SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Boundary extensions of symmetric spaces in equivariant KK-theory

Candidate:
Torstein Ulsnaes

Advisors:
Prof.dr. A. Doelman
Prof.dr. L. Dabrowski
Co-advisor:
Prof. Bram Mesland

Academic Year 2023-24

Preface

This thesis is the product of a four year PhD at SISSA (Trieste) and Leiden university under the supervision of Prof. Bram Mesland at the university of Leiden.

Acknowledgements

I would like to thank my advisor, Prof. Bram Mesland for help and guidance during the years of my PhD, Prof. Ludwik Dabrowski and Prof. Antonio Lerario for guiding me through the formalities of the PhD, and Prof. Rafael Torres for and helping me get in contact with Bram.

I'm grateful to Prof. Francesca Arici, Prof. Adam Rennie and Prof. Haluk Sengun for their helpful feedback and comments on the thesis and much more. I'm also grateful to Prof. Nigel Higson for helpful discussions.

Finally, I'd like to thank the many PhD's and postdocs at Leiden and SISSA which have enriched my PhD journey.

Contents

Introduction

Classical harmonic analysis gives a way to associate to any continuous function on the circle S^1 a harmonic function on the Poincaré disk. This is done by integrating f against the Poisson kernel $P : S^1 \times \mathbb{D} \to \mathbb{R}$, to produce a function $F_f \in C(\mathbb{D})$ given by

$$F_f(x) = \int_{S^1} P(x,v) f(v) dv.$$

The function F_f is called the Poisson integral of f. Any bounded harmonic function on $\mathbb{D}$ extending continuously to $\partial\mathbb{D} = S^1$ is the Poisson integral of a unique continuous function on S^1 ([53][Section 6.3]).

The function F_f extends to a continuous function on the closed ball $\overline{\mathbb{D}}$, which restricts to f on $\partial\mathbb{D} \simeq S^1$. From these observations we may deduce that that any continuous function on the closed ball $f \in C(\overline{\mathbb{D}})$ can be decomposed uniquely into a sum

$$f = f_0 + f_h$$

where $f_0 \in C_0(\mathbb{D})$ is a bounded continuous function on $\mathbb{D}$ vanishing at on the boundary S^1, while f_h is a bounded harmonic function on $\mathbb{D}$ extending continuously to the boundary given by the Poisson integral of $f|_{\partial\mathbb{D}}$. This in turn implies that the assignment $s : C(\partial\mathbb{D}) \to C(\overline{\mathbb{D}})$, which sends a function $f \in C(S^1)$ to its Poisson integral $F_f \in C(\overline{\mathbb{D}})$, determines a right inverse of the quotient map

$$q : C(\overline{\mathbb{D}}) \to C(\partial\overline{\mathbb{D}}) \qquad q(f) = f|_{\partial\overline{\mathbb{D}}}$$

given by restricting a function to the boundary. With G the isometry group of $\mathbb{D}$, the map s is what is called a semisplitting of the extension

$$0 \to C_0(\mathbb{D}) \to C(\overline{\mathbb{D}}) \to C(\partial\overline{\mathbb{D}}) \to 0.$$

The map s also commutes with the natural action of G on $\overline{\mathbb{D}}$. The same procedure produces G-equivariant semisplittings of extensions of higher dimensional real hyperbolic spaces

$$0 \to C_0(\mathbb{H}^n) \to C(\overline{\mathbb{H}^n}) \to C(\partial\overline{\mathbb{H}^n}) \to 0$$

and it is shown in [36, Theorem 2.4] how these G-equivariant semisplittings give us a class in $\mathrm{KK}_1^G(C(S^1), C_0(\mathbb{D}))$ representing the extension, where $\mathrm{KK}_1^G(C(S^1), C_0(\mathbb{D}))$ is the equivariant Kasparov group of the pair $(C(S^{n-1}), C_0(\mathbb{H}^n))$.

The spaces $\mathbb{H}^n$ are examples of symmetric spaces on noncompact type of rank 1. It soon became clear to the author that the construction of such an equivariant semisplitting could be done for any extension

$$0 \to C_0(X) \to C(\overline{X}) \to C(X(\infty)) \to 0$$

where X is a symmetric space of noncompact type of rank 1 and $\overline{X} = X \cup X(\infty)$ is a compactification of X called the geodesic compactification. The splitting $s : C(X(\infty)) \to C(\overline{X})$ is now constructed using so called Patterson–Sullivan densities on $X(\infty)$, which is a special family of measures $(\mu_x)_{x \in X}$ indexed by the points in X, and the splitting takes the form

$$s(f)(x) = \int_{X(\infty} f(v) d\mu_x(v) \qquad x \in X.$$

For a symmetric space X of rank 2 or higher, Albuquerque ([1]) used an idea of Patterson to produce a similar family of measures $(\mu_x)_{x \in X}$ on $X(\infty)$ and it is natural to ask whether one could use these densities to produce a splitting

$$s : C(X(\infty)) \to C(\overline{X}) \qquad s(f)(x) = \int_{x \in X(\infty)} f(v) d\mu_x(v), \quad (x \in X).$$

This turns out to be impossible. Indeed the function $s(f)$ cannot be extend continuously to the geodesic boundary for all functions $f \in C(X(\infty))$. We are thus forced to looking at other compactifications of X where the function $s(f)$ for $f \in C(X(\infty))$ could be extended. This eventually leads to the main result of this thesis (Theorem 4.6), which, for a symmetric space X of noncompact type. gives an extension

$$0 \to C_0(Y) \to C(\overline{X}^F) \to C_0(\partial_F X) \to 0$$

that can be split by the map s determined by

$$s(f)(x) = \int_{X(\infty)} f(v) d\mu_x(v).$$

The compactification $\overline{X}^F$ is called the (maximal) Furstenberg compactification of X, $\partial_F X \subset \partial X^F$ is the Furstenberg boundary of X (which is not the boundary of the Furstenberg compactification) and $Y = \overline{X}^F \backslash \partial_F X$. Theorem 4.6 then gives us a KK_G^1-cycle corresponding to the extension. In the case where X has rank 1, we have $\overline{X}^F = \overline{X} = X \cup X(\infty)$ and $\partial_F X = X(\infty)$ so this is indeed a generalization of the original example of [36, Theorem 2.4].

Organization of the thesis

The thesis consists of four chapters. In Chapter I we introduce the background material needed for the subsequent chapters. We cover the very basics of C*-algebras, K-theory and the geodesic compactification of Hadamard manifolds. Some less basic concepts are covered in the later part of the chapter, like equivariant KK-theory and equivariant extensions of C*-algebras. We end Chapter I with the main example. The rest of the thesis will be dedicated to extending this example to a wider class of spaces.

In Chapter II, we introduce symmetric and locally symmetric Riemannian spaces. We look at a way to compactify non-compact symmetric spaces called the geodesic compactification and study its properties. The main takeaway from Chapter II is the existence of a certain map

$$\mu : X \to M_1(X(\infty)) \qquad x \mapsto \mu_x$$

from a symmetric space X to the probability measures on the boundary of the geodesic compactification of X. These maps, called conformal densities, allow us to define a Poisson-like integral on symmetric spaces of non-compact type that have many similarities with the ordinary Poisson integral from harmonic analysis. More explicitly, with X our symmetric space and $X(\infty)$ the boundary of the geodesic compactification of X, we can define a map $C(X(\infty)) \to C_b(X)$ by sending an $f \in C(X(\infty))$ to the function

$$F_f(x) = \int_{X(\infty)} f(v)d\mu_x(v) \qquad x \in X.$$

We end the Chapter with an example of the use of these Poisson-like integrals and show that unlike in classical harmonic analysis, the function F_f does not extend to a function on the whole geodesic compactification.

In Chapter III we introduce three alternative compactifications of a symmetric space X that are (in case X is of noncompact type) all isomorphic to one another. These are the "smallest" compactifications on which the function F_f defined in Chapter II does extend to the whole compactified space. The important feature of these compactifications is that a sequence $x_i \in X$ converges to a boundary point $x_\infty \in \partial X$ if and only if μ_{x_i} converges weakly to some measure in $M_1(X(\infty))$, where μ_x is now a specific choice of conformal density called the Patterson–Sullivan density.

In Chapter IV we finally return to the example at the end of Chapter I, and construct in a similar way a class in KK_1^G representing a certain equivariant extension of C*-algebras

$$0 \to C_0(Y) \to C(\overline{X}^F) \to C(\partial_F X) \to 0$$

We show how an equivariant semisplitting of any extension can be used to construct a concrete realization of an equivariant Kasparov module representing the class of the extension in KK_1^G. We use the Furstenberg compactification $\overline{X}^F$ introduced in Chapter III as this is the compactification where the Poisson-like integral extends to a continuous function on the whole compactification.

Chapter 1

Preliminaries

This chapter is a gentle introduction the theory we will need for the subsequent part of the thesis. The reader familiar with the basics of C*-algebras and K-theory can safely skip ahead to Section 1.7 where we define the geodesic compactification and introduce a fundamental example.

Unless stated otherwise, all algebras will be assumed to be over the complex numbers.

1.1 C*-algebras

Let A be an algebra over $\mathbb{C}$. By an involution on A we mean a map

$$* : A \to A \qquad a \mapsto *(a) := a^*$$

satisfying the following properties for any $a, b \in A$ and $\lambda \in \mathbb{C}$

- $(a+b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$
- $(\lambda a)^* = \overline{\lambda} a^*$.

Definition 1.1. A (complex) pre-C*-algebra is a (complex) algebra A with an involution $* : A \to A$, and a submultiplicative norm (meaning $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$) satisfying

$$\|a^* a\| = \|a\|^2 \quad \forall\, a \in A \tag{1.1}$$

A (semi-)norm satisfying equation (1.1) will be called a C*(-semi-)-norm. A C*-algebra is a complete pre-C*-algebra. We call a C*-algebra unital if there is an element $1 \in A$ such that

$$a1 = 1a = a$$

for all $a \in A$. A C*-algebra A is called commutative if for all $a_1, a_2 \in A$ we have

$$a_1 a_2 = a_2 a_1.$$

We call the innocuous looking equation (1.1) the C*-identity. It has profound consequences for the theory of C*-algebras, which separates it from the theory of, say, Banach *-algebras (complete normed algebras with an isometric involution).

A *morphism or *-homomorphism* $\phi : A \to B$ between two C*-algebras is an algebra homomorphisms which commutes with the involution, i.e.

$$\phi(a^*) = \phi(a)^*.$$

If A and B are unital C*-algebras, then a morphism

$$\phi : A \to B$$

is called unital if it maps the unit in A to the unit in B.

Example 1.2. A *representation* of a C*-algebra A is a morphism $\pi : A \to B(H)$ for some Hilbert space H. The representation is said to be *faithful* if π is injective and *non-degenerate* if $\pi(A)H$ is dense in H.

The prototypical example of a C*-algebra is the algebra of bounded linear maps on a fixed complex Hilbert space H, denoted $B(H)$, with the supremum norm given on $T \in B(H)$ by

$$\|T\| := \sup_{\|x\| \leq 1} \|T(x)\|.$$

The involution in $B(H)$ is given by sending a map to its adjoint.

It is convenient to have a unit in the C*-algebra. The next example gives the to most common ways to add a unit to a non-unital C*-algebra.

Example 1.3. Let A be a nonunital C*-algebra. The *unitization* of A, denoted A^+, is the universal C*-algebra satisfying the following property:

It is the smallest unital C*-algebra containing A as an ideal in the sense that any morphism $f : A \to B$ from A to a unital C*-algebra B lifts to a unique unital morphism $\hat{f} : A^+ \to B$.

The *multiplier algebra* of A, denoted $M(A)$, is the universal C*-algebra satisfying the following property:

It is the largest C*-algebra containing A as an essential ideal (Definition 1.9). Equivalently, let B be any C*-algebra containing A as an essential ideal. Then there is a unique map $B \to M(A)$ such that the inclusion $\iota_A : A \to M(A)$ factors through

$$A \to B \to M(A).$$

We will see concrete realisations of these two algebras later when we define Hilbert C*-modules in Section 1.4, but let us mention that for a locally compact (but noncompact) Hausdorff space X we have

$$C_0(X)^+ = C(X \cup \{\infty\}),$$

where $X \cup \{\infty\}$ is the one-point compactification or the Alexandroff compactification of X and

$$M(C_0(X)) = C(\beta X)$$

with βX the Stone–Cech compactification of X.

The next two results are some of the many consequences of equation (1.1).

Theorem 1.4 ((Gelfand-Naimark) [41, Theorem 3.4.1])**.** *Any C*-algebra is a C*-subalgebra of $B(H)$ for some Hilbert space H.*

Proposition 1.5 ([41] Cor. 2.1.2)**.** *Let A be an algebra with an involution $* : A \to A$. Then there is at most one norm on A, making it a C*-algebra.*

Example 1.6. Let A be any C*-algebra. Then define a norm on the n'fold direct sum of A (treated as a vector space over $\mathbb{C}$)

$$\underbrace{A \oplus \cdots \oplus A}_{n}$$

by

$$\|(b_1, \ldots, b_n)\|_2 := \|b_1^* b_1 + \cdots + b_n^* b_n^2\|^{1/2}.$$

Now define the algebra of A-valued $n \times n$-matrices $M_n(A)$. This is an algebra over $\mathbb{C}$ with an involution given by the complex conjugate

$$(a_{i,j})^* = (a_{j,i}^*),$$

and product given by the usual matrix multiplication:

$$(a_{i,j})(b_{ij}) = (\sum_{k=1}^{n} a_{ik} b_{kj}).$$

The norm

$$\|(a_{i,j})\| := \sup_{\|(b_1,\ldots,b_n)\|_2 = 1} \left\| (\sum_{j=1}^{n} (a_{1,j} b_j), \ldots, \sum_{j=1}^{n} (a_{n,j} b_j) \right\|_2$$

determines a C*-norm on $M_n(A)$ making it into a C*-algebra, hence it is the unique C*-norm on $M_n(A)$.

Any morphism $\phi : A \to B$ induces a map

$$\phi : M_n(A) \to M_n(B) \qquad (a_{i,j}) \mapsto \phi(a_{i,j}) \tag{1.2}$$

which can be shown to be a *-homomorphism of C*-algebras.

Proposition 1.5 tells us that a C*-norm on A is uniquely determined by the algebraic structure of A. This gives a close connection between the algebraic properties of A and the topological properties of A, which makes it possible to translate theorems of algebra into the language of C*-algebras. For example, here is a C*-analogue of the classical Wedderburn theorem[1]

Example 1.7. Any finite dimensional C*-algebra A (i.e. $A \subset B(H)$ for a finite dimensional Hilbert space H), is isomorphic to a direct sum of full matrix algebras

$$A \simeq \bigoplus_{i=n}^{k} M_{n_i}(\mathbb{C}).$$

Example 1.8. Let us give another important example of C*-algebras, namely commutative C*-algebras $C_0(X)$, of complex-valued functions on a locally compact Hausdorff space X vanishing at infinity. Recall that a function f on a locally compact space X is said to vanish at infinity if for any $\epsilon > 0$ there is a compact subset $K_\epsilon \subset X$ such that

$$|f(x)| < \epsilon \qquad \forall\, x \notin K_\epsilon.$$

We can multiply and add two functions $f, h \in C_0(X)$ pointwise as follows

$$(fh)(x) = f(x)h(x) \qquad (f+h)(x) = f(x) + h(x), \qquad x \in X.$$

Similarly we define scalar multiplication by $(\lambda f)(x) = \lambda f(x) \qquad x \in X, \lambda \in \mathbb{C}$. The norm on $C_0(X)$ making it a C*-algebra, called the supremum norm, is defined by

$$\|f\| := \sup_{x\in X} |f(x)|. \tag{1.3}$$

Note that $C_0(X)$ is unital if and only if X is compact, in which case $C_0(X) = C(X)$, with unit given by the constant function $X \ni x \mapsto 1 \in \mathbb{C}$.

If X is any topological space, then the algebra $C_0(X)$ can be defined just as in Example 1.8 and does produce a C*-algebra with respect to the supremum norm (eq. (1.3)). The reason we restrict ourselves to locally compact Hausdorff spaces is that given a commutative C*-algebra $C_0(X)$ there always exists a (unique up to homeomorphism) locally compact Hausdorff space Y and an isomorphism of C*-algebras

[1] Another example is Kadison's transitivity theorem which generalizes Jacobson's transitivity theorem, but we will not need it here.

$$C_0(Y) \simeq C_0(X).$$

Given a commutative C*-algebra A, we can always write $A = C_0(Y)$ for some locally compact Hausdorff space Y. For this reason, from now on X will always denote a locally compact Hausdorff space. The space Y is called the Gelfand dual (or the spectrum) of A which, as a set, consists of all *-homomorphisms

$$C_0(Y) \to \mathbb{C}.$$

The assignment

$$Y \mapsto C_0(Y)$$

determines a contravariant functor from the category of locally compact Hausdorff spaces with morphisms given by proper continuous maps, to the category of commutative C*-algebras, by sending a proper map $f : X \to X'$ to the map

$$C_0(X') \to C_0(X) \qquad h \mapsto h \circ f \qquad h \in C_0(X').$$

Note that we need properness of f to ensure that $f \circ h$ vanishes at infinity on X. The Gelfand transform determines a contravariant equivalence of categories between the unital commutative C*-algebras with unital morphisms and the category of compact Hausdorff spaces. In the non-unital case we need to take care of what morphisms we allow [2]. If $f : X \to Y$ is a proper map, then the induced map

$$C_0(Y) \to C_0(X) \qquad h \mapsto h \circ f$$

sends approximate units in $C_0(Y)$ to approximate units in $C_0(X)$ [3], so we cannot find a map $X \times Y \to X$ corresponding to the inclusion into the first factor

$$C(X) \to C(X) \oplus C(Y) = C(X \sqcup Y).$$

Definition 1.9. Let A be a C*-algebra. A closed $*$-invariant subalgebra of $I \subset A$ is called a C*-subalgebra of A. If $I \subset A$ is a C*-subalgebra for which

$$aI, Ia \subset I \qquad \forall\, a \in A$$

then I is called an ideal of A.

An ideal $I \subset A$ is called essential, if for any other ideal $J \subset A$, we have

$$I \cap J = \{0\} \Rightarrow J = 0.$$

[2]See for instance [39, p. 9] for a category which is equivalent to the commutative C*-algebra category.

[3]An approximate unit of a C*-algebra A is a net e_α of positive elements such that for all $x \in A$ we have $\|e_i x - x\| \to 0$ as $i \to \infty$.

An example of an essential ideal is $C_0((0,1))$ inside $C([0,1])$. A typical non-example is the following: Let $I \subset A$ be an ideal of A and B any (non-trivial) C*-algebra, then $I \oplus \{0\} \subset A \oplus B$ is a non-essential ideal.

Let us collect some basic properties of C*-algebras.

Proposition 1.10 ([41] Chapter 2)**.** *Let A, B be C*-algebras, and $\phi : A \to B$ a morphism of C*-algebras. Then*

1. *The map ϕ is contractive, meaning $\|\phi(a)\| \leq \|a\|$;*
2. $\mathrm{Ker}(\phi)$ *is an ideal in A;*
3. *The inclusion $\phi(A) \subset B$ is a closed C*-subalgebra of B;*
4. *There is an isomorphism*
$$A/\mathrm{Ker}(\phi) \simeq \phi(A);$$
5. *If A and B are two C*-algebras, then their direct sum $A \oplus B$ is a C*-algebra with respect to the operations*
$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2)\\ (a_1, b_1)(a_2, b_2) &= (a_1 a_2, b_1 b_2)\\ (a_1, b_1)^* &= (a_1^*, b_1^*)\\ \lambda(a_1, b_1) &= (\lambda a_1, \lambda b_1)\end{aligned}$$
for $(a_i, b_i) \in A \oplus B$ and $\lambda \in \mathbb{C}$, and norm given by $\|(a,b)\| = \max(\|a\|, \|b\|)$.

Tensor products of C*-algebras are more subtle as there are several choices of norms on the algebraic tensor product of two C*-algebras. We will not delve into the theory of tensor products for C*-algebras here, but refer the interested reader to the very thorough exposition in [11]. Let us just define one norm, which in some interesting cases turns out to be the only pre-C*-norm on the algebraic tensor product.

We denote by $A_1 \odot A_2$ the algebraic tensor product of two C*-algebras A_1, A_2. This is the linear span of the simple tensors $(a_1 \odot a_2)$ with involution
$$(a_1 \odot a_2)^* = a_1^* \odot a_2^*,$$
and product
$$(a_1 \odot a_2)(a_1' \odot a_2') = a_1 a_1' \odot a_2 a_2'.$$
The tensor product of two Hilbert spaces $H_1 \otimes H_2$ is a Hilbert space in its own right with respect to the inner product
$$\langle v \otimes w, v' \otimes w' \rangle := \langle v, v' \rangle \langle w, w' \rangle \qquad v, v' \in H_1, w, w' \in H_2.$$

Definition 1.11. Let A_1 and A_2 be two C*-algebras with faithful representations $\pi_i : A \to B(H_i)$. Then we have an injective *-preserving algebra homomorphism

$$\pi_1 \otimes \pi_2 : A_1 \odot A_2 \to B(H_1 \otimes H_2)$$

given by

$$\pi_1 \otimes \pi_2(a \odot a')(h \otimes h') := \pi_1(a)(h) \otimes \pi_2(a')(h').$$

The *minimal* or *spatial* tensor product, denoted $A_1 \otimes A_2$, is the completion of $A_1 \odot A_2$ in $B(H_1 \otimes H_2)$.

Since $A_1 \odot A_2$ is a subalgebra of $B(H \otimes H)$ closed under involution, it is clear that the minimal tensor product norm is a pre-C*-norm on the algebraic tensor product. It remains to be verified that the norm is independent of choice of faithful representations π_i. We refer the reader to [11, Chap. 3] for the proof of this fact. In case one of the algebras is commutative, this is the only norm on the algebraic tensor product making its completion a C*-algebra.

We will also need the following definition:

Definition 1.12. A short exact sequence of C*-algebras $0 \to B \to E \to A \to 0$ is called an extension of B by A. Note that B then must be isomorphic to an ideal of E.

The theory of extensions of C*-algebras will be covered in some detail later in Section 1.6. For now, let us give a simple example: For a commutative C*-algebra (ref. Ex 1.8) if $V \subset X$ is a closed subset, we can define a C*-subalgebra given by

$$I(V) := \{f \in C_0(X) \mid f|_V = 0\}$$

which is easily seen to be an ideal of $C_0(X)$. All ideals of $C_0(X)$ arise in this way for some closed subset of X. Letting $O = X \backslash V$ be the complement of V then it is easy to show we have an isomorphism

$$I(V) = C_0(O) \subset C_0(X)$$

and the quotient algebra is given by

$$C_0(X)/C_0(O) \simeq C_0(X \backslash O) = C_0(V).$$

This gives us an extension of C*-algebras

$$0 \to C_0(O) \to C_0(X) \to C_0(V) \to 0. \tag{1.4}$$

For example if $\overline{M}$ is a manifold with boundary ∂M and interior M, we get an extension

$$0 \to C_0(M) \to C_0(\overline{M}) \to C_0(\partial M) \to 0.$$

An *operator system* is a closed and $*$-invariant subspace $F \subset A$ of a unital C*-algebra A, such that $1_A \in F$.

Definition 1.13 ([11, Definition 3.7.5])**.** Let E be a unital C*-algebra. An extension

$$0 \to B \to E \xrightarrow{\pi} A \to 0$$

is called locally split if for every finite dimensional operator system $F \subset A$ there exists a unital completely positive map $\sigma : F \to E$ such that $\pi \circ \sigma = id_F$.

Definition 1.14. A C*-algebra A is said to be *nuclear* if for any C*-algebra B there is a unique pre-C*-norm on $A \odot B$.

Definition 1.15. A C*-algebra A is called exact if for any exact sequence of C*-algebras

$$0 \to I \xrightarrow{\iota} E \xrightarrow{p} B \to 0$$

the sequence

$$0 \to I \otimes A \xrightarrow{\iota \otimes id} E \otimes A \xrightarrow{p \otimes id} B \otimes A \to 0$$

is exact, where $\iota \otimes id$ and $p \otimes id$ are the maps given on simple tensors in $I \odot A$ and $B \odot A$ by

$$\iota \otimes id(m \odot a) = \iota(a) \odot a \quad \text{and} \quad p \otimes id(b \odot a) = p(b) \odot a$$

respectively, and extended to the C*-tensor product.

We mention without proof that any nuclear C*-algebra is exact, but the converse is in general not true. One family of examples of exact C*-algebras which are not nuclear are the reduced group C*-algebra of discrete non-amenable subgroups of $GL_n(\mathbb{C})$ (see Theorem 1.37). Further, the following example shows that not all nuclear C*-algebras are commutative.

Example 1.16. Let H be a Hilbert space. A finite rank operator $F : H \to H$ is a linear map of the form

$$F(x) = \sum_{n=1}^{k} y_i \langle x_i, x \rangle$$

for some finite set $x_i, y_i \in H$.

Let $B_{\text{fin}}(H)$ denote the collection of all finite rank operators on H and let $\mathbb{K} := \mathbb{K}(H)$ denote the closure of $B_{\text{fin}}(H)$. The algebra $\mathbb{K}$ can be shown to be nuclear C*-algebra sitting in $B(H)$ as an essential ideal.

Definition 1.17. An operator $T \in \mathbb{K}(H)$ defined in Example 1.16 is called a compact operator. A compact operator is a bounded linear map $T : H \to H$ satisfying any of the following equivalent conditions ([41, Chap. 2.4])

1. For any bounded $U \subset H$, $T(U)$ has compact closure;

2. T is the norm limit of finite rank operators

3. T is in the norm limit of finite sums of rank 1 projections

4. $T \in \bigcap_{I \subset B(H)} I$ where I runs over all ideals of $B(H)$ containing $B_{\text{fin}}(H)$.

Let $\mathbb{K} = \mathbb{K}(l^2(\mathbb{N}))$ be the C*-algebra of compact operators on a separable infinite dimensional Hilbert space (i.e. a Hilbert space with a countable infinite orthonormal basis).

Definition 1.18. A C*-algebra A is called stable if $A \otimes \mathbb{K} \simeq A$. For any C*-algebra B, the C*-algebra $B \otimes \mathbb{K}$ is called the stabilization of B.

The stabilization $B \otimes \mathbb{K}$ is stable, since the tensor product is associative and $\mathbb{K} \otimes \mathbb{K} \simeq \mathbb{K}$.

Positive elements If $A = C(X)$ is a commutative C*-algebra, then the subset $A_+ = \{f \in C(X) \mid f \geq 0\}$ of real-valued positive functions on X has the following properties:

1. Each element $f \in A_+$ has a unique square root in A_+;

2. The set A_+ is a cone in A, meaning it is closed under addition and multiplication by $\mathbb{R}_{\geq 0}$;

3. Every element in A_+ is of the form $|f|^2 = \overline{f} f$ for some function $f \in A$;

4. Every $f \in A$ can be written as

$$f = f_1 - f_2 + i(f_3 - f_4)$$

for some $f_i \in A_+$.

The elements in A_+ are called positive elements of A. For a general C*-algebra we have a similar definition:

Definition 1.19. Let A be any C*-algebra and define

$$A_+ := \{a^* a \mid a \in A\}.$$

The elements in A_+ are called positive, and A_+ is called the positive cone of A.

We have:

Proposition 1.20 ([41] Sec. 2.2)**.** *The positive cone satisfies the following properties*

1. *The set A_+ is a cone in A, meaning it is closed under sums and multiplication by $\mathbb{R}_{\geq 0}$*

2. *Every $a \in A_+$ has a unique square root in A_+, i.e. there is an element $b \in A_+$ such that $b^2 = a$;*

3. *every $a \in A$ can be written as*

$$a = a_1 - a_2 + i(a_3 - a_4)$$

for $a_i \in A_+$.

Now if $\phi : A \to B$ is a morphism of C*-algebras then

$$\phi(a^*a) = \phi(a)^*\phi(a)$$

hence $\phi(A_+) \subset B_+$. Similarly, the induced maps on matrix algebras

$$\phi : M_n(A) \to M_n(B) \qquad \phi[(a_ij)] = [\phi(a_{ij})]$$

also preserves positive elements (being themselves morphisms of C*-algebras).

The following definition gives a weakening of the notion of morphisms of C*-algebras, that is useful in applications (see Example 1.49)

Definition 1.21. A bounded linear map $\phi : A \to B$ between C*-algebras is called positive if

$$\phi(A_+) \subset B_+.$$

It is called contractive if $\|\phi(a)\| \leq \|a\|$ for all $a \in A$.

A positive map is called completely positive if the induced map

$$\phi : M_n(A) \to M_n(B) \qquad [a_{ij}] \mapsto [\phi(a_{ij})],$$

is positive for all $n \in \mathbb{N}$. Similarly it is called a completely positive contractive map if the induced maps

$$\phi : M_n(A) \to M_n(B) \qquad [a_{ij}] \mapsto [\phi(a_{ij})]$$

are positive and contractive for all $n \in \mathbb{N}$.

1.2 Group actions and crossed products

Throughout this thesis, all groups will be assumed to be locally compact and Hausdorff topological groups, and, unless mentioned otherwise, unimodular, meaning the left and right Haar measures agree (Definition 2.21). In this section we will see what happens when a group G acts on a C*-algebra A. A good reference for this material is [44].

Definition 1.22. A group action of a locally compact topological group G on a C*-algebra is a group homomorphism

$$G \to \mathrm{Aut}(A).$$

The action is called continuous if the map

$$G \times A \to A \qquad (g, a) \mapsto \alpha_g(a)$$

is continuous. It is called strongly continuous if for all $a \in A$, the map

$$G \to A \qquad g \mapsto \alpha_g(a)$$

is continuous.

Clearly a continuous action is strongly continuous, but the converse may fail. So it would be better to call it weakly continuous, but we will adhere to the convention of Definition 1.22. Whenever there is a group action on a C*-algebra, it will *always be assumed to be strongly continuous.*

Example 1.23. Assume X is a space with a continuous action of a group G, i.e. the map

$$G \times X \to X \times X \qquad (g, x) \mapsto (x, gx)$$

is continuous. Then the action of G induces a strongly continuous action of G on $C_0(X)$ by

$$(gf)(x) = f(g^{-1}x). \tag{1.5}$$

Definition 1.24. A *G-C*-algebra* or a *C*-dynamical system* (A, α), is a C*-algebra A together with a strongly continuous action $\alpha : G \to \mathrm{Aut}(A)$ of a group G.

A *morphism* of G-C*-algebras

$$\phi : (A, \alpha) \to (B, \beta)$$

is a morphism of C*-algebras

$$\phi : A \to B$$

that commutes with the action of G, that is,

$$\phi(\alpha_g(a)) = \beta_g(\phi(a)).$$

Similar to Definition 1.12, we have the following:

Definition 1.25. A short exact sequence

$$0 \to B \to E \to A \to 0$$

of G-C*-algebras, where each map commutes with the group action, is called an equivariant extension of G-C*-algebras.

We will often write A for a G-C*-algebra (A, α), when the action is clear from the context.

Example 1.26. Let G be a group. A unitary representation of G is a continuous group homomorphism
$$\phi : G \to U(B(H)) \qquad \phi(g) := U_g$$
from G to the group of unitaries on a Hilbert space H. We get a strongly continuous action on $B(H)$ and $\mathbb{K}(H)$ by
$$\alpha_g(T) := U_g T U_g^*.$$
for $T \in B(H)$ or $T \in \mathbb{K}(H)$ respectively.

Definition 1.27. A function $f : G \to A$ is called compactly supported if
$$\mathrm{supp}(f) := \overline{\{g \in G \mid f(g) \neq 0\}} \subset G$$
is compact. The set $C_c(G, A)$ of compactly supported A-valued functions admits the structure of an algebra over $\mathbb{C}$ with respect to the operations given, for any $f, h \in C_c(G, A)$, by

- $(f + h)(g) = f(g) + h(g)$
- $(f \star h)(g) = \int_G f(s)\alpha_s(h(s^{-1}g))ds.$

where the integral is the (Bochner) integral with respect to the Haar measure ds on G. The product $\star : C_c(G, A) \to C_c(G, A)$ is called the convolution product. We can also define an involution on $C_c(G, A)$ by

- $f^*(g) = \alpha_g(f(g^{-1})^*).$

turning $C_c(G, A)$ into a $*$-algebra.

The natural substitute for representations of $C_c(G, A)$ are the integrated forms of covariant representations. Let us go through the definitions.

Definition 1.28. A covariant representation (or covariant pair) for a G-C*-algebra (A, α) is a pair (π, u) where
$$\pi : A \to B(H)$$
is a non-degenerate representation of A, and
$$u : G \to U(H) \qquad g \mapsto u_g$$
is a unitary representation (Example 1.26) of G satisfying the so called *covariance relation*:
$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^*. \tag{1.6}$$

Given a covariant representation of a G-C*-algebra (A, α) there is a natural way to induce a *-preserving algebra homomorphism of $C_c(G, A)$ on $B(H)$ called the integrated form of (ϕ, u).

Definition 1.29. Given a covariant representation (π, u) of (A, α), the integrated form is the representation

$$\pi \rtimes u : C_c(G, A) \to B(H)$$

given for an $f \in C_c(G, A)$ by the Bochner integral

$$(\pi \rtimes u)(f) := \int_{s \in G} \pi(f(s)) u_s ds.$$

The operator $(\pi \rtimes u)(f) := \int_{s \in G} \pi(f(s)) u_s ds$ is the unique operator in $B(H)$ which acts on a vector $v \in H$ by

$$v \mapsto \int_{s \in G} \pi(f(s))(u_s(v)) ds.$$

There may be no C*-norm on $C_c(G, A)$ but we can always find pre-C*-norms whose completion give a C*-algebras (see Definition 1.1). We will now define the two most common norms on $C_c(G, A)$, denoted by $\| \cdot \|_r$ and $\| \cdot \|$.

Recall that the left regular representation of G is the unitary representation

$$\lambda : G \to U(L^2(G)) \qquad g \mapsto \lambda_g$$

where

$$\lambda_g(f)(t) = f(g^{-1}t) \qquad \text{for all } f \in L^2(G).$$

We define $L^2(G, H)$ to be the Hilbert space completion of $C_c(G, H)$ with respect to the inner product

$$\langle f, h \rangle := \int_G \langle f(g), h(g) \rangle dg \qquad (f, h \in C_c(G, H))$$

where dg denotes the Haar measure on G (see [54, Appendix I.4]). Let $\pi : A \to B(H)$ be any faithful non-degenerate representation of our C*-algebra A, then we can extend π to a representation $\hat{\pi} : A \to B(L^2(G, H))$ given by

$$\hat{\pi}(a)(h)(g) := \pi(\alpha_{g^{-1}}(a))h(g) \qquad a \in \mathcal{A}, h \in B(L^2(G, \mathcal{A})), g \in G.$$

A long, but simple computation will show that the pair $(\hat{\pi}, \lambda)$ is a covariant pair for (A, α).

Definition 1.30. The *regular representation* of a G-C*-algebra is the integrated form of the covariant pair $(\hat{\pi}, \lambda)$. That is, it is the representation

$$\hat{\pi} \rtimes \lambda : C_c(G, A) \to B(L^2(G, H))$$

given for any $f \in C_c(G,A)$, $h \in L^2(G,A)$ and $g \in G$ by

$$(\hat{\pi} \rtimes \lambda)(f)(h)(g) = \int_G \pi(\alpha_{g^{-1}}(f(s)))h(g^{-1}s)ds.$$

Definition 1.31. The *reduced crossed product* of a C*-dynamical system (A, α), is the completion of $C_c(G,A)$ with respect to the norm

$$\|f\|_r := \|\hat{\pi} \rtimes \lambda(f)\| \qquad f \in C_c(G,A).$$

The completion of $C_c(G,A)$ is denoted

$$A \rtimes_{r,\alpha} G \qquad \text{or} \qquad A \rtimes_r G.$$

Definition 1.32. The *maximal* or *universal* crossed product is the completion of $C_c(G,A)$ with respect to the norm

$$\|f\| := \sup_{(\pi,u)} \|\pi \rtimes u(f)\|$$

where (π, u) runs over all covariant representations of (A, α). The completion of $C_c(G,A)$ is denoted

$$A \rtimes_\alpha G \qquad \text{or} \qquad A \rtimes G.$$

We refer the reader to the book of Williams [54] for the proof that these are indeed pre-C*-norms on $C_c(G,A)$ satisfying the C*-identity (eq. (1.1)). For crossed products by discrete groups the book of Phillips [46] is also recommended.

Let us state the following property showing the maximality of the universal C*-norm among all "sensible" norms on $C_c(G,A)$:

Lemma 1.33. *Let $\|\cdot\|_t$ be a pre-C*-norm on $C_c(G,A)$ given by a representation of*

$$\pi_t : C_c(G,A) \to B(H)$$

which is norm-decreasing with respect to the L^1 norm on $C_c(G,A)$ i.e. the norm

$$\|f\|_1 := \left\| \int_G f(g)^* f(g) d\mu(g) \right\|.$$

Then for all $f \in C_c(G,A)$ we have $\|f\|_t \leq \|f\|$.

Proof. The assertion follows from [54, Corollary 2.46]. □

The reduced crossed product is in general not the minimal pre-C*-norm on $C_c(G,A)$, however.

Example 1.34 (Group C*-algebras)**.** Let G be a group. Let $id : G \to \mathbb{C}$ be the trivial G action on $\mathbb{C}$ (i.e. the trivial representation of G) $id(g) = 1$ for all $g \in G$. Then the crossed product

$$C_r^*(G) := \mathbb{C} \rtimes_{r,id} G$$

is called the *reduced group C*-algebra* of G (see [54, Example 7.9]), while

$$C^*(G) := \mathbb{C} \rtimes_{id} G$$

is the *(full) group C*-algebra* of G (see [54, Example 2.33]).

Both the reduced and universal crossed product C*-algebras retain some information about the underlying dynamical system. In general, however, one loses information about the underlying dynamics when passing to the associated crossed product C*-algebra, as there are several examples of isomorphic crossed product C*-algebras arising from vastly different topological dynamical systems $(C_0(X), \alpha)$ (see [45] for several examples of this).

1.3 Exactness of the reduced crossed product functor

Given an equivariant morphism $\phi : A \to B$ of two G-C*-algebras, we can define a map

$$\phi : C_c(G, A) \to C_c(G, B) \qquad \phi(f)(g) = \phi(f(g)). \tag{1.7}$$

which induces maps

$$\phi : A \rtimes G \to B \rtimes G \qquad \phi : A \rtimes_r G \to B \rtimes_r G.$$

The equivariance of ϕ makes the induced maps a $*$-homomorphism of the crossed products C*-algebras. A natural question is: Under which conditions an extension of G-C*-algebras

$$0 \to B \to E \to A \to 0$$

gives us an extension of the associated crossed product C*-algebras

$$0 \to B \rtimes_{(r),\beta} G \to E \rtimes_{(r),\gamma} G \to A \rtimes_{(r),\alpha} G?$$

Definition 1.35. A group G is called *exact* (or C*-exact) if for any extension of G-C*-algebras

$$0 \to I \to A \to A/I \to 0$$

the associated sequence of reduced crossed products

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

is also exact, i.e., $- \rtimes_r G$ is an exact functor on the category of G-C*-algebras with equivariant morphisms.

It is known that the full crossed product functor $-\rtimes G$ is always an exact functor for any group G.

Some authors call G exact if $C_r^*(G)$ is an exact C*-algebra, which amounts to the minimal tensor product functor $-\otimes C_r^*(G)$ being exact (Definition 1.15). In general this is weaker than our definition, but we will see the two definitions agree for discrete groups. Here are some of the main theorems regarding exactness of groups:

Theorem 1.36 ([19, Theorem 6]). *Let K be a field, $n \geq 1$ a positive integer and $G \subset GL_n(K)$ any discrete subgroup. Then the reduced group C*-algebra $C_r^*(G)$ is exact.*

It turns out that if Γ is any discrete group we have

$$C_r^*(\Gamma) \text{ is an exact C*-algebra} \quad \Leftrightarrow \quad -\rtimes_{r,\alpha}\Gamma \text{ is an exact functor}$$

The implication $\Leftarrow$ is always true for any locally compact group since if Γ acts trivially on a C*-algebra A, then $A \rtimes_r \Gamma \simeq A \otimes_{min} C_r^*(\Gamma)$. The other direction is proved in [31, Theorem 5.2]. Hence we have the following

Theorem 1.37 ([31, Theorem 5.2]). *If K is any field and Γ any discrete linear subgroup of $GL_n(K)$ then the functor*

$$-\rtimes_r \Gamma$$

is exact.

The authors of [31] prove this theorem by proving the following, slightly stronger statement: For any discrete group Γ and a Γ-equivariant extension $0 \to B \to E \to A \to 0$ the sequence

$$0 \to B \rtimes_r \Gamma \to E \rtimes_r \Gamma \to A \rtimes_r \Gamma \to 0 \tag{1.8}$$

is exact if and only if the sequence

$$0 \to (B \rtimes \Gamma) \otimes C_r^*(\Gamma) \to (E \rtimes \Gamma) \otimes C_r^*(\Gamma) \to (A \rtimes \Gamma) \otimes C_r^*(\Gamma) \to 0 \tag{1.9}$$

is exact. This gives us a way to determine if a sequence of crossed products is exact even in cases where the functor $-\rtimes_r \Gamma$ is not exact. Before listing some of these cases, we will need the following definition:

Definition 1.38 ([11, Definition 4.3.5]). An action of a discrete group Γ on a compact space X is called *amenable* if there exists a net of weak*-continuous maps

$$m_i : X \to \mathrm{Prob}(\Gamma) \qquad x \mapsto m_i^x$$

such that for each $s \in \Gamma$

$$\lim_{i\to\infty} \sup_{x\in X} \|s_\star m_i^x - m_i^{sx}\|_1 = 0$$

where the norm is given for a measure $\mu \in \mathrm{Prob}(\Gamma)$ by

$$\|\mu\|_1 = \sum_{\gamma\in\Gamma} |\mu(\gamma)|$$

Any action of an amenable group on a compact space X is amenable in the sense of Definition 1.38, but there are plenty of amenable actions by non-amenable groups. Amenable actions on a compact space X induce strongly continuous group actions on $C(X)$, and there is a similar notion of amenability on the C*-algebraic level, though they become quite technical. We refer the interested reader to [11] section 4.5 for arbitrary unital C*-algebras and discrete groups. For us the following will suffice as our definition

Proposition 1.39. *Let Γ be a discrete group and A a nuclear Γ-C^*-algebra. Then the action of Γ on A is amenable if and only if*

$$A \rtimes_r \Gamma$$

is nuclear.

We are mostly interested in amenable actions due to the following theorem:

Theorem 1.40 ([11, Theorem 4.3.4])**.** *Let A be any C^*-algebra and Γ a discrete group acting amenably on A. Then*

$$A \rtimes_r \Gamma = A \rtimes \Gamma.$$

So in case the action of Γ is amenable there is a unique pre-C*-norm on $C_c(G, A)$. Let us see an example of amenable actions

Example 1.41. Let G be a unimodular Lie group and $\Gamma \subset G$ a discrete subgroup with $\Gamma\backslash G$ of finite volume with respect to the restricted Haar measure of G. Let $H \subset G$ be a closed subgroup. Then the action of Γ on G/H is amenable if and only if H is amenable. If Γ is an arbitrary discrete subgroup of G then the action of Γ on G/H is amenable if H is amenable (see [55, Corollary 4.3.7]).

As a special case of this, if G is a connected semisimple Lie group with finite center and maximal compact subgroup K, $\Gamma \subset G$ a lattice (meaning $\Gamma\backslash G$ has finite volume) and $P \subset G$ is a parabolic subgroup (Definition 2.47), then the action of Γ on G/P is amenable if and only if P is a minimal parabolic subgroup (as these are the only amenable parabolic subgroups of G).

We will now list a few of the cases where the sequence of equation (1.8) is exact in the next lemma.

Lemma 1.42. *For Γ any discrete group, the sequence (1.8) is exact in the following cases*

1. *The action of Γ is amenable on all C^*-algebras in the sequence.*
2. *A is nuclear and the action of Γ on A is amenable.*
3. *There is a unique C^*-norm on the algebraic tensor product $A \rtimes \Gamma \odot C_r^*(\Gamma)$.*
4. *The sequence $0 \to B \rtimes \Gamma \to E \rtimes \Gamma \to A \rtimes \Gamma \to 0$ is locally split (or semisplit).*

5. *The sequence* $0 \to B \to E \to A \to 0$ *is locally split by* Γ*-equivariant maps (or equivariantly semisplit).*

Proof. **Case 1)** follows from the fact that for any Γ-C*-algebra B with an amenable Γ action we have $B \rtimes_r \Gamma = B \rtimes \Gamma$, and $- \rtimes \Gamma$ is an exact functor.

Case 2): The conditions assure that the crossed product $A \rtimes \Gamma$ is nuclear, hence the claim follows from Case 3 above.

Note that $B \rtimes \Gamma$ is nuclear always implies that B is nuclear. If the action is amenable, the converse also holds. For transformation groupoids given by the action of a discrete group on a locally compact Hausdorff space, Theorem 3.5 of [2] $A \rtimes_r \Gamma$ is nuclear if and only if the action of Γ is amenable.

Case 3) follows from the correspondence of the sequences (1.8) and (1.9) and Corollary 3.7.3 [11], which states that the sequence

$$0 \to B \otimes D \to E \otimes D \to A \otimes D \to 0$$

is exact if the algebraic tensor product $A \odot D$ admits a unique C*-norm, which happens for instance when either A or D are nuclear.

Case 4) If the sequence $0 \to B \to E \to A \to 0$ is locally split (or semisplit), then by Proposition 3.7.6 of [11] the sequence

$$0 \to B \otimes D \to E \otimes D \to A \otimes D \to 0$$

is exact for any C*-algebra D.

Case 5) If the sequence $0 \to B \to E \to A \to 0$ is equivariantly locally split/semisplit, then the sequence $0 \to I \rtimes \Gamma \to A \rtimes \Gamma \to (A/I) \rtimes \Gamma \to 0$ is locally split/semisplit respectively, so we can use Case 4). ☐

1.4 Hilbert C*-modules

If we think of $\mathbb{C}$ as a C*-algebra, a complex Hilbert space H is nothing but a special module over the C*-algebra $\mathbb{C}$ with an inner product taking values in $\mathbb{C}$. The next definition is a natural generalization of the notion of Hilbert spaces where $\mathbb{C}$ is replaced by an arbitrary C*-algebra -

Definition 1.43 ([34] p. 2)**.** A right pre-Hilbert A-module, is a (complex) linear space H, with a right A-module structure, together with a map $\langle -, - \rangle : H \times H \to A$ satisfying

1. $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in H$, $a \in A$ and $\lambda \in \mathbb{C}$,

2. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $x, y, z \in H$, and all $\alpha, \beta \in \mathbb{C}$,

3. $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in H$ and all $a \in A$,

4. $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in H$,

5. $\langle x, x\rangle \geq 0$; and $\langle x, x\rangle = 0 \Rightarrow x = 0$ for all $x \in H$,

If H is complete with respect to the norm

$$\|x\| := \|\langle x, x\rangle\|^{1/2}$$

it is called a right Hilbert A-module.

It should be clear that Definition 1.43 is modelled on the definition Hilbert spaces, in fact since $\mathbb{C}$ is a C*-algebra, it is easy to show that

Example 1.44. A right Hilbert $\mathbb{C}$-module is a Hilbert space.

The analogy between Hilbert spaces and right Hilbert C*-modules is not perfect though. Recall that for a Hilbert space H, any bounded linear map $T : H \to H$ has an adjoint, meaning there is a map $T^* : H \to H$ satisfying

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x, y \in H$. The map T^* is uniquely determined and automatically linear and bounded if T is. For Hilbert C*-modules we have a similar notion

Definition 1.45. Let H be a right Hilbert A-module. By an *operator* on H we mean a bounded $\mathbb{C}$-linear map $L : H \to H$ for which

$$L(xa) = L(x)a \qquad \text{for all } x \in H, a \in A.$$

An operator $L : H \to H$ is called adjointable if there exists another operator L^* such that

$$\langle L(x), y\rangle = \langle x, L^*(y)\rangle \qquad \text{for all } x, y \in H.$$

The set of adjointable operators on H are denoted $\mathcal{L}_A(H)$ or $\mathcal{L}(H)$.

As Definition 1.45 would suggest, there can be non-adjointable operators on Hilbert C*-modules. An example of non-adjointable operators can be found in [34, p. 8]. We have the following:

Lemma 1.46 ([34] p.8)**.** *Let H be a Hilbert A-module. Then $\mathcal{L}(H)$ is a C*-algebra with respect to the norm*

$$\|T\| := \sup_{\substack{x \in H \\ \|x\|=1}} \|T(x)\| = \sup_{\substack{x \in H \\ \|x\|=1}} \sup_{\substack{y \in H \\ \|y\|=1}} \|\langle Tx, y\rangle\|. \tag{1.10}$$

We shall now look at a class of operators which are always adjointable

Definition 1.47. Let A be a C*-algebra and H be a Hilbert A-module (Definition 1.43). The *rank one* operators are the operators of the form

$$\theta_{x,y} : H \to H \qquad \theta_{x,y}(z) = x\langle y, z\rangle \qquad \text{for all } z \in H$$

where $x, y \in H$. The compact operators in H, denoted $\mathbb{K}(H)$ is the closed linear span

$$\mathbb{K}(H) := \overline{\operatorname{Span}\{\theta_{x,y} \mid x, y \in H\}}$$

where the closure is taken with respect to the operator norm defined by eq. (1.10).

A little bit of work shows that for any $x, y \in H$ $\theta_{x,y}$ is adjointable with adjoint given by

$$\theta^*_{x,y} = \theta_{y,x}.$$

It follows that $\mathbb{K}(H) \subset \mathcal{L}(H)$.

Example 1.48. Let us look at a useful example of a Hilbert module. As a byproduct we will get a concrete realization of the multiplier algebra defined in Example 1.3. Let A be a C*-algebra. Define a map

$$\langle -, - \rangle : A \times A \to A \qquad \langle a, b\rangle := a^*b, \quad a, b \in A. \tag{1.11}$$

This can be shown to be an A-valued inner product satisfying properties (1)-(4) of Definition 1.43 hence A is a pre-Hilbert module over itself. Completeness follows from completeness of A using the C*-identity (Equation (1.1)):

$$\|\langle a, a\rangle\|^{1/2} = (\|a\|^2)^{1/2} = \|a\|.$$

So A is a Hilbert A-module with respect to the inner product of equation (1.11).

It turns out that if A is nonunital, we have

$$\mathcal{L}(A) = M(A) \quad \text{and} \quad \mathbb{K}(A) = A.$$

Example 1.49 ([34] p. 7)**.** Let A be a unital C*-algebra and, $B \subset A$ any C*-subalgebra containing the unit of A. If H is a Hilbert A-module, let $\phi : A \to B$ be a map satisfying

1. $\phi|_B = \mathrm{Id}_B$;

2. ϕ is a contractive completely positive map (Definition 1.21);

The map ϕ satisfying property (1) and (2) is called a conditional expectation from A to B. By a result of Tomiyama ([11, Theorem 1.5.10]) if ϕ is contractive and satisfies property (1), it is automatically completely positive and B-linear, meaning

$$\phi(ab) = \phi(a)b \qquad \text{for all } a \in A \text{ and } b \in B.$$

As in Example 1.48 we may treat A as a Hilbert A-module with inner product

$$\langle a_1, a_2 \rangle_A := a_1^* a_2$$

The Hilbert module A becomes a Hilbert B-module with respect to the B-valued inner product

$$\langle x, y \rangle_B := \phi(\langle x, y \rangle_A) \qquad x, y \in H.$$

The complete positivity of ϕ is exactly what is needed to ensure that $\langle -, - \rangle_B$ satisfies property (5) of Definition 1.43, since if $a_i \in H$ $(i = 1, \dots, n)$ is any finite sequence of elements in H, then

$$\phi\left(\langle \sum_{i=1}^{n} a_i, \sum_{j=1}^{n} a_j \rangle\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(\langle a_i, a_j \rangle)$$

which is is positive for all finite sums $\sum_{i=1}^{n} a_i$ in A if and only if ϕ is completely positive (see the proof of [34, Lemma 4.3(i)]).

1.5 K-theory

Operator K-theory is an example of what is called a generalized homology theory on C*-algebras extending the topological K-theory for compact topological spaces through Gelfand duality. We will define both operator K-theory and (compactly supported) topological K-theory in this section and return to this material later when we introduce equivariant KK-theory in the next section. Let us start with topological K-theory.

As previously, X denotes a locally compact Hausdorff space, and all vector bundles will be assumed to be complex and of finite rank. Additionally we will assume any vector bundle $\pi : E \to X$ to be trivial outside a compact set. Note that saying that a bundle $E \to X$ on a locally compact but noncompact Hausdorff space X is trivial outside a compact set, is equivalent to saying that E is the restriction of a bundle $E^\infty :\to X \cup \{\infty\}$ on the one point compactification of X.

We denote by $e^n = X \times \mathbb{C}^n$ the trivial bundle of rank n over X.

Definition 1.50. We define the Whitney sum of two vector bundles $\pi_i : E_i \to X$ $(i = 1, 2)$ to be the vector bundle

$$E_1 \oplus E_2 := \{(v_1, v_2, x) \in E_1 \times E_2 \times X \mid \pi_1(v_1) = \pi_2(v_2) = x\}$$

Definition 1.51 ([22] p. 39)**.** Two vector bundles $E_i \to X$ $(i = 1, 2)$ are called stably isomorphic, denoted $E_1 \simeq_s E_2$, if there is a trivial vector bundle $e^n \to X$ such that

$$E_1 \oplus e^n = E_2 \oplus e^n.$$

Definition 1.52. Given a bundle $\pi : E \to X$ over X and a continuous map $f : Y \to X$, the pullback of E by f is the bundle

$$f^*E \to Y$$

with total space

$$f^*E = \{(v, y) \in E \oplus Y \mid f(y) = \pi(v)\}$$

and bundle map $(f^*\pi) : f^*E \to Y$ given by projection onto the second factor.

It should be clear that the pullback of a bundle preserves the rank of the bundle, and that the pullback of a trivial bundle is trivial. We also have the that for any set $O \subset Y$ and continuous map $\phi : X \to Y$ and bundle $E \to Y$, we have $\phi^*(E|_O) = (\phi^*E)|_{\phi^{-1}(O)}$. Combined, these three properties give us the following:

Lemma 1.53. *Let $\pi : E \to X$ be a bundle, and $K \subset X$ a compact subset such that the restriction$\pi|_{X\backslash K} : E|_{X\backslash K} \to X\backslash K$ is trivial. Let $\phi : Y \to X$ be a proper map. Then ϕ^*E is trivial outside a compact set in Y.*

Denote by $[E]$ and $[E']$ the stable isomorphism class of the bundles E and E' respectively. Then the equivalence class of their sum $[E \oplus E']$ only depends on the stable isomorphism class of E and E', so we may define an operation on the collection of stable isomorphism classes by

$$[E] \oplus [E'] := [E \oplus E'].$$

Denote by $V(X)$ the collection of all stable isomorphism classes of bundles over X. This is an abelian semigroup with respect to Whitney sums, and the class $[e^0]$ (the rank–zero bundle) is the zero element

The next proposition has important consequences for the theory, and is one of the reasons we restrict our attention to vector bundles that are trivial outside a compact set.

Proposition 1.54 ([42, Proposition 1.7.9])**.** *If X is compact, then any vector bundle over X is a subbundle of a trivial bundle e^n for some n. Equivalently, for any bundle $E \to X$, there is a bundle $E' \to X$, and $n \in \mathbb{N}$ such that*

$$E \oplus E' = e^n.$$

One consequence of Proposition 1.54 is that the semigroup $V(X)$ has the cancellation property, meaning

$$[E_1] \oplus [E_2] = [E_1] \oplus [E_3] \Rightarrow [E_2] = [E_3]$$

since we may add a bundle E_1' to each side for which $E_1' \oplus E_1 = e^m$ for some m.

Knowing this, we can form the Grothendieck group $\mathrm{Gr}(V(X))$ as the collection of formal differences

$$\{[E_1] - [E_2] \mid [E_i] \in V(X)\}/\sim$$

with addition defined by $([E_1]-[E_2])+([E_1']-[E_2']) := ([E_1 \oplus E_1'] - [E_2 \oplus E_2'])$ and identifying elements

$$[E_1] - [E_2] \sim [E_1'] - [E_2'] \Leftrightarrow [E_1 \oplus E_2'] = [E_1' \oplus E_2].$$

Definition 1.55. The (unreduced compactly supported complex) topological K-theory group of X is defined to be

$$K^0(X) := Gr(V(X)).$$

Now, to steer things towards operator K-theory, assume that X is compact, so that $C_0(X) = C(X)$ and let

$$M_\infty(\mathbb{C}) = \lim_{n \to \infty} M_n(\mathbb{C})$$

be the algebraic direct limit of $n \times n$-matrices with complex coefficients and connecting morphisms

$$M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \qquad x \mapsto \mathrm{diag}(x, 0).$$

The Serre-Swan theorem tells us that there is a 1-1 correspondence between isomorphism classes of complex vector bundles on X and homotopy classes of projection valued continuous maps $X \to M_\infty(\mathbb{C})$ (see [26, Section 4.1]).

The Whitney sum of two projection valued functions $p, p' \in C(X, M_\infty(\mathbb{C}))$ is given by

$$p \oplus p' = \mathrm{diag}(p, p'),$$

i.e. by the block diagonal matrix with entries p and p' along the diagonal. The stable equivalence relations of vector bundles translate to certain equivalence relations the corresponding projection valued functions, which produces a semigroup under addition. Taking the Grothendieck group gives us the operator K-theory groups for $C(X)$, denoted $K_0(C(X))$ which agrees with $K^0(X)$.

With this example in mind, let A be an arbitrary unital C*-algebra and consider the projections in the algebraic direct limit

$$C(A, M_\infty(\mathbb{C})) = M_\infty(A) = \lim_{n \in \mathbb{N}} M_n(A),$$

where the connecting morphisms are simply the maps

$$M_n(A) \to M_{n+1}(A) \qquad M \mapsto M \oplus 0$$

given by padding a matrix in M_n with zeros. Recall that an element in M_∞ is represented by some element in $M_n(A)$ for some $n \in \mathbb{N}$ (see Example 1.6). For any two elements in $p, q \in M_\infty(A)$ we define their sum $p \oplus q$ to be the block diagonal matrix $\mathrm{diag}(p, q)$.

Projections in $M_\infty(A)$ are, as for C*-algebras, the elements $p \in M_\infty(A)$ for which

$$p = p^2 = p^*.$$

We are now ready to define the equivalence relation:

Definition 1.56. Two projections $p, q \in M_\infty(A)$ are called equivalent, denoted $p \sim q$, if there is a rectangular A valued matrix v such that

$$p = v^*v \qquad q = vv^*.$$

Assuming A is unital, we define:

Definition 1.57. Two projections $p, q \in M_\infty(A)$ are called stably equivalent if there is an $n \in \mathbb{N}$ such that

$$I_n \oplus p \sim I_n \oplus q$$

where I_n is the $n \times n$-identity matrix.

Let $V(A)$ be the collection of stable equivalence classes of projections in $M_\infty(A)$. This is a semigroup with the cancellation property, with respect to addition given by diagonal concatenation of block matrices. We define, just as in the topological case:

Definition 1.58. Let A be a unital C*-algebra, then the K-theory group of A is

$$K_0(A) = \mathrm{Gr}(V(A)).$$

If A is nonunital, then we define

$$K_0(A) = \mathrm{Ker}(\iota_* : K_0(A^+) \to K_0(\mathbb{C}) = \mathbb{Z})$$

where $\iota : A^+ \to \mathbb{C}$ is the canonical unital *-homomorphism from the unitization of A.

Bott periodicity and the six–term exact sequence of K-theory

Definition 1.59. Let A be a C*-algebra. The *suspension* of A is the C*-algebra

$$SA := C_0(\mathbb{R}, A)$$

of continuous functions from $\mathbb{R}$ to A vanishing at infinity. Similarly, we denote then n-fold suspension by

$$S^nA := C_0(\mathbb{R}^n, A).$$

For a *-homomorphism $\phi : A \to B$ we define the mapping cone of ϕ to be

$$C_\phi := \{(a, f) \in A \otimes C([0,1], B) \mid f(0) = \phi(a), f(1) = 0\}.$$

We define higher K-groups as follows

Definition 1.60. Let A be any C*-algebra, then for $n \in \mathbb{N}$ define

$$K_n(A) := K_0(S^n A).$$

Let us go through some of the basic properties of the $K_i(A)$-groups, starting with functoriality: If

$$\phi : A \to B$$

is a *-homomorphism, we get a morphism of the associated matrix C*-algebras

$$M_n(A) \to M_n(B) \qquad (a_{i,j}) \mapsto (\phi(a_{i,j})).$$

This gives us a map

$$\phi_* : V(S^n A) \to V(S^n B) \qquad \phi[p] := [\phi(p)]$$

which can be shown to commute with addition, so by the universal properties of the Grothendieck group induces a map

$$\phi_* : K_n(A) \to K_n(B).$$

Hence we have the following

Proposition 1.61. *For any $n \in \mathbb{N}$, the assignment*

$$A \to K_n(A)$$

is a functor from the category of C-algebras to the category of abelian groups.*

Next, let us state the following fundamental theorem

Proposition 1.62 (Bott periodicity [41, Theorem 7.5.1 7])**.** *For any C*-algebra A there is an isomorphism*

$$\delta : K_2(A) \to K_0(A).$$

Let us see what the K-theory functor does to extensions of C*-algebras

Theorem 1.63 ([41, Theorem 7.5.18])**.** *Given an extension*

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

of C-algebra, we have a 6-term exact sequence of K-groups:*

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\iota_*} & K_0(E) & \xrightarrow{p_*} & K_0(A) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(A) & \xleftarrow[p_*]{} & K_1(E) & \xleftarrow[\iota_*]{} & K_1(B) \end{array}$$

The maps p_* and ι_* are induced by functoriality of K_i. Let us see how the maps ∂, called the connecting morphisms, are constructed in [41]. Let

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

be an extension of C*-algebras. We have natural maps

$$j : B \to C_p \qquad k : SA \to C_p$$

to the mapping cone of p (Definition 1.59) where $j(b) = (\iota(b), 0)$ and $k(f) = (0, f)$ are the inclusion maps. It is shown in [41, Lemma 7.5.12] that $j_* : K_0(B) \to K_0(C_p)$ is an isomorphism. We now define the connecting map to be

$$\partial = (j_*)^{-1} k_* : K_0(SA) = K_1(A) \to K_0(B)$$

Using a similar construction for the extension

$$0 \to SB \to SE \to SA \to 0$$

and Bott periodicity obtain the map

$$\partial : K_0(A) \to K_1(B).$$

1.6 Equivariant extensions and KK-theory

The interplay between extensions of C*-algebras and K-homology dates back to the now classical work of Brown, Douglas and Fillmore ([9], [10]) where the authors, motivated by the study of essentially normal operators, set out to classify extensions of the form

$$0 \to \mathbb{K} \to E \to C(X) \to 0$$

where $\mathbb{K} := \mathbb{K}(l^2(\mathbb{Z}))$ are the compact operators on a separable infinite dimensional Hilbert space. By what seems a coincidence the Ext-group they defined turned out to be isomorphic to the K-homology of the space X, that is the abstract homology theory dual to topological K-theory defined in the previous section. The definition of the Ext-group was later extended to include extensions of the form

$$0 \to \mathbb{K} \to E \to A \to 0$$

for arbitrary stable C*-algebras A (see Definition 1.18) at the expense of making Ext a semigroup. Voiculescu then showed [52] that if A is separable, the semigroup is actually a monoid (i.e. has a zero element) with unit the class of any split extension. The invertible elements of $\mathrm{Ext}(A)$ were later characterized by Arveson in [3] (see also [27, Theorem 3.2.9]) as those extensions for which the associated "Busby invariant" (Def. 1.69) map $\phi : A \to Q(B)$ lifts to a completely positive contractive map $\psi : A \to M(B)$, which is easily proved to be equivalent to having a completely positive contractive splitting of the quotient map in the extension. This is automatic if A is nuclear, by the lifting theorem of Choi and Effros [12].

The theory was greatly generalized by Kasparov in [29] and later in [30] laying the foundations of KK-theory, the bivariant K-theory which bears his name. A new semigroup $\mathrm{Ext}(A, B)$ was defined which extends the definition of $\mathrm{Ext}(A)$ to extensions of the form

$$0 \to B \to E \to A \to 0$$

so $\mathrm{Ext}(A) = \mathrm{Ext}(\mathbb{K}, A)$. Kasparov then proved using his generalized Stinespring dilation theorem, that

$$\mathrm{KK}_1(A, B) = \mathrm{KK}(SA, B) = \mathrm{Ext}(A, B)^0$$

where $\mathrm{Ext}(A, B)^0$ denotes the subgroup of $\mathrm{Ext}(A, B)$ of invertible elements and $SA = C(0, 1) \otimes A$ is the suspension of A. The group KK-groups introduced by Kasparov generalize K-theory and its dual theory K-homology in the sense that (assuming A is separable)

$$\mathrm{KK}(\mathbb{C}, A) = K_0(A) \qquad \mathrm{KK}(A, \mathbb{C}) = K^0(A).$$

The isomorphism uses the characterization of invertible elements in $\mathrm{Ext}(A, B)$ given by the existence of a completely positive splitting $\phi : A \to E$ of the quotient map $p : E \to A$ of the extension. The splitting also gives a concrete realization of the KK^1-cycle representing the extension.

Parallel to this, a theory of equivariant extensions and equivariant KK-theory emerged. The equivariant KK-groups were defined by Kasparov in [30], but it was not linked to an equivariant Ext-group. The equivariant version of the Ext-semigroup was defined in [51] which also gave a characterization of invertibility of elements by means of equivariant completely positive splittings, and proved an isomorphism

$$\mathrm{Ext}_G(A, B)^0 \simeq \mathrm{KK}^1_G(A, B)$$

just as in the non-equivariant case, but this time we need B to be equivariantly $\mathbb{K}_G := \mathbb{K}(\bigoplus_{n\in\mathbb{N}} L^2(G))$-stable rather than just $\mathbb{K}$-stable. We will focus entirely on the equivariant side of the story here, but the non-equivariant case can always be recovered by setting $G = \{0\}$.

1.6.1 Equivariant KK-theory

Let us start by recalling the definition of equivariant KK-theory, and the equivariant extension semigroup Ext_G. For a good reference to KK-theory see for instance [5] where there is a short description on equivariant KK-theory in Chapter VIII.20. The book [27] is also a good reference to KK-theory, which focuses heavily on extensions of C*-algebras, but equivariant KK-theory is not covered.

Throughout this section G will denote a locally compact group and all C*-algebras will be *assumed to be separable*, meaning they have a countable dense subset or equivalently, can be represented on a Hilbert space with a countable basis. Recall that we call a C*-algebra A a G-C*-algebra (Def. 1.24) if it is endowed with a strongly continuous action of G, that is, an action for which the map

$$G \to A \qquad g \mapsto \alpha_g(a)$$

is continuous for all $a \in A$, where $\alpha_g \in \mathrm{Aut}(A)$ denotes the group action of g on A.

If (A, G, α) is a G-C*-algebra, a right Hilbert A-module will be assumed to also have a group action satisfying the following compatibility conditions:

Definition 1.64 ([5, Definition 20.1.1])**.** Let (A, G, α) be a G-C*-algebra. An equivariant right Hilbert A-module is a right Hilbert A-module H (in the sense of Definition 1.43) with an action of G by bounded invertible linear transformations for which

$$G \mapsto \mathbb{R} \quad g \mapsto \|\langle gx, gx\rangle\| \tag{1.12}$$

is continuous and

$$g(xa) = (gx)\alpha_g(a).$$

From now on, any Hilbert module over a G-C*-algebra, will be assumed to be equivariant.

An action of G on H satisfying equation (1.12) is called a *continuous G-action.* As in the case of Hilbert spaces, an action of G on H induces an action of G on the adjointable operators $\mathcal{L}(H)$ by $(gT)(x) = gT(g^{-1}x)$, which is not in general continuous with respect to the operator topology on $\mathcal{L}(H)$, just strictly continuous, i.e. for each $x \in H$ the map $g \mapsto (gT)(x)$ is continuous.

Definition 1.65 ([5] Definition 20.1.2)**.** An operator $T \in \mathcal{L}(H)$ for which $g \mapsto gT$ is operator norm-continuous is called a G-continuous operator.

We are now ready to define equivariant Kasparov modules:

Definition 1.66 ([5, Definition 20.2.1])**.** An odd Kasparov G-module for the G-algebras (A, B) is a triple (H, ϕ, F) where H is a countably generated (equivariant) right Hilbert B-module with a continuous action of G, $\phi : A \to \mathcal{L}(H)$ is an equivariant *-homomorphism and $F \in \mathcal{L}(H)$ is G-continuous operator such that

- $[F, \phi(a)] \in \mathbb{K}(H)$
- $(F^2 - 1)\phi(a) \in \mathbb{K}(H)$
- $(F^* - F)\phi(a) \in \mathbb{K}(H)$
- $(gF - F)\phi(a) \in \mathbb{K}(H)$

for all $a \in A$ and $g \in G$.

An equivariant Kasparov module is call *degenerate* if we have

$$[F, \phi(a)] = (F^2 - 1)\phi(a) = (F^* - F)\phi(a) = (gF - F)\phi(a) = 0.$$

The set of degenerate Kasparov G-modules will be denoted by $\mathbb{D}_G(A, B)$.

Given a Kasparov G-module (H, ϕ, F) for (A, B), let $f_A : C \to A$ and $f^B : B \to D$ be two equivariant *-homomorphisms. Then the pullback of (H, ϕ, F) by f_A is a Kasparov G-module for (C, B) given by

$$f_A^*(H, \phi, F) = (H, \phi \circ f_A, F).$$

Similarly, the pushforward of (H, ϕ, F) by f^B is a Kasparov G-module over (A, D) given by

$$f_*^B(H, \phi, F) = (H \otimes_B D, \phi \otimes 1, F \otimes 1)$$

Two Kasparov G-modules (H_1, ϕ_1, F_1), (H_2, ϕ_2, F_2) for (A, B) are said to be *unitarily equivalent*, denoted by $(H_1, \phi_1, F_1) \simeq_u (H_2, \phi_2, F_2)$, if there is a unitary $u \in L(H_1, H_2)$ intertwining the action of G, ϕ_i and F_i. They are said to be *homotopic* if there is a Kasparov G-module $(\hat{H}, \hat{\phi}, \hat{F})$ for $(A, C([0, 1], B))$ such that

$$(ev_0)_*(\hat{H}, \hat{\phi}, \hat{F}) \simeq_u (H, \phi, F) \qquad \text{and} \qquad (ev_1)_*(\hat{H}, \hat{\phi}, \hat{F}) \simeq_u (H', \phi', F'),$$

where $ev_t : C([0, 1], B) \to B$ is the evaluation at t. We can add two Kasparov modules, just as in the non-equivariant case, by defining

$$(H, \phi, F) \oplus (H', \phi', F') := (H \oplus H', \phi \oplus \phi', F \oplus F') \tag{1.13}$$

The collection of all Kasparov G-modules for (A, B) becomes an abelian semigroup with respect to addition. We denote this semigroup by $E_G(A, B)$. Finally

Definition 1.67. Denote by $\mathrm{KK}_G^1(A, B)$ the quotient of $E_G(A, B)$ by the relation of homotopy equivalence.

Proposition 1.68 ([5, Proposition 20.2.3])**.** $\mathrm{KK}_G^1(A, B)$ *is an abelian group with respect to addition given on Kasparov modules by eq.* (1.13).

Next, let us define the equivariant extension group $\mathrm{Ext}_G(A, B)$ following [51]. An extension of G-C*-algebras

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

is called an equivariant extension of B by A if both ι and p are equivariant, though some authors prefer to call this an equivariant extension of A by B. For ease of notation we will refer to the extensions by its middle algebra and write (E). Two G-extensions (E) and (E') of B by A are said to be isomorphic if there is a $*$-homomorphism $E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

The homomorphism $E \to E'$ is then necessarily an equivariant $*$-isomorphism ([51] Theorem 2.2). As with non-equivariant extensions, there is a 1 to 1 correspondence between isomorphism classes of G-extensions and elements in $\mathrm{Hom}_G(A, Q(B))$, the set of equivariant $*$-homomorphisms from A to the Corona algebra $Q(B) = M(B)/B$. The construction of the extension associated with a given Busby map $\phi : A \to Q(B)$, is given by the pullback diagram

$$\begin{array}{ccc} E_\phi & \xrightarrow{p} & \mathcal{A} \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \phi} \\ M(B) & \xrightarrow{q_B} & Q(B) \end{array}$$

where $q_B : M(B) \to Q(B)$ is the quotient map and T and p are the canonical maps of the pullback construction. Explicitly, we have

$$E_\phi = \{(a, m) \in A \oplus M(B) \mid \phi(a) = q_B(m)\},$$

with p and T the projections onto the first and second factor respectively and the associated extension being

$$0 \to B \to E_\phi \to A$$

where the inclusion $B \to E_\phi$ is induced by the inclusion of $B \to M(B)$.

Conversely, given a G-extension we associate to it a map $\phi : A \to Q(B)$. To define ϕ, we need to first define the map $T : E \to M(B)$ which is given implicitly by the equation

$$\iota(T(e)b) = e\iota(b).$$

The injectivity of ι ensures T is uniquely determined.

The map T restricts to the identity on B, since for any $b, b' \in B$ we have

$$\iota(T(\iota(b'))b) = \iota(b')\iota(b) = \iota(b'b).$$

So $T(\iota(b'))$ acts by left multiplication by b' on B in $M(B)$, which is exactly how b' is imbedded into $M(B)$. Given any splitting s (not necessarily linear) of $p : E \to A$, we define:

Definition 1.69. The Busby map for the extension (E) is the map

$$\phi = q_B \circ T \circ s.$$

To see that ϕ does not depend on the choice of splitting, assume $s_1, s_2 : A \to E$ are two splittings of the quotient map $p : E \to A$ of the extension (E), that is $p \circ s_i = id_A$. We have that for any $a \in A$ $p(s_1(a) - s_2(a)) = 0$, hence $s_1(a) - s_2(a) \in \iota(B)$. Thus

$$q_B \circ T \circ s_1(a) - q_B \circ T \circ s_2(a) = q_B(T(s_1(a) - s_2(a))) = 0$$

since $T(\iota(B))$ is contained in $B \subset M(B)$. Two G-extensions are called *unitarily equivalent* if there is a unitary $u \in M(B)$ with $gu - u \in B$ for all $g \in G$ and a $*$-homomorphism $E \to E'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad_u} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

The map $E \to E'$ is necessarily an isomorphism, but *need not be equivariant*. In what follows we use $\simeq_u$ to indicate unitary equivalence of extensions. The next lemma shows what this amounts to at the level of Busby maps

Lemma 1.70. *Given two equivariant $*$-homomorphisms*

$$\phi_i \in \mathrm{Hom}_G(A, Q(B))$$

their extensions (E_{ϕ_i}) $(i = 1, 2)$ are unitarily equivalent if and only if there is a $u \in M(B)$ such that

- *$gq_B(u) = q_B(u)$ for all $g \in G$;*
- *$Ad(u) \circ \phi_1 = \phi_2$.*

Proof. Assume $u \in M(B)$ is as in the Lemma. We have

$$E_i := E_{\phi_i} = \{(a, m) \in A \oplus M(B) \mid \phi_i(a) = q_B(m)\}$$

but $\phi(a) = q_B(m) \Leftrightarrow \psi(a) = Ad_{q_B(u)}(q_B(m)) = q_B(Ad_u(m))$ hence the map $id \oplus Ad_u : A \oplus M(B) \to A \oplus M(B)$ restricts to an isomorphism $E_1 \to E_2$ which makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad_u} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Conversely, assume the above diagram commutes for some $u \in M(B)$ and an arbitrary isomorphism $f : E_1 \to E_2$. Using the implicit definition of the maps $T_i : E_i \to M(B)$:

$$\iota_i(T_i(e)b) = e\iota_i(b)$$

substituting $\iota_1 = f \circ \iota_2 \circ Ad_{u^*}$ the left hand side becomes

$$\iota_1(T_1(e)b) = (f \circ \iota_2)(Ad_{u^*}(T_1(e))Ad_{u^*}(b)),$$

while the right hand side reads

$$e\iota_1(b) = f \circ \iota_2(T_2(f^{-1}(e))Ad_{u^*}(b)).$$

Putting these together gives us

$$T_1 \circ f = Ad_u \circ T_2.$$

Let $s_1 : A \to E_1$ be any splitting of the quotient maps $p_1 : E_1 \to A$. The map $s_2 = f \circ s_1$, is then a splitting for p_2 and the Busby map thus related by

$$\begin{aligned} \phi_1 &= q_B \circ T_1 \circ s_1 \\ &= q_B \circ (Ad_u \circ T_2 \circ f^{-1}) \circ (f \circ s_2) \\ &= q_B \circ (Ad_u \circ T_2 \circ s_2) \\ &= Ad_{q_B(u)} \circ (q_B \circ T_2 \circ s_2) = Ad_{q_B(u)} \circ \phi_2 \end{aligned}$$

Finally, for a unitary $u \in M(B)$ we have $gu - u \in B$ if and only if $gq_B(u) = q_B(u)$, which concludes the proof. □

Remark 1.71. Since $q_B(u)$ is a unitary in $Q(B)$, the reader may wonder why we did not simply pick a unitary in $Q(B)$ when defining unitary equivalence. There is nothing seriously wrong with this approach, but the resulting Ext-groups (to be defined shortly) have been less used in practice.

More precisely, if in the definition of unitary equivalence we pick an invariant unitary $u \in Q(B)$ rather than one in $M(B)$ we get what is called "weak" unitary equivalence in [5], and another Ext_G group that does not agree with the usual Ext_G-group of Definition

1.72 (see for instance [15] section V.6 for an example using the Cuntz algebra in the non-equivariant case). The reason they differ boils down to the fact that not all unitaries in $Q(B)$ can be lifted to unitaries in $M(B)$.

To define addition of two extensions, we will need to make some extra assumptions on the algebra B. We need to assume B is stable (Def. 1.18), i.e. that there is a *-isomorphism

$$B \simeq B \otimes \mathbb{K}$$

where $\mathbb{K}$ has the trivial G-action. Assuming B is stable, let (E_i) $(i = 1, 2)$ be two G-extensions of B by A. Then we define their sum to be the extension

$$0 \to M_2(B) \xrightarrow{\hat{\iota}} \widehat{E} \xrightarrow{\iota} A \to 0$$

where

$$\widehat{E} = \left\{ \begin{bmatrix} e_1 & b_1 \\ b_2 & e_2 \end{bmatrix} \;\middle|\; e_i \in E_i,\; b_i \in \mathcal{B},\; p_1(e_1) = p_2(e_2) \right\}$$

the quotient map is given by

$$\hat{p} = p_1 \oplus p_2 : \begin{bmatrix} e_1 & b_1 \\ b_2 & e_2 \end{bmatrix} \mapsto p_1(e_1)$$

and the inclusion $\hat{\imath}$ is the obvious one. This is an extension of B by A since when B is stable we have $M_2(B) \simeq B$. At the level of Busby invariants this additive structure takes the form

$$\phi_1 \oplus \phi_2 = \phi'$$

where

$$\phi'(a) = Ad_{q_B(V_1)}(\phi_1(a)) + Ad_{q_B(V_2)}(\phi_2(a))$$

for any choice of G-invariant isometries $V_i \in M(B)$ with $V_1V_1^* + V_2V_2^* = 1$. To prove the existence of such isometries, we use the fact that B is stable, since then we have an imbedding $M(\mathbb{K}) \otimes M(B) \subset M(B \otimes \mathbb{K}) \simeq M(B)$ and any two isometries $W_i \in M(\mathbb{K})$ with $W_1W_1^* + W_2W_2^*$ give isometries $V_i = W_i \otimes 1 \in M(\mathbb{K}) \otimes M(B) \subset M(B)$ which are G-invariant and satisfy $V_1V_1^* + V_2V_2^* = 1$.

Similar to the non-equivariant case, an extension (E_ϕ) associated with the map $\phi \in \mathrm{Hom}_G(A, Q(B))$ is called *degenerate* if ϕ lifts to an equivariant *-homomorphism

$$\hat{\phi} : A \to M(B).$$

This is equivalent to the quotient map $p : E \to A$ being split by an equivariant $*$-homomorphism. To see why, let $s : A \to E$ be a splitting of p assumed without loss of generality to be linear. Then

$$T(s(aa') - s(a)s(a')) = \hat{\phi}(aa') - \hat{\phi}(a)\hat{\phi}(a') = 0.$$

Hence $s(aa') - s(a)s(a') \in \text{Ker}T$. Now $p(s(aa') - s(a)s(a')) = aa' - aa' = 0$ hence

$$s(aa') - s(a)s(a') \in \text{Ker}T \cap B,$$

but since T acts as the identity on B, this means $\ker T \cap B = \{0\}$ and so s is multiplicative. The fact that s is $*$-preserving and equivariant can be proved similarly.

We can now define an equivalence relation on $\text{Hom}_G(A, Q(B))$ by saying $\phi \sim \phi'$ if and only if there are degenerate $*$-homomorphisms $\phi_0, \phi_0' \in \text{Hom}_G(A, Q(B))$ such that[4]

$$\phi \oplus \phi_0 \simeq_u \phi' \oplus \phi_0'.$$

Definition 1.72. The equivariant extension semigroup is defined as

$$\text{Ext}_G(A, B) = \text{Hom}_G(A, Q(B))/\sim$$

and $\text{Ext}_G(A, B)^0$ denotes the subgroup of invertible elements in $\text{Ext}_G(A, B)$.

As in the non-equivariant case, there is a way to characterize G-extensions which are invertible using splittings. Let $\mathbb{K}_G := \mathbb{K}(\bigoplus_{n\in\mathbb{N}} L^2(G))$ with G acting diagonally by the regular representation. Then we have

Theorem 1.73 ([51, Theorem 8.1])**.** *An extension*

$$0 \to B \otimes \mathbb{K}_G \to E \to A \otimes \mathbb{K}_G \to 0 \tag{1.14}$$

is invertible if and only if the sequence

$$0 \to B \otimes \mathbb{K}_G \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G \to A \otimes \mathbb{K}_G \otimes \mathbb{K}_G \to 0 \tag{1.15}$$

obtained by tensoring everything with $\mathbb{K}_G$, is equivariantly semisplit.

Clearly if the extension $0 \to B \otimes \mathbb{K}_G \to E \to A \otimes \mathbb{K}_G \to 0$ is equivariantly semisplit in the first place, by the equivariant completely positive map

$$s : A \otimes \mathbb{K}_G \to E$$

then the sequence of equation (1.15) would also be equivariantly semisplit, by the map

$$s \otimes \text{id} : A \otimes \mathbb{K}_G \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G$$

so being equivariantly semisplit after tensoring with $\mathbb{K}_G$ is weaker than being equivariantly semisplit.

We will need the following definition:

[4] This should be reminiscent of the way Kasparov's KK-groups are defined as operator homotopy equivalence classes of Kasparov modules modulo degenerate modules.

Definition 1.74 ([51, Definition 9.1]). A G-C*-algebra B is called K-proper if for any countably generated Hilbert B-module E with a continuous G-action, there is an isomorphism

$$E \oplus (B \otimes L^2(G)^\infty) \simeq B \otimes L^2(G)^\infty$$

as Hilbert B-modules, where

$$L^2(G)^\infty = \bigoplus_{n \in \mathbb{N}} L^2(G)$$

is the infinite sum of $L^2(G)$.

So a C*-algebra is K-proper if it satisfies an equivariant version of Kasparov's stabilization theorem. The definition is a generalization of proper actions on topological spaces, in the sense that if $B = C_0(X)$ for some locally compact Hausdorff proper G-space X, then B is K-proper.

This is by no means the only candidate for a generalization of proper actions to G-C*-algebras (see [48] [38], [30]). See also [37, Theorem 8.5] for several equivalent definitions of K-proper actions. We are now ready to state the main theorem of [51] which reads

Theorem 1.75 ([51] Theorem9.2). *Assume A, B are G-C^*-algebras with B K-proper. Then*

$$\mathrm{Ext}_G(A, B \otimes \mathbb{K}_G)^0 \simeq \mathrm{KK}^1_G(A, B)$$

We will not repeat the proof here. However, in the next section we will see how to assign a class to an equivariantly semisplit G-extension of commutative C*-algebras (see Example 1.84). Commutative C*-algebras are very far from being stable though, so we will first need make precise what it means for an equivariant KK-cycle to be "associated" with an extension of non-separable and/or non K-proper C*-algebra. If

$$0 \to B \xrightarrow{\iota} E \xrightarrow{p} A \to 0$$

is a G-extension, then by tensoring everything with $\mathbb{K}_G$ we get a G-extension

$$0 \to B \otimes \mathbb{K}_G \xrightarrow{\iota \otimes id} E \otimes \mathbb{K}_G \xrightarrow{p \otimes id} A \otimes \mathbb{K}_G \to 0$$

which in turn gives an element in $\mathrm{Ext}_G(B \otimes \mathbb{K}_G, A \otimes \mathbb{K}_G)^0 = KK^1_G(B, A)$. We refer to this element as the class corresponding to the extension.

1.7 An example from harmonic analysis

We will now introduce an example, studied in [36], where classical harmonic analysis is used to produce an equivariant completely positive splitting of an extension and a class in equivariant KK-theory.

First, we will need to define the geodesic compactification of symmetric spaces of non-positive curvature. Our main reference will be [16, Section 1.7]:

Definition 1.76. A Hadamard manifold is a complete simply connected Riemannian manifold with non-positive sectional curvature.

The important feature of Hadamard manifolds is the following:

Theorem 1.77 (Hadamard)**.** *For any Hadamard manifold X and any point $x_0 \in X$, the exponential map* $\exp : T_{x_0}X \to X$ *is a global diffeomorphism.*

Let X be a Hadamard manifold of dimension n.

Definition 1.78. For each $x \in X$ and $v \in T_xX$, we denote by $\gamma_{x,v} : \mathbb{R} \to X$ the unique geodesic in X with

$$\gamma_{x,v}(0) = x \qquad \text{and} \qquad \gamma'_{x,v}(0) = v$$

The geodesic $\gamma_{x,v}$ is called a directed geodesic (or a geodesic ray) centered at x in direction v.

Now fix a point $x_0 \in X$ and write $\gamma_v := \gamma_{x_0,v}$. We define:

Definition 1.79 ([16, Section 1.27 1.7])**.** The geodesic boundary of X is defined to be

$$X(\infty) := \{\gamma_v \mid \|v\| = 1\}$$

We topologize $X(\infty)$ in such a way that the map

$$X(\infty) \to S^{n-1} \subset T_{x_0}X, \qquad \gamma_v \mapsto v$$

is a homeomorphism, i.e. the pullback or weak topology induced by the map $X(\infty) \to S^{n-1}$.

For any open set $U \subset X(\infty) \simeq S^{n-1} \subset T_{x_0}X$ and $r > 0$, define

$$S_r^U := \left(\bigcup_{v \in U \ t > r} \gamma_v(t) \right) \sqcup U$$

which is a truncated cone of rays centered at x_0 in the direction of vectors in U.

Definition 1.80 ([16] Sec. 1.7)**.** The geodesic compactification of X is (as a set) the disjoint union

$$\overline{X} = X \cup X(\infty).$$

We endow $X \cup X(\infty)$ with the topology generated by the open sets of X together with S_r^U, where U ranges over all open sets in S^{n-1} and r over all positive numbers.

The topology on $X \cup X(\infty)$ given in Definition 1.80 is called the *cone topology*, and makes $X \cup X(\infty)$ homeomorphic to the closed n-ball, with X its interior points and $X(\infty) \simeq S^{n-1}$ its boundary.

It is useful to have the following convergence criterion in mind: Given a sequence of points $(x_i)_{i\in\mathbb{N}}$ in X, then $x_i \to v \in X(\infty)$ if and only if $d(x_i, x_0) \to \infty$ and the vector $v_i \in T_{x_0}X$ pointing in the direction of the geodesic connecting x_0 and x_i converges to v in $S^{n-1} \subset T_{x_0}X$. Thus for any geodesic ray γ_v and any sequence of points $x_i \in \gamma_v(\mathbb{R}^+)$ for which $d(x_0, x_i) \to \infty$, the sequence $(x_i)_{i\in\mathbb{N}}$ converges in the geodesic compactification to $v \in S^{n-1} \simeq X(\infty)$. Not all convergent sequences are of this form though, as there are sequences $x_i \in X$ such that $x_i \to v \in X(\infty)$, but $d(x_i, \gamma_v) := \inf_{t>0} d(x_i, \gamma_v(t)) \to \infty$ as $i \to \infty$.

There is another way to describe the points in $X(\infty)$, sometimes more convenient in practice, which we will now define. Let

$$\mathcal{P} = \{\gamma_{x,v} \mid x \in X, v \in S^{n-1} \subset T_xX\}$$

be the set of all (unit speed) geodesic rays in X (Definition 1.78). Then define an equivalence relation on $\mathcal{P}$ by

$$\gamma \sim \gamma' \Leftrightarrow \sup_{t\in\mathbb{R}^+} d(\gamma(t), \gamma'(t)) < \infty \tag{1.16}$$

We have the following lemma

Lemma 1.81 ([16, Proposition 1.7.3])**.** *For any $x_0 \in X$, the set $\mathcal{P}/\sim$ is in bijection with the geodesic rays of the form*

$$\gamma_{x_0,v} \qquad \text{for } v \in S^{n-1} \subset T_{x_0}X.$$

Example 1.82. It may be helpful to think of what Lemma 1.81 looks like when X is the euclidean space $\mathbb{R}^n$. In this case the Lemma simply states that if γ is any geodesic ray in $\mathbb{R}^n$, then γ is parallel to a unique geodesic ray centered at 0. So Lemma 1.81 is an analogue of Euclid's parallel postulate[5] for Hadamard manifolds.

However, the case of $\mathbb{R}^n$ may leave the reader wondering why we use geodesic rays at all, and not just geodesics. After all in $\mathbb{R}^n$ for two geodesics $\gamma, \gamma' : \mathbb{R} \to \mathbb{R}^n$ (i.e. straight lines) the conditions

$$\sup_{t\in\mathbb{R}^+} d(\gamma(t), \gamma'(t)) < \infty \tag{1.17}$$

and

$$\sup_{t\in\mathbb{R}^-} d(\gamma(t), \gamma'(t)) < \infty \tag{1.18}$$

[5] which is equivalent to the statement that any straight line is parallel to a unique straight line through a point

and

$$\sup_{t\in\mathbb{R}} d(\gamma(t),\gamma'(t)) < \infty$$

are all equivalent to the condition that $d(\gamma(t),\gamma'(t))$ is constant in t. We mention that in case the space X has negative curvature, there are geodesics which satisfy (1.17) but do not satisfy (1.18), and that are not constant in t in either direction. That is, there are geodesics which are "parallel" in the positive direction, but diverge in the negative direction.

Under the bijection established by Lemma 1.81 the set $\mathcal{P}/\sim$ inherits a topology from $X(\infty)$. It is thus possible to view $X(\infty)$ as a quotient of the space of all geodesics in X. One bonus of this description is that the G-action on $X(\infty)$ is now simple to express. It is just

$$g[\gamma_{x,v}] = \gamma_{gx,(dg)v}, \tag{1.19}$$

which is the image of $\gamma_{x,v}$ under the action of G on X.

With X any Riemannian space, we denote by $\mathrm{Iso}(X)$ its isometry group, and $\mathrm{Iso}(X)^0$ the connected component of the isometry group of X.

Proposition 1.83 ([16] p. 30)**.** *Any isometry $\phi : X \to X$ extends to a homeomorphism*

$$\overline{\phi} : X \cup X(\infty) \to X \cup X(\infty).$$

The extension makes $X \cup X(\infty)$ a $G = \mathrm{Iso}(X)$-space. The action of G on $X(\infty)$ is given by equation (1.19)*.*

The case of the hyperbolic spaces Now assume $X = \mathbb{H}^n$ is the real hyperbolic n-space. Then $\mathbb{H}^n$ is a Hadamard manifold, and thus we can define its geodesic compactification

$$\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial\mathbb{H}^n.$$

By Proposition 1.83, the isometry group $G = \mathrm{Iso}(\mathbb{H}^n)^0 = SO^0(1,n)$, acts on $\overline{\mathbb{H}^n}$ by homeomorphisms that extend the isometric action of G on $\mathbb{H}^n$.

Indeed if $n = 2$, then $\mathbb{H}^2$ can be identified with the disk $\mathbb{D} \subset \mathbb{R}^2 \simeq \mathbb{C}$. An orientation preserving isometry acts on $\mathbb{H}^2$ by a Möbius transformation:

$$z \mapsto e^{i\theta}\frac{z-a}{1-\overline{a}z} \qquad a \in \mathbb{D}, \theta \in \mathbb{R}.$$

The Möbius actions act by homeomorphisms on the closed disk $\overline{\mathbb{D}}$ but they are not isometries with respect to the angular metric on S^1 unless $a = 0$. We then get an equivariant extensions of G-C*-algebras

$$0 \to C_0(\mathbb{H}^n) \xrightarrow{\iota} C(\overline{\mathbb{H}^n}) \xrightarrow{q} C(\partial\mathbb{H}^n) \to 0. \tag{1.20}$$

Let now Γ be a discrete torsion-free subgroup $\Gamma \subset G = \mathrm{Iso}(\mathbb{H}^n)^0 = SO(n,1)^0$. Using Theorem 1.37, we get an exact sequence of crossed products

$$0 \to C_0(\mathbb{H}^n) \rtimes_r \Gamma \xrightarrow{\iota \rtimes id} C(\overline{\mathbb{H}^n}) \rtimes_r \Gamma \xrightarrow{q \rtimes id} C(\partial \mathbb{H}^n) \rtimes_r \Gamma \to 0. \tag{1.21}$$

By Theorem 1.63 this induces a six term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{H}^n) \rtimes_r \Gamma) & \longrightarrow & K_0(C(\overline{\mathbb{H}^n}) \rtimes_r \Gamma) & \longrightarrow & K_0(C(\partial\mathbb{H}^n) \rtimes_r \Gamma) \\ {\scriptstyle\partial}\big\uparrow & & & & \big\downarrow{\scriptstyle\partial} \\ K_1(C(\partial\mathbb{H}^n) \rtimes_r \Gamma) & \longleftarrow & K_1(C(\overline{\mathbb{H}^n}) \rtimes_r \Gamma) & \longleftarrow & K_1(C_0(\mathbb{H}^n) \rtimes_r \Gamma) \end{array}$$

For any torsion free discrete subgroup Γ of G, Γ acts properly and freely on $\mathbb{H}^n$ and it is well known that this implies

$$K_i(C_0(\mathbb{H}^n) \rtimes_r \Gamma) = K_i(C_0(\mathbb{H}^n/\Gamma)).$$

In case Γ satisfies certain technical conditions, which are known to hold if Γ is cocompact[6], we also have

$$K_i(C(\overline{\mathbb{H}}^n) \rtimes_r \Gamma) = K_i(C_r^*(\Gamma)).$$

Thus the K-theory of two interesting C*-algebras make their appearance in the above 6-term sequence: One is the continuous functions on the space $\mathbb{H}^n/\Gamma$, which is a classifying space for free and proper Γ actions, the other is the reduced group C*-algebra of Γ (see Ex. 1.34) $C_r^*(\Gamma)$.

The extension of equation (1.20) is equipped with a completely positive equivariant splitting

$$C(\partial\mathbb{H}^n) \to C(\overline{\mathbb{H}^n})$$

defined by integration against the Poisson kernel which for $\mathbb{H}^n$ with the ball model takes the form

$$P : \mathbb{H}^n \times \partial\mathbb{H}^n \to \mathbb{R} \qquad P(x,\xi) = \frac{1-|x|^2}{|x-\xi|^n} \tag{1.22}$$

where $|-|$ is the euclidean distance. It is a well-known fact that all bounded harmonic functions f on $\mathbb{H}^n$ which can be extended continuously to $\partial\mathbb{H}^n$ can be written as an integral

$$f(x) = \int_{\partial\mathbb{H}^n} P(x,\xi)\hat{f}(\xi)d\lambda(\xi),$$

for some function $\hat{f} \in C(\partial\mathbb{H}^n)$. The assignment $\hat{f} \mapsto f$ has the following properties:

[6]The condition on Γ we are tacitly requiring is that it should satisfy the Baum-Connes conjecture with coefficients in $\mathbb{C}$ and $C(\overline{H}^n)$. Then $K(C_0(\mathbb{H}^n) \rtimes_r \Gamma) = K_i^{top}(\Gamma, C_0(\mathbb{H}^n)) = K_i^{top}(\Gamma, \mathbb{C}) = K_i(C_r^*(\Gamma))$ where the first and third equality follows from the Baum-Connes conjecture, and the second is the content of Lemma 5 of [17]

- The function f extends to the whole geodesic compactification $\overline{\mathbb{H}}^n$ and restricts to $\hat{f}$ on $\partial\mathbb{H}^n \simeq S^{n-1}$;
- The map $\hat{f} \mapsto f$ is a G-equivariant completely positive splitting of the quotient map $q : C(\overline{\mathbb{H}^n}) \to C(\partial\mathbb{H}^n)$.

We saw in Section 1.6 how these equivariant semisplittings are used to determine if the extension can be represented by a class in $\mathrm{KK}^1_\Gamma(C(\partial\mathbb{H}^n), C_0(\mathbb{H}^n))$. We will see later in Chapter IV how to create a KK^1_G-cycle representing a G-equivariant extension using a G-equivariant semisplitting. In the case of the extension of equation (1.20) however, we can clearly construct the corresponding Kasparov module as follows:

Example 1.84. We repeat here the construction in Section 3.2 of [36]. Let $G = \mathrm{Iso}(\mathbb{H}^n)^0 = SO(1,n)^0$ and write $\mu_x = P(x,-)dv$ for $x \in \mathbb{H}^n$, where dv is the Lebesgue measure on $\partial\mathbb{H}^n \simeq S^{n-1}$. The family μ_x satisfies

$$\mu_{gx} = g_*\mu_x, \qquad g \in G,\ x \in \mathbb{H}^n$$

and μ_x varies continuously with x in the sense that if $x_i \to x$, then

$$\int_{\partial\overline{\mathbb{H}^n}} f(v)d\mu_{x_i}(v) \to \int_{\partial\overline{\mathbb{H}^n}} f(v)d\mu_x(v)$$

for all $f \in C(\partial\overline{\mathbb{H}^n})$. Now define

$$T_1\mathbb{H}^n = \mathbb{H}^n \times \partial\overline{\mathbb{H}}^n.$$

The map

$$\rho : C_c(T_1\mathbb{H}^n) \to C_c(\mathbb{H}), \qquad \rho(\Psi)(x) := \int_{\partial\overline{\mathbb{H}^n}} \Psi(x,v)d\mu_x(v)$$

is a conditional expectation, hence (see Example 1.49) we get a pre-Hilbert $C_0(\mathbb{H}^n)$-module structure on $C_c(T_1\mathbb{H}^n)$ given by the inner product

$$\langle f, f'\rangle(x) := \rho(\overline{f}f')(x), \qquad x \in \mathbb{H}^n,\ f, f' \in C_c(T_1\mathbb{H}^n).$$

Denote by $L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}$ the completion of $C_c(T_1\mathbb{H}^n)$ with respect to this inner product. As the notation suggests, we think of $L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}$ as sections of a continuous field of Hilbert spaces $(L^2(C(\partial\overline{\mathbb{H}^n})), \mu_x)$, fibered over $\mathbb{H}^n$, which vanish at infinity. We also have an action of $C(\partial\overline{\mathbb{H}^n})$ on $L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}$ by adjointable operators, given by multiplication:

$$(f \cdot \Psi)(x,v) = f(v)\Psi(x,v) \qquad f \in C(\partial\overline{\mathbb{H}^n}), \Psi \in C_c(T_1\mathbb{H}^n).$$

Let $p \in \mathcal{L}(L^2(T_1\mathbb{H}^n, \mu_x)_{x \in C_0(\mathbb{H}^n)})$ be the adjointable operator given by

$$p(\Psi)(x,v) = \int_{\partial\overline{\mathbb{H}^n}} \Psi(x,w)d\mu_x(w).$$

It can be shown that p is a projection, i.e. that $p^* = p^2 = p$. Moreover p commutes with the action of G since

$$\begin{aligned} p(g\Psi)(x,v) &= \int_{\partial\overline{\mathbb{H}^n}} \Psi(g^{-1}x, g^{-1}w)d\mu_x(w) = \Psi(g^{-1}x, g^{-1}w)d\mu_x(w) \\ &= \Psi(g^{-1}x, w)(g^{-1})_* d\mu_x(w) = \Psi(g^{-1}x, w)d\mu_{g^{-1}}(w) = g(p\Psi)(x,v). \end{aligned}$$

The triple

$$(L^2(T_1\mathbb{H}^n, \mu_x)_{C_0(\mathbb{H}^n)}, C(\partial\overline{\mathbb{H}^n}), 2p-1)$$

is an Kasparov G-module representing the class of extension (1.20) in $\mathrm{KK}^1_\Gamma(C(\partial\overline{\mathbb{H}^n}), C_0(\mathbb{H}^n))$ (see [36, Theorem 3.4]).

The space $\mathbb{H}^n$ is an example of a symmetric space of noncompact type of rank 1. In the next sections, we will discuss what happens when we try to extend the construction in Example 1.84 to a larger class of spaces, namely the symmetric spaces of noncompact type (of arbitrary rank).

Chapter 2

Symmetric spaces and conformal densities

2.1 Background on symmetric spaces

One of the standard reference for the theory of symmetric spaces is the book of Helgason [23] from which most of the material in this section is taken. For the general theory of manifolds of nonpositive curvature the book of Eberlein [16] is recommended. The section about Lie theory is taken from the book of Knapp [33] and Borel and Ji [7].

All manifolds in this and later sections will, unless stated otherwise, be assumed to be *complete Riemannian manifolds*. Riemannian globally symmetric spaces are defined in either of the following ways

1. **The geometric way**: As complete Riemannian spaces for which the geodesic symmetries are global isometries.

2. **The group theoretic way**: As Riemannian symmetric pairs (G, K)

3. **The Lie algebraic way**: As orthogonal symmetric Lie algebras $(\mathfrak{g}, s)$.

There are maps to move between the definitions in the direction

$$(1) \longrightarrow (2) \longrightarrow (3) \tag{2.1}$$

but going the other way generally requires some choice. In what follows, we will switch between these three points of view, so let us go through the details of how each are defined and how they are related, starting with the geometric definition, which will be the one we use most often in this work.

2.1.1 The geometric way

Definition 2.1. Given a manifold X, a *geodesic symmetry* or *central symmetry* at $p \in X$ is a pair (s_p, U_p) where $U_p \subset X$ is an open neighborhood of p and $s_p : U_p \to U_p$ is diffeomorphism which flips every geodesic centered at p contained in U_p.

Such geodesic symmetries always exist for any manifold since we can construct them at a point $p \in X$ using the composition

$$\exp_p \circ (-\mathrm{Id}) \circ \exp_p^{-1} : U_p \to U_p$$

which is a local diffeomorphism and "radially isometric", meaning it preserves the distance from p.

Let us for a moment think about what happens for the sphere $X = S^n$. Given a point p in S^n, then a geodesic symmetry at p is simply a rotation by an angle of π about an axis through p. It follows that any geodesic symmetry on S^n is the restriction of a global isometry of S^n. This leads us to the following definition:

Definition 2.2. A *Riemannian globally symmetric space* (henceforth a symmetric space) is a complete Riemannian manifold X for which each geodesic symmetry is the restriction of a global isometry of X. A *Riemannian locally symmetric space*, is a space where each geodesic symmetry is a local isometry.

The following theorem is very useful as it shows any locally symmetric space has a symmetric space as universal cover:

Theorem 2.3 ([23, Theorem IV.6.5.6])**.** *A simply connected locally symmetric space is globally symmetric.*

Though it will not be used in this thesis, we mention that there is another characterization of locally symmetric space found in the literature, which succinctly states that a manifold is locally symmetric if and only if its curvature tensor is invariant under all parallel translates (see [23, Theorem IV.2.1.3]). The "only if" part of the implication is proved by the existence of so-called transvections along geodesics, which are 1-parameter families of isometries on symmetric spaces that implement parallel transports along geodesics. Let us see how they are defined.

Let $\gamma : \mathbb{R} \to X$ be a geodesic centered at some point $p \in X$. For some $\epsilon > 0$ and some geodesic symmetry $s_p : U_p \to U_p$ around p, the segment $\gamma(-\epsilon, \epsilon) \subset U_p$. Now let $q_t = \gamma(t)$ for $t \in (-\epsilon/2, \epsilon/2)$ and $s_{q_t} : U_{q_t} \to U_{q_t}$ a geodesic symmetry. The composition $p_t := s_p \circ s_{q_t} : U_p \cap U_{q_t} \to U_p \cap U_{q_t}$ is a local isometry which acts as translation along γ sending p to $\gamma(2t)$.

Definition 2.4 (Transvections)**.** The local isometries of the form $s_p \circ s_q$, where s_p, s_q are geodesic symmetries, are called local transvections or transvections if s_p, s_q are (global) isometries. If X is globally symmetric the transvections form a subgroup of $\mathrm{Iso}(X)$ called the transvection group.

So why do we care about transvections? As we hinted to earlier, one reason is they implement parallel transport along a geodesic which shows that the curvature tensor must be invariant under parallel transports, as it is invariant under isometries. More precisely, assume X is globally symmetric now. Given a geodesic γ centered at p, the *transvections along* γ form a 1-parameter subgroup of isometries p_t generated by flips along points in $\gamma(\mathbb{R})$, which act on points of γ by translation along γ, and act on vectors on $\gamma(\mathbb{R})$ by parallel transport along γ. Later we will see that the tangent space of X can be identified with a subset $\mathfrak{p} \subset \mathfrak{g}$ of the Lie algebra of $G = \mathrm{Iso}(X)^0$, and the transvections are then the elements of G of the form (see [16, Proposition 2.1.1])

$$\exp(tV)x_0$$

for some fixed point $x_0 \in X$ and $V \in \mathfrak{p} \simeq T_{x_0}X$, hence any geodesic can be written as

$$t \mapsto \exp(tV)x_0 \qquad V \in T_{x_0}X.$$

More generally we have the following:

Proposition 2.5 ([23, Th. IV.7.2])**.** *Let X be any symmetric space, $G = \mathrm{Iso}(X)^0$ and $K = \mathrm{Stab}_{x_0}G \subset G$. The Lie algebra $\mathfrak{g}$ of G decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k}$ the Lie algebra of K, and $\mathfrak{p} \simeq T_{x_0}X$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a linear subspace such that for all $x, y, z \in \mathfrak{a}$ $[x, [y, z]] \in \mathfrak{a}$. Then $\exp(\mathfrak{a})x_0 \subset X$ is a totally geodesic submanifold of X, and every totally geodesic submanifold arises in this way.*

In Proposition 2.5 $\mathfrak{a}$ need not be be a subalgebra. Subspaces in $\mathfrak{p}$ satisfying the conditions in Proposition 2.5 are called Lie triple systems. An immediate consequence of the existence of transvections is the following lemma:

Lemma 2.6 ([16, Proposition 2.1.1])**.** *Let X be a globally symmetric space, then the identity component $\mathrm{Iso}(X)^0 \subset \mathrm{Iso}(X)$ of the isometry group of X acts transitively on X.*

Proof. By completeness any two points $p, q \in X$ can be connected by a geodesic γ, but on $\gamma(\mathbb{R}) \subset X$ the transvection group along γ acts transitively. □

2.1.2 The group theoretic way

Let us now look at the group theoretic definition of symmetric spaces.

Definition 2.7. A *Riemannian symmetric pair* (henceforth a symmetric pair) is a pair of groups (G, K) where G is connected Lie group and $K \subset G$ a closed subgroup satisfying the following:

- there is an involutive automorphism $\theta : G \to G$ such that $(G^\theta)^0 \subset K \subset G^\theta$, where G^θ is fixed point group of θ;

- The subgroup $\mathrm{Ad}_G(K) \subset GL(\mathfrak{g})$ is compact.

A symmetric pair (G, K) is called *effective* if it satisfies the following additional condition

- $Z(G) \cap K$ is discrete.

The second condition in Definition 2.7 tell us that K is compact in $G/Z(G)$ (i.e. $K/(Z(G) \cap K)$ is compact), since $\ker \mathrm{Ad}_G = Z(G)$ when G is connected. We do not require K to be compact in Definition 2.7. For this reason many authors add the condition that the group G has finite center. This is done because G^θ contains the center of G and will be compact if and only if the center is finite (see Proposition 2.19 below).

Proposition 2.8 ([23, Th. VI.2.1] , [23, Th. VI.2.2 (ii)])**.** *Let (G, K) be a symmetric pair. Assume G/K is a symmetric space without compact or Euclidean factors in its de Rahm decomposition. Then K contains a unique maximal compact subgroup $H' \subset K$ which is also a maximal compact subgroup of G. All maximal compact subgroups of a connected semisimple Lie group G are connected and conjugate.*

Note that $K_1 \subset G$ is conjugate to K is the same thing as saying K_1 fixes gK in G/K for some $g \in G$.

Proposition 2.9 (Proposition IV.3.4 [23])**.** *Let (G, K) be a Riemannian symmetric pair. The homogeneous space G/K admits a G-invariant metric, and any such metric makes it into a Riemannian symmetric space. The symmetry s_e at $[e] \subset G/K$ is given by*

$$s_e(gK) = \theta(g)K.$$

If L_g denotes the left action of $g \in G$ on G/K by left multiplication, we have

$$s_e L_{\theta(g)} s_e = L_g \qquad \forall\ g \in G.$$

Proof. We will only show existence of such a metric, and refer to [23, Proposition IV.3.4] for the complete proof. We write $X = G/K$ for the quotient of G by K, $0 := [K] \in X$ for the class of K in X and $d\theta$ for the differential $d\theta : \mathfrak{g} \to \mathfrak{g}$ of θ. Since θ is an involution, so is $d\theta$, hence if we write $\mathfrak{k} = \mathrm{Eig}_{+1}(d\theta)$ and $\mathfrak{p} = \mathrm{Eig}_{-1}(d\theta)$ for the $+1$ eigenspaces and -1 eigenspaces of $d\theta$ respectively, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Now since $K \subset G^\theta$, we get $\mathrm{Ad}(K)d\theta = d\theta\,\mathrm{Ad}(K)$, hence $\mathrm{Ad}(K)$ preserves the eigenspaces of $d\theta$, and restricts to a map on $\mathfrak{p}$ and $\mathfrak{k}$. We thus get a well-defined K-action on $T_0X = \mathfrak{p}$.

Let $B_0 : T_0X \times T_0X \to \mathbb{C}$ be any inner product on $T_0X = \mathfrak{p}$. By compactness of $\mathrm{Ad}(K) \subset GL(\mathfrak{p})$ we may define a K-invariant inner product by averaging with respect to the normalized Haar measure μ on $\mathrm{ad}(K)$ as follows:

$$\langle X, Y \rangle_0 := \int_{T \in \mathrm{ad}(K)} B_0(TX, TY)\, d\mu(T) \qquad X, Y \in \mathfrak{p} = T_0X.$$

The inner product $\langle -, - \rangle_0$ determines a Riemannian metric on G/K by

$$\langle X, Y \rangle_x := \langle dL_{g^{-1}} X, dL_{g^{-1}} Y \rangle_0 \qquad \forall\, X, Y \in T_x X.$$

This is clearly a G-invariant metric (as $L_g \circ L_{g'} = L_{gg'}$) giving G/K the structure of a Riemannian manifold. Write s_0 for the map $s_0(gK) = \theta(g)K$. Then $(ds_0)_0 = d\theta = -I : \mathfrak{p} \to \mathfrak{p}$ is the map flipping every line through the origin and so a geodesic symmetry. Any other symmetry is conjugate to s_0 by some L_g. □

The G-invariant metric of Proposition 2.9 is in general not unique. For instance if $\mathfrak{p}$ is a reducible $\mathrm{Ad}(K)$-module, we can scale the metric on each $\mathrm{Ad}(K)$-invariant component by a scalar and get a different G-invariant metric. However, in the proof of Proposition 2.9, the geodesic symmetry s_p is constructed in a way that does not depend on the choice of invariant metric. Neither does the Riemann curvature tensor [23, Theorem IV.4.2] which is determined entirely by the Lie algebra $\mathfrak{g}$ and given by the formula

$$R(X, Y)Z = -[[X, Y], Z] \tag{2.2}$$

or scaled by 1/4 depending on the convention.

Symmetric spaces with a unique Riemannian metric (up to scaling) are called *irreducible*, and are precisely those non-Euclidean symmetric spaces which admit no non-trivial de Rham decomposition (see the end of Section 2.1.3).

Definition 2.10. For a Lie algebra $\mathfrak{g}$, the Killing form is the (possibly degenerate) bilinear form on $\mathfrak{g}$ given by

$$B_0(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y), \qquad X, Y \in \mathfrak{g}.$$

The following example shows us the "standard" choice of inner product on the homogeneous space G/K with G semisimple:

Example 2.11. Let $X = G/K$ be a noncompact symmetric space without Euclidean factors in its de Rahm decomposition. Denote by $T_0 X = \mathfrak{p}$ the tangent space at 0. The Killing form B_0 (Definition 2.10) restricts to an inner product on $\mathfrak{p}$ and for any $k \in K$ and $X, Y \in \mathfrak{p}$ we have

$$\mathrm{ad}_{\mathrm{Ad}(k)X} Y = [\mathrm{Ad}(k)X, Y] = \mathrm{Ad}(k)([X, \mathrm{Ad}(k)^{-1}(Y)])$$

so $\mathrm{ad}_{\mathrm{ad(k)X}} = \mathrm{Ad}(k) \circ \mathrm{ad}_x \circ \mathrm{Ad}(k)^{-1}$. It follows that

$$\begin{aligned} B_0(\mathrm{Ad}(k)(X), \mathrm{Ad}(k)(Y)) &= \mathrm{Tr}(\mathrm{Ad}(k) \circ \mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{Ad}(k)^{-1}) \\ &= \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = B_0(X, Y). \end{aligned}$$

is indeed $\mathrm{Ad}(K)$-invariant.

Cartan's criterion states that $\mathfrak{g}$ is semisimple if and only if B_0 is non-degenerate when defined on the whole of $\mathfrak{g}$. The group $G = \mathrm{Iso}(X)^0$ is semisimple if X has no Euclidean factors (see the discussion on p. 198 [23]), hence B_0 is non-degenerate on $\mathfrak{p}$.

2.1.3 The Lie algebraic way

We now turn to the Lie algebraic way to define symmetric spaces.

Definition 2.12. Let Θ be an involutive Lie algebra homomorphism. Then Θ is called a Cartan involution if the bilinear form

$$B_\Theta(X,Y) := -B(X,\Theta(Y)) \qquad X,Y \in \mathfrak{g}$$

is a positive definite bilinear form on $\mathfrak{g}$, where B denotes the Killing form on $\mathfrak{g}$ (see Ex. 2.11). The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into eigenspaces of Θ, with $\mathfrak{k} = \mathrm{Eig}_+\Theta$ and $\mathfrak{p} = \mathrm{Eig}_-\Theta$, is called the Cartan decomposition of $\mathfrak{g}$.

Finally, we have

Definition 2.13. An *orthogonal symmetric Lie algebra* $(\mathfrak{g},\Theta)$ is a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ together with an involutive automorphism $\Theta : \mathfrak{g} \to \mathfrak{g}$ such that

- $\mathfrak{k} = \mathfrak{g}^\Theta$ (the fixed point algebra of Θ) is a compact subalgebra of $\mathfrak{g}$
- $\mathfrak{k}$ does not intersect the center of $\mathfrak{g}$.

An orthogonal symmetric Lie algebra is called *irreducible* if ([23] VIII.5.5)

- $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ contains no ideal of $\mathfrak{g}$
- the algebra $\mathrm{ad}_\mathfrak{g}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

An orthogonal symmetric Lie algebra is called *effective* if it satisfies

- $Z(\mathfrak{g}) \cap \mathfrak{k} = 0$.

The next proposition shows exactly what involutions of $\mathfrak{g}$ are Cartan involutions:

Proposition 2.14 ([23, Corollary III.7.1])**.** *Let $\mathfrak{g}$ be a Lie algebra with a decomposition*

$$\mathfrak{g} : \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ a subspace. Then the map

$$s : \mathfrak{g} \to \mathfrak{g} \qquad s(x,y) = (x,-y), \quad x \in \mathfrak{k}, y \in \mathfrak{p}$$

is is an automorphism of $\mathfrak{g}$ if and only if s is a Cartan involution.

It could be helpful to see an example of Cartan involutions:

Example 2.15 ([33, Proposition 6.28]). Any real semisimple Lie algebra $\mathfrak{g}$ is isomorphic to a matrix Lie algebra which is closed under transposition. If $\mathfrak{g}$ is endowed with a Cartan involution Θ, this isomorphism may be chosen so that Θ corresponds to the map

$$\Theta(g) = (-g)^t$$

where t denotes the transpose.

Proposition 2.14 tells us that the automorphism Θ in Definition 2.13 is a Cartan involution in the sense of Definition 2.12.

The Riemannian structure on G/K for $G = \mathrm{Lie}(\mathfrak{g})$ of an irreducible orthogonal symmetric Lie algebra $\mathfrak{g}$ is unique up to scaling. To see why, let $(-,-)$ and $\langle -,- \rangle$ be two K-invariant inner products on $T_0X = \mathfrak{p}$. By Riesz representation, there exists a positive definite matrix A such that $(AX, Y) = \langle X, Y \rangle$. Since both inner products are $\mathrm{Ad}(K)$-invariant, A commutes with $\mathrm{ad}(K)$, which by Schur's lemma implies that $A = \lambda I$ for some scalar λ.

The second condition in Definition 2.13 ensures that any semisimple orthogonal symmetric Lie algebra decomposes as a direct sum

$$\mathfrak{g} = \bigoplus \mathfrak{g}_i$$

of Θ-invariant irreducible orthogonal symmetric Lie algebras, mutually orthogonal with respect to the Killing form.

The three types of irreducible symmetric spaces

Throughout this section $(\mathfrak{g}, \Theta)$ will denote an effective symmetric orthogonal Lie algebra and B the (possibly degenerate) inner product given by the Killing form. As usual we denote by $\mathfrak{k} \subset \mathfrak{g}$ the fixed point subalgebra of Θ and $\mathfrak{p}$ the -1 eigenspace of Θ, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then

Definition 2.16 ([23, p. 230]). The Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is said to be of

- compact type if B is negative definite on $\mathfrak{g}$;
- noncompact type if B is non-degenerate (equiv. $\mathfrak{g}$ is semisimple) and $B|_{\mathfrak{p}}$ is positive definite;
- Euclidean type if $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$.

We say that a symmetric pair (G, K) is of compact/noncompact/Euclidean type if its associated orthogonal Lie algebra is. Similarly, a symmetric space X is of compact/noncompact/Euclidean type if its associated symmetric pair (G, K) is of compact/noncompact/Euclidean type, respectively.

We have the following decomposition theorem

Theorem 2.17 (Theorem V.1.1 [23]). *The Lie algebra $\mathfrak{g}$ decomposes into a direct sum of Θ-stable subalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ such that*

- *The decomposition is orthogonal with respect to the Killing form;*
- $(\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0})$, $(\mathfrak{g}_+, \Theta|_{\mathfrak{g}_+})$, $(\mathfrak{g}_-, \Theta|_{\mathfrak{g}_-})$ *are of Euclidean, compact and noncompact type respectively.*

Note that Theorem 2.17 shows that any effective orthogonal Lie algebra $\mathfrak{g}$ is semisimple up to a Euclidean factor $\mathfrak{g}_0$. The corresponding statement for symmetric spaces reads

Theorem 2.18 ([23, Proposition V.4.2]). *Let X be a simply connected symmetric space. Then X admits a decomposition*

$$X = X_0 \times X_+ \times X_-$$

where X_0 is of Euclidean type, X_+ is of compact type and X_- is of noncompact type.

As the name suggests, the symmetric spaces of compact type are indeed compact. It follows from the Hadamard theorem (Theorem 1.77) that both the noncompact type and Euclidean types are diffeomorphic to $\mathbb{R}^n$ for some n. The decomposition in Theorem 2.18 can be further refined by the classical de Rham decomposition of our simply connected symmetric space X. This yields a decomposition of X into *irreducible* components

$$X = X_0 \times X_+^1 \times \cdots \times X_+^{n_+} \times X_-^1 \times \cdots \times X_-^{n_-},$$

where each X_+^i (resp. X_-^i) are now simply connected symmetric spaces associated with irreducible orthogonal Lie algebras of compact (resp. noncompact) type, and X_0 is of Euclidean type. See [23, Proposition VIII.5.5] .

The following proposition shows how to move from an orthogonal symmetric Lie algebra to a symmetric pair:

Proposition 2.19 ([33, VI.6.31]). *Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, Cartan involution $\Theta : \mathfrak{g} \to \mathfrak{g}$ and corresponding Cartan decomposition (Definition 2.12)*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Let $K \subset G$ be the analytic subgroup corresponding to $\mathfrak{k}$. Then

1. *there exists a Lie group automorphism $\theta : G \to G$ with $\theta^2 = Id$ and $d_e\theta = \Theta$;*
2. *The fixed point subgroup G^θ is equal to K;*
3. *The map $K \times \mathfrak{p} \to G$ given by $(k, x) = k\exp(x)$ is a diffeomorphism onto G;*
4. *The subgroup K is closed;*

5. *The subgroup K contains the center of G;*

6. *The subgroup K is compact if and only if the center of G is finite;*

7. *when the center of G is finite K is a maximal compact subgroup.*

In particular, if G has semisimple Lie algebra and G has finite center, then G has a maximal compact subgroup $K \subset G$ given as the fixed points of some involution $\theta : G \to G$.

Proposition 2.19 and Proposition 2.8 show that if (G, K) is of noncompact type we may assume

- G is semisimple;
- $K = G^\theta$ for the Cartan involution of G;
- $K \subset G$ is a maximal compact subgroup given as the stabilizer of some point $x_0 \subset X = G/K$.

In case $G = \mathrm{Iso}(X)^0 =$ with $X = X' \times \mathbb{R}^n$ is a symmetric space with Euclidean factor $\mathbb{R}^n$, the group G is of the form $G = G' \times E_n$, where $G' = \mathrm{Iso}(X')^0$ is semisimple and $E_n = \mathrm{Iso}(\mathbb{R}^n)^0$. So modulo the factor E_n, G can always be assumed to be semisimple.

Definition 2.20. The involution θ defined in Proposition 2.19 is called the Cartan involution (or global Cartan involution) of G.

Let us end this section with a comment on the Haar measures and symmetric pairs. Recall that a (left/right) Haar measure on G is a (left/right) G-invariant Radon measure on G and are unique up to scaling. Define

$$R_g : G \to G \qquad R_g(h) = hg$$

to be the right translation map on G. The pushforward of μ_l by R_g is the measure defined on Borel sets $A \subset G$ by

$$(R_g)_* \mu_l(A) \qquad \mu_l(Ag^{-1})$$

which is also a left Haar measure on G, hence by uniqueness of Haar measure must be a scalar multiple of μ_l. We thus have a well defined map on G given by

$$\Delta_G : G \to \mathbb{R}^+ \qquad \Delta(g)(R_g)_* \mu_r = \mu_r.$$

It can be shown to be a group homomorphism and independent on the choice of Haar measure μ_l. We define:

Definition 2.21. The group homomorphism

$$\Delta_G : G \to \mathbb{R}^+$$

is called the *modular function* of G. A group with trivial modular function, i.e. $\Delta_G(g) = 1$ for all $g \in G$, is called *unimodular.*

Examples of unimodular groups are compact groups, solvable groups, discrete groups, abelian groups and all groups with trivial abelianization as these admit no non-trivial group homomorphisms to abelian groups. Products and semidirect products of unimodular groups are also unimodular. A connected semisimple Lie group G is unimodular as well since it admits an *Iwasawa decomposition* (see Proposition 2.54):

$$G = KAN$$

with K compact, A abelian and N nilpotent (all unimodular). We thus get:

Proposition 2.22. *Let (G, K) be the symmetric pair associated with a symmetric space X. Then G is unimodular.*

Proof. We have seen (Theorem 2.17) that the Lie algebra of G is semisimple up to a Euclidean factor of $E_n = \mathrm{Iso}(\mathbb{R}^n)$, thus

$$G = G' \oplus E_n$$

for some semisimple G'. Being a product of unimodular groups, G is unimodular. □

As a consequence we have the following important corollary:

Corollary 2.23. *Let (G, K) be the symmetric pair associated with a symmetric space X. If $H \subset G$ is any closed unimodular subgroup, then*

$$G/H$$

admits a G-invariant measure.

Proof. This follows from the fact that for any locally compact group G and closed subgroup $H \subset G$ the homogeneous space G/H admits a G-invariant measure if and only if Δ_H is the restriction of the modular function Δ_G of G to H. i.e. if their modular functions of G and H agree on H (see [47] Section 3.1). □

Assuming now (G, K) is a symmetric pair associated with a symmetric space X and $\Gamma \subset G$ any discrete subgroup.

Definition 2.24. A subset $F \subset G$ is called a *strict fundamental domain* for the action of Γ on G if

- F is a Borel subset of G;
- the quotient map $\pi : G \to G/\Gamma$ restricts to a bijection on F.

Regarding strict fundamental domains, we have the following, which holds for any locally compact group:

Lemma 2.25 ([40, Lemma 4.1.1]). *Any discrete subgroup $\Gamma \subset G$ admits a strict fundamental domain.*

The existence of a strict fundamental domain for Γ gives us the following natural choice of G-invariant measure on G/Γ.

Definition 2.26. Let $\pi : G \to G/\Gamma$ be the natural quotient map and μ_G a left Haar measure on G. Then define a measure on G/Γ by $\mu_{G/\Gamma}(A) = \mu_G(\pi^{-1}(A) \cap F)$ where F is any choice of strict fundamental domain for Γ.

The measure in Definition 2.26 is often referred to as the restriction of the Haar measure of G and is clearly G-invariant on G/Γ. If $K \subset G$ is compact we can play the same game for

$$K\backslash G/\Gamma$$

by averaging the measure $\mu_{G/\Gamma}$ over K to get a measure on $K\backslash G/\Gamma$.

Comparing definitions of symmetric spaces

Let us now see how the assignments in equation (2.1) on page 51 are constructed. When we say a symmetric pair or orthogonal symmetric Lie algebra is *associated with* a symmetric space or a symmetric pair respectively, we mean they are constructed in the way described in this section.

Using the same numbering as in equation (2.1), we have:

(1) $\longrightarrow$ **(2)**: Given a symmetric space X defined the geometric way, we can get a symmetric pair (G, K) by setting $G = \text{Iso}(X)^0$, the connected component of the identity of the isometry group of X, and $K = \text{Stab}_G(x_0)$ for some point $x_0 \in X$. The group G is known to admit a smooth structure making it into a Lie group acting smoothly on X (see [23] p. 211) and K is a closed subgroup, hence also a Lie group. The geodesic symmetry (Definition 2.1) $s_{x_0} : X \to X$ is a global isometry on X, and we can define a map $\theta : G \to G$ by

$$\theta(g) := s_{x_0} g s_{x_0}.$$

Note that s_{x_0} may not be in $\text{Iso}(X)^0$, but conjugation by s_{x_0} preserves $\text{Iso}(X)^0$, so this map is well defined. Since $s_{x_0} k s_{x_0}$ fixes x_0 for all $k \in K$ we have

$$G_0^\theta \subset K \subset G^\theta$$

hence (G, K) is a symmetric pair with respect to the involution θ.

(2) $\longrightarrow$ **(3)**: To go from a symmetric pair to an orthogonal symmetric Lie algebra is done in the only sensible way one can think of. Given a symmetric pair (G, K) with involution θ, let $\mathfrak{g}$ be the Lie algebra of G, and $\Theta = d_e\theta$ the differential of θ at e. The proof that $(\mathfrak{g}, \Theta)$ is an orthogonal symmetric Lie algebra (ref. Definition 2.13) can be found in [23, Theorem IV.3.3] .

Clearly many symmetric pairs can give the same orthogonal symmetric Lie algebra, as many Lie groups have the same Lie algebra, but the assignment of an orthogonal symmetric Lie algebra to a symmetric pair is canonical in the sense that it does not require any choice.

It should be emphasised that if $G = \mathrm{Iso}(X)^0$ and $K = \mathrm{stab}_G(x_0)$ is a symmetric pair associated with a symmetric space X, then the pair (G, K) (resp. orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$) has the following properties, not shared by all symmetric pairs (resp. orthogonal symmetric Lie algebras) :

- The group $K \subset G$ is compact (resp. $\mathfrak{k} = \mathrm{Eig}_{+1}(\Theta) \subset \mathfrak{g}$ is compact)
- The pair (G, K) is effective, i.e. $Z(G) \cap K$ is discrete (resp. $Z(\mathfrak{g}) \cap \mathfrak{k} = 0$).
- The involution θ on G is unique (reps. Θ is unique on $\mathfrak{g}$)[1]

Thus the assignment $X \mapsto (G, K)$ sending a symmetric space to its associated symmetric pair is not surjective. For this reason, as we are mostly interested in the geometric realization of symmetric pairs and orthogonal symmetric Lie algebras, we will restrict ourselves to effective symmetric symmetric pairs (G, K) with K compact, and effective orthogonal symmetric Lie algebras.

The added complexity in the definitions of symmetric pairs and orthogonal Lie algebras may seem a bit gratuitous at this point, as we are only interested in their associated symmetric space. To properly motivate symmetric pairs would take us too far from the scope of the thesis, however we mention that in many cases (like when G reductive and $K \subset G$ is maximal compact), the symmetric pair (G, K) is an example of a *Gelfand pair*, meaning that the subalgebra

$$C_c(K\backslash G/K) \subset C_c(G)$$

of K-bi-invariant compactly supported functions on G is commutative with respect to the convolution product (see Definition 1.27). Gelfand pairs are actively studied for their applications to representation theory and to exotic group C*-algebras among others (see [14]).

Cartan subalgebras

As in the previous section, let $\mathfrak{g} = (\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra (Definition 2.13) associated with a symmetric space of noncompact type $X = G/K$ with

[1] This is a property of all effective symmetric pairs.

$G = \text{Iso}(X)^0$ and $K = \text{Stab}_G(x_0)$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition (Definition) 2.12). Then any subalgebra $\mathfrak{a} \subset \mathfrak{p}$ is necessarily abelian, since for any $x, y \in \mathfrak{a}$, we have

$$\Theta([x,y]) = [\Theta(x), \Theta(y)] = [-x,-y] = [x,y] \tag{2.3}$$

so $[x,y] \in \mathfrak{k} \cap \mathfrak{a} = \{0\}$ for all $x, y \in \mathfrak{a}$.

Definition 2.27 (Cartan subalgebra)**.** A Cartan subalgebra of $\mathfrak{g}$ is a maximal (abelian) subalgebra $\mathfrak{a} \subset \mathfrak{p}$.

Let us mention that Definition 2.27 is the one used when dealing with symmetric spaces (see [7] p. 45), while books dealing with Lie theory tend to use another definition which in general is nonequivalent to ours. See for instance [33]. The Lie theoretic definition is as follows:

Definition 2.28 (Cartan subalgebra (Lie theory))**.** Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal subalgebra, and $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ (the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$). Let $\mathfrak{t} \subset \mathfrak{m}$ be the center of $\mathfrak{m}$. Then the Cartan subalgebra of $\mathfrak{g}$ is the algebra $\mathfrak{t} \oplus \mathfrak{a}$.

Whenever $\mathfrak{t} \neq \{0\}$ the two definitions of Cartan subalgebras are not equivalent. We emphasise that when we speak of a Cartan subalgebra in the subsequent part of the thesis, *we will always mean a Cartan subalgebra in the sense of Definition 2.27.*

Definition 2.29. The (real) rank of G (or $X = G/K$) is the dimension of A and is denoted by $rk_{\mathbb{R}}(G)$ or simply $rk(G)$ (or $rk(X)$).

Example 2.30. The symmetric spaces of noncompact type of rank 1 are either [47]

- $\mathbb{H}^n_{\mathbb{R}}$ the n-dimension real hyperbolic space;
- $\mathbb{H}^n_{\mathbb{C}}$ the n-dimension complex hyperbolic space;
- $\mathbb{H}^n_{\mathbb{H}}$ the n-dimension quaternionic hyperbolic space;

or the exceptional case

- The Cayley plane, or octonionic hyperbolic plane, $\mathbb{P}^2(O)$.

2.2 Some Lie theory

We will go over some some of the Lie theory required for the subsequent part of the Chapter. For completeness, let us state the following definition:

Definition 2.31. A Lie group is a smooth manifold G where the group operations are diffeomorphisms of G.

Definition 2.32. A (real) Lie algebra, is an algebra $\mathfrak{g}$ together with a map, called the Lie bracket,

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which is bilinear over $\mathbb{R}$ and satisfies, for any $x, y, z \in \mathfrak{g}$

1. $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$;
2. $[x,x] = 0$.

The first property of Definition 2.32 shows that $\mathfrak{g}$ is in general not associative with respect to the product given by the Lie bracket.

Example 2.33. Given an associative algebra A over $\mathbb{R}$, A becomes a Lie algebra with respect to the Lie bracket

$$[x,y] := xy - yx \qquad x, y \in A.$$

Definition 2.34. A Lie algebra $\mathfrak{g}$ is called semisimple if the Killing form $B_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ (Definition 2.10) is non-degenerate.

2.2.1 Roots and Weyl chambers

Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition (Definition 2.12). Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal subalgebra (i.e. a Cartan subalgebra). As we saw in equation 2.3, $\mathfrak{a}$ must be abelian and it is shown in [33, Lemma VI.4.6.45] that the elements in $\mathfrak{a}$ act on $\mathfrak{g}$ by the adjoint representation as semisimple operators and that we can find a basis of common eigenvectors in $\mathfrak{g}$ making $\mathrm{ad}(\mathfrak{a})$ into diagonal matrices with real entries.

Fix $H \in \mathfrak{a}$ and let $x \in \mathfrak{g}$ be an eigenvector of H. Since $\mathrm{ad}(H + H') = \mathrm{ad}(H) + \mathrm{ad}(H')$ for all $H, H' \in \mathfrak{a}$, we see that that the map

$$\lambda : \mathfrak{a} \to \mathbb{C} \qquad \lambda(H) = \mathrm{Eig}_H(x)$$

i.e. the map sending H to its eigenvalue at x, is a linear functional on $\mathfrak{a}$. We define

Definition 2.35. The restricted roots (henceforth roots) of $\mathfrak{g}$ associated with $\mathfrak{a}$ are the functionals $\lambda \in \mathfrak{a}^\star$ for which

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$$

is non-trivial. Denote the set of all roots by Λ.

The eigenspaces of the $\mathrm{ad}(\mathfrak{a})$-action, yields a orthogonal decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha \tag{2.4}$$

Definition 2.36. The decomposition of Equation 2.4 is called the root space decomposition of $\mathfrak{g}$ with respect to the restricted roots of $\mathfrak{a}$.

The kernels of all the roots Λ are linear hyperspaces in $\mathfrak{a}$ which partition $\mathfrak{a}$ into open cones. These are the connected components of the space

$$\mathfrak{a} \setminus \bigcup_{\lambda \in \Lambda} \ker_\lambda. \tag{2.5}$$

Definition 2.37. The connected components of Equation 2.5 are called the Weyl chambers of $\mathfrak{a}$. Pick a Weyl chamber and call it the positive Weyl chamber. We denote this positive Weyl chamber by $\mathfrak{a}^+$.

The positive Weyl chamber is thus an open cone $\mathfrak{a}^+$ in $\mathfrak{a}$. In case $\mathfrak{g} = \mathfrak{k} \oplus k$ is the Lie algebra associated with a symmetric space of noncompact type with corresponding Lie group $G = \mathrm{Iso}(X)^0$, the exponential $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism $\mathfrak{p} \to X = G/K$ and we denote by $A = \exp(\mathfrak{a})$. By a slight abuse of terminology we will also refer to the image of the positive Weyl chamber under exp as the positive Weyl chamber of A, and denote it by A^+.

Definition 2.38. The positive roots of $\mathfrak{a}$ is the subset

$$\Lambda^+ := \{\lambda \in \Lambda \mid \lambda(\mathfrak{a}^+) \geq 0\}.$$

We then define:

Definition 2.39. The simple roots of $\mathfrak{a}$ is the collection of positive roots $\lambda \in \Lambda^+$ which cannot be written as a sum of two positive roots. Denote the set of simple roots by Σ

The simple roots play the role of a basis for the root system in the sense that any positive root can be written as a sum of simple roots (and any root can be written as a difference of two positive roots), though the expansion is not unique.

Definition 2.40. The Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{a}$, denoted $W = W(\mathfrak{g}, \mathfrak{a})$ is the group generated by reflections about the hyperspaces $\ker\lambda$ where λ runs over all restricted roots of $\mathfrak{g}$.

We have the following proposition, which is a consequence of [23, Corollary VII.2.13]:

Proposition 2.41. *With G the semisimple Lie group with Lie algebra $\mathfrak{g}$, and $K = \exp(\mathfrak{k})$ a maximal compact subgroup we have*

$$W = Z_K(\mathfrak{a})/N_K(\mathfrak{a})$$

where $Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K and $N_K(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in K.

The Weyl group W is finite, but this is not at all obvious at this point. The next proposition implies W is finite since the set of Weyl chambers is finite.

Proposition 2.42 ([23, Theorem VII.2.12]). *The action of W on $\mathfrak{a}$ is simply transitive on the set of Weyl chambers of $\mathfrak{a}$.*

A simply transitive action means that for any two Weyl chambers chambers $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{a}$ there is exactly one Weyl group element $w \in W$ such that

$$w\mathfrak{a}_1 = \mathfrak{a}_2$$

in other words, $\mathfrak{a}^+$ is a strict fundamental domain (Definition 2.24) for the action of W on $\mathfrak{a} \setminus \bigcup_{\lambda \in \Lambda} \ker\lambda$. In fact $\overline{\mathfrak{a}^+}$ is a strict fundamental domain for the W-action on $\mathfrak{a}$ ([7] p. 189).

2.2.2 Positive Weyl chambers at infinity and G-orbits in $X(\infty)$

In Section 1.7 we constructed a compactification for any Hadamard manifold X called the geodesic compactification $X \cup X(\infty)$. Let us see what this looks like for symmetric spaces of noncompact type. We deduce by Theorem 2.18 and Definition 2.16 that symmetric spaces of noncompact type are nonpositively curved simply connected complete Riemannian manifolds, hence give examples of Hadamard manifolds.

When studying the geodesic compactification of symmetric spaces of noncompact type, there is a marked distinction between the rank 1 and higher rank case. Here are some properties of rank 1 symmetric spaces

- $G = \mathrm{Iso}(X)^0$ (and $K \subset G$ a maximal compact) acts transitively on $X(\infty)$
- The classical harmonic theory of the Poincaré disc can be extended using geodesic boundaries and so-called Patterson–Sullivan densities (or harmonic densities) [47].
- The assignment $X(\infty) \ni x \mapsto \mathrm{Stab}_G(x) \subset G$ is a 1-1 correspondence between parabolic subgroups of G and points in $X(\infty)$.
- For every two points $x, y \in X(\infty)$ there is a geodesic γ such that $\lim_{t\to\infty} \gamma(t) = x$ while $\lim_{t\to-\infty} \gamma(t) = y$.

All the above properties fail spectacularly for higher ranks. For instance one can show that in the higher rank case, the largest number of distinct maximal flats $A_1, \ldots, A_n \subset X$ for which $\gamma \subset \bigcap A_i$ depends only on the equivalence class $[\gamma] \in X(\infty)$ of the geodesic γ. This number is invariant under the action of G since if $\gamma \subset \bigcap A_i$, then $g\gamma \subset \bigcap gA_i$. If G was to act transitively, every geodesic would have to lie in the same number of maximal flats. This is never the case for higher rank symmetric space. We illustrate why with an example:

Example 2.43. Let $X = \mathbb{H}^2 \times \mathbb{H}^2$ and $x_0 \in \mathbb{H}^2$ be an arbitrary point. Let $A \subset X$ be a maximal flat, and $v_1, v_2 \subset T_{x_0}\mathbb{H}^2$ be any two vectors of length 1 in the norm given by the Riemannian metric. Let $\gamma_i(t) = \exp(tv_i)x_0$ be the corresponding geodesics. Any two vectors $(v_1, 0)$ and $(0, v_2)$ in $T_{(x_0,x_0)}X$ span a flat 2-dimensional subalgebra. This follows from the curvature formula of Equation 2.2 and the fact that

$$[(v_1, 0), (0, v_2)] = 0.$$

So $A = \exp(\text{span}\{(v_1, 0), (0, v_2)\}) \subset X$ is a maximal flat totally geodesic subspace. From this observation we can deduce that all maximal flat submanifolds of X containing (x_0, x_0) are of the form

$$A = \gamma_1(\mathbb{R}) \times \gamma_2(\mathbb{R})$$

for two geodesics γ_i centered at x_0. A geodesic in A centered at (x_0, x_0) must take the form

$$\gamma_{\alpha,\beta}(t) := (\exp(\alpha t v_1)x_0, \exp(\beta t v_2)x_0)$$

for some real numbers $\alpha, \beta \geq 0$ and $\alpha+\beta = 1$. If $\alpha, \beta > 0$ then the vector $\alpha v_1+\beta v_2$ uniquely determines v_1 and v_2, so it sits in a unique maximal flat A tangential to v_1 and v_2. If for instance $\beta = 0$ then $\alpha v_1 + \beta v_2 = v_1$ does not say anything about the second coordinate, so a geodesic in this direction would be contained in any maximal flat tangential to v_1.

As usual we let $G = \text{Iso}(X)^0$ and $K = \text{Stab}_G(x_0) \subset G$ a maximal compact given as the stabilizer of some point $x_0 \in X$. We denote by $\mathfrak{g}$ the Lie algebra of G, and recall that it is semisimple since X has no Euclidean factors being of noncompact type. We can decompose $\mathfrak{g}$ by the Cartan involution inherited by G as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with $\exp : \mathfrak{p} \to G/K$ a diffeomorphism (see Proposition 2.19).

Denote by $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$, $A = \exp(\mathfrak{a}) \subset G/K = X$ is a maximal flat totally geodesic submanifold of X (by Proposition 2.5), and we write A^+ and $\mathfrak{a}^+$ for the positive Weyl chambers of A and $\mathfrak{a}$ respectively.

Definition 2.44. The positive Weyl chamber at infinity, denote $A^+(\infty)$, is the set of limit points of A^+ in the geodesic boundary $X(\infty)$ (Definition 1.79). Under the identification $\mathfrak{p} \simeq T_{x_0}X$ it can be identified with the unit vectors in $\mathfrak{a}^+$. The closure $\overline{A^+(\infty)}$ is called the closed positive Weyl chamber at infinity.

Since $A \simeq \mathbb{R}^n$, the positive Weyl chamber at infinity $A^+(\infty)$ will be a connected subset homeomorphic to a ball in $X(\infty)$. It is very useful however to think of $A^+(\infty)$ as a simplicial complex in S^{n-1} bounded by a family of "walls" given by limit points of $\exp(\ker(\alpha)) \subset A$, with $\alpha \in \Lambda^+$ running over all positive root. The reason we want to keep track of the "walls" is that they determine the G-orbit of the point in $\overline{A^+(\infty)}$. To see how, we need the following:

Proposition 2.45. *The closed positive Weyl chamber at infinity* $\overline{A^+(\infty)}$ *is a strict fundamental domain for the action of* G *on* $X(\infty)$.

The proof is straightforward, but it requires the use of polar decomposition of X which we have not yet defined, so we omit it here (see [7, p. 44]).

We have the following definition:

Definition 2.46. The *regular boundary* $X_{reg}(\infty)$ of $X \cup X(\infty)$ is the G-orbit of $A^+(\infty)$ in $X(\infty)$

We deduce from the fact that $A^+(\infty)$ is open and dense in $\overline{A^+(\infty)}$ that the regular boundary $X_{reg}(\infty) \subset X(\infty)$ is also open and dense.

We now collect some properties of the G-orbit structure of $X(\infty)$. Let 2^Σ denote power set of the set of simple roots. Before describing the G-orbits in $X(\infty)$ we will need the following definition:

Definition 2.47 ([7, Proposition I.2.6])**.** A (proper) subgroup $P \subset G$ is called parabolic, if it is the stabilizer in G of a point on the geodesic boundary $X(\infty)$.

Let $P_0 \subset G$ be a minimal parabolic subgroup, in the sense that it contains no other proper parabolic subgroups of G.

Lemma 2.48. *There is a 1-1 correspondence between conjugacy classes of parabolic subgroups containing* P_0 *and subsets* $I \in 2^\Sigma$.

Proof. See Corollary IV.11.17 and Proposition IV.14.18 of [6] □

We denote by $P_I \subset G$ the unique parabolic subgroup containing P_0 corresponding to the set $I \subset 2^\Sigma$. For algebraic groups over $\mathbb{C}$ a more common definition of parabolic subgroups are those closed subgroups $H \subset G$ for which G/H is a complete variety (or equivalently a projective variety).

The following proposition lists some of the properties of parabolic subgroups found in [33, Section V.7.]:

Proposition 2.49. *Let* $P \subset G$ *be a parabolic subgroup of* G *as in Definition 2.47. Then* P *admits a decomposition*

$$P = M_P A_P N_P$$

where:

- M_P *is reductive;*
- $A_P \subset A$ *is abelian;*
- N_P *is nilpotent;*

- *M_P normalizes N_p and centralizes A_P;*
- *$M_P \subset K$ if and only if P is minimal;*
- *$A_P = A$ if and only if P is minimal;*
- *P is the normalizer of N_P;*
- *The normalizer of P in G is P.*
- *$P \cap K = M_P \cap K$ is a maximal compact in P*

When P is minimal we write $P = MAN$ without any subscripts. For each proper subset $I \in 2^\Sigma$ we also get a subset of the boundary of $\overline{A^+(\infty)}$ as follows: Let

$$A_I^+(\infty) = \{\exp(x) \in \overline{A^+(\infty)} \mid x \in \overline{\mathfrak{a}^+},\ \|x\| = 1,\ \text{and } \alpha(x) = 0 \Leftrightarrow \alpha \in I\}.$$

This is the limit in $X(\infty)$ of a subcone in A^+. It is equal to the interior points of $\bigcap_{\alpha\in I} \ker_\alpha \cap \overline{A^+(\infty)}$ in case it is not discrete and equal to the whole set if discrete. There is a more concise way to describe this subset, namely (see [7, Corollary I.2.17])

$$A_I^+(\infty) = \{v \in \overline{A^+(\infty)} \mid \mathrm{Stab}_G(v) = P_I\}.$$

Finally, we are ready to describe the G-orbits in $X(\infty)$:

Lemma 2.50. *The closed positive Weyl chamber at infinity $\overline{A^+(\infty)}$ is the disjoint union*

$$\overline{A^+(\infty)} = \bigsqcup_{I\in 2^\Sigma} A_I^+(\infty)$$

with the convention that $A_\emptyset^+(\infty) = A^+(\infty)$.

For each I we have a G-equivariant homeomorphism

$$GA_I^+(\infty) \simeq A_I^+(\infty) \times G/P_I = A_I^+(\infty) \times (KM_P/M_P).$$

Proof. This follows again from Proposition I.2.16 of [7] which reads

$$X(\infty) = \bigsqcup_{P\subset G} A_P^+(\infty)$$

where $A_p^+(\infty) = \{v \in X(\infty) \mid \mathrm{Stab}_G(v) = P\}$ and P runs over all parabolic subgroups in G. But since $\mathrm{Stab}_G(gv) = g\mathrm{Stab}_G(v)g^{-1}$ we get that

$$\overline{A^+(\infty)} = \bigsqcup_{I\in 2^\Sigma} A_I^+(\infty)$$

since $\overline{A^+(\infty)}$ is a fundamental domain for the G action on $X(\infty)$ and 2^Σ is in 1-1 correspondence with conjugacy classes of parabolic subgroups. Lastly, for each fixed $I \in 2^\Sigma$ the map

$$f_I : GA_I^+(\infty) \to A_I^+(\infty) \times G/P_I \qquad f(ga) = (a, gP_I)$$

is easily seen to be a G-equivariant homeomorphism with inverse $f_I^{-1}(a, gP_I) = ga$. □

Thus, the regular boundary has a nice description as a trivial G-bundle:

$$X_{reg}(\infty) \simeq A^+(\infty) \times G/P_0$$

where $P_0 \subset G$ is a minimal parabolic subgroup and comes with a projection

$$p_r : X_{reg}(\infty) \to G/P_0.$$

We will see later (Equation 2.7 in the next section) that there is a G-equivariant isomorphism

$$G/P_I = (KM_P)/M_P$$

which is equivariant with respect to a certain G-action on $(KM_P)/M_P$ given by the Iwasawa projections (Definition 2.57). This G-action restricts to the usual left multiplication action on $K \subset G$, hence we see that K acts transitively on each G orbit in $X(\infty)$.

2.2.3 Lie group decomposition theorems

In this section we introduce some of the main decomposition theorems for semisimple Lie groups that will be used later. Let $G = \mathrm{Iso}(X)^0$ for a symmetric space X of noncompact type and $K \subset G$ a maximal compact subgroup. Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{k} \subset \mathfrak{g}$ the Lie algebra of K.

Theorem 2.51. *Any semisimple Lie algebra $\mathfrak{g}$ admits a Cartan involution (Definition 2.20) which is unique up to inner automorphisms of $\mathfrak{g}$, i.e. if $\Theta, \Theta' : \mathfrak{g} \to \mathfrak{g}$ are two Cartan involutions, then there exists a $g \in G$ such that $\mathrm{Ad}(g)$ is in $Int(\mathfrak{g}) = \mathrm{Ad}(G)^0$ and $\Theta'(x) = (\mathrm{Ad}(g) \circ \Theta \circ \mathrm{Ad}(g^{-1}))(X)$ for all $X \in \mathfrak{g}$.*

Proof. This is Corollary VI.6.18 and Corollary VI.6.19 of [33]. □

We let $\mathfrak{a} \subset \mathfrak{p}$ be denote a maximal abelian subalgebra, and $A = \exp(\mathfrak{a})$ the corresponding subgroup. We may switch between thinking of A as a subgroup of G and as a submanifold of G/K, as $A \cap K = \emptyset$.

Since $\mathfrak{p}$ is not a Lie subalgebra of $\mathfrak{g}$, $\exp(\mathfrak{p})$ is not a subgroup of G, so we have no exact analogue of the Cartan decomposition for $\mathfrak{g}$ on the Lie group G. The closest thing we have is the diffeomorphism $K \times \mathfrak{p} \to G$ given in 2.19.

The next decomposition we will need, when defining polar coordinates, is a corollary to the following proposition,

Proposition 2.52 ([33, Theorem VII.7.39]). *Every element in G has a decomposition as k_1ak_2 with $k_1, k_2 \in K$ and $a \in A$. In this decomposition, a is uniquely determined up to conjugation by a member of the Weyl group $W(G, A)$. If $a = \exp(H)$ for $H \in \mathfrak{a}$ such that $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$ (i.e. a is in a Weyl chamber), then k_1 is unique up to right multiplication by a member of $M = Z_K(A)$ (the centralizer in K of A).*

We let A^+ denote a choice of positive Weyl chamber in $A \subset G$ (see Sec. 2.2.1) and $\overline{A^+}$ the its closure in G (or equivalently in G/K). The maximal compact K acts on the space

$$K/M \times \overline{A^+}$$

by left multiplication on K/M and trivially on $\overline{A^+}$. We have the following corollary

Corollary 2.53 (Polar coordinates on G/K). *There is a well defined surjective continuous, K-equivariant map $\phi : K/M \times \overline{A}^+ \to X = G/K$ given by*

$$\phi(kM, a) = kaK$$

with following properties

- *ϕ restricts to an imbedding $K/M \times A^+ \to X$ with open dense image.*
- *$\phi(kM, a) = \phi(k'M, a')$ implies $a = a'$.*

It follows that every element $x \in X$ can be written as $x = ka$, where $a \in \overline{A^+}$ is unique, and if a is not on the boundary of $\overline{A}^+$, k is unique, modulo $M = Z_K(A)$.

Proof. The Weyl group $W(G, A)$ permutes the positive Weyl chambers of A and $\overline{A}^+$ is a fundamental domain for $W(G, A)$ action on A, hence Theorem 2.52 gives us that $G = K\overline{A^+}K$ and so

$$X = G/K = K\overline{A^+}.$$

The other claims follow readily from Proposition 2.52. □

Proposition 2.54 ([33] Theorem 6.46). *Let $P = MAN$ be a minimal parabolic subgroup of G. Then the map*

$$G \simeq K \times A \times N \qquad (k, a, n) \mapsto kan \in G$$

where K is our maximal compact subgroup is a diffeomorphism. The decomposition $G = KAN$ is called the Iwasawa decomposition.

Definition 2.55. The assignments

$$\overline{k} : G \to K$$
$$\overline{a} : G \to A$$
$$\overline{n} : G \to N$$

determined by Proposition 2.54 are called the Iwasawa projections onto K, A and N respectively.

Note that Iwasawa decomposition shares two subgroups in common with the minimal parabolic $P = MAN$ and we have $M \subset K$. This gives a very useful isomorphism

$$G/P = K/M \tag{2.6}$$

which holds (only!) when P is minimal. This isomorphism is G-equivariant with respect to the G-action on K/M given by

$$g \cdot kM := \overline{k}(g)kM.$$

The next proposition give us a decomposition of G generalizing the Iwasawa decomposition, or horospherical decomposition):

Proposition 2.56 ([7, Equation I.1.20])**.** *Let $P = M_P A_P N_P \subset G$ be any parabolic subgroup. Then the map*

$$KM_P \times A_P \times N_P \to G \qquad (n, a, mk) \mapsto namk \in G$$

is a diffeomorphism. The decomposition

$$G = KM_P A_P N_P$$

is called the generalized Iwasawa decomposition, or the horospherical decomposition of G.

Definition 2.57. The assignments

$$\begin{aligned} \overline{k}_P &: G \to KM_P \\ \overline{a}_P &: G \to A_P \\ \overline{n}_P &: G \to N_P \end{aligned}$$

determined by Proposition 2.56 are called the generalized Iwasawa projections.

Using this generalized Iwasawa decomposition, Equation 2.6 now looks like

$$G/P = (KM_P)/M_P. \tag{2.7}$$

Just as for Equation 2.6 the identification is given by sending $g = namk$

$$gP \mapsto mkM_P.$$

The action of G on $(KM_P)/M_P$ is given by

$$g \cdot kmM_P := \overline{k}_P(g)kmM_P.$$

The following spaces play a central role in the book of Borel and Ji:

Definition 2.58. Let $I \subset \Sigma$ be a subset of simple roots and $P_I = M_I A_I N_I$ the corresponding parabolic given by Proposition 2.49. Then the boundary symmetric space associated with I is

$$X_I = M_I/(M_I \cap K).$$

A corollary of Proposition 2.56 is

Corollary 2.59 ([7, p. 35])**.** *Let $I \subset \Sigma$ be a subset of simple roots, and $P_I = M_I A_I N_I$ the corresponding parabolic subgroup of G. Then the symmetric space X decomposes as*

$$N_I \times A_I \times X_I \simeq G/K = X$$

by the map

$$(n, a, m(M_I \cap K)) \mapsto namK.$$

2.3 The Dirichlet problem on symmetric spaces

The existence of solutions of partial differential equations on a bounded region with prescribed (often continuous) boundary values go under the name Dirichlet problems. For the Laplace-Beltrami operator Δ on the hyperbolic disk $\mathbb{D}$ the problem can be stated as follows: Given a continuous function f on S^1 does there exist a harmonic function on $\mathbb{D} \simeq \mathbb{H}^2$ extending f?

One can also generalize this question to the higher dimensional hyperbolic n-space $\mathbb{H}^n$ with boundaries the $(n-1)$-sphere in the obvious way. In both cases the problem has a positive answer: A function $f \in C(S^{n-1})$ determines a unique harmonic function in $C(\mathbb{H}^n)$ extending f given by its Poisson integral

$$F_f(x) := \int_{S^{n-1}} f(y)P(x,y)d\lambda(y)$$

where λ is the normalized Lebesgue measure on S^{n-1} and $P(x, y)$ is the Poisson kernel (Equation 1.22). Note that the hyperbolic spaces are examples of symmetric spaces of noncompact type of rank 1, and it is a fact that any such space admits a positive solution to the Dirichlet problem for Δ by means of so-called harmonic densities. Harmonic densities are examples of conformal densities which we will describe in the next section. But first, for completeness, let us mention the following fundamental result due to Furstenberg [18], (see also [20, Th. 12.10]) characterizing the space of bounded harmonic functions on X. Let $H : G \to \mathfrak{a}$ be given by $H(kan) = \log(a)$, i.e. the inverse of exp of the A-part of the Iwasawa decomposition $G = KAN$ of G. We have denoted by $b^* \in \mathfrak{a}^*$ the dual of the *barycenter* of $\mathfrak{a}^+$ which we will defined in the next section (equation (2.8) on page 74). For now let's just think of b^* as a special functional on $\mathfrak{a}$.

Theorem 2.60 ([18, Chap. V, Theorem 3.5.4]). *With X a symmetric space of noncompact type, let*

$$H_p := \{f \in C_b(X) \mid \Delta(f) = 0\}.$$

be the bounded harmonic functions on X where Δ is the Laplace-Beltrami operator. Let $P_0 \subset G$ be a minimal parabolic subgroup of G, then there is a G-equivariant bijection

$$L^\infty(G/P_0) \simeq H_p$$

determined by the Poisson integral formula

$$L^\infty(G/P_0) \ni f \mapsto \left(gx_0 \mapsto \int_{G/P_0} f(v)P(g,v)d\mu(v) \right)$$

where m is the unique K-invariant probability measure on G/P_0 and

$$P(g,v) = e^{-b^*(H(g^{-1}k))}$$

Theorem 2.60 puts no constraints on the rank of the space X, however if the rank is 1 (and only then), the space G/P_0 coincides with the whole geodesic boundary $X(\infty)$. As we will see later, this has important consequences for the boundary behaviour of the Poisson integral

$$gx_0 \mapsto \int_{G/P_0} f(v)P(g,v)dm(v).$$

2.4 Conformal densities

Conformal densities are certain families of measures on the geodesic boundary $X(\infty)$ associated with discrete subgroups of $G = \mathrm{Iso}(X)^0$. They were originally constructed by Patterson [43] for Fuchsian groups acting on the hyperbolic plane, then generalized by Sullivan to higher dimensional hyperbolic spaces [49], and finally by Albuquerque to arbitrary higher rank noncompact symmetric spaces in [1].

The conformal densities are defined for arbitrary discrete subgroups of the isometry group of X, however since we will not need this level of generality here, we will quickly restrict ourselves to the case where the discrete subgroup $\Gamma \subset G$ is a lattice, meaning $\Gamma\backslash G$ has finite volume with respect to the restricted Haar measure of G (ref. Definition 2.26).

As before, let X be a symmetric space of noncompact type, and $\Gamma \subset G = \mathrm{Iso}(X)^0$ a discrete subgroup. Keeping the notation of section 2.2.1, let

$$b := \sum_{\alpha \in \Lambda^+} m_\alpha H_\alpha \in \mathfrak{a}^+ \tag{2.8}$$

where $m_\alpha = \dim(\mathfrak{g}_\alpha)$ is the dimension of the root space of α (Equation 2.4), while H_α is the vector in $\mathfrak{a}$ dual to the root α. Since $\mathfrak{a}^+$ is a cone, b is also in the positive Weyl chamber. We call this vector the *barycenter* of $\mathfrak{a}^+$. Normalizing

$$b_1 := b/||b|| \tag{2.9}$$

with respect to the norm induced by the Killing form, we get an element in $\mathfrak{a}^+(\infty) \simeq A^+(\infty) \subset X(\infty)$. The vector b_1 is a unit vector in $T_{x_0}X$ pointing in the direction of maximal "maximal divergence" of geodesic rays, or the direction of maximal volume growth, as does any vector in its K-orbit. This is because b_1 turns out to be an extremal value of the function

$$H \mapsto \prod_{\alpha\in\Sigma^+} (\sinh(\alpha(H))^{m_\alpha} \qquad H \in \mathfrak{a} \tag{2.10}$$

among the unit vector of $\mathfrak{a}^+$. The function in equation 2.10 measures the rate of change of the volume form on X, see [24, Proposition I.5.1]. The special orbit $Gb_1 \subset X(\infty)$ in $X(\infty)$ is denoted by $\partial_F X$. We can describe $\partial_F X$ also in terms of Brownian motions on X: Limits of random walks in X almost certainly lie in $\partial_F X$. Again, since b_1 is in the interior of $A^+(\infty)$, it is a regular boundary point we get the following

Proposition 2.61. *There is a G-equivariant isomorphism*

$$\partial_F X \simeq G/P_0$$

where P_0 is any minimal parabolic subgroup.

Proof. Since b_1 is in the interior of the positive Weyl chamber at infinity $A^+(\infty)$, Lemma 2.50 tells us that the stabilizer of a point on Gb_1 is a minimal parabolic, hence $Gb_1 = \partial X_F = G/P_0$. □

Let us define:

Definition 2.62. The *Busemann function* is the map $h : X \times X(\infty) \to \mathbb{R}$ given by

$$h_v(x) = \lim_{t\to\infty} d(x, \gamma_v(t)) - t$$

where γ_v is the unique geodesic ray centered at x_0 with $\gamma_v'(0) = v$

To see that Busemann function is well defined, let $f(t) = d(x, \gamma_v(t)) - t$. By the triangle inequality we have that for $s < t$

$$d(x, \gamma_v(t)) \leq d(x, \gamma_v(s)) + t - s$$

hence

$$(d(x, \gamma_v(t)) - t) - (d(x, \gamma_v(s)) - s) \leq 0$$

so $f(t) - f(s) \leq 0$ and f is monotone non-increasing. Similarly, the reverse triangle inequality yields

$$\begin{aligned} |f(t)| &= |d(x, \gamma_v(t)) - t| \\ &= |d(x, \gamma_v(t)) - d(\gamma_v(t), x_0)| \\ &\leq d(x, x_0) \end{aligned}$$

hence f is also bounded so converges as $t \to \infty$.

The Busemann function shows up everywhere when dealing with asymptotic behaviour in negatively curved symmetric spaces. It is useful to think of the Busemann function as defining a distance between points in X and points in ∂X. To motivate this, let X be a symmetric space of noncompact type with a point $x_0 \in X$ given. Let γ be a geodesic ray centered at x_0. Let $y = \gamma(t_0)$ ($t_0 \in \mathbb{R}^+$) be a point on $\gamma(\mathbb{R}^+)$, and B_r be the ball of radius r (in the metric of X) centered at $\gamma(t_0 + r)$.

Definition 2.63. The limit of B_r as $r \to \infty$ is called the *horosphere* centered at $[\gamma] \in \partial X$ with radius y.

See Section 1.10 of [16] for the proof of the following proposition

Proposition 2.64. *Let $y \in X$ and $v \in \partial X$. The horosphere centered at v of radius y is the level set*

$$h_v^{-1}(h_v(y)).$$

Let B be the horosphere centered at v of radius x_0. Then the value $h_v(y)$ is the negative of the distance between y and B (if y is inside B) or equal to this distance (if y is outside B).

Example 2.65. Let us compute the Busemann function in the case $X = \mathbb{R}^n$ with the Euclidean metric. Let $v \in \partial\mathbb{R}^n \simeq S^{n-1}$ be a point in the geodesic boundary represented by a geodesic ray (a straight line) γ_v centered at 0. The horosphere centered at v of radius x_0 is nothing but the hyperplane through x_0 orthogonal to γ. Thus, using the expression for the Busemann function in Proposition 2.64, the Busemann functions are given by

$$h_v(y) = ||y||cos(\alpha)$$

where α is the angle between the line γ and the line segment from 0 to y.

In Example 2.69 below, we will see another way to determine the Busemann function on symmetric spaces of noncompact type of rank 1. We are now ready to define conformal densities. The following definition holds for arbitrary discrete subgroups $\Gamma \subset G$.

Definition 2.66 ([?, Definition 1.1]). A Γ-conformal density on $X(\infty)$ of dimension $s \geq 0$ is a map

$$\mu : X \to M^+(X(\infty)) \qquad x \mapsto \mu_x$$

from X to the positive Borel measures on $X(\infty)$ satisfying the following properties

1. μ is Γ-equivariant (meaning $\gamma_* \mu_x = \mu_{\gamma x}$ for all $x \in X$ and $\gamma \in \Gamma$) and continuous with respect to the weak topology on $M^+(X(\infty))$

2. For all $x \in X$ the measures μ_{x_0} and μ_x are equivalent and

$$\frac{d\mu_x}{d\mu_{x_0}}(v) = e^{-sh_v(x)}$$

   where $h_v(x)$ is the Busemann function (Definition 2.62).

Definition 2.67 (critical exponent). The critical exponent, denoted $\beta := \beta(\Gamma)$, of a discrete subgroup $\Gamma \subset G$ is the infimum of all dimensions of all Γ-conformal densities.

The subgroup Γ is said to be of divergence type if $\sum_{\gamma \in \Gamma} e^{-\beta d(x_0, \gamma x_0)} = \infty$.

It is not clear from Definition 2.67 that such an infimum is finite. This is proved in [?, Corollary 3.9]. We can determine the critical exponent in any of the following ways:

- By the equation $\beta(\Gamma) = \inf_{s \in \mathbb{R}^+} \sum_{\gamma \in \Gamma} e^{-sd(x_0, \gamma x_0)} < \infty$
- By the limit $\beta(\Gamma) = \limsup_{t \to \infty} \frac{\log |\Gamma x_0 \cap B(x_0, t)|}{t}$
- (If Γ is a lattice) by $\beta(\Gamma) = ||b||$.

The third point shows that for lattices the critical exponent $\beta(\Gamma)$ only depends on the group G. Since we will exclusively be dealing with lattices and all lattices are of divergence type ([?, Proposition D]) we will henceforth assume Γ is of divergence type.

We summarize the main results in [1] regarding conformal densities for lattices in the following proposition

Proposition 2.68. *Given a lattice $\Gamma \subset G$, there exists a Γ-conformal density μ_x of dimension $\beta(\Gamma)$ such that for all $x \in X$*

1. *μ_x is a probability measure in the Lebesgue class on $\partial_F X = Gb_1 \subset X_{reg}(\infty)$;*
2. *$x \mapsto \mu_x$ is G-equivariant, that is $\mu_{gx} = g_* \mu_x$ for all $g \in G$;*
3. *μ_x is the unique $K_x = \mathrm{Stab}_G(x)$-invariant measure on $\partial_F X$.*

Furthermore, the conformal density μ is the unique Γ-conformal density supported on $X_{reg}(\infty)$ of dimension $\beta(\Gamma)$.

The density in Proposition 2.68 is constructed using an idea of Patterson [43]. We will show how to create these densities and refer to [1] for the proof of its properties. The construction works for any non-elementary group (i.e. groups which do not stabilize any finite subset of X) with only minor adjustments if it is not of divergence type. Define for any $s > \beta(\Gamma)$ a function

$$\psi(s) = \sum_{\gamma \in \Gamma} e^{-sd(x_0, \gamma x_0)}$$

and probability measures

$$\mu_{x_0,s} := \frac{1}{\psi(s)} \sum_{\gamma \in \Gamma} e^{-sd(x_0,\gamma x_0)} \delta_{\gamma x_0} \qquad s > \beta(\Gamma)$$

where δ_y denotes the Dirac point measure at y. By a compactness argument we can find a convergent sequence s_i with $s_1 \geq s_2 \geq \dots$ such that $s_i \to \beta(\Gamma)$ and μ_{x_0,s_i} converges to some probability measure μ_{x_0} on $X \cup X(\infty)$.

Now, if $g \in G$, it is easy to check the assignment $x \mapsto \mu_{x,s}$ is G-equivariant:

$$\begin{aligned} g_*\mu_{x_0,s} :&= \frac{1}{\psi(s)} \sum_{\gamma \in \Gamma} e^{-sd(x_0,\gamma x_0)} \delta_{g\gamma x_0} = \frac{1}{\psi(s)} \sum_{\gamma \in \Gamma} e^{-sd(x_0,g^{-1}\gamma x_0)} \delta_{\gamma x_0} \\ &= \frac{1}{\psi(s)} \sum_{\gamma \in \Gamma} e^{-sd(gx_0,\gamma x_0)} \delta_{\gamma x_0} = \mu_{gx_0,s}. \end{aligned}$$

Hence $g_*\mu_x = \mu_{gx}$. We also have

$$\frac{1}{\psi(s)} \sum_{\gamma \in \Gamma} e^{-sd(gx,\gamma x_0)} \delta_{\gamma x_0} = \frac{1}{\psi(s)} \sum_{\gamma \in \Gamma} e^{-s(d(gx,\gamma x_0) - d(x,\gamma x_0))} e^{-sd(x,\gamma x_0)} \delta_{\gamma x_0}.$$

Define $\Psi : X \cup X(\infty) \to \mathbb{R}^+$ to be the function

$$\Psi(x) = \begin{cases} d(gx_0, x) - d(x_0, x) & x \in X \\ h_x(gx_0, x_0) & x \in X(\infty). \end{cases}$$

This function is continuous on $X \cup X(\infty)$ and gives an expression for the Radon-Nikodym derivative

$$g_*\mu_{x_0,s} = e^{-s\Psi} \mu_{x_0,s}.$$

Taking limits as $s \to \beta = \beta(\Gamma)$ yields

$$g_*\mu_{x_0} = e^{-\beta\Psi} \mu_{x_0}.$$

Now pick a ball $B_r(x_0)$ of radius r centered at x_0. Since $B_r(x_0)$ is bounded the set $B_r(x_0) \cap \Gamma x_0$ is finite, hence when $s_i \to \infty$, as $\psi(s_i) \to \infty$, we have

$$\mu_{x_0,s_i}(B_r(x_0)) = \frac{1}{\psi(s)} \sum_{\gamma\in\Gamma} e^{-sd(gx,\gamma x_0)} \delta_{\gamma x_0}(B_r(x_0)) \leq \frac{1}{\psi(s_i)} |B_r(x_0) \cap \Gamma x_0| \to 0.$$

This shows the support of μ_{x_0} must lie on the boundary of $X \cup X(\infty)$. Letting x vary, the family μ_x is thus a Γ-conformal density on $X(\infty)$ of dimension β, which is actually G-equivariant.

We only used the fact that Γ is of divergence type to ensure that $\operatorname{supp} \mu_x$ is in $X(\infty)$ since $\psi(s)$ blows up as $s \to \beta$. This requirement can be dropped if we scale ψ by a slowly increasing function in s as is done in [43].

Looking at the limit of $\mu_{x_0} = \lim_{s\to\beta(\Gamma)} \mu_{x_0,s}$ since all $\mu_{x_0,s}$ are $K = \operatorname{Stab}_G(x_0)$-invariant, so is the limit measure μ_{x_0}. Since K acts transitively on the Furstenberg boundary $\partial_F X = G/P_0$ the measure μ_{x_0} is the unique K-invariant probability measure on $\partial_F X$. Similarly for any $x \in X$ the measure μ_x is the unique $K_x = \operatorname{Stab}_G(x)$-invariant probability measure on $\partial_F X$. This shows that the Patterson–Sullivan densities for a lattice Γ, does not depend on Γ.

Example 2.69. Let us see how these conformal densities can be computed for a familiar example. The simplest examples of symmetric spaces of noncompact type are those of rank 1. As we have seen, these are entirely classified, and are either $\mathbb{H}^n$ (the hyperbolic n-space over $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$) or the exceptional case which is the Cayley plane.

For $X = \mathbb{H}^n = \mathbb{H}^n_{\mathbb{R}}$, we have $G = \operatorname{Iso}(X)^0 = SO(1,n)^0$ and $K = SO(n)$. The Iwasawa decomposition of G takes the form (see [47, Lemma 2.4])

$$G = KAN$$

where $K = 1 \oplus SO(n) \subset G$ is a maximal compact subgroup,

$$K = \left\{ \begin{bmatrix} I_1 & 0 \\ 0 & S \end{bmatrix} \;\middle|\; S \in SO(n) \right\} \simeq SO(n)$$

$$A = \left\{ \begin{bmatrix} \cosh(t) & 0 & -\sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

and

$$N = U \left\{ \begin{bmatrix} 1 & u & \frac{-||u||^2}{2} \\ 0 & I_{n-1} & -u^t \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; u \in \mathbb{R}^{n-1} \right\} U \simeq \mathbb{R}^{n-1},$$

where

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & I_{n-1} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \cosh(t) & 0 & -\sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix} \begin{bmatrix} \cosh(s) & 0 & -\sinh(s) \\ 0 & I_{n-1} & 0 \\ \sinh(s) & 0 & \cosh(s) \end{bmatrix} = \begin{bmatrix} \cosh(s-t) & 0 & -\sinh(s+t) \\ 0 & I_{n-1} & 0 \\ \sinh(s+t) & 0 & \cosh(s+t) \end{bmatrix}$$

So we have $A \simeq \mathbb{R}$ and the restricted root system is as trivial as it gets. We could also deduce $A \simeq \mathbb{R}$ from the isomorphism:

$$UAU = \left\{ \begin{bmatrix} e^t & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \middle| t \in \mathbb{R} \right\} \simeq \mathbb{R}.$$

There are two roots and one positive root, corresponding to the choice of positive direction of the identification $A \simeq \mathbb{R}$. Adhering to the notation in [47] we write a_t for the matrix

$$\begin{bmatrix} \cosh(t) & 0 & -\sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix}$$

Since G has rank 1, there is a unique conjugacy class of parabolic subgroups. A standard choice of parabolic subgroup is given by

$$P = MAN$$

where

$$M = U \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| g \in SO(n-1) \right\} U.$$

Since P is minimal the Langlands decomposition tells us that $M \subset K$, hence we have $PK = KP = K(MAN) = KAN = G$. This gives

$$\mathbb{H}^n = G/K = (KAN)/K = PK/K = P/K = MAN/K = AN.$$

Any point in $\mathbb{H}^n$ can thus be written uniquely as $x = a_t n x_0$ (note that we have not used that $\mathbb{H}^n$ is rank 1, just that P is minimal).

Using the G-equivariant isomorphism

$$\partial\mathbb{H}^n \simeq G/P$$

with G acting by left multiplication on G/P, we pick $\xi_0 = eP \subset G/P$ and $x_0 = [e] \subset G/K$. Then for any point $x = na_t x_0 \in \mathbb{H}^n$ we have

$$h_v(x) = t.$$

Now using the fact that the critical exponent of any lattice in G is given by $\delta = 2n$ ([47, Proposition 3.9]) we can write down the Radon-Nikodym derivatives of the Patterson–Sullivan densities of a lattice in G explicitly as follows. With x and ξ as above. Let μ_x be the Patterson–Sullivan density of a lattice $\Gamma \subset G$, then we have

$$\frac{d\mu_x}{d\mu_{x_0}}(\xi) = e^{-2nh_\xi(x)} = e^{-2nt}.$$

2.5 An example where the Poisson integral does not extend continuously

Let us return to the example of section 1.7, now with $\mathbb{H}^n$ replaced by a symmetric space X of noncompact type.

At the end of Section 2.3, we listed some special properties of rank 1 symmetric spaces of noncompact type. The following lemma is yet another important perk of being rank 1:

Lemma 2.70. *Let X be a symmetric space of noncompact type of rank 1 and $\mu : X \to M_1(\partial X)$ the Patterson–Sullivan density of any lattice $\Gamma \subset G = Iso(X)^0$. Then the assignment*

$$f \mapsto F_f \qquad F_f(x) = \int_{\partial X} f(v) d\mu_x(v) \ (x \in X)$$

determines a G-equivariant completely positive splitting of the extension

$$0 \to C_0(X) \to C(X \cup X(\infty)) \to C(X(\infty)) \to 0.$$

Since the Patterson–Sullivan densities have been defined also for higher rank symmetric spaces, the question of whether a similar splitting can be constructed in higher ranks naturally presents itself. However by Proposition 2.68, the support of these higher rank densities is contained in a single G-orbit $\partial_F X$ in $X(\infty)$. If two functions $f_1, f_2 \in C(X(\infty))$ agree on this orbit, their Poisson integrals

$$F_{f_i}(x) = \int_{\partial_F X} f_i(v) d\mu_x(v)$$

would also agree. This is thus not a splitting of the extension in Equation 2.11. One could try to circumvent this issue by defining $X(\infty)_0 = X(\infty) \backslash \partial_F X$ and looking at the extension

$$0 \to C_0(X \cup X(\infty)_0) \to C(X \cup X(\infty)) \to C(\partial_F X) \to 0. \tag{2.11}$$

At least now the map $f \mapsto F_f$ is injective from $C(\partial_F X)$. There is however a fundamental issue here, as seen in the following proposition

Proposition 2.71. *Let X be a noncompact symmetric space of rank ≥ 2. Then there exists a function $f \in C(\partial_F X)$ such that the Poisson integral*

$$F_f(x) = \int_{\partial_F X} f(v) d\mu_x(v) \tag{2.12}$$

does not extend to a continuous function on $X(\infty)$.

Proof. Let $f \in C(\partial_f X)$. In Theorem 2.3 of [25] and the subsequent remark, the authors prove that $F_f(x_i) \to f(w)$ if x_i converges to w along a geodesic ray centered at x_0. This can also be deduced from [13, Theorem 2.4]. If F_f could be extended continuously to $X(\infty)$, it would thus have to be constant on each positive Weyl chamber at infinity $A^+(\infty)$.

Now let $A \subset X$ be a maximal flat submanifold and $A_i^+ \subset A$ $(i = 1, 2)$ two adjacent Weyl chambers, i.e. $\overline{A_1^+} \cap \overline{A_2^+}$ is of dimension $\dim(A) - 1$. Then $\overline{A_1^+(\infty)} \cap \overline{A_2^+(\infty)} \neq \emptyset$. Let $f \in C(\partial_F X)$ be any function that separates the points $\{w_i\} = A_i^+(\infty) \cap \partial_F X$. Then $F_f|_{A_i^+(\infty)}$ is just the constant function with value $f(w_i)$. Since by assumption $f(w_1) \neq f(w_2)$ F_f is not continuous at $\overline{A_1^+(\infty)} \cap \overline{A_2^+(\infty)}$. □

The proposition tells us that for many functions $f \in C(\partial_F X)$ the Poisson integral F_f cannot be extended to the whole geodesic compactification when the rank of X is greater than 1.

We should warn the reader that it is claimed in [13, Theorem 2.4] that if x_i is a sequence in X converging to a point w in $X(\infty)$ then μ_{x_i} converges in the weak topology on $M_1(G/P_0)$ to certain measure μ_w supported on a subset of G/P_0. In the case where $w \in X_{reg}(\infty)$, then μ_w is the point measure with support $A_w^+(\infty) \cap \partial_F X$, where $A_w^+(\infty)$ is the positive Weyl chamber at infinity containing w. It follows from their result that the Poisson integral is constant on positive Weyl chambers at infinity. They also claim the measure μ_w is independent of choice of convergent sequence $x_i \to w$.

Theorem 2.4 of [13] implies the Poisson integral (Equation 2.12) admits a continuous extension to the whole compactification, but also that it is constant on all positive Weyl chambers at infinity, and as we have seen in the proof of Proposition 2.71 this results in a contradiction. To see why Theorem 2.4 of [13] implies there is a well-defined continuous extension of F_f for every function $f \in C(G/P_0)$, let us prove the following lemma:

Lemma 2.72. *Let $X \subset (V, d)$ be any subset of a metric space V and $X_0 \subset X$ a subset such that $X \subset \overline{X_0}$. Assume $f \in C(X_0)$ and that there is a function $\overline{f} : \overline{X} \to \mathbb{C}$ such that for any convergent sequence $x_i \to x$, with $x_i \in X_0$, we have*

$$\overline{f}(x) = \lim_i f(x_i).$$

Then $\overline{f} \in C(X)$.

Proof. Since X is a metric space with respect to the restricted metric of V, it is sequential, i.e. the topology is determined by its convergent sequences, which implies $\overline{f}$ is continuous if and only if it maps convergent sequences to convergent sequences. Let $y_i \to y$ be any convergent sequence of points in X. Pick $x_i \in X_0$ such that

$$\begin{aligned} d(x_n, y_n) &< 1/n \\ ||f(x_n) - \overline{f}(y_n)|| &< 1/n. \end{aligned}$$

This can be done since X_0 is dense in X (and by the assumptions on $\overline{f}$ in the lemma). Then a triangle inequality argument shows that the sequence

$$x_1, y_1, x_2, y_2, \dots$$

is also Cauchy. Since y_i converges in X, this Cauchy sequence has limit y, hence x_i also converges to y. It follows that

$$\begin{aligned} ||\overline{f}(y) - \overline{f}(y_n)|| &\leq ||f(x_n) - \overline{f}(y_n)|| + ||\overline{f}(y) - f(x_n)|| \\ &< 1/n + ||\overline{f}(y) - f(x_n)|| \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

which concludes the proof. □

With Lemma 2.72 at our disposal, we start to see what is the issue with extending the Poisson integral for arbitrary functions on $C(\partial_F X)$: If μ_{x_i} converges weakly to the same measure μ_w independently of choice of convergent sequence $x_i \to w$, then the Poisson integral converges

$$F_f(x_i) = \int_{\partial_F X} f(s) d\mu_{x_i}(s) \xrightarrow{i\to\infty} \int_{\partial_F X} f(s) d\mu_w(s)$$

for any sequence $x_i \to w$. Combining this with Lemma 2.72 would imply that F_f extends continuously to $X \cup X(\infty)$, which would contradict Proposition 2.71.

In the next chapter we will introduce another compactification of X where the Poisson integral F_f of Equation 2.12 does extend to a continuous function on the compactified space, for every $f \in C(\partial_F X)$.

Chapter 3

Compactifications of symmetric spaces

3.1 The (maximal) Furstenberg compactification

As we have seen, there are serious problems with extending the Poisson integral to the whole geodesic boundary. In this section we will introduce a new compactification of X on which the Poisson integral does extend continuously. This compactification is called the maximal Furstenberg compactification and was first constructed by Furstenberg in [18]. There are several other compactifications isomorphic to it, like the maximal Satake compactification or a certain Martin compactification which, depending on the application, may be easier to work with.

Let us start with the following definition

Definition 3.1. A compactification $\overline{X}^A$ dominates a compactification $\overline{X}^B$ if the identity map on X extends to a (necessarily equivariant) surjection $\overline{X}^A \to \overline{X}^B$.

The next lemma holds with minor changes for any semi-simple G with finite center (see [18] Chapter II).

Lemma 3.2. *Given a minimal parabolic subgroup $P_0 \subset G$, with μ_{x_0} the unique K-invariant probability measure on G/P_0, the map*

$$x = gx_0 \mapsto g_*\mu_{x_0}$$

determines a G-equivariant imbedding of X into $M_1(G/P_0)$ (the space of probability measures on $G/P_0 = K/M$).

Proof. Let us show the map is injective. As usual μ_{x_0} is the unique K-invariant probability measure on G/P. Using the Cartan decomposition $G = K\overline{A}^+K$ we may write $g \in G$ as $g = kak'$ for $k, k' \in K$ and $a \in \overline{A^+}$. Then

$$\begin{aligned} g_*\mu_{x_0} = \mu_{x_0} &\Leftrightarrow k_*a_*k'_*\mu_{x_0} = \mu_{x_0} \\ &\Leftrightarrow k_*a_*k'_*\mu_{x_0} = k_*\mu_{x_0} && (\text{since } k_*\mu_{x_0} = \mu_{x_0}) \\ &\Leftrightarrow a_*k'_*\mu_{x_0} = \mu_{x_0} \\ &\Leftrightarrow k'_*\mu_{x_0} = a_*^{-1}\mu_{x_0} \end{aligned}$$

It follows that

$$g_*\mu_{x_0} = \mu_{x_0} \quad \Leftrightarrow \quad a_*\mu_{x_0} = \mu_{x_0}$$

hence we can, by uniqueness of the $\overline{A^+}$-component of the polar decomposition (Corollary 2.53), reduce the question to proving the implication

$$a_*\mu_{x_0} = \mu_{x_0} \;\Leftrightarrow\; a = 0.$$

Using the expression for the Radon-Nikodym derivative $\frac{dg_*\mu_{x_0}}{d\mu_{x_0}}$ (see [1] p. 26), this reduces to proving

$$\frac{d(a_*\mu_{x_0})}{d\mu_{x_0}}(kM) = e^{\langle b, H(k^{-1}a)\rangle} = 1 \qquad \forall\, kM \in K/M$$

where $H : G = NAK \to \mathfrak{a}$ is the logarithm of the A-part of the Iwasawa decomposition $G = NAK$[1]. This in turn is equivalent to

$$\langle b, H(k^{-1}a)\rangle = 0 \qquad \forall \;\, kM \in K/M.$$

Putting $\hat{k} = e$ yields

$$\langle b, \log(a)\rangle = 0$$

but this holds if and only if

$$\log(a) \in \bigcap_{\alpha\in\Lambda^+} \mathrm{Ker}(\alpha)$$

which implies that $a = 0$. To see why, note that $\bigcap_{\alpha\in\Lambda^+}\mathrm{Ker}(\alpha) = 0$ unless the root system is of type A_1 (i.e. has a single positive root). The only symmetric space of noncompact type with restricted root system of type A_1 are of rank 1. In this case K acts transitively

[1] We defined the Iwasawa decomposition in Proposition 2.54 as $G = KAN$ with corresponding Iwasawa projection $\overline{a} : G = KAN \to A$. Note that $G = G^{-1} = (KAN)^{-1} = NAK$ and since AN is a subgroup G we have $AN = NA$. This gives similar Iwasawa decompositions $G = NAK = ANK = KNA$ but it should be mentioned that in general $G \neq AKN = NKA$ (see [40] Ex. 7.1.5). Each decomposition has their own corresponding Iwasawa projections. It is easy to show we can express the map $H : G = NAK \to \mathfrak{a}$ using the original Iwasawa projections (Definition 2.55) $\overline{a} : G = KAN \to A$ as follows $H(g) = -log(\overline{a}(g^{-1})$

on unit sphere $S^n \subset T_0X$. It follows that for some k we have $H(k^{-1}a) = ||a||b$. Hence also here we must have $a = 0$.

Now to show the assignment is continuous, let $x = gx_0$ for some $g \in G$. Then for any $f \in C(G/P)$ we have

$$\mu_x(f) = \int_{G/P} f(kM)e^{\rho(H(k^{-1}g))}d\mu_{x_0}(kM).$$

Since $e^{\rho(H(k^{-1}g))}$ depends continuously on g, the assignment $x \mapsto \mu_x$ is continuous with respect to the weak topology on $M_1(G/P)$. □

Lemma 3.2 shows that X can be equivariantly imbedded into $M_1(G/P_0)$, which is a compact metrizable space with respect to the weak topology.

Definition 3.3. The *(maximal) Furstenberg compactification* $\overline{X}^F$ is the weak closure of X in $M_1(G/P_0)$.

The boundary of this compactification will be denoted by ∂X^F, not to be confused with the Furstenberg boundary $\partial_F X = G/P_0$ of Proposition 2.61). As it is the closure of an imbedding of X of $M_1(G/P_0)$, it is clear that is metrizable, in fact it can be shown to be homeomorphic to a closed ball.

If x_i is any sequence in X converging to a point $v \in \partial_F X$, then [25, Theorem 2.3] tells us that $\mu_{x_i} \to \delta_v$, the point measure at v. This gives us a G-equivariant imbedding

$$G/P_0 = \partial_F X \subset \partial X^F$$

as a compact G-orbit in ∂X^F. In fact we will see later (Proposition 3.30) that this is the unique compact (or closed) G-orbit in $\overline{X}^F$.

Proposition 3.4. *Let $f \in C(G/P_0)$, then the map*

$$x \mapsto \int_{G/P_0} f(v)d\mu_x(v)$$

admits a continuous extension to $\overline{X}^F$.

Proof. This follows directly from the construction of $\overline{X}^F$. If x_i converges to a boundary point x_∞ in $\overline{X}^F$ this is equivalent to $\mu_{x_i} \to \mu_s$ (weakly) for some measure $\mu_s \in M_1(G/P_0)$, that is

$$\int_{\partial_F X} f(v)d\mu_{x_i}(v) \to \int_{\partial_F X} f(v)d\mu_s(v)$$

for all $f \in C(\partial_F X)$. Since this is independent of choice of convergence sequence $x_i \to s$, Lemma 2.72 implies the extension is continuous. □

We also denote by F_f the extension of F_f to $\overline{X}^F$. It is known that the Furstenberg compactification is isomorphic to other compactifications, like the maximal Satake compactification and the Martin compactification with parameter $\lambda = \lambda_0$ (defined in Definition (3.15)). The fortunate feature of the Furstenberg compactification is that the continuous extension of the Poisson integral to the whole compactification can be readily proven as we did in Proposition 3.4.

There are however some downsides. One of which is that the Furstenberg compactification is only defined if $X \to M_1(G/P_0)$ is injective, which leaves out any symmetric space with a euclidean factor (since G/P_0 is compact P_0 necessarily contains any $\mathbb{R}^n$-factor of G, hence $\mathbb{R}$ acts trivially on G/P_0 and must lie in the kernel of the map $X \mapsto M_1(G/P_0)$). One may argue that these spaces are not very interesting anyway as we may compactify the $\mathbb{R}$-factor separately.

Let us conclude our discussion on the Furstenberg compactification by explaining the term "maximal", which we have occasionally prepended to the Furstenberg compactification without much justification. Originally the Furstenberg compactification was defined as the imbedding into $M_1(G/P_0)$ as we did above, but one could also use larger parabolic subgroups P_I. Under certain conditions ([7] Prop. I.6.16) the map

$$X \to M_1(G/P_I)$$

defined exactly as before, would also be injective and yield another compactification. In this way we get a family of compactifications, indexed by certain "admissible" subsets $I \subset \Sigma$ of simple roots. Our "maximal" Furstenberg compactification is readily seen to dominate all other such compactifications in the sense of Definition 3.1, since P_0 is minimal. For more on this see [7].

3.2 The Laplace operator and Casimir elements for symmetric spaces

Let us take a short detour and introduce our favorite differential operator, the Laplace operator:

Definition 3.5 ([24, p. 31])**.** Given a Riemannian manifold (M, g), the Laplace-Beltrami (or simply the Laplace) operator Δ on M is the differential operator Δ, given in local coordinates $(x_1, ..., x_n)$ at $p \in M$ for an $f \in C^2(M)$ by

$$\Delta(f) = \frac{1}{\sqrt{|g|}} \sum_{j=1}^{n} \partial_j \sum_{i=1}^{n} g^{ij} \sqrt{|g|} \partial_i f$$

where $g_{ij} = \langle \partial_i, \partial_j \rangle$ is the matrix representation of the Riemannian metric g in the coordinates $(x_1, ..., x_n)$, g^{ij} the inverse matrix of g_{ij}, and $|g| = \det(g_{ij})$.

When dealing with symmetric pairs (G, K) we have not yet specified a G-invariant metric on G/K, so it is convenient to have a notion of a "Laplace like" operator that does not hinge upon this choice of metric. With this in mind, we define the *Laplacian operators* as follows.

Definition 3.6 ([18, Definition 4.3])**.** Let G be a semisimple Lie group with finite center and $K \subset G$ a maximal compact subgroup. A differential operator on G/K is called a Laplacian if

1. $\Delta(1) = 0$ for the constant function 1;
2. Δ is an elliptic second order differential operator;
3. Δ is G-invariant, meaning $g[\Delta(f)] = \Delta(gf)$ for all $g \in G$, where $gf(x) = f(g^{-1}x)$.

The Laplace operator is of course an example of a Laplacian operator in the sense of Definition 3.6. We define

Definition 3.7. With M and Δ as in Definition 3.5, a function $f \in C^2(M)$ is harmonic if

$$\Delta(f) = 0.$$

A very useful property of harmonic functions on rank 1 symmetric spaces is given by the following proposition:

Proposition 3.8 ([24, Theorem II.5.28])**.** *Let X be a symmetric space of rank one (or $\mathbb{R}^n$) and let $u \in C^2(X \times X)$ be a function satisfying*

$$\Delta_x u(x, y) = \Delta_y u(x, y) \qquad \text{on } X \times X$$

where Δ denotes the Laplace-Beltrami operator on X. Then for any $(x, y) \in X \times X$ and $r \geq 0$

$$\int_{S_r(x)} u(s, y)ds = \int_{S_r(y)} u(x, t)dt.$$

So, if we have a harmonic function $f \in C^2(X)$ on a rank 1 symmetric space X (or $\mathbb{R}^n$), then by Proposition 3.8 with $u(x, y) = f(x)$ we get

$$f(x) = \frac{1}{c(r)} \int_{S_r(x)} f(t)dt \tag{3.1}$$

where $c(r) = \int_{S_r(x)} 1ds$ is the area of the sphere of radius r centered at x in X. In case $X = \mathbb{R}^n$, if a function f satisfies equation (3.1) for any x, it is known to be harmonic, hence in this case equation (3.1) characterizes harmonic functions.

One could also call a function harmonic if $\Delta(f) = 0$ with respect to any Laplacian operator (Def. 3.6). These functions can then be characterized by an integral equation similar to equation (3.1) (see [18, Theorem 4.4]).

Lastly, let us relate the Laplace-Beltrami operators to a certain element in the universal enveloping Lie algebra of $\mathrm{u}(\mathfrak{g})$ of $\mathfrak{g}$ called the Casimir element. For ease of notation, let us assume $\mathfrak{g}$ is a complex Lie algebra (if it is real, replace it with the complexification $\mathfrak{g}^{\mathbb{C}}$ and using the natural map $\mathrm{u}(\mathfrak{g}) \to \mathrm{u}(\mathfrak{g})^{\mathbb{C}}$ given by Proposition 3.10 below).

First we need to define:

Definition 3.9. The universal enveloping Lie algebra, denoted $\mathrm{u}(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$ is the quotient of the tensor algebra

$$T(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

by the relation

$$x \otimes y - y \otimes x \sim [x, y].$$

The universal enveloping Lie algebra $\mathrm{u}(\mathfrak{g})$ of Definition 3.9 comes equipped with an inclusion

$$\iota : \mathfrak{g} \to \mathrm{u}(\mathfrak{g})$$

defined by $\iota(x) = x$ for all $x \in \mathfrak{g}$. The terms "enveloping" and "inclusion" indicates that the Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathrm{u}(\mathfrak{g})$, and indeed the map $\iota : \mathfrak{g} \to \mathrm{u}(\mathfrak{g})$ is always injective. As with any universal object, it is uniquely determined by some universal properties, which for $\mathrm{u}(\mathfrak{g})$ read.

Definition 3.10 ([33, Proposition III.1.3.3])**.** The universal enveloping Lie algebra $\mathrm{u}(\mathfrak{g})$ and the canonical inclusion $\iota : \mathfrak{g} \to \mathrm{u}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ satisfy the following universal property:

For any unital associative algebra A and linear mapping $\pi : \mathfrak{g} \to A$ such that

$$\pi(x)\pi(y) - \pi(y)\pi(x) = \pi([x, y]) \qquad \forall\ X, Y \in \mathfrak{g},$$

there exists a unique unital algebra homomorphism $\hat{\pi} : \mathrm{u}(\mathfrak{g}) \to A$ such that $\hat{\pi} \circ \iota = \pi$.

Let $\mathbb{D}(G)$ and $\mathbb{D}(G/K)$ denote the algebra of differential operators on G and G/K, respectively which are invariant under the action of G by left multiplication. Recall that for a Lie group G the Lie algebra $\mathfrak{g}$ acts on $C^\infty(G)$ as a differential operator by

$$(x \cdot f)(g) := \left.\frac{d}{dt}\right|_{t=0} f(ge^{tx}) \qquad x \in \mathfrak{g}, g \in G. \tag{3.2}$$

Using the universal property of Proposition 3.10 we can extend the action of $\mathfrak{g}$ on $C^\infty(G)$ to an action of $\mathrm{u}(\mathfrak{g})$ on $C^\infty(G)$.

This action is trivially invariant under the usual left action of G on $C(G)$ by left multiplication, hence we have a well-defined map

$$\mathrm{u}(\mathfrak{g}) \to \mathbb{D}(G)$$

Proposition 3.11 ([33, p. 180])**.** *The map* $\mathrm{u}(\mathfrak{g}) \to \mathbb{D}(G)$ *is an algebra isomorphism.*

Now for a symmetric space (or any homogeneous space), $X = G/K$, a function $f \in C^\infty(G/K)$ determines a right K-invariant function $\hat{f} \in C^\infty(G)$ by $\hat{f} = f \circ q$, where $q : G \to G/K$ is the quotient map. However, if $x \in \mathfrak{g}$, the function

$$F = (x \cdot \hat{f}) \in C^\infty(G) \tag{3.3}$$

may no longer be right K-invariant. In fact for $k \in K$ we have

$$F(gk) = \frac{d}{dt}\bigg|_{t=0} \hat{f}(gke^{tx}) = \frac{d}{dt}\bigg|_{t=0} \hat{f}(gke^{tx}k^{-1}k) = \frac{d}{dt}\bigg|_{t=0} \hat{f}(ge^{t\mathrm{Ad}_k(x)}) \tag{3.4}$$

by right K-invariance of $\hat{f}$. Thus to ensure $x \in \mathfrak{g}$ descends to a well-defined differential operator on $C^\infty(G/K)$ we would have to assume $\mathrm{Ad}_k(x) = x$ for all $k \in K$.

Denote by $\mathbb{D}(G)^K$ the right K-invariant differential operators in $\mathbb{D}(G)$. Combining equation (3.4) with Proposition 3.9 tells us that we have an isomorphism

$$\mathbb{D}(G)^K \simeq \{x \in \mathrm{u}(\mathfrak{g}) \mid \mathrm{Ad}_k(x) = x, \quad \text{for all } k \in K\}.$$

Thus in particular the center $Z(\mathrm{u}(\mathfrak{g}))$ determines a well-defined family of G-invariant differential operators on G/K, by the action given in equation (3.3). Not all elements in $Z(\mathrm{u}(\mathfrak{g}))$ are Laplacian operators in the sense of Definition 3.6 though, as they can be non-elliptic and of any order. Now if (G, K) is a symmetric pair with G semisimple, which happens if $G = \mathrm{Iso}(X)^0$ for a symmetric space with no euclidean factors, then the Killing form (Ex. 2.11) is non-degenerate, which gives us an identification $\mathfrak{g} \mapsto \mathfrak{g}^*$ by $x \mapsto \langle x, - \rangle$ for $x \in \mathfrak{g}$, and so

$$\mathrm{End}_{\mathbb{C}}(\mathfrak{g}) \simeq \mathfrak{g} \otimes \mathfrak{g}^* \simeq \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{u}(\mathfrak{g}) \tag{3.5}$$

All the isomorphisms in equation (3.5) are G-equivariant maps with respect to the adjoint action of G on $\mathfrak{g}$ and thus the image of $\mathrm{Id} \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ in $\mathrm{u}(\mathfrak{g})$ is a G-bi-invariant differential operator on G. We now define:

Definition 3.12. The Casimir element Ω of $\mathrm{u}(\mathfrak{g})$ is the element given in a basis $x_1, ..., x_n$ of $\mathfrak{g}$ by

$$\Omega := \sum_{i,j} B(x_i, x_j) x_j \otimes x_i$$

where B is a choice of inner product on $\mathfrak{g}$.

Note that this is the image of $2\mathrm{Id} \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ in $Z(\mathfrak{g}) \subset \mathrm{u}(\mathfrak{g})$ given by chain of maps in (3.5), hence is G-invariant. The Casimir element is invariant of choice of basis x_i ([33, Prop. V.5.24]) and can be defined similarly with respect to arbitrary inner products on $\mathfrak{g}$.

Proposition 3.13 ([24, Ex. A.4 p.331])**.** *Assume G/K is a noncompact symmetric space. Then under the action of $Z(\mathrm{u}(\mathfrak{g}))$ on G/K given by equation* (3.3) *the Casimir element corresponds to the Laplace-Beltrami operator.*

3.3 The Martin compactification

The Martin compactification has its root in the study of the asymptotic behaviour of Green's functions. In the case of $\mathbb{H}^2$ and the Laplace operator, classical harmonic analysis tells us that any bounded harmonic function can be written as an integral of a function on $\partial\mathbb{H}^2 \simeq S^1$.

Recall that the Furstenberg boundary $\partial_F X \simeq G/P_0$ determines the space of all bounded harmonic functions on a symmetric space of noncompact type $X = G/K$ by sending $f \mapsto (x \mapsto \int_{\partial_F X} f(v) d\mu_x(v))$ where μ_x is the unique $K_x = \mathrm{Stab}_G(x)$-invariant probability measure on G/P_0.

However in higher ranks, for dimension reasons, the Furstenberg boundary $\partial_F X$ is not the boundary of any "nice" G-compactification of X. The idea of Martin in [35] was to find a compactification of X for which the classical harmonic analysis on $\mathbb{H}^2$ could carry over. To achieve this, he noted that the Poisson kernel can be written as a limit

$$P(x,\xi) = \lim_{y \to \xi} \frac{G(x,y)}{G(x_0,y)} \tag{3.6}$$

of Green's functions for the Laplace operator Δ on $\mathbb{H}^2$. Thus he was led to the idea of constructing an ideal boundary $\partial X^M(0)$, consisting of all possible functions obtained as limits of fractions of Green's functions as in equation (3.6). The union $X \cup X^M(0)$ was topologized in a natural way and, almost by construction, a theory similar to the classical theory of Poisson integrals could be defined for this compactification.

Later this was generalized by looking at Green's functions of differential equations of the form

$$(\Delta + \lambda)(f) = 0, \qquad \lambda \in \mathbb{R}$$

and a similar boundary $\partial X^M(\lambda)$ was constructed.

We will now define this compactification in more detail and show that it is isomorphic to the Furstenberg compactification whenever they are both defined.

A *Green's function* is a kernel of an integral operator which acts as an inverse of a differential operator on a domain with prescribed boundary values. In particular, if L is a differential operator in the variable x, then

$$\left(\int (LG(x,y)) f(y) d\lambda(y) \right) = f(x).$$

There is a priori no reason to assume that such a G should exist. For a $\lambda \in \mathbb{R}$, we will denote by

$$L_\lambda = \Delta + \lambda \qquad L_\lambda(f) = \Delta(f) + \lambda f$$

where Δ is the Laplace-Beltrami operator on X. Let G_λ be a Green's function associated with L_λ which vanishes at infinity. The spaces

$$C_\lambda(X) = \{f \in C^\infty(X) \mid L_\lambda(f) = 0, f > 0\}$$

of L_λ are convex cones and, when non-empty, will be referred to as the positive eigenspace of L_λ. The extremal points of C_λ are called minimal eigenfunctions or simply minimal functions. We have the following:

Theorem 3.14 ([20, p. 97])**.** *There is a value $\lambda_0 \geq 0$ for which $C_\lambda \neq 0$ if and only if $\lambda \leq \lambda_0$.*

We define

Definition 3.15. The value λ_0 of Theorem 3.14 is called the *bottom of the spectrum* of Δ.

The reason we call it the bottom of the spectrum is that it coincides with the bottom of the L^2-spectrum of Δ (see for instance [50, Th. 2.2]). We recall that the L^2-spectrum of Δ does not have any eigenvalues, only an absolutely continuous part [4].

On symmetric spaces of noncompact type, there is a simple way to determine the bottom of the spectrum λ_0, namely.

Proposition 3.16 ([20, p.97])**.** *For a symmetric space $X = G/K$ a symmetric space of noncompact type we have*

$$\lambda_0(X) = \frac{||b||^2}{4}$$

where b is the barycenter of $\mathfrak{a}^+$ (eq. (2.8) on page 74).

Example 3.17. Note that the barycenter is not determined by the root system of the Lie group G since it keeps track of the multiplicities of the root (i.e. the dimension of the root spaces $\mathfrak{g}_\alpha$ in equation (2.4)) so we could have Lie groups G of equal rank and isomorphic root spaces with barycenters of different length. The easiest example of this phenomenon are the real hyperbolic spaces

$$\mathbb{H}^n = SO^0(1,n)/SO(n)$$

which all have the same root system (namely A_1) and are all of rank 1. This means there are two roots $\{\alpha, -\alpha\} \subset \mathfrak{a}^* \simeq \mathbb{R}$, one positive root $\alpha \in \mathfrak{a}^*$ and the Weyl group $W = \mathbb{Z}_2$ acts by flipping the two root vectors. In the root space decomposition of the Lie algebra

$$\mathfrak{so}(1.n) = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

the dimension of $\mathfrak{g}_\alpha$ is the same as the dimension of $\mathfrak{g}_{-\alpha}$ (since the Cartan involution determines a bijection $\Theta : \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$), and the dimension of $\mathfrak{g}_0 = Z_K(\mathfrak{a}) \oplus \mathfrak{a} = \mathfrak{m} \oplus \mathfrak{a}$ is $\dim(\mathfrak{m}) + \dim(\mathfrak{a})$, where $\mathfrak{m}$ is the Lie algebra of the M component of a minimal parabolic subgroup $P = MAN$. We have seen in Example 2.69 that $M \simeq SO(n-1)$, so we can determine

$$\begin{aligned} \dim(M) &= \frac{(n-1)(n-2)}{2} \\ \dim(\mathfrak{a}) &= 1 \\ \dim(SO(n,1)^0) &= \frac{n(n+1)}{2} \end{aligned}$$

which yields

$$\begin{aligned} \dim(\mathfrak{g}_\alpha) &= \frac{\dim(\mathfrak{g}) - \dim(\mathfrak{g}_0)}{2} \\ &= \frac{n(n+1) - [(n-1)(n-2) + 2]}{4} \\ &= \frac{4n}{4} = n. \end{aligned}$$

Now the bottom of the spectrum of the Laplace operator Δ on $\mathbb{H}^n$ is given by the formula in Proposition 3.16:

$$\lambda_0(\mathbb{H}^n) = \frac{\|b\|^2}{4} = \frac{n^2}{4}.$$

For any $\lambda \leq \lambda_0$ there exists a unique positive Green's function G_λ vanishing at infinity [28, Theorem 16.6.1]. Define

$$K^\lambda(x,y) \begin{cases} \frac{G_\lambda(x,y)}{G_\lambda(x_0,y)} & y \neq x_0 \\ 0 & y = x_0, x \neq x_0 \\ 1 & y = x = x_0 \end{cases}$$

where $x, y \in X$ and as usual x_0 denotes the equivalence class of K in $X = G/K$. For any fixed x, the function $K^\lambda(x,y)$ is continuous in y except at $y = x_0$. By uniqueness of the Green's function the map

$$y \mapsto K^\lambda(-,y)$$

is injective. It can also be shown that the family of functions $\{K(-,y)\}_{y \in X}$ are uniformly continuous in y, hence are precompact in $C(X)$ with the topology of uniform convergence on compact sets.

Definition 3.18. The Martin compactification with parameter $\lambda \leq \lambda_0$ is the closure of $\{K(-,y)\}_{y\in X}$ in $C(X)$. Denote this closure by $\overline{X}^M(\lambda)$.

So as a set, the Martin boundary $\partial X^M(\lambda)$ consists of equivalence classes of unbounded sequences $y_i \in X$ for which $K^\lambda(-,y_i)$ converges uniformly on compact sets. We follow [35] and define

Definition 3.19. A sequence $x_i \in X$ for which $K(-,x_i)$ converges in $\overline{X}^M(\lambda)$ is called a fundamental sequence.

The limit of an unbounded fundamental sequence is denoted by $K^\lambda(-,\xi)$, where ξ is thought to represent a point on the boundary $\partial X^M(\lambda)$. These limit functions are called the Martin kernels and play the role of the Poisson kernel in the Martin compactification.

There is remarkably little variation in isomorphism type of the compactifications $\overline{X}^M(\lambda)$ as λ varies. In fact, all $\overline{X}^M(\lambda)$ for $\lambda < \lambda_0$ turn out to be isomorphic to the closure of X in

$$\overline{X}^M(\lambda_0) \times X \cup X(\infty)$$

under the diagonal imbedding. This is precisely the smallest compactification dominating both $X \cup X(\infty)$ and $\overline{X}^M(\lambda_0)$.

We will now restrict our attention to the case where $\lambda = \lambda_0$ as this will turn out to be isomorphic to the maximal Furstenberg compactification defined earlier. Our overarching goal is to describe in detail the compactifications on which the Poisson integral

$$F_f : x \mapsto \int_{G/P} f(v) d\mu_x(v)$$

extends for any $f \in C(G/P)$. The parameter $\lambda = \lambda_0$ is in a sense the "smallest" compactification for which this is possible, as evidenced by the construction of the Furstenberg compactification. Note that for any compactification that dominates $\overline{X}^F$ (Definition 3.1), we can extend the Poisson integral by composing it with the surjection onto $\overline{X}^F$. Let us prove the following theorem:

Theorem 3.20. *Let X be a symmetric space of noncompact type. Then there is an equivariant isomorphism*

$$\overline{X}^M(\lambda_0) \xrightarrow{\sim} \overline{X}^F$$

extending the identity map on X.

We will follow [20] and prove Theorem 3.20 by comparison of convergence criteria. Before proving it, we will need some preliminary lemmas. Let $H_\alpha \in \mathfrak{a}$ denote the dual vector of a root α, and

$$\mathfrak{a}_I = \operatorname{span}\{H_\alpha \mid \alpha \in I\} \subset \mathfrak{a} \tag{3.7}$$

$$\mathfrak{a}^I = (\mathfrak{a}_I)^\perp \subset \mathfrak{a} \tag{3.8}$$

so that $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}^I$. We denote by A^I and A_I the exponential of $\mathfrak{a}^I$ and $\mathfrak{a}_I$ respectively.

The polar decomposition $X = K\overline{A^+}$ allows us to write any $x \in X$ as

$$x = ka_I a^I$$

where $a^I \in A^I$, $a_I \in A_I$ and $k \in K$. We denote by P_I the parabolic subgroup associated with the subset $I \subset \Sigma$, and $P_I = M_I A_I N_I$ its Langlands decomposition.

We denote by $S_I = P_I \cap K = M_I \cap K$. This is a maximal compact subgroup in both M_I and P_I which coincides with the centralizer of A_I in K (see [7] p.31).

As mentioned above, the notion of a fundamental sequence in [20] is slightly different from the one in [7]. We have used the definition of the latter reference which gives a necessary and sufficient condition for convergence. In [20] however, the authors use a stronger notion of fundamental sequence that is sufficient but not necessary for convergence to boundary points. One can go from our definition of fundamental sequence to their definition by passing to a subsequence, but we would like to find a convergence criterion for sequences that is sufficient *and necessary* in polar coordinates.

To keep track of which definition we are using we will call the one in [20] strong fundamental sequences as they are fundamental sequences in the sense of Definition 3.19. Here is the definition:

Definition 3.21 (strong fundamental sequence [20] III.3.35)**.** A sequence $x_i \in X$ is called *strong fundamental* if there exists a subset $I \subset \Sigma$ of simple roots (Definition 2.39) such that we can write

$$x_i = k_i a_i^I a^i$$

with $a_i^I \in A^I \cap \overline{A^+}$, $a_i \in \overline{A^+}$, and $k_i \in K$ with

- $a_i^I \to a^I$ for some $a^I \in A^I$
- $\alpha(a_i) \to 0$ for all $\alpha \in A^I$
- $\alpha(a_i) \to \infty$ for all $\alpha \notin I$
- $k_n \to k$ for some $k \in K$.

Let us see how the authors produce a strong convergent subsequence from a fundamental sequence converging to a boundary poitn in $\overline{X}^M(\lambda_0)$:

Let $x_n \in X$ be a fundamental sequence in $\overline{X}^M$, assumed to converge to a boundary point $x_\infty \in \partial X(\lambda_0)$. Let $I = \{\alpha \in \Sigma \mid \sup_i \alpha(a_i) < \infty\}$ and as in equation (3.7), decompose $\mathfrak{a}$ as the direct sum

$$\mathfrak{a} = \mathfrak{a}^I \oplus \mathfrak{a}_I.$$

By passing several times to subsequences the authors assume that k_n converges, $\alpha(a_i)$ converges for all $\alpha \in I$ and $\alpha(a_i) \to \infty$ for all $\alpha \notin I$, which yields the desired strong fundamental sequence. Using the decomposition of $\mathfrak{a}$, we may write $a_n = a_n^I a_{I,n}$, and the last conditions imply $a_n^I \to a^I$ in $\mathfrak{a}^I$.

However, [20, Corollary VII.7.32] tells us that both the limit a^I, and the set I are intrinsic to the limit point, so in particular shared by all strong fundamental subsequences of x_n. If a_n^I did not converge we could produce two subsequences that converge to distinct points a^I and $\hat{a}^I$ in $\mathfrak{a}^I$. This would imply x_n has two subsequences converging to distinct limit points, yielding a contradiction as x_n is assumed to converge.

Similarly, if $\alpha(a_{n,I}) \not\to \infty$ for some $\alpha \notin I$, we could pass to a subsequence where $\alpha(a_{n,I})$ is bounded. We could then enlarge I by α and run into the same issues.

Hence the only thing that requires passing to a subsequence is the convergence of k_n. Let us summarize the discussion in the following lemma:

Lemma 3.22. *An unbounded sequence x_n in X converges to a boundary point $x_\infty \in \partial X^M(\lambda_0)$ if and only if there is an $I \in \Sigma$ such that we can write*

$$x_n = k_n a_n^I a_i$$

with $k_n \in K$, $a_n^I \in A^I \cap \overline{A^+}$ and $a_i \in \overline{A^+}$, and

- $a_n^I \to a$ *in* A^I
- $\alpha(a_i) \to 0$ *for all* $\alpha \in I$
- $\alpha(a_i) \to \infty$ *for all* $\alpha \in \Delta \backslash I$.
- $[k_n] \to [k]$ *in* $K/(S_I \cap a S^I a^{-1})$.

Furthermore, this I is uniquely determined by the limit point x_∞.

For other necessary and sufficient conditions for unbounded sequences in X to converge in $\overline{X}^M(\lambda_0)$ see [7] p. 124. We can now state the following:

Lemma 3.23. *An unbounded sequence x_n is fundamental if and only if there is a subset $I \subset \Sigma$ and a decomposition $x_n = k_n a_n^I a_i$ such that*

- $a_i^I \to a^I$ *for some* $a^I \in A^I$
- $\alpha(a_i) = 0$ *for all* $\alpha \in I$
- $\alpha(a_i) \to \infty$ *for all* $\alpha \notin I$
- $[k_i]$ *converges in* $K/(S_I \cap a^I S_I (a^I)^{-1})$.

Hence two fundamental sequences x_n and x'_n have the same limit if and only if $I = I'$, $\lim a_i^I = \lim (a'_i)^I$ and $k^{-1}k' \in S_I \cap a^I S_I (a^I)^{-1}$.

Proof. This is Proposition VII.7.31 of [20]. ◻

This gives us a complete characterization of fundamental sequences in X in terms of their polar decomposition. Similarly, we have

Lemma 3.24. *A sequence $x_i \in X \subset \overline{X}^F$ converging to a boundary point $x_\infty \in \partial X^F$ if and only if there is a subset $I \subset \Sigma$ such that $x_i = k_i a_i^I a^i$ with*

- $a_i^I \to a^I$ *for some* $a^I \in A^I$
- $\alpha(a_i) = 0$ *for all* $\alpha \in I$
- $\alpha(a_i) \to \infty$ *for all* $\alpha \notin I$
- $[k_i]$ *converges in* $K/(S_I \cap a^I S_I (a^I)^{-1})$.

Proof. This is Prop. IX.9.46 [20]. □

We can now prove Theorem 3.20:

Th. 3.20. The conditions for a sequence $\{x_i\} \subset X$ to converge to a boundary point is stated in Lemma 3.24 for the Furstenberg compactification and Lemma 3.22 for the Martin compactification and are clearly identical. We conclude that the identity map on X extends to an isomorphism between these two compactifications. □

3.4 The Chabauty compactification

We will here describe a third compactification isomorphic to the maximal Furstenberg and Martin compactification with $\lambda = \lambda_0$. It is called the Chabauty compactification. Where the Martin compactification has a differential geometric flavour, and the Furstenberg compactification is defined using measure theory, the Chabauty compactification is defined almost entirely by group theoretic means.

Let X be a symmetric space of noncompact type and $\mathcal{P} = \mathcal{P}(G)$ the collection of all closed subgroups of $G = \mathrm{Iso}(X)^0$. The set $\mathcal{P}$ can be endowed with the topology induced by the following neighborhood basis about a closed subgroup $C \in \mathcal{P}$

$$V_{K,U}(C) := \{D \in \mathcal{P} \mid K \cap D \subset CU,\ C \cap K \subset DU\}$$

where U runs over a neighborhood basis of $e \in G$ and K runs over all compact subgroups of G. We have the following theorem

Theorem 3.25 ([8, Th. 1, §5.3, Chapter VIII])**.** *The space $\mathcal{P}$ with the topology induced by $V_{K,U}$, is compact and Hausdorff.*

See §5.6, Chapter VIII of the same reference for the Hausdorff part. Now for a symmetric space of noncompact type $X = G/K$ we have a natural imbedding

$$X \to \mathcal{P} \qquad x \mapsto \mathrm{Stab}_G(x) =: K_x. \tag{3.9}$$

Definition 3.26. The Chabauty compactification $\overline{X}^C$ of X is defined to be the closure of X in $\mathcal{P}$ under the imbedding given by equation (3.9).

The following lemma gives a way to check if a sequence in the Chabauty compactification converges

Lemma 3.27 ([21, Lemma 2])**.** *A sequence of closed subgroups $F_i \in \mathcal{P}$ converges to $F \in \mathcal{P}$ if and only if*

- *if $x_i \in F_i$ is a sequence such that $x_i \to x$ in G, then $x \in F$*
- *each $x \in F$ there is a sequence $x_i \in F_i$ such that $x_i \to x$ in G.*

As mentioned, we have the following theorem, valid for any symmetric space X of noncompact type:

Theorem 3.28 ([21])**.** *Let X be a symmetric space of noncompact type. Then the identity map on X extends to G-equivariant isomorphisms*

$$\overline{X}^C \leftrightarrow \overline{X}^F \leftrightarrow \overline{X}^M(\lambda_0).$$

3.4.1 G-orbits and a natural extension

Let us look at the G-orbit structure of $\overline{X}^F$. Recall our standing assumptions that X is a symmetric space of noncompact type and $G = \mathrm{Iso}(X)^0$ is the isometry group (a semisimple Lie group) with maximal compact subgroup $K \subset G$. Let $I \subset \Sigma$ be a subset of simple roots of G, and let P_I be the corresponding parabolic subgroup. In case $I = \emptyset$ we have $P_\emptyset$ is a minimal parabolic subgroup of G. For any I the parabolic subgroup P_I admits a Langlands decomposition (Proposition 2.49)

$$P_I = M_I A_I N_I$$

and we define:

Definition 3.29. For any parabolic subgroup $P \subset G$, the boundary symmetric space X_P associated with P is defined to be ([7] eq. I.4.31)

$$X_P := M_P/(K \cap M_P)$$

and is thus isomorphic to the M_P-orbit of $x_0 = K \subset G/K$ in X. For $I \subset \Sigma$ we denote by X_I the boundary symmetric space of the parabolic subgroup P_I.

Let X_P be the boundary symmetric space of a parabolic subgroup $P \subset G$. Similar to the Iwasawa projections, we define the map

$$\overline{m} : P \to M_P \qquad \overline{m}(man) = m$$

of projection onto the M_P-factor in the Langlads decomposition of P. Then the boundary symmetric space X_P has a natural P-action given by

$$p \cdot m(M_P \cap K) = \overline{m}(p)m(M_P \cap K) \qquad \forall\, p \in P. \tag{3.10}$$

Next, for any $k \in K$ we get a map $X_P \to X_{^kP}$, where $^kP = kPk^{=1}$ by

$$k \cdot m(M_P \cap K) = {}^k m(M_{^kP} \cap K) = {}^k m({}^k M_P \cap K) \qquad \forall\, k \in K. \tag{3.11}$$

Now for P any parabolic subgroup of G with Langlands decomposition $P = M_P A_P N_P$, using the generalized Iwasawa decomposition of Proposition 2.56

$$G = KM_P \times A_P \times N_P$$

and the associated projection (Definition 2.57)

$$\overline{m} : G \to KM_P$$

we see that we can write any element in g as a product $g = kman$, with $km \in KM_P$, $a \in A_P$, $n \in N_P$. Combining equation (3.10) and equation (3.11) we get a G-action the collection of all boundary symmetric spaces given for an element $g = kp \in G$ by (see for instance [7] Proposition I.10.8)

$$g \cdot m(M_P \cap K) = (\overline{m}(p)m)^k(M_{P^k} \cap K). \tag{3.12}$$

Let us prove it is well-defined. The generalized Iwasawa decomposition tells us that the product km is uniquely determined, but the elements k and m are only determined up to an element in $M_p \cap K$. More precisely, $km = k'm'$ for $k, k' \in K$ and $m, m' \in M_P$ if and only if there is an $s \in M_P \cap K$ such that

$$ks = k' \qquad \text{and} \qquad s^{-1}m = m'.$$

Thus if we write $K_P = M_P \cap K$ we have

$${}^{k'}K_P = {}^{ks}K_P = k\left(sK_Ps^{-1}\right)k^{-1} = kK_Pk^{-1} = {}^kK_P.$$

Then for any $\hat{m}M_P \cap K \in X_P$ we get

$$\begin{aligned} g \cdot \hat{m}K_P &= {}^k(m\hat{m})^k K_P = km\hat{m}k^{-1}kK_Pk^{-1} \\ &= km\hat{m}k^{-1}k'K_P(k')^{-1} = k'm'\hat{m}sK_P(k')^{-1} \\ &= k'm'\hat{m}K_P(k')^{-1} = k'm'\hat{m}s(k')^{-1}k'K_P(k')^{-1} \\ &= {}^{k'}(m'\hat{m})^{k'}K_P = {}^{k'}(m'\hat{m})K_{k'P} \end{aligned}$$

showing that the G action of equation (3.12) is well-defined.

We are now ready to write the G-orbit structure of the Furstenberg compactification $\overline{X}^F$ as

$$\overline{X}^F = X \bigsqcup_{I\subset\Sigma} GX_I = X \bigsqcup_{I\subset\Sigma} KX_I \tag{3.13}$$

where $X_I = M_I/(M_I \cap K)$ are the boundary symmetric spaces associated with P_I. Since all X_I are are not closed unless $I = \emptyset$ in which case it is a single point, we get the following

Proposition 3.30. *There is a unique closed G-orbit in $\overline{X}^F$. It is given by*

$$GX_\emptyset \simeq G/P_0 \simeq \partial_F X$$

where $P_0 \subset G$ is a minimal parabolic.

We can thus define

$$X_0 = \overline{X}^F \backslash \partial_F X$$

and get an extension

$$0 \to C_0(X_0) \to C(\overline{X}^F) \to C(\partial_F X) \to 0.$$

From the theory in the preceding sections we have the following

Proposition 3.31. *The assignment $f \mapsto F_f$ (eq. (2.12)) from $C(\partial_F X)$ to $C(\overline{X}^F)$ determines an equivariant completely positive contractive splitting of the extension*

$$0 \to C_0(X_0) \to C_0(\overline{X}^F) \to C(\partial_F X) \to 0.$$

Proof. The map

$$F_f(x) = \int_{\partial_F X} f(v)d\mu_x(v)$$

extends continuously to a function on $\overline{X}^F$ which agrees with f on $\partial_F X$. Since μ_x are all probability measures, this assignment is easily seen to be a completely positive contraction. The G-equivariance of the density $x \mapsto \mu_x$ makes it G-equivariant. □

3.5 A worked example

Since all higher rank symmetric spaces of noncompact type are of dimension 4 or higher, trying to visualize their compactifications can be challenging and inevitably lead to some sort of compromise between rigour and clarity of exposition.

We have opted to look at a 3-dimensional example, at the expense of having to add a euclidean factor to our space. This poses no issues when defining the geodesic compactification.

The space we will be looking at is $X = \mathbb{H}^2 \times \mathbb{R}$, with isometry group $G = SL_2(\mathbb{R}) \times \mathbb{R}$ and maximal compact subgroup $K = SO(2) \times \{0\}$.

First, let us see what goes wrong in the case of the Furstenberg compactification: Recall that it is defined by first imbedding X into $M_1(G/P)$ where P is a minimal parabolic subgroup of G. The theory developed in [18] is not well suited for this example as G is not semisimple.

The situation is also quite unappealing for the Chabauty compactification, which is defined for the imbedding of X into the space $\mathcal{P}$ of all closed subgroups of G, by the assignment

$$x \mapsto \mathrm{Stab}_G(x).$$

if $x = (x_1, x_2) \in \mathbb{H}^2 \times \mathbb{R}$ and $G = SL_2(\mathbb{R}) \times \mathbb{R}$, then

$$\mathrm{Stab}_G(x) = \mathrm{Stab}_{SL_2(\mathbb{R})}(x_1)$$

hence the function will be constant along the $\mathbb{R}$-factor. The closure of the image of X in $\mathcal{P}$ is simply the closure of the image of the $\mathbb{H}^2$-factor.

One could circumvent this issue by looking at a slightly larger group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, namely

$$G = SL_2(\mathbb{R}) \times (\mathbb{R} \rtimes \mathbb{Z}_2)$$

allowing also reflections along the $\mathbb{R}$ factor, in which case

$$\mathrm{Stab}_G(x_1, x_2) = \mathrm{Stab}_{SL_2(\mathbb{R})}(x_1) \times \mathrm{Stab}_{\mathbb{R} \rtimes \mathbb{Z}_2}(x_2).$$

Note that since $\mathrm{Stab}_{\mathbb{R} \rtimes \mathbb{Z}_2}(x_2) = \mathbb{Z}_2$ would be the reflection about x_2, this would indeed make the assignment $x \mapsto \mathrm{Stab}_G(x)$ injective.

If we are given a sequence $(h, r_i) \in \mathbb{H}^2 \times \mathbb{R}$ for some fixed $h \in \mathbb{H}^2$ and $r_i \in \mathbb{R}$ with $r_i \to \infty$, then the stabilizer of (h, r_i) will be the group

$$SO_2 \times \mathbb{Z}_2$$

where $\mathbb{Z}_2$ is the subgroup of reflections about r_i. Using the convergence criterion in Lemma 3.27 it is easy to see that

$$\mathrm{Stab}_G(h, r_i) \to \mathrm{Stab}_{SL_2}(h) \times \{0\}.$$

For a general sequence $(h_i, r_i) \in \mathbb{H}^2 \times \mathbb{R}$, using Lemma 3.27 we can show that the limit takes either of two forms $K \times \{0\}$ for some compact subgroup $K \subset G$ if r_i is unbounded or $K \times \mathbb{Z}_2$ if r_i is bounded. Only the latter limit points see the $\mathbb{R}$-factor. In this way we get that the Chabauty compactification of $\mathbb{H}^2 \times \mathbb{R}$ will be a closed ball with a neighborhood of the north and south pole in $\partial\mathbb{H}^2 \rtimes \mathbb{R}$ identified. Thus the Chabauty compactification is homeomorphic to a solid torus. See also the Remark 2.1 of [32].

For the case of the geodesic compactification however, things work out without many issues as it is defined for any Hadamard manifold (see Section 1.7). The geodesic compactification of $\mathbb{H}^2 \times \mathbb{R}$ can be visualized as a globe (see figure 3.1) with the equator identified with $\partial_F X$. The north and south pole are our irregular boundary points. A positive Weyl chamber at infinity $A^+(\infty)$ is an open half circle with boundaries the two irregular points. There is a single positive root on A given as the dual of the vector in A orthogonal to the axis $(\mathrm{Id}, \mathbb{R})$ depicted by the red line in Figure 3.1

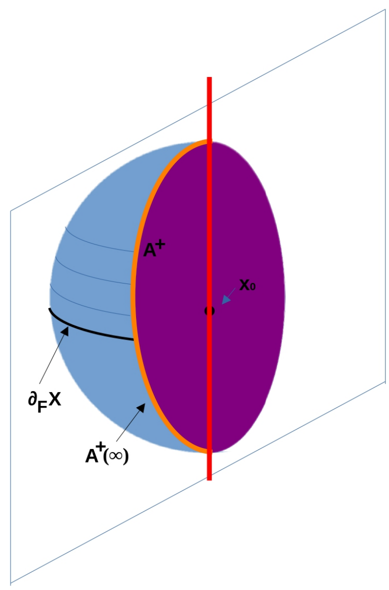

Figure 3.1: The geodesic compactification of $\mathbb{H}^2 \times \mathbb{R}$

The orbits of G on $X(\infty)$ are the circles parallel to $\partial_F X$ or the two fixed points, which are both fixed points of G.

The polar coordinates (Cor. 2.53) can also be visualized easily in this example. With $\overline{A^+}$ the closure of A^+ in X (see Figure 3.1) any element in X can be written as a product

$$x = ka \qquad a \in \overline{A^+},\ k \in K = SO(2) \times \{0\}.$$

The properties of this coordinate system is now easy to visualize. Since the K-component acts by rotation about the axis connecting the north and the south pole, and this axis of rotation is exactly $\partial\overline{A^+}$, the K-component of the coordinate is unique if and only if x is not on $\partial\overline{A^+} \subset X$ while the $\overline{A^+}$-component is always unique.

The Patterson–Sullivan densities of a lattice in X give measures which are supported on $\partial_F X$. These "don't see" the $\mathbb{R}$-direction, meaning that if $g = (I, r) \in G = SL_2(\mathbb{R}) \times \mathbb{R}$ $\mu_x = \mu_{rx}$ for all $r \in \mathbb{R}$. Thus if we have a sequence of points $x_i \in X$ converging to infinity along the north-south axis then $\mu_{x_i} \to \mu_{x_0}$ (the unique $K = \mathrm{Stab}_G(x_0)$-invariant measure).

Chapter 4

The Kasparov module of the boundary extension

4.1 The KK^1_G-class of the boundary extension

In Section 1.6 we saw the intimate connection between equivariant extensions and KK^1_G-cycles given by Theorem 1.75 which stated that for A and B G-C*-algebras with B K-proper (Definition 1.74) we have an isomorphism

$$\mathrm{Ext}_G(A, B \otimes \mathbb{K}_G)^0 \simeq \mathrm{KK}^1_G(A, B).$$

The space

$$\mathbb{K}_G = \mathbb{K}\left(\bigoplus_{n\in\mathbb{N}} L^2(G)\right)$$

is given the G-action with respect to the diagonal action of the regular representation on $\bigoplus_{n\in\mathbb{N}} L^2(G)$ and $\mathrm{Ext}_G(A, B\otimes\mathbb{K}_G)^0$ is the subgroup of invertible extensions in $\mathrm{Ext}_G(A, B\otimes \mathbb{K}_G)$ (Definition 1.72).

We mentioned that in case an equivariant extension

$$0 \to B \to E \to A \to 0 \tag{4.1}$$

of (separable) G-C*-algebras does not satisfy the condition in Theorem 1.75, the equivariant extension obtained by tensoring everything with $\mathbb{K}_G$

$$0 \to B \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G \to A \otimes \mathbb{K}_G \to 0 \tag{4.2}$$

does satisfy the conditions in the theorem, and we say that the KK^1_G-class associated with the extension of equation (4.1) is the class associated with the extension of equation (4.2) under the isomorphism given by Theorem 1.75.

We are now going to go through the way in which one produces a KK^1_G-cycle from an equivariant semisplit extension. Then relate it to the usual KK^1_G-groups defined in Section 1.6. Let

$$0 \to B \to E \to A \to 0 \tag{4.3}$$

be a G-equivariant extension of G-C*-algebras with an equivariant completely positive contractive splitting $s : A \to E$. As a consequence of Lemma 1.73 if the extension 4.3 is equivariantly semisplit, then the extension is invertible in $\mathrm{Ext}_G(A, B \otimes \mathbb{K}_G)$ (though the converse is not in general true, unless A and B are $\mathbb{K}_G$-stable).

We saw in Section 1.6 that the extension 4.3 is equivariantly semisplit if and only if the associated Busby map (which is necessarily G-equivariant) lifts to a G-equivariant completely positive contractive map

$$\hat{\phi} : A \to M(B).$$

Assuming the Busby map lifts to $M(B)$, we will now construct an explicit realization of the cycle in $\mathrm{KK}^1_G(A, B)$ representing the class of the extension 4.3. We call a completely positive map $\rho : A \to \mathcal{L}(F)$ from A to a Hilbert B-module F *strict* if for some approximate unit $e_i \in A$, there exists a $p \in \mathcal{L}(F)$ such that

$$\|(\rho(e_i) - p)x\| \to 0 \qquad \|(\rho(e_i)^* - p^*)x\| \to 0$$

for all $x \in F$. The points $\rho(e_i)$ is said to converge strictly to p un $\mathcal{L}(F)$. The following theorem, due to Kasparov, generalizes both the Stinespring dilation theorem and the classical GNS-construction.

Theorem 4.1 ([34, Theorem 5.6])**.** *Let A, B be C^*-algebras, F a right Hilbert B-module and $\rho : A \to \mathcal{L}(F)$ a strict completely positive map. Then*

1. *there exists a Hilbert B-module F_ρ, a *-homomorphism $\pi_\rho : A \to \mathcal{L}(F_\rho)$ and an element $v_p \in \mathcal{L}(F, F_\rho)$ such that*

$$\rho(a) = v_\rho^* \pi_\rho(a) v_\rho \qquad a \in A$$

$$\phi_\rho(A) v_\rho F \textit{ is dense in } F_\rho.$$

2. *If H is a Hilbert B-module, $\pi : A \to \mathcal{L}(H)$ is a *-homomorphism $w \in \mathcal{L}(F, H)$ and*

$$\rho(a) = w^* \pi(a) w \qquad a \in A$$

$$\pi(a) w F \textit{ is dense in } H$$

then there is a unitary $u \in \mathcal{L}(F_\rho, H)$ such that

$$\pi(a) = u \pi_\rho u^* \qquad (a \in A)$$

and $w = u v_\rho$.

Strictness is automatic in many cases of interest, in particular if A is unital or if ρ is nondegenerate, since then ρ lifts to a strictly continuous unital map $\rho : M(A) \to \mathcal{L}(F)$ (see [34, Corollary 5.7]).

If in Theorem 4.1 we assume A and B are G-C*-algebras and ρ is G-equivariant, then F_ρ can be chosen to be a Hilbert B-G-module and π_ρ and v_ρ G-equivariant. It may be helpful to get an idea of how this special triple $(F_\rho, \pi_\rho, v_\rho)$ is constructed. The space F_ρ is the completion of the algebraic tensor product

$$A \odot F$$

with respect to the (possibly degenerate and incomplete) B-valued inner product, given on simple tensors by

$$\langle a \otimes x, a' \otimes x' \rangle(b) := \langle \rho(a^* a')x, x' \rangle \qquad (a, a' \in A, x, x' \in B). \tag{4.4}$$

We quotient out the zero elements of the inner product of equation 4.4 and then complete $A \odot F$ to an Hilbert B-module, denoted $F_\rho = A \otimes_\rho B$. The C*-algebra B acts on $A \otimes_\rho F$ by acting on the right on F and the representation $\pi_\rho : A \to A \otimes_\rho F$ is given by left multiplication on the A component. The map $v_\rho^* : \mathcal{L}(F, A \otimes_\rho F)$ is the extension of the map given on simple tensors $a \otimes x \in A \otimes_\rho F$ by

$$v^*(a \otimes x)_\rho := \rho(a)x$$

and its adjoint is

$$v_\rho : F \to A \otimes_\rho F \to F \qquad v_\rho(x) = \lim_i (e_i \otimes x)$$

where e_i is any countable approximate unit for A[1]. If ρ is nondegenerate (meaning $\rho(e_i) \to 1$ strictly) or unital then we have $v_\rho(x) = 1 \otimes x$.

Now for any $a \in A$

$$v_\rho^* \pi_\rho(a) v_\rho(x) = \rho(a)x$$

thus $\rho(a) = \mathrm{Ad}_{v_\rho} \circ \pi_\rho(a)$. A short computation shows that when ρ is nondegenerate we have

$$v_\rho^* v_\rho(x) = v_\rho^*(1 \otimes x) = \rho(1)x = x$$

thus in this case v_ρ is an isometry and

$$p := v_\rho v_\rho^*(a \otimes x) = v_\rho \rho(a)x = 1 \otimes \rho(a)x$$

is a projection in $\mathcal{L}(A \otimes_\rho F)$.

[1] for the existence of this limit we need ρ to be strict!

For a non-strict completely positive we can use [27, Lemma 3.2.8] which states that any completely positive map $\phi : A \to B$ with B unital, has a (unique) extension

$$\hat{\phi} : A^+ \to B$$

to the unitization A^+ of A such that $\hat{\phi}(1) = 1$. If ϕ is contractive, $\hat{\phi}$ will also be contractive. This map is now unital, hence nondegenerate and we can use Theorem 4.1.

There is a different description of the groups $KK^1_G(A, B)$ that is easier to relate to extensions, namely:

Definition 4.2 ([51, p. 11])**.** A A-B KK^1-cocycle is a triple (π, v, p) where

- $\pi : A \to M(B)$ is a *-homomorphism;
- $v : G \to U(M(B))$ is a strictly continuous map with $v(e) = 1$ and $v_{gh} = v_g\beta_g(v_h$ where β is the G-action on B extended to $M(B)$ (where $v_g := v(g)$);
- $p \in M(B)$ is a a projection

satisfying

1. $\mathrm{Ad}v_g \circ \beta_g \circ \pi(a) = \pi(\alpha_g(a))$ for all $a \in A$ and all $g \in G$;
2. $\mathrm{Ad}v_g \circ \beta_g(p) - p \in B \otimes \mathbb{K}$;
3. $p\pi(a) - \pi(a)p \in B$, for all $a \in A$.

Let $\mathbb{E}(A, B)$ denote the set of all A-B KK^1-cocycles. Then we have:

Proposition 4.3. *If (π, v, p) is a A-B KK^1-cocycle then the triple $(B, \pi, (2p - 1))$ is a Kasparov G-module for (A, B) and the map*

$$\mathbb{E}(A, B) \to \mathrm{KK}^1_G(A, B) \qquad (\pi, v, p) \mapsto (B, \pi, (2p - 1))$$

is surjective.

Proof. This follows from Theorem [51, Theorem 4.3] □

In [51] the author defines an equivalence relation on the collection $\mathbb{E}(A, B)$, but we will cheat a little and say that two A-B KK^1-cocycles are homotopic if they map to the same element in $KK^1_G(A, B)$ with the map given in Proposition 4.3. Denote by $[(\pi, v, p)]$ the equivalence class of a A-B KK^1-cocycle under the equivalence relation of being mapped to the same $KK^1_G(A, B)$ class. We similarly define an additive structure on the collection of all homotopy classes of A-B KK^1-cocycle by pulling back the group structure on $\mathrm{KK}^1_G(A, B)$. Then:

Definition 4.4. Let $\mathrm{kK}^1_G(A,B)$ denote the abelian group of A-B KK^1-cocycle $\mathbb{E}(A,B)$ modulo the relation of mapping to the same element in $KK^1_G(A,B)$ with the group structure inherited by $KK^1_G(A,B)$.

Given an extension

$$0 \to B \to E \to A \to 0$$

and assume the Busby map lifts to a completely positive equivariant contraction $\rho : A \to M(B)$ We have the following, which is a consequence Theorem 5.3 and Theorem 4.3 of [51]:

Theorem 4.5. *Let $\rho : A \to M(B)$ be a completely positive contractive G-equivariant lift of the Busby map of the G-equivariant extension*

$$0 \to B \to E \to A \to 0.$$

With the notation of Theorem 4.1, let $p = v_\rho v_\rho^*$. Then ρ is nondegenerate, p is a projection in $M(B)$ and the triple

$$(F_\rho, \pi_\rho, (2p-1))$$

is a Kasparov G module for (A,B) representing the extension. If ρ is not nondegenerate, replace A by the unitization A^+ and using [27, Lemma 3.2.8] replace ρ with the unique unital extension $\hat{\rho} : M(A) \to M(B)$, the class of

$$(F_{\hat{\rho}}, \pi_{\hat{\rho}}, (2p-1))$$

now represents the extension.

Proof. This can be proved by following a sequence of isomorphisms

$$\mathrm{Ext}_G(A \otimes \mathbb{K}_G, B \otimes \mathbb{K}_G)^0 \to \mathrm{Ext}_{G,t}(A \otimes \mathbb{K}_G, B \otimes \mathbb{K}_G)^0 \to \mathrm{kK}^1_G(A,B) \to \mathrm{KK}^1_G(A,B)$$

constructed in [51]. The group $\mathrm{Ext}_{G,\alpha}(A,B)^0$ are the invertible elements of a "twisted" extension group. □

Returning to our G-extension

$$0 \to C_0(\overline{X}^F \backslash \partial_F X) \to C(\overline{X}^F) \to C(\partial_F X) \to 0 \tag{4.5}$$

we have seen that the Poisson integral on the Furstenberg boundary $\partial_F X$ extends to a continuous function on the Furstenberg compactification $\overline{X}^F$. We will now follow [36] Section 3.2 and define a KK^1_G-cycle corresponding to the extension. To simplify the notation, let $Y = \overline{X}^M \backslash \partial_F X$. We will also denote by

$$\phi : C(\partial_F X) \to C_b(Y)$$

the lift of the Busby function. If we write $F_f \in C(\overline{X}^F)$ for the Poisson integral of f, that is

$$F_f(y) = \int_{\partial_F X} f(v) d\mu_y(v)$$

it is easy to verify that for $f \in C(\partial_F X)$ and $h \in C_0(Y)$ we have

$$s(f)(h)(y) = F_f(y)h(y)$$

i.e. $s(f)$ is the multiplication by F_f map. We now use Theorem 4.5 to produce a Kasparov G-module representing the extension.

Let $E_0 = C_c(\partial_F X \times Y) = C(\partial_F X) \otimes C_c(Y)$, and define the $C_c(Y)$-valued inner product inner product on simple tensors by

$$\langle f \otimes h, f' \otimes h' \rangle(y) := s(\overline{f} f')(y)\overline{h}(y)h'(y) = F_{\overline{f} f'}(y)\overline{h}(y)h'(y)$$

Let E be the resulting Hilbert C*-module obtained by removing the zero length vectors and completing E_0 with respect to this inner product. Then E is a Hilbert $C_0(Y)$-module with respect to right multiplication and the map

$$\pi : C(\partial_F X) \to E \qquad \pi(f)h(x, y) = f(y)h(x, y) \qquad f \in C(\partial_F X),\ h \in E_0$$

determines a *-homomorphism to the adjointable operators on E. Since s is unital, the elements $v_s : C_0(Y) \to E$ defined in Theorem 4.1 is an isometry, here given by

$$v_s(f) = 1 \otimes f \in E_0$$

with adjoint

$$v_s^*(f \otimes h) = s(f)h \qquad f \otimes h \in E_0.$$

This gives us a projection

$$p = v_s v_s^* : E_0 \to E_0 \qquad p(f \otimes h)(x, y) = 1 \otimes s(f)h$$

At the level of functions on $E_0 = C_c(Y \times \partial_F X)$, the map p is given by

$$p(f)(x, y) = \int_{\partial_F X} f(v, y) d\mu_y(v).$$

where μ_y is either the Patterson–Sullivan density at $y \in X$, or its limit measure on the boundary point $y \in \partial X^F$ of the Furstenberg compactification.

The triple (E, π, p) is the triple given by Theorem 4.1 for the splitting and thus we have by Theorem 4.5 the following theorem:

Theorem 4.6. *The triple $(E, \pi, 2p-1)$ is a G-equivariant KK^1_G-cycle for $(C(\partial_F X), C_0(Y))$ representing the class of the boundary extension 4.5.*

In case $X = \mathbb{H}^n$, $\overline{\mathbb{H}^n}^F$ is isomorphic to the geodesic compactification $\overline{\mathbb{H}^n}$ and

$$\partial_F X = \partial\overline{\mathbb{H}^n}$$

so the extension 4.5 reduces to the familiar boundary extension

$$0 \to C_0(\mathbb{H}^n) \to C_0(\overline{\mathbb{H}^n}) \to C(\partial\overline{\mathbb{H}^n}) \to 0.$$

The reader is invited to verify that the triple $(E, \pi, 2p-1)$ given by Theorem 4.6 is precisely Kasparov module constructed in [36, Theorem 3.4] (see Example 1.84).

4.2 Concluding remarks

The results in the previous section rely on the existence of the Furstenberg compactification, that is, we have to assume the symmetric space is of noncompact type, or else the isometry group $G = \mathrm{Iso}(X)^0$ will not be semisimple and the whole theory developed in [18] is no longer applicable.

We saw in Chapter III that the Chabauty compactification, the Martin compactification with parameter $\lambda = \lambda_0$ and the (maximal) Furstenberg compactification are all equivariantly isomorphic for symmetric spaces of noncompact type. In Section 3.5 we tweak the Chabauty compactification into a compactification of $\mathbb{H}^2 \rtimes \mathbb{R}$ by working with the full isometry group, to produce a space homeomorphic to a solid torus.

For the case of the Chabauty compactification, this does produce an extension

$$0 \to C_0(Y_0) \to C(\overline{X}^C) \to C(\partial_F X) \to 0$$

with $Y_0 = \overline{X}^C \backslash \partial_F X$, but it is still unclear to the author whether the Poisson integral construction can be used to produce a splitting of this extension.

Bibliography

[1] P. Albuquerque. Patterson-Sullivan Theory in Higher Rank Symmetric Spaces. *Geometric & Functional Analysis GAFA*, 9(1):1–28, Mar. 1999.

[2] C. Anantharaman-Delaroche. Amenability and exactness for groups, group actions and operator algebras. page 38, 2007.

[3] W. Arveson. Notes on extensions of C*-algebras. *Duke Mathematical Journal*, 44(2):329–355, June 1977.

[4] W. Ballmann, M. Mukherjee, and P. Polymerakis. On the Spectrum of Certain Hadamard Manifolds. *Symmetry, Integrability and Geometry: Methods and Applications*, July 2023.

[5] B. Blackadar. *K-Theory for Operator Algebras.* Mathematical Sciences Research Institute Publications. Springer-Verlag, New York, 1986.

[6] A. Borel. *Linear Algebraic Groups.* Springer Science & Business Media, 1991.

[7] A. Borel and L. Ji. Compactifications of Locally Symmetric Spaces. *Journal of Differential Geometry*, 73(2):263–317, June 2006.

[8] N. Bourbaki. *Integration II: Chapters 7–9.* Springer, Berlin, Heidelberg, 2004.

[9] L. Brown, R. Douglas, and P. Fillmore. Unitary equivalence modulo the compact operators and extensions of C*-algebras. volume 345, pages 58–128. 1973.

[10] L. G. Brown, R. G. Douglas, and P. A. Fillmore. Extensions of C*-algebras and K-Homology. *Annals of Mathematics*, 105(2):265–324, 1977. Publisher: Annals of Mathematics.

[11] N. P. Brown and N. Ozawa. *C*-algebras and Finite-dimensional Approximations.* American Mathematical Soc., 2008.

[12] M.-D. Choi and E. Effros. The Completely Positive Lifting Problem for C*-algebras. *Annals of Mathematics. Second Series*, 104, Nov. 1976.

[13] C. Connell and B. Farb. The Degree Theorem in Higher Rank. *Journal of Differential Geometry*, 65(1):19–59, Oct. 2003.

[14] A. Dabeler. *Exotic group C*-algebras of higher rank Lie groups*. PhD thesis, 2023.

[15] K. R. Davidson. *C*-algebras by Example*. American Mathematical Soc., 1996.

[16] P. Eberlein. *Geometry of Nonpositively Curved Manifolds*. University of Chicago Press, 1996.

[17] H. Emerson and R. Meyer. Euler characteristics and Gysin sequences for group actions on boundaries. *Mathematische Annalen*, 334(4):853–904, Apr. 2006.

[18] H. Furstenberg. A Poisson Formula for Semi-Simple Lie Groups. *Annals of Mathematics*, 77(2):335–386, 1963. Publisher: Annals of Mathematics.

[19] E. Guentner, N. Higson, and S. Weinberger. The Novikov Conjecture for Linear Groups. *Publications mathématiques de l'IHÉS*, 101(1):243–268, June 2005.

[20] Y. Guivarc'h, L. Ji, and J. C. Taylor. *Compactifications of Symmetric Spaces*. Springer Science & Business Media, Dec. 2012.

[21] Y. Guivarch and B. Remy. Group-theoretic compactification of Bruhat–Tits buildings. *Annales Scientifiques de l'École Normale Supérieure*, 39(6):871–920, Nov. 2006.

[22] A. Hatcher. *Vector bundles and K-theory*. 2017.

[23] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, Feb. 1979.

[24] S. Helgason. *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions*. American Mathematical Society, 1984.

[25] S. Helgason and A. Korányi. A Fatou-type theorem for harmonic functions on symmetric spaces. *Bulletin of the American Mathematical Society*, 74(2):258–263, 1968.

[26] N. Higson and J. Roe. *Analytic K-Homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, New York, Dec. 2000.

[27] K. K. Jensen and K. Thomsen. *Elements of KK-Theory*. Mathematics: Theory & Applications. Birkhäuser Basel, 1991.

[28] F. I. Karpelevich. The geometry of geodesics and the eigenfunctions of the Beltrami–Laplace operator on symmetric spaces. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 14:48–185, 1965.

[29] G. Kasparov. The operator K-functor and extensions of C*-algebras. *Mathematics of the USSR-Izvestiya*, 16:513, 1981.

[30] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Inventiones Mathematicae*, 91(1):147–201, 1988.

[31] E. Kirchberg and S. Wassermann. Exact groups and continuous bundles of C*-algebras. *Mathematische Annalen*, 315(2):169–203, 1999.

[32] B. Kloeckner and R. Mazzeo. On the asymptotic behavior of minimal surfaces in $H^2 \times R$. *Indiana University Mathematics Journal*, pages 631–658, 2017.

[33] A. W. Knapp. *Lie Groups Beyond an Introduction.* Birkhäuser, Boston, MA, 1996.

[34] E. C. Lance. *Hilbert C*-Modules: A Toolkit for Operator Algebraists.* Cambridge University Press, Mar. 1995.

[35] R. S. Martin. Minimal Positive Harmonic Functions. *Transactions of the American Mathematical Society*, 49(1):137–172, 1941.

[36] B. Mesland and M. H. Sengun. Hecke operators in KK-theory and the K-homology of Bianchi groups. *Journal of Noncommutative Geometry*, 4(1):125–189, 2020.

[37] R. Meyer. Equivariant Kasparov theory and generalized homomorphisms. 21(3):201–228, Nov. 2000. arXiv: math/0001094.

[38] R. Meyer. Generalized Fixed Point Algebras and Square-Integrable Group Actions. *Journal of Functional Analysis*, 186(1):167–195, Oct. 2001.

[39] R. Meyer. Categorical aspects of bivariant K-theory. In *K-theory and Noncommutative Geometry.* European Mathematical Society, 2008.

[40] D. W. Morris. *Introduction to Arithmetic Groups*, volume 2. Lieu de publication inconnu: Deductive Press, 2015.

[41] G. J. Murphy. *C*-Algebras and Operator Theory.* Academic Press, 2014.

[42] E. Park. *Complex Topological K-Theory.* Cambridge University Press, Mar. 2008.

[43] S. J. Patterson. The limit set of a Fuchsian group. pages 241–273, 1976.

[44] G. K. Pedersen and G. K. Pedersen. *C*-algebras and Their Automorphism Groups.* Academic Press, 1979. Google-Books-ID: yBCoAAAAIAAJ.

[45] N. C. Phillips. Examples of different minimal diffeomorphisms giving the same C*-algebras. *Israel Journal of Mathematics*, 160(1):189–217, 2007.

[46] N. C. Phillips. An introduction to crossed product C*-algebras and minimal dynamics., 2017.

[47] J.-F. Quint. An overview of Patterson-Sullivan theory. *Workshop The barycenter method, FIM, Zurich*, 2006.

[48] M. A. Rieffel. Proper Actions of Groups on C*-Algebras. In H. Araki and R. V. Kadison, editors, *Mappings of Operator Algebras: proceedings of the fourth Japan-US Joint Seminar on Operator Algebras, University of Pennsylvania,from May 23 through May 27 1988*, Progress in Mathematics, pages 141–182. Birkhäuser, Boston, MA, 1991.

[49] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Publications mathématiques de l'IHÉS*, 50(1):171–202, Dec. 1979.

[50] D. Sullivan. Related aspects of positivity in Riemannian geometry. *Journal of Differential Geometry*, 25(3):327–351, 1987.

[51] K. Thomsen. Equivariant KK-Theory and C*-Extensions. *K-Theory*, 19, 2000.

[52] D. Voiculescu. A non-commutative Weyl-von Neumann theorem. *Rev. roum. math. pures et appl.*, 21:97–113, 1976.

[53] A. Vretblad. *Fourier Analysis and Its Applications*, volume 223 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2003.

[54] D. P. Williams. Crossed Products of C*-Algebras. page 528, 2007.

[55] R. Zimmer. *Ergodic Theory and Semisimple Groups*, volume 81 of *Monographs in Mathematics*. Springer, 1984.