

Physics Area - PhD course in
Statistical Physics

Statistical Spectroscopy of Quantum Many-Body Systems

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Abstract

A central theme in the field of quantum many-body physics is the characterization of thermal properties, which can be accessed, for instance, through the analysis of the Hamiltonian spectrum. This thesis approaches the problem from both numerical and analytical viewpoints, treating the spectrum as a statistical object whose correlations encode predictive information about the physics of the underlying system.

In the first part of this thesis, we review the relation between thermalization, quantum integrability and chaos in one-dimensional quantum many-body systems. We focus on the statistics of energy levels and establish whether quantum integrability can be assessed from a statistical viewpoint.

We then consider the problem of intermediate statistics between purely integrable and non-integrable quantum many-body systems. The first contribution of this thesis is an answer to the problem of discriminating between the spectrum of a genuine integrable system and a statistical mixture of, possibly exponentially many, non-ergodic sectors. We show that the spectral decimation algorithm can capture both qualitative and quantitative features of a ‘characteristic symmetry sector’ from an unbiased analysis of numerically available spectral data. When applied to physically relevant systems, the algorithm is able to identify traces of Hilbert space fragmentation and disorder-induced many-body localization.

The second part of this thesis is dedicated to a class of Hamiltonians of particular interest to statistical spectroscopy. Such Hamiltonians involve only the permutation of sites without the presence of external interactions. Nonetheless, they generate a highly tunable and numerically implementable class of models. For instance, it is easy to engineer integrability and its breaking while preserving essential structures such as the block-diagonal decomposition. These Hamiltonians are naturally suited to generating unitary random walks on the permutation group, which can, for instance, be compared with their classical counterparts.

After a detailed discussion of the numerical implementation of these Hamiltonians and their spectral properties, we depart from the analysis of the system’s static properties to answer a genuine out-of-equilibrium question: do quantum coherences accelerate mixing in the

computational basis? We consider an exactly solvable class of quantum stochastic walks which, in the absence of quantum coherences, admit a limit coinciding with a classical continuous-time Markov chain. While the trace distance of the associated density matrices from the maximally mixed state remains identical to that in the classical limit, we show that the probability distance in the presence of quantum coherences, which defines mixing in the computational basis, is upper bounded by the classical limit. An analysis of the slowest mode identifies a necessary condition under which quantum coherences and classical mixing become of the same order, thereby defining a slow-mode time scale. Finally, a numerical analysis of quantum stochastic walks based on random transpositions corroborates the analytical findings, in particular the accelerated mixing. We also identify a closed-form expression for the ratio of quantum to classical mixing times, which yields a collapse over numerically accessible system sizes.

Overall, this thesis presents a modern approach to the theory of statistical spectroscopy by proposing analytically grounded and numerically implementable protocols to ascertain the thermal properties of a one-dimensional quantum many-body system from the analysis of the spectrum within numerically accessible volumes. It also shows that the analysis of the statistical properties of Hamiltonians can be of practical aid in the study of out-of-equilibrium properties of quantum many-body systems, such as mixing in quantum stochastic walks.

List of Publications

This thesis is based in part on the following publications¹:

- ¹F. He, A. Hutsalyuk, G. Mussardo, and A. Stampiggi*, *Statistical signatures of integrable and non-integrable quantum hamiltonians*, Journal of Statistical Mechanics: Theory and Experiment **2026**, 023101 (2026).
- ²F. He, A. Hutsalyuk, G. Mussardo, and A. Stampiggi*, *Spectral decimation of quantum many-body hamiltonians*, Phys. Rev. B **113**, 245121 (2026).
- ³F. He, A. Hutsalyuk, G. Mussardo, and A. Stampiggi*, *Quantum stochastic walks on the permutation group*, 2026, arXiv:2609.13387 [cond-mat.stat-mech].

Other works completed during the Ph.D. program, but not included in the present thesis, are:

- ¹G. Mussardo, A. Stampiggi*, and A. Trombettoni, *Reflection and transmission amplitudes in a digital quantum simulation*, EPJ Quantum Technology **11** (2024).
- ²A. Stampiggi, *Discrete-to-continuum approach for the analytic continuation of one-particle propagator on the circle*, 2025, arXiv:2503.14600 [quant-ph].
- ³A. Hutsalyuk, M. Lájer, G. Mussardo, and A. Stampiggi*, *Variational method in quantum field theory*, Journal of High Energy Physics **2026** (2026).

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Contents

Abstract	ii
List of Publications	iv
Introduction	1
1 Hamiltonian Statistics	9
1.1 Gap Statistics	10
1.1.1 Unfolding the Spectrum	12
1.1.2 Histogram Resolution	14
1.2 Random Matrices as Prototypes of Quantum Chaos	15
1.2.1 Gaussian Orthogonal Ensemble	15
1.2.2 Statistics of Eigenvectors	23
1.2.3 Hamiltonian Lines	25
1.2.4 Wigner's Semicircle	27
1.2.5 Wigner-Dyson distribution	31
1.2.6 Higher order spacing distributions	34
1.3 Quantum Integrability	37
1.3.1 Classical Systems with a Finite Number of Degrees of Freedom	37
1.3.2 Classical Systems with Infinite Degrees of Freedom	39
1.3.3 Integrable Quantum Field Theories	40
1.3.4 The Role of Explicit Symmetries	42

1.3.5	Many-Body Integrability and The Bethe Ansatz	45
1.3.6	Poisson distribution for Level Spacings in Integrable Systems .	53
1.4	Statistical Integrability	59
1.5	A Class of Quantum Integrable Hamiltonians	62
1.5.1	Locality of the Hamiltonians	66
1.5.2	The basis of momenta	71
2	Statistical Mixtures	75
2.1	Statistical Mixtures	76
2.2	Characteristic Symmetry Sector	81
2.3	Decimation Algorithm	83
2.3.1	Prelude: Rejection Sampling	83
2.3.2	Spectral Decimation Algorithm	86
2.3.3	Higher order spacings	90
2.4	Implementation	90
2.5	Outcomes of Decimation	93
2.6	Stability of the Decimation Algorithm	94
2.6.1	Choice of the Extraction fraction	95
2.6.2	Sources of systematic error	95
2.6.3	Variance of the Output after Resamplings	97
2.7	Decimation of Statistical Mixtures	97
2.8	Applications to Quantum Many-Body Hamiltonians	101
2.8.1	Hilbert-Space Fragmentation	101
2.8.2	Many-Body Localization	104
2.8.3	Characteristic Symmetry Entropy	107
3	Permutation Hamiltonians and Walks on S_L	112
3.1	Permutation Hamiltonians	113
3.1.1	Generalities on the Permutation Group	113

3.1.2	Hamiltonians on Graphs	115
3.1.3	The Role of Boundary Conditions	119
3.2	Examples of Permutation Hamiltonians	121
3.2.1	Numerical Results	123
3.3	Random Walks on S_L and the Card Shuffling Problem	125
3.4	Classical continuous-time random walk	131
3.4.1	Examples of Transition Matrices	132
3.4.2	Inverse participation ratio in a classical scenario	136
3.4.3	Random-Time Measurements and Average Mixing	139
3.5	Unitary Quantum Walks	140
3.5.1	Quantum Dephasing Time	142
3.6	Quantum Stochastic Walks	144
3.6.1	The case of S_2	150
3.6.2	Dynamics of the Slowest Mode	151
3.6.3	Numerical Results for Random Transpositions	153
	Conclusions	156
	Future Perspectives	160
	End Matter	163
A	Higher Order Spacings in Random Matrix Theory	164
A.1	Higher order spacing probability distributions	171
B	Permutation Group and Its IRs	175
C	Mixing of random q-cycle walks	180
C.1	Exact treatment of a lower bound	184
D	IPR of Class Operators	188

Introduction

An exact solution of a finite quantum many-body problem is formally contained in the spectral decomposition of its Hamiltonian. Yet, this statement conceals the central conundrum of many-body physics: for a system with L local degrees of freedom, the dimension of the Hilbert space grows exponentially, and with it the amount of information needed to reconstruct generic observables. The spectral basis is therefore the natural solution of the many-body problem, and at the same time an object that rapidly becomes inaccessible.

Statistical spectroscopy [1–3] acknowledges the inherent complexity of the spectrum and formulates a different question. Rather than resolving every energy level and eigenvector, it asks which collective behaviors of the spectrum are sufficient to characterize physically meaningful properties. Correlations between nearby levels distinguish random-matrix universality from integrability; degeneracies and crossings reveal dynamical constraints; and the organization of levels into symmetry sectors reflects the decomposition of the Hilbert space. From this viewpoint, the spectrum ceases to be a set of isolated numerical values, but acquires depth through its statistical correlations.

This thesis aims to frame statistical spectroscopy as a theory in which methods from statistical physics are used to study equilibrium and out-of-equilibrium properties of quantum many-body systems. Each of the three chapters addresses a different question, which we summarize below.

What can spectral statistics reveal about chaos and integrability? In Chapter 1 and in the later parts of this Introduction we review this question from both historical and modern viewpoints. We discuss the signatures of integrability and non-integrability in a twofold way. We first survey analytical approaches, such as exact results from exactly solvable models and the theory of random matrices, and then describe protocols to analyze numerical spectra. We report a criterion to motivate a statistical formulation of integrability, originally due to Andrea de

Luca and Giuseppe Mussardo, and was later published in [3]. This argument shows that the number of statistically generated integrals of motion has to be no less than polynomial in the number of physical degrees of freedom of the system.

What happens when the spectrum conceals independent symmetry sectors? In Chapter 2 we address the issue of determining from numerically accessible spectra whether a system is genuinely integrable or conceals a finite – possibly exponential – number of non-integrable symmetry sectors, some of which contain a finite number of energy levels. We describe the spectral decimation algorithm, whose purpose is to separate accidental inter-sector crossings from genuine intra-sector correlations using numerically obtained energy data. The spectral decimation measures quantitatively the size of a ‘characteristic symmetry sector’ and is able to reveal non-integrable correlations within a seemingly integrable spectrum. We test the algorithm against spectra of random matrix theory and apply it to the study of quantum many-body systems featuring Hilbert-space fragmentation and disorder-induced many-body localization.

Can the analysis of spectral and symmetry structures predict out-of-equilibrium mixing? In Chapter 3 we present a class of quantum Hamiltonians whose constituents permute the on-site degrees of freedom of the system. These ‘permutation Hamiltonians’ share a common Hilbert-space structure which can be understood within the representation theory of the permutation group. Besides the high degree of mathematical control, these Hamiltonians can be engineered to cross over between ergodic and non-ergodic dynamics. We present an extensive study of the spectral properties of some representative models contained within this class. Then, we move on to the study of out-of-equilibrium properties of these Hamiltonians, in relation to the well-known combinatorial problem of card shuffling, which can be naturally interpreted as a Markov process on the permutation group. Permutation Hamiltonians become, in this case, generators of a unitary quantum walk, and coincide with the transition matrix of the classical continuous-time walk. Finally, we ask whether the presence of quantum coherences can generate a speedup in the mixing of computational basis states. The answer is affirmative, and we derive an exact upper bound and corroborate it with a numerical study.

To set the stage, we begin with two introductory sections outlining the many-body problem and some historical milestones in the field of statistical spectroscopy.

Generalities of the Many-Body Problem

We consider quantum systems defined on a finite-dimensional lattice of L sites. On each site there exists a local Hilbert space $\mathbb{C}^q$ of dimension q . It then follows that the full Hilbert space $\mathcal{H}$ of the quantum system has dimension $d = q^L$. On this space, a Hamiltonian H is the generator of time evolution. More specifically, each state $|\psi\rangle$ of $\mathcal{H}$ evolves in time according to $|\psi(t)\rangle = e^{-iHt} |\psi\rangle$, where we have set $\hbar = 1$.

We focus on closed quantum systems and on the evolution of pure states, under the assumption that probability is conserved. This naturally selects Hermitian Hamiltonians, which can be represented as $d \times d$ matrices in any basis of $\mathcal{H}$. A first choice of basis is the “computational” one, consisting of unit vectors $|q_1, \dots, q_L\rangle$, with $q_j = \{1, \dots, q\}$. We write any state vector $|\psi\rangle = \sum_{j=1}^d c_j^\psi |j\rangle$, and the Hamiltonian has entries $H_{j,j'} = \langle j'|H|j\rangle$. From the representation of H in the computational basis, one can, for instance, obtain the spectrum through exact numerical diagonalization (ED). We label a vector of the eigenbasis as $|E_n\rangle$. It is characterized by an eigenvalue E_n and a projection of the wavefunction $\langle j|E_n\rangle$. The collection of all the latter coefficients defines the “rotation matrix” of entries $R_n^j = \langle j|E_n\rangle$.

The spectral basis gives information not only about the equilibrium properties – energy levels and their correlations – but also about dynamical properties of quantum many-body systems. For instance, one can compute the transition amplitudes

$$\mathcal{A}(\psi, \phi, t) = \langle \phi | e^{-iHt} | \psi \rangle = \sum_{j,k=1}^d \left(c_k^\phi \right)^* c_j^\psi \langle k | e^{-iHt} | j \rangle, \quad (1)$$

from which the Born transition probability $Q(\psi, \phi, t) = |\mathcal{A}(\psi, \phi, t)|^2$ can be accessed. More generally, the expectation values of any operator $\mathcal{O}$ whose elements in the computational basis are known², can be computed:

$$\langle \mathcal{O} \rangle_{\psi,t} = \sum_{n=1}^d |\tilde{c}_n^\psi|^2 \langle E_n | \mathcal{O} | E_n \rangle + \sum_{n \neq m} \left(\tilde{c}_m^\psi \right)^* \tilde{c}_n^\psi \langle E_m | \mathcal{O} | E_n \rangle e^{i(E_m - E_n)t}. \quad (2)$$

We have separated the long-time-average, or “diagonal”, component from the oscillating part, the typical interference pattern characteristic of quantum dynamics.

From a numerical standpoint, the recipe is quite straightforward: i) one first writes the Hamiltonian in the computational basis and ii) diagonalizes it to obtain E_n

²Clearly, if the elements of $\mathcal{O}$ are known in the computational basis, $\langle k | \mathcal{O} | j \rangle = O_{k,j}$, the respective transition elements between energy eigenstates follow from the application of the rotation matrix R . In other words $\langle E_m | \mathcal{O} | E_n \rangle = \sum_{k,j} (R_m^k)^\dagger O_{k,j} R_n^j$.

and R_n^j . This prescription provides a *solution to the many-body problem*, and not necessarily a fully exact solution. When the Hamiltonian spectrum is non-degenerate, the matrix R_n^j is unique up to multiplication by $U(1)$ phases on each individual row. On the other hand, if some energies E_n are degenerate, then the subspace of vectors $|E_{n_1}\rangle, \dots, |E_{n_m}\rangle$ is a generic m -dimensional basis of the subspace associated with energy $E = E_{n_1} = \dots = E_{n_m}$. To distinguish such eigenstates in an ideal experiment, one would need additional conserved quantities which are independent and in involution with the Hamiltonian. From the knowledge of their eigenvalues and overlaps with H , one could then write eigenstates which simultaneously diagonalize all conserved charges, i.e. $|E_{n_1}, \{Q_j\}\rangle, \dots, |E_{n_m}, \{Q'_j\}\rangle$.

To understand this concept, it suffices to consider a two-level system with a single energy level E and parity symmetry. Let us label $|+\rangle$ and $|-\rangle$ the two distinct (orthogonal) parity states. Depending on the particular implementation of the numerical diagonalization algorithm, one can find the following vectors

$$\begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix} = \begin{bmatrix} a & e^{i\alpha}\sqrt{1-a^2} \\ e^{i\beta}\sqrt{1-b^2} & b \end{bmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}, \quad (3)$$

with the phases satisfying $e^{i(\beta-\alpha)} = -\frac{b\sqrt{1-a^2}}{a\sqrt{1-b^2}}$ (both a and b are assumed real-valued and less than one). This example shows that numerical diagonalization in the presence of degenerate spectra can provide a generic rotation matrix which does not automatically resolve other conserved charges.

In practice, ED is severely constrained by the Hilbert space dimension $d = q^L$, which grows exponentially with the system size L , and by the runtime of diagonalization algorithms. In the worst-case scenario, when all eigenvectors and eigenvalues are needed, an exponential (in L) number of operations is required – e.g., the Jacobi algorithm, which diagonalizes dense matrices, has a runtime of $O(d^3)$ operations [4]. It is not surprising that numerical diagonalization algorithms scale with d . Indeed, just by counting the number of eigenvalues and eigenvectors to determine, one observes that full diagonalization requires the estimation of $O(d^2)$ real numbers.

Moreover, if the local Hilbert space dimension q increases, even smaller system sizes can be probed. A natural way to access larger system sizes is to consider Hamiltonians commuting with discrete symmetries. Then H becomes reducible in the basis of the irreducible representations of the symmetry operators, and acquires a block-diagonal form. A large part of this thesis is dedicated to the analysis of the possibility that some of these symmetries are unknown, i.e. difficult to ascertain from the inspection of the Hamiltonian. In these cases, spectral statistics can be used as a

probe of the signatures of more exotic emergent symmetries of many-body systems. The many-body Hamiltonian problem is thus polynomial in the Hilbert space dimension d and hence exponential in the lattice size L . Generic Hamiltonian problems are therefore intractable both analytically and numerically for large L . Moreover, as we showed before, observables and transition probabilities depend on the interference of many spectral levels, not only those which are consecutive, but also the ones which are positioned in different regions of the spectrum. Aside from long-range correlations of the eigenvalues, which characterize the interference term $e^{i(E_m - E_n)t}$, eigenvectors also play a role.

While ED is often not sufficient to resolve the thermodynamic properties of quantum many-body systems, the exponentially many (in L) data contained in the spectral basis offer fertile ground for a statistical analysis. The foundational belief of the field of *statistical spectroscopy* – a term coined by Porter [1] – is that the statistical properties of a few levels in the bulk spectrum, at finite L , can provide information about the thermodynamic properties of a many-body system.

It may sound absurd to apply a statistical approach to the Hamiltonian problem; after all, the spectral basis is deterministic – up to numerical uncertainties of the ED, which are however inconsequential. On the other hand, it is reasonable that in most cases characteristic properties of the many-body problem, like those of typical observables, are mildly dependent on the properties of individual spectral lines, while better captured by quantifiers of collective properties, like statistical correlations. Of course, there is rich physics beyond the regime where statistical physics is effective. Many-body scars are an example [5–7]: exceptional eigenstates with atypical out-of-equilibrium properties – in particular, characterized by the failure of thermalization, as defined in the following section.

It would be a hopeless task to develop tools with applicability to all many-body Hamiltonian systems, without reference to particular models. In the main body of the thesis, we shall broadly focus on *one-dimensional* quantum many-body systems, especially those whose local degree of freedom is a spin-1/2 variable (qubits), but qudit Hamiltonians will also be considered. Secondly, numerical spectra will always be obtained from ED, since we focus on the bulk properties of the spectrum. Other useful frameworks and algorithms for the many-body problem, like tensor networks and matrix product states [8], will not be considered here, as they are most effective in the investigation of the low-lying properties of the spectrum. Much of the analytical insight will come from two separate classes of models that turn out to be both well established and to feature radically different equilibrium and out-of-equilibrium

properties: random matrix ensembles (particularly the Gaussian orthogonal ensemble) and exactly solvable Hamiltonians – which we will also call “integrable”.

Two Approaches to the Many-Body Problem: Equilibrium and Out-of-Equilibrium

As we discussed in the previous section, the many-body problem mainly consists of the information contained in the spectral basis; from that all dynamics can be extracted once the matrix elements of operators are known. This problem is exponentially hard and in general intractable both analytically and numerically – we refer the reader to [9] for a review of numerical approaches to the study of thermalization of disordered many-body systems. It is therefore sensible to restrict the scope of the search to a more modest question, typically: does the quantum system achieve thermal equilibrium? Or, in other words: do observables, in the thermodynamic limit, reach or fail to reach the diagonal ensemble in the long-time limit?

The natural setup to answer this question is to study directly the out-of-equilibrium properties of the system. However, historically, this problem was first addressed in the setting of equilibrium properties of the many-body problem. Wigner first proposed in the 1950s that the statistics of consecutive spacings of heavy atomic nuclei can be modeled through a model of random matrices – we refer the reader to the monograph of Porter [1] and the references therein. Atomic nuclei are examples of strongly correlated quantum systems, and the energy levels they occupy are those in the highly excited band, which we will refer to as the “bulk spectrum”. In the 1970s and 1980s the properties of semiclassical systems were analyzed: Berry [10] pointed out that semiclassical wavefunctions can be divided into two sets, regular and irregular, depending on their semiclassical phase-space behavior. This distinction was also apparent in the correlations between consecutive energy levels. Regular spectra, or “semiclassically integrable” spectra, are characterized by statistically uncorrelated energy levels – this is the content of the conjecture of Berry and Tabor [11] – while irregular spectra, which are often called “ergodic” or “chaotic”, are instead correlated and feature level repulsion [12]. For the latter case, Bohigas, Giannoni and Schmit showed that the spectral correlations predicted by the theory of random matrices agree with those of Sinai’s billiard, and proposed a conjecture extending this correspondence to time-reversal-invariant systems whose classical analogues are strongly chaotic systems [13].

In the 1990s, a new paradigm for studying quantum thermalization emerged with the pioneering work of Deutsch [14] and Srednicki [15]. The “eigenstate thermalization hypothesis” (ETH) addresses the typical behavior of ergodic systems and is a statement about the statistical properties of expectation values of operators, Eq. (2), in typical states. The quantity under scrutiny is the diagonal component, which converges to the long-time average

$$\overline{\mathcal{O}}_\psi(t) = \frac{1}{t} \int_0^t d\tau \langle \mathcal{O} \rangle_{\psi, \tau} \xrightarrow{t \rightarrow \infty} \sum_n |\tilde{c}_n^\psi|^2 \langle E_n | \mathcal{O} | E_n \rangle, \quad (4)$$

and the question to be addressed is: “is its value *typical* or is it susceptible to the chosen initial state?”. The ETH prediction for ergodic systems consists of the following behavior of matrix elements of $\mathcal{O}$ in the spectral basis:

$$\langle E_m | \mathcal{O} | E_n \rangle = \mathcal{O}(\overline{E}) \delta_{m,n} + e^{-S(\overline{E})/2} f_{\mathcal{O}}(\overline{E}, \omega_{m,n}) R_{m,n}, \quad (5)$$

where $\overline{E} = (E_m + E_n)/2$ and $\omega_{m,n} = (E_m - E_n)/2$ are respectively the mean energy and the energy difference. The diagonal part of the ETH ansatz is solely dependent on a smooth function $\mathcal{O}(\overline{E})$ of the average energy $\overline{E}$, while the energy difference $\omega_{m,n}$ only affects the off-diagonal component. The off-diagonal component presents three terms: an entropic contribution (thermodynamic), a matrix of random entries $R_{m,n}$ and a smooth function $f_{\mathcal{O}}(\overline{E}, \omega)$. The latter quantity has a typical form: for small frequencies, ω features a plateau, while it decays to 0 as $\omega \rightarrow \infty$. The Thouless energy ω_{Th} is the energy scale separating these two regimes. Its inverse is the Thouless time $t_{\text{Th}} = \omega_{\text{Th}}^{-1}$, which for diffusive dynamics is proportional to the squared volume L^2 . Assuming the ETH ansatz to hold, one can expand Eq. (5) around the mean energy $\langle E \rangle = \langle \psi | H | \psi \rangle$ [16] (assuming a variance $\delta E_\psi^2 = \langle \psi | H^2 | \psi \rangle - \langle E \rangle^2$ sufficiently small) to obtain a *microcanonical* description of the long-time average

$$\lim_{t \rightarrow \infty} \overline{\mathcal{O}}_\psi(t) = \frac{1}{\mathcal{N}_{\langle E \rangle, \delta E^2}} \sum_{|E_m - \langle E \rangle| < \delta E} \mathcal{O}_{m,m} := \text{Tr}[\rho_{\text{MC}} \mathcal{O}], \quad (6)$$

with corrections that decay with volume. This latter result, which is founded on ETH, shows that the equilibrium value of a (few-body) observable is an ensemble property (ρ_{MC} with a number of states $\mathcal{N}_{\langle E \rangle, \delta E^2}$). Thus, it suffices to know the average energy and its variance to be able to capture the equilibrium properties of an observable for ergodic systems.

Eq. (6) can be used as a definition of quantum ergodicity, where observables thermalize

to their microcanonical value. When the condition is not met, i.e.

$$\lim_{L \rightarrow \infty} \lim_{t \rightarrow \infty} \langle \mathcal{O} \rangle_{\psi, t} \neq \text{Tr}[\rho_{\text{MC}} \mathcal{O}] \quad (7)$$

the system fails to equilibrate to the microcanonical ensemble and is said to be *non-ergodic*. A central open problem in the theory of quantum many-body systems is to establish whether strongly interacting systems in the presence of disorder – the simplest case being a random on-site density/magnetic interaction – reach or fail to reach thermal equilibrium. This phenomenon goes under the name of many-body localization (MBL) and an extensive body of work has covered various facets of this physical phenomenon – see e.g. the reviews [9, 17–19].

The distinction between ergodic and non-ergodic behavior can not be made a priori by a simple inspection of the Hamiltonian. One-dimensional systems equipped with an extensive number of local and quasi-local integrals of motion, which we label *quantum integrable*, constitute a subset of beautiful models [20] which fail to thermalize [21]. For integrable models, it is known that the ensemble characterizing the dynamics is the *generalized Gibbs ensemble* ρ_{GGE} [22], which depends not only on the Hamiltonian expectation values but also on those of the conserved charges.

The ergodicity or non-ergodicity of a model can also be established on the grounds of its spectral (equilibrium) properties. In the next chapter, we review perhaps the most intuitive and historically relevant diagnostic tools for distinguishing between quantum chaotic and integrable systems: gap statistics.

Chapter 1

Hamiltonian Statistics

This chapter reviews the spectral statistics of quantum many-body Hamiltonians from both numerical and analytical viewpoints. Historically, this approach has received particular attention as a numerically accessible probe of the ergodic or non-ergodic nature of quantum systems. As we shall discuss, one-dimensional quantum systems feature different spectral signatures, which can be used as numerical probes of the presence or absence of thermalization. On the one hand, quantum chaotic systems are characterized by the absence of local and quasi-local conserved charges – with the exception of the Hamiltonian itself and discrete countable symmetries. The equilibrium statistical properties of these systems are well described by the theory of random matrices (RMT). On the other hand, systems which contain an extensive and independent set of quasi-local integrals of motion are said to be quantum integrable. In contrast to RMT, there are no comprehensive theories for a ‘minimal model’ of integrability. Instead, a wide array of techniques, from the Jordan-Wigner mapping to more sophisticated Bethe-Ansatz-inspired methods, offers exact solutions to the Hamiltonian problem at finite volume and in the thermodynamic limit. This chapter is organized as follows. First, we discuss the properties of the Hamiltonian spectrum; then, the statistics of random matrices and of quantum integrable models. This is followed by a method to statistically generate, from knowledge of the Hamiltonian, an extensive set of mutually independent commuting charges. The material of this chapter is organized as follows. Section 1.1 provides an original synthesis of gap statistics, drawing on Refs [3, 23]; Section 1.2 reviews RMT, following [3, 24, 25]; Section 1.3 reviews integrable models, drawing from Refs [3, 26–28]; Section 1.4 discusses the necessary conditions for statistical integrability introduced in Ref. [29]; and Section 1.5 presents the construction of a class of quantum-integrable Hamiltonians originally developed by Giuseppe Mussardo and Andrea De Luca and later published in Ref.

[3].

1.1. Gap Statistics

At volume L , the $d \times d$ Hamiltonian H can be diagonalized numerically through ED. The *ordered spectrum* is the sequence $\{E_1, E_2, \dots, E_d\}$ such that $E_i \leq E_{i+1}, \forall i$. The empirical *density of states* (EDOS) $\rho(E)$ is defined as

$$\rho(E) = \sum_{j=1}^d \delta(E - E_j). \quad (1.1)$$

Similarly, the empirical cumulative density of states (ECDOS) counts the number of energy levels less than or equal to E :

$$c(E) = \sum_{j=1}^d \theta(E - E_j), \quad (1.2)$$

where θ is the Heaviside step function. Consecutive energy differences, also called *folded gaps* are defined as

$$\Delta E_j = E_{j+1} - E_j \geq 0, \quad (1.3)$$

and folded k -gaps are similarly defined:

$$\Delta_k E_j = E_{j+k} - E_j = \sum_{n=1}^k \Delta E_{j-n+1}. \quad (1.4)$$

The EDOS and ECDOS are *deterministic* functions. It seems paradoxical that they can be treated with tools of statistical analysis. However, in analyzing the behavior of the spectrum of H across different volumes L , energy levels tend to distribute, as L increases, according to a smooth and continuous density of states (DOS). To allow for this comparison, the Hamiltonian H must be rescaled. This is necessary because energy is typically extensive in L . This rescaling can be performed at the operator level, e.g. by dividing the Hamiltonian by the number of terms in it or by applying an affine transformation to the spectrum:

$$E_j \rightarrow \frac{E_j - E_1}{E_d - E_1}. \quad (1.5)$$

The rescaled spectrum lies in the interval $[0, 1]$ and it can be compared across different volumes. This affine transformation changes only the support of the DOS and CDOS.

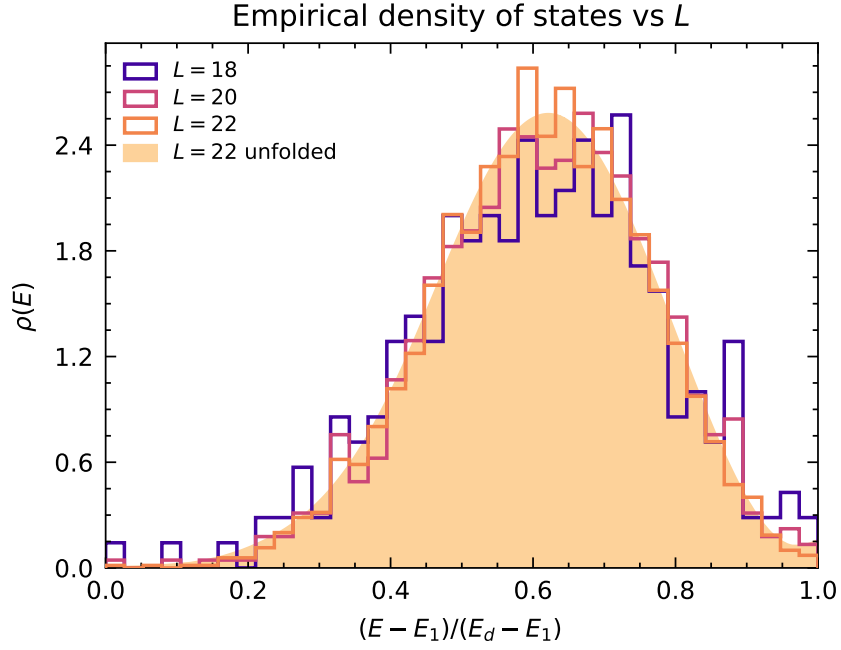


Figure 1.1: EDOS vs L for a Hamiltonian system, obtained with ED. The Hamiltonian diagonalized is the spin-1/2 Heisenberg chain with periodic boundary conditions in the zero-magnetization and zero-total-momentum sector. As L increases, the density of states approaches a smooth continuous curve (filled region), obtained through the unfolding method of Bernstein polynomials.

Also relevant to the present study is that the gap statistics will not be affected by this rescaling, and energy correlations are preserved.

The number of eigenvalues in the spectrum is $d \sim O(e^L)$, and it is therefore expected that an interval dE would typically contain an exponential number of energy levels. The frequency histogram of the EDOS, with resolution dE , would smear the δ -peaks, and the DOS can be treated as a continuous curve $\rho(E)$; see Fig. 1.1. The DOS is subject to fluctuations and is, in principle, not monotonic. On the other hand, the cumulative density $c(E)$ is guaranteed to be a monotonic function and, for the purposes of extracting gap statistics, is the starting quantity to analyze.

The probability density function (PDF) of consecutive gaps can discriminate between quantum chaotic and integrable systems, and can be directly accessed from the E_j 's, through the following protocol:

- 1) one orders the spectrum (*sorting*);
- 2) the energy differences are computed;
- 3) the consecutive gaps are plotted in a frequency histogram, from which the empirical PDF is extracted.

This protocol, applied to the spectrum of a quantum harmonic oscillator of characteristic frequency ω , reveals that the gap PDF converges to $\delta(\Delta E - \omega)$. The position of the peak is $\Delta E = \omega$ and depends on the particular oscillator. However, the functional form is the same. It is thus convenient to modify step ii), i.e. to rescale all gaps ΔE by the average $\overline{\Delta E} = \sum_{j=1}^d \Delta E_j / d$, so that the average of the PDF is 1. This generates the *standard* gap PDF.

However, when this procedure is applied to the spectrum of a many-body system, points ii) and iii) need to be handled with care. Concerning ii), the energy differences ΔE_j depend on the DOS, and regions with smaller DOS have comparatively greater gaps because they are less populated, while regions with greater DOS are affected by the inverse phenomenon. In other words, different regions of width dE have different averages $\overline{\Delta E}$. This nontrivial dependence on the sampled energy window, if not accounted for, generates standard gap PDFs which present distorted features; see Fig. 1.2. The technique of spectral unfolding allows one to mitigate this issue. Another source of distortion in the gap PDF comes from the presence of *exact* spectral degeneracies (up to numerical precision). When they are a finite fraction of the spectrum, i.e. they do not decrease as L increases, they lead to an overabundance of zero gaps in the PDF. This issue cannot be mitigated by the unfolding procedure described below, because exact degeneracies are typically due to unaccounted discrete symmetries. Perhaps less relevant to qualitative analyses, but critical when using the gap PDF as a predictive statistical tool – see the next chapter – is the resolution of the frequency histogram in iii): too small a resolution leads to enhanced fluctuations, while too large a dE causes oversmoothing of the same fluctuations. The two errors have different origins: the first is due to undersampling in the histogram bin, leading to large variances in the counts, while the second is attributed to too coarse an approximation of the true mean PDF.

1.1.1. Unfolding the Spectrum

We describe the unfolding procedure used to obtain consecutive spacings from the rescaled energy spectrum, defined as

$$s_i = e_i - e_{i-1}, \quad e_i = c(E_i) = \int_{-\infty}^{E_i} dE \rho(E), \quad (1.6)$$

where ρ is the DOS. Fitting ρ directly is impractical here, first because it is subject to fluctuations, especially for very small system sizes. Second, we recognize that e_i is the CDOS, which is a much better quantity to fit, as it is monotonic. Thus, we seek

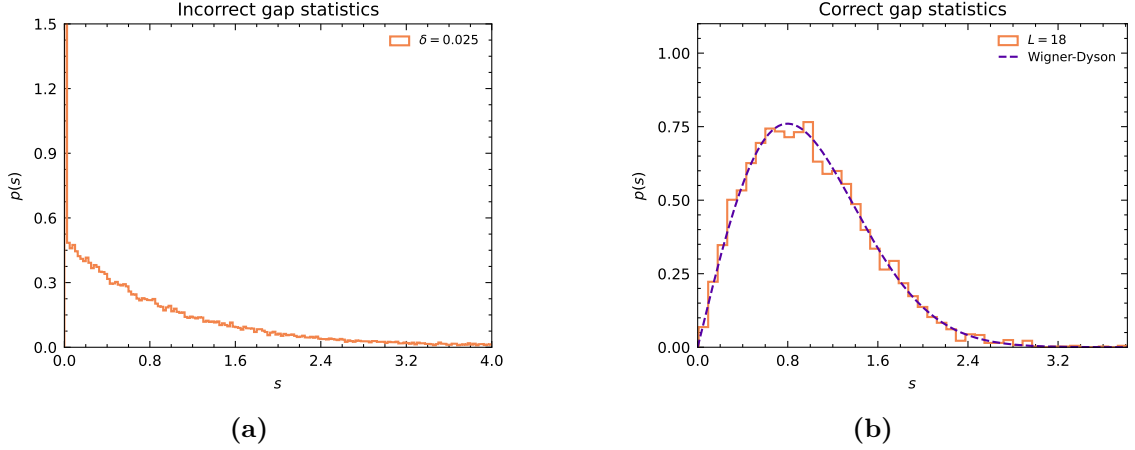


Figure 1.2: Gap distribution of the same translationally invariant Hamiltonian. (a) Gap distribution of the raw (folded) spectrum, without accounting for discrete symmetries such as translation invariance and using an excessively fine resolution. (b) Gap distribution of a symmetry sector, with unfolding and Freedman-Diaconis histogram resolution.

to fit the empirical CDOS $\tilde{c}_i = \sum_{j=1}^d \theta(E_i - E_j)$. Since rescaled energies lie in the unit interval, the problem is reduced to fitting the data $(i/d, \tilde{c}_i)$ in the interval $[0, 1]$.

The most important property that the fit must preserve is the monotonicity of the $\tilde{c}$'s, which guarantees that ρ will always be positive. A linear polynomial fit with basis functions x^n does not provide generic monotonic $\tilde{c}$'s. In $[0, 1]$ there exists a basis of polynomials, called Bernstein polynomials $b_{\nu,n}(x)$:

$$b_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \quad \nu \in \{0, 1, \dots, n\}, \quad (1.7)$$

which is of order n .

Bernstein polynomials satisfy:

$$\frac{d}{dx} b_{\nu,n}(x) = n [b_{\nu-1,n-1}(x) - b_{\nu,n-1}(x)]. \quad (1.8)$$

If one then considers a generic polynomial approximation of degree n of a monotonic function $f(x)$, i.e.

$$f_n(x) = \sum_{\nu=0}^n \gamma_\nu b_{\nu,n}(x), \quad (1.9)$$

it is easy to show that

$$\gamma_0 \leq \gamma_1 \leq \dots \leq \gamma_n \rightarrow \frac{d}{dx} f_n(x) \geq 0. \quad (1.10)$$

Introducing auxiliary constants $\beta_i \geq 0$ such that $\gamma_k = \sum_{i=0}^k \beta_i$, one can easily perform

a constrained linear least-squares fit to obtain the β_i 's and hence the best monotonic approximation of a monotonic $f(x)$ through a polynomial of degree n . This is exactly the strategy we use to obtain the CDE and, as a byproduct, consecutive gaps.

1.1.2. Histogram Resolution

A central problem in the numerical estimation of continuous PDFs is the estimation of the proper histogram bin width. If the bin width is too large, the histogram fails to capture the intrinsic variations of the PDF, while if it is too small, fluctuations in the bin counts prevent the reproduction of a smooth curve. For these reasons, choosing an appropriate bin width is not only of qualitative but also of quantitative importance.

Let us consider a sample of d independently and identically distributed (i.i.d.) elements drawn from the PDF $p(s)$. For its discrete representation, we homogeneously partition the interval $s \in [0, \infty)$ into bins of width δ , such that bin j comprises values $s \in [\delta j, \delta(j+1)]$. Let d_j be the number of samples falling in bin j . The empirical histogram PDF is

$$\tilde{\pi}_j = \frac{d_j}{\delta d}, \quad (1.11)$$

where $\tilde{\pi}_j$ is, in principle, a random variable because d_j is the sum of binomial counts. Indeed, the expectation of $\tilde{\pi}_j$ is the mean of the continuous PDF over the bin:

$$\mathbb{E}[\tilde{\pi}_j] = \pi_j = \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} ds p(s). \quad (1.12)$$

Its variance, on the other hand, is, for $d_j \gg 1$,

$$\text{var}[\tilde{\pi}_j] = \frac{\text{var}[d_j]}{(\delta d)^2} = \frac{d_j}{(\delta d)^2}. \quad (1.13)$$

The optimal bin-width is that which minimizes the fluctuations between $\tilde{\pi}_j$, π_j and $p(s)$ in each bin, over the whole histogram. This quantity is called the integrated mean squared error (IMSE), defined as

$$\text{IMSE} = \sum_j \delta (\tilde{\pi}_j - \pi_j)^2 + \sum_j \int_{j\delta}^{(j+1)\delta} ds (p(s) - \pi_j)^2. \quad (1.14)$$

The reason for the second term is as follows. Denoting the midpoint value by $s_j = \delta(j + 1/2)$, π_j approximates $p(s_j)$ to order δ^2 , which is subleading in the IMSE.

We evaluate the two terms separately. The first is quantified by the variance of $\tilde{\pi}_j$, which comes from the Bernoulli distribution of d_j . When summed over bins, it gives:

$$\sigma_{\text{hist}}^2 = \sum_j \delta \frac{d_j}{(\delta d)^2} = \frac{1}{\delta d}. \quad (1.15)$$

On the other hand, the variance of the integral mean is computed by expanding $q(s)$ around the midpoint δ_j :

$$\sigma_{\text{mean}}^2 = \frac{\mathcal{I}_{1,2}[p]}{12} \delta^2, \quad \mathcal{I}_{n,m}[p] = \int_0^\infty d\sigma \left(\frac{d^n p(\sigma)}{d\sigma^n} \right)^m. \quad (1.16)$$

The minimum of their sum yields the Freedman-Diaconis rule [30]:

$$\delta = \left(\frac{6}{\mathcal{I}_{1,2}[p]d} \right)^{1/3}. \quad (1.17)$$

For the Poisson distribution, $\mathcal{I}_{1,2} = 1/2$.

1.2. Random Matrices as Prototypes of Quantum Chaos

The core of RMT is the study of ensembles of matrices whose entries are drawn from certain probability distributions. A natural question to ask is then: “what are the statistical properties of a few of their eigenvalues or eigenvectors?”

It turns out that for real Hermitian $N \times N$ matrices, whose diagonal and off-diagonal entries are i.i.d. distributed according to $\mathcal{N}(0, 2\sigma^2)$ and $\mathcal{N}(0, \sigma^2)$, respectively¹, the large- N statistical properties of n eigenvalues of H become independent of the probability density. In other words, the joint probability density function (JPDF) of n eigenvalues of H tends to the n -point correlation function of the GOE [24].

1.2.1. Gaussian Orthogonal Ensemble

The GOE of dimension N is made up of real-valued symmetric matrices H which:

¹By $h \sim \mathcal{N}(\mu, \sigma^2)$, we mean that h is a random variable drawn from a Gaussian distribution of mean μ and variance σ^2 , i.e. that the PDF of h is $\exp(-(h - \mu)^2/(2\sigma^2))$.

A) Are invariant under conjugation by any rotation matrix W :

$$H \in \text{GOE} \rightarrow WHW^T \in \text{GOE}, \quad \forall \{W \mid WW^T = W^T W = \mathbf{1}\}. \quad (1.18)$$

Thus, a GOE matrix looks the same upon rotation of the basis with real coefficients.

B) The diagonal and off-diagonal elements $H_{k,j}$, $k \leq j$ are statistically independent.

Denote by $P(H)dH$ the probability that a Hamiltonian H lies within the volume element $dH = \prod_{k \leq j} dH_{k,j}$. The two requirements of $H \in \text{GOE}$ amount to the following statements about $P(H)$:

A) The probability $P(H')$, with $H' = WHW^T$ must satisfy

$$P(H')dH' = P(H)dH. \quad (1.19)$$

B) Because entries are statistically independent, $P(H)$ factorizes into $N(N+1)/2$ independent functions

$$P(H) = \prod_{k \leq j} f_{k,j}(H_{k,j}). \quad (1.20)$$

Time-Reversal For the kind of systems that we consider, we naturally assume the dynamics to be time-reversal invariant. With $U(t)$ the time-evolution operator, we seek a transformation

$$TU(t)T^{-1} = U(-t), \quad \forall t. \quad (1.21)$$

Since $U(t) = e^{-iHt}$, a choice of unitary T would lead to negative energies, while their positivity is preserved by choosing an anti-unitary operator such that

$$T(-i)T^{-1} = i, \quad THT^{-1} = H. \quad (1.22)$$

Any antiunitary operator can be written in terms of a unitary operator K ($KK^\dagger = K^\dagger K = \mathbf{1}$) and a charge-conjugacy operator C :

$$T = KC, \quad \langle \phi | C | \psi \rangle = \langle \psi | \phi \rangle. \quad (1.23)$$

Under time reversal, the Hamiltonian H is self-dual. Indeed, if we require that the expectation of operators A be invariant under time reversal,

$$(\phi, A\psi) = (\psi^R, A^R \phi^R), \quad (1.24)$$

one would only need to map all observables to

$$A^R = T A T^{-1}. \quad (1.25)$$

Indeed, $H = H^R$, confirming that the Hamiltonian is self-dual.

Let us now consider a unitary transformation of the states $|\psi\rangle \rightarrow |\psi'\rangle = U |\psi\rangle$. This rotation naturally changes the time-reversal operator

$$T' = U T U^\dagger, \quad (1.26)$$

in such a way that all observables in the primed basis transform equivalently:

$$(A')^R = U A^R U^\dagger. \quad (1.27)$$

Time-reversal is an antiunitary involution, i.e.

$$T^2 = \alpha \mathbf{1}, \quad |\alpha| = 1. \quad (1.28)$$

However $T^2 = K K^*$, and together with the fact that $K^T K^* = \mathbf{1}$, we obtain

$$K = \alpha K^T = \alpha (\alpha K^T)^T = \alpha^2 K. \quad (1.29)$$

This has an immediate effect on the physical spin of the system, encoded in the vector $\mathbf{J}$. We label the states of an irreducible representation as $|j, m\rangle$, and time reversal acts conventionally as

$$T |j, m\rangle = (-1)^{j-m} |j, -m\rangle, \quad T^2 |j, m\rangle = (-1)^{2j} |j, m\rangle. \quad (1.30)$$

However, as we showed, K being unitary constrains α to be either $(+1)$ or (-1) . Thus, if

- $\alpha = 1$, i.e. $K K^* = \mathbf{1}$, then the total angular momentum $\mathbf{J}$ of the system is integer, otherwise if
- $\alpha = -1$, i.e. $K K^* = -\mathbf{1}$, then the total angular momentum $\mathbf{J}$ of the system is half-integer.

The GOE ensemble of matrices appears quite often in physics. In particular, for a rotationally invariant system, the Hamiltonian H_{GOE} admits a basis where its entries are real-valued. Indeed, because of rotational invariance, $[H, J_k] = 0$, for $k = 1, 2, 3$.

A choice of the unitary operator K is

$$K = e^{i\pi J_2}, \quad (1.31)$$

where J_2 is chosen to be purely imaginary. With this choice, we have that

$$H_{\text{GOE}}^R = H_{\text{GOE}}^* = H_{\text{GOE}}^T, \quad (1.32)$$

and because $H_{\text{GOE}}^R = H_{\text{GOE}}$, we obtain that the entries are real-valued. Thus, matrices drawn from the GOE ensemble are naturally rotationally invariant.

Invariants We now come to the question: to what degree are we free to specify the functional form of the distribution $P(H)$ of the GOE ensemble? Surprisingly, postulates A) and B) completely fix $P(H)$.

To observe this, we recall two lemmas:

- 1) All the invariants of an $(N \times N)$ matrix H under a nonsingular similarity transformation A , i.e.

$$H \rightarrow H' = AHA^{-1}, \quad (1.33)$$

can be expressed as functions of the first N powers of H :

$$\text{Tr } H^j = \sum_{n=1}^N E_n^j. \quad (1.34)$$

This is the content of Weyl's lemma.

- 2) We anticipate that the postulate of statistical independence excludes all powers except the first two, and these occur only in an exponential. We will invoke the following result to help us derive that. Consider three differentiable functions satisfying the following relation

$$f_1(xy) = f_2(x) + f_3(y); \quad (1.35)$$

then they are of the form

$$f_k(z) = a \log z + b_k, \quad b_3 = b_1 + b_2. \quad (1.36)$$

This result can be proven, for instance, by first differentiating w.r.t. x and integrating w.r.t. y .

Let us begin by considering the consequences of statistical independence. From definition A), we have that for any rotation W , a GOE matrix remains in the ensemble:

$$H \in \text{GOE} \rightarrow H' = W^{-1}HW \in \text{GOE}. \quad (1.37)$$

We choose a particular rotation matrix, which depends on a parameter θ :

$$W(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 & \dots & 0 \\ -\sin \theta & \cos \theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \quad (1.38)$$

It is clear that if we assume $H' \in \text{GOE}$, its distribution will generally be independent of θ . Thus, we are led to the following variation equation

$$\frac{\partial H}{\partial \theta} = \frac{\partial U^T}{\partial \theta} H' U + U^T H' \frac{\partial U}{\partial \theta} = AH + HA^T, \quad (1.39)$$

where

$$A = \frac{\partial U^T}{\partial \theta} U = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \quad (1.40)$$

If the PDF $P(H)$ is invariant w.r.t. any transformation U , it must be independent of θ , i.e.

$$\sum_{j \leq k} \frac{d \log f_{k,j}}{dH_{k,j}} \frac{\partial H_{k,j}}{\partial \theta} = 0. \quad (1.41)$$

We write down this equation explicitly for the GOE:

$$\begin{aligned} 0 = & 2 \left(\frac{d \log f_{2,2}}{dH_{2,2}} - \frac{d \log f_{1,1}}{dH_{1,1}} \right) H_{1,2} + \frac{d \log f_{1,2}}{dH_{1,2}} (H_{1,1} - H_{2,2}) \\ & + \sum_{k=3}^N \left(\frac{d \log f_{2,k}}{dH_{2,k}} H_{1,k} - \frac{d \log f_{1,k}}{dH_{1,k}} H_{2,k} \right). \end{aligned} \quad (1.42)$$

This equation is satisfied by requiring that each term in the parentheses is a constant

and that they sum up to zero:

$$\frac{d \log f_{2,k}}{dH_{2,k}} H_{1,k} - \frac{d \log f_{1,k}}{dH_{1,k}} H_{2,k} = C_k. \quad (1.43)$$

We can divide by $H_{1,k}H_{2,k}$ and obtain a function that satisfies 2). However, the function $f_1(xy)$ depends on $H_{1,k}H_{2,k}$ and not on its logarithm; therefore, the only compatible solution is for C_k to vanish. This means that

$$\frac{1}{H_{2,k}} \frac{d \log f_{2,k}}{dH_{2,k}} = \frac{1}{H_{1,k}} \frac{d \log f_{1,k}}{dH_{1,k}} = -\frac{1}{a}, \quad (1.44)$$

which upon integration gives a Gaussian function

$$f_{1,k} = \exp \left(-\frac{(H_{1,k})^2}{2a} \right). \quad (1.45)$$

To satisfy Eq. (1.42), we must require that the constant for $f_{k,k}$ is actually half that of the off-diagonal elements:

$$f_{k,k} = \exp \left(-\frac{(H_{k,k})^2}{4a} \right). \quad (1.46)$$

While the particular choice of U is quite restrictive, one may consider more generic U 's and obtain the product form

$$P(H) = \exp \left(-a \operatorname{Tr} H^2 + b \operatorname{Tr} H + c \right), \quad (1.47)$$

where $a > 0$ and $b, c \in \mathbb{R}$ to ensure normalizability. Thus, the GOE PDF – as well as those for the unitary and symplectic ensembles – depends only on two moments of the trace, which are respectively related to the average energy and its variance. Assuming that the average energy vanishes, one obtains a cleaner normalized form of the JPDF [25]

$$P(H) = \prod_k \exp \left(-\frac{1}{\sqrt{2\pi}} \frac{(H_{k,k})^2}{2} \right) \prod_{j < k} \exp \left(-\frac{1}{\sqrt{2\pi}} (H_{j,k})^2 \right). \quad (1.48)$$

PDF as a Function of Spectral Quantities For the GOE ensemble, the independent matrix entries are exactly $N(N-1)/2 + N = N(N+1)/2$, corresponding to the off-diagonal and diagonal elements. The PDF $P(H)$, written as Eq. (1.48), involves the $H_{j,k}$, but any change of variables is possible. The most natural one is to map the $\{H_{j,k}\} \rightarrow \{E_m, \mathbf{v}_n\}$, where E_m is the m -th eigenvalue of H with

eigenvector $\mathbf{v}_m$ (in the computational basis). This change of variables is allowed because the degrees of freedom match: there are N eigenvalues and N eigenvectors – which collectively only have $N(N-1)/2$ free parameters: out of the N^2 parameters, one imposes $N(N-1)/2$ orthogonality conditions and N normalization conditions, compactly $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{i,j}$. Overall, the total number of eigenvalue and eigenvector components is $N(N+1)/2$. Thus, we aim to evaluate

$$P(E, O) = P(H) \frac{\prod_{j \leq k} dH_{j,k}}{dO \prod_j dE_j}. \quad (1.49)$$

It is useful first to discuss the volume element dO [25] of $N \times N$ orthogonal matrices satisfying $OO^T = \mathbf{1}$. Its volume on the Stiefel manifold is

$$\text{Vol}(\mathbb{V}_N) = \int_{\mathbb{V}_N} dO = \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)}, \quad (1.50)$$

where the function $\Gamma_m(z)$ is defined as

$$\Gamma_m(z) = \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma\left(z - \frac{j-1}{2}\right). \quad (1.51)$$

We define conventionally the Haar measure on the orthogonal group as

$$\mathcal{D}O = \frac{dO}{\text{Vol}(\mathbb{V}_N)}, \quad (1.52)$$

which is invariant under orthogonal conjugation and defines a probability measure on the space of $N \times N$ orthogonal matrices.

To compute $P(E, O)$, we need the Jacobian of the transformation from the set of matrix elements to the spectral basis. While in principle this Jacobian $J(H \rightarrow \{E, O\})$ depends on both energies E and eigenvectors $\mathbf{v}$, it turns out that for GOE matrices it is a function of the energies only, and coincides with the Vandermonde determinant

$$J(H \rightarrow \{E, O\}) = \prod_{j>k} (E_j - E_k). \quad (1.53)$$

Because the GOE ensemble is rotationally invariant, $P(H)$ is a function of the traces of the powers of H . This implies that the volume element $\mathcal{D}O$ is the only eigenvector-dependent term in $P(E, O)$, which then factorizes as

$$P(E, O) \mathcal{D}O \prod_k dE_k = (\mathcal{D}O) P(E) \prod_k dE_k, \quad (1.54)$$

where $P(E)$ is the JPDF of the eigenvalues of the GOE ensemble. Because of rotational invariance, the JPDF of the eigenvalues can be written as

$$P(E) = \frac{\pi^{N^2/2}}{\Gamma_N(N/2)} \phi \left(\sum_j E_j, \dots, \sum_j E_j^N \right) \prod_{j>k} (E_j - E_k), \quad (1.55)$$

where ϕ is a function of the trace of the first N powers of H . Comparing with Eq. (1.48), we obtain, for the *ordered spectrum*

$$P(E) = \frac{1}{Z_N} e^{-\frac{1}{2} \sum_j E_j^2} \prod_{j>k} (E_j - E_k), \quad Z_N = \frac{(2\pi)^{N/2}}{N!} \prod_{j=1}^N \frac{\Gamma(1 + \frac{j}{2})}{\Gamma(3/2)}. \quad (1.56)$$

Information-Theoretic Derivation of the JPDF The discussion presented thus far clearly exhibits the beauty of RMT: from a small set of assumptions on the Hamiltonians and their ensemble, one completely fixes the JPDF of the eigenvalues. We present here a different yet illuminating derivation, using the tools of information theory.

Let $\mathcal{I}(P(H))$ be the amount of information carried by the JPDF $P(H)$. For discrete events occurring with probability p_j , the information is encoded in the Shannon entropy

$$\mathcal{I} = - \sum_j p_j \log p_j; \quad (1.57)$$

Thus, for continuous variables, it is reasonable to write

$$\mathcal{I}(P(H)) = - \int dH P(H) \log P(H), \quad dH = \prod_{i \leq j} dH_{i,j}. \quad (1.58)$$

We seek to fix $P(H)$ based on a finite set of constraints. The selection criterion is minimization with respect to the information $\mathcal{I}$: $P(H)$ must carry the minimum amount of information compatible with the constraints, or equivalently maximize the entropy of $P(H)$. The constraints are naturally selected by specifying a few moments of $P(H)$:

$$\langle f_i(H) \rangle = \int dH P(H) f_i(H) = k_i. \quad (1.59)$$

We regard $\mathcal{I}$ as an information-theoretic free energy, subject to the Lagrange multi-

pliers

$$\begin{aligned} \mathcal{I}(P(H)|\{k_i\}) = & - \int dH P(H) \log P(H) - \alpha \left(1 - \int dH P(H) \right) \\ & - \sum_i \lambda_i \left(k_i - \int dH P(H) f_i(H) \right). \end{aligned} \quad (1.60)$$

Varying w.r.t. $P(H)$ yields the extremum equation

$$0 = 1 - \alpha + \log P(H) - \sum_i \lambda_i f_i(H), \quad (1.61)$$

and therefore one obtains the following general form for $P(H)$:

$$P(H) = \frac{e^{\sum_i \lambda_i f_i(H)}}{\int dH e^{\sum_i \lambda_i f_i(H)}}, \quad (1.62)$$

compatible with the principle of maximum entropy. We observe that RMT only specifies the first two moments of the distribution:

$$k_1 = \text{Tr } H, \quad k_2 = \text{Tr } H^2, \quad (1.63)$$

which indeed yields Eq. (1.48).

1.2.2. Statistics of Eigenvectors

The constitutive definition of the GOE ensemble is rotational invariance. This implies that the statistical properties of eigenvectors and energy eigenvalues are actually independent. Thus, a GOE matrix has the property of “looking” the same – or, more precisely, of featuring the same statistical correlations – in any basis. We then ask a sensible question: what is the typical distribution of the eigenvectors of a GOE matrix?

We restrict ourselves to real-valued Hermitian matrices; therefore, the eigenvector components are real-valued – that is, the rotation matrix O is orthogonal. The sole constraint on the components of an eigenvector $\mathbf{v}$ is normalization

$$P_{\text{GOE}}(\mathbf{v}) = C_N \delta \left(1 - \sum_n v_n^2 \right). \quad (1.64)$$

We are interested in the marginal distribution of a single component, say v_1^2 (the

distribution is the same for all other components), defined as

$$P_{\text{GOE}}(y) = \int dc_1 \dots dc_N \delta(y - c_1^2) P_{\text{GOE}}(\mathbf{v}). \quad (1.65)$$

The computation can be performed with the following steps:

i) First, we define the auxiliary function

$$P_{\text{GOE}}(y, t) = C_N \int dc_1 \dots dc_N \delta(y - c_1^2) \delta\left(t - \sum_n v_n^2\right) \quad (1.66)$$

such that $P_{\text{GOE}}(y) \equiv P_{\text{GOE}}(y, 1)$.

ii) Then, we take the Laplace transform w.r.t t

$$\begin{aligned} \int_0^\infty ds e^{-st} P_{\text{GOE}}(y, t) &= C_N \left(\int_0^\infty dx \delta(y - x^2) e^{-sx^2} \right) \left(\int_0^\infty dx e^{-sx^2} \right)^{N-1} \\ &\propto \frac{e^{-sy}}{\sqrt{y} \sqrt{s^{N-1}}} \end{aligned} \quad (1.67)$$

iii) We invert the Laplace transform to obtain, setting $t = 1$:

$$P_{\text{GOE}}(y) = \frac{\theta(1-y)}{\sqrt{\pi}} \frac{\Gamma(N/2)}{\Gamma\left(\frac{N-1}{2}\right)} \frac{(1-y)^{\frac{N-3}{2}}}{\sqrt{y}}. \quad (1.68)$$

iv) We observe that the average $\langle y \rangle_{\text{GOE}} \sim 1/N$, and therefore, it makes sense to consider the PDF of the rescaled variable $\eta = yN$, which is

$$P_{\text{GOE}}(\eta) \underset{N \rightarrow \infty}{\sim} \frac{1}{\sqrt{2\pi\eta}} e^{-\eta}. \quad (1.69)$$

This latter equation is the Porter-Thomas distribution: for GOE matrices, the distribution of eigenvector components follows an exponential law, modulated by an inverse-square-root behavior: very small components are much more likely to occur than large ones. To construct a random GOE eigenvector, one needs to perform N independent draws from $P_{\text{GOE}}(\eta)$ and rescale the vector by its norm.

We conclude with a final remark: why is this the distribution of eigenvector components? Strictly speaking, this is just the distribution of vectors for a rotationally invariant system. However, one may construct a GOE matrix from knowledge of $P(E)$ and $P(\eta)$ alone, as follows:

- First, one draws N energies from the JPDF $P(E)$ and constructs the diagonal matrix

$$D = \text{diag}(E_1, \dots, E_N), \quad (1.70)$$

where by convention the energies are ordered ($E_i \leq E_j$ if $i \leq j$).

- Second, one constructs a rotation matrix O by drawing $N(N-1)/2$ independent variables from $P(\eta)$, imposing recursively the normalization and the orthogonality of eigenvectors.
- Lastly, one constructs the matrix

$$H = O^T D O, \quad (1.71)$$

which is by definition a GOE matrix.

It then follows that O can be constructed from $P(\eta)$ only: in other words, given a set of eigenvectors drawn from $P(\eta)$, they can always be associated with a GOE matrix whose spectrum is given by D .

1.2.3. Hamiltonian Lines

Consider a quantum Hamiltonian $H(\lambda)$ that depends on a real parameter λ . The corresponding energy levels $E_i(\lambda)$ can be represented as curves in the (E, λ) plane. In general, two arbitrary curves in a plane – obtained by varying a single parameter – are expected to intersect. However, this is typically not the case when the curves represent eigenvalues of a quantum Hamiltonian. In fact, degeneracy between two energy levels generally requires the variation of two independent parameters, not just one (see, for instance, [31, 32]). Let us present the familiar argument for this statement together with a simple derivation of the behaviour of the probability density of the gap.

Let us imagine that there exists a value λ_* where two energy levels $E_1(\lambda_*)$ and $E_2(\lambda_*)$ get arbitrarily close. Making a proper rotation in the Hilbert space, in the vicinity of this value of λ , we can restrict our attention to the 2×2 block matrix, which involves these two levels, ignoring the rest of the spectrum

$$H = \begin{bmatrix} E_1 & V \\ V^* & E_2 \end{bmatrix}. \quad (1.72)$$

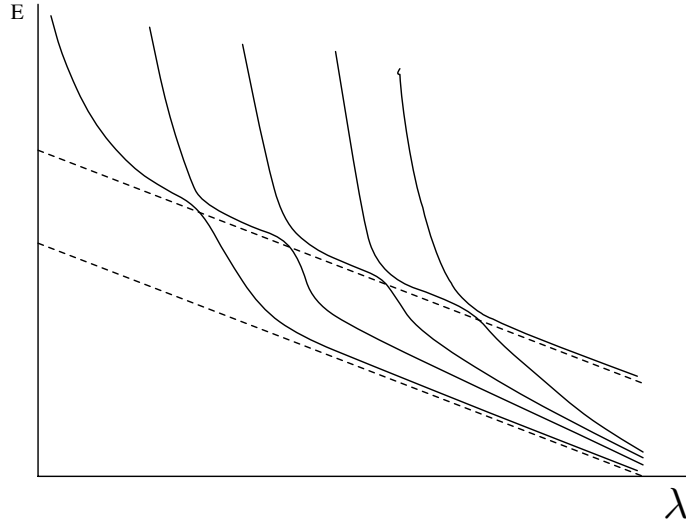


Figure 1.3: Typical example of level repulsion in a quantum non-integrable Hamiltonian which depends on a parameter λ .

After diagonalization, we have the true energy levels of this two-level system

$$\hat{E}_{1,2} = \frac{E_1 + E_2}{2} \pm \frac{1}{2} \sqrt{(E_1 - E_2)^2 + |V|^2}. \quad (1.73)$$

Therefore, in order to have a crossing of the two levels, we need to impose two conditions: (i) $E_1 = E_2$ and (ii) $V = 0$. Taking the first of them for granted, i.e. $E_1 = E_2$, whether or not the second condition is satisfied generally delineates two distinct classes of Hamiltonians: those describing integrable systems (when $V = 0$ for a symmetry reason, e.g. the two states transform according to two different irreducible representations of the symmetry group) and those characterizing chaotic ones ($V \neq 0$). In the latter case, nearby eigenstates “hybridize” due to the absence of conservation laws, which creates avoided crossings and leads to level repulsion.

Indeed, with $V \neq 0$, let us estimate the probability density of finding a level at a distance Δ from a given one

$$\hat{P}(\Delta) \propto \int dE_- \mathcal{D}V \delta\left(\Delta - \sqrt{(E_1 - E_2)^2 + |V|^2}\right), \quad (1.74)$$

where $E_- = E_1 - E_2$. For the integration over the variable V , we have two cases, depending on whether V is real or complex:

$$\mathcal{D}V = \begin{cases} dV, & V \text{ real,} \\ dV_1 dV_2, & V \text{ complex with } V = V_1 + iV_2. \end{cases} \quad (1.75)$$

Integrating over E_- , we are left with

$$\hat{P}(\Delta) \propto \int \mathcal{D}V \sqrt{1 - \frac{|V|^2}{\Delta^2}} \theta(\Delta - |V|) \quad (1.76)$$

which results in

$$\hat{P}(\Delta) \propto \begin{cases} \Delta, & V \text{ real,} \\ \Delta^2, & V \text{ complex.} \end{cases} \quad (1.77)$$

1.2.4. Wigner's Semicircle

We now begin the study of the properties of eigenvalues, that is their statistical correlations. Because the JPDF $P(E)$ depends on all the eigenvalues simultaneously, we may ask questions regarding the first marginals, i.e. the 1-point and 2-point connected functions.

The 1-point connected function

$$\rho(E) = \int dE_2 \dots dE_N P^{\text{unordered}}(E, E_2, \dots, E_N), \quad (1.78)$$

is nothing but the DOS and the two-point correlator

$$R_2(E, E') = N \int dE_3 \dots dE_N P^{\text{unordered}}(E, E', E_3, \dots, E_N) \quad (1.79)$$

quantifies the correlation between any two unordered energy levels – it is possible to use ordered energies in the definition, but the notation becomes cumbersome.

We are concerned in this section with the evaluation of the DOS, which we anticipate follows a semicircle distribution (a.k.a. Wigner's semicircle). We observe that $P(E)$ may be written in terms of a Coulomb gas generating function as follows. First, we define the rescaled energies

$$x_i = E_i / \sqrt{N}. \quad (1.80)$$

In terms of this rescaling, we may rewrite

$$P(E) \propto e^{-N^2 W}, \quad W = \frac{1}{2N} \sum_i x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \log |x_i - x_j|. \quad (1.81)$$

$P(E)$ is naturally described by a one-dimensional fluid of particles at positions $\{x_1, \dots, x_N\}$, interacting through an effective interaction W . The particles are static, and they are naturally attracted to the center of a quadratic potential located at

$x = 0$ (the average energy). However, they cannot be arbitrarily close because of the logarithmic repulsion, which prevents the equilibrium configuration from being localized at zero. Therefore, the level repulsion of GOE matrices originates from the Vandermonde determinant.

We are interested in the limit $N \rightarrow \infty$, which corresponds simultaneously to the thermodynamic limit and to the limit of zero effective temperature $\beta = 1/N^2 \rightarrow 0$. To compute the DOS, we utilize the EDOS as counting function

$$n(x) = \frac{1}{N} \sum_j \delta(x - x_j). \quad (1.82)$$

Then, the DOS is nothing more than

$$\rho(x) = \int dx_1 \dots dx_N n(x) e^{-N^2 W[\{x\}]}. \quad (1.83)$$

The r.h.s. has the form of a path integral where $\{x_1, \dots, x_N\}$ describes the trajectory of the particle.

We do not assume that $n(x)$ is a function with a fixed shape; rather, we would like to average over the possible configurations of $n(x)$ and derive its saddle point. To this end, we introduce the functional identity

$$1 = \int \mathcal{D}n(x) \delta\left(n(x) - \frac{1}{N} \sum_j \delta(x - x_j)\right). \quad (1.84)$$

We observe that W can be rewritten using the function n alone, since it amounts to a counting function. The quadratic term is simple:

$$\frac{1}{N} \sum_i x_i^2 = \int_{\mathbb{R}} dx n(x) x^2. \quad (1.85)$$

The logarithmic term requires care: while, in general, $\log|x_i - x_j|$ may become arbitrarily large and negative, it will never be infinite because the probability that two eigenvalues are coincident must be zero. Indeed, if two eigenvalues are coincident, the Vandermonde determinant vanishes, and therefore the probability $P(H)$ of the corresponding GOE matrix is zero. However, when introducing the counting function, the quantity $\log|x - x'|$ may be infinite. Therefore, we apply a *coarse-graining* procedure and declare that the difference $|x - x'|$ may never be smaller than a set threshold, possibly dependent on x itself. We label the minimal value of the difference as $\Delta(x)$.

Therefore, before introducing $n(x)$ in the logarithmic interaction term, we perform point splitting:

$$\begin{aligned} \frac{1}{2N^2} \sum_{i \neq j} \log |x_i - x_j| &= \frac{1}{2N^2} \left[\sum_{i,j} \log |x_i - x_j| - \sum_i \log \Delta(x_i) \right] \\ &= \int dx dx' n(x) n(x') \log |x - x'| - \frac{1}{2N} \int dx n(x) \log \Delta(x). \end{aligned} \quad (1.86)$$

In other words, the second term of the equation absorbs the divergence of the functional integration, generating a finite result.

Concerning the choice of $\Delta(x)$, there is a natural ansatz which takes into account two facts: the cutoff should decrease in regions with greater $n(x)$, i.e. it should be minimal in the bulk spectrum, and N charges spanning a finite interval have an average spacing $O(1/N)$. These two assumptions justify setting

$$\Delta(x) = \frac{c}{Nn(x)}, \quad (1.87)$$

where the constant c is rather inconsequential and can be absorbed in the normalization of the path integral, which in turn is fixed by requiring that $\rho(x)$ is normalized.

We now have to evaluate the *combinatorial* term associated with

$$I_N[n(x)] = \int_{\mathbb{R}} \prod_k dx_k \delta \left(n(x) - \frac{1}{N} \sum_{j=1}^N \delta(x - x_j) \right), \quad (1.88)$$

which counts the number of microstates compatible with a given density $n(x)$. Let us introduce the integration over a dummy density $\hat{n}(x)$ and obtain

$$\begin{aligned} I_N[n(x)] &= \int \mathcal{D}[\hat{n}(x)] \int_{\mathbb{R}} \prod_k dx_k \exp \left(iN \int dx \hat{n}(x) n(x) \right. \\ &\quad \left. - i \sum_{j=1}^N \int dx \hat{n}(x) \delta(x - x_j) \right) \\ &= \int \mathcal{D}[\hat{n}(x)] \exp \left(iN \int dx \hat{n}(x) n(x) \right) \\ &\quad \times \left[\int_{\mathbb{R}} dy \exp \left(-i \int dx \hat{n}(x) \delta(x - y) \right) \right]^N \\ &:= \int \mathcal{D}[\hat{n}(x)] e^{NS[\hat{n}(x)|n(x)]}, \end{aligned} \quad (1.89)$$

where we defined the *entropy*

$$S[\hat{n}(x)|n(x)] = i \int dx \hat{n}(x)n(x) + \log \int_{\mathbb{R}} dy e^{-i\hat{n}(y)}. \quad (1.90)$$

For large N , the value which extremizes the entropy is the stationary distribution obtained by differentiating with respect to $\hat{n}(x)$:

$$0 = \frac{\delta S}{\delta \hat{n}(x)} = in(x) - i \frac{e^{-i\hat{n}(x)}}{\int_{\mathbb{R}} dy e^{-i\hat{n}(y)}}, \quad (1.91)$$

which gives

$$i\hat{n}(x) = -\log n(x) - \log \int_{\mathbb{R}} dy e^{-i\hat{n}(y)}. \quad (1.92)$$

Since $\int dx n(x) = 1$, substituting this into the entropy gives

$$S[n(x)] = - \int dx n(x) \log n(x), \quad (1.93)$$

which precisely coincides with the Shannon entropy of the profile. What we observe is that the factor $I_N[n(x)] \sim e^{NS}$ is subleading compared with the $e^{-N^2 W}$ term and can therefore be discarded in the optimization of n .

However, it is rather crucial that $n(x)$ is properly normalized. To this end, we introduce the following multiplier

$$\delta \left(1 - \int dx n(x) \right) = \int_{\mathbb{R}} \frac{dk}{2\pi} \exp \left(ik \left(1 - \int dx n(x) \right) \right). \quad (1.94)$$

In order for this multiplier not to be suppressed, we need to rescale $k = N^2 \kappa$. In other words, we must require that the optimal profile satisfy

$$0 = \frac{\delta}{\delta n(x)} \left(\int dx n(x) \frac{x^2}{2} - \int dx dx' n(x)n(x') \log |x - x'| - \kappa \int dx n(x) \right), \quad (1.95)$$

leading to the integral equation

$$\frac{x^2}{2} - \int dx' n(x') \log |x - x'| - \kappa = 0, \quad (1.96)$$

subject to the condition $n(x) \geq 0$ and $\int dx n(x) = 1$.

Let us inspect the limit $x \rightarrow \infty$. What we observe is that there is a fundamental

incompatibility of the equation

$$\frac{x^2}{2} - \log x - \kappa \sim 0, \quad (1.97)$$

which implies that the support of $n(x)$ is a finite interval (a, b) . The solution can be obtained by first differentiating the equation – using the principal value for the weak derivative – and then evaluating the free energy for arbitrary (a, b) and choosing them to minimize it, which results in $a = -b = \sqrt{2}$. Performing these steps gives rise to Wigner’s semicircle law [25]:

$$\rho(x) = \frac{1}{\pi} \sqrt{2 - x^2} = \frac{1}{\pi} \sqrt{2 - \frac{E^2}{N}}. \quad (1.98)$$

1.2.5. Wigner-Dyson distribution

We shall begin with the discussion of a 2×2 GOE matrix, and compute the PDF of the spacing $s = E_2 - E_1$. This computation is relevant because the PDF thus obtained captures all the qualitative features of the $N \times N$ matrix in the limit $N \rightarrow \infty$ – see e.g. [24] for the precise computation of upper and lower bounds on the PDF.

We thus consider the matrix

$$H_s = \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix}, \quad (1.99)$$

where x_1 and $x_2 \sim \mathcal{N}(0, 1)$, while $x_3 \sim \mathcal{N}(0, 1/2)$. The eigenvalues E_i are solutions of the characteristic equation

$$E^2 - \text{Tr}(H_s)E + \det H_s = 0, \quad (1.100)$$

which gives the solutions

$$E_{1,2} = \frac{1}{2} (x_1 + x_2 \pm s), \quad s = \sqrt{(x_1 - x_2)^2 + 4x_3^2}. \quad (1.101)$$

Since s is a composite random variable, its PDF is

$$p(s) = \int_{-\infty}^{\infty} dx_1 dx_2 dx_3 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-x_3^2}}{\sqrt{\pi}} \delta\left(s - \sqrt{(x_1 - x_2)^2 + 4x_3^2}\right). \quad (1.102)$$

This integral is easily solved in polar coordinates

$$r \cos \theta = x_1 - x_2, \quad r \sin \theta = 2x_3, \quad \psi = x_1 + x_2. \quad (1.103)$$

The determinant of the transformation is $(-r/4)$. Thus, we obtain

$$p(s) = \frac{s}{2} e^{-s^2/4}. \quad (1.104)$$

However, the $p(s)$ in the current form has average $\langle s \rangle = \sqrt{\pi}$. It is therefore convenient to bring the PDF in the standard form

$$p(s) \rightarrow \langle s \rangle p(s/\langle s \rangle), \quad (1.105)$$

yielding the Wigner-Dyson distribution or Wigner's surmise

$$p_{\text{WD}}(s) = \frac{\pi s}{2} e^{-\pi s^2/4}, \quad (1.106)$$

whose plot is shown in Fig. 1.2(b).

Wigner's surmise The PDF $p_{\text{WD}}(s)$ can also be derived from a heuristic conjecture by Wigner [33, 34], which goes as follows. For a random sequence, the probability that a level will be in the small interval $I = (E + s, E + s + ds)$ is proportional to ds and will be independent of whether or not there is a level at E . However, if there is level repulsion – which is justified a posteriori by the presence of the Vandermonde determinant in $P(E)$ – the argument must be modified, since the probability we are looking for concerns the occurrence of the two events A and B :

A: no-level in $(E, E + s)$;

B: one level in $(E + s, E + s + ds)$.

Therefore, we are concerned with the joint probability of the events $A \cup B$, given by $P(A \cup B) = P(A/B)P(B)$, where $P(A/B)$ is the conditional probability of A given B . Hence, we have $p(s)ds = P(A \cup B)$ and the level spacing probability density $p(s)$ is such that

$$p(s)ds = \text{Prob}(\text{one level in } dI / \text{no level in } I) \text{Prob}(\text{no level in } I). \quad (1.107)$$

In terms of $p(s)$, the probability of having no level in I is given by

$$\text{Prob}(\text{no level in } I) = \int_s^\infty ds' p(s'), \quad (1.108)$$

namely, we need to have all level spacings larger than s and not to have a level in the interval $(E, E + s)$. Hence, with the notation

$$\mu(s)ds = \text{Prob}(\text{ one level in } dI / \text{ no level in } I), \quad (1.109)$$

we have

$$p(s) = \mu(s) \int_s^\infty ds' p(s'). \quad (1.110)$$

The differential equation that comes from this relation can be easily solved

$$p(s) = \mathcal{N} \mu(s) e^{-\int_0^s dt \mu(t)}. \quad (1.111)$$

This probability distribution must satisfy two conditions:

1. normalization

$$\int ds p(s) = 1; \quad (1.112)$$

2. Unit mean gap

$$\int ds s p(s) = 1. \quad (1.113)$$

The probability distribution $p(s)$ is fully determined once the quantity $\mu(s)$ is specified. However, it should be noted that the resulting $p(s)$ is expected to accurately capture the behaviour only in the limit of small s . With this remark in mind, if we consider the case where $\mu = \text{const}$, imposing the two conditions above yields the Poisson distribution

$$p(s) = e^{-s}, \quad (1.114)$$

This is a distribution which we will meet in Section 1.3.6 in relation to the energy-level distribution of integrable models. On the other hand, taking for $\mu(s)$ a linear repulsive law for the levels, $\mu(s) \simeq \alpha s$, and imposing the two conditions above, the unit-mean-gap condition fixes α to the value $\alpha = \pi/2$, so that we recover the GOE gap distribution.

It is important to note that choosing $\mu(s) \simeq \alpha s^2$ does *not* reproduce the full gap distribution of other ensembles (i.e. the unitary and symplectic ones). Instead, it yields a different distribution that agrees with them only in the small- s limit.

Finally, assuming that $\mu(s)$ has a small s behavior $\mu(s) \simeq \alpha(1 + \nu)s^\nu$, one gets the Brody set of distributions [35]

$$p_B(s) = (1 + \nu)as^\nu \exp(-as^{1+\nu}), \quad a = \left[\Gamma\left(\frac{2 + \nu}{1 + \nu}\right) \right]^{1+\nu}, \quad (1.115)$$

where ν is the so-called “Brody parameter”. The interesting feature of this family of distributions is that it interpolates between the Poisson distribution ($\nu = 0$) and the GOE one ($\nu = 1$).

1.2.6. Higher order spacing distributions

Important information about the global distribution of energy levels is encoded in their correlation functions. The n -point correlation function of an $N \times N$ random matrix is given by [36–38]

$$R_n(x_1, \dots, x_n) = \frac{N!}{(N - n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_{n+1} \cdots dx_N P_N(x_1, \dots, x_N). \quad (1.116)$$

This expression gives the probability density of finding a level (regardless of its labeling) around each of the points $x_1, x_2, \dots, x_n$, once we integrate out the remaining levels. Exact evaluations of these distributions can be found in the classical works of Mehta [24, 39]. Unfolding the energy spectrum and going to the $N \rightarrow \infty$ limit, correlation functions depend only on the eigenvalue differences $x_i - x_j$, for $n \geq 2$. In particular, setting $s = x_1 - x_2$, for the pair-correlation $R_2^{(\beta)}(s)$ we have (see Figure 1.4)²

$$R_2^{(\beta)}(s) = \begin{cases} 1 - K^2(s) - \frac{dK}{ds}J(s), & (\text{GOE}), \beta = 1, \\ 1 - K^2(s), & (\text{GUE}), \beta = 2, \end{cases} \quad (1.117)$$

where

$$K(s) = \frac{\sin(\pi s)}{\pi s}, \quad J(s) = \int_s^\infty dt K(t). \quad (1.118)$$

²In this subsection, we also present results for the Gaussian Unitary Ensemble (GUE), which is a rotation-invariant ensemble where H is Hermitian but generally not real-valued. Therefore, rotation matrices are generic unitary matrices, rather than simply orthogonal matrices as in the case of the GOE. It is conventionally associated with an exponent $\beta = 2$, corresponding to

$$P^{(\beta)}(E_1, \dots, E_N) \propto \int \prod_{j=1}^N dE_j e^{-\frac{1}{2}\lambda_j^2} \prod_{1 \leq k < l \leq N} |\lambda_l - \lambda_k|^\beta.$$

The GOE ensemble has $\beta = 1$, the GUE ensemble has $\beta = 2$, while the Gaussian Symplectic Ensemble has $\beta = 4$.

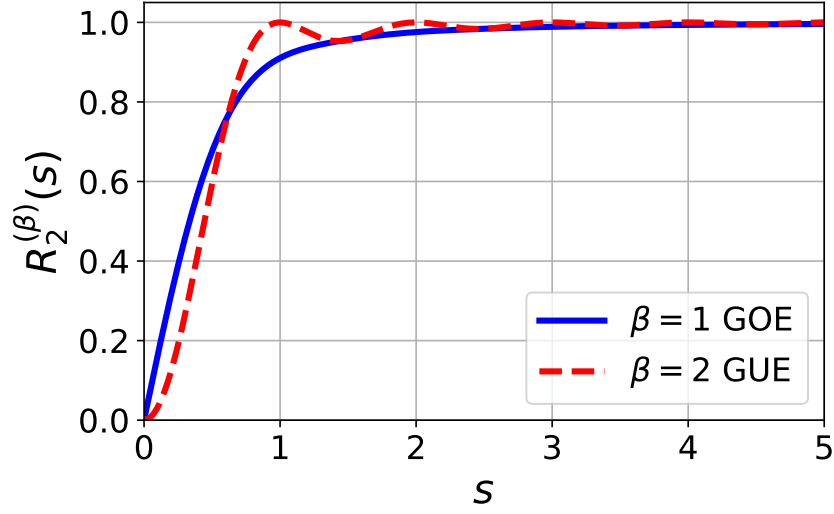


Figure 1.4: 2-point correlation function $R_2^{(\beta)}(s)$ for the GUE and GOE. Note that both curves vanish at the origin, reflecting level repulsion, and rapidly saturate to the asymptotic value 1, which characterizes the spectral rigidity.

Quantities directly related to the correlation functions are the higher-order spacing distributions $P_k^{(\beta)}(s)$, where we recall that the k -step spacing is defined as

$$s_{n,k} = e_{n+k} - e_n. \quad (1.119)$$

For those gaps, the average is

$$\langle s_{n,k} \rangle = \sum_{i=1}^k s_i = k \langle s_i \rangle = k, \quad (1.120)$$

since $\langle s_i \rangle = 1$. Moreover, we have the identity

$$R_2^{(\beta)}(s) = \sum_{k=1}^{\infty} P_k^{(\beta)}(s), \quad (1.121)$$

since $R_2^{(\beta)}(s)$ is the probability distribution for having a spacing s between *any* two eigenvalues of the unfolded spectrum.

Following Mehta [24], if we denote by $E_\beta(j; s)$ the probability that an interval of length s contains exactly j eigenvalues for the Dyson ensemble with index β , then the probability density $P_k^{(\beta)}(s)$ of the k th neighbour spacing (i.e., the distance between

two eigenvalues with $k - 1$ intervening levels) is given by

$$P_k^{(\beta)}(s) = \frac{d^2}{ds^2} \left[\sum_{j=0}^{k-1} (k-j) E_\beta(j; s) \right] . \quad (1.122)$$

This formula follows from the following considerations:

1. **Gap probability.** The quantity $E_\beta(j; s)$ is a *gap probability* for the interval $(0, s)$; namely, it enforces that exactly j points fall within the interval.

- Differentiating once in s introduces the condition that a point is located at the right boundary of the interval.
- Differentiating a second time enforces that this boundary point is the *nearest eigenvalue* at a distance s from the origin, i.e. it turns the gap probability into a spacing density.

Thus, the second derivative in (1.122) is the probabilistic mechanism that converts an occupancy probability into a spacing distribution.

2. **Origin of the combinatorial weight $(k-j)$.** Suppose there are j points inside $(0, s)$. Then the interval $[0, s]$ spans a block of $k+2$ consecutive eigenvalues (the two endpoints plus the k interior ones). Among the $k+1$ gaps formed by these $k+2$ levels, precisely $(k+1-j)$ of them can serve as the “anchoring” gap coinciding with the interval $[0, s]$. This simple combinatorial count yields the weight $(k+1-j)$ and, shifting $k \rightarrow k-1$, we end up with the factor $(k-j)$ in (1.122).

For the GUE, the quantities defined above admit a Fredholm determinantal representation

$$D(z, s) = \text{Det}(1 - (1-z)\mathcal{K}_{[0,s]}) , \quad (1.123)$$

where the integral operator $\mathcal{K}_{[0,s]}$ is defined in $L^2([0, s])$ in terms of the kernel $K(x, y) \equiv K(x-y)$ of Eq. (1.118) as

$$(\mathcal{K}_{[0,s]} f)(x) = \int_0^s K(x, y) f(y) dy . \quad (1.124)$$

Indeed, we have

$$E_2(k; s) = \frac{1}{k!} \left. \frac{d^k}{dz^k} D(z, s) \right|_{z=0} . \quad (1.125)$$

As discussed in Appendix A, the calculation of the Fredholm determinant (1.123)

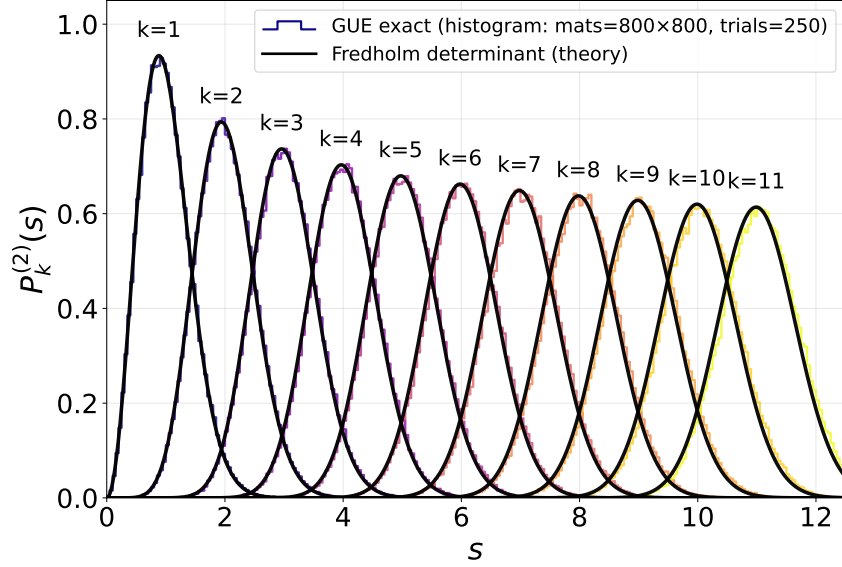


Figure 1.5: Higher order spacing probability distribution $P_k^{(2)}(s)$ ($k = 1, 2, \dots, 11$) for GUE. The histogram is an average over 250 realizations of GUE spectra, each from an 800×800 matrix. The k distribution is sharply peaked at $s \simeq k$, reflecting level repulsion and spectral rigidity.

through its eigenvalues can be implemented numerically very efficiently [40, 41]. The corresponding probability distribution for the higher spacings is shown in Figure 1.5. As discussed in Appendix A, analogous formulas also hold for the GOE, substituting the determinant with the Pfaffian.

1.3. Quantum Integrability

To understand the role of integrability in one-dimensional quantum many-body systems, we first discuss the notion of integrability in classical dynamics and in quantum field theories. These two cases allow us to identify the minimal set of assumptions that an integrable quantum many-body system has to satisfy; we then proceed with the discussion of concrete models.

1.3.1. Classical Systems with a Finite Number of Degrees of Freedom

Let us consider a classical system with n physical particles in d dimensions. The number of degrees of freedom is $N = nd$ and the phase space of generalized coordinates

and momenta, (q_i, p_i) , is $2N$ -dimensional. In classical systems, integrability has a well-defined meaning and there exist techniques to solve explicitly the equations of motion.

In classical mechanics, a system is said to be *integrable* if it possesses N independent integrals of motion $Q_i(q, p)$ which are in mutual involution – that is, their Poisson bracket vanishes:

$$\{Q_i, Q_j\} = 0, \quad \forall i, j. \quad (1.126)$$

When time independent, the Hamiltonian is the integral of motion associated with temporal translations.

Such integrals of motion are conserved quantities and therefore are not individually modified by the dynamics. An initial configuration with given values of the $Q_i(q, p)$ establishes N relations satisfied by the q_i 's and p_i 's at any time t . Thus, the submanifold of configurations explored by the dynamics has dimension $D = 2N - N = N$ and, under quite general conditions [42], has the topology of an N -dimensional torus. Such a manifold can be parametrized by a set of *canonical* action-angle variables (I_i, φ_i) . Denoting by γ_i the i -th irreducible cycle of the torus, the action variables I_i are defined as

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot d\mathbf{q}, \quad (1.127)$$

and are functions of the integrals of motion Q_i . Conversely, one can write integrals of motion as functions of the action variables

$$Q_i = Q_i(I_1, \dots, I_N). \quad (1.128)$$

Of course, the Hamiltonian $H = H(I_1, \dots, I_N)$ is an integral of motion and therefore a function of the action variables. Angle variables φ_i are naturally defined as the conjugates of the I_i 's, i.e. through Hamilton's equations

$$\dot{I}_i = -\frac{\partial H}{\partial \varphi_i} = 0, \quad \dot{\varphi}_i = \frac{\partial H}{\partial I_i} := \omega_i, \quad (1.129)$$

where $\dot{f} = \frac{df}{dt}$. The solutions show that the I_i 's are constant and that the angle variables are linear

$$\varphi_i = \varphi_{i,0} + \omega_i t. \quad (1.130)$$

Classical ergodic systems are characterized by dynamical trajectories that densely visit the phase space. On the other hand, classical integrable models constrain phase-space dynamics to N -dimensional tori, and therefore they are prototypes of

non-ergodic systems.

Despite this feature, it is worth stressing that for large N it may be difficult to visually distinguish the motion of an integrable model from that of a non-integrable one. The reason is that all single-valued functions $f(\mathbf{q}, \mathbf{p}, t)$ of the system, expressed in terms of the action-angle variables, are periodic functions of φ_i and therefore admit a decomposition into Fourier modes. Schematically, we can write such a decomposition as

$$f(\mathbf{q}, \mathbf{p}, t) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} f_{n_1, \dots, n_N} e^{it(n_1\omega_1 + \cdots + n_N\omega_N)}. \quad (1.131)$$

Since the frequencies ω_i are typically incommensurable, this function is not strictly periodic for any finite t , but only at $t \rightarrow \infty$. At finite N , one may observe characteristic revivals, which are reminiscent of a periodic structure. Nonetheless, the corresponding visual intricacy of motion may hide the integrability structure of the model, especially when the number of degrees of freedom tends to infinity, $N \rightarrow \infty$. Examples of dynamical systems with infinitely many degrees of freedom are field theories, both in the relativistic and non-relativistic versions.

1.3.2. Classical Systems with Infinite Degrees of Freedom

A theory whose Hamiltonian is expressed in terms of fields $\phi_a(\mathbf{x}, t)$ is an example of dynamics with an infinite number of degrees of freedom. The fields $\phi_a(\mathbf{x}, t)$ are generally time- and position-dependent functions and are labeled by an internal index a . In the present discussion, we restrict ourselves to field theories in one spatial and one temporal dimension (abbreviated hereafter as $(1+1)$ -d)³, whose dynamics is encoded in the Lagrangian density $\mathcal{L}(\phi_a, \partial_t \phi_a, \partial_x \phi_a)$. Conjugate momenta are defined as

$$\pi_a(x, t) = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi_a)}, \quad (1.132)$$

and satisfy the equal-time Poisson bracket relations

$$\{\phi_a(x, t), \pi_b(x', t')\} = \delta_{a,b} \delta(x' - x) \delta(t' - t). \quad (1.133)$$

³Indeed, a general theorem by Coleman and Mandula [43] rules out the possibility of nontrivial integrable *quantum* field theories in spatial dimensions higher than 1. By nontrivial, we mean theories whose scattering matrix does not describe free bosons or fermions.

The Hamiltonian density is defined through the functional Legendre transform

$$\mathcal{H}(x, t) = \sum_a \phi_a(x, t) \pi_a(x, t) - \mathcal{L}(x, t). \quad (1.134)$$

Integrating over the spatial direction gives the Hamiltonian, which is the generator of time-evolution

$$H = \int dx \mathcal{H}(x, t). \quad (1.135)$$

In analogy to finite-dimensional classical systems, we can extend the notion of integrability to field theories: we require the presence of *infinitely* many functionally independent conserved charges $Q_i(\phi_a, \pi_a)$, which satisfy Eq. (1.126). Because these quantities are conserved, their time derivatives must vanish:

$$\frac{dQ_i}{dt} = \frac{\partial_t Q_i}{\partial t} + \{Q_i, H\} = 0. \quad (1.136)$$

As for the Hamiltonian, these charges admit a decomposition in terms of a local density $\rho_i(x, t)$ associated with a current $j_i(x, t)$, such that

$$Q_i = \int dx \rho_i(x, t), \quad \partial_t \rho_i(x, t) + \partial_x j_i(x, t) = 0. \quad (1.137)$$

For models which are known to be classically integrable, such as the Sin(h)-Gordon model, their action-angle variables and their explicit solutions of the equation of motion are provided by the Inverse Scattering Methods [44, 45]. Explicit expressions of the conserved densities $\rho_i(x, t)$ for the Sinh-Gordon model can be found e.g. in [46]. When properly generalized, these classical integrable systems yield non-trivial examples of quantum integrable models [47]. But how can one identify and possibly classify *all* quantum integrable systems with infinite degrees of freedom? To the best of our knowledge, a clear and well-defined answer exists only in the context of relativistic quantum field theories.

1.3.3. Integrable Quantum Field Theories

The simplest approach to defining a quantum field theory from a classical Lagrangian is to take the equations of motion of the classical field theory and promote them to the *on-shell* description of the theory. The field can be decomposed into modes which, if assumed *off-shell*, describe the *quantum fluctuations* of the field. However, the brackets of quantum many-body dynamics are governed not by Poisson brackets but by commutators, which ensure the validity of the Heisenberg equation – i.e. the

time-evolution equation of operators. Naturally, quantum field theories (QFTs) may also admit conserved charges, which are observables satisfying the involution

$$[Q_a, Q_b] = 0. \quad (1.138)$$

It is worth emphasizing for now that the existence of an *infinite set* of mutually commuting conserved charges implies that all scattering processes – regardless of the number of particles involved – are purely elastic and factorizable [48]. This holds despite the fact that relativistic field theories generally allow particle production whenever sufficient energy is available in the center-of-mass frame. For kinematical reasons⁴, nontrivial examples of integrable relativistic quantum field theories exist only in $(1 + 1)$ dimensions. The list includes purely bosonic models, such as the Sinh-Gordon and, more generally, Toda models [47, 50–60], fermionic systems, such as the Gross-Neveu model [48], as well as supersymmetric versions of all these examples (see, for instance, [49, 61–63]). This list increases even further if one also adds theories with soliton excitations, such as the Sine-Gordon model or various sigma models based on group manifolds [48, 64, 65].

The elasticity property of scattering amplitudes has an immediate consequence for relativistic QFTs: if the scattering is to be elastic, all production and decay processes must be forbidden as a consequence of the peculiar values of the parameters of the model. Conversely, if for a given model we are able to show the existence of production processes, this fact alone automatically provides explicit proof that such a model is *not* integrable. This criterion has the advantage of being checkable for any process involving a finite number of particles in a *finite* number of steps [57].

For relativistic quantum field theories which are uniquely defined by diagrammatic rules (as in *Diagrammar* by 't Hooft and Veltman [66]), we then have the following *integrable sieve*: given a class of models – specified by the number of fields present in the Lagrangian and its symmetry – the systematic analysis of all production processes acts, as a matter of fact, as a *sieve*, in the sense that at the end of this procedure, we are left with very special Lagrangians which – by construction – are integrable. On the basis of this algorithmic procedure, it has been possible, for instance, to exclude from the list of integrable models all Lagrangians with purely polynomial interactions, namely Landau-Ginzburg (LG) theories. On the other hand,

⁴The eigenvalues of the local charges on multi-particle states, which are their common eigenvectors, can be expressed as the sum of higher powers of their momenta. Applying the charges on states with different momenta, one can arbitrarily shift the point in space-time where the interactions take place [49]. In two or more spatial dimensions, the only integrable theories with localized particle excitations are the free ones.

it has led to the definition of new quantum integrable field theories using models which are generalizations of the Sinh-Gordon model, namely the affine simply-laced Toda field theories [47, 50–60, 67, 68]: these theories are built from the simple roots of the ADE Lie algebras, and the Sinh-Gordon model (corresponding to the algebra A_1) is indeed their simplest representative. For the identification of non-relativistic local and continuous field theories based on the considerations made above, see [69, 70].

1.3.4. The Role of Explicit Symmetries

The class of integrable models presented so far is characterized by the presence of conserved quantities, which by Noether’s theorem are associated with symmetries. However, these symmetries need not be explicit, and it may in fact be very difficult to spot them by inspecting the Hamiltonian itself.

The most common symmetries of a physical problem are explicit symmetries. They may be global – usually associated with a discrete group, such as charge conjugation and isospin – or local. In the latter case, the prototypes are gauge theories based on continuous groups derived from Lie algebras. We briefly discuss symmetries based on discrete Abelian groups $\mathbb{Z}_n$, mentioning relevant physical models.

- **$\mathbb{Z}_2$ Field Theories.** Consider a real scalar bosonic field $\varphi(x, t)$ in $(1 + 1)$ dimensions, subject to two different dynamics associated with the Hamiltonian densities

$$\begin{aligned}\mathcal{H}_{LG} &= \frac{1}{2} \left[(\pi)^2 + (\partial_x \varphi)^2 + m^2 \varphi^2 \right] + \frac{g}{4!} \varphi^4, \\ \mathcal{H}_{ShG} &= \frac{1}{2} \left[(\pi)^2 + (\partial_x \varphi)^2 \right] + \frac{m^2}{2\gamma} (\cosh(\gamma \varphi) - 1)\end{aligned}\tag{1.139}$$

the first refers to the φ^4 Landau-Ginzburg theory while the second refers to the Sinh-Gordon theory⁵. Both systems are invariant under the $\mathbb{Z}_2$ symmetry $\varphi \rightarrow -\varphi$ which therefore splits their Hilbert space into even and odd sectors,

⁵Upon quantization, the various quantities presented in these expressions must be regarded as defined by proper normal ordering.

so that the matrix of their Hamiltonian takes the block form

$$H = \begin{bmatrix} \text{blue square} & \\ & \text{red square} \end{bmatrix}. \quad (1.140)$$

This $\mathbb{Z}_2$ symmetry is of course the global symmetry of both models and defines the chirality sectors.

However, the Hamiltonian of the Sinh-Gordon theory possesses an infinite set of hidden, conserved local charges Q_a , which are absent in the Landau-Ginzburg theory. These charges define a dynamical symmetry unique to the Sinh-Gordon model and not present in the Landau-Ginzburg case. As a consequence, while the Hilbert space of the Landau-Ginzburg theory decomposes only into two sectors – corresponding to even and odd parity – the Hilbert space of the Sinh-Gordon model splits into an infinite number of subspaces, each characterized by the quantum numbers associated with these conserved charges. Ultimately, this implies that the Hamiltonian of the Sinh-Gordon model has a block-diagonal structure composed of infinitely many blocks of varying dimension, both in the even and in the odd sector of the global $\mathbb{Z}_2$ symmetry:

$$H_{ShG} = \begin{bmatrix} H_0 & 0 & 0 & \cdots \\ 0 & H_1 & 0 & \cdots \\ 0 & 0 & H_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{with } \dim(H_n) = d_n. \quad (1.141)$$

Moreover, in this case the matrix elements of the Hamiltonian density and other local operators can be computed exactly [71, 72].

- **$\mathbb{Z}_3$ Field Theory.** A less trivial example is provided by a vector of two scalar fields defined in $(1+1)$ dimensions, $\phi(x, t) = (\phi_1(x, t), \phi_2(x, t))$, which can also be combined into the complex field $\Phi = \phi_1 + i\phi_2$, subject to two different

dynamics associated with the Hamiltonian densities⁶

$$\begin{aligned}\mathcal{H}_{LG} &= \frac{1}{2} \left[(\Pi \Pi^\dagger) + (\partial_x \Phi \partial_x \Phi^\dagger) + m^2 \Phi \Phi^\dagger \right] + \frac{g_1}{3!} (\Phi^3 + \Phi^{\dagger 3}) + \frac{g_2}{4!} (\Phi \Phi^\dagger)^2, \\ \mathcal{H}_{Toda} &= \frac{1}{2} \sum_{i=1}^2 \left[(\pi_i)^2 + (\partial_x \phi_i)^2 \right] + \frac{m^2}{\beta} \sum_{k=0}^2 e^{\beta \alpha_k \cdot \phi}\end{aligned}\tag{1.142}$$

the first refers to a complex Landau-Ginzburg theory while the second refers to an Affine Toda Field Theory associated with the root system A_2 where, in the last expression, $\alpha_1 = (1, -1/\sqrt{3})$ and $\alpha_2 = (0, 2/\sqrt{3})$ are the simple roots of the Lie algebra A_2 while $\alpha_0 = -(\alpha_1 + \alpha_2) = (-1, -1/\sqrt{3})$ is the (minus) maximal root of this Lie algebra.

Both theories are invariant under a $\mathbb{Z}_3$ symmetry: this is explicitly manifest in H_{LG} of the Landau-Ginzburg theory since it corresponds to the field transformations

$$\Phi \rightarrow e^{2\pi i/3} \Phi, \quad \Phi^\dagger \rightarrow e^{-2\pi i/3} \Phi^\dagger \tag{1.143}$$

while, in the Affine Toda Field Theory, the three roots of the A_2 algebra are invariant under their $\mathbb{Z}_3$ cyclic rotation

$$\alpha_0 \rightarrow \alpha_1, \quad \alpha_1 \rightarrow \alpha_2, \quad \alpha_2 \rightarrow \alpha_0. \tag{1.144}$$

Hence, the potential

$$V(\phi) = \sum_{k=0}^2 e^{\beta \alpha_k \cdot \phi} \tag{1.145}$$

is invariant under these cyclic rotations of the exponentials or, equivalently, under the transformations of the fields (1.143) given above. This $\mathbb{Z}_3$ symmetry is the global symmetry of both models, and therefore their Hamiltonian splits into three different blocks corresponding to the sectors with $\mathbb{Z}_3$ charges $\mathcal{C} =$

⁶In these expressions, $\pi_i = \partial_t \phi_i$, while $\Pi = \partial_t \Phi$.

$(0, e^{2\pi i/3}, e^{-2\pi i/3})$. This splitting is visually represented as

$$H = \begin{bmatrix} \text{blue square} & & \\ & \text{red square} & \\ & & \text{green square} \end{bmatrix}. \quad (1.146)$$

However, the Hamiltonian of the Affine Toda Field theory possesses an infinite set of hidden, conserved local charges Q_a [73, 74], which are instead absent in the Landau-Ginzburg theory. These charges define a dynamical symmetry unique to the Affine Toda Field theory and are not present in the Landau-Ginzburg case. As a consequence, while the Hilbert space of the Landau-Ginzburg theory decomposes only into the three sectors mentioned above, the Hilbert space of the Affine Toda Field theory splits into an infinite number of subspaces, each characterized by the quantum numbers associated with these conserved charges. Hence, the Hamiltonian of the Affine Toda Field theory ultimately has a block-diagonal structure composed of infinitely many blocks of varying dimension in the three charge sectors of the global $\mathbb{Z}_3$ symmetry.

$$H_{\text{Toda}} = \begin{bmatrix} H_0 & 0 & 0 & 0 & \cdots \\ 0 & H_1 & 0 & 0 & \cdots \\ 0 & 0 & H_2 & 0 & \cdots \\ 0 & 0 & 0 & H_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad \text{with } \dim(H_n) = d_n. \quad (1.147)$$

1.3.5. Many-Body Integrability and The Bethe Ansatz

It may occur that a quantum many-body system possesses more conserved charges than those given by the explicit symmetries of the Hamiltonian. A subset of these models, supported by these charges, can be solved exactly with various techniques. The most powerful is the Bethe Ansatz, which is nothing more than an ansatz for the factorizability of the many-body wavefunction. The exact solution comes

from the imposition of periodic boundary conditions, which allows one to find the eigenenergies as functions of a nonlinear system of equations. A second method is a mapping from local spin variables to nonlocal spinless fermions – the Jordan-Wigner transformation. Some models can be mapped to systems of free fermions, which are naturally non-interacting.

In the following, we start by discussing spin models and generalizations thereof. We focus mainly on spin-1/2 systems, where the local Hilbert space is two-dimensional. The simplest local operators in these models are Hermitian 2×2 matrices, which admit a linear decomposition in the Pauli basis:

$$\mathcal{O} = a_0 \mathbf{1} + a_X X + a_Y Y + a_Z Z, \quad (1.148)$$

with

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.149)$$

Spin-1/2 matrices are simply related to their respective Pauli matrices by a rescaling. For example, $S_X = \frac{1}{2}X$, and so on. Of importance are the ladder operators

$$S_{\pm} = S_X \pm iS_Y, \quad (1.150)$$

which increase (respectively, lower) the spin by one unit. Finally, spin matrices satisfy the algebra closure

$$[S_n^a, S_m^b] = i\delta_{n,m}\epsilon_c^{a,b}S_n^c, \quad (1.151)$$

where the lowercase indicates the site, and the uppercase the matrix type.

XX Model The simplest spin chain is the XX one, defined as

$$H_{XX}(J, h) = J \sum_{j=1}^L [S_j^x S_{j+1}^x + S_j^y S_{j+1}^y] - 2h \sum_{n=1}^L S_n^z. \quad (1.152)$$

As we shall show, this model is indeed the simplest many-body system, as it can be mapped to a Hamiltonian of free fermions via the Jordan-Wigner transformation [26–28]. It also follows that this Hamiltonian is not *strongly* interacting, despite the presence of nearest-neighbour interactions.

It is convenient to rewrite the Hamiltonian using ladder operators. Using

$$S_n^x = \frac{1}{2}(S_n^+ + S_n^-), \quad S_n^y = -\frac{i}{2}(S_n^+ - S_n^-), \quad S_n^z = S_n^+ S_n^- - \frac{1}{2}, \quad (1.153)$$

and the relations $\{S_n^+, S_n^-\} = 1$ and $(S_n^\pm)^2 = 0$ (while for $n \neq m$, $[S_n^\pm, S_m^\pm] = [S_n^\pm, S_m^\mp] = 0$), we can rewrite the Hamiltonian as

$$H_{\text{XX}}(J, h) = \frac{J}{2} \sum_{j=1}^L [S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+] - 2h \sum_{n=1}^L \left(S_n^+ S_n^- - \frac{1}{2} \right). \quad (1.154)$$

The two terms, corresponding to the kinetic and potential terms, separately conserve the total spin when H_{XX} acts on a state of fixed total magnetization M .

In other words, H_{XX} admits a block-diagonal form with $(L+1)$ blocks of total magnetization $M \in \{-L/2, \dots, L/2\}$, of corresponding dimension

$$d_{L,M} = \binom{L}{\frac{L}{2} - |M|}. \quad (1.155)$$

The generator of this conserved charge is

$$S_Z = \sum_{n=1}^L S_n^z = \sum_{n=1}^L \left(S_n^+ S_n^- - \frac{1}{2} \right), \quad (1.156)$$

and because there is a single generator of this global symmetry, we will often call it $U(1)$.

We now describe the Jordan-Wigner mapping. Denote by c_j and $c_j^\dagger$ the annihilation and creation operators, which are nonlocal with respect to the spin basis:

$$c_j = \exp \left(i\pi \sum_{m=1}^{j-1} S_m^+ S_m^- \right) S_j^-, \quad c_j^\dagger = S_j^+ \exp \left(-i\pi \sum_{m=1}^{j-1} S_m^+ S_m^- \right). \quad (1.157)$$

Using the anti-commutation relations of the ladder operators, we can show that $\{c_j, c_{j'}^\dagger\} = \delta_{j,j'}$, and $\{c_j, c_{j'}\} = \{c_j^\dagger, c_{j'}^\dagger\} = 0$. This is the operator algebra of spinless fermions.

The Pauli Z matrix satisfies the identity

$$Z_n = 2S_n^z = -\exp(\pm i\pi S_n^+ S_n^-), \quad (1.158)$$

which allows the rewriting

$$c_j = \left(\prod_{m=1}^{j-1} (-Z_m) \right) S_j^-, \quad c_j^\dagger = S_j^+ \left(\prod_{m=1}^{j-1} (-Z_m) \right). \quad (1.159)$$

Since $Z_m^2 = 1$ and spin operators commute for different sites, we have

$$c_n c_n^\dagger = S_n^- S_n^+, \quad c_n^\dagger c_n = S_n^+ S_n^-, \quad (1.160)$$

thus we can express spin operators in terms of fermionic ones

$$S_j^- = \exp \left(-i\pi \sum_{m=1}^{j-1} c_m^\dagger c_m \right) c_j, \quad S_j^+ = c_j^\dagger \exp \left(i\pi \sum_{m=1}^{j-1} c_m^\dagger c_m \right), \quad S_j^+ S_j^- = c_j^\dagger c_j. \quad (1.161)$$

We notice an important identity: $S_n^- Z_n = S_n^- = -Z_n S_n^-$, which allows writing

$$c_{j+1}^\dagger c_j = S_{j+1}^+ S_j^- \quad (1.162)$$

For the endpoints, denote by $\hat{n}_j = c_j^\dagger c_j$ the occupancy operator and by $\hat{N} = \sum_j \hat{n}_j$ the total fermion-number operator. We have

$$\begin{aligned} c_L c_1 &= (-1)^{\hat{N}} S_L^- S_1^-, & c_L^\dagger c_1^\dagger &= -(-1)^{\hat{N}} S_L^+ S_1^+, \\ c_L^\dagger c_1 &= -(-1)^{\hat{N}} S_L^+ S_1^-, & c_L c_1^\dagger &= (-1)^{\hat{N}} S_L^- S_1^+. \end{aligned} \quad (1.163)$$

Thus, the XX Hamiltonian is rewritten as

$$H_{\text{XX}} = \frac{J}{2} \sum_{j=1}^{L-1} (c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) - 2h \sum_{j=1}^L \hat{n}_j - \frac{J}{2} (-1)^{\hat{N}} (c_1^\dagger c_L + c_L^\dagger c_1) + hL. \quad (1.164)$$

The first two terms correspond to the *bulk* kinetic and potential terms, while the third one is a boundary term which alternates signs depending on whether the number of fermions in the chain is even or odd. The final term is an overall constant term which factors out of the unitary dynamics $e^{-iH_{\text{XX}}t}$ and can be safely discarded.

Also, because the potential term is constant for each site, this term is a constant proportional to the number of fermions in each sector. Therefore, only the kinetic term survives. The Hamiltonian is invariant under the translation $j \rightarrow j+1$, corresponding to the discrete symmetry $\mathbb{Z}_L$. Thus, within each magnetization sector, we have L blocks corresponding to the different momentum sectors.

For the case of $M = \pm L/2$, there is only one state, corresponding to a chain with either no fermions or L fermions. In both cases, no dynamics can occur. The case $M = -(L-1)/2$ corresponds to the one-particle sector, and the L blocks are

one-dimensional, corresponding to the Fourier modes

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-ikj} |j\rangle. \quad (1.165)$$

These modes diagonalize the Hamiltonian with eigenvalue $E_k = \cos k$. Higher-particle sectors with F fermions are *factorized*, i.e. a suitable basis is the antisymmetrization of the F momenta. The total energy is additive: $E_{k_1, \dots, k_F} = \sum_{M=1}^F \cos k_M$.

XXZ Model The XXZ spin chain is the simplest generalization of the XX spin chain that contains interaction terms in the fermion mapping. The Hamiltonian of the model is

$$H_{\text{XXZ}}(J, \Delta, h) = J \sum_{j=1}^L \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right] - 2h \sum_{n=1}^L S_n^z. \quad (1.166)$$

As with the XX chain, we perform a Jordan-Wigner transformation to obtain the corresponding fermionic chain

$$\begin{aligned} H_{\text{XXZ}} = & \frac{J}{2} \sum_{j=1}^{L-1} \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) + J\Delta \sum_{j=1}^L \hat{n}_j \hat{n}_{j+1} \\ & - (2h + J\Delta) \sum_{j=1}^L \hat{n}_j - \frac{J}{2} (-1)^{\hat{N}} \left(c_1^\dagger c_L + c_L^\dagger c_1 \right) + \frac{J\Delta L}{4} + hL. \end{aligned} \quad (1.167)$$

The boundary term $-\frac{1}{2}(-1)^{\hat{N}} (c_1^\dagger c_L + c_L^\dagger c_1)$ contributes as an $O(1/L)$ correction in the thermodynamic limit, which we can therefore neglect. In the following, we then restrict to the bulk Hamiltonian where $h = 0$:

$$H = -\frac{1}{2} \sum_{j=1}^{L-1} \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) + \Delta \sum_{j=1}^L \hat{n}_j \hat{n}_{j+1} - \Delta \sum_{j=1}^L \hat{n}_j. \quad (1.168)$$

This Hamiltonian is a very simple generalization of the XX chain in which nearest-neighbour density interactions are present. It is also apparent that the inclusion of density-density interactions preserves the $U(1)$ symmetry of the model, and therefore the XXZ chain also presents the same block-diagonal structure as the XX chain.

We will now show a very important property of this Hamiltonian: that the many-body wave function is factorizable into two-body terms.

It is thus worth studying the two-particle sector first. We observe that for two localized states with $x_1 \ll x_2$, particles move freely in the presence of a constant on-site potential. Therefore, plane waves can be assumed as solutions at large

separations:

$$\psi_{\mathbf{x}}(\mathbf{p}) \propto A(p_1, p_2) e^{-i(p_1 x_1 + p_2 x_2)} + A(p_2, p_1) e^{-i(p_1 x_2 + p_2 x_1)}. \quad (1.169)$$

We restrict ourselves to the fundamental domain $1 \leq x_1 < x_2 \leq L$. When $|x_1 - x_2| > 2$, the contact term is absent, and $\psi_{\mathbf{x}}(\mathbf{p})$ is an eigenfunction with energy

$$E(\mathbf{p}) = \cos p_1 + \cos p_2 - 2\Delta. \quad (1.170)$$

When $x_2 = x_1 + 1$, the contact term is nonzero. The wave-function is

$$|\Psi(\mathbf{p})\rangle = \sum_{1 \leq x_1 < x_2 \leq L} \psi_{\mathbf{x}}(\mathbf{p}) |x_1, x_2\rangle. \quad (1.171)$$

Applying H to $|\Psi(\mathbf{p})\rangle$, no kinetic contribution with $p_1 x_1 + p_2 x_1$ must be present, and this should cancel with the contact term. This condition is satisfied by

$$0 = -\frac{1}{2} [\psi_{x_1, x_1} + \psi_{x_1+1, x_1+1}] + \Delta \psi_{x_1, x_1+1}. \quad (1.172)$$

Define the scattering matrix $S(p_1, p_2)$ as the ratio of the amplitudes

$$S(p_1, p_2) = \frac{A(p_2, p_1)}{A(p_1, p_2)} = -\frac{e^{-i(p_1+p_2)} + 1 - 2\Delta e^{-ip_2}}{e^{-i(p_1+p_2)} + 1 - 2\Delta e^{-ip_1}}. \quad (1.173)$$

It is immediately apparent that, when $\Delta \rightarrow 0$, the S -matrix becomes $S = -1$. Denote the free two-particle wavefunction by $\phi_{x_1, x_2}(\mathbf{p}) = e^{-i(p_1 x_1 + p_2 x_2)} - e^{-i(p_1 x_2 + p_2 x_1)}$. When $\Delta \neq 0$, the S -matrix is corrected by a linear term in Δ :

$$\begin{aligned} \psi_{\mathbf{x}}(\mathbf{p}) &\propto \phi_{x_1, x_2}(\mathbf{p}) + 2\Delta \left(e^{-i(p_1(x_1+1) + p_2 x_2)} \right. \\ &\quad \left. - e^{-i(p_1 x_2 + p_2(x_1+1))} \right) \\ &= \phi_{x_1, x_2}(\mathbf{p}) + 2\Delta \phi_{x_1+1, x_2}(\mathbf{p}). \end{aligned} \quad (1.174)$$

This relation is unique and does not hold for generic many-body Hamiltonians, as it is nonperturbative in the strength of the perturbation.

In the many-body case (F fermions), the many-body wavefunction which solves the

Hamiltonian problem is given by the Bethe Ansatz [75]

$$\begin{aligned}\psi_{\mathbf{x}}(\mathbf{p}) &\propto \sum_{P \in \mathcal{S}_F} A(P(\mathbf{p})) e^{i \sum_{j=1}^F p_{P(j)} x_j} \\ &= \sum_{P \in \mathcal{S}_F} \exp \left(i \sum_{j=1}^F p_{P(j)} x_j + \frac{1}{2} \sum_{j < \ell}^F S(p_{P(j)}, p_{P(\ell)}) \right).\end{aligned}\quad (1.175)$$

where in this case

$$A(P(\mathbf{p})) = (-1)^{\text{sgn}(P)} \prod_{j < \ell} e^{-i(p_{P(j)} + p_{P(\ell)})} + 1 - 2\Delta e^{-ip_{P(j)}}. \quad (1.176)$$

The many-body wavefunction cannot be written simply as the sum of free wavefunctions. Introduce $C = \prod_{j < \ell} (e^{-i(p_j + p_\ell)} + 1)$ and $f_{j\ell}(P) = e^{-ip_{P(j)}} / (e^{-i(p_{P(j)} + p_{P(\ell)})} + 1)$. We rewrite

$$A(P(\mathbf{p})) = (-1)^{\text{sgn}(P)} C \prod_{j < \ell} (1 - 2\Delta f_{j,\ell}(P)). \quad (1.177)$$

We notice that the factor C is permutation-independent and factors out of the wavefunction. When $\Delta = 0$, only the permutation sign survives, and this corresponds to the free wavefunction $\phi_{\mathbf{x}}(\mathbf{p})$. At $O(\Delta)$, the wavefunction reads

$$\psi_{\mathbf{x}}(\mathbf{p}) = \phi_{\mathbf{x}}(\mathbf{p}) - 2\Delta \sum_{j < \ell} \sum_{P \in \mathcal{S}_F} (-1)^{\text{sgn}(P)} f_{j\ell}(P) e^{-i \sum_k p_{P(k)} x_k} + O(\Delta^2). \quad (1.178)$$

A closed-form solution to the Hamiltonian problem can be obtained when periodic boundary conditions are applied:

$$\psi_{\mathbf{x} + Le_M}(\mathbf{p}) = \psi_{\mathbf{x}}(\mathbf{p}), \quad (1.179)$$

where $M \in \{1, \dots, F\}$ is any of the particle indices. Thus, if particle position x_i is shifted by L , this is equivalent to the same configuration upon interaction with all other particles in the respective free regions:

$$\psi(x_1, x_2, x_3, \dots, x_F) = \prod_{j \neq 1} S(p_1, p_j) \psi(x_1 + L, x_2, x_3, \dots, x_F); \quad (1.180)$$

however, the phases of the two wavefunctions differ by $e^{ip_1 L}$, since the particles are in two different regions of phase space. Consequently, the quantization condition reads

$$1 = e^{-ip_1 L} \prod_{j \neq i} S(p_i, p_j). \quad (1.181)$$

Taking the logarithms, we obtain

$$mLp_i + i \sum_{j \neq i} \ln S(p_i, p_j) = 2\pi n_i, \quad \forall i. \quad (1.182)$$

The set $\{n_1, \dots, n_F\}$ contains the Bethe numbers and in turn gives the values of the p_i . Naturally, not all sets of Bethe numbers are associated with a solution in terms of the p_i for a given S . Once the solutions $\{p_i\}$ are known, one can compute the total energy and momentum.

Thus, the Bethe Ansatz provides a powerful technique for solving a many-body problem from an analytical wavefunction form. However, the factorizability of the wavefunction is a rather peculiar property of the Hamiltonian, and in principle one cannot predict whether the coordinate Bethe Ansatz holds.

XXX Model The limit $\Delta = 1$ and $h = 0$ of the XXZ spin chain is the original model solved by Bethe, the XXX spin chain (also known as the Heisenberg model):

$$H_{\text{XXX}}(J) = J \sum_{j=1}^L \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z \right] = J \sum_{j=1}^L \mathbf{S}_j \cdot \mathbf{S}_{j+1}. \quad (1.183)$$

This Hamiltonian is a sum of *permutation* operators in the spin basis. To observe this, denote by $P_{i,j}$ the operator which exchanges spins $i \leftrightarrow j$. Then, it is straightforward to show that

$$\mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{4} P_{i,j} - \frac{1}{2}. \quad (1.184)$$

Thus, Eq. (1.183) is the homogeneous sum of nearest-neighbour permutations.

This model features an enhancement of the global symmetry from $U(1)$ to $SU(2)$, since the Hamiltonian commutes not only with the projection of the total angular momentum operator along the Z axis, but with the full operator

$$\mathbf{S} = \sum_{j=1}^L \mathbf{S}_j. \quad (1.185)$$

From textbook treatments of quantum mechanics [76] and group theory [77], the total Hilbert space can be decomposed into the irreducible representations of $SU(2)$ identified by the value $S(S+1)$ of $\mathbf{S}^2$ and the $(2S+1)$ states of the irreducible representation labeled by the eigenvalues of S^z . Correspondingly, the matrix of the Hamiltonian in Eq. (1.183) splits into blocks of different sizes. With periodic boundary conditions, an additional global symmetry of the model is translation by one site, $\mathbb{Z}_L$, which leads to the conservation of the one-site shift operator T , which

satisfies

$$T\mathbf{S}_i T^{-1} = \mathbf{S}_{i+1}, \quad (\mathbf{S}_{i+L} = \mathbf{S}_i). \quad (1.186)$$

Since T is a matrix satisfying the cyclic relation $T^L = \mathbf{1}$, it admits a set of L distinct eigenvalues $\delta = e^{ip}$, where $p_n = 2\pi n/L$, $n = \{0, \dots, L-1\}$. Thus, the Hilbert space splits into L eigenspaces of the momentum operator P corresponding to the quantum numbers p_n . Hence, $SU(2)$ and translation are the global symmetries of the Heisenberg model (1.183).

If the operators $\mathbf{S}_i$ transform according to the spin-1/2 irreducible representation of $SU(2)$, the Heisenberg model also admits a dynamical symmetry generated by the infinite number of conserved quantities obtained from the derivatives of the logarithm of the transfer matrix $\tau(\lambda)$ with respect to the spectral parameter λ [78]:

$$Q_n = \frac{d^n}{d\lambda^n} \log \tau(\lambda). \quad (1.187)$$

The presence of this dynamical symmetry (absent for all other values of the spin of the operators $\mathbf{S}_i$) further splits the Hilbert space and reduces the Hamiltonian to blocks of smaller size, ultimately one-dimensional, corresponding to the states which are solutions of the Bethe Ansatz equations, i.e. the common eigenvectors of all the conserved charges Q_n , including the Hamiltonian.

1.3.6. Poisson distribution for Level Spacings in Integrable Systems

Through several examples, we provided a glimpse of the beauty and intricacy of integrable models: to ascertain their exact solvability, one needs to develop a technique that gives both wavefunctions and energies in closed form. In the many-body case, the Bethe Ansatz in its various formulations provides powerful machinery that, however, may be complicated to use.

It is often the case that one wishes first to understand numerically whether the system under consideration is integrable, and level statistics provide perhaps the most accessible quantity, obtained directly from the spectrum without relying on the eigenvector structure.

For the *regular* spectrum in the semiclassical limit of integrable systems, Berry and Tabor [11] showed that *generic* integrable systems feature uncorrelated spectra, i.e.

the rescaled consecutive spacings follow the Poisson (or exponential) distribution:

$$\mathcal{P}(s) = e^{-s}. \quad (1.188)$$

For generic strongly interacting and integrable quantum many-body systems, such as the XXZ or Heisenberg chains, the level statistics follow the Poisson probability density function (PDF); see e.g. Fig. 1.2. However, it is quite cavalier to state that a system is integrable because the level statistics *look* Poissonian, as ED is limited to probing small system sizes. Moreover, the ambiguity of the unfolding procedure may invite criticism of one's numerical approach.

Perhaps most crucially, the Poisson distribution is a generic feature of many physical processes. For example, one can generate energies according to the semicircle law by implementing a rejection-sampling technique and observe that the distribution of energy gaps is Poissonian. This is true for any DOS generated artificially without starting from a Hamiltonian. Indeed, providing only the DOS as input without encoding the correlations between energies gives independent sampled quantities that, by *definition*, follow the Poisson distribution.

Moreover, the presence of global symmetries generates energy levels which may be arbitrarily close because the global symmetry protects them with two distinct quantum numbers. Thus, one is often tempted to perform a procedure that deletes energy levels which may be ϵ -close. Naturally, all the aforementioned numerical nuances may call the assessment of quantum integrability into serious question, leading to the belief that only an analytical proof is the ultimate certificate of quantum integrability.

Berry-Tabor Theorem for Semi-Classical Integrable Systems Berry and Tabor [11], following the previous conjecture of Percival [79], convincingly argued that, in integrable models, the extensive (and basically independent) quantum numbers of the conserved charges make energy levels essentially uncorrelated: crossings are allowed, there is no level repulsion, and the sequence of spacings behaves like a set of independent and identically distributed (i.i.d.) random variables. For an integrable system with a classical analogue possessing d degrees of freedom, semiclassical quantization yields energy eigenvalues of the form

$$E(\mathbf{n}) = H(I(\mathbf{n})), \quad I_j = \hbar(n_j + \alpha_j), \quad (1.189)$$

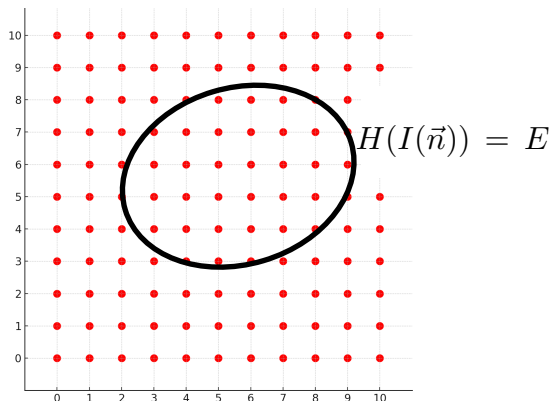


Figure 1.6: Energy-level curve $H(I(\mathbf{n})) = E$ in the lattice of the quantum numbers $\mathbf{n}$. The curve passes through points with very different quantum numbers.

where $\mathbf{n} \in \mathbb{Z}^d$. It is worth stressing that essentially the same mathematical structure also holds in the Bethe Ansatz formulation of quantum integrable models – there, the Bethe roots are solutions of a nonlinear equation. Since the mapping from integer vectors $\mathbf{n}$ to energies is essentially arithmetic and uncorrelated, i.e. levels with very different quantum numbers can have very similar energy values (Figure 1.6), the local statistics of $E(\mathbf{n})$ resemble those of a random sequence with Poissonian gaps. The problem is closely connected to geometric probability, specifically to the enumeration of lattice points lying inside closed contours [80].

The geometrical origin of the problem and the emergence of the Poisson distribution are clear from the following example inspired by the Mikado pick-up sticks: imagine that the N uncorrelated energy levels of an integrable model are parametrized in terms of the N random straight lines in the (ℓ, E) -plane (Figure 1.7)

$$E_i(\ell) = a_i \ell + b_i, \quad i = \{1, \dots, N\}, \quad (1.190)$$

where (a_i, b_i) are i.i.d. random variables drawn from a continuous joint density $f_{A,B}$. As ℓ varies, the set $\{E_i(\ell)\}$ will exhibit many true crossings (independent slopes imply no hybridization), exactly mirroring the abundance of crossings observed in integrable quantum systems. Let us now fix a vertical slice at $\ell = \ell_0$ and consider the N ordinates

$$E_i := E_i(\ell_0) = a_i \ell_0 + b_i. \quad (1.191)$$

Because the pairs (a_i, b_i) are i.i.d., the E_i are themselves i.i.d. with a continuous density

$$f_Y(E; \ell_0) = \int_{\mathbb{R}} da \, f_{A,B}(a, E - a\ell_0). \quad (1.192)$$

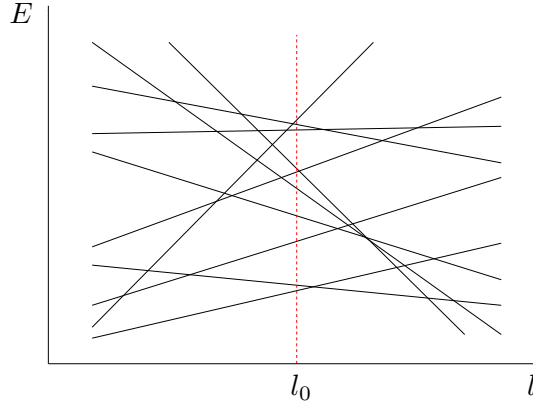


Figure 1.7: Energy lines as Mikado pick-up sticks in terms of a parameter l . The probability distribution at an arbitrary point of the parameter l is a purely geometrical problem.

As outlined in Sec. 1.1.1, order the energies $E_1 < \dots < E_N$, define their spacings as $\Delta_i := E_{i+1} - E_i$ and rescale by the local mean spacing $1/\bar{\rho}$. After this rescaling, the distribution of the unfolded gaps reduces to a standard order-statistics problem for i.i.d. samples with density $f_Y(\cdot; l_0)$, a setting treated in full generality by Pyke [81]. Indeed, by Pyke’s spacing theorem, we have

- Conditioned on a location E , the (unfolded) nearest-neighbor gap is exponential with rate $f_Y(E; \ell_0)$.
- A “typical” spacing is located near E with weight $f_Y(E; \ell_0)dE$.

Mixing these exponentials over y gives the limiting spacing density

$$g(s) = \int_0^1 dx \underbrace{f(x)}_{\text{location density}} \underbrace{[f(x)e^{-sf(x)}]}_{\text{conditional spacing density}} = \int_0^1 dx f(x)^2 e^{-sf(x)}. \quad (1.193)$$

Two immediate consequences follow:

- If $f_Y(\cdot; \ell_0)$ is (locally) flat, then $g(s | \ell_0) \approx e^{-s}$ (pure Poisson).
- In general, g is a mixture of exponentials and still exhibits no level repulsion since the value at $s = 0$ is always different from zero

$$g(0) = \int dx f(x)^2 > 0. \quad (1.194)$$

An alternative derivation of the Poisson distribution is provided by the Wigner surmise, given in Eq. (1.111). If one assumes that the occurrences of energy levels in

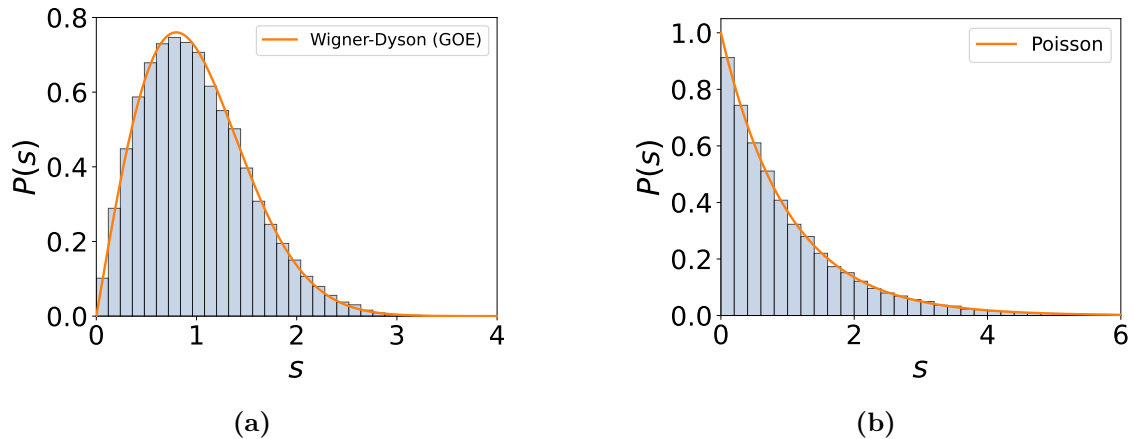


Figure 1.8: Level spacing distributions: (a) GOE Wigner-Dyson; (b) Poisson. In both figures is also present the discrete PDF obtained with a resolution δ .

two disjoint intervals are statistically independent, then the PDF of the consecutive gaps is Poissonian. This is in sharp contrast to the Wigner-Dyson case, where strong correlations between levels preclude such independence and instead give rise to level repulsion. The curves for the Poisson and Wigner-Dyson probability distributions are shown in Figure 1.8. For the Poisson distribution, the maximum value occurs at $s = 0$, indicating a very strong level of clustering. Therefore, its spectrum generally presents many closely spaced pairs, triples, quadruplets, etc. of levels.

As emphasized by Berry and Tabor [11], it is worth noting that the spacing distribution of one of the simplest and most familiar integrable systems – the multi-dimensional harmonic oscillator – does not conform, however, to Poisson’s law. This exception stems from the arithmetic structure of its spectrum, which is generally given by linear combinations of the oscillator frequencies with integer coefficients

$$E_{\mathbf{n}} = \sum_j \hbar \omega_j (n_j + 1/2). \quad (1.195)$$

If the frequency ratios ω_i/ω_j are rational, many different $\mathbf{n}$ yield the same energy, giving rise to large degeneracies, larger than those predicted by the Poisson distribution. Indeed, these “resonances” destroy the randomness needed for Poisson statistics. If, instead, the frequency ratios are irrational but algebraically related, the spectrum is still highly structured, with stronger correlations and clustering effects. Even after unfolding, the spectrum reflects the lattice arithmetic of linear forms in integers, not the independence assumed in the aforementioned Pyke’s theorem. We refer the reader to the original references [11, 31, 32] for a discussion of this subtle but important point.

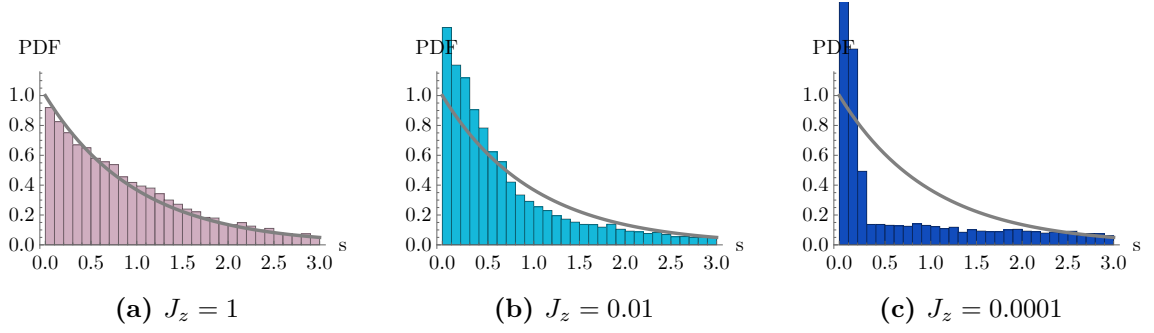


Figure 1.9: Level spacing distributions of the integrable XYZ spin chain on $L = 14$ sites with $J_x = 0.5$ and $J_y = 0.7$, with both couplings kept fixed while J_z varies. Graph (a) corresponds to $J_z = 1$, while graphs (b) and (c) are close to the value $J_z = 0$ of the free-fermion limit, when the original model coincides with the XY model. The Poisson statistics at $J_z = 1$ turn into the distribution of the free-fermion model at $J_z = 0$, where the spectrum is given by a set of harmonic oscillators. The solid line is the Poisson distribution e^{-s} .

Free-fermion systems are, of course, a typical example of a set of free oscillators, and the same considerations apply: see Figure 1.9 for the evolution of the probability distribution of the gaps in the integrable interacting XYZ model as J_z varies,

$$H_{\text{XYZ}} = \sum_{j=1}^L \left(J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right), \quad (1.196)$$

which reduces to a free fermion system, i.e. the XY model, in the limit $J_z = 0$.

Higher order spacing distributions in integrable models It is quite easy to find the probability distributions $\mathcal{P}_k(s)$ of higher-order spacings for integrable Hamiltonians. Indeed, given that each gap s_i is uncorrelated, the $\mathcal{P}_k(s)$ are simply the convolution of k Poisson distributions

$$\mathcal{P}_k(s) = (\mathcal{P}_1 \star \mathcal{P}_1 \star \mathcal{P}_1 \cdots \star \mathcal{P}_1)(s). \quad (1.197)$$

This convolution can be evaluated using the Fourier transform: putting

$$\hat{\mathcal{P}}_1(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \mathcal{P}_1(x) e^{ipx} = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + ip}, \quad (1.198)$$

we have indeed

$$\mathcal{P}_k(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \left(\hat{\mathcal{P}}_1(p) \right)^k e^{-ips} = \frac{s^{k-1}}{(k-1)!} e^{-s}. \quad (1.199)$$

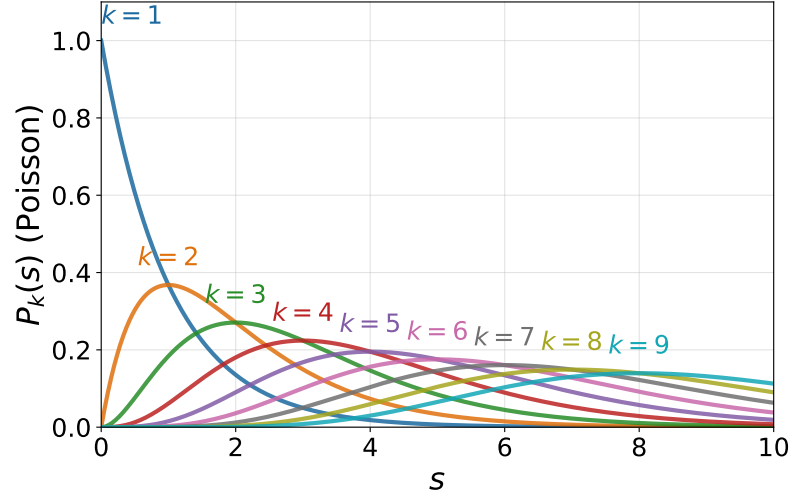


Figure 1.10: The probability distributions of higher-order level spacings for the Poisson case.

For the mean and variance of these distributions, we have

$$\mathbb{E}_k[s] = k, \quad \text{var}_k[s] = k. \quad (1.200)$$

As shown in Figure 1.10, the behavior of these distributions is markedly different from that observed in chaotic systems. In particular, as k increases, the variance of the curves also grows, causing them to broaden progressively. Since for the Poisson distribution the 2-point correlation function $R_2(s)$ is identically equal to 1, we have the identity

$$R_2(s) = \sum_{k=1}^{\infty} \mathcal{P}_k(s) = 1. \quad (1.201)$$

1.4. Statistical Integrability

In what follows, we focus on generic quantum systems that do not necessarily have a (semi)classical limit, and we are mostly interested in the Hamiltonians H of the systems explicitly represented as Hermitian matrices, i.e. $N \times N$ arrays of complex numbers, where we will eventually consider the $N \rightarrow \infty$ limit. In the absence of alternative formulations of these models – such as representations in terms of coordinates and momenta – or a lucky insight, perhaps regarding a Bethe Ansatz solution, the question arises: what constitutes an appropriate criterion for determining whether the system under consideration is quantum integrable?

We adopt as a starting point a criterion similar to the one proposed by Caux and Mossel (CM) [29]. Namely, we assume that our Hamiltonian H_N is a member of a sequence of Hamiltonians of increasing dimensions $(H_{N_1}, H_{N_2}, \dots, H_{N_k}, \dots)$, with $N_1 < N_2 < \dots < N_k$, for which it is possible to define a sequence of sets of operators $(\{Q_a^{(N_1)}\}, \{Q_a^{(N_2)}\}, \dots, \{Q_a^{(N_k)}\})$ such that

1. All operators $\{Q_a^{(N_j)}\}$ commute with each other and with their Hamiltonian H_{N_j} ;
2. the operators in $\{Q_a^{(N_j)}\}$ are algebraically independent;
3. since all these charges $\{Q_a^{(N_j)}\}$ are on the same footing (i.e. any of them can play the role of quantum Hamiltonian), they must share the same statistical features for any N and j , as, for instance, the statistical distribution of their unfolded spectra;
4. the cardinality $\mathcal{C}^{(N_j)}$ of the set $\{Q_a^{(N_j)}\}$ becomes unbounded in the infinite size limit $N_j \rightarrow \infty$.

An open question concerns, of course, the number n of conserved charges $\{Q_a^{(N_j)}\}$ in any given N_j -dimensional Hilbert space: is it, for instance, the dimension N_j of the Hilbert space itself? Or, if the Hilbert space is constructed as the tensor product of M_j q -dimensional linear spaces, i.e. $N_j = q^{M_j}$, is it instead the number M_j of these vector spaces? In the next section, we will see that this last option seems to be selected by a statistical argument. At any rate, the number n of conserved charges must be *maximal*, meaning that we should be able to distinguish the set of Hamiltonian eigenvectors in terms of the quantum numbers of the conserved charges.

Concerning algebraic independence, it is easy to see that if any Hamiltonian H in the sequence of Hamiltonians $(H_{N_1}, H_{N_2}, \dots, H_{N_k}, \dots)$ has a non-degenerate set of eigenvalues E_i ($i = 1, 2, \dots, N$), then any operator Q which commutes with H is functionally dependent on it. Indeed, if $[Q, H] = 0$, then Q also commutes with all powers of H , i.e. $[Q, H^k] = 0$ ($k = 1, 2, \dots, N$). Hence, Q can be expressed as $Q = \mathcal{P}(H)$, where $\mathcal{P}$ is a polynomial of order $(N - 1)$.

To see this, the operator H admits the spectral decomposition

$$H = \sum_{i=1}^N E_i |E_i\rangle \langle E_i| = \sum_{i=1}^N E_i P_i, \quad (1.202)$$

where the projectors P_i on the one-dimensional eigenstate $|E_i\rangle$ can be written as

$$P_i = \prod_{j \neq i} \frac{H - E_j}{E_i - E_j} . \quad (1.203)$$

Since Q commutes with H and its eigenstates $|E_i\rangle$ are not degenerate, they are also eigenstates of Q with eigenvalues q_i

$$Q |E_i\rangle = q_i |E_i\rangle . \quad (1.204)$$

Thus Q is diagonal in the basis of H and therefore it can be expressed in terms of H and its powers H^k as

$$Q = \sum_{i=1}^N q_i P_i = \sum_{i=1}^N q_i \prod_{j \neq i} \frac{H - E_j}{E_i - E_j} . \quad (1.205)$$

Equivalently, since all eigenvalues E_i are nondegenerate, the Vandermonde matrix made of the eigenvalues of $\mathbf{1}$ and H^k ($k = 1, \dots, N-1$)

$$V_N = \begin{pmatrix} 1 & E_1 & E_1^2 & \dots & E_1^{N-1} \\ 1 & E_2 & E_2^2 & \dots & E_2^{N-1} \\ 1 & E_3 & E_3^2 & \dots & E_3^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & E_N & E_N^2 & \dots & E_N^{N-1} \end{pmatrix} \quad (1.206)$$

has the nonzero determinant $\|V_N\| = \prod_{i < j} (E_i - E_j)$. Hence, $\mathbf{1}, H, H^2, \dots, H^{N-1}$ form a complete basis for the diagonal matrices, and therefore any operator Q simultaneously diagonalizable with H can be written as a linear combination of the identity operator $\mathbf{1}$ and higher powers of H , i.e. Q is functionally dependent on H , as stated by a well-known result of von Neumann [82]:

$$[Q, H] = 0 \implies Q = \sum_{k=0}^{N-1} p_k H^k = \mathcal{P}(H), \quad (1.207)$$

where the coefficients p_k can be explicitly computed by comparing this expression with the one in Eq. (1.205).

The argument does not apply if the spectrum of H is instead generally degenerate. In the following section, we see in detail how, in this case, one can show that there exists a maximal set of functionally independent operators Q_a which commute with H and themselves and whose number scales as $\log N$.

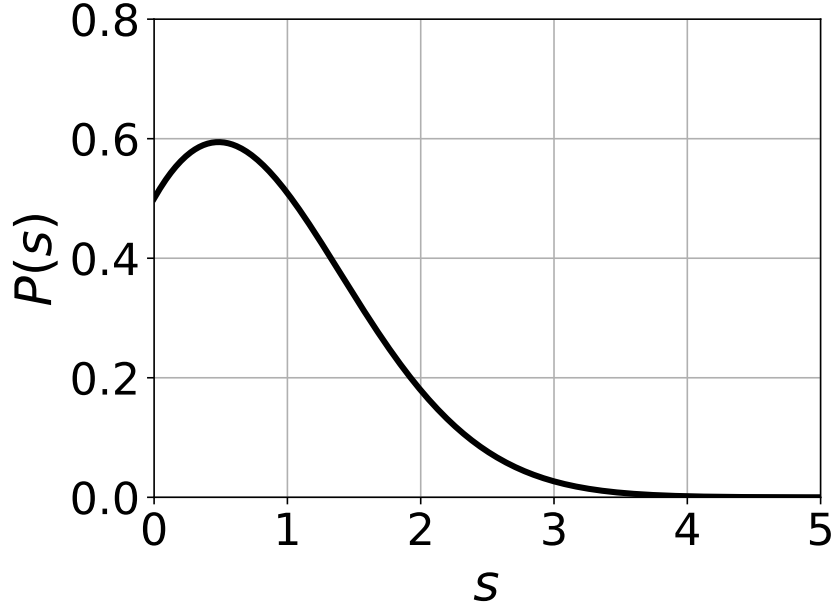


Figure 1.11: An example of a level-spacing distribution $P(s)$ with a nonzero probability of zero gaps in the spectrum.

1.5. A Class of Quantum Integrable Hamiltonians

⁷To establish a probabilistic argument ensuring the existence of a set of operators that commute with H but are not functionally dependent on it, it suffices, for the present purpose, to assume the existence of a nonzero probability $\hat{p}_0$ of having vanishing spacings of H , i.e., $P(s = 0) = \hat{p}_0 \neq 0$, as exemplified by the distribution shown in Fig. 1.11. The symmetries present in (unresolved) integrable models may induce degeneracies in the spectrum and, therefore, a nonzero probability of vanishing gaps. Let us see how we can define a maximal set of conserved charges which commute with H and themselves. To this end, we regard H as part of an ensemble of Hamiltonians whose level spacings are described by the same probability distribution $P(s)$. Hence, we can generate a string of numbers s_i according to the probability distribution $P(s)$. Once we choose an arbitrary value E_1 as a starting point, we can proceed by defining all successive energy levels E_i by the recursive relation

$$E_{i+1} = E_i + s_i, \quad E_2 = E_1 + s_1. \quad (1.208)$$

⁷The argument presented in this Section is originally due to Giuseppe Mussardo and Andrea De Luca, unpublished, and was later published in [3]

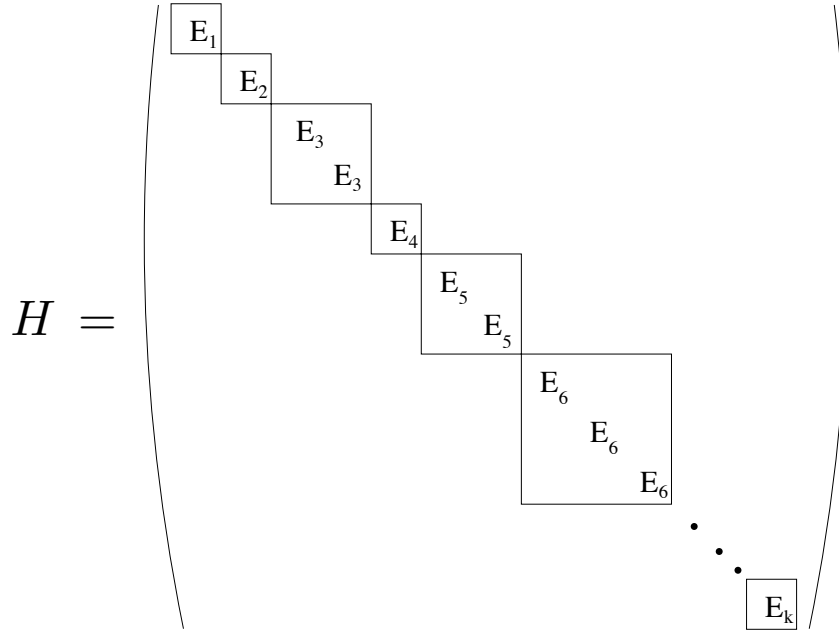


Figure 1.12: Block form of a Hamiltonian with a degenerate spectrum according to the probability distribution $P(s)$ of its level spacings and an energy resolution δE .

We will collect the sequence of all gaps s_i in a string $\mathcal{S}$ of real numbers. Imagining that we have an energy resolution δE (so that energy levels which differ by δE can be considered as effectively degenerate), the string $\mathcal{S}$ of the gaps takes the following form

$$\mathcal{S} = (x_1, x_2, 0, x_3, 0, x_4, 0, x_5, 0, 0, x_6, x_7, \dots), \quad (1.209)$$

and contains a certain number of zeros (with probability $p_0 = P(\delta E)$), together with a sequence of positive real numbers $x > 0$ (with total probability $p = 1 - p_0$). The presence of zeros in the sequence $\mathcal{S}$ induces degeneracies in the energy levels E_i defined by the recursive relation (1.208): the corresponding diagonal matrix form of the Hamiltonian will be made of diagonal blocks of equal eigenvalues, as in Figure 1.12.

In an $N \times N$ truncated representation of the Hamiltonian, it is quite simple to compute how many blocks B_S of size S are present, on average, in the matrix H_N at leading order in N . We have to use the laws of probability:

1. B_1 is the expected number of 1×1 blocks, alias the number of non-degenerate eigenvalues of H_N . Since there is a non-degenerate eigenvalue each time the two consecutive values of s_i are both different from 0, and each non-zero value x_i is independently generated with probability p , we have $B_1 \simeq Np^2$.

2. B_2 is the expected number of 2×2 blocks in H_N . We have two degenerate eigenvalues each time we encounter the string $(\dots, x_1, 0, x_2, \dots)$ in the sequence of the s_i 's, with $x_1 \neq 0$ and $x_2 \neq 0$. Hence, employing the independence of the corresponding probabilities, we have $B_2 \simeq Np^2p_0$.
3. Generalizing the previous argument, to obtain the expected number B_k of $k \times k$ blocks in H_N , one has to consider the probability of getting a string made of $(k-1)$ 0's, sandwiched between two non-zero numbers x_l and x_{l+1} , i.e.

$$(\dots, x_l, \overbrace{0, 0, 0, \dots, 0, 0, 0}^{k-\text{times}}, x_{l+1}, \dots).$$

Hence, $B_k \simeq Np^2p_0^{k-1}$.

The above considerations can be refined by taking into account the edge effects of the blocks. One can write down recursive equations for their probabilistic values and the final exact expressions for the expected number of the various blocks are given by

$$\begin{aligned} B_1 &= 2p + (N-2)p^2; \\ B_2 &= 2(1-p)p + (N-3)p^2(1-p); \\ B_k &= 2(1-p)^{k-1}p + (N-k-1)p^2(1-p)^{k-1}; \\ &\vdots \\ B_{N-1} &= 2(1-p)^{N-2}p; \\ B_N &= (1-p)^{N-1}. \end{aligned} \tag{1.210}$$

Statistically, an $N \times N$ matrix H_N cannot have blocks of arbitrary size: in fact, there will be, on average, no blocks of size l when $(N-l-1)p^2p_0^{l-1} < 1$, a fact which is reasonably confirmed by simulations (see Table 1.1).

When the Hamiltonian is degenerate, we are unable to express an arbitrary operator Q_N which commutes with H_N only in terms of H_N and its powers H_N^k ($k = 1, 2, \dots, N - N_B$, where N_B is the number of blocks): the only operators expressible as $Q_N = \sum_{k=0}^{N-N_B} \alpha_k H_N^k$ are those which have precisely the same block form as H_N . In order to find a maximal set of independent commuting operators Q_k ($k = 1, 2, \dots, n$) such that

$$[Q_i, Q_k] = 0, \tag{1.211}$$

and in which Q_1 identifies the original H , we can argue as follows. In integrable models, all conserved charges are essentially on an equal footing – i.e. each of them can, in principle, serve as the Hamiltonian. It follows that the level spacings of every

$N = 30000$	$B_1^{ex} = 24290$	$B_2^{ex} = 2469$	$B_3^{ex} = 220$	$B_4^{ex} = 19$	$B_5^{ex} = 6$	$B_6^{ex} = 1$
	$B_1^{th} = 24300$	$B_2^{th} = 2429$	$B_3^{th} = 242$	$B_4^{th} = 24$	$B_5^{th} = 2$	$B_6^{th} = 0.2$
$N = 50000$	$B_1^{ex} = 40417$	$B_2^{ex} = 4071$	$B_3^{ex} = 421$	$B_4^{ex} = 38$	$B_5^{ex} = 4$	$B_6^{ex} = 1$
	$B_1^{th} = 40500$	$B_2^{th} = 4050$	$B_3^{th} = 405$	$B_4^{th} = 40$	$B_5^{th} = 4$	$B_6^{th} = 0.4$
$N = 100000$	$B_1^{ex} = 80798$	$B_2^{ex} = 8102$	$B_3^{ex} = 866$	$B_4^{ex} = 87$	$B_5^{ex} = 8$	$B_6^{ex} = 2$
	$B_1^{th} = 81000$	$B_2^{th} = 8100$	$B_3^{th} = 810$	$B_4^{th} = 81$	$B_5^{ex} = 8$	$B_6^{ex} = 0.8$

Table 1.1: Comparison of the number of blocks B_k^{ex} obtained from N randomly generated gaps with $p_0 = 0.1$ and the numbers B_k^{th} given by the theoretical formula (1.210).

conserved charge must exhibit the same level-spacing statistics $P(s)$ as those of the original Hamiltonian; otherwise, the symmetry underlying their equivalence would be violated. Indeed, if one such charge, say Q_r , were to display, for instance, a spacing distribution with zero probability of vanishing gaps, then, upon adopting Q_r as the Hamiltonian and invoking the result of the previous section, we would be forced to conclude that all other conserved charges Q_i are functions of Q_r , and hence not independent – a contradiction.

Once it is established that all Q_i share the same level-spacing statistics, determining their total number n and constructing them explicitly becomes straightforward. The essential role of each Q_i is to lift the degeneracies of the others while preserving their mutual symmetry: since they possess the same number of degenerate-eigenvalue blocks, it is the relative arrangement of these blocks that must differ. This requirement leads to a simple condition on the corresponding sequences $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ of level spacings associated with the n conserved charges Q_i generated according to the probability distribution $P(s)$ with cutoff δE :

$$\begin{aligned}
Q_1 &\implies \mathcal{S}_1 = (x_1, x_2, 0, x_3, 0, x_4, 0, x_5, x_6, \dots) \\
Q_2 &\implies \mathcal{S}_2 = (y_1, 0, y_2, y_3, y_4, 0, y_5, y_6, 0, \dots) \\
Q_3 &\implies \mathcal{S}_3 = (0, z_1, z_2, 0, z_3, z_4, 0, z_5, z_6, \dots) \\
&\dots \quad \dots \quad \dots
\end{aligned} \tag{1.212}$$

Once we organize these sequences in a matrix, putting one on top of the other,

$$V = \begin{pmatrix} x_1 & x_2 & 0 & x_3 & 0 & x_4 & 0 & x_5 & x_6 & \dots \\ y_1 & 0 & y_2 & y_3 & y_4 & 0 & y_5 & y_6 & 0 & \dots \\ 0 & z_1 & z_2 & 0 & z_3 & z_4 & 0 & z_5 & z_6 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.213}$$

it is easy to see that they will all remain degenerate if there is an entire column made of 0's. The probability that this event will not happen is given by

$$\hat{P} = (1 - p_0^n)^N . \quad (1.214)$$

This is the probability that not all elements in a column are 0, raised to the power of the N columns. If we want to be sure of this event, we can impose that

$$\hat{P} > 1 - \epsilon \quad (1.215)$$

with $\epsilon \rightarrow 0$, and in this way we find that the number of independent conserved charges Q_i needed to resolve all relative degeneracies scales only as $\log N$

$$n \simeq \frac{1}{\log p_0} (\log \epsilon - \log N) \quad (1.216)$$

($\log p_0 < 0$). So, for instance, taking $p_0 = 10^{-3}$, $\epsilon = 10^{-6}$ and $N = 10^9$, one would need only 4 extra conserved charges in addition to the original Hamiltonian. These extra charges completely resolve the degeneracy of Q_1 and simultaneously their own degeneracies. From the above construction, it is, moreover, obvious that all these operators are linearly and functionally independent of each other. Computationally speaking, it is very easy to find n of such operators: once a sequence of random numbers is generated with the probability distribution $P(s)$, it is sufficient to generate a few of its permutations and to check that the corresponding operators associated with the new sequences do not have an overlapping column of all zeros.

The logarithmic dependence of the number of independent conserved charges is a welcome feature of the above construction: in fact, as anticipated, if we associate the above Hamiltonian with a quantum system of size L , with q states per site, the system will have a total number of states equal to $N = q^L$. Therefore, the number of conserved charges found with the above construction scales only with L and *not* with the dimension of the Hilbert space. Let us, however, mention that a drawback of this stochastic construction is that it is quite difficult to disentangle the local nature of both the Hamiltonian and the associated conserved charges.

1.5.1. Locality of the Hamiltonians

It is worthwhile to comment on the locality of interactions and their implications for the structure of their matrix representation. By local models, we mean those that are local either in the spatial coordinate x (for continuous systems) or at the

lattice site i (for discrete systems). In a generic basis of the Hilbert space, the matrix representation of a local Hamiltonian is typically dense, meaning that almost all of its entries are nonzero (as will be illustrated by an explicit example below). However, for any local theory, there exists a particular basis – referred to here as the local basis – in which the Hamiltonian is represented by a sparse matrix, i.e. one in which the vast majority of entries vanish. This sparsity is the defining structural feature of local Hamiltonians.

To clarify the concept of the local basis, let us consider the paradigmatic example of the one-dimensional quantum Ising model. Its quantum Hamiltonian for a lattice of L sites is given by

$$H \equiv \sum_{i=1}^L \left(\sigma_z^i \sigma_z^{i+1} + h \sigma_x^i \right) = \sum_{i=1}^L H_i , \quad (1.217)$$

where σ_a are the usual Pauli matrices. The last equality makes the local nature of this model evident: the Hamiltonian has been written as a sum of operators involving only two lattice sites. The matrix representation of this Hamiltonian can be obtained once we fix a basis, with a typical choice being the set of common eigenstates of the σ_z^i operators. They can be written as:

$$|\uparrow\uparrow \dots \uparrow\rangle, \quad |\uparrow\uparrow \dots \downarrow\rangle, \quad \dots \quad \Rightarrow \quad |m_1 m_2 \dots m_L\rangle , \quad (1.218)$$

where each $m_i \in \{\uparrow, \downarrow\}$ corresponds to the two possible eigenstates of σ_z^i :

$$\sigma_z^i |\uparrow\rangle = |\uparrow\rangle, \quad \sigma_z^i |\downarrow\rangle = -|\downarrow\rangle \quad \Rightarrow \quad \sigma_z^i |m_i\rangle = m_i |m_i\rangle .$$

Therefore, the Hilbert space is made up of $N = 2^L$ elements, with a possible basis given by the one above. We call this the real-space basis, characterized by the fact that its elements are simply tensor products of the states of each site. In this basis, the matrix elements of the Hamiltonian density H_i are given by

$$\mathcal{H}_{s,s'} = \langle m_1 \dots m_N | H_i | m'_1 \dots m'_N \rangle = (m_i m_{i+1} \delta_{m_i, m'_i} + h \delta_{m_i, -m'_i}) \prod_{k \neq i} \delta_{m_k, m'_k} , \quad (1.219)$$

where s, s' label the full set of indices $m_1, \dots, m_L$. From this expression, it is easy to deduce that on each row of the matrix there are $L + 1$ non-zero entries and, therefore, the total number of non-zero elements of the $N \times N$ matrix H_N is $\mathcal{N} = (L + 1)N$. Since the total number of matrix elements is N^2 , the density of non-zero elements is given by

$$\rho = \frac{\mathcal{N}}{N^2} = \left(\frac{\log N}{\log 2} + 1 \right) \frac{1}{N} , \quad (1.220)$$

while the density of the zeros of the matrix H_N is

$$\rho_0 = 1 - \rho = 1 - \left(\frac{\log N}{\log 2} + 1 \right) \frac{1}{N} . \quad (1.221)$$

Therefore, for large values of N , the Hamiltonian matrix H_N is a *sparse* matrix, i.e. a matrix with a very large number of zeros and very few non-zero entries.

As a matter of fact, this statement is quite general for any local quantum Hamiltonian. Consider, for instance, one coming from quantum field theory: the fraction of nonzero elements of the Hamiltonian is $O(e^{-L})$, i.e. exponentially small in the thermodynamic limit $L \rightarrow \infty$.

Take for simplicity a $(1+1)$ bosonic field theory, whose Hamiltonian can be written as

$$H = \int \mathcal{H}(x) dx = \int \left[\frac{1}{2} \Pi^2(x, t) + \frac{1}{2} (\partial_x \hat{\varphi}(x, t))^2 + V(\hat{\varphi}(x, t)) \right] dx , \quad (1.222)$$

where $\Pi(x, t) = \frac{\partial \hat{\varphi}}{\partial t}$ is the canonical conjugate of the operator $\hat{\varphi}(x, t)$ and they satisfy the equal-time commutation relation

$$[\hat{\varphi}(x, t), \Pi(y, t)] = i\delta(x - y) . \quad (1.223)$$

The action of $\Pi(x)$ on the basis in which the operator $\hat{\varphi}(x)$ is diagonal is given by

$$\Pi(x) \rightarrow -i \frac{\delta}{\delta \varphi(x)} . \quad (1.224)$$

To construct the Hilbert space of such a theory, we can choose the local basis given by the coordinate representation: in this representation, at any given time t , the states $|\varphi\rangle$ are given by the values of the field $\hat{\varphi}(x, t)$ at position x :

$$\begin{aligned} \hat{\varphi}(x) |\varphi\rangle &= \varphi(x) |\varphi\rangle , \\ |\varphi\rangle &= \prod_x |\varphi(x)\rangle , \\ \langle \varphi' | \varphi \rangle &= \prod_x \delta(\varphi'(x) - \varphi(x)) , \\ \prod_x \int_{-\infty}^{\infty} d\varphi(x) |\varphi\rangle \langle \varphi| &= 1 . \end{aligned} \quad (1.225)$$

In order to set up a matrix representation of the Hamiltonian (1.222) with indices which are discrete rather than continuous, it is convenient to discretize both space (in terms of a lattice with spacing a) and the values of the field $\varphi(x)$ (in terms of a

set of q values spaced by ϵ). This means that x will be restricted to the lattice points

$$x = ma \ , \ m = 1, 2, \dots, L \ , \quad (1.226)$$

while, at these points, the field φ will take the q values

$$\varphi_m = \left(-\frac{q-1}{2}, -\frac{q+1}{2}, \dots, 0, \dots, \frac{q-3}{2}, \frac{q-1}{2} \right) \epsilon \ . \quad (1.227)$$

In such a scheme, the local basis is spanned by the $N = q^L$ vectors associated with the values of the field at the L lattice points

$$| \varphi \rangle = | \varphi_1, \varphi_2, \dots, \varphi_L \rangle \ , \quad \varphi_i \in \left[-\frac{(q-1)}{2}, \frac{(q-1)}{2} \right] \epsilon \ , \quad (1.228)$$

while the Hamiltonian (1.222) becomes

$$\begin{aligned} H_N &= a \sum_{m=1}^L \left[\frac{1}{2} \Pi_m^2 + \frac{1}{2} \left(\frac{\varphi_{m+1} - \varphi_{m-1}}{2a} \right)^2 + V(\varphi_m) \right] \\ &= a \sum_{m=1}^L \left[\frac{1}{2} \Pi_m^2 + \frac{\varphi_{m+1}^2 + \varphi_{m-1}^2 - 2\varphi_{m-1}\varphi_{m+1}}{8a^2} + V(\varphi_m) \right] \ . \end{aligned}$$

The last two terms in this Hamiltonian act locally on the states (1.228), whereas the operator Π^2 induces hopping between the states. To show this, observe that for any function $f(\{\varphi_i\})$ (where the φ_i 's are measured in units of ϵ), we have

$$\Pi_m f(\{\varphi_i\}) = -i \frac{\partial f[\{\varphi_i\}]}{\partial \varphi_m} = -i \frac{f[\varphi_m + 1, \{\varphi_k\}] - f[\varphi_m - 1, \{\varphi_k\}]}{2\epsilon} \ , \quad (1.229)$$

where, in the last expression, $\{\varphi_k\}$ denotes all values of the field other than φ_m . Iterating this definition, for the action of Π_m^2 we have

$$\begin{aligned} \Pi_m^2 f[\{\varphi_i\}] &= -\frac{\partial^2 f[\{\varphi_i\}]}{\partial \varphi_m^2} \\ &= -\frac{f[\varphi_m + 2, \{\varphi_k\}] + f[\varphi_m - 2, \{\varphi_k\}] - 2f[\varphi_m, \{\varphi_k\}]}{4\epsilon^2} \ . \end{aligned} \quad (1.230)$$

Therefore, for the matrix elements of this operator on the local basis, we have

$$\begin{aligned} &\langle \varphi'_L, \dots, \varphi'_m, \dots, \varphi'_1 | \Pi_m^2 | \varphi_1, \dots, \varphi_m, \dots, \varphi_L \rangle \\ &= C \delta_{\varphi_1, \varphi'_1} \cdots \delta_{\varphi_{m-1}, \varphi'_{m-1}} \delta_{\varphi_{m+1}, \varphi'_{m+1}} \cdots \delta_{\varphi_L, \varphi'_L} \\ &\quad \times \left[\delta_{\varphi_m+2, \varphi'_m} + \delta_{\varphi_m-2, \varphi'_m} - 2\delta_{\varphi_m, \varphi'_m} \right] \ . \end{aligned}$$

where C is a normalization factor. Hence, this operator induces a hopping term among the states (1.228). In conclusion, the Hamiltonian of any scalar field theory can be considered as a suitable generalization of the Hamiltonian of the Ising model, where $\Pi^2(i)$ plays the role of σ_i^x and induces a spin-flip among the field values.

Let us now count the number of nonzero terms present in the $N \times N$ Hamiltonian H_N of the bosonic quantum field theory: in each row, there are $3L$ nonzero terms (L of them are the diagonal terms, where all values of φ are the same for the bra and ket states, while $2L$ are those where the bra and ket states differ in the value of the field φ_m by ± 2). Since there are N rows, the total number of nonzero values of such a matrix closely follows the previous computation for the Ising case and is equal to $\mathcal{N} = 3LN$, i.e. the density of nonzero values is given by

$$\rho = \frac{\mathcal{N}}{N^2} = \frac{3}{\log q} \frac{1}{N} \log N . \quad (1.231)$$

while the density of zeros is

$$\rho_0 = 1 - \rho = 1 - \frac{3}{\log q} \frac{1}{N} \log N . \quad (1.232)$$

Therefore, for large values of N , as in the Ising case, the Hamiltonian matrix H_N has a very large number of zero entries, i.e. it is a *sparse* matrix.

Let us finally discuss the nature of the matrices associated with local conserved charges in an integrable quantum field theory. Here, local conserved charges Q_s are derived from the conservation laws of local densities, and Q_i can be expressed as

$$Q_i = \int dx \rho_i(x, t) . \quad (1.233)$$

The densities ρ_i are, in general, functions of the following operators: the canonical conjugate field Π , the field $\hat{\varphi}(x)$, and higher powers and spatial derivatives of both. In particular, there is a maximum power $\Pi^k(x)$ ($k \leq i$) present in ρ_i . While all expressions in ρ_i involving the field $\hat{\varphi}(x)$ and its derivatives act locally on the basis (1.228), the higher powers $\Pi^k(x)$ of the conjugate field will induce a hopping term spanning at most k values in the local basis. Therefore, repeating the analysis above, one arrives at the conclusion that, in the local basis, the matrices of the local conserved charges are also sparse, with densities of nonzero and zero entries given by

$$\rho^{(s)} = \frac{k+1}{N \log q} \log N , \quad \rho_0^{(s)} = 1 - \rho^{(s)} = 1 - \frac{k+1}{N \log q} \log N . \quad (1.234)$$

We conclude this subsection with a remark on the nonlocal character of the projectors onto energy subspaces. As is evident from their explicit form in Eq. (1.203), these operators involve progressively higher powers of H (up to the highest one). Since a generic k -th power of H is an operator with support extending over k sites, it follows directly that these projectors are intrinsically nonlocal.

1.5.2. The basis of momenta

The conclusions of the previous section are quite reasonable, but after changing the basis with a unitary matrix U , the resulting matrix of the Hamiltonian (1.217) in the new basis can be obtained as

$$\mathcal{H}' = U^\dagger \mathcal{H} U \quad (1.235)$$

and will generally no longer be sparse; i.e. for an arbitrary unitary matrix U , any notion of locality will be lost.

Let us focus our attention on a much more common case: will the Hamiltonian still be sparse in the basis of momenta? We will show that there is a subtle issue related to the definition of this change of basis. To be more specific, let us place the Ising Hamiltonian on a circle by introducing periodic boundary conditions. This means that site $L + 1$ is identified with the first one. In this case, if the system is homogeneous, it becomes translation invariant (in units of lattice spacing). Namely, there is the one-site shift operator T , defined by its action on the basis vectors

$$T |m_1, \dots, m_L\rangle = |m_L, m_1, \dots, m_{L-1}\rangle \quad (1.236)$$

which commutes with the Hamiltonian

$$[H, T] = 0. \quad (1.237)$$

Clearly, we have:

$$T^L = \mathbf{1} \quad (1.238)$$

and therefore, we can define the total momentum P as:

$$T = e^{iP} \quad (1.239)$$

with

$$[H, P] = 0. \quad (1.240)$$

Since T is a unitary operator, using (1.238), we easily deduce the usual structure of the spectrum of the momentum operator P , i.e. $\{\frac{2\pi n}{L}, n = 0, \dots, L-1\}$. The basis of momenta can be considered as the basis of the eigenstates of the total momentum. However, these eigenvalues are degenerate and so many definitions are possible. Two significant examples will explain the ambiguity. Let us construct a complete set of eigenstates of P in the following way. Let us pick a state $|v\rangle = |m_1, \dots, m_L\rangle$ in the real-space basis. We obtain an invariant subspace for T by considering the set of states:

$$I_v = \text{Span}\{|v\rangle, T|v\rangle, T^2|v\rangle, \dots, T^{L_v-1}|v\rangle\}$$

where L_v has been defined as:⁸

$$L_v \equiv \min\{n : T^n|v\rangle = |v\rangle\}.$$

The eigenstates of P inside I_v can be written as:

$$|\tilde{v}_n\rangle = \sum_{k=0}^{L_v-1} e^{\frac{2\pi i n k}{L_v}} T^k |v\rangle \Rightarrow P|\tilde{v}_n\rangle = \frac{2\pi n}{L_v} |\tilde{v}_n\rangle. \quad (1.241)$$

A full basis of eigenstates for P can be obtained by repeating this procedure for different states $|v\rangle$: we call this basis the *rigid-translation Fourier basis* (RTFB). It is easy to understand that the resulting matrix in this new basis is still sparse. In fact, each state is a superposition of at most L states, and it follows that the corresponding transformation U contains at most L non-zero entries in each row and column. Using (1.235), we conclude that the Hamiltonian matrix in this basis is still sparse, having at most L^3 nonzero entries in each row ($L^3 \ll 2^L$).

Let us now change the point of view and consider what happens in the second-quantized framework, where there is a set of operators which satisfy (where the $\pm$ stands for the fermionic and bosonic case)

$$[a_i, a_j^\dagger]_{\pm} = \delta_{ij}. \quad (1.242)$$

They create and annihilate a (free) excitation at position i . The real-space basis can be written in this formalism as

$$(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots |\Omega\rangle \quad (1.243)$$

where $|\Omega\rangle$ is the reference vacuum state of the theory. The same formalism can be

⁸The minimum exists because the set contains at least L . Moreover, L_v is a divisor of L .

adopted in the Ising case by setting⁹ $a_i^\dagger \simeq S_i^+$ and taking $|\Omega\rangle \equiv |\downarrow \dots \downarrow\rangle$. Here we can define an excitation with well-defined momentum by setting

$$\mathbf{a}_k^\dagger = \sum_{j=0}^{L-1} e^{ikj} a_j^\dagger . \quad (1.244)$$

Inserting this expression into (1.243), we get a new basis of eigenstates of the total momentum P

$$P \mathbf{a}_{k_1}^\dagger \mathbf{a}_{k_2}^\dagger \dots |\Omega\rangle = \left(\sum_i k_i \right) \mathbf{a}_{k_1}^\dagger \mathbf{a}_{k_2}^\dagger \dots |\Omega\rangle . \quad (1.245)$$

We call this the *single-particle Fourier basis* (SPFB). However, once we restrict ourselves to a subspace where P is defined (appearing as a block for the Hamiltonian matrix), there is a strong difference between this case and the one defined in (1.243): in fact, here not only does the full state have a defined momentum, but each excitation does as well. To better understand this difference, let us consider a two-particle case (it will be clear that there is no difference for the one-particle case). Let us consider a state $|v\rangle = S_{x_1}^\dagger S_{x_2}^\dagger |\Omega\rangle = |x_1, x_2\rangle$, which is the state with only two up spins in positions x_1, x_2 . In the RTFB, this state will appear in L states $|\tilde{v}_n\rangle$, with $n = 0, \dots, L-1$, obtained as superpositions of the rigid translations $T^k |v\rangle$, where the distance $|x_2 - x_1|$ always remains the same. Instead, in the SPFB, each two-particle state has a non-zero matrix element with $|x_1, x_2\rangle$:

$$\langle x_1, x_2 | \mathbf{a}_{k_1}^\dagger \mathbf{a}_{k_2}^\dagger |\Omega\rangle = \sum_{j_1, j_2} e^{i(k_1 j_1 + k_2 j_2)} \langle x_1, x_2 | j_1, j_2 \rangle = e^{i(k_1 x_1 + k_2 x_2)} + e^{i(k_1 x_2 + k_2 x_1)} .$$

As the number of particles increases up to M , in the RTFB case, we always have one summation with $\simeq L$ terms. Instead, in the SPFB, we will have M summations, corresponding to $\simeq L^M$ terms.

The conclusions we can draw from these considerations are as follows:

- A local Hamiltonian will appear as a sparse matrix in the real-space basis.
- If we consider the Fourier basis obtained as a superposition of the rigid translations of the real-space basis, the Hamiltonian will appear again as a sparse matrix, albeit with a larger density of nonzero entries. This is true because the change of basis we are considering is sparse.
- If we consider the Fourier basis obtained by taking the Fourier transform of the free single particles, the change of basis in each block of defined total momentum

⁹This transformation can be made more rigorous using the Jordan-Wigner transformation.

will not be sparse at all. Thus, in the general case, the Hamiltonian matrix will be characterized by dense blocks of fixed total momentum. Therefore, except for this trivial symmetry, it will not be sparse at all.

Chapter 2

Statistical Mixtures

Quantum integrability and chaos can be understood as two prototypes of quantum thermalization: while ergodic systems often thermalize at long times, quantum integrable systems do not. However, Hamiltonian systems with intermediate behavior between ergodicity and non-ergodicity exist, the simplest example consisting of a chaotic Hamiltonian with many symmetry sectors. Sometimes, these symmetries are not evident from the structure of the Hamiltonian itself and can be regarded as ‘hidden’. Examples are the commutant algebra of Hilbert-space-fragmented Hamiltonians or local integrals of motion in disordered quantum many-body systems. In the absence of analytical solutions which hint at a deeper emergent symmetry structure, the analysis of the numerical spectrum can point towards level repulsion, indicating chaotic and ergodic behavior, or level attraction. In the latter case, the system may be genuinely integrable or a ‘statistical mixture’ of (possibly exponentially many) emergent symmetry sectors protected by a set of commuting operators. This chapter is structured as follows. First, we introduce statistical mixtures and analyze the probability distribution of their consecutive gaps, which will allow us to identify the ‘characteristic symmetry sector’ (CSS) as the typical set of statistically correlated levels. We describe the ‘decimation algorithm’, a numerical procedure which determines the size of the CSS and gives information about its statistics. We discuss the stability of the algorithm and benchmark it against known examples. Finally, we apply the decimation algorithm to the study of Hamiltonian systems featuring Hilbert-space fragmentation and disorder-induced many-body localization. Parts of the material of this chapter are taken from [3] and [23].

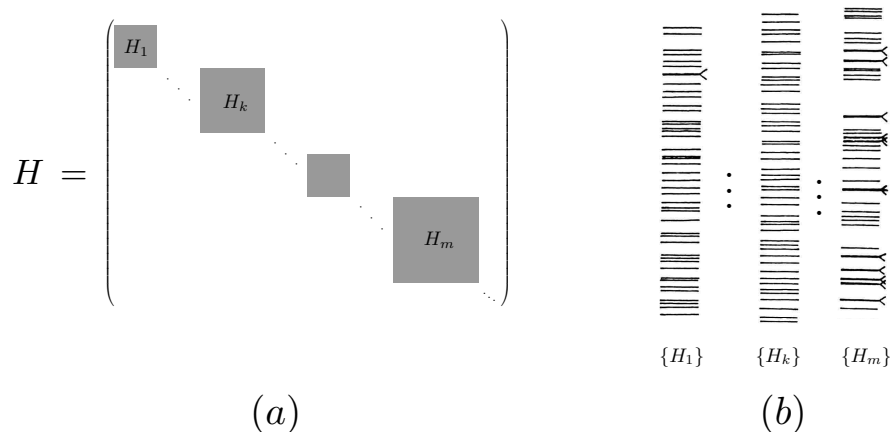


Figure 2.1: (a) Hidden block form of the Hamiltonian matrix; (b) spectra originating from the various blocks.

2.1. Statistical Mixtures

We focus on spectra obtained from a set of unrelated level sequences that have been superimposed, a case that presents a more subtle and potentially deceptive scenario than the quantum spectra discussed in the previous chapter. This case occurs when the Hamiltonian has a block structure due to some global symmetry that has not been previously identified. The same situation also occurs in the presence of fragmentation of the Hilbert space. Thus, imagine that our Hamiltonian has some hidden block form, as shown in Figure 2.1. Each block of the Hamiltonian produces its own spectrum, which may individually follow either Wigner-Dyson or Poisson statistics. However, if one is unaware of the underlying block structure and simply diagonalizes the full Hamiltonian H , the only accessible object is the global spectrum.

This immediately raises the central question: once the spectrum of H is unfolded, what is the level-spacing distribution obtained in such a case, where the observed sequence of levels is, in fact, a superposition of distinct but hidden spectra? This problem has been addressed by several authors [2, 24, 39, 83, 84] and in the following, we remind the final formula according to the derivation given by Mehta [24].

Let us first set some notation. Let us define $E(s)$ as the probability that a *randomly chosen interval* of length s in the spectrum does not contain levels, i.e. it is empty. Hence, differentiating (1.108), we have

$$-E'(s) = \Pr \left([0, s] \text{ empty AND a level at } s \right). \quad (2.1)$$

This is the probability density that the interval is empty and a level sits at the boundary. Differentiating again, we have

$$P(s) = \frac{d^2}{ds^2} E(s) \quad (2.2)$$

where $P(s)$ is the probability density that the *nearest neighbour* lies at a distance s . Ultimately, $P(s)$ is the PDF that we wish to compute for the superposition.

Let us now derive the level-spacing distribution for a mixture of spectra. We denote by ρ_i the level density of the i -th sequence, such that $p_i(\rho_i s) \rho_i ds$ is the probability that a spacing in the i -th sequence has a value between s and $s + ds$. Each function $p_i(x)$ satisfies the two conditions

$$\int_0^\infty p_i(x) dx = 1, \quad \int_0^\infty x p_i(x) dx = 1. \quad (2.3)$$

Introducing the three auxiliary quantities $\Psi_i(x)$, $R_i(x)$ and $E_i(s)$, defined as

$$\begin{aligned} \Psi_i(x) &= \int_0^x p_i(y) dy = 1 - \int_x^\infty p_i(y) dy = 1 - \int_0^\infty p_i(x+y) dy, \\ R_i(x) &= 1 - \Psi_i(x) = \int_x^\infty p_i(y) dy = \int_0^\infty p_i(x+y) dy, \\ E_i(x) &= \int_x^\infty R_i(y) dy = \int_x^\infty dy \int_y^\infty p_i(z) dz = \int_0^\infty \int_0^\infty p_i(x+y+z) dy dz, \end{aligned} \quad (2.4)$$

we have that $\Psi_i(\rho_i s)$ is the probability that a spacing in the i -th sequence is less than or equal to s , while $E_i(\rho_i s)$ is the probability that a given interval of length s does not contain any level of the i -th sequence.

Superposing n sequences¹, the total density is given by

$$\rho = \sum_i^n \rho_i. \quad (2.5)$$

As for the individual sequences, for the superposed sequence, we define the probability $P(x)$, such that $P(\rho s) \rho ds$ is the probability that a spacing lies between s and $s + ds$.

¹It is important to stress that the sequences superimpose in the bulk. If the sequences are separated by a wide gap, then the PDF of the mixtures is just the average PDF of the individual distribution. In the following, we will not discuss the latter scenario.

As above, we also define

$$\begin{aligned}
\Psi(x) &= \int_0^x P(y) dy = 1 - \int_x^\infty P(y) dy = 1 - \int_0^\infty P(x+y) dy , \\
R(x) &= 1 - \Psi(x) = \int_x^\infty P(y) dy = \int_0^\infty P(x+y) dy , \\
E(x) &= \int_x^\infty R(y) dy = \int_x^\infty dy \int_y^\infty P(z) dz = \int_0^\infty \int_0^\infty P(x+y+z) dy dz ,
\end{aligned} \tag{2.6}$$

$E(\rho s)$ is the probability that a given interval of length s does not contain any level and, for the uncorrelated nature of the superposition, we have

$$E(\rho s) = \prod_{i=1}^n E_i(\rho_i s) . \tag{2.7}$$

Setting $x = \rho s$ and

$$f_i = \frac{\rho_i}{\rho}, \quad \sum_{i=1}^n f_i = 1 , \tag{2.8}$$

we have

$$E(x) = \prod_{i=1}^n E_i(f_i x) , \tag{2.9}$$

and for the final expression of the spacing probability $P(x)$, we have

$$\begin{aligned}
P(x) &= \frac{d^2}{dx^2} E(x) \\
&= E(x) \left\{ \sum_{i=1}^n f_i^2 \frac{p_i(f_i x)}{E_i(f_i x)} + \left[\sum_{i=1}^n f_i \frac{R_i(f_i x)}{E_i(f_i x)} \right]^2 - \sum_{i=1}^n \left(f_i \frac{R_i(f_i x)}{E_i(f_i x)} \right)^2 \right\} .
\end{aligned} \tag{2.10}$$

For a Poisson distribution, we have

$$R_P(u) = E_P(u) = p(u) = e^{-u} , \tag{2.11}$$

while for a GOE distribution we have

$$\begin{aligned}
R_{GOE}(u) &= e^{-u^2 \pi/4} , \\
E_{GOE}(u) &= \operatorname{erfc} \left(\frac{\sqrt{\pi}}{2} u \right) ,
\end{aligned} \tag{2.12}$$

where

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-y^2} dy .$$

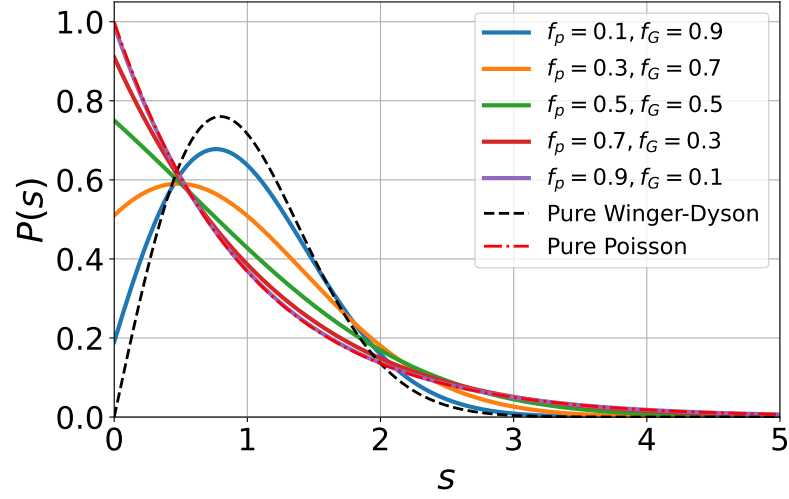


Figure 2.2: $P(s)$ for a mixture of a Poisson distribution with fraction f_1 and a GOE distribution with fraction $f_2 = 1 - f_1$.

Similarly for a GUE distribution we have

$$\begin{aligned} R_{GUE}(u) &= \operatorname{erfc}\left(\frac{2}{\sqrt{\pi}}u\right) + \frac{4}{\pi}ue^{-4u^2/\pi}, \\ E_{GUE}(u) &= e^{-4u^2/\pi} - u \operatorname{erfc}\left(\frac{2}{\sqrt{\pi}}u\right). \end{aligned} \quad (2.13)$$

Let us now analyze closely some relevant examples:

- If all n component spectra are Poisson distributed, their superposition likewise yields a Poissonian level-spacing distribution, $P(s) = e^{-s}$, consistent with the intuitive expectation that the random superposition of independent random sequences produces a sequence that remains random in character.
- If the spectrum is made up of two parts, the first with a Poisson distribution with a fraction f_1 and the second with a GOE with a fraction $f_2 = 1 - f_1$, varying f_1 , we have the set of curves shown in Figure 2.2. The larger the value of f_1 , the more pronounced the Poissonian character of the resulting distribution becomes.
- If the spectrum consists of two GOE components with variable relative fractions, the corresponding level-spacing distributions $P(s)$ are shown in Figure 2.3 (a). Note that, although for a pure GOE one has $P_1(0) = 0$, in the case of a superposition of two GOE spectra, the probability density at the origin instead satisfies $P(0) \neq 0$. Furthermore, as the fraction f_1 is increased towards the

symmetric partition $f_1 = f_2 = 1/2$, the value of the curve at the origin changes systematically. As will be discussed hereafter, this represents a generic feature of mixtures of GOE spectra.

- Let us now consider a spectrum made of a mixture of n GOE spectra. Since

$$R(0) = E(0) = 1 , \quad (2.14)$$

we have that at the origin, the probability distribution $P(s)$ is usually different from zero

$$P(0) = 1 - \sum_{i=1}^n f_i^2 , \quad (2.15)$$

with derivative

$$P'(0) = -1 + 3 \sum_{i=1}^n f_i^2 + \left(\frac{\pi}{2} - 2 \right) \sum_{i=1}^n f_i^3 . \quad (2.16)$$

In the case of n sequences with equal fractions $f_i = 1/n$, the resulting distribution approaches the Poissonian form and becomes increasingly indistinguishable from it as n grows, see Figure 2.3 (b). Indeed, for $n \rightarrow \infty$, we have $P(0) \rightarrow 1$ and, moreover, for $P(x)$, posing $z = x/n$ we have

$$P(nz) = E_{WD}^n(z) \left[\frac{1}{n} \frac{P_1(z)}{E_{WD}(z)} + \left(1 - \frac{1}{n} \right) \frac{R_{WD}^2(z)}{E_{WD}^2(z)} \right] . \quad (2.17)$$

In the limit $n \rightarrow \infty$, i.e. $z \rightarrow 0$, the expression within the square brackets becomes equal to 1, while for the prefactor, expanding it in a series in z , we have

$$E_{WD}(z) \simeq 1 - z + \dots = 1 - \frac{x}{n} + \dots \quad (2.18)$$

and therefore

$$\lim_{n \rightarrow \infty} P(x) = \lim_{n \rightarrow \infty} \left(1 - \frac{x}{n} \right)^n = e^{-x} . \quad (2.19)$$

In summary, a sufficiently fragmented non-integrable Hamiltonian, composed of many independent Wigner-Dyson subsequences, can mimic a Poissonian spectrum so closely that it becomes practically indistinguishable from that of a genuinely integrable system. This circumstance raises the nontrivial problem of how to unambiguously discriminate between genuine integrability and an apparent Poissonian spectrum arising from the superposition of many chaotic subsequences. In Appendix A.1, we study the statistics of higher-order spacings of statistical mixtures generated by RMT ensembles.

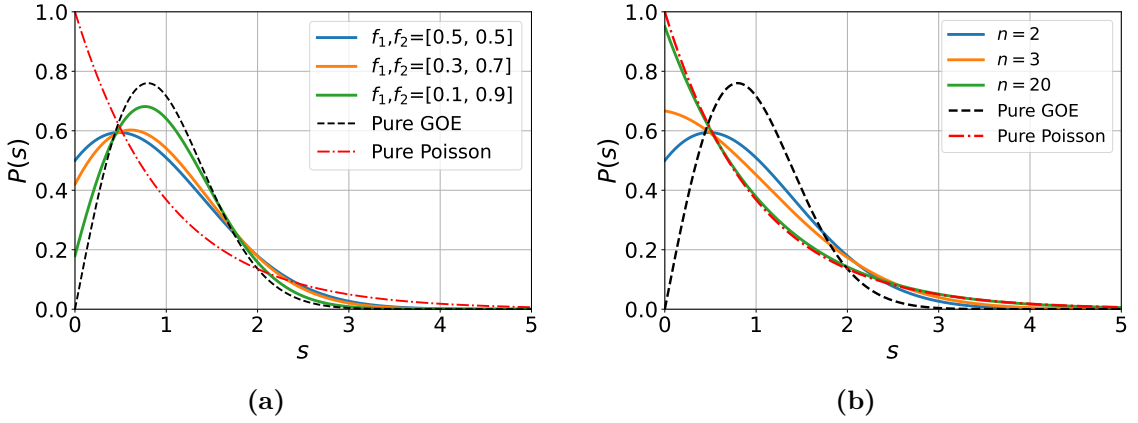


Figure 2.3: (a) $P(s)$ for a mixture of two GOE distributions with fraction f_1 and f_2 . (b) $P(s)$ for a mixture of n GOE distributions with equal fraction $f_i = 1/n$, for $n = 2$, $n = 3$ and $n = 20$.

2.2. Characteristic Symmetry Sector

The simplest model of a statistical mixture is that of a Hamiltonian system where each energy level belongs to one of N disjoint “sectors”. In principle, these sectors may be generated by a symmetry with N irreducible representations in the physical Hilbert space $\mathcal{H}$ or by some constraints. Each sector $\mathcal{H}_i$ has dimension d_i , and it contains as many energy levels $E_j^{(i)}$, distributed according to the density $\rho_i(E)$. In the previous section, we showed how the exact PDF $P(s)$ can be derived. It is immediately apparent that the formula becomes quite convoluted, since both the $p_i(s)$ ’s and their integrals appear. However, the relevant information about the superposition is encoded in the region $s \approx 0$. Therefore, we ask the question: can an approximate form of $P(s)$ be obtained which describes well the region where gaps are small?

We restrict ourselves here to the assumption that Poisson gaps come only from levels of distinct blocks; thus, we require each $p_i(s)$ to exhibit level repulsion, modeled through a Brody-like small-gap behavior: $p_i(s) \sim s^{\nu_i}$ as $s \sim 0$, for some $\nu_i > 0$. The simplest prototypes of such PDFs are the Brody-like distributions [35]

$$p_B(S, \rho', \nu) = \rho'(1 + \nu) a s^\nu \exp(-a(S\rho')^{1+\nu}), \quad a = \left[\Gamma\left(\frac{2+\nu}{1+\nu}\right) \right]^{1+\nu}, \quad (2.20)$$

which interpolates between the GOE ($\nu = 1$) and Poisson ($\nu = 0$) gap distributions. The Brody-like distribution is properly normalized and of mean $1/\mu'$. For these

distributions, one can compute the integrals explicitly

$$P_B(S, \rho', \nu) = e^{-(S\rho')^{1+\nu}a}, \quad E_B(S, \rho', \nu) = \frac{\Gamma\left(\frac{1}{1+\nu}, (S\rho')^{1+\nu}a\right)}{\rho\Gamma\left(\frac{1}{1+\nu}\right)}, \quad (2.21)$$

where $\Gamma(\alpha, z) = \int_z^\infty dt e^{-t} t^{\alpha-1}$ denotes the incomplete gamma function. The small- S expansions

$$P_B(S, \rho, \nu) \sim 1 - \left(S\rho\Gamma\left(\frac{2+\nu}{1+\nu}\right)\right)^{1+\nu}, \quad E_B(S, \rho, \nu) \sim 1 - s + \frac{(S\rho)^{2+\nu}}{2+\nu}\Gamma\left(\frac{2+\nu}{1+\nu}\right)^{1+\nu} \quad (2.22)$$

can be inserted in Eq. (2.10) to obtain the approximation

$$p_N(s) \sim \sum_{i=1}^N \left(\frac{\rho_i}{\rho}\right)^2 p_i\left(\frac{\rho_i}{\rho}s\right) e^{-s(1-\rho_i/\rho)} + M e^{-s}, \quad M = 1 - \sum_{i=1}^N \left(\frac{\rho_i}{\rho}\right)^2, \quad (2.23)$$

where $p_i(s)$ is the standardized distribution with average 1. In the presence of one spectrum, this formula reduces to $p_1(s)$ when $\rho = 1$, and in the limit $N \rightarrow \infty$, as $\rho_i \sim 1/N$, only the Poisson component survives, irrespective of the p_i 's. Concerning the validity of the approximation, even when N is small, the approximation qualitatively describes the exact solution [2, 24, 83] in the interval $s \lesssim 0.5$; see, for example, the plots in Section 2.7.

Generally, this Poisson component comprises a fraction M of the mixture, which also corresponds to the value of the PDF near the origin. The remaining component, of size $(1 - M)$, is the correlated component of the mixture, the “characteristic symmetry sector” (CSS). Spectral decimation, applied to a mixture, separates the Poisson component from the CSS. Its output, a spectrum of size d_{out} , will then be distributed according to the correlated component of Eq. (2.23). The size of the CSS extracted through spectral decimation is a Gaussian random variable with an expectation value

$$\mathbb{E}[d_{\text{out}}] = d(1 - M) = \sum_i \frac{d_i^2}{d}. \quad (2.24)$$

This is the central result of this chapter: d_{out} is the size of the CSS where all energies are statistically dependent. If all spectra have the same dimensions $d_i = d/N$, then it is easy to see that $\mathbb{E}[d_{\text{out}}] = d/N$ as well. On the other hand, if the densities are not equal, d_{out} is dominated by the largest sectors. Finally, if the system has uncorrelated energy levels – which is the case for strongly interacting integrable systems and free ones – $\mathbb{E}[d_{\text{out}}] = 1$, since $d_i = 1$. Information about the variance of d_{out} will be given later. In other words, in the absence of explicit symmetry generators that allow

us to sift the energies of the N blocks, the spectrum of the mixture only retains information about a CSS with a “size-biased” dimension, equivalently characterized by the frequency with which two consecutive energies belong to the same sub-block. The CSS is a natural probe of the two-level spectral correlations. However, it also influences dynamical quantities which probe long-range correlations, such as the spectral form factor. In Section 2.7, this latter dynamical quantity is studied as the size of the CSS varies.

2.3. Decimation Algorithm

The problem we aim to address in this section is the following: given a Hamiltonian matrix H and a chosen resolution δE for constructing histograms, can we determine whether the energy levels of H follow a genuinely Poissonian distribution (and are therefore indicative of integrability), or whether we are merely misled by the superposition of a large number of components that produces an apparently Poissonian spectrum? To answer this question, we proposed in [3] a twofold protocol: (i) a Monte Carlo decimation procedure designed to filter out levels that may be truly Poissonian; (ii) a comparison between the higher-order statistical distributions of the original spectrum and the corresponding Poissonian predictions.

We will mainly focus on (i). Concerning (ii), an extensive treatment of higher-order spacings can be found in Section 1.3.6 and Appendix A. We begin by reviewing the Rejection Sampling (RS) subroutine, which will be used extensively in the Spectral Decimation (SD) algorithm. Then, we will provide the details of the SD and its heuristic interpretation.

2.3.1. Prelude: Rejection Sampling

The RS algorithm provides a method to construct a target distribution $\tilde{q}(s)$ from samples $\{s\}$ drawn according to an original distribution $q(s)$. This is achieved by generating an *empirical* distribution $\pi(s)$, which converges to $\tilde{q}(s)$ in the limit of infinitely many samples. Such a construction is possible only if the target distribution is entirely supported within the original one. In particular, if the target distribution $\tilde{q}(s)$ has $\tilde{d}$ elements and the original distribution $q(s)$ has $d > \tilde{d}$, then $\tilde{q}(s)$ can be exactly reproduced from $q(s)$ without distortion, provided that $\tilde{q}(s)\tilde{d} \leq q(s)d$ for any $s \geq 0$; see Figure 2.4. Thus, the objective is to generate a target distribution $\tilde{q}(s)$ supported on $\tilde{d}$ elements from an original distribution $q(s)$ defined on d elements.

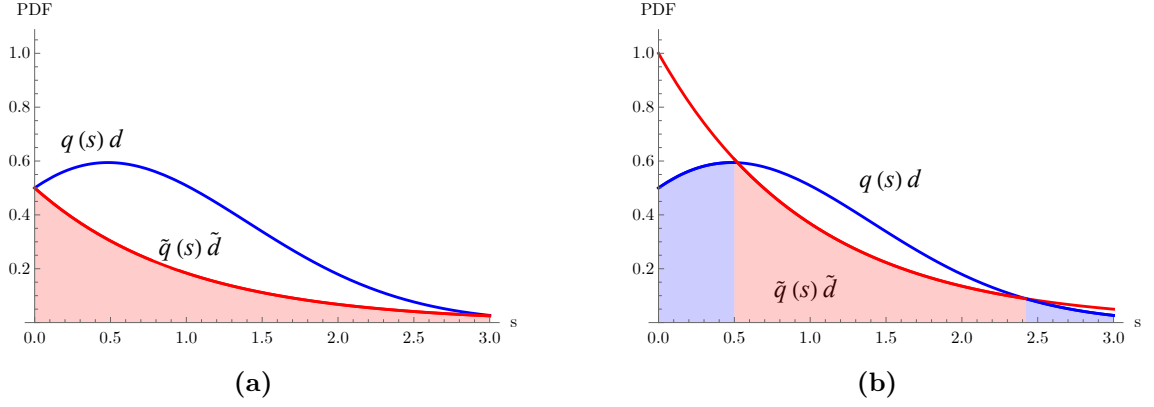


Figure 2.4: The Rejection Sampling algorithm permits the reconstruction of the target curve $\tilde{q}(s)$, defined on $\tilde{d}$ elements, from the original curve $q(s)$, defined on d elements, only if the rescaled curve $\tilde{q}(s)\tilde{d}$ is entirely contained within $q(s)d$ (as illustrated by the red curve (a)). If this condition is not satisfied, the resulting probability distribution coincides with the portion lying above the coloured region (b).

Whether the empirical distribution $\pi(s)$ matches the target is not guaranteed by the algorithm, but by the particular choice of $q(s)$, d and $\tilde{d}$.

In what follows, starting from a given list of gaps, we sequentially extract a prescribed number of them and record, for each extraction, the step index $t = 1, 2, 3, \dots$ at which it occurs. Let $\mathcal{S} = \{s\}$ denote the initial set of gaps and let $\mathcal{S}_t$ denote the set of gaps available at step t (with $\mathcal{S}_1 = \mathcal{S}$). Let $\mathcal{E}_t$ be the set of gaps extracted up to and including step t , $\mathcal{A}_t \subseteq \mathcal{E}_t$ the set of accepted gaps and $\mathcal{R}_t \subseteq \mathcal{E}_t$ the subset of rejected gaps. These sets satisfy

$$\mathcal{E}_t = \mathcal{A}_t \cup \mathcal{R}_t, \quad \mathcal{A}_t \cap \mathcal{R}_t = \emptyset, \quad (2.25)$$

$$\mathcal{S}_t = \mathcal{S} \setminus \mathcal{E}_{t-1}, \quad \mathcal{S}_{t+1} = \mathcal{S} \setminus \mathcal{E}_t, \quad (2.26)$$

$$\mathcal{E}_0 = \mathcal{A}_0 = \mathcal{R}_0 = \emptyset, \quad |\mathcal{E}_t| = t \quad \text{for } t = 1, 2, 3, \dots \quad (2.27)$$

By construction, one gap is extracted per step, and this leads to $|\mathcal{E}_t| = t$.

At step t , let $\tilde{s}_t$ be a value extracted uniformly from $\mathcal{S}_t$. It will be accepted if a uniformly distributed random number u taken from the interval $(0, 1]$ satisfies

$$u \leq r(\tilde{s}_t) = M \frac{\tilde{q}(\tilde{s}_t)}{q(\tilde{s}_t)}, \quad (2.28)$$

where M denotes a constant ensuring the validity of the inequality $M\tilde{q}(x) \leq q(x)$ for any x . Hence, the parameter r serves as the “acceptance threshold”, while M has

the meaning of the acceptance probability² per trial. Hence, if Eq. (2.28) is satisfied, $\tilde{s}_t$ will be added to $\mathcal{A}_t$, forming the new $\mathcal{A}_{t+1}$; otherwise, it will be added to $\mathcal{R}_t$, making the new $\mathcal{R}_{t+1}$.

Since the initial distribution contains d elements, at most d iterations can be executed. If, at some iteration $t_* \leq d$, the required $\tilde{d}$ elements are identified, the algorithm terminates and is deemed *successful*. Conversely, if all d iterations are exhausted without locating the $\tilde{d}$, the algorithm is said to *fail*.

In our implementation, the constant M is chosen to be $M = \min\{q(0), 1\}$. A crucial observation is that if the original distribution is the Wigner-Dyson probability density function (hence $M = 0$), while the target distribution satisfies $\tilde{q}(0) \neq 0$, then the RS algorithm invariably fails. On the other hand, if the original and target distributions coincide, i.e. $q = \tilde{q}$, then choosing $M = 1$, the RS algorithm reproduces the same distribution as uniform random sampling.

A closely related modification of the RS is the well-known *Metropolis-Hastings (MH) algorithm*, a general method for sampling from a probability distribution $q(s)$ defined over a finite set $\{s\}$ of d elements. The goal is to obtain a representative subset of $\tilde{d} < d$ samples. The algorithm proceeds as follows. Starting from an initial element $s_0 \in \{s\}$, one generates at iteration t a candidate element $\tilde{s}$ from a proposal distribution $p(\tilde{s} | s_t)$. The candidate is then accepted with probability

$$\alpha(s_t \rightarrow \tilde{s}) = \min \left\{ 1, \frac{q(\tilde{s}) p(s_t | \tilde{s})}{q(s_t) p(\tilde{s} | s_t)} \right\}. \quad (2.29)$$

If the candidate is accepted, the new state is set to $s_{t+1} = \tilde{s}$; otherwise, the chain remains at $s_{t+1} = s_t$. In the simplest symmetric case (the one which we will use later), where the proposal distribution satisfies $p(\tilde{s} | s_t) = p(s_t | \tilde{s})$, the acceptance probability reduces to

$$\alpha(s_t \rightarrow \tilde{s}) = \min \left\{ 1, \frac{q(\tilde{s})}{q(s_t)} \right\}. \quad (2.30)$$

It follows that if the candidate $\tilde{s}$ is more probable than the current state s_t , i.e. $q(\tilde{s}) \geq q(s_t)$, then the move is always accepted. Conversely, if $q(\tilde{s}) < q(s_t)$, the move is accepted with probability $q(\tilde{s})/q(s_t)$. This conditional acceptance mechanism guarantees that the Markov chain constructed by the MH algorithm has $q(s)$ as its stationary distribution, thereby allowing one to extract a representative sample of size $\tilde{d}$.

²Indeed, the total probability of accepting a proposed x is the expectation of $r(x)$ over the initial distribution $q(x)$, $\eta = \mathbb{E}_{x \in q}[r(x)]$. Therefore, $\eta = \int r(x)q(x)dx = \int M \frac{\tilde{q}(x)}{q(x)}q(x)dx = M \int \tilde{q}(x)dx = M$.

2.3.2. Spectral Decimation Algorithm

The *spectral decimation* algorithm is designed to test whether the energy spectrum of d levels is uncorrelated – i.e. whether the gap statistics follow a genuine Poisson distribution $P(s) = e^{-s}$ – or instead represents a mixture of spectra made, for instance, of several non-integrable spectra characterized by Wigner–Dyson statistics. In the latter case, the algorithm further enables the determination of the number of contributing spectra in the mixture.

Input and Halting Conditions

We assume that energies are already unfolded and therefore gaps are simply $s_i = e_{i+1} - e_i$: these are the input data of the algorithm. In principle, the number of levels d may be exponentially large, making a direct test of whether all gaps follow Poisson statistics both difficult and computationally inefficient. Fortunately, a smaller representative sample of $\tilde{d}$ elements can always be extracted from the same underlying distribution using the Metropolis-Hastings algorithm described above. Hence, the analysis may be performed either on the complete set of unfolded gaps or on a suitably chosen subset. Hereafter, we shall use the symbol d interchangeably to denote either the complete set of unfolded gaps or a representative subset thereof.

In our implementation of the SD algorithm, $q(s)$ denotes the initial empirical distribution of the d gaps, while the target distribution $\tilde{q}(s)$ is chosen to be the Poisson law $\tilde{q}(s) = e^{-s}$. We can run the SD algorithm using an *extraction fraction* f , which controls the rate of decimation. Namely, if d_k gaps are present at iteration k , with $d_1 = d$, then $d_k^e = f d_k$ is the target number of gaps at iteration k required to follow the Poisson distribution. If such a number of gaps is found, the remaining gaps $d_k^r = (1 - f)d_k = (1 - f)^k d$ will be those used in the next iteration, $d_{k+1} := d_k^r$. See Figure 2.5 for a schematic visualization of the tree structure of the algorithm.

Hence, the algorithm runs as follows: given a final number³ d_{halt} of gaps and a decimation fraction f (typically on the order of $f \sim 10^{-2}$), at the k -th step of decimation, with $d_k^r > d_{\text{halt}}$, there are two possible outcomes:

- *Successful extraction:* if, after $t_* < d_k$ iterations, the target number $d_k^e = f d_k$ of gaps consistent with the Poisson law is found, then the remaining gaps d_k^r will be used in the next iteration, $d_{k+1} := d_k^r$, until they eventually reach the

³For the validity of our statistical analysis, it is essential that both d and d_{halt} be sufficiently large. In practice, typical values are $d \sim 10^6$ and $d_{\text{halt}} \sim 10^3$.

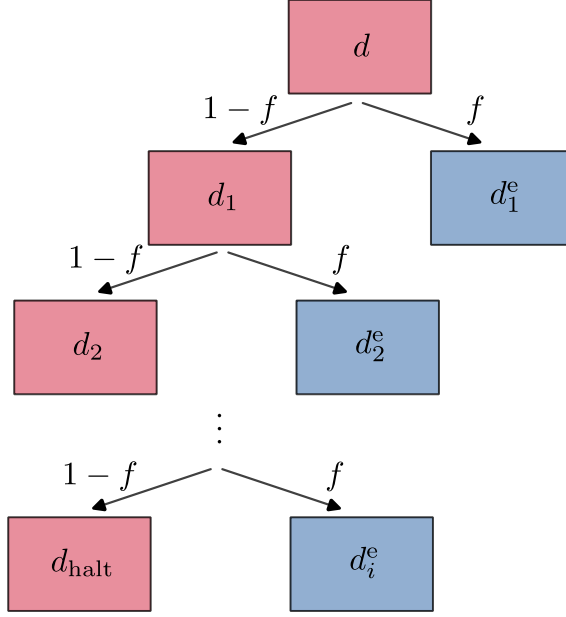


Figure 2.5: Tree structure of the spectral decimation algorithm. At each iteration, a fraction f is decimated from the spectrum. The algorithm ultimately stops when a size d_{halt} is reached, unless a halting condition is met.

number d_{halt} chosen as the final number of gaps at which the algorithm stops.

- *Unsuccessful extraction:* if all d_k iterations are exhausted without identifying the required $d_k^e = f d_k$ gaps satisfying the Poisson distribution, then the algorithm halts.

Let us discuss the interpretation of these two outputs using the example of an initial distribution of the gaps obtained by joining N non-integrable Wigner-Dyson spectra, each of length δ . The probability density of zero gaps is equal to [2]

$$q(0) = 1 - \frac{1}{N} , \quad (2.31)$$

and is therefore almost indistinguishable from a Poissonian distribution for sufficiently large N . The initial sample will then have $d = N\delta$ gaps. Imagine that we now apply the RS algorithm to extract a fraction $f \ll q(0)$ of the original gaps. In an infinitesimal interval $[0, \eta]$ near the origin, the number of zero gaps

$$d^{(\eta)} = dq(0)\eta \quad (2.32)$$

will then be decimated to

$$d_1^{(\eta)} = dq(0)\eta - df\eta = d \left(1 - \frac{1}{N} - f \right) \eta. \quad (2.33)$$

Crucially, $d_1^{(\eta)} < d^{(\eta)}$, and this justifies the name “decimation” for the algorithm. After the m -th decimation, the number of zero gaps will be

$$d_m^{(\eta)} = d \left(1 - \frac{1}{N} - f \sum_{k=0}^{m-1} (1-f)^k \right) \eta = d \left((1-f)^m - \frac{1}{N} \right) \eta. \quad (2.34)$$

The algorithm is said to have decimated all zero gaps if

$$m \geq m^* = -\frac{\log N}{\log(1-f)}. \quad (2.35)$$

The latter equation clarifies the role of the extraction fraction f in controlling the rate of decimation. Moreover, at iteration $m = m^* + 1$, the SD will terminate because we have exhausted all possible zero gaps necessary to obtain a Poisson distribution; this may occur before the final halting condition is fulfilled, i.e. when $(1-f)^{m^*} d > d_{\text{halt}}$.

Physical Interpretation

By construction, the SD algorithm terminates in two possible ways and it is useful to analyse their implications.

In the first case, namely if the SD algorithm reaches the halting threshold d_{halt} , it provides a positive, though probabilistic, indication of the uncorrelated nature of the spectrum. In this case, the decimation has reduced the original set of d gaps to a subset of size d_{halt} . For $d_{\text{halt}} \ll d$, this implies that, to the best of our numerical resolution, the original spectrum is uncorrelated. More precisely, the number of uncorrelated gaps is at least $d - d_{\text{halt}}$, namely those identified by the SD procedure. One may then define the *fraction of uncorrelated gaps* as

$$f_u = (1 - d_{\text{halt}}/d) . \quad (2.36)$$

This fraction can be fixed in advance as a target for the SD algorithm. In this sense, the spectral decimation acts as a *certificate* for the presence of uncorrelated energies.

Let us now discuss the second way the algorithm can terminate. In this case the algorithm may also halt prematurely at step m of the decimation tree. In this event, the RS procedure provides positive – albeit probabilistic – evidence for the presence of multiple independent spectra. Suppose the algorithm halts at iteration m , which implies that iteration $(m-1)$ was successful. This occurs because, after the m -th application of RS, the reservoir of zero gaps is exhausted. Consequently, the number

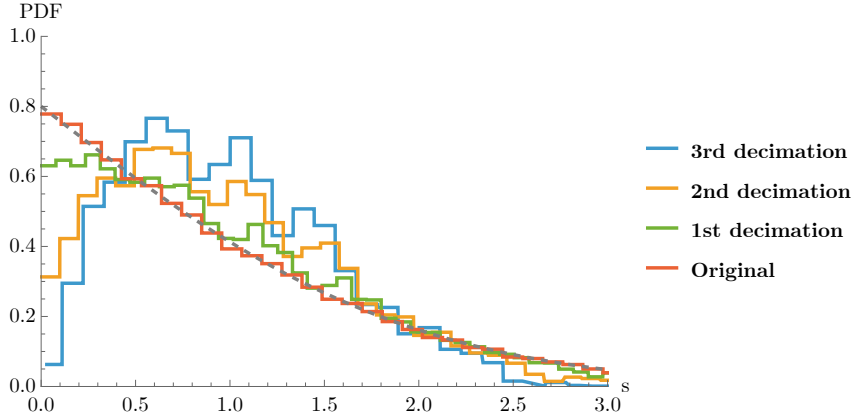


Figure 2.6: Application of spectral decimation. The empirical PDF is obtained by superposing five Wigner–Dyson distributions generated from GOE spectra of size $d = 10^5$. The dashed line represents the analytical PDF from [2, 83], Eq. (2.17). The decimation output after three iterations ($f = 1/2$) is also shown: the probability of zero gaps decreases as they are progressively depleted.

of independent spectra N can be estimated as

$$\left(\frac{1}{1-f}\right)^{m-1} \leq N \leq \left(\frac{1}{1-f}\right)^m. \quad (2.37)$$

This bound can be made arbitrarily narrow by selecting a sufficiently small extraction fraction f , although, as we will discuss, if f is too small, finite-size fluctuations may lead to underestimates of N . Conversely, when the SD algorithm reaches the halting threshold $\tilde{d}_h$ after m iterations, one can establish a lower bound on the number of copies

$$N \geq \left(\frac{1}{1-f}\right)^m. \quad (2.38)$$

This second mode of termination is illustrated in Figure 2.6, which shows the decimation of five Wigner–Dyson spectra generated from GOE ensembles of size $d = 10^5$. With $f = 1/2$, the algorithm halts after three iterations when the RS algorithm fails.

Notice that selecting different values of f inevitably leads to different failure iterations m_f of the SD algorithm. Hence, the number of iterations itself is not physically meaningful. Reversing the argument, however, the iterations cannot be arbitrarily large if the spectrum consists of N independent Wigner–Dyson components. Indeed, the right-hand side of the previous equation shows that the SD algorithm admits at

most

$$m = O\left(-\frac{\log N}{\log(1-f)}\right) \quad (2.39)$$

iterations, which does not depend on the initial d .

However, the dependence on d is taken into account by the computational cost of one iteration. Assuming a single RS acceptance/rejection decision to be $O(1)$, the m -th iteration of the SD will have polynomial runtime. The best-case scenario is when exactly $d_m^e = f(1-f)^{m-1}d$ calls are made, i.e. $O(d_m^e)$, while the worst-case scenario is when the entire initial set of gaps is scanned, $O(d_m)$. As we already mentioned, the empirical distribution can be sampled without distortions by a simple RS involving $O(\tilde{d})$ calls, where $\tilde{d}$ is arbitrarily fixed.

In summary, both SD outcomes provide information with clear statistical interpretations that can be made arbitrarily precise by fine-tuning the input parameters f and $\tilde{d}_h$.

2.3.3. Higher order spacings

An additional and highly effective diagnostic for distinguishing between a genuine Poisson distribution and a Poisson-like distribution arising from a mixture of spectra is provided by the analysis of higher-order level spacings. In the previous sections, we computed these distributions both for the purely Poisson case and for spectral mixtures, and significant discrepancies emerge that allow one to discriminate between the two situations. For instance, Figure 2.7 displays the higher-order spacing distributions for a mixture of 10 spectra compared to those of a purely Poisson ensemble. The two sets of curves are clearly distinguishable, at least for a moderate number of spectra in the mixture. Naturally, as the number of superposed spectra tends to infinity, these differences become increasingly subtle and eventually imperceptible, in which case it is more advantageous to rely on the Monte Carlo decimation method.

2.4. Implementation

In this section, we summarize the SD algorithm, before addressing its applications. Given a sample of size d , an iteration m of the decimation algorithm, with extraction fraction f and halting threshold d_{halt} , proceeds as follows:

1. the empirical PDF π_j of the available “remainder” gaps, $\mathcal{R}_m$ of $d_R(m) =$

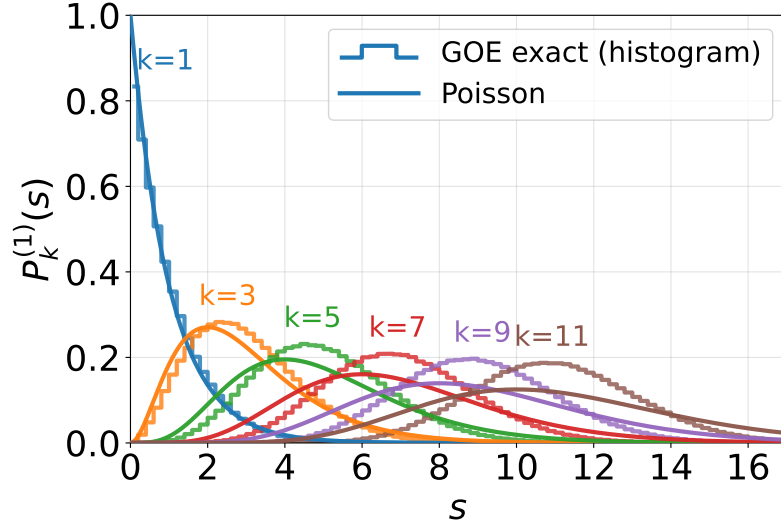


Figure 2.7: k -th level-spacing probability distribution $P_k^{(1)}(s)$ for a Poisson distribution and a superposition of 10 GOE spectra, where the histogram is an average over 200 realizations of 10 GOE spectra, each from an exact diagonalization of a 400×400 matrix.

$(1 - f)^m d$ elements, is computed with bin-width

$$\delta(m) = \left(\frac{12}{d_R(m)} \right)^{1/3}. \quad (2.40)$$

At the first iteration, $m = 0$, the set of remainder gaps coincides with the original set $\mathcal{R}_0$ of d gaps. The bin-width δ is chosen according to the Freedman-Diaconis rule for the Poisson PDF [30], which minimizes the integrated mean squared error between the empirical and true PDF – see Section 1.1.2 for a derivation of the formula.

2. To the set $\mathcal{R}_m$, one applies rejection-sampling (RS) with target probability $\tilde{q}(s) = e^{-s}$ and acceptance ratio

$$M(m) = \min \{ \pi_0, 1 \}. \quad (2.41)$$

Here $M(m)$ is the fraction of Poisson gaps, up to order $O(\delta)$, at iteration m , estimated from the zero-bin empirical PDF π_0 . One therefore obtains a set $\mathcal{A}_m$ of accepted gaps which follow the Poisson distribution.

3. A set of $d_E(m) = f d_R(m)$ Poisson gaps is extracted uniformly from $\mathcal{A}_m$. This yields a subset $\mathcal{E}_m$ of d_E Poisson gaps at each iteration, which is removed from

the set of available gaps:

$$\mathcal{R}_{m+1} = \mathcal{R}_m - \mathcal{E}_m. \quad (2.42)$$

4. The set $\mathcal{R}_{m+1}$ of $d_R(m+1) = (1-f)^{m+1}d$ remainder gaps can be utilized as the input for the iteration $(m+1)$.

Halting Conditions

- H1. If after 1., $d_R(m) \leq d_{\text{halt}}$, the algorithm ends and the global spectrum is statistically Poissonian with uncorrelated fraction f_u .
- H2. If after 3. and $d_R(m) > d_{\text{halt}}$, the number of accepted gaps, whose expected value is $\mathbb{E}[d_A(m)] = M(m)d_R(m)$, is less than $d_E(m) = fd_R(m)$, the algorithm halts because the requested fraction has not been found. In this case, the algorithm ends and the global spectrum is not statistically Poissonian with uncorrelated fraction f_u .

Generalizations. The spectral decimation algorithm can be generalized to sample statistics other than those of consecutive gaps. Another choice is the r -ratio

$$r_j = \frac{\min\{E_{j+1} - E_j, E_{j+2} - E_{j+1}\}}{\max\{E_{j+1} - E_j, E_{j+2} - E_{j+1}\}}. \quad (2.43)$$

While the decimation procedure can be applied to different spectral observables (e.g. gaps or r -ratios), the statistical-mixture approximation leading to Eq. (2.24) is strictly derived for independent spacings. Since r -ratios involve correlated triplets of levels [84], their mixture structure does not generically obey the same size-biased law. Therefore, the CSS extracted from r -ratios need not coincide quantitatively with that extracted from gaps, and its derivation is an interesting problem that lies beyond the scope of this thesis. We noticed that the CSS size in fragmented Hamiltonians (Section 2.8.1) and in toy models of statistical mixtures (Section 2.7) differs from Eq. (2.24) when computed with the r -ratio. On the other hand, for disordered quantum Hamiltonians (Section 2.8.2), the CSS sizes found through the r -ratio and gap statistics present the same qualitative behavior, while still featuring quantitative deviations, especially at large disorder couplings. Nevertheless, the r -ratio remains a valuable numerical tool, as it is inherently independent of the local density of states, bypassing the need for spectral unfolding.

2.5. Outcomes of Decimation

From a statistical perspective, Poissonian spacings signal the absence of level repulsion, either because the system is integrable or because levels from different symmetry sectors cross freely. Spectral decimation exploits this distinction by iteratively removing those spacings that are statistically indistinguishable from a Poisson process. Since level repulsion is preserved under this procedure, correlations originating within the same dynamical sector survive the decimation, while accidental inter-sector crossings are progressively eliminated. Repeating this process isolates a characteristic subset of correlated levels, whose size reflects the dominant symmetry structure of the spectrum.

Thus, the purpose of the SD is to extract, via consecutive extractions, the Poisson component of a set of initial d unfolded gaps obtained from the spectrum of a quantum many-body Hamiltonian – either the full spectrum or a fraction of it. By construction, it removes the largest fraction of Poisson gaps, leaving a set of d_{out} remainder gaps. Moreover, it is a self-averaging algorithm: if the inputs are spectra obtained from n different realizations of the same Hamiltonian (for example, one featuring disorder), it is possible not only to decimate individual samples, but also to collect the gaps across all samples. The sample obtained by combining the gaps in this way is called a “pooled sample”. The d_{out} obtained by averaging the sizes of the decimated samples is equivalent to the decimated size of the pooled sample divided by n . While the decimation of one sample is more intuitive and sometimes the only option, when possible it is always best to pool many realizations together, as larger samples reduce the statistical fluctuations.

Aside from the initial set of d gaps, two other inputs are required by the algorithm: the first is an “extraction fraction” f , i.e. the fraction of Poisson gaps that are removed from the current sample. While f is, in general, arbitrary, too small a fraction can lead to systematic underestimates of d_{out} , while too large a fraction can lead to the exact opposite – we will discuss later how to choose f properly to ensure the algorithm’s stability. The second parameter of the algorithm is the halting size d_{halt} , which is associated with an operational confidence level $f_u = 1 - d_{\text{halt}}/d$. When the sample has been decimated to a size d_{halt} , at least a fraction f_u of the initial set of gaps is Poisson distributed. d_{halt} prevents the algorithm from running indefinitely. When pooling n realizations together, the halting number can be set to $d_{\text{halt}} = n$, implying that if the pooled sample has been decimated to $d_{\text{out}} \leq n$, then the individual samples will have sub-blocks of dimension one, on average.

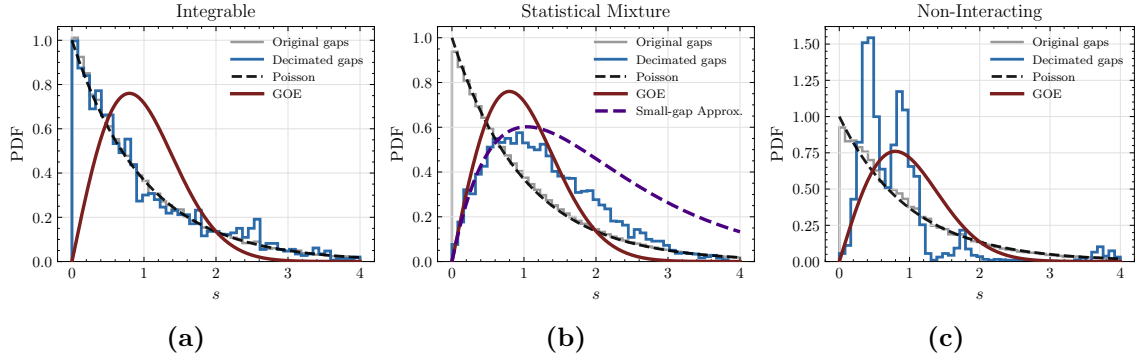


Figure 2.8: Outcomes of the decimation when the parent Hamiltonian is (a) integrable, (b) a statistical mixture of chaotic spectra, or (c) non-interacting. In (a), the periodic spin-1/2 XXX model is diagonalized for $L = 22$ sites in the $SU(2)$ irreducible representation. The remainder gaps are Poissonian, and the decimation succeeds for any d_{halt} . In (b), the parent Hamiltonian is a mixture of 24 random GOE (Gaussian Orthogonal Ensemble) matrices of the same size, $d = 10^4$. Here, the decimation ends at the size of one GOE block, and the gap statistics clearly show signs of level repulsion. The small-gap expansion closely follows the decimated gap statistics for $s \lesssim 1$. Panel (c) shows the decimated spectrum of an $L = 10$ free-fermion chain with random on-site energies that are i.i.d. and uniformly distributed in $[0, 1]$. The gap statistics of the decimated gaps present an excess in the zero-gap probability compared to the pure Poisson case, due to the i.i.d. nature of the frequencies.

When the spectrum is that of a mixture, the algorithm ends before meeting the halting threshold: $d_{\text{out}} > d_{\text{halt}}$. In this case, Eq. (2.24) provides the expectation of d_{out} . On the other hand, if the algorithm ends after meeting the threshold d_{halt} , one concludes that, with operational confidence f_u , the spectrum is composed of uncorrelated energy levels. If the remainder-gap statistics are compatible with a Poisson distribution, this provides a strong hint of the integrability of the quantum many-body system. It may also happen that the remainder-gap statistics feature distinct peaks: in this case, it is likely that the spectrum is free or weakly interacting. These outcomes are reported in Fig. 2.8.

2.6. Stability of the Decimation Algorithm

Here, we discuss in detail the choice of the extraction fraction f and sources of systematic errors. Finally, we offer a theoretical prediction of the variance of the decimation algorithm as it is repeated on the same spectrum many times (resampled).

2.6.1. Choice of the Extraction fraction

The extraction fraction f is a free parameter of the spectral decimation algorithm. However, if f is chosen too small, gaps can be deemed Poissonian even if the underlying spectrum is that of a random matrix, such as a GOE. Therefore, for a spectrum of d elements, one fixes f by looking at the first-bin value of a GOE.

For a GOE matrix, the bulk gap PDF is well approximated by the Wigner–Dyson surmise $p_{\text{GOE}}(s) = \frac{\pi}{2} s e^{-s^2 \pi/4}$. However, it is exact only for 2×2 GOE matrices, not for arbitrary sizes; see [24] for a derivation of lower and upper bounds on the GOE CDF. This discrepancy is noticeable at small s , where the true GOE PDF behaves as $\pi s/6$; hence, the value of the zero-bin PDF is

$$\pi_{0,\text{GOE}}(\delta) = \frac{\pi^2}{12} \delta, \quad (2.44)$$

and its variance is

$$\sigma_{\text{GOE},0}^2 = \frac{\pi_{0,\text{GOE}}(1 - \delta \pi_{0,\text{GOE}})}{d\delta}. \quad (2.45)$$

In practice, one chooses $f = \pi_{0,\text{GOE}}(\delta) + z_q \sigma_{\text{GOE},0}$, such that if, at a given iteration, a small-gap probability is compatible with a GOE, then the algorithm halts. Assuming d is sufficiently large, z_q is the probit of a normal distribution $\mathcal{N}(0, 1)$ of a certain quantile q . If, at a certain iteration, the algorithm ends, it means that the underlying remainder distribution is GOE-like, with a zero-bin value compatible with $\pi_{0,\text{GOE}}$ at confidence level q . For example, when $q = 0.95$ (two-sided test), $z_q = 1.96$. This choice of f guarantees that any GOE gap distribution will cause the decimation algorithm to halt, thereby returning its actual size.

In the case of systems with randomness, we set $f = \pi_{0,\text{GOE}}(\delta) + 1.96 \sigma_{\text{GOE},0}$ for sample-by-sample decimations and $f = \pi_{0,\text{GOE}}(\delta)$ when samples are pooled together, as fluctuations are averaged in each bin.

2.6.2. Sources of systematic error

The choices of a finite sample d and resolution δ introduce numerical artifacts, independent of f . This is especially relevant when the initial gaps are i.i.d. Poisson gaps, as this case is a benchmark of the stability of the spectral decimation algorithm.

Indeed, in this case, the RS coincides theoretically with uniform sampling, as $\alpha(s) = p(0) = 1$. This implies that the ratio of accepted values, in each bin, α_j^{RS} is a

constant. However, this does not occur in practice, because of the inherent variance of the Poisson sample. This phenomenon is more evident when either small initial samples are considered or at later stages of the decimation.

For bin j of width δ , the mean Poisson probability is

$$\pi_j^P = \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} ds e^{-s} = e^{-j\delta} \frac{1 - e^{-\delta}}{\delta}, \quad (2.46)$$

and the expected number of counts is $\mathbb{E}[d_j] = d_j^P = d\delta\pi_j^P$, with variance $\text{var}[d_j] = d_j^P(1 - \delta\pi_j^P)$. The actual number of counts in bin j follows a binomial distribution: $d_j \sim \text{Bin}(d, \delta\pi_j^P)$.

Therefore, the empirical acceptance ratio $\alpha_j = \mathbb{E}[d_j]/d_j$ has variance $\text{var}[\alpha_j] = 1/d_j$, implying that the standard deviation behaves as:

$$\sigma_\alpha(s) = \frac{e^{s/2}}{\sqrt{d\delta}}. \quad (2.47)$$

As $\alpha(s)$ is a probability and cannot take values larger than 1, statistical noise becomes stronger as s increases. In particular, if one chooses a threshold $\sigma_\alpha(s) = 1 - \epsilon$, one obtains noisy acceptance probabilities starting from

$$s_{\text{noise}}(\epsilon) = \log(d\delta) + 2\log(1 - \epsilon). \quad (2.48)$$

At the value $s_{\text{noise}}(0) = \log(d\delta)$, the acceptance probability has the wildest oscillations, and it is possible that true Poisson gaps are refused only because of statistical noise.

A practical solution to this problem is to perform a normality test between the empirical counts and the expected Poisson counts. As this should only stabilize true Poisson samples, it is implemented according to the following logic:

- One checks if the first bin counts are compatible with the Poisson estimate;
 - If the condition is true, one proceeds with the checks for the other bins. If a bin, say j , has counts compatible with the Poisson expectation, then the acceptance ratio is set to $\alpha_j \equiv 1$.
 - If the condition is false, implying that the counts are not compatible with the Poisson estimate, the acceptance ratio for that bin is left untouched.

2.6.3. Variance of the Output after Resamplings

Let us consider the application of multiple decimations to the same spectrum of size d . Let the expectation value of the output be $d_{\text{out}} = D$, implying that the Poisson fraction is $M = 1 - D/d$. As the decimation algorithm is probabilistic, what is the variance of d_{out} obtained after resampling the spectrum R times?

When the spectrum has been decimated to the size D , the number of extracted Poisson gaps in the first bin has expectation $\mathbb{E}[d_0] = Md\delta_D$, where δ_D is the optimal bin-width according to the Freedman-Diaconis rule.

The same quantity d_0 can be computed by tracking the number of Poisson gaps extracted at each iteration. At iteration m_* , the spectrum has been decimated to D , thus $D = d(1 - f)^{m_*}$ and

$$m_* = \frac{\ln(D/d)}{\ln(1 - f)}. \quad (2.49)$$

Hence, d_0 will be the sum of the Poisson gaps extracted at every iteration up to m_* , on average:

$$\begin{aligned} \mathbb{E}[d_0] &= \sum_{m=0}^{m_*} d_E(m)\delta(m) \\ &= 12^{1/3} \frac{f}{1 - (1 - f)^{2/3}} (d^{2/3} - D^{2/3}) \underset{f \rightarrow 0}{\sim} \frac{3}{2} 12^{1/3} (d^{2/3} - D^{2/3}). \end{aligned} \quad (2.50)$$

It is easily checked that

$$\text{var}[d_0] = \left(\frac{d\mathbb{E}[d_0]}{dD} \right)^2 \text{var}[D] = \delta_D^2 \text{var}[D]. \quad (2.51)$$

As d_0 is the sum of Bernoulli variables in the first bin, its variance is $\text{var}[d_0] = \mathbb{E}[d_0]$. This equality leads to

$$\text{var}[D] = \frac{Md}{\delta_D} = \left(1 - \frac{D}{d}\right) \frac{dD^{1/3}}{12^{1/3}}. \quad (2.52)$$

2.7. Decimation of Statistical Mixtures

This section provides a check of the stability of the decimation algorithm. The setup considers a generic statistical mixture with N GOE spectra, each of size $d_i = 10^4$. Gap statistics for $N = 2, 4$, and 8 are reported in Fig. 2.9 and compared with the small- s approximation of Section 2.1.

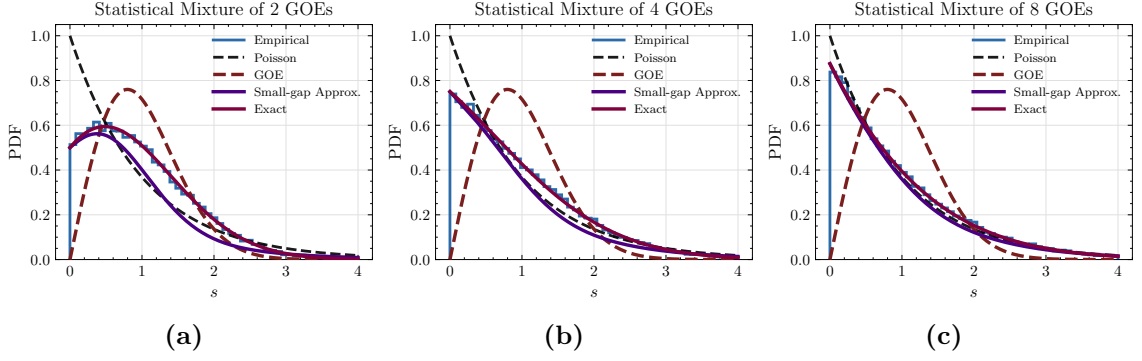


Figure 2.9: Statistical mixtures of N GOEs of the same size, $d_{\text{GOE}} = 10^4$, with $N = 2$ (a), $N = 4$ (b), and $N = 8$ (c), respectively. At moderately small $s \lesssim 0.5$, the small- s approximation (see Section 2.1) qualitatively describes the gap distribution, whose exact PDF [2, 24, 83] is shown. In particular, it predicts the correct value of the PDF near the origin. As N increases, the Poisson component of the mixture becomes dominant relative to the non-Poissonian component, which features level repulsion, and the approximation becomes better at describing the mixture for larger values of s .

One-Sample Analysis. For a mixture of N GOE spectra of the same dimension d_{GOE} , Eq. (2.24) predicts the spectral-decimation output $d_{\text{out},N}^{(1)}$ to be compatible with d_{GOE} . However, $d_{\text{out},N}^{(1)}$ is a random variable, as can be easily checked by repeating the decimation on the same spectrum multiple times (resampling).

For one sample, the Poisson fraction, whose expectation is $\mathbb{E}[M] = 1 - 1/N$, is itself a random variable, quantified by the number of counts in the first bin: $d_0 = M\delta d$. As $\text{var}[d_0] = d_0$, one obtains that $\text{var}[M] = \frac{M}{\delta d}$. Hence

$$\text{var}[d_{\text{out},N}^{(1)}] = \left(1 - \frac{1}{N}\right) \frac{d}{\delta} = 12^{-1/3} \left(1 - \frac{1}{N}\right) (Nd_{\text{GOE}})^{4/3}. \quad (2.53)$$

For sufficiently large N , the standard deviation grows as $N^{1/3}$, implying that $d_{\text{out},N}^{(1)}$ can differ substantially from d_{GOE} while still being compatible with it.

This prediction can be easily checked with a numerical experiment; see Fig. 2.10. Knowledge of one sample, even if it is resampled many times, is not sufficient to estimate d_{GOE} with arbitrary accuracy. To this end, the analysis of multiple samples is needed.

Analysis of Multiple Samples The analysis of t samples of mixtures with the same N can be performed by generating (tN) GOE spectra of dimension d_{GOE} . It is also possible to diagonalize a sufficiently large number of GOE matrices – in our case, 512 – and construct the mixture by superposing N randomly chosen GOE spectra from the original pool. For computational efficiency, we resort to the latter approach

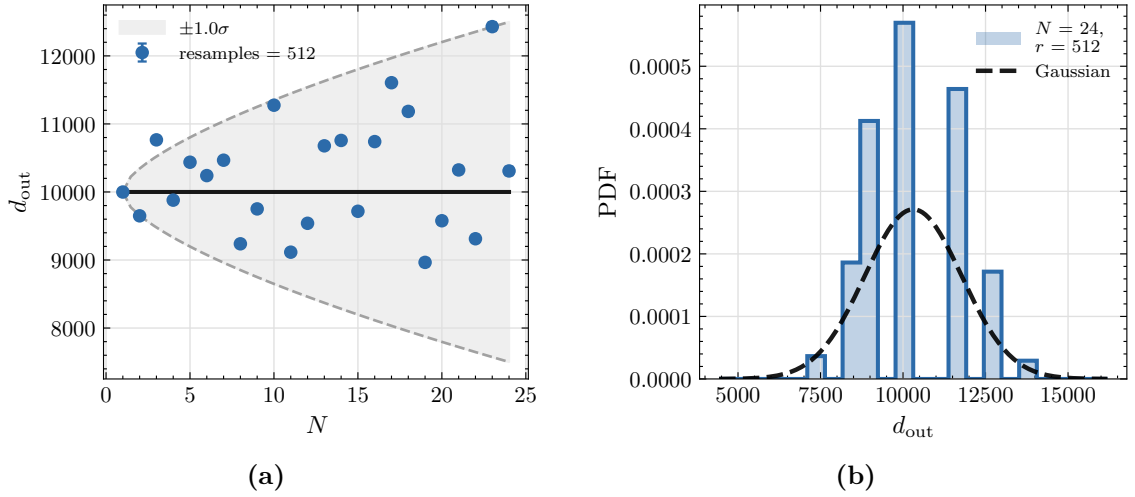


Figure 2.10: Decimation of a single realization of a statistical mixture, with $d_{\text{GOE}} = 1$. (a) After 512 resamples for each $N \leq 24$, d_{out} is plotted against the expected value d_{GOE} . Almost all data points fall within one standard deviation from the mean. (b) Distribution of d_{out} for $N = 24$, resampled $r = 512$ times. Overall, the distribution follows a Gaussian with variance predicted from the spectral decimation, Eq. (2.48). Because of the discrete extraction fraction values, d_{out} can only assume discrete values, thus the peaks in the PDF.

to generate the t samples.

The goal of this analysis is to assess whether the decimation can predict d_{GOE} accurately, up to the finite-size artifacts which are intrinsic to the statistics. These errors can be significantly reduced by pooling the t samples together since we are interested only in the average d_{out} .

Fig. 2.11 shows that d_{GOE} can be accurately estimated from the decimation procedure and that the pooling technique is equivalent to averaging the d_{out} 's of individual samples. For each N , the outcomes of the decimation follow a Gaussian distribution with a mean of d_{GOE} and variance given by Eq. (2.53).

Dynamical Signatures of the CSS We conclude this Section by considering the behavior of dynamical probes of quantum ergodicity, such as the spectral form factor (SFF), defined as

$$K(t) = \left\langle \sum_{m,n} e^{it(E_m - E_n)} \right\rangle. \quad (2.54)$$

The SFF is a measure of spectral rigidity, also known as the two-level form factor [24], which depends not only on short-range correlations but also on additional properties of the Hamiltonian, including long-range spectral correlations and global density-of-states effects.

We consider Hamiltonians H_N whose spectra are obtained as statistical mixtures of

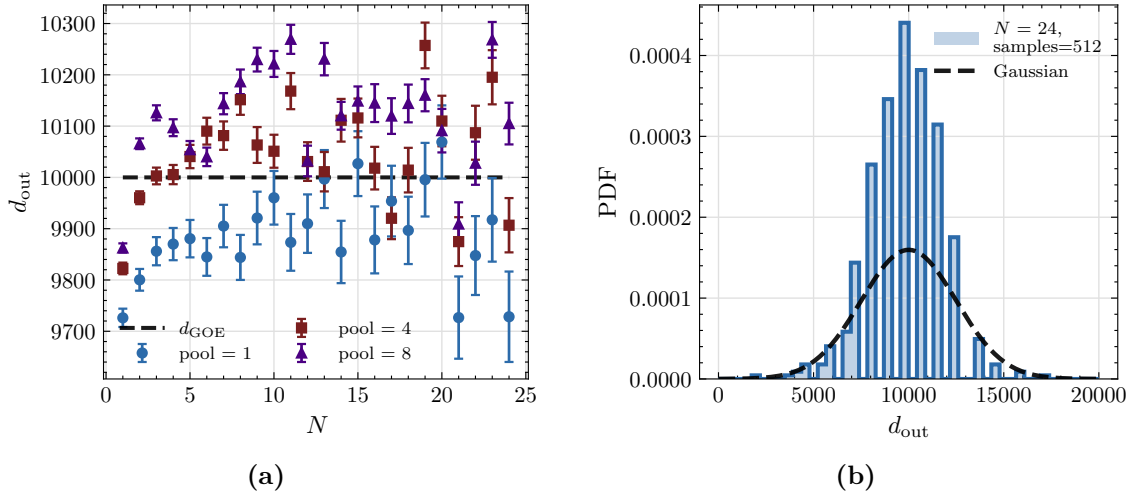


Figure 2.11: Result of the decimation procedure on statistical mixtures generated by N identical GOEs of size $d_{\text{GOE}} = 10^4$. (a) Decimation-algorithm output, d_{out} , as a function of the number N . Each data point is obtained by decimating 512 mixtures, individually or by pooling them in groups of p . For each pooling p , the data are contained within a window of 3% of the true value d_{GOE} . (b) Focusing on $N = 24$ and $p = 1$, we check the distribution of the decimation outputs and observe that they are distributed according to a Gaussian with mean d_{GOE} and the variance in Eq. (2.53). As both the initial size and extraction fraction are fixed, the possible d_{out} are discrete, hence the peaks.

N independent GOE spectra, each of size 10^4 . Since all ensembles share the same semicircular density of states, we focus on $m = 10^3$ consecutive unfolded levels in the bulk and compute the corresponding SFF for different values of N . The results are further averaged over disorder realizations and locally smoothed in time in order to resolve the ramp structure. The plot for $N \leq 8$ is given in Fig. 2.12.

The effective size of the CSS in this construction is $d(N) = m/N$. Fig. 2.12 shows a clear and systematic dependence of the SFF on the CSS size. As $d(N)$ decreases from its maximal value at $N = 1$ — corresponding to a pure GOE spectrum — two characteristic effects emerge:

- the ramp of the SFF becomes progressively steeper;
- the depth of the correlation hole is strongly suppressed.

Together, these effects continuously drive the SFF from the GOE behavior toward the Poisson limit $K_{\text{Poisson}}(t) = 1$, which is recovered as $N \rightarrow \infty$, namely when the spectrum becomes completely uncorrelated.

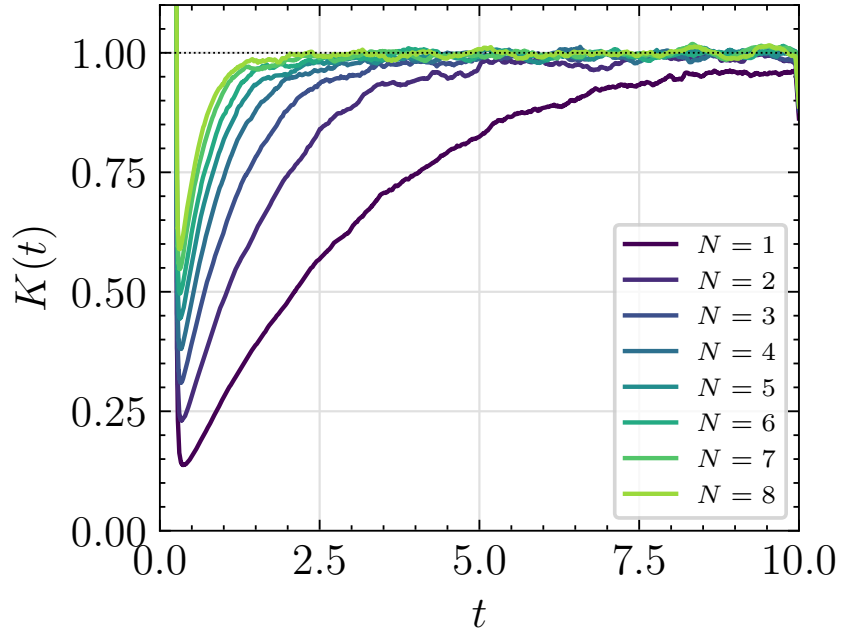


Figure 2.12: Normalized spectral form factor (SFF) of the Hamiltonians H_N , obtained as mixtures of N GOEs of size 10^4 . For each line, $m = 10^3$ eigenvalues are sampled in the bulk of the density of states – Wigner’s semicircle. We observe that, as N increases (equivalently, as the size of the CSS, $d(N) = n/M$, decreases), the SFF interpolates between the GOE curve ($m = 1$) and the Poisson curve ($m \rightarrow \infty$).

2.8. Applications to Quantum Many-Body Hamiltonians

In this Section, we consider realistic quantum many-body Hamiltonians in which the Hilbert space presents disconnected sectors, as in the case of Hilbert-space fragmentation, or disorder, as in systems featuring many-body localization. Emergent symmetries, due either to the presence of many Krylov subspaces or to local integrals of motion, can cause the global statistics to appear Poissonian. We investigate whether the CSS presents the same Poisson statistics or instead reveals the presence of a statistical mixture, and whether its size can inform us about the underlying emergent symmetry.

2.8.1. Hilbert-Space Fragmentation

Hilbert-space fragmentation has emerged as a distinct and conceptually rich form of ergodicity breaking in quantum many-body systems, one that is neither rooted in integrability nor driven by quenched disorder [6, 85–90] (see also [91–109] for

an extensive list of models). In fragmented models, the physically natural (often product-state) basis reveals that the Hamiltonian dynamics does not explore the full Hilbert space but instead decomposes it into an exponential number of dynamically disconnected Krylov subspaces. Unlike conventional symmetry sectors – whose number grows at most polynomially with system size – these Krylov subspaces are generated by microscopic kinetic constraints or conserved dipole-like quantities and can span a broad spectrum of dimensionalities, from finite or logarithmically growing manifolds to subspaces that remain exponentially large. As emphasized in the recent review of fragmentation phenomena [6], this structural decomposition leads to a rich variety of dynamical behaviours: certain subspaces support effectively integrable dynamics with Poissonian spectral statistics, while others exhibit non-integrable, quantum-chaotic behaviour consistent with GOE statistics. Consequently, Hilbert-space fragmentation provides a controlled setting in which integrable and chaotic sectors coexist within a single Hamiltonian, and the global spectral statistics becomes an unreliable indicator of the underlying dynamical properties.

A salient implication of this phenomenon, particularly relevant for our present study, is that conventional diagnostics, such as level-spacing ratios or eigenstate entanglement scaling, can be profoundly obscured by the superposition of spectra originating from multiple dynamically isolated sectors. In weakly fragmented systems, only a measure-zero subset of eigenstates violates ETH, and the majority of sectors behave thermally, often yielding GOE statistics once symmetries are resolved. In strongly fragmented systems, by contrast, the proliferation of small-dimensional Krylov subspaces produces extensive degeneracies and unconventional spectral statistics, typically mimicking Poisson behavior even when the dynamics within individual large sectors remain non-integrable. This mixing problem underscores the need for refined spectroscopic tools capable of disentangling the true dynamical character of each sector. The spectral decimation protocol is precisely designed to meet this challenge: by progressively thinning the spectrum and isolating statistically consistent subsequences, it enables us to diagnose hidden non-integrable sectors in weakly fragmented models and to characterize the emergent integrability structure in strongly fragmented ones. In this sense, Hilbert-space fragmentation provides an ideal testbed for demonstrating the power of spectral decimation in resolving complex ergodicity-breaking mechanisms.

As an example of HSF, we consider the spin-1/2 pair-flip (PF) model introduced in

[110, 111] of Hamiltonian

$$H_{\text{PF}} = \sum_{i=1}^L \sum_{\alpha, \beta=1}^2 \left[g_{i,j}^{\alpha, \beta} |\alpha_i \alpha_{i+1}\rangle \langle \beta_i \beta_{i+1}| + h.c. \right] + \sum_{i=1}^L \sum_{\alpha=1}^2 h_i^{\alpha} |\alpha_i\rangle \langle \alpha_i|, \quad (2.55)$$

where $g_{i,j}^{\alpha, \beta}$ and h_i^{α} are arbitrary constants.

This model features HSF for any $g_{i,j}^{\alpha, \beta}$ and h_i^{α} in the computational basis, i.e. the total Hilbert space is decomposed into $(L + 1)$ disconnected Krylov spaces, each associated with a $U(1)$ charge. As pointed out in [110], each Krylov subspace can be labeled by an “irreducible string”: for a given state, it is obtained by deleting all possible next-nearest-neighbor pairs, proceeding from left to right and repeating the process if new pairs arise after a deletion. For example, if we denote down- and up-spin states by 1 and 2, then the irreducible string for the state $|1221\rangle$ is $\emptyset$, while it is 2 for the state $|22211\rangle$. Each dynamically disconnected Krylov subspace is uniquely characterized by an irreducible string and is spanned by the basis states connected by all allowed pair-flip moves. For example, for the irreducible string $\emptyset$ with $L = 4$, one can start from the initial state $|1221\rangle$ and perform all possible pair flips to generate the basis of the corresponding Krylov subspace, reproducing the action of the Hamiltonian. One can easily notice that no pair flip allows transitions between states belonging to subspaces with different irreducible strings; therefore, the Hilbert space is fragmented. Alternatively, it is also possible to start from the 2^L states of the computational basis and identify their irreducible strings with a “pushdown automaton” [110].

The Hilbert space of this PF model can be expressed as a direct sum of dynamically connected subspaces labeled by an irreducible string x , i.e.,

$$\mathcal{H} = \bigoplus \mathcal{H}_x, \quad \mathcal{H}_x = \text{span}\{|\omega\rangle : \text{irr}(\omega) = x\}, \quad (2.56)$$

where $|\omega\rangle$ represents the quantum state in Hilbert space, and x denotes the corresponding irreducible string of the state $|\omega\rangle$. In the spin-1/2 case, if we denote the total number of 1s and 2s in the irreducible strings by $k = 2M$, it follows that all the possible irreducible strings for a chain of even length L are

$$\emptyset, 12, 21, 1212, 2121, 121212, 212121, \dots \quad (2.57)$$

Each irreducible string x of length k carries a $U(1)$ conserved charge $Q(x) = \sum_i^{L-1} (-1)^i (N_i^{(1)} - N_i^{(2)}) = \pm k$, where $N_i^{(1)}$ and $N_i^{(2)}$ count the numbers of down and up spins at site i . If we define an effective magnetization as $M = -Q(x)/2$, then the

dimension of the Krylov subspace corresponding to H_x is

$$D_{L,M} = \binom{L}{\frac{L}{2} - |M|}. \quad (2.58)$$

Therefore, the entire Hilbert space of the PF model is decomposed into $(L + 1)$ Krylov subspaces. The largest block has dimension $D_{L,>}$, which is equal to $D_{L,0}$ for even L and $D_{L,1/2}$ for odd L . The average block size is $\langle D_L \rangle = 2^L / (L + 1)$.

As the pair-flip model is, in effect, a statistical mixture, the size of the CSS when all blocks are non-integrable is, according to Eq. (2.24),

$$\mathbb{E}[d_{\text{out}}^{\text{PF}}] = 2^{-L} \sum_M D_{L,M}^2 = 2^{-L} \binom{2L}{L} \sim \frac{2^L}{\sqrt{\pi L}}. \quad (2.59)$$

As a particular realization of the PF Hamiltonian, we choose the binary random couplings $|g_{i,j}^{\alpha,\beta}| = J(1 + \delta_{\alpha,\beta})/2$ and h_i^α to be uniformly i.i.d. in $[-w, w]$ for some disorder strength such that $w/J \ll 1$. With this choice of coefficients, the Krylov subsectors exhibit almost perfect GOE-like level spacing and gap ratio statistics, and therefore serve as an ideal testbed for spectral decimation.

For each L , we pool several realizations of the PF Hamiltonian and perform the gap decimation. Fig. 2.13 shows remarkable agreement of the spectral-decimation algorithm with Eq. (2.24). This pooling approach linearly enlarges the input of the spectral-decimation algorithm and, at the same time, removes sample-by-sample fluctuations. Therefore, we have demonstrated that the size of the CSS in a generic mixture, Eq. (2.24), can be probed through the spectral-decimation algorithm, even when HSF occurs.

2.8.2. Many-Body Localization

Many-body localization (MBL) [9, 17, 18, 112–117] has emerged as a paradigmatic mechanism of ergodicity breaking in disordered quantum systems, providing a robust alternative to thermalization in highly excited states [118–143]. Unlike conventional isolated many-body systems, whose long-time behavior is governed by the Eigenstate Thermalization Hypothesis (ETH), MBL systems fail to act as their own thermal reservoirs: they retain a persistent memory of their initial conditions and never approach a Gibbs ensemble. This phenomenology is reflected directly in the structure of the many-body eigenstates: instead of exhibiting volume-law entanglement as predicted by ETH, MBL eigenstates obey area-law scaling even at finite energy

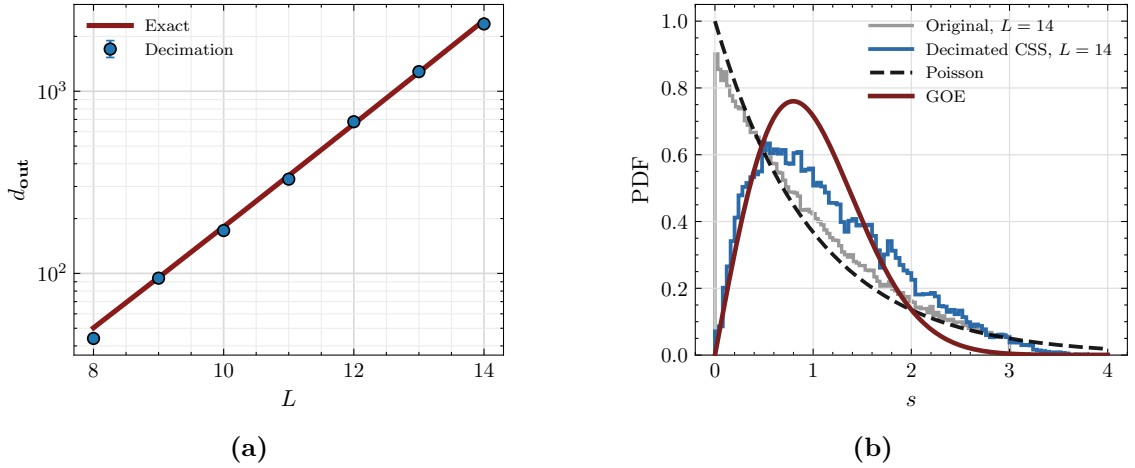


Figure 2.13: Application of spectral decimation to the pair-flip model of Eq. (2.55). (a) The decimation output, d_{out} , closely follows the theoretical prediction in Eq. (2.24). (b) Although the global gap distribution closely follows Poisson statistics, the distribution of the remainder gaps that survive the decimation features clear signs of level repulsion; therefore, the CSS is non-integrable.

density, signaling the emergence of an extensive set of quasi-local integrals of motion (LIOMs or ℓ -bits) [19, 144–150] (see also [151]). This emergent integrability underpins all aspects of the MBL phase, and it distinguishes MBL sharply from both non-interacting Anderson insulators and integrable models described by Generalised Gibbs Ensembles.

A central challenge in contemporary studies of MBL lies in characterising the transition between thermal and localized phases – a problem that remains subtle due to finite-size effects, sample-to-sample fluctuations, and the coexistence of thermal and localized regions (avalanches). Conventional diagnostics such as the level-spacing ratio, entanglement entropy, or transport coefficients capture some aspects of this transition, but they are often obscured by Griffiths effects and by the slow approach to the thermodynamic limit. Here, statistical spectroscopy offers a powerful complementary perspective. Because MBL is intrinsically a form of emergent integrability, its onset leaves sharp fingerprints in the spectral statistics: GOE-like distributions characteristic of chaotic dynamics give way to Poisson statistics as LIOMs proliferate. Yet the precise manner in which this crossover occurs – particularly in finite systems where mixed or “rare-region-dominated” spectra are common – requires more refined tools than simple gap-ratio analysis. The spectral-decimation protocol is tailored precisely for this purpose: by extracting statistically consistent subsequences of the spectrum, it exposes the latent integrable structure hidden beneath disorder-induced spectral mixing, thereby providing a clean, unbiased probe of the emergent integrability that defines the MBL phase.

As a paradigmatic case of MBL, we analyze the spin-1/2 disordered XXZ model [115, 152–154] of Hamiltonian

$$H = J \sum_{i=1}^L \mathbf{S}_i \cdot \mathbf{S}_{i+1} + J(\Delta - 1) \sum_{i=1}^L S_i^Z S_{i+1}^Z + \sum_{i=1}^L h_i S_i^Z. \quad (2.60)$$

We set $J = 1$ and $\Delta = 1$. The h_i 's are random i.i.d. variables uniformly distributed in $[-W, W]$, with W the strength of disorder. The model features $U(1)$ symmetry with a generator $\widehat{M} = \sum_i \sigma_i^Z$. Throughout this thesis, we focus on the $M = 0$ sector for even L and $M = 1/2$ for odd L . The dimension of the Hilbert space is given by Eq. (2.58).

We now summarize the results of the spectral decimation applied to Eq. (2.60), offering both qualitative and quantitative analyses. For each value of $L = \{9, \dots, 16\}$, we consider at least $N = 1000$ samples for each (W, L) . Then, for each (W, L) , we pool all samples and perform the decimation on this larger set, resampled $r = 25$ times.

Qualitative Features of the CSS

Overall, the CSS remains close to the Hilbert space dimension for $W \lesssim 2$ and then decreases exponentially with an (W, L) -dependent slope. Crucially, even at moderate volumes, we observe in the strong disorder region, $W = 10$, that the CSS approaches values on the order of a few percent of $d(L)$, up to 1%. In that region, energy levels tend to form small CSS, implying that global spectral correlations are weakening. The operators enabling these emergent symmetries are the LIOMs and the CSS may be interpreted as an estimator of the effective symmetry sectors generated by them. Thus, spectral decimation provides an independent spectral diagnostic consistent with the presence of LIOMs in the disordered XXZ Hamiltonian.

We emphasize that the extracted CSS size is robust under moderate variations of the halting threshold d_{halt} , provided that the extraction fraction f is chosen according to Section 2.6 so that it does not introduce systematic over- and underestimates of the decimation output. As explained in Sec. 2.5, the self-averaging pooling procedure suppresses finite-sample fluctuations, rendering the CSS stable against such variations. We also compared the CSS size obtained using the r -ratios instead of the gaps, as an internal test of the consistency of the spectral decimation. Qualitatively, the CSS size presents similar L and W dependence, but its quantitative values differ, as outlined in Section 2.5.

Fig. 2.14 shows the underlying distributions of the CSS gaps and r -ratios at weak

and strong disorder. We observe that, at weak disorder, both the gap and r -ratio statistics are GOE-like. At very large disorder, the leading term of the Hamiltonian is $H_Z = \sum_{i=1}^L h_i S_i^Z$, whose spectrum is that of a sum of L independent, identically and uniformly distributed on-site energies. In the limit $L \rightarrow \infty$, the statistics of the “truncated oscillator” approach the Poisson distribution, similarly to the $N \rightarrow \infty$ limit of the mixture (see Section 2.1). As already pointed out in [11], the level statistics of the truncated oscillator with few frequencies, corresponding in our case to finite L , feature characteristic peaks due to equidistant energy levels. Moreover, these peaks cannot be explained, for example, by a Brody parameter, which only interpolates between GOE and Poisson statistics. The CSS features the same phenomenology at strong disorder. While this is evident from the gap statistics (especially when compared to the free-fermion statistics in Fig. 2.8(c)), we notice that this feature is absent from the r -ratio statistics, which mildly deviate from $p(r) = 2/(1+r)^2$.

The analysis of the CSS statistics provides new insight into the ‘emergent integrability’ of the one-dimensional disordered XXZ Hamiltonian. It should not be understood as the emergence of generic integrable dynamics, like that of the same model in the absence of disorder, the strongly interacting XXX chain, but rather as the emergence of asymptotically free dynamics, like that of weakly interacting fermions.

2.8.3. Characteristic Symmetry Entropy

For any (W, L) , the CSS size $d_{\text{out}}(W, L)$ obtained by running multiple times (resampling) the gap decimation of the pooled samples varies exponentially with W and L . Therefore, the CSS size cannot be used as a sensible observable for a finite-size scaling (FSS) analysis.

We start by noticing that the asymptotic form of the Hilbert-space dimension is

$$\log d(L) \sim (\log 2)L - \frac{1}{2} \log L + \frac{1}{2} \log \frac{2}{\pi}, \quad (2.61)$$

one can consider $\frac{1}{L} \log d(L)$ as an intensive quantity, up to $O((\log L)/L)$ corrections. Therefore, we study the ratio of the original dimension to the CSS dimension. This quantity expresses the effective number of blocks with the CSS dimension – assuming them to be uniform. The logarithm of this ratio can be rescaled to define the intensive ‘characteristic symmetry entropy’ (CSE):

$$\Sigma(W, L) = \frac{1}{L \log 2} (\log d(L) - \log d_{\text{out}}(W, L)), \quad (2.62)$$

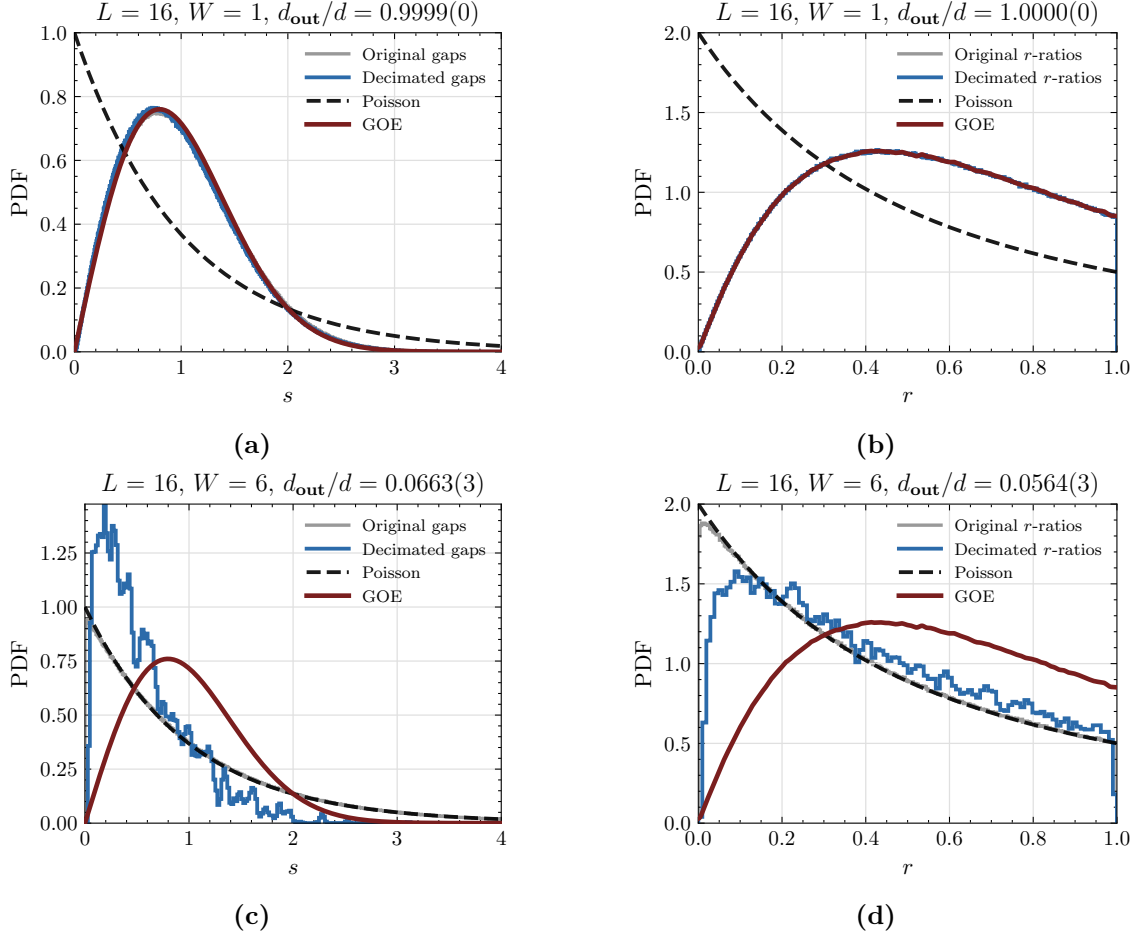


Figure 2.14: Histograms of gaps and r -ratios of the one-dimensional disordered Heisenberg chain for $L = 16$ at weak ($W = 0.5$) and strong ($W = 6.5$) disorder, in an energy window covering $\Delta E = 90\%$ of the full spectrum. At weak disorder, one observes excellent agreement of the distributions of gaps (a) and r -ratios (b) with the Wigner–Dyson distribution of GOE random matrices. The value of d_{out}/d is approximately 1 in both cases, with that obtained from the r -ratios being subject to smaller statistical fluctuations. At strong disorder, on the other hand, while the r -ratio statistics (d) seem compatible with a semi-Poisson distribution (the Poisson distribution will eventually be restored as $W \rightarrow \infty$), the gap statistics (c) clearly deviate from the Poisson distribution and accumulate toward $s \rightarrow 0$. Sharp peaks are characteristic of effectively free or weakly interacting spectra, in which the spectra are regular and energy differences can be expressed as differences of linear combinations of one-particle energies. This distinction between the statistics of free and strongly interacting integrable systems is not captured by the r -ratio distribution, which is the same in both cases. We also observe that the CSS size extracted from r -ratio decimation deviates quantitatively from that of gap decimation, as discussed in Section 2.5.

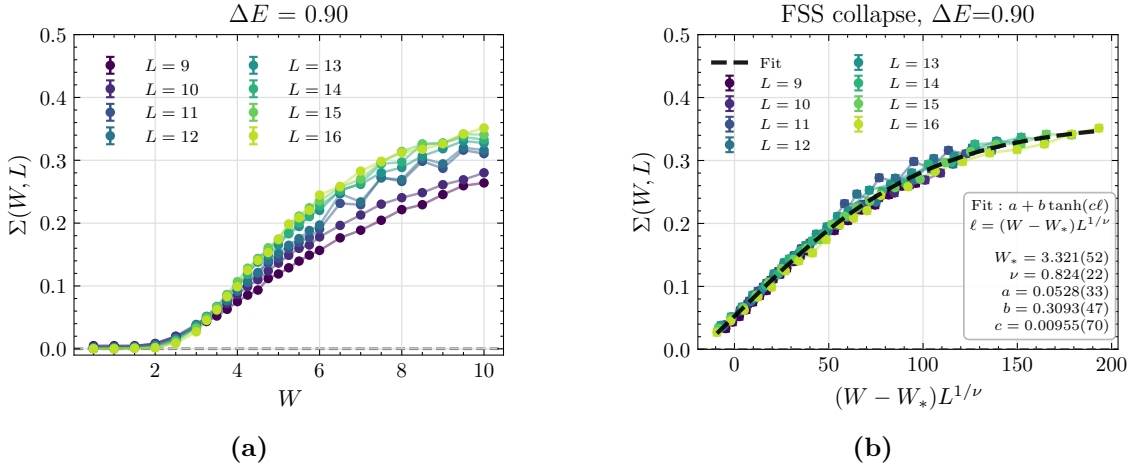


Figure 2.15: (a) Plot of the CSE over a large window of the spectrum ($\Delta E = 0.9$), varying the disorder strength W for different volumes L . For $W \lesssim 3$, the CSE remains close to 0, indicating that the system is close to GOE, while it increases for larger values of W . For the disorder strengths and volumes studied, the CSE does not approach unity, even though the CSS may be as low as 1% of the Hilbert-space dimension. As such, the CSE is a sensible probe of the fine spectral structure of disordered quantum Hamiltonians. (b) FSS collapse of the various $\Sigma(W, L)$ curves. The fitted parameters, as well as the fitting form, are reported in the box at the bottom right.

where $d_{\text{out}}(W, L)$ is the average of the output of the spectral decimation. Vanishing $\Sigma(W, L)$ corresponds to fully chaotic (GOE-like) behavior, while larger values indicate increasing effective fragmentation.

We notice that decimations performed with gaps or with r -ratios produce qualitatively similar CSE values for the disordered XXZ Hamiltonian. We also observe that the CSE never saturates to unity for the system sizes and disorder parameters considered, reaching a maximum value of approximately 0.4; see Fig. 2.15(a). This immediately raises the question of whether the system sizes under consideration represent a legitimate thermodynamic regime in which true MBL scaling dominates. This issue cannot be resolved by exact diagonalization alone, and it can be clarified by a renormalization-group study of the CSS, a task that we will not pursue in this thesis.

Finite-Size Scaling across Multiple Energy Windows

Below, we present an FSS analysis of the CSE. Its purpose is not to draw conclusions about the physics of MBL, but rather to show that the CSE is an observable that exhibits an FSS collapse. Indeed, the validity of this analysis is based on a phenomenological ansatz and not on a renormalization-group scaling form. Moreover, it is likely that the system sizes utilized are not representative of a true thermodynamic

phase, but of a crossover regime. For these reasons, the following analysis will only be indicative rather than conclusive.

While conventional studies of MBL focus on thermal states in a small energy window ΔE , to compensate for the limited system sizes at our disposal, we will also consider wider windows, ranging from the bulk to the full spectrum, where the spectral decimation becomes more statistically accurate. As customary, we introduce a scaling variable

$$\ell = L^{1/\nu}(W - W_*), \quad (2.63)$$

where conventionally W_* is the critical disorder parameter and ν is the dynamical exponent characterizing the divergence of the correlation length: $\xi \sim |W - W_*|^{-\nu}$. It is also worth noting that this ansatz may, in principle, be wrong, as renormalization-group studies [155–157] argue that the transition may be of the Kosterlitz–Thouless (KT) type, where $\nu = \infty$. Therefore, the scaling parameters should be interpreted as effective crossover quantities rather than true thermodynamic exponents. We tried fitting with both a KT law, $\log \xi \sim (-\sqrt{W - W_*})$, and a power law, but found that the KT FSS ansatz does not provide a collapse for the system sizes under study.

Overall, the CSE presents three qualitatively distinct regions: for weak disorder, $W \lesssim 3$, it is approximately zero; it then undergoes a crossover, $3 \lesssim W \lesssim 6$, with an L -dependent slope toward a saturation region, $W \gtrsim 10$. This suggests that the FSS form may be a sigmoid or a hyperbolic tangent. Indeed, the latter option offers a collapse of Σ versus W at different values of L :

$$\Sigma(W, L) = a + b \tanh(c\ell), \quad (2.64)$$

where a , b and c are numerical constants. The FSS ansatz is robust in the disorder window $3 \leq W \leq 10$, offering a collapse of the CSE, see Fig. 2.15(b). Varying the energy window, we observe that the estimated critical point lies in the range $3 \lesssim W_* \lesssim 4$, and that the critical exponent $\nu < 1$, see Fig. 2.16.

While the value of W_* for small ΔE agrees with other estimates from exact-diagonalization studies [153], the value of ν violates the Harris bound ($\nu > 2/D$, where D is the spatial dimension). Nevertheless, among the ansatz forms tested, the power-law form provides the best collapse of the available data, and we report its parameters in Fig. 2.16.

Nonetheless, the CSE can be used as an observable that is easily accessible from numerical data. Moreover, it can be computed from either the gaps, which require unfolding, or the r -ratios, which gauge three-level correlations but do not require

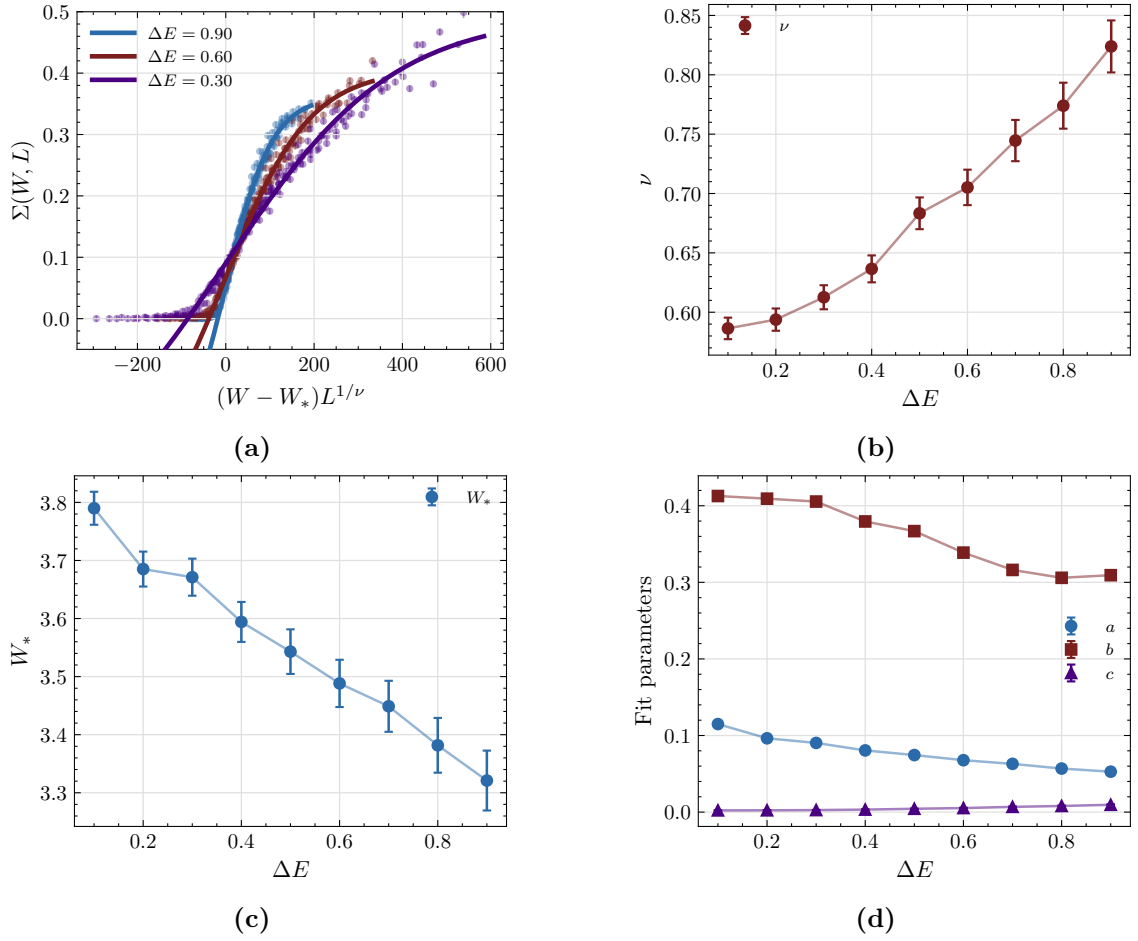


Figure 2.16: Results of finite-size scaling (FSS) of the characteristic symmetry entropy (CSE). (a) The FSS of the CSE is reported for a few energy windows ΔE , showing that, as ΔE increases, the steepness of the curves, dictated by the critical exponent ν , diminishes. (b) The same critical exponent ν is plotted against ΔE for the energy windows studied. (c) A similar plot for W_* . Finally, the fit constants vary monotonically with the energy window (d).

fitting the density of states. Thus, it can be considered a complementary probe of eigenvalue correlations, such as the average r -ratio. Moreover, the CSE is consistent with a power-law FSS ansatz, which brings it to a volume-independent collapse.

Chapter 3

Permutation Hamiltonians and Walks on S_L

Permutation Hamiltonians (PHs) are a class of quantum Hamiltonians whose constituents are elements of the permutation group of L sites. They span the space of interactions that only permute the contents of the sites. Nonetheless, they represent a large class of systems, which feature ergodicity in some realizations and non-ergodicity in others. They are suitable for the study of the spectrum through ED, since they share a common basis of irreducible representations in which the diagonal blocks can be constructed algorithmically. This chapter investigates both the spectral and out-of-equilibrium properties of PHs. After introducing the relevant structure of the permutation group, we discuss the representation and spectral statistics of several families of PHs. The second part of this chapter considers quantum generalizations of classical random walks on the permutation group. Unitary quantum walks are shown not to provide a direct analogue of classical mixing, since reversible evolution does not generally converge to the uniform distribution. Finally, we introduce a quantum stochastic walk that combines classical dissipative dynamics with coherent Hamiltonian evolution. Although coherence does not modify the trace-distance relaxation of the full density matrix, it accelerates the approach of computational-basis probabilities to uniformity. We prove that the instantaneous distance of these probabilities from the uniform distribution is bounded above by the corresponding classical distance, analyze the effect of coherence on the slowest irreducible mode, and corroborate the resulting mixing-time estimates numerically. Both parts of this chapter frame statistical spectroscopy not only as a tool directly useful to the analysis of equilibrium spectral properties, but it also provides a complementary perspective on dephasing, whereas the mixing times of the open quantum walks are obtained from the exact

slow modes of the corresponding generator. The material presented in this chapter is taken from [3, 158].

3.1. Permutation Hamiltonians

In this section, we describe Hamiltonians based on permutations, i.e. the elementary interaction features the swap of sites. They can be generated on graphs and can feature different boundary conditions.

3.1.1. Generalities on the Permutation Group

In what follows, we focus on the archetype of all finite groups: the permutation group. The purpose of this presentation is not to provide a comprehensive treatment of this group but rather to supply the essential background needed for the numerical analysis presented later in this section. Further details on the structure of the permutation group are collected in Appendix B.

The permutation group of L elements, denoted S_L , is a finite group of order $|S_L| = L!$ generated by $L - 1$ elementary (adjacent) transpositions $P_{i,i+1}$, $i \in \{1, \dots, L - 1\}$. A transposition $P_{i,j}$ is an element of the permutation group that exchanges, among L distinct objects, those labelled i and j . In terms of adjacent transpositions, $P_{i,j}$ can be obtained recursively:

$$P_{i,i+2} = P_{i+1,i+2}P_{i,i+1}P_{i+1,i+2} . \quad (3.1)$$

All elements of S_L admit a “word” in terms of generic transpositions $P_{i,j}$, which may not be unique. Moreover, an elementary word contains only adjacent transpositions $P_{i,i+1}$. This word is said to be minimal if it contains the smallest number of $P_{i,i+1}$. For example, for $P_{i,j}$, the minimal word has $2|i - j| - 1$ adjacent transpositions.

Each element of the permutation group belongs to a conjugacy class $\mathcal{C}_\alpha$, which is identified by the structure of its cycles. For example, for $L = 9$, the conjugacy class $[3^2, 2, 1]$ contains two different cycles in which three elements are permuted (3-cycles), a 2-cycle, and an element that is not permuted (a one-cycle). Transpositions are therefore 2-cycles. The number of conjugacy classes equals the number $\mathcal{P}(L)$ of

integer partitions of L , whose asymptotic behaviour is given by

$$\mathcal{P}(L) \simeq \frac{1}{4L\sqrt{3}} \exp\left(\pi\sqrt{\frac{2L}{3}}\right). \quad (3.2)$$

The conjugacy classes of the permutation group can be represented by Young diagrams, which are also in one-to-one correspondence with the group's irreducible representations. Young diagrams provide a natural and powerful framework for studying irreducible representations, as they encode the combinatorial rules required to compute fundamental quantities. Among the most significant is the dimension of a representation, given by the celebrated hook-length formula, recalled in Appendix B. For example, the fully symmetric representation $[L]$ and the fully antisymmetric representation $[1^L]$ are both one-dimensional.

A permutation Hamiltonian is a generic linear superposition of permutation elements

$$H = \sum_{P \in S_L} \alpha_P (P + P^{-1}). \quad (3.3)$$

The inclusion of the inverse guarantees the Hermiticity of the operator. Furthermore, it is not necessary to consider higher powers of permutation elements, as the symmetric group S_L is, by definition, closed under multiplication and already contains all such products. The coupling constants α_P may be treated either as fixed real parameters or as random variables drawn from a prescribed probability distribution.

We consider Hilbert spaces in which each site can be locally occupied by one of n bosonic states, distinguished by their “colors”. Locally, the state will be denoted as $|v_a\rangle_i$, where i refers to the site, while $a = 1, 2, \dots, n$ refers to the color index. Consequently, each site transforms according to the fundamental representation of $SU(n)$, and permutation Hamiltonians act by permuting the sites and their superpositions. For a chain of length L , the Hilbert space has dimension $d = n^L$, which can be decomposed into symmetry sectors, i.e. “color sectors” $\{L_1, L_2, \dots, L_n\}$, where L_j is the number of particles of color $j \in \{1, \dots, n\}$, subject to $\sum_{j=1}^n L_j = L$. Within these color sectors, permutation Hamiltonians remain invariant. More precisely, let $U \in SU(n)$ denote an $n \times n$ unitary transformation representing color mixing. Its tensor product $U^{\otimes L}$ acts on the full Hilbert space as a $d \times d$ matrix and commutes with the Hamiltonian, i.e. $(U^{\otimes L})^{-1} H U^{\otimes L} = H$.

In a given color sector, transpositions $P_{i,j}$ are represented in the computational basis – namely, the product basis in which each site is assigned a specific color – as sparse matrices with a single nonzero entry per row. Remarkably, within these color sectors,

the Hamiltonian admits a further decomposition into irreducible representations of S_L . Since permutations among sites carrying the same color leave the state invariant, bosonic many-body wavefunctions can be labelled by irreducible representations of the subgroup $S_{L_1} \times S_{L_2} \times \dots \times S_{L_n} \subset S_L$, where L_i denotes the number of sites of color i . These representations are preserved by the Hamiltonian, which thus acquires the fragmented block structure shown schematically below:

$$H = \begin{bmatrix} \boxed{\begin{matrix} \square & & \\ & \square & \\ & & \ddots \end{matrix}} & & \\ & \boxed{\begin{matrix} \square & & \\ & \square & \\ & & \ddots \end{matrix}} & \\ & & \ddots \end{bmatrix}. \quad (3.4)$$

Within each irreducible representation, the adjacent transpositions $P_{i,i+1}$ can be represented as real, unitary, and Hermitian matrices [159]. Further details regarding their numerical implementation are provided in Appendix B.

3.1.2. Hamiltonians on Graphs

In permutation Hamiltonians, the interactions act solely on the sites, irrespective of the specific structure of the local Hilbert space. This property makes it possible to investigate a wide variety of models while preserving both the dimensionality and the intrinsic structure of the symmetry blocks.

As a first example, consider Hamiltonians defined on graphs G . Here, the nodes represent sites, and the links correspond to permutations. The degree of a node is defined as the number of links (or edges) incident upon that node. A simple undirected edge connecting sites i and j is represented by the transposition $P_{i,j}$, and the Hamiltonian can be directly constructed from the adjacency matrix A_{ij} of the graph, defined as follows: for a graph $G = (V, E)$ with $|V| = L$ vertices, the adjacency matrix is the $L \times L$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between vertices } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

For undirected graphs without self-loops (the case we are interested in), A is real and symmetric, with vanishing diagonal entries. The Hamiltonian H_G on a graph G

is then given by

$$H_G = \sum_{i < j} J_{ij} A_{ij} P_{ij} , \quad (3.5)$$

where the coupling constants J_{ij} are either fixed real numbers or random variables drawn from a given probability distribution.

Sutherland Permutation Hamiltonians

The minimum number of edges that generates a graph with L nodes is $L - 1$, and the Hamiltonian

$$H_2^o = \sum_{i=1}^{L-1} J_i P_{i,i+1} \quad (3.6)$$

corresponds to the Sutherland permutation model with open boundary conditions [160]. The same Hamiltonian with periodic boundary conditions

$$H_2^p = \sum_{i=1}^L J_i P_{i,i+1 \bmod L} , \quad (3.7)$$

is the graph on which all nodes have degree 2. In both Hamiltonians, the coupling constants J_i may be either fixed real parameters or random variables drawn from a prescribed probability distribution. Consider the homogeneous case in which $J_i = J, \forall i$. If $|a\rangle_i$ and $|b\rangle_i$ denote two generic colors at the site i , the individual operator $P_{i,i+1}$ has the following set of eigenvalues and eigenvectors

$$P_{i,i+1} \left(\frac{|v_i^a v_{i+1}^b\rangle \pm |v_i^b v_{i+1}^a\rangle}{\sqrt{2}} \right) = \pm \left(\frac{|v_i^a v_{i+1}^b\rangle \pm |v_i^b v_{i+1}^a\rangle}{\sqrt{2}} \right) . \quad (3.8)$$

Therefore, the symmetric combination of colors of the neighbouring sites has the highest eigenvalue, while the antisymmetric one (if $a \neq b$) has the minimum eigenvalue. Hence, if $J < 0$, the ground-state eigenvector of the Sutherland permutation Hamiltonian is given by the fully symmetric representation,

$$|SG\rangle = \sum_{P \in S_L} |v_{P(1)}^{a_1} v_{P(2)}^{a_2} \dots v_{P(L)}^{a_L}\rangle . \quad (3.9)$$

The situation is different if J is instead positive, $J > 0$. Indeed, the minimum energy $E^* = -JL$ is obtained when each permutation operator $P_{i,i+1}$ can simultaneously take the value -1 ; this is possible only in a color sector containing L distinct colors, which requires $n \geq L$. If this is the case, the corresponding state belongs to the totally antisymmetric IR (corresponding to the vertically longest Young tableau)

and can be written as a Slater determinant built in terms of any set of L different colors $|v_a\rangle_i$:

$$|v^{(1)}, \dots, v^{(L)}\rangle_- = \frac{1}{\sqrt{L!}} \begin{vmatrix} |v^{(1)}\rangle_1 & \dots & |v^{(L)}\rangle_1 \\ |v^{(1)}\rangle_2 & \dots & |v^{(L)}\rangle_2 \\ \dots & \dots & \dots \\ |v^{(1)}\rangle_L & \dots & |v^{(L)}\rangle_L \end{vmatrix} . \quad (3.10)$$

However, for all other color sectors, it is impossible to satisfy the minimum of every local $P_{i,i+1}$; to find the ground-state energy and wavefunction, one must employ a nested Bethe ansatz approach [160].

Class Operator Hamiltonians

On the other hand, the maximum degree that a node can have is $L - 1$. It gives rise to the fully connected Hamiltonian

$$H_2^{\text{class}} = \sum_{i < j} P_{i,j} . \quad (3.11)$$

Such a Hamiltonian is also called a “class operator” [161]. Indeed, it is invariant under the left and right multiplication of any group element g

$$H_2^{\text{class}} = g^{-1} H_2^{\text{class}} g . \quad (3.12)$$

It is easy to construct other class operator Hamiltonians. Indeed, if a conjugacy class $\mathcal{C}_i$ consists of the group elements $\{g_1, g_2, \dots, g_{d_i}\}$, we can define the Hamiltonian $\mathcal{C}_i$ as

$$\mathcal{C}_i = \frac{1}{d_i} (g_1 + g_2 + \dots + g_{d_i}) . \quad (3.13)$$

For all Hamiltonians constructed in this way, we have $g^{-1} \mathcal{C}_i g$ for all group elements g . Hence, the $\mathcal{C}_i$ commutes with all elements of S_L . On the other hand, any subset $\mathcal{C}$ of S_L , which commutes with all elements of the group, must be expressed as a linear combination of these class operators, $\mathcal{C} = c_1 \mathcal{C}_1 + c_2 \mathcal{C}_2 + \dots$. The subspace, composed of sums of the class operators, belongs to the center $\mathbb{Z}$ of the group. There is a close connection between the algebra of the class operators and the Verlinde algebra [162]. Indeed, since the product $\mathcal{C}_i \mathcal{C}_j$ commutes with every element of the group, it must belong to the center of the group; therefore, there must exist constants c_{ij}^k such that

$$\mathcal{C}_i \mathcal{C}_j = \sum_k c_{ij}^k \mathcal{C}_k . \quad (3.14)$$

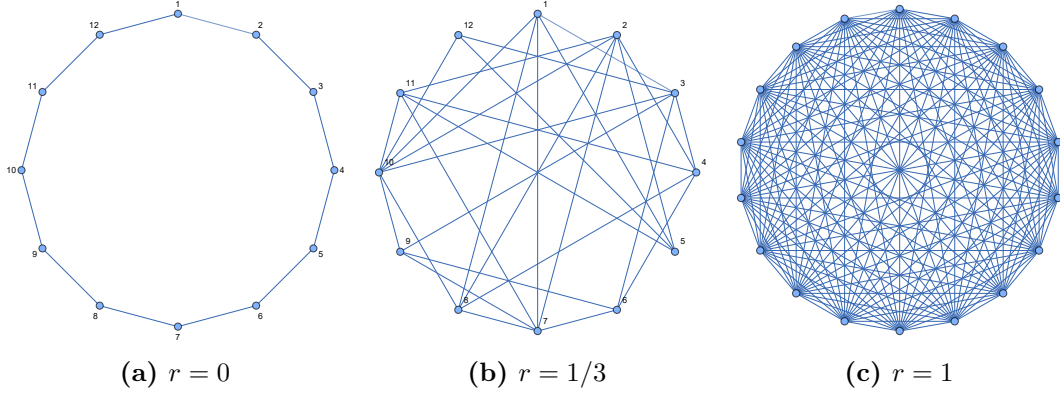


Figure 3.1: Different permutation Hamiltonians on graphs. While the cases $r = 0$ and $r = 1$ are integrable, the intermediate cases are not.

In view of this equation, $\mathcal{C}_i$ can be regarded as a linear mapping $\mathbb{Z} \rightarrow \mathbb{Z}$ with associated matrices, whose entries are $(\mathcal{C}_i)_j^k = c_{ij}^k$. From the associativity condition of the algebra (3.14), all these matrices commute and, therefore, can be simultaneously diagonalized. Denoting by $\mathcal{P}^I$ the projector operator onto the irreducible representation I of the permutation group, we have

$$\mathcal{C}_i \mathcal{P}^I = \left(\frac{\chi_i^I}{d_I} \right) \mathcal{P}^I, \quad (3.15)$$

where χ_i^I is the character of the class i in the Irreducible Representation I while $d_I = \dim I$. In summary, the common eigenvectors of the $\mathcal{C}_i$ are the projector operators $\mathcal{P}^I$ and the eigenvalues $\lambda_i^I = \chi_i^I / \chi_0^I$ are proportional to the characters.

Random Graphs

The Hamiltonians discussed so far are “rigid” in the sense that their structure is fully determined by the underlying graph. However, one may also consider *random* graphs, characterized by an average node degree parameterized by r :

$$\mathcal{I}(r) = (1 - r)2 + r(L - 1). \quad (3.16)$$

The parameter runs in $[0, 1]$, and both 0 and 1 have only one possible Hamiltonian. For generic r , if $[x]$ denotes the integer part of the real number x and $[rL/2] = E$ is the number of edges, the number of Hamiltonians H_G that one can generate on such random graphs is $E!/L!(E - L)!$. Examples of these graphs for varying r are provided in Figure 3.1, while some realizations for $r = 0.05$ are given in Figure 3.2.

Except for the limiting cases $r = 0$ and $r = 1$, where the Hamiltonian is integrable,

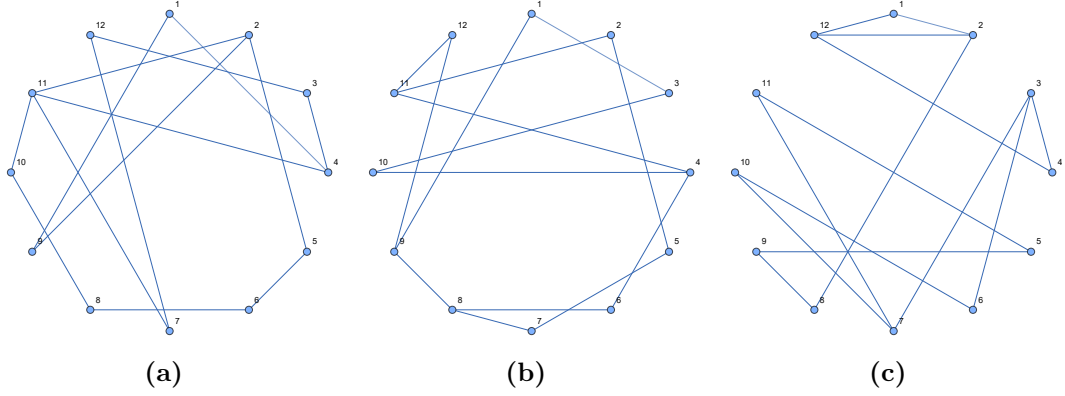


Figure 3.2: Different realizations of permutation Hamiltonians on graphs with the parameter fixed at $r = 0.05$.

intermediate values of r generally yield non-integrable dynamics. The Hamiltonian $\hat{H}$ is obtained as the average Hamiltonian of all the Hamiltonians H_G of the random graphs with a given parameter r :

$$\hat{H} = \sum_G H_G , \quad (3.17)$$

has an unfolded spectrum of averaged energies that is equally spaced, with a Gaussian distribution around the gap value $s = 1$.

3.1.3. The Role of Boundary Conditions

Let us now consider a one-dimensional Hamiltonian. For a fixed matrix size, it is computationally easier to diagonalize a Hamiltonian with open boundary conditions than with periodic boundary conditions, as the corresponding matrix is generally sparser. Nevertheless, diagonalizing Hamiltonians with periodic boundary conditions remains computationally feasible, as we are going to discuss. In a homogeneous system, periodic boundary conditions imply translational invariance. To obtain accurate level statistics, we know that it is essential to factor out global symmetries; translational invariance, when present, is one such symmetry that must be accounted for. For permutation Hamiltonians, the translation operator is given by

$$\mathbb{T} = P_{1,2} P_{2,3} \dots P_{L-2,L-1} P_{L-1,L} = \prod_{j=1}^{L-1} P_{j,j+1} . \quad (3.18)$$

As the generator of the cyclic group $\mathbb{Z}_L$, the translation operator has eigenvalues that are $e^{i\delta}$, where $\delta = 2\pi\kappa/L$ and $\kappa \in \{0, \dots, L-1\}$. Within a given irreducible representation α , the dimension m_κ^α of the sector corresponding to momentum κ

cannot be easily predicted *a priori*, although its average value is d_α/L , where d_α is the dimension of the representation.

It is possible to compute the eigenvalues associated with each momentum sector in a memory-efficient manner for any given irreducible representation. This algorithm avoids explicitly constructing the full translation matrix $\mathbb{T}$; instead, it requires only the application of individual transpositions—highly sparse operators—on a vector. The iterative algorithm proceeds as follows:

1. **Initialize with a sample vector** $|v_i\rangle$. This vector may be chosen as a standard basis vector $|e_i\rangle^\ell = \delta_i^\ell$, though doing so often results in linear dependence among the generated vectors after only a few iterations—significantly fewer than m_κ . To mitigate this, a random vector $|v_i\rangle$, with components randomly chosen in the standard basis, is preferable. For $\kappa = 0$ and $\kappa = L/2$ (when L is even), the eigenvectors are real, so $|v_i\rangle$ can have real coefficients. For all other values of κ , the eigenvectors are complex, and the coefficients of $|v_i\rangle$ should be complex as well.
2. **Project onto the momentum sector** by extracting the κ -component $|\kappa'_i\rangle$ from $|v_i\rangle$ using Eq. (1.241).
3. **Check the linear independence** of $|\kappa'_i\rangle$ with respect to the previously obtained vectors $|\kappa'_j\rangle$ for $j < i$. If $|\kappa'_i\rangle$ is linearly independent, retain its orthogonal component and denote it $|\kappa_i\rangle$. Otherwise, terminate the algorithm. The resulting set of vectors $M_\kappa = \{|\kappa_1\rangle, \dots, |\kappa_{m_\kappa}\rangle\}$ then spans the eigenspace associated with momentum κ .

Once the matrices M_κ have been constructed for a given irreducible representation α , the Hamiltonian can be projected onto the κ -momentum sector. While the full Hamiltonian has dimension $d_\alpha \times d_\alpha$, the projected Hamiltonian $H^{(\kappa)}$ has a reduced dimension $m_\kappa^\alpha \times m_\kappa^\alpha$, given by:

$$H^{(\kappa)} = M_\kappa H M_\kappa^\dagger. \quad (3.19)$$

This projected matrix is typically denser and complex-valued (yet remains Hermitian), but the number of entries is reduced by a factor $O(L^2)$.

With moderate computational resources, it is possible to carry out exact diagonalization within sufficiently large irreducible representations and still obtain meaningful statistical results – see Table 3.1 for illustrative examples.

Irreducible Representation	Dimension of Representation	Dimension of 0-Momentum Sector
$[11^2], L = 22$	58786	2652
$[6^3], L = 18$	87516	4862
$[4^4], L = 16$	24024	1522

Table 3.1: Dimensions of the largest irreducible representations studied for $n = 2, 3, 4$. The dimension of the zero-momentum sector has been determined using the algorithmic method of Section 3.1.

Moreover, a generic translation-invariant Hamiltonian of support k can be written as

$$H_k = \frac{1}{L} \sum_{j=0}^{L-1} \mathbb{T}^j h_{1,\dots,k} \mathbb{T}^{-j} , \quad (3.20)$$

where $h_{1,\dots,k}$ acts on the first k sites. In a given momentum sector, H_k has matrix elements

$$\langle v_\kappa | H_k^{(\kappa)} | u_\kappa \rangle = \langle v_\kappa | h_{1,\dots,k} | u_\kappa \rangle , \quad H_k^{(\kappa)} = M_\kappa h_{1,\dots,k} M_\kappa^\dagger . \quad (3.21)$$

This last equation is quite remarkable because it implies that only one term of the Hamiltonian, $h_{1,\dots,k}$, has to be stored in the computational memory. Moreover, the matrix M_κ does not depend on the particular form of H_k , but only on the irreducible representation chosen; therefore, it can be stored and used for different permutation Hamiltonians.

3.2. Examples of Permutation Hamiltonians

We focus on permutation Hamiltonians with periodic boundary conditions and analyse them within a given momentum sector, as described previously. The two simplest Hamiltonians with two-site interactions are exemplified by the Sutherland permutation model with periodic boundary conditions, denoted by H_2 , which takes the form

$$h_2 = P_{1,2} , \quad (3.22)$$

and the fully connected class operator

$$h_2^{\text{class}} = \frac{1}{L-1} \sum_{j>1} P_{1,j} . \quad (3.23)$$

The Sutherland permutation model is known to be integrable for all $SU(n)$ [160],

and its reduction to $n = 2$ coincides with the antiferromagnetic Heisenberg chain. On the other hand, the fully-connected class operator is “trivially integrable”, because in each irreducible representation, the Hamiltonian is proportional to the identity by Schur’s lemma.

We can generalize both models to arbitrary support by considering the operator $\mathcal{P}_{i_1, \dots, i_k}$, which sums over all the k -cycles of the sites $i_1, \dots, i_k$. $\mathcal{P}_{i_1, i_2, i_3}$ is given by

$$\mathcal{P}_{i_1, i_2, i_3} = P_{i_1, i_2, i_3} + P_{i_1, i_3, i_2} , \quad (3.24)$$

which visually can be represented as

$$\mathcal{P}_{i_1, i_2, i_3} = \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_3 \quad i_2 \end{array} + \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_3 \quad i_2 \end{array} . \quad (3.25)$$

Analogously, for $\mathcal{P}_{i_1, i_2, i_3, i_4}$, we have the graphical representation

$$\mathcal{P}_{i_1, i_2, i_3, i_4} = \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \end{array} + \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \end{array} + \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \end{array} + \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \end{array} + \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \end{array} + \begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2 \\ \swarrow \quad \searrow \\ i_3 \end{array} . \quad (3.26)$$

Each $\mathcal{P}_{i_1, \dots, i_k}$ has $(k - 1)!$ terms, and for each element in the sum, its inverse is present to preserve Hermiticity. We can then define k -site Hamiltonians H_k^{site} with

$$h_k^{\text{site}} = \mathcal{P}_{1, 2, \dots, k} \quad (3.27)$$

and k -class Hamiltonians H_k^{class} with

$$h_k^{\text{class}} = \frac{1}{(k - 1)!} \sum_{i_2 \neq i_3 \neq \dots \neq i_k > 1} \mathcal{P}_{1, i_2, \dots, i_k} . \quad (3.28)$$

As before, the H_k^{class} are trivially integrable, whereas numerical evidence indicates

that the H_k^{site} are generically non-integrable for $k = 3, \dots, L - 1$ in any $SU(n)$, since their level statistics follow the Wigner–Dyson distribution – see the plots at the end of this section.

The last Hamiltonians we consider are long-range, all-to-all ones. We start from the Inozemtsev hyperbolic chain model [163–166]

$$h_W = \sum_{j=2}^L \wp_{L,\pi/\tau}(j-1) P_{1,j} , \quad (3.29)$$

where $\wp$ is the Weierstrass function defined by its two periods L and $1/\tau$

$$\wp_{L,\pi/\tau}(z) = \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z} - \{0\}} \left[\left(z - mL - i \frac{n\pi}{\tau} \right)^{-2} - \left(mL + i \frac{n\pi}{\tau} \right)^{-2} \right] . \quad (3.30)$$

We are interested in two limits. The first one is when $\tau \rightarrow 0$: in this case the effective Hamiltonian reduces to the trigonometric Haldane-Shastry model [167, 168] with

$$h_{HS} = \left(\frac{\pi}{L} \right)^2 \sum_{j=2}^L \left(\sin \frac{\pi(j-1)}{L} \right)^{-2} P_{1,j} . \quad (3.31)$$

On the other hand, if we take the limit $L \rightarrow \infty$, we obtain the hyperbolic Inozemtsev chain

$$h_{hI} = \tau^2 \sum_{j=2}^L (\sinh \tau(j-1))^{-2} P_{1,j} . \quad (3.32)$$

One also retrieves the Heisenberg model as $\tau \rightarrow \infty$ [163].

An important feature of this family of Inozemtsev models is that [165, 169, 170], while they admit exact solutions, their level statistics are not necessarily Poissonian. In particular, the Haldane–Shastry model exhibits almost equispaced degenerate energy levels, reminiscent of those of the multidimensional harmonic oscillator. By contrast, the hyperbolic Inozemtsev chain displays Poissonian level statistics. One may also consider the Haldane–Shastry chain with an arbitrary, non-extensive period relative to the system size; in this case, the level statistics follow the Wigner–Dyson distribution. To the best of our knowledge, no other local models exhibit the same degree of pathological behaviour as the Inozemtsev chain with regard to level statistics.

3.2.1. Numerical Results

We now present the level statistics obtained through the unfolding procedure. The largest irreducible representations for $SU(n)$, with $n = 2, 3$, and 4 , are those in

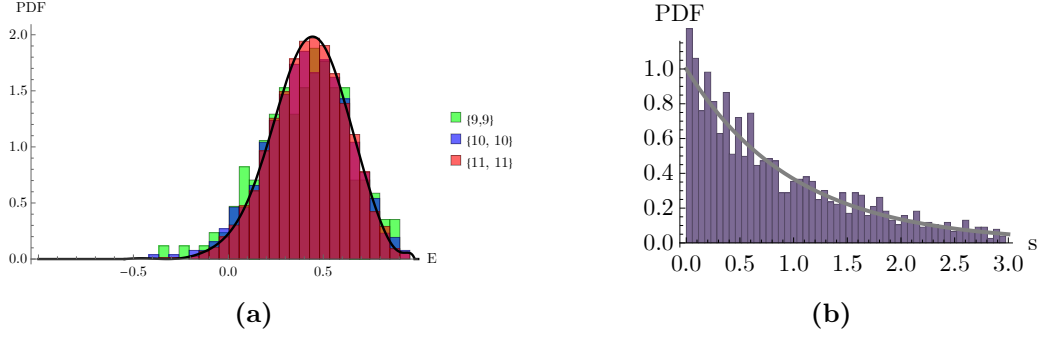


Figure 3.3: On the left, the fit of the density of states using Chebyshev polynomials for a zero-magnetization symmetry sector of the antiferromagnetic Heisenberg chain of various lengths: 18 (green), 20 (blue), and 22 (red). This method is an alternative to fitting the DOS with Bernstein polynomials – although it does not preserve the positivity of the curve. The empirical density collapses to a continuous function as the length increases. On the right, the gap statistics obtained after unfolding are Poissonian because the model is integrable.

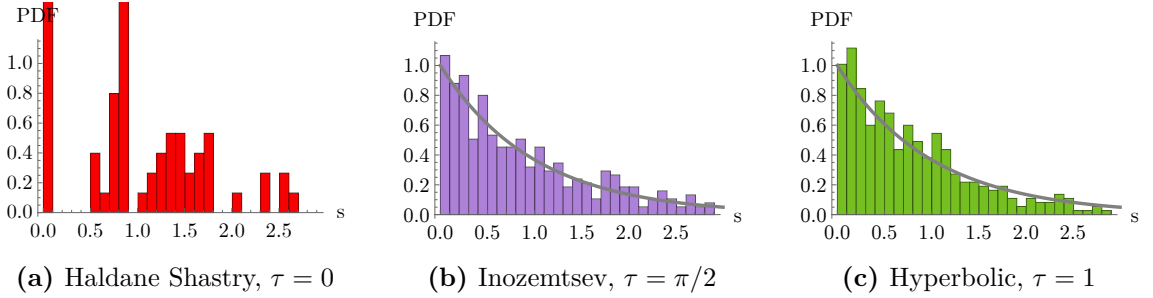


Figure 3.4: Level statistics of the Inozemtsev chain and its limits, for $L = 15$ in the irreducible representation [5³].

Table 3.1.

We start from the $SU(2)$ antiferromagnetic Heisenberg chain, as already reported in Figure 3.3. All momentum sectors exhibit the same Poissonian behaviour. On the other hand, in Figure 3.4 we present level statistics for the Inozemtsev chain and its limits. For the Haldane-Shastry case, we observe two peaks at $s = 0$, indicating a degenerate spectrum, and at $s = 1$, the hallmark of equally spaced levels¹.

For k -site Hamiltonians, we diagonalize momentum sectors separately for $k = 3, 4$ in the representations of $SU(3)$ and $SU(4)$, respectively. While for a generic momentum sector the Hamiltonians are not integrable and their statistics resemble Wigner-Dyson, the zero-momentum sector is intermediate, with a zero-gap probability around $1/2$. Following [1], we expect at least two degenerate, non-integrable spectra. This is

¹More precisely, they are close to being equally spaced in the thermodynamic limit; see details in [169, 170]

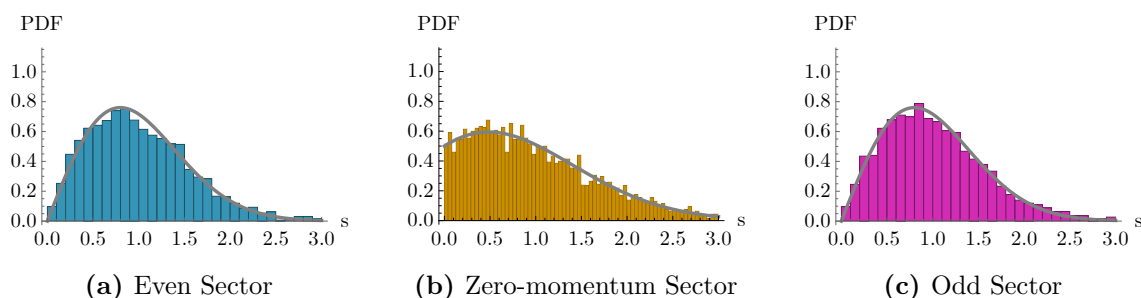


Figure 3.5: Analysis of the zero-momentum sector of the 3-site Hamiltonian H_3 in the irreducible representation $[6^3]$ of $SU(3)$ ($L = 18$). According to the reflection operator, the sector splits into even and odd sectors, which together generate the intermediate statistics for the superposition of 2 GOEs.

indeed the case, since there exists this additional operator

$$S = \prod_{j=1}^{L/2-1} P_{j,L-j+1} , \quad (3.33)$$

which operates a reflection around the center of the chain. Since $S^2 = \text{id}$, reflections generate a Z_2 subgroup of the zero-momentum sector (also the π -momentum one if the chain is even). We label the two symmetry sectors “even” and “odd”, according to the usual convention for parity. In these sectors, the Hamiltonian is non-integrable, see Figure 3.5. This numerical analysis excludes the possibility of integrable k -site Hamiltonians apart from $k = 2$ and $k = L$. Moreover, presently we do not have indications of integrable Hamiltonians made of three-site permutations.

3.3. Random Walks on S_L and the Card Shuffling Problem

We dedicate the rest of the chapter to the study of walks defined on S_L . The structure of the permutation group makes the comparison between classical and quantum dynamics transparent and allows us to ask several questions regarding the out-of-equilibrium properties, one of which is the following: “Which of the two dynamics gives faster mixing on S_L ?” Before addressing the question from the quantum viewpoint, we dedicate the next two sections to the discussion of the discrete- and continuous-time classical cases.

A natural benchmark for any discussion of quantum shuffling is the classical random-transposition shuffle. One starts with an ordered deck of L distinct objects (such

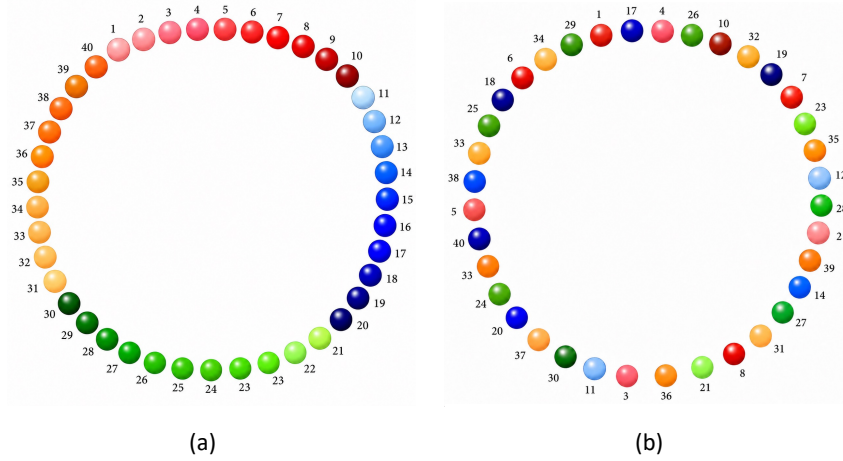


Figure 3.6: (a) Ordered and (b) Random configurations of 40 balls

as balls of different colors) and, at each step, chooses two labels $a, b \in \{1, \dots, L\}$ independently and uniformly, with the possibility of repetitions. If $a \neq b$, the two balls are transposed; if $a = b$, nothing happens. The central question is: “How many random transpositions k^* are necessary before the set is essentially random?”

The answer, due to Diaconis and Shahshahani (DS) [171], reveals that the value k^* is sharply concentrated – in a sense which will be made clear later – around the value

$$k^* \sim \frac{1}{2} L \log L. \quad (3.34)$$

To put this result in the right perspective, we first need to discuss a few notions concerning probability measures on a finite group. It is noteworthy, however, at this stage, to emphasize the exponential separation of scales at play: the number of distinct configurations of the objects is exponential ($L!$), while the number of shuffles necessary to generate a statistically uniform configuration from an ordered one is only $O(L \log L)$.

Probability Measures and Total Variation Distance. Let G be a finite group with $|G|$ elements. A probability measure P on a finite group G is a real-valued function $P : G \rightarrow [0, 1]$ such that $\sum_{g \in G} P(g) = 1$. Hence $P(g)$ represents the probability of choosing the group element g ².

²The function P is not a homomorphism of the group, i.e. $P(g)P(h) \neq P(gh)$. In other words, a step of the walk ending at $gh = g'$ from the identity is not equivalent to a sequence of two jumps $e \rightarrow h \rightarrow gh$. Indeed, one can simply imagine $P(gh)$ to be vanishing while $P(g)$ and $P(h)$ are finite; from this setup, the inequality is clear.

Given two probability measures P and Q on G , we define their distance $D(P, Q)$ as

$$D(P, Q) = \|P - Q\|_{\text{TV}} = \frac{1}{2} \sum_{g \in G} |P(g) - Q(g)|. \quad (3.35)$$

This norm is not the only possible choice of distance in probability space, but, for the purposes of quantifying mixing times on S_L [171], this choice is equivalent to many other definitions, which can be found, for instance, in [172]. From the definition, it follows that

$$0 \leq \|P - Q\|_{\text{TV}} \leq 1. \quad (3.36)$$

$D(P, Q)$ measures the maximal statistical distinguishability between the two distributions. Indeed, if $\|P - Q\|_{\text{TV}} = 0$, then the distributions are identical, whereas if $\|P - Q\|_{\text{TV}} = 1$, then the distributions are maximally distinguishable.

It is rather easy to show that the following bounds hold

$$D_2^2(P, Q) \leq 4 (D(P, Q))^2 \leq |G| D_2^2(P, Q), \quad (3.37)$$

where $|G|$ is the cardinality of the group and

$$D_2^2(P, Q) = \|P - Q\|_2^2 = \sum_g |P(g) - Q(g)|^2 \quad (3.38)$$

is the L_2 -norm. The bound from below comes from $|P(g) - Q(g)| \leq 1$, which in turn implies $|P(g) - Q(g)|^2 < |P(g) - Q(g)|$, while the second follows from the Cauchy–Schwarz inequality³.

DS probability measure. In the case of Diaconis and Shahshahani, the group G under consideration is S_L , the symmetric group of L objects, of cardinality $|S_L| = L!$ — see Appendix B for a summary of properties. The target probability Q is the uniform distribution $U(g) = 1/L!$ and P is obtained from random transpositions. Both a and b are object labels, and the swap $P_{a,b}$ is a *transposition* of a and b . Therefore, such swaps define a measure T such that the probability that $a = b$ (the identity permutation e) is $T(e) = 1/L$. Thus $1 - T(e) = (L - 1)/L$ is the probability that $a \neq b$, and therefore a transposition τ is applied to the L objects. Since the number of distinct transpositions is $L(L - 1)/2$, it follows that the probability of a

³The usual Cauchy–Schwarz inequality on $\mathbb{R}^{|G|}$ is a bound on the dot product of two vectors

$$|\langle \mathbf{u} | \mathbf{v} \rangle|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2.$$

In our case, we choose $u(g) = |P(g) - U(g)|$ and $v(g) = U(g)$.

single transposition is $T(\tau) = 2/L^2$. Clearly, the probability of any other element of the group $\sigma \notin \{\tau, e\}$ is zero: $T(\sigma) = 0$.

Thus T is correctly normalized as a probability measure. The main properties of the DS measure, which are extremely important for the nature of the problem we are discussing, are that all transpositions are equally likely and therefore the probability measure is constant on each conjugacy class.

Random walk on a group and convolution. Repeated random transpositions correspond to a random walk (RW) on the group and give rise to convolutions of probability measures P^{*k} . In general, if P_1 and P_2 are probability measures on the group S_L , where $P_1(\sigma)$ is the probability of performing permutation σ in the first step, while $P_2(\tau)$ is the probability of performing permutation τ in the second step. After two independent steps, the resulting permutation is the product

$$\gamma = \tau\sigma.$$

The probability of ending at this permutation γ is obtained by summing over all possible intermediate decompositions.⁴ This defines the convolution of the probability measures:

$$(P_2 * P_1)(\gamma) = \sum_{\sigma \in S_L} P_2(\gamma\sigma^{-1})P_1(\sigma).$$

In the case of DS shuffling, after k random transpositions, the convolution

$$T^{*k} = \underbrace{T * T * \dots * T}_{k \text{ times}} \quad (3.39)$$

describes the distribution after k independent random transpositions. This distribution can be seen as a random walk on S_L , with $L!$ states. The transition matrix is nothing more than $M_{g \rightarrow h} = T(hg^{-1})$, which is easily seen to be bistochastic, i.e. $\sum_g M_{g \rightarrow h} = \sum_h M_{g \rightarrow h} = 1$.

Heuristic solution of the DS problem. Based on the definitions given above, we can now precisely formulate the DS problem. The question is to identify the value of the iteration k such that

$$\|T^{*k} - U\|_{\text{TV}} < \epsilon \quad (3.40)$$

where ϵ is an arbitrarily small threshold. Physically, it can be interpreted as the iteration at which the convolved probability distribution is ϵ -close to a purely uniform

⁴This sum over intermediate configurations makes the interpretation of a probability convolution as a path integral on S_L immediate.

distribution; that is, if repeated measurements of the walk after k^* steps are performed, one would obtain a uniform distribution of outcomes.

The solution Eq. (3.34) given by Diaconis and Shahshahani has a simple and powerful heuristic explanation in terms of the coupon collector problem, see [173]. In this problem, there are L different coupons, and at each step one coupon is sampled uniformly at random from the same pool. The question concerns how long it will take to see every coupon at least once. At the beginning, new coupons are easy to find. But near the end, when only a few coupons are missing, most samples are repetitions. If only one coupon remains unseen, the probability of finding it at the next draw is $1/L$, so the expected waiting time for the final coupon is L . More generally, if j coupons remain unseen, the probability of obtaining a new one is j/L , and the expected waiting time is L/j . Summing over the expected time scales gives

$$t = L \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{L} \right) \sim L \log L. \quad (3.41)$$

Thus, random coverage of L objects requires about $L \log L$ samples.

For random transpositions, the coupons are the object *labels*. Each transposition touches two labels, and therefore uniformization cannot happen until all labels have been touched at least once. After k random transpositions, approximately $2k$ labels have been sampled, with repetitions. However it may occur that some cards are selected many times, while others are never selected at all. This is precisely analogous to the coupon collector problem. In order for the sequence of extracted labels to forget its initial order, essentially every label must have been touched. If a label has never been touched, it remains fixed in its original position, giving visible memory of the identity permutation. Therefore, the coupon collector estimate gives $2k^* \sim L \log L$ and hence Eq. (3.34).

The coupon collector problem not only identifies the correct time scale, but also provides a lower bound for the mixing time; see Appendix C. It is crucial to stress that, well before this threshold, the distribution remains dramatically non-uniform: a large fraction of the balls have not yet been touched, and the configuration still retains a clear memory of the initial state. Yet, the system exhibits a genuine *cutoff* phenomenon at $k \sim k^*$: the transition from order to randomness occurs abruptly, within a narrow window of width $O(L)$ centered around k^* (see Fig. 3.7). Beyond this point, all macroscopic correlations have essentially vanished, and the distribution is effectively uniform. In other words, at around k^* , the remaining large-scale structure disappears almost all at once. Randomness therefore emerges like a phase transition.

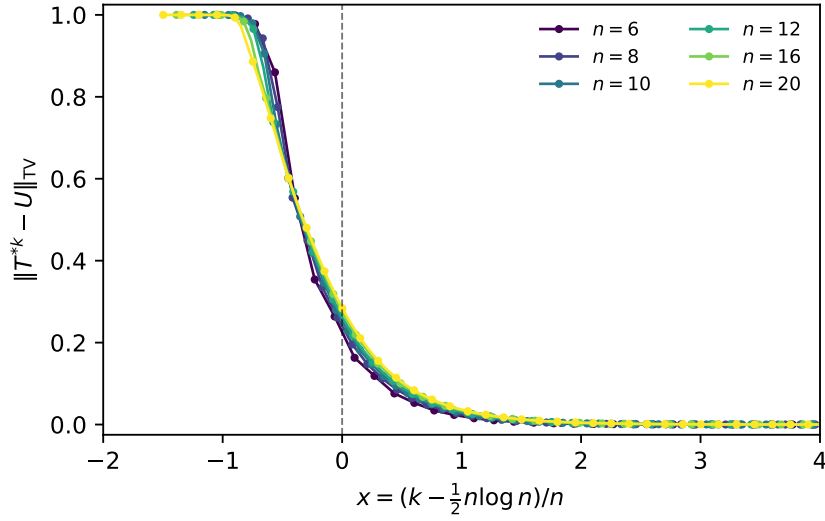


Figure 3.7: Evolution of the measure $\|T^{*k} - U\|_{\text{TV}}$ versus the rescaled variable $x = (k - (1/2)L \log L)/L$. There is a sharp transition around $x \simeq 0$. The region $x < 0$ corresponds to an “ordered phase”, while the region $x > 0$ corresponds to a “random” one.

The actual proof of the theorem, however, requires tools from group theory. In particular, the coupon collector problem clearly identifies a *lower bound* for the mixing time. This argument is presented in Appendix C. The proof of the upper bound, on the other hand, requires results from group theory, such as Frobenius’s formula [171, 172].

Cyclic permutations. The DS theorem concerns the random walk on S_L generated by random *transpositions*. A natural generalization is obtained by replacing transpositions with uniformly chosen q -cycles, i.e. permutations which permute q objects leaving none of the permuted entities in the original position.

We recall that the symmetric group S_L can be decomposed into disjoint conjugacy classes $\mathcal{C}_\mu$, $S_L = \cup_\mu \mathcal{C}_\mu$. Correspondingly, $|S_L| = \sum_\mu N_\mu$, where $N_\mu = |\mathcal{C}_\mu|$ is the cardinality. Classes can be labeled by Young diagrams μ : arrangements of L empty boxes on a two-dimensional grid satisfying the constraint that the numbers of boxes in successive rows decrease. There are as many Young diagrams as there are partitions of the integer L . Young diagrams label not only conjugacy classes but also the irreducible representations λ (irreps) of the group.

A q -cycle has a parity sign $(-1)^{q-1}$. This creates an important subtlety. If q is even, the walk alternates at each subsequent step between elements of different parity. This means that the full permutation group can be explored by the walk. On the other hand, if q is odd, then a cycle does not change the parity of the permutation element. If the starting state of the Markov chain is the identity e , which by definition has

positive parity, one only spans the alternating subspace of S_L , A_L .

From now on, we will tacitly assume that, when an odd- q cycle walk is considered, the probability measure is restricted to A_L , while for an even- q cycle walk it is restricted to S_L . The number of q -cycles is simply the number of possible choices of q out of L , times a combinatorial factor accounting for the permutations of $(q - 1)$ labels [174]:

$$N_q = \binom{L}{q} (q - 1)!. \quad (3.42)$$

A generalization of the “lazy walk” introduced before, which used transpositions, is a probability measure μ such that $\mu(e) = p$ and $\mu(\sigma) = (1 - p)/N_q$ when σ is a q -cycle. Naturally, any other group element that is not a q -cycle has vanishing probability. As in the case of transpositions, we observe that the walk yields identical probabilities for elements of the same conjugacy class.

Concerning the mixing of a q -cycle walk, it is intuitive that it should be faster as q increases. Indeed, the coupon-collector argument, now applied to the simultaneous choice of q labels, suggests that the mixing time is

$$k^*(q) = \frac{L}{q} (\log L + c), \quad c > 0, \quad (3.43)$$

see for instance [175] for a general proof of the result.

3.4. Classical continuous-time random walk

The DS random-transposition shuffle provides a standard example of a discrete random walk on a symmetric group S_L , in which the system either remains unchanged or undergoes a random transposition. The original problem is naturally framed as a discrete-time Markov chain, in which transitions occur at equally spaced times. Since the ultimate goal of this thesis is to address the mixing time of a quantum system evolving in continuous time, we devote this section to the generalization of the discrete-time Markov chain associated with the DS problem for random q -cycles to a continuous-time random walk (CCTRW).

For a CCTRW, a random q -cycle permutation is not applied deterministically, but randomly. The occurrence of transitions is governed by a Poisson counting process of rate γ . Conventionally setting $t = 0$ as the time of measurement, let $N(t)$ be the number of transitions. Then, the probability of observing m events in any interval of length t is $\mathbb{P}[N(t) = m] = \frac{(\gamma t)^m}{m!} e^{-\gamma t}$.

The fundamental object is the *transition matrix* H_q , an $L! \times L!$ matrix whose (σ, σ') element is

$$H_q(\sigma \rightarrow \sigma') = \mu(\sigma' \sigma^{-1}). \quad (3.44)$$

If we focus only on the application of random q -cycles, $H_q(\sigma \rightarrow \sigma')$ is nonzero only if the element $(\sigma' \sigma^{-1})$ belongs to the conjugacy class of q -cycles. Naturally, the matrix H_q is bi-stochastic, i.e. $\sum_{\sigma} H_q(\sigma \rightarrow \sigma') = \sum_{\sigma'} H_q(\sigma \rightarrow \sigma') = 1$.

It follows that H_q can be written in a state space in which each basis element is a permutation $|g\rangle$; these elements are naturally orthogonal, $\langle g|g'\rangle = \delta_{g,g'}$. If we allow for superpositions, this state space becomes the Hilbert space of the permutation Hamiltonian H_q , which for class operators becomes Eq. (3.13):

$$H_q = \frac{1}{N_q} \sum_{g \in \mathcal{C}_q} R(g), \quad (3.45)$$

where $R(g)$ is the representation of g in the regular representation

$$R(g) = \sum_{s \in S_L} |gs\rangle \langle s|. \quad (3.46)$$

The master equation governing the evolution of the CCTRW is obtained as follows. Let $p(\sigma', t)$ be the probability that element σ' is visited at time t . Then, with probability $\gamma dt H_q(\sigma \rightarrow \sigma')$, σ' is reached through a transition from σ . With the complementary probability, σ' has not been subjected to a transition. This implies that

$$p(\sigma', t + dt) = (1 - \gamma dt) p(\sigma', t) + \sum_{\sigma} \gamma dt H_q(\sigma \rightarrow \sigma') p(\sigma, t). \quad (3.47)$$

Denote by $\mathbf{p}(t)$ the vectorization of the $L!$ probability functions. It is immediate to see that, given an initial configuration $\mathbf{p}(0)$, the probability distribution at time t is governed by

$$\mathbf{p}(t) = e^{\mathcal{L}_q t} \mathbf{p}(0), \quad \mathcal{L}_q = \gamma(H_q - \mathbf{1}), \quad (3.48)$$

where $\mathcal{L}_q$ is the stochastic generator of the CCTRW. It is then clear that the same cut-off phenomenon exhibited by the DS discrete random walk will be present in the continuous-time generalization.

3.4.1. Examples of Transition Matrices

The simplest transition matrices are those of the lazy walk with q -cycles in Eq. (3.45). The transition matrix H_q is such that $R(g)H_qR(g^{-1}) = H_q$ for all $g \in S_L$; in other

words, $[R(g), H_q] = 0$, and for this reason it is called a “class operator”. Naturally, H_q is a real-valued, sparse $L! \times L!$ matrix. From the group-theory perspective, it is reducible and, by Schur’s lemma, it is diagonal in each irrep block. There are precisely $\mathcal{P}(L)$ distinct blocks, each associated with an irrep μ of dimension d_μ (which can be computed, for instance, through the hook-length formula). Moreover, each block μ has a degeneracy $m_\mu = d_\mu$. Indeed, for any finite group, the orthogonality relations relate the dimension of the regular representation to those of the irreps. For the permutation group, $L! = \sum_\mu d_\mu^2$.

The CRW associated with the DS random-transposition problem has the transition matrix

$$P_{DS} = \frac{1}{L} \mathbf{1} + \frac{2}{L^2} \sum_{i < j} P_{i,j} = \frac{1}{L} \mathbf{1} + \frac{L-1}{L} H_2, \quad (3.49)$$

and therefore it is immediate to see that the associated CCTRW with transition matrix P_{DS} coincides with that of H_q up to a rescaling of γ :

$$\mathcal{L}_{DS} = \gamma(P_{DS} - \mathbf{1}) = \gamma \left(1 - \frac{1}{L}\right) (H_2 - \mathbf{1}). \quad (3.50)$$

Thus, we can consider H_2 instead of P_{DS} . More generally, this reasoning justifies the earlier assumption of setting $p = 0$ for the probability of the identity in the lazy shuffle for generic q -cycles.

Stationary distribution. Eq. (3.48) implies that, at long times, the leading eigenvalue of H_q dominates the stationary distribution. This eigenvalue is associated with the fully symmetric representation $[L]$. Let $|\sigma\rangle$ be an element of the $L!$ basis associated with the permutation element $\sigma \in S_L$. Because $[L]$ is one-dimensional, the only vector

$$|[L]\rangle = \frac{1}{\sqrt{L!}} \sum_{g \in S_L} |g\rangle \quad (3.51)$$

is the sought eigenvector and the eigenvalue equation is

$$H_q |[L]\rangle = |[L]\rangle. \quad (3.52)$$

This immediately implies that there exists a stationary distribution at long times and, assuming that the initial distribution is localized at the identity, $p(g) = \delta_{g,e}$, we have

$$p_\infty(g) = \frac{1}{L!}. \quad (3.53)$$

Thus, the stationary distribution of the CCTRW is uniform, like that of its discrete-

time counterpart.

Projectors. Other irreps contribute subleading eigenvalues to the stationary distribution. It is worthwhile to recall how, for the regular representation, one can access these eigenvalues.

Let μ be an irrep of dimension d_μ . Then for any function $f : G \rightarrow \mathbb{C}$, one defines the Fourier transform

$$\hat{f}(\mu) = \sum_{g \in G} f(g) \mu(g), \quad (3.54)$$

together with its inverse:

$$f(g) = \frac{1}{|G|} \sum_{\mu} d_{\mu} \operatorname{tr} \left(\hat{f}(\mu) \mu(g^{-1}) \right). \quad (3.55)$$

Notice that while f is a complex scalar function, $\hat{f}(\mu)$ is generally a complex-valued matrix of size d_μ .

We construct the projector P_μ from the regular representation to μ as follows. We notice that if we choose $f \equiv f_\mu(g) = d_\mu \chi_\mu(g)/|G|$, where $\chi_\mu(g) = \operatorname{tr} \mu(g)$ is the character associated with g , the Fourier transform is $\hat{f}_\mu(\lambda) = \delta_{\lambda, \mu} \mathbf{1}_\mu$, because of Schur's orthogonality relation

$$\frac{d_\mu}{|G|} \sum_g \chi_\mu(g^{-1}) \lambda(g) = \delta_{\lambda, \mu} \mathbf{1}_\mu. \quad (3.56)$$

This implies that we can construct P_μ on the regular representation R as a direct sum of irreps:

$$P_\mu = \frac{d_\mu}{|G|} \sum_{g \in G} \chi_\mu(g^{-1}) R(g), \quad (3.57)$$

which is the central idempotent of the group algebra.

Therefore, for the transition matrices H_q , the projection onto the irreps μ is performed by the P_μ , where the matrices $R(\sigma)$ are defined as $R(\sigma) |\sigma'\rangle = |\sigma\sigma'\rangle$. By Schur's lemma, H_q is *diagonal* in each irrep, and therefore the eigenvalues are the characters associated with the q -cycle class:

$$H_q = \sum_{\mu} E_q(\mu) P_\mu, \quad E_q(\mu) = \frac{\chi_\mu^q}{d_\mu}. \quad (3.58)$$

Remarkably, for such class operators H_q the structure of the overlaps between permutation states can be written explicitly in terms of characters, which for S_L are

real-valued:

$$\begin{aligned}\langle h|H_q|g\rangle &= \sum_{\mu} \sum_{s \in S_L} E_q(\mu) \frac{d_{\mu}}{L!} \chi_{\mu}(s) \langle h|R(s)|g\rangle \\ &= \sum_{\mu} E_q(\mu) \frac{d_{\mu}}{L!} \chi_{\mu}(hg^{-1}).\end{aligned}\tag{3.59}$$

Subleading Eigenvalues. We consider the first subleading eigenvalue, associated with the irrep $[L-1, 1]$, of dimension $(L-1)$. The character $\chi_{[L-1,1]}^q = L - q - 1$ and therefore $E_{[L-1,1]} = (L - q - 1)/(L - 1)$. This implies that, for any element g , we have

$$p_t(g) = \frac{1}{L!} + e^{-q\gamma t/(L-1)} \frac{L-1}{L!} \chi_{[L-1,1]}(g) + \dots\tag{3.60}$$

It follows that the variation distance is at leading order

$$|p_t - U| \sim \frac{1}{2} e^{-q\gamma t/(L-1)} (L-1) \frac{1}{L!} \sum_g |\chi_{[L-1,1]}(g)|.\tag{3.61}$$

The character $\chi_{[L-1,1]}(g)$ can be computed by considering the “natural representation” $V = [L] \oplus [L-1, 1]$, which is L -dimensional, with the action of a permutation element given by $P(g)|j\rangle = |g(j)\rangle$. Naturally, $\chi_V(g) = \sum_j \langle j|P(g)|j\rangle = \text{Fix}(g)$ is the number of objects *untouched* by the permutation g . Since $\chi_V = \chi_{[L]} + \chi_{[L-1,1]}$, it follows that

$$\chi_{[L-1,1]}(g) = \text{Fix}(g) - 1.\tag{3.62}$$

For each fixed j , there are $(L-1)!$ permutations that satisfy the rule that j is moved to another position. This means that $\sum_g \text{Fix}(g) = L!$. On the other hand, the number of group elements for which $\text{Fix}(g) = 0$ is called the number of *derangements* D_L (i.e. none of the objects are found in their original positions), and $D_L \sim L!/e$. This implies that $\sum_g |\chi_{[L-1,1]}(g)| = 2D_L$ and

$$|p_t - U| \sim e^{-q\gamma t/(L-1)} \frac{(L-1)}{L!} D_L.\tag{3.63}$$

Given a tolerance ϵ , the mixing time [176–178] is

$$t_{\text{mix}} > \frac{(n-1)}{q\gamma} \log \frac{(L-1)}{\epsilon e},\tag{3.64}$$

which is consistent with the DS bound $k_*(q) = \frac{L \log L}{q} + cL$ with $c > 0$.

Cutoff phenomenon. As mentioned in Sec. 3.4, the discrete DS RW features a sharp cutoff phenomenon when crossing the mixing-time threshold. This phenomenon

is present in the classical case and absent in the quantum one, because quantum dynamics is affected by quasi-periodic revivals due to the interference structure of amplitudes.

For any q -cycle Hamiltonian with $q \geq 2$ (we restrict q to even for the RW over S_L), the theoretical sharp cutoff is $t_{\text{mix}}^{(q)} = \frac{1}{q}L \log L$. More precisely, for any $\epsilon > 0$ and for sufficiently large L , there exists a $c_{\epsilon,q} > 0$ such that [175]

$$D_{TV}(p(t), U) < \epsilon \quad \text{at} \quad t_{\text{mix}}^{(q)}(\epsilon, L) = t_c^{(q)} + c_{\epsilon,q}L. \quad (3.65)$$

Thus, there exists a time window of width $O(L)$ in which, for many values of ϵ , D_{TV} transitions from signaling distinguishability ($D_{TV} \sim 1$) to indistinguishability ($D_{TV} \sim 0$). This is the cutoff phenomenon [178–181].

In exact numerical simulations, it is convenient to study the variable

$$x_\epsilon^{(q)}(L) = \frac{\gamma q t_{\text{mix}}^{(q)}(\epsilon, L)}{L} \log L - \log L, \quad (3.66)$$

which, for sufficiently large L , will plateau at the constant $c_{\epsilon,q}$:

$$x_\epsilon^{(q)}(L) = \gamma q c_{\epsilon,q}, \quad (3.67)$$

and therefore can serve as a probe of the linear correction to the mixing time. In Fig. 3.8 we report our numerical simulations. Due to the finite range of L studied, the variable $x_\epsilon^{(q)}(L)$ will not approach a constant, but polynomial corrections in $1/L$ are also present. We found that it was sufficient to include corrections up to order $1/L^2$ to properly fit $x_\epsilon^{(q)}(L)$ in the range $L \in [12, 25]$ – see Fig. 3.8(c)-(d).

3.4.2. Inverse participation ratio in a classical scenario

A CCTRW with transition matrix H_q has the property that, given two elements g and g' in the same class μ , their probabilities must be the same. The probability of a class is therefore, assuming $p_0(g) = \delta_{g,e}$,

$$p_t(\mu) = \sum_{g \in C_\mu} p_t(g) = \frac{N_\mu}{L!} \sum_{\lambda} d_\lambda \chi_\lambda^\mu e^{\gamma(H_q(\lambda)-1)t}. \quad (3.68)$$

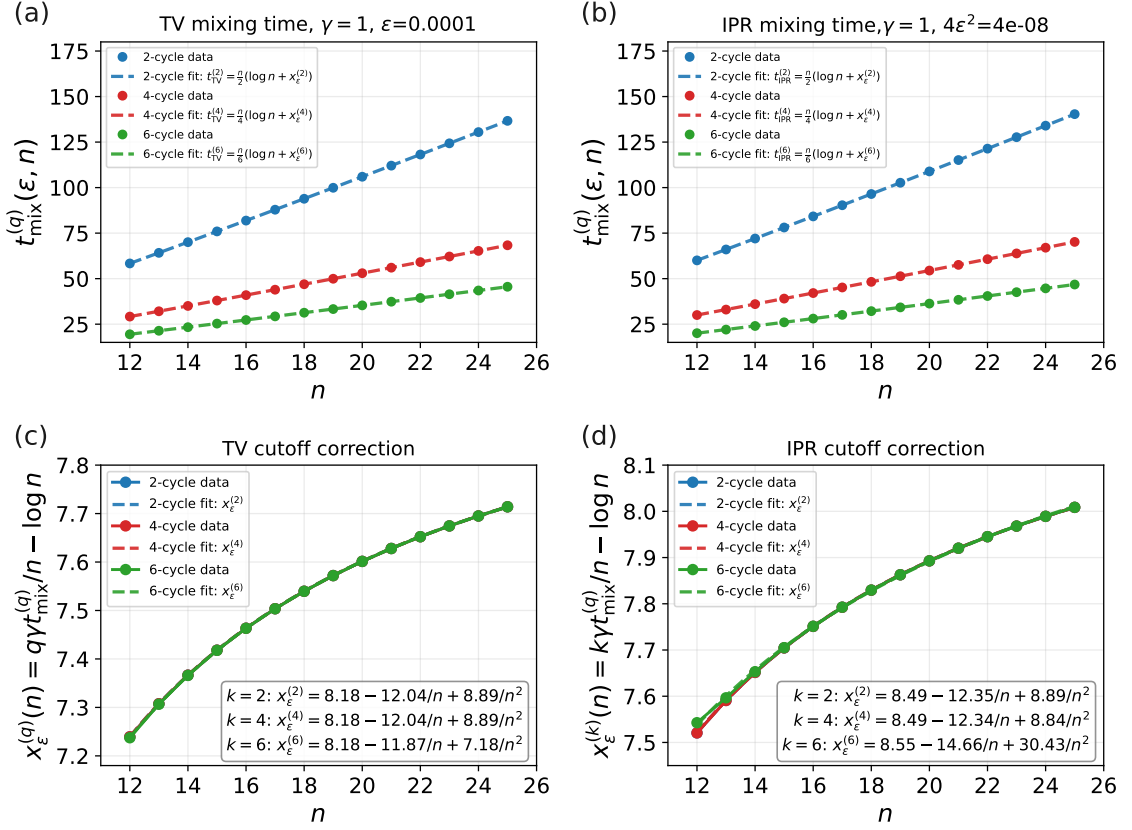


Figure 3.8: Mixing times determined by (a) $D_{\text{TV}}(p(t), \pi)$ and (b) $D_{\text{IPR}}(t)$ over $L \in [12, 25]$, where the threshold is set to $\epsilon = 10^{-4}$ for the probability and $4\epsilon^2$ for the IPR. The three lines are generated by 2-, 4-, and 6-cycle generators; both the exact numerical data (dots) and their fits (dashed lines) are displayed. The corresponding linear correction prefactors for the cutoff are shown in (c) and (d).

The leading contribution comes from the fully symmetric irrep $\lambda = [L]$ and the long-time asymptotics is

$$p_t(\mu) \underset{t \rightarrow \infty}{\sim} \frac{N_{\mu}}{L!}. \quad (3.69)$$

Indeed, since $[L]$ is one-dimensional, $\chi_{[L]}^{\mu} = 1$ for all classes. This result implies that all classes independently approach the uniform distribution asymptotically.

While the $\|\cdot\|_{\text{TV}}$ norm relates to $p_t(g)$, the L_2 -norm, which both upper- and lower-bounds the $\|\cdot\|_{\text{TV}}$ norm, can be written as

$$D_2(p_t, U) = \text{IPR}(t) - \frac{1}{L!}, \quad (3.70)$$

where IPR is the *inverse participation ratio*, also known as the *collision probability* or *coincidence probability*:

$$\text{IPR}(t) = \sum_g (p_t(g))^2. \quad (3.71)$$

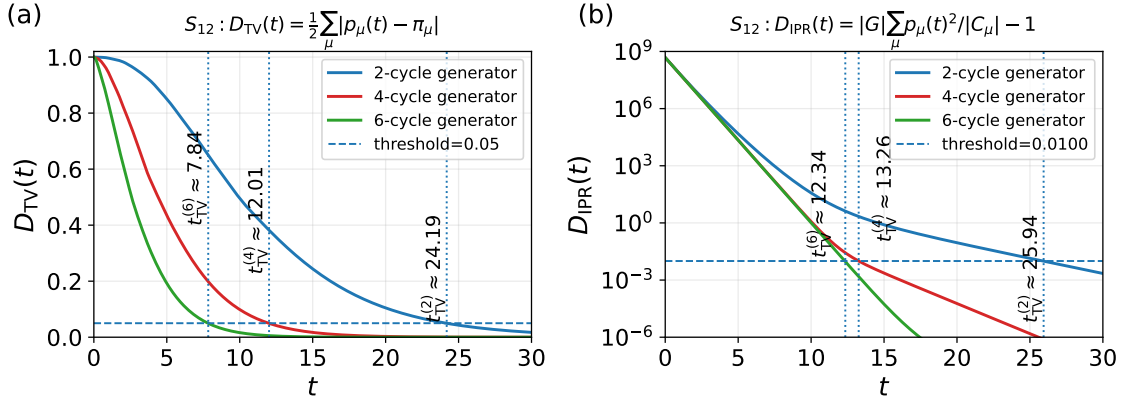


Figure 3.9: Global distance to the stationary distribution for (a) the probability distribution and (b) the IPR, versus t at $L = 12$, where the threshold is set to $\epsilon = 0.05$ for the probability distance and $4\epsilon^2$ for $D_{IPR}(t)$. The three curves are distance measures over time generated by 2-, 4-, and 6-cycle class generators. The mixing times $t_{\text{mixing}}^{(q)}$ are marked with vertical dashed lines. The larger the cycle q , the faster the distances decay (this trend is the opposite of that for the quantum random walk).

The IPR is a measure of “spreading” or “uniformization” and emerges naturally in the study of quantum dynamics. It is also the first nontrivial moment of the more general r -th inverse participation ratio:

$$\text{IPR}_r(t) = \sum_g (p_t(g))^r. \quad (3.72)$$

The case $r = 1$ expresses the conservation of probability, while $r = 2$ is generally a nontrivial function of t .

At long times, for the CCTRW, we have $\text{IPR}(t) \rightarrow 1/L!$, and at short times, for $p_0(g) = \delta_{g,e}$, $\text{IPR}(0) = 1$. From the DS problem, we know that on a scale $t_U = O(L \log L)$, the IPR becomes ϵ -close to the uniform value. However, unlike the quantum case, it *does not oscillate* around the infinite-time value. We also have, for every conjugacy class,

$$\text{IPR}(t, \mu) = \sum_{g \in \mu} (p_t(g))^2 \underset{t \rightarrow \infty}{\sim} \frac{N_{\mu}}{(L!)^2}. \quad (3.73)$$

In the classical setting, there is no qualitative difference between considering the IPR and the DS norm (Fig. 3.9).

3.4.3. Random-Time Measurements and Average Mixing

The probability measures and their associated distances considered so far are *instantaneous*, in the sense that they involve information about measurements at time t . In a CCTRW, these probabilities provide both the simplest and the most effective way to study the dynamics of the Markov process, because Eq. (3.48) admits a well-defined stationary distribution. We now introduce another measure, based on random measurements over an interval $[0, t]$, which offers a better comparison when the walk is considered in a quantum setting – as will be discussed in the next section.

For $M \rightarrow \infty$ measurements uniformly distributed over an interval $[0, t]$, the limiting probability distribution coincides with the *running average* (RA) of the probability:

$$\bar{p}_t(g) = \frac{1}{t} \int_0^t dt' p_{t'}(g), \quad (3.74)$$

with $p_t(g)$ given by Eq. (3.48). Naturally, if the stationary distribution coincides with the uniform one, the *long-time average* (LTA) $\bar{p}_\infty$ will also be uniform.

Thus, we can generalize the DS measure in Eq. (3.35) to one involving the RA:

$$\bar{D}(P_t, Q) = \frac{1}{2} \|\bar{P}_t - Q\|_{\text{TV}}. \quad (3.75)$$

To recover the equivalent of the DS mixing time, we observe that by convexity

$$\bar{D}(p_t, U) \leq \frac{1}{t} \int_0^t dt' D(p_{t'}, U). \quad (3.76)$$

After setting a threshold ϵ for $D(p_t, U)$ one observes that

$$\bar{D}(p_t, U) \leq \frac{L \log L}{t} + \epsilon \left(1 - \frac{L \log L}{t} \right), \quad (3.77)$$

and therefore the expected average mixing time is

$$\bar{t}_{\text{mix}} \sim \frac{L \log L}{\epsilon}. \quad (3.78)$$

This result shows that the instantaneous and average mixing times in the classical case coincide in the worst-case scenario. This estimate is sufficient for the purposes of our work, since it implies that we should not expect an average mixing time larger than the instantaneous one for a CCTRW on the permutation group. In Fig. 3.10, we observe that the average distance $\bar{D}$ does not feature the same cutoff phenomenon as

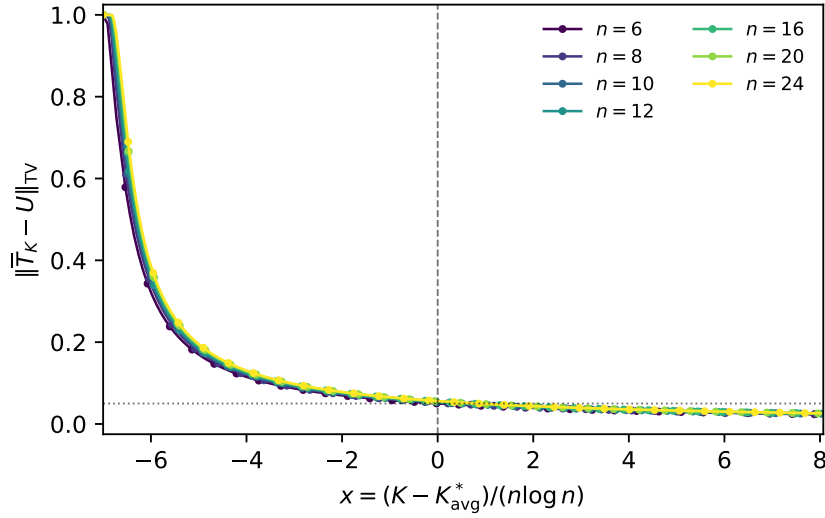


Figure 3.10: Evolution of the average DS measure versus the rescaled variable $x = (K - K_{\text{avg}}^*)/L \log L$. Contrary to the instantaneous distance in Fig. 3.7, its averaged analogue does not exhibit the cutoff phenomenon. Still, the measure vanishes rapidly. The dotted horizontal line corresponds to the threshold $\epsilon = 0.05$.

the instantaneous distance. The reason for this lies in the widths of the subleading corrections to the two mixing times. For the instantaneous one, the width is $O(L)$, while for the average one it is of order $O(L \log L)$.

3.5. Unitary Quantum Walks

Both the RW and its continuous-time generalization describe the evolution of a system on a state space of dimension $D_L = L!$ with transition matrix H_q given by Eq. (3.45). The same H_q has a natural interpretation as a permutation Hamiltonian (PH) of a quantum mechanical system [3]. This identification will make the comparison of the classical RW with its quantum counterpart (QW) immediate.

However, we remark that, for a quantum system evolving unitarily, one cannot expect irreversible mixing like that of the CCTRW counterpart. This occurs because no information is lost, and the von Neumann entropy does not increase over time. Nonetheless, we present in this section the treatment of the QW to discuss the particular ways in which a system fails to mix and the common probes and signatures.

The quantum mechanical system under consideration is made up of L sites, and the on-site Hilbert space has dimension L . The particles are L spinless fermions with L distinct colors $i = \{1, \dots, L\}$. The dimension of the total Hilbert space is $D_L = L!$, and each basis state $|g\rangle$ is labeled by a permutation element $g \in S_L$. The

state $|e\rangle = |1 \dots L\rangle$, associated with the identity, is the “ordered state”.

We focus on the time-evolution of basis states generated by the unitary operator

$$\mathcal{U}(t) = \exp(-iHt), \quad (3.79)$$

i.e. given any initial state $|\psi(0)\rangle$, its trajectory in the Hilbert space is given by

$$|\psi(t)\rangle = \mathcal{U}(t)|\psi(0)\rangle. \quad (3.80)$$

To properly define the QW, we provide the form of the transition probability between elements of the (computational) basis, which for PHs with L distinct colors are the group elements:

$$Q_q(g, h, t) = \left| \langle h | e^{-iH_q t} | g \rangle \right|^2. \quad (3.81)$$

Compared to Eq. (3.48), one observes that $Q(t)$ does not generally approach a stationary distribution at large times. Indeed, quantum mechanics is subject to periodic or quasi-periodic revivals of observables. For example, class Hamiltonians of the type H_q have a spectrum of rational energies, and therefore there exists a time at which the unitary operator satisfies $\mathcal{U}(t_P) = \mathcal{U}(0)$. Since the recurrence time usually scales exponentially with L , the thermodynamic limit is often assumed when studying the behavior of observables [9].

For this reason, the limiting distribution that can be studied for a QW is not the instantaneous one, but rather the one involving the average $\overline{Q}$. It suffices to write the Hamiltonian in its spectral form, $H = \sum_n E_n |E_n\rangle \langle E_n|$, to see that the LTA is given by

$$\overline{Q}_\infty(g) = \sum_E |\langle g | \Pi_E | e \rangle|^2, \quad (3.82)$$

where Π_E is the projector onto the eigenspace of energy E . Thus, to compare classical and quantum walks on S_L , the natural measure to study is the averaged one Eq. (3.75).

For the CCTRW, the LTA distribution exists and corresponds to the uniform one. For a QW, on the other hand, it is not guaranteed that the LTA distribution is uniform. This is true, for instance, for the QW generated by the two-cycle H_2 , where we observe that the distance $\overline{D}$ from the uniform distribution asymptotically approaches a finite value and does not decrease with time; see Fig. 3.11. For this reason, one often evaluates the *quantum mixing time* satisfying

$$\overline{D}(Q_t, \overline{Q}_\infty) \leq \epsilon. \quad (3.83)$$

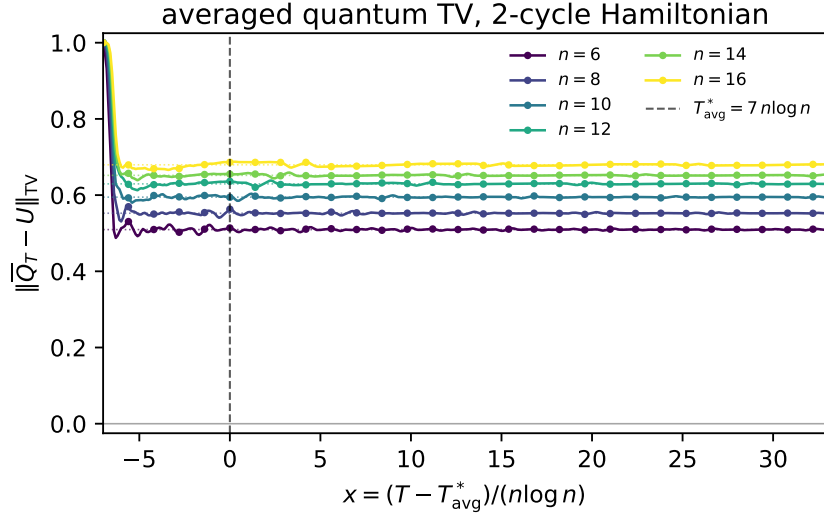


Figure 3.11: Evolution of the average DS measure versus the rescaled variable $x = (K - K_{\text{avg}}^*)/L \log L$ for the QW generated by the two-cycle Hamiltonian H_2 . Contrary to the classical case, the distance of the QW saturates to a finite L -dependent value at long times.

For example, it can be shown [182] that the average mixing time of the QW associated with dense random networks defined on graphs of N_L nodes scales as $O(N_L^{3/2})$.

Because the QW generally does not approach the uniform distribution, uniformization and its comparison with the corresponding CCTRW require the introduction of new probes. Indeed, the measure in Eq. (3.75) is well-defined for any ϵ only for the CCTRW, while for the QW there exists, in principle, a minimum value of $\epsilon > 0$ below which the average mixing time is ill-defined – and can be assumed to be infinite.

3.5.1. Quantum Dephasing Time

While Eq. (3.83) can be used as a measure of mixing for unitary evolution, one can also study higher moments of the probability distribution. The first nontrivial moment is the collision probability, or IPR:

$$\text{IPR}_\psi(t) = \sum_{h \in S_L} \left| \langle h | e^{-iHt} | \psi \rangle \right|^4, \quad (3.84)$$

where $|\psi\rangle$ is the initial state.

The IPR is a standard basis-dependent diagnostic in quantum many-body physics that quantifies the degree of “spreading” over time of a state in a given basis [183–187]. It is particularly relevant to the study of dynamics in the presence of random

interactions and localization physics.

If $\text{IPR}(t) = 1$, the probability is localized in one group element, while if $\text{IPR}(t) = 1/L!$, then the only possible distribution is that of a uniform state. However, quantum dynamics generally prevents *complete* uniform spreading, i.e. the LTA of $\text{IPR}(t)$ is greater than $1/L!$ – as can already be evinced from Fig. 3.11.

Like the probability Q , the IPR does not converge to a finite value as $t \rightarrow \infty$, and therefore it does not, by itself, solve the problem of defining a common measure for the QW and CCTRW to probe mixing. Nonetheless, we can observe that, by expanding at short time scales, the small-time behavior of the IPR for the ordered state is

$$\begin{aligned} \text{IPR}(t) &\sim 1 - 2 \text{Var}_e[H] t^2 + O(t^4), \\ \text{Var}_e[H] &= \langle e | H^2 | e \rangle - (\langle e | H | e \rangle)^2 = \sum_n |\langle E_n | e \rangle|^2 (E_n^2 - E_n \langle H \rangle_e). \end{aligned} \quad (3.85)$$

Thus, the IPR deviates from unity quadratically, and the coefficient is proportional to the variance of the energy in the initial state.

We define a purely quantum time scale, the *dephasing time*, as

$$t_{\text{deph}} \sim \frac{1}{\sqrt{\text{Var}_e[H]}}. \quad (3.86)$$

For $t \ll t_{\text{deph}}$, the IPR has negligible deviations from unity, so the initial state in the computational basis remains localized around its original vector. For $t \gtrsim t_{\text{deph}}$, one observes an exponential descent towards smaller values.

Such a time scale is not only informative about dephasing, but also computationally accessible for moderate system sizes. Indeed, only one application of the Hamiltonian to a single basis state is needed.

For PHs, there is an exact coincidence between the computational-basis variance $\text{Var}_e[H]$ and the typical spectral variance

$$\sigma_E^2 = \int dE \rho(E) E^2 - \left(\int dE \rho(E) E \right)^2, \quad (3.87)$$

where $\rho(E) = D_L^{-1} \sum_j \delta(E - E_j)$ is the empirical density of states. Indeed, for every PH, the only matrix element connecting g to itself is the identity permutation, which has coefficient c_e . Hence

$$\text{Var}_e[H] = \sum_{g \neq e} |c_g|^2. \quad (3.88)$$

On the other hand, in the regular representation, we have $\text{Tr } R(g) = L! \delta_{g,e}$, and therefore $\text{Tr } H = L! c_e$ and $\text{Tr } H^2 = L! \sum_g |c_g|^2$. Therefore,

$$\sigma_E^2 = \left(\frac{1}{L!} \text{Tr } H^2 - \left(\frac{1}{L!} \text{Tr } H \right)^2 \right) = \text{Var}_e[H]. \quad (3.89)$$

In particular, for q -class PHs, we observe that the dephasing time is

$$t_{\text{deph},q} = \sqrt{N_q} \underset{L \gg 1}{\sim} \sqrt{\frac{L^q}{q}}. \quad (3.90)$$

Thus, the IPR leaves the unit plateau at around a time scale of $O(L^{q/2})$. We also stress that the dephasing time is a *local* scale, which probes simply the first layer of connectivity of the identity generated by a single application of H . Thus, it does not contain information about the exploration of the full group.

In Fig. 3.12, we report the IPR for different q -cycles at different system sizes. As L increases, the plateau of the IPR extends. To quantify the time scale of this decay numerically, we compute the first crossing time between the IPR and its LTA (dashed line in the plot). The time scales extracted in this way present the same scaling as the quantum dephasing time $t_{\text{deph},q}$ – see Appendix D for details. Nonetheless, we stress that the time extracted in this way does not necessarily imply that the walk has explored a macroscopic fraction of all $L!$ configurations. It only quantifies the first instance when phases have dephased sufficiently to cross the diagonal value.

3.6. Quantum Stochastic Walks

The main obstruction to applying the definition of classical mixing to the QW is the reversible nature of the evolution: it is not guaranteed that the uniform distribution is reached at long times, either instantaneously or on average. The natural understanding of this phenomenon comes from information theory: a CCTRW eventually loses information, reaching a distribution of maximum entropy – the uniform distribution – while unitary evolution is reversible, and therefore the von Neumann entropy associated with the density matrix $\rho(t) = |\psi(t)\rangle \langle \psi(t)|$ is zero at all times.

Thus, within the theory of quantum stochastic walks [188], we consider a protocol that satisfies, by construction, the requirement that the asymptotic probability is uniform.

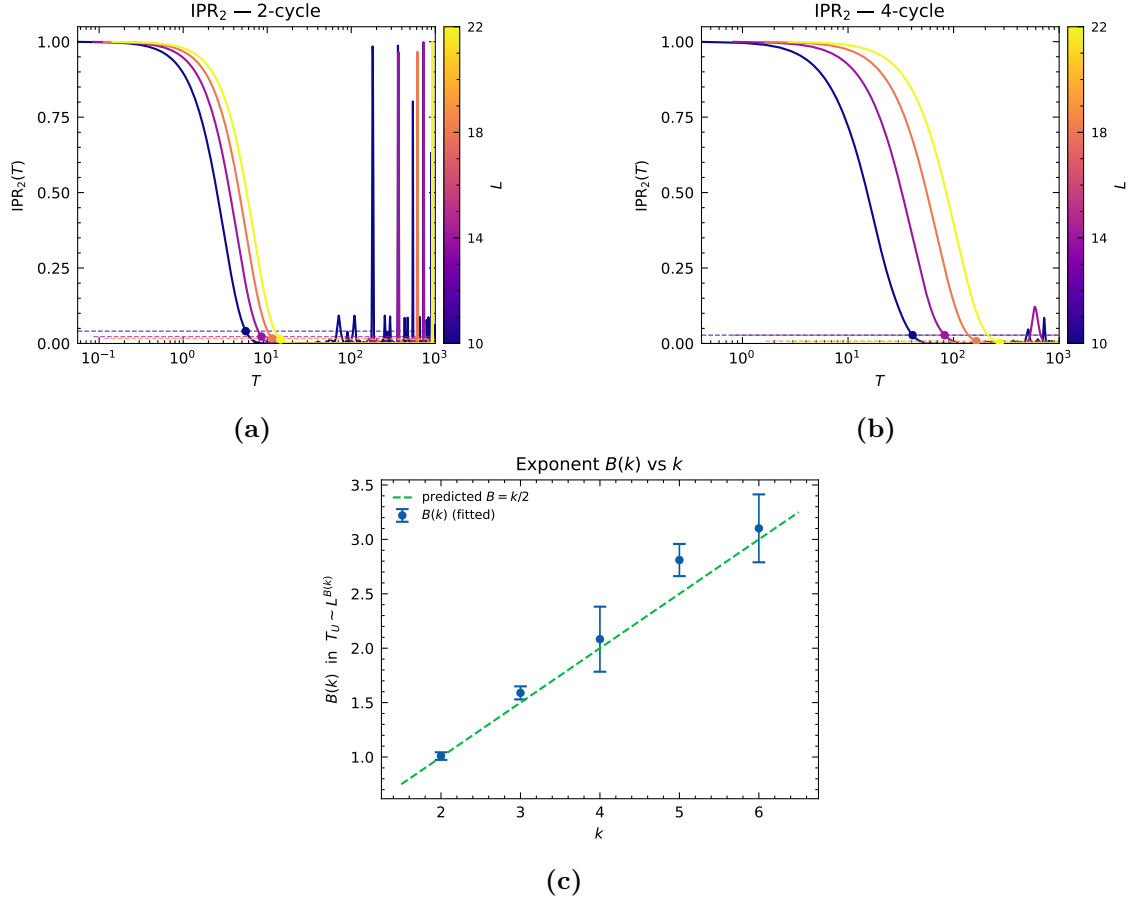


Figure 3.12: IPRs of class operators for $L \in [10, 23]$. (a) The IPR of the transposition Hamiltonian is shown for various values of L . Dots correspond to the uniformization time T_U at the first intersection with the long-time average. (b) IPR of the 4-class Hamiltonian. These class Hamiltonians feature rational eigenvalues; therefore, the IPRs are periodic. As L and the class q increase, we observe that the IPR moves more slowly away from unity. Also, characteristic revivals are pushed further in time as L and q increase. (c) Fitted coefficient $B(q)$ of $T_U(L, q) \sim L^{B(q)}$ as the q -class Hamiltonian varies. In contrast to the CRW result, the coefficient $B(q)$ increases with q and aligns with the theoretical prediction $B(q) \sim q/2$.

We first show how the CCTRW with Poisson rate γ can be naturally cast in the form of an open quantum system (OQS) process. To this end, we extend the states considered from pure states to statistical mixtures represented by density matrices ρ . Ordinarily, ρ is of the form $\rho = \sum_{s,g} p_t(s,g) |s\rangle \langle g|$ with $\sum_s p_t(s,s) = 1$. We restrict ourselves to density matrices in an $L! \times L!$ subspace, i.e. those of the form $\rho = \sum_g p_t(g) |g\rangle \langle g|$. Such restricted density matrices form a regular representation R of S_L . The maximally mixed state (MMS) is

$$\rho_{\text{MMS}} = \frac{1}{L!} \sum_g |g\rangle \langle g| = \frac{\mathbf{1}}{L!}. \quad (3.91)$$

To understand the emergence of classicality, we can observe that the CCTRW over q -cycles is mathematically equivalent to the OQS setup. Consider the initial state $\rho_0 = |e\rangle \langle e|$ and study the following Poisson process: at intervals $\Delta t \sim \text{Exp}(\gamma)$ the following transition occurs

$$\rho_{j+1} = L_g \rho_j L_g^\dagger, \quad L_g = \sum_{s \in S_L} |gs\rangle \langle s| = R(g), \quad (3.92)$$

where g is randomly chosen from the conjugacy class of q -cycles. The jump operator L_g maps $|h\rangle \langle h|$ to $|gh\rangle \langle gh|$.

For infinitesimal times dt , we have the first-order equation

$$\rho(t+dt) = (1 - \gamma dt) \rho(t) + \frac{\gamma}{N_q} dt \sum_{s \in \mathcal{C}_q} L_s \rho(t) L_s^\dagger, \quad (3.93)$$

which corresponds to the following Lindblad equation:

$$\frac{d\rho}{dt} = \gamma (\Phi_q(\rho) - \rho), \quad \Phi_q(\cdot) = \frac{1}{N_q} \sum_{s \in \mathcal{C}_q} R(s)(\cdot) R(s)^\dagger. \quad (3.94)$$

We recall that $p_t(g) = \text{Tr}(\rho(t) |g\rangle \langle g|)$ and therefore the corresponding equation for the probability is

$$\begin{aligned} p_{t+dt}(g) &= (1 - \gamma dt) p_t(g) + \frac{\gamma}{N_q} dt \sum_{s \in \mathcal{C}_q} \text{Tr}(\rho(t) L_s^\dagger |g\rangle \langle g| L_s) \\ &= (1 - \gamma dt) p_t(g) + \frac{\gamma}{N_q} dt \sum_{s \in \mathcal{C}_q} p_t(s^{-1}g). \end{aligned} \quad (3.95)$$

This equation coincides with Eq. (3.48), and therefore the CCTRW is recovered as a particular limit of open quantum dynamics. By appealing to the results of Section 3.4, the stationary state is $\rho_\infty = \rho_{\text{MMS}}$.

Eq. (3.94) describes an essentially classical process in which no superpositions are created at any time. We can create quantum superpositions by considering evolution generated by another m -class operator:

$$\frac{d\rho}{dt} = -i\alpha[H_m, \rho] + \gamma(\Phi_q(\rho) - \rho), \quad \rho_0 = |e\rangle\langle e|. \quad (3.96)$$

Denoting by $\mathcal{Q} = \alpha/\gamma$ the dimensionless strength, when $\mathcal{Q} \gg 1$, the dynamics is dominated by quantum evolution and coincides with the QW with Hamiltonian H_m , while when $\mathcal{Q} \rightarrow 0$, the dynamics is that of a CCTRW.

The OQS dynamics described by Eq. (3.96) possesses several properties of physical and mathematical relevance, which allow us to write its exact solution. First, $\rho_{\alpha,\gamma}(t)$ lives in the regular subspace; moreover, the two terms of the Lindblad equation lie in the center of the group by construction. Therefore, the probabilities of elements belonging to the same conjugacy class are the same at any time, i.e.

$$p_{\alpha,\gamma}(hgh^{-1}, t) = p_{\alpha,\gamma}(g, t), \quad (3.97)$$

where the elements g and hgh^{-1} belong to the same conjugacy class of S_L . The second property is related to the *trace distance* of $\rho_{\alpha,\gamma}(t)$ at long times, defined as

$$T(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1 = \frac{1}{2} \text{Tr} \left[\sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)} \right]. \quad (3.98)$$

This distance measures the distinguishability of any two density matrices. We first observe that since

$$\rho_\gamma(t) = e^{\mathcal{L}t}[\rho(0)], \quad \mathcal{L}[\rho] = \gamma(\Phi(\rho) - \rho), \quad (3.99)$$

is the formal solution to the CCTRW ($\alpha = 0$) limit in Eq. (3.94), the solution of the OQS dynamics in Eq. (3.96) is

$$\rho_{\alpha,\gamma}(t) = \mathcal{U}_\alpha(t) \rho_\gamma(t) \mathcal{U}_\alpha^\dagger(t), \quad \mathcal{U}_\alpha(t) = e^{-i\alpha H_m t}. \quad (3.100)$$

Using the invariance under unitary transformations of the trace distance, we therefore have that

$$T\left(\rho_{\alpha,\gamma}(t), \frac{\mathbf{1}}{L!}\right) = T\left(\rho_\gamma(t), \frac{\mathbf{1}}{L!}\right), \quad (3.101)$$

In other words, the trace distance is independent of α . Thus, the addition of superpositions (through H) in Eq. (3.96) does not affect the convergence of ρ to the uniform distribution.

What about the distance of the probability distributions? The probability distribution generated by Eq. (3.96) is

$$p_{\alpha,\gamma}(g, t) = \sum_{h \in S_L} Q_{\alpha,m}(g, h, t) p_{\gamma}(h, t), \quad (3.102)$$

where $p_{\gamma}(h, t)$ is given by Eq. (3.48) and $Q_m(g, h, t)$ is the transition probability of the QW in Eq. (3.81). This result is therefore a convolution of two separate probabilities, one associated with the CCTRW of q -cycles and the other with a QW of m -cycles. Let us denote by

$$\Delta(\rho) = \sum_g |g\rangle \langle g| \rho |g\rangle \langle g| = \sum_g p(g) |g\rangle \langle g| \quad (3.103)$$

the dephasing map in the computational basis. We can write the DS measure in Eq. (3.35) as

$$D(\alpha, t) = T \left(\Delta(\rho_{\alpha,\gamma}(t)), \Delta \left(\frac{\mathbf{1}}{L!} \right) \right). \quad (3.104)$$

Because Δ is a contraction mapping, since it projects density matrices onto the diagonal ($|g\rangle \langle g|$) subspace by eliminating cross terms ($|s\rangle \langle g|$), we have the inequality

$$T \left(\Delta(\rho_{\alpha,\gamma}(t)), \Delta \left(\frac{\mathbf{1}}{L!} \right) \right) \leq T \left(\rho_{\alpha,\gamma}(t), \frac{\mathbf{1}}{L!} \right). \quad (3.105)$$

From trace invariance, Eq. (3.101), we observe that the right-hand side of the inequality is the DS measure of the CCTRW, $D(0, t)$, and therefore we have the upper bound

$$D(\alpha, t) \leq D(0, t), \quad (3.106)$$

Thus, the addition of a Hamiltonian can never make the computational-basis probability distribution less mixed than the classical one evaluated at the same time.

As we shall see from the numerical analysis in the next sections, the inequality Eq. (3.106) seems not to be saturated by computational-basis initial states $\rho(0) = |e\rangle \langle e|$ evolved with (m, q) -cycles. In contrast, insensitivity of the mixing time to the coherent coupling strength has been shown to hold for a class of initial states in open quadratic fermionic systems [189]. The setup considered in the latter closely aligns to the one considered in this paper: in absence of quantum coherence, the dissipator dynamics reduces to the master equation of a uniform random walk on the L -dimensional hypercube [190], when the initial state is a density matrix diagonal in the basis of fermion occupation numbers.

Exact convolution structure. As anticipated, the convolution structure of the total probability

$$p_{\alpha,\gamma}(t) = Q_{\alpha t} * p_{\gamma}(t), \quad (3.107)$$

and the constancy over conjugacy classes make the estimation of the Fourier transform immediate. Combining the notions of Sections 3.4 and 3.5, we find that the group Fourier transform of $p_{\alpha,\gamma}(t)$ reads

$$\hat{p}_{\alpha,\gamma}(\lambda, t) = \kappa_{\lambda}(\alpha t) e^{-\gamma(1-E_q(\lambda))t} \mathbf{1}_{d_{\lambda}}, \quad (3.108)$$

where $\kappa_{\lambda}(\alpha t)$ is the Fourier transform of the QW kernel, given by

$$\kappa_{\lambda}(\alpha t) = \frac{1}{d_{\lambda}} \sum_g \chi_{\lambda}(g) Q_{\alpha t}(g), \quad Q_{\theta}(g) = \left| \frac{1}{L!} \sum_{\lambda} d_{\lambda} e^{-i\theta E_m(\lambda)} \chi_{\lambda}(g) \right|^2. \quad (3.109)$$

Small α limit. When $\alpha \ll \gamma$, we can perturbatively expand the evolution of the density matrix. With $H_m = \sum_{s \in \mathcal{C}_m} R(s)/N_m$, we have that the probability matrix deviates from the identity as

$$Q_{\alpha t} = \mathbf{1} + \frac{(\alpha t)^2}{N_m} (H_m - \mathbf{1}) + O((\alpha t)^4). \quad (3.110)$$

No linear corrections are expected since Q is a squared amplitude. Therefore, the first corrections to the probability $p_{\alpha,\gamma}(t)$ are

$$p_{\alpha,\gamma} \underset{\alpha \ll \gamma}{\sim} p_{\gamma}(t) + \frac{(\alpha t)^2}{N_m} (H_m - \mathbf{1}) p_{\gamma}(t) + O((\alpha t)^4). \quad (3.111)$$

At leading order, the unitary evolution acts on the probabilities as an additional *classical* m -cycle step. In Fourier space the quantum kernel reads

$$\kappa_{\lambda}(\alpha t) = 1 - \frac{(\alpha t)^2}{N_m} (1 - E_m(\lambda)) + O((\alpha t)^4), \quad (3.112)$$

and therefore the irrep-resolved mode for the full probability is

$$\hat{p}_{\alpha,\gamma}(\lambda, t) \underset{\alpha \ll \gamma}{\sim} e^{-\gamma(1-E_q(\lambda))t} \left(1 - \frac{(\alpha t)^2}{N_m} (1 - E_m(\lambda)) \right) + O((\alpha t)^4). \quad (3.113)$$

3.6.1. The case of S_2

S_2 is a group of cardinality 2, and therefore one can use the basis of Pauli operators to resolve the dynamics of the OQS. There is a single transposition element $\tau = P_{1,2}$. Denote the Pauli matrices as

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3.114)$$

Clearly, X is bistochastic and defines the regular representation of τ . We therefore take

$$H = R(\tau) = X, \quad \Phi(\rho) = X\rho X. \quad (3.115)$$

Starting from the identity $\rho_0 = |e\rangle\langle e|$, the evolution is exact since $X^2 = \mathbf{1}$:

$$\rho_{\alpha,\gamma}(t) = \frac{1}{2} \left[\mathbf{1} + e^{-2\gamma t} (\cos(2\alpha t)Z + \sin(2\alpha t)Y) \right]. \quad (3.116)$$

Its trace distance follows exactly the exponential decay

$$T\left(\rho_{\alpha,\gamma}(t), \frac{\mathbf{1}}{2}\right) = \frac{1}{2}e^{-2\gamma t}, \quad (3.117)$$

independent of α . The mixing time is therefore

$$t_{\text{mix}}(\gamma, \epsilon) = \frac{1}{2\gamma} \left(\log \frac{1}{2\epsilon} \right), \quad \epsilon < \frac{1}{2}. \quad (3.118)$$

Similarly, the probabilities are

$$p_{\alpha,\gamma}(e, t) = \frac{1}{2} + \frac{1}{2}e^{-2\gamma t} \cos(2\alpha t), \quad p_{\alpha,\gamma}(\tau, t) = \frac{1}{2} - \frac{1}{2}e^{-2\gamma t} \cos(2\alpha t) \quad (3.119)$$

and the DS distance reads

$$D(\alpha, t) = \frac{1}{2}e^{-2\gamma t} |\cos(2\alpha t)|. \quad (3.120)$$

From this minimal example, we observe the mechanism at play: while the trace distance remains α -independent, the coherences generated by quantum dynamics, here in the nonzero elements $|e\rangle\langle\tau|$ and $|\tau\rangle\langle e|$, allow the information to be distributed over a larger space.

Thus, the case of S_2 is the minimal example in which the addition of quantum coherences accelerates mixing in the computational basis.

3.6.2. Dynamics of the Slowest Mode

Because of the damping factor in Eq. (3.108), for sufficiently large times, the irrep $[L - 1, 1]$ is the slowest one to decay. In analogy to Section 3.4, we discuss the contribution of this irrep to the mixing time. While Eq. (3.106) may be impractical to study analytically, we provide here an analysis of the slow-mode estimate. Using the expansion of the total probability

$$p_{\alpha,\gamma}(g, t) = \frac{1}{L!} \sum_{\lambda} d_{\lambda} \hat{p}_{\alpha,\gamma}(\lambda, t) \chi_{\lambda}(g) = \frac{1}{L!} + \frac{L-1}{L!} \hat{p}_{\alpha,\gamma}([L-1, 1], t) \chi_{[L-1, 1]}(g) + \dots, \quad (3.121)$$

we notice that, because of the exponential damping, the irrep $\lambda_S = [L - 1, 1]$ is the slowest mode to converge. In the purely classical case $\alpha = 0$, the analysis of this irrep is sufficient to inform us about the mixing time scale – see Section 3.4. We therefore aim to compute the QW kernel, which coincides with the ratio

$$\frac{\hat{p}_{\alpha,\gamma}([L-1, 1], t)}{\hat{p}_{0,\gamma}([L-1, 1], t)} = \kappa_{[L-1, 1]}(\alpha t), \quad (3.122)$$

to characterize the dynamics of the slowest mode.

Using the definition of the Kronecker coefficient associated with the irreps (λ, ρ, σ) ,

$$g_{\lambda,\rho,\sigma} = \frac{1}{L!} \sum_{g \in S_L} \chi_{\lambda}(g) \chi_{\rho}(g) \chi_{\sigma}(g), \quad (3.123)$$

we obtain the following expression for $\kappa_{[L-1, 1]}(\alpha t)$:

$$\kappa_{[L-1, 1]}(\alpha t) = \sum_{\rho, \sigma} \frac{g_{[L-1, 1], \rho, \sigma} d_{\rho} d_{\sigma}}{L!(L-1)} \cos(X_m(\rho, \sigma) \alpha t), \quad X_m(\rho, \sigma) = E_m(\rho) - E_m(\sigma). \quad (3.124)$$

In contrast to the classical case, the case $\alpha \neq 0$ involves mixing of irreps.

Nonetheless, the Kronecker coefficient can be given a probabilistic interpretation. From the identity

$$\sum_{\rho, \sigma} d_{\rho} d_{\sigma} g_{\lambda, \rho, \sigma} = L! d_{\lambda}, \quad (3.125)$$

we observe that the weight

$$w_{\rho, \sigma}^{\lambda} = \frac{d_{\rho} d_{\sigma} g_{\lambda, \rho, \sigma}}{L! d_{\lambda}} \leq 1, \quad (3.126)$$

is such that $\sum_{\rho, \sigma} w_{\rho, \sigma}^{\lambda} = 1$ and can therefore be interpreted as a probability measure

on the joint space of two irreps. The marginals with respect to a single irrep yield the Plancherel measure $\sum_{\sigma} w_{\rho, \sigma}^{\lambda} = d_{\rho}^2/L!$. Therefore, the slow-mode ratio becomes

$$\kappa_{\lambda_S}(\alpha t) = \mathbb{E}_S[e^{i\alpha t X_m}] = \mathbb{E}_S[\cos(\alpha t X_m)] \quad (3.127)$$

and thus coincides with the characteristic function of $X_m(\rho, \sigma)$.

For nontrivial cycles, we have the following exact identities, which follow from character multiplication and column orthogonality:

$$\mathbb{E}_S[E_m(\rho)] = 0, \quad \mathbb{E}_S[E_m(\rho)^2] = \frac{1}{N_m}, \quad \mathbb{E}_S[E_m(\rho)E_m(\sigma)] = \frac{E_m(\lambda_S)}{N_m}. \quad (3.128)$$

Since $E_m(\lambda_S) = (L - m - 1)/(L - 1)$, we obtain that the variance of X_m is

$$\text{Var}_S[X_m] = 2 \left[\frac{1}{N_m} - \frac{E_m(\lambda_S)}{N_m} \right] = \frac{2m}{(L - 1)N_m} \underset{L \gg 1}{\sim} \frac{2m^2}{L^{m+1}}. \quad (3.129)$$

This allows us to compute the statistical correlation of any two energies

$$\text{Corr}_S[E_m(\rho), E_m(\sigma)] = \frac{\text{Cov}_S[E_m(\rho), E_m(\sigma)]}{\sqrt{\mathbb{E}_S[E_m(\rho)^2]\mathbb{E}_S[E_m(\sigma)^2]}} = 1 - \frac{m}{L - 1}, \quad (3.130)$$

which tends to unity (perfect correlation) as L increases.

Using the relation $\cos(x) \geq 1 - x^2/2$, we find that

$$\kappa_S(\alpha t) \geq 1 - \frac{(\alpha t)^2}{2} \text{Var}_S[X_m] = 1 - \frac{m(\alpha t)^2}{(L - 1)N_m} \underset{L \gg 1}{\sim} 1 - \frac{m^2}{L^{m+1}} \alpha^2 t^2. \quad (3.131)$$

If we define the quantum slow dephasing time as

$$t_S(\alpha) = \frac{L^{\frac{m+1}{2}}}{\alpha m}, \quad (3.132)$$

then the bound Eq. (3.131) takes the form

$$\kappa_S(\alpha t) \geq 1 - \left(\frac{t}{t_S(\alpha)} \right)^2. \quad (3.133)$$

While the analysis of $\kappa_S(\alpha t)$ is not by itself sufficient to extract a precise ratio

$$r(\alpha, \gamma, t) = \frac{D(\alpha, t)}{D(0, t)}, \quad (3.134)$$

we observe that coherent dynamics cannot substantially suppress the slow mode before the quantum slow dephasing time unless $t \sim t_S(\alpha)$. This suggests a necessary criterion for the impact of quantum coherence on classical mixing. The classical slow-mode mixing occurs in a time window $t_{\text{mix},S}^{(0)} = \frac{L}{\gamma q} \log \frac{L}{\epsilon \epsilon}$. Therefore, equating $t_{\text{mix},S}^{(0)} = t_S(\alpha)$ gives an estimate for the ‘critical’ value of the dimensionless ratio $\mathcal{Q} = \alpha/\gamma$ at which quantum coherences generate an $O(1)$ modification of the slow mode relative to the classical mixing time:

$$\mathcal{Q}_* \sim \frac{q}{m} \frac{L^{(m-1)/2}}{\log \frac{L}{\epsilon \epsilon}}. \quad (3.135)$$

For example, in the case of random transpositions, one would need to set $\mathcal{Q}_* \sim \sqrt{L}/\log \frac{L}{\epsilon \epsilon}$.

3.6.3. Numerical Results for Random Transpositions

We now investigate numerically the quantum-to-classical ratio $r(\alpha, \gamma, t) = \frac{D(\alpha, t)}{D(0, t)}$ for random transpositions $m = q = 2$. The invariance of the probabilities within each class allows for an efficient estimate of the probability distance D for values of $L \lesssim 25$. Without loss of generality, we fix $\gamma = 1$ throughout the rest of this section.

We begin by considering the α dependence of the probability variation distance for $L = 23$. Instead of summing over the $L!$ group elements, we utilize the fact that the probability is constant over conjugacy classes, and therefore the sum is reduced to $\mathcal{P}(23) = 1255$ terms. In Fig. 3.13(a), the probability distance is shown for various values of $\mathcal{Q}$. The corresponding critical value is $\mathcal{Q}_* \approx 0.943$ for the threshold $\epsilon = 0.05$. Similarly, Fig. 3.13(b) shows the quantum-to-classical ratio r as a function of γt . We observe that, as α increases, the ratio rapidly decreases from 1 at shorter times.

This numerical result corroborates the analytical bound in Eq. (3.106) and, remarkably, shows that its effect is actually stronger than simply providing oscillations on top of a classical exponential profile. Indeed, this suggests analyzing the mixing time at the threshold ϵ as a function of α .

We analyze the mixing time for $L = 10, 15, 20, 23$ on a common grid of 401 values of $\mathcal{Q} = \alpha/\gamma$. For every L considered, the data are accurately described by

$$\frac{t_{\text{mix}}^{(L)}(\mathcal{Q}, \epsilon)}{t_{\text{mix}}^{(L)}(0, \epsilon)} = \sqrt{\frac{1}{1 + X_L^2}}, \quad X_L = \frac{\mathcal{Q}}{\sqrt{a_L}}, \quad (3.136)$$

see Fig. 3.14. The coefficients a_L are size dependent and consistent with the following

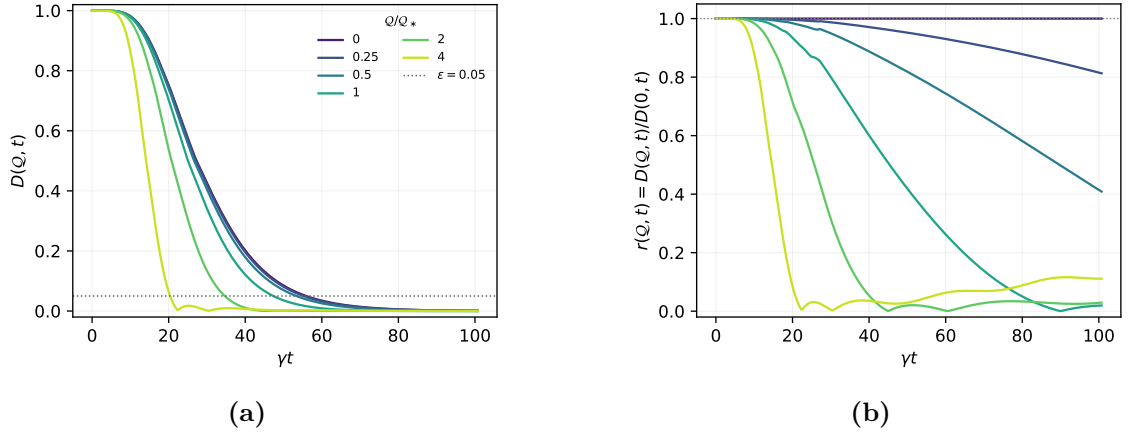


Figure 3.13: Probability distance between the QSW dynamics in Eq. (3.102) and the uniform distribution for random transpositions ($m = q = 2$) and $L = 23$. In (a), the probability distance is plotted for several values of $\alpha = Q\gamma$, with $\gamma = 1$. According to the criterion in Eq. (3.135), we start to observe appreciable deviations at $Q_* \approx 0.943$. (b) Plot of the quantum-to-classical ratio $r(\alpha, t) = D(\alpha, t)/D(0, t)$. For any value of α , because of Eq. (3.106), the ratio is smaller than one. We observe that, as α increases, the first time at which the ratio deviates from unity decreases.

L	a_L^{fit}	a_L^{fit}/a_L	R^2
10	1.1465	0.9614	0.9999264
15	1.5731	0.9721	0.9999434
20	1.9759	0.9762	0.9999504
23	2.2094	0.9773	0.9999520

Table 3.2: Fitted data parameters from Fig. 3.14, together with the scaling-fit parameters, which approach unity as L increases. The R^2 coefficient of the fit is also shown.

form

$$a_L = \left(\frac{2}{L} \log \left(\frac{(L-1)D_L}{L! \gamma \epsilon} \right) \right)^{-1}. \quad (3.137)$$

From the numerically-extracted best fit parameter, Table 3.2 shows that the rescaled combination $\tilde{a}_L = a_L^{\text{fit}}/a_L$ approaches unity as L increases.

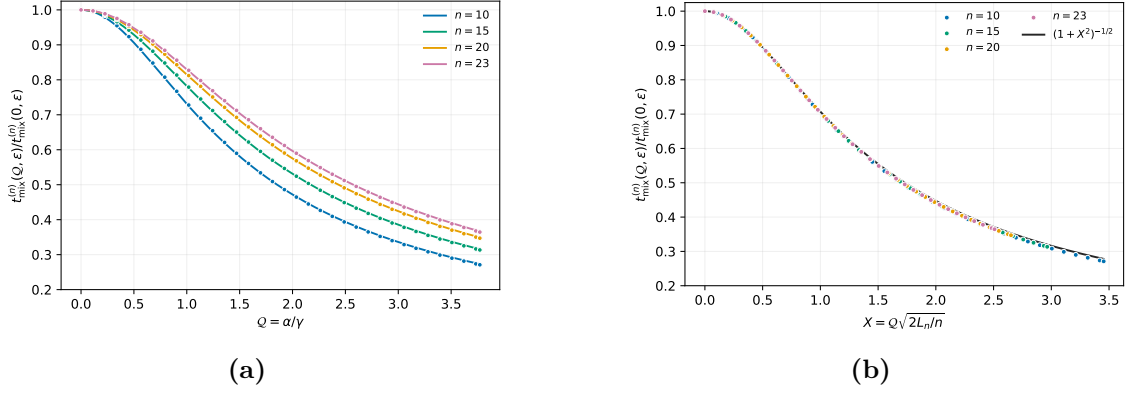


Figure 3.14: (a) Mixing time at $\epsilon = 0.05$ for several symmetric groups computed as a ratio to the classical one at $Q = 0$. Solid curves are independent fits to Eq. (3.136); dots are a sparse subset of the 401 numerical points used in each fit. (b) Collapse of the numerically estimated mixing times for varying L onto the scaling variable $X_L = Q\sqrt{2\Lambda_L/L}$, with $\Lambda_L = \log\left(\frac{(L-1)D_L}{L!\gamma^\epsilon}\right)$.

Conclusions

This thesis aimed at presenting a modern approach to statistical spectroscopy, which leverages the correlations within the spectrum of the Hamiltonian to determine equilibrium and out-of-equilibrium properties of quantum many-body systems. Returning to the three questions introduced in the Introduction of this thesis, we present here a discussion of the results and methods.

Chapter 1: what can spectral statistics reveal about chaos and integrability? Determining whether an infinite Hermitian matrix defines an integrable or chaotic quantum system is a central problem in many-body physics. In this chapter, we have approached this question from a statistical perspective, focusing on the spectral properties of the Hamiltonian. After a review of the spectral properties of integrable and random matrix models, we surveyed the principles of statistical integrability, where the fundamental ingredient was the presence of a degenerate spectrum. A class of quantum integrable Hamiltonians can be constructed under the consistency principle that if degeneracies occur, a sufficient number of independent quantum conserved charges must exist, in a number which is linear in the number of degrees of freedom of the many-body problem.

Chapter 2: what happens when the spectrum conceals independent symmetry sectors? While the hallmark of integrability is traditionally identified with Poissonian statistics of the unfolded level spacings, we have emphasized that similar behavior may also emerge from mixtures of distinct spectral components. This ambiguity motivates the search for sharper criteria capable of distinguishing the spectrum of a genuinely integrable model from that of a system composed of non-integrable subsectors.

Under this rationale, we applied the spectral decimation procedure [3, 23] to one-dimensional quantum many-body Hamiltonians featuring nontrivial forms of emergent symmetries. We argued that the underlying framework in which they can be understood is that of statistical mixtures, for which we derived the size of a characteristic

symmetry sector (CSS). The CSS is the typical sector with non-Poissonian statistics, characterized either by level repulsion or by isolated peaks, and its size is a size-biased average of the symmetry sectors present in the Hamiltonian.

For systems featuring Hilbert space fragmentation, in particular the pair-flip model, we showed that the CSS follows the relation predicted by the statistical mixture approximation and demonstrated that, while the overall global statistics may be close to Poissonian, the statistics of the CSS clearly feature level repulsion.

For systems undergoing many-body localization, we focused on the one-dimensional disordered Heisenberg chain and showed that the CSS can be used as a probe of the emergent symmetries generated by the local integrals of motion. We clarified the nature of the emergent integrability by looking at the statistics of the CSS at large disorder, which show clear peaks reminiscent of those of weakly interacting or almost free systems. The size of the CSS can be transformed into a novel order parameter, the characteristic symmetry entropy (CSE), which is computationally inexpensive since it is derived from the Hamiltonian spectrum. Like other observables, such as the r -ratio [116, 191, 192] or the zeros of the renormalization group β -function [155, 156, 193–198], it features universal finite-size scaling collapse. An interesting future direction is to derive the CSE finite-size scaling form from an explicit renormalization-group description of LIOM formation and clarify whether the transition is of the Kosterlitz–Thouless type or is governed by a different infinite-randomness scenario.

In conclusion, we demonstrated that the CSS and the CSE can be used as both qualitative and quantitative probes of the statistical properties of many-body spectra and can offer guidance in discriminating between strongly interacting integrable systems, free ones and statistical mixtures of non-integrable components. Moreover, as spectral decimation operates directly on spectral statistics without relying on entanglement or dynamical probes, it provides a computationally inexpensive and broadly applicable diagnostic for emergent integrability.

Chapter 3: can the analysis of spectral and symmetry structures predict out-of-equilibrium mixing? Permutation Hamiltonians offer a natural environment to address fundamental questions in quantum many-body physics. They are a controllable, tunable and highly versatile class of quantum systems that allows the investigation of ergodic and non-ergodic, equilibrium and out-of-equilibrium properties of many-body systems, while retaining the essential structure of the Hilbert space, deeply related to the representation theory of the permutation group. We extensively studied the construction, numerical implementation and graph interpretation of permutation Hamiltonians. Within that class, we were able to study with a

high degree of accuracy the spectral statistics and determine that certain multi-site permutation Hamiltonians are generically non-integrable, while the limiting cases (the all-to-all and 2-site models) are exactly solvable.

We then approached a fundamental question in the equilibration problem of quantum many-body systems: the estimation of the mixing time after which the dynamics has effectively explored the totality of the Hilbert space. While the card shuffling problem emerges naturally in the setting of combinatorics, it admits a formulation in terms of a Markov process on the permutation group S_L . In this work, we focused on an essential probe of the out-of-equilibrium physics of the problem, the mixing time, originally proved by Diaconis and Shahshahani [171, 172] to be $O(n \log n)$ in the classical case. The same time-scale emerges when the continuous-time generalization of the Markov process, with rate γ , is considered.

Within the framework of permutation Hamiltonians [3], we identified the transition matrix of the Markov process as the Hamiltonian of an associated quantum walk. Unlike the classical counterpart, the unitary quantum walk on S_L does not lose information and has vanishing von Neumann entropy. This implies that classical mixing is not generically allowed within purely unitary dynamics.

By casting the continuous-time random walk evolution as a particular limit of a quantum stochastic process [188], we considered the effect of quantum coherences on classical mixing by introducing a Hamiltonian in the corresponding Lindblad equation. We proved an upper bound lemma stating that any nonzero quantum coherence strength α implies a probability distance at any time in the evolution that is no greater than in the classical case $\alpha = 0$. An analysis of the slowest mode, associated with the first nontrivial irreducible representation $[L - 1, 1]$, identifies a critical value of the parameter α^* such that quantum coherences occur on the same time scale as classical mixing. These results are also corroborated by a numerical analysis for $L \leq 23$, which demonstrates the upper bound in action and, for a set threshold ϵ , shows that the mixing time decreases with increasing α . For the case of random transpositions, we numerically identify a scaling of the mixing-time ratios which depends on the quantum-classical ratio $\mathcal{Q} = \alpha/\gamma$.

In conclusion, this thesis has addressed statistical spectroscopy from both operational and methodological points of view. Regarding the former, we presented a method to filter correlated levels from statistical mixtures with many uncorrelated subsectors, the SD algorithm. We also considered the equilibrium and non-equilibrium properties of PHs. Statistical spectroscopy naturally addresses the equilibrium properties upon resolution of the symmetries, whereas it informs about out-of-equilibrium problems

such as mixing times of quantum stochastic walks. Overall, the contents and findings of this thesis highlight the role of statistics in quantum many-body physics.

Future Perspectives

The findings and methods developed in this thesis emphasize a central aspect of probability: that it can arise in seemingly unexpected ways, even when studying systems which are fundamentally deterministic. Its use can transform an inherent computational limitation into a new tool to probe physically-relevant properties.

The ubiquity of probability and statistics teaches, in our view, an important lesson: it proves to be a natural language when the information available is too large to be individually resolved. Moreover, it offers both established tools and a distinct framework to analyze complex systems.

Spectroscopy is only one lens which puts the rich and faceted world of quantum many-body physics into focus. Our aim for future work leverages the analytical and computational frameworks developed in this thesis to inquire into the dynamics of quantum many-body systems. Below, we outline some directions where probability may be of use in the understanding of quantum many-body physics.

Statistical Complexity. A central theme in the study of dynamics and out-of-equilibrium properties of quantum many-body systems is quantifying the amount of resources needed to operate a protocol, be it state preparation, time-evolution, or post-selection filtering. These issues fall under the wide umbrella of complexity theory and are of both theoretical [199] and experimental [200] relevance.

At the same time, quantum systems with a polynomial number of degrees of freedom feature an exponentially large state space. Thus, the information contained within a single observable comes generally from exponentially many sources. As remarked in the thesis, the tools of statistics and probability offer an operational framework to quantify such effects in estimating equilibrium properties of Hamiltonian systems.

A natural synergy of the two viewpoints is providing a statistical answer to the following physical question: given a reference state which can be efficiently prepared and two different interactions, what can be said about the relative complexity of the

two quench protocols?

The question is wide in scope, and clearly requires further specification of the scalar to be estimated. For example, one may seek to assess the complexity of evaluating the long-time value of an observable from short-time data — which may be obtained for instance from tensor-network techniques. Then, complexity inferred from posterior data can be analyzed with the tools of statistical inference. Another interesting viewpoint is to benchmark a quench protocol with different definitions of complexity. These two points mirror the suggestions given in [201], where classical non-linear dynamics was analyzed with the tools of statistical complexity. An application of statistical complexity to the study of equilibration in quantum many-body physics has been recently studied in connection to ETH [202].

Our proposal somehow leverages the large body of literature encompassing the field of statistical complexity, proposing the use of inferential statistics on simulated or experimental data streams. Closely related inference protocols have been implemented in trapped-ion quantum simulators, where Hamiltonian and Lindbladian Learning is combined with propagation of uncertainties into the simulated observables [203].

Canonical Approaches to Out-of-Equilibrium Dynamics. The thesis partially discussed a relevant subset of quantum many-body systems under active interest: Hamiltonian systems featuring random interactions. The key open question regards the thermalization properties of these systems. ED techniques have been extensively utilized, but most of the conclusions that can be drawn from their analysis are about crossover phenomena between ergodicity and non-ergodicity [9].

A possible direction towards the understanding of thermalization phenomena in disordered one-dimensional quantum many-body systems may come from analytic or asymptotic solutions, which encompass a vast literature – see e.g. [17, 19].

Inspired by this open problem, a natural question is: can arguments of statistical mechanics be used to lay the foundations for a canonical characterization of dynamics of typical states in disordered quantum systems?

Unpublished work on the single-body case seems to point towards a positive answer to the question: the path-integral representation of quantum amplitudes can be arranged in a cumulant expansion. Future directions would be a comparison with the conventional perturbative expansion, the analysis of the disorder average and the connection with the Anderson transition in generic dimension. Finally, the extension to the many-body case would constitute the main progression towards the definitive answer to the initial question.

Non-Reciprocal Quantum Dynamics. Classical systems often offer important physical inspiration to formulate curious questions about quantum systems. Recently, non-reciprocal active matter systems have attracted attention [204]. Of particular relevance are toy models such as the non-reciprocal Ising [205] and XY [206]. They were shown to exhibit rich physics encompassing static order disruption and stable long-range order.

Hamiltonian evolution of quantum many-body systems is often generated by symmetric interactions between the constituents. Despite this predominance, influential quantum-technology frameworks, such as gate-based quantum computation [207] and experimental platforms of reconfigurable Rydberg atom arrays [208], naturally feature non-reciprocal interactions in the guise of asymmetric operations. Concerning models of unitary quantum dynamics, the quantum East Model [209] realizes non-reciprocal interactions through spin-projection operators and features a transition between fast and slow thermalization throughout the spectrum.

A natural connection to the contents of Chapter 3 on out-of-equilibrium mixing, would be to understand the role of non-reciprocity in equilibration and critical behavior of open quantum many-body systems dynamics. Similar questions have been addressed in chiral dissipation setups, experimentally realizable with a cold-atom platform [210]. A promising starting point would be the investigation of free (Kitaev chain) and interacting (XXZ spin chain) Hamiltonians coupled to an external bath, whose dissipator introduces an effective non-reciprocal interaction and the corresponding modification of the critical behavior.

End Matter

Statement of Use of Generative Artificial Intelligence Tools

No generative-AI or large-language-model system was used to produce the scientific content of the thesis, perform calculations, review the literature, or draft its text. The manuscript was written by the author and underwent several rounds of scientific and linguistic revision with Prof. Giuseppe Mussardo, followed by further proofreading on the author's part.

Code Availability Statement

The decimation algorithm is available on Zenodo [211].

Appendix A

Higher Order Spacings in Random Matrix Theory

As we discussed in the main text, evaluating the higher spacing probability distribution for superposed spectra requires the numerical computation of the determinant of the Gaussian ensemble. In what follows, we will present two numerical strategies for calculating the determinants of GUE and GOE: the eigenvalue expansion method and the contour integral method. Both rely on Nyström-type discretization. Detailed implementation guidelines and technical aspects can be found in [40, 41].

GUE $\beta = 2$

The bulk eigenvalue correlation for GUE is governed by a determinantal point process [212]. The probability generating function (PGF) for the number of energy levels in $[0, s]$ is (see [213])

$$D(z; s) = \det \left(I - (1 - z) \mathcal{K}_{[0, s]} \right), \quad (\text{A.1})$$

where $\mathcal{K}_{[0, s]}$ is the integral operator on $L^2([0, s])$ with kernel

$$K(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}, \quad x, y \in [0, s], \quad (\text{A.2})$$

i.e.,

$$(\mathcal{K}f)(x) = \int_0^s ds \, K(x, y) f(y). \quad (\text{A.3})$$

In order to determine the higher spacing probability distribution $P_k^{(2)}(s)$, we can make use of the following two numerical methods:

(a) Eigenvalue product expansion

- We first discretize the sine-kernel (A.2) using the Nyström-type method, which approximates the integral operator on $L^2([0, s])$ by a finite $m \times m$ symmetric matrix K_m , i.e.,

$$D(z; s) = \det \left(I - (1 - z)K_m \right) \approx \det \left(I - (1 - z) \left(\sqrt{\omega_i} K(x_i, y_j) \sqrt{\omega_j} \right)_{i,j=1}^m \right). \quad (\text{A.4})$$

The $\{\omega_i\}$ ($i = 1, 2, \dots, m$) are Gauss–Legendre [40] quadrature weights.

- Second, we compute the eigenvalues $\{\lambda_j(s)\}$ ($0 \leq \lambda_j(s) \leq 1$) of K_m . The counting probability E_k can now be expressed as

$$\begin{aligned} D(z; s) &= \sum_{k=0}^m E_2(k; s) z^k = \prod_{j=1}^m (1 - (1 - z)\lambda_j(s)) \\ &= E_2(0; s) \sum_{k=1}^m e_k^{(m)}(a_1, a_2, \dots, a_m) z^k, \end{aligned} \quad (\text{A.5})$$

where

$$E_2(0; s) = \prod_{j=1}^m (1 - \lambda_j), \quad a_j = \frac{\lambda_j}{1 - \lambda_j}. \quad (\text{A.6})$$

The e_k are elementary symmetric polynomials, defined as

$$e_k^{(m)}(x_1, x_2, \dots, x_m) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} x_{i_1} x_{i_2} \dots x_{i_k}, \quad (\text{A.7})$$

with the convention $e_0^{(m)} = 1$ and $e_k^{(m)} = 0$ for $k < 0$ and $k > m$. Hence, we obtain the counting probability

$$E_2(k; s) = E_2(0; s) e_k^{(m)}(a_1, a_2, \dots, a_m) \quad k = 1, 2, \dots, m. \quad (\text{A.8})$$

In addition, we can use the recursive equations of the elementary symmetric polynomials to compute them efficiently:

$$e_k^{(n)}(x_1, \dots, x_n) = e_k^{(n-1)}(x_1, \dots, x_{n-1}) + x_n e_{k-1}^{(n-1)}(x_1, \dots, x_{n-1}). \quad (\text{A.9})$$

- Finally, the higher spacing distribution function $P_k^{(2)}(s)$ can be obtained by taking the second derivative of $\{E_2(k; s)\}$ with respect to s .

The eigenvalue product expansion method is effective for small k , but it suffers from instability for large k . The main reason is that it requires full

diagonalization of the kernel matrix K_m , but all the eigenvalues are small and $0 \leq \lambda_j(s) \leq 1$. Therefore, the counting probabilities $\{E_2(k; s)\}_{k \geq 0}$ are very sensitive to the precision of the eigenvalues, while, for large k , they involve many products of eigenvalues and hence demand high numerical accuracy in the eigenvalues.

(b) Contour integral method

In this method, no diagonalization of the kernel matrix is needed. The numerical steps are as follows:

- We first discretize the sine-kernel (A.2) using the Nyström-type method, as in the eigenvalue product expansion method, which leads to

$$D(z; s) = \det \left(I - (1 - z)K_m \right) \approx \det \left(I - (1 - z) \left(\sqrt{\omega_i} K(x_i, y_j) \sqrt{\omega_j} \right)_{i,j=1}^m \right). \quad (\text{A.10})$$

- Second, we calculate $E_2(k; s)$ using Cauchy integrals at $z = 0$,

$$E_2(k; s) = \frac{1}{k!} \frac{d^k}{dz^k} D(z; s) \Big|_{z=0} = \frac{1}{k!} D^{(k)}(0; s), \quad (\text{A.11})$$

$$D^{(k)}(0; s) = \frac{k!}{2\pi i} \oint_{|z|=\rho} \frac{D(z; s)}{z^{k+1}} dz = \frac{k!}{2\pi \rho^k} \int_0^{2\pi} e^{-ik\theta} D(\rho e^{i\theta}; s) d\theta, \quad (\text{A.12})$$

Using the trapezoidal rule with N equispaced angles,

$$\theta_l = \frac{2\pi l}{N}, \quad l = 0, 1, \dots, N-1,$$

the counting probability is

$$E_2(k; s) \approx \frac{1}{N \rho^k} \sum_{l=0}^{N-1} D(\rho e^{i\theta_l}; s) e^{-ik\theta_l}, \quad (\text{A.13})$$

where the determinant $D(\rho e^{i\theta_l}; s)$ can be computed via the LU decomposition. The quantity $E_2(k; s)$ is exactly the discrete/fast Fourier transform of the samples $D(\rho e^{i\theta_l}; s)$. For numerical stability, the contour radius $\rho \in [0, 1]$ is chosen properly to ensure that ρ^{-k} is well behaved and the contour $|z| = \rho$ does not pass through any zeros of $D(\rho e^{i\theta}; s)$.

- Finally, the $P_k^{(2)}(s)$ can be numerically calculated.

The contour integral method is not as intuitive as the eigenvalue product method, but it is stable for large k and permits precise control. The only requirement is the stable evaluation of the determinant over the complex plane. Therefore, it is better to use the contour integral method to evaluate higher-order $P_k^{(2)}(s)$.

GOE $\beta = 1$

For the GOE ensemble, the bulk eigenvalue correlations form a Pfaffian point process [214–216]. The corresponding PGF is

$$D(z; s) = \text{Pf}\left(J - (1 - z)JK_{[0,s]}\right). \quad (\text{A.14})$$

where J is a 2×2 canonical skew-symmetric matrix and $\mathcal{K}_{[0,s]}$ is a matrix-valued operator on $L^2([0, s])$ with kernel

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\mathcal{K})_{x,y} = \begin{pmatrix} S(x, y) & (DS)(x, y) \\ (IS)(x, y) - \epsilon(x, y) & S(y, x) \end{pmatrix}, \quad x, y \in [0, s], \quad (\text{A.15})$$

in the $N \rightarrow \infty$ limit. Here, the kernel matrix has entries

$$S(x, y) = K(x, y), \quad (DS)(x, y) = \frac{\partial}{\partial x} K(x, y), \quad (\text{A.16})$$

$$(IS)(x, y) = \int_y^x K(t, y) dt, \quad \epsilon(x, y) = \frac{1}{2} \text{sign}(x - y), \quad (\text{A.17})$$

where $K(x, y)$ is the sine-kernel (A.2).

In the bulk scaling limit, as proved by Mehta in [24], one can introduce two generating functions

$$D_{\pm}(z; s) = \det\left(1 - (1 - z)\mathcal{K}_{[-s/2, s/2]}^{\pm}\right), \quad (\text{A.18})$$

where $\mathcal{K}^{\pm}$ denotes a scalar Fredholm operator acting on $L^2([-s/2, s/2])$ with kernels

$$K^{\pm}(x, y) = \frac{1}{2} \left(K(x, y) \pm K(x, -y) \right), \quad x, y \in [-s/2, s/2]. \quad (\text{A.19})$$

These kernels are just the orthogonal decomposition of the ordinary sine kernel operator into the even and odd subspaces after a shift from $L^2([0, s])$ to $L^2([-s/2, s/2])$

(the sine kernel is shift invariant). We then define two reference counting probabilities in the even and odd subspaces

$$E_{\pm}(k; s) = \frac{1}{k!} \frac{d^k}{dz^k} D_{\pm}(z; s) \Big|_{z=0}, \quad k = 0, 1, 2, \dots \quad (\text{A.20})$$

The probabilities $E_1(k; s)$ can therefore be computed from $E_+(k; s)$ and $E_-(k; s)$ using the even/odd recursion [24]:

$$E_1(0; s) = E_+(0; s), \quad (\text{A.21})$$

$$E_1(2k-1; s) = E_-(k-1; s) - E_1(2k-2; s), \quad k \geq 1, \quad (\text{A.22})$$

$$E_1(2k; s) = E_+(k; s) - E_1(2k-1; s), \quad k \geq 1. \quad (\text{A.23})$$

Once the required counting probabilities $\{E_1(k; s)\}_{k \geq 0}$ are computed, one can obtain the k -level spacing probability distribution $P_k^{(1)}(s)$. The two numerical strategies used for GUE – the eigenvalue product expansion and contour integral methods – carry over unchanged here for the calculation of $E_{\pm}(k; s)$ according to $D_{\pm}(z; s)$.

Poisson Counting

The counting probability for a Poisson process is

$$E_{\text{Poisson}}(k; s) = \frac{s^k e^{-s}}{k!}. \quad (\text{A.24})$$

Mixtures

The counting probability distribution functions $E_{\beta}(k; s)$ for GUE, GOE and Poisson processes can be computed using the numerical methods discussed in this appendix. Hence, we can obtain any k -th spacing probability distribution $P_k(s)$ for superposed spectra with arbitrary fractions using selection rules. In Figure A.1, we show examples of the higher-spacing probability distribution for (i) the superposition of 3 GOE spectra and (ii) a Poisson-GOE mixture, both with unequal fractions.

Convolution method

The exact computation of k -th level spacing distributions differs significantly from a natural approximation, and this discrepancy is worth explaining. To start with, we

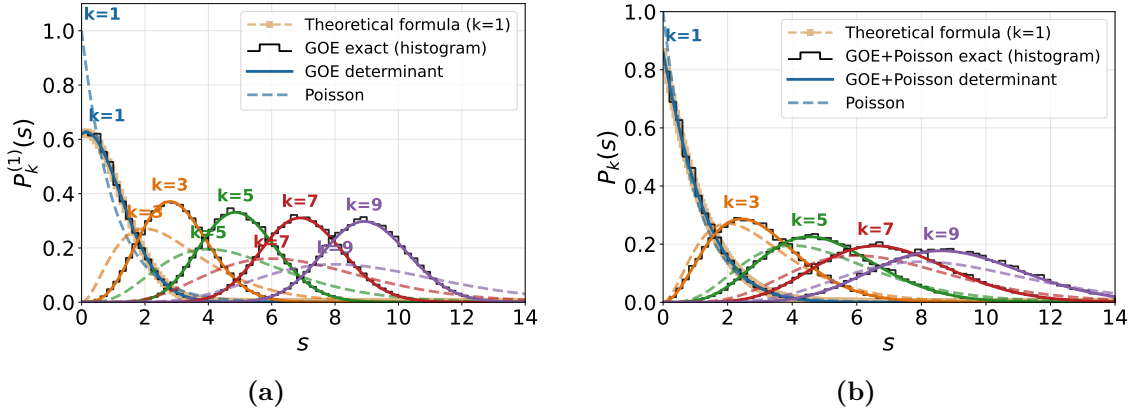


Figure A.1: k -th level spacing probability distribution $P_k(s)$ for a mixture of (a) 3 GOE spectra with fractions $f_1 = 0.2$, $f_2 = 0.3$, $f_3 = 0.5$; (b) 1 Poisson spectrum and 2 GOE spectra with fractions $f_{\text{Poisson}} = 0.5$, GOE: $f_1 = f_2 = 0.25$. The exact GOE and GOE+Poisson results are averages over 10 realizations of (a) 3 GOE spectra with total size $N_{\text{tot}} = 4000$ and (b) 1 Poisson and 2 GOE spectra with total size $N_{\text{tot}} = 4000$.

obviously have

$$s_{n,k} = (e_{n,k} - e_{n,k-1}) + (e_{n,k-1} - e_{n,k-2}) + \cdots (e_{n+1} - e_n) = \sum_{i=1}^k s_i, \quad (\text{A.25})$$

which, for the average, implies

$$\langle s_{n,k} \rangle = \sum_{i=1}^k s_i = k \langle s_i \rangle = k, \quad (\text{A.26})$$

since $\langle s_i \rangle = 1$. The approximation for the probability distributions of these higher-order spacings relies on treating each individual level spacing as statistically independent of the others. Under this assumption, the probability distribution $P_k(s)$ of the k -step spacing is obtained as the convolution of k Wigner–Dyson nearest-neighbour spacing distributions

$$P_k(s) = (P_1^{WD} \star P_1^{WD} \star P_1^{WD} \cdots \star P_1^{WD})(s) \quad (\text{A.27})$$

namely

$$P_k(s) = \int_0^s \cdots \int_0^s \left(\prod_{i=1}^k P_1^{WD}(x_i) \right) \delta \left(s - \sum_{i=1}^k x_i \right) dx_1 dx_2 \cdots dx_n. \quad (\text{A.28})$$

Its analytic expression for $k = 2$ in the GOE ($\beta = 1$) is given by

$$P_{k=2}(s) = \frac{\pi}{16} e^{-\frac{\pi s^2}{4}} \left[4s + \sqrt{2} e^{\frac{\pi s^2}{8}} (\pi s^2 - 4) \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{\pi}{2}} s \right) \right] \quad (\text{A.29})$$

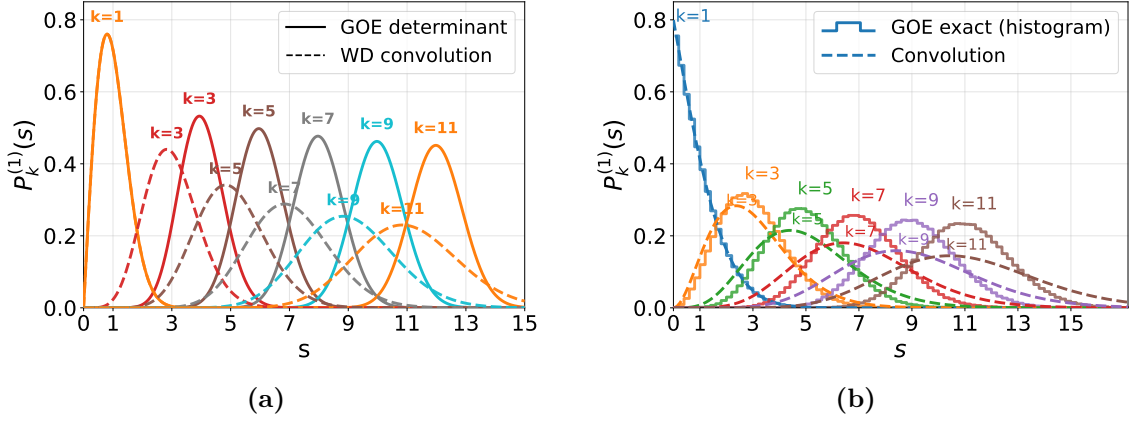


Figure A.2: (a) k -th level spacing probability distribution $P_k^{(1)}(s)$ for a single GOE determinant and Wigner-Dyson convolution. (b) k -th spacing probability distribution $P_k^{(1)}(s)$ for a superposition of 5 GOE spectra with equal fractions $f = 1/5 = 0.2$, where the histogram is an average over 200 realizations of 5 superposed GOE spectra, each from an exact diagonalization of a 400×400 matrix.

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt .$$

For higher k , one can evaluate the convolution using the Fourier transform: putting

$$\hat{P}_1^{WD}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_1^{WD}(x) e^{ipx} dx \quad (\text{A.30})$$

we have indeed

$$P_k(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{P}_1^{WD}(p) \right)^k e^{-ipx} dp . \quad (\text{A.31})$$

For large k , by the Central Limit Theorem, the higher-spacing probability distribution is given by the normal distribution

$$P_k(s) \simeq \frac{1}{2\pi k \sigma_1^2(\beta)} \exp \left[-\frac{(s - k)^2}{2k \sigma_1^2(\beta)} \right] \quad (\text{A.32})$$

where

$$\sigma_1^2(\beta) = \begin{cases} \frac{4}{\pi} - 1 = 0.27234... & , \quad \beta = 1 , \\ \frac{3\pi}{8} - 1 = 0.17810... & , \quad \beta = 2. \end{cases} \quad (\text{A.33})$$

This “independent-gaps” approximation, although simple, essentially neglects residual correlations among adjacent spacings. The variance of the corresponding curves in the “independent-gaps” approximation is predicted to grow as $\sqrt{k}$.

In Figure A.2, we compare the k -th level spacing probability distributions for a

single GOE spectrum and a superposition of 5 GOE spectra with convolution-based predictions obtained from Wigner-Dyson distributions and exact RMT predictions. As k increases, the convolution curves broaden rapidly; by contrast, the level spacing distributions for the single and superposed GOE matrices remain noticeably rigid.

A.1. Higher order spacing probability distributions

For the Gaussian ensembles in RMT, the Fredholm-determinant form of a probability-generating function for GUE and GOE allows us to calculate the higher-order spacing probability distribution precisely; the GUE determinant has already been introduced in Section 1.2.6. In this section, we consider a superposition of several spectra drawn from the same GOE, for which the higher-order spacing probability distribution can be obtained exactly via the Fredholm determinant.

We define a counting function $N([0, s])$, which counts how many energy levels are present in the bulk interval $[0, s]$. For the Fredholm determinant operator $\mathcal{K}$, the probability generating function (PGF) for the number of levels in $[0, s]$ can be written as

$$D(z; s) \equiv \mathbb{E} \left[z^{N([0, s])} \right] = \det \left(I - (1 - z) \mathcal{K}_{[0, s]} \right), \quad (\text{A.34})$$

where $\mathbb{E}(X)$ represents the expectation/mean value of the random variable X and $\mathcal{K}$ is defined on $L^2([0, s])$. Expanding about $z = 0$, the coefficients of the power series give us counting probabilities:

$$D(z; s) = \sum_{k=0}^{\infty} E_{\beta}(k; s) z^k, \quad (\text{A.35})$$

$$E_{\beta}(k; s) = \Pr \left(N([0, s]) = k \right) = \frac{1}{k!} \left. \frac{d^k}{dz^k} D(z; s) \right|_{z=0}. \quad (\text{A.36})$$

The k -th level spacing distribution function is

$$P_k^{(\beta)}(s) = \frac{d^2}{ds^2} \left(\sum_{j=0}^{k-1} (k-j) E_{\beta}(j; s) \right). \quad (\text{A.37})$$

We now consider a superposition of n spectra with fractions $f_1, f_2, \dots, f_n$, $\sum_{i=1}^n f_i = 1$. The counting function that counts the number of levels in $[0, s]$ for a collection of

these n spectra is

$$N_{\text{sup}}([0, s]) = \sum_{i=1}^n N_i([0, f_i s]) , \quad (\text{A.38})$$

i.e. each spectrum i contributes in the interval $[0, f_i s]$ to the collective spectrum in $[0, s]$. Due to the independence of each individual spectrum, the overall PGF can be factorized into the product of the PGFs corresponding to each component:

$$D_{\text{sup}}(z; s) \equiv \mathbb{E} \left[z^{N_{\text{sup}}([0, s])} \right] = \prod_i^n \det \left(I - (1 - z) \mathcal{K}_{[0, f_i s]} \right) . \quad (\text{A.39})$$

Therefore, the superposed k -level counting probability $E_{\beta}^{\text{sup}}(k; s)$ ($k = 0, 1, 2, \dots$) can be written as

$$E_{\beta}^{\text{sup}}(k; s) = \frac{1}{k!} \frac{\text{d}^k}{\text{d}z^k} D_{\text{sup}}(z; s) \Big|_{z=0} = \sum_{k_1+k_2+\dots+k_n=k} \prod_{i=1}^m E_{\beta}^{(i)}(k_i; f_i s) . \quad (\text{A.40})$$

One can quickly check that Eq. (A.40) reduces to Eq. (2.7) when $k = 0$. A similar example for the rules (A.39) and (A.40) can be found in [217]. The k -th level spacing probability distribution is

$$P_k^{(\beta)}(s) = \frac{\text{d}^2}{\text{d}s^2} \left(\sum_{j=0}^{k-1} (k-j) E_{\beta}^{\text{sup}}(j; s) \right) , \quad (\text{A.41})$$

which is the exact formula we rely on to calculate the higher-order spacing probability distribution for superposed spectra.

To make the selection rule (A.40) clearer, we consider a mixture of 3 spectra drawn from the same ensemble – e.g. GUE or GOE – with fractions f_1, f_2, f_3 . The counting probabilities for different k are

$$\begin{aligned} k = 0 : \quad & E_{\beta}^{\text{sup}}(0; s) = E_{\beta}^{(1)}(0; f_1 s) E_{\beta}^{(2)}(0; f_2 s) E_{\beta}^{(3)}(0; f_3 s) , \\ k = 1 : \quad & E_{\beta}^{\text{sup}}(1; s) = E_{\beta}^{(1)}(1; f_1 s) E_{\beta}^{(2)}(0; f_2 s) E_{\beta}^{(3)}(0; f_3 s) \\ & + E_{\beta}^{(1)}(0; f_1 s) E_{\beta}^{(2)}(1; f_2 s) E_{\beta}^{(3)}(0; f_3 s) \\ & + E_{\beta}^{(1)}(0; f_1 s) E_{\beta}^{(2)}(0; f_2 s) E_{\beta}^{(3)}(1; f_3 s) , \\ k = 2 : \quad & E_{\beta}^{\text{sup}}(2; s) = \dots \end{aligned} \quad (\text{A.42})$$

The k -th level spacing probability distribution is

$$\begin{aligned}
P_1^{(\beta)}(s) &= \frac{d^2}{ds^2} E_\beta^{\text{sup}}(0; s) , \\
P_2^{(\beta)}(s) &= \frac{d^2}{ds^2} \left[2E_\beta^{\text{sup}}(0; s) + E_\beta^{\text{sup}}(1; s) \right] , \\
P_3^{(\beta)}(s) &= \frac{d^2}{ds^2} \left[3E_\beta^{\text{sup}}(0; s) + 2E_\beta^{\text{sup}}(1; s) + E_\beta^{\text{sup}}(2; s) \right] , \\
&\dots , \\
P_k^{(\beta)}(s) &= P_{k-1}^{(\beta)} + \frac{d^2}{ds^2} \sum_{j=0}^{k-1} E_\beta^{\text{sup}}(j; s) \quad (k = 1, 2, 3, \dots) , \tag{A.43}
\end{aligned}$$

where we take $P_0^{(\beta)}(s) = 0$.

In the overall PGF in Eq. (A.39) for superposed spectra, we did not specify the determinant kernel $\mathcal{K}_{[0,s]}$, which can be the kernel of any Gaussian ensemble ($\beta = 1, 2$) or Poisson distribution (one should note that the PGF for GOE is a Pfaffian). The k -th-level counting probability in Eq. (A.40) and the probability distribution in Eq. (A.41) are therefore universal for any superposition of independent spectra drawn from the same Gaussian ensemble. Since the individual spectra composing the mixture are independent of each other, the product rule in Eq. (A.39) is expected to be universal. Hence, it is valid regardless of whether its constituents are GOE or GUE bulk spectra (with respect to their own determinant kernel) or a Gaussian-Poisson hybrid.

To calculate exactly the k -th-level spacing distribution for a joint spectrum, it is necessary to (i) evaluate the counting probability $E_\beta(k; f_i s)$ for each individual component at its own fraction; (ii) use Eq. (A.40) to combine them; and (iii) obtain the k -th-level spacing probability distribution from Eq. (A.41). To illustrate this procedure, we show in Figure A.3 the higher-order spacing probability distribution of 3 superposed GUE spectra with unequal fractions. The exact numerical method that we adopted for computing the k -th-level spacing probability distribution is discussed in Appendix A, where one can also find several examples of superposed GOE spectra.

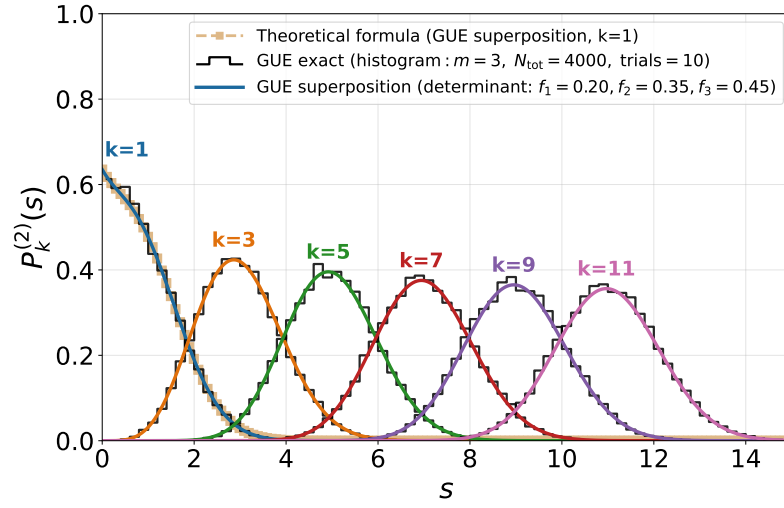


Figure A.3: k -th level spacing probability distribution $P_k^{(1)}(s)$ for superposed 3 GUE spectra with fraction $f_1 = 0.2, f_2 = 0.35, f_3 = 0.45$. The histogram is an average over 10 exact diagonalization realizations for three GUE matrices with total size 4000×4000 . The theoretical formula is Eq.(2.10) for GUE.

Appendix B

Permutation Group and Its IRs

We devote this appendix to presenting more information about the permutation group of L elements $\mathcal{S}_L$ and its irreducible representations (IRs) in the Hilbert space of permutation Hamiltonians, i.e. those in which the local Hilbert space consists of one particle with one of n distinct colours. More details can be found in classic textbooks [161, 190, 218–220] and classical references [221, 222].

Each element P of $\mathcal{S}_L$ belongs to a conjugacy class $\mathcal{C}_\alpha$, which is uniquely described by a partition of L . A cycle of length n permutes n elements and is assigned a “parity” $(-1)^{n-1}$. The permutation group thus splits into two parity classes: the even permutations, which form the alternating subgroup A_L , and the odd permutations, which form the coset $\mathcal{S}_L \setminus A_L$. This parity can be checked readily by computing the action of P on the fully antisymmetric combination generated by the vector $|1, \dots, L\rangle$, corresponding to the only vector of the fully antisymmetric representation $[1^L]$.

Let us now describe how the generators $P_{i,i+1}$ are realized as orthogonal matrices in a given irreducible representation β , following the standard procedure of [161].

A particular “standard” basis for an irreducible representation has vectors $|s_i\rangle$ that are characterized by $L - 1$ “quantum numbers”: $|L_i\rangle = |\beta, \beta_1^i, \dots, \beta_{L-2}^i\rangle$. Here β_j^i is the irreducible representation of $\mathcal{S}_{L-j}$ under which the standard vector $|s_i\rangle$ transforms when $\mathcal{S}_L$ is restricted to the subgroup $\mathcal{S}_{L-j}$ generated by the first $(L - j)$ elementary transpositions.

Visually, a standard vector $|s_i\rangle$ is represented by a standard Young tableau, obtained by filling an empty diagram with the integers $\{1, \dots, L\}$ according to the following rule: if the last j boxes are removed from the tableau, the corresponding diagram corresponds to one of the β_j^i . The number of such tableaux is exactly d_β . Another

possible enumeration of the standard vectors, equivalent to the Young tableaux, is given by Yamanouchi symbols. The j th entry of the symbol indicates the row of the original Young diagram. For that row, the rightmost box will be dropped in the reduction $\mathcal{S}_{L-j} \rightarrow \mathcal{S}_{L-j-1}$.

The number of tableaux obtained through this rule coincides with the dimension d_β of the representation, which can also be readily computed through the hook formula, as noted in the main text. The hooks can be directly computed from the Young diagram in this way. For each box i of the diagram, its hook is defined as

$$H_i = 1 + R_i + B_i, \quad (\text{B.1})$$

where R_i is the number of boxes to the right of i , and B_i is the number below it. Then the formula reads

$$d_\beta = \frac{L!}{\prod_{i \in \text{boxes}} H_i}. \quad (\text{B.2})$$

For example, both the symmetric, $[L]$, and the antisymmetric, $[1^L]$, have dimension one.

Young provided a theorem to construct the elementary nearest-neighbour transpositions $P_{i\ i+1}$ in a given irreducible representation β . The matrices $IR_\beta(P_{i\ i+1}) \equiv P_{i\ i+1}^\beta$ are chosen to be real and unitary (orthogonal). Crucially, the construction refers to the standard basis – also known as the “orthogonal” basis in the literature. It is known that for the regular representation the rotation matrix between the physical and standard bases is given by symmetrizers and antisymmetrizers read from the Young tableaux – see the Supplementary Material of [159].

For a fixed standard basis vector $|s_j\rangle$, the j -th row of $P_{i\ i+1}^\beta$ will contain at most two entries. Therefore, transposition matrices are sparse, and the maximum number of nonzero entries in total is $2/d_\beta$. For the j -th row, we find in the basis a vector $|s_k\rangle$ whose tableau coincides with that of $|s_j\rangle$ upon the exchange of the integer entries i and $i+1$. Between the two tableaux, one computes the axial distance [161] $\ell_{j,k}^i$, and the entries of the j th and k th rows form an $SU(2)$ matrix

$$\begin{bmatrix} \langle s_j | P_{i\ i+1}^\beta | s_j \rangle & \langle s_k | P_{i\ i+1}^\beta | s_j \rangle \\ \langle s_j | P_{i\ i+1}^\beta | s_k \rangle & \langle s_k | P_{i\ i+1}^\beta | s_k \rangle \end{bmatrix} = \begin{bmatrix} -(\ell_{j,k}^i)^{-1} & \left[1 - (\ell_{j,k}^i)^{-2}\right]^{\frac{1}{2}} \\ \left[1 - (\ell_{j,k}^i)^{-2}\right]^{\frac{1}{2}} & (\ell_{j,k}^i)^{-1} \end{bmatrix}. \quad (\text{B.3})$$

We conclude this appendix by commenting on the structure of the Hilbert space of dimension $d = n^L$. The decomposition presented in Section 3.1 is special because

it is left invariant by the action of a permutation Hamiltonian. More generally, an operator on the Hilbert space can be decomposed into the irreducible representations of $SU(n)^{\otimes L}$, i.e. by “spins” or equivalently by a Young diagram of $SU(n)$. Confusion may arise since Young diagrams are also used for $\mathcal{S}_L$. Generic $SU(n)$ Young diagrams are labelled by the partitions of L , with at most n parts, $p(L)|_n$. To differentiate these irreducible representations from those of $\mathcal{S}_L$, we fill a Young diagram of $SU(n)$ with a light gray colour and denote its associated partition by double square brackets, e.g.

$$[[3, 2, 1]]_n = \begin{array}{|c|c|c|} \hline \text{gray} & \text{gray} & \text{gray} \\ \hline \text{gray} & \text{gray} & \\ \hline \text{gray} & & \\ \hline \end{array}^n$$

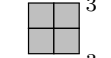
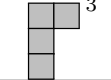
denotes an irreducible representation with $L = 6$ and generic $n \leq L$.

An irreducible representation $\mathfrak{IR}_\alpha$ of $SU(n)$ has dimension $\mathbf{f}_\alpha$ and multiplicity $\mathbf{m}_\alpha$. These coefficients can be read from the Young diagram [159]. To compute $\mathbf{m}_\alpha$, it is sufficient to divide $L!$ by the product of the hooks read from the Young diagram. For $\mathbf{f}_\alpha$, fill all the diagonal boxes with n , all the boxes to the right of the diagonal with $n + \gamma$, where γ is the number of boxes from the diagonal counted horizontally, and all the boxes to the left of the diagonal with $n - \delta$, where δ is the number of boxes from the diagonal counted vertically. To compute $\mathbf{f}_\alpha$, multiply the entries obtained in this way and divide by the product of the hooks. Summing over all irreducible representations yields the dimension of the Hilbert space: $d = n^L = \sum_\alpha \mathbf{m}_\alpha \mathbf{f}_\alpha$.

As mentioned above, for the permutation Hamiltonian, one can find other invariant subspaces, each labelled by the colour sector $\{L_1, L_2, \dots, L_n\}$, $\sum_i L_i = L$. These subspaces are spanned by the vectors of the “physical basis” $|i_1, \dots, i_L\rangle$, where L_1 sites (not necessarily adjacent) are occupied by colour 1, L_2 sites by colour 2, and so on. Each subspace has dimension $f_{L_1, L_2, \dots, L_n} = L! / (L_1! \dots L_n!)$. Naturally, the number of colour sectors and their dimensions must match n^L .

The number of distinguishable colour sectors equals $\mathcal{P}(L)|_n$, and each of them appears with a nontrivial multiplicity, since H puts all colours on the same footing. To find the multiplicity $m_{L_1, \dots, L_n}$, with $L_1 \geq L_2 \geq \dots \geq L_n$, denote by ℓ_s the number of L_i such that $L_i = s$. Then, the multiplicity is the number of ways n “boxes” can be filled with the indistinguishable numbers $s = \{0, 1, \dots, L\}$, each appearing ℓ_s times:

$$m_{L_1, \dots, L_n} = \frac{n!}{\ell_0! \ell_1! \dots \ell_N!}. \quad (\text{B.4})$$

Irreducible Representation	f_α	m_α
 ³	1	15
 ³	3	15
 ³	2	6
 ³	3	3

(a) $SU(n)$

Colour Sector	f_{n_1, n_2, n_3}	m_{n_1, n_2, n_3}
$\{4, 0, 0\}$	1	3
$\{3, 1, 0\}$	4	6
$\{2, 2, 0\}$	6	3
$\{2, 1, 1\}$	12	3

(b) Colour Sector

Table B.1: Decomposition of the Hilbert space for $L = 4$, $n = 3$ according to (a) $SU(n)$ representations and (b) colour sectors.

Summing over partitions, we retrieve the dimension of the Hilbert space

$$d = n^L = \sum_{\{L_1, \dots, L_n\} \in \mathcal{P}(L)|_n} m_{L_1, \dots, L_n} f_{L_1, \dots, L_n}. \quad (\text{B.5})$$

In Table B.1, we provide an example of the two aforementioned decompositions for $n = 3$ and $L = 4$.

As we mentioned in Section 3.1, colour sectors can be further split into irreducible representations of $\mathcal{S}_L$. The reason is that many-body wavefunctions of L bosons in the colour sector $\{L_1, \dots, L_n\}$ transform according to irreducible representations of $\mathcal{S}_{\{L_1, \dots, L_n\}} = \mathcal{S}_{L_1} \otimes \dots \otimes \mathcal{S}_{L_n}$, which is a subgroup of $\mathcal{S}_L$. It is well known that the irreducible representations of $\mathcal{S}_{\{L_1, \dots, L_n\}}$ are those of $\mathcal{S}_L$. They can be found through the outer product formula [161]. For the case at hand, the particles are bosons and do not have mixed statistics with respect to $\mathcal{S}_L$; the outer product formula admits a convenient algorithm, which we provide in Figure B.1.

The outer product formula provides, in essence, the symmetry transformation of an L -boson wavefunction subject to the constraints given by the colour sector $\{L_1, \dots, L_n\}$. Therefore, not all irreducible representations appear in the decomposition, but only those with a number of rows $r \leq n$. While the irreducible representations α appearing in a certain outer product formula can be predicted on the basis of symmetry, it is not straightforward to obtain the multiplicities m_α – this is exactly the purpose of the algorithm in Figure B.1. An obvious check is that the dimension of the colour sector should coincide with $\sum_\alpha m_\alpha d_\alpha$.

For $SU(L)$, the outer product decomposition of the colour sector $\{1, 1, \dots, 1\}$, where all particles are of different colours, coincides with the regular representation of $\mathcal{S}_L$, where all irreducible representations α appear $m_\alpha = d_\alpha$ times, such that $L! = |\mathcal{S}_L| = \sum_\alpha d_\alpha^2$. For small n , on the other hand, very interesting representations

Input: A partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ and a fully symmetric partition (r) .
Output: The partitions occurring in the outer product $\lambda \otimes (r)$.

```

 $\mathcal{C} \leftarrow \{(\lambda, \emptyset)\}$ 
for  $b = 1, \dots, r$  do
   $\mathcal{C}_{\text{new}} \leftarrow \emptyset$ 
  for each  $(\mu, H) \in \mathcal{C}$  do
     $i_{\min} \leftarrow 1$  if  $H = \emptyset$ ; otherwise  $i_{\min} \leftarrow \text{row}(\text{last}(H))$ 
    for  $i = i_{\min}, \dots, \ell(\mu) + 1$  do
       $c \leftarrow \mu_i + 1$  if  $i \leq \ell(\mu)$ , and  $c \leftarrow 1$  otherwise
      if  $(i = 1 \text{ or } \mu_{i-1} \geq c)$  and  $c \notin \text{cols}(H)$  then
        Add the box  $(i, c)$  to  $\mu$ , obtaining  $\mu'$ 
         $\mathcal{C}_{\text{new}} \leftarrow \mathcal{C}_{\text{new}} \cup \{(\mu', H \parallel (i, c))\}$ 
   $\mathcal{C} \leftarrow \mathcal{C}_{\text{new}}$ 
return  $\{\mu : (\mu, H) \in \mathcal{C}\}$ .

```

Here H records the added boxes. Restricting their row indices to be nondecreasing avoids duplicate insertion orders, while $c \notin \text{cols}(H)$ enforces the horizontal-strip rule.

Figure B.1: Pseudocode for the outer product of a partition λ with a fully symmetric partition (r) . The algorithm enumerates the Young diagrams obtained from λ by adding a horizontal r -strip. Adapted from the Wolfram Mathematica implementation in Ref. [3].

are those for which $L = mn$. There is one irreducible representation in the colour sector $\{m, m, \dots, m\}$ (i.e. where all particle colours are equally distributed), namely $[m^n]$. This is the representation in which the ground state of the antiferromagnetic Heisenberg and Sutherland permutation Hamiltonians lies, and one may regard its dimension as that of the Hilbert space up to subleading corrections. Using the hook formula, one finds asymptotically:

$$d_{[m^n]} = (nm)! \prod_{j=0}^{n-1} \frac{j!}{(n+j)!} \underset{n \gg 1}{\sim} m^{nm} \frac{G(m+1)}{n^{m(m-1)/2}}, \quad (\text{B.6})$$

where $G(z) = \prod_{k=0}^z \Gamma(k+1)$ is the Barnes G function.

Appendix C

Mixing of random q -cycle walks

This appendix details the Diaconis-Shahshahani theorem for q -cycles discussed in Chapter 3.3. We limit ourselves to the analysis of the lower bound, which already contains the relevant mixing time. For a general proof of the upper bound, we refer to [171, 172]. We briefly recall its statement.

Let us consider a random walk on S_n , the permutation group of n elements. Let the transition matrix H_q be the operator involving the sum over all cycles of length q ($\mathcal{C}_{[q, 1^{n-q}]}$ of cardinality N_q):

$$H_q = \frac{1}{N_q} \sum_{g \in \mathcal{C}_{[q, 1^{n-q}]}} R(g). \quad (\text{C.1})$$

Here, $R(g)$ is the element g in its regular representation R . This transition matrix generates a discrete-time Markov chain in which, at each step, a state $|g\rangle$ is mapped to $|g'\rangle$, connected to it by a q -cycle permutation uniformly chosen from its conjugacy class. We aim to compute the DS measure and determine the mixing time k^* set by the threshold ϵ , i.e.

$$||P^{k^*} - U|| = \frac{1}{2} \sum_g \left| P^{k^*} - \frac{1}{n!} \right| < \epsilon. \quad (\text{C.2})$$

Before addressing the problem, it is convenient to recall several facts about the regular representation and the Fourier transform on S_n .

Properties of the regular representation. Let ρ be an irrep of S_n of dimension d_ρ . The regular representation $(R(g) |h\rangle = |gh\rangle)$ has characters

$$r(g) = \text{Tr } R(g) = n! \delta_{g,e}. \quad (\text{C.3})$$

Moreover, it is reducible, each irrep ρ has multiplicity d_ρ , and there are therefore finitely many irreps, since

$$\sum_{\rho} d_{\rho} \chi_{\rho}(g) = n! \delta_{g,e}. \quad (\text{C.4})$$

Fourier transform. By construction, the matrices $R(g)$ are orthogonal, i.e. real and unitary. Given a block ρ , it is also unitary, and its matrix elements form an orthogonal basis for the set of all functions on S_n . Let f be a function on S_n ; we then define it through the *inverse Fourier transform*

$$f(g) = \frac{1}{|S_n|} \sum_{\rho} d_{\rho} \text{Tr} \left(\rho(g^{-1}) F(\rho) \right) = \frac{1}{|S_n|} \text{Tr} \left(R(g^{-1}) F(R) \right). \quad (\text{C.5})$$

Here, $F(\rho)$ is said to be the (matrix) Fourier transform of f on the group S_n in irrep ρ . A similar relation to the Plancherel formula holds for any two functions f and h :

$$\sum_{g \in S_n} f(g^{-1}) h(g) = \frac{1}{|S_n|} \sum_{\rho} d_{\rho} \text{Tr} (H(\rho) F(\rho)). \quad (\text{C.6})$$

These formulas are straightforward to prove. Indeed, we recognize that both formulas are linear in f and that it is therefore sufficient to prove them for the function $f(g) = \delta_{g,t}$. Thus, $F(\rho) = \rho(t)$ and both results follow from Eq. (C.4). For real functions and unitary representations, the Plancherel theorem can be written as

$$\sum_{g \in S_n} f(g) h(g) = \frac{1}{|S_n|} \sum_{\rho} d_{\rho} \text{Tr} (H(\rho) (F(\rho))^*), \quad (\text{C.7})$$

where $(\cdot)^*$ denotes complex conjugation.

Schur's lemma. Finally, we recall the content of a useful lemma that will allow an analytical treatment of the mixing problem. Let ρ^1 and ρ^2 be two irreducible representations and f a linear map between the two vector spaces V_1 and V_2 on which the irreps act. Then, if

$$\rho^2(g) \circ f = f \circ \rho^1(g), \quad \forall g \in S_n, \implies f = \text{const } \mathbf{1}. \quad (\text{C.8})$$

If ρ^1 and ρ^2 are not equivalent, the constant is zero.

Upper bound lemma. We now begin with the analytical treatment of the DS measure. Squaring the DS measure, Eq. (C.2), one obtains a useful bound in terms

of the Cauchy-Schwarz inequality

$$4\|p-U\|^2 \leq n! \sum_s |Q(s) - 1/n!|^2 = \sum_\rho d_\rho \operatorname{Tr}(P(\rho)P^*(\rho)) - 1 = \sum_{\rho^*} d_\rho \operatorname{Tr}(P(\rho)P^*(\rho)), \quad (\text{C.9})$$

where ρ^* in the last sum conventionally denotes all irreps except the trivial (symmetric) one. Indeed, in the symmetric representation, the Fourier transform is 1 for all elements of S_n , and it cancels with the (-1) term. The intermediate identity is also obtained through the application of the Plancherel formula.

We remark that the use of the Cauchy-Schwarz inequality is not as crude as it may first appear. Indeed, it is applied when p is close to uniform, i.e. when $\|p - U\|$ approaches a constant value. One can also show that, in several examples [171], the upper bound matches a lower bound up to one or two terms.

Lower bounds on the DS measure. We state for completeness two lower bounds, both containing the irrep-resolved quantity

$$\Delta_{p,U} = \sum_{\rho^*} d_\rho \operatorname{Tr}(P(\rho)P^*(\rho)). \quad (\text{C.10})$$

Such bounds read

$$\begin{aligned} \|p - U\| &\geq \frac{\Delta_{p,U}}{2n!}, \\ \|p - U\|^2 &\geq \frac{\Delta_{p,U}}{4n!}. \end{aligned} \quad (\text{C.11})$$

Random q -cycles. We consider now the case of a walk on S_n generated by elements belonging to the q -cycle conjugacy class, with even q . For odd q , it suffices to restrict the measure to the alternating group of S_n , A_n . It is essential that the probability measure $p(\tau) = 1/N_q$ be supported only on the conjugacy class $[q, 1^{n-q}]$, for it implies the property

$$P(\rho) = \sum_{g \in S_n} p(g)\rho(g) = \frac{1}{N_q} \sum_{g \in \mathcal{C}_q} \rho(g) = H_q(\rho). \quad (\text{C.12})$$

The operator $H_q(\rho)$ is a class operator on the irreducible representation ρ . On the regular representation on which the random walk acts, we have

$$P(R) = H_q = \frac{1}{N_q} \sum_{g \in \mathcal{C}_q} R(g), \quad (\text{C.13})$$

where H_q is the class Hamiltonian of q -cycles and the transition matrix of the random walk. Thus, $P(\rho)$ commutes with every element $\rho(g)$ of the group in the irrep ρ , and therefore it is proportional to the identity by Schur's lemma. By taking the trace,

we obtain the value of the constant

$$P(\rho) = r_\rho(q) \mathbf{1}_{d_\rho}, \quad r_\rho(q) = \frac{\chi_\rho^q}{d_\rho}. \quad (\text{C.14})$$

We apply the upper bound lemma to this result:

$$\|p^* - U\|^2 \leq \frac{1}{4} \sum_{\rho^*} d_\rho^2 r_\rho(q)^{2k}, \quad (\text{C.15})$$

where we used the fact that the characters χ_ρ are real-valued.

Emergence of the $n \log n$ scale from the upper bound lemma. The largest eigenvalues of H_q are those near the fully symmetric representation. Let us consider the natural n -dimensional representation

$$V = [n] \oplus [n-1, 1]. \quad (\text{C.16})$$

This is an n -dimensional representation in which each state $|j\rangle$ is the *position label* of one of the n objects, and the action of any group element is the position $g(j)$ of the permutation g :

$$V(g) |j\rangle = |g(j)\rangle. \quad (\text{C.17})$$

Thus $d_{[n-1,1]} = n-1$.

Naturally, $\chi_V(g) = \sum_j \langle j | V(g) | j \rangle = \text{Fix}(g)$ is the number of objects *untouched* by the permutation g . Since $\chi_V = \chi_{[n]} + \chi_{[n-1,1]}$, it follows that

$$\chi_{[n-1,1]}(g) = \text{Fix}(g) - 1. \quad (\text{C.18})$$

For each fixed j , there are $(n-1)!$ permutations which satisfy the rule that j is moved to another position. This means that $\sum_g \text{Fix}(g) = n!$. On the other hand, the number of group elements for which $\text{Fix}(g) = 0$ is called the number of *derangements* D_n (i.e. no object is found in its original place), and $D_n \sim n!/e$. A q -cycle leaves $(n-q)$ objects invariant and thus has $(n-q)$ fixed points; therefore,

$$\chi_{[n-1,1]}^q = n - q - 1, \implies r_{[n-1,1]}(q) = 1 - \frac{q}{n-1}. \quad (\text{C.19})$$

For large enough n , the contribution to the upper-bound sum of $[n-1, 1]$ is

$$d_{[n-1,1]}^2 r_\rho(q)^{2k} \simeq n^2 \exp\left(-2k \frac{q}{n}\right) \simeq \exp\left(-2\left(k \frac{q}{n} - \log n\right)\right). \quad (\text{C.20})$$

Therefore, this term becomes $O(\epsilon)$ when

$$k = k^* = \frac{n}{q} \log n + O(n). \quad (\text{C.21})$$

This analysis identifies the cutoff scale, but does not by itself prove that all the other representations are negligible.

C.1. Exact treatment of a lower bound

The following argument is the standard fixed-point lower bound for random transpositions, adapted here to uniformly distributed q -cycles. Closely related formulations appear in [172, Chapter 3D]; see also [175] for the random q -cycle walk.

The coupon collector idea can be turned into a rigorous argument that shows that, for $n \rightarrow \infty$ and for any $c > 0$

$$\|p_q^{k_-} - U\| \geq 1 - 6e^{-c} + o(1), \quad k_- = \frac{n}{q}(\log n - c), \quad (\text{C.22})$$

establishing that a time

$$k_- = \frac{n}{q}(\log n - c) \quad (\text{C.23})$$

is unlikely to generate mixing (if $c > 0$).

We begin by stating an equivalent definition of the DS measure

$$\|p - U\| = \max_{A \subset S_n} |p(A) - U(A)|, \quad (\text{C.24})$$

where A is a subset of S_n . In particular, we aim at finding an event A such that $p^{*k_-}(A) \simeq 1$ and $U(A) \simeq 0$. Finding such an event would naturally provide a *lower bound* on the DS measure and therefore prove the unlikelihood of mixing at k_- .

Coupon collector problem as a problem of random variables

We begin by considering the coupon collector problem, i.e. we ask whether a label j has been found after k applications of random q -cycles. Define the following indicator function

$$I_j(k) = \begin{cases} 1, & \text{if label } j \text{ has never appeared in any of the } k \text{ cycles } \{g_\ell\}, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{C.25})$$

Denote also by $X_k = \sum_{j=1}^n I_j(k)$ the number of untouched labels.

By itself X_k is not sufficient to define an event. Indeed, let $\sigma_k = g_k \dots g_1$ be the final permutation and $\text{Fix}(\sigma_k)$ the number of fixed points. If a label j has not been touched in any of the permutations, it will be a fixed point of σ_k , therefore

$$X_k \leq \text{Fix}(\sigma_k). \quad (\text{C.26})$$

We now prove several important properties of X_k :

- **Average.** Since $I_j(k)$ is a boolean variable, $\mathbb{P}[I_j(k) = 1] = \mathbb{E}[I_j(k)]$ and it evaluates to

$$\mathbb{E}[I_j(k)] = \left(1 - \frac{q}{n}\right)^k = a_n^k. \quad (\text{C.27})$$

Thus, it follows that

$$\mathbb{E}[X_k] = m_n = na_n^k. \quad (\text{C.28})$$

- **Variance.** $I_j(k)$ is a Bernoulli variable with probability a_n^k . It follows that

$$\text{Var}[I_j(k)] = a_n^k(1 - a_n^k) \leq a_n^k = \mathbb{E}[I_j(k)]. \quad (\text{C.29})$$

Moreover, any two labels j_1 and j_2 are not independent but correlated. The probability that two variables are not touched is

$$\begin{aligned} \mathbb{P}(j_1 \text{ and } j_2 \text{ not touched}) &= \mathbb{E}[I_{j_1}(k)I_{j_2}(k)] \\ &= \left(\frac{\binom{n-2}{q}}{\binom{n}{q}}\right)^k = \left(\frac{(n-q)(n-q-1)}{n(n-1)}\right)^k = b_n^k. \end{aligned} \quad (\text{C.30})$$

Because $a_n^2 - b_n = \frac{q(n-q)}{n^2(n-1)}$ it follows that

$$\text{Cov}[I_{j_1}(k)I_{j_2}(k)] = b_n^k - a_n^{2k} < 0. \quad (\text{C.31})$$

Thus, such indicator functions are negatively correlated. For the total sum X_k , one finds

$$\begin{aligned} \text{Var}[X_k] &= \sum_j \text{Var}[I_j(k)] + 2 \sum_{j_1 < j_2} \text{Cov}[I_{j_1}(k)I_{j_2}(k)] \\ &\leq \sum_j \text{Var}[I_j(k)] \leq na_n^k = \mathbb{E}[X_k] = m_n. \end{aligned} \quad (\text{C.32})$$

- **Markov's inequality and Chebyshev's inequality.** Since the random variable X_k is non-negative, one may choose an arbitrary value $0 < a \leq 1$ and

decompose

$$\mathbb{E}[X_k] = \mathbb{P}(X_k \geq a)\mathbb{E}[X_k|X_k \geq a] + \mathbb{P}(X_k < a)\mathbb{E}[X_k|X_k < a]. \quad (\text{C.33})$$

Since X_k is non-negative, it can attain values lower than a and therefore $\mathbb{P}(X_k < a)\mathbb{E}[X_k|X_k < a] \geq 0$. Moreover, we have that $\mathbb{P}(X_k \geq a)\mathbb{E}[X_k|X_k \geq a] \geq a\mathbb{P}(X_k \geq a)$. Combining these results leads to Markov's inequality

$$\mathbb{E}[X_k] \geq a\mathbb{P}(X_k \geq a). \quad (\text{C.34})$$

Similarly, we obtain Chebyshev's inequality as a special case of Markov's inequality applied to the probability $|X_k - \mathbb{E}[X_k]| > a$:

$$\mathbb{P}(|X_k - \mathbb{E}[X_k]| > a) = \mathbb{P}(|X_k - \mathbb{E}[X_k]|^2 > a^2) \leq \frac{\text{Var}[X_k]}{a^2}. \quad (\text{C.35})$$

Equipped with this knowledge, we can fix the time $k_- = \frac{n}{q}(\log n - c)$ and show that

$$m_n(k_-) = e^c + o(1). \quad (\text{C.36})$$

Applying Chebyshev's inequality to the value $a = m_n(k_-)/2$ shows that

$$\mathbb{P}(|X_k - m_n| > m_n/2) = \mathbb{P}(X_k < m_n/2) \leq 4 \frac{\text{Var}[X_k]}{m_n^2} \leq \frac{4}{m_n} = 4e^{-c}. \quad (\text{C.37})$$

Conversely, it is very likely that the variable X_k is larger than half its mean when $c > 0$.

We are now ready to construct the event, which is

$$A_{n,c} = \{ \sigma \in S_n \mid \text{Fix}(\sigma) \geq \frac{m_n}{2} \}. \quad (\text{C.38})$$

From Eq. (C.26) it follows that

$$\{ X_k \geq \frac{m_n}{2} \} \subseteq A_{n,c}, \quad (\text{C.39})$$

and therefore

$$p_q^{k_-}(A_{n,c}) \geq \mathbb{P}\left(X_k \geq \frac{m_n}{2}\right) \geq 1 - 4e^{-c} + o(1). \quad (\text{C.40})$$

This shows that the event is likely to occur.

What about the same event under the uniform distribution? The probability that a uniformly chosen permutation does not involve site j is $1/n$ – there are $(n-1)!$

permutations which leave j untouched out of $n!$. Therefore, the average number of fixed points for a permutation drawn from the uniform distribution is

$$\mathbb{E}_U[\text{Fix}] = n \frac{1}{n} = 1. \quad (\text{C.41})$$

Applying Markov's inequality we have

$$U_{q,k_-}(A_{n,c}) = \mathbb{P}\left(\text{Fix} \geq \frac{m_n}{2}\right) \leq \frac{2}{m_n} = 2e^{-c} + o(1). \quad (\text{C.42})$$

These two considerations can be summarized in the following lower bound for the DS distance

$$\|p_q^{k_-} - U\| \geq p_q^{k_-}(A_{n,c}) - U_{q,k_-}(A_{n,c}) \geq 1 - 6e^{-c} + o(1). \quad (\text{C.43})$$

Thus, for any finite $c > 0$, when $n \rightarrow \infty$, the random walk is unlikely to mix. Therefore, setting $c \rightarrow 0$, we have the following bound

$$t_{\text{mix}}(q) \geq \frac{n}{q} \log n. \quad (\text{C.44})$$

Appendix D

IPR of Class Operators

In Section 3.4 and Section 3.5 we argued that class operators can be exactly solved in terms of the characters of S_n . In this Appendix, we shall show that a statistical analysis of such characters can inform about the value of the quantum dephasing time for the unitary QW.

In the main part of the manuscript, we defined QWs through Hamiltonians which coincide with the transition matrix of a CCTRW. We first observe how the dynamical properties of the two dynamics are related to different spectral regions of the operator H .

For the CCTRW, long-time relaxation is governed by the spectral outliers closest to the stationary eigenvalue. For the unitary quantum walk, decoherence is driven by interference among a macroscopic number of energy levels; thus, it is more susceptible to the statistics of the bulk spectrum of the Hamiltonian. This distinction becomes particularly transparent when computing the IPR.

Indeed, we recall that for the class operators H_q , the IPR of the QW reads

$$\text{IPR}(t) = \sum_{g \in S_n} \left| \sum_{\mu} \frac{d_{\mu}}{n!} \chi_{\mu}(g) e^{-iE_q(\mu)t} \right|^4, \quad E_q(\mu) = \frac{\chi_{\mu}^{[q]}}{d_{\mu}}, \quad (\text{D.1})$$

where the sum runs over all the irreps of S_n . The sum can be expanded and reads

$$\begin{aligned} \text{IPR}(t) = & \sum_{\mu_1, \dots, \mu_4} \frac{d_{\mu_1} d_{\mu_2} d_{\mu_3} d_{\mu_4}}{(n!)^4} \left(\sum_{g \in S_n} \chi_{\mu_1}(g) \chi_{\mu_2}(g) \chi_{\mu_3}(g) \chi_{\mu_4}(g) \right) \\ & \times \exp \left(-it \left(\frac{\chi_{\mu_1}^{[q]}}{d_{\mu_1}} - \frac{\chi_{\mu_2}^{[q]}}{d_{\mu_2}} + \frac{\chi_{\mu_3}^{[q]}}{d_{\mu_3}} - \frac{\chi_{\mu_4}^{[q]}}{d_{\mu_4}} \right) \right). \end{aligned} \quad (\text{D.2})$$

The dependence on the group element g can be eliminated by introducing the conventional inner product on the group [161]:

$$\langle f, h \rangle_{S_n} = \frac{1}{n!} \sum_{g \in S_n} f(g) h^*(g). \quad (\text{D.3})$$

Then $\sum_{g \in S_n} \chi_{\mu_1}(g) \chi_{\mu_2}(g) \chi_{\mu_3}(g) \chi_{\mu_4}(g) = n! \langle \chi_{\mu_1} \chi_{\mu_2}, \chi_{\mu_3} \chi_{\mu_4} \rangle_{S_n}$. However, the individual functions in the inner product admit a representation in terms of the Kronecker coefficients of S_n :

$$\chi_{\mu}(g) \chi_{\nu}(g) = \sum_{\rho} g_{\mu\nu}^{\rho} \chi_{\rho}(g), \quad (\text{D.4})$$

and therefore the sum can be written compactly as

$$\frac{1}{n!} \sum_{g \in S_n} \chi_{\mu_1}(g) \chi_{\mu_2}(g) \chi_{\mu_3}(g) \chi_{\mu_4}(g) = \sum_{\rho} g_{\mu_1\mu_2}^{\rho} g_{\mu_3\mu_4}^{\rho}, \quad (\text{D.5})$$

after applying the character orthogonality relations [161].

Define Ω as the set of distinct energy differences. For each value Δ , we have

$$\Omega_{\Delta} = \{ \mu_1, \mu_2, \mu_3, \mu_4 \mid E_{\mu_1} - E_{\mu_2} + E_{\mu_3} - E_{\mu_4} = \Delta \}. \quad (\text{D.6})$$

The IPR can then be rewritten as

$$\begin{aligned} \text{IPR}(t) &= \frac{1}{n!} \sum_{\Delta \in \Omega} c_{\Delta} \cos(\Delta t), \\ c_{\Delta} &= \sum_{\mu_1, \dots, \mu_4} \frac{d_{\mu_1} d_{\mu_2} d_{\mu_3} d_{\mu_4}}{(n!)^2} \sum_{\rho} g_{\mu_1\mu_2}^{\rho} g_{\mu_3\mu_4}^{\rho}, \\ \Delta &= \frac{\chi_{\mu_1}^{[q]}}{d_{\mu_1}} - \frac{\chi_{\mu_2}^{[q]}}{d_{\mu_2}} + \frac{\chi_{\mu_3}^{[q]}}{d_{\mu_3}} - \frac{\chi_{\mu_4}^{[q]}}{d_{\mu_4}}. \end{aligned} \quad (\text{D.7})$$

In terms of Δ , large Δ 's correspond to fast oscillatory modes, while smaller Δ 's are responsible to long-time dephasing. To quantify such separation of time-scales, one can study, instead of the instantaneous IPR, its smearing

$$\langle \text{IPR}(t) \rangle_{\tau} = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} dT' \text{IPR}_2(T') = \frac{1}{n!} \sum_{\Delta \in \Omega} c_{\Delta} \text{sinc}\left(\frac{\Delta\tau}{2}\right) \cos(\Delta t), \quad (\text{D.8})$$

where $\text{sinc}(x) = \sin(x)/x$. Setting $\tau = 1$ and $T = k \in \{0, 1, 2, \dots\}$ one reduces the continuous-time dynamics of the QW to a discrete-time one. In practice, one applies random measurements on an interval centered at integer t with unit width. The sinc function offers a smooth regularization, suppressing oscillations at frequencies

$\Delta = O(1)$ while preserving slower modes.

For H_q , the IPR is the sum of two contributions: the LTA $\overline{\text{IPR}} = c_0/n!$, which consists of the irreps satisfying $E_{\mu_1} + E_{\mu_3} = E_{\mu_2} + E_{\mu_4}$ and the off-diagonal $\widetilde{\text{IPR}}(t)$ one. The dephasing time t_{deph} can be computed through the first zero of $\widetilde{\text{IPR}}(t)$, i.e. the first-hitting time of the LTA. We now show how the spectral analysis of the modes allows to estimate the dephasing time Eq. (3.90) of the main text.

The treatment of the IPR simplifies in the thermodynamic limit, $n \rightarrow \infty$. There, irreps ρ can be treated as random variables distributed according to the Plancherel measure $d\mu[\rho] = d_\rho^2/n!$ and t_U can be extracted with statistical arguments as follows. The Δ entering IPR can be divided in modes with a definite finite- n scaling:

- **Standing modes**, $\Delta = O(1)$. A particular representative is $\Delta = 0$, e.g. when $\rho = \rho'$ and $\sigma = \sigma'$. These contribute to the LTA, which is time-independent.
- **Bulk modes**. These modes are typical and characterize the *bulk* of the Plancherel measure: diagrams whose dimension is $d_\rho \sim \sqrt{n!/\mathcal{P}(n)}$. The particular shape of the distribution was derived by Kerov [223]. The q -class characters follow a central-limit theorem [224]: $\frac{\chi_\rho^{[q]}}{d_\rho} \xrightarrow{L \rightarrow \infty} \mathcal{N}(0, qn^{-q})$. Hence, typical gaps are of order $O(1/\sqrt{n^q})$ for H_q .
- **Outlier modes**. For q -class PH, these occur when the representations are close to the trivial one and populate the tails of the Plancherel measure. For the particular case of $\rho = [n]$ and $\rho' = [n-1, 1]$, their gap is $\delta_{\rho\rho'} = q/(n-1) = O(1/n)$. However, outlier modes become superexponentially suppressed in the Plancherel measure. Indeed, from the hook formula, their dimension is polynomial rather than factorial (bulk).

The dephasing time obtained from the statistical behavior of eigenvalues in the bulk gives the quantum dephasing time for class operators, as defined in Section 3.5. However, it neglects the structure of the eigenvalue coefficients c_Δ . A statistical approach to the evaluation of the IPR, accounting for the correlation between eigenvectors and eigenvalues, is an interesting research direction of interest in the study of nonequilibrium properties in integrable and chaotic quantum many-body systems.

Recurrence Time. To conclude this Appendix, we briefly discuss an important property of such class operators, i.e. the period of the unitary generator $e^{-iH_q t}$. The energy associated to each irrep ρ is *rational*: $E_\rho^{[q]} = \chi_\rho^{[q]}/d_\rho$. This implies that the

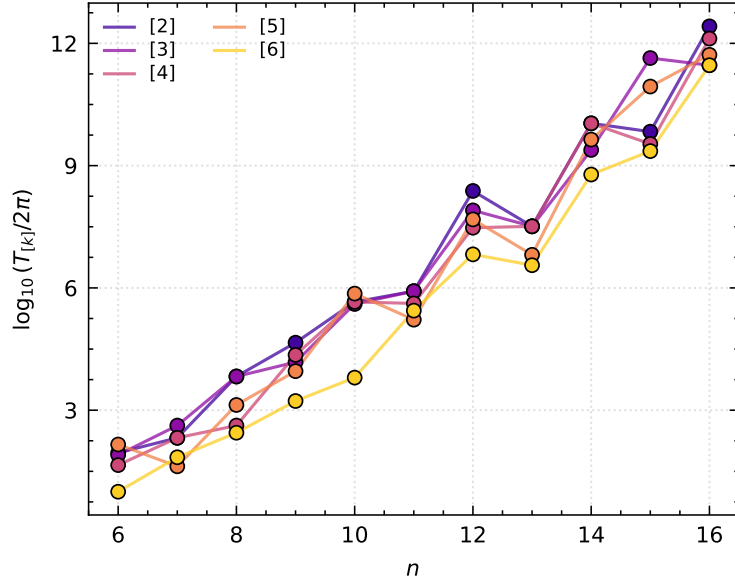


Figure D.1: Dependence of the period $T_{[k]}(L)$ of k -class PH with the number of sites n . This period likely scales as $T_{[k]}(n) = O(e^n)$.

unitary time evolution is periodic of time $T_{[q]}(n)$. The period is a function of the characters, and can be found as follows:

- first, the irreps with nonzero character are selected. Let this set be denoted as $\{\rho\}^* = \{\rho \text{ s. t. } \chi_\rho^* \neq 0\}$. Indeed, if a character is zero, so is its associated energy. Zero energies do not generate interference patterns.
- Second, we compute the least common multiple (LCM) of the d_ρ 's and the greatest common divisor (GCD) of the $\chi_\rho^{[k]}$ in $\{\rho\}^*$.
- The period is finally given by

$$T_{[q]}(n) = 2\pi \frac{\text{LCM}(d_\rho^*)}{\text{GCD}(\chi_\rho^{[q],*})}. \quad (\text{D.9})$$

In Fig. D.1 we compute such period for $n \in [6, 16]$, which is likely of order $T_{[q]}(n) = O(e^n)$. This is confirmed by seemingly straight lines in the log-plot. This result is consistent with the law $\sum_\rho d_\rho^2 = n!$ in the regular representation, meaning that typical bulk irreps have dimension $d_\rho \sim n!/\mathcal{P}(n)$, where $\mathcal{P}(n)$ is the number of partitions of n , coincident with the number of irreducible representations. We also observe that for moderate systems sizes $n \geq 10$, the order of magnitude of such recurrence time is 10^5 .

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