have

$$\begin{aligned}\left[1-x_n^+(\omega)\right]^{-(\alpha_1-1)} - [1-x_{n-1}^+(\sigma\omega)]^{-(\alpha_1-1)} \geq & \Bigg[(\alpha_1-1)\left[\frac{1}{2\alpha(\omega)}u_{\alpha(\omega)}(x_n^+(\omega))\right](1-x_n^+(\omega))^{\alpha(\omega)-\alpha_1} \\ & -\frac{\alpha_1(\alpha_1-1)}{2}\left[\frac{1}{2\alpha(\omega)}+u_{\alpha(\omega)}(x_n^+(\omega))\right]^2(1-x_n^+(\omega))^{2\alpha(\omega)-\alpha_1-1}\Bigg].\end{aligned}$$

Iterating this step on the sequence $x_k^+(\sigma^{n-k}\omega)$ for $k=0$ to $k=n-1$ and letting $x_0^+(\sigma^n\omega)=0$ we have

$$\begin{aligned}\frac{1}{\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq 1+(\alpha_1-1)\Bigg[&\sum_{k=1}^{n}\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)}+u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))\right](1-x_k^+(\sigma^{n-k}\omega))^{\alpha(\sigma^{n-k}\omega)-\alpha_1} \\ & -\frac{\alpha_1}{2}\sum_{k=1}^{n}\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)}+u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))\right]^2(1-x_k^+(\sigma^{n-k}\omega))^{2\alpha(\sigma^{n-k}\omega)-\alpha_1-1}\Bigg].\end{aligned} \tag{3.15}$$

Applying (3.11) on every element of the right hand side of (3.15) we obtain

$$\begin{aligned}\frac{1}{\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq (\alpha_1-1)\sum_{k=1}^{n}\Bigg[&\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)}+u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))\right](1-x_k^+(\alpha_1))^{\alpha(\sigma^{n-k}\omega)-\alpha_1}. \\ & -\frac{\alpha_1}{2}\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)}+u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))\right]^2(1-x_k^+(\alpha_2))^{2\alpha(\sigma^{n-k}\omega)-\alpha_1-1}\Bigg].\end{aligned} \tag{3.16}$$

We further apply the weak estimate (3.11) for every element of the right hand side to obtain

$$\begin{aligned}\frac{1}{\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq & \sum_{k=1}^{n}\Bigg((\alpha_1-1)\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)}+u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))\right]\left[\frac{c_k(\alpha_1)}{k^{\frac{1}{\alpha_1-1}}}\right]^{\alpha(\sigma^{n-k}\omega)-\alpha_1} \\ & -\frac{\alpha_1(\alpha_1-1)}{2}\left[\frac{1}{2\alpha(\sigma^{n-k}\omega)}+u_{\alpha(\sigma^{n-k}\omega)}(x_k^+(\sigma^{n-k}\omega))\right]^2\left[\frac{c_k(\alpha_2)}{k^{\frac{1}{\alpha_2-1}}}\right]^{2\alpha(\sigma^{n-k}\omega)-\alpha_1-1}\Bigg) \\ = & \sum_{k=1}^{n} A_k(\omega).\end{aligned}$$

This implies

$$\frac{(\log n)^q}{n\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq \frac{(\log n)^q}{n}\sum_{k=1}^{n} A_k(\omega). \tag{3.17}$$

where $(A_k(\omega))_{k\geq 1}$ is a sequence of random variable as defined in 3.3. Fix arbitrary $0<c<c(\nu)$. By Assumption **(A)** there exist $N(c)\in\mathbb{N}$ independent of ω such that

$$\frac{(\log n)^q}{n}\sum_{k=1}^{n} E_\nu A_k \geq \frac{c+c(\nu)}{2}$$

for all $n > N_1 = N(c)$. Together with Lemma 3.2.5 we have that for all $n \geq N_1$:

$$\begin{aligned} P\left\{\frac{(\log n)^q}{n}\sum_{k=1}^{n} A_k < c\right\} &= P\left\{\frac{(\log n)^q}{n}\left\{\sum_{k=1}^{n} A_k - \sum_{k=1}^{n} E_\nu A_k\right\} < c - \frac{(\log n)^q}{n}\sum_{k=1}^{n} E_\nu A_k\right\} \\ &\leq P\left\{\frac{(\log n)^q}{n}\left|\sum_{k=1}^{n} A_k - \sum_{k=1}^{n} E_\nu A_k\right| > \frac{c(\nu)-c}{2}\right\} \\ &\leq \exp\left[-\frac{2n[c(\nu)-c]^2}{(\log n)^{2q}s^2}\right] \end{aligned}$$

where $s = (1/2 + M) + (1/2 + M)^2, M \in \mathbb{R}$. We rewrite the events above as

$$\Xi_i = \left\{\omega \in \Omega \middle| \frac{(\log(N(c)+i))^q}{N(c)+i}\sum_{k=1}^{N(c)+i} A_k(\omega) < c\right\}, i \in \mathbb{N}.$$

There exist constant $0 < v < 1$ $u > 0$ and $C < \infty$ such that

$$\sum_{n>N_1} \exp\left[-\frac{2n[c(\nu)-c]^2}{(\log n)^{2q}s^2}\right] \leq \sum_{n\geq N_1} \exp\left[-un^v\right] \leq \exp\left[-uN_1^v\right] < \infty.$$

This implies that the sum of probabilities of events $(\Xi_n)_n$ is bounded. Applying Borel-Cantelli lemma yield

$$P(\cup_{m=1}^{\infty} \cap_{n\geq m} \Xi_n^c) = 1.$$

Thus the intersection $\Omega' = \cup_{m=1}^{\infty} \cap_{n\geq m} \Xi_n^c$ is a full measure set and random variable

$$n_1(\omega) := \inf\{m > N_1 | \forall n > m, \frac{(\log n)^q}{n}\sum_{k=1}^{n} A_k(\omega) \geq c\}$$

is finite on Ω'. If $n \geq n_1(\omega)$, then equation (3.17) implies that $\frac{(\log n)^q}{n\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq c$. Thus,

$$\liminf_n \frac{(\log n)^q}{n\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq c.$$

We now choose a sequence of $c_k \to c(\nu)$ and repeat the process to construct a full measure set $\Omega^{(k)}$, with $\liminf_n \frac{(\log n)^q}{n\left[1-x_n^+(\omega)\right]^{\alpha_1-1}} \geq c_k$ for every $\omega \in \Omega^{(k)}$. The intersection $\cap_k \Omega^{(k)}$ has a full measure and for every $\omega \in \cap_k \Omega^{(k)}$ holds

$$\limsup_n \frac{n^{\frac{1}{\alpha_1-1}}(1-x_n^+(\omega))}{\left[(\log n)^{\frac{q}{\alpha_1-1}}\right]} \leq \frac{1}{\left[c(\nu)\right]^{\frac{1}{\alpha_1-1}}}.$$

For $m > N_1$ fix $0 < v < 1$, we have that

$$P\left\{n_1(\omega) > m\right\} \leq \sum_{n>m} \exp\left[-\frac{2n[c(\nu)-c]^2}{(\log n)^{2q}s^2}\right] \leq \sum_{n\geq m} \exp\left[-un^v\right] \leq C\exp\left[-um^v\right]$$

for suitable constants $C > 0$ and $u > 0$. □

We assume **(A)** in the following corollaries.

Corollary 3.2.7. *For every $0 < c < c(\nu)$ there are constants $C > 0$, $u > 0$, $v > 0$ and a random variable $n_1 : \Omega \to \mathbb{N}$ such that*

$$y_n^-(\omega) \geq -\frac{C}{2\alpha_1}\left[\frac{(\log n)^q}{cn}\right]^{\frac{\alpha_1}{\alpha_1-1}}, \quad y_n^+(\omega) \geq \frac{C}{2\alpha_2}\left[\frac{(\log n)^q}{cn}\right]^{\frac{\alpha_2}{\alpha_1-1}}, \tag{3.18}$$

for all $n > n_1(\omega)$ and $P\{n_1(\omega) > n\} \leq C\exp[-un^v]$ for all $n \in \mathbb{N}$.

Proof. Recall that for points $x \in [-1, 0)$ the map is defined as $g(-x) = -g(x)$. Therefore, using $g_\omega(y_n^-(\omega)) = x_{n-1}^+(\sigma\omega)$ and $y_n^-(\omega)$ is in the neighbourhood of 0 by definition g_ω we have

$$y_n^-(\omega) = -\frac{1}{2\alpha(\omega)}\left(1 - x_{n-1}^+(\sigma\omega)\right)^{\alpha(\omega)}$$

Let $0 < c < c(\nu)$ by Lemma 3.2.6 we have

$$y_n^-(\omega) \geq -\frac{1}{2\alpha(\omega)}\left[\frac{(\log(n-1))^q}{c(n-1)}\right]^{\frac{\alpha_1}{\alpha_1-1}}$$

for all $n > n_1(\sigma\omega)$ and $P\{n_1(\omega) > n\} \leq C\exp[-un^v]$ for all $n \in \mathbb{N}$. Since $\alpha(\omega) \geq \alpha_1$ this completes the proof for $y_n^-(\omega)$. The proof for $y_n^-(\omega)$ is analogous the only difference is we use $\alpha(\omega) \leq \alpha_2$. ☐

Corollary 3.2.8. *There are constants $C > 0$, $u > 0$,$v > 0$, and a random variable $n_1 : \Omega \to \mathbb{N}$ such that*

$$m(\delta_n^\pm(\omega)) \leq C\left[\frac{(\log n)^q}{n}\right]^{\frac{\alpha_1}{\alpha_1-1}} \quad \text{and} \quad m(\Delta_n^\pm(\omega)) \leq C\left[\frac{(\log n)^q}{n}\right]^{\frac{1}{\alpha_1-1}}, \tag{3.19}$$

for all $n > n_1(\omega)$ and $P\{n_1(\omega) > n\} \leq C\exp[-un^v]$.

Proof of Corollary. Let $0 < c < c(\nu) < c'$. By Lemma 3.2.6, we have for all $n > \bar{n}_1(\omega) := \max\{n_1(\omega), n_2(\omega)\}$,

$$\begin{aligned} m(\Delta_n^+(\omega)) &= x_{n+1}^+(\omega) - x_n^+(\omega) = (1 - x_n^+(\omega)) - (1 - x_{n+1}^+(\omega)) \\ &\leq \left[\frac{(\log n)^q}{cn}\right]^{\frac{1}{\alpha_1-1}}. \end{aligned}$$

Similarly, by corollary 3.2.7, for all $n > \bar{n}_1(\omega)$ we have

$$\begin{aligned} &m(\delta_n^-(\omega)) = y_n^-(\omega) - y_{n-1}^-(\omega) \\ &\leq \frac{1}{2\alpha_1}\left[\frac{(\log(n-2))^q}{c(n-2)}\right]^{\frac{\alpha_1}{\alpha_1-1}} \lesssim \left[\frac{(\log n)^q}{n}\right]^{\frac{\alpha_1}{\alpha_1-1}}. \end{aligned}$$

☐

3.3 Distortion Estimates

Definition 3.3.1. Define $s : \Delta \times \Delta \to \mathbb{Z}_+ \cup \{\infty\}$ as $s(z_1, z_2) = 0$ if z_1 and z_2 lie in different towers Δ_ω and if $z_1, z_2 \in \Delta_\omega$, then

$$s(z_1, z_2) = \min\{n \geq 0 | \left(F_\omega^{R_\omega}\right)^n (z_1) \text{ and } \left(F_\omega^{R_\omega}\right)^n (z_2) \text{ lie in distinct elements}\}$$

Proposition 3.3.2. *There are constants $\mathfrak{D} > 0$ and $0 < \gamma < 1$ such that for all ω and $x, y \in \delta^{\pm}_{i,j}(\omega)$*

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)}\right| \leq \mathfrak{D}\gamma^{s(F_\omega^{R_\omega}(x,0), F_\omega^{R_\omega}(y,0))}.$$

Let U_{1^-} and U_{-1^+} be sufficiently small, one-sided neighbourhoods of 1 and -1 such that the expansion in (3.2) holds for g_{α_1} and g_{α_2}.

$$U_{0^+} = \left(g_{\alpha_1}^{-1}\left(U_{-1^+}\right)\right) \cap \left(g_{\alpha_2}^{-1}\left(U_{-1^+}\right)\right) \quad \text{and} \quad U_{0^-} = \left(g_{\alpha_1}^{-1}\left(U_{1^-}\right)\right) \cap \left(g_{\alpha_2}^{-1}\left(U_{1^-}\right)\right).$$

Since g_ω is C^2 for all ω on $I \backslash \left(U_{0^+} \cup U_{0^-} \cup U_{-1^+} \cup U_{1^-}\right)$, it suffices to restrict our prove to this sets. We start by proving the simpler version of the result and then we use the tower map to get the above result.

Lemma 3.3.3. *There exists a constant $\mathfrak{D} > 0$ such that for all $0 \leq m < n$, all $x, y \in \delta^{\pm}_n(\omega)$ and all $\omega \in \Omega$*

$$\log \frac{\left|\left(g_{\sigma^m\omega}^{n-m}\right)'\left(g_\omega^m(x)\right)\right|}{\left|\left(g_{\sigma^m\omega}^{n-m}\right)'\left(g_\omega^m(y)\right)\right|} \leq \mathfrak{D}|g_\omega^n(x) - g_\omega^n(y)|.$$

We first prove the following preliminary result.

Sublemma 3.3.4. *There exists a constant $K > 0$ such that for all $1 \leq i < n$, all $x \in \delta^{\pm}_n(\omega)$ and all $\omega \in \Omega$*

$$\mathfrak{D}_i(\omega) := \frac{\left|g''_{\sigma^i\omega}(g_\omega^i(x))\right|}{\left|g'_{\sigma^i\omega}(g_\omega^i(x))\right|} \lesssim (1 - g_{\alpha_1}^i(x))^{\alpha_2 - 2} \quad \text{and} \quad \mathfrak{D}_0(\omega) := \frac{|g''_\omega(x)|}{|g'_\omega(x)|} \lesssim |x|^{-1}.$$

Proof of Sublemma 3.3.4. Let $x \in \delta^-_n(\omega) \subset U_{0^-}$. Then $g_\omega^i(x) \in \Delta^+_{n-i}(\sigma^i\omega) \subset U_{1^-}$ and $0 < (1 - g_\omega^i(x)) < 1$. Using the expansion (3.2) noting that $g(-1) = -g(x)$ we have that there are functions $v_{\sigma^i\omega}(x), w_{\sigma^i\omega}(x) \to 0$ such that

$$\begin{aligned}\frac{\left|g''_{\sigma^i\omega}(g_\omega^i(x))\right|}{\left|g'_{\sigma^i\omega}(g_\omega^i(x))\right|} &= \frac{\left|\left(w_{\sigma^i\omega}(x) - (\alpha(\sigma^i\omega) - 1)/2\right)(1 - g_\omega^i(x))^{\alpha(\sigma^i\omega)-2}\right|}{\left|1 + (1/2 + v_{\sigma^i\omega}(x))\,(1 - g_\omega^i(x))^{\alpha(\sigma^i\omega)-1}\right|} \\ &\lesssim \left|\left(w_{\sigma^i\omega}(x) - (\alpha(\sigma^i\omega) - 1)/2\right)(1 - g_\omega^i(x))^{\alpha(\sigma^i\omega)-2}\right|.\end{aligned}$$

Lemma 3.11 implies $1 - g_{\alpha_1}^i(x) < 1 - g_\omega^i(x) < 1 - g_{\alpha_2}^i(x)$. So

$$\frac{\left|g''_{\sigma^i\omega}(g_\omega^i(x))\right|}{\left|g'_{\sigma^i\omega}(g_\omega^i(x))\right|} \leq \sup_{x \in I, \beta \in [\alpha_1, \alpha_2]} \{|w_\beta(x) - (\alpha_1 - 1)/2|\,(1 - g_{\alpha_1}^i(x))^{\alpha_2 - 2}\} \lesssim (1 - g_{\alpha_1}^i(x))^{\alpha_2 - 2}.$$

For $\mathfrak{D}_0(\omega)$ we have by definition of our map (3.1) that

$$\frac{|g''_\omega(x)|}{|g'_\omega(x)|} = \frac{\left|\left(\frac{1}{\alpha(\omega)}(\frac{1}{\alpha(\omega)}-1)(2\alpha(\omega))^{\frac{1}{\alpha(\omega)}} + w_\omega(x)\right) x^{\frac{1}{\alpha(\omega)}-2}\right|}{\left|\left((2\alpha(\omega))^{\frac{1}{\alpha(\omega)}}/\alpha(\omega) + v_\omega(x)\right) x^{\frac{1}{\alpha(\omega)}-1}\right|}$$
$$\leq \frac{\sup_{\beta\in[\alpha_1,\alpha_2]}\{\left|\frac{1}{\alpha_1}(\frac{1}{\alpha_1}-1)(2\alpha_2)^{\frac{1}{\alpha_1}} + w_\beta(x)\right|\}|x^{-1}|}{\left|(2\alpha_1)^{\frac{1}{\alpha_2}}/\alpha_2 + v_\omega(x)\right|} \lesssim |x|^{-1}.$$

□

Sublemma 3.3.5. *There exist $0 < C < \infty$ such that for all $n \in \mathbb{N}$ and $\omega \in \Omega$*

$$\frac{x^-_{n-2}(\sigma\omega) - x^-_{n-1}(\sigma\omega)}{x^-_{n-1}(\sigma\omega) - x^-_n(\sigma\omega)} < C.$$

Proof. By the definition of maps g_ω the first four pre-images of 0 satisfy:

$$\sup_{\omega\in\Omega} x^-_i(\omega) < \inf_{\omega\in\Omega} x^-_{i-1}(\omega), \quad i = 1,2,3,4.$$

(Recall that $x^-_n(\omega) = g^{-n}_\omega(0)$). Thus there exists $\tau > 0$ such that for any $\omega \in \Omega$ we have:

$$|x^-_1(\omega) - x^-_3(\omega)| < \tau|x^-_{3i}(\omega) - x^-_{3i+1}(\omega)| \text{ for } i = 0, 3. \tag{3.20}$$

By construction $g^n_\omega : [x^-_{n+4}(\omega), x^-_n(\omega)) \to [x^-_4(\omega), x^-_0(\omega))$ is a diffeomoprhism for $n \geq 1$. Moreover, $g^n_\omega\left([x^-_{n+4}(\omega), x^-_n(\omega))\right)$ contains τ-scaled neighbourhood of $g^n_\omega\left([x^-_{n+3}(\omega), x^-_{n+1}(\omega)\right)$ by (3.20). Since g_ω has negative schwarzian derivative on $I\backslash(\Delta_{0+}\cup\Delta_{0-})$, there exists M depending only on τ (by Koebe Principle) such that

$$\left|\frac{Dg^n_\omega(x)}{Dg^n_\omega(y)} - 1\right| \leq M \quad \forall x, y \in [x^-_{n+3}(\omega), x^-_{n+1}(\omega)] \quad \text{and} \quad n \geq 1.$$

Hence, for any $n \in \mathbb{N}$ and $\omega \in \Omega$ we have

$$\frac{x^-_{n+3}(\omega) - x^-_{n+2}(\omega)}{x^-_{n+2}(\omega) - x^-_{n+1}(\omega)} \leq M\frac{|x^-_3(\sigma^n\omega) - x^-_2(\sigma^n\omega)|}{|x^-_2(\sigma^n\omega) - x^-_1(\sigma^n\omega)|},$$

where the last term is bounded by a constant independent of ω by (3.20). □

Now we are ready to prove Lemma 3.3.3.

Proof of Lemma 3.3.3. By the chain rule, we can write

$$\log\frac{\left|g''_{\sigma^i\omega}(g^i_\omega(x))\right|}{\left|g'_{\sigma^i\omega}(g^i_\omega(x))\right|} = \log\prod_{i=m}^{n-1}\frac{\left|g'_{\sigma^i\omega}(g^i_\omega(x))\right|}{\left|g'_{\sigma^i\omega}(g^i_\omega(y))\right|} = \sum_{i=m}^{n-1}\log\frac{\left|g'_{\sigma^i\omega}(g^i_\omega(x))\right|}{\left|g'_{\sigma^i\omega}(g^i_\omega(y))\right|}. \tag{3.21}$$

As both $g^i_\omega(x), g^i_\omega(y)$ are in the same smoothness component $\Delta^\pm_{n-i}(\sigma^i\omega)$of $g_{\sigma^i\omega}$, we have by the Mean Value Theorem that there exists $u_i(\sigma^i\omega) \in (g^i_\omega(x), g^i_\omega(y))$ such that

$$\log\frac{\left|g'_{\sigma^i\omega}(g^i_\omega(x))\right|}{\left|g'_{\sigma^i\omega}(g^i_\omega(y))\right|} = \log\left|g'_{\sigma^i\omega}(g^i_\omega(x))\right| - \log\left|g'_{\sigma^i\omega}(g^i_\omega(y))\right| = \frac{\left|g''_{\sigma^i\omega}(u_i(\sigma^i\omega))\right|}{\left|g'_{\sigma^i\omega}(u_i(\sigma^i\omega))\right|}|g^i_\omega(x) - g^i_\omega(y)|.$$

Letting $\mathfrak{D}_i(\omega) := \left|g''_{\sigma^i\omega}(u_i(\sigma^i\omega))\right|/\left|g'_{\sigma^i\omega}(u_i(\sigma^i\omega))\right|$ substituting the above equation into (3.21) we get

$$\log \frac{\left|\left(g^{n-m}_{\sigma^m\omega}\right)'\left(g^m_\omega(x)\right)\right|}{\left|\left(g^{n-m}_{\sigma^m\omega}\right)'\left(g^m_\omega(y)\right)\right|} = \sum_{i=m}^{n-1} \mathfrak{D}_i(\omega)|g^i_\omega(x) - g^i_\omega(y)| \le \sum_{i=0}^{n-1} \mathfrak{D}_i(\omega)|g^i_\omega(x) - g^i_\omega(y)|. \tag{3.22}$$

Now it remains to estimate (3.22) by $\mathfrak{D}|g^n(x) - g^n(y)|$. This will be done in two steps. In the first step we find a uniform bound for the sum $\widehat{\mathfrak{D}}$ independent of m, n and ω. In the second step using the uniform bounds we improve the bounds and show the desired estimates with $\mathfrak{D} := \widehat{\mathfrak{D}}^2/|\Delta_0^-|$.

Assume that $x, y \in \delta_n^+(\omega)$, the estimates for $\delta_n^-(\omega)$ are the same. For $1 \le i < n$ we have that $g^i_\omega(x), g^i_\omega(y), u_i(\sigma^i\omega) \in \Delta^-_{n-i}(\sigma^i\omega)$ and therefore we can bound (3.22) by

$$\begin{aligned}\sum_{i=0}^{n-1} \mathfrak{D}_i(\omega)|g^i_\omega(x) - g^i_\omega(y)| &\le \mathfrak{D}_0(\omega)|x-y| + \sum_{i=1}^{n-1} \mathfrak{D}_i(\omega)|g^i_\omega(x) - g^i_\omega(y)| \\ &\le \mathfrak{D}_0(\omega)\delta_n^+(\omega) + \sum_{i=0}^{n-1} \mathfrak{D}_i(\omega)|\Delta^-_{n-i}(\sigma^i\omega)|.\end{aligned} \tag{3.23}$$

By Sublemma 3.3.4 we have for the first term $i = 0$

$$\begin{aligned}\mathfrak{D}_0(\omega)|\delta_n^+(\omega)| &\lesssim u_0(\omega)^{-1}|\delta_n^+(\omega)| \lesssim \frac{1}{y_n^+(\omega)}|\delta_n^+(\omega)| \\ &\lesssim \frac{y^+_{n-1}(\omega) - y^+_n(\omega)}{y^+_n(\omega)} \lesssim \frac{y^+_{n-1}(\omega)}{y^+_n(\omega)} - 1 \lesssim \left(1 + \frac{x^-_{n-2}(\sigma\omega) - x^-_{n-1}(\sigma\omega)}{x^-_{n-1}(\sigma\omega) - x^-_n(\sigma\omega)}\right)^{\alpha(\omega)} - 1\end{aligned} \tag{3.24}$$

for all $n \ge 2$. This is bounded by Sublemma 3.3.5.

For the other terms, we use triangular inequality, Lemma 3.3.4 and weaker estimates in (3.11)

$$\begin{aligned}\mathfrak{D}_i(\omega)|\Delta^-_{n-i}(\sigma^i\omega)| &\lesssim (1 - g^i_{\alpha_1}(x))^{\alpha_2 - 2}|\Delta^-_{n-i}(\sigma^i\omega)| \\ &\lesssim (1 - g^i_{\alpha_1}(x))^{\alpha_2 - 2}\left(|x^-_{n-i}(\sigma^i\omega) - 1| + |1 - x^-_{n-i+1}(\sigma^i\omega)|\right) \\ &\lesssim (n-i)^{-\frac{\alpha_2 - 2}{\alpha_1 - 1}}(n-i)^{-\frac{1}{\alpha_2 - 1}}.\end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) back into (3.23) we get for

$$\mathfrak{D}_0(\omega)|\delta_n^+(\omega)| + \sum_{i=1}^{n-1} \mathfrak{D}_i(\omega)|\Delta^-_{n-i}(\sigma^i\omega)| \lesssim 1 + \sum_{i=1}^{n-1}(n-i)^{-(1+\frac{1}{\alpha_2 - 1})} := \widehat{\mathfrak{D}} < \infty \tag{3.26}$$

where $\widehat{\mathfrak{D}}$ is *independent of* m, n and ω. Substituting (3.26) back into (3.23) and then into (3.22) we get

$$\log \left|\frac{(g^{n-m}_{\sigma^m\omega})'(g^m_\omega(x))}{(g^{n-m}_{\sigma^m\omega})'(g^m_\omega(y))}\right| \le \widehat{\mathfrak{D}} \tag{3.27}$$

which completes the first part of the proof. We now use this bound to improve our estimates as follows. The Mean Value Theorem, (3.27) implies that the diffemorphisms $g^n_\omega : \delta_n^+(\omega) \to \Delta_0^-(\sigma^n\omega)$ and $g^{n-m}_{\sigma^m\omega} : \Delta^-_{n-m}(\sigma^m\omega) \to \Delta_0^-(\sigma^n\omega)$ all have uniformly bounded distortion in the sense that for every $x, y \in \delta_n^+(\omega)$ and $1 \le m < n$ we have

$$\frac{|x-y|}{|\delta_n^+(\omega)|} \le e^{\widehat{\mathfrak{D}}}\frac{|g^n_\omega(x) - g^n_\omega(y)|}{|\Delta_0^-(\sigma^n\omega)|} \tag{3.28}$$

and

$$\frac{|g_\omega^m(x)-g_\omega^m(y)|}{|\Delta_{n-m}^+(\sigma^m\omega)|} \le e^{\widehat{\mathfrak{D}}} \frac{|g_{\sigma^m\omega}^{n-m}(g_\omega^m(x))-g_{\sigma^m\omega}^{n-m}(g_\omega^m(y))|}{|\Delta_0^-(\sigma^n\omega)|} = e^{\widehat{\mathfrak{D}}} \frac{|g_\omega^n(x)-g_\omega^n(y)|}{|\Delta_0^-(\sigma^n\omega)|}. \tag{3.29}$$

Therefore

$$|x-y| \le \frac{e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-(\sigma^n\omega)|}|g_\omega^n(x)-g_\omega^n(y)||\delta_n^+(\omega)| \tag{3.30}$$

and

$$|g_\omega^m(x)-g_\omega^m(y)| \le \frac{e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-|}|g_\omega^n(x)-g_\omega^n(y)||\Delta_{n-m}^-(\sigma^m\omega)|. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.22) (with $i=m$), and letting $\mathfrak{D} := \widehat{\mathfrak{D}}e^{\widehat{\mathfrak{D}}}/|\Delta_0^-|$, we get

$$\begin{aligned}
&\frac{\left(g_{\sigma^m\omega}^{n-m}\right)'(g_\omega^m(x))}{\left(g_{\sigma^m\omega}^{n-m}\right)'(g_\omega^m(y))} \le \mathfrak{D}_0(\omega)|x-y| + \sum_{i=1}^{n-1}\mathfrak{D}_i(\omega)|g_{\sigma^i\omega}^i(x)-g_{\sigma^i\omega}^i(y)| \\
&= \frac{e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-(\sigma^n\omega)|}\left[\mathfrak{D}_0(\omega)|\delta_n^+(\omega)| + \sum_{i=1}^{n-1}\mathfrak{D}_i(\omega)|\Delta_{n-i}^-(\sigma^i\omega)|\right]|g_\omega^n(x)-g_\omega^n(y)| \\
&\le \frac{\widehat{\mathfrak{D}}e^{\widehat{\mathfrak{D}}}}{|\Delta_0^-(\sigma^n\omega)|}|g_\omega^n(x)-g_\omega^n(y)| = \mathfrak{D}|g_\omega^n(x)-g_\omega^n(y)|.
\end{aligned}$$

Notice that the last inequality follows from (3.26). This completes the proof. □

Proof of Proposition 3.3.2. Let $x, y \in \delta_{i,j}(\omega)$. We have that

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)}\right| = \left|\frac{JG_\omega^{R_\omega}(x,\omega)}{JG_\omega^{R_\omega}(y,\omega)}\right| = \left|\frac{(g_\omega^{R_\omega})'(x)}{(g_\omega^{R_\omega})'(y)}\right|. \tag{3.32}$$

By the construction there exists $\gamma \in (0,1)$ such that $\left(g_\omega^{R_\omega}\right)' > \gamma^{-1}$. This implies that

$$\gamma^{-1} \le \min\left(g_\omega^{R_\omega}\right)' \le \left|\frac{\left(g_\omega^{R_\omega}\right)(x) - \left(g_\omega^{R_\omega}\right)(y)}{x-y}\right| \le \frac{1}{|x-y|}. \tag{3.33}$$

Suppose $S(x,y)=n$. then by induction on (3.33) we have

$$|x-y| \le \gamma^n. \tag{3.34}$$

The inequality $|x-1| \le \frac{e^C-1}{C}|\log x|$ for $e^{-C} < x < e^C$, Lemma 3.3.3 and (3.34) implies

$$\left|\frac{(g_\omega^{R_\omega})'(x)}{(g_\omega^{R_\omega})'(y)} - 1\right| \le \frac{e^C-1}{C}\left|\log\frac{(g_\omega^{R_\omega})'(x)}{(g_\omega^{R_\omega})'(y)}\right| \le \tilde{C}\left|g_\omega^{R_\omega}(x) - g_\omega^{R_\omega}(y)\right| \le \mathfrak{D}\gamma^{s(F_\omega^{R_\omega}(x,0),F_\omega^{R_\omega}(y,0))}.$$

This proves (P2). □

3.4 Tail Estimate

In this section we derive tails estimates for the tower for the first family. The main results is the following

Proposition 3.4.1. *There are constants $C>0, u>0, v>0$, a full measure set $G\subset\Omega$ and a random variable $n_1:\Omega\to\mathbb{N}$ such that*

$$m(R_\omega>n)=\sum_{i+j>n} m(\delta_{i,j}(\omega))\lesssim(\log n)^{\frac{q\alpha_1}{\alpha_1-1}}\cdot n^{-\frac{\alpha_1}{\alpha_1-1}},$$

for all $n>n_1(\omega)$ and $P\{n_1(\omega)>n\}\le C\exp\left[-un^v\right]$. In particular, we have

$$\int_\Omega m\{x\in\Lambda\big|R_\omega=n\}dP\lesssim(\log n)^{\frac{q\alpha_1}{\alpha_1-1}}\cdot n^{-\frac{\alpha_2}{\alpha_2-1}-1}.$$

The following lemma will be used repeatedly in the estimates later.

Lemma 3.4.2. *For any $\zeta>0$ holds*

$$\sum_{i=1}^{n-1}\frac{1}{i^{\zeta+1}\cdot(n-i)^{\zeta+1}}\le\frac{2^{\zeta+1}}{n^{\zeta+1}}\sum_{i=1}^{n-1}\left(\frac{1}{i^{\zeta+1}}+\frac{1}{(n-i)^{\zeta+1}}\right). \tag{3.35}$$

The proof follows from the standard inequality $(n-i)^{\zeta+1}+i^{\zeta+1}\ge(\frac{n}{2})^{\zeta+1}$ for $\zeta>0$.

Lemma 3.4.3. *For the deterministic map g_{α_2},*

$$\sum_{i+j=n} m(\delta^-_{i,j}(\alpha_2))\lesssim\left[2^{\frac{\alpha_2}{\alpha_2-1}+2}\sum_{i=1}^{\infty} i^{-\frac{\alpha_2}{\alpha_2-1}-1}\right]n^{-\frac{\alpha_2}{\alpha_2-1}-1}.$$

Proof. Bounded distortion implies

$$m(\delta^-_{i,j}(\alpha_2))\le\frac{\mathcal{D}}{m(\Delta_0^+(\alpha_2))}m(\delta_i^-(\alpha_2))\cdot m(\delta_j^+(\alpha_2)).$$

By the asymptotic estimate of y_n^- in Lemma 3.2.3, we have that

$$m(\delta_n^-(\alpha_2))\lesssim n^{-(\frac{\alpha_2}{\alpha_2-1}+1)}. \tag{3.36}$$

Combining the above two inequalities we have

$$\sum_{i+j=n} m(\delta^-_{i,j}(\alpha_2))\lesssim\sum_{i+j=n}(ij)^{-\frac{\alpha_2}{\alpha_2-1}-1}\lesssim\sum_{i=1}^{n}(i(n-i))^{-\frac{\alpha_2}{\alpha_2-1}-1}.$$

Now applying Lemma 3.4.2 we obtain the result. □

Proof of Proposition 3.4.1. Recall that $g^i_\omega\delta^-_{i,j}(\omega)=\delta^+_j(\sigma^i\omega)$ and $g^i_\omega\delta^-_i(\omega)=\Delta^+_0(\sigma^i\omega)$. Bounded distortion implies

$$\frac{1}{\mathcal{D}}\frac{m(\delta^+_j(\sigma^i\omega))}{m(\Delta^+_0(\sigma^i\omega))}\le\frac{m(\delta^-_{i,j}(\omega))}{m(\delta^-_i(\omega))}\le\mathcal{D}\frac{m(\delta^+_j(\sigma^i\omega))}{m(\Delta^+_0(\sigma^i\omega))}.$$

Therefore,

$$m(\delta_{i,j}^{-}(\omega)) \leq \frac{\mathcal{D}}{m(\Delta_0^{+}(\sigma^i\omega))} m(\delta_i^{-}(\omega)) \cdot m(\delta_j^{+}(\sigma^i\omega)).$$

For $n \in \mathbb{N}$ let $n_2(\omega) = \max_{0\leq i\leq n} n_1(\sigma^i\omega)$. Notice that n_2 is almost everywhere finite, i.e. for a.e. $\omega \in \Omega$ there exits $n_2(\omega) < n$, and

$$P\{n_2(\omega) > n\} \leq \sum_{i=0}^{n} P\{n_1(\sigma^i\omega) > n\} \leq (n+1)e^{-un^v}.$$

which decays at stretched exponential rate as n increases. Thus, by Proposition 3.2.8 and inequality (3.36) for all $\omega \in \Omega$ and for any $n \in \mathbb{N}$ such that $n > 2n_2(\omega)$ we have

$$m(\delta_{i,j}^{-}(\omega)) \lesssim (\log n)^{\frac{q\alpha_1}{\alpha_1-1}} \cdot \left(\frac{2}{n}\right)^{\frac{\alpha_1}{\alpha_1-1}} j^{-\frac{\alpha_1}{\alpha_2-1}} \text{ for } i \geq n/2 \text{ and } i+j=n. \tag{3.37}$$

Similarly, for small is we have

$$m(\delta_{i,j}^{-}(\omega)) \lesssim (\log n)^{\frac{q\alpha_1}{\alpha_1-1}} \cdot \left(\frac{2}{n}\right)^{\frac{\alpha_1}{\alpha_1-1}} i^{-\frac{\alpha_1}{\alpha_2-1}} \text{ for } 1 \leq i < n/2 \text{ and } i+j=n. \tag{3.38}$$

Using (3.37) and (3.38) we obtain

$$\sum_{i+j=n} m(\delta_{i,j}(\omega)) \leq (\log n)^{\frac{q\alpha_1}{\alpha_1-1}} \cdot \left(\frac{2}{n}\right)^{\frac{\alpha_1}{\alpha_2-1}} \sum_{i=1}^{[n/2]} i^{-\frac{\alpha_1}{\alpha_2-1}} \lesssim (\log n)^{\frac{q\alpha_1}{\alpha_1-1}} \cdot \left(\frac{2}{n}\right)^{\frac{\alpha_1}{\alpha_2-1}}. \tag{3.39}$$

where in the last estimate we used $1 < \alpha_1 < \alpha_2 < 2$ we have $\alpha_1/(\alpha_2-1) > 1$.
This further implies that, for any $n > 2n_2(\omega)$

$$m(R_\omega > n) = \sum_{k>n} \sum_{i+j=k} m(\delta_{i,j}(\omega)) \lesssim \sum_{k>n} (\log k)^{\frac{q\alpha_1}{\alpha_1-1}} k^{-\frac{\alpha_1}{\alpha_1-1}} \leq C(\log n)^{\frac{q\alpha_1}{\alpha_1-1}} n^{1-\frac{\alpha_1}{\alpha_1-1}}.$$

for some $C > 0$.
We have by equation (3.39) that

$$\begin{aligned}
\int_\Omega m\{x \in \Lambda \big| R_\omega = n\} dP &= \int_\Omega \sum_{i+j=n} m(\delta_{i,j}(\omega)) dP \\
&\lesssim \int_{\{n_2(\omega)<n\}} \sum_{i+j=n} m(\delta_{i,j}(\omega)) dP + \int_{\{n_2(\omega)>n\}} \sum_{i+j=n} m(\delta_{i,j}(\omega)) \\
&\lesssim (\log n)^{\frac{q\alpha_1}{\alpha_1-1}} n^{-\frac{\alpha_1}{\alpha_1-1}} + C\exp\left[-un^v\right].
\end{aligned}$$

which proves the claim. ◻

Corollary 3.4.4. *There exist $M < \infty$ such that $m(\Delta_\omega) < M$ for all $\omega \in \Omega$.*

Proof. By definition

$$m(\Delta_\omega) = \sum_{\ell=0}^{\infty} \sum_{k=2}^{\infty} \sum_{i+j=\ell+k} m\left(\delta_{ij}^{-}(\sigma^{-\ell}\omega)\right)$$

Applying inequality 3.36 to the inner sum (over $i+j$) in bracket we get

$$m(\Delta_\omega) \lesssim \sum_{\ell=0}^{\infty} \sum_{k=2}^{\infty} \frac{(\log(\ell+k))^{\frac{q\alpha_1}{\alpha_1-1}}}{(\ell+k)^{\frac{\alpha_2}{\alpha_2-1}+1}} \lesssim \sum_{\ell=0}^{\infty} \sum_{k\geq\ell+2} \frac{(\log k)^{\frac{q\alpha_1}{\alpha_1-1}}}{k^{\frac{\alpha_2}{\alpha_2-1}}} \leq \sum_{\ell\geq 2} \frac{(\log \ell)^{\frac{q\alpha_1}{\alpha_1-1}}}{\ell^{\frac{\alpha_2}{\alpha_2-1}}}$$

Thus, there exists $C > 0$ depending only on q, α_2 and α_1 such that $m(\Delta_\omega) \leq C$. ◻

3.4.1 Decay of Correlation

Now we are ready to establish decay of correlations for the random system. This procedure is somewhat standard by now. But we give the main steps for the sake of completeness.
Since Lebesgue measure is equivariant measure for the random system, it is equivariant for the fibred system $((F_{\sigma^k\omega})_* m = m)$. Thus, the random variable $K_\omega : \Omega \to \mathbb{R}^+$ in [Theorem 4.1, [BBR19]] is equal to 1. Let $0 < \gamma < 1$ be as in Proposition 3.3.2 and let

$$\mathcal{F}_\gamma^+ = \{\varphi_\omega : \Delta_\omega \to \mathbb{R}^+ | \exists C_\varphi > 0, \forall J_\omega \in \mathcal{P}_\omega \text{ either } \varphi_\omega | J_\omega \equiv 0 \text{ or } \varphi_\omega | J_\omega > 0 \\ \text{and } \left| \log \frac{\varphi_\omega(x)}{\varphi_\omega(y)} \right| \le C_\varphi \gamma^{s(x,y)} \ \forall x, y \in I_\omega\} \tag{3.40}$$

and

$$\mathcal{F}_\gamma^1 = \{\varphi_\omega \in \mathcal{L}_\infty | \exists C_\varphi > 0, |\varphi_\omega(x) - \varphi_\omega(y)| \le C_\varphi \gamma^{s(x,y)} \ \forall x, y \in \Delta_\omega\}. \tag{3.41}$$

Now we show that [Theorem 4., [BBR19]] is applicable and hence Theorem 3.1.4 follows.
Let $\varphi \in L^\infty(\mathbb{T})$ and $\psi \in \mathcal{C}^\eta(\mathbb{T})$. Define

$$\varphi_\omega = \varphi \circ \pi_\omega, \psi_\omega = \psi \circ \pi_\omega : \Delta_\omega \to \mathbb{R}$$

where $\pi_\omega : \Delta_\omega \to \Lambda_\omega$ is the tower projection defined as $\pi_\omega(x, l) = g^l_{\sigma^{-1}\omega}(x)$.
Together with change of variable formula, we have

$$\begin{aligned} \int (\varphi \circ g^n_\omega \psi dm) &= \int_{\Lambda_\omega} \left((\varphi \circ \pi_{\sigma^n\omega}) \circ (\pi^{-1}_{\sigma^n\omega} \circ g^n_\omega \circ \pi_\omega) \circ \pi^{-1}_\omega \right) \left((\psi \circ \pi_\omega) \circ \pi^{-1}_\omega \right) dm \\ &= \int_{\Delta_\omega} (\varphi_{\sigma^n\omega} \circ F^n_\omega) \psi_\omega dm. \end{aligned}$$

To obtain "past" and "future" correlations we apply [Theorem 4.2, [BBR19]]. It suffices to show that $\psi_\omega \in \mathcal{F}^1$ and $\varphi_\omega \in \mathcal{L}_\infty$. Observe that $(\mathcal{L}_\infty, \|.\|_{\mathcal{L}_\infty})$ and $(\mathcal{F}^1, \|.\|_{\mathcal{F}})$ are Banach spaces with the norms $\|\varphi\|_{\mathcal{L}_\infty} = C'_\varphi$ and $\|\varphi\|_{\mathcal{L}_\infty} = \max\{C_\varphi, C'_\varphi\}$. Since $|g^{R_\omega}| > 2$, $|x - y| \le (\frac{1}{2})^{s(x,y)}$. Together with the fact that $\psi \in \mathcal{C}^\eta(\mathbb{T})$ is we have that $\psi_\omega \in \mathcal{F}^1$. As for $\nu_\omega-$ a.e $(x, \ell) \in \Delta_\omega$ $|\varphi_\omega| \le \|\varphi\|_{L^\infty}$ and the projection π_ω is nonsingular, we have that $\varphi_\omega \in \mathcal{L}_\infty$.

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