Editors' Suggestion

Bridging Carrollian and celestial holography

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Gravity in $4d$ asymptotically flat spacetime constitutes the archetypal example of a gravitational system with leaky boundary conditions. Pursuing our previous analysis of [Carrollian Perspective on Celestial Holography, Phys. Rev. Lett. **129**, 071602 (2022)], we argue that the holographic description of such a system requires the coupling of the dual theory living at null infinity to some external sources encoding the radiation reaching the conformal boundary and responsible for the nonconservation of the charges. In particular, we show that the sourced Ward identities of a conformal Carrollian field theory living at null infinity reproduce the Bondi-van der Burg-Metzner-Sachs flux-balance laws. We also derive the general form of low-point correlation functions for conformal Carrollian field theories and exhibit a new branch of solutions, which is argued to be the relevant one for holographic purposes. We then relate our Carrollian approach to the celestial holography proposal by mapping the Carrollian Ward identities to those constraining celestial operators through a suitable integral transform.

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I. INTRODUCTION

The program of *flat-space holography* consists in building a holographic duality between gravity in asymptotically flat spacetime and a lower-dimensional field theory. The motivations are twofold. Firstly, from a purely theoretical perspective, this program is enshrined in a broader context that aims at understanding how general is the holographic principle. Does it extend beyond the framework of the celebrated AdS/CFT correspondence [1–3]? Secondly, asymptotically flat spacetimes provide realistic models to describe a huge range of physical processes occurring in our Universe, all the way up to astrophysical scales which are below the cosmological scale.

In spite of repeated early efforts from various points of views [4–9], it became clear very soon that flat-space holography would not merely immediately follow from AdS/CFT via a simple limit procedure where the anti–de Sitter (AdS) radius is sent to infinity. But fortunately, the last decades have also taught us that the symmetries of flat spacetime turn out to be way richer and more subtle than originally thought [10–15]. Building on the constraints imposed by the Bondi-van der Burg-Metzner-Sachs (BMS) symmetries, two bottom-up roads towards flat-space holography have emerged over the years: *Carrollian* and *celestial* holography.

The Carrollian approach to flat space holography proposes that the role of the dual theory is played by a *conformal Carrollian field theory* (or Carrollian CFT for short) that lives on the codimension-one boundary of spacetime (null infinity $\mathscr{I}$). Alternatively, one could refer to this theory as a BMS-invariant field theory, as it has been known for quite some time that the BMS group is isomorphic to the conformal Carroll group [16], but it is fair to say that the "Carrollian" name, due to Lévy-Leblond [17], has struck physicists' imagination.

This first approach has proven to be very successful in the context of three-dimensional ($3d$) gravity. Among other results, one can find: (*i*) a matching between the entropy of asymptotically flat cosmological solutions and entropy computed with a Cardy-like formula for a Carrollian CFT [18,19]; (*ii*) a computation of the entanglement entropy in the Carrollian CFT from the bulk geometry using some extension of the Ryu-Takayanagi prescription [20–22]; (*iii*) the form of the correlation functions in the dual theory [23–27]; and (*iv*) an effective action for the dual

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Carrollian CFT [28–31]. This construction of effective action has been repeated in $4d$ gravity and shown to describe the dynamics of nonradiative spacetime [32]. A complementary approach was adopted earlier in [33] where an action encoding the radiative modes at $\mathscr{I}$ was explicitly written. Let us also mention that Carrollian holography is well-understood in the fluid/gravity correspondence for both $3d$ and $4d$ spacetimes using a suitable flat-limit procedure [34–38]. The dual fluid is a Carrollian fluid [39–43] at $\mathscr{I}$ whose properties are deduced by taking the ultrarelativistic limit of a relativistic fluid.

The main difficulties inherent to the Carrollian holographic approach are related to two key differences compared to what one is used to encounter in AdS/CFT. (*i*) The conformal boundary $\mathscr{I}$ is a *null* hypersurface, implying that the dual theory involves some Carrollian or "ultralocal" physics [44–49] which intrinsically defines the unconventional features of Carrollian CFTs [41,50–55,60], about which very little is known so far. This contrasts with the AdS/CFT correspondence where the boundary is timelike and the dual theory is an honest CFT whose proprieties have been studied for decades. (*ii*) The gravitational charges at null infinity are generically nonconserved due to the presence of matter or gravitational radiation [10,11,56–59]. This constitutes the plain vanilla example of a gravitational system with *leaky boundary conditions*. Describing the nonconservation of the charges from the point of view of the dual theory requires the development of new techniques such as the coupling with external sources [60]. On the contrary, Dirichlet-type of boundary conditions are usually imposed in AdS, preventing leaks through the conformal boundary, which thus acts as a reflective cavity. Let us however mention that leaky boundary conditions can also be imposed in AdS by relaxing the usual boundary conditions and turning on boundary sources [61–65]. This relaxation is necessary in order to recover a radiative phase space and BMS symmetries from a large radius limit of AdS.

In the celestial holography paradigm, the proposed dual theory to gravity in flat space is a *celestial CFT* (or CCFT for short) living on a codimension-two boundary, the celestial sphere. It is anchored in the fact that $\mathcal{S}$-matrix elements in the bulk, once rewritten in the boost eigenstate basis, enjoy conformal invariance in a manifest way [13,66–71]. The advantages of this approach is that one can use some of the very powerful CFT techniques to study the celestial dual, such as operator product expansions (OPEs) [72–83], conformal block decomposition [84–88], state-operator correspondence [89], null states and conformal multiplets [90–93], crossing symmetry [94] or shadow formalism [70,95–97]. This CFT language has recently allowed one to uncover new $w_{1+\infty}$ symmetries in CCFT [98] (see also [99–101]). This remarkable finding suggests that gravity might exhibit much more symmetries than one could have expected. Patterns of these symmetries have been found in the subleading orders of the gravitational solution space in [102,103]. However, there is always a price to pay for the emergence of CFT-like features in flat space, and this is manifested in some of the exotic features that CCFTs exhibit. It is indeed not clear for the moment exactly to which extent they differ from standard (e.g. unitary and compact) CFTs or what could be an axiomatic definition of CCFTs. Moreover, the fact that the dual theory is codimension-two with respect to the bulk makes the link with the AdS/CFT correspondence more nebulous, since it would require more involved steps than simply getting celestial correlators from a flat limit of those in AdS (see however [66,69,104–108] for connections to AdS and [109] to wedge holography). Finally, because the celestial encoding favors conformal transformations over time translations (which we recall are not lost but rather reshuffled into shifts in the conformal dimension of celestial operators), the dynamics of the gravitational theory such as the Bondi mass loss formula is not easily interpreted in the celestial CFT.

Though these two approaches to flat space holography seem in apparent tension, it has been argued in [60] (see also [55]) that they are in fact complementary to each other, as depicted in Fig. 1. Explicit links between them can be established; in particular, Carrollian source operators $\sigma_{(k,\bar{k})}$ living at null infinity can be mapped to CCFT operators $\mathcal{O}_{\Delta,J}$ living on the celestial sphere after using an appropriate integral transform. This allows one to relate the correlation functions between the sourced Carrollian CFT and those of the CCFT. In particular, the Ward identities of the sourced Carrollian CFT are then found to be equivalent to those of the CCFT encoding the bulk soft theorem. In this paper, we will pursue our previous analysis of [60] and provide more details about the interplay between the Carrollian and the celestial approaches to flat space holography.

A. Summary of the paper

The aims of this paper are threefold. The first objective is to review the key ingredients needed to bridge Carrollian and celestial holography. These both build up on the fact that the $\mathcal{S}$-matrix is in a sense holographic by nature, namely scattering amplitudes in the bulk can be identified with correlation functions at the boundary. Depending on the choice of basis for the fields, position space or Mellin space, one ends up with the Carrollian or the celestial approach. We will also review the asymptotic symmetry analysis of both electrodynamics and gravity. These will constitute the two concrete examples that we shall discuss in our setup.

After this review part, the second objective is to improve the Carrollian holography proposal. As mentioned above, one of the main obstructions to this approach [obstruction (*ii*)] is the nonconservation of the asymptotic charges Q at null infinity due to the presence of radiation, as illustrated in the left part of Fig. 1. We argue that, from the point of

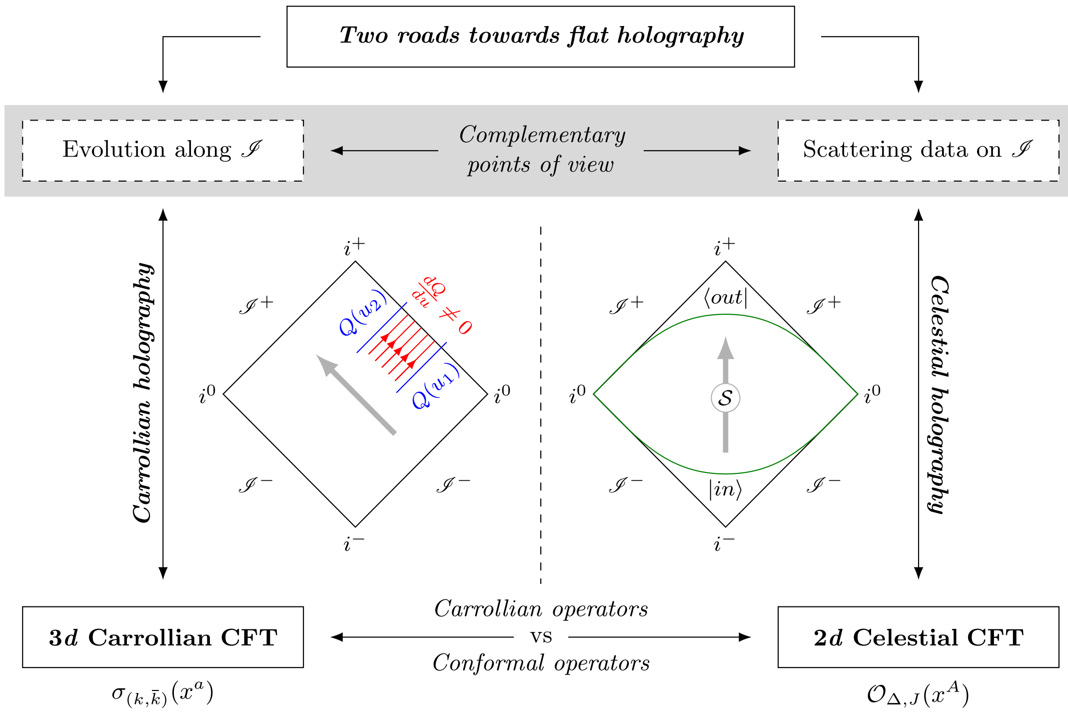

FIG. 1. Holographic nature of null infinity—two equivalent visions: (*i*) On the left picture, $\mathscr{I}^+$ is seen as a boundary along which there is a Carrollian time evolution. This naturally leads to the Carrollian holography proposal where the putative dual theory is a $3d$ Carrollian CFT. This picture is well adapted to describe the dynamics of the system through the flux-balance laws encoding the nonconservation of the charges Q. (*ii*) On the right picture, $\mathscr{I}^+$ is seen as a portion of a Cauchy hypersuface at late time. This naturally leads to the celestial holography proposal where the putative dual theory is a $2d$ CCFT. This second approach is particularly natural when considering a scattering process in flat spacetime.

view of the dual theory, these dissipative properties can be encoded by coupling the theory to some external sources. We discuss the general framework needed to treat symmetries in that context. This allows us to write flux-balance laws and Ward identities for a sourced field theory. Applying these considerations to a sourced Carrollian CFT, we show that the sourced Ward identities are able to describe the asymptotic dynamics of $4d$ asymptotically flat spacetimes given an appropriate holographic map between the bulk and boundary quantities. In particular, the external sources are argued to encode holographically the radiation in the bulk, which in gravity is contained in the asymptotic shear and the Bondi news tensor.

While the insertion of external sources locally spoils the symmetries of the theory, leading to nonconservation of the currents and dissipation, one can still extract some useful constraints implied by the symmetries on the correlators by using some holographic inputs. A crucial ingredient to implement this constraint is to consider that the Carrollian CFT is not living on $\mathscr{I}^+$ and $\mathscr{I}^-$ separately, but on the whole conformal boundary $\hat{\mathscr{I}} = \mathscr{I}^+ \sqcup \mathscr{I}^-$ obtained by gluing antipodally $\mathscr{I}^+$ and $\mathscr{I}^-$ along $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$, which provides a geometric implementation of the antipodal matching from the Carrollian point of view, see Fig. 2.

After providing details of this construction, we show that the sourced Ward identities of the Carrollian CFT integrated over $\hat{\mathscr{I}}$ determine the general form of low-point (1-, 2- and 3-point) correlation functions of this theory. In particular, for the 2-point function, we exhibit a new branch of solutions that does not seem to have appeared in previous literature. We then argue that this new branch is precisely the one that is relevant for holographic Carrollian CFT. We relate it to the usual bulk propagator via Fourier transform.

The third objective is to provide further details on the relation between the Carrollian and the celestial approaches to flat space holography. We show that the Carrollian source operators are mapped on celestial operators through an integral transform, coined as the $\mathcal{B}$-transform, which is the combination of a Mellin and a Fourier transform. This allows us to relate the correlation functions in the two theories. After providing the properties of the $\mathcal{B}$-transform, we demonstrate the equivalence between the sourced Ward identities in the Carrollian CFT that holographically encode the asymptotic bulk dynamics and the Ward identities in the CCFT that encode the leading and subleading soft theorems. For each of these two points, we consider both electrodynamics and gravity in the bulk.

B. Organization of the paper

The remaining of the paper is organized as follows. In Sec. II, we review the scattering of massless particles in flat

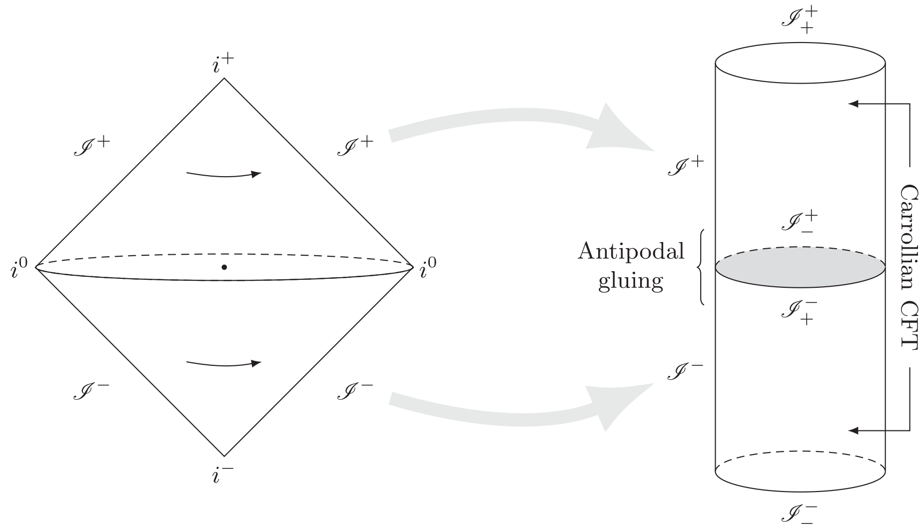

FIG. 2. Geometric implementation of the antipodal matching.

spacetime, which serves to introduce important notations and conventions for the remaining of the article. In Sec. III, we review the asymptotic symmetry analysis of electrodynamics and gravity in $4d$ asymptotically flat spacetime. In Sec. IV we develop a framework to treat the symmetries and their associated flux-balance laws/Ward identities in presence of external sources. We then specify this formalism to sourced theories exhibiting $U(1)$ or conformal Carrollian symmetries. In Sec. V we provide more insights about the nature of the proposed $3d$ holographic sourced conformal Carrollian field theory. We derive the low-point functions for this theory using the sourced Ward identities. We also propose a holographic correspondence between bulk metric and boundary Carrollian stress tensor, which allows us to show that the sourced Ward identities of the Carrollian CFT encode the BMS flux-balance laws. In Sec. VI, we relate the sourced conformal Carrollian field theory with the celestial CFT. We provide the integral transform that maps Carrollian source operators to celestial operators and show that the BMS Ward identities in both theories are equivalent. The low-point functions are also related with each other. Finally, in Sec. VII, we conclude our analysis with some comments and future directions. This paper is also complemented with some appendices; Appendix A describes our coordinate conventions, Appendix B reviews the isomorphism between global conformal Carroll algebra in three dimensions and the Poincaré algebra in four dimensions, and finally Appendix C provides details on the derivation of the classical constraints on the Carrollian stress tensor.

II. MASSLESS SCATTERING IN FLAT SPACETIME

This section is a review of salient features of scattering of massless fields in flat spacetime. We consider a scattering of massless bosonic spin-s fields through the $\mathcal{S}$-matrix approach via perturbation theory in Minkowski spacetime. We describe the asymptotic free states in terms of plane wave and conformal primary wave function bases. The boundary operators are obtained by taking the large radius expansion of the bulk fields as they approach null infinity. Reciprocally, the bulk fields can be reconstructed from the boundary operators using the Kirchhoff-d'Adhémar formula that defines a boundary-to-bulk propagator. Finally, we discuss various integral transforms that relate position space, momentum space and Mellin space. This allows one to relate formulate the scattering problem in position and Mellin spaces, which will be suitable for the subsequent discussion on flat space holography. To write this section, we abundantly used references [13,68,70,110–113] where complementary material can be found.

A. Bulk operators

1. *Massless fields in Minkowski spacetime and plane wave expansion*

Let us consider a free massless bosonic spin-s Fronsdal field $\phi_I^{(s)}$ in Minkowski spacetime [114–116] ($s = 0, 1, 2, \dots$). $I = (\mu_1 \mu_2 \dots \mu_s)$ is a symmetrized multi-index notation, $\mu_i \in \{0, \dots, 3\}$, and spacetime is covered by standard Cartesian coordinates $X^\mu = \{t, \vec{x}\}$. The massless spin-s field $\phi_I^{(s)}$ can be conveniently put in the De Donder traceless gauge

$$\partial^\nu \phi^{(s)}_{\nu\mu_2\dots\mu_s}(X) = 0, \qquad \eta^{\mu\nu}\phi^{(s)}_{\mu\nu\mu_3\dots\mu_s}(X) = 0, \qquad (2.1)$$

in which the equations of motion reduce to

$$\partial^\mu \partial_\mu \phi_I^{(s)}(X) = 0 \qquad (2.2)$$

and the residual gauge transformations are

$$\delta_\lambda \phi^{(s)}_{\mu_1\ldots\mu_s}(X) = \partial_{(\mu_1}\lambda_{\mu_2\ldots\mu_s)}(X)$$
$$\text{such that } \partial^\nu \lambda_{\nu\mu_3\ldots\mu_s}(X) = 0 \quad \text{and} \quad \eta^{\mu\nu}\lambda_{\mu\nu\mu_4\ldots\mu_s}(X) = 0. \tag{2.3}$$

The field can be quantized in the Heisenberg representation and expanded in Fourier modes. Each mode is labeled by an on shell null 4-momentum vector (see e.g. [13])

$$p^\mu(\omega, w, \bar{w}) = \omega q^\mu(w, \bar{w}), \qquad q^\mu(w, \bar{w}) = \frac{1}{\sqrt{2}}(1 + w\bar{w}, w + \bar{w}, -i(w - \bar{w}), 1 - w\bar{w}), \tag{2.4}$$

parametrized by the light cone energy $\omega > 0$ and coordinates $(w, \bar{w})$ on the complex plane. Let $\varepsilon^\pm_\mu(\vec{q})$ be the polarization covectors,

$$\varepsilon^+_\mu(\vec{q}) = \partial_w q_\mu = \frac{1}{\sqrt{2}}(-\bar{w}, 1, -i, -\bar{w}),$$
$$\varepsilon^-_\mu(\vec{q}) = [\varepsilon^+_\mu(\vec{q})]^* = \partial_{\bar{w}} q_\mu = \frac{1}{\sqrt{2}}(-w, 1, i, -w). \tag{2.5}$$

We can complete these into a null co-tetrad $\mathcal{N} = \{q_\mu, n_\mu, \varepsilon^+_\mu, \varepsilon^-_\mu\}$ satisfying

$$q^\mu n_\mu = -1, \qquad \varepsilon^\mu_+ \varepsilon^-_\mu = 1, \qquad q^\mu \varepsilon^\pm_\mu = 0 = n^\mu \varepsilon^\pm_\mu, \tag{2.6}$$

by setting $n_\mu \equiv \partial_w \partial_{\bar{w}} q_\mu$. The spin-$s$ field in De Donder gauge (2.1) can be expanded in Fourier modes as

$$\phi^{(s)}_I(X) = K_{(s)} \sum_{\alpha=\pm} \int \frac{d^3p}{(2\pi)^3 2p^0} [\varepsilon^{*\alpha}_I(\vec{q}) a^{(s)}_\alpha(\vec{p}) e^{ip^\mu X_\mu} + \varepsilon^\alpha_I(\vec{q}) a^{(s)}_\alpha(\vec{p})^\dagger e^{-ip^\mu X_\mu}] \tag{2.7}$$

after choosing a Lorentz-invariant measure in momentum space and defining the polarization tensors

$$\varepsilon^\pm_{\mu_1\ldots\mu_s}(\vec{q}) = \varepsilon^\pm_{\mu_1}(\vec{q})\varepsilon^\pm_{\mu_2}(\vec{q})\ldots\varepsilon^\pm_{\mu_s}(\vec{q}), \tag{2.8}$$

which are fully symmetric and transverse tensors, i.e. $q^{\mu_i}\varepsilon^\pm_{\mu_1\ldots\mu_i\ldots\mu_s}(\vec{q})$ for any $i = 1, \ldots, s$. The overall constant $K_{(s)} \in \mathbb{R}^+_0$ may depend on the coupling constant for the relevant spin (e.g. the elementary charge e for the spin-1 field or the Newton-Cavendish constant G for the spin-2 field). One usually takes

$$K_{(1)} = e\sqrt{\hbar}, \qquad K_{(2)} = \sqrt{32\pi G\hbar}. \tag{2.9}$$

Ladder operators obey the usual commutation relations

$$[a^{(s)}_\alpha(\vec{p}), a^{(s)}_{\alpha'}(\vec{p}')^\dagger] = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}')\delta_{\alpha,\alpha'} \tag{2.10}$$

as induced by the canonical commutation relations of the field (2.7).

Importantly, a residual gauge transformation (2.3) driven by $\lambda_{\mu_2\ldots\mu_s}(X)$ admitting a well-defined Fourier transform acts on the field (2.7) by addition of a term of the form

$$\partial_{(\mu_1}\lambda_{\mu_2\ldots\mu_s)}(X) = \int \frac{d^3p}{(2\pi)^3 2p^0} [p_{(\mu_1}\hat{\lambda}_{\mu_2\ldots\mu_s)}(\vec{p}) e^{ip^\mu X_\mu} + p_{(\mu_1}\hat{\lambda}^*_{\mu_2\ldots\mu_s)}(\vec{p}) e^{-ip^\mu X_\mu}]. \tag{2.11}$$

This impacts the polarization tensors as

$$\delta_\lambda \varepsilon^\alpha_{\mu_1\ldots\mu_s}(\vec{q}) = q_{(\mu_1}\tilde{\lambda}^\alpha_{\mu_2\ldots\mu_s)}(\vec{p}) \tag{2.12}$$

writing $\hat{\lambda}_{\mu_2\ldots\mu_s}(\vec{p}) \equiv \frac{K_{(s)}}{\omega}\tilde{\lambda}^{\alpha}_{\mu_2\ldots\mu_s}(\vec{p})a^{(s)}_{\alpha}(\vec{p})$ and $\eta^{\mu\nu}\tilde{\lambda}^{\alpha}_{\mu\nu\mu_4\ldots\mu_s} = 0$, but the ladder operators $a^{(s)}_{\alpha}$ are left invariant by these gauge transformations. However, as we will review later, asymptotic gauge symmetries (whose generating parameters do not have finite energy) do act nontrivially on the operators.

Transforming the integration measure for the parametrization (2.4), we can write

$$\phi^{(s)}_I(X) = \frac{K_{(s)}}{16\pi^3}\sum_{\alpha=\pm}\int \omega d\omega d^2w[a^{(s)}_{\alpha}(\omega, w, \bar{w})\varphi^{*\alpha}_I(\omega, w, \bar{w}|X) + a^{(s)}_{\alpha}(\omega, w, \bar{w})^{\dagger}\varphi^{\alpha}_I(\omega, w, \bar{w}|X)], \tag{2.13}$$

where $d^2w = idwd\bar{w}$ denotes the integration measure on the complex plane with local holomorphic coordinates $(w, \bar{w})$ and

$$\varphi^{*\alpha}_I(\omega, w, \bar{w}|X) \equiv \varepsilon^{*\alpha}_I(w, \bar{w})e^{i\omega q^{\mu}X_{\mu}} \tag{2.14}$$

are the basis vectors of plane waves. In this parametrization, the canonical commutation relations (2.10) become

$$[a^{(s)}_{\alpha}(\omega, w, \bar{w}), a^{(s)}_{\alpha'}(\omega', w', \bar{w}')^{\dagger}] = 16\pi^3\omega^{-1}\delta(\omega-\omega')\delta^{(2)}(w-w')\delta_{\alpha,\alpha'}. \tag{2.15}$$

At the quantum level, $a^{(s)}_{+}(\omega, w, \bar{w})^{\dagger}$ (resp. $a^{(s)}_{-}(\omega, w, \bar{w})^{\dagger}$) creates a massless particle of spin s and helicity $J = +s$ (resp. $J = -s$), with energy ω and a null momentum pointing towards the direction $q^{\mu}(w, \bar{w})$.

Poincaré transformations $X'^{\mu} = \Lambda^{\mu}{}_{\nu}X^{\nu} + t^{\mu}$ act on the gauge field as

$$\phi^{(s)}_{\mu_1\mu_2\ldots\mu_s}(X) \mapsto \phi'^{(s)}_{\mu_1\mu_2\ldots\mu_s}(X') = \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2}\ldots\Lambda_{\mu_s}{}^{\nu_s}\phi^{(s)}_{\nu_1\nu_2\ldots\nu_s}(X). \tag{2.16}$$

Lorentz transformations induce a $SL(2,\mathbb{C})$ Möbius transformation

$$w \mapsto w'(w) = \frac{aw+b}{cw+d} \tag{2.17}$$

with $ad - bc = 1$ on the complex coordinates determining the direction of the null momentum q^{μ} with the embedding (2.4) of the Riemann sphere into the light cone. The expression of the matrix $\Lambda^{\mu}{}_{\nu}$ in terms of the Möbius parameters (a, b, c, d) can be found e.g. in [117]. Since p^{μ} is a Lorentz vector, one has

$$\omega' = \left|\frac{\partial w'}{\partial w}\right|^{-1}\omega, \quad q^{\mu}(w', \bar{w}') = \left|\frac{\partial w'}{\partial w}\right|\Lambda^{\mu}{}_{\nu}q^{\nu}(w, \bar{w}). \tag{2.18}$$

The second equation states the fact that a Lorentz transformation on the light cone induces the corresponding Möbius transformation on the Riemann sphere via the embedding (2.4). Owing to (2.5) and (2.18), one can show that $\varepsilon^{\pm}_{\mu}(w, \bar{w})$ do not transform homogeneously under the action of (2.17) but the supplementary terms are part of the residual gauge freedom (2.12), i.e.

$$\varepsilon'^{\pm}_{\mu}(w', \bar{w}') = \left(\frac{\partial w'}{\partial w}\right)^{\mp\frac{1}{2}}\left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{\pm\frac{1}{2}}\Lambda_{\mu}{}^{\nu}\varepsilon^{\pm}_{\nu}(w, \bar{w}) + \mathscr{A}(w, \bar{w})\Lambda_{\mu}{}^{\nu}q_{\nu}(w, \bar{w}), \tag{2.19}$$

where $\mathscr{A}$ is a fixed function of $(w, \bar{w})$. For the expansion (2.13) and (2.14) recalling that the integration measure and the plane wave are Lorentz invariant, we deduce from (2.19) that the transformation of ladder operators under the Poincaré group is

$$a'^{(s)}_{\pm}(\omega', w', \bar{w}') = \left(\frac{\partial w'}{\partial w}\right)^{-\frac{J}{2}}\left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{\frac{J}{2}}e^{-i\omega q^{\mu}(w,\bar{w})\Lambda^{\nu}{}_{\mu}t_{\nu}}a^{(s)}_{\pm}(\omega, w, \bar{w}). \tag{2.20}$$

In particular, one recovers that the ladder operators are eigenvectors of translations, which is expected since they are assumed to create/annihilate energy eigenstates. The infinitesimal version of (2.20) can be obtained by setting $X'^{\mu} = X^{\mu} - \epsilon\xi^{\mu}$, with $\xi^{\mu} = \varpi^{\mu}{}_{\nu}X^{\nu} + \tau^{\mu}$ ($\varpi_{\mu\nu} = \varpi_{[\mu\nu]}$) and $w'(w) = w - \epsilon\mathcal{Y}^{w}(w)$, and retaining only the linear terms in ϵ. One concludes that

$$\delta_{\xi(\mathcal{T},\mathcal{Y})} a_{\pm}^{(s)}(\omega,w,\bar{w}) = \left[-i\omega\mathcal{T} + \mathcal{Y}^w\partial_w + \mathcal{Y}^{\bar{w}}\partial_{\bar{w}} + \frac{J}{2}\partial_w\mathcal{Y}^w - \frac{J}{2}\partial_{\bar{w}}\mathcal{Y}^{\bar{w}} - \frac{\omega}{2}(\partial_w\mathcal{Y}^w + \partial_{\bar{w}}\mathcal{Y}^{\bar{w}})\partial_\omega\right] a_{\pm}^{(s)}(\omega,w,\bar{w}), \tag{2.21}$$

where $\xi(\mathcal{T},\mathcal{Y})$ is now parametrized by the function $\mathcal{T}(w,\bar{w}) = -q^\mu(w,\bar{w})\tau_\mu$ and the vector $\mathcal{Y} = \mathcal{Y}^w(w)\partial_w + \mathcal{Y}^{\bar{w}}(\bar{w})\partial_{\bar{w}}$ on the Riemann sphere, while acting in momentum space covered by coordinates $(\omega,w,\bar{w})$.

2. Mellin transform and conformal primary basis

In (2.13), we expanded the massless spin-s field in the plane wave basis (2.14) whose elements are energy eigenstates. Another convenient choice is the conformal primary wave function basis [66,68–70] whose elements are boost eigenstates. This basis trades the energy parameter ω for the eigenvalue Δ of the Lorentz boost along the direction fixed by the null momentum p^μ.

The map between the two bases is given by the Mellin transform defined as

$$F(\Delta) = \mathcal{M}[f(\omega),\Delta] \equiv \int_0^{+\infty} d\omega\omega^{\Delta-1} f(\omega) \tag{2.22}$$

for $f\colon \mathbb{R}^+ \to \mathbb{C}$ where $\Delta = c + i\nu$ is generically complex [see e.g. [118] for the precise assumptions that make the integral (2.22) well-defined]. According to the Mellin inversion theorem, if $F(\Delta)$ is analytic in the complex strip defined by $c \in]a,b[\subset \mathbb{R}$ and the integral

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\nu\omega^{-(c+i\nu)} F(c+i\nu) \tag{2.23}$$

converges for any $c \in]a,b[$, then it defines the inverse Mellin transform $f(\omega) \equiv \mathcal{M}^{-1}[F(\Delta),\omega]$ acting on $F(\Delta)$.

The Mellin transform of the plane waves (2.14), for $\omega > 0$, yields (see e.g. [66,68,70])

$$\begin{aligned} V_I^{*\alpha}(\Delta,w,\bar{w}|X) &= \lim_{\epsilon\to 0^+} \varepsilon_I^{*\alpha}(w,\bar{w}) \int_0^{+\infty} d\omega\omega^{\Delta-1} e^{i\omega q^\mu X_\mu - \epsilon\omega} \\ &= \lim_{\epsilon\to 0^+} \varepsilon_I^{*\alpha}(w,\bar{w}) \frac{i^\Delta \Gamma[\Delta]}{(q^\mu X_\mu + i\epsilon)^\Delta}, \end{aligned} \tag{2.24}$$

where $\epsilon > 0$ is a regulator that can be arbitrarily close to zero. The computation of the integral (2.24) is made particularly easy if one uses Ramanujan's master theorem, stating that if $f(\omega)$ is a complex-valued function on the positive real axis admitting a power-series expansion of the form $f(\omega) = \sum_{k=0}^{+\infty} \frac{\varphi(k)}{k!}(-\omega)^k$ in some neighborhood of the origin, then its Mellin transform is simply given by $\mathcal{M}[f(\omega),\Delta] = \Gamma[\Delta]\varphi(-\Delta)$ in a certain range of validity $\Delta \in \mathcal{R} \subset \mathbb{C}$. For the plane wave, the regulation of the integral (2.24) by a decaying exponential function $\sim e^{-\epsilon\omega}$ allows it to converge in the whole half complex plane $\mathcal{R} = \{c + i\nu | c > 0, \nu \in \mathbb{R}\}$. Using the identity

$$\int_0^{+\infty} d\omega\omega^{i\nu-1} = 2\pi\delta(\nu), \tag{2.25}$$

one can show [70] that the statement that plane waves form a delta-function normalizable basis for the Klein-Gordon inner product translates, after Mellin transform, into requiring that Δ lays on the principal continuous series of the irreducible unitary representations of the Lorentz group, i.e. $\Delta = 1 + i\nu$.

Let us first consider the expansion of the bulk field (2.13), where we recall that the polarization is taken as products of (2.5). In terms of the Mellin representatives (2.24), such an expansion reads [110,119]

$$\phi_I^{(s)}(X) = \frac{K_{(s)}}{32\pi^4} \sum_{\alpha=\pm} \int d\nu d^2w [a_{2-\Delta,\alpha}^{(s)}(w,\bar{w}) V_I^{*\alpha}(\Delta,w,\bar{w}|X) + a_{2-\Delta,\alpha}^{(s)}(w,\bar{w})^\dagger V_I^\alpha(\Delta,w,\bar{w}|X)], \tag{2.26}$$

where we defined the ladder operators in the Mellin basis as in [93]

$$\begin{aligned} a_{\Delta,\alpha}^{(s)}(w,\bar{w}) &\equiv \mathcal{M}[a_\alpha^{(s)}(\omega,w,\bar{w}),\Delta] = \int_0^{+\infty} d\omega\omega^{\Delta-1} a_\alpha^{(s)}(\omega,w,\bar{w}), \\ a_{\Delta,\alpha}^{(s)}(w,\bar{w})^\dagger &\equiv \mathcal{M}[a_\alpha^{(s)}(\omega,w,\bar{w})^\dagger,\Delta] = \int_0^{+\infty} d\omega\omega^{\Delta-1} a_\alpha^{(s)}(\omega,w,\bar{w})^\dagger. \end{aligned} \tag{2.27}$$

These last relations can be inverted as

$$a_\alpha^{(s)}(\omega, w, \bar{w}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\nu \omega^{-\Delta} a_{\Delta,\alpha}^{(s)}(w, \bar{w}),$$
$$a_\alpha^{(s)}(\omega, w, \bar{w})^\dagger = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\nu \omega^{-\Delta} a_{\Delta,\alpha}^{(s)}(w, \bar{w})^\dagger. \quad (2.28)$$

Crucially, applying the Mellin transforms (2.27)–(2.20), it has been observed that these operators transform as

$$a'^{(s)}_{\Delta,\alpha}(w', \bar{w}') = \left(\frac{\partial w'}{\partial w}\right)^{-\frac{\Delta+J}{2}} \left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{-\frac{\Delta-J}{2}} a_{\Delta,\alpha}^{(s)}(w, \bar{w}), \quad (2.29)$$

under the action of the Lorentz group, which is precisely the definition of a conformal primary field of conformal dimension Δ and $2d$ spin $J = \alpha s$ (see e.g. [120]). The action of the Poincaré translations in the Mellin basis is given by [121]

$$a'^{(s)}_{\Delta,\alpha}(w', \bar{w}') = e^{-iq^\mu(w,\bar{w})t_\mu \hat{\partial}_\Delta} a_{\Delta,\alpha}^{(s)}(w, \bar{w}) = \sum_{n=0}^{+\infty} \frac{(-iq^\mu(w,\bar{w})t_\mu)^n}{n!} a^{(s)}_{\Delta+n,\alpha}(w, \bar{w}) \quad (2.30)$$

defining the discrete derivative operator $\widehat{\partial}_\Delta F(\Delta) \equiv F(\Delta + 1)$. The infinitesimal action of Poincaré transformations in the Mellin basis reads as

$$\delta_{\xi(\mathcal{T},\mathcal{Y})} a_{\Delta,\alpha}^{(s)}(w, \bar{w}) = \left(-i\mathcal{T}\hat{\partial}_\Delta + \mathcal{Y}^w \partial_w + \mathcal{Y}^{\bar{w}}\partial_{\bar{w}} + \frac{\Delta + J}{2}\partial_w \mathcal{Y}^w + \frac{\Delta - J}{2}\partial_{\bar{w}}\mathcal{Y}^{\bar{w}}\right) a_{\Delta,\alpha}^{(s)}(w, \bar{w}). \quad (2.31)$$

Remark—The functions $V_I^\alpha(\Delta, w, \bar{w}|X)$ defined in (2.24) are not strictly speaking boost eigenstates since they do not transform covariantly under the action of the Lorentz group; this is due to the transformation law of the polarization vector (2.19). Instead, one can define the conformal primary wave functions [70] by rescaling the functions $V_I^\alpha(\Delta, w, \bar{w}|X)$ and performing a gauge transformation

$$A_I^\alpha(\Delta, w, \bar{w}|X) = c^{(s)}(\Delta) V_I^\alpha(\Delta, w, \bar{w}|X) + \partial_{(\mu_1} f^{(s)\alpha}_{\mu_2\ldots\mu_s)}(\Delta, w, \bar{w}|X). \quad (2.32)$$

The precise expressions of scaling factor $c^{(s)}(\Delta)$ and gauge parameter $f^{(s)\alpha}_{\mu_2\ldots\mu_s}(\Delta, w, \bar{w}|X)$ will depend on the spin s (explicit expressions for $s = 1, 2$ can be found e.g. in [70,119]). It is convenient to use the compact form [112]

$$A_I^\pm(\Delta, w, \bar{w}|X) = m^\pm_{\mu_1}(w, \bar{w}|X)\ldots m^\pm_{\mu_s}(w, \bar{w}|X)\frac{1}{(q_\nu X^\nu)^\Delta}, \quad (2.33)$$

for any (integer) spin s, where $m^\pm_\mu(w, \bar{w}|X)$ represent the modified polarization covectors

$$m^\pm_\mu(w, \bar{w}|X) \equiv \varepsilon^\pm_\mu(w, \bar{w}) - \frac{\varepsilon^\pm_\nu X^\nu}{q_\rho X^\rho} q_\mu(w, \bar{w}). \quad (2.34)$$

One can show that the set of functions (2.33) are actual boost eigenstates, because now (2.34) transforms homogeneously under $SL(2, \mathbb{C})$ transformations (2.17), i.e.

$$m'^\pm_\mu(w', \bar{w}'|X') = \left(\frac{\partial w'}{\partial w}\right)^{\mp\frac{1}{2}} \left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{\pm\frac{1}{2}} \Lambda_\mu{}^\nu m^\pm_\nu(w, \bar{w}|X). \quad (2.35)$$

While the basis elements $V_I^\alpha(\Delta, w, \bar{w}|X)$ only satisfy the De Donder gauge-fixing conditions (2.1) because $q^\mu \varepsilon^\pm_\mu = 0$, the new basis functions $A_I^\alpha(\Delta, w, \bar{w}|X)$ also satisfy the radial (or Fock-Schwinger) gauge condition $X^\mu A^\alpha_{\mu\mu_2\ldots\mu_s}(\Delta, w, \bar{w}|X) = 0$ because $X^\mu m^\pm_\mu = 0$ [70]. The price to pay for trading $\varepsilon^\pm_\mu$ for $m^\pm_\mu$ is thus the breaking of manifest invariance under translations, because the supplementary gauge fixing is sensitive to the choice of an origin. Notice that the fixation of the radial gauge is compatible with the De Donder gauge, as argued in [122]. The so-called celestial operators are defined as [119]

$$\mathcal{O}^{(s)}_{\Delta,\alpha}(w, \bar{w}) \equiv \langle A_I^\alpha(\Delta, w, \bar{w}|X), \phi_I^{(s)}(X)\rangle, \quad (2.36)$$

where $\langle\cdot,\cdot\rangle$ denotes the relevant spin-s inner product. For scalar fields ($s = 0$), it will be the usual Klein-Gordon inner product; more involved expressions are needed for $s = 1, 2$ which can be constructed from symplectic structures available in the literature (see [119] and references therein). Notice that the operators $\mathcal{O}^{(s)}_{\Delta,\alpha}(w, \bar{w})$ and $a^{(s)}_{\Delta,\alpha}(w, \bar{w})$ are proportional to each other [93,119] (see also [123] for more details). Hence, we will persist in using consistently the Mellin ladder operators $a^{(s)}_{\Delta,\alpha}(w, \bar{w})$ in the rest of the paper.

B. Boundary operators

1. From bulk to boundary: The large-r expansion

In this section, we review how to construct boundary operators by performing a large-r expansion around $\mathscr{I}^+$ and $\mathscr{I}^-$ [13].

We first analyze the late-time behavior of free massless fields around $\mathscr{I}^+$; these will ultimately approximate outgoing free fields of the $\mathcal{S}$-matrix. We start from the Fourier expansion

$$\phi_I^{(s)}(X) = \frac{K_{(s)}}{16\pi^3}\int \omega d\omega d^2w[\epsilon_I^{*\alpha}(w,\bar{w})a_\alpha^{(s)}(\omega,w,\bar{w})e^{i\omega q^\mu X_\mu} + \epsilon_I^{\alpha}(w,\bar{w})a_\alpha^{(s)}(\omega,w,\bar{w})^\dagger e^{-i\omega q^\mu X_\mu}] \tag{2.37}$$

and use retarded Bondi coordinates $\{u,r,z,\bar{z}\}$ with flat boundary representatives, as defined in Appendix A, to perform an asymptotic expansion in r near $\mathscr{I}^+ = \{r\to+\infty\}$. Rewriting X^μ as (A8) in terms of retarded Bondi coordinates one has $q^\mu X_\mu = -u - r|z-w|^2$. We now introduce polar coordinates in the complex plane as $z - w = \rho e^{i\vartheta}$. The integration measure becomes $d^2w \to 2\rho d\rho d\vartheta$ and altogether (2.37) can be rewritten as

$$\phi_I^{(s)}(X) = \frac{K_{(s)}}{8\pi^3}\int_0^{+\infty} d\omega\omega \int_0^{+\infty} d\rho\rho \int_0^{2\pi} d\vartheta[\epsilon_I^{*\alpha}(w,\bar{w})a_\alpha^{(s)}(\omega,w,\bar{w})e^{-i\omega u - i\omega r\rho^2} + \text{h.c.}], \tag{2.38}$$

where "h.c." stands for Hermitian conjugation. The expansion of (2.38) in the limit $r\to+\infty$ (as approaching $\mathscr{I}^+$) is given by the stationary phase approximation of the ρ-integral around the point $\rho = 0$ corresponding to the situation where q and x are collinear. Evaluating the integral in ρ for large-r gives the property [113]

$$\rho e^{-i\omega r\rho^2} = -\frac{i}{2r\omega}\delta(\rho) + \mathcal{O}(r^{-2}). \tag{2.39}$$

In Bondi retarded coordinates, the expressions of the elements of the cotetrad $\mathcal{N}$ are

$$\begin{aligned} &q_\mu dX^\mu = -du - |z-w|^2 dr - r(\bar{z}-\bar{w})dz - r(z-w)d\bar{z}, \qquad n_\mu dX^\mu = -dr, \\ &\epsilon_\mu^+ dX^\mu = (\bar{z}-\bar{w})dr + rd\bar{z}, \qquad \epsilon_\mu^- dX^\mu = (z-w)dr + rdz, \end{aligned} \tag{2.40}$$

and, in the collinear limit $w = z$, we have $\epsilon_\mu^+ dX^\mu = rd\bar{z}$ and $\epsilon_\mu^- dX^\mu = rdz$. Using (2.39), the leading components of $\phi_I^{(s)}(X)$ near $\mathscr{I}^+$ are thus

$$\phi_{z\ldots z}^{(s)}(X) = -\frac{iK_{(s)}}{8\pi^2}r^{s-1}\int_0^{+\infty} d\omega[a_+^{(s)}(\omega,z,\bar{z})e^{-i\omega u} - a_-^{(s)}(\omega,z,\bar{z})^\dagger e^{+i\omega u}] + \mathcal{O}(r^{s-2}), \tag{2.41}$$

and its complex-conjugated component $\phi_{\bar{z}\ldots\bar{z}}^{(s)}(X)$ as well as components of the form $\phi_{rz\ldots z}^{(s)}(X)$, $\phi_{rrz\ldots z}^{(s)}(X)$ etc., all generically of order $\mathcal{O}(r^{s-1})$ as well. Indeed, the subleading $\mathcal{O}(r^{-2})$ terms in (2.39) contribute in the radial component of in (2.40), recalling that $dr = -r^2 d(r^{-1})$, producing a contribution at leading order. One can extract the *boundary value* of the field by

$$\bar{\phi}_{z\ldots z}^{(s)}(u,z,\bar{z})dz\otimes\cdots\otimes dz + \bar{\phi}_{\bar{z}\ldots\bar{z}}^{(s)}(u,z,\bar{z})d\bar{z}\otimes\cdots\otimes d\bar{z} \equiv \lim_{r\to+\infty}\iota^*(r^{1-s}\phi_{\mu_1\ldots\mu_s}^{(s)}dX^{\mu_1}\otimes\ldots\otimes dX^{\mu_s}), \tag{2.42}$$

where ι^* denotes the pullback on the constant r hypersurfaces, hence deleting the leading radial components. Importantly, the resulting fields are independent of the residual gauge ambiguity (2.11) and (2.12) as can be seen from the fact that $q_\mu dX^\mu = \mathcal{O}(1)$ in the collinear limit while $\epsilon_\mu^\pm dX^\mu = \mathcal{O}(r)$ (for a discussion on this phenomenon, see [124]).

To summarize, the asymptotic behavior of the field $\phi_I^{(s)}(X)$ is encoded by the boundary value (2.42): its Fourier expansion can be directly read off from (2.41) as [13,125]

$$\bar{\phi}_{z\ldots z}^{(s)}(u,z,\bar{z}) = -\frac{iK_{(s)}}{8\pi^2}\int_0^{+\infty} d\omega[a_+^{(s)}(\omega,z,\bar{z})e^{-i\omega u} - a_-^{(s)}(\omega,z,\bar{z})^\dagger e^{+i\omega u}], \tag{2.43}$$

while the Fourier modes for $\bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}} = (\bar{\phi}^{(s)}_{z\ldots z})^\dagger$ are obtained from (2.43) by exchanging $a^{(s)}_{\pm} \to a^{(s)}_{\mp}$.

Remark—As stated below Eq. (2.42), boundary values (2.43) of the fields are left invariant under residual gauge transformations (2.3) whose parameters $\lambda_{\mu_2\ldots\mu_s}(X)$ admit a Fourier decomposition (2.11). Notice that there exist gauge transformations which do not admit such Fourier decomposition and do modify the boundary value of the fields. For instance, picking $s=1$, the action of a gauge transformation $\delta_\lambda \phi^{(1)}_\mu = \partial_\mu \lambda$ of parameter λ satisfying $\partial^\mu \partial_\mu \lambda = 0$, which is solved asymptotically by $\lambda(u,r,z,\bar{z}) = \lambda^{(0)}(z,\bar{z}) + u\partial_z \partial_{\bar{z}} \lambda^{(0)} r^{-1} \ln r + \mathcal{O}(r^{-1})$, would shift the boundary gauge field by

$$\delta_\lambda \bar{\phi}^{(1)}_z(u,z,\bar{z}) = \partial_z \lambda^{(0)}(z,\bar{z}). \tag{2.44}$$

Another point of view is to reinterpret this shift as the following change in the zero mode of the ladder operators

$$\delta^S_\lambda a^{(s)}_+(\omega,z,\bar{z}) = -\delta^S_\lambda a^{(s)}_-(\omega,z,\bar{z})^\dagger \equiv \frac{8\pi^2 i}{K_{(1)}} \partial_z \lambda^{(0)}(z,\bar{z}) \delta(\omega). \tag{2.45}$$

Hence, the operator δ^S_λ (where S stands for "soft") only affects the "Goldstone mode" (i.e. the zero-energy mode transforming in a pure inhomogeneous way) but leaves the bulk field (2.37) unchanged since $\omega\delta(\omega) \simeq 0$ in the sense of distributions. Notice that this mode has to be distinguished from the leading soft-photon mode [126]

$$\mathcal{N}^{(0)}_z(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} du \partial_u \bar{\phi}^{(1)}_z(u,z,\bar{z}) = \bar{\phi}^{(1)}_z(u,z,\bar{z})|_{u\to+\infty} - \bar{\phi}^{(1)}_z(u,z,\bar{z})|_{u\to-\infty} \tag{2.46}$$

whose expression in terms of ladder operators is obtained using (2.43) and $\int_{-\infty}^{+\infty} du e^{\pm i\omega u} = 2\pi\delta(\omega)$ and gives

$$\mathcal{N}^{(0)}_z(z,\bar{z}) = -\frac{K_{(1)}}{8\pi} \lim_{\omega\to 0^+} [\omega a_+(\omega,z,\bar{z}) + \omega a_-(\omega,z,\bar{z})^\dagger]. \tag{2.47}$$

The nontriviality of the result corresponds to the presence of a pole in the ladder operators in Fourier space. The discussion is easily extended to any spin $s \geq 1$.

The commutation relation (2.15) implies (see e.g. [125–127])

$$\begin{aligned} [\bar{\phi}^{(s)}_{z\ldots z}(u_1,z_1,\bar{z}_1), \bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(u_2,z_2,\bar{z}_2)] &= -\frac{i}{4} K^2_{(s)} \mathrm{sign}(u_1-u_2) \delta^{(2)}(z_1-z_2), \\ [\partial_{u_1}\bar{\phi}^{(s)}_{z\ldots z}(u_1,z_1,\bar{z}_1), \bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(u_2,z_2,\bar{z}_2)] &= -\frac{i}{2} K^2_{(s)} \delta(u_1-u_2) \delta^{(2)}(z_1-z_2). \end{aligned} \tag{2.48}$$

In our notations, $\mathrm{sign}(x)$ is the sign distribution related to the Heaviside and Dirac distributions $\Theta(x)$, $\delta(x)$ as

$$\mathrm{sign}(x) = 2\Theta(x) - 1, \qquad \mathrm{sign}'(x) = 2\delta(x), \tag{2.49}$$

for the particular choice $\Theta(0) = \frac{1}{2}$. Owing to (2.18), the Fourier transform of (2.20) gives

$$\bar{\phi}'^{(s)}_{z\ldots z}(u',z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-\frac{1+J}{2}} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\frac{1-J}{2}} \bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}) \tag{2.50}$$

in the collinear limit $w=z$, where $u' = |\frac{\partial z'}{\partial z}|(u - q^\mu(z,\bar{z})\Lambda_\mu{}^\nu t_\nu)$ and $J=s$. The complex conjugated component $\bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}$ transforms in the same way but with the flipped helicity $J=-s$. This translates infinitesimally into

$$\begin{aligned} \delta_{\xi(\mathcal{T},\mathcal{Y})} \bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}) = &\left(\mathcal{T} + \frac{u}{2}(\partial_z \mathcal{Y}^z + \partial_{\bar{z}} \mathcal{Y}^{\bar{z}})\right) \partial_u \bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}) \\ &+ \left(\mathcal{Y}^z \partial_z + \mathcal{Y}^{\bar{z}} \partial_{\bar{z}} + \frac{1+J}{2} \partial_z \mathcal{Y}^z + \frac{1-J}{2} \partial_{\bar{z}} \mathcal{Y}^{\bar{z}}\right) \bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}). \end{aligned} \tag{2.51}$$

The same analysis can be performed for operators in the vicinity of past null infinity $\mathscr{I}^-$. The first step consists in trading the retarded Bondi coordinates $\{u, r, z, \bar{z}\}$ for the advanced Bondi coordinates $\{v, r', z', \bar{z}'\}$, also with flat boundary representative (see Appendix A) in order to perform the large-r' expansion around $\mathscr{I}^- = \{r' = +\infty\}$. According to the change of coordinates (A13), this amounts effectively to make the permutation $\{u \mapsto v, r \mapsto -r', z \mapsto z'\}$ in all expressions. This induces a sign flip in the polarization covectors $\varepsilon^\pm_\mu$ in (2.40). As now $q^\mu X_\mu = -v + r'|z' - w|^2$, another global minus sign is induced by the evaluation of the $(w, \bar{w})$ integral in (2.13) in the stationary phase approximation around the collinear configuration $z' = w$, see (2.39). Therefore, the large-r' limit of the field goes in a parallel way as before and provides the following expansion of operators near $\mathscr{I}^-$:

$$\phi^{(s)}_{z\ldots z}(X) = (-1)^s \frac{iK_{(s)}}{8\pi^2} r'^{s-1} \int_0^{+\infty} d\omega [a^{(s)}_+(\omega, z', \bar{z}') e^{-i\omega v} - a^{(s)}_-(\omega, z', \bar{z}')^\dagger e^{+i\omega v}] + \mathcal{O}(r'^{s-2}). \tag{2.52}$$

Because the considered coordinate systems $\{u, r, z, \bar{z}\}$ and $\{v, r', z', \bar{z}'\}$ both interpolate between $\mathscr{I}^-$ and $\mathscr{I}^+$, the most natural choice to extract the boundary value at $\mathscr{I}^-$ is

$$\bar{\phi}^{(s)}_{z\ldots z}(v, z, \bar{z}) dz \otimes \cdots \otimes dz + \bar{\phi}^{(s)}_{\bar{z}\ldots \bar{z}}(v, z, \bar{z}) d\bar{z} \otimes \cdots \otimes d\bar{z} \equiv \lim_{r \to -\infty} \iota^* (r^{1-s} \phi^{(s)}_{\mu_1 \ldots \mu_s} dX^{\mu_1} \otimes \ldots \otimes dX^{\mu_s}) \tag{2.53}$$

as before, where the primes on the z's have been dropped because of (A13). This implies

$$\bar{\phi}^{(s)}_{z\ldots z}(v, z, \bar{z}) = -\frac{iK_{(s)}}{8\pi^2} \int_0^{+\infty} d\omega [a^{(s)}_+(\omega, z, \bar{z}) e^{-i\omega v} - a^{(s)}_-(\omega, z, \bar{z})^\dagger e^{i\omega v}], \tag{2.54}$$

and in particular $\bar{\phi}^{(s)}_{z\ldots z}(u, z, \bar{z}) \equiv \bar{\phi}^{(s)}_{z\ldots z}(v, z, \bar{z})$, which is also consistent with the change of coordinates (A13). This finally shows that the boundary value at $\mathscr{I}^-$ also transforms as (2.50) and (2.51) up to the mere replacement $u \mapsto v$.

2. *From boundary to bulk: The Kirchhoff-d'Adhémar formula*

Up to this stage, we have reviewed how a bulk free field induces a boundary field in the asymptotic region. We now make the link with the Kirchhoff-d'Adhémar formula [128,129] that describes how to reconstruct the bulk free field from its boundary value via a boundary-to-bulk propagator.

Inverting the Fourier transform (2.43), we get

$$\begin{aligned} a^{(s)}_+(\omega, z, \bar{z}) &= \frac{4\pi i}{K_{(s)}} \int_{-\infty}^{+\infty} du e^{i\omega u} \bar{\phi}^{(s)}_{z\ldots z}(u, z, \bar{z}), \\ a^{(s)}_-(\omega, z, \bar{z})^\dagger &= -\frac{4\pi i}{K_{(s)}} \int_{-\infty}^{+\infty} du e^{-i\omega u} \bar{\phi}^{(s)}_{z\ldots z}(u, z, \bar{z}), \end{aligned} \tag{2.55}$$

for $\omega > 0$. Rewriting the expression for the field (2.13) as

$$\phi^{(s)}_I(X) = \frac{K_{(s)}}{16\pi^3} \int \omega d\omega d^2 w \varepsilon^{*+}_I(w, \bar{w}) [a^{(s)}_+(\omega, w, \bar{w}) e^{i\omega q^\mu X_\mu} + a^{(s)}_-(\omega, w, \bar{w})^\dagger e^{-i\omega q^\mu X_\mu}] + \text{h.c.} \tag{2.56}$$

and inserting (2.55) gives

$$\begin{aligned} \phi^{(s)}_I(X) &= \frac{i}{4\pi^2} \int \omega d\omega d^2 w \varepsilon^{*+}_I(w, \bar{w}) \int_{-\infty}^{+\infty} d\tilde{u} [e^{i\omega(q^\mu X_\mu + \tilde{u})} - e^{-i\omega(q^\mu X_\mu + \tilde{u})}] \bar{\phi}^{(s)}_{z\ldots z}(\tilde{u}, w, \bar{w}) + \text{h.c.} \\ &= \frac{i}{4\pi^2} \int d^2 w \varepsilon^{*+}_I(w, \bar{w}) \int_{-\infty}^{+\infty} d\tilde{u} \int_{-\infty}^{+\infty} d\omega \omega e^{i\omega(q^\mu X_\mu + \tilde{u})} \bar{\phi}^{(s)}_{z\ldots z}(\tilde{u}, w, \bar{w}) + \text{h.c.} \end{aligned} \tag{2.57}$$

One concludes, making use of the identity $\int_{-\infty}^{+\infty} dx x e^{ipx} = -2\pi i \partial_p \delta(p)$, that

$$\phi_I^{(s)}(X) = \frac{1}{2\pi}\int d^2w d\tilde{u}\varepsilon_I^{*+}(w,\bar{w})[\partial_{\tilde{u}}\delta(\tilde{u}+q^\mu X_\mu)\bar{\phi}^{(s)}_{z...z}(\tilde{u},w,\bar{w})] + \text{h.c.} \tag{2.58}$$

In this sense, the boundary-to-bulk propagator is $\mathcal{P}(\tilde{u},w,\bar{w}|X) = \partial_{\tilde{u}}\delta(q^\mu X_\mu + \tilde{u})$. Integrating out the δ-distribution, we obtain the Kirchhoff-d'Adhémar formula [128,129]

$$\phi_I^{(s)}(X) = -\frac{1}{2\pi}\int d^2w\varepsilon_I^{*+}(w,\bar{w})\partial_{\tilde{u}}\bar{\phi}^{(s)}_{z...z}(\tilde{u}=-q^\mu X_\mu,w,\bar{w}) + \text{h.c.} \tag{2.59}$$

which allows one to reconstruct the bulk field $\phi_I^{(s)}(X)$ from its boundary value at $\mathscr{I}^+$. A similar relation holds for the boundary value at $\mathscr{I}^-$.

Remark—Consistently with the remark of the previous section, we recover from (2.59) that a shift of the form (2.44) does not alter the bulk field. Notice that, by contrast, changing the value of the pole in the Fourier transform $a_\pm^{(s)}(\omega,z,\bar{z}) \mapsto a_\pm^{(s)}(\omega,z,\bar{z}) + \Delta_\eta a_\pm^{(s)}(\omega,z,\bar{z})$ with

$$\Delta_\eta a_+^{(s)}(\omega,z,\bar{z}) = -\Delta_\eta a_-^{(s)}(\omega,z,\bar{z})^\dagger \equiv -\frac{4\pi}{K_{(s)}}\eta(z,\bar{z})\frac{1}{\omega}, \tag{2.60}$$

which corresponds to shifting the boundary field by $\Delta_\eta\bar{\phi}^{(s)}_{z...z}(u,z,\bar{z}) = \frac{1}{2}\eta(z,\bar{z})\text{sign}(u)$, does affect the bulk field as

$$\Delta_\eta\phi_I^{(s)}(X) = -\frac{1}{2\pi}\int d^2w\varepsilon_I^{*+}(w,\bar{w})\eta(w,\bar{w})\delta(q^\mu X_\mu) + \text{h.c.} \tag{2.61}$$

C. From null infinity to the celestial sphere

1. From position space to Mellin basis

Combining the results of the previous sections, we can construct a dictionary between boundary operators $\bar{\phi}^{(s)}_{z...z}(u,z,\bar{z})$ [respectively $\bar{\phi}^{(s)}_{z...z}(v,z,\bar{z})$] evolving in retarded (resp. advanced) time and the ladder operators in the Mellin basis $a^{(s)}_{\Delta,\alpha}(z,\bar{z})$ and $a^{(s)}_{\Delta,\alpha}(z,\bar{z})^\dagger$ living on the celestial sphere. Injecting (2.55) into (2.27), and using (2.24) to compute the integral on ω, yields

$$\begin{aligned} a^{(s)}_{\Delta,+}(z,\bar{z}) &= \frac{4\pi i}{K_{(s)}}\lim_{\epsilon\to 0^+}\int_0^{+\infty} d\omega\omega^{\Delta-1}\int_{-\infty}^{+\infty} du e^{i\omega u-\omega\epsilon}\bar{\phi}^{(s)}_{z...z}(u,z,\bar{z}) \\ &= \frac{4\pi}{K_{(s)}} i^{\Delta+1}\Gamma[\Delta]\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} du(u+i\epsilon)^{-\Delta}\bar{\phi}^{(s)}_{z...z}(u,z,\bar{z}), \end{aligned} \tag{2.62}$$

where $\Delta = c + i\nu$, $c > 0$. The integral transform (2.62) coincides with the one discussed in the extrapolate-style dictionary presented in [93]. It essentially trades the time dependence u of the operators in position space for the conformal dimension Δ of the operators in Mellin space. This motivates introducing

$$\boxed{\mathcal{B}_\pm[f(u),\Delta] \equiv \kappa^\pm_\Delta \lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} du(u\pm i\epsilon)^{-\Delta}f(u)} \tag{2.63}$$

for any smooth functions f defined on $\mathscr{I}^+$, with $\kappa^\pm_\Delta = 4\pi(\pm i)^{\Delta+1}\Gamma[\Delta]$. The integral (2.63), here referred to as the $\mathcal{B}$-transform, is merely the composition of a Fourier transform (which maps $u \mapsto \omega$) and a Mellin transform (which maps $\omega \mapsto \Delta$), see (2.62). One should stress that the $\mathcal{B}_+$-transform (resp. $\mathcal{B}_-$) alone is *not* invertible since it projects out the positive (resp. negative) frequency modes of f, i.e. it annihilates the second (resp. first) term in the decomposition

$$f(u) = \int_0^{+\infty} d\omega[f_+(\omega)e^{-i\omega u} + f_-(\omega)e^{i\omega u}]. \tag{2.64}$$

This is easily seen from the first line of (2.62) where the integral in u effectively gets rid of all positive frequency modes of $\bar{\phi}^{(s)}_{z...z}(u,z,\bar{z})$. However $\mathcal{B}_+$ (resp. $\mathcal{B}_-$) is invertible when restricted to functions with negative (resp. positive) frequency only.

This comes from the fact that composing a Fourier and a Mellin transform is only possible if the integral over frequencies is taken over $\mathbb{R}^+$, which amounts to use "unilateral" (hence improper) Fourier transforms that are also not invertible. The physical meaning of this choice of integration domain is again that the asymptotic excitations of $\phi_I^{(s)}$ have strictly positive energy, $\omega > 0$.

Keeping this in mind, we have

$$a^{(s)}_{\Delta,+}(z,\bar{z}) = \frac{1}{K_{(s)}}\mathcal{B}_+[\bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}),\Delta], \qquad a^{(s)}_{\Delta,-}(z,\bar{z})^\dagger = \frac{1}{K_{(s)}}\mathcal{B}_-[\bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}),\Delta], \tag{2.65}$$

and

$$a^{(s)}_{\Delta,-}(z,\bar{z}) = \frac{1}{K_{(s)}}\mathcal{B}_+[\bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(u,z,\bar{z}),\Delta], \qquad a^{(s)}_{\Delta,+}(z,\bar{z})^\dagger = \frac{1}{K_{(s)}}\mathcal{B}_-[\bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(u,z,\bar{z}),\Delta]. \tag{2.66}$$

One can write similar expressions at $\mathscr{I}^-$ for $\bar{\phi}^{(s)}_{z\ldots z}(v,z,\bar{z})$ expanded as (2.54) by making the replacement $u \mapsto v$.

2. From Mellin basis to position space

We now want to discuss how to reconstruct the boundary field $\bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z})$ from the celestial operators $a^{(s)}_{\Delta,\pm}(z,\bar{z})$. To achieve this, let us first define

$$\boxed{\mathcal{B}^{-1}_\pm[F(\Delta),u] \equiv -\frac{1}{16\pi^3}\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} d\nu \frac{(\pm i)^\Delta \Gamma[1-\Delta]}{(u\mp i\epsilon)^{1-\Delta}}F(\Delta).} \tag{2.67}$$

These operators are effectively obtained by the successive action of two integral transforms

$$\mathcal{B}^{-1}_\pm[F(\Delta),u] = \frac{\mp i}{16\pi^3}\int_0^{+\infty} d\omega e^{\mp i\omega u}\int_{-\infty}^{+\infty} d\nu \omega^{-\Delta}F(\Delta). \tag{2.68}$$

The second integral in (2.68) is the inverse Mellin transform (2.23), while the first integral means that functions in the image of $\mathcal{B}^{-1}_+$ (resp. $\mathcal{B}^{-1}_+$) have negative (resp. positive) frequencies only. Now, one always has

$$\mathcal{B}_\pm \circ \mathcal{B}^{-1}_\pm[F(\Delta)] = F(\Delta). \tag{2.69}$$

This relation can checked explicitly by providing a realization of the δ distribution on conformal weights as

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} du \frac{i^\Delta \Gamma[\Delta]}{(u+i\epsilon)^\Delta}\frac{(-i)^{1-\Delta'}\Gamma[1-\Delta']}{(u-i\epsilon)^{1-\Delta'}} = 4\pi^2\delta(\nu-\nu'), \tag{2.70}$$

where $\Delta = c + i\nu$ and $\Delta' = c + i\nu'$. This identity is the analog of (2.25) but adapted for the $\mathcal{B}$-transform instead of the Mellin transform; it can be derived by first rewriting each of the fractions in the integrand of (2.70) as Mellin transforms of plane waves, then performing the integral in u to obtain a δ-function and concluding using (2.25).

However, as we already emphasized, the transforms $\mathcal{B}_\pm$ are not invertible. Rather, if we decompose $f(u)$ in positive/negative frequency modes as in (2.64), then

$$\mathcal{B}^{-1}_\pm \circ \mathcal{B}_\pm[f(u)] = \int_0^{+\infty} d\omega e^{\mp i\omega u} f_\pm(\omega), \tag{2.71}$$

i.e. $\mathcal{B}^{-1}_\pm \circ \mathcal{B}_\pm$ are the projectors on negative/positive frequency modes. This follows from

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} d\nu \frac{i^\Delta\Gamma[\Delta]}{(u+i\epsilon)^\Delta}\frac{(-i)^{1-\Delta}\Gamma[1-\Delta]}{(u'-i\epsilon)^{1-\Delta}} = 2\pi\int_0^{+\infty} d\omega e^{i\omega(u-u')}, \tag{2.72}$$

which is not $\delta(u-u')$ because of the restricted domain of integration. This is responsible for the fact that $\mathcal{B}^{-1}_\pm \circ \mathcal{B}_\pm$ is a projection rather than the identity map. Equation (2.72) can be derived by rewriting the second fraction in the integrand as the Mellin transform of a plane wave and recognizing the resulting integral over ν as an inverse Mellin transform acting on the first fraction.

From all these remarks, we conclude that the relations (2.65) are inverted as follows:

$$\bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z}) = K_{(s)}\mathcal{B}^{-1}_+[a^{(s)}_{\Delta,+}(z,\bar{z}),u] + K_{(s)}\mathcal{B}^{-1}_-[a^{(s)}_{\Delta,-}(z,\bar{z})^\dagger,u]. \tag{2.73}$$

with a similar expression for $\bar{\phi}^{(s)}_{z\ldots z}(v,z,\bar{z})$ obtained by making the replacement $u \mapsto v$.

D. $\mathcal{S}$-matrix in the three scattering bases

We now turn to the scattering of interacting massless spin-s fields in flat spacetime. This discussion will be repeated in the three scattering bases [113]: momentum space, Mellin space and position space. Even though the $\mathcal{S}$-matrix is known to be trivial for higher spins $s > 2$ [130–134] (see e.g. [135] for a recent review), it will be valuable to keep the spin arbitrary throughout to highlight some general structure. In fact, some patterns of an infrared triangle [13] have been shown to persist for higher-spin theories [136–139], which makes the analysis for $s > 2$ interesting by itself. Moreover, there exist nontrivial (interacting) chiral higher-spin theories [140–146] (see [147,148] for the relevance of these theories in the context of celestial OPEs). The presence of interactions in those theories does not necessarily mean that the $\mathcal{S}$-matrix is nontrivial; typically, the usual theorems apply and, as a result of the higher-spin symmetries, scattering amplitudes vanish. However there are examples of chiral higher-spin theories with a nontrivial $\mathcal{S}$-matrix [149,150].

1. Asymptotically free fields for null scattering processes

As usual in the $\mathcal{S}$-matrix picture, the interacting theory is taken to be asymptotically free. More explicitly, in retarded coordinates (A8), we suppose that the bulk operator $\phi_I^{(s)}$ of the full interacting theory can be expanded around $\mathscr{I}^+$ as

$$\phi_I^{(s)}(X) = \phi_I^{(s)}(X)^{\text{out}} + \mathcal{O}(r^{s-2}), \tag{2.74}$$

and in advanced coordinates (A11) around $\mathscr{I}^-$ as

$$\phi_I^{(s)}(X) = \phi_I^{(s)}(X)^{\text{in}} + \mathcal{O}(r^{s-2}). \tag{2.75}$$

Here $\phi_I^{(s)}(X)^{\text{out/in}}$ are the free fields discussed in the previous sections and $\mathcal{O}(r^{s-2})$ stands for the contributions of the interactions in the bulk that are not seen near the boundary.

Under these assumptions, the boundary values (2.43) and (2.54) of the interacting field at $\mathscr{I}^+$ and $\mathscr{I}^-$ are finite and coincide with the boundary values of $\phi_I^{(s)}(X)^{\text{out/in}}$,

$$\bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z})^{\text{out}} = -\frac{iK_{(s)}}{8\pi^2}\int_0^{+\infty} d\omega[a_+^{(s)\text{out}}(\omega,z,\bar{z})e^{-i\omega u} - a_-^{(s)\text{out}}(\omega,z,\bar{z})^\dagger e^{i\omega u}], \tag{2.76}$$

$$\bar{\phi}^{(s)}_{z\ldots z}(v,z,\bar{z})^{\text{in}} = -\frac{iK_{(s)}}{8\pi^2}\int_0^{+\infty} d\omega[a_+^{(s)\text{in}}(\omega,z,\bar{z})e^{-i\omega v} - a_-^{(s)\text{in}}(\omega,z,\bar{z})^\dagger e^{i\omega v}]. \tag{2.77}$$

We therefore asymptotically end up with two free theories related by a nontrivial interaction process. In the free theory, our conventions (2.42)–(2.53) mean that incoming and outgoing boundary values are equal, up to the mere replacement $u \mapsto v$. In presence of interactions, the antipodal matching conditions of [126,151] read

$$\bar{\phi}^{(s)}_{z\ldots z}(u,z,\bar{z})^{\text{out}}|_{\mathscr{I}^+_-} = \bar{\phi}^{(s)}_{z\ldots z}(v,z,\bar{z})^{\text{in}}|_{\mathscr{I}^-_+}, \tag{2.78}$$

where $\mathscr{I}^+_- = \{X\in\mathscr{I}^+ | u\to-\infty\}$ and $\mathscr{I}^-_+ = \{X\in\mathscr{I}^- | v\to+\infty\}$. This is because the coordinates $z = z'$ define antipodal directions on the celestial sphere between $\mathscr{I}^-$ and $\mathscr{I}^+$.

In Sec. II C, three descriptions of these fields have been presented: the direct interpretation in position space and their decomposition in terms of ladder operators both in Fourier and Mellin spaces. In the following, we relate the asymptotic states in these three equivalent descriptions by integral transforms.

2. Scattering amplitudes in momentum space

In momentum space, we can construct an incoming or outgoing state, respectively denoted as $|\omega,z,\bar{z},s,\alpha\rangle$ and $\langle\omega,z,\bar{z},s,\alpha|$, representing a particle of light-cone energy ω, spin s and helicity $J = \alpha s = \pm s$ coming from or heading to the point $(z,\bar{z})$ on the celestial sphere, by acting on the respective vacua with the ladder operators $a_\alpha^{(s)\text{in/out}}(\omega,z,\bar{z})$, i.e.

$$\begin{aligned}\langle\omega,z,\bar{z},s,\alpha| &= -\frac{iK_{(s)}}{4\pi}\langle 0|a_\alpha^{(s)\text{out}}(\omega,z,\bar{z}),\\ |\omega,z,\bar{z},s,\alpha\rangle &= \frac{iK_{(s)}}{4\pi}a_\alpha^{(s)\text{in}}(\omega,z,\bar{z})^\dagger|0\rangle.\end{aligned} \tag{2.79}$$

The perhaps unusual choice of normalization of energy eigenstates in (2.79) is inspired by the relations (2.55) and will be motivated in Sec. II D 4. The scattering amplitudes in momentum space involving N massless particles—n of which are outgoing—are given by the $\mathcal{S}$-matrix elements $\mathcal{A}_N(p_1;\ldots;p_N) = \langle\text{out}|\text{in}\rangle_{\text{mom}}$ with

$$\langle\text{out}| = \langle\omega_1,z_1,\bar{z}_1,s_1,\alpha_1| \otimes \ldots \otimes \langle\omega_n,z_n,\bar{z}_n,s_n,\alpha_n| \tag{2.80}$$

and

$$|\text{in}\rangle = |\omega_{n+1}, z_{n+1}, \bar{z}_{n+1}, s_{n+1}, \alpha_{n+1}\rangle \otimes \ldots \otimes |\omega_N, z_N, \bar{z}_N, s_N, \alpha_N\rangle. \tag{2.81}$$

3. Scattering amplitudes in Mellin space

Alternatively, as it was pointed out in [68,70] (see [111] for a review), one can express the $\mathcal{S}$-matrix elements in Mellin space using the change of representation discussed in Sec. II A 2. The Mellin transform (2.27) turns the in/out asymptotic ladder operators in momentum space to asymptotic operators in the Mellin basis. Correspondingly, we obtain the asymptotic states on the celestial sphere from the usual momentum eigenstates by

$$\langle\Delta, z, \bar{z}, s, \alpha| = -\frac{iK_{(s)}}{4\pi}\langle 0|a^{(s)\text{out}}_{\Delta,\alpha}(z, \bar{z}) = \int_0^{+\infty} d\omega\omega^{\Delta-1}\langle\omega, z, \bar{z}, s, \alpha| \tag{2.82}$$

for outgoing particles, and

$$|\Delta, z, \bar{z}, s, \alpha\rangle = \frac{iK_{(s)}}{4\pi} a^{(s)\text{in}}_{\Delta,\alpha}(z, \bar{z})^\dagger|0\rangle = \int_0^{+\infty} d\omega\omega^{\Delta-1}|\omega, z, \bar{z}, s, \alpha\rangle \tag{2.83}$$

for incoming particles. Because of (2.29), states defined as (2.82) and (2.83) are boost eigenstates. The so-called "celestial amplitudes" $\mathcal{M}_N = \langle\text{out}|\text{in}\rangle_{\text{boost}}$ involving N inserted particles on the celestial sphere are now obtained as (dropping the spin indices)

$$\mathcal{M}_N(\Delta_1, z_1, \bar{z}_1; \ldots; \Delta_N, z_N, \bar{z}_N) = \int_0^{+\infty} d\omega_1\omega_1^{\Delta_1-1} \ldots \int_0^{+\infty} d\omega_N\omega_N^{\Delta_N-1}\mathcal{A}_N(p_1; \ldots; p_N). \tag{2.84}$$

4. Scattering amplitudes in position space

Finally, we can define the asymptotic quantum states directly in position space. Outgoing states at $\mathscr{I}^+$ are naturally defined as

$$\begin{aligned}\langle u, z, \bar{z}, s, +| &= \langle 0|\bar{\phi}^{(s)}_{z\ldots z}(u, z, \bar{z}) = \frac{1}{2\pi}\int_0^{+\infty} d\omega e^{-i\omega u}\langle\omega, z, \bar{z}, s, +|,\\ \langle u, z, \bar{z}, s, -| &= \langle 0|\bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(u, z, \bar{z}) = \frac{1}{2\pi}\int_0^{+\infty} d\omega e^{-i\omega u}\langle\omega, z, \bar{z}, s, -|,\end{aligned} \tag{2.85}$$

because of (2.76) and (2.77). The boundary field $\bar{\phi}^{(s)}_{z\ldots z}(u, z, \bar{z})$ creates outgoing spin-s particles with positive helicity and destroys outgoing spin-s particles with negative helicity, while $\bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(u, z, \bar{z}) = \bar{\phi}^{(s)}_{z\ldots z}(u, z, \bar{z})^\dagger$ acts in the opposite way. The normalization of (2.79) allows us to represent the creation and annihilation operators in position space (2.85) without extra normalization factors. Playing the same game at $\mathscr{I}^-$, we construct the incoming states in position space as

$$\begin{aligned}|v, z, \bar{z}, s, +\rangle &= \bar{\phi}^{(s)}_{z\ldots z}(v, z, \bar{z})^\dagger|0\rangle = \frac{1}{2\pi}\int_0^{+\infty} d\omega e^{i\omega v}|\omega, z, \bar{z}, s, +\rangle,\\ |v, z, \bar{z}, s, -\rangle &= \bar{\phi}^{(s)}_{\bar{z}\ldots\bar{z}}(v, z, \bar{z})^\dagger|0\rangle = \frac{1}{2\pi}\int_0^{+\infty} d\omega e^{i\omega v}|\omega, z, \bar{z}, s, -\rangle.\end{aligned} \tag{2.86}$$

We then introduce "position space amplitudes" $\mathcal{C}_N = \langle\text{out}|\text{in}\rangle_{\text{pos}}$, which are obtained from the usual momentum representation of the $\mathcal{S}$-matrix as

$$\boxed{\mathcal{C}_N(u_1,z_1,\bar{z}_1;\ldots;u_n,z_n,\bar{z}_n;v_{n+1},z_{n+1},\bar{z}_{n+1};\ldots;v_N,z_N,\bar{z}_N) = \frac{1}{(2\pi)^N}\prod_{k=1}^{n}\int_0^{+\infty} d\omega_k e^{-i\omega_k u_k}\prod_{\ell=n+1}^{N}\int_0^{+\infty} d\omega_\ell e^{i\omega_\ell v_\ell}\mathcal{A}_N(p_1;\ldots;p_N).} \tag{2.87}$$

These amplitudes can be translated in the Mellin representation thanks to the $\mathcal{B}$-transform (2.63) by noticing that

$$\langle \Delta, z, \bar{z}, s, \alpha| = \frac{1}{4\pi i}\mathcal{B}_+[\langle u, z, \bar{z}, s, \alpha|, \Delta], \qquad |\Delta, z, \bar{z}, s, \alpha\rangle = -\frac{1}{4\pi i}\mathcal{B}_-[|v, z, \bar{z}, s, \alpha\rangle, \Delta], \tag{2.88}$$

owing to the definitions (2.65), (2.82), and (2.83). Hence

$$\mathcal{M}_N(\Delta_1, z_1, \bar{z}_1; \ldots; \Delta_N, z_N, \bar{z}_N) = \frac{(-1)^{N-n}}{(4\pi i)^N}\mathcal{B}_-^{(N-n)}[\mathcal{B}_+^{(n)}[\mathcal{C}_N(u_1, z_1, \bar{z}_1; \ldots; v_N, z_N, \bar{z}_N), \{\Delta_1, \ldots, \Delta_n\}], \{\Delta_{n+1}, \ldots, \Delta_N\}], \tag{2.89}$$

where $\mathcal{B}_+^{(k)}$ represents k successive applications of the $\mathcal{B}$-transform.

5. Low-point amplitudes

Let us make more concrete the action of the various integral transforms introduced so far by considering propagation of one particle, ignoring quantum-loop effects. In the momentum basis, the tree-level amplitude $\mathcal{A}_2$ is

$$\mathcal{A}_2(\omega_1, z_1, \bar{z}_1; \omega_2, z_2, \bar{z}_2) = K_{(s)}^2 \pi \frac{\delta(\omega_1 - \omega_2)}{\omega_1}\delta^{(2)}(z_1 - z_2)\delta_{\alpha_1,\alpha_2}, \tag{2.90}$$

for $\omega_i > 0$. This simply describes the travel of a particle of helicity $J = \alpha s$ inserted at $\mathscr{I}^-$ crossing Minkowski spacetime towards $\mathscr{I}^+$.

Recalling the definition (2.82) and (2.83) and using the transformation (2.84), the two-point amplitude in Mellin space can be deduced from (2.90) and is given by [71]

$$\begin{aligned}\mathcal{M}_2(\Delta_1, z_1, \bar{z}_1; \Delta_2, z_2, \bar{z}_2) &= \int_0^{+\infty} d\omega_1 \omega_1^{\Delta_1 - 1}\int_0^{+\infty} d\omega_2 \omega_2^{\Delta_2 - 1}\mathcal{A}_2(\omega_1, z_1, \bar{z}_1; \omega_2, z_2, \bar{z}_2)\\ &= K_{(s)}^2\pi\int_0^{+\infty} d\omega\omega^{\Delta_1+\Delta_2-3}\delta^{(2)}(z_1 - z_2)\delta_{\alpha_1,\alpha_2} = 2\pi^2 K_{(s)}^2\delta(\nu_1+\nu_2)\delta^{(2)}(z_1 - z_2)\delta_{\alpha_1,\alpha_2},\end{aligned} \tag{2.91}$$

where $\Delta_i = 1 + i\nu_i$ is assumed for the integral over ω to be converging, as prescribed by (2.25).

The position space amplitude $\mathcal{C}_2$ can also be computed from (2.90) as

$$\begin{aligned}\mathcal{C}_2(u_1, z_1, \bar{z}_1; u_2, z_2, \bar{z}_2) &= \frac{1}{4\pi^2}\int_0^{+\infty} d\omega_1\int_0^{+\infty} d\omega_2 e^{-i\omega_1 u_1}e^{i\omega_2 u_2}\mathcal{A}_2(\omega_1, z_1, \bar{z}_1; \omega_2, z_2, \bar{z}_2)\\ &= \frac{K_{(s)}^2}{4\pi}\underbrace{\int_0^{+\infty}\frac{d\omega}{\omega}e^{-i\omega(u_1-u_2)}}_{\equiv \mathcal{I}_0(u_1-u_2)}\delta^{(2)}(z_1 - z_2)\delta_{\alpha_1,\alpha_2},\end{aligned} \tag{2.92}$$

owing to (2.87) and using (A13) to trade v for u. The integral $\mathcal{I}_0(u_1 - u_2)$ is divergent but can nevertheless be regulated noticing that $\mathcal{I}_0(u_1 - u_2) = \lim_{\beta\to 0^+}\mathcal{I}_\beta(u_1 - u_2)$ with the definition

$$\mathscr{I}_\beta(x) = \lim_{\epsilon\to 0^+}\int_0^{+\infty} d\omega\omega^{\beta-1}e^{-i\omega x - \omega\epsilon} = \lim_{\epsilon\to 0^+}\frac{\Gamma[\beta](-i)^\beta}{(x - i\epsilon)^\beta}. \tag{2.93}$$

In the limit $\beta \to 0^+$, we obtain

$$\mathcal{I}_\beta(x) = \frac{1}{\beta} - \left[\gamma + \ln|x| + \frac{i\pi}{2}\mathrm{sign}(x)\right] + \mathcal{O}(\beta), \tag{2.94}$$

where γ is the Euler-Mascheroni constant. So (2.92) evaluates to

$$\boxed{\mathcal{C}_2(u_1;u_2) = \frac{K_{(s)}^2}{4\pi}\left[\frac{1}{\beta} - \left(\gamma + \ln|u_1-u_2| + \frac{i\pi}{2}\mathrm{sign}(u_1-u_2)\right)\right]\delta^{(2)}(z_1-z_2)\delta_{\alpha_1,\alpha_2} + \mathcal{O}(\beta).} \tag{2.95}$$

We show in Sec. V D that (2.95) is a Poincaré invariant object as a consequence of (2.50); importantly, the pole in $1/\beta$ shall be kept because it is essential to ensure boost invariance. Let us stress that, fundamentally, this divergence can be interpreted as an infrared pole. Indeed, if we choose to regularize the integral $\mathcal{I}_0(x)$ as

$$\mathcal{I}_0(x) = \lim_{\epsilon\to 0^+}\int_\epsilon^{+\infty}\frac{d\omega}{\omega}e^{-i\omega x} \equiv \lim_{\epsilon\to 0^+}E_1(ix\epsilon), \tag{2.96}$$

where $E_1(z)$ represents the principal branch $\mathrm{Arg}(z)\in]-\pi,\pi[$ of the complex exponential integral, for any $z\notin\mathbb{R}_-$, we have the following series [152]:

$$E_1(z) = -\gamma - \ln z - \sum_{k=1}^{+\infty}\frac{1}{k!}\frac{(-z)^k}{k} \Rightarrow \mathcal{I}_0(x) = \ln\frac{1}{\epsilon} - \gamma - \ln|x| - \mathrm{sign}(x)\frac{i\pi}{2} + \mathcal{O}(\epsilon) \tag{2.97}$$

in the limit $\epsilon\to 0^+$. Boost invariance requires that $\ln(\epsilon^{-1})$ behaves around zero as the pole of the Euler gamma function, hence $\beta^{-1}\simeq\ln(\epsilon^{-1})$ and both treatments of the divergence agree on the final result.

Let us now observe thanks to (2.85) and (2.86) that

$$\mathcal{C}_2(u_1,z_1,\bar{z}_1;u_2,z_2,\bar{z}_2) \equiv \langle 0|\bar{\phi}^{(s)}_{z\ldots z}(u_1,z_1,\bar{z}_1)\bar{\phi}^{(s)}_{z\ldots z}(u_2,z_2,\bar{z}_2)^\dagger|0\rangle \tag{2.98}$$

picking $\alpha=+$ for definiteness (similar considerations apply, *mutatis mutandis*, for $\alpha=-$). In Sec. V, we will interpret this object as a (holographic) boundary two-point function; it results from the double limit of the bulk two-point function where one point is sent to $\mathscr{I}^+$ and the other one to $\mathscr{I}^-$. Although (2.98) has an infrared divergence, we directly observe from (2.94) that the difference $\mathcal{I}_\beta(u_1-u_2)-\mathcal{I}_\beta(u_2-u_1)$ is finite in the limit $\beta\to 0^+$ and reproduces the commutation relation (2.48), i.e.

$$\begin{aligned}\langle 0|[\bar{\phi}^{(s)}_{z\ldots z}(u_1,z_1,\bar{z}_1),\bar{\phi}^{(s)}_{z\ldots z}(u_2,z_2,\bar{z}_2)^\dagger]|0\rangle &= \lim_{\beta\to 0^+}\frac{K_{(s)}^2}{4\pi}[\mathcal{I}_\beta(u_1-u_2)-\mathcal{I}_\beta(u_2-u_1)]\delta^{(2)}(z_1-z_2)\\ &= -\frac{iK_{(s)}^2}{4}\mathrm{sign}(u_1-u_2)\delta^{(2)}(z_1-z_2).\end{aligned} \tag{2.99}$$

The behavior (2.95) of the boundary two-point function (2.98) has also been established independently in [153].

Finally, let us close the loop by showing that applying twice the $\mathcal{B}$-transform as prescribed in (2.89) gives us back the celestial amplitude. Keeping (2.88) in mind, we first compute

$$\begin{aligned}\mathcal{B}_+[\mathcal{I}_0(u_1-u_2),\Delta_1] &= 4\pi i^{\Delta_1+1}\Gamma[\Delta_1]\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}\frac{du_1}{(u_1+i\epsilon)^{\Delta_1}}\mathcal{I}_0(u_1-u_2)\\ &= 4\pi i\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}du_1\int_0^{+\infty}\omega^{\Delta_1-1}e^{i\omega u_1-\omega\epsilon}\int_0^{+\infty}\frac{d\omega'}{\omega'}e^{-i\omega'(u_1-u_2)}\\ &= 8\pi^2 i\lim_{\epsilon\to 0^+}\int_0^{+\infty}d\omega\omega^{\Delta_1-2}e^{i\omega u_2-\omega\epsilon} = 8\pi^2 i\lim_{\epsilon\to 0^+}\frac{i^{\Delta_1-1}\Gamma[\Delta_1-1]}{(u_2+i\epsilon)^{\Delta_1-1}}.\end{aligned} \tag{2.100}$$

Computing the second $\mathcal{B}$-transform leads to

$$\mathcal{B}_-[\mathcal{B}_+[\mathcal{I}_0(u_1-u_2),\Delta_1],\Delta_2] = 32\pi^3\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}du_2\frac{i^{\Delta_1-1}\Gamma[\Delta_1-1]}{(u_2+i\epsilon)^{\Delta_1-1}}\frac{(-i)^{\Delta_2}\Gamma[\Delta_2]}{(u_2-i\epsilon)^{\Delta_2}} = 128\pi^5\delta(\nu_1+\nu_2), \tag{2.101}$$

as a corollary of (2.70) assuming $\Delta_i = 1+i\nu_i$. Therefore, using the dictionary (2.89),

$$\begin{aligned}&\mathcal{M}_2(\Delta_1, z_1, \bar{z}_1; \Delta_2, z_2, \bar{z}_2)\\&\quad = \frac{1}{(4\pi)^2}\mathcal{B}_-[\mathcal{B}_+[\mathcal{C}_2(u_1, z_1, \bar{z}_1; u_2, z_2, \bar{z}_2), \Delta_1], \Delta_2]\\&\quad = 2\pi^2 K^2_{(s)}\delta(\nu_1+\nu_2)\delta^{(2)}(z_1-z_2)\delta_{\alpha_1,\alpha_2}, \qquad (2.102)\end{aligned}$$

we recover (2.91) as it should.

A similar treatment could be applied to three-point amplitudes but kinematic constraints imply that the latter have to vanish in Lorentzian signature $(-,+,+,+)$. This issue can be circumvented by working in split signature $(-,+,-,+)$ in which z and $\bar{z}$ are no longer related by complex conjugation (see e.g. [154]) and the description of amplitudes in position space can be adapted to this framework. These considerations, as well as the computation of the higher-point amplitudes ($N \geq 4$) will be discussed elsewhere.

6. The three bases for scattering amplitudes in flat spacetime

From the above discussion, it follows that one can work in three different spaces to discuss the boundary fields:

(i) The position space involving the boundary field $\bar{\phi}^{(s)}(u/v, z, \bar{z})$,

(ii) The Fourier space providing the usual description by ladder operators $a_{\pm}^{(s)\mathrm{out/in}}(\omega, z, \bar{z})$,

(iii) The Mellin space with the celestial ladder operators $a_{\Delta,\pm}^{(s)\mathrm{out/in}}(z, \bar{z})$,

see Fig. 3 for a summary. This observation was already pointed out in [113] where it was argued that the three spaces, or "bases," are useful, depending on the question of interest. For instance, to discuss infrared issues, Fourier space will be suited to write the soft theorems, Mellin space will be more convenient to discuss symmetries through some Ward identities, and position space will be appropriate to highlight the memory effects.

We observe that the $\mathcal{S}$-matrix is holographic by nature, in the sense that its elements can be reinterpreted as correlations between boundary operators. Interestingly, as we

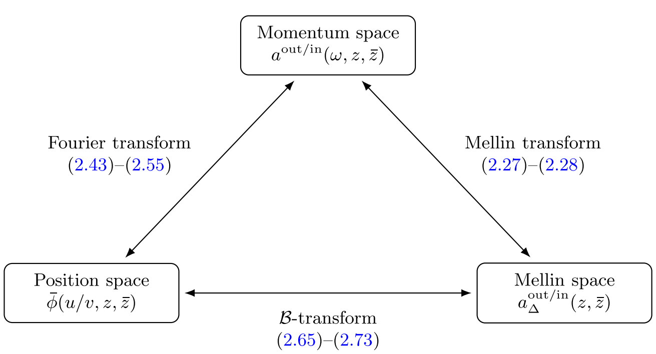

FIG. 3. Interplay between the three bases of scattering in flat spacetime.

will argue in Sec. V, the boundary operators in position space can be interpreted as operators sourcing a dual Carrollian CFT. The $\mathcal{S}$-matrix elements are then just seen as correlation functions for these operators. This points towards the Carrollian approach of flat space holography. Similarly, the boundary operators in Mellin space can be interpreted as operators in the CCFT, pointing towards the celestial holography proposal. Therefore, one can already deduce that the $\mathcal{B}$-transform discussed in Sec. II C relates the Carrollian CFT and the CCFT. It trades the null time dependence of the Carrollian operators for the conformal dimension of the CCFT operators. We will come back to this observation in Sec. VI when discussing the link between Carrollian and celestial holographies.

III. ASYMPTOTIC SYMMETRIES IN FLAT SPACETIME

This section reviews the asymptotic symmetry analysis of electrodynamics and gravity. This allows us to fix the notations and conventions that will be useful in order to discuss the holographic Carrollian correspondence in Sec. V. The BMS asymptotic symmetries of gravity will play the role of global spacetime symmetries in the dual Carrollian CFT, while the $U(1)$ asymptotic symmetries of electrodynamics will provide an example of global internal symmetry.

A. Scalar electrodynamics

We start by reviewing the asymptotic symmetry structure of electrodynamics on a flat background (see [126,155] for the original references and [13] for a review). We denote the electromagnetic potential as $A_\mu(X)$ and the matter current as $\mathcal{J}_\mu(X)$. For definiteness, we focus on complex scalar matter field $\phi(X)$ carrying a charge Qe, where $Q \in \mathbb{R}$ and e is the elementary charge. The electromagnetic current is thus

$$\mathcal{J}_\mu = iQe(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*). \qquad (3.1)$$

In the retarded coordinates $\{u, r, z, \bar{z}\}$, radiative boundary conditions for the potential A_μ near future null infinity are

$$\begin{aligned}&A_z(u, r, z, \bar{z}) = A_z^{(0)}(u, z, \bar{z}) + \mathcal{O}(r^{-1}),\\&A_r(u, r, z, \bar{z}) = \mathcal{O}(r^{-2}), \qquad A_u(u, r, z, \bar{z}) = \mathcal{O}(r^{-1}), \qquad (3.2)\end{aligned}$$

consistently with Sec. II. Comparing with (2.43), the role of the boundary field is played here by $A_z^{(0)}(u, z, \bar{z})$. These boundary conditions are preserved by gauge transformations with $\lambda(u, r, z, \bar{z}) = \lambda^{(0)}(z, \bar{z}) + \mathcal{O}(r^{-1})$ such that $\delta_\lambda A_z^{(0)} = \partial_z\lambda^{(0)}$. On the matter side, the massless scalar field decays as

$$\phi(u,r,z,\bar{z}) = \frac{\phi^{(0)}(u,z,\bar{z})}{r} + \mathcal{O}(r^{-2}) \tag{3.3}$$

at future null infinity. The boundary field $\phi^{(0)}(u,z,\bar{z})$ is left unconstrained by the massless Klein-Gordon equation and transforms as

$$\delta_\lambda \phi^{(0)}(u,z,\bar{z}) = -i\lambda^{(0)}(z,\bar{z}) Q e \phi^{(0)}(u,z,\bar{z}) \tag{3.4}$$

under the gauge symmetry.

Expanding the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and current $\mathcal{J}_\mu$ in $1/r$ near $\mathscr{I}^+$ taking (3.2) into account, we have

$$\begin{aligned} F_{uz}(u,r,z,\bar{z}) &= F_{uz}^{(0)}(u,z,\bar{z}) + \mathcal{O}(r^{-1}), \\ F_{ru}(u,r,z,\bar{z}) &= F_{ru}^{(2)}(u,z,\bar{z}) r^{-2} + \mathcal{O}(r^{-3}), \\ F_{z\bar{z}}(u,r,z,\bar{z}) &= F_{z\bar{z}}^{(0)}(u,z,\bar{z}) + \mathcal{O}(r^{-1}), \\ \mathcal{J}_u(u,r,z,\bar{z}) &= \mathcal{J}_u^{(2)}(u,z,\bar{z}) r^{-2} + \mathcal{O}(r^{-3}). \end{aligned} \tag{3.5}$$

In addition, one requires that the magnetic field is vanishing at $\mathscr{I}^+_\pm$ (see [156] for a consideration of magnetic contributions at $\mathscr{I}^+_\pm$), i.e.

$$F_{z\bar{z}}^{(0)}|_{\mathscr{I}^+_\pm} = (\partial_z A_{\bar{z}}^{(0)} - \partial_{\bar{z}} A_z^{(0)})|_{\mathscr{I}^+_\pm} = 0. \tag{3.6}$$

The leading equation of motion is

$$\partial_u F_{ru}^{(2)} + \partial_{\bar{z}} F_{uz}^{(0)} + \partial_z F_{u\bar{z}}^{(0)} + e^2 \mathcal{J}_u^{(2)} = 0. \tag{3.7}$$

The surface charges associated with the residual gauge symmetry driven by $\lambda^{(0)}$ can be derived as [13,126,155]

$$Q_\lambda[A] = -\frac{1}{e^2}\int_\Sigma d^2 z \lambda^{(0)}(z,\bar{z}) F_{ru}^{(2)}(u,z,\bar{z}), \tag{3.8}$$

where Σ is a $\{u = \text{const.}\}$ cut of $\mathscr{I}^+$. As retarded time evolves along $\mathscr{I}^+$, these charges are not conserved because of the presence of electromagnetic radiation, encoded in the leading of piece of the Maxwell field $A_z^{(0)}$, and the null matter current $\mathcal{J}_u^{(2)}$. Indeed, using (3.7), one has the following flux-balance law:

$$\frac{dQ_\lambda}{du} = \frac{1}{e^2}\int_\Sigma d^2 z \lambda^{(0)} (\partial_{\bar{z}} F_{uz}^{(0)} + \partial_z F_{u\bar{z}}^{(0)} + e^2 \mathcal{J}_u^{(2)}) \equiv \int_\Sigma d^2 z F_\lambda. \tag{3.9}$$

In order to distinguish between hard (finite energy) and soft (zero energy) radiative excitations of the gauge field, one can split it as [126]

$$A_z^{(0)}(u,z,\bar{z}) = \tilde{A}_z^{(0)}(u,z,\bar{z}) + \partial_z \chi(z,\bar{z}), \tag{3.10}$$

where $\chi(z,\bar{z})$ is related to the early and late-time values of the boundary field as

$$\partial_z \chi(z,\bar{z}) = \frac{1}{2}(A_z^{(0)}|_{\mathscr{I}^+_+} + A_z^{(0)}|_{\mathscr{I}^+_-}). \tag{3.11}$$

Because $\chi(z,\bar{z})$ transforms by an inhomogeneous shift ($\delta_\lambda \chi = \lambda^{(0)}$), it is interpreted as the Goldstone mode of asymptotic symmetry breaking while $\tilde{A}_z^{(0)}(u,z,\bar{z})$ is taken to be gauge-invariant (i.e. $\delta_\lambda \tilde{A}_z^{(0)} = 0$). The latter encodes the hard modes of the boundary value $A_z^{(0)}$ of the gauge field A_μ, which are captured by the Fourier expansion (2.43) for $s = 1$. In particular, it transforms as (2.51) with $J = 1$ under Poincaré symmetries. Defining $\mathcal{N}_z^{(0)} \equiv \int_{-\infty}^{+\infty} du F_{uz}^{(0)}$ and recalling the boundary condition (3.6), we have

$$\partial_z \mathcal{N}_{\bar{z}}^{(0)} - \partial_{\bar{z}} \mathcal{N}_z^{(0)} = \int_{-\infty}^{+\infty} du \partial_u F_{z\bar{z}}^{(0)} = F_{z\bar{z}}^{(0)}|_{\mathscr{I}^+_+} - F_{z\bar{z}}^{(0)}|_{\mathscr{I}^+_-} = 0, \tag{3.12}$$

where we used the Bianchi identity $\partial_{[\mu} F_{\nu\rho]} = 0$ expressed for the leading components of the Faraday tensor. This equation is solved if there exists a scalar field $N(z,\bar{z})$ such that

$$\partial_z N(z,\bar{z}) \equiv \frac{1}{e^2}\int_{-\infty}^{+\infty} du F_{uz}^{(0)}(u,z,\bar{z}) = \frac{1}{e^2}(A_z^{(0)}|_{\mathscr{I}^+_+} - A_z^{(0)}|_{\mathscr{I}^+_-}). \tag{3.13}$$

Notice that the so-defined $N(z,\bar{z})$, referred to as the memory mode, is gauge invariant. Hence the radiative variables are organized in hard and soft fields as

$$\begin{aligned} \Gamma^H &= \{\tilde{A}_z^{(0)}, \partial_u \tilde{A}_z^{(0)}, \tilde{A}_{\bar{z}}^{(0)}, \partial_u \tilde{A}_{\bar{z}}^{(0)}\}, \\ \Gamma^S &= \{\partial_z \chi, \partial_z N, \partial_{\bar{z}} \chi, \partial_{\bar{z}} N\}. \end{aligned} \tag{3.14}$$

The evaluation of the symplectic structure at the boundary $\mathscr{I}^+$ has been detailed e.g. in [13,126] and gives rise to the following Poisson brackets:

$$\begin{aligned} &\{\partial_{u_1} \tilde{A}_{z_1}^{(0)}(u_1,z_1,\bar{z}_1), \tilde{A}_{\bar{z}_2}^{(0)}(u_2,z_2,\bar{z}_2)\} \\ &\quad = -\frac{e^2}{2}\delta(u_1 - u_2)\delta^{(2)}(z_1 - z_2), \end{aligned} \tag{3.15}$$

$$\{\partial_{z_1} N(z_1,\bar{z}_1), \partial_{\bar{z}_2} \chi(z_2,\bar{z}_2)\} = -\frac{1}{2}\delta^{(2)}(z_1 - z_2), \tag{3.16}$$

for the hard and soft symplectic pairs $(\tilde{A}_z^{(0)}, \partial_u \tilde{A}_{\bar{z}}^{(0)})$ and $(\partial_z \chi, \partial_{\bar{z}} N)$. The quantum version of the bracket (3.15) is nothing but (2.48) for the choice of normalization (2.9). Because the theory is linear, the hard flux only encompasses matter contributions while the flux terms depending on the gauge field are soft, i.e.

$$\mathcal{F}^H_\lambda = \int_{\mathscr{I}^+} dud^2z\lambda^{(0)}\mathcal{J}^{(2)}_u, \tag{3.17}$$

$$\begin{aligned}\mathcal{F}^S_\lambda &= \frac{1}{e^2}\int_{\mathscr{I}^+} dud^2z\lambda^{(0)}(\partial_{\bar{z}}F^{(0)}_{uz} + \partial_z F^{(0)}_{u\bar{z}}) \\ &= 2\int_\Sigma d^2z\lambda^{(0)}\partial_z\partial_{\bar{z}}N,\end{aligned} \tag{3.18}$$

where the second equality of (3.18) holds because of (3.12) and (3.13). From (3.15) and (3.16) together with the canonical commutation relation $\{\phi^{(0)}(u,z,\bar{z}), \partial_{u'}\phi^{(0)}(u',z',\bar{z}')\} = \delta(u-u')\delta^{(2)}(z-z')$ for the matter field, one deduces the very important properties

$$\begin{aligned}&\{\mathcal{F}^H_\lambda, \Gamma^S\} = 0 = \{\mathcal{F}^S_\lambda, \Gamma^H\}, \\ &\{\mathcal{F}^H_\lambda, \tilde{A}^{(0)}_z\} = 0, \qquad \{\mathcal{F}^S_\lambda, \partial_z\chi\} = \partial_z\lambda^{(0)}, \\ &\{\mathcal{F}^H_\lambda, \phi^{(0)}\} = \delta_\lambda\phi^{(0)}, \qquad \{\mathcal{F}^S_\lambda, \phi^{(0)}\} = 0.\end{aligned} \tag{3.19}$$

Because N and $\mathcal{J}_\mu$ are gauge-invariant, we also have that hard and soft fluxes represent separately the $U(1)$ algebra, i.e. $\delta_{\lambda_1}\mathcal{F}^{H/S}_{\lambda_2} = \{\mathcal{F}^{H/S}_{\lambda_1}, \mathcal{F}^{H/S}_{\lambda_2}\} = 0$.

A similar analysis can be performed at $\mathscr{I}^-$, choosing the advanced Bondi coordinates $\{v,r,z,\bar{z}\}$ as defined in Appendix A. The equation of motion (3.7) becomes

$$-\partial_v F^{(2)}_{rv} + \partial_{\bar{z}}F^{(0)}_{vz} + \partial_z F^{(0)}_{v\bar{z}} + e^2\mathcal{J}^{(2)}_v = 0. \tag{3.20}$$

The induced surface charges are

$$Q_\lambda[A] = -\frac{1}{e^2}\int_\Sigma d^2z\lambda^{(0)}(z,\bar{z})F^{(2)}_{rv}(v,z,\bar{z}), \tag{3.21}$$

with no sign change with respect to (3.8). The fluxes get a sign flip because of the corresponding sign flip in the equation of motion (3.20), i.e.

$$\begin{aligned}\frac{dQ_\lambda}{dv} &= -\frac{1}{e^2}\int_\Sigma d^2z\lambda^{(0)}(\partial_{\bar{z}}F^{(0)}_{vz} + \partial_z F^{(0)}_{v\bar{z}} + e^2\mathcal{J}^{(2)}_v) \\ &= \int_\Sigma d^2z F_\lambda.\end{aligned} \tag{3.22}$$

Using an homologous split between hard and soft modes in the boundary value $A^{(0)}_z(v,z,\bar{z})$ of the gauge field, it can be shown that the property (3.19) also holds at past null infinity without any relative change of sign. Notice finally that the convention (2.78) implies $A^{(0)}_z(u\to-\infty,z,\bar{z}) = A^{(0)}_z(v\to+\infty,z,\bar{z})$, i.e. the antipodal matching condition around spatial infinity as stated in [157].

B. Gravity

We now switch to gravity in 4d asymptotically flat spacetimes. We follow the same notations and conventions than [60,158,159]. In retarded Bondi coordinates $\{u,r,x^A\}$, $x^A = (z,\bar{z})$, the solution space of four-dimensional asymptotically flat metrics reads as

$$\begin{aligned}ds^2 = &\left(\frac{2M}{r} + \mathcal{O}(r^{-2})\right)du^2 - 2(1+\mathcal{O}(r^{-2}))dudr + (r^2\mathring{q}_{AB} + rC_{AB} + \mathcal{O}(r^0))dx^Adx^B \\ &+ \left(\frac{1}{2}\partial^B C_{AB} + \frac{2}{3r}\left(N_A + \frac{1}{4}C_{AB}\partial_C C^{BC}\right) + \mathcal{O}(r^{-2})\right)dudx^A,\end{aligned} \tag{3.23}$$

where $\mathring{q}_{AB}dx^Adx^B = 2dzd\bar{z}$ is the flat metric on the punctured complex plane $\mathbb{C}^*$. The topology of future null infinity is taken to be $\mathscr{I}^+ \simeq \mathbb{R}\times\mathbb{C}^*$ in order to allow meromorphic superrotations among the set of asymptotic symmetries (see e.g. [158,160,161] for details on the precise setup). Indices A, B are lowered and raised by $\mathring{q}_{AB}$ and its inverse. The asymptotic shear C_{AB} is a symmetric trace-free tensor ($\mathring{q}^{AB}C_{AB} = 0$). The Bondi mass aspect $M(u,z,\bar{z})$ and the angular momentum aspect $N_A(u,z,\bar{z})$ satisfy the time evolution/constraint equations

$$\begin{aligned}\partial_u M &= -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}\partial_A\partial_B N^{AB} - 4\pi G T^{m(2)}_{uu}, \\ \partial_u N_A &= \partial_A M + \frac{1}{16}\partial_A(N_{BC}C^{BC}) - \frac{1}{4}N^{BC}\partial_A C_{BC} - 8\pi G T^{m(2)}_{uA} \\ &\quad - \frac{1}{4}\partial_B(C^{BC}N_{AC} - N^{BC}C_{AC}) - \frac{1}{4}\partial_B\partial^B\partial^C C_{AC} + \frac{1}{4}\partial_B\partial_A\partial_C C^{BC},\end{aligned} \tag{3.24}$$

with $N_{AB} = \partial_u C_{AB}$ the Bondi news tensor encoding the outgoing radiation and $T^m_{\mu\nu}$ the null matter stress tensor whose expansion near null infinity is taken to be

$$T^m_{uu}(u,r,z,\bar z) = T^{m(2)}_{uu}(u,z,\bar z)\frac{1}{r^2} + \mathcal{O}(r^{-3}),$$
$$T^m_{uA}(u,r,z,\bar z) = T^{m(2)}_{uA}(u,z,\bar z)\frac{1}{r^2} + \mathcal{O}(r^{-3}). \tag{3.25}$$

The diffeomorphisms preserving the solution space (3.23) are generated by vectors fields $\xi = \xi^u\partial_u + \xi^z\partial_z + \xi^{\bar z}\partial_{\bar z} + \xi^r\partial_r$ whose leading-order components read

$$\xi^u = \mathcal{T} + \frac{u}{2}(\partial_z\mathcal{Y}^z + \partial_{\bar z}\mathcal{Y}^{\bar z}),$$
$$\xi^z = \mathcal{Y}^z + \mathcal{O}(r^{-1}), \qquad \xi^{\bar z} = \mathcal{Y}^{\bar z} + \mathcal{O}(r^{-1}),$$
$$\xi^r = -\frac{r}{2}(\partial_z\mathcal{Y}^z + \partial_{\bar z}\mathcal{Y}^{\bar z}) + \mathcal{O}(r^0), \tag{3.26}$$

where $\mathcal{T} = \mathcal{T}(z,\bar z)$ is the supertranslation parameter and $\mathcal{Y}^z = \mathcal{Y}^z(z)$, $\mathcal{Y}^{\bar z} = \mathcal{Y}^{\bar z}(\bar z)$ are the superrotation parameters satisfying the conformal Killing equation

$$\partial_{\bar z}\mathcal{Y}^z = 0, \qquad \partial_z\mathcal{Y}^{\bar z} = 0. \tag{3.27}$$

Using the modified Lie bracket $[\xi_1,\xi_2]_\star \equiv [\xi_1,\xi_2] - \delta_{\xi_1}\xi_2 + \delta_{\xi_2}\xi_1$, where the last two terms take into account the field-dependence of (3.26) at subleading orders in r [12], these asymptotic Killing vectors satisfy the commutation relations

$$[\xi(\mathcal{T}_1,\mathcal{Y}^z_1,\mathcal{Y}^{\bar z}_1),\xi(\mathcal{T}_2,\mathcal{Y}^z_2,\mathcal{Y}^{\bar z}_2)]_\star = \xi(\mathcal{T}_{12},\mathcal{Y}^z_{12},\mathcal{Y}^{\bar z}_{12}), \tag{3.28}$$

with

$$\mathcal{T}_{12} = \mathcal{Y}^z_1\partial_z\mathcal{T}_2 - \frac{1}{2}\partial_z\mathcal{Y}^z_1\mathcal{T}_2 + \text{c.c.} - (1\leftrightarrow 2),$$
$$\mathcal{Y}^z_{12} = \mathcal{Y}^z_1\partial_z\mathcal{Y}^z_2 - (1\leftrightarrow 2), \qquad \mathcal{Y}^{\bar z}_{12} = \mathcal{Y}^{\bar z}_1\partial_{\bar z}\mathcal{Y}^{\bar z}_2 - (1\leftrightarrow 2). \tag{3.29}$$

This corresponds to the (extended) BMS algebra [12,162,163]. The six globally well-defined solutions of (3.27) and the four linearly-independent solutions of $(\partial_A\partial_B\mathcal{T})^{TF} = 0$ generate the Poincaré subgroup. The latter condition, where TF means the trace-free part with respect to the boundary metric $\mathring{q}_{AB}$, is generally solved as $\mathcal{T}(z,\bar z) = b_\mu q^\mu(z,\bar z)$ where b^μ are the constant parameters of an infinitesimal bulk translation in Cartesian coordinates.

The infinitesimal transformation of the solution space is induced by the Lie derivative of the bulk metric along BMS generators (3.26). The most crucial transformation for the purpose of this paper is the variation of the asymptotic shear, which reads as

$$\delta_\xi C_{zz} = \delta^H_\xi C_{zz} + \delta^S_\xi C_{zz},$$
$$\delta^H_\xi C_{zz} = \left[\left(\mathcal{T} + \frac{u}{2}(\partial_z\mathcal{Y}^z + \partial_{\bar z}\mathcal{Y}^{\bar z})\right)\partial_u + \mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar z}\partial_{\bar z} + \frac{3}{2}\partial_z\mathcal{Y}^z - \frac{1}{2}\partial_{\bar z}\mathcal{Y}^{\bar z}\right]C_{zz},$$
$$\delta^S_\xi C_{zz} = -2\partial_z^2\mathcal{T} - u\partial_z^3\mathcal{Y}^z, \tag{3.30}$$

together with the complex conjugate relations for $C_{\bar z\bar z}$. The hard and soft parts of the transformation are denoted by $\delta^H_\xi C_{AB}$ and $\delta^S_\xi C_{AB}$, respectively, and are built up from the respective homogeneous and inhomogeneous terms appearing in the induced variation.

Following the discussion of [62,65,102,103,158,159,164–166], the BMS charges are taken to be

$$Q_\xi[g] = \frac{1}{16\pi G}\int_\Sigma d^2z(4\mathcal{T}\bar M + 2\mathcal{Y}^A\bar N_A), \tag{3.31}$$

where the redefined mass $\bar M$ and angular momentum aspect $\bar N_A$ are given by

$$\bar M = M + \frac{1}{8}N_{AB}C^{AB},$$
$$\bar N_A = N_A - u\partial_A\bar M + \frac{1}{4}C_{AB}\partial_C C^{BC} + \frac{3}{32}\partial_A(C_{BC}C^{BC})$$
$$+ \frac{u}{4}\partial^B\left[\left(\partial_B\partial_C - \frac{1}{2}N_{BC}\right)C_A{}^C\right] - \frac{u}{4}\partial^B\left[\left(\partial_A\partial_C - \frac{1}{2}N_{AC}\right)C_B{}^C\right]. \tag{3.32}$$

The latter can be rewritten in a more elegant way in terms of Newman-Penrose coefficients [167,168] as the compact expressions $\bar M = -\frac{1}{2}(\Psi^0_2 + \bar\Psi^0_2)$, $\bar N_z = -\Psi^0_1 + u\partial_z\Psi^0_2$ and $\bar N_{\bar z} = \bar N^*_z$; see [159]. Using the time evolution/constraint equations (3.24), these charges satisfy the flux-balance laws

$$\frac{dQ_\xi}{du} = \int_\Sigma d^2 z F_{\xi(\mathcal{T},\mathcal{Y})}, \tag{3.33}$$

where the local fluxes (respectively associated with supertranslations and superrotations) read as

$$F_{\xi(\mathcal{T},0)} = \frac{1}{16\pi G}\mathcal{T}\left[\partial_z^2 N_{\bar{z}\bar{z}} + \frac{1}{2}C_{\bar{z}\bar{z}}\partial_u N_{zz} + \text{c.c.}\right] - \mathcal{T}T^{m(2)}_{uu}, \tag{3.34}$$

$$\begin{aligned} F_{\xi(0,\mathcal{Y})} = \frac{1}{16\pi G}\mathcal{Y}^z\left[-u\partial_z^3 N_{\bar{z}\bar{z}} + C_{zz}\partial_z N_{\bar{z}\bar{z}} - \frac{u}{2}\partial_z C_{zz}\partial_u N_{\bar{z}\bar{z}} - \frac{u}{2}C_{zz}\partial_z\partial_u N_{\bar{z}\bar{z}}\right] \\ - \mathcal{Y}^z T^{m(2)}_{uz} + \frac{u}{2}\mathcal{Y}^z\partial_z T^{m(2)}_{uu} + \text{c.c.} \end{aligned} \tag{3.35}$$

As in electrodynamics, a careful split between hard and soft boundary degrees of freedom can be performed, taking into account the factorization of the radiative phase space between hard and soft sectors [159,165,169]. At quantum level, this split allows for the BMS Ward identities to correctly reproduce the leading and subleading soft theorems [125,170] at all orders of perturbation [159,171]. For completeness, we provide here a mere summary of the construction and refer to [159] for details.

For the analysis of the radiative phase spaces including superrotations, the following early and late time behavior of the radiative fields are imposed [62,65,158,165,169]:

$$C_{AB}|_{\mathscr{I}^+_\pm} = (u + C_\pm)N^{\text{vac}}_{AB} - 2(\partial_A\partial_B C_\pm)^{TF} + o(u^{-1}), \qquad N_{AB} = N^{\text{vac}}_{AB} + o(u^{-2}), \tag{3.36}$$

where $C_\pm(z,\bar{z})$ correspond to the values of the supertranslation field for $u \to \pm\infty$ and whose difference encodes the displacement memory effect, and $N^{\text{vac}}_{AB}(z,\bar{z})$ is the vacuum news tensor [164,172], identified with the trace-free part of the Geroch tensor [165,173,174]. The latter can be expressed as

$$N^{\text{vac}}_{zz} = \frac{1}{2}(\partial_z\varphi)^2 - \partial_z^2\varphi, \qquad N^{\text{vac}}_{\bar{z}\bar{z}} = \frac{1}{2}(\partial_{\bar{z}}\bar{\varphi})^2 - \partial_{\bar{z}}^2\bar{\varphi}, \tag{3.37}$$

where $\varphi(z)$, $\bar{\varphi}(\bar{z})$ are the holomorphic superboost fields encoding the velocity kick (or refraction) memory effect [164]. For conformal primary fields $\psi_{k,\bar{k}}(z,\bar{z})$ of weights $(k,\bar{k})$ that transform as

$$\delta_{\mathcal{Y}}\psi_{(k,\bar{k})}(z,\bar{z}) = (\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} + k\partial_z\mathcal{Y}^z + \bar{k}\partial_{\bar{z}}\mathcal{Y}^{\bar{z}})\psi_{(k,\bar{k})}(z,\bar{z}) \tag{3.38}$$

under meromorphic superrotations, it is convenient to introduce the conformally covariant derivative operators [160,165]

$$\begin{aligned} \mathscr{D}_z\colon (k,\bar{k}) \to (k+1,\bar{k})\colon \psi_{(k,\bar{k})} \mapsto \mathscr{D}_z\psi_{(k,\bar{k})} = [\partial_z - k\partial_z\varphi]\psi_{(k,\bar{k})}, \\ \mathscr{D}_{\bar{z}}\colon (k,\bar{k}) \to (k,\bar{k}+1)\colon \psi_{(k,\bar{k})} \mapsto \mathscr{D}_{\bar{z}}\psi_{(k,\bar{k})} = [\partial_{\bar{z}} - \bar{k}\partial_{\bar{z}}\bar{\varphi}]\psi_{(k,\bar{k})}, \end{aligned} \tag{3.39}$$

satisfying $[\mathscr{D}_z, \mathscr{D}_{\bar{z}}]\psi_{(k,\bar{k})} = 0$. With these definition, the split between hard and soft variables works as follows. Defining

$$C_{zz} \equiv uN^{\text{vac}}_{zz} + C^{(\circ)}_{zz} + \tilde{C}_{zz}, \qquad N_{zz} = N^{\text{vac}}_{zz} + \tilde{N}_{zz}, \tag{3.40}$$

where $C^{(\circ)}_{zz}(z,\bar{z}) = -2\mathscr{D}_z^2 C^{(\circ)}(z,\bar{z})$ for $C^{(\circ)} = \frac{1}{2}(C_+ + C_-)$, the Goldstone mode of supertranslation. The falloff conditions (3.36) imply that $\tilde{N}_{zz} = \partial_u\tilde{C}_{zz} = o(u^{-2})$ and that the asymptotic shear is "purely electric" at early and late times

$$(\mathscr{D}_z^2 C_{\bar{z}\bar{z}} - \mathscr{D}_{\bar{z}}^2 C_{zz})|_{\mathscr{I}^+_\pm} = 0 \Rightarrow (\mathscr{D}_z^2\tilde{C}_{\bar{z}\bar{z}} - \mathscr{D}_{\bar{z}}^2\tilde{C}_{zz})|_{\mathscr{I}^+_\pm} = 0. \tag{3.41}$$

The implication comes from the fact that $\mathscr{D}_z$ and $\mathscr{D}_{\bar{z}}$ commute, hence $\mathscr{D}_z C^{(\circ)}_{\bar{z}\bar{z}} - \mathscr{D}_{\bar{z}} C^{(\circ)}_{zz} = 0$ and by definition $\mathscr{D}_z N^{\text{vac}}_{\bar{z}\bar{z}} = 0 = \mathscr{D}_{\bar{z}} N^{\text{vac}}_{zz}$. The condition (3.41) is the analog of (3.12) for gravity and is solved by

$$\tilde{C}_{zz}|_{\mathscr{I}^+_\pm} = \mp 2\mathscr{D}_z^2 N^{(\circ)}, \qquad N^{(\circ)}(z,\bar{z}) = \frac{1}{2}(C_+ - C_-). \tag{3.42}$$

The hard variables of the radiative phase space are collectively denoted as

$$\Gamma^H = \{\tilde{C}_{zz}, \tilde{N}_{zz}, \tilde{C}_{\bar{z}\bar{z}}, \tilde{N}_{\bar{z}\bar{z}}\}. \tag{3.43}$$

On the other hand, soft variables are identified as

$$\Gamma^S = \{C^{(\circ)}_{zz}, C^{(\circ)}_{\bar{z}\bar{z}}, \mathcal{N}^{(0)}_{zz}, \mathcal{N}^{(0)}_{\bar{z}\bar{z}}, \Pi_{zz}, \Pi_{\bar{z}\bar{z}}, N^{\mathrm{vac}}_{zz}, N^{\mathrm{vac}}_{\bar{z}\bar{z}}\}, \qquad \Pi_{zz} \equiv 2\mathcal{N}^{(1)}_{zz} + C^{(0)}\mathcal{N}^{(0)}_{zz}, \tag{3.44}$$

where $\mathcal{N}^{(0)}_{zz}$ and $\mathcal{N}^{(1)}_{zz}$ are respectively the leading and subleading soft news [125,170] given by

$$\mathcal{N}^{(0)}_{zz}(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} du \tilde{N}_{zz}(u,z,\bar{z}), \qquad \mathcal{N}^{(1)}_{zz}(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} duu \tilde{N}_{zz}(u,z,\bar{z}). \tag{3.45}$$

The boundary condition $\tilde{N}_{AB} = o(u^{-2})$ implies that both integrals converge and in particular $\mathcal{N}^{(0)}_{zz}(z,\bar{z}) = -4\mathscr{D}_z^2 N^{(\circ)}(z,\bar{z})$ because of (3.42). The transformations of all these objects, which can be worked out from (3.30), are reproduced in Eq. (3.6) of [159]. The fields in (3.43) are functions of $(u,z,\bar{z})$ and transform homogeneously under the action of extended BMS symmetries. For instance,

$$\delta_{\xi(\mathcal{T},\mathcal{Y})}\tilde{C}_{zz} = \left(\mathcal{T} + \frac{u}{2}(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}})\right)\tilde{N}_{zz} + \left(\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} + \frac{3}{2}\partial_z\mathcal{Y}^z - \frac{1}{2}\partial_{\bar{z}}\mathcal{Y}^{\bar{z}}\right)\tilde{C}_{zz}. \tag{3.46}$$

The quantized modes of $\tilde{C}_{zz}(u,z,\bar{z})$ are simply captured by the expansion (2.43) for $s=2$. The fields in (3.44), on the other hand, are functions of $(z,\bar{z})$ only, defined at the corners of $\mathscr{I}^+$ and their transformation laws

$$\delta_{\xi(\mathcal{T},\mathcal{Y})}C^{(\circ)}_{zz} = \left(\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} + \frac{3}{2}\partial_z\mathcal{Y}^z - \frac{1}{2}\partial_{\bar{z}}\mathcal{Y}^{\bar{z}}\right)C^{(\circ)}_{zz} - 2\mathscr{D}_z^2\mathcal{T}, \tag{3.47}$$

$$\delta_{\xi(\mathcal{T},\mathcal{Y})}N^{\mathrm{vac}}_{zz} = (\mathcal{Y}^z\partial_z + 2\partial_z\mathcal{Y}^z)N^{\mathrm{vac}}_{zz} - \partial_z^3\mathcal{Y}^z, \tag{3.48}$$

account for the inhomogeneous pieces in (3.30). As shown in [159,169], (3.43) and (3.44) constitute Darboux variables parametrizing the radiative phase space of asymptotically flat gravity at $\mathscr{I}^+$. Using them, the suitable split of the integrated BMS fluxes $\mathcal{F}_{\xi(\mathcal{T},\mathcal{Y})} = \int_{\mathscr{I}^+} dud^2z F_{\xi(\mathcal{T},\mathcal{Y})}$ computed from (3.34) and (3.35) into pure hard and soft parts has been proposed to be

$$\begin{aligned}
\mathcal{F}^H_{\xi(\mathcal{T},0)} &= -\frac{1}{16\pi G}\int_{\mathscr{I}^+} dud^2z\mathcal{T}[\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}}] - \int_{\mathscr{I}^+} dud^2z\mathcal{T}T^{m(2)}_{uu},\\
\mathcal{F}^S_{\xi(\mathcal{T},0)} &= \frac{1}{8\pi G}\int_\Sigma d^2z\mathcal{T}[\mathscr{D}_z^2\mathcal{N}^{(0)}_{\bar{z}\bar{z}}],\\
\mathcal{F}^H_{\xi(0,\mathcal{Y})} &= \frac{1}{16\pi G}\int_{\mathscr{I}^+} dud^2z\mathcal{Y}^z\left[\frac{3}{2}\tilde{C}_{zz}\partial_z\tilde{N}_{\bar{z}\bar{z}} + \frac{1}{2}\tilde{N}_{zz}\partial_z\tilde{C}_{zz} + \frac{u}{2}\partial_z(\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}})\right]\\
&\quad - \int_{\mathscr{I}^+} dud^2z\mathcal{Y}^z\left[T^{m(2)}_{uz} + \frac{u}{2}\partial_z\mathcal{Y}^z T^{m(2)}_{uu}\right] + \text{c.c.},\\
\mathcal{F}^S_{\xi(0,\mathcal{Y})} &= \frac{1}{16\pi G}\int_\Sigma d^2z\mathcal{Y}^z\left[-\mathscr{D}_z^3\mathcal{N}^{(1)}_{\bar{z}\bar{z}} + \frac{3}{2}C^{(\circ)}_{zz}\mathscr{D}\mathcal{N}^{(0)}_{\bar{z}\bar{z}} + \frac{1}{2}\mathcal{N}^{(0)}_{\bar{z}\bar{z}}\mathscr{D}_z C^{(\circ)}_{zz}\right] + \text{c.c.},
\end{aligned} \tag{3.49}$$

where (3.41) was used to simplify the expressions of soft fluxes. Unlike the case of electromagnetism, the nonlinearity of Einstein theory implies that hard fluxes do include pure gravitational terms, notably the first term in $F^H_{\xi(1,0)}$, responsible for the Bondi mass loss. Inverting the symplectic structure yields the following Poisson brackets

$$\begin{aligned}
\{\tilde{N}_{z_1z_1}(u_1,z_1,\bar{z}_1), \tilde{C}_{\bar{z}_2\bar{z}_2}(u_2,z_2,\bar{z}_2)\} &= -16\pi G\delta(u_1-u_2)\delta^{(2)}(z_1-z_2),\\
\{\mathcal{N}^{(0)}_{z_1z_1}(z_1,\bar{z}_1), C^{(0)}_{\bar{z}_2\bar{z}_2}(z_2,\bar{z}_2)\} &= -16\pi G\delta^{(2)}(z_1-z_2),
\end{aligned} \tag{3.50}$$

$$\{\Pi_{z_1 z_1}(z_1,\bar{z}_1), N^{\rm vac}_{\bar{z}_2\bar{z}_2}(z_2,\bar{z}_2)\} = -16\pi G \delta^{(2)}(z_1 - z_2), \tag{3.51}$$

from which one can derive the very important property that the hard and soft parts of the fluxes act independently on (3.43) and (3.44) as

$$\begin{aligned} \{\mathcal{F}^H_{\xi(\mathcal{T},\mathcal{Y})}, \Gamma^H\} &= \delta_{\xi(\mathcal{T},\mathcal{Y})}\Gamma^H, \qquad \{\mathcal{F}^H_{\xi(\mathcal{T},\mathcal{Y})}, \Gamma^S\} = 0, \\ \{\mathcal{F}^S_{\xi(\mathcal{T},\mathcal{Y})}, \Gamma^S\} &= \delta_{\xi(\mathcal{T},\mathcal{Y})}\Gamma^S, \qquad \{\mathcal{F}^S_{\xi(\mathcal{T},\mathcal{Y})}, \Gamma^H\} = 0, \end{aligned} \tag{3.52}$$

and form separately two representations of the extended BMS algebra (3.28) and (3.29) [158,159].

As usual, a similar analysis can be performed at past null infinity $\mathscr{I}^-$ by trading the retarded coordinates for the advanced ones. At the vicinity of $\mathscr{I}^-$, the solution space is expanded as

$$\begin{aligned} ds^2 = &\left(\frac{2M^{(-)}}{r'} + \mathcal{O}(r'^{-2})\right)dv^2 + 2(1+\mathcal{O}(r'^{-2}))dvdr' + (r'^2 \mathring{q}_{AB} + r' C^{(-)}_{AB} + \mathcal{O}(r'^0))dx^A dx^B \\ &- \left(-\frac{1}{2}\partial^B C^{(-)}_{AB} - \frac{2}{3r'}\left(N^{(-)}_A + \frac{1}{4}C^{(-)}_{AB}\partial_C C^{BC}_{(-)}\right) + \mathcal{O}(r'^{-2})\right)dvdx^A, \end{aligned} \tag{3.53}$$

still denoting $x^A = (z', \bar{z}') = (z, \bar{z})$. The Bondi time evolution/constraint equations for the Bondi mass aspect $M^{(-)}(v, z, \bar{z})$ and the angular momentum aspect $N^{(-)}_A(v, z, \bar{z})$ at $\mathscr{I}^-$ read as

$$\begin{aligned} \partial_v M^{(-)} &= \frac{1}{8}N^{(-)}_{AB}N^{AB}_{(-)} + \frac{1}{4}\partial_A\partial_B N^{AB}_{(-)} + 4\pi G T^{m(2,-)}_{vv}, \\ \partial_v N^{(-)}_A &= -D_A M^{(-)} + \frac{1}{16}\partial_A(N^{(-)}_{BC}C^{BC}_{(-)}) - \frac{1}{4}N^{BC}_{(-)}D_A C^{(-)}_{BC} - 8\pi G T^{m(2,-)}_{vA} \\ &\quad - \frac{1}{4}\partial_B(C^{BC}_{(-)}N^{(-)}_{AC} - N^{BC}_{(-)}C^{(-)}_{AC}) + \frac{1}{4}\partial_B\partial^B\partial^C C^{(-)}_{AC} - \frac{1}{4}\partial_B\partial_A\partial_C C^{BC}_{(-)}, \end{aligned} \tag{3.54}$$

with the (past) Bondi news tensor $N^{(-)}_{AB} = \partial_v C^{(-)}_{AB}$ encoding the incoming radiation. Under the BMS symmetries acting at $\mathscr{I}^-$

$$\xi|_{\mathscr{I}^-} = \left(\mathcal{T}(z,\bar{z}) + \frac{v}{2}(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}})\right)\partial_v + \mathcal{Y}^z(z)\partial_z + \mathcal{Y}^{\bar{z}}(\bar{z})\partial_{\bar{z}}, \tag{3.55}$$

the variation of the shear is still given by (3.30) with $u \mapsto v$ and a sign flip in $\delta^S_\xi C_{zz}$. The covariant phase-space analysis and a similar split between hard and soft boundary degrees of freedom yield again (3.52). Around spatial infinity, the requirement (2.78) now implies $C_{zz}(u \to -\infty, z, \bar{z}) = -C^{(-)}_{zz}(v \to +\infty, z, \bar{z})$, where the minus sign is due to $r' = -r$ and the convention (2.53). This agrees with the antipodal matching in the sense of [151,170].

IV. SOURCED QUANTUM FIELD THEORY

As discussed in Sec. III the charges associated with the asymptotic symmetries are not conserved due to the radiation leaking through $\mathscr{I}$. We argued in [60] (see also Sec. V) that the nonconservation of the bulk charges can be holographically interpreted as a coupling of the dual theory with some external sources. In this section, we discuss a general framework that allows us to deal with symmetries in presence of external sources. The details on the newly introduced notions and the resulting construction will be presented in the upcoming paper [175] (see also [176,177]). We argue that, at the classical level, the inclusion of external sources in field theories allows us to derive a generalized version of Noether's theorem giving an interpretation of flux-balance laws for the Noether currents from symmetry principle. We then obtain the quantum analogue of this result by writing the sourced Ward identities using a path integral formulation. Finally, we exemplify these sourced Ward identities for a sourced theory exhibiting $U(1)$ symmetries and BMS symmetries.

A. Generalized Noether symmetries

In this section, we briefly discuss a generalization of Noether's theorem that allows us to write some flux-balance laws associated with (generalized) symmetries in presence of external sources. We consider a theory living on a n-dimensional manifold $\mathscr{M}$ with coordinates x^a and boundary $\partial\mathscr{M}$. The dynamical fields are denoted by $\Psi^i(x)$ and the external sources by $\sigma^m(x)$. The latter are defined as local functions with no associated equations of motion, contrarily to $\Psi^i(x)$. The action reads as

$$S[\Psi|\sigma] = \int_{\mathscr{M}} d^n x L[\Psi|\sigma]. \tag{4.1}$$

To derive flux-balance laws for the Noether charges from first principles, it will be useful to allow variations of the sources on the phase space. The variation of the action reads as

$$\delta S = \int_{\mathscr{M}} d^n x \frac{\delta S}{\delta \Psi^i} \delta\Psi^i + \int_{\mathscr{M}} d^n x \frac{\delta S}{\delta \sigma^m} \delta\sigma^m, \tag{4.2}$$

discarding the boundary terms if the fields and sources are sufficiently decaying while approaching $\partial\mathscr{M}$. For a fixed set of sources $\sigma^m(x)$, the action is stationary for arbitrary variations $\delta\Psi^i$ if and only if the equations of motion $\frac{\delta S}{\delta \Psi^i} = 0$ are obeyed by the dynamical fields $\Psi^i(x)$.

If the theory admits some Noether symmetries $\delta_K \Psi^i = K^i[\Psi]$ in absence of external sources, turning on the sources will usually break these symmetries. However, the Noetherian symmetries $\delta_K \Psi^i = K^i[\Psi]$ of the theory without source can be promoted to generalized symmetries of the sourced theory in the sense of [176] (notice that the present notion of generalized symmetries shall not be confused with the concept of higher-form symmetries discussed in quantum field theory and that goes under the same name, see e.g. [178]). The detailed analysis of the implications of the existence of such symmetries requires some care in the definition of a generalized symmetry and is beyond the scope of this article. This analysis will be presented elsewhere [175]. For the purpose of the present work we will only rely on the following features, which we will take as a naive definition for a generalized symmetry; the joint action on both the fields and sources

$$\delta_K \Psi^i = K^i[\Psi|\sigma], \qquad \delta_K \sigma^m = K^m[\sigma] \tag{4.3}$$

is then a symmetry in the sense that it is required to be a symmetry of the sourced equations of motion

$$\frac{\delta S}{\delta \Psi^i} = 0 \Rightarrow \delta_K \left(\frac{\delta S}{\delta \Psi^i} \right) = 0 \tag{4.4}$$

but is not required to preserve the action

$$\delta_K L = \partial_a B_K^a + V_K[\sigma], \qquad V_K[\sigma = 0] = 0. \tag{4.5}$$

As the second equation of (4.3) expresses that the external sources transform among themselves, this translates into the fact that the generalized symmetries break the invariance of the action in (4.5) only by terms depending on the sources which vanish when $\sigma = 0$. Consequently, in the absence of source ($\sigma = 0$), the symmetries are fully restored as rightful variational or Noether symmetries. Hence one can say that generalized symmetries are symmetries of the sourced equations of motion extending the Noetherian symmetries of the source-free action. In this framework, writing the usual Noether current as $\boldsymbol{j}_K = j_K^a (d^{n-1}x)_a$, the following relation can be obtained

$$\partial_a j_K^a = K^i \frac{\delta S}{\delta \Psi^i} + F_K, \tag{4.6}$$

where $\boldsymbol{F}_K = F_K(d^n x)$ is a flux term whose explicit form is given by [175]

$$F_K = K^m \frac{\delta L}{\delta \sigma^m} - V_K. \tag{4.7}$$

On shell, (4.6) leads to the flux-balance law,

$$\boxed{d\boldsymbol{j}_K = \boldsymbol{F}_K[\Psi|\sigma].} \tag{4.8}$$

The latter generalizes the first Noether theorem in presence of external sources: the conservation of the Noether current $\boldsymbol{j}_K$ associated with a symmetry of characteristics K is broken by the flux term $\boldsymbol{F}_K$. If one turns off the sources, we have $F_K[\Psi|\sigma = 0] = 0$ and one recovers the standard conservation law $d\boldsymbol{j}_K = 0$.

B. Sourced Ward identities

Let us now derive Ward identities associated with the generalized symmetries (4.3). The derivation follows the usual steps by properly taking into account the presence of external sources (see e.g. [120] for the standard derivation of Ward identities from the path integral). For a fixed source σ, the partition function reads as

$$\mathcal{Z}_\sigma[J_i] = \int \mathcal{D}[\Psi]_\sigma \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A), \tag{4.9}$$

where we allow the path integral measure $\mathcal{D}[\Psi]_\sigma$ to depend upon the sources σ^m and we use the convenient notation $J_A \Psi^A \equiv \int_{\mathscr{M}} d^n x J_i(x) \Psi^i(x)$. Here $J_i(x)$ are classical sources introduced in the definition of the partition function. They are not meant to be quantized and will be sent to zero when evaluating the correlation function. We stress that they should be distinguished from the sources σ^m that have a physical meaning and break the conservation of Noether currents: the sources σ^m will be integrated in the final path integral and will generate a flux.

Then, we suggest to consider that the partition function of the full sourced quantum theory should be computed by integrating (4.9) over the sources as

$$\mathcal{Z}[J_i, J_m] = \int \mathcal{D}[\sigma] \mathcal{Z}_\sigma[J_i] \exp \frac{i}{\hbar} J_M \sigma^M. \tag{4.10}$$

We used again the shorthand $J_M \sigma^M \equiv \int_{\mathscr{M}} d^n x J_m(x) \sigma^m(x)$ and introduced a second bunch of classical sources J_m as part of the definition of the partition function. The sources σ^m are therefore promoted as *source operators* and will play an important role in the Carrollian holographic correspondence discussed in Sec. V B. A very similar procedure was adopted in [179] in the AdS/CFT context where the path integral was performed over the boundary sources. We will further comment on this in the discussion closing the paper (Sec. VII). Notice that, even though the source operators σ^m and field operators Ψ^i seem on the same footing in (4.10), they are distinguished by their role played in the generalized symmetries introduced in Sec. IV A. In the absence of source, the symmetries of the partition functions are restored. We now investigate the implications of generalized symmetries on the correlation functions when the sources are turned on. Inserting the off shell relation (4.6) in the path integral, we get

$$\begin{aligned} &\frac{\partial}{\partial x^a} \int \mathcal{D}[\sigma] \int \mathcal{D}[\Psi]_\sigma j_K^a(x) \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M) \\ &= \int \mathcal{D}[\sigma] \int \mathcal{D}[\Psi]_\sigma \left(K^i[\Psi(x)|\sigma(x)] \frac{\delta S}{\delta \Psi^i(x)} + F_K(x) \right) \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M). \end{aligned} \tag{4.11}$$

The first term in the right-hand side can be reworked by noticing that

$$\begin{aligned} &\frac{\hbar}{i} \frac{\delta}{\delta \Psi^i(x)} \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M) \\ &= \frac{\delta}{\delta \Psi^i(x)} (S[\Psi|\sigma] + J_A \Psi^A) \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M) \\ &= \left(\frac{\delta S}{\delta \Psi^i(x)} + J_i(x) \right) \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M), \end{aligned} \tag{4.12}$$

and assuming that $\delta(0) = 0$ and the path integral is invariant under field translation, i.e. $\mathcal{D}[\Psi + \delta\Psi]_\sigma = \mathcal{D}[\Psi]_\sigma$ for any variation $\delta\Psi^i$:

$$\begin{aligned} &\int \mathcal{D}[\sigma] \int \mathcal{D}[\Psi]_\sigma \left(K^i[\Psi(x)|\sigma(x)] \frac{\delta S}{\delta \Psi^i(x)} \right) \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M) \\ &= - \int \mathcal{D}[\sigma] \int \mathcal{D}[\Psi]_\sigma J_i(x) K^i[\Psi(x)|\sigma(x)] \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M). \end{aligned} \tag{4.13}$$

As a conclusion, (4.11) is simply

$$\begin{aligned} &\frac{\partial}{\partial x^a} \int \mathcal{D}[\sigma] \int \mathcal{D}[\Psi]_\sigma j_K^a(x) \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M) \\ &= \int \mathcal{D}[\sigma] \int \mathcal{D}[\Psi]_\sigma [-J_i(x) K^i[\Psi(x)|\sigma(x)] + F_K(x)] \exp \frac{i}{\hbar} (S[\Psi|\sigma] + J_A \Psi^A + J_M \sigma^M). \end{aligned} \tag{4.14}$$

Acting on the left-hand side with N successive derivatives with respect to $J_i(x)$ and setting $J_i(x) = 0 = J_m(x)$ afterwards gives

$$\boxed{\frac{\partial}{\partial x^a} \langle j_K^a(x) X_N^\Psi \rangle = \frac{\hbar}{i} \sum_{k=1}^N \delta^{(n)}(x - x_k) \delta_{K^{i_k}} \langle X_N^\Psi \rangle + \langle F_K(x) X_N^\Psi \rangle,} \tag{4.15}$$

where we have introduced

$$X_N^\Psi \equiv \Psi^{i_1}(x_1)\ldots\Psi^{i_N}(x_N),$$
$$\delta_{K^{i_k}} X_N^\Psi = \Psi^{i_1}(x_1)\ldots K^{i_k}[\Psi(x_k)]\ldots\Psi^{i_N}(x_N) \tag{4.16}$$

to denote a collection of N quantum insertions and their transformation under the symmetry of characteristics K and recalled that, by definition,

$$\langle X_N^\Psi\rangle \equiv \left(\frac{\hbar}{i}\right)^N \frac{\delta}{\delta J_{i_1}(x_1)}\cdots\frac{\delta}{\delta J_{i_N}(x_N)}\mathcal{Z}[J]\bigg|_{J=0}. \tag{4.17}$$

Now acting on the left-hand side of (4.14) with N successive derivatives with respect to the other $J_m(x)$ yields

$$\boxed{\frac{\partial}{\partial x^a}\langle j_K^a(x)X_N^\sigma\rangle = \langle F_K(x)X_N^\sigma\rangle,} \tag{4.18}$$

where

$$X_N^\sigma \equiv \sigma^{m_1}(x_1)\ldots\sigma^{m_N}(x_N) \tag{4.19}$$

denotes a collection of N insertions of quantum sources. Equations (4.15) and (4.18) are the local version of the infinitesimal Ward identities in presence of external sources. With no field insertion in the correlators, one finds

$$\partial_a\langle j_K^a(x)\rangle = \langle F_K(x)\rangle, \tag{4.20}$$

which reproduces the classical flux-balance equation (4.8). Integrating (4.15) on the whole manifold $\mathscr{M}$ with boundary $\partial\mathscr{M}$ gives

$$\boxed{\sum_{k=1}^N \delta_{K^{i_k}}\langle X_N^\Psi\rangle = \frac{i}{\hbar}\left\langle\left(\int_{\mathscr{M}}\boldsymbol{F}_K - \int_{\partial\mathscr{M}}\boldsymbol{j}_K\right)X_N^\Psi\right\rangle.} \tag{4.21}$$

The standard textbook result is recovered by setting the sources to zero and assuming that the Noether currents vanish at the boundary, so that the right-hand side of (4.21) vanishes. Finally integrating (4.18) on $\mathscr{M}$ leads to

$$\boxed{\left\langle\left(\int_{\mathscr{M}}\boldsymbol{F}_K - \int_{\partial\mathscr{M}}\boldsymbol{j}_K\right)X_N^\sigma\right\rangle = 0.} \tag{4.22}$$

Remark—At this stage, and since the partition function (4.10) can be rewritten as

$$\mathcal{Z}[J_i,J_m] = \int\mathcal{D}[\sigma]\int\mathcal{D}[\Psi]_\sigma \exp\frac{i}{\hbar}(S[\Psi|\sigma]+J_A\Psi^A+J_M\sigma^M), \tag{4.23}$$

one might wonder what is responsible for the asymmetry between the Ward identities for the fields (4.15) and the sources (4.18). In the following lines, we wish to highlight that, besides a possible dependence of the measure $\mathcal{D}[\Psi]_\sigma$ in the sources, this asymmetry can ultimately be traced back to the fact that the generalized symmetries treat fields and sources differently. To convince oneself of this point, it is instructive to consider the subcase where the measures $\mathcal{D}[\sigma]$ and $\mathcal{D}[\Psi]$ are assumed to commute and use this property to rewrite the Ward identities in a form that puts sources and fields on an equal footing. To achieve this, we only need to reintroduce the explicit form of the flux (4.7), which has not yet been used in the derivation of the local version of the Ward identities (4.15) and (4.18). Injecting (4.7) into (4.14), we have

$$\begin{aligned}&\frac{\partial}{\partial x^a}\int\mathcal{D}[\sigma]\int\mathcal{D}[\Psi]_\sigma j_K^a(x)\exp\frac{i}{\hbar}(S[\Psi|\sigma]+J_A\Psi^A+J_M\sigma^M)\\&=\int\mathcal{D}[\sigma]\int\mathcal{D}[\Psi]_\sigma\left(-K^i[\Psi(x)|\sigma(x)]J_i(x)+K^m[\sigma(x)]\frac{\delta S}{\delta\sigma^m(x)}-V_K[\sigma(x)]\right)\times\cdots\\&\quad\times\exp\frac{i}{\hbar}(S[\Psi|\sigma]+J_A\Psi^A+J_M\sigma^M).\end{aligned} \tag{4.24}$$

Assuming that one can commute the measures $\mathcal{D}[\sigma]$ and $\mathcal{D}[\Psi]$ and that the path integral for the sources is invariant under translation as well, i.e. $\mathcal{D}[\sigma+\delta\sigma]=\mathcal{D}[\sigma]$, the second term in the right-hand side can be reworked as

$$\begin{aligned}&\frac{\partial}{\partial x^a}\int\mathcal{D}[\sigma]\int\mathcal{D}[\Psi]_\sigma j_K^a(x)\exp\frac{i}{\hbar}(S[\Psi|\sigma]+J_A\Psi^A+J_M\sigma^M)\\&=\int\mathcal{D}[\sigma]\int\mathcal{D}[\Psi]_\sigma(-K^i[\Psi(x)|\sigma(x)]J_i(x)-K^m[\sigma(x)]J_m(x)-V_K[\sigma(x)])\times\cdots\\&\quad\times\exp\frac{i}{\hbar}(S[\Psi|\sigma]+J_A\Psi^A+J_M\sigma^M).\end{aligned} \tag{4.25}$$

Deriving N times with respect to $J_i(x)$ and then setting $J_m = 0 = J_i$ gives

$$\frac{\partial}{\partial x^a}\langle j^a_K(x) X^\Psi_N\rangle = \frac{\hbar}{i}\sum_{k=1}^{N}\delta^{(n)}(x-x_k)\delta_{K^{i_k}}\langle X^\Psi_N\rangle - \langle V_K(x) X^\Psi_N\rangle. \tag{4.26}$$

Similarly, deriving N times with respect to $J_m(x)$ and then setting $J_m = 0 = J_i$ gives

$$\frac{\partial}{\partial x^a}\langle j^a_K(x) X^\sigma_N\rangle = \frac{\hbar}{i}\sum_{k=1}^{N}\delta^{(n)}(x-x_k)\delta_{K^{m_k}}\langle X^\sigma_N\rangle - \langle V_K(x) X^\sigma_N\rangle. \tag{4.27}$$

Under the above assumptions, these expressions are equivalent to (4.15) and (4.18) respectively. They highlight that the asymmetry between fields and sources in the treatment of generalized symmetries is always present as a result of the appearance of $V_K[\sigma(x)]$ in the right-hand sides. In particular, these Ward identities are distinctly different from the usual Ward identities of a Noetherian symmetry.

C. Sourced $U(1)$ Ward identities

As a warm-up, we apply the above framework to the simple case of a field theory exhibiting a $U(1)$ symmetry. We assume that the fields Ψ_Q inserted in the correlators transform as

$$\delta_\lambda \Psi_Q = -i\lambda Q \Psi_Q \tag{4.28}$$

under the $U(1)$ symmetry parametrized by some arbitrary function $\lambda(x)$. The infinitesimal Ward identity (4.15) in presence of external sources then simply reads as

$$\begin{aligned}&\frac{\partial}{\partial x^a}\langle j^a_\lambda(x)\Psi^{i_1}_{Q_{i_1}}(x_1)\ldots\Psi^{i_N}_{Q_{i_N}}(x_N)\rangle - \hbar\sum_{k=1}^{N}\lambda(x_k)Q_i\delta^{(n)}(x-x_k)\langle\Psi^{i_1}_{Q_{i_1}}(x_1)\ldots\Psi^{i_N}_{Q_{i_N}}(x_N)\rangle\\ &\quad = \langle F_\lambda(x)\Psi^{i_1}_{Q_{i_1}}(x_1)\ldots\Psi^{i_N}_{Q_{i_N}}(x_N)\rangle.\end{aligned} \tag{4.29}$$

While this example is quite trivial, it will be relevant when discussing electrodynamics from the Carrollian holographic perspective.

D. Sourced conformal Carrollian Ward identities

We now make the same exercise for the case of $3d$ conformal Carrollian field theory. After reviewing conformal Carrollian symmetries and introducing associated primary fields, we write the sourced Ward identities involving the Carrollian momenta. Notice importantly that we make here a slight abuse of terminology, as the very meaning of what exactly *is* a quantum Carrollian field theory is still largely unknown. For us, this term will refer to a $3d$ theory which enjoys conformal Carrollian symmetries.

1. Conformal Carrollian symmetries

A Carrollian structure [16,173,180–191] on a $3d$ manifold $\mathscr{I}$ is a couple (q_{ab}, n^a), where q_{ab} is a degenerate metric with signature $(0,+,+)$ and n^a is a vector field in the kernel of the metric, i.e. $q_{ab}n^a = 0$. For convenience, we choose coordinates $(u, z, \bar z)$ on $\mathscr{I}$ such that $q_{ab}dx^a dx^b = 0du^2 + 2dzd\bar z$ and $n^a\partial_a = \partial_u$. We still denote $x^A = (z, \bar z)$, $A = 1, 2$. The conformal Carrollian symmetries are generated by vector fields $\bar\xi = \bar\xi^a\partial_a$ satisfying

$$\mathscr{L}_{\bar\xi} q_{ab} = 2\alpha q_{ab}, \qquad \mathscr{L}_{\bar\xi} n^a = -\alpha n^a, \tag{4.30}$$

where α is a function on $\mathscr{I}$. The solution $\bar\xi$ of (4.30) is given by

$$\bar\xi = \left[\mathcal{T} + \frac{u}{2}(\partial_z\mathcal{Y}^z + \partial_{\bar z}\mathcal{Y}^{\bar z})\right]\partial_u + \mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar z}\partial_{\bar z}, \tag{4.31}$$

with $\mathcal{T} = \mathcal{T}(z,\bar z)$, $\mathcal{Y}^z = \mathcal{Y}^z(z)$ and $\mathcal{Y}^{\bar z} = \mathcal{Y}^{\bar z}(\bar z)$. These vector fields are called conformal Carrollian Killing vectors and $\alpha = \frac{1}{2}(\partial_z\mathcal{Y}^z + \partial_{\bar z}\mathcal{Y}^{\bar z})$. They precisely coincide with the restriction to $\mathscr{I}$ of the asymptotic Killing vector fields (2.26). Furthermore, their standard Lie bracket on $\mathscr{I}$, $[\bar\xi(\mathcal{T}_1, \mathcal{Y}^z_1, \mathcal{Y}^{\bar z}_1), \bar\xi(\mathcal{T}_2, \mathcal{Y}^z_2, \mathcal{Y}^{\bar z}_2)] = \bar\xi(\mathcal{T}_{12}, \mathcal{Y}^z_{12}, \mathcal{Y}^{\bar z}_{12})$, reproduces (3.29). This shows that the conformal Carroll algebra in $3d$ is isomorphic to the BMS algebra in $4d$ [16,182].

We will call the global conformal Carrollian algebra the subalgebra generated by

(1) Carrollian translations, $P_a = \partial_a$ or $P_0 \equiv \partial_u$, $P_1 = \partial_z$, $P_2 = \partial_{\bar z}$.
(2) Carrollian rotation, $J \equiv x_1\partial_2 - x_2\partial_1 = -z\partial_z + \bar z\partial_{\bar z}$.
(3) Carrollian boosts, $B_A \equiv x_A\partial_u$ or $B_1 = \bar z\partial_u$, $B_2 = z\partial_u$.
(4) Carrollian dilatation, $D \equiv x^a\partial_a$ or $D = u\partial_u + z\partial_z + \bar z\partial_{\bar z}$.
(5) Carrollian special conformal transformations, $K_0 \equiv -x^A x_A\partial_u = -2z\bar z\partial_u$, and $K_A \equiv 2x_A D - x^B x_B\partial_A$, or $K_1 = 2u\bar z\partial_u + 2\bar z^2\partial_{\bar z}$ and $K_2 = 2uz\partial_u + 2z^2\partial_z$.

Here, $x_A = q_{AB}x^B$. In this basis, the commutation relations (3.29) for the global conformal Carrollian subalgebra [192,193] can be split as follows: first, one easily checks that B_1, B_2 P_0 and K_0 form an Abelian subalgebra

$$[B_A, B_B] = 0, \quad [B_A, P_0] = [B_A, K_0] = 0, \quad [K_0, P_0] = 0. \tag{4.32}$$

The six remaining generators form a Lorentz subalgebra, however in an unusual basis:

$$\begin{aligned}
&[D, P_{0,A}] = -P_{0,A}, \qquad [D, K_{0,A}] = K_{0,A}, \qquad [D, B_A] = [D, J] = 0, \\
&[J, G_A] = q_{2A} G_1 - q_{1A} G_2, \qquad G \in \{P, B, K\}, \qquad [J, P_0] = [J, K_0] = 0, \\
&[B_A, P_B] = -q_{AB} P_0, \qquad [B_A, K_B] = -q_{AB} K_0, \\
&[K_0, P_A] = 2B_A, \qquad [K_A, P_0] = -2B_A, \qquad [K_A, P_B] = -2q_{AB} D - 2J\delta_{1,[A}\delta_{B],2}.
\end{aligned} \tag{4.33}$$

We present in Appendix B the concrete relations defining the isomorphism between the global conformal Carrollian algebra in $3d$ and the Poincaré algebra in $4d$.

In conformal field theory, primary fields are required to transform consistently under the action of the infinite-dimensional Witt algebra $\mathfrak{Witt}$. Quasiprimaries on the other hand only need to behave well under the action of the finite-dimensional subalgebra of Möbius transformations $\mathfrak{sl}(2,\mathbb{C})$ (the "global" conformal algebra). The BMS algebra $\mathfrak{bms}_4$ should play in the Carrollian context a role similar to the Witt algebra of conformal field theory with the Poincaré algebra $\mathfrak{iso}(3,1)$ (the "global" conformal Carrollian algebra in $3d$) playing the role of Möbius transformations. Crucially, the embeddings of the Poincaré (resp. Möbius) algebra inside the BMS (resp. Witt) algebra are not unique and correspond to an extra piece of geometry. In the asymptotically flat case, these are the gravity vacua discussed in [172,181]. These embeddings can be locally realized by a choice of Poincaré operators [187,189] (generalizing Möbius operators [194] of conformal geometry) with vanishing "curvature" (corresponding to the curvature of a Cartan connection). From these considerations, we will say that a field is a conformal Carrollian primary (also referred to as Carrollian tensor [34,40,166,185]) if it transforms infinitesimally as

$$\delta_{\bar{\xi}} \Phi_{(k,\bar{k})} = \left[\left(\mathcal{T} + \frac{u}{2} (\partial_z \mathcal{Y}^z + \partial_{\bar{z}} \mathcal{Y}^{\bar{z}}) \right) \partial_u + \mathcal{Y}^z \partial_z + \mathcal{Y}^{\bar{z}} \partial_{\bar{z}} + k \partial_z \mathcal{Y}^z + \bar{k} \partial_{\bar{z}} \mathcal{Y}^{\bar{z}} \right] \Phi_{(k,\bar{k})} \tag{4.34}$$

under full conformal Carroll symmetries (4.30). Here the Carrollian weights $(k, \bar{k})$ are some integers or half-integers. Quasiconformal Carrollian primary fields are only required to transform properly as (4.34) under the global subalgebra displayed above. Importantly, recalling that $\partial_u \bar{\xi}^u = \frac{1}{2}(\partial_z \mathcal{Y}^z + \partial_{\bar{z}} \mathcal{Y}^{\bar{z}})$ and $[\delta_{\bar{\xi}}, d] = 0$, it can be deduced from (4.34) that $\partial_u \Phi_{(k,\bar{k})}(u, z, \bar{z})$ is also transforming as a conformal Carrollian primary with weights $(k + \frac{1}{2}, \bar{k} + \frac{1}{2})$.

2. Classical flux-balance laws

Let us now specify the flux-balance law (4.8) for a $3d$ conformal Carrollian field theory. We assume that Noether currents associated with the global conformal Carrollian symmetries (4.31) take the following Brown-York expression [195,196]

$$j^a_{\bar{\xi}} = \mathcal{C}^a{}_b \bar{\xi}^b, \tag{4.35}$$

where $\mathcal{C}^a{}_b$ is the Carrollian stress tensor. Its components are called the Carrollian momenta [34,39–41,102,166,196–199] and denoted as

$$\mathcal{C}^a{}_b = \begin{bmatrix} \mathcal{M} & \mathcal{N}_B \\ \mathcal{B}^A & \mathcal{A}^A{}_B \end{bmatrix}. \tag{4.36}$$

If the conformal Carrollian field theory under consideration is sourced, the currents (4.35) are no longer meant to be conserved but still obey some flux-balance laws as in (4.8) or in coordinates

$$\partial_a j^a_{\bar{\xi}} = F_a[\sigma] \bar{\xi}^a, \tag{4.37}$$

where σ denotes again the external sources coupled to the theory. Here we assumed that the flux can be written as $F_{\bar{\xi}} = F_a \bar{\xi}^a$, which is sufficient for the holographic purposes discussed in this paper. The flux-balance law (4.37) is obeyed for the generators of the global Carroll subalgebra (isomorphic to Poincaré algebra) if and only if $\mathcal{C}^a{}_b$ satisfies the following constraints:

	Generator	Constraint
Carrollian translations	∂_a	$\partial_a \mathcal{C}^a{}_b = F_b$
Carrollian rotation	$-z\partial_z + \bar{z}\partial_{\bar{z}}$	$\mathcal{C}^z{}_z - \mathcal{C}^{\bar{z}}{}_{\bar{z}} = 0$
Carrollian boosts	$x^A \partial_u$	$\mathcal{C}^A{}_u = 0$
Carrollian dilatation	$x^a \partial_a$	$\mathcal{C}^a{}_a = 0$

(4.38)

In Appendix C, we show that the Carrollian special conformal transformations K_0, K_A do not impose further constraints. Furthermore, the above global conformal Carrollian symmetries are enough to completely constrain $\mathcal{C}^a{}_b$, i.e. (4.37) is automatically satisfied by the supertranslation (and superrotation) currents provided (4.38) holds. Notice also that the term $\mathcal{C}^a{}_b \partial_a \bar{\xi}^b$ does not contribute to the left-hand side of (4.37) as a consequence of (4.31) and (4.38); this is in line with the hypothesis of flux being linear in the symmetry parameters.

In terms of the Carrollian momenta (4.36), the constraints (4.38) read as

$$\boxed{\begin{array}{c} \partial_u \mathcal{M} = F_u, \quad \mathcal{B}^A = 0, \\ \partial_u \mathcal{N}_z - \frac{1}{2}\partial_z \mathcal{M} + \partial_{\bar{z}} \mathcal{A}^{\bar{z}}{}_z = F_z, \quad \mathcal{A}^z{}_z + \frac{1}{2}\mathcal{M} = 0, \\ \partial_u \mathcal{N}_{\bar{z}} - \frac{1}{2}\partial_{\bar{z}} \mathcal{M} + \partial_z \mathcal{A}^z{}_{\bar{z}} = F_{\bar{z}}, \quad \mathcal{A}^{\bar{z}}{}_{\bar{z}} + \frac{1}{2}\mathcal{M} = 0, \end{array}} \tag{4.39}$$

In the left column, the constraints take the form of flux-balance equations for the Carrollian momenta $\mathcal{M}$ and $\mathcal{N}_A$ that are sourced by the fluxes F_a. In the right column, the constraints completely fix the Carrollian momenta $\mathcal{B}^A$, $\mathcal{A}^z{}_z$ and $\mathcal{A}^{\bar{z}}{}_{\bar{z}}$. Notice that $\mathcal{A}^z{}_{\bar{z}}$ and $\mathcal{A}^{\bar{z}}{}_z$ are not fixed by the symmetries.

3. Sourced Ward identities

At the quantum level, the analog of the constraints (4.38) will be provided by the sourced infinitesimal Ward identities (4.15) for the specific case of a $3d$ conformal Carrollian field theory. Similarly to the $2d$ CFT case (see for instance [120]), the Ward identities of a $3d$ conformal Carrollian field theory can be rewritten very simply in terms of $\mathcal{C}^a{}_b$. We assume that the operators $\Psi^i(x)$ inserted in the correlators are (quasi-)conformal Carrollian primary fields transforming as (4.34). The sourced Ward identities (4.15) for the (global) conformal Carrollian symmetries imply

$$\begin{aligned} &\partial_a \langle \mathcal{C}^a{}_b X^\Psi_N \rangle + \frac{\hbar}{i} \sum_i \delta^{(3)}(x - x_i) \frac{\partial}{\partial x^b_i} \langle X^\Psi_N \rangle = \langle F_b X^\Psi_N \rangle, \\ &\langle (\mathcal{C}^z{}_z - \mathcal{C}^{\bar{z}}{}_{\bar{z}}) X^\Psi_N \rangle + \frac{\hbar}{i} \sum_i \delta^{(3)}(x - x_i)(k_i - \bar{k}_i) \langle X^\Psi_N \rangle = 0, \\ &\langle \mathcal{C}^A{}_u X^\Psi_N \rangle = 0, \\ &\langle \mathcal{C}^a{}_a X^\Psi_N \rangle + \frac{\hbar}{i} \sum_i \delta^{(3)}(x - x_i)(k_i + \bar{k}_i) \langle X^\Psi_N \rangle = 0, \end{aligned} \tag{4.40}$$

where X^Ψ_N is defined in (4.16). The derivation of (4.40) is pretty similar to the classical case (4.38) (see Appendix C) and will not be repeated. Now, taking linear combinations of (4.40), we express the Ward identities in terms of the Carrollian momenta:

$$\boxed{\begin{aligned} &\partial_u \langle \mathcal{M} X^\Psi_N \rangle + \tfrac{\hbar}{i} \sum_i \delta^{(3)}(x - x_i) \partial_{u_i} \langle X^\Psi_N \rangle = \langle F_u X^\Psi_N \rangle, \\ &\partial_u \langle \mathcal{N}_z X^\Psi_N \rangle - \tfrac{1}{2} \partial_z \langle \mathcal{M} X^\Psi_N \rangle + \partial_{\bar{z}} \langle \mathcal{A}^{\bar{z}}{}_z X^\Psi_N \rangle + \tfrac{\hbar}{i} \sum_i [\delta^{(3)}(x - x_i) \partial_{z_i} \langle X^\Psi_N \rangle - \partial_z (\delta^{(3)}(x - x_i) k_i \langle X^\Psi_N \rangle)] = \langle F_z X^\Psi_N \rangle, \\ &\partial_u \langle \mathcal{N}_{\bar{z}} X^\Psi_N \rangle - \tfrac{1}{2} \partial_{\bar{z}} \langle \mathcal{M} X^\Psi_N \rangle + \partial_z \langle \mathcal{A}^z{}_{\bar{z}} X^\Psi_N \rangle + \tfrac{\hbar}{i} \sum_i [\delta^{(3)}(x - x_i) \partial_{\bar{z}_i} \langle X^\Psi_N \rangle - \partial_{\bar{z}} (\delta^{(3)}(x - x_i) \bar{k}_i \langle X^\Psi_N \rangle)] = \langle F_{\bar{z}} X^\Psi_N \rangle, \\ &\langle \mathcal{B}^A X^\Psi_N \rangle = 0, \\ &\left\langle \left(\mathcal{A}^z{}_z + \tfrac{1}{2} \mathcal{M} \right) X^\Psi_N \right\rangle + \tfrac{\hbar}{i} \sum_i \delta^{(3)}(x - x_i) k_i \langle X^\Psi_N \rangle = 0, \\ &\left\langle \left(\mathcal{A}^{\bar{z}}{}_{\bar{z}} + \tfrac{1}{2} \mathcal{M} \right) X^\Psi_N \right\rangle + \tfrac{\hbar}{i} \sum_i \delta^{(3)}(x - x_i) \bar{k}_i \langle X^\Psi_N \rangle = 0. \end{aligned}} \tag{4.41}$$

With no field insertion in the correlators, the expectation values of the operators reproduce the classical relations (4.39).

V. HOLOGRAPHIC CONFORMAL CARROLLIAN FIELD THEORY

In this section, we discuss the main ingredients needed for a holographic description of asymptotically flat spacetime in terms of a dual sourced conformal Carrollian field theory. First, we argue that the dual theory lives on $\hat{\mathscr{I}} = \mathscr{I}^- \sqcup \mathscr{I}^+$ where the gluing between $\mathscr{I}^-$ and $\mathscr{I}^+$ is obtained by identifying antipodally $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$. We will make this gluing precise by introducing the geometry of time-ordered conformal Carrollian manifolds. We then propose an identification between scattering elements in position space and Carrollian correlation functions by relating the bulk quantities introduced in Sec. III and the boundary objects discussed in Sec. IV. Eventually, we deduce the explicit form of low-point correlation functions of a Carrollian CFT. In particular, we find a new branch of solutions of the two-point function and argue that this is the appropriate one for holographic purposes.

A. Time-ordered conformal Carrollian manifolds

In this section, we introduce some geometrical description of the manifold on which the dual sourced Carrollian CFT is living. As reviewed in Sec. IV D 1, a Carrollian structure on a manifold $\mathscr{M}$ is made of a pair (q_{ab}, n^a) where $q_{ab}n^b = 0$ and $\mathscr{L}_n q_{ab} \propto q_{ab}$. A conformal Carrollian structure is then defined as an equivalence class $[q_{ab}, n^a]$ for the equivalence relation $(q_{ab}, n^a) \sim (\Omega^2 q_{ab}, \Omega^{-1} n^a)$ where Ω is a nowhere vanishing function on $\mathscr{M}$. Equipped with such a structure, $(\mathscr{M}, [q_{ab}, n^a])$ is called a conformal Carrollian manifold.

To obtain the universal structure [173,181] at, say, future null infinity, we would need to add two hypotheses to our definition of conformal Carrollian manifold. First, fix the topology of $\mathscr{M}$ as $\mathbb{R} \times \Sigma$ where $\mathbb{R}$ is spanned by the flow of n^a and Σ is the space of null generators (for our purpose, we choose Σ to be the one-puncture complex plane, see e.g. [158,160]). Second, require that the vector field n^a is complete and nowhere vanishing. However, as we will argue later, the Carrollian CFT is not living at $\mathscr{I}^+$ or $\mathscr{I}^-$ separately, but should really be seen as living on the whole conformal boundary of asymptotically flat spacetimes. Therefore, we would like to give a geometrical notion of "past" and "future" null infinity after gluing $\mathscr{I}^-$ with $\mathscr{I}^+$. The idea is to single out the separating surface $\Sigma_0 \simeq \Sigma$ as the unique locus where the Carrollian vector n^a vanishes. This provides a definition of "time-ordered" conformal Carrollian manifold, which is a conformal Carrollian manifold $(\mathscr{M}, [q_{ab}, n^a])$ satisfying all the hypotheses above except that n^a now vanishes on a codimension one surface Σ_0. For a chosen connection ∇_a defined on $\mathscr{M}$ (nothing will in fact depend on this choice), we demand

$$\nabla_a n^b|_{\Sigma_0} = 0, \qquad \nabla_a \nabla_b n^c|_{\Sigma_0} \neq 0. \tag{5.1}$$

Now, around Σ_0, we can always choose local coordinates (s, x^A) such that

$$s|_{\Sigma_0} = 0, \qquad n^a \partial_a = \tilde{f}(s, x^A)\partial_s \tag{5.2}$$

for some function $\tilde{f}$. Since n^a vanishes on Σ_0, the constraints (5.1) only involves partial derivatives in the coordinates and are satisfied if $\partial_s \tilde{f} \to 0$ and $\partial_s^2 \tilde{f} \to 2f(x^A)$ as $s \to 0$ for some other function f on Σ. The latter condition is solved by $\tilde{f}(s, x^A) = s^2 f(x^A) + \mathcal{O}(s^3)$ and we have

$$n^a \partial_a = (s^2 f + \mathcal{O}(s^3))\partial_s. \tag{5.3}$$

Notice that the orientation of n^a does not change across Σ_0. This provides a notion of global ordering of the points in $\mathscr{M}$: points are "in the past" of $\mathscr{M}$ if the flow generated by n^a takes them towards the separating surface Σ_0 and "in the future" if the flow takes them away from it. Points of Σ_0 are fixed points of the flow of n^a, they are neither in the past nor in the future of $\mathscr{M}$.

Going back to the conformal boundary of asymptotically flat spacetimes, we are led to consider

$$\hat{\mathscr{I}} \equiv \mathscr{I}^- \sqcup \mathscr{I}^+ \tag{5.4}$$

as conformal Carrollian manifold $\mathscr{M}$, where $\sqcup$ represents a union by gluing of the two conformal boundaries. According to the above discussion, we will take $\hat{\mathscr{I}}$ to be a three-dimensional time-ordered conformal Carrollian manifold. The separating surface along which the gluing is performed is

$$\Sigma_0 \simeq \mathscr{I}^+_- \simeq \mathscr{I}^-_+, \tag{5.5}$$

i.e. $\mathscr{I}^+$ is glued to $\mathscr{I}^-$ by identifying continuously the past surface $\mathscr{I}^+_-$ of $\mathscr{I}^+$ and the future surface $\mathscr{I}^-_+$ of $\mathscr{I}^-$ around spatial infinity. On $\mathscr{I}^+$, we can always choose local coordinates (u, x^A) such that u is the retarded time and $n^a\partial_a = \partial_u$. Analogously on $\mathscr{I}^-$, there exists a local coordinate system (v, x^A) such that v is the advanced time and $n^a\partial_a = \partial_v$. Let us recall that our coordinate choices are such that the angular coordinates $x^A = (z, \bar{z})$ are antipodally identified between $\mathscr{I}^+$ and $\mathscr{I}^-$ (see Appendix A). Since n vanishes on Σ_0 (defined as the transverse Σ in the limit $u \to -\infty$ or $v \to +\infty$), both coordinate systems cannot be extended on the whole $\hat{\mathscr{I}}$. On both sides, we parametrize

$$\begin{cases} u = \tan(\pi(s - \frac{1}{2})), & s > 0 \text{ on } \mathscr{I}^+, \\ v = \tan(\pi(s - \frac{1}{2})), & s < 0 \text{ on } \mathscr{I}^-. \end{cases} \tag{5.6}$$

Figure 4 depicts the situation.

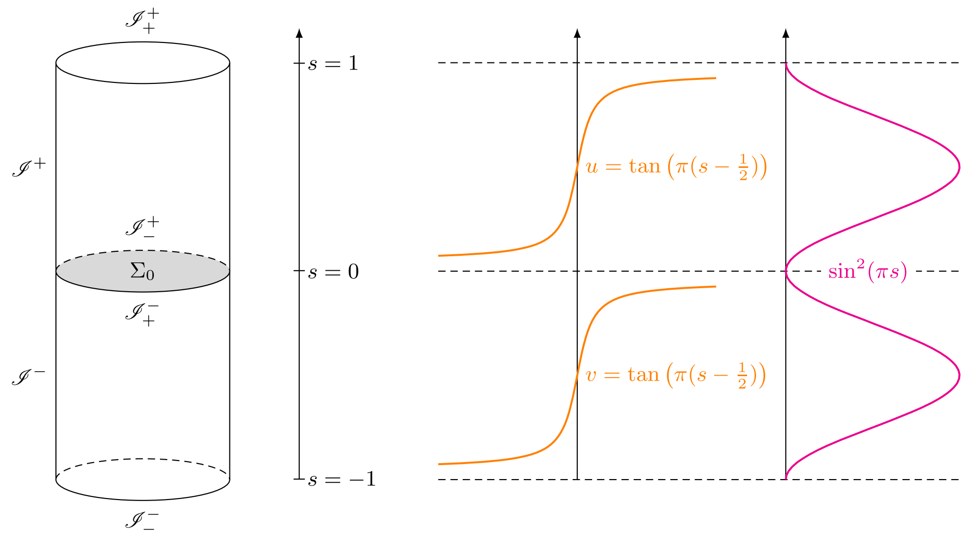

FIG. 4. Time-ordered conformal boundary $\hat{\mathscr{I}}$ of asymptotically flat spacetimes.

The time reparametrizations (5.6) define a global "time" coordinate

$$s\colon \hat{\mathscr{I}} \to]-1, +1[, \tag{5.7}$$

where the separating surface is located at $\Sigma_0 = \{s = 0\} \subset \hat{\mathscr{I}}$. We have $s = -\pi^{-1}\mathrm{arccot} u$ (and similarly for v), hence the Carrollian vector reads now as

$$n^a\partial_a = \frac{1}{\pi}\sin^2(\pi s)\partial_s. \tag{5.8}$$

In this coordinate system, it is manifest that $n \equiv 0$ if and only if $s = 0$ (i.e. on the separating surface) on $\hat{\mathscr{I}}$. On $\mathscr{I}^+$ and $\mathscr{I}^-$, conformal Carrollian automorphisms act infinitesimally as (4.31) or

$$\begin{cases} \bar{\xi}^+(u, x^A) = (\mathcal{T}^+ + \frac{u}{2}\partial_B\mathcal{Y}^B_+)\partial_u + \mathcal{Y}^A_+\partial_A, & \text{on } \mathscr{I}^+, \\ \bar{\xi}^-(v, x^A) = (\mathcal{T}^- + \frac{v}{2}\partial_B\mathcal{Y}^B_-)\partial_v + \mathcal{Y}^A_-\partial_A, & \text{on } \mathscr{I}^-. \end{cases} \tag{5.9}$$

Using (5.6) and (5.8), we can define the unique global smooth infinitesimal automorphism of the time-ordered geometry $\hat{\mathscr{I}}$ as

$$\bar{\xi}(s, x^A) = \frac{1}{\pi}\left(\mathcal{T} - \frac{1}{2}\cot(\pi s)\partial_B\mathcal{Y}^B\right)\sin^2(\pi s)\partial_s + \mathcal{Y}^A\partial_A, \tag{5.10}$$

where $\mathcal{T}(x^A)$, $\mathcal{Y}^A(x^B)$ are smooth functions on the 2-sphere such that

$$\mathcal{T}(x^A) = \mathcal{T}^+(x^A) = \mathcal{T}^-(x^A),$$
$$\mathcal{Y}^A_+(x^B) = \mathcal{Y}^A(x^B) = \mathcal{Y}^A_-(x^B). \tag{5.11}$$

In this picture, the antipodal matching for BMS generators advocated in [125,151,170] and confirmed later in [14,15,200–205] by a phase space analysis at spacelike infinity, corresponds to the requirement that $\bar{\xi}$ given by (5.10) is smooth at Σ_0. A final remark is that the diagonal BMS group selected in this way preserves the location of the separating surface since

$$\bar{\xi}|_{\Sigma_0} = \bar{\xi}(s = 0, x^A) = \mathcal{Y}^A\partial_A \in \mathfrak{diff}(\Sigma_0). \tag{5.12}$$

For the sake of simplicity, in what follows, we will keep the coordinate u (resp. v) to denote the Carrollian time on $\mathscr{I}^+$ (resp. $\mathscr{I}^-$). It will be important to remember that combinations such as $u - v$ are in fact invariant under the action of the diagonal BMS group (5.11).

B. Carrollian holographic correspondence

Having discussed the geometry of the conformal boundary, we are now ready to develop our proposal on how a putative conformal Carrollian theory living on $\hat{\mathscr{I}}$ can encode a scattering of massless particles.

1. Boundary operators as conformal Carrollian fields

As already anticipated in Sec. II D 6, we will argue that $\mathcal{S}$-matrix elements written in position space can naturally be interpreted as correlation functions of operators sourcing the Carrollian CFT living on $\hat{\mathscr{I}}$. The latter will be taken as the source operators $\sigma^m(x)$ introduced in Sec. IV B, assumed to transform as conformal Carrollian primaries (4.34), namely

$$\delta_{\bar{\xi}}\sigma_{(k,\bar{k})}(x) = \left[\left(\mathcal{T} + \frac{u}{2}(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}})\right)\partial_u + \mathcal{Y}^z\partial_z + \mathcal{Y}^z\partial_{\bar{z}} + k\partial_z\mathcal{Y}^z + \bar{k}\partial_{\bar{z}}\mathcal{Y}^{\bar{z}}\right]\sigma_{(k,\bar{k})}(x). \tag{5.13}$$

For instance, it is suggestive to write boundary correlators in the form

$$\langle \sigma^{\text{out}}_{(k_1,\bar{k}_1)}(x_1)\ldots\sigma^{\text{out}}_{(k_n,\bar{k}_n)}(x_n)\sigma^{\text{in}}_{(k_{n+1},\bar{k}_{n+1})}(x_{n+1})\ldots\sigma^{\text{in}}_{(k_N,\bar{k}_N)}(x_N)\rangle, \tag{5.14}$$

where $\sigma^{\text{out}}_{(k_1,\bar{k}_1)}(x_1)\ldots\sigma^{\text{out}}_{(k_n,\bar{k}_n)}(x_n)$ are n insertions of source operators at $\mathscr{I}^+$ and $\sigma^{\text{in}}_{(k_{n+1},\bar{k}_{n+1})}(x_{n+1})\ldots\sigma^{\text{in}}_{(k_N,\bar{k}_N)}(x_N)$ are $N-n$ insertions at $\mathscr{I}^-$. For outgoing insertions, $x_i = (u_i, z_i, \bar{z}_i)$ while for incoming insertions, $x_j = (v_j, z_j, \bar{z}_j)$. The boundary values (2.43) and (2.54) of the operators in position space introduced in Sec. II are natural candidates for conformal Carrollian primaries sourcing the dual theory (for $s = 2$, these are quasiprimaries), i.e.

$$\sigma^{\text{out}}_{(k,\bar{k})}(u,z,\bar{z}) \equiv \lim_{r\to+\infty}(r^{1-s}\phi^{(s)\text{out}}_{z\ldots z}(u,r,z,\bar{z})) = \bar{\phi}^{(s)\text{out}}_{z\ldots z}(u,z,\bar{z}), \tag{5.15}$$

for "out" insertions in retarded Bondi coordinates and the similar identification

$$\sigma^{\text{in}}_{(k,\bar{k})}(v,z,\bar{z}) \equiv \lim_{r\to-\infty}(r^{1-s}\phi^{(s)\text{in}}_{z\ldots z}(v,r,z,\bar{z})^\dagger) = \bar{\phi}^{(s)\text{in}}_{z\ldots z}(v,z,\bar{z})^\dagger \tag{5.16}$$

for "in" insertions in advanced Bondi coordinates and the Hermitian conjugated contributions for reversed helicities. From (2.51), the conformal Carrollian weights $k, \bar{k}$ in (5.13) are fixed in terms of the helicity J as

$$k = \frac{1\pm J}{2}, \qquad \bar{k} = \frac{1\mp J}{2}, \tag{5.17}$$

the upper sign for outgoing fields and the lower sign for incoming fields.

Aside of the source operators, the Carrollian CFT is populated by fields $\Psi^i(x)$. Among them, one finds the Carrollian stress tensor $\mathcal{C}^a{}_b$ that gathers the Carrollian momenta (4.36) and will be identified with the gravitational momenta in the bulk, irrespective of the presence of radiation. Furthermore, the propagating massless fields are not only described by their boundary values but also by a tower of subleading pieces such as the components $F^{(2)}_{ur}$ of the Faraday tensor for spin-1 field or the subleading tensors $D_{AB}, E_{AB}, \ldots$ in the expansion of the angular part of the Bondi metric. These are all natural candidates for the fields of the Carrollian CFT whose evolution is modified by the presence of source operators that holographically encode the radiative degrees of freedom.

Remark—The fact that the algebraic constraints (5.17) hold, i.e. that the conformal Carrollian operators constructed from the boundary value of bulk quantum spin-s fields can be labeled only with one quantum number J, comes from the conformal compactification process and its impact on the fall-offs near $\mathscr{I}^+$ imposed for radiative fields. Indeed, under the combined action of Weyl rescalings and BMS symmetries, the boundary value of a spin-s field with helicity $J = \pm s$ and Weyl weight W will transform infinitesimally as

$$\begin{aligned}\delta_{\bar{\xi},\Omega}\bar{\phi}^W_{z\ldots z} &= [\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} + s\partial_z\mathcal{Y}^z - W\Omega]\bar{\phi}^W_{z\ldots z},\\ \delta_{\bar{\xi},\Omega}\bar{\phi}^W_{\bar{z}\ldots\bar{z}} &= [\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} + s\partial_{\bar{z}}\mathcal{Y}^{\bar{z}} - W\Omega]\bar{\phi}^W_{\bar{z}\ldots\bar{z}}.\end{aligned} \tag{5.18}$$

Fixing the representative of the conformal boundary metric to be the flat metric implies $\delta_{\bar{\xi},\Omega}\mathring{q}_{z\bar{z}} = \partial_z\mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}} - 2\Omega = 0$, so that

$$\begin{aligned}\delta_{\bar{\xi},\Omega=\alpha}\bar{\phi}^W_{z\ldots z} &= \left[\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} + \left(s - \frac{1}{2}W\right)\partial_z\mathcal{Y}^z - \frac{1}{2}W\partial_{\bar{z}}\mathcal{Y}^{\bar{z}}\right]\bar{\phi}^W_{z\ldots z},\\ \delta_{\bar{\xi},\Omega=\alpha}\bar{\phi}^W_{\bar{z}\ldots\bar{z}} &= \left[\mathcal{Y}^z\partial_z + \mathcal{Y}^{\bar{z}}\partial_{\bar{z}} - \frac{1}{2}W\partial_z\mathcal{Y}^z + \left(s - \frac{1}{2}W\right)\partial_{\bar{z}}\mathcal{Y}^{\bar{z}}\right]\bar{\phi}^W_{\bar{z}\ldots\bar{z}}.\end{aligned} \tag{5.19}$$

Using (5.15) and by identification with the transformation (5.13), we find

$$
\begin{aligned}
&\bar{\phi}^W_{z\ldots z} \mapsto \sigma_{(k,\bar{k})} \quad \text{with} \quad k = s - \frac{1}{2}W, \quad \bar{k} = -\frac{1}{2}W, \\
&\bar{\phi}^W_{\bar{z}\ldots\bar{z}} \mapsto \sigma_{(k,\bar{k})} \quad \text{with} \quad k = -\frac{1}{2}W, \quad \bar{k} = s - \frac{1}{2}W. \qquad (5.20)
\end{aligned}
$$

Finally, the radiative falloff (2.41) fixes the Weyl weight W of the boundary value of the field because, under a boundary Weyl rescaling induced by $r \to \Omega^{-1} r$, the boundary spin-s field scales as $\bar{\phi}^W_{z\ldots z} \to \Omega^{1-s}\bar{\phi}^W_{z\ldots z}$ so $W = s - 1$, which gives again (5.17) with the upper choice of sign. The reasoning can also be applied for incoming fields defined as (5.16), the Hermitian conjugation being responsible for the sign flip in (5.17).

Owing to (5.15) and (5.16), we propose the following holographic identification between conformal Carrollian correlators (5.14) and $\mathcal{S}$-matrix elements written in position space (2.87):

$$
\boxed{\begin{array}{c}
\langle \sigma^{\text{out}}_{(k_1,\bar{k}_1)}(x_1)\ldots\sigma^{\text{out}}_{(k_n,\bar{k}_n)}(x_n)\sigma^{\text{in}}_{(k_{n+1},\bar{k}_{n+1})}(x_{n+1})\ldots\sigma^{\text{in}}_{(k_N,\bar{k}_N)}(x_N)\rangle \\
\equiv \\
\frac{1}{(2\pi)^N}\prod_{k=1}^n \int_0^{+\infty} d\omega_k e^{-i\omega_k u_k} \prod_{\ell=n+1}^N \int_0^{+\infty} d\omega_\ell e^{i\omega_\ell v_\ell} \mathcal{A}_N(p_1;\ldots;p_N),
\end{array}} \tag{5.21}
$$

or equivalently

$$
\begin{aligned}
&\langle \sigma^{\text{out}}_{(k_1,\bar{k}_1)}(x_1)\ldots\sigma^{\text{out}}_{(k_n,\bar{k}_n)}(x_n)\sigma^{\text{in}}_{(k_{n+1},\bar{k}_{n+1})}(x_{n+1})\ldots\sigma^{\text{in}}_{(k_N,\bar{k}_N)}(x_N)\rangle \\
&\equiv \\
&\langle 0|\bar{\phi}^{(s)}_{I_1}(x_1)^{\text{out}}\ldots\bar{\phi}^{(s)}_{I_n}(x_n)^{\text{out}}\bar{\phi}^{(s)}_{I_{n+1}}(x_{n+1})^{\text{in}\dagger}\ldots\bar{\phi}^{(s)}_{I_N}(x_N)^{\text{in}\dagger}|0\rangle,
\end{aligned} \tag{5.22}
$$

where $I_i = z\ldots z$ if the helicity is positive ($J_i = +s$), and $I_i = \bar{z}\ldots\bar{z}$ otherwise, and Carrollian weights $(k_i, \bar{k}_i)$ are fixed by bulk helicities according to (5.17).

Particularizing (5.22) to the insertion of two fields only, we obtain, by virtue of the regulated expression (2.95), that the Carrollian two-point function is given by

$$
\langle \sigma^{\text{out}}_{(k_1,\bar{k}_1)}(u, z_1, \bar{z}_1)\sigma^{\text{in}}_{(k_2,\bar{k}_2)}(v, z_2, \bar{z}_2)\rangle = \frac{K^2_{(s)}}{4\pi}\left[\frac{1}{\beta} - \left(\gamma + \ln|u-v| + \frac{i\pi}{2}\text{sign}(u-v)\right)\right]\delta^{(2)}(z_1 - z_2)\delta_{k^+_{12},2}\delta_{k^-_{12},0} \tag{5.23}
$$

denoting $k^\pm_{12} \equiv \sum_i (k_i \pm \bar{k}_i)$. There is correlation if k^+_{12} and k^-_{12} obey two algebraic conditions: the first condition $k^+_{12} = 2$ is imposed by (5.17), i.e. the fact that the inserted fields are identified with boundary values of bulk radiative fields, while the second implies the conservation of helicity $J_1 = J_2$. As above, $\beta \in \mathbb{R}^+_0$ is an infrared regulator than can be arbitrarily close to zero, encoding the fact that position space amplitudes are divergent in the low-energy regime. Finally, from the identification (5.15) and recalling (2.48), one gets that the source operators (5.15) obey the following canonical commutation relations:

$$
[\sigma^{\text{out}}_{(k_1,\bar{k}_1)}(u_1, z_1, \bar{z}_1), \sigma^{\text{out}}_{(k_2,\bar{k}_2)}(u_2, z_2, \bar{z}_2)] = -\frac{i}{2}K^2_{(s)}\text{sign}(u_1 - u_2)\delta^{(2)}(z_1 - z_2)\delta_{k^+_{12},2}\delta_{k^-_{12},0}. \tag{5.24}
$$

Analogous considerations hold for "in" insertions.

2. Electrodynamics

We now detail some holographic aspects of electrodynamics reviewed in Sec. III A in terms of the dual sourced Carrollian CFT. For the sake of conciseness, we give all definitions for $\mathscr{I}^+$ and drop the "out" subscripts to lighten the notations when the coordinate dependence is clear. The analogous relations can be easily deduced for $\mathscr{I}^-$.

The dual conformal Carrollian theory is invariant under $U(1)$ transformations. The source operators that are considered in this section are denoted as $\sigma_{(k,\bar{k}),Q}$ and bear conformal Carrollian weights $(k, \bar{k})$ describing their transformations under boundary diffeomorphisms (4.31) and a "weight" Q under the internal $U(1)$ symmetry, which is interpreted as the electric charge of the bulk field. Following (5.15), we introduce

$$\sigma_z^{(A)}(u,z,\bar z) \equiv \sigma_{(1,0),0}(u,z,\bar z) = A_z^{(0)}(u,z,\bar z), \qquad \sigma_{\bar z}^{(A)}(u,z,\bar z) = (\sigma_z^{(A)})^\dagger,$$
$$\sigma^{(\phi)}(u,z,\bar z) \equiv \sigma_{(\frac12,\frac12),Q}(u,z,\bar z) = \phi^{(0)}(u,z,\bar z) = \lim_{r\to\infty} r\phi(u,r,z,\bar z), \tag{5.25}$$

coinciding with the boundary values of the bulk gauge field A_μ and some massless scalar matter field ϕ at $\mathscr{I}^+$. These fields are conformal Carrollian primaries with respective weights (1,0) and $(\frac12,\frac12)$ by virtue of (5.17), on which the global $U(1)$ symmetry acts as

$$\delta_\lambda \sigma_z^{(A)}(u,z,\bar z) = \partial_z \lambda^{(0)}(z,\bar z), \qquad \delta_\lambda \sigma^{(\phi)}(u,z,\bar z) = -ieQ\lambda^{(0)}(z,\bar z)\sigma^{(\phi)}(u,z,\bar z). \tag{5.26}$$

The homogeneous transformation determines the charge under the $U(1)$ transformation (which is proportional to the electric charge of the field) and the inhomogeneous transformation vanishes for parameters $\lambda^{(0)} \in \mathbb{R}$. The Noether current for the $U(1)$ symmetry is given by

$$\langle j^u_\lambda \rangle \equiv -\frac{1}{e^2}\lambda^{(0)} F^{(2)}_{ru}, \qquad \langle j^z_\lambda\rangle \equiv 0 \equiv \langle j^{\bar z}_\lambda\rangle \tag{5.27}$$

and the corresponding fluxes are

$$F_\lambda = \lambda^{(0)}\mathcal{J}^{(2)}_u(\sigma^{(\phi)}) - \frac{1}{e^2}(\partial_u\sigma_z^{(A)}\partial_{\bar z}\lambda^{(0)} + \partial_u\sigma_{\bar z}^{(A)}\partial_z\lambda^{(0)}). \tag{5.28}$$

The factors are adjusted to precisely match with the charges (3.8) obtained by the bulk computation. With the identifications (5.27) and (5.28), one can then check explicitly that the time evolution equations in the sourced Ward identities (4.29) reproduce the asymptotic Maxwell equation (3.7) when there is no insertion in the correlators. Notice finally that the fluxes (5.28) vanish identically if $\tilde A_z^{(0)} = 0 = \tilde A_{\bar z}^{(0)}$ and $\mathcal{J}^{(2)}_u = 0$.

3. Gravity

We now argue that a quantum conformal Carrollian field theory coupled with external sources is a viable candidate to describe holographically gravity in $4d$ asymptotically flat spacetimes reviewed in Sec. III B. We propose the following correspondence between Carrollian momenta (4.36) (left-hand side) and gravitational data (3.32) (right-hand side) at $\mathscr{I}^+$:

$$\langle \mathcal{C}^u{}_u\rangle \equiv \frac{\bar M}{4\pi G}, \qquad \langle \mathcal{C}^u{}_A\rangle \equiv \frac{1}{8\pi G}(\bar N_A + u\partial_A \bar M),$$
$$\langle \mathcal{C}^A{}_B\rangle + \frac12 \delta^A{}_B\langle\mathcal{C}^u{}_u\rangle \equiv 0. \tag{5.29}$$

The factors are fixed by demanding that the gravitational charges (3.31) correspond to the Noether currents (4.35) of the conformal Carrollian field theory integrated on a section $u = \text{const}$. The correspondence (5.29) is inspired by the AdS/CFT dictionary where the holographic stress-energy tensor of the CFT is identified with some subleading order pieces in the expansion of the bulk metric [206,207]. Indeed, recall that the Carrollian momenta are nothing but the components of an ultra-relativistic stress tensor living at null infinity.

Following the identification (5.15), we identify the source operators in the Carrollian CFT as the asymptotic shear

$$\sigma^{(g)}_{zz}(u,z,\bar z) \equiv \sigma_{(\frac32,-\frac12)}(u,z,\bar z) = C_{zz}(u,z,\bar z),$$
$$\sigma^{(g)}_{\bar z\bar z}(u,z,\bar z) \equiv \sigma_{(-\frac12,\frac32)}(u,z,\bar z) = C_{\bar z\bar z}(u,z,\bar z), \tag{5.30}$$

and similarly at $\mathscr{I}^-$. They are quasi-conformal Carrollian primary fields because of (3.30). The homogeneous part of the transformation determines the Carrollian weights of the field to be $k = \frac32$ and $\bar k = -\frac12$. In the presence of matter, boundary values of radiative null fields will also be considered as source operators $\sigma^{(\phi)}$ with weights $(k,\bar k)$ fixed by the helicity of the bulk field as (5.17). All these sources are responsible for the dissipation in the conformal Carrollian field theory through the fluxes $F_{\bar\xi} = F_a\bar\xi^a$ with

$$F_u = \frac{1}{16\pi G}\left[\partial_z^2\partial_u\sigma^{(g)}_{\bar z\bar z} + \frac12\sigma^{(g)}_{\bar z\bar z}\partial_u^2\sigma^{(g)}_{zz} + \text{H.c.}\right] - T^{m(2)}_{uu}(\sigma^{(\phi)}),$$
$$F_z = \frac{1}{16\pi G}\left[-u\partial_z^3\partial_u\sigma^{(g)}_{\bar z\bar z} + \sigma^{(g)}_{zz}\partial_z\partial_u\sigma^{(g)}_{\bar z\bar z} - \frac{u}{2}(\partial_z\sigma^{(g)}_{zz}\partial_u^2\sigma^{(g)}_{\bar z\bar z} + \sigma^{(g)}_{zz}\partial_z\partial_u^2\sigma^{(g)}_{\bar z\bar z})\right] - T^{m(2)}_{uz}(\sigma^{(\phi)}) + \frac{u}{2}\partial_z T^{m(2)}_{uu}(\sigma^{(\phi)}),$$
$$F_{\bar z} = (F_z)^\dagger. \tag{5.31}$$

Taking the identifications (5.29) and (5.31) into account, one can then check explicitly that the time evolution equations in the sourced Ward identities (4.41) reproduce the gravitational retarded time evolution equations (3.24) when there is no insertion in the correlators. Notice that imposing the BMS-invariant conditions $\tilde{C}_{zz} = 0 = \tilde{C}_{\bar{z}\bar{z}}$ and $T^{m(2)}_{uu} = 0 = T^{m(2)}_{uz} = T^{m(2)}_{u\bar{z}}$ makes the flux vanish identically.

C. Holographic Ward identities for a massless scattering

In this section, we specify the holographic sourced Ward identities discussed in Secs. IV C and IV D to the case where the bulk process consists of a massless scattering.

Let us consider a scattering of null particles in some asymptotically-flat spacetime at null infinity. Through the holographic identification (5.21), the scattering amplitudes are encoded by correlation functions $\langle X^\sigma_N \rangle$ of source operators in a putative dual conformal Carrollian theory living on $\hat{\mathscr{I}}$. The integrated version of the sourced Ward identity (4.22) is specified as

$$\left\langle \left(\int_{\hat{\mathscr{I}}} \boldsymbol{F}_{\bar{\xi}} - \int_{\mathscr{I}^+_+} \boldsymbol{j}_{\bar{\xi}} + \int_{\mathscr{I}^-_-} \boldsymbol{j}_{\bar{\xi}} \right) X^\sigma_N \right\rangle = 0, \tag{5.32}$$

for any conformal Carrollian vector $\bar{\xi}$, where $\boldsymbol{j}_{\bar{\xi}}|_{\mathscr{I}^-_-}, \boldsymbol{j}_{\bar{\xi}}|_{\mathscr{I}^+_+}$ are sensitive to motions of massive particles and the flux $\boldsymbol{F}_{\bar{\xi}}$ is given by (5.31). Hypothesizing that the process under consideration only encompasses massless fields and nothing arrives at past and future timelike infinities, the Noether current $\boldsymbol{j}_{\bar{\xi}}$ vanishes at $\mathscr{I}^-_-$ and $\mathscr{I}^+_+$. Hence (5.32) becomes

$$\boxed{\langle \mathcal{F}_{\bar{\xi}} X^\sigma_N \rangle = 0,} \quad \text{where } \mathcal{F}_{\bar{\xi}} = \int_{\hat{\mathscr{I}}} \boldsymbol{F}_{\bar{\xi}}. \tag{5.33}$$

Taking into account that the insertion of the flux operator $\mathcal{F}_{\bar{\xi}}$ generates the transformation of the source operators in X^σ_N, we have

$$\boxed{\delta_{\bar{\xi}} \langle X^\sigma_N \rangle = 0,} \tag{5.34}$$

hence recovering the invariance of the correlators under conformal Carrollian symmetries. Notice that with no source inserted ($N = 0$), we obtain the natural result $\langle \mathcal{F}_{\bar{\xi}} \rangle = 0$. The consequence of the relation (5.34) has been studied e.g. in [23,26,193]. In the next section, we will revisit how to deduce the generic form of correlators from these symmetry constraints.

Prior to that, let us mention that the derivation can also be made for electrodynamics. In that case, the conformal Carrollian correlators obey

$$\langle \mathcal{F}_\lambda X^\sigma_N \rangle = 0 \Leftrightarrow \delta_\lambda \langle X^\sigma_N \rangle = 0 \tag{5.35}$$

for any gauge parameter λ. In particular, this constrains the two-point function to satisfy

$$\delta_\lambda \langle X^\sigma_2 \rangle = 0 \Rightarrow (\lambda^{(0)}(z_1, \bar{z}_1) Q_1 - \lambda^{(0)}(z_2, \bar{z}_2) Q_2) \langle X^\sigma_2 \rangle = 0 \tag{5.36}$$

using (5.15), (5.16), and (5.26). For the particular transformation $\lambda^{(0)} = c \in \mathbb{R}_0$, this imposes the algebraic constraint $Q_1 = Q_2$, which is nothing but the statement of conservation of electric charge. For a generic function $\lambda^{(0)}(z, \bar{z})$, this further imposes $\langle X^\sigma_2 \rangle \propto \delta^{(2)}(z_1 - z_2) \delta_{Q_1, Q_2}$, which is consistent with conformal Carrollian invariance (5.34) as we will show in the next section.

D. Conformal Carrollian invariant correlation functions

In this section, we deduce the explicit form of the two- and three-point correlation functions in Carrollian CFT from the Ward identities (5.34). The computation of higher-point functions is left to a future endeavor.

Let $\Phi_{(k_1, \bar{k}_1)}(x_1)$ and $\Phi_{(k_2, \bar{k}_2)}(x_2)$ be two quasiconformal-Carrollian primary operators. We want to study the constrains on the two-point function $\langle X_2 \rangle \equiv \langle \Phi_{(k_1, \bar{k}_1)}(x_1) \Phi_{(k_2, \bar{k}_2)}(x_2) \rangle$ implied by $\delta_{\bar{\xi}} \langle X_2 \rangle = 0$, where $\bar{\xi}$ denotes an element of the global part of the conformal Carroll algebra (or equivalently Poincaré algebra), which transforms the inserted fields as (4.34). Invariance under Carrollian translations generated by $P_a = \partial_a$ gives

$$\frac{\partial}{\partial x^a_1} \langle X_2 \rangle + \frac{\partial}{\partial x^a_2} \langle X_2 \rangle = 0 \Rightarrow \langle X_2 \rangle = \langle X_2 \rangle (u_{12}, z_{12}, \bar{z}_{12}), \tag{5.37}$$

where $u_{12} \equiv u_1 - u_2$ and $z_{12} \equiv z_1 - z_2$. Invariance under Carrollian boosts $B_A = x_A \partial_u$ gives

$$x^{12}_A \partial_{u_{12}} \langle X_2 \rangle = 0 \Rightarrow \langle X_2 \rangle = f_{(2)}(z_{12}, \bar{z}_{12}) + g_{(2)}(u_{12}) \delta^{(2)}(z_{12}). \tag{5.38}$$

The general solution thus involves two distinct branches [55,193]. The time-independent one $\langle X_2 \rangle^{ti}$ is meant to be invariant under the stabilizer group of $u =$ const. cuts of the conformal Carrollian manifold (i.e. the conformal group in two dimensions), while the other branch $\langle X_2 \rangle^{td}$ involves explicitly the time direction but at the price to reduce the angular dependence to contact terms, which can be expected for an "ultralocal" theory of fields.

1. Time-independent branch

Selecting first the time-independent branch by setting $g_{(2)} \equiv 0$, the invariance under the Carrollian dilatation $D = x^a \partial_a = u\partial_u + z\partial_z + \bar{z}\partial_{\bar{z}}$ imposes

$$z_{12}\partial_{z_{12}}f_{(2)} + \bar{z}_{12}\partial_{\bar{z}_{12}}f_{(2)} + k_{12}^{+}f_{(2)} = 0 \Rightarrow f_{(2)} = \frac{\tilde{c}_{(2)}}{z_{12}^{a}\bar{z}_{12}^{b}}, \quad a + b = k_{12}^{+}. \tag{5.39}$$

Invariance under Carrollian rotation $J = -z\partial_z + \bar{z}\partial_{\bar{z}}$ also imposes

$$z_{12}\partial_{z_{12}}f_{(2)} - \bar{z}_{12}\partial_{\bar{z}_{12}}f_{(2)} + k_{12}^{-}f_{(2)} = 0 \Rightarrow a - b = k_{12}^{-}, \tag{5.40}$$

which then implies that $a = \sum_i k_i$ and $b = \sum_i \bar{k}_i$. Because of time-independence, K_0 brings no additional constraint. K_1 and K_2 respectively impose that $k_1 = k_2$ and $\bar{k}_1 = \bar{k}_2$, which allows us to conclude that [55,193]

$$\langle X_2 \rangle^{ti} = f_{(2)} = \frac{\tilde{c}_{(2)}}{(z_1 - z_2)^{k_1+k_2}(\bar{z}_1 - \bar{z}_2)^{\bar{k}_1+\bar{k}_2}}\delta_{k_1,k_2}\delta_{\bar{k}_1,\bar{k}_2}. \tag{5.41}$$

This is exactly the standard two-point function for a $2d$ CFT. However, although this branch is allowed from a symmetry analysis, it is not related to dynamical bulk events such as scattering processes since it has no time dependence.

2. Time-dependent branch

We are thus rather interested in the time-dependent branch $\langle X_2 \rangle^{td}$ where we set $f_{(2)} \equiv 0$. The Ward identity encoding the invariance under Carrollian rotation J simply adds one algebraic constraint as

$$g_{(2)}(z_{12}\partial_{z_{12}} - \bar{z}_{12}\partial_{\bar{z}_{12}} + k_{12}^{-})\delta^{(2)}(z_{12}) = k_{12}^{-}g_{(2)}\delta^{(2)}(z_{12}) = 0 \Rightarrow k_{12}^{-} = 0. \tag{5.42}$$

Now, invariance under Carrollian dilatation D gives

$$u_{12}\partial_{u_{12}}\langle X_2 \rangle^{td} + z_{12}\partial_{z_{12}}\langle X_2 \rangle^{td} + \bar{z}_{12}\partial_{\bar{z}_{12}}\langle X_2 \rangle^{td} + k_{12}^{+}\langle X_2 \rangle^{td} = 0, \tag{5.43}$$

which, using the fact that $x\delta'(x) \simeq -\delta(x)$ in the sense of distributions, becomes

$$u_{12}g'_{(2)}(u_{12}) + (k_{12}^{+} - 2)g_{(2)}(u_{12}) = 0. \tag{5.44}$$

When (5.44) is satisfied, we can check that the Ward identities for K_0, K_1, and K_2 do not bring additional constraints. Indeed, the proof for K_0 is immediate, because

$$(|z_1|^2\partial_{u_1} + |z_2|^2\partial_{u_2})\langle X_2 \rangle^{td} \propto (|z_1|^2 - |z_2|^2)\delta^{(2)}(z_1 - z_2) = 0. \tag{5.45}$$

The Ward identity for K_1 is equivalent to the following differential constraint:

$$\begin{aligned} & u_{12}z_1\partial_{u_{12}}\langle X_2 \rangle^{td} + (z_1^2 - z_2^2)\partial_{z_{12}}\langle X_2 \rangle^{td} + 2z_1\langle X_2 \rangle^{td}\sum_i k_i = 0 \\ & \Rightarrow (2 - k_{12}^{-})z_1\langle X_2 \rangle^{td} + (z_1 + z_2)z_{12}\partial_{z_{12}}\langle X_2 \rangle^{td} + 2z_1\langle X_2 \rangle^{td}\sum_i k_i = 0 \\ & \Rightarrow \left(2 - \sum_i k_i - \sum_i \bar{k}_i - 2 + 2\sum_i k_i\right) z_1\langle X_2 \rangle^{td} = \left(\sum_i k_i - \sum_i \bar{k}_i\right) z_1\langle X_2 \rangle^{td} = 0, \end{aligned} \tag{5.46}$$

using successively (5.44) and the constraint $k_{12}^{-} = 0$. This concludes the derivation since the invariance under the last special conformal transformation K_2 is proven in an analogous way. The deal now consists in solving carefully the master equation (5.44), which offers a few surprises.

3. Continuous set of functional solutions

For generic values of $k_{12}^{+} \neq 2$, the general functional solution of this equation is

$$g_{(2)}(u_{12}) \propto \frac{1}{u_{12}^{k_{12}^{+}-2}}, \tag{5.47}$$

leading to the following correlator

$$\boxed{\langle X_2\rangle^{td,f} = \frac{c_{f,(2)}}{(u_1-u_2)^{k_{12}^+-2}}\delta^{(2)}(z_1-z_2)\delta_{k_{12}^-,0}.} \tag{5.48}$$

This class of solutions is consistent with previous analyses [55,193,208].

When $k_{12}^+ = 2$, the above solution seems to be time independent again. But if the insertions represent boundary values of bulk-scattering fields, this algebraic constraint is precisely implied by (5.17). The only hope to keep a time-dependent solution from (5.48) that can match e.g. with (5.23) in the limit $k_{12}^+ \to 2$ is to have a particular dependency in k_{12}^+ in the overall constant, left unfixed by the symmetries.

Going to Fourier space, $g_{(2)}(u_{12}) = \int_{-\infty}^{+\infty} d\omega G(\omega) e^{-i\omega u_{12}}$ and posing $\beta \equiv k_{12}^+ - 2$, (5.44) becomes

$$\int_{-\infty}^{+\infty} d\omega[-\omega G'(\omega) + (\beta-1)G(\omega)]e^{-i\omega u_{12}} = 0. \tag{5.49}$$

Assuming that $\beta > 0$, the distribution

$$G_\beta(\omega) = 2\pi c\omega^{\beta-1}\Theta(\omega) \tag{5.50}$$

is solution of (5.49), where c is a constant. Indeed, enforcing that $\omega \geq 0$ to interpret it afterwards as the (light cone) energy of particles,

$$\int_{-\infty}^{+\infty} d\omega[-\omega G'_\beta(\omega) + (\beta-1)G_\beta(\omega)]e^{-i\omega u_{12}} = -2\pi c\int_{-\infty}^{+\infty} d\omega\omega^\beta\delta(\omega)e^{-i\omega u_{12}} = 0. \tag{5.51}$$

Inverting the Fourier transform for $\beta > 0$ gives

$$g_{(2)}(u_{12}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega G_\beta(\omega)e^{-i\omega u_{12}} = c\int_0^{+\infty} d\omega\omega^{\beta-1}e^{-i\omega u_{12}}, \tag{5.52}$$

which involves the integral (2.93). This fixes the dependency of $c_{f,(2)}$ as

$$g_{(2)}(u_{12}) = \frac{c\Gamma[k_{12}^+-2]}{u_{12}^{k_{12}^+-2}} \xrightarrow{k_{12}^+\to 2} c\left[\frac{1}{k_{12}^+-2} - (\gamma + \ln|u_{12}|)\right] + \mathcal{O}(k_{12}^+-2) \tag{5.53}$$

only focusing on the functional branch by fixing $c \in \mathbb{R}$. As we discussed in Sec. II D 4, the pole is related to an infrared divergence, which seems entangled with the large-r limit of the bulk fields inducing the algebraic constraints (5.17). As we shall now see, this is however not the unique time-dependent solution for $k_{12}^+ = 2$.

4. Discrete set of distributional solutions

When $k_{12}^+ = 2 + n$ for $n \in \mathbb{N}$, the solution of (5.44) is enriched by a discrete set of distributional solutions. For $n = 0$, using the fact that $x\delta(x) \simeq 0$ in the sense of distributions, we see that $g_{(2)}(u_{12}) \propto f_{(0)}(u_{12}) \equiv \mathrm{sign}(u_{12})$ is a solution. We thus have

$$\langle X_2\rangle^{td,d} = c_{d,(2)}\mathrm{sign}(u_1-u_2)\delta^{(2)}(z_1-z_2)\delta_{k_{12}^-,0}\delta_{k_{12}^+,2}, \tag{5.54}$$

among the possible solutions of the Carrollian Ward identity for $k_{12}^+ = 2$. For any $n \in \mathbb{N}_0$, it can be shown by recurrence that (5.44) is solved by the successive distributional derivatives of the sign function, denoted by $f_{(n)}(u_{12}) \equiv \frac{d^n}{du_{12}^n} f_{(0)}(u_{12})$. Indeed, if $f_{(n)}(u_{12})$ is solution of (5.44) for $k_+^{12} - 2 = n$, i.e. $u_{12}f'_{(n)}(u_{12}) + nf_{(n)}(u_{12}) = 0$, then

$$u_{12}f'_{(n+1)}(u_{12}) + (n+1)f_{(n+1)}(u_{12}) = (u_{12}f'_{(n)}(u_{12}))' + nf'_{(n)}(u_{12}) = 0. \tag{5.55}$$

So in general, we can write

$$\boxed{\langle X_2\rangle^{td,d}_{(n)} = c_{d,(2)}\frac{d^n}{du_1^n}\mathrm{sign}(u_1-u_2)\delta^{(2)}(z_1-z_2)\delta_{k_{12}^-,0}\delta_{k_{12}^+,2+n}, \quad \forall\ n \in \mathbb{N}.} \tag{5.56}$$

To the best of our knowledge, this particular solution has not been derived in the previous Carrollian literature. Together with (5.53), this proves the Carrollian invariance of the boundary two-point function (5.53) and the commutator (5.24). More precisely, the holographic dictionary (5.15), (5.22) and the considerations of Sec. II D 5 favors the following particular linear combination of (5.53) and (5.54):

$$\boxed{\langle X_2\rangle^{td} \propto \left[\frac{1}{k_{12}^+-2} - \left(\gamma + \ln|u_1-u_2| + \frac{i\pi}{2}\mathrm{sign}(u_1-u_2)\right)\right]\delta^{(2)}(z_1-z_2)\delta_{k_{12}^+,2}\delta_{k_{12}^-,0}.} \tag{5.57}$$

We thus regard the above expression as the form of the two-point function for source operators in any holographic Carrollian CFT. The expression (2.95) of the two-point amplitude in position space fixes the overall constant in (5.57) to $(4\pi)^{-1}K_{(s)}^2$.

Remark—Acting with n time derivatives on the correlators increments $k_{12}^+ \to k_{12}^+ + n$ (because each time derivative increases Carrollian weights by $\frac{1}{2}$), starting from the correlators of boundary value fields with $k_{12}^+ = 2$. For each $n \in \mathbb{N}_0$, we thus have two branches to consider. For instance,

$$\begin{aligned}
\langle \partial_{u_1}\Phi_{z_1 z_1}(u_1,z_1,\bar z_1)\Phi_{\bar z_2\bar z_2}(u_2,z_2,\bar z_2)\rangle &\propto \frac{\delta^{(2)}(z_1-z_2)}{u_1-u_2} \quad \text{or} \quad \delta(u_1-u_2)\delta^{(2)}(z_1-z_2),\\
\langle \partial_{u_1}\Phi_{z_1 z_1}(u_1,z_1,\bar z_1)\partial_{u_2}\Phi_{\bar z_2\bar z_2}(u_2,z_2,\bar z_2)\rangle &\propto \frac{\delta^{(2)}(z_1-z_2)}{(u_1-u_2)^2} \quad \text{or} \quad \delta'(u_1-u_2)\delta^{(2)}(z_1-z_2),
\end{aligned} \tag{5.58}$$

for $k_{12}^+ = 3$ and 4. Both choices lead to time-dependent correlation functions which are possibly well-behaved for the $\mathcal{B}$-transform. However, the compatibility with the holographic dictionary imposes to choose the distributional branch for (expectation values of) commutators and the functional branch for the correlation functions. As a curiosity, we remark that the coexistence of inverse power law in time and distributional dependencies is reminiscent of what happens in $2d$ CFT, where the analogous branches can be mapped onto each other by means of shadow transforms.

Let us conclude this section by giving some comments on the three-point function $\langle X_3\rangle$. It has already been pointed out that the time-dependent three-point function $\langle X_3\rangle^{td}$ is identically zero in a conformal Carrollian quantum theory [55,193,208]. This can be seen as a consequence of the invariance under Carrollian time translation P_0, boosts B_A and special conformal transformation K_0, or, in other words, invariance under Poincaré translations (see Appendix B for the dictionary). One has the following algebraic constraints:

$$\begin{aligned}
\sum_{i=1}^3 \partial_{u_i}\langle X_3\rangle^{td} &= \sum_{i=1}^3 z_i\partial_{u_i}\langle X_3\rangle^{td} = \sum_{i=1}^3 \bar z_i\partial_{u_i}\langle X_3\rangle^{td}\\
&= \sum_{i=1}^3 |z_i|^2\partial_{u_i}\langle X_3\rangle^{td} = 0
\end{aligned} \tag{5.59}$$

solved by $\partial_{u_i}\langle X_3\rangle^{td} = 0$, for all $i = 1, 2, 3$ and the time-dependent three-point function is identically zero. Finally, by similar arguments, one can easily show that this is also the case for the one-point function, namely $\langle X_1\rangle = \langle \Phi_{(k,\bar k)}\rangle = 0$, except for the identity operator.

VI. RELATION WITH CELESTIAL HOLOGRAPHY

In Sec. V, we discussed the Carrollian holography proposal by providing some kinematical properties of the dual Carrollian CFT and its relation with gravity in the bulk. The goal of this section is to make contact between the Carrollian approach and the celestial holography paradigm. In order to do so, we start by recalling some of the well-established symmetry constraints for celestial CFT induced from the bulk analysis, namely the BMS Ward identities encoding bulk soft theorems. We then relate Carrollian CFT and celestial CFT by mapping the Carrollian source operators, their correlation functions and associated Ward identities to those of the CCFT.

A. Ward identities of 2d celestial CFT currents

As already reviewed in Sec. II D 3, for a massless scattering, one can express the $\mathcal{S}$-matrix elements in a boost-eigenstate basis made of the conformal primary wave functions by performing a Mellin transform on the amplitudes in energy eigenstates, see (2.84). The key ingredient of celestial holography is to identify the $\mathcal{S}$-matrix elements written in this conformal basis with correlation functions of $2d$ celestial CFT, namely

$$\begin{aligned}
&\langle \mathcal{O}^{\rm out}_{(\Delta_1,J_1)}(z_1,\bar z_1)\dots\mathcal{O}^{\rm out}_{(\Delta_n,J_n)}(z_n,\bar z_n)\mathcal{O}^{\rm in}_{(\Delta_{n+1},J_{n+1})}(z_{n+1},\bar z_{n+1})\dots\mathcal{O}^{\rm in}_{(\Delta_N,J_N)}(z_N,\bar z_N)\rangle_{\rm CCFT}\\
&\equiv \langle {\rm out}|{\rm in}\rangle_{\rm boost} = \int_0^{+\infty} d\omega_1\omega_1^{\Delta_1-1}\int_0^{+\infty} d\omega_2\omega_2^{\Delta_2-1}\dots\int_0^{+\infty} d\omega_N\omega_N^{\Delta_N-1}\mathcal{A}_N(p_1;\dots;p_N),
\end{aligned} \tag{6.1}$$

where we recall that $\mathcal{A}_N$ denotes the amplitude of N-particle scattering (n of which are outgoing) in the usual energy-eigenstate basis. CCFT operators are characterized by a pair of numbers; Δ is the conformal dimension corresponding to the boost eigenvalue in the bulk and J is the $2d$ spin identified with the bulk helicity. Very often, one trades (Δ, J) for the conformal weights $(h, \bar{h})$ defined as

$$h = \frac{\Delta + J}{2}, \qquad \bar{h} = \frac{\Delta - J}{2}. \tag{6.2}$$

CCFT operators can also wear a number associated with additional global symmetries, e.g. the electric charge Q for the $U(1)$ symmetry, which is sometimes dropped to simplify the notation. In the following, we will use the shorthand notation

$$\mathcal{X}_N = \mathcal{O}^{\text{out}}_{(\Delta_1,J_1)}(z_1,\bar{z}_1)\ldots\mathcal{O}^{\text{out}}_{(\Delta_n,J_n)}(z_n,\bar{z}_n)\mathcal{O}^{\text{in}}_{(\Delta_{n+1},J_{n+1})}(z_{n+1},\bar{z}_{n+1})\ldots\mathcal{O}^{\text{in}}_{(\Delta_N,J_N)}(z_N,\bar{z}_N) \tag{6.3}$$

for N insertions in CCFT correlators.

Using the identification (6.1), soft theorems in the bulk can be rewritten as Ward identities associated to "conformally soft" currents (i.e. with $\Delta \in \mathbb{Z}$ operators) in CCFT [110]. The soft-photon theorem can be rewritten as a $U(1)$ Kac-Moody Ward identity [69,110,126]

$$\langle J(z)\mathcal{X}_N\rangle = \hbar \sum_{q=1}^{N} \frac{e\eta_q Q_q}{z - z_q}\langle \mathcal{X}_N\rangle, \tag{6.4}$$

where $J(z)$ is a $U(1)$ Kac-Moody current of conformal weights (1,0) and $\eta_q = \pm 1$ for outgoing/incoming fields. Note that this sign disappears in the all out convention. Similarly, the leading soft graviton theorem can be encoded in the supertranslation Ward identities of the CCFT [110,125,151]

$$\langle P(z,\bar{z})\mathcal{X}_N\rangle + \hbar \sum_{q=1}^{N} \frac{\eta_q}{z - z_q}\widehat{\partial}_{\Delta_q}\langle \mathcal{X}_N\rangle = 0, \tag{6.5}$$

where $P(z,\bar{z})$ is the supertranslation current with conformal weights $(\frac{3}{2},\frac{1}{2})$ and $\widehat{\partial}_\Delta$ shifts the conformal dimension Δ by unit increment. The subleading soft graviton theorem is described by the CCFT Ward identities of superrotations [69,72,76,209]:

$$\langle T(z)\mathcal{X}_N\rangle + \hbar \sum_{q=1}^{N}\left[\frac{\partial_q}{z - z_q} + \frac{h_q}{(z - z_q)^2}\right]\langle \mathcal{X}_N\rangle = 0, \tag{6.6}$$

where $T(z)$ is the holomorphic celestial stress tensor with conformal weights (2, 0) (similar results hold for the antiholomorphic stress tensor).

The low-point correlation functions in the CCFT can be deduced from the bulk amplitudes using (6.1). Alternatively, they can be deduced by studying the constraints implied by (6.5) and (6.6) [121,210]. The two-point function reads explicitly as

$$\begin{aligned}
&\mathcal{M}(\Delta_1, z_1, \bar{z}_1; \Delta_2, z_2, \bar{z}_2) \\
&\quad \equiv \langle \mathcal{O}^{\text{out}}_{(\Delta_1,J_1)}(z_1,\bar{z}_1)\mathcal{O}^{\text{in}}_{(\Delta_2,J_2)}(z_2,\bar{z}_2)\rangle \\
&\quad = (2\pi)^4 \mathscr{C}_{(2)}\delta(\nu_1 + \nu_2)\delta^{(2)}(z_1 - z_2)\delta_{J_1,J_2},
\end{aligned} \tag{6.7}$$

with $\Delta_q = c + i\nu_q$, while the 3-point correlation function vanishes in Lorentzian signature for the bulk spacetime. The latter can be made nonvanishing by formulating the CCFT correlators with complexified $(z,\bar{z})$, i.e. $\bar{z} \neq z^*$, which amounts to consider holographic duals of bulk amplitudes written in the split metric signature $(-,+,-,+)$ [71].

Another important information in the CCFT that one can deduce is the form of the OPEs. The latter can be obtained from the bulk amplitudes using (6.1) and taking the collinear limit for the particles, which amounts to take the limit $(z_1,\bar{z}_1) \to (z_2,\bar{z}_2)$. The knowledge of the OPEs allows one to deduce new symmetries for scattering amplitudes, which includes the $w_{1+\infty}$ algebra [98,100,211]. An interesting observation for the current discussion is that the OPEs between the CCFT currents $P(z,\bar{z})$ and $T(z)$ can be deduced from the BMS charge algebra [158], the latter being interpreted as an algebra for Noether currents in the Carrollian CFT at null infinity. This constitutes a first important insight suggesting that the Carrollian CFT and the CCFT can be related to one another. We explore this idea in further details in the following section.

B. From Carrollian to celestial holography

In this section, we show that the celestial Ward identities associated to large gauge and BMS symmetries can be recovered from Carrollian correlation functions involving (quasi)conformal Carrollian primary source operators of specific weights.

1. Soft-photon theorem

We consider again here the holographic description of massless scalar electrodynamics and we use the objects defined in Sec. V B 2. The goal now is to deduce the celestial Ward identity (6.4) from the $U(1)$ invariance of Carrollian correlators (5.35) after some $\mathcal{B}$-transforms. We are giving more details about the simpler $U(1)$ case, as the gravity case proceeds similarly.

First, it is useful to express the soft-photon current $J(z)$ in terms of conformal Carrollian fields. Recalling (5.25), we write $\dot{\sigma}_z^{(A)} \equiv \partial_u \sigma_z^{(A)}$, which carries Carrollian weights $(\frac{3}{2}, \frac{1}{2})$ and $\dot{\sigma}_{\bar{z}}^{(A)} = (\dot{\sigma}_z^{(A)})^\dagger$. The soft flux (3.18) then reads

$$\mathcal{F}_\lambda^S = \int_{\hat{\mathscr{I}}} d^3x F_\lambda^S(s, z, \bar{z}) = \frac{1}{e^2} \int_\Sigma d^2z \lambda^{(0)} \left(\int_{-\infty}^{+\infty} du \partial_{\bar{z}} \dot{\sigma}_z^{(A)} - \int_{-\infty}^{+\infty} dv \partial_{\bar{z}} \dot{\sigma}_z^{(A)} \right) + \text{H.c.}, \tag{6.8}$$

where all objects have been promoted to quantum operators. Now using the electricity condition (3.12) after performing the time integrals explicitly,

$$\begin{aligned} \mathcal{F}_\lambda^S &= \frac{2}{e^2} \int_\Sigma d^2z \lambda^{(0)} \left(\int_{-\infty}^{+\infty} du \partial_{\bar{z}} \dot{\sigma}_z^{(A)} - \int_{-\infty}^{+\infty} dv \partial_{\bar{z}} \dot{\sigma}_z^{(A)} \right) \\ &= -\frac{2}{e^2} \int_\Sigma d^2z \partial_{\bar{z}} \lambda^{(0)} \left(\int_{-\infty}^{+\infty} du \dot{\sigma}_z^{(A)} - \int_{-\infty}^{+\infty} dv \dot{\sigma}_z^{(A)} \right) = \frac{1}{2\pi} \int_\Sigma d^2z \partial_{\bar{z}} \lambda^{(0)} J(z), \end{aligned} \tag{6.9}$$

using the expression of the soft-photon current [126,155]

$$J(z) \equiv -\frac{4\pi}{e^2} \left(\int_{-\infty}^{+\infty} du \dot{\sigma}_z^{(A)} - \int_{-\infty}^{+\infty} dv \dot{\sigma}_z^{(A)} \right). \tag{6.10}$$

This operator inserts a soft photon of positive helicity. Notice that by using the electricity condition to trade $\partial_{\bar{z}} \dot{\sigma}_z^{(A)}$ in favor of $\partial_z \dot{\sigma}_{\bar{z}}^{(A)}$, one can insert a soft photon of negative helicity instead by means of the Hermitian conjugated operator $\bar{J}(\bar{z})$. Starting from the Ward identity (5.35), splitting the flux $\mathcal{F}_\lambda = \mathcal{F}_\lambda^H + \mathcal{F}_\lambda^S$ as in (3.17) and (3.18) and using $\langle \mathcal{F}_\lambda^H X_N^\sigma \rangle = i\hbar \delta_\lambda^H \langle X_N^\sigma \rangle$ as a consequence of (3.19), we find

$$\frac{1}{i\hbar} \langle \mathcal{F}_\lambda^S X_N^\sigma \rangle + \delta_\lambda^H \langle X_N^\sigma \rangle = \frac{1}{2\pi i\hbar} \int_\Sigma d^2z \partial_{\bar{z}} \lambda^{(0)} \langle J(z) X_N^\sigma \rangle + \delta_\lambda^H \langle X_N^\sigma \rangle = 0, \tag{6.11}$$

owing to (6.9). Here $\delta_\lambda^H \langle X_N^\sigma \rangle$ represents the homogeneous part of the $U(1)$ transformation of the source operators, i.e.

$$\delta_\lambda^H \sigma^{\text{out/in}}_{(k_j, \bar{k}_j), Q_j}(u_j/v_j, z_j, \bar{z}_j) = \mp\, ieQ_j \lambda^{(0)}(z_j, \bar{z}_j) \sigma^{\text{out/in}}_{(k_j, \bar{k}_j), Q_j}(u_j/v_j, z_j, \bar{z}_j), \tag{6.12}$$

obtained explicitly from (4.28), (5.15), and (5.16). Particularizing for $\lambda^{(0)}(z, \bar{z}) = \frac{1}{z-w}$ and using the property

$$\delta^{(2)}(z - w) = \frac{1}{2\pi} \partial_{\bar{z}} \left(\frac{1}{z - w} \right), \tag{6.13}$$

we have

$$\frac{1}{i\hbar} \langle J(w) X_N^\sigma \rangle + ie \sum_{q=1}^N \frac{\eta_q Q_q}{w - z_q} \langle X_N^\sigma \rangle = 0, \tag{6.14}$$

where $\eta_q = \pm 1$ for incoming/outgoing insertions. The last step needed to translate this result into the celestial picture amounts to relating Carrollian outgoing and incoming source operators to the celestial operators by means of the $\mathcal{B}$-transform as in (2.65), i.e.

$$\boxed{\begin{aligned} \mathcal{O}^{\text{out}}_{(\Delta_i, J_i), Q_i}(z_i, \bar{z}_i) &= \kappa_\Delta^+ \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{du_i}{(u_i + i\epsilon)^{\Delta_i}} \sigma^{\text{out}}_{(k_i, \bar{k}_i), Q_i}(u_i, z_i, \bar{z}_i), \\ \mathcal{O}^{\text{in}}_{(\Delta_j, J_j), Q_j}(z_j, \bar{z}_j) &= \kappa_\Delta^- \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{dv_j}{(v_j - i\epsilon)^{\Delta_j}} \sigma^{\text{in}}_{(k_j, \bar{k}_j), Q_j}(v_j, z_j, \bar{z}_j), \end{aligned}} \tag{6.15}$$

which is also consistent with the extrapolate-style dictionary of [93]. Importantly, let us recall that, in our picture, the Carrollian sources that are $\mathcal{B}$-transformed in (6.15) are identified with the boundary values of bulk massless fields, namely implying that the $2d$ spins are implicitly fixed by Carrollian weights as (5.17). Denoting as before the set of celestial insertions by $\mathcal{X}_N$, one finally checks that (6.14) becomes (6.4), which is nothing but the celestial encoding of Weinberg's soft photon theorem.

Remark—Let us stress that, in this last step to relate Carrollian and celestial results, our proposal to exchange time and conformal dimension by means of (6.15) differs from the proposal of [55] to use the "modified Mellin transform" introduced in [208]. By construction, the $\mathcal{B}$-transform (2.63), defined in Sec. II C as the combination of Fourier and Mellin transforms acting on ladder operators, maps boundary values of bulk scattering fields onto celestial operators and vice versa thanks to the inversion formula (2.73). On the other hand, the modified Mellin transform maps functions in Fourier space onto functions depending both on (retarded) time and conformal dimension, which thus cannot be interpreted as boundary values of scattering fields. Nevertheless, since both integral transforms provide amplitudes which solve conformal Carrollian Ward identities, the link between the $\mathcal{B}$-transform and the modified Mellin transform would be worth exploring.

2. Soft graviton theorems

Now we turn our interest to scattering processes involving gravitons in asymptotically flat spacetime. Let $\langle X_N^\sigma\rangle$ be a conformal Carrollian correlator with N insertions, among which can be found either boundary values for the gravitational field [i.e. quasiconformal Carrollian primary source operators $\sigma_{zz}^{(g)}$ of weights $(\frac{3}{2},-\frac{1}{2})$] or null matter fields (i.e. conformal Carrollian primary source operators $\sigma^{(\phi)}$).

From the considerations of Sec. V C, we impose (5.33) for each conformal Carrollian transformation (4.31). The splitting in hard and soft variables induces a corresponding separation in the integrated fluxes as in (3.49). Considering first a supertranslation (by setting $\mathcal{Y}^z=0=\mathcal{Y}^{\bar z}$) and defining $\dot\sigma_{zz}^{(g)}\equiv\partial_u\sigma^{(g)}_{(\frac{3}{2},-\frac{1}{2})}$ and $\dot\sigma_{\bar z\bar z}^{(g)}=(\dot\sigma_{zz}^{(g)})^\dagger$, we promote the soft flux to the following quantum operator

$$\begin{aligned}\mathcal{F}^S_{\bar\xi(\mathcal{T},0)}&=\int_{\hat{\mathscr{I}}}d^3xF^S_{(\mathcal{T},0)}(s,z,\bar z)\\&=\frac{1}{16\pi G}\int d^2z\mathcal{T}\left(\int_{-\infty}^{+\infty}du\mathscr{D}_{\bar z}^2\dot\sigma_{zz}^{(g)}+\int_{-\infty}^{+\infty}dv\mathscr{D}_{\bar z}^2\dot\sigma_{zz}^{(g)}\right)+\text{H.c.}\\&=\frac{1}{8\pi G}\int_\Sigma d^2z\mathcal{T}\left(\int_{-\infty}^{+\infty}du\mathscr{D}_{\bar z}^2\dot\sigma_{zz}^{(g)}+\int_{-\infty}^{+\infty}dv\mathscr{D}_{\bar z}^2\dot\sigma_{zz}^{(g)}\right)=-\frac{1}{2\pi}\int d^2z\partial_{\bar z}\mathcal{T}P(z,\bar z).\end{aligned}\tag{6.16}$$

The second equality involves the electricity condition (3.41) after performing the time integrals, the third equality uses an integration by parts on the angles and the following definition the supertranslation current [125,151]

$$P(z,\bar z)\equiv\frac{1}{4G}\left(\int_{-\infty}^{+\infty}du\mathscr{D}_{\bar z}\dot\sigma_{zz}^{(g)}+\int_{-\infty}^{+\infty}dv\mathscr{D}_{\bar z}\dot\sigma_{zz}^{(g)}\right),\tag{6.17}$$

which inserts of a soft graviton of positive helicity. Notice again that by using the electricity condition to remove $\mathscr{D}_{\bar z}\dot\sigma_{zz}^{(g)}$ in favor of $\mathscr{D}_z\dot\sigma_{\bar z\bar z}^{(g)}$, we would insert a soft graviton of negative helicity instead. With this definition, (5.33) becomes

$$\frac{1}{i\hbar}\langle\mathcal{F}_{\bar\xi(\mathcal{T},0)}X_N^\sigma\rangle+\delta^H_{\bar\xi}\langle X_N^\sigma\rangle=-\frac{1}{2\pi i\hbar}\int_\Sigma d^2z\partial_{\bar z}\mathcal{T}\langle P(z,\bar z)X_N^\sigma\rangle+\delta^H_{\bar\xi}\langle X_N^\sigma\rangle=0,\tag{6.18}$$

where we used the factorization property (3.52) and $\delta^H_{\bar\xi}$ reproduces the homogeneous transformation (5.13). We have

$$-\frac{1}{2\pi i\hbar}\int_\Sigma d^2z\partial_{\bar z}\mathcal{T}\langle P(z,\bar z)X_N^\sigma\rangle+\left(\sum_{i=1}^n\mathcal{T}(z_i,\bar z_i)\partial_{u_i}+\sum_{j=n+1}^N\mathcal{T}(z_j,\bar z_j)\partial_{v_j}\right)\langle X_N^\sigma\rangle=0,\tag{6.19}$$

assuming that the first n fields are holographically identified with outgoing radiative modes. We now perform the $\mathcal{B}$-transforms

$$\mathcal{O}^{\text{out}}_{(\Delta_i,J_i)}(z_i,\bar{z}_i) = \kappa^+_\Delta \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} \frac{du_i}{(u_i+i\epsilon)^{\Delta_i}} \sigma^{\text{out}}_{(k_i,\bar{k}_i)}(u_i,z_i,\bar{z}_i),$$
$$\mathcal{O}^{\text{in}}_{(\Delta_j,J_j)}(z_j,\bar{z}_j) = \kappa^-_\Delta \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} \frac{dv_j}{(v_j-i\epsilon)^{\Delta_j}} \sigma^{\text{in}}_{(k_j,\bar{k}_j)}(v_j,z_j,\bar{z}_j), \tag{6.20}$$

such that

$$\delta^H_{\bar{\xi}(\mathcal{T},0)} \mathcal{O}^{\text{out/in}}_{(\Delta_j,J_j)}(z_j,\bar{z}_j) = \mp\, i\mathcal{T}(z_j,\bar{z}_j)\widehat{\partial}_{\Delta_j} \mathcal{O}^{\text{out/in}}_{(\Delta_j,J_j)}(z_j,\bar{z}_j), \tag{6.21}$$

in accordance with e.g. (2.31) and the holographic map (5.15) and (5.16). Notice that one also has

$$\delta^H_{\bar{\xi}(0,\mathcal{Y})} \mathcal{O}^{\text{out/in}}_{(\Delta_j,J_j)}(z_j,\bar{z}_j) = (\mathcal{Y}^{z_j}(z_j)\partial_{z_j} + \bar{\mathcal{Y}}^{\bar{z}_j}(\bar{z}_j)\partial_{\bar{z}_j} + h_j\partial_{z_j}\mathcal{Y}^{z_j} + \bar{h}_j\partial_{\bar{z}_j}\bar{\mathcal{Y}}^{\bar{z}_j}) \mathcal{O}^{\text{out/in}}_{(\Delta_j,J_j)}(z_j,\bar{z}_j), \tag{6.22}$$

where $h_j, \bar{h}_j$ are fixed as (6.2). Then particularizing (6.19) for $\mathcal{T}(z,\bar{z}) = \frac{1}{z-w}$, trading X^σ_N for $\mathcal{X}_N$ through (6.20) and using (6.13), we recover (6.5), namely the celestial encoding of the (leading) soft-graviton theorem.

The case of a holomorphic superrotation (setting $\mathcal{T} = 0 = \mathcal{Y}^{\bar{z}}$, the antiholomorphic case being analogous) taken as $\mathcal{Y}^z(z) = \frac{1}{z-w}$, can be considered in a similar fashion. The related soft flux is promoted as the following quantum operator

$$\mathcal{F}^S_{\bar{\xi}(0,\mathcal{Y})} = \int_{\hat{\mathscr{I}}} d^3x F^S_{\bar{\xi}(0,\mathcal{Y})}(s,z,\bar{z}) = -iT(w) \tag{6.23}$$

for the holomorphic stress tensor $T(z)$ [158,159]

$$T(z) \equiv \frac{i}{16\pi G} \int \frac{d^2w}{z-w} \left[\int_{-\infty}^{+\infty} du \left(u\mathscr{D}^3_w \dot{\sigma}^{(g)}_{\bar{w}\bar{w}} + \frac{3}{2}\sigma^{(g,0)}_{ww}\mathscr{D}_w\dot{\sigma}^{(g)}_{\bar{w}\bar{w}} + \frac{1}{2}\dot{\sigma}^{(g)}_{\bar{w}\bar{w}}\mathscr{D}_w\sigma^{(g,0)}_{ww} \right) + (u\mapsto v) \right], \tag{6.24}$$

where $\sigma^{(g,0)}_{ww}(w,\bar{w})$ denotes the quantum operator representing the soft variable $C^{(0)}_{ww}$ defined by (3.40). Then proceeding the same way as before and recalling (6.22), one recovers the $2d$ CFT Ward identity (6.6) from (5.34) after performing the integral transforms (6.20).

VII. DISCUSSION

In this paper, we have provided more details about the Carrollian approach to flat space holography. Let us summarize the steps of this proposal. First, consider gravity in $4d$ asymptotically flat spacetimes without radiation. In this case, the putative dual theory is an honest Carrollian CFT without external source. The insertions of operators in the correlators of this theory, denoted by Ψ^i, are typically components of the Carrollian momenta. This situation is similar to what is usually considered in AdS/CFT with Dirichlet boundary conditions, where the correlation functions in the CFT involve the holographic stress tensor. It is also very reminiscent of the situation arising in $3d$ asymptotically flat spacetimes where the bulk theory is topological.

The second step consists in introducing the radiation in $4d$ asymptotically flat spacetimes. In this case, a first observation is that the BMS charges are no longer conserved due to the radiation reaching null infinity. Therefore, if one identifies the BMS charges in the bulk theory with the Noether charges of the putative dual Carrollian CFT, something has to spoil the global symmetries in the dual theory to yield the nonconservation. In [60] and in the present paper, we have argued that the right setup to spoil the symmetries and encode the radiation at null infinity is to consider a sourced Carrollian CFT. The situation would be very similar to the case of AdS/CFT if one considered leaky boundary conditions instead of conservative boundary conditions such as the standard Dirichlet boundary conditions. In that case, the boundary metric, which plays the role of source, is allowed to fluctuate on the phase space and becomes a field of the dual theory. In the flat case, for pure gravity, the source operators σ^m correspond to the asymptotic shear and encode insertions of gravitons at null infinity. The correlators of source operators are therefore identified with $\mathcal{S}$-matrix elements in the bulk. In this sence, from the Carrollian perspective, the $\mathcal{S}$-matrix is described by the source sector of the theory. Obviously, this sector does not exist in the $3d$ case since there is no propagating degree of freedom and no scattering process occurring.

In the last part of the paper, we have then shown that the source sector of the Carrollian CFT could be related to the celestial CFT. More precisely, the source operators in the Carrollian CFT are mapped on the operators of the CCFT.

We end up this manuscript by providing future potentially interesting directions for this work.

(i) The formalism of sourced quantum field theory introduced in Sec. IV was applied in this work to flat space holography and its associated asymptotic dynamics of the bulk spacetime. We believe that this setup describing a sourced system could be applied to a much broader scope than the one presented here. Indeed, any gravitational system with leaky boundary conditions could in principle be holographically described using this formalism. For instance, it would be worth applying this framework in the case of hypersurfaces at finite distance (see e.g. [196,199,212–217]), such as black hole horizons, to deduce some insights on holography for finite spacetime regions.

(ii) As mentioned in the introduction, most of the results obtained in Carrollian holography are deduced from AdS/CFT by taking a flat limit in the bulk, leading to an ultrarelativistic limit at the boundary. However, as highlighted in [61–65,218], one has to start from leaky boundary conditions in $4d$ AdS if one wants to obtain radiative spacetimes in the limit. This contrasts with the standard Dirichlet boundary conditions that are usually considered in AdS/CFT. Therefore, the first step would be to obtain a holographic description of AdS spacetimes with leaky boundary conditions by coupling the dual CFT with some external sources and using the formalism of Sec. IV. Then, taking a flat limit in the bulk will imply an ultrarelativistic limit at the boundary. The hope is that one might get an explicit realization of the sourced Carrollian CFT using this procedure. It would also be interesting to revisit in this context the flat limit procedure of scattering amplitudes in AdS in the spirit of the works done in [219–222] and relate it to our setup.

(iii) The newly uncovered $w_{1+\infty}$ symmetries in the CCFT [98] have not yet been given a clear interpretation in the Carrollian CFT. As suggested by the recent analysis of [103], the information on these symmetries might be encoded in the subleading orders of the bulk metric. In our analysis, we have only considered Carrollian stress tensor or source operators. However, nothing would prevent us to consider Carrollian fields that are holographically identified with subleading orders in the expansion of the bulk metric. It might be instructive to revisit these $w_{1+\infty}$ symmetries in those terms and provide an interpretation at null infinity.

(iv) Finally, it would be a great progress if one could provide an explicit example of Carrollian CFT living at null infinity that would holographically capture some features of gravity in the bulk (for the celestial approach, see e.g. [223–226] for recent top-down models). A good starting point is the BMS geometric action constructed in [32] which, together with [227], furnishes an effective description of the dual Carrollian CFT for nonradiative spacetimes. The source sector of the Carrollian CFT is not yet known but we believe that the analysis provided in the present paper imposes strong constraints on it. For instance, the Carrollian weights of the source operators are completely fixed via (5.17). Moreover, the 2-point correlation function for source operators is identified with the bulk 2-point amplitude as in (5.23). In particular, this tells us that the propagator of source operators is u-dependent, which suggests that the source sector of the Carrollian CFT is a timelike electric-type of Carrollian theory [41,46,190,228,229].

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APPENDIX A: BONDI COORDINATES FOR MINKOWSKI SPACETIME

In the main text, we make an extensive use of the parametrization of Minkowski space by Bondi coordinates. We found convenient to work with a flat representative of the conformal boundary metric, which amounts to perform a boundary Weyl rescaling with respect to the usual choice of round boundary representative. The aim of this appendix is to install our conventions and notations regarding this choice of coordinates.

1. Bondi coordinates with round boundary representative

In retarded Bondi coordinates $\{u_\circ, r_\circ, z_\circ, \bar{z}_\circ\}$ ($u_\circ \in \mathbb{R}$, $r_\circ \in \mathbb{R}^+$, $z_\circ \in \mathbb{C}$), the Minkowski line elements reads as

$$ds^2 = -du_\circ^2 - 2du_\circ dr_\circ + \frac{4r_\circ^2}{(1+z_\circ\bar{z}_\circ)^2} dz_\circ d\bar{z}_\circ. \tag{A1}$$

Cuts of constant $u_\circ$ of future null infinity $\mathscr{I}^+ = \{r_\circ \to +\infty\}$ are spheres on which the line element is the unit round-sphere metric in stereographic coordinates $z_\circ = e^{i\varphi}\cot\frac{\theta}{2}$. The change of coordinates from the Cartesian chart $X^\mu = \{t, \vec{x}\}$ is given by

$$X^\mu = u_\circ \delta_0^\mu + r_\circ \frac{\sqrt{2}}{1+z_\circ\bar{z}_\circ} q^\mu(z_\circ, \bar{z}_\circ), \tag{A2}$$

where

$$q^\mu(z,\bar{z}) \equiv \frac{1}{\sqrt{2}}(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}) \tag{A3}$$

is the standard parametrization of a null direction pointing toward the direction $(z_\circ, \bar{z}_\circ)$ on the celestial sphere. In a similar way, one defines advanced Bondi coordinates $\{v_\circ, r'_\circ, z'_\circ, \bar{z}'_\circ\}$ $(v_\circ \in \mathbb{R},\ r'_\circ \in \mathbb{R}^+,\ z'_\circ \in \mathbb{C})$ in which the Minkowski line element is

$$ds^2 = -dv_\circ^2 + 2dv_\circ dr'_\circ + \frac{4r_{\circ'}^2}{(1+z'_\circ\bar{z}'_\circ)^2} dz'_\circ d\bar{z}'_\circ. \tag{A4}$$

In terms of the Cartesian coordinates, we have

$$X^\mu = v_\circ \delta_0^\mu - r'_\circ \frac{\sqrt{2}}{1+z'_\circ\bar{z}'_\circ} q^\mu(z'_\circ, \bar{z}'_\circ). \tag{A5}$$

For any finite value of $r_\circ, r'_\circ$, both Bondi coordinate systems are related as

$$v_\circ = u_\circ + 2r_\circ, \qquad r'_\circ = r_\circ, \qquad z'_\circ = -\frac{1}{\bar{z}_\circ}, \tag{A6}$$

where in particular the holomorphic coordinates are related by the antipodal map $(\theta,\varphi) \mapsto (\pi-\theta, \varphi+\pi)$, or $z_\circ \mapsto -\frac{1}{\bar{z}_\circ}$. This particularly convenient choice [151] allows one to label a light ray crossing the spacetime with the same value of the angular coordinates in both coordinate systems. In other words, a light ray originating from a point $z'_\circ = a$ on the celestial sphere at past null infinity will pierce again the celestial sphere at future null infinity at the antipodal point identified by $z_\circ = a$.

2. Bondi coordinates with flat boundary representative

We now introduce the coordinate system $\{u, r, z, \bar{z}\}$ by trading the unit round-sphere metric on the boundary for the flat complex plane metric. The diffeomorphism that implements the boundary Weyl rescaling has been worked out e.g. in [230] and reads as

$$r = \frac{\sqrt{2}}{1+z_\circ\bar{z}_\circ} r_\circ + \frac{u_\circ}{\sqrt{2}}, \qquad u = \frac{1+z_\circ\bar{z}_\circ}{\sqrt{2}} u_\circ - \frac{z_\circ\bar{z}_\circ u_\circ^2}{2r},$$
$$z = z_\circ - \frac{z_\circ u_\circ}{\sqrt{2}r}, \tag{A7}$$

for retarded coordinates. Here $u, r \in \mathbb{R}$ and $z \in \mathbb{C}$. The relation with Cartesian coordinates is now given by

$$X^\mu = u\partial_z\partial_{\bar{z}} q^\mu(z,\bar{z}) + r q^\mu(z,\bar{z}) \tag{A8}$$

and the Minkowski line element reads as

$$ds^2 = -2dudr + 2r^2 dz d\bar{z}. \tag{A9}$$

These coordinates are such that $\mathscr{I}^+$ and $\mathscr{I}^-$ are respectively obtained by taking the limits $r \to +\infty$ and $r \to -\infty$. Cuts of $\mathscr{I}^+$ are now complex planes endowed with a flat metric. Lines obtained by keeping $(u, z, \bar{z})$ fixed form a null geodesic congruence extending from past to future null infinity. The matching between past and future null infinities is therefore here immediate.

Similarly, we can also flatten the boundary representative in advanced Bondi coordinates to reach the coordinates $\{v, r', z', \bar{z}'\}$. The diffeomorphism implementing the boundary Weyl rescaling is given explicitly by

$$r' = \frac{\sqrt{2}}{1+z_\circ\bar{z}_\circ} r'_\circ - \frac{v_\circ}{\sqrt{2}}, \qquad v = \frac{1+z'_\circ\bar{z}'_\circ}{\sqrt{2}} v_\circ + \frac{z'_\circ\bar{z}'_\circ v_\circ^2}{2r'},$$
$$z' = z'_\circ + \frac{z'_\circ v_\circ}{\sqrt{2}r'}. \tag{A10}$$

In these coordinates, related to the Cartesian coordinates via

$$X^\mu = v\partial_{z'}\partial_{\bar{z}'} q^\mu(z',\bar{z}') - r' q^\mu(z',\bar{z}'), \tag{A11}$$

the Minkowski line element is

$$ds^2 = 2dvdr' + 2r'^2 dz' d\bar{z}'. \tag{A12}$$

Here $\mathscr{I}^+$ is reached in the limit $r' \to -\infty$ while $\mathscr{I}^-$ is obtained by taking $r' \to +\infty$. Inverting (A7) recalling that $r_\circ \in \mathbb{R}^+$ and taking into account the diffeomorphism (A6), one can show that (A10) yields

$$v = u, \qquad r' = -r, \qquad z' = z. \tag{A13}$$

The Bondi coordinates with flat boundary representative are asymptotically related to those with round boundary representative as follows. In the limit where $r_\circ$ goes to infinity in the retarded Bondi coordinates, we deduce from (A7) that

$$u = u_\circ \frac{1+z_\circ\bar{z}_\circ}{\sqrt{2}} + \mathcal{O}(r_\circ^{-1}), \qquad r = r_\circ \frac{\sqrt{2}}{1+z_\circ\bar{z}_\circ} + \mathcal{O}(r_\circ^0),$$
$$z = z_\circ + \mathcal{O}(r_\circ^{-1}), \tag{A14}$$

while, in the limit where $r'_\circ \equiv r_\circ$ goes to infinity in the advanced Bondi coordinates, (A10) together with the matching (A13) imply

$$v = v_\circ \frac{1 + z'_\circ \bar{z}'_\circ}{\sqrt{2}} + \mathcal{O}(r_\circ^{-1}), \qquad r = -r_\circ \frac{\sqrt{2}}{1 + z'_\circ \bar{z}'_\circ} + \mathcal{O}(r_\circ^0),$$
$$z = z'_\circ + \mathcal{O}(r_\circ^{-1}). \tag{A15}$$

In this sense, the coordinate system $\{u, r, z, \bar{z}\}$ with flat boundary representative interpolates between the advanced and retarded coordinate systems with boundary round representatives at large distances. In particular points situated at $z_\circ$ on the future celestial sphere are identified with points at $z'_\circ = -\frac{1}{\bar{z}_\circ}$ on the past celestial sphere through a null ray defined with constant $(u, z, \bar{z})$ such that $z \equiv z_\circ$ at future null infinity. Similar considerations apply for the advanced coordinate system up to performing the simple change of coordinates (A13).

APPENDIX B: ISOMORPHISM BETWEEN GLOBAL CONFORMAL CARROLLIAN AND POINCARÉ ALGEBRAS

In Cartesian coordinates $X^\mu = \{t, x^i\}$, the Poincaré generators on Minkowski spacetime are

(i) Translations: $\mathtt{P}_0 = \partial_t$, $\mathtt{P}_i = \partial_{x^i}$, $i = 1, 2, 3$.
(ii) Rotations: $\mathtt{R}_i = x^j \partial_{x^k} - x^k \partial_{x^j}$, $(i, j, k) = (1,2,3)$, (2, 1, 3), (3, 1, 2).
(iii) Special Lorentz transformations: $\mathtt{B}_i = x^i \partial_t + t \partial_{x^i}$, $i = 1, 2, 3$.

They satisfy the well-known $\mathfrak{iso}(3,1)$ algebra

$$[\mathtt{P}_i, \mathtt{P}_j] = 0 = [\mathtt{P}_0, \mathtt{P}_i], \qquad [\mathtt{R}_i, \mathtt{P}_j] = -\varepsilon_{ijk}\mathtt{P}_k, \qquad [\mathtt{R}_i, \mathtt{P}_0] = 0, [\mathtt{B}_i, \mathtt{P}_j] = -\delta_{ij}\mathtt{P}_0,$$
$$[\mathtt{B}_i, \mathtt{P}_0] = -\mathtt{P}_i, \qquad [\mathtt{R}_i, \mathtt{R}_j] = -\varepsilon_{ijk}\mathtt{R}_k, \qquad [\mathtt{R}_i, \mathtt{B}_j] = -\varepsilon_{ijk}\mathtt{B}_k, \qquad [\mathtt{B}_i, \mathtt{B}_j] = \varepsilon_{ijk}\mathtt{R}_k. \tag{B1}$$

Performing the change of coordinates to retarded Bondi gauge with flat conformal frame at the boundary, one finds that the Poincaré generators can be expressed as (4.31) on $\mathscr{I}^+$ with particular functions $\mathcal{T}, \mathcal{Y}^z$ and $\mathcal{Y}^{\bar{z}}$ given in Table I.

From the intrinsic point of view, these generators do not necessarily seem natural but can be related to the standard generators of a conformal Carrollian symmetry algebra (see Sec. IV D 1) thanks to the following isomorphism:

$$\mathfrak{CCarr}_3 \simeq \mathfrak{Conf}_2 \niplus \mathbb{R}^4 \simeq \mathfrak{so}(3,1) \niplus \mathbb{R}^4 \equiv \mathfrak{iso}(3,1). \tag{B2}$$

The dictionary is as follows:

$$P_0 = \frac{1}{\sqrt{2}}(\bar{\mathtt{P}}_0 + \bar{\mathtt{P}}_3), \qquad P_1 = -\frac{i}{2}(\bar{\mathtt{R}}_1 + i\bar{\mathtt{R}}_2 + i\bar{\mathtt{B}}_1 + \bar{\mathtt{B}}_2), \qquad P_2 = \frac{i}{2}(\bar{\mathtt{R}}_1 - i\bar{\mathtt{R}}_2 - i\bar{\mathtt{B}}_1 + \bar{\mathtt{B}}_2),$$
$$J = i\bar{\mathtt{R}}_3, D = \bar{\mathtt{B}}_3, \qquad B_1 = -\frac{1}{\sqrt{2}}(\bar{\mathtt{P}}_1 - i\bar{\mathtt{P}}_2), \qquad B_2 = -\frac{1}{\sqrt{2}}(\bar{\mathtt{P}}_1 + i\bar{\mathtt{P}}_2),$$
$$K_0 = -\sqrt{2}(\bar{\mathtt{P}}_0 - \bar{\mathtt{P}}_3), \qquad K_1 = -i(\bar{\mathtt{R}}_1 + i\bar{\mathtt{R}}_2 - i\bar{\mathtt{B}}_1 - \bar{\mathtt{B}}_2), \qquad K_2 = i(\bar{\mathtt{R}}_1 - i\bar{\mathtt{R}}_2 + i\bar{\mathtt{B}}_1 - \bar{\mathtt{B}}_2). \tag{B3}$$

The bar over the Poincaré generators means again their restriction to future null infinity. Applying the redefinitions (B3) on (B1) gives the algebra (4.32) and (4.33).

TABLE I. Poincaré generators in retarded Bondi gauge.

Generator	$\mathcal{T}(z,\bar{z})$	$\mathcal{Y}^z(z)$	$\mathcal{Y}^{\bar{z}}(\bar{z})$
$\mathtt{P}_0$	$\frac{1}{\sqrt{2}}(1+z\bar{z})$	0	0
$\mathtt{P}_1$	$-\frac{1}{\sqrt{2}}(z+\bar{z})$	0	0
$\mathtt{P}_2$	$\frac{1}{\sqrt{2}}i(z-\bar{z})$	0	0
$\mathtt{P}_3$	$\frac{1}{\sqrt{2}}(1-z\bar{z})$	0	0
$\mathtt{R}_1$	0	$\frac{1}{2}i(1-z^2)$	$-\frac{1}{2}i(1-\bar{z}^2)$
$\mathtt{R}_2$	0	$\frac{1}{2}(1+z^2)$	$\frac{1}{2}(1+\bar{z}^2)$
$\mathtt{R}_3$	0	iz	$-i\bar{z}$
$\mathtt{B}_1$	0	$\frac{1}{2}(1-z^2)$	$\frac{1}{2}(1-\bar{z}^2)$
$\mathtt{B}_2$	0	$\frac{1}{2}i(1+z^2)$	$-\frac{1}{2}i(1+\bar{z}^2)$
$\mathtt{B}_3$	0	z	$\bar{z}$

APPENDIX C: CONSTRAINTS ON THE CARROLLIAN STRESS TENSOR

In this appendix, we detail the proof of the identities (4.38) obeyed by the Carrollian stress tensor $\mathcal{C}^a{}_b$ at the classical level. All of them stem from the flux-balance law (4.37).

For Carrollian translations, $\bar{\xi}^a = \delta^a_b$, hence $\partial_a \mathcal{C}^a{}_b = F_b$ is immediate from (4.37). Invariance under Carrollian rotation imposes

$$0 = \partial_a(-\mathcal{C}^a{}_z z + \mathcal{C}^a{}_{\bar{z}}\bar{z}) + zF_z - \bar{z}F_{\bar{z}} = -z(\partial_a \mathcal{C}^a{}_z - F_z) + \bar{z}(\partial_a \mathcal{C}^a{}_{\bar{z}} - F_{\bar{z}}) - \mathcal{C}^z{}_z + \mathcal{C}^{\bar{z}}{}_{\bar{z}}, \tag{C1}$$

which gives $\mathcal{C}^z{}_z = \mathcal{C}^{\bar{z}}{}_{\bar{z}}$ using the invariance by translation, i.e. the first condition in (4.38). For the Carrollian boosts,

$$0 = \partial_a(\mathcal{C}^a{}_u x^A) - x^A F_u = (\partial_a \mathcal{C}^a{}_u - F_u)x^A + \mathcal{C}^A{}_u \Rightarrow \mathcal{C}^A{}_u = 0. \tag{C2}$$

The invariance under Carrollian dilatations finally gives

$$0 = \partial_a(\mathcal{C}^a{}_b x^b) - F_b x^b = (\partial_a \mathcal{C}^a{}_b - F_b)x^b + \mathcal{C}^a{}_a \Rightarrow \mathcal{C}^a{}_a = 0, \tag{C3}$$

which concludes the demonstration of (4.38).

It remains to show that the Carrollian special conformal transformations $K_0 = -2z\bar{z}\partial_u$, $K_1 = 2u\bar{z}\partial_u + 2\bar{z}^2\partial_{\bar{z}}$ and $K_2 = 2uz\partial_u + 2z^2\partial_z$ do not impose further constraints. Let us prove this statement for K_2 (the proof for K_0 and K_1 is similar). The flux-balance law is particularized as

$$\begin{aligned} 0 &= \partial_a(\mathcal{C}^a{}_u uz + \mathcal{C}^a{}_z z^2) - F_u uz - F_z z^2 \\ &= (\partial_a \mathcal{C}^a{}_u - F_u)uz + (\partial_a \mathcal{C}^a{}_z - F_z)z^2 + \mathcal{C}^u{}_u z + \mathcal{C}^z{}_u u + 2\mathcal{C}^z{}_z z \\ &= (\mathcal{C}^u{}_u + \mathcal{C}^z{}_z + \mathcal{C}^{\bar{z}}{}_{\bar{z}})z = 0. \end{aligned} \tag{C4}$$

The second equality holds by virtue of the invariance by translation and boosts, the third one holds thanks to the invariance by rotation while the last one uses the invariance by dilatation.

Let us finally prove that the global conformal Carrollian symmetries are enough to completely constrain $\mathcal{C}^a{}_b$, i.e. (4.37) is automatically satisfied by the pure supertranslation and superrotation currents provided (4.38) holds. Considering first a generic supertranslation $\bar{\xi}^u = \mathcal{T}(z, \bar{z})$, $\bar{\xi}^A = 0$, we have

$$0 = \partial_a(\mathcal{C}^a{}_u \mathcal{T}) - F_u \mathcal{T} = (\partial_a \mathcal{C}^a{}_u - F_u)\mathcal{T} + \mathcal{C}^A{}_u \partial_A \mathcal{T} = 0 \tag{C5}$$

using successively the invariance by Carrollian translations and boosts. For superrotations $\bar{\xi}^u = \frac{u}{2}\partial_A \mathcal{Y}^A$, $\bar{\xi}^z = \mathcal{Y}^z(z)$ and $\bar{\xi}^{\bar{z}} = \mathcal{Y}^{\bar{z}}(\bar{z})$, we have

$$\begin{aligned} &\partial_a\left(\mathcal{C}^a{}_u \frac{u}{2}\partial_A \mathcal{Y}^A + \mathcal{C}^a{}_A \mathcal{Y}^A\right) - F_u \frac{u}{2}\partial_A \mathcal{Y}^A - F_A \mathcal{Y}^A \\ &= \frac{u}{2}\partial_A \mathcal{Y}^A(\partial_a \mathcal{C}^a{}_u - F_u) + \mathcal{Y}^A(\partial_a \mathcal{C}^a{}_A - F_A) + \frac{1}{2}\mathcal{C}^u{}_u \partial_A \mathcal{Y}^A + \frac{u}{2}\mathcal{C}^B{}_u \partial_B \partial_A \mathcal{Y}^A + \mathcal{C}^B{}_A \partial_B \mathcal{Y}^A \\ &= \frac{1}{2}\mathcal{C}^u{}_u(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}}) + \mathcal{C}^z{}_z \partial_z \mathcal{Y}^z + \mathcal{C}^{\bar{z}}{}_{\bar{z}}\partial_{\bar{z}}\mathcal{Y}^{\bar{z}} \\ &= \frac{1}{2}\mathcal{C}^u{}_u(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}}) + \frac{1}{2}(\mathcal{C}^z{}_z + \mathcal{C}^{\bar{z}}{}_{\bar{z}})\partial_z \mathcal{Y}^z + \frac{1}{2}(\mathcal{C}^z{}_z + \mathcal{C}^{\bar{z}}{}_{\bar{z}})\partial_{\bar{z}}\mathcal{Y}^{\bar{z}} \\ &= \frac{1}{2}(\mathcal{C}^u{}_u + \mathcal{C}^z{}_z + \mathcal{C}^{\bar{z}}{}_{\bar{z}})(\partial_z \mathcal{Y}^z + \partial_{\bar{z}}\mathcal{Y}^{\bar{z}}) = 0. \end{aligned} \tag{C6}$$

The third equality uses the invariance by translations and boosts. It also uses the fact that superrotations are holomorphic. The fourth equality uses the invariance by rotation while the last one invokes the invariance by dilatation. This concludes the proof of (4.38).

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