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Minimal hypersurfaces in $U(m)$-invariant scalar flat Kähler manifolds

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Abstract

In this thesis, we study minimal hypersurfaces with $U(m)$-invariant Kähler metrics. We provide new method to construct AE scalar flat Kähler manifolds which contains stable minimal hypersurface. In complex dimension 2, we compute the ADM mass and volume of the stable minimal hypersphere. Once we have both volume of stable minimal hypersphere and ADM mass, we have compared them and check that it satisfy the Riemannian Penrose Inequality.

Contents

Introduction

Our work in this thesis is concerned to prove the Penrose Inequality for a special class of Kähler manifolds namely $U(m)$-invariant Kähler manifolds. The Penrose Inequality was conjectured by Roger Penrose [26] in 1973, which is the generalization of the Positive Mass Theorem. The Penrose Inequality estimates the mass of a spacetime in terms of the volume of its black holes. Following is the statement of a well known *conjecture*, the Riemannian Penrose Inequality.

Theorem 0.0.1. Let (M^n, g) be a complete AE manifold with non-negative scalar curvature, which has an outermost minimal hypersurface Σ. Then the ADM mass m_{ADM} satisfies the following

$$m_{ADM} \geq \frac{1}{2}\left(\frac{V(\Sigma)}{V_E(\Sigma_1)}\right)^{\frac{n-2}{n-1}}, \tag{0.0.1}$$

where $V_E(\Sigma)$ and $V_E(\Sigma_1)$ are volumes of outermost minimal hypersurface and standard unit hypersphere respectively. Moreover, equality holds if and only if M^n is isometric to a spatial Schwarzschild manifold outside its horizon.

The notion of ADM mass was introduced by Arnowitt *et.al* [2] in the context of Hamiltonian formulation of general relativity. The first proof of the Riemannian Penrose Inequality in dimension three was made by Huisken and Ilmanen [17]. They used the inverse mean curvature flow for a largest connected component of apparent horizen.

Bray [7, 8] proved Theorem 0.0.1 using conformal flow and allowing multiple connected components, first in dimension three and later up to dimension eight.
Hein and Lebrun proved the version of original Penrose Inequality for asymptotically Euclidean (in short AE) Kähler manifolds. They replaced the outermost minimal hypersurface in the original inequality by $2m-2$ real dimension submanifolds and gave the lower bounds

of the ADM mass. Following is the Kähler version of the Penrose type inequality.

Theorem 0.0.2. Let (M, g, J) be an AE Kähler manifold of a complex dimension m with a scalar curvature $R \geq 0$. Then (M,J) carries a canonical divisor D that is expressed as a sum $\sum n_i D_i$ of compact complex hyper-surfaces with positive integer coefficients together with the property that $\cup D_i \neq \emptyset$ whenever $(M, J) \neq \mathbb{C}^m$. In term of this divisor, we have

$$m_{ADM} \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j n_j vol(D_j), \tag{0.0.2}$$

where the equality holds if and only if (M^{2m}, g, J) is a scalar flat Kähler.

By Theorem 0.0.1 and Theorem 0.0.2 we have two different type of inequalities. We like to compare these two inequalities and understand the following problem.
Assume that there exist AE Kähler manifold M with nonnegative scalar curvature which contain both the outermost stable minimal hypersphere and the canonical divisor. Then which of the inequality in (0.0.1) *and* (0.0.2) *is better than the other one ?*

Now, having this problem in mind, one can ask the questions, i.e. is there exists AE scalar flat Kähler manifold which contain stable minimal outermost hypersurfaces? How we can construct AE scalar flat Kähler manifolds? The discussions in this thesis provide a new method to construct such Kähler manifolds.

In Chapter 1, we recall basic notions and provide a brief review of the general theory of the Penrose Inequality in Riemannian and Kähler case.

In Chapter 2, we consider the family of canonical hyperspheres $\Sigma_{e^t}^{2m-1}$ in $U(m)$-invariant Kähler manifold $(M_{a,b}, \partial\bar{\partial} f(t))$ which is defined as follows,

$$M_{a,b} = \{(z_1, ...z_m) \in \mathbb{C}^m : -\infty < a < t < b < +\infty\},$$

where

$$t = \log S, \quad S = \sum_{i=1}^{m} |z_i|^2$$

for any $(z_1, ..., z_m) \in M_{a,b}$. The family of canonical hypersphere is defined as follows,

$$\Sigma_{e^t}^{2m-1} = \left\{ (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m : e^t = \sum_{i=1}^{m} |z_i|^2 \right\} \subset M_{a,b},$$

We compute the mean curvature of of Σ_{e^t} with any $U(m)$-invariant Kähler metric. We provide two different proofs of the mean curvature formula: the first one is a direct computation of the second fundamental form in dimension four, while the second proof is more geometric and based on the first variation formula of area. Moreover, it is for higher dimensions which also works in dimension four. Following is the main result of Chapter 2.

Theorem 0.0.3. The mean curvature and the volume of $\Sigma_{e^t}^{2m-1} \subset (M_{a,b}, \omega = \sqrt{-1}\partial\bar{\partial}f(t))$ w.r.t outward normal vector are respectively given by ,

$$H(t, f(t)) = \frac{-1}{(2m-1)(2)^{\frac{3}{2}}\sqrt{e^t}f_t(f_{tt})^{\frac{3}{2}}}(2(m-1)f_{tt}^2 + f_{ttt}f_t)\,,$$
$$V(\Sigma^{2m-1}) = (2)^m\sqrt{f_{tt}}f_t^{m-1}V_E(\Sigma_1^{2m-1})\,,$$

where $V_E(\Sigma_1^{2m-1})$ is the Euclidean volume of a unit hypersphere.

Since the Burns metric is $U(m)$-invariant and foliated by 3-spheres outside the exceptional divisor, we can apply our formula of mean curvature to check that either it contains a minimal 3-sphere or not . For more detail about these metrics, the reader is refer to [29, 19]. These metrics on $Bl_0(\mathbb{C}^m)$ are scalar flat AE and is known as Burns-Simanca metrics. These metrics play an important role in the construction of the constant scalar curvature Kähler metrics on the blowup of compact manifolds [1]. As a consequences of our main result Theorem 0.0.3, we obtain the following results.

Corollary 0.0.4. The blowup of $\mathbb{C}^2$ does not contain a minimal hypersphere in the family of canonical hypersphere Σ_S with Burns metric.

Corollary 0.0.5. The space $Bl_0\mathbb{C}^2/\Gamma_2$, where $\Gamma_2 = \mathbb{Z}/2\mathbb{Z}$ with Eguchi Hanson metric does not contain a minimal hypersphere in the family of canonical hyperspheres Σ_S.

Corollary 0.0.6. The projective space $\mathbb{C}P^m$ with Fubini Study metric contains a minimal hypersphere in the family of canonical hyperspheres Σ_S.

Even we know the fact that the space $\mathbb{C}P^m$ is not AE, we still for curiosity want to know that either the minimal hyperspheres contain stable one. But thanks to the known fact due to Simons, that a compact manifold with positive Ricci curvature does not contain stable

minimal hypersurfaces [30]. After checking the examples (known to the author), we did not find even one AE manifold which contains a stable minimal hypersurface in Σ_S.

In Chapter 3, we prove that every $U(m)$-invariant Kähler manifold $M_{a,b}$ with nonnegative scalar curvature satisfy a system of nonlinear differential equations subjected to some constrains.

Theorem 0.0.7. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a Kähler metric with non negative scalar curvature, where $f : M_{a,b} \to \mathbb{R}$ is a smooth function. Then $R \geq 0$ if and only if

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t \leq m(m-1)(1-x)x\,, \end{cases} \tag{0.0.3}$$

where $x = \frac{f_{tt}}{f_t}$ and $y = (2m-1)x + \frac{x_t}{x}$. In particular if the scalar curvature vanishes, then we have

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t = m(m-1)(1-x)x\,. \end{cases}$$

In this thesis, all the results are for scalar flat metrics, so whenever we refer to system (0.0.3) , we mean to consider the equality case.

One natural question arises about the converse of Theorem 0.0.7. More precisely, if we have a solution $(x(t), y(t))$ of the system (0.0.3), are we able to construct a scalar flat Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ in such a way that $x = \frac{f_{tt}}{f_t} > 0$ and $y = (2m-1)x + \frac{x_t}{x}$? Moreover, if we are able to construct such metric, is it unique? Are there exist minimal hyperspheres.? In particular, stable minimal hyperspheres? The answer is given in the following.

Theorem 0.0.8. Let (x, y) be a solution of system (0.0.3). Then the following assertion hold true.

- Let $f(t)$ and $u(t)$ be two solutions of the ordinary differential equation $x = \frac{f_{tt}}{f_t}$ on some small interval I, with given initial conditions as follow

$$\begin{cases} x = \frac{f_{tt}}{f_t} \\ f(t_0) = \alpha \\ f_t(t_0) = \beta\,, \end{cases} \quad \text{and} \quad \begin{cases} x = \frac{u_{tt}}{u_t} \\ u(t_0) = a \\ u_t(t_0) = b\,. \end{cases}$$

Then the Kähler metrics ω_f and ω_u has the following relation $\omega_f = \mu\omega_u$.

- The AE scalar flat Kähler manifold $M_{a,\infty}$ contains a minimal hypersphere $\Sigma_{e^{t_0}}$ in the family of canonical hyperspheres Σ_{e^t} if and only if $y(t_0) = 0$ for some t_0.
- The minimal hypersphere Σ_{S_0} is stable if and only if $0 < x(t_0) \leq 1$.

Once we found a family of scalar flat Kähler metrics from a solution of the system (0.0.3), naturally we try to find a way to understand the solutions of the system (0.0.3), and end up with the following theorem.

Theorem 0.0.9. All the solutions (x, y) of system of nonlinear differential equations (0.0.3) are contained in level set of the function

$$f(x, y) := \frac{(y + (1 - m)x - m)^m}{(mx - y + m - 1)^{m-1}}\,, \tag{0.0.4}$$

which we denote by L_λ .

Since corresponding to solutions (x, y) of system (0.0.3), we have scalar flat Kähler metrics, so we would like to translate all the information given in Theorem 0.0.8 to the new setting, i.e. in terms of the level set.

Theorem 0.0.10. The AE Kähler manifold $(\mathbb{C}^m \setminus B_{R_\lambda}(0), \omega_\lambda)$ with scalar flat Kähler metric ω_λ corresponding to the level set L_λ contains a minimal hypersphere Σ_{S_0} of radius $S_0 = e^{t_0}$ in the family of canonical hyperspheres Σ_S if and only if

$$\lambda(x_0) = \frac{((1 - m)x_0 - m)^m}{(mx_0 + m - 1)^{(m-1)}}\,,$$

where $x(t_0) = x_0$.
Moreover, the minimal hypersphere Σ_{S_0} is stable if and only if

$$\lambda \in \left[(-1)^m(2m - 1), \frac{(-m)^m}{(m - 1)^{m-1}}\right[.$$

For the higher dimension, we do not know the domain of the metric but for a complex dimension 2, we explicitly know the radius of the ball B_{R_λ}, and it behaves in the following way :

$$R_\lambda = \begin{cases} < \infty & \lambda \to 3 \\ 0 & \lambda \to 4\,. \end{cases}$$

For $m = 2$ and $x(t) > 0$, the level sets L_λ of the function $f(x, y)$ are given in the following graph.

$$\lambda = \frac{(y - x - 2)^2}{(2x - y + 1)}\,. \tag{0.0.5}$$

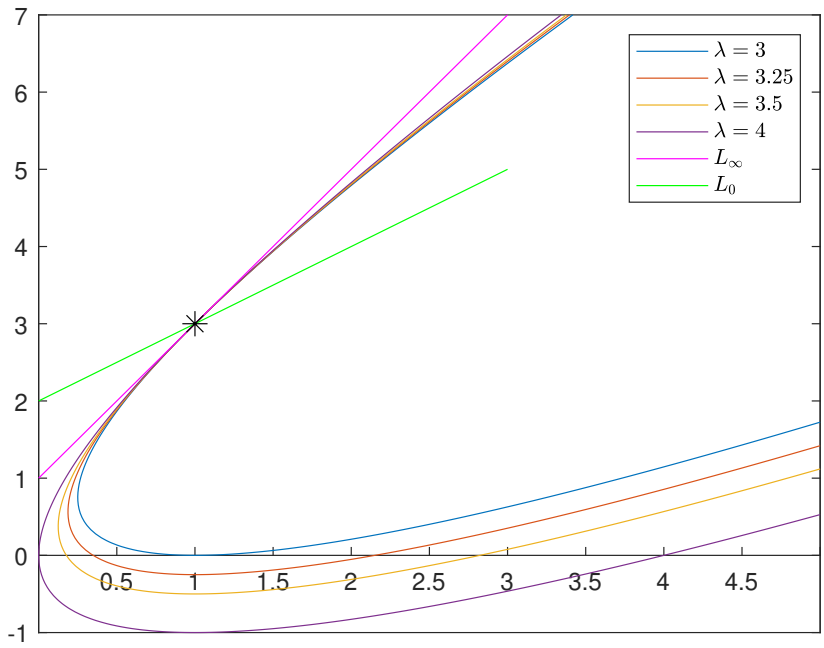


Figure 1: Level curves

Figure 1 represent the level sets L_λ for some λ. It is clear from the figure that when $\lambda \to 4$, L_4 approaches to the origin. The figure contains only those level sets which are interested for us, i.e. for which $x > 0$. We notice that each level set contains the point $(1, 3)$, which means that all the scalar flat Kähler metrics corresponding to the level sets are AE. The level set L_0 and L_∞ contain the Burns and Eguchi Hanson metric respectively if $x \to 1^-$. Furthermore, The level set L_4 passing through the origin is a level set of the scalar flat Kähler metric (PMY) given in [14].

Remark 0.0.11. Figure 1 suggests that there are two metrics on each level set approaching to the Euclidean metric.

Theorem 0.0.12. For complex dimension $m = 2$, if $x(t) > 0$ for all t, then L_λ contains two AE Kähler metrics on $M_{R_\lambda,\infty}$ in which one metric contains two minimal hypersphere if and

only if

$$\lambda = \frac{(x_0+2)^2}{(2x_0+2)}\,,$$

for some $x_0 > 0$. Moreover, one of the the minimal hypersphere is stable if and only if

$$\lambda \in [3,4[\,.$$

The following result is concerning with the volume of minimal hyperspheres.

Theorem 0.0.13. For $m = 2$, the AE Kähler manifold $M_{R_\lambda,\infty}$ contains minimal hyperspheres at $x_0(\lambda)$, where

$$x_0(\lambda) = \lambda - 2 \pm \sqrt{(\lambda-3)\lambda}\,, \tag{0.0.6}$$

for $\lambda \in [3,4)$.

Volume of Σ_{e^t}: $\quad V(\Sigma^3_{e^t}) = (2e^{\nu_-})^{\frac{3}{2}}\sqrt{x}V_E(\Sigma^3_1)$ where

$$\nu_-(x) = -\log\left(1+\sqrt{\frac{4(x-1)+\lambda}{\lambda}}\right).$$

The volume of the minimal stable hypersphere behaves in the following way:

$$V(x_0(\lambda)) = \begin{cases} V_E(\Sigma^3_1) & \lambda \to 3 \\ 0 & \lambda \to 4 \end{cases}$$

Theorem 0.0.14. For $m = 2$, the ADM mass of the the AE Kähler manifold $M_{R_\lambda,\infty}$ is $\frac{\lambda}{2}$.

Theorem 0.0.15. For $m = 2$, the AE Kähler manifold $M_{R_\lambda,\infty}$ satisfies the Riemannian Penrose inequality,

$$m_{ADM} \geq \frac{1}{2}\left(\frac{V_{\Sigma^3(x_0)}}{V_E(\Sigma^3_1)}\right)^{\frac{2}{3}}.$$

$$\begin{aligned}\frac{\lambda}{2} &\geq e^{\nu}(x)^{\frac{1}{3}} \\ &= \frac{\left(\lambda-2-\sqrt{\lambda(\lambda-3)}\right)^{\frac{1}{3}}}{\left(1+\sqrt{\frac{4\left(\lambda-3-\sqrt{\lambda(\lambda-3)}\right)+\lambda}{\lambda}}\right)}\end{aligned}$$

Chapter 1

Preliminaries

1.1 Basic notions of Riemannian manifolds

We start this section by recalling the definition of a Riemannian metric and variation of hypersurfaces in Riemannian manifold. The general theory for existence of compact minimal hypersurfaces and its stability can be found in [3, 5, 11, 13, 23, 24, 28, 34].

Definition 1.1.1. A **Riemannian metric** g on a smooth manifold M is a correspondence which assign to each point $p \in M$ a positive definite, symmetric bilinear form g_p on the tangent space

$$g_p : T_pM \times T_pM \to \mathbb{R}\,.$$

In local coordinate system $\boldsymbol{x} = (x_1, x_2, \dots x_n)$, with basis $(\partial_{x_1}, \partial_{x_2} \dots \partial_{x_n})$ of T_pM,

$$g_{ij} = g(\partial_{x_i}, \partial_{x_j})\,.$$

A smooth manifold M is called a Riemannian manifold if there is a Riemannian metric g on M.

Definition 1.1.2. An affine connection ∇ on M is a mapping

$$\nabla : \boldsymbol{V}(M) \times \boldsymbol{V}(M) \to \boldsymbol{V}(M)\,,$$

satisfying the following properties with $\nabla(X, Y) = \nabla_Y(X)$:

1. $\nabla_{fX+gY}(Z)) = f\nabla_X(Z) + g\nabla_Y(Z)$;

2. $\nabla_X(Y+Z) = \nabla_X(Y) + \nabla_X(Z)$;

3. $\nabla_X(fY) = f\nabla_X(Y) + X(f)fY$.

The following result characterizes that there is a unique connection associated to every Riemannian manifold which we call a Levi-Civita connection.

Theorem 1.1.3. Let M be a Riemannian manifold. Then there exist a unique affine connection ∇ satisfying the following properties for all $X, Y, Z \in \mathbf{V}(M)$:

1. $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$;

2. $X(g(Y, Z)) = g(\nabla_X(Y), Z)) + g(X, \nabla_X Z)$,

where $[X, Y]$ is a Lie bracket of vector fields.

Definition 1.1.4. Let $f : \Sigma \to M$ be an immersion and g is a Riemannian metric on N. Then f induces a Riemannian metric on Σ given by

$$g_\Sigma(X, Y)_p := g(df_p(X), df_p(Y))_{f(p)} \quad \forall p \in \Sigma, \quad \forall\, X, Y \in T_p\Sigma.$$

This immersion is called an isometric immersion.

Definition 1.1.5. The second fundamental form is defined by

$$\Pi_\eta(X) = g\left(\nabla_X(X), \eta\right). \tag{1.1.1}$$

Definition 1.1.6. The mean curvature H of immersed manifold Σ is the g_Σ-trace of the second fundamental form Π.

Definition 1.1.7. A submanifold Σ of M is called totally geodesic if the second fundamental form $\Pi_\eta(X) = 0$, $\forall X \in T_p\Sigma$.

Definition 1.1.8. A submanifold Σ is said to be minimal if its mean curvature H vanished.

Its clear from the above definitions that the class of minimal submanifolds is larger than the class of totally geodesic submanifolds.
Now if we have a smooth family of immersion $F : (-\epsilon, \epsilon) \times \Sigma \to M$ such that $F_0 = \Sigma$ and $\partial_t F_t = f_t \eta$, where $F_t = F(t, x)$, $\Sigma_t = F_t(\Sigma)$ and η is a unit normal vector to Σ, then we say that F_t is a **variation**.

Remark 1.1.9. For any compacted supported function f on Σ, there is a variation F_t with $f_t|_{t=0} = f$. If we set $\tilde{F}_t = exp_x(tf(x)\eta)$ where $exp_x : T_x\Sigma \to \Sigma$, then $\partial_t \tilde{F} = f\eta$.

Associated to the variation F_t we define the area functional $A(t) = Area(\Sigma_t)$. The change in the area functional up to second order is given in the following theorem.

Theorem 1.1.10. Consider the immersion $F : (-\epsilon, \epsilon) \times \Sigma \to M$. We have

$$\frac{d}{dt}|_{t=0} A(\Sigma_t) = -dim(\Sigma) \int_\Sigma Hf\,, \tag{1.1.2}$$

$$\frac{d^2}{d^2t}|_{t=0} A(\Sigma_t) = \int_\Sigma (|\nabla_\Sigma f| - (|\Pi|^2 + Ric_g(\eta, \eta))f^2) + H^2 f^2 + Hf'\,, \tag{1.1.3}$$

where H and Π are mean curvature and second fundamental form of Σ respectively.

It is clear from (1.1.2) that the minimal hypersurfaces are critical points of the area functional.

Definition 1.1.11. A minimal hypersurface Σ is called stable if $\frac{d^2}{d^2t}|_{t=0} A(t) \geq 0$.

Or equivalently, a minimal hypersurface Σ is said to be stable if and only if

$$\int_\Sigma (|\nabla_\Sigma f| \geq \int_\Sigma (|\Pi|^2 + Ric_g(\eta, \eta))f^2). \tag{1.1.4}$$

1.2 Kähler manifolds

In this section, we recall some notions in complex Kähler geometry that we use in the upcoming Chapters (for further details see [31]) .

Definition 1.2.1. . A complex manifold is a smooth manifold M such that the transition maps $\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \subset \mathbb{C}^n \to \Phi_i(U_i \cap U_j) \subset \mathbb{C}^n$ are holomorphic for any pair of $i, j \in I$ with $U_i \cap U_j \neq \emptyset$.

Each pair (U_i, Φ_i) is called complex chart and the whole collection is called complex atlas.

Definition 1.2.2. An endomorphism $J : T_P M \to T_P M$ is said to be an almost complex structure on a smooth manifold M if $J^2 = -1$.

Definition 1.2.3. An almost complex structure J on a smooth manifold M is integrable if it arises from the holomorphic charts.

Definition 1.2.4. The Nijenhuis tensor $N(J) : TM \times TM \to TM$ is defined by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

for $X, Y \in TM$.

Any complex structure on M induces a canonical integrable almost complex structure J. The question is whether the converse of this is possible or not? The answer is positive and is given in the following theorem.

Theorem 1.2.5. An almost complex structure J is integrable if and only if $N(J) = 0$, where N is the Nijenhuis tensor.

Definition 1.2.6. Let (M, J) be a complex manifold with complex structure J. The complexified tangent bundle of M is defined as $T_{\mathbb{C}}M = TM \bigotimes_{\mathbb{R}} \mathbb{C}$.

The complex structure J can be extend to $T_{\mathbb{C}}M$ and decompose the complexified tangent bundle point wise into eigen-spaces of i and $-i$, i.e.

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M = \{X \in T_{\mathbb{C}}M : JX = iX\}$ and $T^{0,1}M = \{X \in T_{\mathbb{C}}M : JX = -iX\}$.
The definition of J can be extend to the real cotangent bundle T^*M by,

$$J\alpha(X) = -\alpha(JX),$$

for $\alpha \in T^*M$ and $X \in TM$. Similarly, we can complexified the cotangent bundle, and the complex structure J gives the decomposition of cotangent bundle

$$T^*_{\mathbb{C}}M = (T^{1,0}M)^* \oplus (T^{0,1})^*M.$$

In local coordinates $(z_1, \ldots, z_m)$ where $z_i = x_i + \sqrt{-1}y_i$, if we define

$$\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i}) \quad \text{and} \quad \partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i}),$$

then $T^{1,0}M$ is spanned by $\{\partial_{z_i}\}$ while $T^{0,1}M$ is spanned by $\{\partial_{\bar{z}_i}\}$. The basis for the dual spaces $(T^{1,0}M)^*$ and $(T^{0,1})^*M$ are given by 1-forms respectively $dz_i = dx_i + idy_i$ and $d\bar{z}_i = dx_i - idy_i$. The complex structure J on $T^{1,0}M$ and $T^{0,1}M$ is defined by $J(\partial_{z_i}) = i\partial_{z_i}$ and $J(\partial_{\bar{z}_i}) = -i\partial_{\bar{z}_i}$ respectively while on $(T^{1,0}M)^*$ and $(T^{0,1})^*M$ it is defined by $J(dz_i) = idz_i$ and $J(d\bar{z_i}) = id\bar{z}_i$. We noted that the complex structure gives natural decomposition of complexified exterior power of cotangent bundle

$$\bigwedge^k T^*_{\mathbb{C}}M := \bigwedge^k T^*M \otimes \mathbb{C},$$

where $T^*_{\mathbb{C}}M = (T^{1,0}M)^* \oplus (T^{0,1}M)^*$. Indeed we have

$$\bigwedge^k T^*_{\mathbb{C}}M := \oplus_{p+q=k} \left(\bigwedge^p \left(T^{1,0}M\right)^* \oplus \bigwedge^q (T^{0,1}M)^* \right).$$

We denote the space of (p,q)-forms by $A^{(p,q)}(M)$, which is locally spanned by $dz_i \wedge, \ldots, dz_p \wedge d\bar{z}_{p+1}, \ldots d\bar{z}_q$. Thus we have

$$A^k(M) = \oplus_{p+q=k} A^{(p,q)}(M).$$

The exterior derivative is a map $d : A^k(M) \to A^{k+1}(M)$ such that $d^2 = 0$. On a complex manifold M, the decomposition of forms gives rise to decomposition of exterior differential $d = \partial + \bar{\partial}$ where

$$\partial : A^{(p,q)}(M) \to A^{(p+1,q)}(M),$$

$$\bar{\partial} : A^{(p,q)}(M) \to A^{(p,q+1)}(M),$$

with the relations that $d^2 = 0$, $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

Definition 1.2.7. Let (M, g, J) be a Riemannian manifold with complex structure J. The metric g is called Hermitian if g is compatible with complex structure J, i.e. if $g(JX, JY) = g(X, Y)$, for all tangent vectors X, Y.

Given a Hermetian metric g, if we define $\omega(X,Y) = g(JX,Y)$ for all tangent vectors X, Y, then ω is anti-symmetric real 1-1 form.
In local coordinates $\boldsymbol{z} = (z_1, z_2 \dots z_m)$, the Hermitian condition implies that

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = g\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right) = 0.$$

So that the Hermitian metric g is determined by

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right).$$

We can express g and the associated 2-form ω as

$$g = g_{i\bar{j}} dz_i \odot d\bar{z}_j,$$

$$\omega = i g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Definition 1.2.8. A Hermitian manifold (M, g, J) is said to be a Kähler manifold if the associated 2-form ω is closed, i.e. if $d\omega = 0$.

Since the Kähler form ω is closed real form it defines a cohomology class $[\omega] \in^2 (M, R)$. The most important result is that on compact manifold Kähler metrics in fixed cohomology class can be parameterized by single real valued function.

Theorem 1.2.9. Let M be a compact Kähler manifold with Kähler form ω. Then for any other Kähler form $\bar{\omega} \in [\omega] \in H^2(M, R)$, there exist a smooth real function f such that $\omega = \bar{\omega} + i\partial\bar{\partial} f$.

The next theorem characterizes some properties of Kähler manifolds, in which the most important one is that the Kähler form ω can be described locally by real valued function.

Theorem 1.2.10. Let M be a complex manifold with a compatible Riemannian metric g and Levi-Civita connection ∇. Then the following are equivalent.

1. $d\omega = 0$.

2. $\nabla J = 0$.

3. For each point $p \in M$, there is a smooth real function f in a neighbourhood of p such that $\omega = i\partial\bar{\partial}f$.

Its clear that every Kähler manifold has underlying Riemannian manifold structure. The next Lemma tells us that there is an interesting relationship between the volume element of Riemannian manifold and m-from of Kähler form.

Lemma 1.2.11. If ω is a Kähler form, then $\frac{\omega^m}{m!}$ is the volume element of the Riemannian metric defined by the Kähler form.

Proof. The computation is really pointwise. Let us fixing point $p \in M$ and holomorphic coordinates $z_i = x_i + \sqrt{-1}y_i$. The set $\{\partial_{x_i}, \partial_{y_i} : i = 1, \ldots, m\}$ form an orthonormal fram for the real tangent space T_pM and $\{dx_i, dy_i : i = 1, \ldots, m\}$ is its coframe. The fram and cofram for the complex tangent space are denoted by

$$\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i}), \quad dz_i = dx_i + idy_i\,.$$

In this notation, the Kähler form and the associated metric are given as follow

$$\omega = \sqrt{-1}\sum_{i=1}^{m} dz_i \wedge d\bar{z}_i\,, \tag{1.2.1}$$

$$g = \sum_{i=1}^{m} dz_i \odot d\bar{z}_i = 2(dx_i \otimes dx_i + dy_i \otimes dy_i)\,. \tag{1.2.2}$$

The volume element of the Riemannian metric is given by

$$dV = \sqrt{g}dx_1dy_{y_1}\cdots dy_{y_m} = 2^m dx_1dy_{y_1}\cdots dx_m dy_{y_m}\,. \tag{1.2.3}$$

Thus we have

$$\omega^m = m!(\sqrt{-1})^m dz_1 \wedge d\bar{z}_1 \cdots dz_m \wedge d\bar{z}_m\,. \tag{1.2.4}$$

By using the relation $dz_i \wedge d\bar{z}_i = -2\sqrt{-1}dx_i \wedge dy_i$ in (1.2.4), we have

$$\frac{\omega^m}{m!} = 2^m dx_1dy_{y_1}\cdots dx_m dy_{y_m}. \tag{1.2.5}$$

We see that (1.2.3) and (1.2.5) conclude the proof. □

The Ricci and scalar curvature of the Kähler metric $\omega = ig_{i\bar{j}}dz_i \wedge d\bar{z}_j$ can be computed respectively by the following formulae,

$$R_{i\bar{j}} = -\partial_i \bar{\partial}_j \log(\det(g_{i\bar{j}})\,, \tag{1.2.6}$$

$$R = -g^{i\bar{j}}\partial_i \bar{\partial} \log(\det(g_{i\bar{j}}))\,. \tag{1.2.7}$$

1.3 Generalities of Penrose Inequality

In this section, we provide a brief discussion about the Riemannian Penrose Inequality an important case of a conjecture made by Roger Penrose. We consider space like slices (M^3, g, Π) of a space time where g is the positive definite metric on M^3 and Π is the second fundamental form of M^3 in the space time.

Definition 1.3.1. [16] A complete connected non-compact Riemannian manifold (M, g) of dimension $n \geq 3$ is said to be AE if there is a compact set $K \in M$ such that the complement of K is disjoint union of ends, where each end is diffeomorphic to the complement of a closed ball in $\mathbb{R}^n$. Moreover in AE coordinates $(x_1, \ldots, x_n)$ the metric g satisfy

$$g_{i\bar{j}} = \delta_{i\bar{j}} + O(|x|^{-\tau})\,, \tag{1.3.1}$$

with

$$g_{i\bar{j},k} = O(|x|^{\tau-1}). \tag{1.3.2}$$

where $|x| = \sqrt{\sum_{i=1}^{n} x_i}$.

In other words, the metric g becomes Euclidean metric plus terms that fall off rapidly at infinity. An asymptotically locally Euclidean manifolds (in short ALE) are those where ends are asymptotic to flat cones. The ALE manifolds with zero scalar curvature are very important as they provide counter example to the generalized positive mass conjecture [15] by Hawking and Pope.

Definition 1.3.2. A complete connected non-compact Riemannian manifold (M, g) of dimension $n \geq 3$ is said to be asymptotically locally Euclidean if there is a compact set $K \in M$ such

that the complement of K is disjoint union of ends, and there is a a diffeomorphism between $M - K$ and $(\mathbb{R}^n - B^n)/\Gamma$ where B is a ball around the origin in $\mathbb{R}^n$. This diffeomorphism gives a specific set of coordinates at infinity, such that

$$g_{i\bar{j}} = \delta_{i\bar{j}} + O(|x|^{\tau}),$$

with

$$g_{i\bar{j},k} = O(|x|^{\tau-1}),$$

where $\Gamma \subset SO(n)$ which acts freely on the unit sphere.

Remark 1.3.3. If $\Gamma = 1$, then ALE manifold is AE.

To the end of every asymptotically Euclidean Riemannian manifold there is an associated quantity called the ADM mass. The notion of ADM mass was introduced by Arnowitt Deser and Misner [2] in the context of Hamiltonian formulation of general relativity.

Definition 1.3.4. For AE manifold, the ADM mass is denoted by m and defined as

$$m_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \sum_{i,j} (g_{ij,i} - g_{ii,j})\eta_j d\mu,$$

where S_r is the coordinate sphere of radius r, η_j is unit normal to S_r and $d\mu$ is the area element of S_r in coordinate chart.

The definition of ADM mass seems to be depend on the choice of coordinates. But Bartnik [4] and chruśceil [10] independently proved that if we impose weak fall of conditions of the following type:

1. the scalar curvature of the metric has the property $\int Rd < \infty$;
2. the component of the metric satisfy $g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\tau}$ for some $\tau > \frac{n-2}{2}$ and some $\alpha \in (0,1)$ at each end in some asymptotic coordinates,

then the ADM mass is finite and does not depend on the choice of coordinates.

Theorem 1.3.5. (The Riemannian Positive Mass Theorem)
Let (M, g) be any asymptotically Euclidean manifold with non negative scalar curvature $R \geq 0$. Then the ADM mass $m_{ADM} \geq 0$ and the equality holds if and only if (M, g) is isometric to the Euclidean space.

The Riemannian Positive Mass Theorem was first proved by Schoen and Yau [27] for an n-dimensional manifold with $3 \leq n \leq 7$. Witten [33] in 1981, proved the Theorem 1.3.5 for spin manifolds in any dimension. Lohkamp [20] extend the proof to any dimension without spin assumption. Hawking and Pop conjectured [15] that a similar Positive Mass Theorem hold for ALE manifolds, but later on LeBrun [19] found counterexamples to the conjecture in 1988.

The Riemannian Penrose inequality is generalization of the Positive Mass Theorem which gives the relation of the ADM mass of an end with the area of the of the surfaces representing black holes. In the time symmetric case black holes are represented by compact minimal surfaces, which are called apparent horizen. Before moving on to the statement of the Riemannain Penrose inequality, we present some definitions and as an example, consider the Schwarzchild manifold.

Definition 1.3.6. Let (M^3, g) be totally geodesic submanifold in the spcace time, the apparent horizon of (M^3, g) is the smallest surface Σ so that any closed minimal surface Σ' is "inside" of Σ (with respect to the AE end).

Remark 1.3.7. There is a chosen end of M^3 and Σ' is "inside" of Σ is defined with respect to this end.

Remark 1.3.8. Since the AE manifolds is allowed to have many ends so there could be multiple horizen associated to each end.

Definition 1.3.9. A surface Σ is said to be outer minimizing if every surface which enclose it has (strictly) greater area. Mathematically, an outer minimizing surface is stable minimal surface.

Since the apparent horizon does not contained in any other surface, so they are outer minimizing (stable surfaces). From the stability argument [25] it follows that outermost minimal surfaces must have the topology of sphere.

Example 1.3.10. The Schwarzchild manifold
The Schwarzchild metric is defined on $R^3 \setminus B_{2m}(0)$, and is given by

$$\tilde{g}_m = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{S^2}. \tag{1.3.3}$$

The metric $\tilde{g}_m$ approaches the Euclidean metric as $r \to \infty$. The parameter m is positive constant and equals to the total mass of the manifold. The coordinates sphere $r = 2m$ is a single minimal sphere of the Schwarzchild manifold. The coordinates sphere $r = 2m$ is called the apparent horizen. The Schwarzchild is spherically symmetric and Ricci flat.

Definition 1.3.11. A manifold M^3 is spherically symmetric if there is an action of rotation group $SO(3)$ on M and the orbits of this action are 2-sphere.

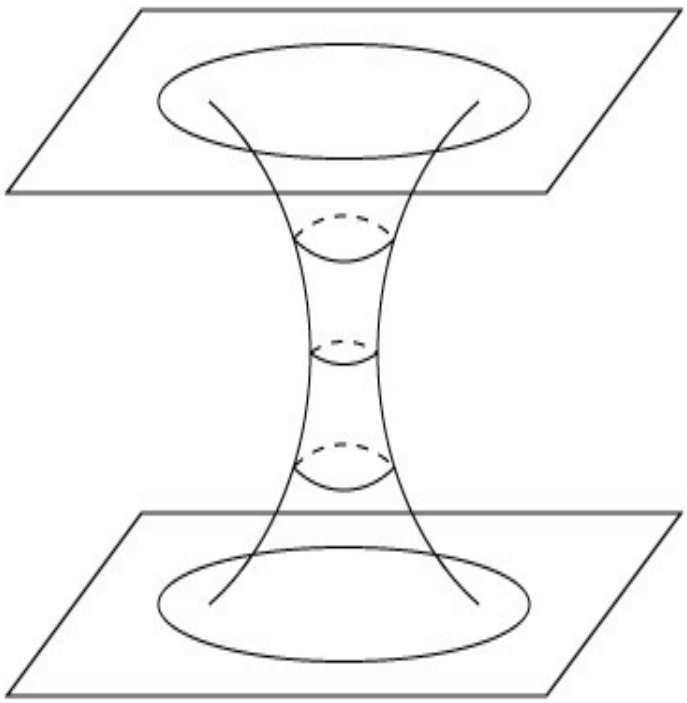


Figure 1.1: The Schwarzchild manifold

Remark 1.3.12. From (1.3.3), the metric coefficient becomes infinite at $r = 0$ and at $r = 2m$. It seems that that the coordinate sphere is not part of the manifold. But it turn out that the singularity $r = 2m$ is only coordinate singularity and can be remove by using the change of coordinates.
Define

$$r^* = \int \frac{dr}{1 - \frac{2m}{r}} = r + 2m \log(r - 2m)\,.$$

Then we have

$$\left(1 - \frac{2m}{r}\right) dr^* = \left(1 - \frac{2m}{r}\right)^{-1} dr^2\,.$$

Now in this coordinates (1.3.3) can be written as

$$\tilde{g}_m = (1 - \frac{2m}{r})(dr^*)^2 + r^2 g_{S^2}\,.$$

Clearly $r = 2m$ is no more a singular point. Thus the change of coordinates allows a smooth extension of $\tilde{g}_m$ to the boundary.

Theorem 1.3.13. Let (M^3, g) be a complete, smooth, asymptotically Euclidean manifold with non negative scalar curvature which has an outermost minimal hypersurface Σ. Then

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}}. \tag{1.3.4}$$

with equality if and only if $(M^3,)$ is isometric to the Schwarzchild metric $(R^3 \setminus 0, \tilde{g}_m)$ of mass m outside their respective horizons.

The first proof of the Riemannian Penrose Inequality was made by Huisken and Ilmanen [17] by using inverse mean curvature flow, for largest connected component of apparent horizen. Bray [7] proved Theorem 1.3.13 using conformal flow and allowing multiple connected components, later in 2009 he extends the proof upto dimensions 8 , [8].

1.4 The Penrose Inequality for Kähler manifolds

In 2016, Hein and Lebrun [16] proved that the lower bound of the mass of the end of AE Kähler manifolds can be given in terms of the volume of the canonical divisor. Moreover

he proved an explicit formula for the mass of ALE Kähler manifolds. In the case when the ALE Kähler manifold is scalar flat, he proved that the mass is a topological invariant of the underlying smooth manifold. In order to present his formula of mass for ALE manifold we recall some definitions.

Definition 1.4.1. The de Rham cohomology group of the manifold M defined as

$$H^m(M) = \frac{\text{closed } m\text{-forms on M}}{\text{exact } m\text{-forms on M}},$$

where the set of closed m-forms are the kernal of $d : A^m(M) \to A^{m+1}(M)$ and the set of m-exact forms are image of $d : A^{m-1}(M) \to A^m(M)$. The de Rham cohomology group for compactly supported forms is denoted by $H_c^m(M)$.

There is a natural map between the deRham cohomology group of compact supported forms and $H^2(M)$

$$H_c^2(M) \to H^2(M).$$

Lemma 1.4.2. Let (M, g) be any ALE manifold. Then the natural map

$$\phi : H_c^2(M) \to H^2(M) \tag{1.4.1}$$

is an isomorphism.

If the manifold is complex, in particular oriented, then we have.

Theorem 1.4.3. (Poincaré duality for non compact manifold)
For noncompact oriented manifold M of real dimension $2m$,

$$H_c^2(M) \sim (H^{2m-2}(M))^*.$$

Now we are ready to state the formula of mass for ALE Kähler manifolds proved in [16].

Theorem 1.4.4. Let (M, g, J) be an ALE Kähler manifold of a complex dimension m. Then the mass $\mathbf{m}$ is given by

$$\mathbf{m} = -\frac{\langle \phi^{-1}(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)} \int R_g dV \tag{1.4.2}$$

where R_g and dV are respectively the scalar curvature and volume form of g, c_1 is the first chern class of (M, J), $[\omega] \in H^2(M)$ is the Kähler class of g, and $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^2_c(M)$ and $H^{2n-1}(M)$. Moreover, If the ALE Kähler metric is assumed to be scalar flat, then the mass is a topological invariant, determined completely by the smooth manifold, together with the first chern class and the Kähler class $[\omega]$ of the metric.

Remark 1.4.5. In general if we have an AE manifold and we find the mass of an end, by the Positive Mass Theorem at least we can have the idea (measure) that it should be positive. But the Positive Mass Theorem is not true for ALE manifolds, so there is a question why should one belief that (1.4.2) is good definition of mass. The answer of this question is very well explained in [16] and here we present the argument. We recall a well known result ([9],[6]) for a compact Kähler manifold that the total scalar curvature is a topological invariant determined by the first chern class of the complex structure and the Kähler class of the metric, i.e.

$$\int_M R dV = \frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle.$$

The equation (1.4.2) can be rewritten as,

$$\int_M R dV = \frac{4(2m-1)}{(m-1)!} \mathbf{m} + \frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle \tag{1.4.3}$$

The essence of the formula in Theorem 1.4.4 is that it measure the deviation of ALE manifold from being compact.

Remark 1.4.6. Having the understanding of (1.4.3), one can immediately notice the known fact that ALE Ricci flat manifolds has zero mass.

Hein and Lebrun proved that the Positive Mass Theorem also hold for Kähler manifolds even if the manifold is non spin, which one would expect due to [20, 27].

Theorem 1.4.7. The Positive Mass Theorem for Kähler manifolds
Let (M, J) be an AE Kähler manifold with non negative scalar curvature $R \geq 0$. Then

$$m \geq 0.$$

The equality holds if and only if (M, J) is a Euclidean space.

Theorem 1.4.8. The Penrose type Inequality for Kähler manifolds
Let (M, g, J) be an AE Kähler manifold of a complex dimension m with scalar curvature $R \geq 0$. Then (M,J) carries a canonical divisor D that is expressed as a sum $\sum n_i D_i$ of compact complex hyper-surfaces with positive integer coefficients, and with the property that $\cup D_i \neq \emptyset$ whenever $(M, J) \neq \mathbb{C}^m$. In term of this divisor,

$$m \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j n_j vol(D_j), \tag{1.4.4}$$

where the equality holds if and only if (M^{2n}, g, J) is a scalar flat Kähler.

Remark 1.4.9. The Hein and Lebrun version of Penrose type Inequality claim that the lower bound of the mass can be obtained if the compact stable minimal hypersurface in Penrose inequality is replaced by $2m-2$ dimensional real manifold.

Chapter 2

Mean curvature and volume of hyperspheres

In this chapter, we consider some special class of hypersurfaces and study mean curvature and volume of of each hypersphere the family of canonical hyperspheres in $\mathbb{C}^m$ with $U(m)$ invariant Kähler metrics. We give a brief introduction of the $U(m)$-invariant Kähler metrics that will often use in this thesis. The most vital part of this chapter is the computation of the mean curvature and volume of the hypersphere in the family of canonical hyperspheres. We pick some known example where we can apply our formula and check if there exist any minimal hypersphere in Σ_S.

2.1 $U(m)$-invariant Kähler metric on $\mathbb{C}^m \setminus 0$

We consider those Kähler metrics for which the potential function $f : (0, \infty) \to \mathbb{R}$ is invariant in local coordinates $\{(z_1, z_2, \ldots, z_m)\}$ and depends only on $S = |z_1|^2 + |z_2|^2 + \cdots + |z_m|^2$. For more details the reader is referred to [14, 18, 22, 32] where $U(m)$-invariant Kähler metrics has been studied.

In this section, we discuss all interesting geometric quantities of the $U(m)$-invariant Kähler metric which will be used latter, i.e. volume element Ricci curvature and scalar curvature. We will mostly use three types of coordinate S, t and Θ.

Remark 2.1.1. First we set the notations for this section. We consider $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ and the metric associated to ω is $g = g_{i\bar{j}} dz_i \odot d\bar{z}_j$. For instance if

$$\omega = \sqrt{-1} \sum_{i=1}^{m} dz_i \wedge d\bar{z}_i,$$

then the associated metric g in real coordinate can be written as

$$g = \sum_{i=1}^{m} dz_i \odot d\bar{z}_i = 2(dx_i \otimes dx_i + dy_i \otimes dy_i).$$

Therefore whenever we use the real coordinates we should be comfortable with the factor 2.

We consider the real $(1,1)$ form

$$\omega = \sqrt{-1} \partial\bar{\partial} f(S) = \sqrt{-1}(f_S \partial\bar{\partial} S + f_{SS} \partial S \wedge \bar{\partial} S), \tag{2.1.1}$$

where $S = |z_1|^2 + |z_2|^2 + \cdots + |z_m|^2$ and f_S denotes the derivative of f with respect to S.

The metric:

The metric g associated to the Kähler form (2.1.1) is given as follows,

$$g = \begin{bmatrix} f_S + f_{SS}|z_1|^2 & z_1\bar{z}_2 f_{SS} & \cdots & z_1\bar{z_m} f_{SS} \\ z_2\bar{z}_1 f_{SS} & f_S + f_{SS}|z_2|^2 & \cdots & z_2\bar{z_m} f_{SS} \\ \vdots & \vdots & \ddots & \vdots \\ z_m\bar{z}_1 f_{SS} & z_m\bar{z}_2 f_{SS} & \cdots & f_S + f_{SS}|z_m|^2 \end{bmatrix}$$

By direct computation one can see that

$$\det(g) = (f_S)^{m-1}(f_S + S f_{SS}), \tag{2.1.2}$$

Clearly $\det(g) > 0$ if and only if $f_S > 0$ and $f_S + S f_{SS} > 0$. We summarize that the form (2.1.1) defines Kähler metric if and only if $f_S > 0$ and $f_S + S f_{SS} > 0$.

The component of the inverse metric of g

$$g^{i\bar{j}} = \frac{(f_S)^{m-2}}{\det(g)}[(f_S + S f_{SS})\delta_{i\bar{j}} - f_{SS} z_i \bar{z}_j].$$

Remark 2.1.2. In our notation, the potential function of Euclidean metric is $f(S) = \frac{S}{2}$.

The Volume form is denoted and defined by

$$\frac{\omega^m}{m!} = (\sqrt{-1})^m((f_S)^{m-1}(f_S + Sf_{SS})dz_1 \wedge d\bar{z_1} \wedge ... \wedge dz_m \wedge d\bar{z_m} \,. \tag{2.1.3}$$

By Lemma 1.2.11 we know that (2.1.3) is a volume form of the Riemannian metric and in real coordinates it can be written as

$$\frac{\omega^m}{m!} = (2f_S)^{m-1}(2(f_S + Sf_{SS})dx_1 \wedge dy_i \cdots dx_m \wedge dy_m \,.$$

$$\frac{\omega^m}{m!} = (f_S)^{m-1}((f_S + Sf_{SS})dV_E \,, \tag{2.1.4}$$

where dV_E is the Euclidean volume form.

The Ricci and scalar curvature

If we denote $v(S) = \log(\det(g))$, then the Ricci curvature and scalar curvature can be computed respectively by the following formulae

$$R_{ij} = -\partial\bar{\partial}v(S) = -(v_S\delta_{i\bar{j}} + v_{SS}z_i z_{\bar{j}}), \tag{2.1.5}$$

$$R = -\frac{S^{1-m}}{\det(g)}[S^m(f_S)^{m-1}v_S]_S \,. \tag{2.1.6}$$

The logarithmic coordinates

It is convenient to consider the change of logarithmic coordinates $t = \log S$ and $f(t) = f(S)$ for the computation purpose. The Kähler form given in (2.1.1) can be written in coordinates t as follows

$$\omega = \sqrt{-1}(f_{tt}\partial t \wedge \bar{\partial}t + f_t\partial\bar{\partial}t). \tag{2.1.7}$$

Indeed we have $f_S = \frac{f_t}{S}$ and $f_{SS} = \frac{1}{S^2}(f_{tt} - f_t)$.

The determinant given equation (2.1.2) becomes

$$\det(g) = e^{-mt}f_t^{m-1}f_{tt}.$$

The Ricci and scalar curvature in t

If we define $v(t) = \log(e^{-mt}f_t^{m-1}f_{tt})$, then the Ricci form and scalar curvature are given respectively as follows:

$$R_{i\bar{j}}(\omega) = -\sqrt{-1}\partial_i\bar{\partial}_{\bar{j}}v(t),$$

$$R = -\triangle v(t). \tag{2.1.8}$$

We know that if we fix the Kähler metric ω om M and ϕ be any smooth real function on M, then there is $\tau \in (-\epsilon, \epsilon)$ for small enough ϵ such that $\omega_\tau = \omega + \tau\sqrt{-1}\partial\bar{\partial}\phi$ is still Kähler metric on M, and we have

$$\triangle\phi = \frac{1}{\omega^m}\frac{d}{d\tau}|_{\tau=0}(\omega + \tau\sqrt{-1}\partial\bar{\partial}\phi)^m\,. \tag{2.1.9}$$

The purpose of following Lemma is to give the explicit formula for scalar curvature equation which involve only derivatives of the potential function $f(t)$.

Lemma 2.1.3. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a $U(m)$ invariant Kähler metric on $\mathbb{C}^m \setminus \{0\}$ and $\phi = \phi(t)$. Then we have

$$\triangle\phi(t) = (m-1)\frac{\phi_t}{f_t} + \frac{\phi_{tt}}{f_{tt}}\,. \tag{2.1.10}$$

Proof. In coordinates t equation (2.1.3) becomes,

$$\omega^m = m!e^{-mt}f_t^{m-1}f_{tt}dz_1 \wedge d\bar{z_1}... \wedge d\bar{z_m}.$$

Now $\phi = \phi(t)$, we have

$$\omega_\tau^m = (\sqrt{-1}\partial\bar{\partial}(f(t)) + \tau\phi(t))^m = m!(e^{-mt}(f_t + \tau\phi_t)^{m-1}(f_{tt} + \tau\phi_{tt}))\,,$$

and thus

$$\begin{aligned}\frac{d}{d\tau}\omega_\tau^m\Big|_{\tau=0} &= \frac{d}{d\tau}(\omega + \tau\sqrt{-1}\partial\bar{\partial}\phi)^m\Big|_{\tau=0}\\ &= m!\left((m-1)e^{-mt}f_t^{m-2}f_{tt}\phi_t + e^{-mt}f_t^{m-1}\phi_{tt}\right)\,.\end{aligned}$$

Using (2.1.9), we obtain

$$\triangle\phi(t) = (m-1)\frac{\phi_t}{f_t} + \frac{\phi_{tt}}{f_{tt}}\,. \tag{2.1.11}$$

□

Proposition 2.1.4. The scalar curvature of $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ is

$$R = \frac{m(m-1)}{f_t} - \frac{2(m-1)f_{ttt}}{f_t f_{tt}} + \frac{f_{ttt}^2}{f_{tt}^3} - \frac{f_{tttt}}{f_{tt}^2} - \frac{(m-1)(m-2)f_{tt}}{f_t^2}\,. \tag{2.1.12}$$

Proof. By (2.1.8) and Lemma (2.1.3), the scalar curvature is

$$R = -\triangle v(t) = -\left((m-1)\frac{v_t}{f_t} + \frac{v_{tt}}{f_t}\right), \tag{2.1.13}$$

where $v(t) = \log(e^{-mt} f_t^{m-1} f_{tt})$. Putting the values of ν_t and v_{tt} in (2.1.13), we have

$$R = \frac{m(m-1)}{f_t} - \frac{2(m-1)f_{ttt}}{f_t f_{tt}} + \frac{f_{ttt}^2}{f_{tt}^3} - \frac{f_{tttt}}{f_{tt}^2} - \frac{(m-1)(m-2)f_{tt}}{f_t^2}. \tag{2.1.14}$$

□

In general the scalar equation for Kähler metrics is nonlinear fourth order PDE but for the U(m)-invariant Kähler metrics it reduce to nonlinear fourth order ODE. We also recall the condition that make ω to be a Kähler-Einstein metric, i.e. the solution of the following equation

$$Ric(\omega) - \lambda\omega = 0, \tag{2.1.15}$$

for some real constant λ.
Since $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$, we have

$$Ric(\omega) - \lambda\omega = -\sqrt{-1}\partial\bar{\partial}\left(\log(e^{-mt} f_t^{m-1} f_{tt}) + \lambda f(t)\right).$$

By equation (2.1.7), it follows easily that all spherical symmetric solutions of the equation $\partial\bar{\partial}\psi = 0$ is of the form,

$$\psi = \begin{cases} c_1 t + c_2 & \text{if} \quad m = 1 \\ c & \text{if} \quad m \geq 2, \end{cases}$$

for some constant $c, c_1, c_2 \in \mathbb{R}$. Thus ω is Kähler-Einstein metric if and only if

$$f_t^{m-1} f_{tt} = e^{\psi(t) + mt - \lambda f}.$$

Hence we have proved the following results.

Proposition 2.1.5. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a Kähler form on $\mathbb{C}\backslash 0$. Then ω is Kähler-Einstein metric if and only if there exist $c_1, c_2 \in \mathbb{R}$ such that $f(t)$ satisfies

$$f_{tt} = e^{c_1 t + c_2 - \lambda f}.$$

Proposition 2.1.6. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a Kähler form on $\mathbb{C}^m \setminus 0$ with $m \geq 2$. Then ω is Kähler-Einstein if and only if there exist $c \in \mathbb{R}$ such that $f(t)$ satisfies

$$f_t^{m-1} f_{tt} = e^{c+mt-\lambda f}.$$

Example 2.1.7. Let $f(t) = \frac{e^{\beta t}}{\beta}$ with $\beta \neq 0$. Then we have

$$f_{tt} = e^{\beta t}.$$

Thus the Kähler form defined by

$$\omega_\beta = \frac{\sqrt{-1}}{\beta^2}\partial\bar{\partial}e^{\beta t},$$

are all Ricci flat on $\mathbb{C} \setminus 0$.

Example 2.1.8. Let $f(t) = \frac{t^2}{2}$. Then $f_{tt} = 1$, and the Kähler form

$$\omega_0 = \sqrt{-1}\partial\bar{\partial}\left(\frac{t^2}{2}\right),$$

is Ricci flat on $\mathbb{C} \setminus 0$.

– **Notation:** For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ we denote $z_j = \rho_j e^{i\theta_j}$ the polar coordinates on $\mathbb{C}$. We denote the new coordinates by

$$\Theta = (\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m).$$

The metric corresponding to the Kähler form given in (2.1.1) is

$$g = f_{SS}(\partial S \odot \bar{\partial} S) + f_S \partial\bar{\partial} S, \tag{2.1.16}$$

with the convention that $\partial S \odot \bar{\partial} S = \partial S \otimes \bar{\partial} S + \bar{\partial} S \otimes \partial S$.

In the coordinates Θ, we have $S = \sum_j \rho_j^2$ and the metric g associated to (2.1.16) becomes

$$g = \sum_{k,j=1}^{m} f_{SS}\rho_j\rho_k(d\rho_j \odot d\rho_k + \rho_j\rho_k d\theta_k \odot d\theta_j) + f_S(d\rho_j \odot d\rho_j + \rho_j^2 d\theta_j \odot d\theta_j). \tag{2.1.17}$$

The matrix corresponding of the metric g in this new coordinate is a block matrix

$$g = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where A and B denotes the matrix representation of the metric g with respect to $(\partial_{\rho_1}, \ldots, \partial_{\rho_m})$ and $(\partial_{\theta_1}, \ldots, \partial_{\theta_m})$ respectively. The matrix A and B are as follows

$$A = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{im} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{bmatrix}, \quad B = \begin{bmatrix} \rho_1^2 g_{11} & \rho_1\rho_2 g_{12} & \cdots & \rho_1\rho_m g_{1m} \\ \rho_1\rho_2 g_{21} & \rho_2^2 g_{22} & \cdots & \rho_2\rho_m g_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ \rho_1\rho_m g_{m1} & \rho_2\rho_m g_{m2} & \cdots & \rho_m^2 g_{mm} \end{bmatrix},$$

where $g_{ii} = 2f_{SS}\rho_i^2 + 2f_S$ and $g_{ij} = 2f_{SS}\rho_i\rho_j$. Clearly the determinant of the matrix g is

$$\det(g) = \det(A) \cdot \det(B).$$

By direct computation we have

$$\det(A) = 2^m (f_S)^{m-1}(f_{SS}S + f_S), \quad \det(B) = 2^m \left(\prod_{i=1}^m \rho_i\right)^2 (f_S)^{m-1}(f_{SS}S + f_S)$$

which gives

$$\det(g) = 2^{2m} \left(\prod_{i=1}^m \rho_i\right)^2 (f_S)^{2m-2}(f_{SS}S + f_S)^2. \tag{2.1.18}$$

In local coordinates $(\rho_1, ..., \rho_m, \theta_1, ..., \theta_m)$, the volume form 2.1.4 becomes

$$dV = \left(\prod_{i=1}^m \rho_i\right) (2f_S)^{m-1}(2(f_{SS}S + f_S)d\rho_1 \wedge ... \wedge d\rho_m \wedge d\theta_1 \wedge ... \wedge d\theta_m\,. \tag{2.1.19}$$

2.2 Computing mean curvature: the 4-dimensional case

We consider the family of canonical hyperspheres denoted by Σ_S and defined as follows,

$$\Sigma_S = \{(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m : S = |z_1|^2 + |z_2|^2 + \cdots + |z_m|^2\} \subset \mathbb{C}^m.$$

We want to compute the mean curvature of this family Σ_S with $U(m)$-invariant Kähler metrics discussed in previous section. This section contains only 4 dimensional case, while in the next section we generalize to higher dimension. The approach to obtain the mean curvature of Σ_S is direct computation of the second fundamental form of Σ_S . The main result of this section is following,

Theorem 2.2.1. The mean curvature of the family $\Sigma_S \subset \mathbb{C}^2$ with radial Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(S)$ and outward normal vector η is a function that depends on the potential function of ω and is given by

$$H(S, f(S)) = \frac{-1}{3\sqrt{S}\alpha^{\frac{3}{2}} f_S}(\alpha^2 + 2f_S(f_{SSS}S^2 + 3f_{SS}S + f_S)),$$

where $\alpha = 2(f_{SS}S + f_S)$.

In order to compute the mean curvature the first thing is to know the normal vector to the hypersphere Σ_S. For this we consider Σ_S as a level set of the function $h : \mathbb{C}^2 \to \mathbb{R}$ defined by

$$h(\rho_1, \theta_1, \rho_2, \theta_2) = \rho_1^2 + \rho_2^2 .$$

Lemma 2.2.2. The normal vector to Σ_S is $\eta = \rho_1\partial_{\rho_1} + \rho_2\partial_{\rho_2}$, with length $E = \sqrt{2S(f_{SS}S + f_S)}$.

Proof. We compute the gradient of h, which is normal to the level set of the function h

$$\begin{aligned}
\nabla h = g^{ij}\partial_i h\partial_j &= g^{11}\partial_{\rho_1}h\partial_{\rho_1} + g^{12}\partial_{\rho_1}h\partial_{\rho_2} + g^{22}\partial_{\rho_2}h\partial_2 + g^{21}\partial_{\rho_2}h\partial_{\rho_1} \\
&= 2(g^{11}\rho_1 + g^{21}\rho_2)\partial_{\rho_1} + 2(g^{12}\rho_1 + g^{22}\rho_2)\partial_{\rho_2} \\
&= \frac{\rho_1}{f_{SS}S + f_S}\partial_{\rho_1} + \frac{\rho_2}{f_{SS}S + f_S}\partial_{\rho_2}.
\end{aligned}$$

Since $2(g^{11}\rho_1 + g^{21}\rho_2) = \frac{\rho_1}{f_{SS}S+f_S}$ and $2(g^{12}\rho_1 + g^{22}\rho_2) = \frac{\rho_2}{f_{SS}S+f_S}$
the norm of ∇h is

$$\|\nabla h\| = \sqrt{g(\nabla h, \nabla h)} = \sqrt{\frac{2S}{f_{SS}S + f_S}}.$$

The normal vector is given by

$$\tilde{\eta} = \frac{\nabla h}{\|\nabla h\|} = \frac{1}{E}(\rho_1\partial_{\rho_1} + \rho_2\partial_{\rho_2}).$$

That is $\tilde{\eta} = \frac{\eta}{E}$, where $\eta = \rho_1\partial_{\rho_1} + \rho_2\partial_{\rho_2}$ and $E = \sqrt{2S(f_{SS}S + f_S)}$. □

Once we have the normal vector we can restrict the Kähler metric on $\mathbb{C}^2 \setminus 0$ to the family of hyperspheres Σ_S, i.e. on the tangent space $T_p\Sigma_S = \{X \in \mathbb{R}^4 : g(X, \eta) = 0\}$. Since the tangent space is independent of the metric so the basis for $T_p\Sigma_S$ can be obtained by using the Euclidean metric.

Lemma 2.2.3. Th restriction of the metric g to Σ_S is given by

$$g_{\Sigma_S} = \begin{bmatrix} 2Sf_S & 0 & 0 \\ 0 & 2S(f_{SS}S + f_S) & 0 \\ 0 & 0 & 2f_S\rho_1^2\rho_2^2 \end{bmatrix}$$

Proof. We choose basis $e_1 = -\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}, e_2 = \partial_{\theta_1} + \partial_{\theta_2}, e_3 = \rho_2^2\partial_{\theta_1} - \rho_1^2\partial_{\theta_2}$ for $T_pS_R^3$,

$$\begin{aligned} g_{\Sigma_S}(e_1, e_1) &= g(-\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}, -\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}) \\ &= \rho_2^2 g_{11} + \rho_1^2 g_{22} - 2\rho_1\rho_2 g_{21} \\ &= \rho_2^2(2f_{SS}\rho_1^2 + 2f_S) + \rho_1^2(2f_{SS}\rho_2^2 + 2f_S) - 4\rho_1^2\rho_2^2 f_{SS} \\ &= 2f_S(\rho_1^2 + \rho_2^2) = 2f_S S \end{aligned}$$

$$\begin{aligned} g_{\Sigma_S}(e_2, e_2) &= g(\partial_{\theta_1} + \partial_{\theta_2}, \partial_{\theta_1} + \partial_{\theta_2}) \\ &= g_{33} + 2g_{34} + g_{44} = \rho_1^2 a + 2\rho_1\rho_2 b + \rho_2^2 d \\ &= \rho_1^2(2f_{SS}\rho_1^2 + 2f_S) + 4\rho_1^2\rho_2^2 f_{SS} + \rho_2^2(2f_{SS}\rho_2^2 + 2f_S) \\ &= 2f_{SS}(\rho_1^2 + \rho_2^2)^2 + 2f_S(\rho_1^2 + \rho_2^2) \\ &= 2f_{SS}S^2 + 2f_S S \\ &= 2S(f_{SS}S + 2f_S) \end{aligned}$$

$$\begin{aligned} g_{\Sigma_S}(e_3, e_3) &= g(\rho_2^2\partial_{\theta_1} - \rho_1^2\partial_{\theta_2}, \rho_2^2\partial_{\theta_1} - \rho_1^2\partial_{\theta_2}) \\ &= \rho_2^4 g_{33} + \rho_1^4 g_{44} - 2\rho_1^2\rho_2^2 g_{34} \\ &= \rho_2^4\rho_1^2(2f_{SS}\rho_1^2 + 2f_S) + \rho_1^4\rho_2^2(2f_{SS}\rho_2^2 + 2f_S) - 4\rho_1^4\rho_2^4 f_{SS} \\ &= 2f_S\rho_2^2\rho_1^2 S \end{aligned}$$

$$
\begin{aligned}
g_{\Sigma_S}(e_2, e_3) &= g(\partial_{\theta_1} + \partial_{\theta_2}, \rho_2^2 \partial_{\theta_1} - \rho_1^2 \partial_{\theta_2}) \\
&= \rho_2^2 g_{33} - \rho_1^2 g_{44} \\
&= \rho_1^2 \rho_2^2 (2 f_{SS} \rho_1^2 + 2 f_S) - \rho_1^2 \rho_2^2 (2 f_{SS} \rho_2^2 + 2 f_S) = 0
\end{aligned}
$$

Since ∂_{ρ_i} and ∂_{θ_j} are orthogonal, therefore $g(e_1, e_2) = (e_1, e_3) = 0$. Thus the restricted metric on the family of sphere is

$$
g_{\Sigma_S} = \begin{bmatrix} 2Sf_S & 0 & 0 \\ 0 & 2S(f_{SS}S + f_S) & 0 \\ 0 & 0 & 2f_S \rho_1^2 \rho_2^2 S \end{bmatrix}
$$

with inverse

$$
g_{\Sigma_S}^{-1} = \begin{bmatrix} \frac{1}{2Sf_S} & 0 & 0 \\ 0 & \frac{1}{2S(f_{SS}S + f_S)} & 0 \\ 0 & 0 & \frac{1}{2f_S \rho_1^2 \rho_2^2 S} \end{bmatrix}
$$

. □

So far, we have got the induced metric on Σ_S. The next step is the computation of the coefficients of the second fundamental form of the family of hyperspheres Σ_S. For this purpose we need to compute the connection of the metric. Since the restricted metric to Σ_S is diagonal, we only need the connections given in the following Lemma,

Lemma 2.2.4. The connections of the metric on $\mathbb{C}^m \setminus 0$

$$
\begin{aligned}
\nabla e_1(e_1) &= -E\tilde{\eta} + \rho_2^2 \Gamma_{11}^k \partial_k + \rho_1^2 \Gamma_{22}^k \partial_k - 2\rho_1 \rho_2 \Gamma_{12}^k \partial_k \\
\nabla_{e_2}(e_2) &= \Gamma_{33}^k \partial_k + 2\Gamma_{34}^k \partial_k + \Gamma_{44}^k \partial_k \\
\nabla_{e_3}(e_3) &= \rho_2^4 \Gamma_{33}^k \partial_k + \rho_1^4 \Gamma_{44}^k \partial_k - 2\rho_1^2 \rho_2^2 \Gamma_{34}^k \partial_k
\end{aligned}
$$

Proof. We know that $\nabla_{\partial_i}(\partial_j) = \Gamma_{ij}^k \partial_k$ and so

$$\begin{aligned}
\nabla e_1(e_1) &= \nabla_{(-\rho_2\partial_{\rho_1}+\rho_1\partial_{\rho_2})}(-\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}) \\
&= \nabla_{-\rho_2\partial_{\rho_1}}(-\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}) + \nabla_{\rho_1\partial_{\rho_2}}(-\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}) \\
&= -\rho_2\nabla_{\partial_{\rho_1}}(-\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}) + (\rho_1)\nabla_{\partial_{\rho_2}}(-\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}) \\
&= -\rho_2(-\rho_2\nabla_{\partial_{\rho_1}}(\partial_{\rho_1}) - \nabla_{\partial_{\rho_1}}(\rho_2)\partial_{\rho_1} + \rho_1\nabla_{\partial_{\rho_1}}(\partial_{\rho_2}) + \partial_{\rho_2}) + \rho_1(-\rho_2\nabla_{\partial_{\rho_2}}(\partial_{\rho_1}) - \nabla_{\partial_{\rho_2}}(\rho_2)\partial_{\rho_1} \\
&+ \rho_1\nabla_{\partial_{\rho_2}}(\partial_{\rho_2}))) \\
&= -\rho_2\partial_{\rho_2} + \rho_2^2\Gamma_{11}^k\partial_k - \rho_2^2\rho_2\Gamma_{12}^k\partial_k - \rho_2\partial_{\rho_2} - \rho_2\rho_1\Gamma_{12}^k\partial_k - \rho_1\partial_{\rho_1} + \rho_1^2\Gamma_{22}^k \\
&= -\rho_2\partial_{\rho_2} - \rho_1\partial_{\rho_1} + \rho_2^2\Gamma_{11}^k\partial_k + \rho_1^2\Gamma_{22}^k\partial_k - 2\rho_1\rho_2\Gamma_{12}^k\partial_k \\
&= -E\tilde{\eta} + \rho_2^2\Gamma_{11}^k\partial_k + \rho_1^2\Gamma_{22}^k\partial_k - 2\rho_1\rho_2\Gamma_{12}^k\partial_k.
\end{aligned}$$

In the same way we get,

$$\begin{aligned}
\nabla_{e_2}(e_2) &= \Gamma_{33}^k\partial_k + 2\Gamma_{34}^k\partial_k + \Gamma_{44}^k\partial_k \\
\nabla_{e_3}(e_3) &= \rho_2^4\Gamma_{33}^k\partial_k + \rho_1^4\Gamma_{44}^k\partial_k - 2\rho_1^2\rho_2^2\Gamma_{34}^k\partial_k
\end{aligned}$$

□

In the next lemma we list all possible Christoffel symbols.

Lemma 2.2.5. The Christoffel Symbols of $(\mathbb{C}^2 \setminus 0, g)$ in coordinates basis $\{\partial_{\rho_1}, \partial_{\rho_2}, \partial_{\theta_1}, \partial_{\theta_2}\}$

are given as

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(g_{22}\partial_{\rho_1}g_{11} - g_{12}(2\partial_{\rho_1}g_{12} - \partial_{\rho_2}g_{11}))\\
\Gamma^2_{11} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(-g_{12}\partial_{\rho_1}g_{11} + g_{11}(2\partial_{\rho_1}g_{12} - \partial_{\rho_2}g_{11}))\\
\Gamma^3_{11} &= 0 = \Gamma^4_{11}\\
\Gamma^1_{12} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(g_{22}\partial_{\rho_2}g_{11} - g_{12}\partial_{\rho_1}g_{22})\\
\Gamma^2_{12} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(-g_{12}\partial_{\rho_2}g_{11} + g_{22}\partial_{\rho_1}g_{22})\\
\Gamma^3_{12} &= 0 = \Gamma^4_{12}\\
\Gamma^1_{13} &= 0 = \Gamma^2_{13}\\
\Gamma^3_{13} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{g_{22}}{\rho_1^2}\partial_{\rho_1}(\rho_1^2 g_{11}) - \frac{g_{12}}{\rho_1\rho_2}(\partial_{\rho_1}(\rho_1\rho_2 g_{12}))\\
\Gamma^4_{13} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{-g_{12}}{\rho_1\rho_2}\partial_{\rho_1}(\rho_1^2 g_{11}) + \frac{g_{11}}{\rho_1^2}(\partial_{\rho_1}(\rho_1\rho_2 g_{12})\\
\Gamma^1_{14} &= 0 = \Gamma^2_{14}\\
\Gamma^3_{14} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{g_{22}}{\rho_1^2}\partial_{\rho_1}(\rho_1\rho_2 g_{12}) + \frac{a}{\rho_2}\partial_{\rho_1}(\rho_2^2 g_{22}))\\
\Gamma^4_{14} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{-g_{12}}{\rho_1\rho_2}\partial_{\rho_1}(\rho_1\rho_2 g_{12}) + \frac{g_{11}}{\rho_2}(\partial_{\rho_1}(\rho_2^2 g_{22})))\\
\Gamma^1_{22} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(g_{22}(2\partial_{\rho_2}g_{12} - \partial_{\rho_1}g_{22}) + g_{11}\partial_{\rho_2}g_{22})\\
\Gamma^2_{22} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(-g_{12}(\partial_{\rho_2}g_{12} - \partial_{\rho_1}g_{22}) + g_{11}\partial_{\rho_2}g_{22})\\
\Gamma^3_{22} &= 0 = \Gamma^4_{22}\\
\Gamma^1_{23} &= 0 = \Gamma^2_{23}\\
\Gamma^3_{23} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{g_{22}}{\rho_1^2}\partial_{\rho_2}(\rho_1^2 g_{11}) - \frac{g_{12}}{\rho_1\rho_2}\partial_{\rho_2}(\rho_1\rho_2 g_{12}))\\
\Gamma^3_{23} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{g_{22}}{\rho_1^2}\partial_{\rho_2}(\rho_1^2 g_{11}) - \frac{b}{\rho_1\rho_2}\partial_{\rho_2}(\rho_1\rho_2 g_{12}))
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^4_{23} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{-g_{12}}{\rho_1\rho_2}\partial_{\rho_2}(\rho_1^2 g_{11}) + \frac{g_{11}}{\rho_2^2}(\partial_{\rho_2}(\rho_2\rho_1 g_{12})))\\
\Gamma^1_{24} &= 0 = \Gamma^2_{24}\\
\Gamma^3_{24} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{g_{22}}{\rho_1^2}\partial_{\rho_2}(\rho_1\rho_2 g_{12}) - \frac{g_{12}}{\rho_1\rho_2}\partial_{\rho_2}(\rho_2^2 g_{22}))\\
\Gamma^1_{24} &= 0 = \Gamma^2_{24}\\
\Gamma^4_{24} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(\frac{-g_{12}}{\rho_1\rho_2}(\partial_{\rho_2}(\rho_1\rho_2 g_{12}) + \frac{a}{\rho_2^2}(\partial_{\rho_2}(\rho_2^2 g_{22})\\
\Gamma^1_{33} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(-g_{22}\partial_{\rho_1}(\rho_1^2 g_{11}) + g_{12}\partial_{\rho_2}(\rho_1^2 g_{11}))\\
\Gamma^2_{33} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(g_{12}\partial_{\rho_1}(\rho_1^2 g_{11}) - g_{11}\partial_{\rho_2}(\rho_1^2 g_{11}))\\
\Gamma^3_{33} &= 0 = \Gamma^4_{33}\\
\Gamma^1_{34} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(-g_{22}\partial_{\rho_1}(\rho_1\rho_2 g_{22}) + g_{22}\partial_{\rho_2}(\rho_1\rho_2 g_{22}))\\
\Gamma^2_{34} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(g_{12}\partial_{\rho_1}(\rho_1\rho_2 g_{12}) - a\partial_{\rho_2}(\rho_1\rho_2 g_{12}))\\
\Gamma^3_{34} &= 0 = \Gamma^4_{34}\\
\Gamma^1_{44} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(-g_{12}\partial_{\rho_1}(\rho_2^2 g_{22}) + g_{12}\partial_{\rho_2}(\rho_2^2 g_{22}))\\
\Gamma^2_{44} &= \frac{1}{2(g_{11}g_{22}-g_{12}^2)}(g_{12}\partial_{\rho_1}(\rho_2^2 g_{22}) - g_{11}\partial_{\rho_2}(\rho_2^2 g_{22}))\\
\Gamma^3_{44} &= 0 = \Gamma^4_{44}\,.
\end{aligned}
$$

We have collected all the ingredients which we need to compute the coefficient of the second fundamental form.

Proposition 2.2.6. The second fundamental form of Σ_S is

$$
\Pi = \begin{bmatrix} g(\nabla_{e_1}(e_1),\eta) & 0 & 0 \\ 0 & g(\nabla_{e_2}(e_2),\eta)(& g(\nabla_{e_3}(e_3),\eta) \\ 0 & g(\nabla_{e_3}(e_2),\eta) & g(\nabla_{e_3}(e_3),\eta) \end{bmatrix}
$$

where $E = \sqrt{2S(f_{SS}S + f_S)}$

$$g(\nabla_{e_1}(e_1), \eta) = -E + \frac{1}{E}[(\rho_2^2\Gamma_{11}^1 + \rho_1^2\Gamma_{22}^1 - 2\rho_1\rho_2\Gamma_{12}^1)]\rho_1 g_{11} + (\rho_2^2\Gamma_{11}^2 + \rho_1^2\Gamma_{22}^2 - 2\rho_1\rho_2\Gamma_{12}^2)\rho_2 g_{22}$$
$$+ (\rho_2^2\Gamma_{11}^2 + \rho_1^2\Gamma_{22}^2 - 2\rho_1\rho_2\Gamma_{12}^2)\rho_1 g_{12} + (\rho_2^2\Gamma_{11}^1 + \rho_1^2\Gamma_{22}^1 - 2\rho_1\rho_2\Gamma_{12}^1)\rho_2 g_{12}]$$

$$g(\nabla_{e_2}(e_2), \eta) = \frac{1}{E}[(\Gamma_{33}^1 + \Gamma_{44}^1 + 2\Gamma_{34}^1)]\rho_1 g_{11} + (\Gamma_{33}^2 + \Gamma_{44}^2 + 2\Gamma_{34}^2)\rho_2 g_{22} + (\Gamma_{33}^1 + \Gamma_{44}^1 + 2\Gamma_{34}^1)\rho_2 g_{12}$$
$$+ (\Gamma_{33}^2 + \Gamma_{44}^2 + 2\Gamma_{34}^2)\rho_1 g_{12}]$$

$$g(\nabla_{e_3}(e_3), \eta) = \frac{1}{E}[(\rho_2^4\Gamma_{33}^1 + \rho_1^4\Gamma_{44}^1 - 2\rho_1^2\rho_2^2\Gamma_{34}^1)\rho_1 g_{11} + (\rho_2^4\Gamma_{33}^2 + \rho_1^4\Gamma_{44}^2 - 2\rho_1^2\rho_2^2\Gamma_{34}^2)\rho_2 g_{22}$$
$$+ (\rho_2^4\Gamma_{33}^2 + \rho_1^4\Gamma_{44}^2 - 2\rho_1^2\rho_2^2\Gamma_{34}^2)\rho_1 g_{12} + (\rho_2^4\Gamma_{33}^1 + \rho_1^4\Gamma_{44}^1 - 2\rho_1^2\rho_2^2\Gamma_{34}^1)\rho_2 g_{12}]$$

and $E = \sqrt{2S(f_{SS}S + f_S)}$.

Proposition 2.2.7. The eigenvalues of the second fundamental form Π are

$$\lambda_1 = \frac{-1}{2f_S S^{\frac{1}{2}}}(\sqrt{\alpha}) = \lambda_3$$

$$\lambda_2 = \frac{-1}{\alpha^{\frac{3}{2}}}(2f_{SSS}S^{\frac{3}{2}} + 6f_{SS}S^{\frac{1}{2}} + \frac{2f_S}{S^{\frac{1}{2}}})$$

where $\alpha = 2f_{SS}S + 2f_S$.

Proof.

$$g_{\Sigma_S}^{-1}\Pi = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

where

$$\lambda_1 = \frac{1}{2Sf_S} g(\nabla_{e_1}(e_1), \eta)$$
$$\lambda_2 = \frac{1}{2S(f_{SS}S + f_S)} g(\nabla_{e_2}(e_2), \eta)$$
$$\lambda_3 = \frac{1}{2f_S\rho_1^2\rho_2^2} g(\nabla_{e_3}(e_3), \eta)$$

Since eigenvalues does not depend on the chosen basis so for simplification we take $\lim \rho_2 \to 0$,

$$\begin{aligned}
\lambda_1 &= \lim_{\rho_2\to 0} \frac{1}{2Sf_S} g(\nabla_{e_1}(e_1), \eta) \\
&= \lim_{\rho_2\to 0} \frac{1}{2Sf_S}(-E + \frac{1}{E}[(\rho_2^2\Gamma_{11}^1 + \rho_1^2\Gamma_{22}^1 - 2\rho_1\rho_2\Gamma_{12}^1)]\rho_1 g_{11} + (\rho_2^2\Gamma_{11}^2 + \rho_1^2\Gamma_{22}^2 - 2\rho_1\rho_2\Gamma_{12}^2)\rho_2 g_{22} \\
&+ (\rho_2^2\Gamma_{11}^2 + \rho_1^2\Gamma_{22}^2 - 2\rho_1\rho_2\Gamma_{12}^2)\rho_1 g_{12} + (\rho_2^2\Gamma_{11}^1 + \rho_1^2\Gamma_{22}^1 - 2\rho_1\rho_2\Gamma_{12}^1)\rho_2 g_{12}]] \\
&= \frac{-1}{2f_S\rho_1}(\sqrt{(2f_{SS}S + 2f_S)})
\end{aligned}$$

$$\begin{aligned}
\lambda_2 &= \lim_{\rho_2\to 0} \frac{1}{2S(f_{SS}S + f_S)} g(\nabla_{e_2}(e_2), \eta) \\
&= \lim_{\rho_2\to 0} (\frac{1}{2S(f_{SS}S + f_S)} \frac{1}{E}[(\Gamma_{33}^1 + \Gamma_{44}^1 + 2\Gamma_{34}^1)\rho_1 a + (\Gamma_{33}^2 + \Gamma_{44}^2 + 2\Gamma_{34}^2)\rho_2 d \\
&+ (\Gamma_{33}^1 + \Gamma_{44}^1 + 2\Gamma_{34}^1)\rho_2 b + (\Gamma_{33}^2 + \Gamma_{44}^2 + 2\Gamma_{34}^2)\rho_1 b] \\
&= \frac{-1}{(2f_{SS}S + 2f_S)^{\frac{3}{2}}}(2f_{SSS}\rho_1^3 + 6f_{SS}\rho_1 + \frac{2f_S}{\rho_1})
\end{aligned}$$

$$\begin{aligned}
\lambda_3 &= \lim_{\rho_2\to 0} \frac{1}{2f_S\rho_1^2\rho_2^2} g\left(\nabla_{e_3}(e_3), \eta\right) \\
&= \lim_{\rho_2\to 0} \frac{1}{2f_S\rho_1^2\rho_2^2} \frac{1}{E}[(\rho_2^4\Gamma_{33}^1 + \rho_1^4\Gamma_{44}^1 - 2\rho_1^2\rho_2^2\Gamma_{34}^1)\rho_1 g_{11} + (\rho_2^4\Gamma_{33}^2 + \rho_1^4\Gamma_{44}^2 - 2\rho_1^2\rho_2^2\Gamma_{34}^2)\rho_2 g_{22} \\
&+ (\rho_2^4\Gamma_{33}^2 + \rho_1^4\Gamma_{44}^2 - 2\rho_1^2\rho_2^2\Gamma_{34}^2)\rho_1 g_{12} + (\rho_2^4\Gamma_{33}^1 + \rho_1^4\Gamma_{44}^1 - 2\rho_1^2\rho_2^2\Gamma_{34}^1)\rho_2 g_{12}] \\
&= \frac{-1}{2f_S\rho_1}(\sqrt{(2f_{SS}S + 2f_S)})
\end{aligned}$$

Since now we have $S = \rho_1^2$ so we get

$$\begin{aligned}
\lambda_1 &= \frac{-1}{2f_S S^{\frac{1}{2}}}(\sqrt{\alpha}) = \lambda_3 \\
\lambda_2 &= \frac{-1}{\alpha^{\frac{3}{2}}}\left(2f_{SSS}S^{\frac{3}{2}} + 6f_{SS}S^{\frac{1}{2}} + \frac{2f_S}{S^{\frac{1}{2}}}\right)
\end{aligned}$$

where $\alpha = 2f_{SS}S + 2f_S$. □

Corollary 2.2.8. Considering the $U(m)$-invariant metric the family of canonical hypersphere does not contain any totally geodesic hypersphere.

Proof. Since positivity of (2.1.2) forces $\alpha = 2f_{SS}S + 2f_S$ to be positive. Therefore the second fundamental form can not vanish identically. □

Once we get the eigenvalues of the second fundamental, the mean curvature formula is just the average of the eigenvalues, i.e.

$$\begin{aligned} H(S, f(S)) &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \\ &= \frac{-1}{3\sqrt{S}a^{\frac{3}{2}}f_S}\left(a^2 + 2f_S(f_{SSS}S^2 + 3f_{SS}S + f_S)\right), \end{aligned}$$

which completes the proof of the Theorem 2.2.1.

2.3 The variational approach: the higher dimensional case

In this section, we extend the proof of Theorem 2.2.1 to higher complex dimensions m, by using the first variation formula given in Theorem 1.1.2. The change in the area functional is measured in terms of mean curvature. The area of Σ_S can be computed by restricting the volume form given in 2.1.4 to Σ_S to get the volume form on Σ_S, and then integrate. Before going to compute the area we prove the following,

Lemma 2.3.1. The mean curvature of Σ_S with $U(m)$-invariant metric is constant.

In order to restrict the 2.1.4 to Σ_S, we need the unit normal vector to Σ. The following Lemma gives us the expression of the unit normal vector.

Lemma 2.3.2. The unit normal vector to Σ_S^{2m-1} is

$$\tilde{\eta} = \frac{R}{E}\partial_R,$$

where $E = \sqrt{2S(f_{SS}S + f_S)}$.

Proof. The proof is straightforward computation similar to Lemma 2.2.2. □

Proposition 2.3.3. The area of the family of hyperspheres Σ^{2m-1} is given as follows

$$A_{\Sigma^{2m-1}} = S^{(m-\frac{1}{2})}(2f_S)^{m-1}\sqrt{2(f_{SS}S + f_S)}V_E(\Sigma_1^{2m-1}).$$

where $V_E(\Sigma_1^{2m-1})$ is Euclidean volume of unit hypersphere.

Proof. The contraction of 2.1.4 with unit normal gives,

$$dV_{\Sigma^{2m-1}} = (2f_S)^{m-1}\sqrt{2(f_S + Sf_{SS}}\, dV_E|_{\Sigma_S}. \tag{2.3.1}$$

where dV_E is Euclidean Volume form of $\mathbb{C}^m$. After integration of the above equation we get the area

$$\begin{aligned} A_{\Sigma^{2m-1}} &= (2f_S)^{m-1}\sqrt{2(f_{SS}S + f_S)}\int_{\Sigma_S} dV_E|_{\Sigma_S} \\ &= (2f_S)^{m-1}\sqrt{2(f_{SS}S + f_S)}\, V_E(\Sigma_S) \\ &= R^{2m-1}(2f_S)^{m-1}\sqrt{2(f_{SS}S + f_S)}\, V_E(\Sigma_1)\,, \end{aligned}$$

where $V_E(\Sigma_1)$ is the area of unit hypersphere with Euclidean metric. □

Remark 2.3.4. The the area of unit hypersphere is given by

$$V_E(\Sigma_1) = (2\pi)^m \prod_{k=1}^{m-1} \frac{1}{2k}\,.$$

Once we have the area of family of canonical hyperspheres Σ_S^{2n-1}, we want to understand the change in the area function under variation of the hypersphere Σ_S^{2m-1} of radius R in the normal direction. The variation of Σ_S^{2m-1} in normal direction ∂_R gives hypersphere of new radius $\widetilde{R}$. If $\widetilde{S} = \widetilde{R}^2$, then we denote and define the new hypersphere with radius $\widetilde{R}$ by

$$\Sigma_{\widetilde{S}}^{2m-1} = \Sigma_S^{2m-1} + t\partial_R\,.$$

Let $\tilde{p} \in \Sigma_{\widetilde{S}}^{2m-1}$. Then $\tilde{p} = p + t\partial_R$ for some t, where $p = (R, 0, 0, ..., 0) = R\partial_R \in \Sigma_S^{2m-1}$, and so the Euclidean distance is

$$|\tilde{p}|^2 = \langle\, (R+t)\partial_R, (R+t)\partial_R\,\rangle = (R+t)^2.$$

This shows that that the new radius $\widetilde{R} = R + t$, for some t.

The next lemma measures the change in area while moving Σ_S^{2m-1} in normal direction.

Lemma 2.3.5. Let $\Sigma_{\widetilde{S}}^{2m-1}$ be the above variation of Σ_S^{2m-1}. Then the first variation of area is given by

$$\frac{d}{dt}A(\Sigma_{\widetilde{S}}^{2m-1})\Big|_{t=0} = \frac{2^{m-1}S^{m-1}f_S^{m-2}}{\sqrt{\alpha(S)}}\left((m-1)(\alpha(S))^2 + 2f_S(f_{SSS}S^2 + 3f_{SS}S + f_S)\right)V_E(\Sigma_1^{2m-1}).$$

Proof. By Proposition 2.3.3, we have

$$\begin{aligned} A(\Sigma_{\widetilde{S}}^{2m-1}) &= 2^{m-1}V_E(\Sigma_1^{2m-1})\widetilde{R}^{2m-1}(f_S(\widetilde{S}))^{m-1}\sqrt{\alpha(\widetilde{S})} \\ &= 2^{m-1}V_E(\Sigma_1^{2m-1})K(\widetilde{S}), \end{aligned} \tag{2.3.2}$$

where $\alpha(\widetilde{S}) = 2\widetilde{S}\, f_{SS}(\widetilde{S}) + 2f_S(\widetilde{S})$. Differentiating $K(\widetilde{S})$ in (2.3.2), we obtain

$$\begin{aligned} \frac{d}{dt}K(\widetilde{S}) &= (2m-1)\widetilde{R}^{2m-2}(f_S(\widetilde{S}))^{m-1}\sqrt{\alpha(\widetilde{S})} \\ &\quad + (m-1)\widetilde{R}^{2m-1}\sqrt{\alpha(\widetilde{S})}(f_S(\widetilde{S}))^{m-2}\frac{d}{dt}f_S(\widetilde{S}) \\ &\quad + \frac{1}{2\sqrt{\alpha}}\widetilde{R}^{2m-1}f_S(\widetilde{S})^{m-1}\frac{d}{dt}\alpha(\widetilde{S}) \end{aligned}$$

Since at $t=0$, $\widetilde{R} = R$ and $f_S(\widetilde{S}) = f_S(S) = f_S$, so we have

$$\begin{aligned} \frac{d}{dt}K(\widetilde{S})\Big|_{t=0} &= \frac{1}{\sqrt{a(S)}}\Big[(2m-1)R^{2m-2}(f_S)^{m-1}\alpha(S) + (m-1)R^{2m-1}f_S^{m-2}\alpha(S)\frac{d}{dt}f_S(\widetilde{S})\Big|_{t=0} \\ &\quad + \frac{1}{2}\, R^{2m-1}(f_S)^{m-1}\frac{d}{dt}\alpha(\widetilde{S})\Big|_{t=0}\Big]. \end{aligned} \tag{2.3.3}$$

By chain rule we know that

$$\begin{aligned} \frac{d}{dt}f_S(\widetilde{S}) &= \frac{d}{d\widetilde{S}}f_S(\widetilde{S})\frac{d}{dt}(\widetilde{S}) \\ &= \frac{d}{dS}f_S(\widetilde{S})\frac{dS}{d\widetilde{S}}\frac{d\widetilde{S}}{dt}. \end{aligned} \tag{2.3.4}$$

We recall that $\widetilde{S} = \widetilde{R}^2 = (R+t)^2$ and so we have $\frac{d\widetilde{S}}{dS} = 1 + \frac{t}{R}$ and $\frac{d\widetilde{S}}{dt} = 2t + 2R$. Thus (2.3.4) implies that

$$\frac{d}{dt}f_S(\widetilde{S})\Big|_{t=0} = f_{SS}2R\,.$$

Similarly we obtain

$$\frac{d}{dt}\alpha(\widetilde{S})|_{t=0} = 8f_{SS}R + 4R^3 f_{SSS}\,.$$

Then (2.3.3) becomes

$$\begin{aligned} \frac{d}{dt}K(\widetilde{S})\Big|_{t=0} &= \frac{1}{\sqrt{a(S)}}\Big[(2m-1)R^{2m-2}(f_S)^{m-1}\alpha(S) + 2(m-1)f_S^{m-2}f_{SS}\alpha(S)\, R^{2m} + \\ &\quad 4(f_S)^{m-1}f_{SS}R^{2m} + 2R^{2m+2}(f_S)^{m-1}f_{SSS}\Big]. \end{aligned}$$

After replacing $\alpha = 2(f_{SS}R^2 + f_S)$ we obtain

$$\frac{d}{dt}K(\widetilde{S})\Big|_{t=0} = \frac{1}{\sqrt{\alpha(S)}}[(8m-2)R^{2m}f_{SS}f_S^{m-1} + 2R^{2m+2}f_S^{m-1}f_{SSS} + 2(2m-1)R^{2m-2}f_S^m$$
$$+ 4(m-1)R^{2m+2}f_S^{m-2}f_{SS}^2].$$

After simplification and replacing $R^2 = S$, we have

$$\frac{d}{dt}A(\Sigma_{\widetilde{S}}^{2m-1})\Big|_{t=0} = \frac{2^{m-1}S^{m-1}f_S^{m-2}}{\sqrt{\alpha(S)}}\left((m-1)(\alpha(S))^2 + 2f_S(f_{SSS}S^2 + 3f_{SS}S + f_S)\right),.$$

□

By (1.1.2), the first variation formula for the area function is given by

$$\frac{d}{dt}A(\Sigma_{\widetilde{S}}^{2m-1})\Big|_{t=0} = -(2m-1)\int_{\Sigma_S^{2m-1}} Hg(\partial_R, \tilde{\eta})dV_{\Sigma_S^{2m-1}},$$

where H is the mean curvature of Σ_S^{2m-1}. Since the mean curvature only depends on radius, we have

$$\frac{d}{dt}A(\Sigma_{\widetilde{S}}^{2m-1})\Big|_{t=0} = -(2m-1)\frac{EH}{R}\int_{\Sigma_S^{2m-1}} dV_{\Sigma_S^{2m-1}}$$
$$= -(2m-1)\frac{EH}{R}A(\Sigma_S^{2m-1}),$$

which further implies that

$$H = \frac{-R}{(2m-1)A(\Sigma_S^{2m-1})E}\cdot\frac{d}{dt}A(\Sigma_{\widetilde{S}}^{2m-1})\Big|_{t=0}.$$

Proposition 2.3.3 and Lemma 2.3.5 implies the following result.

Theorem 2.3.6. The mean curvature of the family $\Sigma_S^{2m-1} \subset \mathbb{C}^m$ with radial Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(S)$ with outward normal vector η, is a function that depend on the potential function of Kähler metric given as follows,

$$H = \frac{-1}{(2m-1)\alpha^{\frac{3}{2}}\sqrt{S}f_S}((m-1)\alpha^2 + 2f_S(f_{SSS}S^2 + 3Sf_{SS} + f_S)).$$

Remark 2.3.7. The variataional approach of the proof of mean curvature is more geometric as compared to the direct computation of the second fundamental form of Σ_S.

2.4 Examples

In this section, we apply the formula in Theorem 2.2.1 for different metrics.

2.4.1 The Fubini Study metric

The complex projective space $\mathbb{C}P^m$ is endowed with the Fubini Study metric. The Kähler form is given by

$$\omega_{FS} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(|z_1|^2 + \cdots + |z_{m+1}|^2)\,, \tag{2.4.1}$$

where $[z_1, \ldots, z_{m+1}]$ are homogeneous coordinates of $\mathbb{C}P^m$. In the chart $U_1 = \{(z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1} : z_1 \neq 0\}$ with local coordinates $\{\xi = (\xi_1, ..., \xi_m) : \xi_i = \frac{z_i}{z_1}\}$, we can relate to this metric a local Kähler potential given by

$$\begin{aligned}\omega_{FS} &= \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1 + |\xi|^2)\,,\\ &= \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1 + S)\,,\end{aligned}$$

where $S = |\xi|^2$.

Now, when the Fubini Study metric is in the form of $U(m)$-invariant metric, we look at the family of canonical hyperspheres $\Sigma_S \subset \mathbb{C}^m$ with the Fubini Study metric. Notice that Σ_S can be covered by $m + 1$ charts but since the mean curvature is constant, its enough to consider only one chart. The potential function we use for a Fubini Study metric is given by

$$f(S) = \frac{1}{2}\log(1 + S).$$

By the formula in Theorem 2.3.6, we get

$$H(S, f(S)) = \frac{S - m}{3\sqrt{S}}\,.$$

Thus $(\mathbb{C}P^m, \omega_{FS})$ contains a minimal hypersphere of radius m in the family of canonical hypersphere Σ_S.

2.4.2 Kähler metric on the blow up of $\mathbb{C}^m$ at the origin

Let us consider the blow up of $\mathbb{C}^m$ at the origin. We denote the blow up of $\mathbb{C}^m$ at the origin by $Bl_0\mathbb{C}^m$ and defined as

$$Bl_0\mathbb{C}^m = \{((z_1, z_2, \ldots, z_m), [t_1, t_2, \ldots, t_m]) \in \mathbb{C}^m \times \mathbb{C}P^{m-1} : z_i t_j - z_j t_i = 0\} \subset \mathbb{C}^m \times \mathbb{C}P^{m-1}.$$

There is a natural projection map $\pi_1 : Bl_0\mathbb{C}^m \to \mathbb{C}^m$ defined by

$$\pi_1((z_1, z_2, \ldots, z_m), [t_1, t_2, \ldots, z_m]) = (z_1, z_2, \ldots, z_m).$$

The inverse image $\pi_1^{-1}(p)$ of $p \in \mathbb{C}^m$ is a line passing through that point p.
The **exceptional divisor** E is defined as the inverse image of the origin, i.e. $\pi^{-1}(0) = \mathbb{C}P^{m-1}$.
The map π_1 restrict to a biholomorphism

$$\pi_1 : Bl_0\mathbb{C}^m \setminus E \to \mathbb{C}^m \setminus 0.$$

A system of charts that cover the exceptional divisor is given as follows: for every $i = 1, 2, \ldots, m$,

$$U_i^1 = \{((z_1, z_2, \ldots, z_m), [t_1, t_2, \ldots, z_m]) : t_i \neq 0, z_j = z_i t_j\}.$$

The coordinate map $\Phi_i : U_i^1 \to \mathbb{C}^m$ is defined as

$$((z_1, z_2, \ldots, z_m), [t_1, t_2, \ldots, t_m]) \to \left(z_j, \frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_m}{t_i}\right),$$

with inverse map $\Phi_i^{-1} : \mathbb{C}^m \to U_i^1$

$$(z_1, z_2, \ldots, z_m) \to ((z_1 z_i, z_i z_2, \ldots, z_i, \ldots, z_i z_m), [z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_m]). \tag{2.4.2}$$

For every $i = 1, 2, \ldots, m$ the charts U_i^1 intersects the exceptional divisor E as

$$E \cap U_i^1 = \{z_i = 0\}.$$

Remark 2.4.1. The superscript of U_i^1 represent the charts for first blow up $Bl_0\mathbb{C}^m$.

Lemma 2.4.2. The pull back of the smooth form $\omega = \sqrt{-1}\partial\bar{\partial}\log(S)$ on $\mathbb{C}^m \setminus 0$ extend to the Fubini Study metric on the exceptional divisor$E = \mathbb{C}P^{m-1}$.

Proof. Given the smooth form

$$\omega = \sqrt{-1}\partial\bar{\partial}\log(S)$$

on $\mathbb{C}^m \setminus \{0\}$ where $S = \sum_{i=1}^m |z_i|^2$. Then the pull back $\pi_1^*\omega$ is given in local coordinates (2.4.2) by,

$$\begin{aligned}\pi_1^*\omega &= \partial\bar{\partial}\log(|z_i|^2(|z_1^2| + |z_2|^2 \cdots + |z_{i-1}|^2 + 1 + |z_{i+1}|^2 + \cdots + |z_m|^2) \\ &= \partial\bar{\partial}\log(|z_1|^2 + |z_2|^2 \cdots + |z_{i-1}|^2 + 1 + |z_{i+1}|^2 + \cdots + |z_m|^2). \end{aligned} \tag{2.4.3}$$

Clearly (2.4.3) is the Fubini Study metric on the exceptional divisor E in homogeneous coordinates $[z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_m]$. □

Let $g : \mathbb{C}^m \to \mathbb{R}$ be a smooth function that depends on $S = \sum_{i=1}^m |z_i|^2$. Then the smooth form

$$\omega = \sqrt{-1}\partial\bar{\partial}(\log S + g(S))\,, \tag{2.4.4}$$

which gives Kähler metric on $\mathbb{C}^m \setminus \{0\}$ if and only if $\frac{1}{S} + g_S > 0$ and $g_S + Sg_{SS} > 0$. The next proposition explains when the Kähler form (2.4.4) on $\mathbb{C}^m \setminus 0$ can be extend to the blowup of $Bl_0\mathbb{C}^m$.

Proposition 2.4.3. The smooth form $\omega = \sqrt{-1}\partial\bar{\partial}(\log S + g(S))$ on $\mathbb{C}^m \setminus \{0\}$ extend to Kähler metric on the $Bl_0\mathbb{C}^m$ if and only if $g_S(0) > 0$, and $\frac{1}{S} + g_S > 0$, $g_S + Sg_{SS} > 0$.

Proof. For the sake of simplicity, we prove only the case when $m = 2$. The general case follow from the same argument.
Given the projection map,

$$\pi_1 : Bl_0\mathbb{C}^2 \to \mathbb{C}^2$$

On the chart U_1, we have $S = |z_1|^2(1 + |z_2|^2)$ and $E \cap U_1^1 = \{z_1 = 0\}$. The pull back of the Kähler metric (2.4.4) to the $Bl_0\mathbb{C}^2$ is given in coordinates (2.4.2) by

$$\pi_1^*\omega = \begin{bmatrix} (1 + |z_2|^2)(g_S + Sg_{SS}) & z_1\bar{z}_2(g_S + Sg_{SS}) \\ z_2\bar{z}_1(g_S + Sg_{SS}) & |z_1|^2(g_S + |z_1|^2|z_2|^2 g_{SS}) + \frac{1}{1+|z_2|^2} \end{bmatrix}.$$

The restriction of $\pi_1^*\omega$ to the exceptional divisor E is given below,

$$\pi_1^*\omega|_E = \begin{bmatrix} (1+|z_2|^2)g_S(0) & 0 \\ 0 & \frac{1}{1+|z_2|^2} \end{bmatrix}$$

Clearly $\pi_1^*\omega|_E$ is positive definite if and only if $g_S(0) > 0$.
In the same way on U_2^1, the pull back $\pi_1^*\omega$

$$\pi_1^*\omega = \begin{bmatrix} \frac{1}{1+|z_1|^2} + |z_2|^2(g_S + |z_1|^2|z_2|^2 g_{SS}) & z_1\bar{z}_2(g_S + Sg_{SS}) \\ z_2\bar{z}_1(g_S + Sg_{SS}) & (1+|z_1|^2)(g_S + Sg_{SS}) \end{bmatrix}$$

can be restrict to the exceptional divisor as follows

$$\pi_1^*\omega|_E = \begin{bmatrix} \frac{1}{1+|z_1|^2} & 0 \\ 0 & (1+|z_1|^2)g_S \end{bmatrix}$$

which is positive definite if and only if $g_S(0) > 0$. □

Remark 2.4.4. If $g_S(0) = 0$, then $\pi_1^*\omega|_E$ defines metric only along the exceptional divisor. So the condition $g_S(0) \neq 0$ guaranty the non degeneracy of the metric orthogonal to the exceptional divisor. The other two conditions $\frac{1}{S} + g_S > 0$, $g_S + Sg_{SS} > 0$ are because ω has to be Kähler metric on $\mathbb{C}^2 \setminus 0$.

Remark 2.4.5. By (2.1.6) the scalar curvature of the Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}(\log S + S)$ on the $Bl_0\mathbb{C}^m$ is given by

$$R = \frac{(-2+m)(-1+m)}{(1+S)^2}. \tag{2.4.5}$$

Remark 2.4.6. For $m = 2$, the Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}(\log S + S)$ is scalar flat. This metric is known as Burns metric (see for example [19, 29]).

Remark 2.4.7. Consider the local coordinates $(z_1, \ldots, z_m)$ on $Bl_0\mathbb{C}^m \setminus E$ such that $S = \sum_{i=1}^m |z_i|^2$. Then the Kähler metric can be written as

$$\omega = \delta_{i\bar{j}}(1 + \frac{1}{S}) + \frac{-1}{S^2} z_i z_{\bar{j}} dz_i \wedge d\bar{z}_j = \delta_{i\bar{j}} + \delta_{i\bar{j}}\frac{1}{S} - \frac{-1}{S^2} z_i z_{\bar{j}} dz_i \wedge d\bar{z}_j.$$

Thus for a large S, the Burns metric looks like Euclidean metric

$$g_{i\bar{j}} = \delta_{i\bar{j}} + O\left(\frac{1}{S}\right).$$

By comparing with Definition 1.3.1, the decay rate τ is 2.

Proposition 2.4.8. $\left(Bl_0\mathbb{C}^2, \omega = \sqrt{-1}\partial\bar{\partial}(\log(S) + S\right)$ does not contain any minimal hypersphere in the family of canonical hyperspheres Σ_S .

Proof. By Theorem 2.2.1, the mean curvature of the family of canonical hyperspheres Σ_S is given by

$$\begin{aligned} H(S, \log(S) + S) &= \frac{-(3S+1)}{3\sqrt{2S}(S+1)} \\ &= \frac{-1}{\sqrt{2S}} - \frac{2S}{\sqrt{2S}(S+1)}. \end{aligned}$$

Clearly H does not vanishes for any S. Therefore $Bl_0\mathbb{C}^2$ does not contain any minimal hypersphere in Σ_S. □

Remark 2.4.9. By Proposition 2.2.7, we can compute the principal curvature of Σ_S with Burns metric,

$$\lambda_1 = \lambda_3 = \frac{-\sqrt{S}}{\sqrt{2}(S+1)} \tag{2.4.6}$$

$$\lambda_2 = \frac{-1}{\sqrt{2S}} \tag{2.4.7}$$

The principal curvatures of Σ_S with Burns metric behaves like the principal curvatures of Euclidean sphere, i.e. $\lambda_i \to \frac{-1}{\sqrt{2S}}$ for large S, which is compatible with the fact that the Burns metric is AE. But if $S \to 0$, then Σ_S does not shrink to a point from all direction like the Euclidean sphere. By (2.4.6) and (2.4.7), it is easy to see that as $S \to 0$, $\lambda_1 = \lambda_3 \to 0$ and $\lambda_2 \to \infty$.

As $S \to 0$, one of the principal directions collapse and all the hypersurfaces converges to to the exceptional divisor, which is holomorphic submanifold of $Bl_0(\mathbb{C}^2)$, so one would naturally expect that the principal curvature vanish there .

Proposition 2.4.10. The basis $\{e_1, e_2, e_3\}$ of Σ_S can be lift to the basis $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ of $\pi_1^{-1}(\Sigma_S)$ where $\tilde{e_1} = -\bar{\rho_2}\partial_{\rho_1} + \partial_\lambda$, $\tilde{e_2} = \partial_{\theta_1}$, $\tilde{e_3} = \rho_2^2\partial_{\theta_1} - (\rho_2^2 + \rho_1^2)\partial_\mu$. Moreover $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\} \to \{\partial_\lambda, \partial_{\theta_1}\}$ as $S \to 0$.

Proof. Given the projection map,

$$\pi_1 : Bl_0\mathbb{C}^2 \to \mathbb{C}^2$$

and $\Sigma_S \subset \mathbb{C}^2$. We have basis the following basis

$$e_1 = -\rho_2\partial_{\rho_1} + \rho_1\partial_{\rho_2}, \quad e_2 = \partial_{\theta_1} + \partial_{\theta_2}, \quad e_3 = \rho_2^2\partial_{\theta_1} - \rho_1^2\partial_{\theta_2}$$

for Σ_S. We can lift these basis to $\pi^{-1}(\Sigma_S)$ in $Bl_0\mathbb{C}^2$ via the map

$$F : \mathbb{C}^2_{(z_1,z_2)} \to U_1^1,$$

$$(\rho_1, \rho_2, \theta_1, \theta_2) \to (\rho_1, \lambda, \theta_1, \mu) := (\rho_1, \frac{\rho_2}{\rho_1}, \theta_1, \theta_2 - \theta_1)\,.$$

The differential of F is given as follows;

$$dF : T\mathbb{C}^2_{(z_1,z_2)} \to TU_1$$

$$dF(\gamma(t)) = \frac{d}{dt}F(\gamma(t)|_{t=0}.$$

Now we have $dF(\partial\rho_2) = \frac{1}{\bar{\rho_1}}\partial_\lambda$, $dF(\partial_{\theta_1}) = \partial_{\theta_1} - \partial_\mu$ and $dF(\partial_{\theta_2}) = \partial_\mu$. We have lift the $\{e_1, e_2, e_3\}$ to $\pi^{-1}(\Sigma_S)$ as follows:

$$\begin{aligned}\tilde{e_1} = dF(e_1) &= dF(-\bar{\rho_2}\partial_{\rho_1} + \bar{\rho_1}\partial_{\rho_2}) \\ &= -\bar{\rho_2}\partial_{\rho_1} + \partial_\lambda\end{aligned}$$

$$\tilde{e_2} = dF(e_2) = dF(\partial_{\theta_1} + \partial_{\theta_2}) = \partial_{\theta_1}$$

$$\begin{aligned}\tilde{e_3} = dF(e_3) &= dF(\rho_2^2\partial_{\theta_1} - \rho_1^2\partial_{\theta_2}) \\ &= \rho_2^2(\partial_{\theta_1} - \partial_\mu) - \rho_1^2\partial_\mu \\ &= \rho_2^2\partial_{\theta_1} - (\rho_2^2 + \rho_1^2)\partial_\mu\end{aligned}$$

Clearly when $\rho_1, \rho_2 \to 0$ we get $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\} \to \{\partial_\lambda, \partial_{\theta_1}\}$.

□

2.4.3 The Eguchi Hanson metric

Consider smooth form

$$\omega = \sqrt{-1}\partial\bar{\partial}(\sqrt{S^2+1} + \log S - \log(\sqrt{S^2+1}+1)) \tag{2.4.8}$$

on $\mathbb{C}^2 \setminus 0$ where $S = |z_1|^2 + |z_2|^2$. The smooth form (2.4.8) extends to the Kähler metric on on $Bl_0\mathbb{C}^2/\Gamma_2$ where $\Gamma_2 = \mathbb{Z}/2\mathbb{Z}$. The metric associated to (2.4.8) is known in literature as the Eguchi-Hanson metric ([12],[21])

Remark 2.4.11. Consider the smooth form $\omega = \sqrt{-1}\partial\bar{\partial}(\log(S) + g(S))$ on $\mathbb{C}^2 \setminus 0$ where $g(S) = \sqrt{S^2+1} - \log(\sqrt{S^2+1}+1)$. By Lemma 2.4.3, Since $g_S(0) = 0$, therefore ω can not be extend to the metirc on whole $Bl_0\mathbb{C}^2$.

Remark 2.4.12. For large S, we have $f(S) = S + O\left(\frac{1}{S}\right)$. Since $\sqrt{S^2+1} \sim S$ and $\log(\sqrt{S^2+1}+1) \sim \log(S+1) \sim \log(S) + \frac{1}{S}$. Therefore for large S we have,

$$\omega = \partial\bar{\partial}f(S) = \partial\bar{\partial}(S + O\left(\frac{1}{S}\right) = \delta_{i\bar{j}} + O\left(\frac{1}{S^2}\right).$$

This shows that the Eguchi Hanson metric fall off to Euclidean metric and the decay rate τ is 4.

Remark 2.4.13. The Eguchi Hanson metric

$$\omega = \sqrt{-1}\partial\bar{\partial}(\sqrt{S^2+1} + \log S - \log(\sqrt{S^2+1}+1))$$

is Ricci flat. Since

$$\log(\det(g)) = \log(f_S(f_S + Sf_{SS})) = 1$$

Thus $R_{i\bar{j}} = -\partial\bar{\partial}\log(\det(g)) = 0 \quad \forall \quad i,j = 1,2$.

Proposition 2.4.14. $Bl_0\mathbb{C}^2/\Gamma_2$ with Eguchi Hanson metric does not contains any minimal hypersphere Σ_S.

Proof. For $f(S) = \sqrt{S^2+1} + \log S - \log(\sqrt{S^2+1}+1)$, by Theorem 2.2.1 we have

$$H(S) = -\frac{2S}{\sqrt{2}(1+S^2)^{\frac{3}{4}}} - \frac{2S^{\frac{3}{2}}(2+S^2)}{(1+S^2)^2}.$$

Clearly H does not vanishes for any S □

Chapter 3

Existence of AE scalar flat Kähler metrics

In this chapter, we prove that if the scalar curvature is non negative then the scalar curvature equation can be reduced to system of nonlinear of ODE. Moreover, we prove the existence of AE scalar flat Kähler metric on $\mathbb{C}^m \setminus B_R(0)$.

3.1 Scalar curvature and system of ODE

In this section, we prove that if we have $U(m)$-invaraint Kähler metrics $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ with non-negative scalar curvature, then we can construct $x = \frac{f_{tt}}{f_t}$ and $y = (2m-1)x + \frac{x_t}{x}$., which are solutions of some system of ODE. We also discuss the converse direction.

Theorem 3.1.1. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a Kähler metric with non negative scalar curvature. Then $R \geq 0$ if and only if

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t \leq m(m-1)(1-x)x\,. \end{cases}$$

where $x = \frac{f_{tt}}{f_t}$ and $y = (2m-1)x + \frac{x_t}{x}$. In particular if the scalar curvature vanishes, then we have

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t = m(m-1)(1-x)x\,. \end{cases} \tag{3.1.1}$$

Proof. Set $\nu = \log f_t$. Then we have $x = \nu_t > 0$, since $f_t > 0$ and $f_{tt} > 0$. Moreover, (2.1.14) implies that

$$\begin{aligned} f_{tt}R &= m(m-1)(1-\nu_t)\nu_t - (m-1)(m-2)\nu_t^2 - 2(m-1)(\nu_{tt}+\nu_t^2) \\ &\quad + \frac{(\nu_{tt}+\nu_t)^2}{\nu_t^2} - \frac{\nu_{ttt}+3\nu_t\nu_{tt}+\nu_t^3}{\nu_t}, \end{aligned}$$

or equivalently

$$f_{tt}R = m(m-1)(1-\nu_t)\nu_t - (2m-1)\nu_{tt} - \left(\frac{\nu_{tt}}{\nu_t}\right)_t . \tag{3.1.2}$$

Since $f_{tt} > 0$ and $R = 0$, so we have

$$m(m-1)(1-\nu_t)\nu_t - (2m-1)\nu_{tt} - \left(\frac{\nu_{tt}}{\nu_t}\right)_t = 0 .$$

As $x = \nu_t$ and $y = (2m-1)x + \frac{x_t}{x}$, we have

$$m(m-1)(1-x) - (2m-1)x_t - \left(\frac{x_t}{x}\right)_t = 0 .$$

Thus we obtain

$$m(m-1)(1-x) = (2m-1)x_t + \left(\frac{x_t}{x}\right)_t = y_t .$$

Also $y = (2m-1)x + \frac{x_t}{x}$ implies that

$$x_t = xy - (2m-1)x^2 .$$

□

From Theorem 3.1.1 we have seen that, given a scalar flat Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ we can construct integral curve $\gamma(t) = (x(t), y(t))$ which satisfy the system (3.1.1). A natural question arises about the converse of Theorem 3.1.1. More precisely, if we have solution $(x(t), y(t))$ of the system (3.1.1), are we able to construct a scalar flat Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ in such a way that $x = \frac{f_{tt}}{f_t} > 0$ and $y = (2m-1)x + \frac{x_t}{x}$. Moreover, is it unique? Assume that $(x(t), y(t))$ is solution to the system (3.1.1) such that $x = (\log f_t)_t$ and $y = (2m-1)x + \frac{x_t}{x}$. Then by integrating $x = (\log f_t)_t$ we get

$$f_t = e^{\int x(t)+C},$$

and we can construct a scalar flat Kähler metric as follows,

$$\begin{aligned}\omega &= \sqrt{-1}(f_t\partial\bar{\partial}t + f_{tt}\partial t \wedge \bar{\partial}t)\\ &= \sqrt{-1}f_t(\partial\bar{\partial}t + x(t)\partial t \wedge \bar{\partial}t)\end{aligned} \tag{3.1.3}$$

Clearly from 3.1.3, we see that the metric we construct from solution $(x(t), y(t))$ of the system (3.1.1) is not unique. In fact corresponding to one solution $(x(t), y(t))$ of the system (3.1.1), we get family of Kähler metrics. The following proposition tells us how these family of Kähler metrics are related to each other.

Proposition 3.1.2. Let $f(t)$ and $u(t)$ be two solutions of the ordinary differential equation $x = \frac{f_{tt}}{f_t}$ on some small interval I, with given initial conditions as follow

$$\begin{cases} x = \frac{f_{tt}}{f_t} \\ f(t_0) = \alpha \\ f_t(t_0) = \beta\,, \end{cases} \quad \text{and} \quad \begin{cases} x = \frac{u_{tt}}{u_t} \\ u(t_0) = a \\ u_t(t_0) = b\,. \end{cases}$$

Then the Kähler metrics ω_f and ω_u has the following relation $\omega_f = \mu\omega_u$

Proof. By Cauchy Theorem, the solution to the ordinary differential equation must be unique. Therefore $f(t) = \mu u(t) + \alpha - \mu a$ with $\mu = \frac{\beta}{b}$ and by 3.1.3 we get $\omega_f = \mu\omega_u$. □

We conclude that from the solution (x, y) of the system (3.1.1) we can construct Kähler metrics $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ but not uniquely. Now we are interested in finding solutions (x, y) of the system (3.1.1)

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t = m(m-1)(1-x)x\,. \end{cases} \tag{3.1.4}$$

such that $x > 0$.

Definition 3.1.3. Consider the nonlinear system of ODE

$$\begin{cases} x_t = f(x, y) \\ y_t = g(x, y). \end{cases} \tag{3.1.5}$$

A point (x_0, y_0) is called equilibrium point the system if $f(x_0, y_0) = 0 = g(x_0, y_0)$.

Definition 3.1.4. An equilibrium point (x_0, y_0) is asymptotically stable if for any given solution $(x(t), y(t))$ of (3.1.5) with initial condition sufficiently close to (x_0, y_0) then

$$\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$$

In general for nonlinear system it is not trivial to find analytic solution but one can linearize the non-system by finding Jacobian matrix of the system around the equilibrium point and the eigenvalues helps to determine the type of the equilibrium point.

Definition 3.1.5. If all the eigenvalues of the Jacobian matrix of the system (3.1.5) at the equilibrium point (x_0, y_0) are negative then the point is asymptotically stable.

Remark 3.1.6. There are many types of equilibrium point and their type can be determine by the eigenvalues of the Jacobian matrix. But due to lack of time we can not present the complete classification of equilibrium points of the nonlinear system and behaviour of the solutions near them.

Lemma 3.1.7. The equilibrium point $(1, 2m-1)$ of the system (3.1.1) is asymptotically stable.

Proof. The Jacobian of the system (3.1.1) is given by

$$J = \begin{bmatrix} y - 2(2m-1)x & x \\ m(m-1)(1-2x) & 0 \end{bmatrix}.$$

$$J|_{(1,2m-1)} = \begin{bmatrix} -(2m-1) & 1 \\ -m(m-1) & 0 \end{bmatrix}.$$

The eigenvalues are solutions of the following equation,

$$\det(J|_{(1,2m-1)} - kI) = k^2 + (2m-1)k + m^2 - m = (k+m)(k+(m-1)).$$

Clearly $k = -m, 1-m$ are negative for all m. □

Remark 3.1.8. Notice that the for the Euclidean metric $(1, 2m-1)$ is solution to the system (3.1.1). By Lemma (3.1.7) we learnt that the scalar flat Kähler metrics corresponding to the solutions of the system (3.1.1) are asymptotically Euclidean.

In order to find solutions of the system (3.1.1) it is convenient to define an affine diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(x, y) = (\tilde{x}, \tilde{y}) = (mx - y + m - 1, y + (1 - m)x - m)\,, \tag{3.1.6}$$

with the inverse given by

$$\Phi^{-1}(\tilde{x}, \tilde{y}) = (\tilde{x} + \tilde{y} + 1, (m - 1)\tilde{x} + m\tilde{y} + (2m - 1)).$$

Lemma 3.1.9. The diffeomorphism defined in (3.1.6) transform system (3.1.1) to the following system

$$\begin{cases} \tilde{x}_t = -m\tilde{x}(1 + \tilde{x} + \tilde{y}) \\ \tilde{y}_t = (1 - m)\tilde{y}(1 + \tilde{x} + \tilde{y})\,. \end{cases} \tag{3.1.7}$$

Proof. By the affine diffeomorphism Φ we have

$$\begin{cases} \tilde{x} = mx - y + m - 1 \\ \tilde{y} = y + (1 - m)x - m \end{cases} \iff \begin{cases} x = \tilde{x} + \tilde{y} + 1 \\ y = (m - 1)\tilde{x} + m\tilde{y} + (2m - 1)\,. \end{cases}$$

Thus we get $x_t = \tilde{x}_t + \tilde{y}_t$. But by (3.1.1), $x_t = x(y - (2m - 1)x) = x(-m\tilde{x} - (m - 1)\tilde{y})$. So we have

$$\tilde{x}_t + \tilde{y}_t = x(-m\tilde{x} - (m - 1)\tilde{y})\,. \tag{3.1.8}$$

Similarly, we obtain

$$(m - 1)\tilde{x}_t + m\tilde{y}_t = m(m - 1)x(-\tilde{x} - \tilde{y})\,. \tag{3.1.9}$$

From (3.1.8) and (3.1.9), we obtain the system of ordinary differential equations

$$\begin{cases} \tilde{x}_t = -m\tilde{x}(1 + \tilde{x} + \tilde{y}) \\ \tilde{y}_t = (1 - m)\tilde{y}(1 + \tilde{x} + \tilde{y}). \end{cases}$$

□

Now we are interested in finding solutions of the the system (3.1.7) which lies above the line $\tilde{x} + \tilde{y} = -1$, since each of them corresponds to a scalar flat $U(m)$-symmetric Kähler metric on $\mathbb{C}^m \setminus 0$ up to constant. Before going to the solutions, we present several examples where we construct solution of the system (3.1.1) for scalar flat Kähler metrics.

Example 3.1.10. For the Euclidean metric, we have the constant integral curve $\gamma(t) = (x(t), y(t)) = (1, 2m-1)$. Applying the affine diffeomorphism Φ, we get $(\tilde{x}, \tilde{y}) = (0, 0)$.

Example 3.1.11. Consider the Burns metric $(Bl_0\mathbb{C}^2, \omega)$ with the scalar flat metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$, where the potential function is given by

$$f(t) = t + e^t\,.$$

We construct the integral curve $\gamma(t) = (x(t), y(t))$ as

$$\begin{aligned} x(t) &= \frac{e^t}{1+e^t}\,, \\ y(t) &= \frac{3e^t}{1+e^t} + \frac{1}{1+e^t}\,, \end{aligned}$$

which satisfy the system

$$\begin{cases} x_t = xy - 3x^2 = \frac{e^t}{(1+e^t)^2} \\ y_t = 2(1-x)x = \frac{2e^t}{(1+e^t)^2}. \end{cases}$$

Next we show that the curve $\Phi \circ \gamma$ is solution of the system (3.1.7). The diffeomorphism Φ transform $\gamma(t) = (x(t), y(t))$ as follows

$$\Phi \circ \gamma = \Phi(x(t), y(t)) = (\tilde{x}(t), \tilde{y}(t)) = (0, \frac{-1}{1+e^t}).$$

Clearly $(\tilde{x}, \tilde{y})$ satisfies the system

$$\begin{cases} \tilde{x}_t = 0 \\ \tilde{y}_t = -\tilde{y}(1+\tilde{y}) = \frac{1}{(1+e^t)^2}\,, \end{cases}$$

Example 3.1.12. Consider the Eguchi Hanson metric $(Bl_0\mathbb{C}^2/\Gamma_2, \omega)$ where $\Gamma_2 = \mathbb{Z}/2\mathbb{Z}$ and the potential function of scalar flat metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ is given by

$$f(t) = t + \sqrt{1+e^{2t}} - \log(1 + \sqrt{1+e^{2t}})\,.$$

We construct the curves

$$\begin{cases} x(t) = \frac{e^{2t}}{1+e^{2t}} \\ y(t) = \frac{2+3e^{2t}}{1+e^{2t}}, \end{cases}$$

which are solution of the following system

$$\begin{cases} x_t = xy - 3x^2 = \frac{2e^{2t}}{(1+e^{2t})^2} \\ y_t = 2x(1-x) = \frac{2e^{2t}}{(1+e^{2t})^2} \,. \end{cases}$$

Applying diffeomorphism Φ, we get the new curves

$$\begin{cases} \tilde{x}(t) = \frac{-1}{1+e^{2t}} \\ \tilde{y}(t) = 0\,, \end{cases}$$

which satisfies the system

$$\begin{cases} \tilde{x}_t = -2\tilde{x}(1+\tilde{x}) = \frac{2e^{2t}}{(1+e^{2t})^2} \\ \tilde{y}_t = 0\,. \end{cases}$$

3.2 Solution along separatices

In this section, we discuss solution of the system (3.1.7) along $\tilde{x} = 0$ or $\tilde{y} = 0$. We reconstruct the scalar flat Kähler metric corresponding to these solutions. Recall that the solution of the system (3.1.7) correspond to rotationally symmetric scalar flat Kähler metrics on $\mathbb{C}^m \setminus 0$ and the integral curves lies above the line $\tilde{x} + \tilde{y} = -1$. Clearly all points on the line $\tilde{x} + \tilde{y} = -1$ are singular points of the system (3.1.7). Since the eigenvalues of the Jacobian J are non zero at origin, so origin is an isolated hyperbolic singular point for $\Phi_* V$, and any other points is regular. The solutions along the separatices can be described quite explicitly and we obtain the expression of the metric given in the following result.

Proposition 3.2.1. Let $m \geq 2$ and $k \in \{m-1, m\}$. Then there exist constants $A > 0$ and $B > -1$ such that

$$f_S = A\sqrt[k]{1 + BS^{-k}},$$

and $\omega = \sqrt{-1}\partial\bar{\partial} f(S)$ defines a Kähler metric on $\mathbb{C}^m \setminus 0$ in the domain $|z|^2 > \sqrt[k]{-B}$ or $|z|^2 > 0$ according to $-1 < B < 0$, $B = 0$ or $B > 0$ respectively. Moreover, ω is scalar flat for all k, A, B, and is Ricci flat whenever $k = m$.

Proof. Consider system (3.1.7), the integral curve $\tilde{x}(t)$ along the axis $\tilde{y} = 0$ with $\tilde{x}(0) = \tilde{x}_0$ which lies above the line $\tilde{x} + \tilde{y} = -1$ can be find by taking integration of the following differential equation

$$\frac{d\tilde{x}}{dt} = -m\tilde{x}(\tilde{x} + \tilde{y} + 1)\,.$$

By separating the variables we get,

$$\frac{\tilde{x}}{1+\tilde{x}} = \frac{\tilde{x}_0}{1+\tilde{x_0}} e^{-mt}\,. \tag{3.2.1}$$

Now along the axis $\tilde{y} = 0$, we have $x = 1 + \tilde{x}$, and by using (3.2.1) we obtain

$$x = \frac{1}{1 - \frac{\tilde{x}_0}{1+\tilde{x_0}} e^{-mt}}\,. \tag{3.2.2}$$

Recall that $\nu_t = x$, so by integrating (3.2.2), we get

$$\nu(t) = t + \frac{1}{m}\log(1 - \frac{\tilde{x}_0}{1+\tilde{x_0}} e^{-mt}) + c\,.$$

Since $\nu = \log(f_t)$, so we have

$$f_t = e^{t+c}\sqrt[m]{1 - \frac{\tilde{x}_0}{1+\tilde{x_0}} e^{-mt}}, \tag{3.2.3}$$

for some constant c. We noted that the solutions with $-1 < \tilde{x}_0 \leq 0$ are defined for all $t > 0$, while solutions with $\tilde{x}_0 > 0$ are defined just for $t > \log \sqrt[m]{\frac{\tilde{x}_0}{1+\tilde{x}_0}}$. Now $f_t = f_S e^t$, we have

$$f_S = A\sqrt[m]{1 + BS^{-m}}\,.$$

where $A = e^c$ and $B = -\frac{\tilde{x}_0}{1+\tilde{x}_0}$. Here A is any positive constant and $B > 0$ for $-1 < \tilde{x}_0 \leq 0$, where $-1 < B < 0$ for $\tilde{x}_0 > 0$. According to these two cases f_S is defined for $S > 0$, or just for $S > \sqrt[m]{-B}$. By (3.2.3) we have

$$f_t^{m-1} f_{tt} = e^{mt+mc}\,. \tag{3.2.4}$$

Comparing (3.2.4) with Proposition 2.1.6, the Kähler metric ω is Ricci flat for any A and B. Similarly, considering the integral curve along the axis $\tilde{x} = 0$ with $\tilde{y}(0) = \tilde{y}_0$, which has to be greater than -1, we find

$$\nu_t = \frac{1}{1 - \frac{\tilde{y}_0}{1+\tilde{y}_0} e^{(1-m)t}}$$

Similarly, in this case solutions with $-1 < \tilde{y}_0 \leq 0$ are defined for all $t > 0$ while solutions with $\tilde{y}_0 > 0$ are defined only for $t > \sqrt[m-1]{\frac{\tilde{y}_0}{1+\tilde{y}_0}}$. Finally, in this case we have

$$f_S = A \sqrt[m-1]{1 + BS^{1-m}}$$

where A any positive number and $B = -\frac{\tilde{y}_0}{1+\tilde{y}_0}$. Notice that A is any positive constant and $B > 0$ for $-1 < \tilde{y}_0 \leq 0$ while $-1 < B < 0$ for $\tilde{y}_0 > 0$. Similar to the previous case, f_S is defined for $S > 0$, or just for $S > \sqrt[m-1]{-B}$, which completes the proof. □

Remark 3.2.2. For $m = 2$ and $k = 1$, along the line $\tilde{x} = 0$ with $\tilde{y}(0) = \frac{-1}{2}$, we obtained the Burns metric on the $Bl_0\mathbb{C}^2$. For $m = k = 2$, Along the line $\tilde{y} = 0$ with $\tilde{x}(0) = \frac{-1}{2}$, we get the Eguchi Hanson metric on $Bl_0\mathbb{C}^2/\Gamma_2$.

By simple calculations we find that the metric of Proposition 3.2.1 is explicitly given by

$$\omega = \sqrt{-1}A\sqrt[k]{1 + BS^{-k}}\left(\partial\bar{\partial}S - \frac{BS^{-k-1}}{1 + BS^{-k}}\partial S \wedge \bar{\partial}S\right),$$

or equivalently

$$\omega = \sqrt{-1}A\sqrt[k]{S^k + B}\left(\partial\bar{\partial}\log S + \frac{S^k}{B + S^k}\partial\log S \wedge \bar{\partial}\log S\right).$$

In the next result, we extend the Kähler metric on $\mathbb{C}^m \setminus 0$ in Proposition 3.2.1 to Kähler metric on $M_k^m = Bl_0\mathbb{C}^m/\Gamma_k$, where $\Gamma_k = \mathbb{Z}/k\mathbb{Z}$.

Proposition 3.2.3. Assume that $B > 0$. Then the Kähler metric on $\mathbb{C}^m \setminus 0$ given by

$$\omega = \sqrt{-1}A\sqrt[k]{S^k + B}\left(\partial\bar{\partial}\log S + \frac{S^k}{B + S^k}\partial\log S \wedge \bar{\partial}\log S\right) \tag{3.2.5}$$

induces the Kähler metric on $M_k^m = Bl_0\mathbb{C}^m/\Gamma_k$, where $\Gamma_k = \mathbb{Z}/k\mathbb{Z}$.

Proof. Recall that the blow up of $\mathbb{C}^m$ at the origin is given by

$$Bl_0\mathbb{C}^m = \{((z_1, \ldots, z_m), [t_1, \ldots, t_m]) \in \mathbb{C}^m \times \mathbb{C}P^{m-1} : z_i t_j - z_j t_i = 0\},$$

with the system of charts

$$U_i^1 = \{((z_1, \ldots, z_m), [t_1, \ldots, z_m]) : t_i \neq 0, z_j = z_i t_j\},$$

for $i, j = 1, \ldots, m$. These charts are biholomorphic to $\mathbb{C}^m$ via the map $\Phi_i : U_i^1 \to \mathbb{C}^m$ defined as

$$((z_1, z_2, \ldots, z_m), [t_1, t_2, \ldots, z_m]) \to \left(z_j, \frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_m}{t_i} \right),$$

with inverse

$$(z_1, z_2, \ldots, z_m) \to ((z_1, z_1 z_2, \ldots, z_1 z_m), [1, z_2, \ldots, z_m]). \tag{3.2.6}$$

In (3.2.6) we have

$$S = |z_i|^2(|z_1^2| + \cdots + |z_{i-1}|^2 + 1 + |z_{i+1}|^2 + \cdots + |z_m|^2).$$

For simplicity we fix U_1, and denote $z_1 = \lambda$ and $\tilde{z} = (z_2, \ldots, z_m) \in \mathbb{C}^{m-1}$. Then $S = |\lambda|^2(1 + |\tilde{z}|^2)$ and the metric in (3.2.5) can be written as

$$\omega = \sqrt{-1} A \sqrt[k]{(|\lambda|^2(1 + |\tilde{z}|^2)^k + B} \left(\partial\bar{\partial} \log(1 + |\tilde{z}|^2) + \frac{|\lambda|^{2k}(1 + |\tilde{z}|^2)^k}{B + |\lambda|^{2k}(1 + |\tilde{z}|^2)^k} \partial \log S \wedge \bar{\partial} \log S \right). \tag{3.2.7}$$

Since k is a positive integer, we have

$$\begin{aligned} |\lambda|^{2k} \partial \log S \wedge \bar{\partial} \log S = |\lambda|^{2k-2} [d\lambda \wedge d\bar{\lambda} + \lambda d\bar{\lambda} \wedge \partial \log(1 + |\tilde{z}|^2) \\ + \bar{\lambda} d\lambda \wedge \bar{\partial} \log(1 + |\tilde{z}|^2) + |\lambda|^2 \partial \log(1 + |\tilde{z}|^2) \wedge \bar{\partial} \log(1 + |\tilde{z}|^2)]. \end{aligned}$$

It is easy to see that ω is smooth on $Bl_0\mathbb{C}^m$ but it is degenerate at point $\lambda = 0$ in the direction of λ as soon as $k \geq 2$. Therefore (3.2.7) does not defines Kähler metric on $Bl_0\mathbb{C}^m$.
On the contrary if we consider the quotient $M_k^m = Bl_0\mathbb{C}^m/\Gamma_k$ with the cyclic group $\Gamma_k = \mathbb{Z}/k\mathbb{Z}$. The action of the group Γ_k on $Bl_0\mathbb{C}^m$ is defined as $[\gamma].(z, [t]) = (e^{2\pi i\gamma} z, [t])$. The quotient map $Bl_0\mathbb{C}^m \to M_k^m$ is defined by $(\lambda, \tilde{z}) \to (\lambda^k, \tilde{z})$. Therefore in local coordinates

$(\mu, \tilde{z})$ we have

$$\begin{aligned}
\omega &= A\sqrt[k]{|\mu|^2(1+|\tilde{z}|^2)^k + B}\Bigg[\partial\bar{\partial}\log((1+|\tilde{z}|^2) \\
&+ \frac{|\mu|^2(1+|\tilde{z}|^2)^k}{k^2(B+|\mu|^2(1+|\tilde{z}|^2)^k)}\partial(\log|\mu|^2(1+|\tilde{z}|^2)^k)\wedge\bar{\partial}(\log|\mu|^2(1+|\tilde{z}|^2)^k))\Bigg] \\
&= A\sqrt[k]{|\mu|^2(1+|\tilde{z}|^2)^k + B}\Bigg[\partial\bar{\partial}\log((1+|\tilde{z}|^2) \\
&+ \frac{(1+|\tilde{z}|^2)^k}{k^2(B+|\mu|^2(1+|\tilde{z}|^2)^k)}[d\mu\wedge d\bar{\mu} + \mu d\bar{\mu}\wedge\partial\log(1+|\tilde{z}|^2)^k \\
&+ \bar{\mu}d\mu\wedge\bar{\partial}\log(1+|\tilde{z}|^2)^k + |\mu|^2\partial\log(1+|\tilde{z}|^2)^k\wedge\bar{\partial}\log(1+|\tilde{z}|^2)^k]\Bigg]
\end{aligned}$$

Clearly ω is smooth and if $\mu = 0$ we have,

$$\omega = A\sqrt[k]{B}\left[\partial\bar{\partial}\log(1+|\tilde{z}|^2) + \frac{(1+|\tilde{z}|^2)^k}{k^2B}d\mu\wedge d\bar{\mu}\right]$$

which is positive definite. Therefore (3.2.5) defines a Kähler metric on the blow up of $\mathbb{C}^m$ at origin quotient by the cyclic group Γ_k . □

3.3 General solutions

In the previous section, we have discussed the solutions of the system of ODE (3.1.7) along the lines $\tilde{x} = 0$ and $\tilde{y} = 0$ and construct the corresponding family of of scalar flat Kähler metrics. As we will see in the next chapter that scalar flat Kähler metrics discussed in Proposition 3.1.1 does not contain any minimal hyperspheres in the family of canonical hyperspheres, so we need to find more solutions of the system (3.1.7), in particular scalar flat Kähler metrics which contains minimal hypersphere .The minimality and stability of hyperspheres in the family of canonical hyperspheres Σ_S will be discussed in details later.

In this section we focus on scalar flat Kähler metrics which correspond to the solution of the system (3.1.7) more generally. We reduce system (3.1.7) to one equation as follows

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{(m-1)}{m}\frac{\tilde{y}}{\tilde{x}},$$

and solve it explicitly. By separation of variable we get,

$$\frac{d\tilde{y}}{\tilde{y}} = \frac{(m-1)}{m}\frac{d\tilde{x}}{\tilde{x}}\,,$$

and so the explicit solution is

$$\tilde{y} = \lambda(\tilde{x})^{\frac{m-1}{m}}\,.$$

This can be written as

$$\frac{(\tilde{y})^m}{(\tilde{x})^{m-1}} = \lambda\,. \tag{3.3.1}$$

Using the substitution, $\tilde{x} = mx - y + m - 1, \tilde{y} = y + (1-m)x - m$, in (3.3.1), we have

$$\frac{(y + (1-m)x - m)^m}{(mx - y + m - 1)^{m-1}} = \lambda\,. \tag{3.3.2}$$

This further implies that the solution (x, y) of the system (3.1.1) has to satisfy (3.3.2). Now we define a function

$$f(x,y) := \frac{(y + (1-m)x - m)^m}{(mx - y + m - 1)^{m-1}}\,. \tag{3.3.3}$$

The equation (3.3.2) can be seen as level set of the function $f(x, y)$, i.e. $L_\lambda(f) = \{(x, y) : f(x, y) = \lambda\}$. By Lemma (3.1.7) we know that all solutions (x, y) of the system (3.1.1) approaches $(1, 2m-1)$ and it is easy to see that each L_λ contains the point $(1, 2m-1)$. Remember that we are interested in only those solutions (x, y) where $x > 0$.

Thus from the above discussion we prove the following result.

Proposition 3.3.1. Each solutions $(x(t), y(t))$ of the system (3.1.1) is contained in one $L_\lambda(f)$.

By Proposition 3.1.2 and Lemma (3.1.7), we know that corresponding to the integral curve $(x(t), y(t))$ satisfying system (3.1.1), we have a family of AE scalar flat Kähler metrics and thus we have the following result.

Theorem 3.3.2. There exist AE scalar flat Kähler metric $\omega_\lambda = \sqrt{-1}\partial\bar{\partial}f_\lambda(t)$ up to constant on $\mathbb{C}^m \setminus B_{R_\lambda}(0)$ corresponding to each level curve $L_\lambda(f)$.

The level sets $L_0(f)$ and $L_\infty(f)$ are the lines $y = (m-1)x + m$ and $y = mx + (m-1)$ respectively. Each of these line pass through $(1, 2m-1)$.

Example 3.3.3. By Example 3.1.11, we have the integral curves of the Burns metric given as follows,

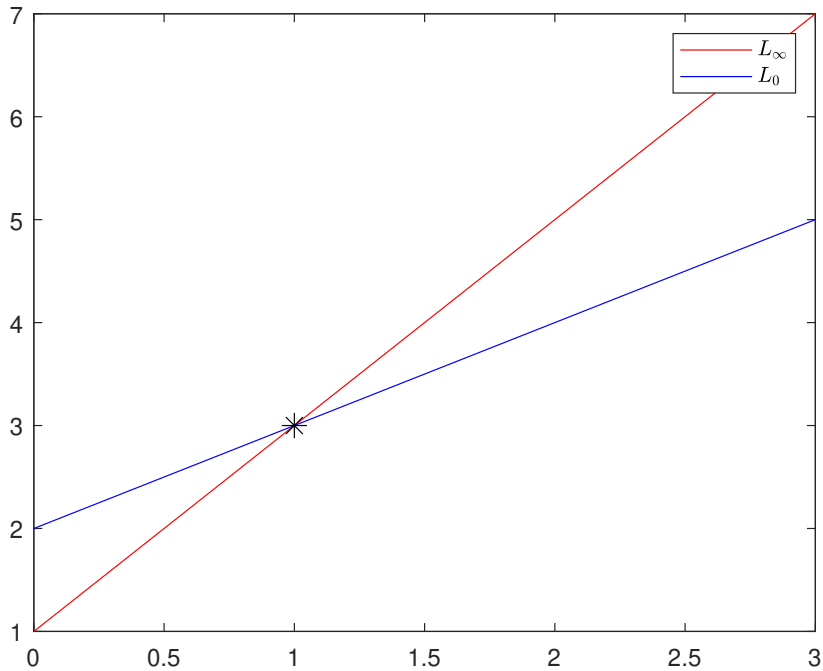
$$x(t) = \frac{e^t}{1+e^t}$$
$$y(t) = \frac{3e^t+1}{1+e^t}.$$

Clearly $(x(t), y(t))$ satisfy the linear equation $y = 2x+1$. Thus the Burns metric is contained in the level set $L_\infty(f)$.

Example 3.3.4. By Example 3.1.12, we have the integral curves for Eguchi Hanson metric given below

$$\begin{cases} x = \frac{e^{2t}}{1+e^{2t}} \\ y = \frac{2+3e^{2t}}{1+e^{2t}}, \end{cases}$$

Clearly $(x(t), y(t))$ satisfy the linear equation $y = x+2$. Thus The Eguchi Hanson metric is contained in $L_0(f)$.



Chapter 4

Stable minimal hyperspheres in scalar flat Kähler manifolds

In this chapter, we consider the family of canonical hyperspheres Σ_S^{2m-1} with rotationally symmetric Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$. We discuss the possible conditions for the family contains a minimal stable hypersphere. We also prove that the AE scalar flat Kähler metrics corresponding to the level set $L_\lambda(f)$ contains minimal hypersphere in Σ_S when λ belongs to some certain interval.

4.1 Stable minimal hyperspheres

In this section we consider the Kähler metric ω and discuss the condition for the minimal hyperspheres $\Sigma_{S_0}^{2m-1}$ in the family of canonical hyperspheres.

By considering the change of coordinates $t = \log S$, the formula for mean curvature and area of Σ_S^{2m-1} given in Theorem 2.3.6 and Proposition 2.3.3 respectively changes as follow

$$H(t, f(t)) = \frac{-1}{(2m-1)\sqrt{2}f_t(f_{tt})^{\frac{3}{2}}}(2(m-1)f_{tt}^2 + f_{ttt}f_t)\,, \tag{4.1.1}$$

$$V(\Sigma^{2m-1}) = \sqrt{2f_{tt}}(2f_t)^{m-1}V_E(\Sigma_1^{2m-1})\,. \tag{4.1.2}$$

Or equivalently

$$V(\Sigma^{2m-1}) = (2f_t)^{\frac{2m-1}{2}}\sqrt{x(t)}V_E(\Sigma_1^{2m-1})\,,$$

where $x(t) = \frac{f_{tt}}{f_t}$.
Consider the hypersphere of radius $S_0 = e^{t_0}$ in Σ_S^{2m-1}. Clearly $\Sigma_{S_0}^{2m-1}$ is minimal if and only if

$$(2(m-1)f_{tt}^2 + f_{ttt}f_t)\big|_{t=t_0} = 0 \iff f_{ttt}|_{t=t_0} = \left.\frac{-2(m-1)f_{tt}^2}{f_t}\right|_{t=t_0} . \tag{4.1.3}$$

The following lemma describes the condition when this minimal hypersphere is stable.

Lemma 4.1.1. The minimal hypersphere of radius $S_0 = e^{t_0}$ in the family of canonical hyperspheres Σ_S^{2m-1} is stable if and only if

$$\left[f_t^2 f_{tttt} - 2(m-1)(4m-3)f_{tt}^3\right]_{t=t_0} \geq 0 . \tag{4.1.4}$$

Proof. We focus only on the terms which depend on the radius of the hyperspheres. Differentiating (4.1.2) implies that

$$\begin{aligned} A_t &= 2(m-1)\sqrt{f_{tt}}f_{tt}f_t^{m-2} + \frac{f_t^{m-1}f_{ttt}}{2\sqrt{f_{tt}}} \\ &= \frac{f_t^{m-2}}{\sqrt{2f_{tt}}}\left(2(m-1)f_{tt}^2 + f_t f_{ttt}\right) . \end{aligned}$$

Now differentiate again and using (4.1.3) we have

$$\begin{aligned} A_{tt}(t_0) &= \frac{1}{\sqrt{f_{tt}}}((4m-3)f_{tt}f_{ttt} + f_t f_{tttt}) \\ &= \frac{f_t^{m-2}}{\sqrt{f_{tt}}}(-2(m-1)(4m-3)\frac{f_{tt}^3}{f_t} + f_t f_{tttt}) \\ &= \frac{f_t^{m-2}}{\sqrt{f_{tt}}f_t}(-2(m-1)(4m-3)f_{tt}^3 + f_t^2 f_{tttt}) . \end{aligned}$$

Thus the minimal hypersphere of radius S_0 in the family of canonical hyperspheres Σ_S^{2m-1} is stable if and only if

$$\left[f_t^2 f_{tttt} - 2(m-1)(4m-3)f_{tt}^3\right]_{t=t_0} \geq 0 .$$

□

In the above, we discussed the stability condition for the minimal hyperspheres $\Sigma_{S_0}^{2m-1}$ with respect to derivative of potential function of the Kähler metric ω. Now we translate the minimality and stability conditions of hypersphere of radius $S_0 = e^{t_0}$ in the family Σ_S^{2m-1} in terms of $x(t))$ and $y(t)$.

Lemma 4.1.2. Assume that there is a minimal hypersphere of radius $S_0 = e^{t_0}$ in the family of canonical hyperspheres Σ_S^{2m-1}. Then solution of the system (3.1.1) at t_0 is given by $(x(t_0), 0)$, i.e. $y(t_0) = 0$.

Proof. Set $\nu = \log f_t$ then $x = \nu_t$. By minimality, we have

$$\begin{aligned} & H(t_0) = 0 \\ \iff & (2(m-1)f_{tt}^2 + f_{ttt}f_t) = 0 \\ \iff & 2(m-1)((e^\nu)_t)^2 + e^\nu(e^\nu)_{tt} = 0\,. \end{aligned}$$

Now consider

$$\begin{aligned} 2(m-1)((e^\nu)_t)^2 + e^\nu(e^\nu)_{tt} &= 2(m-1)(\nu_t e^\nu)^2 + e^\nu(\nu_{tt}e^\nu + (\nu_t)^2 e^\nu \\ &= (e^\nu)^2((2m-1)(\nu_t)^2 + \nu_{tt}) \\ &= (e^\nu)^2((2m-1)x^2 + x_t) \\ &= (e^\nu)^2\, xy\,, \end{aligned}$$

where the last equality is by using (3.1.1). But $(e^\nu)^2 > 0$ and $x > 0$, so $H(t_0) = 0 \iff y(t_0) = 0$. □

Lemma 4.1.3. The minimal hypersphere of radius $S_0 = e^{t_0}$ in the family of canonical hyperspheres Σ_S^{2m-1} is stable if and only if

$$\left.\frac{dy}{dx}\right|_{t=t_0} \leq 0\,.$$

Proof. By Lemma 4.1.1, the minimal hypersphere of radius $S_0 = e^{t_0}$ is stable if and only if

$$f_{tttt}f_t^2 - C(m)f_{tt}^3 \geq 0\,,$$

where $C(m) = 2(m-1)(4m-3)$.

We recall the notation $\nu = \log f_t$ and $x = \nu_t$. Now we have

$$\begin{aligned}
f_{tttt}f_t^2 - C(m)f_{tt}^3 &= (e^\nu)_{ttt}e^{2\nu} - C(m)(e^\nu)_t)^3 \\
&= e^{2\nu}(\nu_{ttt}e^\nu + \nu_{tt}\nu_t e^\nu + 2(\nu_t)\nu_{tt}e^\nu + (\nu_t)^3 e^\nu) - C(m)\nu_t^3(e^\nu)^3 \\
&= e^{3\nu}(\nu_{ttt} + \nu_{tt}\nu_t + 2(\nu_t)\nu_{tt} - (C(m)-1)(\nu_t)^3) \\
&= e^{3\nu}(x_{tt} + 3xx_t - (C(m)-1)x^3) \\
&= xy_t + x_t y - (4m-5)xx_t - (C(m)-1)x^3) \\
&= x_t(y - (4m-5)x) + xy_t - (C(m)-1)x^3) \\
&= x(y-(4m-5)x)(y-(2m-1)x)) + xy_t - (C(m)-1)x^3) \\
&= x\left((y-(4m-5)x)(y-(2m-1)x) + y_t - (C(m)-1)x^2)\right) \\
&= x(y^2 - 6(m-1)xy + y_t)\,,
\end{aligned}$$

where we have used $x_{tt} = (xy - (2m-1)x^2)_t$. But $x > 0$, we then obtain

$$f_{tttt}f_t^2 - C(m)f_{tt}^3 \geq 0 \iff y_t \geq -y^2 + 6(m-1)xy\,. \tag{4.1.5}$$

At $t = t_0$, by Lemma 4.1.2, $y(t_0) = 0$ and thus (4.1.5) implies that $y_t \geq 0$. Moreover, $x_t = xy - (2m-1)x^2$ implies that $x_t|_{t=t_0} < 0$. Thus we have

$$\left.\frac{dy}{dx}\right|_{t=t_0} = \left.\frac{dy}{dt}\frac{dt}{dx}\right|_{t=t_0} \leq 0\,.$$

□

If we consider the scalar flat Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$, then the condition for the stability of minimal hypersphere of radius $S_0 = e^{t_0}$ is given in the following result.

Lemma 4.1.4. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a scalar flat Kähler metric. A minimal hypersphere of radius $S_0 = e^{t_0}$ in the family of canonical hyperspheres Σ_S^{2m-1} is stable if and only if $0 < x(t_0) \leq 1$.

Proof. By Theorem 3.1.1, we can construct the integral curves $x = \frac{f_{tt}}{f_t} > 0$ and $y = 3x + \frac{x_t}{x}$ which satisfy the system

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t = m(m-1)(1-x)x\,. \end{cases} \tag{4.1.6}$$

We write system (4.1.6) as follow

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{m(m-1)(1-x)}{y-(2m-1)x}\,.$$

By Lemma 4.1.2, $y(t_0) = 0$, and so we have

$$\left.\frac{dy}{dx}\right|_{t=t_0} = \frac{-m(m-1)(1-x(t_0))}{(2m-1)x(t_0)}\,. \tag{4.1.7}$$

By Lemma 4.1.3, Σ_{S_0} is stable if and only if

$$\frac{dy}{dx}|_{t=t_0} \leq 0\,.$$

Hence (4.1.7) implies that $0 < x(t_0) \leq 1$. □

By Lemmas 4.1.2 and 4.1.4, we conclude the following result.

Proposition 4.1.5. Let $\omega = \sqrt{-1}\partial\bar{\partial}f(t)$ be a scalar flat Kähler metric on $\mathbb{C}^m \setminus 0$. Then Σ_{S_0} is a stable minimal hypersphere if and only if $(x(t), y(t))$ satisfy the following system

$$\begin{cases} x_t = xy - (2m-1)x^2 \\ y_t = m(m-1)(1-x)x \\ (x(t_0), y(t_0)) = (x_0, 0)) \quad 0 < x_0 \leq 1\,. \end{cases}$$

Remark 4.1.6. We have constructed the scalar flat Kähler metric along the line $\tilde{y} = 0$ and $\tilde{x}(t_0) = x_0$ in Proposition 3.2.1 which satisfy the system 3.1.1. Since $\tilde{y} = 0$ implies that $y = (m-1)x + m$. But $x > 0$, so we must have $y(t_0) \neq 0$, otherwise $y(t_0) = 0 \iff x(t_0) = \frac{m}{1-m} < 0$, a contradiction. Similar result arises when we consider he case $\tilde{x} = 0$ and $\tilde{y}(t_0) = y_0$. Therefore, by Lemma 4.1.2 the metric in Proposition 3.2.1 does not contain minimal hypersphere Σ_{S_0} in the family of canonical hyperspheres.

4.2 Existence of stable minimal hyperspheres in asymptotically Euclidean manifolds $\mathbb{C}^m \setminus B_{R_\lambda}(0)$

In this section, we discuss that the scalar flat Kähler metric in section 3.3 contains minimal stable hypersphere when the level set of the function given in (3.3.3) intersect the x-axis.

By Proposition 3.3.1, we have seen that the solutions of the system 3.1.1 contained in the level set of the function

$$f(x,y) := \frac{(y+(1-m)x-m)^m}{(mx-y+m-1)^{m-1}}\,. \tag{4.2.1}$$

By Lemma 4.1.2, we know that the hypersphere of radius $S_0 = e^{t_0}$ in $\mathbb{C}^m \setminus B_{R_\lambda}(0)$, is minimal if $y(t_0) = 0$. Thus for $(x(t_0), y(t_0)) = (x_0, 0)$ we have

$$\lambda(x_0) = \frac{((1-m)x_0 - m)^m}{(mx_0 + m - 1)^{m-1}},$$

Lemma 4.2.1. $\big(\mathbb{C}^m \setminus B_{R_\lambda}(0), \omega_\lambda = \sqrt{-1}\partial\bar{\partial} f_\lambda(t)\big)$ where ω_λ is the AE scalar flat Kähler metric corresponding to the level set L_λ. The hypersphere Σ_{S_0} of radius $S_0 = e^{t_0}$ at $t = t_0$ is minimal if and only if

$$\lambda(x_0) = \frac{((1-m)x_0 - m)^m}{(mx_0 + m - 1)^{m-1}},$$

Theorem 4.2.2. $\big(\mathbb{C}^m \setminus B_{R_\lambda}(0), \omega_\lambda = \sqrt{-1}\partial\bar{\partial} f_\lambda(t)\big)$ where ω_λ is the AE scalar flat Kähler metric corresponding to the level set L_λ. The minimal hypersphere Σ_{S_0} of radius $S_0 = e^{t_0}$ at $t = t_0$ is stable if and only if $\lambda \in [(-1)^m(2m-1), \frac{(-m)^m}{(m-1)^{m-1}})$.

Proof. By Lemma 4.2.1, the hypersphere Σ_{S_0} is minimal if and only if

$$\lambda(x_0) = \frac{((1-m)x_0 - m)^m}{(mx_0 + m - 1)^{m-1}}\,.$$

By Lemma 4.1.4, we have that Σ_{S_0} is stable if and only if $0 < x_0 \leq, 1$ which gives $\lambda \in [(-1)^m(2m-1), \frac{(-m)^m}{(m-1)^{m-1}}]$.

□

Remark 4.2.3. It is important to notice that the level curves L_λ for $\lambda \in [(-1)^m(2m-1), \frac{(-m)^m}{(m-1)^{m-1}}]$ cross the x-axis at two points which means that there exist two minimal hyperspheres in $\mathbb{C}^m \setminus B_{R_\lambda(0)}$ in which one is stable and the other is not.

In general it is not easy to find the domain of the metric but in dimension 2 we can compute the radius of the ball B_{R_λ}. In the following proposition we reprove the fact that the scalar flat metrics are AE and find the explicit domain for the metric.

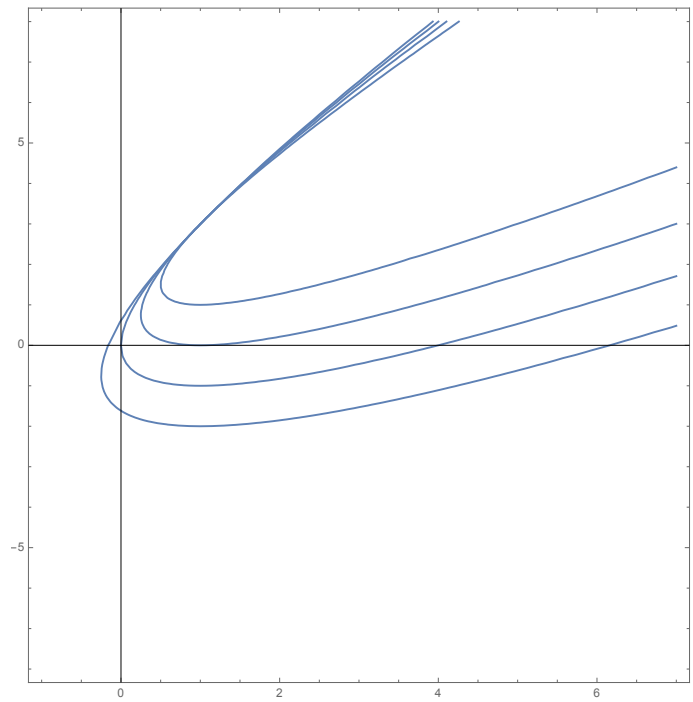


Figure 4.1: Level curves of $f(x, y)$ for $m = 2$

Proposition 4.2.4. Let $m = 2$ and $\lambda \in [3, 4)$ then the scalar flat Kähler metrics corresponding to the $L_\lambda(f)$ are AE and $R_\lambda = e^{T_\lambda}$ where

$$T_\lambda = \frac{1}{2}\left(\frac{-\pi\lambda}{\sqrt{-(-4+\lambda)\lambda}} - \log(-1 + \frac{4}{\lambda})\right).$$

Proof. In complex dimension 2, the equation (3.3.2) becomes

$$(y - x - 2)^2 = \lambda(2x - y + 1),$$

which further implies that

$$y = x + 2 - \frac{\lambda}{2} \pm \sqrt{\lambda(x-1) + \frac{\lambda^2}{4}}\,. \tag{4.2.2}$$

Putting this value of y in the first equation of (3.1.1) we have

$$\begin{aligned} \frac{dx}{dt} &= xy - 3x^2 \\ &= x\left(-2x + 2 - \frac{\lambda}{2} \pm \sqrt{\lambda(x-1) + \frac{\lambda^2}{4}}\right). \end{aligned} \tag{4.2.3}$$

We denote

$$F_{\pm}(x) = x\left(-2x + 2 - \frac{\lambda}{2} \pm \sqrt{\lambda(x-1)) + \frac{\lambda^2}{4}}\right).$$

Obviously, $F_+(1) = 0$ and $F_{\pm}(1 - \frac{\lambda}{4}) = 0$. Now (4.2.3) implies that

$$\frac{dx}{dt} = F_{\pm}(x) \implies dt = \frac{1}{F_{\pm}(x)}dx\,. \tag{4.2.4}$$

By denoting

$$t_+(x) = \int_{\alpha}^{x_\lambda(t)} \frac{1}{F_+(x)}dx \quad \text{where} \quad \alpha \in \left(1 - \frac{\lambda}{4}, 1\right), \tag{4.2.5}$$

$$t_-(x) = \int_{\beta}^{x_\lambda(t)} \frac{1}{F_-(x)}dx \quad \text{where} \quad \beta \in \left(1 - \frac{\lambda}{4}, \infty\right), \tag{4.2.6}$$

the Taylor expansion of $F_+(x)$ around x=1 is given by

$$F_+(x) \simeq -(x-1) - (1 + \frac{1}{\lambda})(x-1)^2 + \frac{(2-\lambda)}{\lambda^2}(x-1)^3 + O(x-1)^4\,.$$

For the choice of α, (4.2.5) implies that

$$t_+(x) = \int_{1-\frac{\lambda}{4}}^{x_\lambda(t)} \left(\frac{-1}{x-1} + O(x-1)\right)dx = -\log(1-x) + O((x-1)^2).$$

Clearly $x \to 1^-$ implies that $t \to \infty$.

On the other hand for $\lambda \in [3, 4)$, (4.2.6) implies that

$$\begin{aligned}\lim_{x\to\infty} t_-(x) &= \lim_{x\to\infty} \int_{1-\frac{\lambda}{4}}^{x_\lambda(t)} \frac{dx}{x\left(-2x + 2 - \frac{\lambda}{2} - \sqrt{\lambda(x-1)) + \frac{\lambda^2}{4}}\right)} \\ &= \frac{1}{2}\left(\frac{-\pi\lambda}{\sqrt{-(-4+\lambda)\lambda}} - \log(-1 + \frac{4}{\lambda})\right) = T_\lambda\,.\end{aligned}$$

Since $t = \log R^2$ so we get $R_\lambda = e^{T_\lambda}$. □

By Theorem 4.2.2, we have that when $m = 2$ the scalar flat Kähler metrics discussed in Section 3.3 contains minimal stable hypersphere if $\lambda \in [3, 4)$. Moreover we have explicitly compute the domain of the metric in Proposition 4.2.4. The following proposition tell us how the domain of the metric changes when λ approaches the extreme of the the interval $[3, 4)$.

Proposition 4.2.5. The radius of the ball B_{R_λ} behaves in the following way:

$$R_\lambda = \begin{cases} < \infty & \text{when} \quad \lambda \to 3 \\ 0 & \text{when} \quad \lambda \to 4\,. \end{cases}$$

Proof. From Proposition 4.2.4, we have

$$\lim_{x\to\infty} t_-(x) = T_\lambda = \frac{1}{2}\left(\frac{-\pi\lambda}{\sqrt{-(-4+\lambda)\lambda}} - \log(-1+\frac{4}{\lambda})\right),$$

which gives

$$R_\lambda = e^{T_\lambda}.$$

Clearly if $\lambda \to 3$ we have, $T_\lambda \to \frac{1}{2}(-\sqrt{3}\pi - \log 3)$ and if $\lambda \to 4$ we get, $T_4 \to -\infty$. Therefore we conclude that as $\lambda \to 4$, the radius of the ball B_{R_λ} shrink to a point 0. □

Fu *et*. al [14] in 2016 presented the following result in which they proved the asymptotic behaviour of the potential function for the AE scalar flat Kähler metric.

Theorem 4.2.6. [14] There exist a one parameter family of functions $t \to f_a(t)$ defined on $\mathbb{R}$ and smoothly depending on the parameter $a > 0$, such that the metric associated to the Kähler form ω_f is complete and scalar flat on $\mathbb{C}^n \setminus 0$. Moreover, the function f_a has the following expansion as $t \to \infty$

$$f_a(t) = \begin{cases} |w|^2 - \frac{na^{n-1}}{(n-1)(n-2)}|w|^{4-2n} + \frac{a^n}{n}|w|^{2-2n} + O(|w|^{-2n}) & \text{for} \quad n \geq 3 \\ |w|^2 + 2a\log|w|^2 + \frac{a^2}{2|w|^2} + O(|w|^{-4}) & \text{for} \quad n = 2,. \end{cases}$$

In particular, the metric is AE at infinity, and as $t \to -\infty$, we have the following expansion

$$f_a(t) = a\log|w|^2 - \frac{2a}{n(n-1)}\log(-\log|w|^2) + O(\frac{1}{\log|w|^2})$$

where $t = \log|w|^2$.

In the next Lemma, we construct of (x, y) for the scalar flat Kähler metric given in Theorem 4.2.6.

Lemma 4.2.7. For $m = 2$, the asymptotic curves for the AE scalar flat Kähler metric given in Theorem 4.2.6 behaves in the following way,

$$\begin{cases} (x(t), y(t)) \to (1, 3) & \text{when} \quad t \to \infty \\ (x(t), y(t)) \to (0, 0) & \text{when} \quad t \to -\infty \end{cases}$$

Proof. At infinity the asymptotic behavior of the potential function is given by

$$f_a(t) = e^t + 2at + \frac{a^2 e^{-t}}{2} + O(e^{-2t}) \,.$$

which gives the asymptotic curves as follows,

$$\begin{aligned} x(t) &= \frac{(f_a)_{tt}}{(f_a)_t} = \frac{e^t + \frac{a^2 e^{-t}}{2} + O(e^{-2t})}{e^t + 2a - \frac{a^2 e^{-t}}{2} + O(e^{-2t})} \\ y(t) &= 3x(t) + \frac{x_t}{x} = 3 - \frac{4a(a - 2e^t)}{a^2 - 4ae^t - 2e^{2t}} - \frac{2a^2}{a^2 + 2e^{2t}} \end{aligned}$$

It is easy to see that as $t \to \infty$ we have $(x(t), y(t)) \to (1, 3)$.
On the other hand near the origin we have the asymptotic behaviour of the potential function given as follows

$$f_a(t) = at - a\log(-t) + O(\frac{1}{t}).$$

The asymptotic curves near the origin are given as follows,

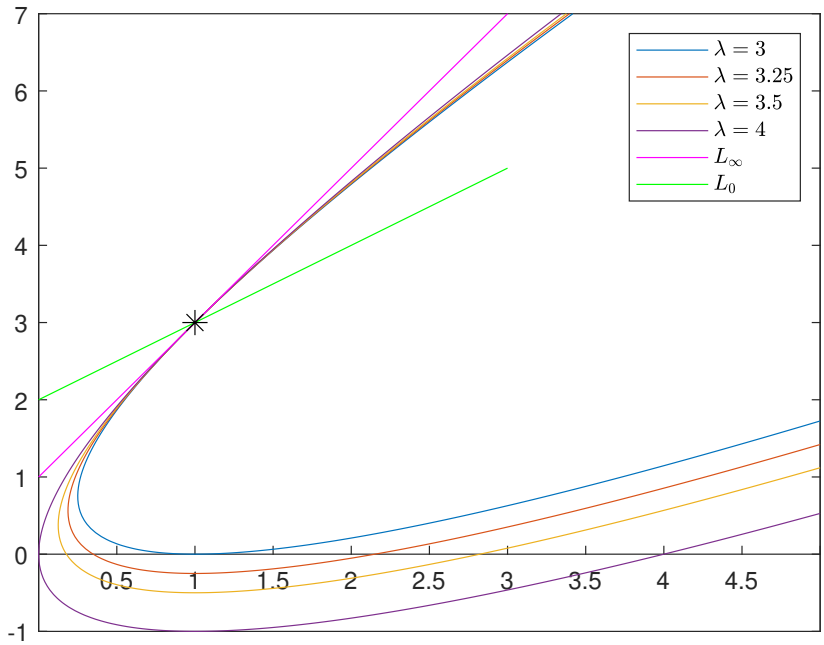
$$\begin{aligned} x(t) &= \frac{1}{t(t-1)} \\ y(t) &= \frac{2}{t-1} - \frac{1}{t} \end{aligned}$$

As $t \to -\infty$ then $(x(t), y(t)) \to (0, 0)$. □

By Lemma 4.2.7 and Proposition 4.2.5 we conclude the AE scalar flat Kähler metric given in Theorem 4.2.6 for $m = 2$ is contained in the level set L_4. In other words we have the following proposition.

Proposition 4.2.8. For $m = 2$ the Kähler metric ω_λ corresponding to the level curve L_λ approaches to ω_{f_a} as $\lambda \to 4$.

The following graph represent the level set L_λ for some λ, and as its clear from the graph when $\lambda \to 4$ and L_4 approaches the origin. The figure contains only those level sets which are interested for us, i.e. for which $(x > 0)$. Notice that every level set contained the point $(1, 3)$, which means that all the scalar flat Kähler metrics corresponding to the level sets are asymptotically Euclidean. The level set L_0 and L_∞ contained the Burns and Eguchi Hanson metric if $x \to 1^{-1}$. The level set L_4 passing through the origin is level set of the scalar flat Kähler metric (PMY) given [14].



4.3 Penrose Inequality in complex dimension 2

So far, we have proved the existence of minimal hyperspheres in scalar flat AE Kähler manifold in any dimension. In this section, we compute the ADM mass and volume of the minimal hypersphere in complex dimension 2.

Proposition 4.3.1. The volume of the hyperspheres Σ_S with $\omega = \sqrt{-1}\partial\bar{\partial}f(t))$ at any point is given by

$$A(t) = \sqrt{x(t)}(2f_t)^{\frac{3}{2}} V_E(\Sigma_1^3). \tag{4.3.1}$$

where $V_E(\Sigma_1^3) = 2\pi^2$ is the Euclidean volume of unit 3-sphere.

From the proof of Theorem 3.1.1 we recall that $\nu = \log f_t$ and $\nu_t = x(t)$, which gives the relation between the derivatives of the potential function as follows,

$$f_t = e^{\nu}, \quad f_{tt} = x(t) f_t \tag{4.3.2}$$

In order to find the volume of Σ_{S_0}, we only need the expression of the potential function f at $x_0(\lambda)$. By equation (4.2.4) we have ,

$$dt = \frac{1}{F_{\pm}(x)} dx\,.$$

Multiply both sides by $x(t)$ and integrating we get,

$$\int_{t_0}^{t} x(t) dt = \int_{x(t_0)}^{x(t)} \frac{x(t)}{F_{\pm}(x)} dx. \tag{4.3.3}$$

Notice that $\nu = \log f_t$ and $\nu_t = x(t)$ implies that

$$\nu(t) = \int_{t_0}^{t} x(t) dt \tag{4.3.4}$$

By (4.3.3) and (4.3.4) we get,

$$\nu(x(t)) = \int_{x(t_0)}^{x(t)} \frac{x(t)}{F_{\pm}(x)} dx = \int_{x(t_0)}^{x(t)} \frac{dx}{2 - 2x - \frac{\lambda}{2} \pm \sqrt{\lambda(x-1) + \frac{\lambda^2}{4}}}$$

We choose different normalization and get,

$$\nu_{-}(x(t)) = \int_{1-\frac{\lambda}{4}}^{x(t)} \frac{x(t)}{F_{-}(x)} dx = -\log\left(1 + \sqrt{\frac{4(x-1)+\lambda}{\lambda}}\right) \tag{4.3.5}$$

$$\nu_{+}(x(t)) = \int_{1-\frac{\lambda}{4}}^{x(t)} \frac{x(t)}{F_{+}(x)} dx = -\log\left(1 - \sqrt{\frac{4(x-1)+\lambda}{\lambda}}\right) \tag{4.3.6}$$

Proposition 4.3.2. Given $(M_{R_\lambda,\infty}, \omega_\lambda)$ where ω_λ is AE scalar flat Kähler metric corresponding to L_λ. Then L_λ contains minimal hypersphere at $x_0(\lambda)$ where

$$x_0(\lambda) = \lambda - 2 \pm \sqrt{(\lambda - 3)\lambda} \tag{4.3.7}$$

where $\lambda \in [3,4)$. Moreover, the volume of these minimal hyperspheres can be computed as follows:

$$V(x_0(\lambda)) = \sqrt{x_0(\lambda)}(2e^{\nu_-(x_0(\lambda))})^{\frac{3}{2}} V_E(\Sigma_1^3) \tag{4.3.8}$$

where

$$\nu_-(x_0(\lambda)) = -\log\left(1 + \frac{\sqrt{4x_0(\lambda)+\lambda-4}}{\lambda}\right).$$

The minimal hypersphere at $x_0(\lambda) = \lambda - 2 - \sqrt{(\lambda-3)\lambda}$ is stable and its volume behaves in the following way:

$$V(x_0(\lambda)) = \begin{cases} V_E(\Sigma_1^3) & \lambda \to 3 \\ 0 & \lambda \to 4 \end{cases}$$

Proof. By Lemma 4.2.1, we know that Σ_{S_0} is minimal if and only if

$$\lambda(x_0) = \frac{(x_0+2)^2}{2x_0+1}.$$

$$x_0(\lambda) = \lambda - 2 \pm \sqrt{(\lambda-3)\lambda}. \tag{4.3.9}$$

Clearly the minimal hypersphere at $x_0(\lambda) = \lambda - 2 - \sqrt{(\lambda-3)\lambda}$ is stable. By (4.3.2) and 4.3.5 we have,

$$V(x_0(\lambda)) = \sqrt{x_0(\lambda)}(2e^{\nu_-(x_0(\lambda))})^{\frac{3}{2}} V_E(\Sigma_1^3) \tag{4.3.10}$$

□

Theorem 4.3.3. The ADM mass of the AE Kähler manifold $M_{R_\lambda,\infty}$ is $\frac{\lambda}{2}$.

Proof. Taylor expansion of $f_t(x)$ and $f_{tt}(x)$ around $x = 1^-$

$$f_t(x) \sim -\frac{\lambda}{2(x-1)} - \frac{1}{2} + \frac{1}{2\lambda}(x-1) - \frac{1}{\lambda^2}(x-1)^2 + O(x-1)^3$$

$$f_{tt}(x) \sim -\frac{\lambda}{2(x-1)} - \left(\frac{1+\lambda}{2}\right) + \left(\frac{-1}{2} + \frac{1}{2\lambda}\right)(x-1) + \frac{\lambda-2}{2\lambda^2}(x-1)^2 + O(x-1)^3$$

$$m_{ADM} = \lim_{x\to 1^-}(f_t(x) - f_{tt}(x)) = \lim_{x\to 1^-}\left(\frac{\lambda}{2} + O(x-1)\right) = \frac{\lambda}{2}$$

□

Theorem 4.3.4. For $m = 2$, the AE Kähler manifold $M_{R_\lambda,\infty}$ satisfies the Riemannian Penrose inequality,

$$m_{ADM} \geq \frac{1}{2}\left(\frac{V_{\Sigma^3(x_0)}}{V_E(\Sigma_1^3)}\right)^{\frac{2}{3}}.$$

$$\begin{aligned}\frac{\lambda}{2} &\geq e^{\nu}(x)^{\frac{1}{3}} \\ &= \frac{\left(\lambda - 2 - \sqrt{\lambda(\lambda-3)}\right)^{\frac{1}{3}}}{\left(1 + \sqrt{\frac{4\left(\lambda-3-\sqrt{\lambda(\lambda-3)}\right)+\lambda}{\lambda}}\right)}\end{aligned}$$

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