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On the motive of the nested Quot scheme of points on a curve

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ABSTRACT

Let C be a smooth curve over an algebraically closed field $\mathbf{k}$, and let E be a locally free sheaf of rank r. We compute, for every $d > 0$, the generating function of the motives $[\mathrm{Quot}_C(E, \boldsymbol{n})] \in K_0(\mathrm{Var}_{\mathbf{k}})$, varying $\boldsymbol{n} = (0 \leq n_1 \leq \cdots \leq n_d)$, where $\mathrm{Quot}_C(E, \boldsymbol{n})$ is the *nested Quot scheme of points*, parametrising 0-dimensional subsequent quotients $E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1$ of fixed length $n_i = \chi(T_i)$. The resulting series, obtained by exploiting the Białynicki-Birula decomposition, factors into a product of shifted motivic zeta functions of C. In particular, it is a rational function.

0. Introduction

Let $K_0(\mathrm{Var}_{\mathbf{k}})$ be the Grothendieck ring of varieties over an algebraically closed field $\mathbf{k}$. If Y is a $\mathbf{k}$-variety, its *motivic zeta function*

$$\zeta_Y(q) = 1 + \sum_{n>0} \left[\mathrm{Sym}^n Y\right] q^n \in K_0(\mathrm{Var}_{\mathbf{k}})[\![q]\!]$$

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is a generating series introduced by Kapranov in [23], where he proved that for smooth curves it is a rational function in q.

In this paper we compute the motive of the *nested Quot scheme of points* $\mathrm{Quot}_C(E,\boldsymbol{n})$ on a smooth curve C, entirely in terms of $\zeta_C(q)$. Here, E is a locally free sheaf on C, and $\boldsymbol{n} = (0 \leq n_1 \leq \cdots \leq n_d)$ is a non-decreasing tuple of integers, for some fixed $d > 0$. The scheme $\mathrm{Quot}_C(E,\boldsymbol{n})$ generalises the classical Quot scheme of Grothendieck (recovered when $d = 1$): it parametrises flags of quotients $E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1$ where T_i is a 0-dimensional sheaf of length n_i.

Our main result, proved in Theorem 4.2 in the main body, is the following.

Theorem A. *Let C be a smooth curve over $\mathbf{k}$, let E be a locally free sheaf of rank r on C. Then*

$$\sum_{0\leq n_1\leq\cdots\leq n_d} \left[\mathrm{Quot}_C(E,\boldsymbol{n})\right] q_1^{n_1}\cdots q_d^{n_d} = \prod_{\alpha=1}^{r}\prod_{i=1}^{d} \zeta_C\left(\mathbb{L}^{\alpha-1} q_i q_{i+1}\cdots q_d\right)$$
$$\in K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1,\ldots,q_d]\!],$$

where $\mathbb{L} = [\mathbb{A}^1_{\mathbf{k}}]$ is the Lefschetz motive. In particular, this generating function is rational in $q_1,\ldots,q_d$.

The statement taken with $d = 1$, thus regarding the motive $[\mathrm{Quot}_C(E,n)]$ of the usual Quot scheme of points, was proved in [1]. Our result is a natural generalisation, which was inspired by Mochizuki's paper on "Filt schemes" [24].

Our formula fits nicely in the philosophical path according to which

"rank r theories factorise in r rank 1 theories".

There are to date a number of examples of this phenomenon in Donaldson–Thomas theory, exhibiting a generating series of rank r invariants as a product of r (suitably shifted) generating series of rank 1 invariants: see for instance [2,28] for enumerative DT invariants, [15] for K-theoretic DT invariants, [6,7] for motivic DT invariants and [26,14] for the parallel pictures in string theory.

The paper is organised as follows. In Section 1 we introduce the *nested Quot scheme* and prove its connectedness. In Section 2 we describe its tangent space and prove that, for a smooth quasiprojective curve, the nested Quot scheme is smooth. Under the assumption that the locally free sheaf is split, in Section 3 we describe a torus framing action and its associated Białynicki-Birula decomposition. In Section 4 we prove that the motive of the nested Quot scheme is independent of the locally free sheaf, and exploit the Białynicki-Birula decomposition to prove Theorem A. Our result readily implies closed formulae for the generating series of Hodge–Deligne polynomials, χ_y-genera, Poincaré polynomials, Euler characteristics, since these are all motivic measures; we provide some explicit formulae in Section 4.4.

After our paper was written, we were informed that our formula for the motive of the nested Quot scheme on a *projective* curve can be alternatively obtained, after some manipulations, from general results on the stack of iterated Hecke correspondences [17, Corollary 4.10] (see also [20, Section 3] for a related computation of the Voevodsky motive with rational coefficients). Our paper provides a direct and self-contained argument for this formula, exploiting the geometry of the nested Quot scheme.

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Conventions. All *schemes* are of finite type over an algebraically closed field $\mathbf{k}$. A *variety* is a reduced separated $\mathbf{k}$-scheme. If Y is a scheme and $Y_1, \ldots, Y_s$ are locally closed subschemes of Y, we say that they form a (locally closed) *stratification*, denoted '$Y = Y_1 \amalg \cdots \amalg Y_s$' with a slight abuse of notation, if the natural morphism of schemes $Y_1 \amalg \cdots \amalg Y_s \to Y$ is bijective. This is crucial in our calculations since this condition implies the identity $[Y] = [Y_1] + \cdots + [Y_s]$ in $K_0(\mathrm{Var}_{\mathbf{k}})$.

1. Nested Quot schemes of points

1.1. The moduli space

Let X be a quasiprojective $\mathbf{k}$-variety and E a coherent sheaf on X. Fix an integer $d > 0$ and a non-decreasing d-tuple $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ of non-negative integers $n_i \in \mathbb{Z}_{\geq 0}$. We define the *nested Quot functor* associated to $(X, E, \boldsymbol{n})$ to be the functor $\mathsf{Quot}_X(E, \boldsymbol{n})\colon \mathrm{Sch}_{\mathbf{k}}^{\mathrm{op}} \to \mathrm{Sets}$ sending a $\mathbf{k}$-scheme B to the set of isomorphism classes of subsequent quotients

$$E_B \twoheadrightarrow \mathcal{T}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{T}_1,$$

where E_B is the pullback of E along $X \times_{\mathbf{k}} B \to X$ and $\mathcal{T}_i \in \mathrm{Coh}(X \times_{\mathbf{k}} B)$ is a B-flat family of 0-dimensional sheaves of length n_i over X for all $i = 1, \ldots, d$. Two 'nested quotients'

$$E_B \twoheadrightarrow \mathcal{T}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{T}_1, \qquad E_B \twoheadrightarrow \mathcal{T}'_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{T}'_1$$

are considered isomorphic when $\ker(E_B \twoheadrightarrow \mathcal{T}_i) = \ker(E_B \twoheadrightarrow \mathcal{T}_i')$ for all $i = 1, \ldots, d$.

The representability of the functor $\mathsf{Quot}_X(E, \boldsymbol{n})$ can be proved adapting the proof of [29, Theorem 4.5.1] or by an explicit induction on d as in [21, Section 2.A.1]. We define $\mathrm{Quot}_X(E, \boldsymbol{n})$ to be the moduli scheme representing the above functor. Its closed points are then in bijection with the set of isomorphism classes of nested quotients

$$E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1,$$

where each $T_i \in \mathrm{Coh}(X)$ is a 0-dimensional quotient of E of length n_i. The nested Quot scheme comes with a closed immersion

$$\mathrm{Quot}_X(E, \boldsymbol{n}) \hookrightarrow \prod_{i=1}^{d} \mathrm{Quot}_X(E, n_i) \tag{1.1}$$

cut out by the nesting condition $\ker(E \twoheadrightarrow T_d) \hookrightarrow \ker(E \twoheadrightarrow T_{d-1}) \hookrightarrow \cdots \hookrightarrow \ker(E \twoheadrightarrow T_1)$. In particular, it is projective as soon as X is projective. If C is a smooth proper curve over $\mathbb{C}$ and $E \in \mathrm{Coh}(C)$ is a locally free sheaf, the cohomology of $\mathrm{Quot}_C(E, \boldsymbol{n})$ was studied by Mochizuki [24].

Example 1.1. The classical Quot scheme $\mathrm{Quot}_X(E, n)$ of length n quotients of E is obtained by setting $\boldsymbol{n} = (n)$, i.e. taking $d = 1$ and $n_d = n$. If we set $\boldsymbol{n} = (1 \leq 2 \leq \cdots \leq d)$, we obtain Mochizuki's *complete Filt scheme* $\mathrm{Filt}(E, d)$, which for $d = 1$ reduce to $\mathrm{Filt}(E, 1) = \mathbb{P}(E)$ [24]. When $E = \mathscr{O}_X$, we use the notation $\mathrm{Hilb}^{\boldsymbol{n}}(X)$ to denote $\mathrm{Quot}_X(\mathscr{O}_X, \boldsymbol{n})$. This space is the *nested Hilbert scheme of points*, studied extensively by Cheah [9,8,10].

1.2. Support map and nested punctual Quot scheme

Fix a variety X, a coherent sheaf E and a d-tuple of non-negative integers $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ for some $d > 0$. Composing the embedding (1.1) with the usual Quot-to-Chow morphisms yields the *support map*

$$\mathsf{h}_{E,\boldsymbol{n}} \colon \mathrm{Quot}_X(E, \boldsymbol{n}) \hookrightarrow \prod_{i=1}^{d} \mathrm{Quot}_X(E, n_i) \to \prod_{i=1}^{d} \mathrm{Sym}^{n_i}(X) \tag{1.2}$$

recording the 0-cycles $([\mathrm{Supp}\, T_i] \in \mathrm{Sym}^{n_i}(X))_{1 \leq i \leq d}$ attached to a d-tuple $(E \twoheadrightarrow T_i)_{1 \leq i \leq d}$. Here, $\mathrm{Sym}^m X = X^m / \mathfrak{S}_m$ is the m-th symmetric power of X.

We make the following definition.

Definition 1.2 *(Nested punctual Quot scheme).* Let X be a variety, $x \in X$ a point, $E \in \mathrm{Coh}(X)$ a coherent sheaf, $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ a tuple of non-negative integers. The *nested punctual Quot scheme* attached to $(X, E, \boldsymbol{n}, x)$ is the closed subscheme

$$\mathrm{Quot}_X(E,\boldsymbol{n})_x \subset \mathrm{Quot}_X(E,\boldsymbol{n}),$$

defined as the preimage of the cycle $(n_1x,\ldots,n_dx)$ along the support map $\mathsf{h}_{E,\boldsymbol{n}}$.

The name 'punctual' refers, as for the classical Quot schemes, to the fact that all quotients are entirely supported at a single point. We will not need the following result.

Lemma 1.3. *Let X be a smooth quasiprojective variety of dimension m, and let E be a locally free sheaf of rank r on X. For every d-tuple $\boldsymbol{n}=(n_1\leq\cdots\leq n_d)$, and for every $x\in X$, one has a non-canonical isomorphism*

$$\mathrm{Quot}_X(E,\boldsymbol{n})_x \cong \mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},\boldsymbol{n})_0.$$

Proof. The result follows from the isomorphism $\mathrm{Quot}_X(E,k)_x \xrightarrow{\sim} \mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},k)_0$ relating the classical punctual Quot schemes, which is proved in full detail in [27, Section 2.1] exploiting a choice of étale coordinates around x (which exist by the smoothness assumption, and which explain the non-canonical nature of the isomorphism). It remains to observe that the induced isomorphism

$$\prod_{i=1}^d \mathrm{Quot}_X(E,n_i)_x \xrightarrow{\sim} \prod_{i=1}^d \mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},n_i)_0$$

maps the subscheme $\mathrm{Quot}_X(E,\boldsymbol{n})_x$ isomorphically onto $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},\boldsymbol{n})_0$. □

1.3. Connectedness

We prove the following connectedness result for the nested Quot scheme. A proof in the case $(r,d,\boldsymbol{n})=(1,1,n)$ of the classical Hilbert scheme was first given by Hartshorne [19], and by Fogarty in the surface case [16]. We shall also exploit Cheah's connectedness result for $\mathrm{Hilb}^{\boldsymbol{n}}(X)$, see [9, Sec. 0.4].

Theorem 1.4. *If X is an irreducible quasiprojective* $\mathbf{k}$*-variety and E is a locally free sheaf on X, then $\mathrm{Quot}_X(E,\boldsymbol{n})$ is connected for every $\boldsymbol{n}=(n_1\leq\cdots\leq n_d)$. In particular, the classical Quot scheme $\mathrm{Quot}_X(E,n)$ is connected for every $n\geq 0$.*

Proof. The proof consists of several steps.

Step 1: We reduce to proving the statement when $E=\mathscr{O}_X^{\oplus r}$ is trivial. Let $x=[E\twoheadrightarrow T_d\twoheadrightarrow\cdots\twoheadrightarrow T_1]\in\mathrm{Quot}_X(E,\boldsymbol{n})$ be a point, where E is arbitrary. Since T_d is 0-dimensional we can find an open neighbourhood $U\subset X$ of the set-theoretic support of T_d such that $E|_U=\mathscr{O}_U^{\oplus r}$ is trivial. The point x then lies in the image of the open immersion $\mathrm{Quot}_U(\mathscr{O}_U^{\oplus r},\boldsymbol{n})\hookrightarrow\mathrm{Quot}_X(E,\boldsymbol{n})$. By assumption, the space $\mathrm{Quot}_U(\mathscr{O}_U^{\oplus r},\boldsymbol{n})$ is connected. Now if $x'=[E\twoheadrightarrow T'_d\twoheadrightarrow\cdots\twoheadrightarrow T'_1]\in\mathrm{Quot}_X(E,\boldsymbol{n})$ is another point, we can find another open subset $U' \subset X$ surrounding the support of T'_d and trivialising E. Since X is irreducible, $U \cap U' \neq \emptyset$, which implies $\mathrm{Quot}_U(\mathscr{O}_U^{\oplus r}, \boldsymbol{n}) \cap \mathrm{Quot}_{U'}(\mathscr{O}_{U'}^{\oplus r}, \boldsymbol{n}) \neq \emptyset$, so x and x' are connected in $\mathrm{Quot}_X(E, \boldsymbol{n})$ by any point in this intersection.

Step 2: The scheme $\mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})$ has a framing $\mathbf{T}$-action with non-empty fixed locus, where $\mathbf{T} = \mathbb{G}_m^r$ (see Proposition 3.1 for an explicit description of this fixed locus: we shall exploit it in the next step). Let $x \in \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})$ be an arbitrary point. Then the closure of its orbit contains a $\mathbf{T}$-fixed point — this will be explained in Section 3. Therefore it is enough to prove that any two $\mathbf{T}$-fixed points $x, x' \in \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})^{\mathbf{T}}$ are connected in $\mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})$.

Step 3: In principle, we should check connectedness for an *arbitrary* pair (x, x') of $\mathbf{T}$-fixed points

$$x = [\mathscr{O}_X^{\oplus r} \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1] \in \prod_{\alpha=1}^{r} \mathrm{Hilb}^{\boldsymbol{n}_\alpha}(X) \subset \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})^{\mathbf{T}},$$

$$x' = [\mathscr{O}_X^{\oplus r} \twoheadrightarrow T'_d \twoheadrightarrow \cdots \twoheadrightarrow T'_1] \in \prod_{\alpha=1}^{r} \mathrm{Hilb}^{\boldsymbol{n}'_\alpha}(X) \subset \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})^{\mathbf{T}},$$

where $\sum_{1 \leq \alpha \leq r} \boldsymbol{n}_\alpha = \boldsymbol{n} = \sum_{1 \leq \alpha \leq r} \boldsymbol{n}'_\alpha$. But since each nested Hilbert scheme $\mathrm{Hilb}^{\boldsymbol{m}}(X)$ is connected (cf. [9, Sec. 0.4]), we can in fact choose a pair of convenient x and x'. We fix them satisfying the condition that $\mathrm{Supp}(T_d), \mathrm{Supp}(T'_d)$ consist of n_d distinct points. When viewed in the full space $\mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})$, the points x and x' both belong to the open subset

$$U \subset \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n}),$$

defined by the cartesian diagram

$$\begin{array}{ccc} U & \longrightarrow & \prod_{i=1}^d (\mathrm{Sym}^{n_i} X \setminus \Delta_{\mathrm{big}}) \\ \big\downarrow & \square & \big\downarrow \text{open} \\ \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n}) & \xrightarrow{\mathsf{h}_{\mathscr{O}_X^{\oplus r}, \boldsymbol{n}}} & \prod_{i=1}^d \mathrm{Sym}^{n_i} X \end{array} \tag{1.3}$$

where $\Delta_{\mathrm{big}} \subset \mathrm{Sym}^{n_i} X$ is the big diagonal and the bottom map is the support map (1.2). In other words, $U \subset \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, \boldsymbol{n})$ is the open subscheme consisting of the flags of quotients $[\mathscr{O}_X^{\oplus r} \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1]$ where each T_i is supported on n_i distinct points. This yields an open immersion

$$U \hookrightarrow \prod_{i=1}^{d} V_i,$$

where $V_i \subset \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, n_i - n_{i-1})$ is the open subscheme consisting of points $[\mathscr{O}_X^{\oplus r} \twoheadrightarrow T_i']$ where the quotients T_i' are supported on $n_i - n_{i-1}$ distinct points (and we set $n_0 = 0$). The scheme V_i is the image of the étale map (cf. [2, Proposition A.3])

$$A_i \xrightarrow{\oplus} \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, n_i - n_{i-1})$$

defined on the open subscheme

$$A_i \subset \mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, 1)^{n_i - n_{i-1}}$$

parametrising quotients $(\mathscr{O}_X^{\oplus r} \twoheadrightarrow \mathscr{O}_{x_k})_k$ with $x_k \neq x_l$ for every $k \neq l$. On the other hand,

$$\mathrm{Quot}_X(\mathscr{O}_X^{\oplus r}, 1)^{n_i - n_{i-1}} \cong \mathbb{P}(\mathscr{O}_X^{\oplus r})^{n_i - n_{i-1}} \cong (X \times_{\mathbf{k}} \mathbb{P}^{r-1})^{n_i - n_{i-1}}$$

is irreducible, hence A_i is irreducible, and in particular V_i is irreducible, being the image of an irreducible space along a continuous map. Therefore $U \hookrightarrow \prod_i V_i$ is also irreducible, in particular connected, which completes the proof. $\square$

2. Tangent space and smoothness in the case of curves

Fix $(X, E, \boldsymbol{n})$ as in the previous section. For any point $x \in \mathrm{Quot}_X(E, \boldsymbol{n})$ representing a d-tuple of nested quotients

$$E \twoheadrightarrow T_d \xrightarrow{p_{d-1}} T_{d-1} \xrightarrow{p_{d-2}} \cdots \xrightarrow{p_2} T_2 \xrightarrow{p_1} T_1$$

we set $K_i = \ker(E \twoheadrightarrow T_i)$. We have a flag of subsheaves

$$K_d \xhookrightarrow{\iota_{d-1}} K_{d-1} \xhookrightarrow{\iota_{d-2}} \cdots \xhookrightarrow{\iota_2} K_2 \xhookrightarrow{\iota_1} K_1 \hookrightarrow E$$

and, for any $i = 1, \ldots, d-1$, maps

$$\begin{aligned} \phi_i \colon \mathrm{Hom}_X(K_i, T_i) &\to \mathrm{Hom}_X(K_{i+1}, T_i), \qquad && g \mapsto g \circ \iota_i \\ \psi_i \colon \mathrm{Hom}_X(K_{i+1}, T_{i+1}) &\to \mathrm{Hom}_X(K_{i+1}, T_i), \qquad && h \mapsto p_i \circ h \end{aligned}$$

which we assemble in a matrix

$$\Delta_x = \begin{pmatrix} -\phi_1 & \psi_1 & 0 & 0 & \cdots & 0 \\ 0 & -\phi_2 & \psi_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_{d-1} & \psi_{d-1} \end{pmatrix}$$

defining a map

$$\Delta_x\colon \bigoplus_{i=1}^{d} \operatorname{Hom}_X(K_i,T_i) \longrightarrow \bigoplus_{i=1}^{d-1} \operatorname{Hom}_X(K_{i+1},T_i).$$

The embedding (1.1) induces a $\mathbf{k}$-linear inclusion of tangent spaces

$$T_x \operatorname{Quot}_X(E,\boldsymbol{n}) \hookrightarrow \bigoplus_{i=1}^{d} \operatorname{Hom}_X(K_i,T_i),$$

which can be described as follows: a d-tuple of maps $(\delta_1,\dots,\delta_d) \in \bigoplus_{i=1}^{d} \operatorname{Hom}_X(K_i,T_i)$ belongs to the tangent space of $\operatorname{Quot}_X(E,\boldsymbol{n})$ at x precisely when the diagram

$$\begin{array}{ccccccccc} K_d & \xhookrightarrow{\iota_{d-1}} & K_{d-1} & \xhookrightarrow{\iota_{d-2}} & \cdots & \xhookrightarrow{\iota_2} & K_2 & \xhookrightarrow{\iota_1} & K_1 \\ \downarrow{\scriptstyle \delta_d} & & \downarrow{\scriptstyle \delta_{d-1}} & & & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_1} \\ T_d & \xrightarrow{p_{d-1}} & T_{d-1} & \xrightarrow{p_{d-2}} & \cdots & \xrightarrow{p_2} & T_2 & \xrightarrow{p_1} & T_1 \end{array} \tag{2.1}$$

commutes. This is formalised in terms of the map Δ_x in the next proposition.

Proposition 2.1. *Set $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$. The tangent space of $\operatorname{Quot}_X(E,\boldsymbol{n})$ at a point $x = [E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1]$ is*

$$T_x \operatorname{Quot}_X(E,\boldsymbol{n}) = \ker\left(\bigoplus_{i=1}^{d} \operatorname{Hom}(K_i,T_i) \xrightarrow{\Delta_x} \bigoplus_{i=1}^{d-1} \operatorname{Hom}(K_{i+1},T_i) \right).$$

In particular, if E is locally free of rank r on a smooth curve C, we have that $\operatorname{Quot}_C(E,\boldsymbol{n})$ is smooth of dimension $r \cdot n_d$.

Proof. Along the same lines of [29, Prop. 4.5.3(i)] it is easy to see that the tangent space is given by the maps making Diagram (2.1) commute, which is equivalent to belonging to the kernel of Δ_x.

Let Q_i be the 0-dimensional sheaf fitting in the exact sequences

$$0 \to K_i \to K_{i-1} \to Q_i \to 0$$
$$0 \to Q_i \to T_i \to T_{i-1} \to 0$$

for every $i = 1,\dots,d$. If $X = C$ is a smooth curve, we have that each K_i is a locally free sheaf of rank r (because torsion free is equivalent to locally free on smooth curves); since Q_i is a 0-dimensional sheaf, we obtain the vanishings

$$\operatorname{Ext}^j_C(K_i,T_i) = \operatorname{Ext}^j_C(K_{i+1},T_i) = \operatorname{Ext}^j_C(K_i,Q_i) = 0, \quad j > 0. \tag{2.2}$$

Therefore each ψ_i is a surjective map, which implies that Δ_x is surjective and that the dimension of the tangent space is computed as

$$\begin{aligned}\dim_{\mathbf{k}} T_x \operatorname{Quot}_C(E,\boldsymbol{n}) &= \dim_{\mathbf{k}}\left(\bigoplus_{i=1}^{d}\operatorname{Hom}_C(K_i,T_i)\right) - \dim_{\mathbf{k}}\left(\bigoplus_{i=1}^{d-1}\operatorname{Hom}_C(K_{i+1},T_i)\right)\\ &= \sum_{i=1}^{d} rn_i - \sum_{i=1}^{d-1} rn_i\\ &= rn_d.\end{aligned}$$

Since the tangent space dimension is constant and $\operatorname{Quot}_C(E,\boldsymbol{n})$ is connected by Theorem 1.4, it is enough to find a smooth open subset $U \subset \operatorname{Quot}_C(E,\boldsymbol{n})$ of dimension rn_d. We shall exploit the fact that the classical Quot scheme $\operatorname{Quot}_C(E,m)$ is smooth of dimension rm, which follows from standard deformation theory and the vanishing $\operatorname{Ext}^1_C(K,T) = \mathrm{H}^1(C, K^\vee \otimes T) = 0$ for an arbitrary point $[K \hookrightarrow E \twoheadrightarrow T] \in \operatorname{Quot}_C(E,m)$.

Let $U \subset \operatorname{Quot}_C(E,\boldsymbol{n})$ be the open subscheme as in Diagram (1.3) (which of course exists for arbitrary E), and write $U \hookrightarrow \prod_{i=1}^{d} V_i$ as in the proof of Theorem 1.4. We know that each $V_i \subset \operatorname{Quot}_C(E, n_i - n_{i-1})$ is smooth of dimension $r\cdot(n_i - n_{i-1})$, therefore U is smooth of dimension rn_d as required. □

Remark 2.2. The smoothness of $\operatorname{Quot}_C(E,\boldsymbol{n})$ was already proved by Mochizuki [24, Prop. 2.1], via a tangent-obstruction theory argument. See also [25] for the classification of smoothness of $\operatorname{Quot}_X(E,\boldsymbol{n})$ when X has arbitrary dimension.

3. Białynicki-Birula decomposition

Let E be a locally free sheaf of rank r on a variety X. Assume that $E = \bigoplus_{\alpha=1}^{r} L_\alpha$ splits into a sum of line bundles on X. Then $\operatorname{Quot}_X(E,\boldsymbol{n})$ admits the action of the algebraic torus $\mathbf{T} = \mathbb{G}_m^r$ as in [4]. Indeed, $\mathbf{T}$ acts diagonally on the product $\prod_{i=1}^{d}\operatorname{Quot}_X(E,n_i)$ and the closed subscheme $\operatorname{Quot}_X(E,\boldsymbol{n})$ is $\mathbf{T}$-invariant. Its fixed locus is determined by a straightforward generalisation of the main result of [4].

Proposition 3.1. *If $E = \bigoplus_{\alpha=1}^{r} L_\alpha$, there is a scheme-theoretic identity*

$$\operatorname{Quot}_X(E,\boldsymbol{n})^{\mathbf{T}} = \coprod_{\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r=\boldsymbol{n}}\ \prod_{\alpha=1}^{r}\operatorname{Quot}_X(L_\alpha,\boldsymbol{n}_\alpha).$$

Proof. We construct a bijection on $\mathbf{k}$-valued points, which is straightforward to verify in families.

Fix tuples $\boldsymbol{n}_\alpha = (n_{\alpha,1} \le \cdots \le n_{\alpha,d})$ such that $n_i = \sum_{1\le\alpha\le r} n_{\alpha,i}$ for every $i = 1,\ldots,d$. An element of the connected component corresponding to $(\boldsymbol{n}_1,\ldots,\boldsymbol{n}_r)$ in the right hand side is a tuple of nested quotients

$$\left([L_\alpha \twoheadrightarrow T_d^{(\alpha)} \twoheadrightarrow \cdots \twoheadrightarrow T_1^{(\alpha)}]\right)_{1\leq\alpha\leq r},$$

where each $T_i^{(\alpha)}$ is the structure sheaf of a finite subscheme of X of length $n_{\alpha,i}$. By Bifet's theorem on the $\mathbf{T}$-fixed locus of ordinary Quot schemes [4], we have that

$$\bigoplus_{1\leq\alpha\leq r}\left(L_\alpha \twoheadrightarrow T_i^{(\alpha)}\right) \in \operatorname{Quot}_X(E, n_i)^{\mathbf{T}} \tag{3.1}$$

for each $i = 1, \ldots, d$, and since each of the original tuples of quotients was nested according to $\boldsymbol{n}$, it follows that also the tuples (3.1) are nested according to $\boldsymbol{n}$, and this proves that (3.1) defines a point in $\operatorname{Quot}_X(E, \boldsymbol{n})^{\mathbf{T}}$.

The reverse inclusion follows by an analogous reasoning relying once more on Bifet's result [4]. □

Remark 3.2. For a locally free sheaf L of rank 1, we naturally have the isomorphism

$$\operatorname{Quot}_X(L, \boldsymbol{n}) \cong \operatorname{Hilb}^{\boldsymbol{n}}(X),$$

where $\operatorname{Hilb}^{\boldsymbol{n}}(X)$ is the nested Hilbert scheme of points, see for example [9]. Moreover, if $X = C$ is a smooth quasiprojective curve, we have (see [9, Sec. 0.2])

$$\operatorname{Hilb}^{\boldsymbol{n}}(C) \cong \operatorname{Sym}^{n_1}(C) \times \operatorname{Sym}^{n_2-n_1}(C) \times \cdots \times \operatorname{Sym}^{n_d-n_{d-1}}(C). \tag{3.2}$$

Assume now $X = C$ is a smooth quasiprojective curve and let $x \in \operatorname{Quot}_C(E, \boldsymbol{n})^{\mathbf{T}}$ be a $\mathbf{T}$-fixed point, corresponding to the tuple

$$\left([L_\alpha \twoheadrightarrow T_d^{(\alpha)} \twoheadrightarrow \cdots \twoheadrightarrow T_1^{(\alpha)}]\right)_\alpha \in \prod_{\alpha=1}^r \operatorname{Quot}_C(L_\alpha, \boldsymbol{n}_\alpha). \tag{3.3}$$

Set $K_i^{(\alpha)} = \ker(L_\alpha \twoheadrightarrow T_i^{(\alpha)})$. The tangent space at x can be written as

$$\begin{aligned} &T_x \operatorname{Quot}_C(E, \boldsymbol{n}) \\ &= \ker\left(\bigoplus_{1\leq\alpha,\beta\leq r}\bigoplus_{i=1}^d \operatorname{Hom}_C\left(K_i^{(\alpha)}, T_i^{(\beta)}\right) \xrightarrow{\Delta_x} \bigoplus_{1\leq\alpha,\beta\leq r}\bigoplus_{i=1}^{d-1} \operatorname{Hom}_C\left(K_{i+1}^{(\alpha)}, T_i^{(\beta)}\right)\right). \end{aligned} \tag{3.4}$$

Denote by $w_1, \ldots, w_r$ the coordinates of the algebraic torus $\mathbf{T}$, which we see as irreducible $\mathbf{T}$-characters. As a $\mathbf{T}$-representation, the tangent space admits the following weight decomposition

$$T_x \operatorname{Quot}_C(E,\boldsymbol{n}) = \ker\left(\bigoplus_{1\le\alpha,\beta\le r}\bigoplus_{i=1}^{d}\operatorname{Hom}_C\big(K_i^{(\alpha)}\otimes w_\alpha, T_i^{(\beta)}\otimes w_\beta\big)\right.$$
$$\left.\xrightarrow{\Delta_x}\bigoplus_{1\le\alpha,\beta\le r}\bigoplus_{i=1}^{d-1}\operatorname{Hom}_C\big(K_{i+1}^{(\alpha)}\otimes w_\alpha, T_i^{(\beta)}\otimes w_\beta\big)\right).$$

We recall the classical result of Białynicki-Birula (see [3, Section 4]), by which we obtain a decomposition of $\operatorname{Quot}_X(E,\boldsymbol{n})$ in the case when E is completely decomposable.

Theorem 3.3 *(Białynicki-Birula). Let X be a smooth projective scheme with a $\mathbb{G}_m$-action and let $\{X_i\}_i$ be the connected components of the $\mathbb{G}_m$-fixed locus $X^{\mathbb{G}_m}\subset X$. Then there exists a locally closed stratification $X=\coprod_i X_i^+$, such that each $X_i^+\to X_i$ is an affine fibre bundle. Moreover, for every closed point $x\in X_i$, the tangent space is given by $T_x(X_i^+) = T_x(X)^{\mathsf{fix}}\oplus T_x(X)^+$, where $T_x(X)^{\mathsf{fix}}$ (resp. $T_x(X)^+$) denotes the $\mathbb{G}_m$-fixed (resp. positive) part of $T_x(X)$. In particular, the relative dimension of $X_i^+\to X_i$ is equal to $\dim T_x(X)^+$ for $x\in X_i$.*

The Białynicki-Birula “strata” are constructed as follows. If t denotes the coordinate of $\mathbb{G}_m$, we have

$$X_i^+ = \left\{\, x\in X \;\middle|\; \lim_{t\to 0} t\cdot x\in X_i \,\right\}.$$

In particular, the properness assumption assures that the closure of each $\mathbb{G}_m$-orbit in X contains the $\mathbb{G}_m$-fixed point $\lim_{t\to 0} t\cdot x$. Recently Jelisiejew–Sienkiewicz [22] generalised Theorem 3.3, proving the X_i^+ always exists even when X is not projective (or even not smooth). However, in the smooth case they cover X as long as the closure of every $\mathbb{G}_m$-orbit contains a fixed point.

We now determine a Białynicki-Birula decomposition for $\operatorname{Quot}_C(E,\boldsymbol{n})$, where C is a smooth *quasiprojective* curve. See Mochizuki's paper [24, Section 2.3.4] for an equivalent construction and tangent space calculation (in the projective case), using a slightly different, but technically equivalent, tangent complex.[1]

Let $\mathbb{G}_m\hookrightarrow\mathbf{T}$ be the generic 1-parameter subtorus given by $w\mapsto(w,w^2,\dots,w^r)$; it is clear that $\operatorname{Quot}_C(E,\boldsymbol{n})^{\mathbf{T}} = \operatorname{Quot}_C(E,\boldsymbol{n})^{\mathbb{G}_m}$. Let

$$Q_{\underline{\boldsymbol{n}}} = \prod_{\alpha=1}^{r}\operatorname{Quot}_C(L_\alpha,\boldsymbol{n}_\alpha)\subset\operatorname{Quot}_C(E,\boldsymbol{n})^{\mathbb{G}_m}$$

be the connected component of the fixed locus corresponding to the r-tuple $\underline{\boldsymbol{n}} = (\boldsymbol{n}_\alpha)_{1\le\alpha\le r}$ decomposing $\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r=\boldsymbol{n}$.

[1] We thank Takuro Mochizuki for kindly sharing with us a note proving that the tangent complex used in [24] is quasi-isomorphic to the one encoded by the map Δ_x.

Proposition 3.4. *Let C be a smooth quasiprojective curve and $E = \bigoplus_{\alpha=1}^r L_\alpha$. Then the nested Quot scheme admits a locally closed stratification*

$$\operatorname{Quot}_C(E,\boldsymbol{n}) = \coprod_{\underline{\boldsymbol{n}}} Q^+_{\underline{\boldsymbol{n}}},$$

where $\underline{\boldsymbol{n}} = (\boldsymbol{n}_\alpha)_{1\le\alpha\le r}$ are such that $\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r = \boldsymbol{n}$ and $Q^+_{\underline{\boldsymbol{n}}} \to Q_{\underline{\boldsymbol{n}}}$ is an affine fibre bundle of relative dimension $\sum_{1\le\alpha\le r}(\alpha-1)n_{\alpha,d}$.

Proof. The strata $Q^+_{\underline{\boldsymbol{n}}}$ are induced by Theorem 3.3 — we just need to show that the closure of every orbit contains a fixed point. Choose a compactification $C \hookrightarrow \overline{C}$, an extension $\overline{L}_\alpha$ of each line bundle L_α and consider the induced open immersion

$$\operatorname{Quot}_C\left(\bigoplus_{\alpha=1}^r L_\alpha, \boldsymbol{n}\right) \hookrightarrow \operatorname{Quot}_{\overline{C}}\left(\bigoplus_{\alpha=1}^r \overline{L}_\alpha, \boldsymbol{n}\right).$$

The closure of every orbit must contain a fixed point in $\operatorname{Quot}_{\overline{C}}\left(\bigoplus_{\alpha=1}^r \overline{L}_\alpha, \boldsymbol{n}\right)$, but the $\mathbb{G}_m$-action does not move the support of a nested quotient, by which we conclude that such a fixed point had to be already contained in $\operatorname{Quot}_C\left(\bigoplus_{\alpha=1}^r L_\alpha, \boldsymbol{n}\right)$.

Let $x \in Q_{\underline{\boldsymbol{n}}}$ be a fixed point as in (3.3). The positive part of the tangent space (3.4) is

$$T_x^+ \operatorname{Quot}_C(E,\boldsymbol{n}) = \ker\left(\bigoplus_{\alpha<\beta}\bigoplus_{i=1}^d \operatorname{Hom}_C\big(K_i^{(\alpha)}, T_i^{(\beta)}\big) \xrightarrow{\Delta_x^+} \bigoplus_{\alpha<\beta}\bigoplus_{i=1}^{d-1} \operatorname{Hom}_C\big(K_{i+1}^{(\alpha)}, T_i^{(\beta)}\big)\right),$$

where Δ_x^+ is the restriction of the map Δ_x. Thanks to the vanishings (2.2), Δ_x^+ is surjective, therefore the relative dimension is computed as

$$\begin{aligned}
\dim_{\mathbf{k}} T_x^+ \operatorname{Quot}_C(E,\boldsymbol{n}) &= \dim_{\mathbf{k}}\left(\bigoplus_{\alpha<\beta}\bigoplus_{i=1}^d \operatorname{Hom}_C\big(K_i^{(\alpha)}, T_i^{(\beta)}\big)\right) \\
&\qquad - \dim_{\mathbf{k}}\left(\bigoplus_{\alpha<\beta}\bigoplus_{i=1}^{d-1} \operatorname{Hom}_C\big(K_{i+1}^{(\alpha)}, T_i^{(\beta)}\big)\right) \\
&= \sum_{\alpha<\beta}\left(\sum_{i=1}^d n_{\beta,i} - \sum_{i=1}^{d-1} n_{\beta,i}\right) \\
&= \sum_{\beta=1}^r (\beta-1) n_{\beta,d}
\end{aligned}$$

where we used $n_{\beta,i} = \dim_{\mathbf{k}} \operatorname{Hom}_C(K_i^{(\alpha)}, T_i^{(\beta)})$ since $K_i^{(\alpha)} = \ker(L_\alpha \twoheadrightarrow T_i^{(\alpha)})$ has rank 1. The proof is complete. □

4. The motive of the nested Quot scheme on a curve

4.1. Grothendieck ring of varieties

Let B be a scheme locally of finite type over $\mathbf{k}$. The *Grothendieck group of B-varieties*, denoted $K_0(\mathrm{Var}_B)$, is defined to be the free abelian group generated by isomorphism classes $[X \to B]$ of finite type B-varieties, modulo the scissor relations, namely the identities $[h\colon X \to B] = [h|_Z\colon Z \to B] + [h|_{X\setminus Z}\colon X \setminus Z \to B]$ whenever $Z \hookrightarrow X$ is a closed B-subvariety of X. The neutral element for the addition operation is the class of the empty variety. The operation

$$[X \to B] \cdot [X' \to B] = [X \times_B X' \to B]$$

defines a ring structure on $K_0(\mathrm{Var}_B)$, with identity $\mathbb{1}_B = [\mathrm{id}\colon B \to B]$. Therefore $K_0(\mathrm{Var}_B)$ is called the *Grothendieck ring of B-varieties*. If $B = \operatorname{Spec} \mathbf{k}$, we write $K_0(\mathrm{Var}_{\mathbf{k}})$ instead of $K_0(\mathrm{Var}_{\operatorname{Spec}\mathbf{k}})$, and we shorten $[X] = [X \to \operatorname{Spec}\mathbf{k}]$ for every $\mathbf{k}$-variety X.

The main rules for calculations in $K_0(\mathrm{Var}_{\mathbf{k}})$ are the following:

(1) If $X \to Y$ is a geometric bijection, i.e. a bijective morphism, then $[X] = [Y]$.
(2) If $X \to Y$ is Zariski locally trivial with fibre F, then $[X] = [Y] \cdot [F]$.

These are, indeed, the only properties that we will use.

The *Lefschetz motive* is the class $\mathbb{L} = [\mathbb{A}^1_{\mathbf{k}}] \in K_0(\mathrm{Var}_{\mathbf{k}})$. It can be used to express, for instance, the class of the projective space, namely $[\mathbb{P}^n_{\mathbf{k}}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n \in K_0(\mathrm{Var}_{\mathbf{k}})$.

4.2. Independence of the vector bundle

The following result generalises [27, Corollary 2.5], which in turn generalises the main theorem of [1] extending it from proper smooth curves to arbitrary smooth varieties.

Proposition 4.1. *Let E be a locally free sheaf of rank r on a $\mathbf{k}$-variety X. For every $\boldsymbol{n}$, the motivic class of $\mathrm{Quot}_X(E,\boldsymbol{n})$ is independent of E, that is*

$$\left[\mathrm{Quot}_X(E,\boldsymbol{n})\right] = \left[\mathrm{Quot}_X(\mathscr{O}_X^{\oplus r},\boldsymbol{n})\right] \in K_0(\mathrm{Var}_{\mathbf{k}}).$$

Proof. Let $(U_k)_{1\le k\le e}$ be a Zariski open cover trivialising E. We can refine it to a locally closed stratification $X = W_1 \amalg \cdots \amalg W_e$ such that $W_k \subset U_k$, so that in particular $E|_{W_k} = \mathscr{O}_{W_k}^{\oplus r}$ for every k. Each W_k is taken with the reduced induced scheme structure.

Let $\mathrm{Quot}_{X,W_k}(E,\boldsymbol{n}) \subset \mathrm{Quot}_X(E,\boldsymbol{n})$ be the preimage of $\mathrm{Sym}^{n_d}(W_k) \subset \mathrm{Sym}^{n_d}(X)$ along the projection

$$\mathrm{pr}_d \circ \mathsf{h}_{E,\boldsymbol{n}} \colon \mathrm{Quot}_X(E, \boldsymbol{n}) \to \prod_{i=1}^{d} \mathrm{Sym}^{n_i}(X) \to \mathrm{Sym}^{n_d}(X),$$

where $\mathsf{h}_{E,\boldsymbol{n}}$ is the support map (1.2). We endow $\mathrm{Quot}_{X,W_k}(E, \boldsymbol{n})$ with the reduced scheme structure. We have a geometric bijection

$$\coprod_{\boldsymbol{n}_1+\cdots+\boldsymbol{n}_e=\boldsymbol{n}} \prod_{k=1}^{e} \mathrm{Quot}_{X,W_k}(E, \boldsymbol{n}_k) \to \mathrm{Quot}_X(E, \boldsymbol{n}),$$

therefore the motive $[\mathrm{Quot}_X(E, \boldsymbol{n})]$ is computed entirely in terms of the motives $[\mathrm{Quot}_{X,W_k}(E, \boldsymbol{n}_k)]$. It is enough to prove that these are independent of E. In the cartesian diagram

$$\begin{array}{ccc} \mathrm{Quot}_{U_k,W_k}(E|_{U_k}, \boldsymbol{n}_k) & \xhookrightarrow{\ j\ } & \mathrm{Quot}_{X,W_k}(E, \boldsymbol{n}_k) \\ \big\downarrow & \square & \big\downarrow \\ \mathrm{Quot}_{U_k}(E|_{U_k}, \boldsymbol{n}_k) & \xhookrightarrow{\text{open}} & \mathrm{Quot}_X(E, \boldsymbol{n}_k) \end{array}$$

the open immersion j is in fact surjective, hence an isomorphism. But we can repeat this process with $\mathscr{O}_X^{\oplus r}$ in the place of E. It follows that

$$\mathrm{Quot}_{X,W_k}(E, \boldsymbol{n}_k) \cong \mathrm{Quot}_{U_k,W_k}(\mathscr{O}_{U_k}^{\oplus r}, \boldsymbol{n}_k) \cong \mathrm{Quot}_{X,W_k}(\mathscr{O}_X^{\oplus r}, \boldsymbol{n}_k),$$

which yields the result. $\square$

4.3. Proof of the main theorem

Let X be a smooth quasiprojective variety and E a locally free sheaf of rank r. Define

$$\mathsf{Z}_{X,r,d}(\boldsymbol{q}) = \sum_{\boldsymbol{n}} \left[\mathrm{Quot}_X(E, \boldsymbol{n})\right] \boldsymbol{q}^{\boldsymbol{n}} \in K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!],$$

where $\boldsymbol{n} = (n_1 \le \cdots \le n_d)$ and we use the multivariable notation $\boldsymbol{q} = (q_1, \ldots, q_d)$ and $\boldsymbol{q}^{\boldsymbol{n}} = \prod_{i=1}^d q^{n_i}$. The notation $\mathsf{Z}_{X,r,d}$ reflects the independence on E that we proved in Proposition 4.1. If $X = C$ is a smooth quasiprojective curve and $r = d = 1$, then $\mathsf{Z}_{C,1,1}(q)$ is simply the Kapranov motivic zeta function

$$\mathsf{Z}_{C,1,1}(q) = \zeta_C(q) = \sum_{n \ge 0} \left[\mathrm{Sym}^n(C)\right] q^n. \tag{4.1}$$

We can now prove our main theorem, first stated in Theorem A in the Introduction.

Theorem 4.2. *Let C be a smooth quasiprojective curve. The generating series $\mathsf{Z}_{C,r,d}(q)$ is a product of shifted motivic zeta functions: there is an identity*

$$\mathsf{Z}_{C,r,d}(\boldsymbol{q}) = \prod_{\alpha=1}^{r}\prod_{i=1}^{d}\zeta_C\left(\mathbb{L}^{\alpha-1}q_iq_{i+1}\cdots q_d\right).$$

In particular, $\mathsf{Z}_{C,r,d}(\boldsymbol{q})$ is a rational function in $q_1,\ldots,q_d$.

Proof. By Proposition 4.1 the motive $[\operatorname{Quot}_C(E,\boldsymbol{n})]$ is independent on the vector bundle E, so we may assume $E=\mathscr{O}_C^{\oplus r}$. In this case, we may compute the motive exploiting the decomposition of $\operatorname{Quot}_C(\mathscr{O}_C^{\oplus r},\boldsymbol{n})$ given by Proposition 3.4. Every stratum is a Zariski locally trivial fibration over a connected component of the fixed locus, with fibre an affine space whose dimension we computed in Proposition 3.4.

In what follows, we denote by $\boldsymbol{n}_\alpha=(n_{\alpha,1}\leq\cdots\leq n_{\alpha,d})$ a nested tuple of non-negative integers and by $\boldsymbol{l}_\alpha=(l_{\alpha,1},\ldots,l_{\alpha,d})$ a tuple of non-negative integers. Clearly the two collections of tuples are in bijection, by means of the correspondence

$$(n_{\alpha,1}\leq\cdots\leq n_{\alpha,d})\longleftrightarrow(n_{\alpha,1},n_{\alpha,2}-n_{\alpha,1},\ldots,n_{\alpha,d}-n_{\alpha,d-1}). \tag{4.2}$$

We compute

$$\begin{aligned}
&\sum_{\boldsymbol{n}}\left[\operatorname{Quot}_C(\mathscr{O}_C^{\oplus r},\boldsymbol{n})\right]\boldsymbol{q}^{\boldsymbol{n}}\\
&=\sum_{\boldsymbol{n}}\boldsymbol{q}^{\boldsymbol{n}}\sum_{\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r=\boldsymbol{n}}\prod_{\alpha=1}^{r}\left[\operatorname{Quot}_C(\mathscr{O}_C,\boldsymbol{n}_\alpha)\right]\cdot\mathbb{L}^{(\alpha-1)n_{\alpha,d}} && \text{by Proposition 3.4}\\
&=\sum_{\boldsymbol{n}_1,\ldots,\boldsymbol{n}_r}\prod_{\alpha=1}^{r}\boldsymbol{q}^{\boldsymbol{n}_\alpha}\left[\operatorname{Hilb}^{\boldsymbol{n}_\alpha}(C)\right]\cdot\mathbb{L}^{(\alpha-1)n_{\alpha,d}}\\
&=\sum_{\boldsymbol{l}_1,\ldots,\boldsymbol{l}_r}\prod_{\alpha=1}^{r}\left(\prod_{i=1}^{d}q_i^{\sum_{j=1}^{i}l_{\alpha,j}}\right)\cdot\left[\operatorname{Hilb}^{\boldsymbol{n}_\alpha}(C)\right]\cdot\mathbb{L}^{(\alpha-1)\sum_{i=1}^{d}l_{\alpha,i}} && \text{by (4.2)}\\
&=\sum_{\boldsymbol{l}_1,\ldots,\boldsymbol{l}_r}\prod_{\alpha=1}^{r}\prod_{i=1}^{d}q_i^{\sum_{j=1}^{i}l_{\alpha,j}}\cdot\left[\operatorname{Sym}^{l_{\alpha,i}}(C)\right]\cdot\mathbb{L}^{(\alpha-1)l_{\alpha,i}} && \text{by (3.2)}\\
&=\sum_{\boldsymbol{l}_1,\ldots,\boldsymbol{l}_r}\prod_{\alpha=1}^{r}\prod_{i=1}^{d}(q_iq_{i+1}\cdots q_d)^{l_{\alpha,i}}\cdot\left[\operatorname{Sym}^{l_{\alpha,i}}(C)\right]\cdot\mathbb{L}^{(\alpha-1)l_{\alpha,i}}\\
&=\prod_{\alpha=1}^{r}\prod_{i=1}^{d}\zeta_C\left(\mathbb{L}^{\alpha-1}q_iq_{i+1}\cdots q_d\right) && \text{by (4.1).}
\end{aligned}$$

The rationality follows by the rationality of the Kapranov zeta function, proved in [23, Theorem 1.1.9]. □

Remark 4.3. We can reformulate our main theorem in terms of the motivic exponential, for which a minimal background is provided in Appendix A. The case $r = d = 1$ yields the classical expression

$$\zeta_C(q) = \operatorname{Exp}_+([C]q).$$

The general case becomes

$$\begin{aligned}\mathsf{Z}_{C,r,d}(\boldsymbol{q}) &= \operatorname{Exp}_+\left([C]\sum_{\alpha=1}^{r}\mathbb{L}^{\alpha-1}\sum_{i=1}^{d}q_iq_{i+1}\cdots q_d\right)\\ &= \operatorname{Exp}_+\left([C\times_{\mathbf{k}}\mathbb{P}^{r-1}_{\mathbf{k}}]\sum_{i=1}^{d}q_iq_{i+1}\cdots q_d\right).\end{aligned}$$

Setting $d = 1$ we recover the calculations of [1,27].

4.4. Hodge–Deligne polynomial

In this subsection we work over $\mathbf{k} = \mathbb{C}$. Ring homomorphisms $K_0(\mathrm{Var}_{\mathbb{C}}) \to R$ are called *motivic measures*. A typical example of a motivic measure is the Hodge–Deligne polynomial

$$\mathsf{E}\colon K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u,v],$$

defined by sending the class $[Y]$ of a smooth projective variety[2] Y to

$$\mathsf{E}(Y;u,v) = \sum_{p,q\geq 0}\dim_{\mathbb{C}}\mathrm{H}^q(Y,\Omega^p_Y)(-u)^p(-v)^q.$$

Notation 4.4. If $f(u,v) = \sum_{i,j}p_{ij}u^iv^j \in \mathbb{Z}[u,v]$, we set

$$(1-q)^{-f(u,v)} = \prod_{i,j}\left(1-u^iv^jq\right)^{-p_{ij}}.$$

This is actually the formula defining the *power structure* on $\mathbb{Z}[u,v]$. The motivic measure E can be proved to be a morphism of rings with power structure, see [18] for full details.

[2] By a beautiful result of Bittner [5], the classes of smooth projective varieties generate $K_0(\mathrm{Var}_{\mathbf{k}})$ as soon as $\operatorname{char}\mathbf{k} = 0$. But of course E can be defined on arbitrary varieties via mixed Hodge structures.

Let C be a smooth projective curve of genus g. We have

$$\begin{aligned}\mathsf{E}(\zeta_C(q)) &= \sum_{n\geq 0}\mathsf{E}(\operatorname{Sym}^n(C);u,v)q^n = (1-q)^{-\mathsf{E}(C;u,v)}\\ &= (1-q)^{-(1-gu-gv+uv)}\\ &= \frac{(1-uq)^g(1-vq)^g}{(1-q)(1-uvq)}.\end{aligned} \tag{4.3}$$

For E a locally free sheaf of rank r over C, define

$$\mathsf{E}_{C,r,d}(\boldsymbol{q}) = \sum_{\boldsymbol{n}}\mathsf{E}(\operatorname{Quot}_C(E,\boldsymbol{n});u,v)\boldsymbol{q}^{\boldsymbol{n}}.$$

As a direct consequence of Theorem 4.2, we obtain the following corollary.

Corollary 4.5. *There is an identity*

$$\mathsf{E}_{C,r,d}(\boldsymbol{q}) = \prod_{\alpha=1}^{r}\prod_{i=1}^{d}\frac{\left(1-u^{\alpha}v^{\alpha-1}q_iq_{i+1}\cdots q_d\right)^g\left(1-u^{\alpha-1}v^{\alpha}q_iq_{i+1}\cdots q_d\right)^g}{\left(1-u^{\alpha-1}v^{\alpha-1}q_iq_{i+1}\cdots q_d\right)\left(1-u^{\alpha}v^{\alpha}q_iq_{i+1}\cdots q_d\right)}.$$

Proof. This follows by combining Theorem 4.2 and Equation (4.3) with one another, after observing that E is multiplicative (being a ring homomorphism) and sends $\mathbb{L}$ to uv. □

The generating function of the signed Poincaré polynomials is obtained from $\mathsf{E}_{C,r,d}(\boldsymbol{q})$ by setting $u=v$. The result confirms a result of L. Chen [11] obtained in the case $C=\mathbb{P}^1$. The generating series of topological Euler characteristics is obtained from $\mathsf{E}_{C,r,d}(\boldsymbol{q})$ by setting $u=v=1$, also in the quasiprojective case. So we obtain

$$\sum_{\boldsymbol{n}} e_{\mathrm{top}}(\operatorname{Quot}_C(E,\boldsymbol{n}))\boldsymbol{q}^{\boldsymbol{n}} = \prod_{i=1}^{d}\left(1-q_iq_{i+1}\cdots q_d\right)^{-r\cdot e_{\mathrm{top}}(C)}.$$

Appendix A. Motivic exponentials

If (Λ,μ,ϵ) is a commutative monoid in the category of $\mathbf{k}$-schemes, where $\mu\colon\Lambda\times\Lambda\to\Lambda$ is the multiplication map and $\epsilon\colon\operatorname{Spec}\mathbf{k}\to\Lambda$ is the identity element, then by [12, Example 3.5 (4)], one has a λ-ring structure σ_μ on the Grothendieck ring

$$K_0(\mathrm{Var}_\Lambda),$$

determined by the operations

$$\sigma_\mu^n\left[Y\xrightarrow{f}\Lambda\right] = \left[\operatorname{Sym}^n Y\xrightarrow{\operatorname{Sym}^n f}\operatorname{Sym}^n\Lambda\xrightarrow{\mu}\Lambda\right].$$

Assume $\Lambda_+ \subset \Lambda$ is a sub-monoid such that $\coprod_{n>0} \Lambda_+^{\times n} \to \Lambda$ is of finite type. Then we can define the *motivic exponential*

$$\mathrm{Exp}_\mu \colon K_0(\mathrm{Var}_{\Lambda_+}) \to K_0(\mathrm{Var}_\Lambda)^\times$$

by setting

$$\mathrm{Exp}_\mu(A) = \sum_{n\geq 0} \sigma_\mu^n(A)$$

for an effective class A, and extending via

$$\mathrm{Exp}_\mu(A - B) = \mathrm{Exp}_\mu(A) \cdot \mathrm{Exp}_\mu(B)^{-1}$$

whenever A and B are effective. The map Exp_μ is injective. See [13, Section 1] for more details.

We use this construction in the case $(\Lambda, \mu, \epsilon) = (\mathbb{N}^d, +, 0)$, and setting $\Lambda_+ = \mathbb{N}^d \setminus 0$. Of course here we are seeing $\mathbb{N}^d$ as the $\mathbf{k}$-scheme $\coprod_{\boldsymbol{n}\in\mathbb{N}^d} \operatorname{Spec} \mathbf{k}$. There is an isomorphism

$$K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!] \xrightarrow{\sim} K_0(\mathrm{Var}_{\mathbb{N}^d})$$

defined by sending

$$\sum_{\boldsymbol{n}\in\mathbb{N}^d} Y_{\boldsymbol{n}} \cdot q_1^{n_1} \cdots q_d^{n_d} \;\mapsto\; \left[\coprod_{\boldsymbol{n}\in\mathbb{N}^d} Y_{\boldsymbol{n}} \to \operatorname{Spec} \mathbf{k}(\boldsymbol{n}) \right]$$

for varieties $Y_{\boldsymbol{n}}$, and extending by linearity. Under this identification, if we let $\mathfrak{m}$ be the ideal generated by $(q_1, \ldots, q_d)$ in $K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!]$, we can see Exp_+ as a group isomorphism

$$\begin{aligned} &\mathrm{Exp}_+ \colon \mathfrak{m} \cdot K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!] \xrightarrow{\sim} 1 + \mathfrak{m} \cdot K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!] \\ &\quad \subset (K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!])^\times \end{aligned}$$

between an additive group (on the left) and a multiplicative group (on the right). In particular, one has the identity

$$\mathrm{Exp}_+\left(\sum_{\ell=1}^s f_\ell(q_1, \ldots, q_d) \right) = \prod_{\ell=1}^s \mathrm{Exp}_+(f_\ell(q_1, \ldots, q_d))$$

for $f_\ell(q_1, \ldots, q_d) \in \mathfrak{m} \cdot K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!]$.

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