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Area functional and relaxation: an approach in dimension 2 and codimension 2 via strict BV-convergence

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Contents

Introduction

In this thesis we address the problem of relaxation of the Cartesian area functional with respect to the strict convergence in BV for maps $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. The relaxation technique allows to extend the notion of non-parametric area of C^1-maps to more general, possibly singular, maps. The existence of discontinuities for a map u can be interpreted at the level of its graph as the presence of "holes" and so computing the relaxed area consists in finding the "most convenient" way to fill these holes, by means of surface area. As opposite to the scalar case, that has been completely understood, the 2-codimensional one, that we are considering here, turns out to be very challenging and many open questions are still left.
The content of the thesis is based on results contained in [3], [4] and [14], which have been obtained during the period of Ph.D. at SISSA (International School for Advanced Studies) in Trieste, in collaboration with Giovanni Bellettini and Riccardo Scala.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $v = (v_1, v_2) : \Omega \to \mathbb{R}^2$ be a map of class $C^1(\Omega; \mathbb{R}^2)$. The area functional $\mathcal{A}(v; \Omega)$ computes the 2-dimensional Hausdorff measure $\mathcal{H}^2$ of the graph

$$G_v := \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\} \tag{0.0.1}$$

of v, a Cartesian 2-manifold in $\Omega \times \mathbb{R}^2 \subset \mathbb{R}^4$, and is given by

$$\mathcal{A}(v; \Omega) := \int_\Omega \sqrt{1 + |\nabla v|^2 + |Jv|^2}\, dx = \int_\Omega |\mathcal{M}(\nabla v)|\, dx, \tag{0.0.2}$$

where $\mathcal{M}(\nabla v) = (1, \nabla v_1, \nabla v_2, Jv)$ and $Jv = \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\frac{\partial v_1}{\partial x_2}$ is the Jacobian determinant of v. As opposite to the case when the map is scalar-valued, the functional $\mathcal{A}(\cdot\,; \Omega)$ is not convex, but only polyconvex in ∇v, and its growth is not linear, due to the presence of $\det(\nabla v)$.
The main motivation for studying relaxation of this functional is to try to extend $\mathcal{A}(\cdot\,; \Omega)$ in a reasonable way out of $C^1(\Omega; \mathbb{R}^2)$: setting for convenience

$$\mathcal{A}(v; \Omega) := +\infty \qquad \forall v \in L^1(\Omega; \mathbb{R}^2) \setminus C^1(\Omega; \mathbb{R}^2),$$

let us consider the sequential lower semicontinuous envelope

$$\overline{\mathcal{A}}_\tau(u; \Omega) := \inf\left\{\liminf_{k\to+\infty} \mathcal{A}(v_k; \Omega) : (v_k) \subset C^1(\Omega; \mathbb{R}^2) \cap S,\ v_k \xrightarrow{\tau} u\right\} \qquad \forall u \in S \tag{0.0.3}$$

of $\mathcal{A}(\cdot\,; \Omega)$ with respect to a metrizable topology τ on a subspace $S \subseteq L^1(\Omega; \mathbb{R}^2)$ containing those $v \in C^1(\Omega; \mathbb{R}^2)$ with $\mathcal{A}(v; \Omega) < +\infty$, and choose this as the extended notion of area.

A typical choice is $S = L^1(\Omega;\mathbb{R}^2)$ and τ the $L^1(\Omega;\mathbb{R}^2)$ topology, i.e., $\overline{\mathcal{A}}_\tau = \overline{\mathcal{A}}_{L^1}$, a case in which little is known[1]. It is not difficult to show that the domain of $\overline{\mathcal{A}}_{L^1}$ is properly contained in $BV(\Omega;\mathbb{R}^2)$, but its characterization for the moment is not available. Also, one can prove that

$$\overline{\mathcal{A}}_{L^1}(u;\Omega) \geq \int_\Omega \sqrt{1+|\nabla u|^2}dx + |D^s u|(\Omega), \tag{0.0.4}$$

but the inequality might be strict, as we can see already for two elementary maps (see u_V in (0.0.5) and u_T in Fig. 1 below). Here ∇u is the approximate gradient of u, $|\cdot|$ is the Frobenius norm, $D^s u$ is the singular part of the distributional gradient Du of u, and $|D^s u|(\Omega)$ stands for the total variation of $D^s u$. Finding the expression of $\overline{\mathcal{A}}_{L^1}(\cdot\,;\Omega)$ is possible, at the moment, only in very special cases. This is also due to its nonlocal behaviour, since for several maps u, the set function

$$U \mapsto \overline{\mathcal{A}}_{L^1}(u;U) := \inf\left\{\liminf_{k\to+\infty} \mathcal{A}(u;U) : (v_k) \subset C^1(U;\mathbb{R}^2),\ v_k \to u \text{ in } L^1(U;\mathbb{R}^2)\right\}$$

is not subadditive with respect to the open set $U \subseteq \Omega$. This happens, for example, for u_T on an open disk B_ℓ, as conjectured in [20], and proven in [1]. A complete picture can be found in [8, 44], where $\overline{\mathcal{A}}_{L^1}(u_T;B_\ell)$ is explicitely computed, taking advantage of the symmetry of the map and of B_ℓ. We refer also to [5] where an upper bound inequality is proved for a triple junction map without symmetry assumptions.
Also for the vortex map $u_V : B_\ell \setminus \{0\} \to \mathbb{S}^1$,

$$u_V(x) := \frac{x}{|x|}, \tag{0.0.5}$$

the above mentioned nonsubadditivity holds. Notice that $u_V \in W^{1,p}(B_\ell;\mathbb{R}^2)$ for $p < 2$. The nonlocal behaviour is hidden in the following results, proved in [1]: we have

$$\overline{\mathcal{A}}_{L^1}(u_V;B_\ell) = \int_{B_\ell} \sqrt{1+|\nabla u_V|^2}dx + \pi \qquad \text{if } \ell \text{ is sufficiently large,} \tag{0.0.6}$$

while

$$\overline{\mathcal{A}}_{L^1}(u_V;B_\ell) < \int_{B_\ell} \sqrt{1+|\nabla u_V|^2}dx + \pi \qquad \text{if } \ell \text{ is sufficiently small.} \tag{0.0.7}$$

The explicit computation of $\overline{\mathcal{A}}_{L^1}(u_V;B_\ell)$ for small values of ℓ has been done in [6], where it is shown that in (0.0.7), in place of π, the singular contribution of $\overline{\mathcal{A}}_{L^1}(u_V;B_\ell)$ is exactly the area of the solution to a Plateau-type problem in codimension 1. It looks like a (half) catenoid constrained to contain a segment (a radius of B_ℓ) and it is the vertical part of a Cartesian current[2] obtained as a limit of the graphs of a recovery sequence. If ℓ is large enough, a minimizer of this Plateau problem has the shape of two half-disks of radius 1, whose total area is π, recovering the result in (0.0.6).

The L^1-topology is rather weak, and so it is convenient in order to show compactness results, in the effort of proving existence of minimizers of some possible weak formulation of

[1] For scalar valued maps it is known that the domain of $\overline{\mathcal{A}}_{L^1}(\cdot;\Omega)$ is $BV(\Omega)$, and on $BV(\Omega)$ the relaxed functional can be represented as the right-hand side of (0.0.4), see [18, 28].

[2] For the theory of Cartesian currents we refer to [25, 26], while for a brief introduction see Chapter 1.

the two-codimensional Cartesian Plateau problem, but also to treat existence of minimizers of relevant energies involving the area functional (see [20]). However, the above discussion illustrates the difficulties of the study of the corresponding relaxation problem. Besides all nonlocality phenomena, the L^1-convergence does not provide any control on the derivatives of v and, of course, neither on the Jacobian determinant.
The aim of this thesis is to study the relaxation of the area in $S = BV(\Omega;\mathbb{R}^2)$ in a different topology, stronger than the L^1-topology, in order to possibly avoid nonlocality and keep some control of the gradient terms. Specifically, we take as τ in (0.0.3) the topology induced by the strict convergence in $BV(\Omega;\mathbb{R}^2)$. We recall that (v_k) converges to u strictly $BV(\Omega;\mathbb{R}^2)$ if $v_k \to u$ in $L^1(\Omega;\mathbb{R}^2)$ and $|Dv_k|(\Omega) \to |Du|(\Omega)$. On the space $W^{1,1}(\Omega;\mathbb{R}^2)$ this notion of convergence is weaker than the strong $W^{1,1}$-convergence, and in general not related with the weak $W^{1,1}$-convergence In advantage, the strict convergence, unlike the ones of Sobolev spaces, still allows to consider relaxation in (0.0.3) for all BV-maps. We are therefore led to consider, for all $u \in BV(\Omega;\mathbb{R}^2)$, the corresponding relaxed area functional $\overline{\mathcal{A}}_\tau = \overline{\mathcal{A}}_{BV}$ (which we call simply BV-relaxed area)

$$\overline{\mathcal{A}}_{BV}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} \mathcal{A}(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ strictly } BV(\Omega;\mathbb{R}^2)\right\}. \tag{0.0.8}$$

One of the main advantages of considering the strict convergence (at least in dimension 2) is related to its inheritance property on one-dimensional slices, where it further behaves like a uniform convergence. This sort of rigidity of the strict convergence allows us to compute explicit integral formulas of the BV-relaxed area for many more maps in comparison to the L^1-case.
The analysis of the BV-relaxed area turns out to be highly related to the study of the BV-relaxed Jacobian total variation $\overline{TVJ}_{BV}$ (see (0.0.13) below), that is a generalized notion of total variation of the Jacobian determinant for a BV function. Roughly, this quantity seems to be the correct object to consider in order to fill completely vertical holes in the graph of a singular map. For this reason, it will appear as singular term in the expression of the BV-relaxed area for maps with 0-dimensional singularities, like vortex-type maps (Chapter 2) or, more in general, 0-homogeneous maps (Chapter 4).

In the last part of Chapter 1, we briefly introduce the formalism of currents. In particular, we recall some results valid for the class of integer multiplicity currents, that will be crucial in the proof of Theorem 3.2.2. Moreover, we recall some useful properties of Cartesian currents [26, 27], with the purpose to establish a connection with a recent approach developed by Mucci in [40], based on the notion of minimal lifting measures in the sense of Jerrard and Jung [31]. Currents represent a powerful geometric tool, especially the Cartesian ones, in order to introduce generalized version of graphs and treat singularities in a manageable geometric sense. However, we will underline some differencies between the two approaches, basically related to the fact that currents are oriented objects, then the way to regard singularities of maps (and so the corresponding way to "fill the holes in the graph") can be different from the point of view of approximation by smooth maps. Similar observations can be found already for the L^1-relaxed area in [6], where the authors point out that the minimal Cartesian current that fills the hole in the graph of u_V has less area than the catenoid constrained to contain a segment, which was described above.

In Chapter 2 (based on results in [3]) we start our analysis with maps $w : B_\ell \setminus \{0\} \to$

$\mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ of the form

$$w(x) = \varphi(u_V(x)) = \varphi\left(\frac{x}{|x|}\right), \tag{0.0.9}$$

with $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ Lipschitz continuous. The vortex map corresponds to the case $\varphi = \text{id}$. To the best of our knowledge, nothing is known about $\overline{\mathcal{A}}_{L^1}(w; B_\ell)$ when $\varphi \neq \text{id}$; in Chapter 6 we shall formulate some conjectures in the case of a double vortex, i.e. in angular coordinates $\varphi(\theta) = e^{2i\theta}$.
We prove in Theorem 2.2.3 that

$$\overline{\mathcal{A}}_{BV}(w; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla w|^2} dx + \pi |\deg(\varphi)|. \tag{0.0.10}$$

In particular,

$$\overline{\mathcal{A}}_{BV}(u_V; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u_V|^2} dx + \pi. \tag{0.0.11}$$

By (1.3.9), for ℓ large enough we find $\overline{\mathcal{A}}_{BV}(u_V; B_\ell) = \overline{\mathcal{A}}_{L^1}(u_V; B_\ell)$ while by (1.3.10), for small values of ℓ we have $\overline{\mathcal{A}}_{BV}(u_V; B_\ell) > \overline{\mathcal{A}}_{L^1}(u_V; B_\ell)$. We also remark that for any radius ℓ, in the computation of $\overline{\mathcal{A}}_{BV}(u_V; B_\ell)$, the minimal surface employed to fill the holes of the graph $G_{u_V} \subset \mathbb{R}^4$ of u_V is the unit two dimensional disk living upon the origin of $\mathbb{R}^2$.

Thereafter, we extend our analysis to a more general class of maps $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. To state our result, denote by $\text{Det}\nabla u$ the distributional Jacobian determinant of u and recall that when it is a Radon measure and $|\text{Det}\nabla u|(\Omega) < +\infty$, then $\text{Det}\nabla u$ can be written as

$$\text{Det}\nabla u = \pi \sum_{i=1}^{m} d_i \delta_{x_i}, \tag{0.0.12}$$

where the points $x_i \in \Omega$ are the topological singularities of u, around which the degree of u is nontrivial and equals $d_i \in \mathbb{Z} \setminus \{0\}$ (see for instance [11]). The main result of Chapter 2 is the following:

Theorem 0.0.1. Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ be with $|\text{Det}\nabla u|(\Omega) < +\infty$, so that (0.0.12) holds. Then

$$\overline{\mathcal{A}}_{BV}(u; \Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx + |\text{Det}\nabla u|(\Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx + \pi \sum_{i=1}^{m} |d_i|.$$

The total variation of $\text{Det}\nabla u$ can be characterized by relaxation. More precisely, for maps $v \in W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^2)$, we introduce the functional $TVJ(v; \Omega) := \int_\Omega |\det\nabla v| dx$, measuring the total variation of the Jacobian determinant of v, and consider

$$\overline{TVJ}_{W^{1,1}}(u; \Omega) := \inf \left\{ \liminf_{k \to +\infty} TVJ(v_k; \Omega) : (v_k) \subset C^1(\Omega; \mathbb{R}^2),\ v_k \to u \text{ in } W^{1,1}(\Omega; \mathbb{R}^2) \right\},$$

for all $u \in W^{1,1}(\Omega; \mathbb{R}^2)$. It is known (see [11]) that for u as in Theorem 0.0.1,

$$\overline{TVJ}_{W^{1,1}}(u; \Omega) = |\text{Det}\nabla u|(\Omega).$$

We shall show in Theorem 2.3.3 that

$$\overline{TVJ}_{W^{1,1}}(u;\Omega) = \overline{TVJ}_{BV}(u;\Omega),$$

where

$$\overline{TVJ}_{BV}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} TVJ(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ strictly } BV(\Omega;\mathbb{R}^2)\right\}. \tag{0.0.13}$$

We notice that the choice of the L^1-convergence in the relaxation is in this case not interesting: if $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ and Ω is simply connected, then $\overline{TVJ}_{L^1}(u;\Omega)$ trivializes and becomes identically zero (see [11, Cor. 5]). Weak notions of Jacobian determinant are needed in order to detect the presence of fractures in the image of singular maps, since the pointwise Jacobian determinant cannot do this job. In fact, for a map u as before, clearly $\det\nabla u = 0$ a.e., but $\mathrm{Det}\nabla u$ is a non-zero measure. We can also interpret it in terms of non trivial relaxed Jacobian total variation in (0.0.13), that in particular is telling us that any limit of TVJ along a smooth approximating sequence for u is non-zero.
Eventually, we consider some piecewise constant maps valued in $\mathbb{S}^1$, in particular the symmetric triple-point map (see Fig. 1). If we call $T_{\alpha\beta\gamma}$ the equilateral triangle with vertices

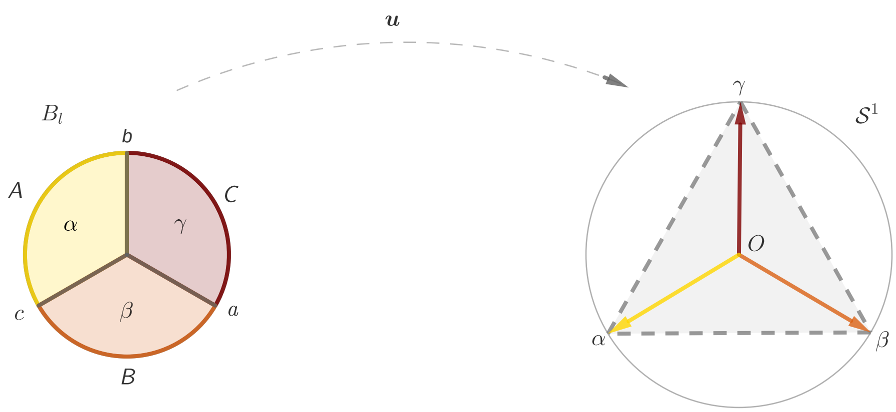

Figure 1

$\alpha, \beta, \gamma \in \mathbb{S}^1$ and $L := |\beta - \alpha|$ its side length, then we shall prove in Theorem 2.4.1 that

$$\overline{\mathcal{A}}_{BV}(u_T; B_\ell) = |B_\ell| + L\mathcal{H}^1(J_{u_T}) + |T_{\alpha\beta\gamma}|,$$

where $|\cdot|$ is the Lebesgue measure and J_{u_T} is the jump set of u_T.
In particular, in view of the results in [1], [8], we find $\overline{\mathcal{A}}_{BV}(u_T;B_\ell) > \overline{\mathcal{A}}_{L^1}(u_T;B_\ell)$. We will also see that the same argument used to prove Theorem 2.4.1 provides a proof also for a symmetric n-uple junction function.

As opposite to $\overline{\mathcal{A}}_{L^1}(u;\cdot)$, we see that $\overline{\mathcal{A}}_{BV}(u;\cdot)$, at least for the maps u taking values in $\mathbb{S}^1$ considered here, is a measure, and admits an integral representation.

In Chapter 3 we deal with maps u jumping on a curve, which are Lipschitz continuous outside of it. The main difference with the previous chapter (and also with Chapter 4) is that in this case the image of u can have non-zero Lebesgue measure. More in details, we start by considering the case of a straight jump, i.e. $u : R = [a,b] \times [-1,1] \to \mathbb{R}^2$ is such that $u \in \mathrm{Lip}(R^\pm; \mathbb{R}^2)$, where $R^+ = \{(t,\sigma) \in R : \sigma > 0\}$ and $R^- = \{(t,\sigma) \in R : \sigma < 0\}$. We briefly say that u is *piecewise Lipschitz in* R. Denoting by $u^\pm$ the trace of $u_{|R^\pm}$, we can consider the affine interpolation surface X^{aff} spanning $\mathrm{graph}(u^\pm)$, namely

$$X^{\mathrm{aff}}(t,s) = (t, su^+(t) + (1-s)u^-(t)) \qquad \forall (t,s) \in [a,b] \times I,$$

where $I := [0,1]$. Then we prove the following

Theorem 0.0.2. Let $u : R \to \mathbb{R}^2$ be piecewise Lipschitz in R. Then

$$\overline{\mathcal{A}}_{BV}(u,R) = \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-) + \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dtds. \tag{0.0.14}$$

The last integral in (0.0.14) is the area of X^{aff}. In other words, the best way to fill the hole in the graph of u upon the jump segment $[a,b]$ is given by the surface X^{aff}.
While the proof of the upper bound inequality for (0.0.14) is quite standard (Proposition 3.2.5), the one of the lower bound is more involved (Proposition 3.2.4), and it requires some tools from the theory of integer multiplicity currents, such as the isoperimetric inequality and the flat norm (briefly recalled in Chapter 1). Of course, the difficulty is concentrated around the jump segment, upon which one has to show that the graph of an approximating smooth sequence (v_k) has (at the limit) area bounded from below by the area of the affine interpolation surface X^{aff}. The properties of the strict convergence (Lemmas 3.1.1 and 3.1.4) enter at the level of vertical slices of the graph of v_k in a neighbourhood of the jump segment, but these results only are not enough to pass to the limit in the area of the graph of v_k. For this purpose, the idea is to make a decomposition of the graph of v_k and of the surface X^{aff} in several tiny strips. The key point is that, when the number of these strips is very high, the boundaries of $\mathrm{graph}(v_k)$ and X^{aff} are decomposed in little pieces which are pairwise uniformly close together, as a consequence of the strict convergence. At the same time, the strips which decompose X^{aff} are very close to a minimal mass current having the same boundary of X^{aff}.
In Remark 3.2.6, we propose an alternative proof of the lower bound inequality in (0.0.14), based on results in [40] with the theory of Cartesian currents, briefly summarized in Chapter 1.
In [10], the authors compute the relaxed area $\overline{\mathcal{A}}_{L^\infty}(u,\Omega)$ with respect to the local uniform convergence out of the jump set, for u as in Proposition 3.2.4. They obtain, as singular contribution, the area of the minimal semicartesian[3] surface spanning the graphs of the two traces. In particular, since X^{aff} is semicartesian and spans $\mathrm{graph}(u^\pm)$ as well (see [10, Definition 2.4]), we have $\overline{\mathcal{A}}_{L^\infty}(u,R) \le \overline{\mathcal{A}}_{BV}(u,R)$. In general, this inequality holds strictly, even if $\mathrm{graph}(u^\pm)$ are coplanar. We can find an example in [10, Remark 8.5], where one can notice that in order to minimize the area of the spanning surface, the approximating sequence needs not keep the total variation of the limit map, which instead is forced to be preserved under strict convergence. Moreover, it can be seen that, in general, $\overline{\mathcal{A}}_{L^\infty}(u,\cdot)$ is not subadditive (take $u = u_T$), while $\overline{\mathcal{A}}_{BV}(u,\cdot)$ is clearly a measure.
Thereafter, we generalize Theorem 0.0.2 where the jump set is a curve α of class C^2

[3]See Remark 3.2.2.

contained in Ω. In this case, one can still build up X^{aff} along the image of α and prove that it is the right object to consider. The analysis presents some technical issues when the curve touches $\partial\Omega$. To this purpose, we shall suppose that Ω is class C^1 and that α hits $\partial\Omega$ transversally.

In Chapter 4, we study $\overline{\mathcal{A}}_{BV}$ for 0-homogeneous maps. Precisely, we say that $u \in BV(B_\ell;\mathbb{R}^2)$ is 0-*homogeneous* (or simply *homogeneous*) if it is of the form

$$u(x) = \gamma\left(\frac{x}{|x|}\right) \qquad \text{a.e } x \in B_\ell, \tag{0.0.15}$$

for some $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$. Notice carefully the difference with definition (0.0.9): we are relaxing the regularity assumption on φ and, in addition, we are not imposing any constraint on its image. In order to ensure the consistency of definition (0.0.15), we shall prove in Proposition 4.3.4 that the homogeneous extension of a map $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ belongs to $BV(B_\ell;\mathbb{R}^2)$. Notice that the maps u_V and u_T are 0-homogeneous, as well as the vortex-type maps in (0.0.9). The aim of this chapter is to prove an integral representation formula for $\overline{\mathcal{A}}_{BV}(u, B_\ell)$, which further shows that $u \in \mathrm{Dom}\left(\overline{\mathcal{A}}_{BV}(\cdot\,; B_\ell)\right)$ for any u as in (0.0.15). This class of functions turns to be very interesting from the geometric point of view, because of their connection with singular planar Plateau problems, arising in the analysis of the relaxed Jacobian total variation. In fact, using the strict BV-convergence, it is possible to define a notion of area enclosed by the image of γ. More explicitely, we consider the relaxation

$$\overline{P}(\gamma) := \inf\left\{\liminf_{n\to+\infty} P(\varphi_n) : \varphi_n \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2), \varphi_n \to \gamma \text{ strictly } BV(\mathbb{S}^1;\mathbb{R}^2)\right\} \tag{0.0.16}$$

of the (singular) Plateau problem

$$P(\varphi) = \inf\left\{\int_{B_1} |Jv|\, dx : \, v \in \mathrm{Lip}(B_1;\mathbb{R}^2), v_{|\partial B_1} = \varphi\right\} \tag{0.0.17}$$

associated to any $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$. The problem in (0.0.17) was already considered by E. Paolini in [42] (see also [24], [21, pag. 338] and references therein for further information on the planar Plateau problem) and it is *singular* in the sense that φ can self intersect. For both (0.0.16) and (0.0.17), we shall establish invariance under domain rescaling and boundary data reparametrization and continuity properties with respect to the strict convergence of the data. Moreover, we prove a characterization of $\overline{P}(\gamma)$ in terms of the original P computed for the Lipschitz curve $\widetilde{\gamma}$ obtained from γ by "filling jumps with segments". The construction of $\widetilde{\gamma}$ can be done by suitably reparametrizing a smooth approximating sequence for γ in the strict convergence (see Lemma 4.3.5).

In the first place, we consider the relevant subclass of homogeneous piecewise constant maps and we compute their BV-relaxed area. In Examples 4.2.1 and 4.2.6, we construct piecewise constant maps, not homogeneous, with infinite BV-relaxed total variation, and so infinite BV-relaxed area. The interesting feature of Example 4.2.1 is that the constructed map takes only 3 distinct values and its L^1-relaxed area is finite. This in particular shows the proper inclusion

$$\mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,;\Omega)) \subsetneq \mathrm{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot\,;\Omega)).$$

In Example 4.2.6, we build a map assuming only 5 distinct values whose minimal lifting current[4] has no vertical part, i.e. the completely vertical lifting measure is zero.
Next, we prove the main result of the chapter, that reads as follows:

Theorem 0.0.3. Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and u be as in Definition 0.0.15. Then

$$\overline{\mathcal{A}}_{BV}(u;B_\ell) = \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell) + \overline{P}(\gamma), \tag{0.0.18}$$

where $D^s u$ is the singular part of the measure Du.

A crucial ingredient in the proof of Theorem 0.0.3 is the computation of $\overline{TVJ}_{BV}(u,B_\ell)$ in terms of the relaxed Plateau problem (0.0.16) (Theorem 4.3.13). It is easy to see that the expression in (0.0.18) defines a finite positive measure on B_ℓ.

The aim of Chapter 5 is to combine the results of the previous chapters to compute the BV-relaxed area for general piecewise Lipschitz maps, whose jump set is a finite family of smooth curves allowed to meet at junction points. More precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded open set of class C^1 and be $\{\Omega_k\}_{k=1,\dots,N}$ a finite partition of Ω made of Lipschitz sets. Suppose that the $\Sigma := \cup_k \partial\Omega_k$ is the support of a finite family of C^2-curves $\alpha_\ell : \overline{I}_\ell \to \overline{\Omega}$, $\ell = 1,\dots,n$, $I_\ell = (a_\ell, b_\ell)$. We suppose that the curves α_ℓ, arc-length parametrized on $\overline{I}_\ell$, are injective on I_ℓ, $\alpha_\ell(I_\ell) \subset \Omega$, and that α_ℓ is of class C^2 up to a_ℓ and b_ℓ (namely $\dot\alpha_\ell$ and $\ddot\alpha_\ell$ are continuous on I_ℓ). Furthermore, we assume that $\alpha_\ell(I_\ell)$ and $\alpha_\ell(I_h)$, for $\ell \neq h$, may intersect only at the endpoints. Finally, we also allow α_ℓ to have endpoints on $\partial\Omega$ (and we assume such endpoints to be distinct for different curves). So, the α_ℓ's can have common endpoints only at the interior of Ω, and we denote these junction points by $\{p_i\}_{i=1,\dots,m}$.

A map $u \in BV(\Omega;\mathbb{R}^2)$ is called piecewise Lipschitz on Ω if its restriction to any Ω_k is Lipschitz. Notice that if p_i is a junction point and Ω^i_k ($k = 1,\dots,N_i$) denote the connected components of $\Omega \setminus \Sigma$ which have p_i as boundary point, then there exists the limit

$$\beta^i_k := \lim_{\substack{x\to p_i \\ x\in\Omega^i_k}} u(x).$$

For the sake of simplicity, we assume that the enumeration $k = 1,\dots,N_i$ respects the counterclockwise order of Ω^i_k's around p_i. For all i we denote by $\widetilde{\gamma}^i$ the Lipschitz curve which parametrizes on $\mathbb{S}^1$ the polygon in $\mathbb{R}^2$ with vertices $\beta^i_1, \beta^i_2, \dots, \beta^i_{N_i}$, in the order. Notice carefully that this can be a self-intersecting polygonal curve. Finally, set $I = [0,1]$. The main result is the following

Theorem 0.0.4 (Relaxation for general piecewise Lipschitz maps). Let $u : \Omega \to \mathbb{R}^2$ be piecewise Lipschitz on Ω. Then

$$\overline{\mathcal{A}}_{BV}(u;\Omega) = \int_{\Omega\setminus\Sigma} |\mathcal{M}(\nabla u)|\,dx + \sum_{\ell=1}^{n} \int_{[a_\ell,b_\ell]\times I} |\partial_t X^{\mathrm{aff}}_{(\ell)} \wedge \partial_s X^{\mathrm{aff}}_{(\ell)}| dt ds + \sum_{i=1}^{m} P(\widetilde{\gamma}^i), \tag{0.0.19}$$

where, for any $\ell = 1,\dots,n$,

$$X^{\mathrm{aff}}_{(\ell)}(t,s) = (t, s u^+_\ell(t) + (1-s)u^-_\ell(t)) \qquad \forall (t,s) \in [a_\ell,b_\ell]\times I, \tag{0.0.20}$$

and $u^\pm_\ell$ are the traces of u on the support of α_ℓ.

[4] The notion of minimal lifting current is given in Section 1.5 of Chapter 1.

Let us examine the expression (0.0.19): the first integral is the classical area out of Σ; next we have a singular contribution composed by two terms, the first one is coming from a 1-dimensional measure concentrated along the image of α_ℓ's, while the second one is 0-dimensional, since it is concentrated at the junction points. In particular, we recover the same structure of the BV-relaxed area for qualitatively different maps, like Sobolev functions valued in $\mathbb{S}^1$ in Theorem 0.0.1 and homogeneous maps in Theorem 0.0.3. All these computations show that the BV-relaxed area is a quite rigid notion of extended area for graphs and suggest that it could be a local object. In other words, the structure of this functional seems to be robust, due to the "stable behaviour" of the surfaces filling the holes in the graph. So far, indeed, we have never recorded interaction phenomena between singularities and the boundary of the domain nor among singularities each other, which are always the cases where nonlocality appears for the L^1-relaxed area, for instance.
The proof of the lower bound inequality in (0.0.19) is almost a straightforward consequence of Corollary 3.2.12 in Chapter 3 about piecewise Lipschitz maps jumping on a family of disjoint curves and continuity properties of the generalized Plateau-type problem (0.0.16), studied in Chapter 4. The proof of the upper bound, instead, is more involved: one would like to apply relaxation results of Chapter 4 around each junction point p_i, where the map u is not homogeneous, in general; so, the idea is to slightly modify the jump set around p_i by straightening the curves α_ℓ and defining a recovery sequence which is homogeneous and piecewise constant in small balls $B_{r/2}(p_i)$ and coincides with u out of $\cup_{i=1}^m B_r(p_i)$. The main difficulty is to show how this modified jump set can be glued in a smooth way with the curves α_ℓ's out of $\cup_{i=1}^m B_{r_i}(p_i)$.

We point out that, at the present stage, we miss the generalization of our results in higher dimension or codimension. On the one hand the strict convergence in BV provides some control on the gradient of u, and consequently, on the distributional determinant. In the case of maps $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, for instance, this notion of convergence might be useful to get some control of the 2×2-subdeterminants of ∇u, but seems too weak to control the higher order minor. On the other hand, even in the case of maps $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^2$, the strict convergence in BV is not sufficient to imply any sort of uniform convergence on two-dimensional slices, which, in our arguments, is crucial to localize the concentrations of $|\det \nabla v_k|$ (where (v_k) is a sequence converging to u).

Finally, in Chapter 6 we collect some open problems and further directions that we would like to explore. First, we give some preliminary ideas in order to show the subaddivity of $\overline{\mathcal{A}}_{BV}(u; \cdot)$ for a generic $u \in \mathrm{Dom}(\overline{\mathcal{A}}_{BV})$, that is the main question left open by our analysis. Moreover, we underline that a further step in the computation of the BV-relaxed area could be to provide density properties in BV for the class of general piecewise Lipschitz maps (or similar kind of maps) with respect to the strict convergence. Next, we try to formulate some questions about the L^1-relaxed area, related to perturbated vortices, vortices of degree $d > 1$, multipoles and symmetric n-ple point maps.

Chapter 1

Definitions and tools

We start this preliminary chapter by recalling some basic tools of Measure Theory and fundamental properties of BV functions. In Section 1.3 we define the *area functional* and its classical extension via relaxation with respect to the L^1-convergence. We introduce also the relaxed area with respect to the strict convergence in BV, that is the main object of this thesis. In Section 1.4 we define some weak notions of Jacobian determinant and its total variation. Finally, in Section 1.5 we present a quick overview on integer multiplicity and Cartesian currents.

1.1 Notation

In the sequel, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, endowed with the Euclidean norm $|\cdot|$. The symbol $B_r(x)$ stands for the ball of radius r centered at x. If $x = 0$, we often write $B_r := B_r(0)$. The symbol Ω always denotes an open set of $\mathbb{R}^n$; we specify whenever Ω is bounded. The topological boundary of Ω is denoted by $\partial\Omega$. For $k = 0, 1, \ldots, \infty$, we use the standard notation $C^k(\Omega; \mathbb{R}^m)$ (and $C^k_c(\Omega; \mathbb{R}^m)$) to denote the space of k-times continuously differentiable maps (and compact support in Ω) valued in $\mathbb{R}^m$. The space of Lipschitz continuous maps is denoted by $\mathrm{Lip}(\Omega; \mathbb{R}^m)$. For $p \in [1, \infty]$, we denote by $L^p(\Omega; \mathbb{R}^m)$ and $W^{1,p}(\Omega; \mathbb{R}^m)$ respectively the Lebesgue space and the Sobolev space of exponent p; we denote the corresponding norms by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$. If $m = 1$, we usually omit the target space $\mathbb{R}$ in the notation. For an integer $M \geq 2$, we set $\mathbb{S}^{M-1} := \{x \in \mathbb{R}^M : |x| = 1\}$, that is the unit sphere in $\mathbb{R}^M$. The n-dimensional Lebesgue and Hausdorff measures are denoted by $\mathscr{L}^n$ and $\mathcal{H}^n$. We write also $|\cdot|$ in place of $\mathscr{L}^n$.

1.2 Radon measures and BV functions

For an exhaustive theory on BV functions we refer to [2]. We start by recalling some basic definitions of measure theory.
Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Denote by $\mathscr{B}(\Omega)$ its Borel σ-algebra and by $\mathscr{B}_c(\Omega)$ the collection of relatively compact Borel subsets of Ω. A positive measure on the space $(\Omega, \mathscr{B}(\Omega))$ is called a *Borel measure*. If a Borel measure is finite on compact subsets of Ω, it is called a *positive Radon measure*. If it is finite on Ω we say simply that it is a *finite positive measure*.
Let $M \geq 1$ be an integer. We say that a set function $\mu : \mathscr{B}_c(\Omega) \to \mathbb{R}^M$ is a *vector Radon*

measure on Ω if it is a (vector) measure[1] on $(K, \mathscr{B}(K))$ for every compact subset $K \subset \Omega$. If μ can be extended to a measure $\mu : \mathscr{B}(\Omega) \to \mathbb{R}^M$, then we say that μ is a *finite vector Radon measure*. In this case, we define the *total variation* of μ as the finite positive measure $|\mu|$ given by

$$|\mu|(B) := \sup \left\{ \sum_{i=1}^{N} |\mu(B_i)| : N \in \mathbb{N},\, B_i \subset B,\, B_i \in \mathscr{B}(\Omega) \text{ pairwise disjoint} \right\} \quad \forall B \in \mathscr{B}(\Omega). \tag{1.2.1}$$

Of course, the total variation can be defined also for a vector Radon measure as in (1.2.1) (where the B_i's are contained in $\mathscr{B}_c(\Omega)$), and it is a positive measure on $\mathscr{B}(\Omega)$, that can be possibly infinite on Ω. We denote by $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^M)$ (resp. $\mathcal{M}(\Omega; \mathbb{R}^M)$) the space of (resp. finite) vector Radon measures valued in $\mathbb{R}^M$. We say that a sequence (μ_h) in $\mathcal{M}(\Omega; \mathbb{R}^M)$ converges to $\mu \in \mathcal{M}(\Omega; \mathbb{R}^M)$ in the weak* topology if $\int_\Omega f \cdot d\mu_h \to \int_\Omega f \cdot d\mu$ for every $f \in C^0_c(\Omega; \mathbb{R}^M)$.
We recall a fundamental result that we will sistematically use in our analysis.

Theorem 1.2.1 (**Reshetnyak**)**.** Let μ_h, μ be finite Radon measures in Ω, taking values in $\mathbb{R}^M$. Suppose that $\mu_h \overset{*}{\rightharpoonup} \mu$ and $|\mu_h|(\Omega) \to |\mu|(\Omega)$. Then

$$\lim_{h \to +\infty} \int_\Omega f\left(x, \frac{\mu_h}{|\mu_h|}(x)\right) d|\mu_h|(x) = \int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) d|\mu|(x)$$

for any continuous bounded function $f : \Omega \times \mathbb{S}^{M-1} \to \mathbb{R}$.

Proof. See for instance [2, Theorem 2.39]. □

Now we can give the definition of BV function. Let $m \geq 1$ be an integer. We say that a function $u \in L^1(\Omega; \mathbb{R}^m)$ is of *bounded variation* if its distributional gradient Du is a finite vector Radon measure with values in $\mathbb{R}^{m \times n}$. The space of all functions $u : \Omega \to \mathbb{R}^m$ of bounded variation is denoted by $BV(\Omega; \mathbb{R}^m)$. If $m = 1$, we write $BV(\Omega) := BV(\Omega; \mathbb{R})$. The total variation measure $|Du|$ can be computed as in (1.2.1) where $|\cdot|$ is the Frobenius norm of a $(m \times n)$ matrix, which in turn coincides with the euclidean norm of $\mathbb{R}^M$ with $M = mn$. The *total variation of* u is by definition the real positive number $|Du|(\Omega)$. For any $u \in BV(\Omega; \mathbb{R}^m)$, by the Lebesgue decomposition theorem, Du can be written as $Du = \nabla u \mathscr{L}^n + D^s u$, where ∇u is the absolutely continuous part and $D^s u$ is the singular part, both with respect to $\mathscr{L}^n$. Moreover, u is approximately differentiable for almost every $x \in \Omega$ and $\nabla u(x)$ is the approximate gradient at x. In particular, for every $B \in \mathscr{B}(\Omega)$ there holds

$$Du(B) = \int_B \nabla u \, dx + D^s u(B), \qquad |Du|(B) = \int_B |\nabla u| \, dx + |D^s u|(B). \tag{1.2.2}$$

[1] From [2, Definition 1.4], a vector measure μ on the space $(X, \mathcal{E})$, where X is a nonempty set and $\mathcal{E}$ is a σ-algebra in X, is a function $\mu : \mathcal{E} \to \mathbb{R}^m$ such that $\mu(\varnothing) = 0$ and for every sequence of pairwise disjoint sets $(E_i) \subset \mathcal{E}$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

The measure $D^s u$ can be further decomposed into the *jump part* $D^J u$ and the *Cantor part* $D^C u$. If $D^C u = 0$, then we say that $u \in SBV(\Omega;\mathbb{R}^m)$, i.e. the space of *special bounded variation* functions on Ω valued in $\mathbb{R}^m$.
We denote by J_u the approximate jump set of u ([2, Definition 3.67]). The structure theorem for BV functions asserts that J_u is $(n-1)$-rectifiable; moreover, for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$ there exists a unit vector $\nu(x)$ which is normal to the approximate tangent space of J_u and one can define the traces $u^+(x) \neq u^-(x)$ as

$$u^+(x) := \operatorname*{aplim}_{y\to x, (y-x)\cdot\nu>0} u(y), \qquad u^-(x) := \operatorname*{aplim}_{y\to x, (y-x)\cdot\nu<0} u(y).$$

We recall the following approximation result by means of smooth functions.

Theorem 1.2.2 (Approximation by smooth functions). Let $u \in BV(\Omega;\mathbb{R}^m)$. Then there exists a sequence $(v_k) \subset C^\infty(\Omega;\mathbb{R}^m) \cap BV(\Omega;\mathbb{R}^m)$ such that

$$v_k \to u \text{ in } L^1(\Omega;\mathbb{R}^m) \quad \text{and} \quad \int_\Omega |\nabla v_k|\, dx \to |Du|(\Omega).$$

Proof. See [2, Theorem 3.9]. □

1.3 Area functional

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $u \in C^1(\Omega;\mathbb{R}^m)$. Denote by G_u the graph of u, which is a Cartesian manifold of dimension n in $\Omega \times \mathbb{R}^m \subset \mathbb{R}^{n+m}$. The area functional $\mathcal{A}(u;\Omega)$ computes the n-dimensional Hausdorff measure $\mathcal{H}^n$ of G_u, namely

$$\mathcal{A}(u;\Omega) := \mathcal{H}^n(G_u) = \int_\Omega |\mathcal{M}(\nabla u)| dx \in [0,+\infty], \tag{1.3.1}$$

where for a matrix $\xi \in \mathbb{R}^{m\times n}$, $\mathcal{M}(\xi)$ is the n-vector[2] of $\mathbb{R}^{n+m}$ whose components are the minors of ξ up to order $\min\{n,m\}$, with the convention that the minor of order 0 is equal to 1. For an n-vector $\eta \in \Lambda^n(\mathbb{R}^{n+m})$, the symbol $|\eta|$ stands for the norm induced by the euclidean one of $\mathbb{R}^{n+m}$ (see [26, Section 2.2.1]).
Notice that if $\min\{n,m\} = 1$, the previous expression defines a *convex* functional, while if $\min\{n,m\} > 1$, it is only *polyconvex* (see [17]).
Notice that

$$|\mathcal{M}(\nabla u)| \geq |\nabla u| \quad \forall u \in C^1(\Omega;\mathbb{R}^m), \tag{1.3.2}$$

but the growth of $|\mathcal{M}(\nabla u)|$ is not linear in the gradient of u, due to the presence of the higher order minors of ∇u.
In the context of Calculus of Variations, it is useful to extend the definition of area functional for less regular maps, possibly discontinuous ones. As briefly mentioned in the Introduction, a traditional way is to proceed by relaxation with respect to the L^1-convergence. This topology is quite natural to consider in the applications, when the energy functional involves an area term, because of compactness properties of sequences with bounded energies: in fact, in this case, by (1.3.2), one would obtain a bound on the total variation

[2] The linear space of n-vectors of $\mathbb{R}^{n+m}$ is denoted by $\Lambda^n(\mathbb{R}^{n+m})$.

along a sequence with bounded area, and so, if also the L^1-norm is bounded, it admits a convergent subsequence in the weak* topology of BV (see [2, Theorem 3.23]).
The procedure by relaxation can be done as follows: first, we set formally

$$\mathcal{A}(u;\Omega) := \begin{cases} \displaystyle\int_\Omega |\mathcal{M}(\nabla u)|dx & \text{if } u \in C^1(\Omega;\mathbb{R}^m) \cap L^1(\Omega;\mathbb{R}^m), \\ +\infty & \text{if } u \in L^1(\Omega;\mathbb{R}^m) \setminus C^1(\Omega;\mathbb{R}^m). \end{cases} \tag{1.3.3}$$

Then define the extended functional $\overline{\mathcal{A}}_{L^1}$ as the lower semicontinuous envelope of (1.3.3) with respect to the L^1-topology. Since this is a metrizable topology, the relaxation procedure is equivalent to define directly the extended area functional for every $u \in L^1(\Omega;\mathbb{R}^m)$ as

$$\overline{\mathcal{A}}_{L^1}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} \mathcal{A}(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^m),\, v_k \to u \text{ in } L^1(\Omega;\mathbb{R}^m)\right\}. \tag{1.3.4}$$

It is not difficult to see that

$$\mathrm{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot\,;\Omega)) := \{u \in L^1(\Omega;\mathbb{R}^m) : \overline{\mathcal{A}}_{L^1}(u;\Omega) < +\infty\} \subset BV(\Omega;\mathbb{R}^m), \tag{1.3.5}$$

where the inclusion holds strictly: for $n = m = 2$, an example is provided by the map $u(x) = \frac{x}{|x|^{3/2}}$ in $\Omega = B_1((1,0))$.
In [1], the authors proved that

$$\overline{\mathcal{A}}_{L^1}(u;\Omega) = \mathcal{A}(u;\Omega) \qquad \forall u \in C^1(\Omega;\mathbb{R}^m) \cap L^1(\Omega;\mathbb{R}^m).$$

Moreover, notice that the expression (1.3.1) is well defined for $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, if $p \geq \min\{n,m\}$. Also in this case, one can prove ([1, corollary 3.13]) that $\overline{\mathcal{A}}_{L^1}(u;\Omega) = \mathcal{A}(u;\Omega)$.
Now, we recall two fundamental results that we will use in the sequel.

Theorem 1.3.1 (Theorem 3.7, [1])**.** For every $u \in BV(\Omega;\mathbb{R}^m)$, we have

$$\overline{\mathcal{A}}_{L^1}(u;\Omega) \geq \int_\Omega \sqrt{1+|\nabla u|^2}dx + |D^s u|(\Omega). \tag{1.3.6}$$

The previous expression holds as an equality for scalar maps, while in general it might be a strict inequality if $m > 1$, due to the presence of the higher order minors of the Jacobian matrix in the definition of area functional (see for instance (1.3.7) below).

Theorem 1.3.2 (Theorem 3.14, [1])**.** Let $(E_i)_{i\in I}$ be a finite partition of $\mathbb{R}^n$, with E_i of locally finite perimeter[3] for every $i \in I$. Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mathscr{L}^n(\partial\Omega) = 0$ and $\mathcal{H}^{n-1}(\partial^* E_i \cap \partial\Omega) = 0$ for every $i \in I$. Let $v \in BV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ be defined by $v(x) = \alpha_i$ for $x \in E_i$, where $(\alpha_i)_{i\in I}$ is a finite family of points of $\mathbb{R}^m$. Suppose that for every $x \in \overline{\Omega}$ there exists $r > 0$ such that $\mathscr{L}^n(B_r(x) \cap E_i) > 0$ for at most two indices i. Then

$$\begin{aligned}\overline{\mathcal{A}}_{L^1}(v;\Omega) &= \mathscr{L}^n(\Omega) + \frac{1}{2}\sum_{i,j\in I} |\alpha_i - \alpha_j|\mathcal{H}^{n-1}(\Omega \cap \partial^* E_i \cap \partial^* E_j) \\ &= \int_\Omega |\mathcal{M}(\nabla v)|dx + |D^s v|(\Omega).\end{aligned}$$

[3] We refer to [2] for details on the theory of sets of finite perimeter. We denote by $\partial^* E$ the reduced boundary of a set E.

Essentially, this theorem states that for a piecewise constant map v without triple (or multiple) points, $\overline{\mathcal{A}}_{L^1}(v;\cdot)$ is a measure, and thus subadditive. In particular, (1.3.6) holds as an equality for v.

1.3.1 Non-subadditivity of $\overline{\mathcal{A}}_{L^1}$

If we regard the L^1-relaxed area as a function of the set variable, then, in general, it is not subadditive. This phenomenon was conjectured by De Giorgi in [20] and proved by Acerbi and Dal Maso in [1]. The authors showed that there exists a map $v \in BV_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and three open sets $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^n$ such that

$$\Omega_3 \subset \Omega_1 \cup \Omega_2 \quad \text{and} \quad \overline{\mathcal{A}}_{L^1}(v;\Omega_3) > \overline{\mathcal{A}}_{L^1}(v;\Omega_1) + \overline{\mathcal{A}}_{L^1}(v;\Omega_2).$$

De Giorgi suggested to consider $v := u_T : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. the symmetric triple point map (see Fig. 1). The authors apply Theorem 1.3.2 to u_T in a suitable annular region around the origin, where no triple points are present. Thanks to the following estimates ([1, Lemmas 4.2 and 4.4])

$$\overline{\mathcal{A}}_{L^1}(u_T;B_\ell) \leq \pi\ell^2 + 4\ell L \qquad \forall \ell > 0, \tag{1.3.7}$$

$$\overline{\mathcal{A}}_{L^1}(u_T;B_\ell) > \pi\ell^2 + 3\ell L \qquad \forall \ell > 0, \tag{1.3.8}$$

where $L := |\alpha - \beta|$ is the side of the target equilateral triangle, one can see that non-subadditivity arises on any disk centered at 0, by choosing a suitable covering of it, made of the union of an annulus and a small disk.
The inequality (1.3.7) has been refined by Bellettini and Paolini in [8], where the authors exhibit an approximating sequence of Lipschitz maps constructed in a disk B_ℓ by solving three (similar) Plateau-type problems entangled at the target plane. The proof of the upper bound of Theorem 2.4.1 is largely inspired by this construction. The result in [8] turns out to be optimal, as shown in [44], where the symmetry of u_T and B_ℓ plays a crucial role. In this work, the author shows also a further example of nonlocal phenomena, arising in thin domains, that means in the case Ω is a tubular neighbourhood of J_{u_T}: the upper bound given by Bellettini and Paolini is not optimal in this case; more surprisingly, the vertical part of the minimal cartesian current filling the holes in the graph of u_T seems not to be contained in $J_{u_T} \times \mathbb{R}^2$. Furthermore, in [5] it is provided an upper bound for a triple point map with no symmetry assumptions, neither in the source disk (the map can jump on C^2-curves meeting at a triple junction), nor in the target triangle (that can be generic).
The lack ofsubadditivity of $\overline{\mathcal{A}}_{L^1}$ appears also among Sobolev functions, as showed in [1, Theorem 5.1] for the vortex map $v := u_V : \mathbb{R}^n \to \mathbb{R}^n$, for $n \geq 3$, but argument works also for $n = 2$. It is defined by $u_V(x) = \frac{x}{|x|}$ for $x \neq 0$, and it belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for $p < n$. They proved that

$$\overline{\mathcal{A}}_{L^1}(u_V;B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u_V|^2}dx + \omega_n \qquad \text{if } \ell \text{ is sufficiently large,} \tag{1.3.9}$$

where ω_n is the Lebesgue measure of the unit ball $B_1 \subset \mathbb{R}^n$, while

$$\overline{\mathcal{A}}_{L^1}(u_V;B_\ell) \leq \int_{B_\ell} \sqrt{1 + |\nabla u_V|^2}dx + C_n\ell \qquad \text{if } \ell \text{ is sufficiently small,} \tag{1.3.10}$$

for some constant $C_n > 0$ depending only on n. For $n = 2$, the explicit computation of $\overline{\mathcal{A}}_{L^1}(u_V; B_\ell)$ for small values of ℓ has been done in [6], again strongly exploiting the radial symmetries, where it is shown that $\overline{\mathcal{A}}_{L^1}(u_V; B_\ell)$ is related to a Plateau-type problem in codimension 1, whose solution is a sort of (half) catenoid constrained to contain a segment. This "catenoid" describes the vertical part of a Cartesian current[4] obtained as a limit of the graphs of a recovery sequence. Specifically, the main result in [6] reads as

$$\overline{\mathcal{A}}_{L^1}(u_V; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u_V|^2} dx + \inf \mathcal{F}_\varphi(h, \psi), \tag{1.3.11}$$

where the infimum is taken over all functions $h \in C^0([0, 2\ell]; [-1, 1])$ with $h(0) = h(2\ell) = 1$, and $\psi \in BV((0, 2\ell) \times (-1, 1))$ with $\psi = 0$ on UG_h, and

$$\begin{aligned} \mathcal{F}_\varphi(h, \psi) = &\int_{(0,2\ell)\times(-1,1)} \sqrt{1 + |\nabla \psi|^2}\, dtds + |D\psi|((0, 2\ell) \times (-1, 1)) \\ &+ \int_{((0,2\ell)\times\{-1,1\})\cup(\{0,2\ell\}\times(-1,1))} |\psi - \varphi| d\mathcal{H}^1 - |UG_h|, \end{aligned} \tag{1.3.12}$$

where $\varphi : \mathbb{R}\times[-1, 1] \to \mathbb{R}$ is $\varphi(t, s) = \sqrt{1 - s^2}$, and UG_h is the region in $[0, 2\ell]\times[-1, 1]$ upon the graph of h. The latter functional accounts for a Plateau problem in non-parametric form with partial free boundary on a plane domain (see also [7] for more details). If ℓ is large enough, a minimizer of $\mathcal{F}_\varphi$ has the shape of two half-disks of radius 1, whose total area is π, recovering the result in (1.3.9).
Besides these two fundamental examples, in [10] it is proved that non-subaddivity of $\mathcal{A}_{L^1}(u; \cdot)$ arises also for $u : R = [a, b] \times [-1, 1] \to \mathbb{R}^2$ of the form

$$u(t, s) = \begin{cases} (f(t), 0) & \text{if } t \in [a, b], s \in [0, 1], \\ (f(t), 1) & \text{if } t \in [a, b], s \in [-1, 0], \end{cases} \tag{1.3.13}$$

for a non costant function $f \in \mathrm{Lip}([a, b])$.

1.3.2 The case $n = m = 2$ and the functional $\overline{\mathcal{A}}_{BV}$

From the previous examples, we learn that non-local phenomena represent a relevant issue, which can not be avoided in the analysis of $\overline{\mathcal{A}}_{L^1}$, even for very elementary maps from the plane to the plane, as the symmetric triple point and the vortex map. In our analysis, we adopt another strategy to attack the problem of extending the area functional: despite the fact that the L^1-topology is the most reasonable choice in the relaxation from the point of view of Calculus of Variations, one can wonder also to study relaxation with respect to a stronger topology than the L^1. Moreover, one of the biggest issues in the analysis of $\overline{\mathcal{A}}_{L^1}(u; \Omega)$ is related to the lack of control on the behaviour of the recovery sequences on $(n-1)$-dimensional slices: to fix this, for instance in [10], for $n = m = 2$ and u jumping on a line, the authors put stronger assumptions on the approximating sequences, and so they are led to consider the relaxation with respect to the local uniform topology out of the jump set of u. Another possibility, in the case u is not generic but has some geometric properties, is

[4] For a complete theory on Cartesian Currents we refer to [25, 26], while for a brief overview see Section 1.5.

to require the same properties also for the approximating sequences. For instance, suppose that $u \in BV(\Omega;\mathbb{R}^m)$ and $|u| = 1$ almost everywhere, then it is reasonable to put the constraint $v_k \in C^1(\Omega;\mathbb{S}^{m-1})$ in (1.3.4). The resulting relaxed area has been computed in the case $n = m = 2$ by Giaquinta, Modica, and Souček (see [26]), whose singular contribution involves the result of an area-minimizing problem in the setting of Cartesian currents. In the case of Sobolev maps, this number is related to the concept of *minimal connection* between singularity points (see [11]). In the special case of the vortex map u_V, this singular contribution is just the area of the lateral surface of a cylinder departing from the circular hole upon the origin and attaching to the boundary of $B_\ell \times \mathbb{R}^2$ ([26, Section 6.2.3]). Both in the previous approaches, the relaxed area is not subadditive.
However, it is worth to remark that, in the context of L^1-relaxation, the behaviour of a recovery sequence on slices can be controlled in some cases by exploiting symmetrization techniques. For instance, a fine symmetrization argument for the symmetric triple point map u_T can be found in [44, Chapter 4].
Furthermore, if u is a Sobolev map, then one can also consider the relaxed area with respect the strong (or weak) convergence of Sobolev spaces. This approach has been explored by De Philippis in [22], and it underlines the strict connection with weak notions of Jacobian determinant (see Section 1.4 below).
Following the same spirit, we want to put on the space $BV(\Omega;\mathbb{R}^m)$ a topology that allows to control also the derivatives of the approximating sequence, not just the area of their subgraphs, in order to gain control also at level of slices, to possibly avoid non-locality issues. Our choice is the *strict convergence* in BV. In the sequel we will focus on the case $n = m = 2$, so we shall study its properties in dimension 1, which will be applied in several slicing arguments.
For seek of clarity, we recall the expression of the classical area functional in the case $n = m = 2$, that can be deduced from (1.3.1), and the definition of strict convergence. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set and $u \in C^1(\Omega;\mathbb{R}^2)$, then

$$\mathcal{A}(u;\Omega) := \mathcal{H}^2(G_u) = \int_\Omega \sqrt{1 + |\nabla u|^2 + (\det\nabla u)^2}\,dx. \tag{1.3.14}$$

Definition 1.3.3 (Strict convergence). Let $u \in BV(\Omega;\mathbb{R}^2)$ and $(u_k) \subset BV(\Omega;\mathbb{R}^2)$. We say that (u_k) converges to u strictly BV, if

$$u_k \xrightarrow{L^1} u \qquad \text{and} \qquad |Du_k|(\Omega) \to |Du|(\Omega).$$

The topology of the strict convergence in BV is metrized by the distance

$$(u, v) \to \|u - v\|_{L^1(\Omega;\mathbb{R}^2)} + \left||Du|(\Omega) - |Dv|(\Omega)\right|, \qquad u, v \in BV(\Omega;\mathbb{R}^2).$$

Therefore, the corresponding relaxed area functional (that we will briefly call *BV-relaxed area*) is defined by

$$\overline{\mathcal{A}}_{BV}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} \mathcal{A}(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ strictly } BV(\Omega;\mathbb{R}^2)\right\}. \tag{1.3.15}$$

Notice that the class of competitors is non-empty thanks to Theorem 1.2.2. Clearly, we have

$$\overline{\mathcal{A}}_{L^1}(\cdot\,;\Omega) \le \overline{\mathcal{A}}_{BV}(\cdot\,;\Omega), \tag{1.3.16}$$

hence

$$\mathrm{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot\,;\Omega)) \subset \mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,;\Omega)). \tag{1.3.17}$$

The inequality (1.3.16) might be strict, in general, as we shall see in Chapter 2 for the vortex map, in formula (2.2.16). Moreover, we will see in Chapter 4 that also the inclusion (1.3.17) is strict, by providing an example among piecewise constant maps.
Of course, a boundedness assumption on the area along a smooth sequence v_k does not imply the existence of a subsequence strictly converging to u, but only weakly*-BV. However, in codimension 1 we have that $\overline{\mathcal{A}}_{BV} = \overline{\mathcal{A}}_{L^1}$.

Remark 1.3.4 (**Weak convergences and strict convergence**)**.** Suppose that $u_k \to u$ strictly $BV(\Omega)$. Then $u_k \rightharpoonup u$ w^*-$BV(\Omega)$, i.e.

$$u_k \xrightarrow{L^1} u \qquad \text{and} \qquad \int_\Omega \varphi \cdot Du_k \to \int_\Omega \varphi \cdot Du \qquad \forall \varphi \in C_c^0(\Omega;\mathbb{R}^2),$$

with $\cdot$ the scalar product in $\mathbb{R}^2$. A similar definition holds for vector valued maps. The converse is not true, already in one dimension: consider the sequence $(f_k) \subset W^{1,1}((0,2\pi))$,

$$f_k(x) := \frac{1}{k}\sin(kx) \qquad \forall x \in (0,2\pi).$$

Then $f_k \rightharpoonup 0$ weakly in $W^{1,1}((0,2\pi))$, so in particular w^*-BV, but the convergence is not strict in BV, since $\|f_k'\|_{L^1((0,2\pi))} = 4$ for all $k \in \mathbb{N}$. We underline that on the space $W^{1,1}(\Omega)$ the strict BV convergence is not comparable with the weak convergence: the following slight modification of [25, Example 4, pag. 42], provides a sequence converging strictly $BV((0,1))$ but not weakly in $W^{1,1}((0,1))$. Consider the sequence $(g_k) \subset L^1((0,1))$ defined by

$$g_k(x) := 2^k \sum_{i=0}^{k-1} \chi_{\left[\frac{i}{k},\frac{i}{k}+\frac{1}{k2^k}\right]}(x) \qquad \forall x \in [0,1],\ \forall k \ge 1,$$

where χ_A is the characteristic function of the set A. Then $\|g_k\|_{L^1} = 1$ for every $k \in \mathbb{N}$. Now, let $f_k \in C([0,1])$ be the primitive of g_k vanishing at 0; then (f_k) converges uniformly to the identity, and $\|f_k'\|_{L^1} = \|g_k\|_{L^1} = 1 = \|\mathrm{id}'\|_{L^1}$ for any $k \in \mathbb{N}$, and so $f_k \to \mathrm{id}$ strictly $BV((0,1))$. On the other hand, (f_k') cannot converge weakly in L^1 since it is not equi-integrable (see [25, Theorem 2, pag. 50]), since g_k tends to concentrate a large mass in arbitrarily small sets, as k becomes large.
Similar examples can be considered also in the case of vector valued maps.

However, the following result (which we will use very often in the sequel) shows that the strict BV convergence implies the uniform one, under certain hypotheses.

Lemma 1.3.5. Let $(\gamma_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ be a sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma \in BV([a,b];\mathbb{R}^2)$. Then, for every compact subset $K \subset [a,b] \setminus J_\gamma$, we have that

$$\gamma_k \to \gamma \quad \text{uniformly in } K \quad \text{as } k \to +\infty. \tag{1.3.18}$$

Proof. By contradiction, up to a not relabeled subsequence, we may suppose

$$\exists \delta > 0 \quad \exists (\tau_k) \subset K \quad \exists k_0 \in \mathbb{N} : \quad |\gamma_k(\tau_k) - \gamma(\tau_k)| > \delta \quad \forall k \geq k_0,$$

and there exists $\bar{\tau} \in K$ such that $\tau_k \to \bar{\tau}$ as $k \to +\infty$, since K is compact. Now, consider an open interval $E \subset [a, b]$ such that[5] $\bar{\tau} \in E$, $\partial E \subset [a, b] \setminus J_\gamma$, and $|\dot{\gamma}|(E) < \frac{\delta}{4}$. Such an interval E exists because $|\dot{\gamma}|(\{\bar{\tau}\}) = 0$. By hypothesis on strict convergence, since $|\dot{\gamma}|(\partial E) = 0$, we have

$$\lim_{k\to+\infty} \int_E |\dot{\gamma}_k| dt = |\dot{\gamma}|(E).$$

So, we can find an index $k_1 \in \mathbb{N}$ such that $k_1 \geq k_0$ and $\int_E |\dot{\gamma}_k| dt < \frac{\delta}{2}$, for every $k \geq k_1$. Moreover, there exists $k_2 \in \mathbb{N}$, $k_2 \geq k_1$, such that $\tau_k \in E$ for every $k \geq k_2$. Now fix $F \subset E$ such that $|F| = |E|$ and $\gamma_{|F}$ can be identified with its natural continuous representative. Pick a point $z \in F$, then

$$\begin{aligned} |\gamma_k(z) - \gamma(z)| &\geq -|\gamma_k(z) - \gamma_k(\tau_k)| + |\gamma_k(\tau_k) - \gamma(\tau_k)| - |\gamma(\tau_k) - \gamma(z)| \\ &\geq -\left| \int_{\tau_k}^{z} |\dot{\gamma}_k| dt \right| + \delta - |\dot{\gamma}|(E) \geq -\int_E |\dot{\gamma}_k| dt + \delta - \frac{\delta}{4} \\ &\geq -\frac{\delta}{2} + \frac{3}{4}\delta = \frac{\delta}{4}. \end{aligned}$$

Therefore, (γ_k) does not converge to γ pointwise at any point of F, which leads to a contradiction with the fact that $\gamma_k \to \gamma$ in $L^1([a, b])$. So, (3.1.4) is proved. □

An immediate consequence of Lemma 1.3.5 is that the uniform convergence takes place on the full interval if $J_\gamma = \varnothing$. Precisely the following holds.

Corollary 1.3.6. Let $(\gamma_k) \subset W^{1,1}([a, b]; \mathbb{R}^2)$ be a sequence converging strictly $BV([a, b]; \mathbb{R}^2)$ to $\gamma \in C([a, b]; \mathbb{R}^2) \cap BV([a, b]; \mathbb{R}^2)$. Then,

$$\gamma_k \to \gamma \quad \text{uniformly} \;\; \text{as } k \to +\infty.$$

Remark 1.3.7. Lemma 1.3.5 is still valid with the same proof when γ_k and γ are valued in $\mathbb{R}^m$ for $m > 2$. On the contrary, it is crucial that the domain is one-dimensional, since counterexamples can be done already in dimension 2: for instance, the sequence (f_k) given by $f_k(x) := \max\{(1 - k|x|), 0\}$, $x \in \mathbb{R}^2$, converges to 0 in $W^{1,1}(\mathbb{R}^2)$ but not uniformly in any neighborhood of the origin.
In Lemma 3.1.4 and Lemma 4.3.5, we shall prove generalized versions of Corollary 1.3.6 to the case $J_\gamma \neq \varnothing$.

1.4 The Jacobian determinant and its total variation

From the definition (1.3.14), a natural energy that is strictly related to the area functional is the total variation of the Jacobian determinant.

[5] If $\bar{\tau} = a$ or $\bar{\tau} = b$, E is a semi-open interval.

Definition 1.4.1 (Total variation of the Jacobian determinant). Let $u \in W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^2)$. We define the total variation of the Jacobian of u as

$$TVJ(u;\Omega) = \int_\Omega |\mathrm{det}\nabla u| dx. \tag{1.4.1}$$

We need to define $TVJ(\cdot\,;\Omega)$ for less regular maps, such as Sobolev maps with exponent $p < 2$, the main example being the vortex map u_V in (0.0.5). This can be accomplished in two ways. The first one is to define the distributional Jacobian determinant $\mathrm{Det}\nabla u$: if[6] $p \in [1,2)$ and $u \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^\infty_{\rm loc}(\Omega;\mathbb{R}^2)$,

$$< \mathrm{Det}\nabla u, \varphi >:= -\frac{1}{2}\int_\Omega \mathrm{adj}\nabla u(x)u(x)\cdot\nabla\varphi(x)dx \qquad \forall\varphi \in C^\infty_c(\Omega), \tag{1.4.2}$$

where $\mathrm{adj}\nabla u := \begin{pmatrix} \frac{\partial u_2}{\partial y} & -\frac{\partial u_1}{\partial y} \\ -\frac{\partial u_2}{\partial x} & \frac{\partial u_1}{\partial x} \end{pmatrix}$. This definition is justified by the property

$$u \in C^2(\Omega;\mathbb{R}^2) \Rightarrow \mathrm{det}\nabla u = \frac{1}{2}\mathrm{div}(\mathrm{adj}\nabla u\, u).$$

Notice that, if $u \in C^2(\Omega;\mathbb{R}^2)$ and $B_r(x) \subset\subset \Omega$, then by the divergence theorem, writing the outward unit normal to $\partial B_r(x)$ as $\nu = (\nu_1,\nu_2)$, and its $\pi/2$-counterclockwise rotation $\nu^\perp = \tau = (\tau_1,\tau_2)$,

$$\begin{aligned} \int_{B_r(x)} \mathrm{det}\nabla u\, dz &= \frac{1}{2}\int_{\partial B_r(x)} (\mathrm{adj}\nabla u\, u)\cdot\nu\ d\mathcal{H}^1 \\ &= \frac{1}{2}\int_{\partial B_r(x)} \Big((\frac{\partial u_2}{\partial y}u_1 - \frac{\partial u_1}{\partial y}u_2)\nu_1 + \big(-\frac{\partial u_2}{\partial x}u_1 + \frac{\partial u_1}{\partial x}u_2)\nu_2 \Big)\, d\mathcal{H}^1 \\ &= \frac{1}{2}\int_{\partial B_r(x)} \Big(u_1(\frac{\partial u_2}{\partial y}, -\frac{\partial u_2}{\partial x})\cdot\nu + u_2\big(-\frac{\partial u_1}{\partial y}, \frac{\partial u_1}{\partial x})\cdot\nu \Big)\, d\mathcal{H}^1 \\ &= \frac{1}{2}\int_{\partial B_r(x)} (u_1\nabla u_2\cdot\tau - u_2\nabla u_1\cdot\tau)\, d\mathcal{H}^1 \\ &= \frac{1}{2}\int_{\partial B_r(x)} \Big(u_1\frac{\partial u_2}{\partial s} - u_2\frac{\partial u_1}{\partial s}\Big) ds, \end{aligned} \tag{1.4.3}$$

where s is the (oriented) line integral variable on $\partial B_r(x)$ and we set $\nabla u_i\cdot\tau := \frac{\partial u_i}{\partial s}$, $i = 1,2$. By [41, Formula (3.7)] (which in turn is a consequence of Theorem 3.2 in [41]), one sees that formula (1.4.3) is valid also for $u \in W^{1,\infty}(\Omega;\mathbb{R}^2)$.

We recall that

$$\mathrm{Det}\nabla u = \mathrm{det}\nabla u \quad \forall u \in W^{1,2}(\Omega;\mathbb{R}^2),$$

while if $p \in [1,2)$ they can differ, for instance $\mathrm{det}\nabla u_V$ is null, whereas $\mathrm{Det}\nabla u_V = \pi\delta_0$ (see [42]). Then one is led to define $TVJ(u;\Omega) = |\mathrm{Det}\nabla u|(\Omega)$, for those u for which $\mathrm{Det}\nabla u$ is a Radon measure with finite total variation in Ω.

The second way is to argue by relaxation. For $p \in [1,2]$ and $u \in W^{1,p}(\Omega;\mathbb{R}^2)$ one sets

$$\overline{TVJ}_{W^{1,p}}(u;\Omega) := \inf\Big\{\liminf_{k\to+\infty} TVJ(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2), v_k \to u \text{ in } W^{1,p}\Big\}. \tag{1.4.4}$$

[6] Alternatively, if $p \geq \frac{4}{3}$, it is enough to require only $u \in W^{1,p}(\Omega;\mathbb{R}^2)$.

It is known that $TVJ(u;\Omega) = \overline{TVJ}_{W^{1,2}}(u;\Omega)$ for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. Moreover, when $p \in [1,2)$, $\overline{TVJ}_{W^{1,p}}(\cdot\,;\Omega)$ coincides with the total variation of the Jacobian distributional determinant of u, provided $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ (see Theorem 2.1.6 below, and [11, Theorem 11 and Remark 12]). The same conclusions do not hold in general, for maps in $W^{1,p}(\Omega;\mathbb{R}^2)$ which do not take values in $\mathbb{S}^1$ (see [11, Open problem 5]). Notice also that relaxation in (1.4.4) can also be done with respect to the weak convergence in $W^{1,p}$ (we do not treat this in the present thesis and refer the reader to [11, 22–24, 39, 42]).

We emphasize that we required C^1-regularity for the approximating sequences in (1.4.4). This ensures that such sequences are contained in $W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^2)$ which is the minimal feature to guarantee that $\det\nabla v_k \in L^1_{\rm loc}(\Omega)$. Replacing the C^1-regularity with the $W^{1,2}_{\rm loc}$-regularity[7] gives rise to the same relaxed functionals; this can be seen by a density argument, since any $v \in W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^2)$ can be approximated by maps $v_k \in C^1(\Omega;\mathbb{R}^2)$ in $W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^2)$ (such a convergence ensures the corresponding convergence of $TVJ(v_k;\Omega)$ to $TVJ(v;\Omega)$). In the same way, one can also replace the C^1-regularity with the C^∞-regularity.

The approach by relaxation can be used also for jumping maps, as we shall see in the next Chapters, via approximation in the strict BV-convergence: Let $u \in BV(\Omega;\mathbb{R}^2)$ and set

$$\overline{TVJ}_{BV}(u;\Omega) := \inf\left\{\liminf_{k\to+\infty} TVJ(v_k;\Omega) : (v_k) \subset C^1(\Omega;\mathbb{R}^2),\ v_k \to u \text{ strictly } BV(\Omega;\mathbb{R}^2)\right\}. \tag{1.4.5}$$

In this way, we can extend Definition 1.4.1 to BV-maps and we can ask if this extension is compatible with 1.4.4 for (a subclass of) Sobolev maps: this will be the content of Theorem 2.3.3, for Sobolev maps valued in $\mathbb{S}^1$. Moreover, the relaxation with respect to the L^1-convergence is possible, but uninteresting in the case of maps with values in $\mathbb{S}^1$, because the resulting relaxed functional turns out to be zero (see [11, Corollary 5]).

1.5 Overview on currents

We shall use the formalism of rectifiable currents in the proof of Proposition 3.2.4. Moreover, in Chapters 3 and 4 we will make several links with a recent approach via Cartesian currents, that was developed in [40]. In this section, we introduce these objects and summarize their fundamental properties. We refer to [26, 27] and [33] for a complete discussion on currents.
Let $U \subseteq \mathbb{R}^N$ be an open set and $k \le N$. The space $\mathcal{D}_k(U)$ of the *k-currents* in U is the dual of the space $\mathcal{D}^k(U)$ of the k-forms with $C^\infty_c(U)$-coefficients. The space $\mathcal{D}_k(U)$ is endowed with the usual weak* convergence, namely $T_j \rightharpoonup T$ iff $T_j(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^k(U)$. For any current $T \in \mathcal{D}_k(U)$, we define its *mass* as

$$|T| := \sup\{T(\omega) : \omega \in \mathcal{D}^k(U), \|\omega(x)\| \le 1, \forall x \in U\},$$

where $\|\xi\|$ stands for the *comass* of the k-covector $\xi \in \Lambda_k(U)$ (see [26, Section 2.2.1]).

Theorem 1.5.1 (Lower semicontinuity of the mass). Let $T_j, T \in \mathcal{D}_k(U)$. If $T_j \rightharpoonup T$ then $|T| \le \liminf_{j\to+\infty} |T_j|$.

[7] As sometimes can be found in literature.

Proof. See [26, Proposition 1, Section 2.2.3]. □

The *boundary* of a k-current $T \in \mathcal{D}_k(U)$ is the $(k-1)$-current $\partial T \in \mathcal{D}_{k-1}(U)$ defined by $\partial T(\eta) := T(d\eta)$ for every $\eta \in \mathcal{D}^{k-1}(U)$, where $d\eta$ is the exterior differential of η.
The *support* of $T \in \mathcal{D}_k(U)$ is defined by

$$\mathrm{spt} T = \bigcap \{K \subset U \text{ closed} : T(\omega) = 0 \quad \forall \omega \in \mathcal{D}^k(U) \quad \text{with} \quad \mathrm{spt}\omega \subset U \setminus K\}.$$

Assume that $T \in \mathcal{D}_k(U)$ is such that $|T|, |\partial T| < +\infty$. Let $f \in \mathrm{Lip}(U, V)$, $V \subset \mathbb{R}^N$, be such that $f_{|\mathrm{spt}T}$ is proper, i.e. $f^{-1}(K) \cap \mathrm{spt}T$ is compact in U for every compact set $K \subset V$. Then the *push-forward* of T through f is the current $f_\sharp T$ defined by $f_\sharp T(\omega) := T(f^\sharp \omega)$ for every $\omega \in \mathcal{D}^k(V)$, where $f^\sharp\omega$ is the pull-back of ω through f. Moreover, there holds $f_\sharp \partial T = \partial f_\sharp T$.

1.5.1 Integer multiplicity currents

A relevant subclass of currents is the one of integer multiplicity currents. Given an oriented k-rectifiable set[8] $M \subset U$ and a multiplicity funtion $\theta : M \to \mathbb{Z}$ locally $\mathcal{H}^k \llcorner M$ summable, we define the current

$$T(\omega) = \int_M \langle \xi(x), \omega(x) \rangle \theta(x) d\mathcal{H}^k \qquad \forall \omega \in \mathcal{D}^k(U),$$

where $\xi(x)$ is the k-vector in U which orients for $\mathcal{H}^k$-almost every $x \in M$ the approximate tangent k-space $T_x M$ of M at x. The product $\langle \cdot, \cdot \rangle$ denotes the duality between vectors and covectors (see [26, Section 2.2.1]). We say that the current T defined as above is an *integer multiplicity (rectifiable) k-current* in U and we denote it by $T := \tau(M, \theta, \xi)$. If θ is identically equal to 1, T reduces to the oriented integration over the rectifiable set M and we denote it simply $T := [\![M]\!]$. Notice that, according to this definition, any oriented smooth k-submanifold of $\mathbb{R}^N$ can be regarded as a current; moreover, the definition of boundary is compatible with the Stokes theorem.
The next compactness theorem is due to Federer and Fleming.

Theorem 1.5.2 (Compactness). Let $(T_j) \subset \mathcal{D}_k(U)$ be a sequence of integer multiplicity currents such that $\sup_{j \in \mathbb{N}} \{|T_j| + |\partial T_j|\} < +\infty$. Then there exists an integer multiplicity current $T \in \mathcal{D}_k(U)$ and a subsequence $\{T_{j'}\}$ such that $T_{j'} \rightharpoonup T$.

The context of integer multiplicity currents is a good setting to solve Plateau problems. Indeed, thanks to their lower semicontinuity and compactness properties, one can easily prove the existence of minimal mass currents, using direct methods. More precisely, suppose for simplicity that $U = \mathbb{R}^N$, then we say that an integer multiplicity current $T \in \mathcal{D}_k(\mathbb{R}^N)$ is *mass-minimizing* in $\mathbb{R}^N$ if T has compact support and $|T| \leq |S|$ for every integer multiplicity current $S \in \mathcal{D}_k(\mathbb{R}^N)$ with $\partial S = \partial T$.

The next theorem ensures the existence of a mass-minimizing current among integer multiplicity ones with fixed boundary.

[8] M is said to be k-rectifiable if it can be written apart for a null $\mathcal{H}^k$-set as disjoint union of Borel subsets of k-dimensional C^1-submanifolds with finite $\mathcal{H}^k$ measure.

Theorem 1.5.3 (Existence of minimal currents). Suppose that $R \in \mathcal{D}_{k-1}(\mathbb{R}^N)$ has compact support and that there exists an integer multiplicity current $Q \in \mathcal{D}_k(\mathbb{R}^N)$ with $\partial Q = R$. Then there exists a mass-minimizing integer multiplicity current $T \in \mathcal{D}_k(\mathbb{R}^N)$ with $\partial T = R$.

Now we define the notion of flat norm, that allows to characterize the weak convergence for compactly supported integer multiplicity currents with bounded mass and boundary mass. Let $T \in \mathcal{D}_k(\mathbb{R}^N)$ of integer multiplicity with compact support and $|\partial T| < +\infty$. We define the *flat norm* of T as

$$\|T\|_F := \inf\{|S| + |R| : T = \partial R + S,\ R \in \mathcal{D}_{k+1}(\mathbb{R}^N) \text{ i.m.},\ S \in \mathcal{D}_k(\mathbb{R}^N) \text{ i.m.}\}. \tag{1.5.1}$$

Theorem 1.5.4 (Flat norm and weak convergence). Let $T, (T_j)_j$ in $\mathcal{D}_k(\mathbb{R}^N)$ be integer multiplicity currents with $\sup_{j\in\mathbb{N}}\{|T_j| + |\partial T_j|\} < +\infty$. Assume that $\mathrm{spt}T_j \subset K$ for every $j \in \mathbb{N}$, for some compact set $K \subset \mathbb{R}^N$. Then $T_j \rightharpoonup T$ if and only if $\|T_j - T\|_F \to 0$ as $j \to +\infty$.

Finally, we recall the *Isoperimetric theorem* for integer multiplicity currents.

Theorem 1.5.5 (Isoperimetric Inequality). Let $k \geq 2$. Suppose that $T \in \mathcal{D}_{k-1}(\mathbb{R}^N)$ is of integer multiplicity, $\mathrm{spt}T$ is compact and $\partial T = 0$. Then there exists $R \in \mathcal{D}_k(\mathbb{R}^N)$ of integer multiplicity, with compact support and $\partial R = T$, such that

$$|R|^{\frac{k-1}{k}} \leq C|T|,$$

where C is a constant depending only on k and N.

Concerning the proofs of Theorems 1.5.2, 1.5.3, 1.5.4, and 1.5.5, we refer to Theorems 7.5.2, 8.3.3, 8.2.1, and 7.9.1 in [33], respectively.

1.5.2 Cartesian currents

In Chapters 3 and 4, we will make use of the theory of Cartesian currents, developed by Giaquinta-Modica-Souček [26, 27], to make a connection with an alternative approach in the study of the area of singular graphs via strict convergence and minimal lifting measures, recently developed by Mucci [40].
Let $\Omega \subset \mathbb{R}^n$ be an open set. The space $\mathrm{cart}(\Omega; \mathbb{R}^m)$ has been introduced to generalize the notion of graph of a map from Ω to $\mathbb{R}^m$. Start by fixing coordinates $x = (x^1, \ldots, x^n)$ in Ω and $y = (y^1, \ldots, y^m)$ in the target space $\mathbb{R}^m$.

Definition 1.5.6 (Cartesian currents). The space $\mathrm{cart}(\Omega \times \mathbb{R}^m)$ of Cartesian currents is the space of all integer multiplicity n-currents T on $U := \Omega \times \mathbb{R}^m \subset \mathbb{R}^{n+m}$ such that $\partial T = 0, |T| < +\infty$, and the following conditions hold:

- $p_\sharp T = [\![\Omega]\!]$, where $p_\sharp T(\omega) := T(p^\sharp \omega)$ for every $\omega \in \mathcal{D}^n(\Omega)$, and $p : U \to \Omega$ is the orthogonal projection on $\mathbb{R}^n$;
- $T^{\bar{0}0} \geq 0$, where $T^{\bar{0}0}$ is the Radon measure defined by $T^{\bar{0}0}(f) := T(f dx^1 \wedge \ldots \wedge dx^n)$ for every $f \in C^0_c(U)$;

- $\|T\|_1 := \sup\{T(|y|f(x,y)dx^1 \wedge \ldots \wedge dx^n) : f \in C_c^\infty(U),\ |f| \leq 1\} < +\infty$.

The key point of the previous definition is that Cartesian currents arise as weak limit of smooth graphs with equibounded area. However, not all Cartesian currents can be obtained in such a way, and the problem of describing the closure of smooth graphs with respect to the weak convergence of currents is still open (see [26, Sec. 4.2.1]). Notice that if $v \in C^1(\Omega;\mathbb{R}^m) \cap L^1(\Omega;\mathbb{R}^m)$ and $\mathcal{H}^n(G_v) < +\infty$, then $[\![G_v]\!] \in \mathrm{cart}(\Omega \times \mathbb{R}^m)$ and $\|G_v\|_1 = \|v\|_{L^1}$. Moreover, the graph of a discontinuous map $v : \Omega \to \mathbb{R}^m$ cannot be regarded as a cartesian current, in general, because its boundary in $\Omega \times \mathbb{R}^m$ can be not trivial. However, there are maps with non-removable discontinuity points whose graph is a Cartesian current: for example, the 0-homogeneous extension of the double-eight curve (see for instance [39]), to which Example 4.2.5 is largely inspired.
A particularly important result is a kind of *structure theorem*, which shows that every $T \in \mathrm{cart}(\Omega \times \mathbb{R}^m)$ can be written in a suitable sense as an integration over a graph with possibly "vertical parts".

Theorem 1.5.7 (Structure of $\mathrm{cart}(\Omega \times \mathbb{R}^m)$)**.** Let $T \in \mathrm{cart}(\Omega \times \mathbb{R}^m)$. Then there exists a map $v_T \in BV(\Omega;\mathbb{R}^m)$ and an integer multiplicity current $S_T \in \mathcal{D}_n(\Omega \times \mathbb{R}^m)$ with finite mass, such that $T = [\![G_{v_T}]\!] + S_T$. Moreover, S_T is "vertical", i.e. $S_T(\varphi(x,y)dx^1 \wedge \ldots \wedge dx^n) = 0$ for every $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^m)$.

The proof of this result can be found in [26, Sec. 4.2.3]. See also [1, Theorems 2.3 and 2.5].

1.5.3 Minimal lifting currents

In this subsection, we anticipate some notation and recall useful results contained in [40], to make more clear the connection between our analysis of the BV-relaxed area and Mucci's approach based on minimal lifting measures and cartesian currents.
Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in BV(\Omega;\mathbb{R}^m)$. Jerrard and Jung introduced in [31] the notion of *minimal lifting measure* $\mu[u] \in \mathcal{M}(\Omega \times \mathbb{R}^m;\mathbb{R}^{m\times n})$ associated to u, characterized by the following conditions:

1. if $u \in W^{1,1}(\Omega;\mathbb{R}^m)$, then
$$\mu_i^j[u] = (\mathrm{id} \bowtie u)_\sharp(\partial_i u^j \mathscr{L}^n \llcorner \Omega) \quad \forall i = 1,\ldots,n,\ j = 1,\ldots,m,$$
where $(\mathrm{id} \bowtie u)(x) := (x, u(x))$ is the graph map;
2. if $u_k \to u$ strictly $BV(\Omega;\mathbb{R}^m)$, then
$$\mu[u_k] \stackrel{*}{\rightharpoonup} \mu[u] \quad \text{and} \quad |\mu[u_k]|(\Omega \times \mathbb{R}^m) \to |\mu[u]|(\Omega \times \mathbb{R}^m).$$

$\mu[u]$ is called minimal lifting measure since $p_\sharp|\mu[u]|(\Omega \times \mathbb{R}^m) = |Du|(\Omega)$, where $p : \Omega \times \mathbb{R}^m \to \Omega$ is the orthogonal projection. The existence of $\mu[u]$ is guaranteed by Theorem 1.2.2. Moreover, $\mu[u]$ is unique thanks to the explicit formula (see [31, Theorem 2.2])

$$\int_{\Omega\times\mathbb{R}^m} \phi(x,y)d\mu_i^j[u] = \int_\Omega \left[\int_0^1 \phi(x,u^s(x))ds\right] d(Du)_i^j \quad \forall \phi \in C_c^\infty(\Omega \times \mathbb{R}^m), \tag{1.5.2}$$

where for $s \in [0,1]$, u^s is defined by $u^s(x) := su^+(x) + (1-s)u^-(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$ and it coincides with a precise representative $u(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega \setminus J_u$.
Now we can define the notion of *minimal lifting current* associated to u. For simplicity, let us consider the case $n = m = 2$. Any integer multiplicity current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ is identified by the measures

$$\mu_h[T] := T \llcorner dx, \quad \mu_i^j[T] := T \llcorner dx^{\bar{i}} \wedge dy^j, \quad i,j = 1,2, \quad \mu_v[T] := T \llcorner dy,$$

where $\bar{1} := 2, \bar{2} := 1$ and $dx := dx^1 \wedge dx^2$, $dy := dy^1 \wedge dy^2$. The measure $T \llcorner dx$ is defined by $T \llcorner dx(\varphi) := T(\varphi dx)$ for every $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. In a similar way, one defines $T \llcorner dx^{\bar{i}} \wedge dy^j$ and $T \llcorner dy$. If $T = G_u + S_T \in \mathrm{cart}(\Omega; \mathbb{R}^2)$, then clearly $\mu_h[T] = (\mathrm{id} \bowtie u)_\sharp(\mathscr{L}^2 \llcorner \Omega)$, by the structure Theorem 1.5.7. The next result is proved in [40, Theorem 3.5].

Theorem 1.5.8 (Mucci). Let $u \in BV(\Omega; \mathbb{R}^2)$ and suppose that $\overline{\mathcal{A}}_{BV}(u;\Omega) < +\infty$. Then there exists a unique Cartesian current $T_u = G_u + S_{T_u} \in \mathrm{cart}(\Omega; \mathbb{R}^2)$, obtained by imposing $\mu_i^j[T_u] := \mu_i^j[u]$, $i,j = 1,2$. Moreover $|T_u| \le \overline{\mathcal{A}}_{BV}(u;\Omega)$.

We say that T_u is the *minimal lifting current* associated to u. Theorem 1.5.8 is telling us that the vertical part $\mu_v[T_u]$ of T_u is uniquely determined by requiring that its mixed components coincide with the minimal lifting measures in the sense of Jerrard-Jung. In this case, we say that the measure $\mu_v[u] := \mu_v[T_u]$ is the *completely vertical lifting* of u. If u is smooth, then $T_u = G_u$ and $\mu_v[u] = (\mathrm{id} \bowtie u)_\sharp(\det\nabla u \, \mathscr{L}^2 \llcorner \Omega)$; interestingly, in this case, one can prove that (see [40, Theorem 6.2])

$$|\mu_v[u]|(\Omega \times \mathbb{R}^2) = \int_\Omega |\det\nabla u| dx = TVJ(u;\Omega). \tag{1.5.3}$$

The lower bound $|T_u| \le \overline{\mathcal{A}}_{BV}(u;\Omega)$ for the BV-relaxed area is, in general, not optimal, as pointed out in [40] and as we shall see in Example 4.2.6, even in the case u is piecewise constant.
Finally, the uniqueness of T_u still holds true in higher dimension, but fails in higher codimension (see [40, Sections 7 and 8]).

Chapter 2

Singular maps with values in $\mathbb{S}^1$

We start the study of the BV-relaxed area by considering singular maps that take values in the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. After a brief introductory section, where we recall the notion of multiplicity and degree for Sobolev maps, we start our analysis to maps $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. We recall in Theorem 2.1.6 a structure result for the distributional Jacobian determinant of u in the case it is a finite Radon measure, in particular in the case $\overline{\mathcal{A}}_{BV}(u; \Omega)$ is finite. In Section 2.2 we treat the special case of *vortex-type maps*, which have just one singular point (at the origin) and are the simplest homogeneous maps that generalize the vortex map. In Section 2.3, we extend the analysis to the class $W^{1,1}(\Omega; \mathbb{S}^1)$ and show an integral representation formula for the BV-relaxed area. The last Section 2.4 is dedicated to the case of symmetric piecewise constant maps, which are valued in the ordered vertices of a regular polygon (that we can assume inscribed in $\mathbb{S}^1$). The analysis of these maps is not different from the one of the triple point map u_T, on which we shall focus. In particular, we exhibit an explicit recovery sequence for $\overline{\mathcal{A}}_{BV}(u_T; \Omega)$, mostly inspired to the construction in [8]. The content of this chapter is based on results published in [3].

2.1 Sobolev maps and topological degree

In what follows $B_r(x)$ denotes the open ball of $\mathbb{R}^2$ centered at x of radius $r > 0$.

Definition 2.1.1 (Multiplicity). Given $u \in W^{1,1}(\Omega; \mathbb{R}^2)$, for all measurable sets $A \subseteq \Omega$ and all $y \in \mathbb{R}^2$, we set

$$\mathrm{mult}(u, A, y) := \sharp\{u^{-1}(y) \cap A \cap \mathcal{R}_u\},$$

where $\mathcal{R}_u \subseteq \Omega$ is the set of regular points of u (see [26, pag. 202]). Similarly, if $u \in W^{1,1}(\partial B_r(x); \mathbb{S}^1)$, we define

$$\mathrm{mult}(u, A, y) := \sharp\{u^{-1}(y) \cap A \cap \mathcal{R}_u\},$$

for all measurable sets $A \subseteq \partial B_r(x)$ and all $y \in \mathbb{S}^1$.

Let $u \in W^{1,1}(\Omega; \mathbb{R}^2)$; by [26, Theorem 1, Section 3.1.5], if $\det\nabla u \in L^1(\Omega)$, we have

$$\int_A |\det \nabla u| dx = \int_{\mathbb{R}^2} \mathrm{mult}(u, A, y) dy, \tag{2.1.1}$$

for any measurable set $A \subseteq \Omega$. In particular, $\text{mult}(u, A, \cdot)$ is measurable and finite a.e. in $\mathbb{R}^2$.

If a Lipschitz continuous map $\varphi : \partial B_r(x) \to \mathbb{S}^1$ has constant multiplicity on $\partial B_r(x)$, then we will make use of the simplified notation

$$\text{mult}(\varphi) := \text{mult}(\varphi, \partial B_r(x), \cdot).$$

Definition 2.1.2 (Degree). Given $u \in W^{1,1}(\Omega; \mathbb{R}^2)$ with $\det\nabla u \in L^1(\Omega)$, for all measurable sets $A \subseteq \Omega$, we let

$$\deg(u, A, y) := \sum_{x \in u^{-1}(y) \cap A \cap \mathcal{R}_u} \text{sign}(\det \nabla u(x)), \tag{2.1.2}$$

for those $y \in \mathbb{R}^2$ for which $\text{mult}(u, A, \cdot)$ is finite.

Clearly

$$\text{mult}(u, A, \cdot) \geq |\deg(u, A, \cdot)|. \tag{2.1.3}$$

By [25, Theorem 6, Section 3.1.5], if $\det\nabla u \in L^1(\Omega)$, then

$$\int_A \det \nabla u \, dx = \int_{\mathbb{R}^2} \deg(u, A, y) dy, \tag{2.1.4}$$

for any measurable set $A \subseteq \Omega$, and by (2.1.1) and (2.1.3)

$$\int_\Omega |\det \nabla u| dx \geq \int_{\mathbb{R}^2} |\deg(u, \Omega, y)| dy. \tag{2.1.5}$$

Remark 2.1.3. The notion (2.1.2) of degree is too weak to be related to the trace of u on $\partial\Omega$. However, homological invariance is recovered under stronger hypotheses on u; for instance if u, v are Lipschitz in $\widehat{\Omega} \supset\supset \Omega$ and $u = v$ in $\widehat{\Omega} \setminus \overline{\Omega}$, then $\deg(u, \Omega, \cdot) = \deg(v, \Omega, \cdot)$ a.e. in $\mathbb{R}^2$ (see [26, pag. 233 and 469]). In particular, if $u, v : B_r(x) \to \mathbb{R}^2$ are Lipschitz continuous and $u = v$ on $\partial B_r(x)$, then we might extend u to a Lipschitz map $\overline{u}$ on $\mathbb{R}^2$; the map $\overline{v}$ coinciding with v in $B_r(x)$ and with $\overline{u}$ outside $B_r(x)$ is a Lipschitz extension of v. Hence $\deg(\overline{u}, B_r(x), \cdot) = \deg(\overline{v}, B_r(x), \cdot)$, which implies $\deg(u, B_r(x), \cdot) = \deg(v, B_r(x), \cdot)$.

Definition 2.1.4. For an open disc $B_r(x) \subset \mathbb{R}^2$ and $u \in W^{1,1}(\partial B_r(x); \mathbb{S}^1)$, we define (see (1.4.3))

$$\deg(u) := \frac{1}{2\pi} \int_{\partial B_r(x)} \Big(u_1 \frac{\partial u_2}{\partial s} - u_2 \frac{\partial u_1}{\partial s} \Big) ds \ \in \ \mathbb{Z}. \tag{2.1.6}$$

If $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, $B_r(x) \subset\subset \Omega$, and $u \llcorner \partial B_r(x) \in W^{1,1}(\partial B_r(x); \mathbb{S}^1)$ (which is true for almost every r), we set

$$\deg(u, \partial B_r(x)) := \deg(u \llcorner \partial B_r(x)). \tag{2.1.7}$$

Remark 2.1.5. If $u : B_r(x) \to \mathbb{R}^2$ is Lipschitz continuous and $|u| = 1$ on $\partial B_r(x)$, then $\deg(u, B_r(x), \cdot)$ is constant in $B_1 = B_1(0)$, and coincides with $\deg(u, \partial B_r(x))$. Indeed $\deg(u, B_r(x), \cdot)$ is a constant c in B_1 thanks to [30, Theorem 1.3] (and zero on $\mathbb{R}^2 \setminus B_1$), and then it is sufficient to check that $\deg(u, B_r(x), y) = \deg(u, \partial B_r(x))$, for a.e. $y \in B_1$. By applying (1.4.3) to the left-hand side of (2.1.4) one has

$$\int_{\mathbb{R}^2} \deg(u, B_r(x), y)dy = \int_{B_1} \deg(u, B_r(x), y)dy = \pi c$$
$$= \int_{B_r(x)} \det\nabla u(z)\, dz = \pi\deg(u \llcorner \partial B_r(x)).$$

In this particular case, thanks to (2.1.5), we conclude

$$\int_{B_r(x)} |\det \nabla u(z)| dz \geq \int_{B_1} |\deg(u, \partial B_r(x))| dy = \pi |\deg(u, \partial B_r(x))|. \tag{2.1.8}$$

2.1.1 Singular Sobolev maps with values in $\mathbb{S}^1$

We will make use of the following theorems.

Theorem 2.1.6. Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then

$$\overline{TVJ}_{W^{1,1}}(u; \Omega) < +\infty \Longleftrightarrow \mathrm{Det}\nabla u \quad \text{is a finite Radon measure.}$$

In this case $\overline{TVJ}_{W^{1,1}}(u; \Omega) = |\mathrm{Det}\nabla u|(\Omega)$, and there exists a finite set $\{x_1, \ldots, x_m\}$ of points in Ω such that

$$\mathrm{Det}\nabla u = \pi \sum_{i=1}^{m} d_i \delta_{x_i}, \tag{2.1.9}$$

where $d_i = \deg(u, \partial B_{r_i}(x_i)) \in \mathbb{Z} \setminus \{0\}$ for a.e. $r_i > 0$ small enough. In particular

$$|\mathrm{Det}\nabla u|(\Omega) = \pi \sum_{i=1}^{m} |d_i|.$$

Proof. See for instance [11, Proposition 3, Theorem 11 and Remark 11]. See also [32, Proposition 5.2]. □

Remark 2.1.7. Theorem 2.1.6 provides the existence of a radius $r_i > 0$ such that the number d_i not only is the degree of the trace of u on $\partial B_{r_i}(x_i)$, but also on almost every circumference $\partial B_\rho(x_i)$ with $\rho < r_i$. Moreover, on these circumferences, we may assume that u is continuous, since its trace is still of class $W^{1,1}$. For more details, we refer the reader to [11].

Remark 2.1.8. If $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ and we do not assume the finiteness of $\mathrm{Det}\nabla u$, then one can see that there exist points $\{P_j, N_j\}_{j=1}^{\infty} \in \overline{\Omega}$ such that $\sum_{j=1}^{\infty} |P_j - N_j| < +\infty$ and $\mathrm{Det}\nabla u = \pi \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j})$. This result can be found in [13, Theorem 2.10], see also [12].

Theorem 2.1.9. Let $u \in W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$. Then there exists a sequence in $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ converging to u in $W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$.

Proof. See [37, Theorem 2.1]. □

Theorem 2.1.10. Let $B \subset \mathbb{R}^2$ be a bounded open connected set, and $u \in W^{1,1}(B; \mathbb{S}^1)$. Then there exists a sequence in $C^\infty(B; \mathbb{S}^1)$ converging to u in $W^{1,1}(B; \mathbb{S}^1)$ if and only if $\mathrm{Det}\nabla u = 0$ in the sense of distribution.

Proof. See [43, Theorem 1.5]. □

2.2 Relaxation for vortex-type maps in $W^{1,p}(B_\ell;\mathbb{S}^1)$

In this section we focus on maps $w \in W^{1,1}(B_\ell;\mathbb{S}^1)$ of the form

$$w(x) = \varphi\left(\frac{x}{|x|}\right), \tag{2.2.1}$$

where $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ is a Lipschitz map.

Of course $\det \nabla w = 0$ a.e. on B_ℓ. Moreover, $w \in W^{1,p}(B_\ell;\mathbb{S}^1)$ for every $p \in [1,2)$; indeed, for $x \in B_\ell \setminus \{0\}$, let us write in polar coordinates

$$w(x) = \widetilde{w}(\rho,\theta) = \varphi(\cos\theta,\sin\theta) =: f(\theta) = (f_1(\theta), f_2(\theta)) \qquad \forall \rho \in (0,\ell), \quad \forall \theta \in [0,2\pi). \tag{2.2.2}$$

Then for a.e. $\theta \in [0,2\pi)$ and all $\rho \in (0,\ell)$

$$\nabla_{\rho,\theta}\widetilde{w}(\rho,\theta) = \begin{pmatrix} 0 & f_1'(\theta) \\ 0 & f_2'(\theta) \end{pmatrix}, \qquad |\nabla_{\rho,\theta}\widetilde{w}(\rho,\theta)| = |\partial_\theta \widetilde{w}(\rho,\theta)| = |f'(\theta)|,$$

$$\begin{aligned} \int_{B_\ell} |\nabla w|^p dx &= \int_0^{2\pi}\int_0^\ell \rho\left(|\partial_\rho \widetilde{w}|^2 + \frac{|\partial_\theta \widetilde{w}|^2}{\rho^2}\right)^{\frac{p}{2}} d\rho d\theta \\ &= \int_0^{2\pi}\int_0^\ell \frac{|f'(\theta)|^p}{\rho^{p-1}} d\rho d\theta \le 2\pi \mathrm{lip}(f)^p \int_0^\ell \frac{1}{\rho^{p-1}} d\rho < +\infty; \end{aligned} \tag{2.2.3}$$

in particular

$$\int_{B_\ell} |\nabla w| dx = \ell \int_0^{2\pi} |f'(\theta)| d\theta. \tag{2.2.4}$$

Remark 2.2.1. We have used that f in (2.2.2) is Lipschitz continuous in $[0,2\pi]$. Let us check that $\mathrm{lip}(f) = \mathrm{lip}(\varphi)$ and, moreover, $\mathrm{Var}(f) := \int_0^{2\pi} |f'(\theta)| d\theta = \int_{\mathbb{S}^1} |\nabla^{\mathbb{S}^1}\varphi(y)| d\mathcal{H}^1(y) = \mathrm{Var}(\varphi)$, where

$$\nabla^{\mathbb{S}^1}\varphi(z) := \lim_{\substack{y\to z \\ y\in\mathbb{S}^1\setminus\{z\}}} \frac{\varphi(y)-\varphi(z)}{|y-z|}, \tag{2.2.5}$$

is the (tangential) derivative of φ on $\mathbb{S}^1$, that is well-defined for a.e. $z \in \mathbb{S}^1$ as an element of the tangent space $T_{\varphi(z)}\mathbb{S}^1$ to $\mathbb{S}^1$ at $\varphi(z)$. Fix $y_0 \in \mathbb{S}^1$ where φ is differentiable, and take the unique $\theta_0 \in [0,2\pi)$ such that $y_0 = (\cos\theta_0, \sin\theta_0)$. From (2.2.5), it follows

$$\nabla^{\mathbb{S}^1}\varphi(y_0) = \frac{d}{d\theta}_{|\theta=\theta_0} \varphi(\cos\theta,\sin\theta) = f'(\theta_0), \tag{2.2.6}$$

and therefore $\mathrm{lip}(\varphi) = \mathrm{lip}(f)$. Moreover

$$\mathrm{Var}(\varphi) = \int_{\mathbb{S}^1} |\nabla^{\mathbb{S}^1}\varphi(y)| d\mathcal{H}^1(y) = \int_0^{2\pi} |f'(\theta)| d\theta = \mathrm{Var}(f). \tag{2.2.7}$$

In particular, from (2.2.4), we conclude

$$\int_{B_\ell} |\nabla w| \; dx = \ell \mathrm{Var}(\varphi). \tag{2.2.8}$$

Remark 2.2.2 (**Lifting**). A lifting of φ is a map $\overline{\Phi}:[0,2\pi]\to\mathbb{R}$ such that

$$\varphi(\cos\theta,\sin\theta)=(\cos(\overline{\Phi}(\theta)),\sin(\overline{\Phi}(\theta)))\qquad\forall\theta\in[0,2\pi]. \tag{2.2.9}$$

The function $f(\cdot)=\varphi(\cos(\cdot),\sin(\cdot)):[0,2\pi]\to\mathbb{S}^1$ being continuous on a simply-connected set, always admits a continuous lifting $\overline{\Phi}:[0,2\pi]\to\mathbb{R}$ such that

$$\varphi(\cos\theta,\sin\theta)=f(\theta)=(\cos(\overline{\Phi}(\theta)),\sin(\overline{\Phi}(\theta))).$$

Moreover, since the covering map $t\in\mathbb{R}\mapsto e^{it}\in\mathbb{S}^1$ satisfies $|e^{it_1}-e^{it_2}|\le|t_1-t_2|\le\pi|e^{it_1}-e^{it_2}|$ for all t_1,t_2 with $|t_1-t_2|\le\pi$, any continuous lifting of φ must be Lipschitz, indeed if $|\theta_1-\theta_2|\le\pi$, then

$$\frac{|\overline{\Phi}(\theta_1)-\overline{\Phi}(\theta_2)|}{|\theta_1-\theta_2|}\le\pi\frac{|e^{i\overline{\Phi}(\theta_1)}-e^{i\overline{\Phi}(\theta_2)}|}{|e^{i\theta_1}-e^{i\theta_2}|}=\pi\frac{|\varphi(e^{i\theta_1})-\varphi(e^{i\theta_2})|}{|e^{i\theta_1}-e^{i\theta_2}|};$$

while if $|\theta_1-\theta_2|>\pi$, the left-hand side is bounded by $\frac{2}{\pi}\max_{[0,2\pi]}|\overline{\Phi}|$.

Using the 2π-periodicity of f, we see that $\overline{\Phi}(2\pi)-\overline{\Phi}(0)\in2\pi\mathbb{Z}$; hence $\overline{\Phi}$ can be extended in a Lipschitz way to the whole of $\mathbb{R}$ (this can be done extending periodically its first derivative). It is possible to see that the lifting is unique up to a multiple of 2π: fix a starting point, e.g. $(1,0)\in\mathbb{S}^1$ and set $\varphi(1,0)=:y_0\in\mathbb{S}^1$. Now extract the Argument $\theta(y_0)\in[0,2\pi)$ of y_0, and define $\Phi:\mathbb{R}\to\mathbb{R}$ as

$$\Phi(t):=\theta(y_0)+\int_0^t\lambda_\varphi(s)ds, \tag{2.2.10}$$

where $\lambda_\varphi(s)\in\mathbb{R}$ is uniquely determined by

$$\nabla^{\mathbb{S}^1}\varphi(\cos s,\sin s)=\lambda_\varphi(s)\tau_{\varphi(\cos s,\sin s)}\qquad\text{a.e. } s\in\mathbb{R}, \tag{2.2.11}$$

with

$$\tau_{\varphi(\cos s,\sin s)}=\varphi^\perp(\cos s,\sin s)=\big(-\varphi_2(\cos s,\sin s),\varphi_1(\cos s,\sin s)\big) \tag{2.2.12}$$

the unit tangent vector to $\mathbb{S}^1$ (counter-clockwise oriented) at the point $\varphi(\cos s,\sin s)$. By definition, Φ is Lipschitz in $\mathbb{R}$ since $\text{lip}(\Phi)=\|\lambda_\varphi\|_\infty=\text{lip}(\varphi)$. In order to show the lifting property (2.2.9), take a lifting $\overline{\Phi}:\mathbb{R}\to\mathbb{R}$ of φ. Differentiating the equality $\varphi(\cos s,\sin s)=(\cos(\overline{\Phi}(s)),\sin(\overline{\Phi}(s)))$ gives

$$\lambda_\varphi(s)\tau_{\varphi(\cos s,\sin s)}=\overline{\Phi}'(s)(-\sin(\overline{\Phi}(s)),\cos(\overline{\Phi}(s)))=\overline{\Phi}'(s)\tau_{\varphi(\cos s,\sin s)},\qquad\text{a.e. } s\in\mathbb{R},$$

so that $\overline{\Phi}'=\lambda_\varphi$ a.e. in $\mathbb{R}$. This implies, by (2.2.10), that $\Phi(t)-\overline{\Phi}(t)$ is a constant multiple of 2π. Thus Φ also satisfies (2.2.9), and any lifting of φ is of the form (2.2.10), up to a constant multiple of 2π.

As a further consequence of the previous discussion and of (2.2.11)-(2.2.12), for any lifting $\widetilde{\Phi}$ of φ, and in particular for Φ, the map $\widetilde{f}(\theta)=(\cos(\widetilde{\Phi}(\theta)),\sin(\widetilde{\Phi}(\theta)))$ satisfies the same linear ordinary differential system as f, namely

$$f_1'=-\Phi'f_2,\qquad f_2'=\Phi'f_1\qquad\text{a.e. in }\mathbb{R}. \tag{2.2.13}$$

Finally, from (2.2.13) it follows $\lambda_\varphi=f_1f_2'-f_2f_1'$ a.e. in $\mathbb{R}$, so that by (2.1.6), we get

$$\Phi(2\pi)=\Phi(0)+\int_0^{2\pi}\lambda_\varphi(\theta)d\theta=\Phi(0)+2\pi\deg(\varphi). \tag{2.2.14}$$

Now we prove the following

Theorem 2.2.3 (Relaxation for vortex-type maps). Let $\ell > 0$, and $w : B_\ell \setminus \{0\} \to \mathbb{S}^1$ be as in (2.2.1). Then

$$\overline{\mathcal{A}}_{BV}(w; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla w|^2} dx + \pi |\deg(\varphi)|. \tag{2.2.15}$$

In particular,

$$\overline{\mathcal{A}}_{BV}(u_V; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u_V|^2} dx + \pi. \tag{2.2.16}$$

We divide the proof into two parts, the lower bound (Proposition 2.2.4) and the upper bound (Proposition 2.2.5).

Proposition 2.2.4 (Lower bound). Let $w : B_\ell \setminus \{0\} \to \mathbb{S}^1$ be the map defined in (2.2.1). Suppose that $(v_k) \subset C^1(B_\ell; \mathbb{R}^2) \cap BV(B_\ell; \mathbb{R}^2)$ is such that $v_k \to w$ strictly $BV(B_\ell; \mathbb{R}^2)$. Then

$$\liminf_{k \to +\infty} \mathcal{A}(v_k; B_\ell) \geq \int_{B_\ell} \sqrt{1 + |\nabla w|^2} dx + \pi |\deg(\varphi)|.$$

Proof. We may assume that

$$\liminf_{k \to +\infty} \mathcal{A}(v_k; B_\ell) = \lim_{k \to +\infty} \mathcal{A}(v_k; B_\ell) < +\infty.$$

We define the functions $\psi_k, \psi : (0, \ell) \to [0, +\infty)$ as

$$\psi_k(r) := \int_{\partial B_r} |\nabla v_k| ds, \quad \psi(r) := \liminf_{k \to +\infty} \psi_k(r), \qquad r \in (0, \ell),$$

where s is an arc length parameter on ∂B_r. By Fubini's theorem it follows

$$\int_0^\ell \psi_k(r) dr = \int_{B_\ell} |\nabla v_k| dx,$$

hence, using Fatou's lemma, the strict convergence of (v_k) to w, and (2.2.8),

$$\begin{aligned} \int_0^\ell \psi(r) dr \leq \liminf_{k \to +\infty} \int_0^\ell \psi_k(r) dr = \lim_{k \to +\infty} \int_{B_\ell} |\nabla v_k| dx \\ = \int_{B_\ell} |\nabla w| dx = \ell \mathrm{Var}(\varphi). \end{aligned} \tag{2.2.17}$$

In particular,

$$\psi \text{ is almost everywhere finite in } (0, \ell).$$

Now we claim that

$$\psi = \mathrm{Var}(\varphi) \quad \text{ a.e. in } (0, \ell). \tag{2.2.18}$$

Indeed, without loss of generality we may assume that (v_k) converges to w almost everywhere in B_ℓ, so that for almost every $r \in (0, \ell)$

$$v_k \llcorner \partial B_r \to w \llcorner \partial B_r \qquad \mathcal{H}^1 - \text{a.e. in } \partial B_r. \tag{2.2.19}$$

Now fix $r \in (0,\ell)$ such that (2.2.19) holds; consider the total variation of $v_k \llcorner \partial B_r$, that is the $L^1(\partial B_r)$-norm of the tangential derivative of v_k (as in (2.2.5)):

$$|D(v_k \llcorner \partial B_r)|(\partial B_r) = \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds.$$

Clearly

$$\liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds \le \liminf_{k\to+\infty} \int_{\partial B_r} |\nabla v_k| ds = \psi(r). \tag{2.2.20}$$

Let us extract a subsequence $(v_{k_h}) \subset (v_k)$ depending on r, such that

$$\liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds = \lim_{h\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_{k_h}}{\partial s}\right| ds. \tag{2.2.21}$$

Since ψ is almost everywhere finite, we may suppose that $\psi(r) < +\infty$, so that the sequence $(v_{k_h} \llcorner \partial B_r)$ is bounded in $BV(\partial B_r;\mathbb{R}^2)$. Thus, using (2.2.19), we also have

$$v_{k_h} \llcorner \partial B_r \rightharpoonup w \llcorner \partial B_r \quad \text{weakly}^* \text{ in } BV(\partial B_r;\mathbb{R}^2) \quad \text{as } h \to +\infty. \tag{2.2.22}$$

Now, since ∇w is only tangential, and $|\nabla w(r,\theta)|^2 = \frac{|f'(\theta)|^2}{r^2}$, we get

$$\int_{\partial B_r} \left|\frac{\partial w}{\partial s}\right| ds = \int_{\partial B_r} |\nabla w|\, ds = \int_0^{2\pi} r|f'(\theta)|\frac{1}{r} d\theta = \mathrm{Var}(\varphi). \tag{2.2.23}$$

Hence, using the lower semicontinuity of the variation along $(v_{k_h} \llcorner \partial B_r)$, (2.2.21), and (2.2.20) we infer

$$\begin{aligned}\mathrm{Var}(\varphi) &= \int_{\partial B_r} \left|\frac{\partial w}{\partial s}\right| ds \le \liminf_{h\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_{k_h}}{\partial s}\right| ds \\ &= \lim_{h\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_{k_h}}{\partial s}\right| ds = \liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds \le \psi(r).\end{aligned} \tag{2.2.24}$$

Thus $\psi \ge \mathrm{Var}(\varphi)$ almost everywhere in $(0,\ell)$ and, from (2.2.17), we deduce $\psi = \mathrm{Var}(\varphi)$ almost everywhere in $(0,\ell)$, and so (2.2.18) is proved.

As a consequence of the previous arguments,

$$\begin{aligned}&\forall \varepsilon \in (0,\ell) \quad \exists r_\varepsilon \in (0,\varepsilon) \quad \exists (v_{k_h}) \subset (v_k) \quad \text{s.t.} \\ &v_{k_h} \llcorner \partial B_{r_\varepsilon} \to w \llcorner \partial B_{r_\varepsilon} \quad \text{strictly } BV(\partial B_{r_\varepsilon};\mathbb{R}^2),\end{aligned} \tag{2.2.25}$$

where the subsequence (v_{k_h}) depends on ε. Indeed, proving (2.2.18), we have shown that for almost every $r \in (0,\ell)$, there exists a subsequence (v_{k_h}) satisfying (2.2.22); so, given $\varepsilon \in (0,\ell)$, there exists $r_\varepsilon \in (0,\varepsilon)$ and a subsequence (v_{k_h}) depending on ε, such that

$$v_{k_h} \llcorner \partial B_{r_\varepsilon} \rightharpoonup w \llcorner \partial B_{r_\varepsilon} \quad \text{weakly}^* \text{ in } BV(\partial B_{r_\varepsilon};\mathbb{R}^2). \tag{2.2.26}$$

But from the previous discussion we also deduce

$$\lim_{h\to+\infty} \int_{\partial B_{r_\varepsilon}} \left|\frac{\partial v_{k_h}}{\partial s}\right| ds = \psi(r_\varepsilon) = \mathrm{Var}(\varphi) = \int_{\partial B_{r_\varepsilon}} \left|\frac{\partial w}{\partial s}\right| ds; \tag{2.2.27}$$

thus the convergence in (2.2.26) is actually strict in $BV(\partial B_{r_\varepsilon};\mathbb{R}^2)$.

Now, fix $\varepsilon \in (0,\ell)$ and, for simplicity, denote by (v_h) the subsequence (v_{k_h}) for which (2.2.25) holds. Remember that our approximating maps $v_h = ((v_h)_1,(v_h)_2)$ are of class $C^1(\Omega;\mathbb{R}^2)$, so they might have non-zero Jacobian determinant $Jv_h := \det\nabla v_h$, as opposed to $w = (w_1, w_2)$, whose Jacobian determinant vanishes a.e. in B_ℓ. In particular, we expect the contribution of area given by Jv_h to be non trivial around the origin. Thus, we split the area functional as follows:

$$\mathcal{A}(v_h;B_\ell) = \mathcal{A}(v_h;B_\ell\setminus B_{r_\varepsilon}) + \mathcal{A}(v_h;B_{r_\varepsilon}) \geq \mathcal{A}(v_h;B_\ell\setminus B_{r_\varepsilon}) + \int_{B_{r_\epsilon}} |Jv_h|dx,$$

and notice that, by definition of relaxed functional and [1, Theorem 3.7],

$$\liminf_{h\to+\infty} \mathcal{A}(v_h;B_\ell\setminus B_{r_\varepsilon}) \geq \overline{\mathcal{A}}_{L^1}(u;B_\ell\setminus B_{r_\varepsilon}) \geq \int_{B_\ell\setminus B_{r_\varepsilon}} \sqrt{1+|\nabla w|^2}dx.$$

Hence

$$\begin{aligned}\lim_{h\to+\infty} \mathcal{A}(v_h;B_\ell) &\geq \liminf_{h\to+\infty} \mathcal{A}(v_h;B_\ell\setminus B_{r_\varepsilon}) + \liminf_{h\to+\infty}\int_{B_{r_\epsilon}} |Jv_h|\ dx \\ &\geq \int_{B_\ell\setminus B_{r_\varepsilon}} \sqrt{1+|\nabla w|^2}dx + \liminf_{h\to+\infty}\int_{B_{r_\epsilon}} |Jv_h|\ dx.\end{aligned} \tag{2.2.28}$$

To conclude the proof it is then sufficient to show that

$$\liminf_{h\to+\infty}\int_{B_{r_\varepsilon}} |Jv_h|dx \geq \pi|\deg(\varphi)|. \tag{2.2.29}$$

Define the sequence $w_h : B_\ell \to \mathbb{R}^2$ as

$$w_h(x) := \begin{cases} v_h(x) & \text{if } |x|\leq r_\varepsilon \\ \dfrac{\ell-|x|}{\ell-r_\varepsilon} v_h\left(r_\varepsilon\dfrac{x}{|x|}\right) + \dfrac{|x|-r_\varepsilon}{\ell-r_\varepsilon} w\left(r_\varepsilon\dfrac{x}{|x|}\right) & \text{if } r_\varepsilon<|x|<\ell. \end{cases} \tag{2.2.30}$$

Then w_h is Lipschitz continuous and interpolates $v_h \llcorner \partial B_{r_\varepsilon}$ and $w \llcorner \partial B_{r_\varepsilon}$ in the annulus enclosed by $\partial B_{r_\varepsilon}$ and ∂B_ℓ. Now we show that

$$\lim_{h\to+\infty}\int_{B_\ell\setminus B_{r_\varepsilon}} |Jw_h|dx = 0. \tag{2.2.31}$$

Indeed, passing to polar coordinates in $B_\ell\setminus B_{r_\varepsilon}$:

$$w_h(x) = \widetilde{w}_h(\rho,\theta) = \frac{\ell-\rho}{\ell-r_\varepsilon}\widetilde{v}_h(r_\varepsilon,\theta) + \frac{\rho-r_\varepsilon}{\ell-r_\varepsilon}\widetilde{w}(r_\varepsilon,\theta),$$

where

$$\begin{aligned}&\widetilde{v}_h(r_\varepsilon,\theta) := v_h\left(r_\varepsilon(\cos\theta,\sin\theta)\right) = ((\widetilde{v}_h)_1(r_\varepsilon,\theta),(\widetilde{v}_h)_2(r_\varepsilon,\theta)),\\ &\widetilde{w}(r_\varepsilon,\theta) := w\left(r_\varepsilon(\cos\theta,\sin\theta)\right) = f(\theta).\end{aligned}$$

Making use of (2.2.2) and (2.2.13), we get

$$\partial_\rho \widetilde{w}_h(\rho,\theta) = \frac{1}{\ell - r_\varepsilon}(-\widetilde{v}_h + f),$$
$$\partial_\theta \widetilde{w}_h(\rho,\theta) = \frac{1}{\ell - r_\varepsilon}\left[(\ell-\rho)\partial_\theta \widetilde{v}_h - (\rho - r_\varepsilon)\Phi' f^\perp\right],$$

where $f^\perp := (-f_2, f_1)$, $\widetilde{v}_h$ is evaluated at (r_ε,θ), and f and Φ' are evaluated at θ. Then we can compute:

$$\begin{aligned}\partial_\rho \widetilde{w}_h \wedge \partial_\theta \widetilde{w}_h =& \frac{1}{(\ell - r_\varepsilon)^2}\Big[(\ell-\rho)\Big\{(\widetilde{v}_h)_2\partial_\theta(\widetilde{v}_h)_1 - \partial_\theta(\widetilde{v}_h)_1 f_2\Big\}\\ &+ (\ell-\rho)\Big\{\partial_\theta(\widetilde{v}_h)_2 f_1 - (\widetilde{v}_h)_1\partial_\theta(\widetilde{v}_h)_2\Big\} - (\rho - r_\varepsilon)\Phi'\Big\{(\widetilde{v}_h)_1 f_1 + (\widetilde{v}_h)_2 f_2 - 1\Big\}\Big],\end{aligned}$$

where we use also that $f_1^2 + f_2^2 = 1$. Thus since

$$Jw_h(\rho\cos\theta, \rho\sin\theta) = \partial_\rho \widetilde{w}_h(\rho,\theta) \wedge \frac{1}{\rho}\partial_\theta \widetilde{w}_h(\rho,\theta),$$

by the change of variable formula we get

$$\begin{aligned}\int_{B_\ell\setminus B_{r_\varepsilon}} |Jw_h| dx =& \int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |\partial_\rho \widetilde{w}_h \wedge \partial_\theta \widetilde{w}_h| d\rho d\theta\\ \leq& C_{\ell,\varepsilon}\int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |(\widetilde{v}_h)_2\partial_\theta(\widetilde{v}_h)_1 - \partial_\theta(\widetilde{v}_h)_1 f_2| d\rho d\theta\\ &+ C_{\ell,\varepsilon}\int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |(\widetilde{v}_h)_1\partial_\theta(\widetilde{v}_h)_2 - \partial_\theta(\widetilde{v}_h)_2 f_1| d\rho d\theta\\ &+ C_{\ell,\varepsilon}\mathrm{lip}(\Phi)\int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |(\widetilde{v}_h)_1 f_1 + (\widetilde{v}_h)_2 f_2 - 1| d\rho d\theta,\end{aligned} \tag{2.2.32}$$

where $C_{\ell,\varepsilon}$ is a positive constant depending only on ℓ and ε. Consider the first integral on the right hand side of (2.2.32): its integrand is independent of ρ, and so

$$\begin{aligned}&\int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |(\widetilde{v}_h)_2\partial_\theta(\widetilde{v}_h)_1 - \partial_\theta(\widetilde{v}_h)_1 f_2(\theta)|\, d\rho d\theta\\ =& (\ell - r_\varepsilon)\int_0^{2\pi} |(\widetilde{v}_h)_2(r_\varepsilon,\theta) - f_2(\theta)|\, |\partial_\theta(\widetilde{v}_h)_1(r_\varepsilon,\theta)|\, d\theta\\ \leq& C_{\ell,\varepsilon}\|(v_h)_2 - w_2\|_{L^\infty(\partial B_{r_\varepsilon})}\int_{\partial B_{r_\varepsilon}}\left|\frac{\partial v_h}{\partial s}\right| ds \xrightarrow{k\to+\infty} 0,\end{aligned}$$

where in passing to the limit we used (2.2.25), which implies that the variation of v_h on $\partial B_{r_\varepsilon}$ is necessarily equi-bounded and, together with Proposition 1.3.6, that $v_h \to w$ uniformly on $\partial B_{r_\varepsilon}$. For the second integral, the argument is similar.

As for the third one, by the uniform convergence of (v_h) to w on $\partial B_{r_\varepsilon}$, we can pass to the limit under the integral sign:

$$\int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |(\widetilde{v}_h)_1 f_1 + (\widetilde{v}_h)_2 f_2 - 1| d\rho d\theta \xrightarrow{h\to+\infty} \int_{r_\varepsilon}^{\ell}\int_0^{2\pi} |f_1^2 + f_2^2 - 1| d\rho d\theta = 0.$$

Therefore, (2.2.31) holds.

Now, we write the Jacobian determinant of v_h on B_{r_ε} in the following way:

$$\int_{B_{r_\varepsilon}} |Jv_h|dx = \int_{B_\ell} |Jw_h|dx - \int_{B_\ell \setminus B_{r_\varepsilon}} |Jw_h|dx. \tag{2.2.33}$$

Notice that $w_h = w$ on ∂B_ℓ, so that (see Remarks 2.1.3 and 2.1.5)

$$\deg(w_h, \partial B_\ell) = \deg(w, \partial B_\ell) = \deg(\varphi). \tag{2.2.34}$$

We may suppose that v_h takes values in $\overline{B}_1$, since the limit function w is valued in $\mathbb{S}^1$ (see [1, Lemma 3.3]). So $w_h : \overline{B}_\ell \to \overline{B}_1$ is Lipschitz continuous and maps ∂B_ℓ into ∂B_1. Then, by (2.2.34) and (2.1.8), we have

$$\int_{B_\ell} |Jw_h|dx \geq \pi|\deg(w, \partial B_\ell)| = \pi|\deg(\varphi)|. \tag{2.2.35}$$

Finally, passing to the lower limit as $h \to +\infty$ in (2.2.33), using (2.2.31) and the previous inequality, we deduce estimate (2.2.29), which concludes the proof. □

Proposition 2.2.5 (Upper bound). Let $w : B_\ell \setminus \{0\} \to \mathbb{R}^2$ be the map defined in (2.2.1). Then there exists a sequence $(v_k) \subset C^1(B_\ell; \mathbb{R}^2) \cap BV(B_\ell; \mathbb{R}^2)$ such that $v_k \to w$ strictly $BV(B_\ell; \mathbb{R}^2)$ and

$$\limsup_{k \to +\infty} \mathcal{A}(v_k; B_\ell) \leq \int_{B_\ell} \sqrt{1 + |\nabla w|^2} dx + \pi|\deg(\varphi)|. \tag{2.2.36}$$

Proof. Although v_k needs to be of class C^1, we claim that it suffices to build v_k just Lipschitz continuous. Indeed, assume that $(v_k) \subset W^{1,\infty}(B_\ell; \mathbb{R}^2) \cap C^1(B_\ell; \mathbb{R}^2)$ converges to w strictly $BV(B_\ell; \mathbb{R}^2)$ and (2.2.36) holds. Consider, for all $k \in \mathbb{N}$, a sequence $(v_h^k) \subset C^1(B_\ell; \mathbb{R}^2)$ approaching v_k in $W^{1,2}(B_\ell; \mathbb{R}^2)$ as $h \to +\infty$. In particular, we get the L^1-convergence of all minors of ∇v_h^k to the corresponding ones of ∇v_k. Then, by dominated convergence,

$$\lim_{h \to +\infty} \mathcal{A}(v_h^k; B_\ell) = \mathcal{A}(v_k; B_\ell). \tag{2.2.37}$$

Hence, by a diagonal argument, we find a sequence $(v_{h_k}^k)$ converging to w strictly $BV(B_\ell; \mathbb{R}^2)$ such that (2.2.36) holds for $v_{h_k}^k$ in place of v_k.

Let us consider the map $\overline{\varphi} : \mathbb{S}^1 \to \mathbb{S}^1$ given by

$$\overline{\varphi}(\cos\theta, \sin\theta) := (\cos(d\theta), \sin(d\theta)) \quad \text{where } d := \deg(\varphi). \tag{2.2.38}$$

Then

$$\text{mult}(\overline{\varphi}) = |\deg(\overline{\varphi})|, \qquad \deg(\overline{\varphi}) = \deg(\varphi), \tag{2.2.39}$$

and, in particular, $\text{mult}(\overline{\varphi}) = |\deg(\varphi)|$. Moreover, since the maps φ and $\overline{\varphi}$ have the same degree, we can construct a Lipschitz homotopy $H : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ between them. Precisely, if Φ and $\overline{\Phi}$ are Lipschitz liftings of φ and $\overline{\varphi}$ respectively, we define $\Psi(t, \cdot) :=$

$t\Phi(\cdot)+(1-t)\overline{\Phi}(\cdot)$, which is Lipschitz. Hence one defines the map $H(t,\cdot):[0,2\pi)\to\mathbb{S}^1$ as $H(t,\cdot):=(\cos(\Psi(t,\cdot),\sin(\Psi(t,\cdot))),$ which satisfies

$$H(0,\cdot)=\overline{\varphi}(\cdot),\qquad H(1,\cdot)=\varphi(\cdot). \tag{2.2.40}$$

It remains to show that $H(t,\cdot)$ defines a continuous (and then Lipschitz) map from $\mathbb{S}^1$ to $\mathbb{S}^1$, i.e. that is 2π-periodic: to this aim it is enough to observe that $\Psi(t,2\pi)$ and $\Psi(t,0)$ differ from a constant multiple of 2π and indeed, recalling (2.2.14), we have $\Phi(2\pi)-\Phi(0)=2\pi d=\overline{\Phi}(2\pi)-\overline{\Phi}(0)$, from which easily follows that $\Psi(t,2\pi)-\Psi(t,0)=2\pi d$.

We now define the sequence $(v_k)\subset\mathrm{Lip}(B_\ell;\mathbb{R}^2)$ as $v_k(0):=0$,

$$v_k:=\begin{cases}\overline{v}_k & \text{in } B_{\frac{\ell}{k}}\setminus\{0\},\\ h_k & \text{in } B_{\frac{2\ell}{k}}\setminus B_{\frac{\ell}{k}},\\ w=\varphi\left(\frac{x}{|x|}\right) & \text{in } B_\ell\setminus B_{\frac{2\ell}{k}},\end{cases} \tag{2.2.41}$$

where

$$\overline{v}_k(x):=\frac{k}{\ell}|x|\overline{\varphi}\left(\frac{x}{|x|}\right)\qquad\forall x\in B_{\frac{\ell}{k}},$$

and

$$h_k(x):=H\left(\frac{k}{\ell}|x|-1,\frac{x}{|x|}\right)\qquad\forall x\in B_{\frac{2\ell}{k}}\setminus B_{\frac{\ell}{k}}.$$

Let us check that

$$\int_{B_\ell}|Jv_k|dx=\pi|d|\qquad\forall k\in\mathbb{N}. \tag{2.2.42}$$

Since H and w take values on $\mathbb{S}^1$, we have

$$\int_{B_\ell\setminus B_{\frac{\ell}{k}}}|Jv_k|dx=\int_{B_{\frac{2\ell}{k}}\setminus B_{\frac{\ell}{k}}}|Jh_k|dx+\int_{B_\ell\setminus B_{\frac{2\ell}{k}}}|Jw|dx=0.$$

Moreover, mult$(\overline{v}_k,B_{\frac{\ell}{k}},\cdot)$=mult$(\overline{\varphi})$, and therefore, by (2.1.1),

$$\int_{B_{\frac{\ell}{k}}}|Jv_k|dx=\int_{B_{\frac{\ell}{k}}}|J\overline{v}_k|dx=\int_{B_1}\mathrm{mult}(\overline{v}_k,B_{\frac{\ell}{k}},y)dy=|B_1|\mathrm{mult}(\overline{\varphi})=\pi|d|.$$

We now prove that $v_k\to w$ in $W^{1,p}(B_\ell;\mathbb{R}^2)$ for every $p\in[1,2)$. This, in particular, implies the desired strict convergence in BV. Since $v_k=w$ in $B_\ell\setminus B_{\frac{2\ell}{k}}$, we have to do the computation in $B_{\frac{2\ell}{k}}$:

$$\int_{B_{\frac{2\ell}{k}}}|v_k-w|^p dx\le 2^{p-1}\int_{B_{\frac{2\ell}{k}}}(|v_k|^p+|w|^p)dx\le 2^p|B_{\frac{2\ell}{k}}|\xrightarrow{k\to+\infty}0.$$

In addition

$$|\nabla v_k|=|\nabla h_k|\le 2k\,\mathrm{lip}(H)\quad\text{a.e. in } B_{\frac{2\ell}{k}}\setminus B_{\frac{\ell}{k}},$$

hence

$$\begin{aligned}\int_{B_{\frac{2\ell}{k}}\setminus B_{\frac{\ell}{k}}} |\nabla v_k - \nabla w|^p dx &\le C\left[(2k)^p \mathrm{lip}(H)^p |B_{\frac{2\ell}{k}}| + \int_{B_{\frac{2\ell}{k}}} |\nabla w|^p dx\right] \\ &\le C\left[C\frac{k^p}{k^2} + \int_{B_{\frac{2\ell}{k}}} |\nabla w|^p dx\right] \xrightarrow{k\to+\infty} 0,\end{aligned} \tag{2.2.43}$$

where $C > 0$ is a positive constant independent of k. Finally, setting $\overline{w}(x) := \overline{\varphi}\left(\frac{x}{|x|}\right)$ for $x \in B_\ell \setminus \{0\}$, we have

$$\nabla v_k(x) = \frac{k}{\ell}|x|\nabla\overline{w}(x) + \frac{k}{\ell}\overline{w}(x) \otimes \frac{x}{|x|} \qquad \text{for a.e. } x \in B_{\frac{\ell}{k}}.$$

Whence

$$\begin{aligned}\int_{B_{\frac{\ell}{k}}} |\nabla v_k - \nabla w|^p dx &\le C\int_{B_{\frac{\ell}{k}}} \left(k^p|x|^p|\nabla\overline{w}|^p + k^p\left|\overline{w}(x)\otimes\frac{x}{|x|}\right| + |\nabla w|^p\right) dx \\ &\le C\left[\int_{B_{\frac{\ell}{k}}} |\nabla\overline{w}|^p dx + k^p|B_{\frac{\ell}{k}}| + \int_{B_{\frac{\ell}{k}}} |\nabla w|^p dx\right] \xrightarrow{k\to+\infty} 0.\end{aligned} \tag{2.2.44}$$

Now, we easily get (2.2.36): upon extracting a (not relabelled) subsequence such that (∇v_k) converges almost everywhere to ∇w, by (2.2.42) and dominated convergence theorem we have

$$\begin{aligned}\limsup_{k\to+\infty} \mathcal{A}(v_k; B_\ell) &\le \lim_{k\to+\infty}\int_{B_\ell}\sqrt{1+|\nabla v_k|^2}dx + \lim_{k\to+\infty}\int_{B_\ell}|Jv_k|dx \\ &= \int_{B_\ell}\sqrt{1+|\nabla w|^2}dx + \pi|d|.\end{aligned}$$

□

Remark 2.2.6. In the proof of the upper bound in Proposition 2.2.5 we have shown the $W^{1,p}$ convergence of the recovery sequence to the function w, for $p \in [1,2)$. Hence

$$\overline{\mathcal{A}}_{W^{1,p}}(w; B_\ell) \le \int_{B_\ell}\sqrt{1+|\nabla w|^2}dx + \pi|\deg(\varphi)|.$$

Moreover, since in general $\overline{\mathcal{A}}_{BV}(\cdot\,; B_\ell) \le \overline{\mathcal{A}}_{W^{1,p}}(\cdot\,; B_\ell)$ for all $p \ge 1$, we deduce

$$\overline{\mathcal{A}}_{W^{1,p}}(w; B_\ell) = \int_{B_\ell}\sqrt{1+|\nabla w|^2}dx + \pi|\deg(\varphi)|.$$

Remark 2.2.7. As a consequence of Theorem 2.2.3, if $\varphi \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ has degree 0, then

$$\overline{\mathcal{A}}_{L^1}(w; B_\ell) = \overline{\mathcal{A}}_{BV}(w; B_\ell) = \int_{B_\ell}\sqrt{1+|\nabla w|^2}dx.$$

Indeed, in general $\overline{\mathcal{A}}_{L^1} \le \overline{\mathcal{A}}_{BV}$ and, by [1, Theorem 3.7], $\overline{\mathcal{A}}_{L^1}(w; B_\ell) \ge \int_{B_\ell}\sqrt{1+|\nabla w|^2}dx$.

2.3 Relaxation for maps in $W^{1,1}(\Omega;\mathbb{S}^1)$

The main result of this section is contained in Theorem 2.3.6. In the following lemma we generalize to a generic function in $W^{1,1}(B_\ell;\mathbb{R}^2)$ the argument used to prove (2.2.25), by showing that the strict BV convergence on B_ℓ is inherited to almost every circumference centered at the origin[1]. Unlike (2.2.25) of Proposition 2.2.4, in this more general context we have to make use of Theorem 1.2.1.

Lemma 2.3.1 (Inheritance). Let $(v_k) \subset C^1(B_\ell;\mathbb{R}^2)$, $u \in W^{1,1}(B_\ell;\mathbb{R}^2)$, and suppose that $v_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$. Then, for almost every $r \in (0,\ell)$, there exists a subsequence (v_{k_h}), depending on r, such that

$$v_{k_h} \llcorner \partial B_r \to u \llcorner \partial B_r \quad \text{strictly } BV(\partial B_r;\mathbb{R}^2).$$

Proof. The (tangential) variation of the restriction of u on ∂B_r is well-defined and finite for almost every $r \in (0,1)$ since $u \in W^{1,1}(B_\ell;\mathbb{R}^2)$, and

$$|D(u \llcorner \partial B_r)|(\partial B_r) := \int_{\partial B_r} \left|\frac{\partial u}{\partial s}\right| ds = \int_0^{2\pi} |\partial_\theta \widetilde{u}(r,\theta)| d\theta,$$

where $\widetilde{u} : R := (0,\ell) \times [0,2\pi) \to \mathbb{R}^2$, $\widetilde{u}(\rho,\theta) := u(\rho\cos\theta, \rho\sin\theta)$. We compute

$$\int_R |\partial_\theta \widetilde{u}|\, d\rho d\theta = \int_{B_\ell} |(\nabla u)\tau| dx, \tag{2.3.1}$$

with $\tau(x) := \frac{1}{|x|}(-x_2, x_1), x \neq 0$. Indeed

$$\begin{aligned}
&\int_R |\partial_\theta \widetilde{u}|\, d\rho d\theta \\
&= \int_0^\ell \int_0^{2\pi} \left[\sum_{i=1}^2 \rho^2 \left((\partial_{x_1} u_i)^2 (\sin\theta)^2 + (\partial_{x_2} u_i)^2 (\cos\theta)^2 - 2\partial_{x_1} u_i \partial_{x_2} u_i \cos\theta \sin\theta\right)\right]^{\frac{1}{2}} d\rho d\theta \\
&= \int_{B_\ell} \frac{1}{|x|} \left[\sum_{i=1}^2 \left((\partial_{x_1} u_i)^2 x_2^2 + (\partial_{x_2} u_i)^2 x_1^2 - 2\partial_{x_1} u_i \partial_{x_2} u_i x_1 x_2\right)\right]^{\frac{1}{2}} dx \\
&= \int_{B_\ell} \sqrt{|\nabla u_1 \cdot \tau|^2 + |\nabla u_2 \cdot \tau|^2} dx = \int_{B_\ell} |(\nabla u)\tau| dx.
\end{aligned}$$

In the same way we get

$$\int_R |\partial_\theta \widetilde{v}_k|\, d\rho d\theta = \int_{B_\ell} |(\nabla v_k)\tau| dx.$$

Thanks to Theorem 1.2.1, with the choices $M = 4$, $\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^{2\times 2}$, $f \in C_b((B_\ell \setminus \{0\}) \times \mathbb{S}^3)$,

$$f(x,\sigma) := \sqrt{|\sigma_{\text{hor}} \cdot \tau(x)|^2 + |\sigma_{\text{vert}} \cdot \tau(x)|^2},$$

[1] In Chapter 3, the reader can find the proof of a further generalized version of this result for a generic function in $BV(B_\ell;\mathbb{R}^2)$.

where $\sigma \in \mathbb{S}^3$ and $\sigma_{\text{hor}} := (\sigma_1, \sigma_2), \sigma_{\text{vert}} := (\sigma_3, \sigma_4)$, we obtain

$$\lim_{k\to+\infty} \int_{B_\ell} |(\nabla v_k)\tau| dx = \int_{B_\ell} |(\nabla u)\tau| dx. \tag{2.3.2}$$

Now we notice that for almost every $r \in (0, \ell)$ we have

$$v_k \llcorner \partial B_r \to u \llcorner \partial B_r \quad \text{in } L^1(\partial B_r; \mathbb{R}^2).$$

Then, since $(v_k \llcorner \partial B_r) \subset BV(\partial B_r; \mathbb{R}^2)$ for every $r \in (0, \ell)$, by the lower semicontinuity of the variation we get

$$\int_{\partial B_r} \left|\frac{\partial u}{\partial s}\right| ds \leq \liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds \quad \text{for a.e. } r \in (0, \ell). \tag{2.3.3}$$

Integrating with respect to r and by Fatou's lemma, we obtain

$$\int_R |\partial_\theta \tilde{u}|\, dr d\theta = \int_0^\ell \int_{\partial B_r} \left|\frac{\partial u}{\partial s}\right| ds dr \leq \int_0^\ell \liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds dr \leq \liminf_{k\to+\infty} \int_R |\partial_\theta \widetilde{v}_k|\, dr d\theta. \tag{2.3.4}$$

But we notice that, by (2.3.1) and (2.3.2), we must have all equalities in (2.3.4). In particular,

$$\int_{\partial B_r} \left|\frac{\partial u}{\partial s}\right| ds = \liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds \quad \text{for a.e. } r \in (0, \ell),$$

and we conclude extracting a suitable subsequence (v_{k_h}) of (v_k) depending on r such that

$$\lim_{h\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_{k_h}}{\partial s}\right| ds = \liminf_{k\to+\infty} \int_{\partial B_r} \left|\frac{\partial v_k}{\partial s}\right| ds.$$

□

Definition 2.3.2. Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ and $\overline{TVJ}_{W^{1,1}}(u; \Omega) < +\infty$. We set

$$\overline{TVJ}_{BV}(u; \Omega) := \inf \left\{ \liminf_{k\to+\infty} TVJ(v_k; \Omega) : (v_k) \subset C^1(\Omega, \mathbb{R}^2),\ v_k \to u \text{ strictly } BV \right\}.$$

The proof of Theorem 2.3.6 is essentially a consequence of the following theorem.

Theorem 2.3.3 (Relaxation of TVJ in the strict convergence). Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ be such that $\overline{TVJ}_{W^{1,1}}(u; \Omega) < +\infty$, and write $\text{Det}\nabla u$ as in (2.1.9). Then

$$\overline{TVJ}_{BV}(u; \Omega) = \pi \sum_{i=1}^m |d_i|.$$

In particular, $\overline{TVJ}_{BV}(u; \Omega) = \overline{TVJ}_{W^{1,1}}(u; \Omega) = |\text{Det}\nabla u|(\Omega)$.

As usual, we divide the proof of Theorem 2.3.3 into two parts, the lower bound (Proposition 2.3.4) and the upper bound (Proposition 2.3.5).

Proposition 2.3.4 (Lower bound for $\overline{TVJ}_{BV}$). Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ be such that $\overline{TVJ}_{W^{1,1}}(u;\Omega) < +\infty$, and write $\mathrm{Det}\nabla u$ as in (2.1.9). Then

$$\overline{TVJ}_{BV}(u;\Omega) \geq \pi \sum_{i=1}^{m} |d_i|.$$

Proof. According to Theorem 2.1.6, we choose a radius $\ell > 0$ so that the balls $B_\ell(x_i) \subset \Omega$, $i = 1, \dots, m$, are disjoint. Let $(v_k) \subset C^1(\Omega;\mathbb{R}^2)$ be such that $v_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$ and

$$\lim_{k\to+\infty} \int_\Omega |Jv_k| dx = \overline{TVJ}_{BV}(u;\Omega).$$

To show the thesis it is sufficient to prove that, for all $i = 1, \dots, m$,

$$\lim_{k\to+\infty} \int_{B_\ell(x_i)} |Jv_k| dx \geq \pi d_i,$$

and it suffices to show this inequality for $i = 1$. Let us denote $B_\ell(x_1)$ simply by B_ℓ. Without loss of generality we may assume $x_1 = (0,0)$. Since $u \in W^{1,1}(B_\ell;\mathbb{S}^1)$, it is $W^{1,1}(\partial B_r;\mathbb{S}^1)$, in particular continuous, for almost every $r \in (0,\ell)$. Thus, we can choose $\overline{r} > 0$ small enough so that $u \llcorner \partial B_{\overline{r}} \in W^{1,1}(\partial B_r;\mathbb{S}^1)$. Since the balls $B_\ell(x_i)$, $i = 1, \dots, m$, are disjoint, we also have $\deg(u, \partial B_{\overline{r}}, \cdot) = d_1$. From Theorem 2.1.9 and Lemma 2.3.1, we get that

$$\begin{aligned} &\forall \varepsilon \in (0,\overline{r}) \quad \exists r_\varepsilon \in (0,\varepsilon) \quad \exists (v_{k_h}) \subset (v_k) \quad \exists (u_h) \subset C^\infty(\partial B_{r_\varepsilon};\mathbb{S}^1) \quad \text{s.t.} \\ &u \llcorner \partial B_{r_\varepsilon} \in W^{1,1}(\partial B_{r_\varepsilon};\mathbb{S}^1), \quad u_h \to u \llcorner \partial B_{r_\varepsilon} \quad \text{in } W^{1,1}(\partial B_{r_\varepsilon};\mathbb{S}^1), \\ &\text{and } v_{k_h} \llcorner \partial B_{r_\varepsilon} \to u \llcorner \partial B_{r_\varepsilon} \quad \text{strictly } BV(\partial B_{r_\varepsilon};\mathbb{R}^2). \end{aligned} \tag{2.3.5}$$

In particular, on $\partial B_{r_\varepsilon}$ we have uniform convergence of (u_h) and (v_{k_h}) to u by Corollary 1.3.6. Setting as usual $Jv_{k_h} = \det\nabla v_{k_h}$, write

$$\int_{B_{r_\varepsilon}} |Jv_{k_h}| dx = \int_{B_{\overline{r}}} |Jw_h| dx - \int_{B_{\overline{r}}\setminus B_{r_\varepsilon}} |Jw_h| dx,$$

where $w_h \in \mathrm{Lip}(B_{\overline{r}};\mathbb{R}^2)$ and is given by

$$w_h(x) := \begin{cases} v_{k_h}(x) & \text{if } |x| \leq r_\varepsilon \\ \dfrac{\overline{r}-|x|}{\overline{r}-r_\varepsilon} v_{k_h}\left(r_\varepsilon \dfrac{x}{|x|}\right) + \dfrac{|x|-r_\varepsilon}{\overline{r}-r_\varepsilon} u_h\left(r_\varepsilon \dfrac{x}{|x|}\right) & \text{if } r_\varepsilon < |x| \leq \overline{r}. \end{cases} \tag{2.3.6}$$

Now, since $\|v_{k_h} - u_h\|_{L^\infty(\partial B_{r_\varepsilon})} \to 0$ as $h \to +\infty$, arguing as in the proof of (2.2.31) we have

$$\lim_{h\to+\infty} \int_{B_{\overline{r}}\setminus B_{r_\varepsilon}} |Jw_h| dx = 0. \tag{2.3.7}$$

Moreover, from (2.3.6) we note that

$$\deg(w_h, \partial B_{\overline{r}}) = \deg(u_h, \partial B_{r_\varepsilon}). \tag{2.3.8}$$

Thanks to the uniform convergence of (u_h) to u on $\partial B_{r_\varepsilon}$, for h large enough, u_h and $u \llcorner \partial B_{r_\varepsilon}$ must have the same degree

$$\deg(u_h, \partial B_{r_\varepsilon}) = \deg(u, \partial B_{r_\varepsilon}) = d_1.$$

Then, arguing as in (2.2.35), we obtain that

$$\int_{B_{\overline{r}}} |Jw_h|dx \geq \pi|\deg(w_h, \partial B_{\overline{r}})| = \pi|d_1|,$$

for $h \in \mathbb{N}$ sufficiently large. In conclusion we get

$$\overline{TVJ}_{BV}(u; B_\ell) = \lim_{h\to+\infty} \int_{B_\ell} |Jv_{k_h}|dx \geq \liminf_{h\to+\infty} \int_{B_{r_\varepsilon}} |Jv_{k_h}|dx \geq \liminf_{h\to+\infty} \int_{B_{\overline{r}}} |Jw_h|dx \geq \pi|d_1|. \tag{2.3.9}$$

□

Proposition 2.3.5 (Upper bound for $\overline{TVJ}_{BV}$). Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ be such that $\overline{TVJ}_{W^{1,1}}(u; \Omega) < +\infty$, and write $\mathrm{Det}\nabla u$ as in (2.1.9). Then

$$\overline{TVJ}_{BV}(u; \Omega) \leq \pi \sum_{i=1}^{m} |d_i|.$$

Proof. As in the proof of Proposition 2.3.4 we choose a radius $\ell > 0$ so that the balls $B_\ell(x_i) \subset \Omega$, $i = 1, \dots, m$, are disjoint.

We construct a suitable recovery sequence $(v_k) \subset \mathrm{Lip}(\Omega; \mathbb{R}^2)$ such that

$$\lim_{k\to+\infty} v_k = u \quad \text{in } W^{1,1}(\Omega; \mathbb{R}^2) \tag{2.3.10}$$

and setting $B := \cup_{i=1}^n B_\ell(x_i)$,

$$\lim_{k\to+\infty} \int_{B_\ell(x_i)} |Jv_k|dx = \pi|d_i|, \qquad i = 1, \dots, m, \qquad \text{and} \int_{\Omega\setminus B} |Jv_k|dx = 0. \tag{2.3.11}$$

As in the proof of Proposition 2.3.4, we can find $r_1 \leq \ell$ so that $u \in W^{1,1}(\partial B_{r_1}(x_i); \mathbb{R}^2)$ and $\deg(u, \partial B_{r_1}(x_i)) = d_i$, for all $i = 1, \dots, m$. For every $k \in \mathbb{N}$, we set $B_k := \cup_{i=1}^m B_{2^{-k}r_1}(x_i)$. By Theorem 2.1.10, there exists a sequence $\left(u_n^k\right)_{n\in\mathbb{N}} \subset C^\infty(\Omega \setminus B_k; \mathbb{S}^1)$ such that

$$\lim_{n\to+\infty} u_n^k = u \qquad \text{in } W^{1,1}(\Omega \setminus B_k; \mathbb{S}^1). \tag{2.3.12}$$

Now, for all $k > 1$, we choose $r_k \in (2^{-k}r_1, 2^{-k+1}r_1)$ such that the following conditions hold: for all $i = 1, \dots, m$,

$$\begin{aligned} &u \llcorner \partial B_{r_k}(x_i) \in W^{1,1}(\partial B_{r_k}(x_i); \mathbb{S}^1), \\ &\lim_{n\to+\infty} \|u_n^k \llcorner \partial B_{r_k}(x_i) - u \llcorner \partial B_{r_k}(x_i)\|_{W^{1,1}(\partial B_{r_k}(x_i);\mathbb{S}^1)} = 0. \end{aligned} \tag{2.3.13}$$

In particular, for all $k > 1$ and $i = 1, \dots, m$, we have

$$\lim_{n\to+\infty} \|u_n^k \llcorner \partial B_{r_k}(x_i) - u \llcorner \partial B_{r_k}(x_i)\|_{L^\infty(\partial B_{r_k}(x_i);\mathbb{S}^1)} = 0, \tag{2.3.14}$$

thus, using (2.1.7), (2.3.13) and (2.1.6), we obtain

$$\begin{aligned} &|\deg(u_n^k, \partial B_{r_k}(x_i)) - \deg(u, \partial B_{r_k}(x_i))| \\ &\leq \frac{1}{2\pi} \left(\int_{\partial B_{r_k}(x_i)} \left| (u_n^k)_1 \frac{\partial (u_n^k)_2}{\partial s} - u_1 \frac{\partial u_2}{\partial s} \right| ds + \int_{\partial B_{r_k}(x_i)} \left| (u_n^k)_2 \frac{\partial (u_n^k)_1}{\partial s} - u_2 \frac{\partial u_1}{\partial s} \right| ds \right) \longrightarrow 0 \end{aligned} \tag{2.3.15}$$

as $n \to +\infty$.
Therefore, there exists $m_k \in \mathbb{N}$ such that, for all $i = 1, \dots, m$,

$$\deg(u_n^k, \partial B_{r_k}(x_i)) = \deg(u, \partial B_{r_k}(x_i)) = d_i \qquad \forall n \geq m_k. \tag{2.3.16}$$

Now, using (2.3.12) and (2.3.13), for all $k > 1$ there is $\widetilde{m}_k \in \mathbb{N}$ such that, for all $i = 1, \dots, m$,

$$\|u_n^k - u\|_{W^{1,1}(\Omega\setminus(\cup_{i=1}^m B_{r_k}(x_i));\mathbb{S}^1)} \leq \|u_n^k - u\|_{W^{1,1}(\Omega\setminus B_k;\mathbb{S}^1)} \leq \frac{1}{k} \qquad \forall n \geq \widetilde{m}_k, \tag{2.3.17}$$

$$\|u_n^k \llcorner \partial B_{r_k}(x_i) - u \llcorner \partial B_{r_k}(x_i)\|_{W^{1,1}(\partial B_{r_k}(x_i);\mathbb{S}^1)} \leq \frac{1}{k} \qquad \forall n \geq \widetilde{m}_k. \tag{2.3.18}$$

Setting $n_k := \max\{m_k, \widetilde{m}_k\}$, we define $u_k := u_{n_k}^k$, which satisfies (2.3.16) and (2.3.17) for all $k > 1$. In particular

$$\lim_{k\to+\infty} \|u_k - u\|_{W^{1,1}(\Omega\setminus(\cup_{i=1}^m B_{r_k}(x_i));\mathbb{S}^1)} = 0. \tag{2.3.19}$$

For all $i = 1, \dots, m$, let now $\overline{\varphi}_i : \mathbb{S}^1 \to \mathbb{S}^1$ be the Lipschitz function defined in (2.2.38) with $d = d_i$, which satisfies

$$\mathrm{mult}(\overline{\varphi}_i) = |\deg(\overline{\varphi}_i)| \quad \text{and} \quad \deg(\overline{\varphi}_i) = d_i.$$

Now, for all $i = 1, \dots, m$, $\overline{\varphi}_i$ and $u_k \llcorner \partial B_{r_k}(x_i)$ have the same degree, and so there exists a Lipschitz homotopy[2] $H_{k,i} : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$H_{k,i}(0,y) = \overline{\varphi}_i(y), \quad H_{k,i}(1,y) = u_k(r_k y + x_i), \qquad y \in \mathbb{S}^1.$$

Let us define the sequence $(v_k) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ as follows: $v_k := u_k$ in $\Omega \setminus B$, and, for all $i = 1, \dots, m$, $v_k(x_i) := 0$ and

$$v_k(x) := \begin{cases} \dfrac{|x - x_i|}{r_{k+1}} \overline{\varphi}_i\left(\dfrac{x - x_i}{|x - x_i|}\right) & \text{if } x \in B_{r_{k+1}}(x_i) \setminus \{0\}, \\ h_{k,i}(x) & \text{if } x \in B_{r_k}(x_i) \setminus B_{r_{k+1}}(x_i), \\ u_k(x) & \text{if } x \in B_\ell(x_i) \setminus B_{r_k}(x_i), \end{cases} \tag{2.3.20}$$

where

$$h_{k,i}(x) := H_{k,i}\left(\frac{|x - x_i| - r_{k+1}}{r_k - r_{k+1}}, \frac{x - x_i}{|x - x_i|}\right) \qquad \forall x \in B_{r_k}(x_i) \setminus B_{r_{k+1}}(x_i).$$

Since $H_{k,i}$ and u_k take values in $\mathbb{S}^1$, we have $v_k(x) \in \mathbb{S}^1$ for $x \in \Omega \setminus (\cup_{i=1}^m B_{r_{k+1}}(x_i))$, and so

$$\int_{\Omega\setminus(\cup_{i=1}^m B_{r_{k+1}}(x_i))} |Jv_k| dx = 0.$$

[2] To define it it suffices to consider two liftings of $\overline{\varphi}_1$ and $u_k(r_k \cdot + x_1) \llcorner \mathbb{S}^1$, and linearly interpolate them, as done for H in (2.2.40). Observe that $H_{k,i}$ is Lipschitz since $u_k \llcorner \partial B_{r_k}(x_i)$ is Lipschitz by the choice of r_k.

In particular, the second condition in (2.3.11) holds. Moreover, by definition of v_k, we have that $\mathrm{mult}(v_k, B_{r_{k+1}}(x_i), \cdot)$=$\mathrm{mult}(\overline{\varphi}_i)$, and therefore, by (2.1.1),

$$\int_{B_{r_{k+1}}(x_i)} |Jv_k|dx = \int_{B_1} \mathrm{mult}(v_k, B_{r_{k+1}}(x_i), y)dy = |B_1|\mathrm{mult}(\overline{\varphi}_i) = \pi|d_i|,$$

and also the first condition in (2.3.11) follows.

It remains to show (2.3.10). By (2.3.19) and (2.3.17) we have

$$\int_\Omega |v_k - u|dx \leq \int_{\Omega\setminus(\cup_{i=1}^m B_{r_k}(x_i))} |u_k - u|dx + 2m|B_{r_k}(0)| \to 0 \quad \text{as } k \to +\infty,$$
$$\int_{\Omega\setminus(\cup_{i=1}^m B_{r_k}(x_i))} |\nabla v_k - \nabla u|dx = \int_{\Omega\setminus(\cup_{i=1}^m B_{r_k}(x_i))} |\nabla u_k - \nabla u|dx \to 0 \quad \text{as } k \to +\infty.$$

Now, let us show that, for all $i = 1, \dots, m$,

$$\lim_{k\to+\infty} \|\nabla h_{k,i}\|_{L^1(B_{r_k(x_i)}\setminus B_{r_{k+1}}(x_i))} = 0.$$

Let us make the computation for $i = 1$, the other cases being identical. Set $H_k = H_{k,1}$ and $h_k = h_{k,1}$. Assume without loss of generality that $x_1 = (0, 0)$, and denote $B_r(x_1) = B_r$. By definition of H_k we have

$$\|\partial_t H_k\|_{L^\infty([0,1]\times\mathbb{S}^1)} \leq \|\overline{\varphi}_1\|_{L^\infty(\mathbb{S}^1)} + \|u_k\|_{L^\infty(\partial B_{r_k})} \leq 2 \qquad \forall k \in \mathbb{N}. \tag{2.3.21}$$

Moreover, since $\overline{\varphi}_1$ is Lipschitz,

$$|\nabla_y H_k(t, y)| \leq |\nabla^{\mathbb{S}^1}\overline{\varphi}_1(y)| + r_k|\nabla u_k(r_k y)| \leq C + r_k|\nabla u_k(r_k y)|. \tag{2.3.22}$$

We now compute ∇h_k for $x \in B_{r_k} \setminus B_{r_{k+1}}$:

$$\nabla h_k(x) = \frac{1}{r_k - r_{k+1}}\partial_t H_k\left(\frac{|x| - r_{k+1}}{r_k - r_{k+1}}, \frac{x}{|x|}\right) \otimes \frac{x}{|x|} + \nabla_y H_k\left(\frac{|x| - r_{k+1}}{r_k - r_{k+1}}, \frac{x}{|x|}\right)\nabla\left(\frac{x}{|x|}\right)$$

and we get

$$\begin{aligned}
&\int_{B_{r_k}\setminus B_{r_{k+1}}} |\nabla h_k|dx \\
\leq &\int_{B_{r_k}\setminus B_{r_{k+1}}} \frac{1}{r_k - r_{k+1}}\left|\partial_t H_k\left(\frac{|x| - r_{k+1}}{r_k - r_{k+1}}, \frac{x}{|x|}\right)\right| + \left|\nabla_y H_k\left(\frac{|x| - r_{k+1}}{r_k - r_{k+1}}, \frac{x}{|x|}\right)\right|\left|\nabla\left(\frac{x}{|x|}\right)\right|dx \\
\leq &\frac{1}{r_k - r_{k+1}}\|\partial_t H_k\|_{L^\infty}\left|B_{r_k} \setminus B_{r_{k+1}}\right| + \int_{r_{k+1}}^{r_k}\int_0^{2\pi} \rho\frac{1}{\rho}\left|\nabla_y H_k\left(\frac{\rho - r_{k+1}}{r_k - r_{k+1}}, (\cos\theta, \sin\theta)\right)\right|d\rho d\theta \\
\leq &C(r_k + r_{k+1}) + C(r_k - r_{k+1}) + (r_k - r_{k+1})\int_0^{2\pi} r_k|\nabla u_k(r_k(\cos\theta, \sin\theta))|d\theta \\
\leq &Cr_k + (r_k - r_{k+1})\int_{\partial B_{r_k}} |\nabla u_k|d\mathcal{H}^1 \leq C\left(r_k + (r_k - r_{k+1})\right) \to 0 \quad \text{as } k \to +\infty,
\end{aligned} \tag{2.3.23}$$

where we have used (2.3.18) in the last inequality. Then we conclude

$$\begin{aligned}\int_{B_{r_k}\setminus B_{r_{k+1}}}|\nabla v_k-\nabla u|dx&=\int_{B_{r_k}\setminus B_{r_{k+1}}}|\nabla h_k-\nabla u|dx\\&\le\int_{B_{r_k}\setminus B_{r_{k+1}}}|\nabla h_k|dx+\int_{B_{r_k}\setminus B_{r_{k+1}}}|\nabla u|dx\to 0.\end{aligned}$$

Finally, for $x\in B_{r_{k+1}}$, we have

$$\nabla v_k(x)=\frac{1}{r_{k+1}}\frac{x}{|x|}\otimes\overline{\varphi}_1\left(\frac{x}{|x|}\right)+\frac{1}{r_{k+1}}|x|\nabla\left(\overline{\varphi}_1\left(\frac{x}{|x|}\right)\right).$$

Then, since $\overline{\varphi}_1$ is Lipschitz,

$$|\nabla v_k(x)|\le\frac{C}{r_{k+1}},$$

so we get

$$\int_{B_{r_{k+1}}}|\nabla v_k-\nabla u|dx\le\frac{C}{r_{k+1}}|B_{r_{k+1}}|+\int_{B_{r_{k+1}}}|\nabla u|dx\to 0,$$

and (2.3.10) follows. □

Now, we can prove the main result of this section:

Theorem 2.3.6 (Relaxation for Sobolev maps valued in $\mathbb{S}^1$). Let $u\in W^{1,1}(\Omega;\mathbb{S}^1)$. Suppose that $\mathrm{Det}\nabla u$ is a Radon measure with finite total variation $|\mathrm{Det}\,\nabla u|(\Omega)$. Then

$$\overline{\mathcal{A}}_{BV}(u;\Omega)=\int_\Omega\sqrt{1+|\nabla u|^2}dx+|\mathrm{Det}\nabla u|(\Omega)=\int_\Omega\sqrt{1+|\nabla u|^2}dx+\pi\sum_{i=1}^N|d_i|,\quad(2.3.24)$$

where $N\in\mathbb{N}$ and $d_1,\dots,d_N\in\mathbb{Z}\setminus\{0\}$ are such that $\mathrm{Det}\nabla u=\pi\sum_{i=1}^N d_i\delta_{x_i}$.

Proof. We start with the proof of the lower bound. Arguing as in the proof of Proposition 2.3.4, we may suppose $m=1$, $\Omega=B_\ell$ and $x_1=(0,0)$. Let $(v_k)\subset C^1(B_\ell;\mathbb{R}^2)$ be such that $v_k\to u$ strictly $BV(B_\ell;\mathbb{R}^2)$ and

$$\liminf_{k\to+\infty}\mathcal{A}(v_k;B_\ell)=\lim_{k\to+\infty}\mathcal{A}(v_k;B_\ell)<+\infty.$$

Select $r_1>0$ and $d_1\in\mathbb{Z}$ as in the proof of Proposition 2.3.5. Without loss of generality we can suppose that $r_1=\ell$. So we deduce (2.3.5) and the uniform convergence of (v_k) to u on almost every circumference in B_ℓ. Now write $\mathcal{A}(v_k;B_\ell)=\mathcal{A}(v_k;B_\ell\setminus B_{r_\varepsilon})+\mathcal{A}(v_k;B_{r_\varepsilon})\ge\mathcal{A}(v_k;B_\ell\setminus B_{r_\varepsilon})+\int_{B_{r_\varepsilon}}|Jv_k|\;dx$, so that

$$\begin{aligned}\lim_{k\to+\infty}\mathcal{A}(v_k;B_\ell)&\ge\liminf_{k\to+\infty}\mathcal{A}(v_k;B_\ell\setminus B_{r_\varepsilon})+\liminf_{k\to+\infty}\int_{B_{r_\epsilon}}|Jv_k|\;dx\\&\ge\int_{B_\ell\setminus B_{r_\varepsilon}}\sqrt{1+|\nabla u|^2}dx+\liminf_{k\to+\infty}\int_{B_{r_\epsilon}}|Jv_k|\;dx.\end{aligned}\quad(2.3.25)$$

We now apply (2.3.9) and next pass to the limit as $\varepsilon \to 0^+$ to get the lower bound in (2.3.24), i.e.,

$$\liminf_{k\to+\infty} \mathcal{A}(v_k; B_\ell) \geq \int_\Omega \sqrt{1+|\nabla u|^2}dx + \pi \sum_{i=1}^N |d_i|.$$

Concerning the proof of the upper bound, consider the sequence (v_k) defined in (2.3.20), which converges to u in $W^{1,1}(\Omega;\mathbb{R}^2)$. Then, upon extracting a subsequence such that (∇v_k) converges almost everywhere to ∇u, by (2.3.11) and dominated convergence we have, using the inequality $\sqrt{1+a^2+b^2+c^2} \leq \sqrt{1+a^2+b^2}+|c|$ for $a,b,c \in \mathbb{R}$,

$$\begin{aligned}\limsup_{k\to+\infty} \mathcal{A}(v_k; B_\ell(x_i)) &\leq \lim_{k\to+\infty} \int_{B_\ell(x_i)} \sqrt{1+|\nabla v_k|^2}dx + \lim_{k\to+\infty} \int_{B_\ell(x_i)} |Jv_k|dx \\ &= \int_{B_\ell(x_i)} \sqrt{1+|\nabla u|^2}dx + \pi|d_i|,\end{aligned}$$

that leads to

$$\begin{aligned}\limsup_{k\to+\infty} \mathcal{A}(v_k;\Omega) &\leq \lim_{k\to+\infty} \int_{\Omega\setminus\cup_{i=1}^m B_\ell(x_i)} \sqrt{1+|\nabla v_k|^2}dx + \limsup_{k\to+\infty} \mathcal{A}(v_k; \cup_{i=1}^m B_\ell(x_i)) \\ &= \int_\Omega \sqrt{1+|\nabla u|^2}dx + \pi\sum_{i=1}^m |d_i|.\end{aligned}$$

□

Remark 2.3.7. If $u \in W^{1,p}(\Omega;\mathbb{S}^1)$, $p \in [1,2)$, the recovery sequence defined in (2.3.20) converges to u in $W^{1,p}(\Omega;\mathbb{S}^1)$ as well. Then, the results of Theorem 2.3.3 and Theorem 2.3.6 are still valid if one deals with the relaxation of the area functional with respect to the strong topology of $W^{1,p}(\Omega;\mathbb{S}^1)$.

Remark 2.3.8 (**Relaxation in the local uniform convergence outside singularities**)**.** If u is continuous in $\Omega\setminus\{x_1,\dots,x_m\}$, one can relax the area functional with respect to the uniform convergence out of the singularities $\{x_i\}$, i.e., we require that for every compact set $K \subset \Omega\setminus\{x_1,\dots,x_m\}$ the approximating sequence $(u_k) \subset C^1(\Omega;\mathbb{S}^1)$ satisfies

$$u_k \to u \quad \text{in } L^\infty(K),$$

or, in other words, if $u_k \to u$ in $L^\infty_{\text{loc}}(\Omega\setminus\{x_1,\dots,x_m\};\mathbb{R}^2)$. Therefore we are led to consider

$$\begin{aligned}\overline{\mathcal{A}}_{L^\infty}(u;\Omega) := \inf\Big\{ \liminf_{k\to+\infty} \mathcal{A}(u_k;\Omega) : (u_k) \subset C^1(\Omega;\mathbb{R}^2),\ u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^2) \\ \text{and } u_k \to u \text{ in } L^\infty_{\text{loc}}(\Omega\setminus\{x_1,\dots,x_m\};\mathbb{R}^2)\Big\}.\end{aligned} \tag{2.3.26}$$

It is then possible to show that

$$\overline{\mathcal{A}}_{L^\infty}(u;\Omega) = \int_\Omega \sqrt{1+|\nabla u|^2}dx + \pi\sum_{i=1}^m |d_i|. \tag{2.3.27}$$

Notice that, if one considers the functional $\overline{TVJ}_{L^\infty}$, obtained by relaxing TVJ with this notion of convergence, the counterpart of Theorem 2.3.3 does not hold anymore, since we cannot guarantee a uniform bound on the L^1 norm of ∇v_k, needed to get (2.3.7); however, we gain such a control on $\|\nabla v_k\|_{L^1}$ in the area functional, as soon as the approximating sequence (v_k) has bounded area.

The proof of (2.3.27) is the same of the one of Theorem 2.3.6, with the difference that we can deduce straightforwardly the uniform convergence of (v_k) on almost every circumference in B_{r_1}, without passing through (2.3.5).

2.4 Symmetric piecewise constant $BV(\Omega;\mathbb{S}^1)$ maps

This section is devoted to the proof of Theorem 2.4.1, that shows the explicit expression of the BV-relaxed for the symmetric triple point map. As we shall anticipate also in Remark 2.4.5, we will generalize this result for more general piecewise constant maps in Chapter 4. However, for completeness we report the proof also in this particular case, where the construction of the recovery sequence can be made explicitly.
Let us recall that a symmetric triple point map in $\mathbb{R}^2$ is a map $u = u_T : B_\ell(0) \subset \mathbb{R}^2 \to \mathbb{S}^1$ taking three values $\{\alpha,\beta,\gamma\} \subset \mathbb{S}^1$, vertices of an equilateral triangle, on three non-overlapping $2\pi/3$-angular regions A, B, C with common vertex at the origin and interfaces a, b, c (see Figure 2.1). We denote by $T_{\alpha\beta\gamma} \subset \mathbb{R}^2$ the triangle with vertices $\{\alpha,\beta,\gamma\}$, whose

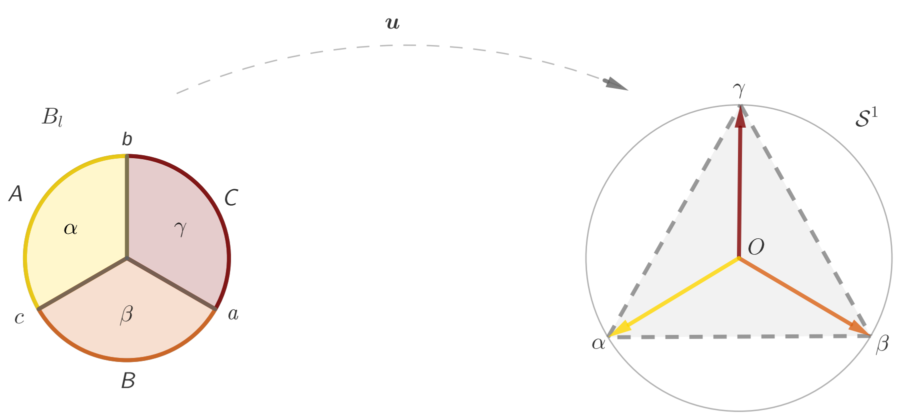

Figure 2.1: The symmetric triple point map: on the left the source disk $B_\ell(0)$, three-sided in the regions A, B, C, where u takes the values α,β,γ, depicted in the $\mathbb{R}^2$ target on the right.

length side is $|\alpha-\beta| =: L = \sqrt{3}$, and by $J_{u_T} = a \cup b \cup c$ the jump set of u. We have $|T_{\alpha\beta\gamma}| = \frac{\sqrt{3}}{4}L^2 = \frac{3\sqrt{3}}{4}$, and $|Du|(B_\ell) = L\mathcal{H}^1(J_u) = 3L\ell$.

Theorem 2.4.1 (Relaxation for the symmetric triple-point map). Let $u_T : B_\ell := B_\ell(0) \to \{\alpha,\beta,\gamma\}$ be the symmetric triple-point map. Then

$$\overline{\mathcal{A}}_{BV}(u_T; B_\ell) = |B_\ell| + L\mathcal{H}^1(J_{u_T}) + |T_{\alpha\beta\gamma}|, \tag{2.4.1}$$

Proof of Theorem 2.4.1: upper bound. For simplicity of notation, in what follows we write

$$\varepsilon \text{ in place of } 1/k,$$

with $k \in \mathbb{N}$.

We construct a recovery sequence $(u^\varepsilon)_\varepsilon \subset \mathrm{Lip}(B_\ell;\mathbb{R}^2)$ as $\varepsilon \to 0^+$. Let us consider the rectangle

$$R := \{(t,s) \in \mathbb{R}^2 : t \in (0,\ell), s \in (0,L)\}$$

and, for $\varepsilon \in (0,\ell)$, the functions $m^\varepsilon : R \to [0,+\infty)$ (whose graph is plotted in Figure 2.2) defined as

$$m^\varepsilon(t,s) := \begin{cases} 0 & t \in [\varepsilon,\ell] \\ 2\frac{\varepsilon-t}{\varepsilon}\frac{sh}{L} & t \in [0,\varepsilon),\ s \in [0,\frac{L}{2}], \\ 2\frac{\varepsilon-t}{\varepsilon}\frac{(L-s)h}{L} & t \in [0,\varepsilon),\ s \in (\frac{L}{2},L], \end{cases} \tag{2.4.2}$$

where $h := \frac{L}{2\sqrt{3}} = \frac{1}{2}$. The number h is the height of each of the three isosceles triangles with common vertex at the origin of the target space that decompose $T_{\alpha\beta\gamma}$ (see Figure 2.1 right). Let us denote by $S^a_\varepsilon, S^b_\varepsilon, S^c_\varepsilon$ three tiny stripes around a, b, c in B_ℓ, of width ε and length $\ell - \frac{\varepsilon}{2\sqrt{3}}$, drawn in Figure 2.3. More explicitely, we have

$$S^b_\varepsilon := \left\{(x,y) \in B_\ell : |x| \le \frac{\varepsilon}{2}, y \ge \frac{\varepsilon}{2\sqrt{3}}\right\}$$

and S^a_ε (S^c_ε) is obtained by clockwisely rotating S^b_ε of an angle $\frac{2\pi}{3}$ ($\frac{4\pi}{3}$ respectively) around the origin.

The idea is to glue m^ε on each strip in order to build three surfaces embedded in $\mathbb{R}^4$ living in three non-collinear copies of $\mathbb{R}^3$, whose total area contribution gives $|T_{\alpha\beta\gamma}|$ in the limit $\varepsilon \to 0^+$.

We introduce the affine diffeomorphism $\psi_\varepsilon : \left[\frac{\varepsilon}{2\sqrt{3}}, \ell\right] \to [0,\ell]$ such that

$$\psi'_\varepsilon(y) = \frac{\ell}{\ell - \frac{\varepsilon}{2\sqrt{3}}} =: k_\varepsilon \to 1 \quad \text{as } \varepsilon \to 0^+.$$

Now we can define u^ε on S^b_ε: we set

$$\xi := \frac{\gamma - \alpha}{L} \in \mathbb{S}^1, \qquad \eta := -\xi^\perp = \beta,$$

(where $\xi^\perp$ is the $\frac{\pi}{2}$-counterclockwise rotation of ξ) and

$$u^\varepsilon(x,y) := \alpha + \left(\frac{L}{2} + \frac{Lx}{\varepsilon}\right)\xi + m^\varepsilon\left(\psi_\varepsilon(y), \frac{L}{2} + \frac{Lx}{\varepsilon}\right)\eta \qquad \forall (x,y) \in S^b_\varepsilon.$$

In a similar way, we define u^ε on S^a_ε and S^c_ε. Setting $T^\varepsilon := \overline{B_{\varepsilon/\sqrt{3}} \setminus (S^a_\varepsilon \cup S^b_\varepsilon \cup S^c_\varepsilon)}$ and $A^\varepsilon := A \setminus (S^a_\varepsilon \cup S^b_\varepsilon \cup S^c_\varepsilon \cup T^\varepsilon)$, $B^\varepsilon := B \setminus (S^a_\varepsilon \cup S^b_\varepsilon \cup S^c_\varepsilon \cup T^\varepsilon)$, $C^\varepsilon := C \setminus (S^a_\varepsilon \cup S^b_\varepsilon \cup S^c_\varepsilon \cup T^\varepsilon)$, we define:

$$u^\varepsilon := \begin{cases} \alpha & \text{in } A^\varepsilon, \\ \beta & \text{in } B^\varepsilon, \\ \gamma & \text{in } C^\varepsilon. \end{cases} \tag{2.4.3}$$

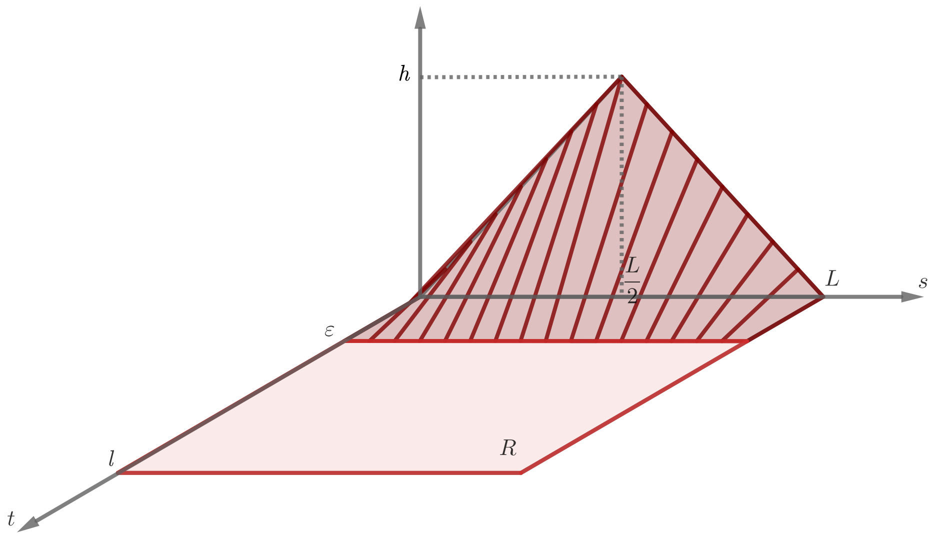

Figure 2.2: The graph of m^ε on the rectangle R.

It remains to define u^ε on the small triangle T^ε. Let us divide it in four triangles $T^a_\varepsilon, T^b_\varepsilon, T^c_\varepsilon, T^0_\varepsilon$ (see Figure 2.4). So, we set $u^\varepsilon = 0$ on T^0_ε and let u^ε be the affine function that equals α (β, γ respectively), in the vertex of T^ε confining with A^ε ($B^\varepsilon, C^\varepsilon$ respectively), and equals 0 on the edge of T^0_ε. A direct check shows that the function u_ε is Lipschitz continuous in B_ℓ.

Let us compute the area of the graph of u^ε on S^b_ε: denoting by $m^\varepsilon_t, m^\varepsilon_s$ the partial derivatives of m^ε, we have

$$\nabla u^\varepsilon(x,y) = \begin{pmatrix} \frac{L}{\varepsilon}\xi_1 + m^\varepsilon_s(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x)\frac{L}{\varepsilon}\eta_1 & m^\varepsilon_t(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x)k_\varepsilon\eta_1 \\ \frac{L}{\varepsilon}\xi_2 + m^\varepsilon_s(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x)\frac{L}{\varepsilon}\eta_2 & m^\varepsilon_t(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x)k_\varepsilon\eta_2. \end{pmatrix} \tag{2.4.4}$$

Recalling that $\xi \cdot \eta = 0$ and $|\xi| = |\eta| = 1$, we can compute the square of the Frobenius norm of ∇u^ε

$$\begin{aligned} |\nabla u^\varepsilon(x,y)|^2 &= \frac{L^2}{\varepsilon^2}\left[\xi_1^2 + (m^\varepsilon_s)^2\eta_1^2 + 2\xi_1\eta_1 m^\varepsilon_s + \xi_2^2 + (m^\varepsilon_s)^2\eta_2^2 + 2\xi_2\eta_2 m^\varepsilon_s\right] + (m^\varepsilon_t)^2k_\varepsilon^2\eta_1^2 \\ &\quad + (m^\varepsilon_t)^2k_\varepsilon^2\eta_2^2 \\ &= \frac{L^2}{\varepsilon^2}(1 + (m^\varepsilon_s)^2) + (m^\varepsilon_t)^2k_\varepsilon^2, \end{aligned} \tag{2.4.5}$$

where m^ε_s and m^ε_t are evaluated at $\left(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x\right)$. Moreover, using that $\xi \cdot \eta^\perp = 1$, we have

$$(\det\nabla u^\varepsilon)^2 = \frac{k_\varepsilon^2L^2}{\varepsilon^2}\left[(\xi_1\eta_2 m^\varepsilon_t + m^\varepsilon_s m^\varepsilon_t\eta_1\eta_2) - (\xi_2\eta_1 m^\varepsilon_t + m^\varepsilon_s m^\varepsilon_t\eta_1\eta_2)\right]^2 = \frac{k_\varepsilon^2L^2}{\varepsilon^2}(m^\varepsilon_t)^2.$$

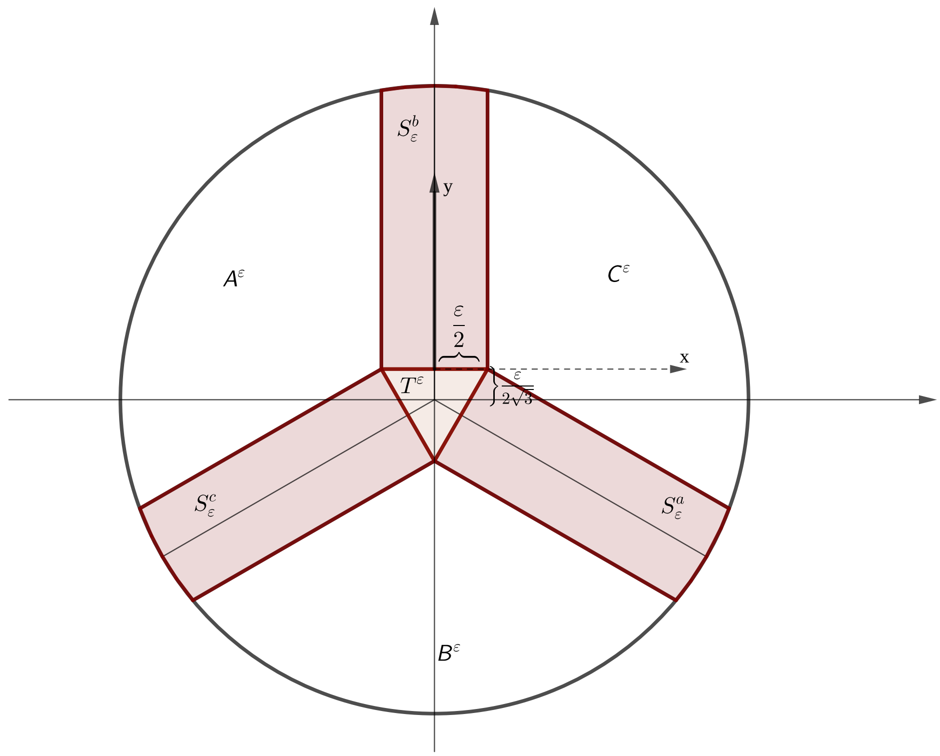

Figure 2.3: The strips $S_\varepsilon^a, S_\varepsilon^b, S_\varepsilon^c$ and the little triangle T^ε in the center.

So we have

$$
\begin{aligned}
&\mathcal{A}(u^\varepsilon; S_\varepsilon^b)\\
&= \int_{S_\varepsilon^b} \sqrt{1 + \frac{L^2}{\varepsilon^2}(1 + (m_s^\varepsilon)^2) + (m_t^\varepsilon)^2 k_\varepsilon^2 + \frac{k_\varepsilon^2 L^2}{\varepsilon^2}(m_t^\varepsilon)^2 dxdy}\\
&= \frac{L}{\varepsilon}\int_{S_\varepsilon^b} \sqrt{1 + m_s^\varepsilon\left(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x\right)^2 + m_t^\varepsilon\left(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x\right)^2 k_\varepsilon^2\left(1 + \frac{\varepsilon^2}{L^2}\right) + O(\varepsilon^2) dxdy}\\
&= \frac{1}{k_\varepsilon}\int_{R\setminus P_\varepsilon} \sqrt{1 + m_s^\varepsilon(t,s)^2 + m_t^\varepsilon(t,s)^2 k_\varepsilon^2\left(1 + \frac{\varepsilon^2}{L^2}\right) + O(\varepsilon^2) dtds},
\end{aligned}
\tag{2.4.6}
$$

where in the last equality we have performed the change of variables

$$
(x, y) = \left(\frac{\varepsilon}{L}\left(s - \frac{L}{2}\right), \psi_\varepsilon^{-1}(t)\right) =: \phi_\varepsilon(t, s)
$$

and we have set $P_\varepsilon = R \setminus \phi_\varepsilon^{-1}(S_\varepsilon^b)$. Notice that $\frac{1}{k_\varepsilon} \to 1$, $k_\varepsilon^2\left(1 + \frac{\varepsilon^2}{L^2}\right) \to 1$ as $\varepsilon \to 0^+$, so that we get

$$
\liminf_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon; S_\varepsilon^b) \le \int_R 1 dtds + \liminf_{\varepsilon\to 0^+}\int_R |m_t^\varepsilon(t,s)| dtds + \liminf_{\varepsilon\to 0^+}\int_R |m_s^\varepsilon(t,s)| dtds. \tag{2.4.7}
$$

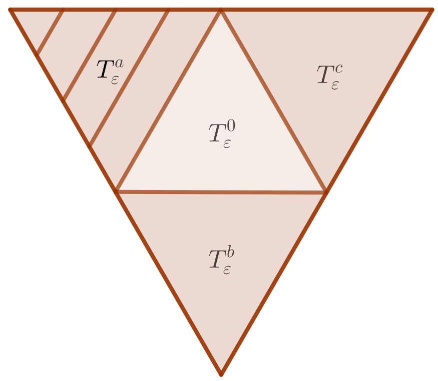

Figure 2.4: The triangle T^ε divided further in the four triangles $T^a_\varepsilon, T^b_\varepsilon, T^c_\varepsilon, T^0_\varepsilon$.

Let us compute explicitely the derivatives of m^ε:

$$m^\varepsilon_t(t,s) = \begin{cases} 0 & t > \varepsilon \\ -2\dfrac{sh}{\varepsilon L} & t < \varepsilon, s < \dfrac{L}{2} \\ -2\dfrac{(L-s)h}{\varepsilon L} & t < \varepsilon, s > \dfrac{L}{2} \end{cases} \qquad m^\varepsilon_s(t,s) = \begin{cases} 0 & t \geq \varepsilon \\ 2\dfrac{\varepsilon - t}{\varepsilon}\dfrac{h}{L} & t < \varepsilon, s < \dfrac{L}{2} \\ -2\dfrac{\varepsilon - t}{\varepsilon}\dfrac{h}{L} & t < \varepsilon, s > \dfrac{L}{2}. \end{cases}$$

Then, we obtain

$$\int_{\{t<\varepsilon, s<\frac{L}{2}\}} |m^\varepsilon_t(t,s)| dtds = \varepsilon \int_0^{\frac{L}{2}} 2\frac{sh}{\varepsilon L} ds = \frac{hL}{4}$$
$$\int_{\{t<\varepsilon, s>\frac{L}{2}\}} |m^\varepsilon_t(t,s)| dtds = \varepsilon \int_{\frac{L}{2}}^{L} 2(L-s)\frac{sh}{\varepsilon L} ds = \frac{hL}{4},$$

so we get

$$\int_R |m^\varepsilon_t(t,s)| dtds = \frac{hL}{4} + \frac{hL}{4} = \frac{hL}{2} \quad \forall \varepsilon > 0. \tag{2.4.8}$$

On the other hand,

$$\int_{\{t<\varepsilon, s<\frac{L}{2}\}} |m^\varepsilon_s(t,s)| dtds = \int_{\{t<\varepsilon, s>\frac{L}{2}\}} |m^\varepsilon_s(t,s)| dtds = \frac{L}{2}\int_0^\varepsilon 2\frac{\varepsilon - t}{\varepsilon}\frac{h}{L} ds = O(\varepsilon),$$

so we get

$$\liminf_{\varepsilon\to 0^+} \int_R |m^\varepsilon_s(t,s)| dtds = 0. \tag{2.4.9}$$

Summarizing, from (2.4.7) we obtain

$$\liminf_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon; S^b_\varepsilon) \leq \ell L + \frac{hL}{2}.$$

In the same way, we can prove that

$$\liminf_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon; S^a_\varepsilon) = \liminf_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon; S^c_\varepsilon) \leq \ell L + \frac{hL}{2}.$$

Clearly, the definition of u^ε on $A^\varepsilon, B^\varepsilon, C^\varepsilon$ provides that

$$\lim_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon; A^\varepsilon \cup B^\varepsilon \cup C^\varepsilon) = |B_\ell| = \pi\ell^2.$$

It remais to show that the area contribution on T^ε is infinitesimal: first notice that

$$\mathcal{A}(u^\varepsilon; T^0_\varepsilon) = |T^0_\varepsilon| = O(\varepsilon^2).$$

Moreover on T^a_ε (respectively $T^b_\varepsilon, T^c_\varepsilon$) u^ε is the affine parameterization of the segment $(\alpha, 0)$ (respectively $(\beta, 0), (\gamma, 0)$) of the target space, therefore on $T^\varepsilon \setminus T^0_\varepsilon$ the area integrand has no Jacobian contribution and so is $O(\varepsilon^{-1})$, giving

$$\mathcal{A}(u^\varepsilon; T^a_\varepsilon) = \mathcal{A}(u^\varepsilon; T^b_\varepsilon) = \mathcal{A}(u^\varepsilon; T^c_\varepsilon) = O(\varepsilon).$$

Then we have

$$\mathcal{A}(u^\varepsilon; T^\varepsilon) = \mathcal{A}(u^\varepsilon; T^0_\varepsilon) + \mathcal{A}(u^\varepsilon; T^a_\varepsilon) + \mathcal{A}(u^\varepsilon; T^b_\varepsilon) + \mathcal{A}(u^\varepsilon; T^c_\varepsilon) = O(\varepsilon^2) + O(\varepsilon).$$

In the end, we conclude

$$\liminf_{\varepsilon\to^+ 0} \mathcal{A}(u^\varepsilon; B_\ell) \leq \pi\ell^2 + 3\ell L + 3\frac{hL}{2},$$

where we recognize that the last quantity on the right-hand side is exactly $|T_{\alpha\beta\gamma}|$.

As a final step, we have to check that (u^ε) converges to u strictly $BV(B_\ell; \mathbb{R}^2)$. Clearly $u^\varepsilon \to u$ in $L^1(B_\ell; \mathbb{R}^2)$. Let us compute the total variation of u^ε: we have

$$|Du^\varepsilon|(B_\ell) = |Du^\varepsilon|(S^a_\varepsilon) + |Du^\varepsilon|(S^b_\varepsilon) + |Du^\varepsilon|(S^c_\varepsilon) + |Du^\varepsilon|(T^\varepsilon).$$

In particular,

$$|Du^\varepsilon|(T^\varepsilon) \leq \mathcal{A}(u^\varepsilon; T^\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Computing the variation on the strip S^b_ε (similarly for the other strips) we find

$$\begin{aligned}
|Du^\varepsilon|(S^b_\varepsilon) &= \int_{S^b_\varepsilon} \sqrt{\frac{L^2}{\varepsilon^2}\left(1 + (m^\varepsilon_s)^2\right) + (m^\varepsilon_t)^2 k^2_\varepsilon}\, dxdy \\
&= \frac{L}{\varepsilon} \int_{S^b_\varepsilon} \sqrt{1 + m^\varepsilon_s \left(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x\right)^2 + m^\varepsilon_t \left(\psi_\varepsilon(y), \frac{L}{2} + \frac{L}{\varepsilon}x\right)^2 k^2_\varepsilon \frac{\varepsilon^2}{L^2}}\, dxdy \\
&= \frac{1}{k_\varepsilon} \int_{R\setminus P_\varepsilon} \sqrt{1 + m^\varepsilon_s(t,s)^2 + m^\varepsilon_t(t,s)^2 k^2_\varepsilon \frac{\varepsilon^2}{L^2}}\, dtds.
\end{aligned}$$

Then, using (2.4.8) and (2.4.9), we conclude

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} |Du^\varepsilon|(S^b_\varepsilon) &\leq \int_R 1 dtds + \limsup_{\varepsilon\to 0^+} \int_R |m^\varepsilon_s(t,s)| dtds + O(\varepsilon) \limsup_{\varepsilon\to 0^+} \int_R |m^\varepsilon_t(t,s)| dtds \\
&= \ell L,
\end{aligned}$$

so that

$$\limsup_{\varepsilon\to 0^+} |Du^\varepsilon|(B_\ell) \leq 3\ell L.$$

By the lower semicontinuity of the variation, we get also

$$\liminf_{\varepsilon\to 0^+} |Du^\varepsilon|(B_\ell) \geq |Du|(B_\ell) = 3\ell L,$$

which shows the desired convergence of (u^ε) to u strictly $BV(B_\ell;\mathbb{R}^2)$. □

Before proving the lower bound, similarly to Lemma 2.3.1, we show that the strict BV convergence is inherited to almost every circumference centered at the origin.

Lemma 2.4.2 (Inheritance). Lemma 2.3.1 holds with u_T in place of u.

Proof. Let $\rho < \ell$ and u be the triple point map; clearly

$$|D(u\llcorner\partial B_\rho)|(\partial B_\rho) = 3L. \tag{2.4.10}$$

On the other hand, since (v_k) converges to u in L^1, for almost every $\rho < \ell$ we have $v_k\llcorner\partial B_\rho \to u\llcorner\partial B_\rho$ in $L^1(\partial B_\rho;\mathbb{R}^2)$, and by lower semicontinuity we infer that

$$|D(u\llcorner\partial B_\rho)|(\partial B_\rho) \leq \liminf_{k\to+\infty}\int_{\partial B_\rho}\left|\frac{\partial v_k}{\partial s}\right| ds \qquad \text{for a.e. } \rho<\ell. \tag{2.4.11}$$

Integrating with respect to $\rho\in(0,\ell)$, by (2.4.10) and Fatou's lemma, we have

$$\begin{aligned}|Du|(B_\ell) = 3\ell L &= \int_0^\ell |D(u\llcorner\partial B_\rho)|(\partial B_\rho)d\rho \\ &\leq \int_0^\ell \liminf_{k\to+\infty}\int_{\partial B_\rho}\left|\frac{\partial v_k}{\partial s}\right| ds d\rho \leq \liminf_{k\to+\infty}\int_{B_\ell}|\nabla v_k|dx.\end{aligned} \tag{2.4.12}$$

By assumption, (v_k) converges to u strictly $BV(B_\ell;\mathbb{R}^2)$, so we have all equalities in (2.4.12), in particular, using (2.4.11),

$$|D(u\llcorner\partial B_\rho)|(\partial B_\rho) = \liminf_{k\to+\infty}\int_{\partial B_\rho}\left|\frac{\partial v_k}{\partial s}\right| ds \qquad \text{for a.e. } \rho<\ell.$$

Upon extracting a suitable subsequence (v_{k_h}) depending on ρ we get the conclusion. □

Proof of Theorem 2.4.1: lower bound. Let $(v_k)\subset C^1(B_\ell;\mathbb{R}^2)$ be a recovery sequence, i.e.,

$$v_k \to u \quad \text{strictly } BV(B_\ell;\mathbb{R}^2) \qquad \text{and} \qquad \lim_{k\to+\infty}\mathcal{A}(v_k;B_\ell) = \overline{\mathcal{A}}_{BV}(u;B_\ell).$$

Fix $\rho\in(0,\ell)$ and a subsequence (v_{k_h}) of (v_k) whose restriction to ∂B_ρ converges to $u\llcorner\partial B_\rho$ strictly $BV(\partial B_\rho;\mathbb{R}^2)$, as in Lemma 2.4.2. For simplicity, let us still denote v_{k_h} by v_k.

Let us split the area functional as

$$\mathcal{A}(v_k;B_\ell) = \mathcal{A}(v_k;B_\ell\setminus B_\rho) + \mathcal{A}(v_k;B_\rho).$$

On $B_\ell\setminus B_\rho$ we still have L^1-convergence of (v_k) to u, but $u\llcorner(B_\ell\setminus B_\rho)$ has no triple points, so by Theorem 3.14 of [1],

$$\begin{aligned}\liminf_{k\to+\infty}\mathcal{A}(v_k;B_\ell\setminus B_\rho) \geq \overline{\mathcal{A}}_{L^1}(u;B_\ell\setminus B_\rho) &= \int_{B_r\setminus B_\rho}|\sqrt{1+|\nabla u|^2}dx + |D^j u|(B_\ell\setminus B_\rho) \\ &= |B_\ell\setminus B_\rho| + 3L(\ell-\rho) = \pi(\ell^2-\rho^2) + 3L(\ell-\rho).\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{k\to+\infty}\mathcal{A}(v_k;B_\ell)\geq&\liminf_{k\to+\infty}\mathcal{A}(v_k;B_\ell\setminus B_\rho)+\liminf_{k\to+\infty}\mathcal{A}(v_k;B_\rho)\\ \geq&\pi(\ell^2-\rho^2)+3L(\ell-\rho)+\liminf_{k\to+\infty}\int_{B_\rho}|Jv_k|dx,\end{aligned}\tag{2.4.13}$$

where as usual $Jv_k := \det\nabla v_k$.

Let us prove that

$$\liminf_{k\to+\infty}\int_{B_\rho}|Jv_k|dx\geq|T_{\alpha\beta\gamma}|,\tag{2.4.14}$$

from which the lower bound in (2.4.1) is obtained by passing to the limit as $\rho\to0^+$ in (2.4.13). Now we observe that, since v_k is Lipschitz on B_ρ, it satisfies the following identity (see (1.4.3)):

$$\int_{B_\rho}Jv_kdx=\frac{1}{2}\int_{\partial B_\rho}\Big((v_k)_1\frac{\partial(v_k)_2}{\partial s}-(v_k)_2\frac{\partial(v_k)_1}{\partial s}\Big)ds\quad\forall k\in\mathbb{N}.$$

Let us parametrize ∂B_ρ from $[0,2\pi)$ and set $\widetilde{v}_k(t):=v_k(s(t))$ for $t\in[0,2\pi)$; then

$$(\dot{\widetilde{v}}_k)_i(t)=\frac{d}{dt}(v_k)_i(s(t))=\rho\frac{\partial(v_k)_i}{\partial s}(s(t)),\qquad i=1,2.$$

Thus we get

$$\int_{\partial B_\rho}\Big((v_k)_1\frac{\partial(v_k)_2}{\partial s}-(v_k)_2\frac{\partial(v_k)_1}{\partial s}\Big)ds=\int_0^{2\pi}\Big((\widetilde{v}_k)_1(t)(\dot{\widetilde{v}}_k)_2(t)-(\widetilde{v}_k)_2(t)(\dot{\widetilde{v}}_k)_1(t)\Big)\,dt.$$

Denoting $\widetilde{v}_k(t)$ simply by $v_k(t)$, we can write

$$\int_{B_\rho}Jv_kdx=\frac{1}{2}\int_0^{2\pi}((v_k)_1(t)(\dot{v}_k)_2(t)-(v_k)_2(t)(\dot{v}_k)_1(t))\,dt.$$

To show (2.4.14) it is sufficient to prove that

$$\liminf_{k\to+\infty}\frac{1}{2}\int_0^{2\pi}((v_k)_1(t)(\dot{v}_k)_2(t)-(v_k)_2(t)(\dot{v}_k)_1(t))\,dt\geq|T_{\alpha\beta\gamma}|,\tag{2.4.15}$$

since obviously

$$\int_{B_\rho}|Jv_k|dx\geq\left|\int_{B_\rho}Jv_kdx\right|.\tag{2.4.16}$$

In order to show (2.4.15), denote by $\theta_1\in[0,2\pi)$ (respectively θ_2,θ_3) the angle of the middle point of the arc $C\cap\partial B_\rho$ (respectively $A\cap\partial B_\rho$, $B\cap\partial B_\rho$) and write

$$\begin{aligned}&\frac{1}{2}\int_0^{2\pi}((v_k)_1(t)(\dot{v}_k)_2(t)-(v_k)_2(t)(\dot{v}_k)_1(t))\,dt\\ =&\frac{1}{2}\int_{\theta_1}^{\theta_2}((v_k)_1(t)(\dot{v}_k)_2(t)-(v_k)_2(t)(\dot{v}_k)_1(t))\,dt\\ &+\frac{1}{2}\int_{\theta_2}^{\theta_3}((v_k)_1(t)(\dot{v}_k)_2(t)-(v_k)_2(t)(\dot{v}_k)_1(t))\,dt\\ &+\frac{1}{2}\int_{\theta_3}^{\theta_1}((v_k)_1(t)(\dot{v}_k)_2(t)-(v_k)_2(t)(\dot{v}_k)_1(t))\,dt.\end{aligned}\tag{2.4.17}$$

Notice that, as a consequence of Lemma 2.4.2, v_k converges to u strictly $BV([\theta_1,\theta_2];\mathbb{R}^2)$. Furthermore, by restricting v_k to $[\theta_1,\theta_1+\delta]$, for a small $\delta>0$, as a consequence of Corollary 1.3.6 we see that v_k converges uniformly to $v\equiv\gamma$ on $[\theta_1,\theta_1+\delta]$. In particular we have

$$\lim_{k\to\infty} v_k(\theta_1)=\gamma.$$

Similarly v_k will tend to α and β in θ_2 and θ_3, respectively. We set

$$L_k:=\int_{\theta_1}^{\theta_2}\left(|\dot v_k(t)|+\frac{1}{k}\right)dt,\qquad z(t)=z_k(t):=\int_{\theta_1}^{t}\left(|\dot v_k(\tau)|+\frac{1}{k}\right)d\tau,\quad t\in[\theta_1,\theta_2].$$

Since z is strictly increasing with derivative bounded from below by $\frac{1}{k}$, we can invert it and denote its inverse $t(z)$. We define $w_k:[0,L_k]\to\mathbb{R}^2$ as

$$w_k(z)=v_k(t(z)).$$

Then we have

$$w_k'(z)=\dot v_k(t(z))\frac{dt}{dz}=\frac{\dot v_k(t(z))}{|\dot v_k(t(z))|+\frac{1}{k}},\quad dt=\frac{1}{|\dot v_k(t(z))|+\frac{1}{k}}dz.$$

Thus, $(w_k)_k$ is uniformly Lipschitz continuous on $[0,L_k]$ (with modulus of derivative bounded by 1), and

$$\begin{aligned}&\frac12\int_{\theta_1}^{\theta_2}((v_k)_1(t)(\dot v_k)_2(t)-(v_k)_2(t)(\dot v_k)_1(t))\,dt\\ =&\frac12\int_0^{L_k}\big((w_k)_1(z)(w_k')_2(z)-(w_k)_2(z)(w_k')_1(z)\big)\,dz.\end{aligned}\tag{2.4.18}$$

We also have

$$\lim_{k\to+\infty}L_k=\lim_{k\to+\infty}\int_{\theta_1}^{\theta_2}\left(|\dot v_k(t)|+\frac1k\right)dt=|Du|\llcorner\{y\in\partial B_\rho:\arg(y)\in[\theta_1,\theta_2]\}=|\gamma-\alpha|=L.$$

We further reparametrize w_k on $[0,L]$ by a multiple of the arc length parameter. Still denoting the obtained function by $(w_k)_k$, we see that w_k is uniformly bounded in $W^{1,\infty}([0,L];\mathbb{R}^2)$ so, upon passing to a (not relabelled) subsequence, we have

$$w_k\overset{*}{\rightharpoonup}w\quad \text{w}^*\text{-}W^{1,\infty}([0,L];\mathbb{R}^2),$$

for some $w\in W^{1,\infty}([0,L];\mathbb{R}^2)$. Hence, we can pass to the limit in (2.4.18), which now reads

$$\begin{aligned}&\frac12\int_0^{L}\big((w_k)_1(z)(w_k')_2(z)-(w_k)_2(z)(w_k')_1(z)\big)\,dz\\ \xrightarrow{k\to+\infty}&\frac12\int_0^{L}\big(w_1(z)w_2'(z)-w_2(z)w_1'(z)\big)\,dz.\end{aligned}\tag{2.4.19}$$

Recalling that

$$\begin{aligned}w(0)&=\lim_{k\to+\infty}w_k(0)=\lim_{k\to+\infty}v_k(\theta_1)=\gamma,\\ w(L)&=\lim_{k\to+\infty}w_k(L)=\lim_{k\to+\infty}w_k(L_k)=\lim_{k\to+\infty}v_k(\theta_2)=\alpha,\end{aligned}$$

we see that w is a 1-Lipschitz curve on $[0, L]$ starting from γ and ending at α; therefore it must coincide with the unit speed parameterization of the segment connecting γ to α, i.e.,

$$w(z) = \gamma + \frac{\alpha - \gamma}{L} z.$$

So, we can easily compute the limit integral in (2.4.19):

$$\begin{aligned}\frac{1}{2}\int_0^L \big(w_1(z)w_2'(z) - w_2(z)w_1'(z)\big)\,dz &= -\frac{1}{2}\int_0^L \left(\gamma + \frac{\alpha-\gamma}{L}z\right)\cdot\frac{(\alpha-\gamma)^\perp}{L}dz \\ &= -\frac{1}{2}\gamma\cdot(\alpha-\gamma)^\perp \\ &= \frac{1}{2}(\gamma_1\alpha_2 - \gamma_2\alpha_1) = |T_{\alpha 0\gamma}|,\end{aligned}$$

where $T_{\alpha 0 \gamma}$ is the triangle with vertices α, γ and the origin 0. We conclude that

$$\lim_{k\to+\infty}\frac{1}{2}\int_{\theta_1}^{\theta_2}\left((v_k)_1(t)(\dot{v}_k)_2(t) - (v_k)_2(t)(\dot{v}_k)_2(t)\right)dt = |T_{\alpha 0\gamma}|.$$

In a similar way, one can prove that

$$\begin{aligned}\lim_{k\to+\infty}\frac{1}{2}\int_{\theta_2}^{\theta_3}\left((v_k)_1(t)(\dot{v}_k)_2(t) - (v_k)_2(t)(\dot{v}_k)_2(t)\right)dt &= |T_{\alpha 0\beta}|,\\ \lim_{k\to+\infty}\frac{1}{2}\int_{\theta_3}^{\theta_1}\left((v_k)_1(t)(\dot{v}_k)_2(t) - (v_k)_2(t)(\dot{v}_k)_2(t)\right)dt &= |T_{\beta 0\gamma}|,\end{aligned}$$

and (2.4.15) follows. □

Remark 2.4.3. A result similar to Theorem 2.4.1 holds, up to trivial modifications, when $u : B_\ell(0) \to \mathbb{S}^1$ is a symmetric n-uple point map, taking (in the order) the values $\alpha_1, \ldots, \alpha_n$ vertices of the regular n-gon $P_{\alpha_1\cdots\alpha_n}$ inscribed in the unit circle, on n non-overlapping $2\pi/n$-angular regions with common vertex at the origin. In formulas, let L be the side of $P_{\alpha_1\cdots\alpha_n}$ and h be the height of each isosceles triangle that decomposes $P_{\alpha_1\cdots\alpha_n}$, then there holds the following

Corollary 2.4.4. Let $u : B_\ell(0) \to \mathbb{S}^1$ be a symmetric n-ple map. Then

$$\overline{\mathcal{A}}_{BV}(u, B_\ell) = |B_\ell| + |Du|(B_\ell) + |P_{\alpha_1\cdots\alpha_n}| = \pi\ell^2 + nL\ell + \frac{n}{2}hL.$$

Remark 2.4.5. We point out that the "orientation preserving" assumption on u is crucial in order to adapt both the upper and the lower bound proofs of Theorem 2.4.1. Indeed, if u does not follow the order of the target vertices, some of the triangles $T_{\alpha_1 0\alpha_2}, \ldots, T_{\alpha_{n-1}0\alpha_n}$ overlap. As a consequence, the sequence (u^ε) may be not optimal anymore and, moreover, the inequality (2.4.16) could be too rough, making the resulting lower bound not optimal as well. In Chapter 4 we will find a way to overcome these issues, by considering a sort of Plateau problem for possibly self-intersecting polygonal curve connecting the α_j's.

Chapter 3

Piecewise Lipschitz maps jumping on a curve

In this chapter we analyze the case of BV-maps which are Lipschitz out of a discontinuity (smooth) curve. After proving some technical properties of strict convergence on one-dimensional slices, we consider first the case of piecewise Lipschitz maps jumping on a segment, for which an explicit integral expression of the BV-relaxed area is provided. The argument in the proof of the lower bound is based on some results of the theory of integer multiplicity currents, briefly sketched in Chapter 1. Thereafter, in Remark 3.2.6, we give an alternative proof, which is derived by results in [40] on minimal lifting currents (see Section 1.5 of Chapter 1). Finally, we extend the validity of the integral formula of the BV-relaxed area to the case the discontinuity set is a curve of class C^2. The results of this chapter are contained in [4].

3.1 Slicing properties of strict convergence

Let $R = [a, b] \times [-1, 1]$. For $(t, \sigma) \in R$, set

$$R_t^{x_1} := \{(x_1, x_2) \in R : x_1 = t\}, \qquad R_\sigma^{x_2} := \{(x_1, x_2) \in R : x_2 = \sigma\}.$$

If $u \in BV(R; \mathbb{R}^2)$, by Lebesgue differentiation theorem and Fubini theorem, for almost every $t \in [a, b]$, the restriction $u \llcorner R_t^{x_1}$ of u on the vertical segment $R_t^{x_1}$ coincides with the trace of u on $\mathcal{H}^1$-almost every point of $R_t^{x_1}$. So, for almost every $t \in [a, b]$, the map $u \llcorner R_t^{x_1}$ is well defined because it is independent of the representative of u. The same argument holds in $R_\sigma^{x_2}$ for almost every $\sigma \in [-1, 1]$.

Lemma 3.1.1 (Inheritance of strict convergence to slices). Let $u \in BV(R; \mathbb{R}^2)$. Suppose that $(v_k) \subset C^1(R; \mathbb{R}^2)$ is a sequence converging to u strictly $BV(R; \mathbb{R}^2)$. Then for almost every $(t, \sigma) \in R$, there exists a subsequence $(v_{k_h}) \subset (v_k)$, depending on t and σ, such that

$$v_{k_h} \llcorner R_t^{x_1} \to u \llcorner R_t^{x_1} \quad \text{strictly } BV(R_t^{x_1}; \mathbb{R}^2), \tag{3.1.1}$$

$$v_{k_h} \llcorner R_\sigma^{x_2} \to u \llcorner R_\sigma^{x_2} \quad \text{strictly } BV(R_\sigma^{x_2}; \mathbb{R}^2). \tag{3.1.2}$$

Proof. For almost every $t \in [a,b]$, in view of the definition of $R_t^{x_1}$, we can define the total variation of $u \llcorner R_t^{x_1}$ as

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) = \sup\left\{-\int_{-1}^{1} u(t,x_2)\cdot g'(x_2)dx_2;\ g \in C_c^1((-1,1);\overline{B}_1(0))\right\}, \quad (3.1.3)$$

where $\overline{B}_1(0) = \{(\xi,\eta) \in \mathbb{R}^2 : \xi^2+\eta^2 \le 1\}$. Let us show that

$$|D_2 u|(R) = \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1})dt, \quad (3.1.4)$$

where $D_2 u := Du\, e_2$ is a Radon measure on R valued in $\mathbb{R}^2$ with finite total variation. Since, for almost every $t \in [a,b]$, $v_k \llcorner R_t^{x_1} \to u \llcorner R_t^{x_1}$ in $L^1(R_t^{x_1};\mathbb{R}^2)$, we have, using (3.1.3),

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) \le \liminf_{k\to+\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t,x_2)|dx_2. \quad (3.1.5)$$

Then, using Fatou lemma and Fubini theorem,

$$\begin{aligned}\int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1})dt &\le \int_a^b \liminf_{k\to+\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t,x_2)|dx_2 dt \quad (3.1.6)\\ &\le \liminf_{k\to+\infty} \int_R |\partial_2 v_k(t,x_2)|dt dx_2 = |D_2 u|(R),\end{aligned}$$

where in the last equality we used Theorem 1.2.1 with $f(x,\nu) = \sqrt{\nu_3^2+\nu_4^2}$, for every $x \in R$, $\nu \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, with

$$\nu = \begin{pmatrix} \nu_1 & \nu_3 \\ \nu_2 & \nu_4 \end{pmatrix}.$$

The converse inequality in (3.1.4) is standard[1]. So, (3.1.4) is proved and (3.1.6) holds as an equality, which implies that also (3.1.5) holds as an equality, namely

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) = \liminf_{k\to+\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t,x_2)|dx_2.$$

Extracting a subsequence $(v_{k_h}) \subset (v_k)$ depending on t, we get

$$v_{k_h} \llcorner R_t^{x_1} \to u \llcorner R_t^{x_1} \quad \text{strictly } BV(R_t^{x_1};\mathbb{R}^2).$$

Finally, repeating the same argument for v_{k_h} on the horizontal slices $\{R_\sigma^{x_2}\}$, we get (3.1.1) for a (not relabeled) sub-subsequence. □

[1] We recall that

$$|D_2 u|(R) = \sup\left\{-\int_R u\cdot\partial_{x_2} g\ dx :\ g \in C_c^1(R;\overline{B}_1(0))\right\}.$$

Now, for $g \in C_c^1(R;\overline{B}_1(0))$, $\int_R u\cdot\partial_y g\ dx = \int_a^b \left(\int_{-1}^1 u(t,x_2)\cdot\partial_{x_2} g(t,x_2)dx_2\right)dt \le \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1})dt$, so $|D_2 u|(R) \le \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1})dt$.

Now, let B_l be the disk of $\mathbb{R}^2$ centered at the origin of radius $l > 0$. We want to prove the analogue of Lemma 3.1.1 in B_l, by slicing with concentric circumferences. If $u \in BV(B_l;\mathbb{R}^2)$, as in the previous case, for almost every $r \in (0,l)$ the restriction $u \llcorner \partial B_r$ is well-defined and independent of the representative of u. In particular, for almost every $r \in (0,l)$, we can define the total variation of $u \llcorner \partial B_r$ as

$$|D(u \llcorner \partial B_r)|(\partial B_r) := \sup\Big\{ -\int_0^{2\pi} \bar{u}(r,\theta)\cdot f'(\theta)d\theta; f \in C^1([0,2\pi];\overline{B}_1(0)), \quad f(0)=f(2\pi), f'(0)=f'(2\pi)\Big\} \tag{3.1.7}$$

which turns out to be finite (see Lemma 3.1.3), giving that $u \llcorner \partial B_r \in BV(\partial B_r;\mathbb{R}^2)$, for almost every $r \in (0,l)$. Here

$$\bar{u}(r,\theta) := u(r\cos\theta, r\sin\theta), \qquad r \in (0,l],\ \theta \in [0,2\pi).$$

We want to relate this quantity with the notion of tangential total variation.

Definition 3.1.2. For $x=(x_1,x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, set $\tau(x) = \frac{1}{|x|}(-x_2,x_1)$. Let $0 < l < L$ and $A_{L,l} := B_L(0)\setminus\overline{B_l(0)}$ be an annulus around 0. We define the tangential total variation of $u \in BV(A_{L,l};\mathbb{R}^2)$ as the total variation of the Radon measure $D_\tau u := Du\tau$, namely

$$|D_\tau u|(A_{L,l}) = |Du\tau|(A_{L,l}) = \sup\Big\{ -\int_{A_{L,l}} u\cdot(\nabla g\tau)\ dx : g \in C^1_c(A_{L,l};\overline{B}_1(0))\Big\}. \tag{3.1.8}$$

The last equality in (3.1.8) is justified since $\tau \in C^\infty(A_{L,l};\mathbb{R}^2)$ satisfies $\mathrm{div}\tau = 0$ everywhere, so for any $g=(g^1,g^2) \in C^1_c(A_{L,l};\mathbb{R}^2)$ we have

$$\begin{aligned} -\int_{A_{L,l}} u\cdot(\nabla g\tau)\ dx &= -\int_{A_{L,l}} u^1\nabla g^1\cdot\tau\ dx - \int_{A_{L,l}} u^2\nabla g^2\cdot\tau\ dx \\ &= -\int_{A_{L,l}} u^1\mathrm{div}(g^1\tau)\ dx - \int_{A_{L,l}} u^2\mathrm{div}(g^2\tau)\ dx \\ &= \int_{A_{L,l}} g^1\tau\cdot dDu^1 + \int_{A_{L,l}} g^2\tau\cdot dDu^2 = \int_{A_{L,l}} g\cdot(dDu)\tau = \langle Du\tau, g\rangle. \end{aligned}$$

This computation shows that $|D_\tau u|(A_{L,l}) \le |Du|(A_{L,l})$, since $|\tau| \le 1$, and also that (3.1.8) is compatible with the case $u \in W^{1,1}(A_{L,l};\mathbb{R}^2)$, where simply $|D_\tau u|(A_{L,l}) = \int_{A_{L,l}} |\nabla u\tau|\ dx$. Moreover, $Du = \frac{Du}{|Du|}|Du|$ by polar decomposition, so that for every $g \in C^1_c(B_l;\mathbb{R}^2)$

$$\langle Du\tau, g\rangle = \int_{A_{L,l}} g\cdot(dDu)\tau = \int_{A_{L,l}} g\cdot\left(\frac{Du}{|Du|}d|Du|\right)\tau = \int_{A_{L,l}} g\cdot\left(\frac{Du}{|Du|}\tau\right)d|Du|,$$

giving that

$$D_\tau u = Du\tau = \frac{Du}{|Du|}\tau|Du|. \tag{3.1.9}$$

Lemma 3.1.3 (Inheritance of strict convergence to circumferences). Let $u \in BV(B_R;\mathbb{R}^2)$ and $(v_k) \subset C^1(B_R;\mathbb{R}^2)$ be a sequence converging to u strictly $BV(B_R;\mathbb{R}^2)$. Then, for almost every $r \in (0,R)$, there exists a subsequence $(v_{k_h}) \subset (v_k)$, depending on r, such that

$$v_{k_h} \llcorner \partial B_r \to u \llcorner \partial B_r \quad \text{strictly } BV(\partial B_r;\mathbb{R}^2) \qquad \text{as } h \to +\infty. \tag{3.1.10}$$

Proof. For almost every $r \in (0,R)$, by Fatou lemma and Fubini theorem, the restriction $v_k \llcorner \partial B_r$ has equi-bounded variation w.r.t. k. Moreover, we may assume that (v_k) converges to u almost everywhere in B_R, so that, for almost every $r \in (0,R)$,

$$v_k \llcorner \partial B_r \to u \llcorner \partial B_r \quad \mathcal{H}^1\text{-a.e. in } \partial B_r. \tag{3.1.11}$$

Now, let $r \in (0,R)$ be such that $v_k \llcorner \partial B_r$ has equi-bounded variation and (3.1.11) holds. Then, there exists a subsequence $(v_{k_h}) \subset (v_k)$ depending on r such that

$$v_{k_h} \llcorner \partial B_r \overset{*}{\rightharpoonup} u \llcorner \partial B_r \quad \text{w}^* - BV(\partial B_r;\mathbb{R}^2).$$

By lower semicontinuity of the variation, we infer that for almost every $r \in (0,R)$, $u \llcorner \partial B_r \in BV(\partial B_r;\mathbb{R}^2)$ and

$$|D(u \llcorner \partial B_r)|(\partial B_r) \le \liminf_{h\to+\infty} \int_{\partial B_r} |\nabla v_{k_h} \tau| \, d\mathcal{H}^1. \tag{3.1.12}$$

Let $0 < l < L \le R$ be such that $v_k \to u$ strictly $BV(A_{L,l},\mathbb{R}^2)$ where, as in Definition 3.1.2, $A_{L,l} := B_L(0) \setminus \overline{B_l(0)}$ (notice that this holds for a.e. l and L); by integration, we get

$$\begin{aligned} \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r)\, dr &\le \int_l^L \left(\liminf_{h\to+\infty} \int_{\partial B_r} |\nabla v_{k_h}\tau| \, d\mathcal{H}^1 \right) dr \\ \le \liminf_{h\to+\infty} \int_l^L \int_{\partial B_r} |\nabla v_{k_h}\tau| \, d\mathcal{H}^1 dr &= \liminf_{h\to+\infty} \int_{A_{L,l}} |\nabla v_{k_h}\tau| \, dx. \end{aligned} \tag{3.1.13}$$

Thanks to Theorem 1.2.1, with the choices $M = 4$, $\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^{2\times 2}$, $f \in C_b(A_{L,l} \times \mathbb{S}^3)$,

$$f(x,\nu) := \sqrt{|\nu_{\text{hor}} \cdot \tau(x)|^2 + |\nu_{\text{vert}} \cdot \tau(x)|^2},$$

where $\nu \in \mathbb{S}^3$ and $\nu_{\text{hor}} := (\nu_1,\nu_3), \nu_{\text{vert}} := (\nu_2,\nu_4)$, we obtain

$$\lim_{k\to+\infty} \int_{A_{L,l}} |\nabla v_k \tau| \, dx = \int_{A_{L,l}} \left| \frac{Du}{|Du|}\tau \right| d|Du| = |D_\tau u|(A_{L,l}), \tag{3.1.14}$$

where in the last equality we have used (3.1.9). So we get

$$|D_\tau u|(B_l) \ge \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r)\, dr.$$

In order to prove the converse inequality, let $g \in C^1_c(A_{L,l};\overline{B}_1(0))$. Then, in polar coordinates, by definition (3.1.7),

$$\int_{A_{L,l}} u \cdot \nabla g \tau \, dx = \int_l^L \int_0^{2\pi} \bar{u}(\rho,\theta) \cdot \partial_\theta \bar{g}(\rho,\theta)\, d\rho d\theta \le \int_l^L |D(u \llcorner \partial B_\rho)|(\partial B_\rho)\, d\rho.$$

So, we have proved that

$$|D_\tau u|(A_{L,l}) = \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r)\ dr.$$

In particular, we deduce that (3.1.13) is a chain of equalities. Then, (3.1.12) holds as an equality and there exists a subsequence $(v_{k_h}) \subset (v_k)$, depending on r, which achieves the full limit. Since l and L are arbitrary, we get the thesis. □

3.1.1 Further properties in dimension 1

For our purposes, we need an improvement of Corollary 1.3.6, where discontinuous functions γ at a single point, or at a finite number of points, are allowed. More precisely, we would like to conclude that, up to further reparametrization, the approximating sequence converges uniformly to a slight modification of the limit map γ; we start with one point discontinuity.

Lemma 3.1.4. Let $I^- := [-1, 0), I^+ := (0, 1]$. Suppose that $(\gamma_k) \subset W^{1,1}([-1, 1]; \mathbb{R}^2)$ is a sequence converging strictly $BV([-1, 1]; \mathbb{R}^2)$ to $\gamma \in BV([-1, 1]; \mathbb{R}^2) \cap W^{1,1}(I^-; \mathbb{R}^2) \cap W^{1,1}(I^+; \mathbb{R}^2)$, with $\gamma^+(0) \neq \gamma^-(0)$. Let $S : [-1/3, 1/3] \to \mathbb{R}^2$ be defined by

$$S(\tau) := \frac{3}{2}\big((1/3 + \tau)\,\gamma^+(0) + (1/3 - \tau)\,\gamma^-(0)\big), \quad \tau \in [-1/3, 1/3].$$

Let $\widetilde{\gamma}^-$ (resp. $\widetilde{\gamma}^+$) be the reparametrization of $\gamma_{|I^-}$ (resp. $\gamma_{|I^+}$) on $[-1, -\frac{1}{3})$ (resp. $(\frac{1}{3}, 1]$) defined by the composition with the increasing linear function taking $[-1, -1/3]$ onto $[-1, 0]$ (resp. $[1/3, 1]$ onto $[0, 1]$). Define

$$\widetilde{\gamma} : [-1, 1] \to \mathbb{R}^2, \qquad \widetilde{\gamma} := \begin{cases} \widetilde{\gamma}^- & \text{in } [-1, -1/3) \\ S & \text{in } [-1/3, 1/3] \\ \widetilde{\gamma}^+ & \text{in } (1/3, 1]. \end{cases} \tag{3.1.15}$$

Then there exist:

(a) a Lipschitz strictly increasing surjective function $h : [-1, 1] \to [-1, 1]$,

(b) a subsequence (k_j) and Lipschitz strictly increasing surjective functions $h_{k_j} : [-1, 1] \to [-1, 1]$ for any $j \in \mathbb{N}$, with $\sup_j \|\dot{h}_{k_j}\|_\infty < +\infty$,

such that

$$\lim_{j \to +\infty} \gamma_{k_j} \circ h_{k_j} = \widetilde{\gamma} \circ h \quad \text{uniformly in } [-1, 1]. \tag{3.1.16}$$

Proof. The lengths L_k of γ_k and L of γ are given by

$$L_k = \int_{-1}^1 |\dot{\gamma}_k|\ d\tau,$$

$$L = |\dot{\gamma}|([-1, 1]) = \int_{-1}^0 |\dot{\gamma}|\ d\tau + |\gamma^+(0) - \gamma^-(0)| + \int_0^1 |\dot{\gamma}|\ d\tau.$$

Since, by assumption, $\gamma_k \to \gamma$ strictly $BV([-1,1];\mathbb{R}^2)$, we have that $L_k \to L$ as $k \to +\infty$. Fix $\eta > 0$ and for all $k \in \mathbb{N}$ define the function[2]

$$s_k : [-1,1] \to [0, L+\eta], \qquad s_k(t) := \frac{L+\eta}{L_k+\eta} \int_{-1}^{t} \Big(|\dot{\gamma}_k(\tau)| + \frac{\eta}{2} \Big)\, d\tau, \tag{3.1.17}$$

with Lipschitz inverse $\alpha_k := s_k^{-1} : [0, L+\eta] \to [-1,1]$. Define

$$\widehat{\gamma}_k : [0, L+\eta] \to \mathbb{R}^2, \qquad \widehat{\gamma}_k(s) := \gamma_k(\alpha_k(s)) \qquad \forall s \in [0, L+\eta]. \tag{3.1.18}$$

Since from (3.1.17)

$$\left| \frac{d\widehat{\gamma}_k}{ds}(s) \right| \le \frac{|\dot{\gamma}_k(\alpha_k(s))|}{|\dot{s}_k(\alpha_k(s))|} \le \frac{L_k+\eta}{L+\eta} \le C \qquad \text{for a.e. } s \in [0, L+\eta],$$

for some constant $C > 0$ independent of k, the sequence $(\widehat{\gamma}_k)$ is bounded in $W^{1,\infty}([0, L+\eta];\mathbb{R}^2)$. Thus, up to a (not relabeled) subsequence, we may assume that there exists $\widehat{\gamma} \in W^{1,\infty}([0, L+\eta];\mathbb{R}^2)$ such that

$$\widehat{\gamma}_k \rightharpoonup \widehat{\gamma} \text{ weakly* in } W^{1,\infty}([0, L+\eta];\mathbb{R}^2) \text{ and uniformly in } [0, L+\eta]. \tag{3.1.19}$$

We observe that for any open interval $J \subseteq [0, L+\eta]$,

$$\int_J |\dot{\widehat{\gamma}}| ds \le \liminf_{k\to+\infty} \int_J |\dot{\widehat{\gamma}}_k| ds \le |J| \liminf_{k\to+\infty} \frac{L_k+\eta}{L+\eta} = |J|,$$

and thus

$$|\dot{\widehat{\gamma}}| \le 1 \text{ a.e. in } [0, L+\eta]. \tag{3.1.20}$$

Now, in order to conclude the proof, we need to show that $\widehat{\gamma}$ is a reparametrization of $\widetilde{\gamma}$. Then the thesis of the lemma will follow by reparametrizing both $\widehat{\gamma}_k$ and $\widehat{\gamma}$ on $[-1,1]$.

Using that (γ_k) strictly converges $BV([-1,1];\mathbb{R}^2)$ to $\gamma \in W^{1,1}(I^-;\mathbb{R}^2) \cap W^{1,1}(I^+;\mathbb{R}^2)$, by Corollary 1.3.6 and a diagonal process, we can find an infinitesimal sequence $(\delta_{k_j}) \subset (0,1]$ such that

$$\|\gamma_{k_j} - \gamma\|_{L^\infty([-1,1]\setminus(-\delta_{k_j},\delta_{k_j}))} \to 0 \tag{3.1.21}$$

and

$$\int_{-1}^{-\delta_{k_j}} |\dot{\gamma}_{k_j}(\tau)|\, d\tau \to \int_{-1}^{0} |\dot{\gamma}(\tau)|\, d\tau, \qquad \int_{\delta_{k_j}}^{1} |\dot{\gamma}_{k_j}(\tau)|\, d\tau \to \int_{0}^{1} |\dot{\gamma}(\tau)|\, d\tau$$

as $j \to +\infty$. In particular,

$$\lim_{j\to+\infty} \gamma_{k_j}(\pm\delta_{k_j}) = \gamma^{\pm}(0) \tag{3.1.22}$$

[2] We need η, since in principle $\dot{\gamma}_k$ could vanish somewhere.

and, setting

$$
\begin{aligned}
r_{k_j}^- &:= s_{k_j}(-\delta_{k_j}) = \frac{L+\eta}{L_{k_j}+\eta}\int_{-1}^{-\delta_{k_j}} \left(|\dot\gamma_{k_j}| + \frac{\eta}{2}\right)\, d\tau,\\
r_{k_j}^+ &:= s_{k_j}(\delta_{k_j}) = \frac{L+\eta}{L_{k_j}+\eta}\left[\int_{-1}^{1} \left(|\dot\gamma_{k_j}| + \frac{\eta}{2}\right)\, d\tau - \int_{\delta_{k_j}}^{1} \left(|\dot\gamma_{k_j}| + \frac{\eta}{2}\right)\, d\tau\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
\lim_{j\to+\infty} r_{k_j}^- &= \frac{\eta}{2} + \int_{-1}^{0} |\dot\gamma|\, d\tau =: r^-,\\
\lim_{j\to+\infty} r_{k_j}^+ &= \frac{\eta}{2} + \int_{-1}^{0} |\dot\gamma|\, d\tau + |\gamma^+(0)-\gamma^-(0)| =: r^+.
\end{aligned}
\tag{3.1.23}
$$

As a consequence of (3.1.19), (3.1.22), and (3.1.23) we get

$$
\gamma_{k_j}(\alpha_{k_j}(r_{k_j}^\pm)) = \widehat\gamma_{k_j}(r_{k_j}^\pm) \to \widehat\gamma(r^\pm) = \gamma^\pm(0).
$$

Therefore the curve $\widehat\gamma$ maps the segment $[r^-, r^+]$ into a curve joining $\gamma^-(0)$ and $\gamma^+(0)$. Now, since $r^+ - r^- = |\gamma^+(0)-\gamma^-(0)|$, from (3.1.20) we conclude that $\widehat\gamma$ coincides with the unit-speed parametrization of the segment joining $\gamma^-(0)$ and $\gamma^+(0)$ on $[r^-, r^+]$. Hence we have shown that

$$
\gamma_{k_j}\circ\alpha_{k_j} \to S\circ\widetilde\alpha \text{ uniformly in } [r^-, r^+] \text{ as } j\to+\infty, \tag{3.1.24}
$$

for the affine increasing reparametrization $\widetilde\alpha : [r^-, r^+] \to [-1/3, 1/3]$.

We now check that $\widehat\gamma = \gamma\circ\alpha$ on $[0, r^-]$ for some increasing bijection $\alpha : [0, r^-] \to [-1, 0]$, and similarly $\widehat\gamma = \gamma\circ\beta$ on $[r^+, L+\eta]$ for some increasing bijection $\beta : [r^+, L+\eta] \to [0, 1]$.

Indeed, the functions $\alpha_k : [0, L+\eta] \to [-1, 1]$ are strictly increasing and satisfy

$$
|\dot\alpha_k(s_k(t))| = \frac{L_k+\eta}{(L+\eta)(|\dot\gamma_k(t)| + \frac{\eta}{2})} \le \frac{C}{\eta},
$$

so that we may assume (up to extracting a further not relabeled subsequence) that

$$
\alpha_{k_j} \rightharpoonup \alpha \text{ weakly* in } W^{1,\infty}([0, L+\eta]) \text{ and uniformly in } [0, L+\eta],
$$

for some nondecreasing map $\alpha \in W^{1,\infty}([0, L+\eta])$. Hence, using (3.1.21), we find out

$$
\widehat\gamma_{k_j}(s) = \gamma_{k_j}(\alpha_{k_j}(s)) \to \gamma(\alpha(s)) \text{ for all } s\in[0, r^-).
$$

This, together with (3.1.19), implies

$$
\widehat\gamma(s) = \gamma\circ\alpha(s) \text{ for all } s\in[0, r^-).
$$

A similar argument shows that this also holds for all $s\in(r^+, L+\eta]$.

Finally, we observe that α is strictly increasing on $[0, r^-)\cup(r^+, L+\eta]$. For, if α is constant on some interval $[s_1, s_2]\subset[0, r^-)$, we have $\lim_{j\to+\infty}\alpha_{k_j}(s_1) = \lim_{h\to+\infty}\alpha_{k_j}(s_2)$ and hence

$$
0 = \lim_{j\to+\infty}\int_{s_1}^{s_2} \dot\alpha_{k_j}(s) ds = \lim_{j\to+\infty}\int_{t_{k_j,1}}^{t_{k_j,2}} d\tau = \lim_{j\to+\infty}(t_{k_j,2} - t_{k_j,1}), \tag{3.1.25}
$$

where $t_{k_j,i}$ are defined by $s_{k_j}(t_{k_j,1}) = s_1$ and $s_{k_j}(t_{k_j,2}) = s_2$. By definition (3.1.17) of s_{k_j} we have

$$0 < s_2 - s_1 = \int_{t_{k_j,1}}^{t_{k_j,2}} \left(|\dot{\gamma}_{k_j}(\tau)| + \frac{\eta}{2}\right) d\tau. \tag{3.1.26}$$

Possibly passing to a (not relabeled) subsequence and using (3.1.25), let $\bar{t} \in [-1, 0]$ be the limit of $(t_{k_j,1})$ and $(t_{k_j,2})$. If $\bar{t} \neq 0$, for any open neighborhood $J \subset (-1, 0)$ of $\bar{t}$, using (3.1.26), we get

$$\int_J |\dot{\gamma}| \, d\tau = \lim_{h \to +\infty} \int_J |\dot{\gamma}_{k_j}| \, d\tau \geq s_2 - s_1,$$

which contradicts the inclusion $\dot{\gamma} \in L^1((-1, 0); \mathbb{R}^2)$. The same argument holds if $\bar{t} = 0$, for J a left neighbourhood of 0 in $(-1, 0)$. We conclude that α is strictly increasing.

Let h_{k_j} be a rescaling of α_{k_j} on $[-1, 1]$; rescaling also α from $[0, r^-]$ to $[-1, -1/3]$, and then from $[r^+, L + \eta]$ to $[1/3, 1]$, using also $\widetilde{\alpha}$ in (3.1.24), we construct a reparametrization $h : [-1, 1] \to [-1, 1]$ such that (3.1.16) holds, and the lemma is proved. □

Lemma 3.1.4 can be readily extended to curves γ with finitely many jump points:

Corollary 3.1.5. Assume that $(\gamma_k) \subset W^{1,1}([0, 2\pi]; \mathbb{R}^2)$ is a sequence that converges strictly $BV([0, 2\pi]; \mathbb{R}^2)$ to a map $\gamma \in SBV([0, 2\pi]; \mathbb{R}^2)$ having finitely many jump points $0 < z_1 < z_2 < \cdots < z_n < 2\pi$. Let $\theta_0 > 0$ be such that the intervals $(z_i - \theta_0, z_i + \theta_0) \subset (0, 2\pi)$ are disjoint, and for all $i = 1, \ldots, n$ let $S_i : [z_i - \theta_0, z_i + \theta_0] \to \mathbb{R}^2$ be defined by

$$S_i(\tau) := \frac{1}{2\theta_0} \left((\tau - z_i + \theta_0) \, \gamma^+(z_i) + (z_i + \theta_0 - \tau) \, \gamma^-(z_i) \right), \quad \tau \in [z_i - \theta_0, z_i + \theta_0].$$

Setting $z_0 := 0$ and $z_{n+1} := 2\pi$, for all $i = 0, \ldots, n$ let $\widetilde{\gamma}_i : [z_i + \theta_0, z_{i+1} - \theta_0] \to \mathbb{R}^2$ be a rescaled reparametrization of $\gamma : [z_i, z_{i+1}] \to \mathbb{R}^2$. Finally, let $\widetilde{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ be the Lipschitz curve defined as

$$\widetilde{\gamma} := \widetilde{\gamma}_0 \star S_1 \star \widetilde{\gamma}_1 \star S_2 \star \widetilde{\gamma}_2 \star \cdots \star S_n \star \widetilde{\gamma}_n, \tag{3.1.27}$$

where $\star$ denotes the arc composition. Then there exist a subsequence (k_j) and Lipschitz increasing surjective functions $h, h_{k_j} : [0, 2\pi] \to [0, 2\pi]$ such that

$$\lim_{j \to +\infty} \gamma_{k_j} \circ h_{k_j} = \widetilde{\gamma} \circ h \quad \text{uniformly in } [0, 2\pi]. \tag{3.1.28}$$

Proof. We skecth the proof which is a direct consequence of the arguments used to prove Lemma 3.1.4. Choose points w_i, $i = 1, \ldots, n-1$ so that $z_i + \theta_0 < w_i < z_{i+1} - \theta_0$, and let $w_0 = 0$ and $w_n = 2\pi$. Then we can apply Lemma 3.1.4 to any interval $[w_i, w_{i+1}]$, and taking a suitable subsequence and concatenating the obtained maps one can easily construct the desired parametrizations. □

3.2 Relaxation on piecewise Lipschitz maps jumping on a curve

Recalling that $R = [a, b] \times [-1, 1]$, consider $R^+ = \{(x_1, x_2) \in R : x_2 > 0\}$ and $R^- = \{(x_1, x_2) \in R : x_2 < 0\}$.

Definition 3.2.1 (**Piecewise Lipschitz map**)**.** We say that a map $u : R \to \mathbb{R}^2$ is piecewise Lipschitz if $u \in BV(R;\mathbb{R}^2)$ and $u \in \mathrm{Lip}(R^-;\mathbb{R}^2) \cap \mathrm{Lip}(R^+;\mathbb{R}^2)$.

Thus $J_u \subseteq [a,b] \times \{0\}$; we define $u^\pm : [a,b] \times \{0\} \to \mathbb{R}^2$ the traces of $u_{|R^\pm}$, which are Lipschitz maps. Set $I = [0,1]$ and define $X^{\mathrm{aff}} : [a,b] \times I \to \mathbb{R}^3$ the affine interpolation surface spanning $\mathrm{graph}(u^\pm) = \{(t, u^\pm(t)) : t \in [a,b]\} \subset \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$, namely

$$X^{\mathrm{aff}}(t,s) = (t, su^+(t) + (1-s)u^-(t)) =: (t, \widehat{X}(t,s)) \qquad \forall (t,s) \in [a,b] \times I. \tag{3.2.1}$$

Remark 3.2.2. For a (semicartesian) map $\Phi : [a,b] \times [c,d] \to \mathbb{R}^3$ of the form $\Phi(t,\sigma) = (t, \phi(t,\sigma)) = (t, \phi_1(t,\sigma), \phi_2(t,\sigma))$, the area integrand is given by

$$|\partial_t \Phi \wedge \partial_\sigma \Phi| = \sqrt{|\partial_\sigma \phi_1|^2 + |\partial_\sigma \phi_2|^2 + (\partial_t \phi_1 \partial_\sigma \phi_2 - \partial_\sigma \phi_1 \partial_t \phi_2)^2} = \sqrt{|\partial_\sigma \phi|^2 + |J\phi|^2}.$$

The main result of this section is the following:

Theorem 3.2.3 (**Relaxed area of piecewise Lipschitz maps: straight jump**)**.** Let $u : R \to \mathbb{R}^2$ be a piecewise Lipschitz map. Then

$$\overline{\mathcal{A}}_{BV}(u,R) = \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-) + \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dtds. \tag{3.2.2}$$

Notice that the Lipschitz regularity of u on $R^\pm$ ensures that the area functional has the classical expression

$$\mathcal{A}(u,R^\pm) = \int_{R^\pm} \sqrt{1 + |\nabla u|^2 + |\det \nabla u|^2} \, dx;$$

therefore, the singular contribution produced by the relaxation in (3.2.2) is given by the area of X^{aff}.

We divide the proof of (3.2.2) in two parts: the lower bound (Proposition 3.2.4) and the upper bound (Proposition 3.2.5).

Proposition 3.2.4 (**Lower bound for** (3.2.2))**.** Let $u : R \to \mathbb{R}^2$ be a piecewise Lipschitz map, and $(v_k) \subset C^1(R;\mathbb{R}^2) \cap BV(R;\mathbb{R}^2)$ be a sequence converging to u strictly $BV(R;\mathbb{R}^2)$. Then

$$\liminf_{k\to+\infty} \mathcal{A}(v_k,R) \geq \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-) + \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dtds. \tag{3.2.3}$$

Proof. Fix $\varepsilon \in (0,1)$. We have

$$\begin{aligned}\liminf_{k\to+\infty} \mathcal{A}(v_k,R) &\geq \liminf_{k\to+\infty} \mathcal{A}(v_k, R \setminus ([a,b]\times[-\varepsilon,\varepsilon])) + \liminf_{k\to+\infty} \mathcal{A}(v_k,[a,b]\times[-\varepsilon,\varepsilon]) \\ &\geq \mathcal{A}(u, R \setminus ([a,b]\times[-\varepsilon,\varepsilon])) + \liminf_{k\to+\infty} \mathcal{A}(v_k,[a,b]\times[-\varepsilon,\varepsilon]),\end{aligned}$$

where in the last inequality we used [1, Theorem 3.7]. Sending ε to 0^+, by dominated convergence it follows $\mathcal{A}(u, R \setminus ([a,b]\times[-\varepsilon,\varepsilon])) \to \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-)$, so (3.2.3) will be proven provided we show that

$$\lim_{\varepsilon\to0^+} \liminf_{k\to+\infty} \mathcal{A}(v_k,[a,b]\times[-\varepsilon,\varepsilon]) \geq \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dtds. \tag{3.2.4}$$

Consider the maps

$$V_k^\varepsilon : R \to \mathbb{R}^3, \quad V_k^\varepsilon(t,\sigma) = (t, v_k(t, \varepsilon\sigma)),$$

and the associated integer multiplicity 2-currents in $\mathbb{R}^3$

$$\mathcal{V}_k^\varepsilon = V_{k\,\sharp}^\varepsilon [\![R]\!].$$

Notice that, neglecting the term $1 + |\partial_{x_1} v_k|^2$, we get

$$\begin{aligned} \mathcal{A}(v_k, [a,b] \times [-\varepsilon, \varepsilon]) &\geq \int_{[a,b]\times[-\varepsilon,\varepsilon]} \sqrt{|\partial_{x_2} v_k|^2 + |Jv_k|^2}\; dx \\ &= \int_R |\partial_t V_k^\varepsilon \wedge \partial_\sigma V_k^\varepsilon| \; dt d\sigma = |\mathcal{V}_k^\varepsilon|, \end{aligned} \tag{3.2.5}$$

where we used Remark 3.2.2, and $|\cdot|$ stands for the mass current. Consider also the maps

$$U_\pm^\varepsilon : R^\pm \to \mathbb{R}^3, \quad U_\pm^\varepsilon(t,\sigma) = (t, u(t, \varepsilon\sigma)), \tag{3.2.6}$$

and the current

$$S_\varepsilon = X_\sharp^{\text{aff}} [\![[a,b] \times I]\!] + U_{+\sharp}^\varepsilon [\![R^+]\!] + U_{-\sharp}^\varepsilon [\![R^-]\!], \tag{3.2.7}$$

see Fig. 3.1. We want now prove the following crucial inequality:

$$\liminf_{k\to+\infty} |\mathcal{V}_k^\varepsilon| \geq |S_\varepsilon|. \tag{3.2.8}$$

To show (3.2.8) we prove that $\mathcal{V}_k^\varepsilon$ are close to suitable currents $\mathcal{M}_n^\varepsilon$ independent of k (see (3.2.19)) which converge to S_ε as $n \to +\infty$.

For any $n \in \mathbb{N}$, $n \geq 1$, consider a partition $\{t_0 = a, t_1, \dots, t_{n+1} = b\}$ of $[a,b]$ in $(n+1)$ intervals $[t_{i-1}, t_i)$, with

$$t_i - t_{i-1} \in \left(\frac{b-a}{2n}, 2\frac{(b-a)}{n} \right). \tag{3.2.9}$$

Moreover, set

$$R_i = [t_{i-1}, t_i) \times [-1,1], \quad R_i^+ = [t_{i-1}, t_i) \times (0,1], \quad R_i^- = [t_{i-1}, t_i) \times [-1, 0),$$

and define the currents

$$\mathcal{V}_{k,i}^\varepsilon = V_{k\,\sharp}^\varepsilon [\![R_i]\!], \quad S_{\varepsilon,i} = X_\sharp^{\text{aff}} [\![[t_{i-1}, t_i) \times I]\!] + U_{+\sharp}^\varepsilon [\![R_i^+]\!] + U_{-\sharp}^\varepsilon [\![R_i^-]\!], \tag{3.2.10}$$

see Fig. 3.1. By definition, we have

$$\begin{aligned} &\mathcal{V}_k^\varepsilon = \sum_{i=1}^{n+1} \mathcal{V}_{k,i}^\varepsilon \quad \text{and} \quad \mathcal{H}^2(\text{spt}\mathcal{V}_{k,i}^\varepsilon \cap \text{spt}\mathcal{V}_{k,j}^\varepsilon) = 0 \quad \text{for } i \neq j, \\ &S_\varepsilon = \sum_{i=1}^{n+1} S_{\varepsilon,i} \quad \text{and} \quad \mathcal{H}^2(\text{spt}S_{\varepsilon,i} \cap \text{spt}S_{\varepsilon,j}) = 0 \quad \text{for } i \neq j. \end{aligned} \tag{3.2.11}$$

Furthermore,

$$\begin{aligned}\partial S_{\varepsilon,i} = &- \Big(U^{\varepsilon}_{-\sharp}[\![\{t_{i-1}\}\times[-1,0)]\!] + X^{\mathrm{aff}}_{\sharp}[\![\{t_{i-1}\}\times I]\!] + U^{\varepsilon}_{+\sharp}[\![\{t_{i-1}\}\times(0,1]]\!]\Big)\\ &- U^{\varepsilon}_{+\sharp}[\![(t_{i-1},t_i)\times\{1\}]\!]\\ &+ \Big(U^{\varepsilon}_{-\sharp}[\![\{t_{i}\}\times[-1,0)]\!] + X^{\mathrm{aff}}_{\sharp}[\![\{t_{i}\}\times I]\!] + U^{\varepsilon}_{+\sharp}[\![\{t_{i}\}\times(0,1]]\!]\Big)\\ &+ U^{\varepsilon}_{-\sharp}[\![(t_{i-1},t_i)\times\{-1\}]\!].\end{aligned} \tag{3.2.12}$$

Now, for fixed $i\in\{1,\dots,n\}$, set

$$\begin{aligned}&\gamma^{u,\varepsilon}_{-,i}(\sigma) = u(t_i,\varepsilon\sigma) && \forall\sigma\in[-1,0),\\ &\gamma^{u,\varepsilon}_{+,i}(\sigma) = u(t_i,\varepsilon\sigma) && \forall\sigma\in(0,1],\\ &\gamma^{0}_{i}(s) = su^+(t_i)+(1-s)u^-(t_i) && \forall s\in I,\\ &\Lambda^{\pm,\varepsilon}_{u,i}(t) = (t,u(t,\pm\varepsilon)) && \forall t\in[t_{i-1},t_i],\end{aligned}$$

and define $\gamma^{u,\varepsilon}_i : [-1,1]\to\mathbb{R}^2$ as in (3.1.15) where $\widetilde{\gamma}^-$, S, and $\widetilde{\gamma}^+$ are replaced by $\gamma^{u,\varepsilon}_{-,i}$, γ^0_i and $\gamma^{u,\varepsilon}_{+,i}$ in the order, after a rescaling on $\left[-1,-\frac{1}{3}\right]$, $\left[-\frac{1}{3},\frac{1}{3}\right]$, and $\left[\frac{1}{3},1\right]$, respectively, as in the statement of Lemma 3.1.4. Also, define $\Gamma^{u,\varepsilon}_i : [-1,1]\to(\{t_i\}\times\mathbb{R}^2)$ as

$$\Gamma^{u,\varepsilon}_i(\sigma) := (t_i,\gamma^{u,\varepsilon}_i(\sigma)) \qquad \forall\sigma\in[-1,1].$$

Using the definition of $U^{\varepsilon}_{\pm}$ and X^{aff}, by (3.2.12) we infer

$$\partial S_{\varepsilon,i} = -\Gamma^{u,\varepsilon}_{i-1\sharp}[\![[-1,1]]\!] - \Lambda^{+,\varepsilon}_{u,i\ \sharp}[\![(t_{i-1},t_i)]\!] + \Gamma^{u,\varepsilon}_{i\ \sharp}[\![[-1,1]]\!] + \Lambda^{-,\varepsilon}_{u,i\ \sharp}[\![(t_{i-1},t_i)]\!]. \tag{3.2.13}$$

Moreover, set

$$\begin{aligned}&\gamma^{\varepsilon}_{k,i}(\sigma) = v_k(t_i,\varepsilon\sigma), \qquad \Gamma^{\varepsilon}_{k,i}(\sigma) = (t_i,\gamma^{\varepsilon}_{k,i}(\sigma)) && \forall\sigma\in[-1,1],\\ &\Lambda^{\pm,\varepsilon}_{k,i}(t) = (t,v_k(t,\pm\varepsilon)) && \forall t\in[t_{i-1},t_i].\end{aligned}$$

By definition of $\mathcal{V}^{\varepsilon}_{k,i}$ in (3.2.10), we also have

$$\partial\mathcal{V}^{\varepsilon}_{k,i} = -\Gamma^{\varepsilon}_{k,i-1\sharp}[\![[-1,1]]\!] - \Lambda^{+,\varepsilon}_{k,i\ \sharp}[\![(t_{i-1},t_i)]\!] + \Gamma^{\varepsilon}_{k,i\sharp}[\![[-1,1]]\!] + \Lambda^{-,\varepsilon}_{k,i\ \sharp}[\![(t_{i-1},t_i)]\!]. \tag{3.2.14}$$

We now define $F^{\varepsilon}_{k,i}\in\mathcal{D}_2(\mathbb{R}^3)$ as a suitable affine interpolation between $\partial\mathcal{V}^{\varepsilon}_{k,i}$ and $\partial S_{\varepsilon,i}$, see Fig. 3.1. First observe that by Lemma 3.1.1, we can suppose that, for our choice of ε and $\{t_1,\dots,t_n\}$, there exists a (not relabeled) subsequence of $(v_k)_k$, such that

$$v_k(t_i,\varepsilon\cdot)\to u(t_i,\varepsilon\cdot) \quad \text{strictly } BV([-1,1];\mathbb{R}^2) \quad \forall i=1,\dots,n, \tag{3.2.15}$$

$$v_k(\cdot,\pm\varepsilon)\to u(\cdot,\pm\varepsilon) \quad \text{strictly } BV([a,b];\mathbb{R}^2). \tag{3.2.16}$$

In particular, by Lemma 3.1.4, we know that there are increasing Lipschitz bijections $h^{\varepsilon}_{k,i}, h^{\varepsilon}_i : [-1,1]\to[-1,1]$ such that $\gamma^{\varepsilon}_{k,i}\circ h^{\varepsilon}_{k,i}\to\gamma^{u,\varepsilon}_i\circ h^{\varepsilon}_i$ uniformly in $[-1,1]$ as $k\to+\infty$.

For $i=1,\dots,n$, we define

$$\begin{aligned}&\Phi^{\varepsilon}_{k,i}(\sigma,s) := s(\Gamma^{\varepsilon}_{k,i}\circ h^{\varepsilon}_{k,i}(\sigma)) + (1-s)(\Gamma^{u,\varepsilon}_i\circ h^{\varepsilon}_i(\sigma)), \qquad (\sigma,s)\in[-1,1]\times I,\\ &\Psi^{\pm,\varepsilon}_{k,i}(t,s) := s\Lambda^{\pm,\varepsilon}_{k,i}(t) + (1-s)\Lambda^{\pm,\varepsilon}_{u,i}(t), \qquad (t,s)\in[t_{i-1},t_i]\times I.\end{aligned}$$

Therefore we set

$$\begin{aligned} F^\varepsilon_{k,i} = &- \Phi^\varepsilon_{k,i-1\,\sharp}[\![[-1,1]\times I]\!] - \Psi^{+,\varepsilon}_{k,i}{}_\sharp [\![[t_{i-1},t_i]\times I]\!] \\ &+ \Phi^\varepsilon_{k,i\,\sharp}[\![[-1,1]\times I]\!] + \Psi^{-,\varepsilon}_{k,i}{}_\sharp [\![[t_{i-1},t_i]\times I]\!]. \end{aligned} \tag{3.2.17}$$

In particular, from (3.2.13) and (3.2.14), a direct check shows that

$$\partial F^\varepsilon_{k,i} = \partial \mathcal{V}^\varepsilon_{k,i} - \partial S_{\varepsilon,i}. \tag{3.2.18}$$

Eventually, we let $M_{\varepsilon,i}$ be an integer multiplicity 2-current of $\mathbb{R}^3$ with minimal mass and boundary $\partial S_{\varepsilon,i}$ (the existence of $M_{\varepsilon,i}$ is guaranteed by Theorem 1.5.3) and set

$$\mathcal{M}^\varepsilon_n := \sum_{i=2}^{n} M_{\varepsilon,i}. \tag{3.2.19}$$

Note carefully that we do not sum over i from 1 to $n+1$, but only from 2 to n. In particular, setting $S^n_\varepsilon = S_\varepsilon - S_{\varepsilon,1} - S_{\varepsilon,n+1}$, we have

$$\partial \mathcal{M}^\varepsilon_n = \partial S^n_\varepsilon = -\Gamma^{u,\varepsilon}_1{}_\sharp[\![[-1,1]]\!] + \Gamma^{u,\varepsilon}_n{}_\sharp[\![[-1,1]]\!] - \Lambda^{+,\varepsilon}_u{}_\sharp[\![[t_1,t_n]]\!] + \Lambda^{-,\varepsilon}_u{}_\sharp[\![[t_1,t_n]]\!], \tag{3.2.20}$$

where

$$\Lambda^{\pm,\varepsilon}_u(t) := (t, u(t,\pm\varepsilon)), \qquad t\in(t_1,t_n).$$

Thus, we have

$$|\mathcal{V}^\varepsilon_{k,i}| \geq |\mathcal{V}^\varepsilon_{k,i} - F^\varepsilon_{k,i}| - |F^\varepsilon_{k,i}| \geq |M_{\varepsilon,i}| - |F^\varepsilon_{k,i}| \quad \text{for i} = 2,\dots,\text{n},$$

where we used the minimality of $M_{\varepsilon,i}$ and (3.2.18). By summing up, using (3.2.11), we get[3]

$$|\mathcal{V}^\varepsilon_k| = \sum_{i=1}^{n+1} |\mathcal{V}^\varepsilon_{k,i}| \geq \sum_{i=2}^{n} |\mathcal{V}^\varepsilon_{k,i}| \geq \sum_{i=2}^{n} |M_{\varepsilon,i}| - \sum_{i=2}^{n} |F^\varepsilon_{k,i}| \geq |\mathcal{M}^\varepsilon_n| - \sum_{i=2}^{n} |F^\varepsilon_{k,i}|. \tag{3.2.21}$$

Therefore,

$$\liminf_{k\to+\infty} |\mathcal{V}^\varepsilon_k| \geq |\mathcal{M}^\varepsilon_n| - \sum_{i=2}^{n} \limsup_{k\to+\infty} |F^\varepsilon_{k,i}|. \tag{3.2.22}$$

In order to obtain (3.2.8), we have to prove that:

(i) $|F^\varepsilon_{k,i}| \to 0$ as $k\to+\infty$ for every $i=2,\dots,n$;

(ii) $\mathcal{M}^\varepsilon_n \rightharpoonup S_\varepsilon$ as $n\to+\infty$,

so that (3.2.8) would follow by lower semicontinuity of the mass and (3.2.22).

(i). Since $\gamma^\varepsilon_{k,i}\circ h^\varepsilon_{k,i} \to \gamma^{u,\varepsilon}_i \circ h^\varepsilon_i$ uniformly in $[-1,1]$ as $k\to+\infty$, also $\Gamma^\varepsilon_{k,i}\circ h^\varepsilon_{k,i} \to \Gamma^{u,\varepsilon}_i \circ h^\varepsilon_i$ uniformly; moreover, by Corollary 1.3.6 and thanks to (3.2.16), $v_k(\cdot,\pm\varepsilon)\to u(\cdot,\pm\varepsilon)$ uniformly on $[t_{i-1},t_i]$, and the same holds for $\Lambda^{\pm,\varepsilon}_{k,i}$ and $\Lambda^{\pm,\varepsilon}_{u,i}$. Finally, by (3.2.15) and (3.2.16), and recalling also Lemma 3.1.4 (b), the L^1-norm of the derivative of $\Gamma^\varepsilon_{k,i}\circ h^\varepsilon_{k,i}$

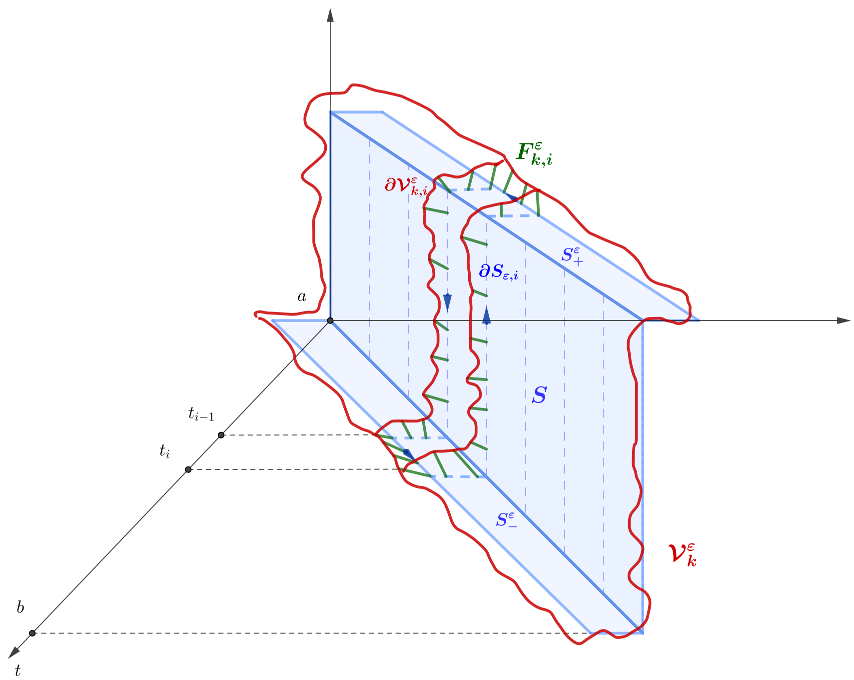

Figure 3.1: Here $S = X^{\mathrm{aff}}_{\sharp}[\![[a,b] \times I]\!]$, $S^{\varepsilon}_{\pm} = U^{\varepsilon}_{\pm\sharp}[\![R^{\pm}]\!]$. The horizontal and vertical axes span the target space $\mathbb{R}^2$. The approximating current $\mathcal{V}^{\varepsilon}_{k}$ is depicted in bold, as well as the boundary of its restriction to R_i, i.e. the current $\partial\mathcal{V}^{\varepsilon}_{k,i}$. The current $\partial S_{\varepsilon,i}$ is depicted with the oriented dotted straight lines, while $F^{\varepsilon}_{k,i}$ is the oriented surface obtained as the union of the short segments connecting $\partial\mathcal{V}^{\varepsilon}_{k,i}$ and $\partial S_{\varepsilon,i}$. Finally, for simplicity, we depict with straight segments the graph of $u^{\pm}$ and the (semi)graph of u on $\{(t,\sigma) : \sigma = \pm\varepsilon\}$, but it is worth to remember that they are graph of Lipschitz maps.

and of $\Lambda^{\pm,\varepsilon}_{k,i}$ is uniformly bounded with respect to k. Hence (i) readily follows from the definition of $F^{\varepsilon}_{k,i}$ in (3.2.17) (see also Remark 4.1.3).

(ii). First observe that $\partial\mathcal{M}^{\varepsilon}_{n}$ has mass uniformly bounded with respect to n. Indeed by (3.2.20)

$$\begin{aligned}|\partial\mathcal{M}^{\varepsilon}_{n}| &= |\partial S^{n}_{\varepsilon}| \\ &\le |\dot\gamma^{u,\varepsilon}_{1}|([-1,1]) + |\dot\gamma^{u,\varepsilon}_{n}|([-1,1]) + \int_a^b \sqrt{1+|\partial_t u(t,\varepsilon)|^2}dt + \int_a^b \sqrt{1+|\partial_t u(t,-\varepsilon)|^2}dt \\ &\le C(\varepsilon, \|u\|_\infty, \mathrm{lip}(u_{|R^+}), \mathrm{lip}(u_{|R^-})).\end{aligned}$$

Moreover, by minimality of $M_{\varepsilon,i}$ and (3.2.11), $|\mathcal{M}^{\varepsilon}_{n}| \le |S^{n}_{\varepsilon}| \le |S_{\varepsilon}|$, hence the sequence $\left(\mathcal{M}^{\varepsilon}_{n}\right)_n$ is compactly supported in $\mathbb{R}^3$ and has bounded mass and bounded boundary mass. Again by minimality of $M_{\varepsilon,i}$, we can assume that there exists a convex compact set $K \subset \mathbb{R}^3$

[3] In (3.2.21) we had to remove the first and last term of the sum, because condition (i) can be false for $i = 1$ and $i = n+1$, since the strict convergence is inherited only on almost every line, as stated in Lemma 3.1.1.

such that $\mathrm{spt}\mathcal{M}_n^\varepsilon \subset K$ for every $n \in \mathbb{N}$. Then, by Theorem 1.5.4, we have

$$\mathcal{M}_n^\varepsilon \rightharpoonup S_\varepsilon \Longleftrightarrow \|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \to 0 \quad \text{as } n \to +\infty,$$

where $\|\cdot\|_F$ stands for the flat norm. Then, we are reduced to show that $\|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \to 0$ as $n \to +\infty$. Notice that

$$\|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \leq \sum_{i=2}^{n} \|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F + \|S_{\varepsilon,1}\|_F + \|S_{\varepsilon,n+1}\|_F, \tag{3.2.23}$$

where, by definition of flat norm (see (1.5.1)),

$$\|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F \leq \inf\{|G_i^\varepsilon| : G_i^\varepsilon \text{ integer multiplicity 3-current s.t. } \partial G_i^\varepsilon = M_{\varepsilon,i} - S_{\varepsilon,i}\}.$$

Observe that the class of competitors in the above minimum problem is non empty, since it contains the affine interpolation current between $M_{\varepsilon,i}$ and $S_{\varepsilon,i}$. So, pick a 3-current G_i^ε such that $\partial G_i^\varepsilon = M_{\varepsilon,i} - S_{\varepsilon,i}$; then

$$|G_i^\varepsilon| \leq C|\partial G_i^\varepsilon|^{\frac{3}{2}}$$

by the Isoperimetric Theorem 1.5.5, for an absolute positive constant $C > 0$. For $i = 2, \ldots, n$, we have

$$\|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F \leq |G_i^\varepsilon| \leq C|\partial G_i^\varepsilon|^{\frac{3}{2}} = C|M_{\varepsilon,i} - S_{\varepsilon,i}|^{\frac{3}{2}} \leq C\left(|M_{\varepsilon,i}|^{\frac{3}{2}} + |S_{\varepsilon,i}|^{\frac{3}{2}}\right) \leq 2C|S_{\varepsilon,i}|^{\frac{3}{2}}, \tag{3.2.24}$$

where in the last inequality we used the minimality of $M_{\varepsilon,i}$. Now let us prove that $|S_{\varepsilon,i}| \leq \frac{C}{n}$ for every $i = 1, \ldots, n+1$, where C is a constant independent of n. We start observing that

$$\begin{aligned}
&|X_\sharp^{\mathrm{aff}} [\![[t_{i-1}, t_i) \times I]\!]| \\
&= \int_{[t_{i-1},t_i]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dtds \\
&= \int_{t_{i-1}}^{t_i} \int_I |(1, s\dot{u}^+ + (1-s)\dot{u}^-) \wedge (0, u^+ - u^-)| \, dtds \\
&\leq \int_{t_{i-1}}^{t_i} \int_I \left(|u^+ - u^-| + \left|(s\dot{u}_1^+ + (1-s)\dot{u}_1^-)(u_2^+ - u_2^-) - (s\dot{u}_2^+ + (1-s)\dot{u}_2^-)(u_1^+ - u_1^-)\right|\right) dtds \\
&\leq \frac{C_1}{n}\|u^+ - u^-\|_{L^\infty(a,b)} + \frac{C_2}{n}\|u^+ - u^-\|_{L^\infty(a,b)} \left(\|\dot{u}^+\|_{L^\infty(a,b)} + \|\dot{u}^-\|_{L^\infty(a,b)}\right) \\
&= \frac{C}{n},
\end{aligned}$$

where we used (3.2.9). Moreover, recalling (3.2.6), we have

$$
\begin{aligned}
&|U^{\varepsilon}_{\pm\sharp}[\![R_i^{\pm}]\!]| = \int_{R_i^{\pm}} |\partial_t U^{\varepsilon}_{\pm} \wedge \partial_\sigma U^{\varepsilon}_{\pm}| \; dt d\sigma \\
&= \int_{R_i^{\pm}} |(1, \partial_t u(t,\varepsilon\sigma)) \wedge (0, \varepsilon\partial_\sigma u(t,\varepsilon\sigma))| \; dt d\sigma \\
&\leq \varepsilon \int_{R_i^{\pm}} |\partial_\sigma u(t,\varepsilon\sigma)| \; dt d\sigma + \varepsilon \int_{R_i^{\pm}} |\partial_t u_1(t,\varepsilon\sigma)\partial_\sigma u_2(t,\varepsilon\sigma) - \partial_t u_2(t,\varepsilon\sigma)\partial_\sigma u_1(t,\varepsilon\sigma)| \; dt d\sigma \\
&\leq \varepsilon \frac{C_3}{n} \left(\|\nabla u\|_{L^\infty(R^{\pm})} + \|\nabla u\|^2_{L^\infty(R^{\pm})} \right) \\
&= \frac{C\varepsilon}{n}.
\end{aligned}
\tag{3.2.25}
$$

Thus,

$$
|S_{\varepsilon,i}| \leq |X^{\mathrm{aff}}_{\sharp}[\![[t_{i-1}, t_i) \times I]\!]| + |U^{\varepsilon}_{+\sharp}[\![R_i^{+}]\!]| + |U^{\varepsilon}_{-\sharp}[\![R_i^{-}]\!]| \leq \frac{C}{n},
$$

as claimed. Finally, by definition of flat norm and the isoperimetric inequality, $\|S_{\varepsilon,i}\|_F \leq |S_{\varepsilon,i}|^{\frac{3}{2}}$ for $i = 1, \ldots, n+1$, so that, from (3.2.24) and (3.2.23), we obtain

$$
\|\mathcal{M}^{\varepsilon}_n - S_\varepsilon\|_F \leq C(n-1)\frac{1}{n^{\frac{3}{2}}} + \frac{C}{n^{\frac{3}{2}}} \leq \frac{C}{n^{\frac{1}{2}}} + \frac{C}{n^{\frac{3}{2}}} \to 0.
$$

This concludes the proof of (ii) and hence of (3.2.8).

We are now in a position to show (3.2.4). From (3.2.5) and (3.2.8),

$$
\liminf_{k\to+\infty} \mathcal{A}(v_k, [a,b]\times[-\varepsilon,\varepsilon]) \geq \liminf_{k\to+\infty} |\mathcal{V}^{\varepsilon}_k| \geq |S_\varepsilon|. \tag{3.2.26}
$$

As in (3.2.25), we have

$$
|U^{\varepsilon}_{\pm\sharp}[\![R^{\pm}]\!]| \leq \varepsilon \left(\|\nabla u\|_{L^\infty(R^{\pm})} + \|\nabla u\|^2_{L^\infty(R^{\pm})} \right) \to 0 \quad \text{as } \varepsilon \to 0^+,
$$

so, from (3.2.26) and (3.2.7), we conclude

$$
\lim_{\varepsilon\to0^+} \liminf_{k\to+\infty} \mathcal{A}(v_k, [a,b]\times[-\varepsilon,\varepsilon]) \geq \lim_{\varepsilon\to0^+} |S_\varepsilon| = |X^{\mathrm{aff}}_{\sharp}[\![[a,b]\times I]\!]| = \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \, dtds.
$$

□

Proposition 3.2.5 (**Upper bound for** (3.2.2))**.** Let $u : R \to \mathbb{R}^2$ be a piecewise Lipschitz map. Then there exists a sequence $(v_k)_k \subset C^1(R;\mathbb{R}^2)$ converging to u strictly $BV(R;\mathbb{R}^2)$ such that

$$
\limsup_{k\to+\infty} \mathcal{A}(v_k, R) \leq \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}| \; dtds. \tag{3.2.27}
$$

Proof. Although v_k needs to be of class C^1, we claim that it suffices to build v_k just Lipschitz continuous. Indeed, assume that $(v_k)_k \subset W^{1,\infty}(R;\mathbb{R}^2)$ converges to u strictly $BV(R;\mathbb{R}^2)$ and (3.2.27) holds. Consider, for all $k \in \mathbb{N}$, a sequence $\left(v^k_h\right)_h \subset C^1(R;\mathbb{R}^2)$

approaching v_k in $W^{1,2}(R;\mathbb{R}^2)$ as $h \to +\infty$. In particular, we get the L^1-convergence of all minors of ∇v_h^k to the corresponding ones of ∇v_k. Then, by dominated convergence,

$$\lim_{h\to+\infty} \mathcal{A}(v_h^k; R) = \mathcal{A}(v_k, R). \tag{3.2.28}$$

Hence, by a diagonal argument, we find a sequence $\left(v_{h_k}^k\right)_k$ converging to u strictly $BV(R;\mathbb{R}^2)$ such that (3.2.27) holds for $v_{h_k}^k$ in place of v_k.

Set for simplicity $\varepsilon = \varepsilon_k = \frac{1}{k}$, and define the sequence $(v_\varepsilon) \subset \mathrm{Lip}(R;\mathbb{R}^2)$ as

$$v_\varepsilon(t,\sigma) := \begin{cases} u(t,\sigma) & (t,\sigma) \in R \setminus ([a,b]\times[-\varepsilon,\varepsilon]), \\ \frac{\varepsilon+\sigma}{2\varepsilon} u(t,\varepsilon) + \frac{\varepsilon-\sigma}{2\varepsilon} u(t,-\varepsilon) & (t,\sigma) \in [a,b]\times(-\varepsilon,\varepsilon). \end{cases} \tag{3.2.29}$$

First, let us check that $v_\varepsilon \to u$ strictly $BV(R;\mathbb{R}^2)$ as $\varepsilon \to 0^+$. Clearly, $v_\varepsilon \to u$ in $L^1(R;\mathbb{R}^2)$. Hence, by lower semicontinuity of the total variation, it is enough to show that

$$\limsup_{\varepsilon\to0^+} \int_R |\nabla v_\varepsilon| dt d\sigma \le |Du|(R),$$

which in turn reduces to prove

$$\limsup_{\varepsilon\to0^+} \int_{[a,b]\times[-\varepsilon,\varepsilon]} |\nabla v_\varepsilon| dt d\sigma \le |Du|([a,b]\times\{0\}),$$

since

$$\begin{aligned}\int_{R\setminus([a,b]\times[-\varepsilon,\varepsilon])} |\nabla v_\varepsilon| dt d\sigma &= \int_{R\setminus([a,b]\times[-\varepsilon,\varepsilon])} |\nabla u| dt d\sigma \\ &\longrightarrow \int_{R^+} |\nabla u| dt d\sigma + \int_{R^-} |\nabla u| dt d\sigma \quad \text{ as } \varepsilon \to 0^+.\end{aligned}$$

For almost every $t \in [a,b]$ and every $\sigma \in [-\varepsilon,\varepsilon]$, one has

$$\partial_t v_\varepsilon(t,\sigma) = \frac{\varepsilon+\sigma}{2\varepsilon}\partial_t u(t,\varepsilon) + \frac{\varepsilon-\sigma}{2\varepsilon}\partial_t u(t,-\varepsilon), \quad \partial_\sigma v_\varepsilon(t,\sigma) = \frac{1}{2\varepsilon}(u(t,\varepsilon) - u(t,-\varepsilon)).$$

Thus, setting $M := \max\{\mathrm{lip}(u_{|R^-}), \mathrm{lip}(u_{|R^+})\}$, we get

$$\begin{aligned}\int_{[a,b]\times[-\varepsilon,\varepsilon]} |\nabla v_\varepsilon| \; dt d\sigma &\le \int_{[a,b]\times[-\varepsilon,\varepsilon]} |\partial_t v_\varepsilon(t,\sigma)| \; dt d\sigma + \int_{[a,b]\times[-\varepsilon,\varepsilon]} |\partial_\sigma v_\varepsilon(t,\sigma)| \; dt d\sigma \\ &\le M \int_{[a,b]\times[-\varepsilon,\varepsilon]} dt d\sigma + \int_{[a,b]\times[-\varepsilon,\varepsilon]} \frac{1}{2\varepsilon} |u(t,\varepsilon) - u(t,-\varepsilon)| \; dt d\sigma \\ &= M(b-a)2\varepsilon + \int_a^b |u(t,\varepsilon) - u(t,-\varepsilon)| \; dt \\ &\xrightarrow{\varepsilon\to0^+} \int_a^b |u^+(t) - u^-(t)| \; dt = |Du|([a,b]\times\{0\}).\end{aligned}$$

Furthermore, since u is piecewise Lipschitz, we have

$$\mathcal{A}(v_\varepsilon; R\setminus[a,b]\times[-\varepsilon,\varepsilon]) = \mathcal{A}(u, R\setminus[a,b]\times[-\varepsilon,\varepsilon]) \to \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-) \quad \text{ as } \varepsilon \to 0^+.$$

So it remains to prove that

$$\limsup_{\varepsilon\to 0^+} \mathcal{A}(v_\varepsilon; [a,b]\times[-\varepsilon,\varepsilon]) \le \int_{[a,b]\times I} |\partial_t X^{\rm aff} \wedge \partial_s X^{\rm aff}|\, dtds. \tag{3.2.30}$$

Let us linearly reparametrize $X^{\rm aff}$ on $R = [a,b]\times[-1,1]$, namely consider Y, having the same image as $X^{\rm aff}$, given by

$$Y(t,\sigma) = (t, \widehat{Y}(t,\sigma)) = \left(t, \frac{1+\sigma}{2}u^+(t) + \frac{1-\sigma}{2}u^-(t)\right), \qquad (t,\sigma)\in R.$$

Now, using the trivial inequality $\sqrt{1+a^2+b^2+c^2} \le 1+|a|+\sqrt{b^2+c^2}$, we find

$$\begin{aligned} &\mathcal{A}(v_\varepsilon; [a,b]\times[-\varepsilon,\varepsilon]) \\ &\le \int_{[a,b]\times[-\varepsilon,\varepsilon]} dtd\sigma + \int_{[a,b]\times[-\varepsilon,\varepsilon]} |\partial_t v_\varepsilon|\, dtd\sigma + \int_{[a,b]\times[-\varepsilon,\varepsilon]} \sqrt{|\partial_\sigma v_\varepsilon|^2 + |Jv_\varepsilon|^2} dtd\sigma \\ &= 2\varepsilon(b-a) + 2\varepsilon \int_R |\partial_t \tilde{v}_\varepsilon|\, dtd\sigma + \int_R \sqrt{|\partial_\sigma \tilde{v}_\varepsilon|^2 + |J\tilde{v}_\varepsilon|^2}\, dtd\sigma, \end{aligned} \tag{3.2.31}$$

where $\tilde{v}_\varepsilon : R \to \mathbb{R}^2$ is defined as $\tilde{v}_\varepsilon(t,\sigma) = v_\varepsilon(t,\varepsilon\sigma)$. A direct computation based in (3.2.29) gives

$$\begin{aligned} \partial_t \tilde{v}_\varepsilon(t,\sigma) &= \frac{1+\sigma}{2}\partial_t u(t,\varepsilon) + \frac{1-\sigma}{2}\partial_t u(t,-\varepsilon) \quad \text{for a.e. } t\in[a,b] \quad \forall \sigma\in[-1,1] \\ \partial_\sigma \tilde{v}_\varepsilon(t,\sigma) &= \varepsilon\partial_\sigma v_\varepsilon(t,\varepsilon\sigma) = \frac{u(t,\varepsilon)-u(t,-\varepsilon)}{2} \quad \text{for a.e. } t\in[a,b] \quad \forall\sigma\in[-1,1]. \end{aligned}$$

Then we have

$$\begin{aligned} \partial_t \tilde{v}_\varepsilon(t,\sigma) &\to \frac{1+\sigma}{2}\dot{u}^+(t) + \frac{1-\sigma}{2}\dot{u}^-(t) = \partial_t \widehat{Y}(t,\sigma) \quad \text{a.e. in } R, \\ \partial_\sigma \tilde{v}_\varepsilon(t,\sigma) &\to \frac{u^+(t)-u^-(t)}{2} = \partial_\sigma \widehat{Y}(t,\sigma) \quad \text{a.e. in } R. \end{aligned}$$

Since $\partial_\sigma \widehat{Y}$ and $\partial_t \widehat{Y}$ are in $L^\infty(R;\mathbb{R}^2)$, by dominated convergence we can pass to the limit in (3.2.31) as $\varepsilon\to 0^+$, so that, using Remark 3.2.2, we obtain (3.2.30). □

Remark 3.2.6. After having proved the upper bound inequality in Proposition 3.2.5, we readily infer that $\overline{\mathcal{A}}_{BV}(u,R) < +\infty$. Hence Proposition 3.2.4 can be deduced from an argument independently developed in [40], based on the theory of Cartesian currents [26]. Indeed, consider $T_u := G_u + S$, where G_u is the 2-current on $R\times\mathbb{R}^2$ carried by the graph of u and S is the 2-current on $R\times\mathbb{R}^2$ given by $S := \tilde{X}_\sharp[\![[a,b]\times I]\!]$, where

$$\tilde{X}(t,s) := (t,0,\widehat{X}(t,s)) = (t,0,su^+(t)+(1-s)u^-(t)), \qquad t\in[a,b], s\in I.$$

Clearly, the mass of T_u is given by

$$\begin{aligned} |T_u| = |G_u| + |S| &= \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-) + \int_{[a,b]\times I} |\partial_t \tilde{X} \wedge \partial_s \tilde{X}|\, dtds \\ &= \mathcal{A}(u,R^+) + \mathcal{A}(u,R^-) + \int_{[a,b]\times I} |\partial_t X^{\rm aff} \wedge \partial_s X^{\rm aff}|\, dtds. \end{aligned}$$

Now we claim that T_u is the minimal lifting current on $R\times\mathbb{R}^2$ associated to u, according to Theorem 1.5.8. Recall that this definition is given by imposing that the mixed components of T_u are the minimal lifting measures $\mu_i^j[u]$ associated to u in the sense of Jerrard and Jung [31]. Once the claim is proven, thanks to Theorem 1.5.8, we have $|T_u| \leq \overline{\mathcal{A}}_{BV}(u;R)$, i.e., inequality (3.2.3).

In order to show the claim, we start to prove that $T_u \in \mathrm{cart}(R\times\mathbb{R}^2)$. For this, it is enough to see that $(\partial T_u)\llcorner(R\times\mathbb{R}^2) = 0$: We get

$$(\partial G_u)\llcorner(R\times\mathbb{R}^2) = \widehat{X}^-_\sharp[\![a,b]\!] - \widehat{X}^+_\sharp[\![a,b]\!] = -\partial\tilde{X}_\sharp[\![a,b]\times I]\!] = -(\partial S)\llcorner(R\times\mathbb{R}^2),$$

where $\widehat{X}^\pm(t) := (t,0,u^\pm(t))$, $t\in[a,b]$. Next, what remains to prove is that the vertical component of T_u is the minimal completely vertical lifting associated to u. To this purpose, denote by $x=(x^1,x^2)$ the (horizontal) variable of R, $y=(y^1,y^2)$ the vertical variable of $\mathbb{R}^2$ and $u=(u^1,u^2)$ the components of u. We have to check that

$$\mu_i^j[T_u] = \mu_i^j[u] \quad \forall i,j=1,2, \tag{3.2.32}$$

where $\mu_i^j[T_u] := T_u\llcorner((-1)^i dx^{\bar{i}}\wedge dy^j)$. By (1.5.2), for every $f\in C_c^\infty(R\times\mathbb{R}^2)$,

$$\int_{R\times\mathbb{R}^2} f(x,y)d\mu_j^i[u] = \int_{R^+\cup R^-} f(x,u(x))\partial_i u^j dx + \int_a^b\left(\int_0^1 f(t,0,\widehat{X}(t,s))ds\right)(u^{j+}-u^{j-})\delta_{i2}dt,$$

where δ_{ij} denotes the Kronecker symbol.
On the other hand, setting $\omega(x,y) := (-1)^i f(x,y)dx^{\bar{i}}\wedge dy^j$, we have

$$\begin{aligned}\int_{R\times\mathbb{R}^2} f(x,y)d\mu_j^i[T_u] &= \int_{R^+\cup R^-} f(x,u(x))\partial_i u^j dx + \int_{\tilde{X}([a,b]\times I)}\omega \\ &= \int_{R^+\cup R^-} f(x,u(x))\partial_i u^j dx + \int_{[a,b]\times I}\omega(\tilde{X}(t,s))d\tilde{X}^{\bar{i}j},\end{aligned}$$

where, if $\tilde{X}=(\tilde{X}_1^1,\tilde{X}_2^1,\tilde{X}_1^2,\tilde{X}_2^2)$, then $d\tilde{X}^{\bar{i}j} = d\tilde{X}_1^{\bar{i}}\wedge d\tilde{X}_2^j$. Notice that $d\tilde{X}^{\bar{i}j}=0$ if $\bar{i}=2$ and $d\tilde{X}^{1j} = (u^{j+}-u^{j-})\,dt\wedge ds$, so we get

$$\begin{aligned}\int_{[a,b]\times I}\omega(\tilde{X}(t,s))d\tilde{X}^{\bar{i}j} &= \int_{[a,b]\times I}(-1)^i f(\tilde{X}(t,s))(u^{j+}-u^{j-})\delta_{i2}\,dt\wedge ds \\ &= \int_a^b\left(\int_0^1 f(t,0,\widehat{X}(t,s))ds\right)(u^{j+}-u^{j-})\delta_{i2}\,dt,\end{aligned}$$

and (3.2.32) follows.

3.2.1 Extension of Theorem 3.2.3

The validity of Theorem 3.2.3 is guaranteed also when the two traces $u^\pm$ of u on $[a,b]\times\{0\}$ coincide on some subset of $[a,b]\times\{0\}$. In particular, (3.2.2) extends to maps u whose jump set J_u is a subset of $[a,b]\times\{0\}$. However, the situation is different when the jump set is curvilineous. Specifically, assume $\Omega\subset\mathbb{R}^2$ is a bounded open and connected set, and:

(H1) $\Sigma = \alpha([a,b]) \subset \Omega$ is a simple curve, arc-length parametrized by $\alpha : [a,b] \to \Omega$ of class C^2 and injective in $[a,b)$;

(H2) If $\alpha(a) = \alpha(b)$, then $\dot\alpha(a^+) = \dot\alpha(b^-)$ and $\ddot\alpha(a^+) = \ddot\alpha(b^-)$;

(H3) $u \in W^{1,\infty}(\Omega \setminus \Sigma; \mathbb{R}^2)$; as usual, we denote by $u^\pm$ the traces of u on Σ, satisfying $u^\pm \in \mathrm{Lip}(\Sigma; \mathbb{R}^2)$.

Again, we introduce the affine interpolation surface $X^{\mathrm{aff}} : [a,b] \times I \to \mathbb{R}^3$ spanning $\mathrm{graph}(u^\pm \circ \alpha) = \{(t, u^\pm(\alpha(t))) : t \in [a,b]\} \subset \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$, namely

$$X^{\mathrm{aff}}(t,s) = (t, su^+(\alpha(t)) + (1-s)u^-(\alpha(t))) \qquad \forall (t,s) \in [a,b] \times I. \tag{3.2.33}$$

Theorem 3.2.7 (**Relaxed area of piecewise Lipschitz maps: curved jump**). Suppose (H1)-(H3). Then

$$\overline{\mathcal{A}}_{BV}(u,\Omega) = \int_{\Omega\setminus\Sigma} |\mathcal{M}(\nabla u)|\, dx + \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}|\, dtds. \tag{3.2.34}$$

Remark 3.2.8. The image of the map X^{aff} sits in $\mathbb{R}^3$ and it is not exactly the interpolation surface which closes the holes in the graph of u, which is instead given by

$$\Psi(t,s) = (\alpha(t), su^+(\alpha(t)) + (1-s)u^-(\alpha(t))) \in \mathbb{R}^4 \qquad \forall t \in [a,b] \times I. \tag{3.2.35}$$

However, since $|\dot\alpha| = 1$,

$$\int_{[a,b]\times I} |\partial_t \Psi \wedge \partial_s \Psi|\, dtds = \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}|\, dtds. \tag{3.2.36}$$

To prove Theorem 3.2.7, we borrow from [9] some notation. We denote by $x = (x_1, x_2)$ coordinates in Ω and by (t,σ) coordinates in $R = [a,b] \times [-1,1]$. Since Σ is simple and of class C^2, we can find $\delta > 0$ and a C^1-diffeomorphism $\Lambda : R_\delta \to \Lambda(R_\delta)$, where $R_\delta = [a,b] \times [-\delta,\delta]$ and $\Lambda(R_\delta) \subset \Omega$ is a curvilineous strip containing Σ of width 2δ. Explicitely we have

$$\Lambda(t,\sigma) = \alpha(t) + \sigma\dot\alpha(t)^\perp \quad \forall (t,\sigma) \in R_\delta, \tag{3.2.37}$$

with $\dot\alpha(t)^\perp$ the counter-clockwise $\frac{\pi}{2}$-rotation of $\dot\alpha(t)$. For $(x_1,x_2) \in \Lambda(R_\delta)$, we can write the inverse $\Lambda^{-1}(x_1,x_2) = (t(x_1,x_2), \sigma(x_1,x_2))$, where:

- $\sigma(x_1,x_2) = d_\Sigma(x_1,x_2)$ is the signed distance[4] of (x_1,x_2) from Σ;
- $t(x_1,x_2)$ is the unique number in $[a,b]$ such that $\alpha(t(x_1,x_2)) = \pi_\Sigma(x_1,x_2)$, where $\pi_\Sigma(x_1,x_2) = (x_1,x_2) - d_\Sigma(x_1,x_2)\nabla d_\Sigma(x_1,x_2)$ is the orthogonal projection on Σ.

Since α is of class C^2, we have that σ is of class C^2 as well and t is of class C^1 on $\overline{\Lambda(R_\delta)}$. Moreover, for $(x_1,x_2) \in \overline{\Lambda(R_\delta)}$, we have

$$|\nabla\sigma(x_1,x_2)| = |\nabla d_\Sigma(x_1,x_2)| = 1, \tag{3.2.38}$$

$$|\nabla t(x_1,x_2)| = 1 + \delta\|\nabla d_\Sigma\|_\infty \le 1 + C\delta. \tag{3.2.39}$$

We divide the proof of Theorem 3.2.3 in two parts, the lower and the upper bound inequalities.

[4]The sign of d_Σ is determined by the orientation induced on Σ by α, so that $d_\Sigma > 0$ in the part of $\Lambda(R_\delta)$ which is pointed by $\dot\alpha^\perp$.

Proposition 3.2.9 (**Lower bound for** (3.2.34))**.** Let $u : \Omega \to \mathbb{R}^2$ as in Theorem 3.2.7 and $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$ be a sequence converging to u strictly $BV(\Omega; \mathbb{R}^2)$. Then (3.2.3) holds with $X^{\rm aff}$ in (3.2.33).

Proof. It is enough to show that

$$\lim_{\varepsilon \to 0^+} \liminf_{k \to +\infty} \mathcal{A}(v_k, \Lambda([a,b] \times [-\varepsilon, \varepsilon])) \geq \int_{[a,b]\times I} |\partial_t X^{\rm aff} \wedge \partial_s X^{\rm aff}| \; dtds. \tag{3.2.40}$$

We start by defining the maps $\Psi^\varepsilon_k : R \to \mathbb{R}^4$ and $\Psi^\varepsilon_\pm : R^\pm \to \mathbb{R}^4$ given by

$$\Psi^\varepsilon_k(t,\sigma) = (\Lambda(t, \varepsilon\sigma), v_k(\Lambda(t,\varepsilon\sigma))), \qquad \Psi^\varepsilon_\pm(t,\sigma) = (\Lambda(t,\varepsilon\sigma), u(\Lambda(t,\varepsilon\sigma))).$$

Introduce the following integer multiplicity 2-currents in $\mathbb{R}^4$:

$$\mathcal{V}^\varepsilon_k = \Psi^\varepsilon_{k\sharp}[\![R]\!], \quad S^\varepsilon = \Psi_\sharp[\![[a,b]\times I]\!] + \Psi^\varepsilon_{-\sharp}[\![R^-]\!] + \Psi^\varepsilon_{+\sharp}[\![R^+]\!],$$

where Ψ is defined in (3.2.35). Using that $Av \wedge Aw = \det A \, v \wedge w$ for any $A \in \mathbb{R}^{2\times 2}$ and $v, w \in \mathbb{R}^2$, by direct computation, we have

$$|\partial_t \Psi^\varepsilon_k \wedge \partial_\sigma \Psi^\varepsilon_k|^2 = \varepsilon^2 |\partial_t \Lambda(t,\varepsilon\sigma) \wedge \partial_\sigma \Lambda(t,\varepsilon\sigma)|^2 \Big[1 + |\nabla v_k(\Lambda(t,\varepsilon\sigma))|^2 + |Jv_k(\Lambda(t,\varepsilon\sigma))|^2\Big].$$

Hence, making the change of variable $x = \Lambda(t, \varepsilon\sigma)$, we obtain

$$\mathcal{A}(v_k, \Lambda([a,b]\times[-\varepsilon,\varepsilon])) = \int_{\Lambda([a,b]\times[-\varepsilon,\varepsilon])} |\mathcal{M}(\nabla v_k)| \; dx = \int_R |\partial_t \Psi^\varepsilon_k \wedge \partial_\sigma \Psi^\varepsilon_k| \; dtd\sigma = |\mathcal{V}^\varepsilon_k|.$$

We notice that $|\Psi^\varepsilon_{\pm\sharp}[\![R^\pm]\!]| \to 0$ as $\varepsilon \to 0^+$, as in (3.2.25), where $\|\nabla u\|_{L^\infty(R^\pm)}$ is replaced with $\|u\|_{W^{1,\infty}(\Omega)}$ and it is used that $|\ddot\alpha| \leq C$. Therefore, recalling also (3.2.36),

$$\lim_{\varepsilon \to 0^+} |S^\varepsilon| = |\Psi_\sharp[\![[a,b]\times I]\!]| = \int_{[a,b]\times I} |\partial_t \Psi \wedge \partial_s \Psi| \; dtds = \int_{[a,b]\times I} |\partial_t X^{\rm aff} \wedge \partial_s X^{\rm aff}| \; dtds.$$

So it is enough to show $\liminf_{k\to+\infty} |\mathcal{V}^\varepsilon_k| \geq |S^\varepsilon|$, which can be proved proceeding as in the proof of Proposition 3.2.4, once we have checked that $v_k \circ \Lambda(\cdot, \varepsilon\cdot) \to u \circ \Lambda(\cdot, \varepsilon\cdot)$ strictly $BV(R; \mathbb{R}^2)$. This is a straightforward computation, and we omit the details. □

Proposition 3.2.10 (**Upper bound for** (3.2.34))**.** Let $u : \Omega \to \mathbb{R}^2$ be as in Theorem 3.2.7. Then, there exists a sequence $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$ converging to u strictly $BV(\Omega; \mathbb{R}^2)$ and such that (3.2.27) holds with $X^{\rm aff}$ in (3.2.33).

Proof. For simplicity, we assume that $\alpha(a) \neq \alpha(b)$ (the case of closed curves is simpler and the following proof can be straightforwardly adapted). We start by fixing $\eta > 0$ small enough and we extend the curve α to $[a-\eta, b+\eta]$ in a C^2-way, so that $\Sigma^\eta := \alpha([a-\eta, b+\eta]) \subset \Omega$, keeping the validity of (H1) on Σ^η . With this extension, we can assume (by choosing a different δ if necessary) that Λ in (3.2.37) is defined on $R^\eta := [a-\eta, b+\eta] \times [-\delta, \delta]$. We observe that

$$u^+(\alpha(t)) = u^-(\alpha(t)) \qquad \text{for all } t \in [a-\eta, a] \cup [b, b+\eta]. \tag{3.2.41}$$

Now, set $\varepsilon = \frac{1}{k}$ and, for k large enough,

$$\Delta_\varepsilon^a := \{x \in \Lambda([a-\varepsilon, a] \times [-\varepsilon, \varepsilon]) : |\sigma(x)| \leq t(x) - a + \varepsilon\},$$
$$\Delta_\varepsilon^b := \{x \in \Lambda([b, b+\varepsilon] \times [-\varepsilon, \varepsilon]) : |\sigma(x)| \leq b + \varepsilon - t(x)\}.$$

We define the recovery sequence $(v_\varepsilon) \subset \mathrm{Lip}(\Omega; \mathbb{R}^2)$ as

$$v_\varepsilon(x) = \begin{cases} \frac{\varepsilon+\sigma(x)}{2\varepsilon} u\big(\Lambda(t(x), \varepsilon)\big) + \frac{\varepsilon-\sigma(x)}{2\varepsilon} u\big(\Lambda(t(x), -\varepsilon)\big) & \text{in } \Lambda([a,b] \times [-\varepsilon, \varepsilon]), \\ u(x) & \text{in } \Omega \setminus \big(\Lambda([a,b] \times [-\varepsilon, \varepsilon])\big) \cup \overline{\Delta}_\varepsilon^a \cup \overline{\Delta}_\varepsilon^b\big). \end{cases} \tag{3.2.42}$$

In order to define v_ε in $\Delta_\varepsilon^a \cup \Delta_\varepsilon^b$ it is sufficient to observe that, by (3.2.41), the restriction of v_ε on $\partial\Delta_\varepsilon^a$ and $\partial\Delta_\varepsilon^b$ is Lipschitz continuous with Lipschitz constant bounded by $\|u\|_{W^{1,\infty}}$. Hence, we can take a Lipschitz extension of v_ε in $\Delta_\varepsilon^a \cup \Delta_\varepsilon^b$ keeping the Lipschitz constant (up to a dimensional factor independent of ε). Thus

$$\int_{\Delta_\varepsilon^a \cup \Delta_\varepsilon^b} |\mathcal{M}(\nabla v_\varepsilon)|\, dx \to 0 \qquad \text{as } \varepsilon \to 0^+. \tag{3.2.43}$$

Let us check that $v_\varepsilon \to u$ strictly $BV(\Omega; \mathbb{R}^2)$ as $\varepsilon \to 0^+$. Clearly, $v_\varepsilon \to u$ in $L^1(\Omega; \mathbb{R}^2)$, since $|\Lambda([a,b] \times [-\varepsilon, \varepsilon])| \to 0$ and $|\Delta_\varepsilon^a \cup \Delta_\varepsilon^b| \to 0$. So, by (3.2.43), as in the proof of Proposition 3.2.5, it is enough to show that

$$\limsup_{\varepsilon \to 0^+} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon|\, dx \leq |Du|(\Sigma) = \int_a^b |u^+(\alpha(t)) - u^-(\alpha(t))|\, dt.$$

Almost everywhere in $\Lambda([a,b] \times [-\varepsilon, \varepsilon])$, we have

$$\begin{aligned} \nabla v_\varepsilon = & \frac{\varepsilon + \sigma}{2\varepsilon} \nabla u(\Lambda(t, \varepsilon)) \partial_t \Lambda(t, \varepsilon) \otimes \nabla t + \frac{\varepsilon - \sigma}{2\varepsilon} \nabla u(\Lambda(t, -\varepsilon)) \partial_t \Lambda(t, -\varepsilon) \otimes \nabla t \\ & + \frac{1}{2\varepsilon} \nabla \sigma \otimes (u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))). \end{aligned}$$

Therefore,

$$\begin{aligned} |\nabla v_\varepsilon| \leq & \frac{1}{2\varepsilon} \Big[(\varepsilon + \sigma) \|\partial_t \Lambda\|_\infty |\nabla u(\Lambda(t, -\varepsilon))| |\nabla t| + (\varepsilon - \sigma) \|\partial_t \Lambda\|_\infty |\nabla u(\Lambda(t, \varepsilon))| |\nabla t| \\ & + |\nabla \sigma| |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \Big] \\ \leq & \frac{1}{2\varepsilon} \Big[2\varepsilon \|u\|_{W^{1,\infty}} \|\partial_t \Lambda\|_\infty (1 + C\varepsilon) + |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \Big], \end{aligned}$$

where we used (3.2.38) and(3.2.39) with ε in place of δ. Thus, we get

$$\begin{aligned} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon|\, dx \leq & C(\delta)(1 + C\varepsilon) |\Lambda([a,b] \times [-\varepsilon, \varepsilon])| \\ & + \frac{1}{2\varepsilon} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))|\, dx \\ = & O_\varepsilon(1) + \frac{1}{2\varepsilon} \int_{\Lambda([a,b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| dx, \end{aligned}$$

where $O_\varepsilon(1)$ is such that $O_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Consider the last integral and perform the change of variable $x = (x_1, x_2) = \Lambda(t, \sigma)$, with

$$|\det\nabla\Lambda(t,\sigma)| = |\partial_t\Lambda \wedge \partial_\sigma\Lambda| = |1 + \sigma\ddot{\alpha} \wedge \dot{\alpha}| = |1 - \kappa_\Sigma\sigma| =: D(\sigma),$$

where κ_Σ is the curvature of Σ. We get

$$\begin{aligned}
&\frac{1}{2\varepsilon}\int_{\Lambda([a,b]\times[-\varepsilon,\varepsilon])} |u(\Lambda(t,\varepsilon)) - u(\Lambda(t,-\varepsilon))|dx \\
&= \frac{1}{2\varepsilon}\int_{[a,b]\times[-\varepsilon,\varepsilon]} |u(\Lambda(t,\varepsilon)) - u(\Lambda(t,-\varepsilon))|D(\sigma)dtd\sigma \\
&\leq \frac{1}{2\varepsilon}\int_a^b\int_{-\varepsilon}^{\varepsilon} |u(\Lambda(t,\varepsilon)) - u(\Lambda(t,-\varepsilon))|\ dtd\sigma + O_\varepsilon(1) \\
&= \int_a^b |u(\Lambda(t,\varepsilon)) - u(\Lambda(t,-\varepsilon))|\ dt + O_\varepsilon(1) \\
&\longrightarrow \int_a^b |u^+(\alpha(t)) - u^-(\alpha(t))|\ dt \quad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

It remains to prove (3.2.27) with X^{aff} in (3.2.33). To this purpose it is enough to show that

$$\liminf_{\varepsilon\to 0^+} \mathcal{A}(v_\varepsilon; \Lambda([a,b]\times[-\varepsilon,\varepsilon])) \leq \int_{[a,b]\times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}|\ dtds.$$

Let us define $\varphi_\varepsilon : R \to \mathbb{R}^2$ as

$$\varphi_\varepsilon(t,\sigma) := \frac{1+\sigma}{2}u(\Lambda(t,\varepsilon)) + \frac{1-\sigma}{2}u(\Lambda((t,-\varepsilon))).$$

Thus, for $x \in \Lambda([a,b]\times[-\varepsilon,\varepsilon])$

$$v_\varepsilon(x) = \varphi_\varepsilon\left(t(x), \frac{\sigma(x)}{\varepsilon}\right)$$

and, almost everywhere in $\Lambda([a,b]\times[-\varepsilon,\varepsilon])$,

$$\nabla v_\varepsilon = \partial_t\varphi_\varepsilon\nabla t + \frac{1}{\varepsilon}\partial_\sigma\varphi_\varepsilon\nabla\sigma, \qquad Jv_\varepsilon = \frac{1}{\varepsilon}|\partial_t\varphi_\varepsilon \wedge \partial_\sigma\varphi_\varepsilon||\nabla t \wedge \nabla\sigma|,$$

where from now on, ∇t and $\nabla\sigma$ are evaluated at x, while $\partial_t\varphi_\varepsilon$ and $\partial_\sigma\varphi_\varepsilon$ are evaluated at $\left(t(x), \frac{\sigma(x)}{\varepsilon}\right)$. Then, we get

$$\begin{aligned}
|\mathcal{M}(\nabla v_\varepsilon)|^2 &= 1 + |\partial_t\varphi_\varepsilon|^2|\nabla t|^2 + \frac{2}{\varepsilon}\partial_t\varphi_\varepsilon\cdot\partial_\sigma\varphi_\varepsilon\nabla t\cdot\nabla\sigma + \frac{1}{\varepsilon^2}\Big[|\partial_\sigma\varphi_\varepsilon|^2|\nabla\sigma|^2 \\
&\quad + |\partial_t\varphi_\varepsilon \wedge \partial_\sigma\varphi_\varepsilon|^2|\nabla t \wedge \nabla\sigma|^2\Big] \\
&\leq 1 + |\partial_t\varphi_\varepsilon|^2(1 + O_\varepsilon(1)) + \frac{2}{\varepsilon}|\partial_t\varphi_\varepsilon\cdot\partial_\sigma\varphi_\varepsilon|(1 + O_\varepsilon(1)) \\
&\quad + \frac{1}{\varepsilon^2}\Big[|\partial_\sigma\varphi_\varepsilon|^2 + |\partial_t\varphi_\varepsilon \wedge \partial_\sigma\varphi_\varepsilon|^2(1 + O_\varepsilon(1))\Big],
\end{aligned}$$

where we used (3.2.38) and(3.2.39) with ε in place of δ. Now, since $O_\varepsilon(1) \sim \varepsilon$ and φ_ε is Lipschitz with Lipschitz constant independent of ε, we obtain

$$\begin{aligned}
&\mathcal{A}(v_\varepsilon; \Lambda([a,b]\times[-\varepsilon,\varepsilon]))\\
\leq &\int_{\Lambda([a,b]\times[-\varepsilon,\varepsilon])} \sqrt{1+|\partial_t\varphi_\varepsilon|^2 + \frac{2}{\varepsilon}|\partial_t\varphi_\varepsilon\cdot\partial_\sigma\varphi_\varepsilon| + \frac{1}{\varepsilon^2}\Big[|\partial_\sigma\varphi_\varepsilon|^2 + |\partial_t\varphi_\varepsilon\wedge\partial_\sigma\varphi_\varepsilon|^2(1+O_\varepsilon(1))\Big]}\, dx\\
&+O_\varepsilon(1)\\
\leq &\int_{[a,b]\times[-\varepsilon,\varepsilon]} \sqrt{1+|\partial_t\varphi_\varepsilon|^2 + \frac{2}{\varepsilon}|\partial_t\varphi_\varepsilon\cdot\partial_\sigma\varphi_\varepsilon| + \frac{1}{\varepsilon^2}\Big[|\partial_\sigma\varphi_\varepsilon|^2 + |\partial_t\varphi_\varepsilon\wedge\partial_\sigma\varphi_\varepsilon|^2(1+O_\varepsilon(1))\Big]} D(\sigma) dt d\sigma\\
&+O_\varepsilon(1),
\end{aligned}$$

where we made the change of variable $x = \Lambda(t,\sigma)$, and so $\partial_t\varphi_\varepsilon$ and $\partial_\sigma\varphi_\varepsilon$ are computed at $\big(t, \frac{\sigma}{\varepsilon}\big)$. Finally, by the change of variable $\frac{\sigma}{\varepsilon} \to \sigma$, we get

$$\begin{aligned}
&\mathcal{A}(v_\varepsilon; \Lambda([a,b]\times[-\varepsilon,\varepsilon]))\\
&\leq \int_R \sqrt{O_\varepsilon(1) + |\partial_\sigma\varphi_\varepsilon(t,\sigma)|^2 + |\partial_t\varphi_\varepsilon(t,\sigma)\wedge\partial_\sigma\varphi_\varepsilon(t,\sigma)|^2(1+O_\varepsilon(1))} D(\varepsilon\sigma)\ dt d\sigma + O_\varepsilon(1)\\
&\longrightarrow \int_{[a,b]\times I} |\partial_t X^{\mathrm{aff}} \wedge \partial_s X^{\mathrm{aff}}|\ dt ds,
\end{aligned}$$

where, to pass to the limit as $\varepsilon \to 0^+$, we apply the dominated convergence theorem (as in the proof of Proposition 3.2.5). □

We observe that Theorem 3.2.7 can be easily extended to the case of curves with one endpoint or both endpoints on $\partial\Omega$. Write:

(H4) Ω is of class C^1, $\alpha : [a,b] \to \overline{\Omega}$ is injective, arc-length parametrized, of class C^2, $\alpha((a,b)) \subset \Omega$, and α hits $\partial\Omega$ transversally at $\alpha(a), \alpha(b)$.

Theorem 3.2.11. Suppose (H3) and (H4). Then (3.2.34) holds with X^{aff} in (3.2.33).

Proof. Lower bound: let $(v_k) \subset C^1(\Omega;\mathbb{R}^2)$ be a sequence converging to u strictly $BV(\Omega;\mathbb{R}^2)$. Fix $0 < \rho < \frac{b-a}{2}$ and notice that $\Lambda([a+\rho, b-\rho]\times[-\varepsilon,\varepsilon]) \subset \Omega$, for $\varepsilon > 0$ small enough. Then it is sufficient to show that

$$\lim_{\varepsilon\to 0^+} \liminf_{k\to+\infty} \mathcal{A}\Big(v_k, \Lambda([a,b]\times[-\varepsilon,\varepsilon])\cap\Omega\Big) \geq \int_{[a+\rho,b-\rho]\times I} |\partial_t X^{\mathrm{aff}}\wedge\partial_s X^{\mathrm{aff}}|\ dt ds; \quad (3.2.44)$$

since the lower bound will follow by the arbitrariness of $\rho > 0$. After writing $\mathcal{A}(v_k, \Lambda([a,b]\times[-\varepsilon,\varepsilon])\cap\Omega) \geq \mathcal{A}(v_k, \Lambda([a+\rho,b-\rho]\times[-\varepsilon,\varepsilon]))$, the proof of (3.2.44) is identical to that of (3.2.40).

Upper bound: let us fix $\eta > 0$ small enough so that $B_{2\eta}(\alpha(a))$ and $B_{2\eta}(\alpha(b))$ are disjoint, and consider $\Omega^\eta := \Omega \cup B_{2\eta}(\alpha(a)) \cup B_{2\eta}(\alpha(b))$. We extend the curve α (still calling α the extension) in $\Omega^\eta \setminus \Omega$ in such a way that it satisfies (H4) in Ω^η, and so that it reaches the boundary of $B_{2\eta}(\alpha(a))\setminus\overline{\Omega}$ and of $B_{2\eta}(\alpha(b))\setminus\overline{\Omega}$ splitting both $B_{2\eta}(\alpha(a))\setminus\overline{\Omega}$ and $B_{2\eta}(\alpha(b))\setminus\overline{\Omega}$ in two connected components. If α is now defined on an interval of the form $[a-\delta, b+\delta]$ with $\delta = \delta(\eta) > \eta$, and if we set $\Sigma^\delta = \alpha([a-\delta, b+\delta])$, we prescribe the traces

u^+ and u^- on Σ^δ in such a way that they are Lipschitz continuous and $u^+ \circ \alpha = u^- \circ \alpha$ on $[a-\delta, a-\eta] \cup [b+\eta, b+\delta]$. Finally we take a Lipschitz extension u^η of u on the four connected components of $B_{2\eta}(\alpha(a)) \setminus \overline{\Omega} \setminus \Sigma^\delta$ and of $B_{2\eta}(\alpha(b)) \setminus \overline{\Omega} \setminus \Sigma^\delta$. It turns out that $u^\eta \in W^{1,\infty}\big((B_{2\eta}(\alpha(a)) \cup B_{2\eta}(\alpha(b))) \setminus \Sigma^\eta; \mathbb{R}^2\big)$, where $\Sigma^\eta = \alpha([a-\eta, b+\eta]) \subset \Omega^\eta$. Since the definition of $(u^\eta)^\pm$ is arbitrary, we can assume that

$$(u^\eta)^\pm(\alpha(t)) = u^\pm(\alpha(a))\Big(1 - \frac{a-t}{\eta}\Big) \qquad \text{for } t \in [a-\eta, a],$$
$$(u^\eta)^\pm(\alpha(t)) = u^\pm(\alpha(b))\Big(1 - \frac{t-b}{\eta}\Big) \qquad \text{for } t \in [b, b+\eta].$$

For $\varepsilon > 0$ small enough, we see that $\Lambda_\varepsilon := \Lambda([a-\eta, b+\eta] \times [-\varepsilon, \varepsilon]) \subset \Omega^\eta$. Hence we define v_k as in the proof of Proposition 3.2.10 with Ω replaced by Ω^η and u replaced by u^η (in particular, $v_\varepsilon = u$ on $\Omega \setminus \Lambda_\varepsilon$). Finally, let us fix $\rho \in (0, \eta)$. We can write

$$\begin{aligned}
\overline{\mathcal{A}}_{BV}(u, \Omega) &\le \liminf_{\varepsilon \to 0^+} \mathcal{A}(v_\varepsilon, \Omega) \\
&\le \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus \Lambda_\varepsilon} |\mathcal{M}(\nabla u)| \, dx + \liminf_{\varepsilon \to 0^+} \int_{\Lambda([a-\rho, b+\rho] \times [-\varepsilon, \varepsilon])} |\mathcal{M}(\nabla v_\varepsilon)| \, dx \\
&= \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \int_{[a-\rho, b+\rho] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \, dt ds,
\end{aligned}$$

where we use that $\Omega \subset \big((\Omega \setminus \Lambda_\varepsilon) \cup \Lambda([a-\rho, b+\rho] \times [-\varepsilon, \varepsilon])\big)$ for $\varepsilon > 0$ small enough. The upper bound then follows by the arbitrariness of ρ. $\square$

Finally, with straightforward modifications of the previous arguments one can show the following:

Corollary 3.2.12. Let Ω have C^1-boundary, let $n \in \mathbb{N}$ and $\alpha_i : [a_i, b_i] \to \overline{\Omega}$, $i = 1, \ldots, n$, be curves satisfying either (H1)-(H2), or (H4). Assume that $\Sigma_i := \alpha_i([a_i, b_i]) \subset \overline{\Omega}$ are mutually disjoint, and let $u \in W^{1,\infty}(\Omega \setminus \Sigma; \mathbb{R}^2)$ satisfy (H3), where $\Sigma := \cup_{i=1}^n \Sigma_i$. Then

$$\overline{\mathcal{A}}_{BV}(u, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| \, dx + \sum_{i=1}^n \int_{[a_i, b_i] \times I} |\partial_t X^{\text{aff}}_{(i)} \wedge \partial_s X^{\text{aff}}_{(i)}| \, dt ds.$$

where $X^{\text{aff}}_{(i)} : [a_i, b_i] \times I \to \mathbb{R}^3$ is the map $X^{\text{aff}}_{(i)}(t, s) = (t, s u^+(\alpha_i(t)) + (1-s) u^-(\alpha_i(t)))$.

Chapter 4

Homogeneous maps

In this chapter we compute $\overline{\mathcal{A}}_{BV}$ for 0-homogeneous maps in $BV(B_r;\mathbb{R}^2)$. We start by treating a particularly relevant subclass, which are the piecewise constant homogeneous maps (that we will called n-uple point maps). After computing the corresponding value of the BV-relaxed area, we construct in Example 4.2.6 a piecewise constant map with infinite BV-relaxed area, whose minimal lifting current has finite mass. Then, we extend the tecniques to general homogeneous maps of bounded variation. In order to do that, we need a preliminar analysis of a sort of planar Plateau problem for self-intersecting curves. The results of Sections 4.1 and 4.2 are contained in [4], while the ones in Section 4.3 can be found in [14].

4.1 Planar Plateau-type problem

Let $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ be a possibly self-intersecting Lipschitz curve. Let us consider, as in [42] (see also [24]), the following planar Plateau-type problem spanning φ:

$$P(\varphi) = \inf \left\{ \int_{B_1} |Jv|\ dx :\ v \in \mathrm{Lip}(B_1;\mathbb{R}^2), v_{|\partial B_1} = \varphi \right\}. \tag{4.1.1}$$

Notice that the class of competitors is non-empty, since it contains the map $v(x) = |x|\varphi\left(\frac{x}{|x|}\right)$ for $x \neq 0$, and $v(0) = 0$. We first observe that P is independent of the radius of the domain of integration. Specifically, for any $r > 0$, let

$$\varphi_r(y) := \varphi\left(\frac{y}{r}\right) \text{ for all } y \in \partial B_r. \tag{4.1.2}$$

Setting $y := rx$, $y \in \overline{B_r}$ and $v_r(y) := v(\frac{y}{r})$, we have

$$\int_{B_1} |Jv|\ dx = \int_{B_r} |Jv_r| dy \qquad\qquad \forall v \in \mathrm{Lip}(B_1;\mathbb{R}^2). \tag{4.1.3}$$

In particular, for any $r > 0$,

$$P(\varphi) = \inf \left\{ \int_{B_r} |Jv|\ dx :\ v \in \mathrm{Lip}(B_r;\mathbb{R}^2), v_{|\partial B_r} = \varphi_r \right\} = P(\varphi_r). \tag{4.1.4}$$

In the next proposition we show that $P(\cdot)$ is invariant under Lipschitz reparameterizations of φ.

Proposition 4.1.1 (Invariance). Let $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ and h be a Lipschitz homeomorphism of $\mathbb{S}^1$. Then

$$P(\varphi \circ h) = P(\varphi).$$

Proof. Since h and the identity map $\mathrm{id} : \mathbb{S}^1 \to \mathbb{S}^1$ have the same degree, they are homotopic in $\mathbb{S}^1$ by Hopf Theorem (see [36, pag. 51]), namely there exists a Lipschitz map[1] $K : [0,1] \times \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$K(0,\cdot) = \mathrm{id}, \quad K(1,\cdot) = h.$$

Define $H : [0,1] \times \mathbb{S}^1 \to \mathbb{R}^2$ as $H(t,\nu) = \varphi(K(t,\nu))$. Then, H is Lipschitz and

$$H(0,\cdot) = \varphi, \quad H(1,\cdot) = \varphi \circ h.$$

Now, suppose $v_k \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ is such that $v_k = \varphi$ on ∂B_1 and

$$\lim_{k\to+\infty} \int_{B_1} |Jv_k| \, dx \to P(\varphi).$$

Define the map $\widetilde{v}_k : B_1 \to \mathbb{R}^2$ as

$$\widetilde{v}_k(x) = \begin{cases} v_k(kx) & \text{for } x \in B_{\frac{1}{k}}, \\ H\left(k|x| - 1, \frac{x}{|x|}\right) & \text{for } x \in B_{\frac{2}{k}} \setminus B_{\frac{1}{k}}, \\ \varphi \circ h\left(\frac{x}{|x|}\right) & \text{for } x \in B_1 \setminus B_{\frac{2}{k}}. \end{cases} \tag{4.1.5}$$

Then $\widetilde{v}_k \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ and $\widetilde{v}_k = \varphi \circ h$ on ∂B_1. Moreover, since H and $\varphi \circ h$ take values in $\varphi(\mathbb{S}^1)$ which is 1-dimensional, by the area formula and (4.1.3) we have

$$\int_{B_1} |J\widetilde{v}_k(x)| \, dx = \int_{B_{\frac{1}{k}}} |Jv_k(kx)| \, dx = \int_{B_1} |Jv_k| \, dx \to P(\varphi)$$

as $k \to +\infty$. In particular $P(\varphi \circ h) \le P(\varphi)$. Exchanging the role of φ and $\varphi \circ h$, we obtain the converse inequality. □

Lemma 4.1.2. Let $\varphi_1, \varphi_2 \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$. Then

$$|P(\varphi_1) - P(\varphi_2)| \le 2\|\varphi_1 - \varphi_2\|_\infty \big(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\big). \tag{4.1.6}$$

Proof. Let $v \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ be such that $v = \varphi_2$ on $\mathbb{S}^1$. We define

$$w(x) = \begin{cases} v_{\frac{1}{2}}(x) = v(2x) & \text{if } |x| < \frac{1}{2}, \\ 2(1-|x|)\varphi_2\left(\frac{x}{|x|}\right) + 2\left(|x| - \frac{1}{2}\right)\varphi_1\left(\frac{x}{|x|}\right) & \text{if } \frac{1}{2} \le |x| \le 1. \end{cases} \tag{4.1.7}$$

Then $w \in \mathrm{Lip}(B_1;\mathbb{R}^2)$, $w(x) = \varphi_2(x/|x|)$ if $x \in \partial B_{\frac{1}{2}}$ and $w = \varphi_1$ on ∂B_1. Let us estimate

$$\int_{B_1 \setminus \overline{B_{\frac{1}{2}}}} |Jw| \, dx.$$

[1]The construction of a Lipschitz homotopy between h and id can be done at the level of liftings, by considering the affine interpolation map (as argued in Proposition 2.2.5).

Writing w in polar coordinates in the annulus $B_1 \setminus \overline{B_{\frac{1}{2}}}$, $\rho \in (\frac{1}{2}, 1)$, $\theta \in [0, 2\pi)$,

$$\bar{w}(\rho,\theta) := w(\rho\cos\theta, \rho\sin\theta) = 2(1-\rho)\bar{\varphi}_2(\theta) + 2\left(\rho - \frac{1}{2}\right)\bar{\varphi}_1(\theta),$$

where $\bar{\varphi}_i(\theta) := \varphi_i(\cos\theta, \sin\theta)$, $i = 1, 2$. Then

$$\begin{aligned}|\partial_\rho \bar{w} \wedge \partial_\theta \bar{w}| &= 4\left|(\bar{\varphi}_1(\theta) - \bar{\varphi}_2(\theta)) \wedge \left((1-\rho)\dot{\bar{\varphi}}_2(\theta) + \left(\rho - \frac{1}{2}\right)\dot{\bar{\varphi}}_1(\theta)\right)\right| \\ &\le 4\,|\bar{\varphi}_1(\theta) - \bar{\varphi}_2(\theta)|\left|(1-\rho)\dot{\bar{\varphi}}_2(\theta) + \left(\rho - \frac{1}{2}\right)\dot{\bar{\varphi}}_1(\theta)\right| \\ &\le 4\|\varphi_1 - \varphi_2\|_\infty \left(|\dot{\bar{\varphi}}_2(\theta)| + |\dot{\bar{\varphi}}_1(\theta)|\right).\end{aligned}$$

Thus, integrating on $B_1 \setminus \overline{B_{\frac{1}{2}}}$,

$$\begin{aligned}\int_{B_1\setminus\overline{B_{\frac{1}{2}}}} |Jw(x)|\, dx &= \int_{\frac{1}{2}}^1 \int_0^{2\pi} \rho \left|\partial_\rho \bar{w} \wedge \frac{\partial_\theta \bar{w}}{\rho}\right|\, d\rho d\theta \\ &\le 2\|\varphi_1 - \varphi_2\|_\infty \int_0^{2\pi} \left(|\dot{\bar{\varphi}}_2(\theta)| + |\dot{\bar{\varphi}}_1(\theta)|\right) d\theta \\ &= 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right).\end{aligned} \tag{4.1.8}$$

Hence

$$P(\varphi_1) \le \int_{B_1} |Jw|\, dx \le \int_{B_{\frac{1}{2}}} |Jv_{\frac{1}{2}}|\, dx + 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right). \tag{4.1.9}$$

Since v is arbitrary and (with the notation in (4.1.2)) $v_{\frac{1}{2}} = (\varphi_2)_{\frac{1}{2}}$ on $\partial B_{\frac{1}{2}}$, using (4.1.4) with $r = \frac{1}{2}$ we can take the infimum on these maps in (4.1.9) and get

$$P(\varphi_1) - P(\varphi_2) \le 2\|\varphi_1 - \varphi_2\|_\infty \left(\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1\right).$$

Exchanging the role of φ_1 and φ_2 we find that also $P(\varphi_2) - P(\varphi_1)$ is bounded by the right-hand side of the previous expression. This concludes the proof. □

Remark 4.1.3. With a similar argument used in the proof of Lemma 4.1.2 it is immediate to obtain that if $[a, b] \subset \mathbb{R}$ is a bounded interval and $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}^2$ are Lipschitz curves, then the following holds: Let $\Phi : [a, b] \times [0, 1] \to \mathbb{R}^2$ be the affine interpolation map $\Phi(t, s) := s\gamma_1(t) + (1-s)\gamma_2(t)$. Then, as in (4.1.8),

$$\int_{[a,b]\times[0,1]} |\Phi_t \wedge \Phi_s|\, dtds \le \|\gamma_1 - \gamma_2\|_\infty (\|\dot{\gamma}_1\|_1 + \|\dot{\gamma}_2\|_1). \tag{4.1.10}$$

Using Lemma 4.1.2 we readily obtain the following continuity property for the minimum of the Plateau-type problem (4.1.1).

Corollary 4.1.4 (Continuity of P). Let $\varphi \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ and suppose that $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ is such that

$$\varphi_k \to \varphi \quad \text{uniformly} \quad \text{and} \quad \sup_{k\in\mathbb{N}} \|\dot{\varphi}_k\|_1 < +\infty.$$

Then $P(\varphi_k) \to P(\varphi)$ as $k \to +\infty$.

In what follows, it is convenient to consider for $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ the relaxation

$$\overline{P}(\gamma) := \inf\left\{\liminf_{k\to+\infty} P(\varphi_k) : \varphi_k \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2), \varphi_k \to \gamma \text{ strictly } BV(\mathbb{S}^1;\mathbb{R}^2)\right\} \tag{4.1.11}$$

of P with respect to the strict convergence in BV of the boundary datum. It is well known that the infimum in (4.1.11) is taken on a non-empty class of approximation maps. Moreover, by (4.1.3), also $\overline{P}$ is invariant by rescaling, i.e. $\overline{P}(\gamma) = \overline{P}(\gamma_r)$.

Lemma 4.1.5. Let $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$. Then $\overline{P}(\varphi) = P(\varphi)$.

Proof. If $(\varphi_k) \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ is a sequence converging to φ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$, then by Corollary 1.3.6 $\varphi_k \to \varphi$ uniformly on $\mathbb{S}^1$ as $k \to +\infty$. Moreover, the strict convergence guarantees that the total variations of φ_k are equibounded. So, thanks to Corollary 4.1.4,

$$P(\varphi_k) \to P(\varphi) \tag{4.1.12}$$

as $k \to +\infty$. Since this holds for any sequence (φ_k) as above, the thesis follows. □

Lemma 4.1.6. Let $\gamma \in SBV(\mathbb{S}^1;\mathbb{R}^2)$ have a finite number of jump points $z_i \in \mathbb{S}^1$, $i = 1,\ldots,n$. Let $\widetilde{\gamma} : \mathbb{S}^1 \to \mathbb{R}^2$ be the Lipschitz map in (3.1.27) (with $\mathbb{S}^1$ identified with $[0,2\pi]$). Then

$$\overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{4.1.13}$$

Proof. Let $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ be a sequence converging strictly to γ. Let us consider a not-relabeled subsequence of $(\varphi_k)_k$; by Corollary 3.1.5 there are a further subsequence $(\varphi_{k_j})_j$ and Lipschitz reparametrizations $\gamma_{k_j} = \varphi_{k_j} \circ h_{k_j} \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ of φ_{k_j} such that $\gamma_{k_j} \to \widetilde{\gamma} \circ h$ uniformly as $j \to +\infty$, for some Lipschitz homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$. Moreover, since by Lemma 3.1.4(b) the reparametrization maps h_{k_j} can be chosen with uniformly bounded Lipschitz constants, it follows that γ_{k_j} have uniformly bounded total variations. Hence it follows from Corollary 4.1.4 that $P(\gamma_{k_j}) \to P(\widetilde{\gamma} \circ h)$ as $j \to +\infty$. On the other hand, by Proposition 4.1.1 we also have $P(\varphi_{k_j}) \to P(\widetilde{\gamma})$ as $j \to +\infty$. Finally, since this argument holds for any subsequence of (φ_k), we conclude that the whole sequence satisfies $P(\varphi_k) \to P(\widetilde{\gamma})$, and therefore $\overline{P}(\gamma) = P(\widetilde{\gamma})$. □

As a consequence of the argument in the proof of Lemma 4.1.6, we easily infer the following continuity property:

Corollary 4.1.7. Let $\gamma \in SBV(\mathbb{S}^1;\mathbb{R}^2)$ and $\widetilde{\gamma}$ be as in Corollary 3.1.5, and assume that $(\varphi_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ is a sequence converging strictly to γ. Then

$$\lim_{k\to+\infty} P(\varphi_k) = \overline{P}(\gamma) = P(\widetilde{\gamma}).$$

Furthermore, we can refine the previous corollary as follows:

Corollary 4.1.8. Let $\gamma, \gamma_k \in SBV(\mathbb{S}^1;\mathbb{R}^2)$, $k \geq 1$, be maps as in Corollary 3.1.5. Assume that (γ_k) converges to γ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$. Then

$$\lim_{k\to+\infty} \overline{P}(\gamma_k) = \overline{P}(\gamma).$$

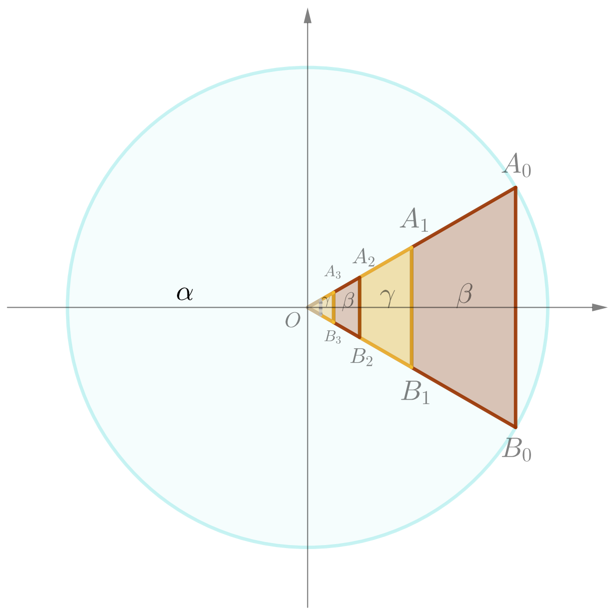

Figure 4.1: The source disc $B_1(0)$ and the values $\{\alpha, \beta, \gamma\}$ of u, with infinitely many triple points.

Proof. By Corollary 4.3.8 and the density of $\mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ in $BV(\mathbb{S}^1; \mathbb{R}^2)$ with respect to the strict convergence, for all $k \geq 1$ we can find $\varphi_k \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ such that

$$\|\gamma_k - \varphi_k\|_1 + \big||\dot{\varphi}_k|(\mathbb{S}^1) - |\dot{\gamma}_k|(\mathbb{S}^1)\big| + \big|P(\varphi_k) - \overline{P}(\gamma_k)\big| \leq \frac{1}{k}.$$

Hence the sequence (φ_k) converges to γ strictly $BV(\mathbb{S}^1; \mathbb{R}^2)$, and by the triangle inequality and Corollary 4.3.8 we conclude

$$\lim_{k \to +\infty} \overline{P}(\gamma_k) = \overline{P}(\gamma).$$

□

4.2 Piecewise constant maps

In this section we study the relaxed area (1.3.15) and the relaxed total variation (1.4.5), on certain piecewise constant maps. We start by exhibiting a BV map taking three values having infinite relaxed total variation of the Jacobian (and hence infinite BV-relaxed area), but finite L^1-relaxed area.

Example 4.2.1. (*BV***-relaxed area and** L^1**-relaxed area**) Let $\alpha, \beta, \gamma \in \mathbb{R}^2$ be three non-collinear vectors. Consider the map $u : B_1(0) \subset \mathbb{R}^2 \to \{\alpha, \beta, \gamma\}$ in Fig. 4.1, obtained by the following procedure: divide the source equilateral triangle $T_{A_0OB_0}$ in two regions with a vertical segment connecting A_1 and B_1, the middle points of the oblique sides of the triangle; assign the value β and γ on the right and on the left as in the figure, and repeat this construction on the equilateral triangle $T_{A_1OB_1}$, and then repeat the argument iteratively on all smaller triangles; finally set $u = \alpha$ in $B_1(0)\backslash T_{A_0OB_0}$. In this way we get an infinite collection of triple points located at $\{A_i, B_i\}_{i\geq 1}$. Then, $u \in BV(B_1(0); \{\alpha, \beta, \gamma\})$, since

$$\begin{aligned} |Du|(B_1(0)) &= \left(1 + 2(1 - \sum_{i=1}^{+\infty} 2^{-2i})\right)|\beta - \alpha| + 2\sum_{i=1}^{+\infty} 2^{-2i}|\alpha - \gamma| + \sum_{i=1}^{+\infty} 2^{-i}|\beta - \gamma| \\ &= \frac{7}{3}|\beta - \alpha| + \frac{2}{3}|\alpha - \gamma| + |\beta - \gamma|. \end{aligned}$$

On the other hand, consider an infinitesimal sequence $(r_i)_{i\geq 1}$ of radii with $0 < r_i < 2^{-(i+1)}$. With an argument similar to [3, Theorem 1.3], we have

$$\overline{TVJ}_{BV}(u, B_{r_i}(A_i)) = |T_{\alpha\beta\gamma}|,$$

$|T_{\alpha\beta\gamma}|$ denoting the Lebesgue measure of the target triangle with vertices α, β, γ, and thus, for every $N \in \mathbb{N}$,

$$\overline{TVJ}_{BV}(u, B_1(0)) \geq \overline{TVJ}_{BV}(u, \cup_{i=1}^{N} B_{r_i}(A_i)) \geq \sum_{i=1}^{N} |T_{\alpha\beta\gamma}| = N|T_{\alpha\beta\gamma}|.$$

Whence

$$\overline{\mathcal{A}}_{BV}(u, B_1(0)) \geq \overline{TVJ}_{BV}(u, B_1(0)) = +\infty. \tag{4.2.1}$$

On the other hand, we claim that

$$\overline{\mathcal{A}}_{L^1}(u, B_1(0)) < +\infty. \tag{4.2.2}$$

Indeed, we can construct a sequence (v_ε) of piecewise constant maps on $B_1(0)$, taking values in $\{\alpha, \beta, \gamma\}$, with uniformly bounded L^1-relaxed area and converging to u in $L^1(B_1(0); \mathbb{R}^2)$: Let $\varepsilon \in (0, 1)$ and consider the intersection with $T_{A_0 0 B_0}$ of a tubular neighbourhood of the segment $\overline{A_i B_i}$ of diameter $\varepsilon 2^{-(i+1)}$, for every $i \in \mathbb{N}$. Then, the map v_ε is obtained by modifying u on these strips in the triangle, by assigning the value α. Now, v_ε is a piecewise constant map valued in $\{\alpha, \beta, \gamma\}$ without triple points, hence, by Theorem 1.3.2,

$$\begin{aligned} &\overline{\mathcal{A}}_{L^1}(v_\varepsilon, B_1(0)) = |B_1(0)| + |Dv_\varepsilon|(B_1(0)) \\ \leq &\pi + \frac{7}{3}|\beta - \alpha| + \frac{2}{3}|\alpha - \gamma| + \left(1 + \frac{\varepsilon}{2}\right)\sum_{i=1}^{+\infty} 2^{-i}(|\beta - \alpha| + |\alpha - \gamma|) \\ \leq &\pi + \frac{23}{6}|\beta - \alpha| + \frac{13}{6}|\alpha - \gamma|. \end{aligned}$$

Clearly, $v_\varepsilon \to u$ in $L^1(B_1(0); \mathbb{R}^2)$ as $\varepsilon \to 0^+$, so by lower semicontinuity

$$\overline{\mathcal{A}}_{L^1}(u, B_1(0)) \leq \pi + \frac{23}{6}|\beta - \alpha| + \frac{13}{6}|\alpha - \gamma| < +\infty.$$

In particular

$$\mathrm{Dom}\Big(\overline{\mathcal{A}}_{BV}(\cdot, B_1(0))\Big) \subsetneq \mathrm{Dom}\Big(\overline{\mathcal{A}}_{L^1}(\cdot, B_1(0))\Big).$$

Remark 4.2.2. Following the notation of [40], one can show (4.2.1) also by considering the measure μ_w^J defined for every $w \in BV(B_1(0);\mathbb{R}^2)$ as

$$\langle \mu_w^J, g\rangle = \frac{1}{2}\int_{J_w}(w^{1-}w^{2+} - w^{1+}w^{2-})\partial_\tau g d\mathcal{H}^1 \quad \forall g \in C_c^\infty(B_1(0)),$$

where $\tau = \nu^\perp$ and ν is the unit normal to J_w, so that $Dw \llcorner J_w = (w^+ - w^-)\otimes \nu\mathcal{H}^1 \llcorner J_w$. If $\overline{\mathcal{A}}_{BV}(w, B_1(0)) < +\infty$, we can consider the minimal lifting current $T_w \in \text{cart}(B_1(0);\mathbb{R}^2)$ associated to w (Theorem 1.5.8), whose vertical part is equal to the completely vertical lifting $\mu_v[w]$ of w. Then, since $|\mu_v[w]|$ is lower semicontinuous with respect to the weak convergence of measures and $|\mu_v[v]|(B_1(0)) = TVJ(v, B_1(0))$ for v smooth (by (1.5.3)), we get

$$|\mu_v[w]|(B_1(0)\times\mathbb{R}^2) \le \overline{TVJ}_{BV}(w, B_1(0)).$$

In particular, if $w \in BV(B_1(0);\mathbb{R}^2)$ is piecewise constant, we have

$$|\mu_w^J|(B_1(0)) \le |\mu_v[w]|(B_1(0)\times\mathbb{R}^2) \le \overline{TVJ}_{BV}(w, B_1(0)), \tag{4.2.3}$$

where the first inequality is a consequence of [40, Corollary 4.3].

Now, if by contradiction $\overline{\mathcal{A}}_{BV}(u, B_1(0))$ is finite for the map u in Example 4.2.1 we have

$$\mu_u^J = \sum_{i=1}^{+\infty}|T_{\alpha\beta\gamma}|(\delta_{A_i} - \delta_{B_i}).$$

In particular $|\mu_u^J|(B_1(0)) = +\infty$, and (4.2.1) follows from (4.2.3). In Example 4.2.6, we construct a piecewise constant map $u \in BV(B_1(0);\mathbb{R}^2)$ taking only five values in $\mathbb{R}^2$ with $\overline{TVJ}_{BV}(u, B_1(0)) = +\infty$ and $\mu_u^J = 0$. In that case, one can see even that $\mu_v[u] = 0$, whence a maximal gap phenomenon occurs between the mass of the current T_u (which is finite and without a vertical contribution) and $\overline{\mathcal{A}}_{BV}(u, B_1(0))$ (which is infinite as well).

4.2.1 Piecewise constant homogeneous maps

We need some tools that allow us to characterize (and compute in some cases) the relaxed functionals for piecewise constant homogeneous maps, which we will called briefly *$n-$uple point maps* ($n \ge 3$). Thus, for $r > 0$, we consider maps $u : B_r := B_r(0) \to \mathbb{R}^2$ of the form

$$u(x) = \gamma\left(\frac{x}{|x|}\right) \quad \text{for a.e. } x \in B_r, \tag{4.2.4}$$

where $\gamma : \mathbb{S}^1 \to \{\alpha_1, \dots, \alpha_n\}$ is piecewise constant and takes the (not necessarily distinct) values $\alpha_1, \dots, \alpha_n \in \mathbb{R}^2$ on the arcs $C_1, \dots, C_n$ in the order (see Fig. 4.2 for $n = 5$). So, u is an $n-$uple point map with one $n-$uple junction at the origin. Now, we can consider the broken line curve $\widetilde{\gamma} \subset \mathbb{R}^2$ (an example of which is in Fig. 4.2) made of the segments connecting α_1 to α_2, α_2 to α_3 and so on, closing up by connecting α_n to α_1. The curve $\widetilde{\gamma}$ can be parametrized as in (3.1.27), and the curves $\widetilde{\gamma}_i$ are constant. Denoting by $L(\gamma)$ the length of $\widetilde{\gamma}$, we have

$$L(\gamma) = \sum_{i=1}^{n}|\alpha_{i+1}-\alpha_i| = |\dot\gamma|(\mathbb{S}^1) = \sup\left\{\sum_{i=1}^{m-1}|\gamma(\nu_{i+1}) - \gamma(\nu_i)| : m \in \mathbb{N}, \{\nu_1, \dots, \nu_m\} \subset \mathbb{S}^1\right\}, \tag{4.2.5}$$

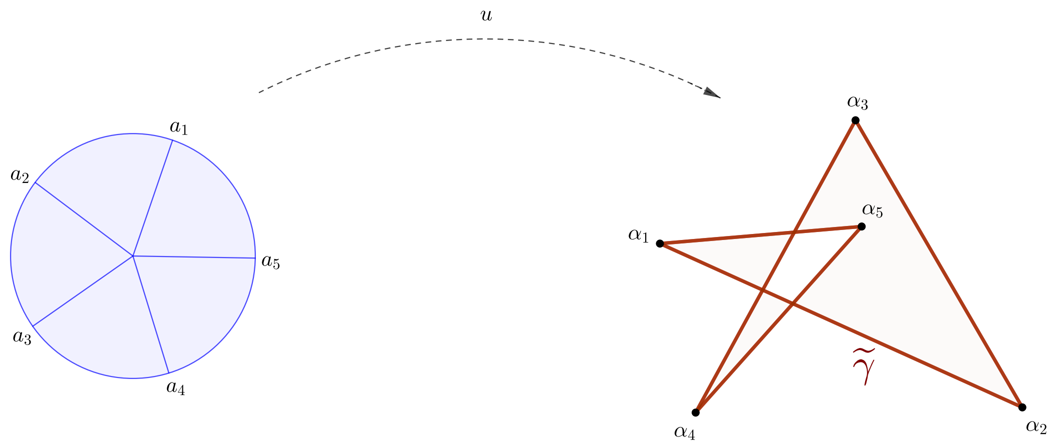

Figure 4.2: An n-uple point map and the corresponding curve γ, for $n = 5$.

with the convention $\alpha_{n+1} := \alpha_1$, Clearly, by definition of u, we have

$$|Du|(B_r) = r|\dot{\gamma}|(\mathbb{S}^1) = rL(\gamma).$$

Thanks to Lemma 4.1.6, for $\overline{P}(\gamma)$ as in (4.1.11) we know that

$$\overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{4.2.6}$$

For a general γ the computation of $\overline{P}(\gamma)$ seems not immediate. For the configuration in Fig. 4.2, we expect it to be the area of the region enclosed by $\widetilde{\gamma}$, with the small internal quadrilateral counted twice.

Theorem 4.2.3 (**Relaxation of TVJ on n-uple point maps**). Let $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}^2$, $\gamma \in BV(\mathbb{S}^1; \{\alpha_1, \dots, \alpha_n\})$ be a function with a finite number of jump points, and let u be as in (4.2.4). Then

$$\overline{TVJ}_{BV}(u, B_r) = \overline{P}(\gamma).$$

Proof. Lower bound: Assume that $(v_k) \subset C^1(B_r; \mathbb{R}^2)$ converges to u strictly $BV(B_r; \mathbb{R}^2)$ and

$$\lim_{k\to+\infty} \int_{B_r} |Jv_k| \; dx = \overline{TVJ}_{BV}(u, B_r).$$

By Lemma 3.1.3, we can fix $\varepsilon \in (0, r)$ and a not-relabeled subsequence depending on ε, such that $v_k \llcorner \partial B_\varepsilon \to u \llcorner \partial B_\varepsilon$ strictly $BV(\partial B_\varepsilon; \mathbb{R}^2)$. Thus, using Corollary 4.3.8 and the rescaling invariance of (4.1.11), we can estimate

$$\overline{TVJ}_{BV}(u, B_r) \geq \liminf_{k\to+\infty} \int_{B_\varepsilon} |Jv_k| \; dx \geq \liminf_{k\to+\infty} P(v_k \llcorner \partial B_\varepsilon) = \overline{P}(u \llcorner \partial B_\varepsilon) = \overline{P}(\gamma). \tag{4.2.7}$$

Upper bound: By an argument similar to the one at the beginning of the proof of Proposition 3.2.5, it will be enough to construct a recovery sequence $(u_k) \subset \mathrm{Lip}(B_r; \mathbb{R}^2)$. Let $\widetilde{\gamma}$ be as above. We start by building a sequence $(\gamma_k)_k$ of Lipschitz reparameterizations of $\widetilde{\gamma}$ which converges strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ to γ. Let us denote by $a_1,\ldots,a_n \in [0,2\pi)$ the angular coordinates of the extremal points of $C_1,\ldots,C_n$, and assume without loss of generality $0 = a_1 < a_2 < \cdots < a_n$. Then

$$\bigcup_{i=1}^{n}[a_i,a_{i+1}] = [0,2\pi],$$

with the convention $a_{n+1} = 2\pi$. Let $(\delta_k)_k$ be an infinitesimal sequence with $0 < \delta_k < \max\{|a_{i+1}-a_i|, i=1,\ldots,n\}$, for instance $\delta_k = \frac{2}{k}$, k large enough. We define the piecewise affine map $\gamma_k : [0,2\pi] \to \mathbb{R}^2$ as

$$\gamma_k(t) = \begin{cases} \alpha_i & \text{if } t \in [a_i+\delta_k/2, a_{i+1}-\delta_k/2], \\ \dfrac{a_{i+1}+\delta_k/2-t}{\delta_k}\alpha_i + \dfrac{t-a_{i+1}+\delta_k/2}{\delta_k}\alpha_{i+1} & \text{if } t \in [a_{i+1}-\delta_k/2, a_{i+1}+\delta_k/2], \end{cases} \tag{4.2.8}$$

for $i = 1,\ldots,n$.

Then $\gamma_k \to \gamma$ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ (a direct computation shows that $|\dot{\gamma}_k|(\mathbb{S}^1) = |\dot{\gamma}|(\mathbb{S}^1)$), γ_k are uniformly bounded in L^∞, and converge almost everywhere to γ. As a consequence, from Corollary 4.3.8,

$$P(\gamma_k) \to \overline{P}(\gamma) \qquad \text{as } k \to +\infty. \tag{4.2.9}$$

Therefore, by (4.1.1) we choose, for all $k > 1$ large enough, a map $v_k \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ such that

$$v_k \llcorner \mathbb{S}^1 = \gamma_k, \qquad \left| P(\gamma_k) - \int_{B_1} |Jv_k|\, dx \right| \le \frac{1}{k}. \tag{4.2.10}$$

Let $c_k > 0$ be the Lipschitz constant of v_k. Defining $v_{k,\rho} \in \mathrm{Lip}(B_\rho;\mathbb{R}^2)$ as $v_{k,\rho}(y) := v_k(\frac{y}{\rho})$ for any $\rho > 0$, it is straightforward that the Lipschitz constant of $v_{k,\rho}$ is c_k/ρ.

We now choose an infinitesimal sequence $(\rho_k) \subset (0,r)$ in such a way that $\lim_{k\to+\infty} c_k\rho_k = 0$. As a consequence we get

$$\int_{B_{\rho_k}} |\nabla v_{k,\rho_k}|\, dx \le \pi c_k \rho_k \to 0 \qquad \text{as } k \to +\infty. \tag{4.2.11}$$

We are now in a position to introduce our recovery sequence: We define $u_k \in \mathrm{Lip}(B_r;\mathbb{R}^2)$ as

$$u_k(x) := \begin{cases} \gamma_k\left(\frac{x}{|x|}\right) & \forall x \in B_r \setminus B_{\rho_k}, \\ v_{k,\rho_k}(x) & \forall x \in B_{\rho_k}. \end{cases} \tag{4.2.12}$$

Using that $\gamma_k \to \gamma$ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ and (4.2.11) we see that $u_k \to u$ strictly $BV(B_r;\mathbb{R}^2)$. Finally, since in $B_r \setminus B_{\rho_k}$ the map u_k depends only on the angular coordinate, its Jacobian determinant vanishes in $B_r \setminus B_{\rho_k}$. Hence

$$\liminf_{k\to+\infty} \int_{B_r} |Ju_k|\, dx = \liminf_{k\to+\infty} \int_{B_{\rho_k}} |Jv_{k,\rho_k}|\, dx = \overline{P}(\gamma), \tag{4.2.13}$$

the convergence being a consequence of (4.1.3), (4.2.10), and (4.2.9). □

As a consequence of Theorem 4.2.3 we deduce:

Theorem 4.2.4 (**Relaxation of $\mathcal{A}$ on n-uple point maps**)**.** Let γ and u be as in Theorem 4.2.3. Then, for any $r > 0$, we have

$$\overline{\mathcal{A}}_{BV}(u, B_r) = \pi r^2 + rL(\gamma) + \overline{P}(\gamma). \tag{4.2.14}$$

Proof. Lower bound: Suppose that $v_k \in C^1(B_r; \mathbb{R}^2)$ is such that

$$v_k \to u \quad \text{strictly } BV(B_r; \mathbb{R}^2) \quad \text{and} \quad \lim_{k\to+\infty} \mathcal{A}(v_k, B_r) = \liminf_{k\to+\infty} \mathcal{A}(v_k, B_r).$$

Now, let $\varepsilon \in (0, r)$ and write $\mathcal{A}(v_k, B_r) = \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \mathcal{A}(v_k, B_\varepsilon) \geq \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k|\, dx$, so that, by [1, Theorem 3.7],

$$\begin{aligned}
\lim_{k\to+\infty} \mathcal{A}(v_k, B_r) &\geq \liminf_{k\to+\infty} \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k|\, dx \\
&\geq |B_r \setminus B_\varepsilon| + r(1-\varepsilon)L(\gamma) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k|\, dx \\
&\geq |B_r \setminus B_\varepsilon| + r(1-\varepsilon)L(\gamma) + \overline{P}(\gamma),
\end{aligned}$$

where in the last line we have applied Theorem 4.2.3 with r replaced by ε. We now pass to the limit as $\varepsilon \to 0^+$ to get the lower bound $\overline{\mathcal{A}}_{BV}(u, B_r) \geq \pi r^2 + rL(\gamma) + \overline{P}(\gamma)$ in (4.2.14).

Upper bound: It is sufficient to consider the sequence $(u_k)_k$ defined in (4.2.12), for which

$$\begin{aligned}
\overline{\mathcal{A}}_{BV}(u, B_r) &\leq \limsup_{k\to+\infty} \mathcal{A}(u_k, B_1) \leq |B_r| + \lim_{k\to+\infty} \int_{B_r} |\nabla u_k|\, dx + \lim_{k\to+\infty} \int_{B_r} |Ju_k|\, dx \\
&= \pi r^2 + rL(\gamma) + \overline{P}(\gamma).
\end{aligned}$$

□

4.2.2 An example of infinite BV-relaxed area

Now, we are in the position to show an example of a piecewise constant map $u \in BV(B_1; \mathbb{R}^2)$ with infinite relaxed Jacobian total variation but vanishing associated minimal vertical lifting measure $\mu_v[u]$. This map is constructed in Example 4.2.6, while the Example 4.2.5 is preparatory.

Example 4.2.5. We want to show here how singular topological phenomena related to the double-eight curve arise also among piecewise constant maps. In Example 4.3.12 one can find the computation of the BV-relaxed area for the homogeneous extension u_8 of the double eight map. In particular, as pointed out in [40], a gap phenomenon occurs for u_8 between the minimal vertical lifting measure and the relaxed Jacobian total variation. We show now that we find such a gap also among piecewise constant maps, by exhibiting a piecewise constant homogeneous map with vanishing minimal vertical lifting measure but with finite non-zero $\overline{TVJ}$. Namely, we are going to define a map $u : B_1 \to \mathbb{R}^2$ assuming five distinct values, for which the resulting closed curve $\widetilde{\gamma}$ has zero degree, but is homotopically non-trivial, since it is, in fact, homeomorphic to the double-eight curve. Let

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subset \mathbb{R}^2$ be the vertices of two equilateral triangles with a common vertex, say α_1 (see Figure 4.3). Fix a partition of $\mathbb{S}^1$ in twelve disjoint non-empty arcs $C_1, \ldots, C_{12}$ (not necessarily of the same length), with extremal points $a_1, \ldots, a_{12}$ in counter-clockwise order. Then, define $\gamma : \mathbb{S}^1 \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ to be constant on the arcs $C_1, \ldots, C_{12}$, precisely equal to, in the order, $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_4, \alpha_5, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \alpha_5, \alpha_4$. Then, the broken line curve $\widetilde{\gamma}$ runs consecutively the triangles $T_{123} := T_{\alpha_1\alpha_2\alpha_3}$ and $T_{145} := T_{\alpha_1\alpha_4\alpha_5}$ twice, and every time with different orientation. Define u as in (4.2.4), obtaining a 12-point map. Now, by applying Theorem 4.2.3 and computing the minimum of the Plateau problem (4.1.1) for $\widetilde{\gamma}$ as in [42, Theorem 5], we obtain

$$\overline{TVJ}_{BV}(u, B_1) = \overline{P}(\gamma) = P(\widetilde{\gamma}) = 2\min\{|T_{123}|, |T_{145}|\}. \tag{4.2.15}$$

Moreover, it is not difficult to see that

$$\mu_u^J = (|T_{123}| + |T_{145}| - |T_{123}| - |T_{145}|)\delta_0 = 0.$$

In this case, we have also $\mu_v[u] = 0$, indeed we can prove that the minimal lifting current T_u associated to u is given by

$$T_u = G_u + S = \sum_{l=1}^{12} [\![\widehat{C}_l]\!] \times [\![c_l]\!] + \sum_{l=1}^{12} [\![0, a_l]\!] \times [\![c_{l-1}, c_l]\!], \tag{4.2.16}$$

where $\widehat{C}_l$ is the circular sector corresponding to C_l and c_l is the assigned value of γ on C_l for $l = 1, \ldots, 12$ (we used the convention $c_0 = c_{12}$). Let us show (4.2.16). One checks that $\mu_i^j[T_u] = \mu_i^j[u]$ for $i, j = 1, 2$ by proceeding as in Remark 3.2.6. So, it remains to prove that $T_u \in \mathrm{cart}(B_1; \mathbb{R}^2)$: it is enough to check that $(\partial T_u) \llcorner B_1 \times \mathbb{R}^2 = 0$. Compute

$$\partial S = \sum_{l=1}^{12} \partial\left([\![0, a_l]\!] \times [\![c_{l-1}, c_l]\!]\right) = \sum_{l=1}^{12} \left(-[\![0]\!] \times [\![c_{l-1}, c_l]\!] + [\![0, a_l]\!] \times [\![c_l]\!] - [\![0, a_l]\!] \times [\![c_{l-1}]\!]\right).$$

Now, since by convention $a_{13} = a_1$,

$$\partial G_u = \sum_{l=1}^{12} \left([\![0, a_{l+1}]\!] \times [\![c_l]\!] - [\![0, a_l]\!] \times [\![c_l]\!]\right) = -\sum_{l=1}^{12} \left([\![0, a_l]\!] \times [\![c_l]\!] - [\![0, a_l]\!] \times [\![c_{l-1}]\!]\right).$$

Moreover, by the choice of $\{c_l\}$,

$$\sum_{l=1}^{12} [\![0]\!] \times [\![c_{l-1}, c_l]\!] = [\![0]\!] \times [\![\alpha_1, \alpha_2]\!] + [\![0]\!] \times [\![\alpha_2, \alpha_3]\!] + \ldots + [\![0]\!] \times [\![\alpha_4, \alpha_1]\!] = 0.$$

Therefore, $\partial G_u = -\partial S$.
Notice that the action of T_u against 2-forms with only vertical differentials is 0, which means that T_u does not have completely vertical part and so $\mu_v[u] = 0$. Roughly, due to cancellations in the part of the boundary of T_u in corrispondence to the origin, the current T_u is not able to detect the hole upon the origin in the graph of u, generated by the presence of the multiple junction.

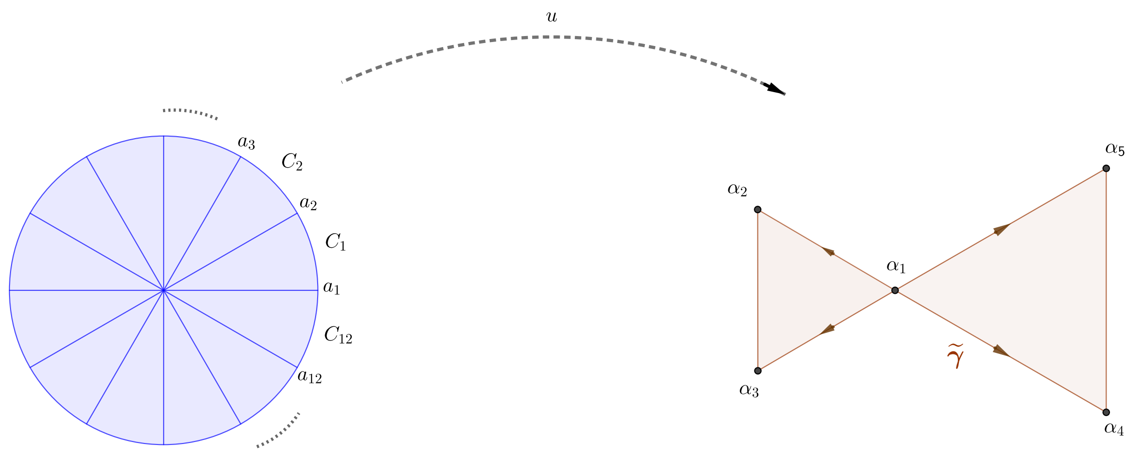

Figure 4.3: The map u and the broken line curve $\widetilde{\gamma}$ of Example 4.2.5.

Example 4.2.6. This example is an adaptation of [39, Theorem 1.3] to the case of piecewise constant maps. Indeed, we construct a piecewise constant map u, taking only five values of $\mathbb{R}^2$, such that

$$\mu_v[u] = 0 \qquad \text{and} \qquad \overline{TVJ}_{BV}(u, B_1) = +\infty.$$

The idea is to replicate the map of Example 4.2.5 infinitely many times on a sequence $\{D_i\}_{i\in\mathbb{N}} \subset B_1$ of disjoint balls, whose measures form an infinitesimal sequence (see Figure 4.4). So, for $i \in \mathbb{N}$, set

$$D_i := B_{r_i}(x_i), \quad \text{with } x_i := \left(-1 + \sum_{j=0}^{i-1} 2^{-j}, 0\right), \quad r_i := 2^{-i-1}.$$

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subset \mathbb{R}^2$ and $\gamma : \mathbb{S}^1 \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be as in Example 4.2.5. Now, define the map $\widehat{\gamma} : \mathbb{S}^1 \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ in the same way as γ, but with different order of the values, in a symmetric way with respect to the vertical axis through α_1, namely, in the same arcs $C_1, \ldots, C_{12}$, $\widehat{\gamma}$ is equal to $\alpha_1, \alpha_5, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_5, \alpha_1, \alpha_2, \alpha_3$. Then, for $i \in \mathbb{N}$, define $u_{|D_i} := u^{(i)}$ as

$$u^{(i)}(x) = \begin{cases} \gamma\left(\dfrac{x - x_i}{|x - x_i|}\right) & \text{if } i \text{ is odd,} \\ \widehat{\gamma}\left(\dfrac{x - x_i}{|x - x_i|}\right) & \text{if } i \text{ is even.} \end{cases}$$

It remains to define u in $B_1 \setminus \cup_{i\in\mathbb{N}} D_i$. Start by considering, for every $i \in \mathbb{N}$, the square Q_i that circumscribes D_i and extend $u^{(i)}$ to Q_i to be constant along horizontal lines. Now, denote by $L_i^{(1)}$ and $L_i^{(2)}$ the vertical left and right sides of ∂Q_i, then extend u to the convex hull of $L_i^{(2)}$ and $L_{i+1}^{(1)}$ to be constant along straight lines which interpolate pointwise the two sides. Finally, extend u in the strip that connects $L_1^{(1)}$ to ∂B_1 to be constant along horizontal lines and set $u = \alpha_1$ in the rest of B_1. (see Figure 4.4). It is not difficult to see that $u \in BV(B_1; \mathbb{R}^2)$, by the choice of the infinitesimal sequence (r_i). Thus, assuming by

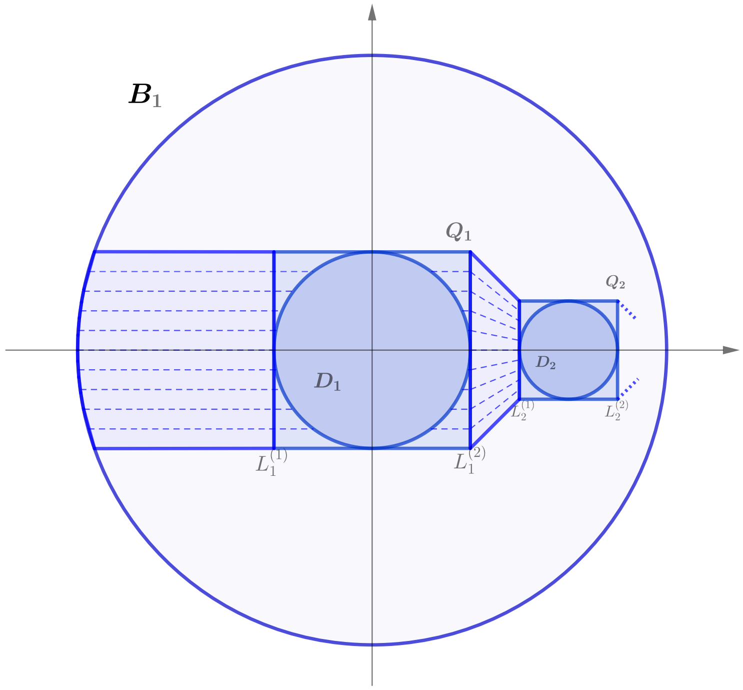

Figure 4.4: The sequence $\{D_i\} \subset B_1$ of disks of Example 4.2.6.

contradiction that $\overline{\mathcal{A}}_{BV}(u, B_1)$ be finite, one can define the current $T_u = G_u + S$ in a similar way as in Example 4.2.5, that is to say, by setting S to be the trivial affine interpolation surface on the jump segments of u. One can prove in the same way that T_u is the current with minimal completely vertical lifting associated to u and $\mu_v[u] = 0$. In particular, $T_u \in \mathrm{cart}(B_1 \times \mathbb{R}^2)$ and has finite mass. On the other hand,

$$\overline{TVJ}_{BV}(u, B_1) \geq \sum_{i=1}^{+\infty} \overline{TVJ}_{BV}(u, D_i) = \sum_{i=1}^{+\infty} 2\min\{|T_{\alpha_1\alpha_2\alpha_3}|, |T_{\alpha_1\alpha_4\alpha_5}|\} = +\infty.$$

In particular $\overline{\mathcal{A}}_{BV}(u, B_1) = +\infty$ as well.

4.3 General homogeneous maps

In this section, we generalize at once the results in Chapter 2 about vortex-type maps and in Section 4.2 about piecewise constant homogeneous maps, by considering general homogeneous maps in $BV(B_\ell; \mathbb{R}^2)$. The results of this section are contained in [14].

Definition 4.3.1. A map $u \in BV(B_\ell; \mathbb{R}^2)$ is 0-homogeneous if it is of the form

$$u(x) = \gamma\left(\frac{x}{|x|}\right) \qquad \text{a.e. } x \in B_\ell \tag{4.3.1}$$

for some $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$. In this case, we say that u is the 0-homogeneous (or simply homogeneous) extension of γ on B_ℓ.

Notice that, according to Definition 4.3.1, the maps u_V and u_T are homogeneous, as well as vortex-type maps (2.2.1) and the maps of the form (4.2.4). The piecewise constant maps of Examples 4.2.1 and 4.2.6, instead, are not homogeneous.
In order to ensure the consistency of Definition 4.3.1, we shall prove in Proposition 4.3.4 that the homogeneous extension of a map $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ belongs to $BV(B_\ell;\mathbb{R}^2)$. In the proof of Lemma 3.1.3 a useful Coarea-type formula is provided:

Lemma 4.3.2. Let $u \in BV(B_\ell;\mathbb{R}^2)$. Then

$$|D_\tau u|(A_{\varepsilon,\ell}) = \int_\varepsilon^\ell |D(u \llcorner \partial B_r)|(\partial B_r)\, dr. \tag{4.3.2}$$

This formula allows us to define a notion of tangential total variation for $u \in BV(B_\ell;\mathbb{R}^2)$ on the whole B_ℓ, since the right hand side of (4.3.2) is monotone non-increasing and equibounded w.r.t. ε.

Definition 4.3.3 (Tangential total variation in B_ℓ). Let τ and $A_{\varepsilon,\ell}$ as in Definition 3.1.2. We define the tangential total variation of $u \in BV(B_\ell;\mathbb{R}^2)$ as

$$|D_\tau u|(B_\ell) := \lim_{\varepsilon\to 0^+} |D_\tau u|(A_{\varepsilon,\ell}) = \int_0^\ell |D(u \llcorner \partial B_r)|(\partial B_r)\, dr. \tag{4.3.3}$$

Proposition 4.3.4. Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and u be defined as in (4.3.1). Then $u \in BV(B_\ell;\mathbb{R}^2)$ and

$$|Du|(B_\ell) = \ell|\dot\gamma|(\mathbb{S}^1). \tag{4.3.4}$$

Moreover,

$$\int_{B_\ell} |\nabla u| dx = \ell \int_{\mathbb{S}^1} |\dot\gamma^a| dy, \qquad |D^s u|(B_\ell) = \ell|\dot\gamma^s|(\mathbb{S}^1). \tag{4.3.5}$$

Proof. Since u does not depend on ρ, by (3.1.7), we have $|D(u\llcorner\partial B_r)|(\partial B_r) = |\dot\gamma|(\mathbb{S}^1)$. So, thanks to (4.3.2), in order to prove (4.3.4) it is enough to show that the variation of u is purely tangential, namely $|Du|(B_\ell) = |D_\tau u|(B_\ell)$. To this purpose, set $\nu(x) = \frac{x}{|x|}$, $x \neq 0$, and define the measure $D_\nu u := Du\nu$ on the annulus $A_{\varepsilon,\ell}$, i.e.

$$\langle D_\nu u, g\rangle = \int_{A_{\varepsilon,\ell}} u^1 \mathrm{div}(g^1\nu) dx + \int_{A_{\varepsilon,\ell}} u^2 \mathrm{div}(g^2\nu) dx \quad \forall g \in C^1_c(A_{\varepsilon,\ell};\mathbb{R}^2).$$

By polar decomposition of vector valued Radon measure, for $i = 1, 2$ we have

$$Du^i = \frac{Du^i}{|Du^i|}|Du^i| = \left(\frac{Du^i}{|Du^i|}\cdot\tau\tau + \frac{Du^i}{|Du^i|}\cdot\nu\nu\right)|Du^i| = Du^i\cdot\tau\tau + Du^i\cdot\nu\nu.$$

Let us prove that $Du^i \cdot \nu = 0$ on $A_{\varepsilon,\ell}$, for $i = 1, 2$. Recall that $\bar{\gamma}(\theta) := \bar{u}(\rho,\theta) = \gamma(\cos\theta, \sin\theta)$. Let $\psi \in C^1_c(A_{\varepsilon,\ell})$, then since $\mathrm{div}\nu = \frac{1}{|x|}$ in $A_{\varepsilon,\ell}$, we get

$$\begin{aligned}
\langle Du^i \cdot \nu, \psi\rangle &= \int_{A_{\varepsilon,\ell}} u^i \mathrm{div}(\psi\nu)dx = \int_{A_{\varepsilon,\ell}} u^i \psi \mathrm{div}\nu dx + \int_{A_{\varepsilon,\ell}} u^i \nabla\psi \cdot \nu dx \\
&= \int_\varepsilon^\ell \int_0^{2\pi} \rho\bar{u}^i(\rho,\theta)\bar{\psi}(\rho,\theta)\frac{1}{\rho} d\rho d\theta + \int_\varepsilon^\ell \int_0^{2\pi} \rho\bar{u}^i(\rho,\theta)\partial_\nu\bar{\psi}(\rho,\theta) d\rho d\theta \\
&= \int_\varepsilon^\ell \int_0^{2\pi} \bar{\gamma}^i(\theta)\bar{\psi}(\rho,\theta) d\rho d\theta + \int_\varepsilon^\ell \bar{\gamma}^i(\theta)\left[\int_0^{2\pi} \rho\partial_\nu\bar{\psi}(\rho,\theta)d\rho\right] d\theta \\
&= \int_\varepsilon^\ell \int_0^{2\pi} \bar{\gamma}^i(\theta)\bar{\psi}(\rho,\theta) d\rho d\theta - \int_\varepsilon^\ell \int_0^{2\pi} \bar{\gamma}^i(\theta)\bar{\psi}(\rho,\theta) d\rho d\theta = 0.
\end{aligned}$$

We infer that $Du = (Du\tau) \otimes \tau$ on $A_{\varepsilon,\ell}$. Now, since $|(Du\tau)\otimes\tau|(A_{\varepsilon,\ell}) \le |D_\tau u|(A_{\varepsilon,\ell}) \le |Du|(A_{\varepsilon,\ell})$, passing to the limit as $\varepsilon \to 0^+$, we conclude that $|Du|(B_\ell) = |D_\tau u|(B_\ell) = \ell|\dot{\gamma}|(\mathbb{S}^1)$.
Finally, in polar coordinates

$$\nabla u(\rho\cos\theta, \rho\sin\theta) = \frac{\dot{\bar{\gamma}}^a(\theta)}{\rho} \qquad \text{a.e. } \rho \in (0,\ell], \theta \in [0,2\pi], \tag{4.3.6}$$

so that

$$\int_{B_\ell} |\nabla u| dx = \int_0^\ell \int_0^{2\pi} \rho \frac{|\dot{\bar{\gamma}}^a(\theta)|}{\rho} d\theta d\rho = \ell \int_{\mathbb{S}^1} |\dot{\gamma}^a| dy$$

and

$$|D^s u|(B_\ell) = |Du|(B_\ell) - \int_{B_\ell} |\nabla u| dx = \ell|\dot{\gamma}|(\mathbb{S}^1) - \ell\int_{\mathbb{S}^1} |\dot{\gamma}^a| dy = \ell|\dot{\gamma}^s|(\mathbb{S}^1).$$

□

4.3.1 Further properties in dimension 1

In order to characterize the BV-relaxed area for u as in (4.3.1), we need to provide a further improvement of Lemma 3.1.4, namely, when γ is just a function of bounded variation.
To this purpose, suppose that $\gamma \in BV([a,b];\mathbb{R}^2)$. Then, it is well known that J_γ is at most countable. So, let $\{t_i\}_{i\in\mathbb{N}}$ be an enumeration[2] of J_γ and $\gamma^\pm(t_i)$ be the traces of γ at t_i. We want to associate to γ a unique continuous curve $\widetilde{\gamma}$ which "completes" the image of γ by means of segments connecting $\gamma^-(t_i)$ to $\gamma^+(t_i)$. In particular, we require that $\widetilde{\gamma}$ has the same total variation L of γ and is compatible with the approximation via strict BV-convergence. Unfortunately, this cannot be done simply by guessing a parametrization of $\widetilde{\gamma}$ starting from the one of γ, as we did in Lemma 3.1.4, but we need an existence result by approximation. Precisely we show the following result.

Lemma 4.3.5. Suppose that $(\gamma_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ is a sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to $\gamma \in BV([a,b];\mathbb{R}^2)$. Then there exist:

(a) a curve $\widetilde{\gamma} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$,

[2] If the number of jumps is finite, then $\{t_i\}$ is definitively constant.

(b) a subsequence (k_j) and Lipschitz strictly increasing surjective functions $h_{k_j} : [a,b] \to [a,b]$ for any $j \in \mathbb{N}$, with $\sup_j \|\dot{h}_{k_j}\|_\infty < +\infty$,

such that

$$\lim_{j \to +\infty} \gamma_{k_j} \circ h_{k_j} = \widetilde{\gamma} \quad \text{uniformly in } [a,b]. \tag{4.3.7}$$

Moreover, $\widetilde{\gamma}$ does not depend on the approximating sequence γ_k, in the sense that if $(\eta_k) \subset W^{1,1}([a,b];\mathbb{R}^2)$ is another sequence converging strictly $BV([a,b];\mathbb{R}^2)$ to γ, then the corresponding $\widetilde{\eta} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ coincides with $\widetilde{\gamma}$.

Proof. The lengths L_k of γ_k and L of γ are given by

$$L_k = \int_a^b |\dot{\gamma}_k| \, d\tau, \qquad L = |\dot{\gamma}|([a,b]).$$

Since, by assumption, $\gamma_k \to \gamma$ strictly $BV([a,b];\mathbb{R}^2)$, we have that $L_k \to L$ as $k \to +\infty$. For every $k \in \mathbb{N}$, define

$$s_k : [a,b] \to [0,L], \qquad s_k(t) := \frac{L}{L_k + b - a} \int_a^t \Big(|\dot{\gamma}_k(\tau)| + 1 \Big) \, d\tau, \tag{4.3.8}$$

with Lipschitz inverse $\alpha_k := s_k^{-1} : [0,L] \to [a,b]$. Notice that

$$\dot{\alpha}_k(s) = \frac{1}{\dot{s}_k(\alpha_k(s))} = \frac{L_k + b - a}{L} \cdot \frac{1}{|\dot{\gamma}_k(\alpha_k(s))| + 1} \le \frac{L_k + b - a}{L} \le C \qquad \text{for a.e. } s \in [0,L], \tag{4.3.9}$$

for some constant $C > 0$ independent of k. Define

$$\bar{\gamma}_k : [0,L] \to \mathbb{R}^2, \qquad \bar{\gamma}_k(s) := \gamma_k(\alpha_k(s)) \qquad \forall s \in [0,L].$$

Since

$$\left| \frac{d\bar{\gamma}_k}{ds}(s) \right| \le \frac{|\dot{\gamma}_k(\alpha_k(s))|}{|\dot{s}_k(\alpha_k(s))|} \le \frac{L_k + b - a}{L} \le C \qquad \text{for a.e. } s \in [0,L],$$

the sequence $(\bar{\gamma}_k)$ is bounded in $W^{1,\infty}([0,L];\mathbb{R}^2)$. Thus, there exists a subsequence $(k_j) \subset (k)$ and $\bar{\gamma} \in W^{1,\infty}([0,L];\mathbb{R}^2)$ such that

$$\bar{\gamma}_{k_j} \rightharpoonup \bar{\gamma} \text{ weakly* in } W^{1,\infty}([0,L];\mathbb{R}^2) \text{ and uniformly in } [0,L]. \tag{4.3.10}$$

Then, we conclude by defining $\widetilde{\gamma}$ and h_k as the composition of $\bar{\gamma}$ and α_k with an affine increasing diffeomorphism $\psi : [a,b] \to [0,L]$.

It remains to show the indipendence of $\bar{\gamma}$ from the sequence γ_k. So, suppose that $\eta_k \in W^{1,1}([a,b];\mathbb{R}^2)$ converges to γ strictly $BV([a,b];\mathbb{R}^2)$. Let $\sigma_k : [a,b] \to [0,L]$ be defined as s_k with η_k in place of γ_k and $\beta_k := \sigma_k^{-1} : [0,L] \to [a,b]$ its (equi-)Lipschitz inverse. As before, we obtain that there exists $(k_h) \subset (k)$ and $\bar{\eta}$ such that

$$\bar{\eta}_{k_h} \rightharpoonup \bar{\eta} \text{ weakly* in } W^{1,\infty}([0,L];\mathbb{R}^2) \text{ and uniformly in } [0,L].$$

Observe that for any open interval $J \subseteq [0,L]$,

$$\int_J |\dot{\bar\gamma}| ds \le \liminf_{k\to+\infty} \int_J |\dot{\bar\gamma}_k| ds \le |J| \liminf_{k\to+\infty} \frac{L_k+b-a}{L} = \frac{L+b-a}{L}|J|,$$

and thus

$$|\dot{\bar\gamma}| \le 1 + \frac{b-a}{L} \text{ a.e. in } [0,L]. \tag{4.3.11}$$

Now, fix $i \in \mathbb{N}$ and take any sequence $(t^{\pm}_{i,j})_j \subset [a,b] \setminus J_\gamma$ such that $t^-_{i,j} \nearrow t^-_i$ and $t^+_{i,j} \searrow t^+_i$ as $j \to +\infty$. By Lemma 1.3.5 and definition of $\gamma^{\pm}$, we have

$$\lim_{j\to+\infty} \gamma_{k_j}(t^{\pm}_{i,j}) = \gamma^{\pm}(t_i). \tag{4.3.12}$$

Setting

$$\begin{aligned} r^-_{i,j} :=& s_{k_j}(t^-_{i,j}) = \frac{L}{L_{k_j}+b-a} \int_a^{t^-_{i,j}} \big(|\dot\gamma_{k_j}|+1\big)\, d\tau, \\ r^+_{i,j} :=& s_{k_j}(t^+_{i,j}) = \frac{L}{L_{k_j}+b-a} \int_a^{t^+_{i,j}} \big(|\dot\gamma_{k_j}|+1\big)\, d\tau, \end{aligned} \tag{4.3.13}$$

we have

$$\begin{aligned} \lim_{j\to+\infty} r^-_{i,j} &= \frac{L}{L+b-a}|\dot\gamma|([a,t_i)) =: s^-(t_i), \\ \lim_{j\to+\infty} r^+_{i,j} &= \frac{L}{L+b-a}|\dot\gamma|([a,t_i]) = \frac{L}{L+b-a}\big[|\dot\gamma|([a,t_i)) + |\gamma^+(t_i)-\gamma^-(t_i)|\big] =: s^+(t_i). \end{aligned} \tag{4.3.14}$$

As a consequence of (4.3.10), (4.3.12), and (4.3.14), we get

$$\bar\gamma(s^{\pm}(t_i)) \leftarrow \bar\gamma_{k_j}(r^{\pm}_{i,j}) = \gamma_{k_j}(\alpha_{k_j}(r^{\pm}_{i,j})) = \gamma_{k_j}(t^{\pm}_{i,j}) \to \gamma^{\pm}(t_i) \quad \text{ as } j \to +\infty.$$

Therefore the curve $\bar\gamma$ maps the segment $[s^-(t_i), s^+(t_i)]$ into a curve joining $\gamma^-(t_i)$ and $\gamma^+(t_i)$. Now, since $s^+(t_i) - s^-(t_i) = \frac{L}{L+b-a}|\gamma^+(t_i) - \gamma^-(t_i)|$, from (4.3.11) we conclude that $\bar\gamma$ coincides with the $\big(1+\frac{b-a}{L}\big)$-speed parametrization ℓ_i of the segment joining $\gamma^-(t_i)$ and $\gamma^+(t_i)$ on $[s^-(t_i), s^+(t_i)]$. Hence we have shown that for every $i \in \mathbb{N}$

$$\gamma_{k_j} \circ \alpha_{k_j} \to \ell_i \text{ uniformly in } [s^-(t_i), s^+(t_i)] \text{ as } j \to +\infty. \tag{4.3.15}$$

An analogous conclusion holds also for η_{k_h}: indeed, let $\sigma_{k_h}(t^{\pm}_{i,h})$ be as in (4.3.13) but with η_{k_h} in place of γ_{k_j}, then it is clear that $\sigma_{k_h}(t^{\pm}_{i,h}) \to s^{\pm}(t_i)$ as $h \to +\infty$ and so

$$\eta_{k_h} \circ \beta_{k_h} \to \ell_i \text{ uniformly in } [s^-(t_i), s^+(t_i)] \text{ as } h \to +\infty. \tag{4.3.16}$$

Therefore, $\bar\eta = \bar\gamma$ on $S = \cup_{i\in\mathbb{N}} S_i$, where $S_i := [s^-(t_i), s^+(t_i)]$. It remains to show that $\bar\eta = \bar\gamma$ on $[0,L] \setminus S$.
By (4.3.9), up to extracting a not relabeled subsequence, we can assume that there exists $\alpha \in W^{1,\infty}([0,L])$ such that

$$\alpha_{k_j} \to \alpha \quad \text{uniformly in } [0,L] \text{ as } j \to +\infty \tag{4.3.17}$$

and, for the same reason, there exists $\beta \in W^{1,\infty}([0,L])$ such that

$$\beta_{k_h} \to \beta \quad \text{uniformly in } [0,L] \text{ as } h \to +\infty. \tag{4.3.18}$$

From Lemma 1.3.5, we deduce that $\bar{\gamma} = \gamma \circ \alpha$ on every compact subset $H \subset [0,L] \setminus S$. But, since α does not depend on the compact H, we deduce that $\bar{\gamma} = \gamma \circ \alpha$ on $[0,L] \setminus S$. In the same way, we infer that $\bar{\eta} = \gamma \circ \beta$ on $[0,L] \setminus S$. Let us show that $\alpha = \beta$ on $[0,L] \setminus S$. Indeed, notice that by definition of s_k,

$$s_k(t) \to s(t) := \frac{L}{L+b-a}(t - a + |\dot{\gamma}|([a,t])) \quad \forall t \in [a,b] \setminus J_\gamma.$$

The map $s : [a,b] \to [0,L]$ is strictly increasing with jumps at each point of J_γ. Notice that the traces of s at every $t_i \in J_\gamma$ are exactly the numbers $s^{\pm}(t_i)$ in (4.3.14). We claim that $\alpha = s^{-1}$ on $[0,L] \setminus S$. Indeed, by (4.3.17) we have that for every $t \in [a,b] \setminus J_\gamma$

$$t = \alpha_{k_j}(s_{k_j}(t)) \to \alpha(s(t)) \quad \text{as } j \to +\infty,$$

then $\alpha = s^{-1}$ on $s([a,b] \setminus J_\gamma) = [0,L] \setminus S$. In the same way, using (4.3.18) one can prove that $\beta = s^{-1}$ on $[0,L] \setminus S$ and we conclude the proof.

□

Remark 4.3.6. From the previous proof, we deduce that the "completed" curve $\widetilde{\gamma}$ does not depend on the subsequence of the approximating sequence γ_k. Moreover, we do not need to discuss the dependence on the reparametrization h_k, because, for our purpose, we shall consider in the sequel the Plateau-type problem (4.1.1) associated to γ_k, which is independent of the reparametrization of the curve.

4.3.2 Relaxation for general homogeneous maps

In this section, we compute the BV-relaxed area for homogeneous maps as in Definition 4.3.1.
First, we want to extend the thesis of Lemma 4.1.6 to the case $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$.

Lemma 4.3.7. Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and $\widetilde{\gamma} : \mathbb{S}^1 \to \mathbb{R}^2$ be the corresponding Lipschitz curve of Lemma 4.3.5. Then

$$\overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{4.3.19}$$

Proof. Let $(\gamma_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ be a sequence converging strictly to γ. Let us consider a not-relabeled subsequence of $(\gamma_k)_k$; by Lemma 4.3.5 there are a further subsequence $(\gamma_{k_j})_j$ and Lipschitz reparametrizations $\widetilde{\gamma}_{k_j} = \gamma_{k_j} \circ h_{k_j} \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ of γ_{k_j} such that $\widetilde{\gamma}_{k_j} \to \widetilde{\gamma}$ uniformly as $j \to +\infty$. Moreover, since by Lemma 4.3.5(b) the homeomorphism h_{k_j} can be chosen with uniformly bounded Lipschitz constant, it follows that $\widetilde{\gamma}_{k_j}$ has uniformly bounded total variation. Hence it follows from Lemma 4.1.4 that $P(\widetilde{\gamma}_{k_j}) \to P(\widetilde{\gamma})$ as $j \to +\infty$. Thanks to (4.1.1), we have also $P(\gamma_{k_j}) \to P(\widetilde{\gamma})$ as $j \to +\infty$. Then, since this argument holds for any subsequence of (γ_k), we conclude that the whole sequence satisfies $P(\gamma_k) \to P(\widetilde{\gamma})$. Finally, since by Lemma 4.3.5 $\widetilde{\gamma}$ does not depend on the approximating sequence, we can repeat the previous argument for another sequence $(\eta_k) \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ converging strictly to γ, obtaining that $P(\eta_k) \to P(\widetilde{\gamma})$. Therefore, we conclude $\overline{P}(\gamma) = P(\widetilde{\gamma})$. □

As a consequence of the argument in the proof of Lemma 4.3.7, we easily infer the following continuity property:

Corollary 4.3.8. Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and $\widetilde{\gamma}$ be as in Lemma 4.3.5, and assume that $(\gamma_k)_k \subset \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$ is a sequence converging strictly to γ. Then

$$\lim_{k\to+\infty} P(\gamma_k) = \overline{P}(\gamma) = P(\widetilde{\gamma}).$$

Now we can pass to treat the relaxation of our functionals. To start with, it is worth to consider the case of homogeneous extension u of a Lipschitz map $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$, namely

$$u(x) = \varphi\left(\frac{x}{|x|}\right) \quad \forall x \in B_\ell \setminus \{(0,0)\}. \tag{4.3.20}$$

In this case, clearly $u \in W^{1,1}(B_\ell;\mathbb{R}^2)$ and $\int_{B_\ell} |\nabla u| dx = \ell \int_{\mathbb{S}^1} |\dot{\varphi}| d\mathcal{H}^1$. The following result extends the validity of [42, Thm.1] also for the relaxation with respect to the strict BV-convergence.

Theorem 4.3.9. Suppose that $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ is Lipschitz continuous and let u be defined as in (4.3.20). Then

$$\overline{TVJ}_{BV}(u, B_\ell) = P(\varphi). \tag{4.3.21}$$

Proof. Let us show the upper bound inequality. Following the proof of Theorem 1 in [42], for $k \geq 2$, a recovery sequence $v_k \in \mathrm{Lip}(B_\ell;\mathbb{R}^2)$ is given by

$$v_k(x) = \begin{cases} u(x) & \text{if } |x| > \ell/k, \\ (v)_{\frac{\ell}{k}}(x) & \text{if } |x| \leq \ell/k, \end{cases} \tag{4.3.22}$$

where $v \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ is any map with $v = \varphi$ on ∂B_1 and $(v)_{\frac{\ell}{k}}(x) := v\left(\frac{k}{\ell}x\right)$ for $x \in B_{\frac{\ell}{k}}$. It is not difficult to see that $v_k \to u$ strongly in $W^{1,1}(B_\ell;\mathbb{R}^2)$ (and hence strictly $BV(B_\ell;\mathbb{R}^2)$). Moreover, by change of variable

$$\int_{B_\ell} |Jv_k| dx = \int_{B_{\frac{\ell}{k}}} |J(v)_{\frac{\ell}{k}}| dx = \int_{B_1} |Jv| dx \quad \forall k \in \mathbb{N}. \tag{4.3.23}$$

Finally, we get

$$\overline{TVJ}_{BV}(u, B_\ell) \leq \liminf_{k\to+\infty} \int_{B_\ell} |Jv_k| dx = \int_{B_1} |Jv| dx$$

for any $v \in \mathrm{Lip}(B_1;\mathbb{R}^2)$ such that $v = \varphi$ on ∂B_1, so we deduce that $\overline{TVJ}_{BV}(u, B_\ell) \leq P(\varphi)$. Now let us prove the lower bound inequality. Assume that $v_k \in C^1(B_\ell;\mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_1;\mathbb{R}^2)$. Then for almost every $\rho < \ell$, there exists a subsequence (v_{k_h}) (depending on ρ) such that its restriction to ∂B_ρ converges strictly $BV(\partial B_\rho;\mathbb{R}^2)$ to $u_{|\partial B_\rho}$. So, fix $\varepsilon < 1$ and a not-relabeled subsequence of (v_k) such that

$$v_{k|\partial B_\varepsilon} \to u_{|\partial B_\varepsilon} \quad \text{strictly } BV(\partial B_\varepsilon;\mathbb{R}^2). \tag{4.3.24}$$

Now, define $w_k : B_\ell \to \mathbb{R}^2$ as

$$w_k(x) = \begin{cases} v_k(x) & \text{if } |x| \leq \varepsilon \\ \dfrac{\ell - |x|}{\ell - \varepsilon} v_k\left(\varepsilon\dfrac{x}{|x|}\right) + \dfrac{|x| - \varepsilon}{\ell - \varepsilon} u\left(\varepsilon\dfrac{x}{|x|}\right) & \text{if } \varepsilon \leq |x| \leq \ell. \end{cases}$$

Then w_k is Lipschitz and $w = u$ on ∂B_ℓ. Moreover, by (4.3.24), the convergence of v_k to u on ∂B_ε is also uniform, so we have (as for the proof of (2.2.31) in Proposition 2.2.4)

$$\lim_{k\to+\infty} \int_{B_\ell\setminus B_\varepsilon} |Jw_k| dx = 0. \tag{4.3.25}$$

Finally, since $w_k = v_k$ in B_ε, by (4.3.25) we get

$$\begin{aligned}\liminf_{k\to+\infty} \int_{B_\ell} |Jv_k| dx &\geq \liminf_{k\to+\infty} \int_{B_\varepsilon} |Jv_k| dx = \liminf_{k\to+\infty} \int_{B_\ell} |Jw_k| dx \\ &\geq P(u \llcorner \partial B_\ell) = P(\varphi_\ell) = P(\varphi),\end{aligned} \tag{4.3.26}$$

where we used (4.1.4). We conclude by taking the infimum in the left hand side. □

Corollary 4.3.10. Let φ and u as in Theorem 4.3.9. Then

$$\overline{\mathcal{A}}_{BV}(u; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u|^2} dx + P(\varphi). \tag{4.3.27}$$

Proof. For the lower bound, suppose that $v_k \in C^1(B_\ell; \mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_\ell; \mathbb{R}^2)$. Now, let $\varepsilon < \ell$ such that (4.3.24) holds, and write $\mathcal{A}(v_k; B_\ell) = \mathcal{A}(v_k; B_\ell \setminus B_\varepsilon) + \mathcal{A}(v_k; B_\varepsilon) \geq \mathcal{A}(v_k; B_\ell \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k| dx$, so that, by [1, Theorem 3.7],

$$\begin{aligned}\lim_{k\to+\infty} \mathcal{A}(v_k; B_\ell) &\geq \liminf_{k\to+\infty} \mathcal{A}(v_k; B_\ell \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k| dx \\ &\geq \int_{B_\ell\setminus B_\varepsilon} \sqrt{1 + |\nabla u|^2} dx + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k| dx.\end{aligned}$$

We now apply (4.3.26) and next pass to the limit as $\varepsilon \to 0^+$ to get the lower bound in (4.3.27).

Concerning the proof of the upper bound for (4.3.27), consider the sequence (v_k) defined in (4.3.22), which converges to u in $W^{1,1}(B_\ell; \mathbb{R}^2)$. Then, upon extracting a subsequence such that (∇v_k) converges almost everywhere to ∇u, by (4.3.23) and dominated convergence we have, using the inequality $\sqrt{1 + a^2 + b^2 + c^2} \leq \sqrt{1 + a^2 + b^2} + |c|$ for $a, b, c \in \mathbb{R}$,

$$\begin{aligned}\overline{\mathcal{A}}_{BV}(u; B_\ell) &\leq \limsup_{k\to+\infty} \mathcal{A}(v_k; B_\ell) \leq \lim_{k\to+\infty} \int_{B_\ell} \sqrt{1 + |\nabla v_k|^2} dx + \lim_{k\to+\infty} \int_{B_\ell} |Jv_k| dx \\ &= \int_{B_\ell} \sqrt{1 + |\nabla u|^2} dx + \int_{B_1} |Jv| dx,\end{aligned}$$

for any $v \in \mathrm{Lip}(B_1; \mathbb{R}^2)$ such that $v = \varphi$ on ∂B_1. Passing to the infimum on the right hand side we obtain the upper bound inequality in (4.3.27). □

Remark 4.3.11. We point out that the result of Corollary 4.3.10 is compatible with Theorem 2.2.3, where φ is valued in $\mathbb{S}^1$, treated in Chapter 2. Indeed, one can argue as in the proof of [42, Theorem 4] to prove that $P(\varphi) = \pi|\mathrm{deg}\varphi|$ for any $\varphi \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$.

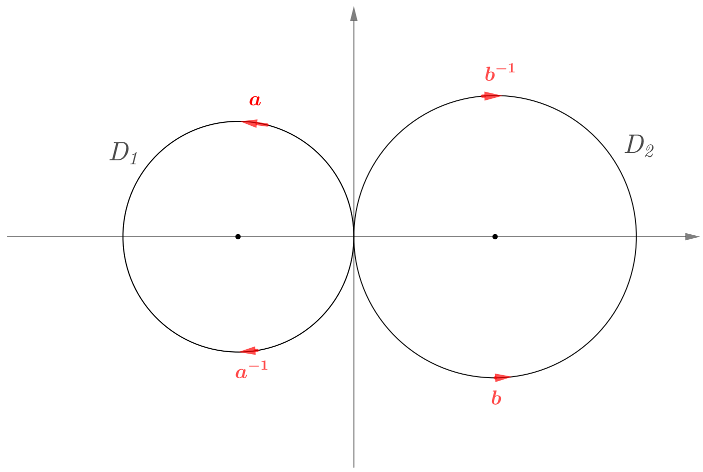

Figure 4.5: The double eight curve φ_8.

Example 4.3.12 (The double eight curve)**.** A very interesting example is the homogeneous extension u_8 of the so called *double eight map* $\varphi_8 \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2)$, defined as $\varphi_8 = a{\cdot}b{\cdot}a^{-1}{\cdot}b^{-1}$, where a, b are the loops in Fig. 4.5. This example was discovered by Malý [35] (see also [25], [23], [39], [42], [22]). Clearly, $\deg(\varphi_8) = 0$, however one can compute as in [42, Thm. 5] (see also [39, Thm. 1.2]) that

$$P(\varphi_8) = \inf\left\{\int_{B_1} |Jv|dx;\ v \in \mathrm{Lip}(B_1;\mathbb{R}^2) : v_{|\partial B_1} = \varphi_8\right\} = 2\min\{|D_1|, |D_2|\}.$$

Therefore, as underlined in [40], since the minimal lifting current T_{u_8} coincides with the graph current G_{u_8}, it has no vertical part, while from Theorem 4.3.9 we have that $\overline{TVJ}(u_8; B_\ell)$ is non-zero. Moreover, $|T_{u_8}| < \overline{\mathcal{A}}_{BV}(u_8; B_\ell)$. In particular, G_{u_8} is a Cartesian current, even if the origin is a non-removable singularity for u_8. Finally, an interesting problem would be the study of $\overline{\mathcal{A}}_{L^1}(u_8; B_\ell)$: since the obstruction generated by φ_8 has a topological nature, we conjecture that $\overline{\mathcal{A}}_{L^1}(u_8; B_\ell) = \overline{\mathcal{A}}_{BV}(u_8; B_\ell)$.

Now, we treat the case $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$. We recall that, by Proposition 4.3.4, its homogeneouos extension u is still $BV(B_\ell;\mathbb{R}^2)$.

Theorem 4.3.13. Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and u as in (4.3.1). Let $\widetilde{\gamma} : \mathbb{S}^1 \to \mathbb{R}^2$ be as in Lemma 4.3.5. Then

$$\overline{TVJ}_{BV}(u; B_\ell) = \overline{P}(\gamma) = P(\widetilde{\gamma}). \tag{4.3.28}$$

Proof. In order to show the upper bound inequality, consider a Lipschitz sequence $\varphi_k : \mathbb{S}^1 \to \mathbb{R}^2$ converging to γ strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ (e.g. a mollifying sequence). Then, by Lemma 4.3.5, there exists a equi-Lipschitz reparameterization $\widetilde{\varphi}_k$ of φ_k that converges to $\widetilde{\gamma}$ uniformly (up to extracting a subsequence). Set

$$u_k(x) = \varphi_k\left(\frac{x}{|x|}\right) \quad \forall x \in B_\ell \setminus \{(0,0)\}, \tag{4.3.29}$$

then $u_k \in W^{1,1}(B_\ell;\mathbb{R}^2)$ and $u_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$, since

$$\|u_k - u\|_{L^1(B_1;\mathbb{R}^2)} \leq \|\varphi_k - \gamma\|_{L^1(\mathbb{S}^1;\mathbb{R}^2)} \to 0,$$
$$\int_{B_\ell} |\nabla u_k| dx = \ell \int_{\mathbb{S}^1} |\dot{\varphi}_k| d\mathcal{H}^1 \to \ell |\dot{\gamma}|(\mathbb{S}^1) = |Du|(B_\ell),$$

where we used Proposition 4.3.4. Now, by lower semicontinuity of $\overline{TVJ}_{BV}(\cdot, B_\ell)$, Theorem 4.3.9, (4.1.1), and Lemma 4.1.4, we have

$$\overline{TVJ}_{BV}(u;B_\ell) \leq \liminf_{k\to+\infty} \overline{TVJ}_{BV}(u_k;B_\ell) = \liminf_{k\to+\infty} P(\varphi_k) = \liminf_{k\to+\infty} P(\widetilde{\varphi}_k) = P(\widetilde{\gamma}).$$

Let us prove the lower bound inequality. Assume that $v_k \in C^1(B_\ell;\mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$ and

$$\lim_{k\to+\infty} \int_{B_\ell} |Jv_k| dx = \overline{TVJ}_{BV}(u;B_\ell).$$

We use Lemma 3.1.3 to fix $\varepsilon < \ell$ and a subsequence $(v_{k_j}) \subset (v_k)$ such that $v_{k_j} \llcorner \partial B_\varepsilon \to u \llcorner \partial B_\varepsilon$ strictly $BV(\partial B_\varepsilon;\mathbb{R}^2)$. According to (4.1.2), we have $u \llcorner \partial B_\varepsilon = \gamma_\varepsilon$. So, let $\widetilde{\gamma}_\varepsilon$ be the Lipschitz curve of Lemma 4.3.5 associated[3] to γ_ε. Using Corollary 4.3.8 and (4.1.4), we conclude

$$\overline{TVJ}_{BV}(u;B_\ell) \geq \liminf_{j\to+\infty} \int_{B_\varepsilon} |Jv_{k_j}| dx \geq \liminf_{j\to+\infty} P(v_{k_j} \llcorner \partial B_\varepsilon) = \overline{P}(\gamma_\varepsilon) = P(\widetilde{\gamma}_\varepsilon) = P(\widetilde{\gamma}). \tag{4.3.30}$$

□

Remark 4.3.14. Setting $\widetilde{u}(x) := \widetilde{\gamma}\left(\frac{x}{|x|}\right)$, then $u \in W^{1,1}(B_\ell;\mathbb{R}^2)$. So, by Theorem 4.3.9 and Theorem 4.3.13, we have

$$\overline{TVJ}_{BV}(\widetilde{u};B_\ell) = \overline{TVJ}_{BV}(u;B_\ell). \tag{4.3.31}$$

We are in the position to state the main result of this section.

Theorem 4.3.15. Let $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$ and u as in Definition 4.3.1. Then

$$\overline{\mathcal{A}}_{BV}(u;B_\ell) = \int_{B_\ell} \sqrt{1+|\nabla u|^2} dx + |D^s u|(B_\ell) + \overline{P}(\gamma). \tag{4.3.32}$$

Proof. For the lower bound, suppose that $v_k \in C^1(B_\ell;\mathbb{R}^2)$ is such that $v_k \to u$ strictly $BV(B_\ell;\mathbb{R}^2)$. Now, let $\varepsilon < \ell$ such that (4.3.24) holds, and write $\mathcal{A}(v_k;B_\ell) = \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \mathcal{A}(v_k;B_\varepsilon) \geq \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k| dx$, so that, by [1, Theorem 3.7],

$$\begin{aligned}\lim_{k\to+\infty} \mathcal{A}(v_k;B_\ell) &\geq \liminf_{k\to+\infty} \mathcal{A}(v_k;B_\ell \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k| dx \\ &\geq \int_{B_\ell\setminus B_\varepsilon} \sqrt{1+|\nabla u|^2} dx + |D^s u|(B_\ell \setminus B_\varepsilon) + \liminf_{k\to+\infty} \int_{B_\epsilon} |Jv_k| dx.\end{aligned}$$

[3]We identify ∂B_ε with $[0, 2\pi\varepsilon]$.

We now apply (4.3.26) and next pass to the limit as $\varepsilon \to 0^+$ to get the lower bound in (4.3.32).

Concerning the proof of the upper bound for (4.3.32), consider the sequence $(u_k) \subset W^{1,1}(B_\ell;\mathbb{R}^2)$ defined in (4.3.29), which converges to u strictly $BV(B_\ell;\mathbb{R}^2)$. Let us prove that

$$\lim_{k\to+\infty}\int_{B_\ell}\sqrt{1+|\nabla u_k|^2}dx = \int_{B_\ell}\sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell). \tag{4.3.33}$$

In polar coordinates, we get

$$\int_{B_\ell}\sqrt{1+|\nabla u_k|^2}dx = \int_0^\ell\int_0^{2\pi}\rho\sqrt{1+\frac{|\dot\varphi_k(\theta)|^2}{\rho^2}}d\theta d\rho.$$

For a fixed $\rho\in(0,\ell)$, consider $f_\rho(\xi)=\rho\sqrt{1+\frac{|\xi|^2}{\rho^2}}$, $\xi\in\mathbb{R}^2$. Then, f_ρ is convex on $\mathbb{R}^2$. Now, if $\mu\in\mathcal{M}([0,2\pi];\mathbb{R}^2)$, one can consider the measure $f_\rho(\mu)\in\mathcal{M}^+([0,2\pi])$ defined as[4]

$$f_\rho(\mu)(A) = \int_A \rho\sqrt{1+\frac{|a(\theta)|^2}{\rho^2}}d\theta + |\mu^s|(A),$$

for any Borel set $A\subseteq[0,2\pi]$, where $\mu^a = a\mathscr{L}^2$ for some $a\in L^1([0,2\pi])$. By [29, Theorem 4], $f_\rho(\cdot)$ is continuous w.r.t. the approximation by convolution. In particular, choosing $\mu := \dot{\overline{\gamma}}\in\mathcal{M}([0,2\pi];\mathbb{R}^2)$ and $A=[0,2\pi]$, for every $\rho\in(0,\ell)$ we have

$$\begin{aligned}\lim_{k\to+\infty} f_\rho(\dot\varphi_k)([0,2\pi]) &= \lim_{k\to+\infty}\int_0^{2\pi}\rho\sqrt{1+\frac{|\dot\varphi_k(\theta)|^2}{\rho^2}}d\theta\\ &= \int_0^{2\pi}\rho\sqrt{1+\frac{|\dot{\overline{\gamma}}^a(\theta)|^2}{\rho^2}}d\theta + |\dot\gamma^s|(\mathbb{S}^1)\\ &= f_\rho(\dot{\overline{\gamma}})([0,2\pi]).\end{aligned}$$

Integrating in $(0,\ell)$, by dominated convergence we infer

$$\begin{aligned}\lim_{k\to+\infty}\int_{B_\ell}\sqrt{1+|\nabla u_k|^2}dx &= \lim_{k\to+\infty}\int_0^\ell\int_0^{2\pi}\rho\sqrt{1+\frac{|\dot\varphi_k(\theta)|^2}{\rho^2}}d\theta d\rho\\ &= \int_0^\ell\int_0^{2\pi}\rho\sqrt{1+\frac{|\dot{\overline{\gamma}}^a(\theta)|^2}{\rho^2}}d\theta d\rho + \ell|\dot\gamma^s|(\mathbb{S}^1)\\ &= \int_{B_\ell}\sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell),\end{aligned}$$

where we used (4.3.6) and (4.3.5). Therefore, we obtain (4.3.33).
Finally, by lower semicontinuity of $\overline{\mathcal{A}}_{BV}(\cdot,B_\ell)$ and by Corollary 4.3.10, we conclude

$$\begin{aligned}\overline{\mathcal{A}}_{BV}(u;B_\ell) \le \liminf_{k\to+\infty}\overline{\mathcal{A}}_{BV}(u_k;B_\ell) &= \lim_{k\to+\infty}\left[\int_{B_\ell}\sqrt{1+|\nabla u_k|^2}dx + P(\varphi_k)\right]\\ &= \int_{B_\ell}\sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell) + \overline{P}(\gamma).\end{aligned}$$

[4]See Theorem 2' in [29]: notice that $f_\rho^* = |\cdot|$ for every $\rho\in(0,\ell)$, where f_ρ^* is the recession function associated to f_ρ.

□

Remark 4.3.16. We point out that, by Lemma 4.3.7, we can write the right hand side of (4.3.32) by substituting $\overline{P}(\gamma)$ with $P(\widetilde{\gamma})$ and get an equivalent expression of $\overline{\mathcal{A}}_{BV}(u;B_\ell)$ in terms of $\widetilde{\gamma}$. The same observation can be done for $\overline{TVJ}_{BV}(u;B_\ell)$.
Furthermore, we notice that, as a function of the set variable, $\overline{TVJ}_{BV}(u;\cdot)$ is a finite positive measure. Precisely, for every open set $A \subset B_\ell$

$$\overline{TVJ}_{BV}(u;A) = \overline{P}(\gamma)\delta_0(A).$$

Indeed, if $0 \in A$ then $B_\varepsilon \subset A$ for some $\varepsilon \in (0,\ell)$ and we can argue as in (4.3.30). On the other hand, suppose that $0 \notin A$ and consider u_k as in (4.3.29). Then, $u_{k|A} \in \mathrm{Lip}(A;\mathbb{R}^2)$ and converges strictly $BV(A;\mathbb{R}^2)$ to $u_{|A}$. Since the image of u_k has zero Lebesgue measure, by lower semicontinuity of $\overline{TVJ}_{BV}(\cdot\,;A)$, we get that $\overline{TVJ}_{BV}(u;A) = 0$.
In the same way, one can prove that for every open set $A \subset B_\ell$

$$\overline{\mathcal{A}}_{BV}(u;A) = \int_A \sqrt{1+|\nabla u|^2}dx + |D^s u|(A) + \overline{P}(\gamma)\delta_0(A).$$

Therefore, also $\overline{\mathcal{A}}_{BV}(u;\,\cdot)$ is a measure and (4.3.32) is an integral representation.

Remark 4.3.17 (On the Plateau problem (4.1.1)**).** Let $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ be Lipschitz. From [15, Theorem 1.3], there exists a least area mapping $v \in W^{1,p}(B_1;\mathbb{R}^2)$, for some $p > 2$, spanning φ, i.e. realizing the infimum of the total variation of the Jacobian determinant in the class of Sobolev maps in $W^{1,p}(B_1;\mathbb{R}^2)$ whose trace on ∂B_1 is φ. In truth, one can prove that the least area mapping is Lipschitz, so that the Plateau problem (4.1.1) attains a minimum. The proof is a consequence of results contained in [16]: interestingly, it seems that one needs to pass through a more general metric result, concerning spaces with upper curvature bounds.

Chapter 5

General piecewise Lipschitz maps

This chapter, which is based on results in [4], combines the tools developed in the previous chapters to compute the BV-relaxed area for an interesting class of maps that we call *piecewise Lipschitz maps*, quickly mentioned in the Introduction. As stated in our main result (Theorem 0.0.4), the relaxed area turns out to be composed by an absolute continuous term and a singular one, that interestingly further splits into two non-trivial pieces, respectively related to the 1-dimensional and 0-dimensional singularities.

5.1 Networks and piecewise Lipschitz maps

Let $\Omega \subset \mathbb{R}^2$ be a connected bounded open set with boundary of class C^1. We say that a collection $\{\Omega_1, \ldots, \Omega_N\}$ of disjoint nonempty open sets is a Lipschitz partition of Ω if $\overline{\Omega} = \cup_{k=1}^N \overline{\Omega}_k$ and for each $k = 1, \ldots, N$, Ω_k is connected and Lipschitz. For a given Lipschitz partition of Ω we can consider its interface $\Sigma := \cup_{k=1}^N \partial\Omega_k$. Also, we can define the (possibly empty) set of interior junction points $\{p_i\}_{i=1}^m$, i.e. points $p_i \in \Omega$ such that there exist $r > 0$ and an integer N_i with $3 \leq N_i \leq N$, such that $B_r(p_i) \subset \Omega$ and $B_s(p_i)$ has nonempty intersection with exactly N_i connected components of Ω, for every $s \in (0, r]$.

We shall consider Lipschitz partitions whose interface is a *network* in the following sense:

Definition 5.1.1 (Network). The interface Σ of a Lipschitz partition of Ω is a network if

$$\Sigma := \bigcup_{\ell=1}^{n} \overline{J}_\ell, \qquad J_\ell = \alpha_\ell(I_\ell), \quad I_\ell = (a_\ell, b_\ell), \tag{5.1.1}$$

where the curves $\alpha_\ell : \overline{I}_\ell := [a_\ell, b_\ell] \to \overline{\Omega}$, $\ell = 1, \ldots, n$, satisfy the following properties:

- α_ℓ is of class C^2, injective with $|\dot{\alpha}_\ell| \equiv 1$ on I_ℓ, and $J_\ell \subset \Omega$;
- $\ell_1 \neq \ell_2 \Rightarrow J_{\ell_1} \cap J_{\ell_2} = \varnothing$;
- $\alpha_\ell(\{a_\ell, b_\ell\}) \subset \{p_1, \ldots, p_m\} \cup \partial\Omega$ for all $\ell = 1, \ldots, n$ such that $\alpha_\ell(a_\ell) \neq \alpha_\ell(b_\ell)$;
- if $x \in \overline{J}_\ell \cap \partial\Omega$, α_ℓ is transversal to $\partial\Omega$ at x;

- $\ell_1 \neq \ell_2 \Rightarrow \overline{J}_{\ell_1} \cap \overline{J}_{\ell_2} \subset \{p_1, \dots, p_m\}$.

From the last condition it follows that if two curves have endpoints on $\partial\Omega$, then these points are distinct.

Definition 5.1.2 (Piecewise Lipschitz map). Let $\{\Omega_k\}_{k=1}^N$ be a Lipschitz partition of Ω whose interface Σ is a network. We say that $u \in BV(\Omega;\mathbb{R}^2)$ is a *piecewise Lipschitz map* if its jump set J_u coincides with Σ and $u \llcorner \Omega_k \in \mathrm{Lip}(\Omega_k;\mathbb{R}^2)$ for any $k = 1, \dots, N$.

Since $u \llcorner \Omega_k \in \mathrm{Lip}(\Omega_k;\mathbb{R}^2)$, the trace of u on $\partial\Omega_k$ is also Lipschitz. In particular, for any $i \in \{1, \dots, m\}$ such that $p_i \in \partial\Omega_k$,

$$\exists \lim_{\substack{x \to p_i \\ x \in \Omega_k}} u(x) =: \beta_i^k \in \mathbb{R}^2.$$

Let $\rho > 0$ be sufficiently small so that $B_\rho(p_i) \subset \Omega$ for $i \in \{1, \dots, m\}$. Let $\ell \in \{1, \dots, n\}$ be such that p_i is an endpoint of $\overline{J}_\ell$; since α_ℓ is of class C^2, for ρ small enough the intersection $\overline{J}_\ell \cap \partial B_\rho(p_i)$ consists either of a single point, or of two points if $\alpha_\ell(a_\ell) = \alpha_\ell(b_\ell) = p_i$. Hence, the map $u \llcorner \partial B_\rho(p_i)$ is piecewise Lipschitz and jumps at any point of $\Sigma \cap \partial B_\rho(p_i)$. In particular, the number of these jump points is, by definition of junction point,

$$N_i = \sharp\big(\Sigma \cap \partial B_\rho(p_i)\big) \geq 3, \qquad i = 1, \dots, m.$$

For $i = 1, \dots, m$, we denote by $\Omega_1^i, \dots, \Omega_{N_i}^i$ the connected components of $\Omega \setminus \Sigma$ whose closure contains p_i, chosen in counterclockwise order around p_i. Since Ω_k is Lipschitz for every $k = 1, \dots, N$, any Ω_k^i has a corner at p_i whose aperture is a positive angle $\theta_i^k \in (0, 2\pi)$.

Lemma 5.1.3 (Circular slices). Let $i \in \{1, \dots, m\}$ be fixed and let $\rho > 0$ be as above. Then the maps $\gamma_\rho^i \in BV(\mathbb{S}^1;\mathbb{R}^2)$ defined by $\gamma_\rho^i(\nu) := u(p_i + \rho\nu)$ converge strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$, as $\rho \to 0^+$, to a piecewise constant map $\gamma^i : \mathbb{S}^1 \to \mathbb{R}^2$ taking, in counterclockwise order, the values $\beta_i^1, \beta_i^2, \dots, \beta_i^{N_i}$ on arcs of size $\theta_i^1, \theta_i^2, \dots, \theta_i^{N_i}$, respectively.

The map γ^i has N_i jumps on $\mathbb{S}^1$ whose angular coordinates are denoted by $a_i^1, a_i^2, \dots, a_i^{N_i}$ (where[1] $a_i^j - a_i^{j-1} = \theta_i^j$, for $j = 1, \dots, N_i + 1$).

Proof. It is easy to see that (γ_ρ^i) converges to γ^i almost everywhere on $\mathbb{S}^1$ as $\rho \to 0^+$. Moreover, γ_ρ^i, for ρ small enough, has exactly N_i jumps at points $a_{i,\rho}^j$ of amplitude $|u^+(p_i + \rho a_{i,\rho}^j) - u^-(p_i + \rho a_{i,\rho}^j)|$ which tend, by continuity of u in $B_\rho(p_i) \setminus \Sigma$, to $|\beta_i^j - \beta_i^{j+1}|$. Also, on the arcs between $a_{i,\rho}^j$ and $a_{i,\rho}^{j+1}$, $|\dot\gamma_\rho^i| \leq L\rho$, where L is the maximum of the Lipschitz constants of u on the sectors Ω_k^i. Hence $|\dot\gamma_\rho^i|(\mathbb{S}^1) \to |\dot\gamma^i|(\mathbb{S}^1)$ and the thesis follows straightforwardly. $\square$

For $\ell = 1, \dots, n$, we denote by $u_{(\ell)}^\pm$ the two traces of u on J_ℓ, and consider the affine interpolation surface $X_{(\ell)}^{\mathrm{aff}} : [a_\ell, b_\ell] \times I \to \mathbb{R}^3$ spanning the graphs of $u_{(\ell)}^-$ and $u_{(\ell^+)}$, given by:

$$X_{(\ell)}^{\mathrm{aff}}(t, s) = (t, s u_{(\ell)}^+(t) + (1 - s) u_{(\ell)}^-(t)), \qquad (t, s) \in [a_\ell, b_\ell] \times I, \tag{5.1.2}$$

where $I := [0, 1]$. For all $i = 1, \dots, m$ we denote by $\widetilde{\gamma}^i$ the (possibly self intersecting) Lipschitz curve which parametrizes on $\mathbb{S}^1$ the polygon in $\mathbb{R}^2$ with vertices $\beta_1^i, \beta_2^i, \dots, \beta_{N_i}^i$, in the order.

[1] With the convention $N_i + 1 = 1$.

5.2 Relaxation for general piecewise Lipschitz maps

We are now ready to prove our main result:

Theorem 5.2.1 (Relaxation for general piecewise Lipschitz maps). Let $u : \Omega \to \mathbb{R}^2$ be piecewise Lipschitz on Ω. Then

$$\overline{\mathcal{A}}_{BV}(u;\Omega) = \int_{\Omega\setminus\Sigma} |\mathcal{M}(\nabla u)|\; dx + \sum_{\ell=1}^{n} \int_{[a_\ell,b_\ell]\times I} |\partial_t X^{\rm aff}_{(\ell)} \wedge \partial_s X^{\rm aff}_{(\ell)}| dtds + \sum_{i=1}^{m} P(\widetilde{\gamma}^i). \quad (5.2.1)$$

Proof. Lower bound: Consider a sequence $(v_k) \subset C^1(\Omega;\mathbb{R}^2)$ converging to u strictly $BV(\Omega;\mathbb{R}^2)$. For any $\rho > 0$ small enough, we take a family of mutually disjoint balls $B_\rho(p_i) \subset \Omega$, $i = 1,\dots,m$. By Lemma 3.1.3, there exists a subsequence $(v_{k_h}) \subset (v_k)$ depending on ρ such that for $i = 1,\dots,m$

$$v_{k_h} \llcorner \partial B_\rho(p_i) \to u \llcorner \partial B_\rho(p_i) \quad \text{strictly } BV(\partial B_\rho(p_i);\mathbb{R}^2). \quad (5.2.2)$$

We may also assume that for $i = 1,\dots,m$

$$\liminf_{k\to+\infty} \int_{B_\rho(p_i)} |Jv_k|\; dx = \lim_{h\to+\infty} \int_{B_\rho(p_i)} |Jv_{k_h}|\; dx.$$

Then

$$\begin{aligned} \mathcal{A}(v_{k_h},\Omega) &= \mathcal{A}(v_{k_h},\Omega\setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) + \sum_{i=1}^m \mathcal{A}(v_{k_h};\overline{B}_\rho(p_i)) \\ &\geq \mathcal{A}(v_{k_h},\Omega\setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) + \sum_{i=1}^m \int_{B_\rho(p_i)} |Jv_{k_h}| dx. \end{aligned}$$

By Corollary 3.2.12, we get

$$\begin{aligned} &\liminf_{h\to+\infty} \mathcal{A}(v_{k_h},\Omega\setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) \\ &\geq \overline{\mathcal{A}}_{BV}(u,\Omega\setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) \\ &= \int_{\Omega\setminus\cup_{i=1}^m B_\rho(p_i)} |\mathcal{M}(\nabla u)| dx + \sum_{\ell=1}^n \int_{[a^\rho_\ell,b^\rho_\ell]\times I} |\partial_t X^{\rm aff}_{(\ell)} \wedge \partial_s X^{\rm aff}_{(\ell)}|\; dtds \\ &\longrightarrow \int_\Omega |\mathcal{M}(\nabla u)| dx + \sum_{\ell=1}^n \int_{[a_\ell,b_\ell]\times I} |\partial_t X^{\rm aff}_{(\ell)} \wedge \partial_s X^{\rm aff}_{(\ell)}| dtds \quad \text{as } \rho\to 0^+, \end{aligned}$$

where $(a^\rho_\ell), (b^\rho_\ell) \subset [a_\ell,b_\ell]$ are respectively a decreasing and increasing sequence of numbers satisfying $a^\rho_\ell \to a_\ell$ and $b^\rho_\ell \to b_\ell$ as $\rho\to 0^+$ and $\alpha_\ell([a^\rho_\ell,b^\rho_\ell]) = \alpha_\ell([a_\ell,b_\ell]) \setminus \cup_{i=1}^m B_\rho(p_i)$.

Let us recall that, by Lemma 4.1.6, $P(\tilde{\gamma}^i) = \overline{P}(\gamma^i)$, with γ^i as in Lemma 5.1.3. So, it remains to show that

$$\liminf_{\rho\to 0^+} \lim_{h\to+\infty} \int_{B_\rho(p_i)} |Jv_{k_h}|\; dx \geq \overline{P}(\gamma^i) \qquad \forall i = 1,\dots,m. \quad (5.2.3)$$

By definition (4.1.11), using (4.1.4) and (5.2.2), we readily conclude that

$$\lim_{h\to+\infty}\int_{B_\rho(p_i)} |Jv_{k_h}|\,dx \geq \overline{P}(\gamma^i_\rho),$$

where γ^i_ρ is defined in Lemma 5.1.3. Then, since γ^i_ρ converge to γ^i strictly $BV(\mathbb{S}^1;\mathbb{R}^2)$ as $\rho\to 0^+$, (5.2.3) follows, thanks to Lemma 5.1.3 and Corollary 4.1.8.

Upper bound: Fix $r>0$ small enough and consider mutually disjoint balls $B_r(p_i)\subset\Omega$, $i=1,\ldots,m$, such that, for every $\ell\in\{1,\ldots,n\}$, $J_\ell\cap\partial B_s(p_i)$, if nonempty, consists either of a single point, or of two points if $\alpha_\ell(a_\ell)=\alpha_\ell(b_\ell)=p_i$, for every $s\in(0,r]$.

Clearly, the difficulty of the proof is concentrated around the junction points p_i. The idea is to modify u on $\cup_{i=1}^m B_r(p_i)$ by constructing a new map u_r (see (5.2.7) and (5.2.19)), which coincides with u out of $\cup_{i=1}^m B_r(p_i)$ and converges to u strictly $BV(\Omega;\mathbb{R}^2)$ as r tends to 0^+. The map u_r will be again a piecewise Lipschitz map with the same set $\{p_i\}$ of junction points, but different jump set Σ_r, with $\Sigma_r\cap B_{r/2}(p_i)$ made of segments, i.e. u_r is of the form (4.2.4) in $B_{r/2}(p_i)$. The difficult point will be to provide that Σ_r is still a union of (pairwise disjoint up to the endpoints) C^2-curves $\widehat{\alpha}_\ell$, in particular that each one hits $\partial B_{r/2}(p_i)$ with vanishing second derivative. At the end, we will apply Theorem 4.2.4 to u_r in $\cup_{i=1}^m B_{r/2}(p_i)$ and Corollary 3.2.12 to u_r in $\Omega\setminus(\cup_{i=1}^m B_{r/2}(p_i))$, and conclude by lower semicontinuity of $\overline{\mathcal{A}}_{BV}(\cdot,\Omega)$.
We start by considering a smooth strictly increasing surjective function $\psi_r:[\frac{r}{2},+\infty)\to[0,+\infty)$ with [2]

$$\psi_r(\rho)=\rho\quad\forall\rho\geq r,\quad \psi_r(\rho)=\Big(\rho-\frac{r}{2}\Big)^3 \text{ in a right neighborhood of } \frac{r}{2},\quad |\psi_r'|\leq C \text{ in } \Big(\frac{r}{2},r\Big) \tag{5.2.4}$$

with $C>0$ independent of r. We define the radial map $\Phi_r:\mathbb{R}^2\setminus B_{\frac{r}{2}}(0)\to\mathbb{R}^2\setminus\{0\}$ as

$$\Phi_r(x)=\psi_r(|x|)\frac{x}{|x|},$$

whose inverse is $\Phi_r^{-1}(y)=f_r(|y|)\frac{y}{|y|}$, where $f_r:=\psi_r^{-1}$, and set

$$\widehat{u}_r(x):=u(p_i+\Phi_r(x-p_i))\qquad \text{for } x\in B_r(p_i)\setminus\overline{B}_{\frac{r}{2}}(p_i),\ i=1,\ldots,m. \tag{5.2.5}$$

The jump set of $\widehat{u}_r$ in $B_r(p_i)\setminus B_{r/2}(p_i)$ is parametrized by the curves

$$\widehat{\alpha}_\ell:=p_i+\Phi_r^{-1}(\alpha_\ell-p_i)\qquad \forall\ell=1,\ldots,n. \tag{5.2.6}$$

Notice carefully that $\widehat{\alpha}_\ell$ is parametrized on the same parameter interval of α_ℓ, but this is not an arc length parametrization for $\widehat{\alpha}_\ell$. Moreover, thanks to the regularity of Φ_r, the map

$$u_r:=\begin{cases} u & \text{in } \Omega\setminus(\cup_{i=1}^m B_r(p_i))\\ \widehat{u}_r & \text{in } B_r(p_i)\setminus B_{\frac{r}{2}}(p_i),\quad i=1,\ldots,m\end{cases} \tag{5.2.7}$$

[2] The exponent must be chosen greater than 2 in order to ensure (5.2.18).

has jump set Σ_r which is parametrized by the curves $\widehat{\alpha}_\ell$, whose supports $\widehat{J}_\ell$ are pairwise disjoint and in turn coincide with the ones of α_ℓ in $\Omega \setminus (\cup_{i=1}^m B_r(p_i))$.

Step 1: Let us first check that the length of $\widehat{\alpha}_\ell$ in $\cup_{i=1}^m (B_r(p_i) \setminus B_{r/2}(p_i))$ is controlled, more precisely, we will show that for each i and ℓ, the length of $\widehat{\alpha}_\ell$ in $B_r(p_i) \setminus B_{r/2}(p_i)$ goes to 0 as $r \to 0^+$. We suppose that $J_\ell \cap \partial B_s(p_i)$, for every $s \le r$, consists of a single point, because the argument adapts also if α_ℓ has two arcs exiting from p_i, simply by considering them separately. To this aim, fix i and ℓ and denote $\alpha_\ell = \alpha$, $J_\ell = J$. Without loss of generality, assume $p_i = 0$, $B_r(0) = B_r$, and suppose that $J \cap B_r$ is parametrized by arc length on $[0, R]$, with $\alpha(0) = 0$ and $\alpha(R) \in \partial B_r$, where $R(r) = R = \mathcal{H}^1(J \cap B_r)$. We can express the gradient of Φ_r^{-1} as follows:

$$\nabla \Phi_r^{-1}(y) = f_r'(|y|)\frac{y}{|y|} \otimes \frac{y}{|y|} + f_r(|y|)\nabla\left(\frac{y}{|y|}\right) = f_r'(|y|)\frac{y}{|y|} \otimes \frac{y}{|y|} + \frac{f_r(|y|)}{|y|}\Pi(y), \quad (5.2.8)$$

where

$$\Pi(y) := \mathrm{Id} - \frac{y \otimes y}{|y|^2},$$

and we used that

$$\nabla\left(\frac{y}{|y|}\right) = \frac{1}{|y|}\Pi(y). \quad (5.2.9)$$

From (5.2.6), we have $\dot{\widehat{\alpha}} = \nabla \Phi_r^{-1}(\alpha)\dot{\alpha}$, and using (5.2.8) and $|\dot{\alpha}| = 1$,

$$|\dot{\widehat{\alpha}}| \le f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}\left|\Pi(\alpha)\dot{\alpha}\right|. \quad (5.2.10)$$

Notice that if r is small, the function $t \mapsto |\alpha(t)| =: \sigma(t)$ is C^1 and invertible from $[0, R]$ to $[0, r]$. Moreover, $\sigma'(t) = \frac{\alpha(t)}{|\alpha(t)|} \cdot \dot{\alpha}(t) \to \frac{\dot{\alpha}(0)}{|\dot{\alpha}(0)|} \cdot \dot{\alpha}(0) = |\dot{\alpha}(0)| = 1$ as $t \to 0^+$. Let us integrate on $[0, R]$ the term $f_r'(|\alpha|)$: performing the change of variable $\sigma(t) = \rho$, we get

$$\int_0^R f_r'(|\alpha(t)|)dt = \int_0^R f_r'\left(\sigma(t)\right)dt = \int_0^r f_r'(\rho)\frac{d\rho}{\sigma'(\sigma^{-1}(\rho))} \le 2\int_0^r f_r'(\rho)d\rho,$$

where in the last inequality we used that, for small r, $\sigma'(\sigma^{-1}(\rho)) \ge \frac{1}{2}$ for every $\rho \in [0, r]$. Sending r to 0^+, we have that $\int_0^R f_r'(|\alpha(t)|)dt \to 0$ by integrability of f' near to the origin.

In order to estimate the second term on the right hand side of (5.2.10), we can use a Taylor expansion of α around 0, writing $\alpha(t) = vt + wt^2 + o(t^2)$, with $v = \dot{\alpha}(0)$, $w = \frac{\ddot{\alpha}(0)}{2}$, and $\lim_{t \to 0^+} o(t^p)/t^p = 0$. We have

$$\Pi(\alpha)\dot{\alpha} = \Pi(vt + wt^2 + o(t^2))(v + 2wt + o_2(t)) = \Pi(v + wt + o_1(t))(v + 2wt + o_2(t)),$$

where $o_1(t) = o(t^2)/t$ and $o_2(t) = o(t)$. Writing $v + 2wt + o_2(t) = v + wt + o_1(t) + wt + o_2(t) - o_1(t)$, we get

$$\Pi(\alpha)\dot{\alpha} = \Pi(v + wt + o_1(t))(v + wt + o_1(t)) + \Pi(v + wt + o_1(t))(wt + o_2(t) - o_1(t)).$$

The first term on the right hand side is 0 and the norm of the second term can be estimated from above by $|w|t + o(t)$. Now, by definition of arc length parameter, $R = \mathcal{H}^1(\mathrm{spt}\alpha \cap B_r(0)) \to 0$ as $r \to 0^+$. Moreover, by Taylor expansion, $|\alpha(t)| > \frac{t}{2}$ for t small enough. Therefore, since $f_r(0) = \frac{r}{2}$, for r small enough we have $\frac{f_r(|\alpha(t)|)}{|\alpha(t)|} \le \frac{2r}{t}$ on $[0, R]$. So, integrating on $[0, R]$ the second term on the right hand side of (5.2.10),

$$\int_0^R \frac{f_r(|\alpha(t)|)}{|\alpha(t)|} |\Pi(\alpha(t))\dot{\alpha}(t)|\, dt \le \int_0^R \frac{2r}{t}(|w|t + o(t))dt \to 0 \qquad \text{as } r \to 0^+.$$

Step 2: Let $\widehat{J} = \widehat{J}_l$ be the support of $\widehat{\alpha}$; let us show that there is a parametrization of $\widehat{J} \cap (B_r \setminus B_{r/2})$ on an interval $[0, L]$, which is of class C^2 up to 0 and with vanishing second derivative at 0. Indeed, set $L := \mathcal{H}^1(\widehat{J} \cap (B_r \setminus B_{r/2}))$ and consider the arc-length parameter $s \in [0, L]$ given by

$$s(t) = \int_0^t |V_r(\alpha(\tau))| d\tau,$$

where

$$V_r(\alpha) := \nabla\Phi_r^{-1}(\alpha)\dot{\alpha}.$$

We compute

$$\frac{d^2}{ds^2}\widehat{\alpha}(t) = \frac{d}{ds}\left(\frac{V_r(\alpha)}{|V_r(\alpha)|}\right) = \Pi(V_r(\alpha))\left(\frac{\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha}) + \nabla\Phi_r^{-1}(\alpha)\ddot{\alpha}}{|V_r(\alpha)|^2}\right). \tag{5.2.11}$$

Here and in what follows, α is evaluated at $t = t(s)$ and $\dot{\alpha}$ and $\ddot{\alpha}$ denote the first and second derivative of α with respect to t. The operation : between a tensor $T = (T_{ijk}) \in \mathbb{R}^{2\times2\times2}$ and a matrix $M = (M_{ij}) \in \mathbb{R}^{2\times2}$ is defined as the vector $T : M \in \mathbb{R}^2$ with components $(T : M)_k = T_{ijk}M_{ij}$ for $k = 1, 2$.
We get

$$\begin{aligned}
\left|\frac{d^2}{ds^2}\widehat{\alpha}(t)\right| &\le \left|\Pi(V_r(\alpha))\left(\frac{\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2}\right)\right| + \frac{|\nabla\Phi_r^{-1}(\alpha)\ddot{\alpha}|}{|V_r(\alpha)|^2} \\
&\le \left|\Pi(V_r(\alpha))\left(\frac{\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2}\right)\right| + C\frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}.
\end{aligned} \tag{5.2.12}$$

where we have used (5.2.8) and that $\ddot{\alpha}$ is bounded.
The Hessian of Φ_r^{-1} can be computed as

$$\begin{aligned}
\nabla^2\Phi_r^{-1}(y) =& f_r''(|y|)\frac{y}{|y|} \otimes \frac{y}{|y|} \otimes \frac{y}{|y|} + f_r'(|y|)\nabla\left(\frac{y}{|y|} \otimes \frac{y}{|y|}\right) + \\
&+ f_r'(|y|)\frac{y}{|y|} \otimes \nabla\left(\frac{y}{|y|}\right) + f_r(|y|)\nabla^2\left(\frac{y}{|y|}\right) \\
=& f_r''(|y|)\frac{y}{|y|} \otimes \frac{y}{|y|} \otimes \frac{y}{|y|} + f_r'(|y|)\nabla\left(\frac{y}{|y|}\right) \otimes \frac{y}{|y|} + \\
&+ 2f_r'(|y|)\frac{y}{|y|} \otimes \nabla\left(\frac{y}{|y|}\right) + f_r(|y|)\nabla\left(\nabla\left(\frac{y}{|y|}\right)\right).
\end{aligned}$$

Then, by (5.2.9), we have

$$\nabla^2\Phi_r^{-1}(\alpha) = f_r''(|\alpha|)\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|} + \left(\frac{f_r'(|\alpha|)}{|\alpha|} - 2\frac{f_r(|\alpha|)}{|\alpha|^2}\right)\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|} + \left(2\frac{f_r'(|\alpha|)}{|\alpha|} - \frac{f_r(|\alpha|)}{|\alpha|^2}\right)\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha).$$

So, for $k = 1, 2$, we have

$$\begin{aligned}
&\left(\nabla^2\Phi_r^{-1}(\alpha) : (\dot\alpha\otimes\dot\alpha)\right)_k \\
=& f_r''(|\alpha|)\left(\left(\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\right) : (\dot\alpha\otimes\dot\alpha)\right)_k \\
&+\left(\frac{f_r'(|\alpha|)}{|\alpha|} - 2\frac{f_r(|\alpha|)}{|\alpha|^2}\right)\left(\left(\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|}\right) : (\dot\alpha\otimes\dot\alpha)\right)_k \qquad (5.2.13)\\
&+\left(2\frac{f_r'(|\alpha|)}{|\alpha|} - \frac{f_r(|\alpha|)}{|\alpha|^2}\right)\left(\left(\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha)\right) : (\dot\alpha\otimes\dot\alpha)\right)_k. \qquad (5.2.14)
\end{aligned}$$

Notice that, since $\Pi(\alpha)$ is symmetric,

$$\Pi(\alpha)_{ij}\alpha_j = 0, \quad \Pi(\alpha)_{ij}\alpha_i = 0, \qquad (5.2.15)$$

where we sum on repeated indeces. So, using (5.2.15) and that, from Taylor expansion, $\dot\alpha(t) = v + 2wt + o(t) = \frac{\alpha(t)}{t} + wt + o(t)$, we have

$$\begin{aligned}
\left(\left(\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|}\right) : (\dot\alpha\otimes\dot\alpha)\right)_k &= \Pi(\alpha)_{ij}\dot\alpha_i\dot\alpha_j\frac{\alpha_k}{|\alpha|} = \Pi(\alpha)_{ij}\left(\frac{\alpha_i}{t} + w_it + o(t)\right)\dot\alpha_j\frac{\alpha_k}{|\alpha|} = \\
&= \Pi(\alpha)_{ij}\left(w_it + o(t)\right)\dot\alpha_j\frac{\alpha_k}{|\alpha|};
\end{aligned}$$

$$\begin{aligned}
\left(\left(\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha)\right) : (\dot\alpha\otimes\dot\alpha)\right)_k &= \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\dot\alpha_i\dot\alpha_j = \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\left(\frac{\alpha_j}{t} + w_jt + o(t)\right)\dot\alpha_i \\
&= \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\left(w_jt + o(t)\right)\dot\alpha_i.
\end{aligned}$$

So, the norm of the sum of (5.2.13) and (5.2.14) can be easily estimated by

$$3\left(\frac{f_r'(|\alpha|)}{|\alpha|} + \frac{f_r(|\alpha|)}{|\alpha|^2}\right)(|w|t + o(t)) \le C\left(f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}\right),$$

where we used that, for t small, $|\alpha(t)| \ge \frac{t}{2}$.
Therefore, (5.2.12) becomes

$$\left|\frac{d^2}{ds^2}\widehat\alpha(t)\right| \le \left|f_r''(|\alpha|)\right|\left|\Pi(V_r(\alpha))\left(\frac{\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|} : (\dot\alpha\otimes\dot\alpha)}{|V_r(\alpha)|^2}\right)\right| + C\frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}. \qquad (5.2.16)$$

Now we treat the first term of the right hand side of (5.2.16). For $j = 1, 2$, by definition of $V_r(\alpha)$, using Taylor expansion and (5.2.15), we have

$$\begin{aligned}(V_r)_j(\alpha) &= f_r'(|\alpha|)\frac{\alpha_i\alpha_j}{|\alpha|^2}\dot\alpha_i + f_r(|\alpha|)\Pi(\alpha)_{ij}\dot\alpha_i \\ &= f_r'(|\alpha|)\frac{\alpha_i\alpha_j}{|\alpha|^2}\left(\frac{\alpha_i}{t} + w_it + o(t)\right) + f_r(|\alpha|)\Pi(\alpha)_{ij}\left(\frac{\alpha_i}{t} + w_it + o(t)\right) \\ &= f_r'(|\alpha|)\left(\frac{\alpha_j}{t} + \frac{\alpha_i\alpha_j}{|\alpha|^2}w_it + o(t)\right) + f_r(|\alpha|)\Pi(\alpha)_{ij}\left(w_it + o(t)\right) \\ &= f_r'(|\alpha|)\left(\frac{\alpha_j}{t} + o(t)\right) + f_r(|\alpha|)O_j(t),\end{aligned} \tag{5.2.17}$$

where in the last equality we used that $\alpha_iw_i = o(t)$, since $v_iw_i = 0$ because $|\dot\alpha| = 1$, and we setted $O_j(t) := \Pi(\alpha)_{ij}(w_it + o(t))$, meaning that $\lim_{t\to0^+}|O_j(t)|/t < +\infty$. Then, we get

$$\alpha = t\left(\frac{V_r(\alpha) - O(t)}{f_r'(|\alpha|)} + o(t)\right).$$

So,

$$\begin{aligned}\Pi(V_r(\alpha))\frac{\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|} : (\dot\alpha\otimes\dot\alpha)}{|V_r(\alpha)|^2} &= \frac{\alpha_i\alpha_j}{|\alpha|^2}\dot\alpha_i\dot\alpha_j\Pi(V_r(\alpha))\frac{\frac{\alpha}{|\alpha|}}{|V_r(\alpha)|^2} \\ &= \frac{\alpha_i\alpha_j}{|\alpha|^2}\dot\alpha_i\dot\alpha_j\frac{t}{|\alpha|}\Pi(V_r(\alpha))\frac{\left(\frac{V_r(\alpha)-O(t)}{f_r'(|\alpha|)} + o(t)\right)}{|V_r(\alpha)|^2} \\ &= \frac{\alpha_i\alpha_j}{|\alpha|^2}\dot\alpha_i\dot\alpha_j\frac{t}{|\alpha|}\Pi(V_r(\alpha))\frac{\left(\frac{O(t)}{f_r'(|\alpha|)} + o(t)\right)}{|V_r(\alpha)|^2},\end{aligned}$$

where we used that $\Pi(V_r(\alpha))V_r(\alpha) = 0$. For t small, we get

$$\left|\Pi(V_r(\alpha))\frac{\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|} : (\dot\alpha\otimes\dot\alpha)}{|V_r(\alpha)|^2}\right| \le 2\frac{\frac{O(t)}{f_r'(|\alpha|)} + o(t)}{|V_r(\alpha)|^2}.$$

Finally, from (5.2.16), we obtain

$$\left|\frac{d^2}{ds^2}\widehat\alpha(t)\right| \le \left|f_r''(|\alpha|)\right|\frac{\frac{O(t)}{f_r'(|\alpha|)} + o(t)}{|V_r(\alpha)|^2} + C\frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}.$$

From the definition of f_r, we have that $f_r(|\alpha(t)|) = \frac{r}{2} + t^{\frac13} + o(t^{\frac13})$ for t near to 0. So, by (5.2.17), we have $|V_r(\alpha(t))| \ge Cf_r'(|\alpha(t)|) = Ct^{-\frac23} + o(t^{-\frac23})$. Then, since $|f_r''(|\alpha(t)|)| = Ct^{-\frac53} + o(t^{-\frac53})$, a straightforward check shows that

$$\frac{d^2}{ds^2}\widehat\alpha(t) \to 0 \quad \text{as } t \to 0^+. \tag{5.2.18}$$

We conclude that the curve $\widehat\alpha$ is C^2 up to 0 with vanishing second derivative, and hence can be extended on the interval $(-\frac{r}{2}, 0)$ to a (not relabeled) curve $\widehat\alpha$ whose support is a straight segment connecting $\widehat\alpha(0)$ to 0 (namely a radius of $B_{r/2}(0)$). Going back to the curves $\widehat{\alpha}_\ell$, we have just proved that we can extend them in $B_{r/2}(p_i)$ with C^2-regularity using a segment along a radius, reaching p_i. In particular, the new supports of $\widehat{\alpha}_\ell$'s form a N^i-junction point around p_i in $B_{r/2}(p_i)$, whose circular sectors $\widehat{C}^i_j$ ($j = 1, \dots, N_i$) have amplitudes $\theta^1_i, \dots, \theta^{N_i}_i$ (according to Lemma 5.1.3). Up to a reparametrization by arc-length of $\widehat{\alpha}_\ell$, we will suppose that $\widehat{\alpha}_\ell : [\widehat{a}_\ell, \widehat{b}_\ell] \to \mathbb{R}^2$ have always derivative of modulus 1.

Step 3: We are ready to extend the map u_r in $B_{r/2}(p_i)$. We eventually observe that, from (5.2.7), $u_r(x) = \gamma^i\left(\frac{2}{r}(x - p_i)\right)$ on $\partial B_{r/2}(p_i)$ (see Lemma 5.1.3), and hence it is constant on any arc with angular coordinate in (a^{j-1}_i, a^j_i). Hence we define

$$u_r(x) := \gamma^i\left(\frac{x - p_i}{|x - p_i|}\right) \qquad x \in B_{\frac{r}{2}}(p_i). \tag{5.2.19}$$

Now, u_r satisfies the hypotheses of Corollary 3.2.12 in $\Omega_r := \Omega \setminus (\cup_{i=1}^m \overline{B}_{r/4}(p_i))$, where all the curves $\widehat{\alpha}_j$ satisfy hypotheses (H3), and they run on a straight segment (along a radius of $B_{r/2}(p_i)$) inside $B_{r/2}(p_i) \setminus B_{r/4}(p_i)$. Then we introduce a sequence of Lipschitz maps $\widetilde{v}_k : \Omega_r \to \mathbb{R}^2$ which are defined as in (3.2.42), where, we recall, $\varepsilon = \frac{1}{k}$, with u_r in place of u and $\Lambda = \mathrm{id}$; in particular, for k large enough, the trace of $\widetilde{v}_k$ on $\partial B_{r/3}(p_i)$ is a piecewise affine map coinciding with γ_k in (4.2.8), with β_i in place of α_i. Thus, if we introduce also the sequence of Lipschitz maps $\widehat{v}_k : B_{r/2}(p_i) \to \mathbb{R}^2$ as in (4.2.12) (with B_r replaced by $B_{r/2}(p_i)$) we see that $\widetilde{v}_k = \widehat{v}_k$ on $\partial B_{r/3}(p_i)$. Therefore we define

$$v^r_k := \begin{cases} \widetilde{v}_k & \text{in } \Omega \setminus (\cup_{i=1}^m B_{r/3}(p_i)) \\ \widehat{v}_k & \text{in } \cup_{i=1}^m B_{r/3}(p_i), \end{cases} \tag{5.2.20}$$

and we readily see that $v^r_k \to u_r$ strictly $BV(\Omega; \mathbb{R}^2)$.
Since the supports of α_ℓ and $\widehat{\alpha}_\ell$ coincide out of $\cup_i B_r(p_i)$, there exist $\widehat{a}^r_\ell, \widehat{b}^r_\ell \in [\widehat{a}_\ell, \widehat{b}_\ell]$ and $a^r_\ell, b^r_\ell \in [a_\ell, b_\ell]$, with $\widehat{a}^r_\ell < \widehat{b}^r_\ell$ and $a^r_\ell < b^r_\ell$, such that

$$\widehat{\alpha}_\ell([\widehat{a}^r_\ell, \widehat{b}^r_\ell]) = \alpha_\ell([a^r_\ell, b^r_\ell]), \quad \widehat{\alpha}_\ell(\widehat{a}^r_\ell) = \alpha_\ell(a^r_\ell), \quad \widehat{\alpha}_\ell(\widehat{b}^r_\ell) = \alpha_\ell(b^r_\ell).$$

In particular, $\widehat{b}^r_\ell - \widehat{a}^r_\ell = b^r_\ell - a^r_\ell$, so up to a translation of the parameter interval of $[\widehat{a}_\ell, \widehat{b}_\ell]$, we can suppose $\widehat{a}^r_\ell = a^r_\ell$ and $\widehat{b}^r_\ell = b^r_\ell$. Clearly, $a^r_\ell \to a_\ell$ non increasingly and $b^r_\ell \to b_\ell$ non decreasingly as $r \to 0^+$.

In view of Corollary 3.2.12 and Theorem 4.2.4 we conclude

$$
\begin{aligned}
\overline{\mathcal{A}}_{BV}(u_r,\Omega) \le & \lim_{k\to+\infty} \mathcal{A}(v_k^r,\Omega) \\
= & \int_{\Omega\setminus(\cup_{i=1}^m B_r(p_i))} |\mathcal{M}(\nabla u)|\,dx + \sum_{\ell=1}^n \int_{[\widehat{a}_\ell,\widehat{b}_\ell]\times I} |\partial_t X_{\ell,r}^{\mathrm{aff}} \wedge \partial_s X_{\ell,r}^{\mathrm{aff}}|\,dtds \\
& + \int_{\cup_{i=1}^m (B_r(p_i)\setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)|\,dx + m\frac{\pi r^2}{9} + \sum_{i=1}^m \overline{P}(\gamma^i) \\
= & \int_{\Omega\setminus(\cup_{i=1}^m B_r(p_i))} |\mathcal{M}(\nabla u)|\,dx + \sum_{\ell=1}^n \int_{[a_\ell^r,b_\ell^r]\times I} |\partial_t X_\ell^{\mathrm{aff}} \wedge \partial_s X_\ell^{\mathrm{aff}}|\,dtds \\
& + \sum_{i=1}^m \overline{P}(\gamma^i) + \int_{\cup_{i=1}^m (B_r(p_i)\setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)|\,dx \\
& + \sum_{\ell=1}^n \int_{([\widehat{a}_\ell^{r/3},a_\ell^r]\cup[b_\ell^r,\widehat{b}_\ell^{r/3}])\times I} |\partial_t X_{\ell,r}^{\mathrm{aff}} \wedge \partial_s X_{\ell,r}^{\mathrm{aff}}|\,dtds \\
& + \frac{r}{3}\sum_{i=1}^m\sum_{j=1}^{N_i} |\beta_i^j - \beta_i^{j+1}| + m\frac{\pi r^2}{9},
\end{aligned}
\tag{5.2.21}
$$

where for all $\ell = 1,\dots,n$ we have $\widehat{a}_\ell \le \widehat{a}_\ell^{r/3} \le a_\ell^r < b_\ell^r \le \widehat{b}_\ell^{r/3} \le \widehat{b}_\ell$, where $\widehat{\alpha}_\ell(\widehat{a}_\ell^{\frac{r}{3}}) \in \partial B_{r/3}(p_i)$, $\widehat{\alpha}_\ell(\widehat{b}_\ell^{\frac{r}{3}}) \in \partial B_{r/3}(p_j)$ for some $i,j \in \{1,\dots,m\}$, unless one of them belongs to $\partial\Omega$, and where $X_{\ell,r}^{\mathrm{aff}}$ is defined as X_ℓ^{aff} with u_r replacing u.

Now, since by (5.2.4) $|\psi_r'| \le C$, u_r is still a piecewise Lipschitz map on Ω, hence, by *Step 1*, the last four terms in (5.2.21) are negligible as $r \to 0^+$. We then conclude, provided that $u_r \to u$ strictly $BV(\Omega;\mathbb{R}^2)$, that

$$
\begin{aligned}
\overline{\mathcal{A}}_{BV}(u,\Omega) \le & \liminf_{r\to 0^+} \overline{\mathcal{A}}_{BV}(u_r,\Omega) \\
\le & \int_\Omega |\mathcal{M}(\nabla u)|\,dx + \sum_{\ell=1}^n \int_{[a_\ell,b_\ell]\times I} |\partial_t X_\ell^{\mathrm{aff}} \wedge \partial_s X_\ell^{\mathrm{aff}}|\,dtds + \sum_{i=1}^m \overline{P}(\gamma^i),
\end{aligned}
$$

that is the thesis. In order to check that $u_r \to u$ strictly $BV(\Omega;\mathbb{R}^2)$ it is sufficient to observe that $u = u_r$ outside $\cup_{i=1}^m B_r(p_i)$ and that

$$
\begin{aligned}
& \limsup_{r\to 0^+} |Du_r|(\cup_{i=1}^m B_r(p_i)) \\
\le & \limsup_{r\to 0^+}\limsup_{k\to+\infty} \int_{\cup_{i=1}^m B_r(p_i)} \sqrt{1+|\nabla v_k^r|^2}\,dx \\
\le & \limsup_{r\to 0^+}\lim_{k\to+\infty} \mathcal{A}(v_k^r;\cup_{i=1}^m B_r(p_i)) \\
= & \limsup_{r\to 0^+}\Big(\int_{\cup_{i=1}^m (B_r(p_i)\setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)|\,dx + m\frac{\pi r^2}{9} \\
& + \sum_{\ell=1}^n \int_{([\widehat{a}_\ell^{r/3},\widehat{a}_\ell^r]\cup[\widehat{b}_\ell^r,\widehat{b}_\ell^{r/3}])\times I} |\partial_t X_{\ell,r}^{\mathrm{aff}} \wedge \partial_s X_{\ell,r}^{\mathrm{aff}}|\,dtds + \frac{r}{3}\sum_{i=1}^m\sum_{j=1}^{N_i} |\alpha_j^i - \alpha_{j+1}^i|\Big) = 0.
\end{aligned}
$$

The proof is complete. □

Chapter 6

Open problems

In this final chapter we briefly collect some open questions and further directions to explore. We start with the BV-relaxed area and the problem of proving its subaddivity, that we expect to be true at least in dimension 2 and codimension 2. Next, we present some problems related to the L^1-relaxed area, trying to formulate and motivate some conjectures.

6.1 On the subaddivity of $\overline{\mathcal{A}}_{BV}(u;\cdot)$

Besides a satisfying characterization of $\mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,;\Omega))$, the main question still left open from our analysis is whether, for $u \in BV(\Omega;\mathbb{R}^2)$, the set function $\overline{\mathcal{A}}_{BV}(u;\cdot)$ is subadditive, and if it gives rise to a measure. The relevant examples in the previous chapters and the existence of a unique minimal lifting current for $u \in \mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,;\Omega))$ give hope to a positive answer.
A possible strategy could be to use a technique from the context of Γ-convergence, the so called *fundamental estimate* (see [19, Chapter 18]), in order to exploit the slicing properties of strict convergence. More in details, assume that $u \in BV(\Omega;\mathbb{R}^2)$ is such that $\overline{\mathcal{A}}_{BV}(u;\Omega) < +\infty$ and let $A', A'', B \subset \Omega$ be open sets, with $A' \subset\subset A''$. Set $S = (A'' \setminus A') \cap B$ and fix recovery sequences (v_k) for $\overline{\mathcal{A}}_{BV}(u;A'')$ and (u_k) for $\overline{\mathcal{A}}_{BV}(u;B)$. We would like to prove an estimate like this: for every $\varepsilon > 0$, there exist $M = M(\varepsilon, A', A'', B) > 0$ and a cut off function φ_k between A' and A'', such that

$$
\begin{aligned}
&\mathcal{A}\left(\varphi_k v_k + (1-\varphi_k)u_k; A' \cup B\right) \\
&\le (1+\varepsilon)\left(\mathcal{A}(v_k;A'') + \mathcal{A}(u_k,B)\right) \\
&\quad + \varepsilon\left(\|u_k\|_{L^1(S;\mathbb{R}^2)} + \|v_k\|_{L^1(S;\mathbb{R}^2)} + \int_S |\nabla u_k| dx + \int_S |\nabla v_k| dx + 1\right) \\
&\quad + M\left(\|u_k - v_k\|_{L^1(S;\mathbb{R}^2)} + \left|\int_S |\nabla v_k| dx - \int_S |\nabla u_k| dx\right|\right).
\end{aligned}
$$

Notice that as $k \to +\infty$, $\varepsilon \to 0^+$, and $A' \nearrow A''$, this inequality would imply

$$\overline{\mathcal{A}}_{BV}(u;A'' \cup B) \le \overline{\mathcal{A}}_{BV}(u;A'') + \overline{\mathcal{A}}_{BV}(u;B).$$

However, it seems hard to control the contribution of the Jacobian determinant of the interpolation map $\varphi_k u_k + (1-\varphi_k)v_k$ on the strip S. More specifically, it is not restrictive to assume that $A' \cap B$, S and $B \setminus A''$ are disjoint, pairwise adjacent rectangles. Let $S = [-\delta, \delta] \times [-h, h]$ and set $\varphi(t) := 0$ if $t < -\delta$, $\varphi(t) := \frac{t-\delta}{2\delta}$, $\varphi(t) := 1$ if $t > \delta$. Define $w_k(t,s) = \varphi(t)u_k(t,s) + (1-\varphi(t))v_k(t,s)$, then $w_k \to u$ strictly $BV(A' \cup B; \mathbb{R}^2)$. For simplicity, let us consider just TVJ: if we compute the expression of Jw_k, we end up with

$$\begin{aligned} Jw_k =& \varphi'(u_k^1 - v_k^1)[\varphi\partial_s u_k^2 + (1-\varphi)\partial_s v_k^2] + \varphi'(u_k^2 - v_k^2)[\varphi\partial_s u_k^1 + (1-\varphi)\partial_s v_k^1] \\ &+ \varphi^2 Ju_k + (1-\varphi)^2 Jv_k + \varphi(1-\varphi)[\partial_t u_k^1 \partial_s v_k^2 + \partial_t v_k^1 \partial_s u_k^2 - \partial_t u_k^2 \partial_s v_k^1 - \partial_t u_k^1 \partial_t v_k^2] \end{aligned}$$

and most of the terms are difficult to treat under the only assumption of strict convergence. However, under the further assumption that $u \in W^{1,1}(\Omega; \mathbb{R}^2)$, we can define in a slightly different way the sequence (w_k) and the situation simplifies a lot. Indeed, we can assume that $u \llcorner \{t = \pm\delta\}$ are continuous and $u_k \llcorner \{t = \pm\delta\}, v_k \llcorner \{t = \pm\delta\} \to u \llcorner \{t = \pm\delta\}$ strictly BV. Set $v \llcorner \{t = \pm\delta\} := v^{\pm\delta}$. Define $w_k(t,s) = \varphi(t)v_k^{\delta}(s) + (1-\varphi(t))u_k^{-\delta}(s)$, then

$$Jw_k = \varphi'(u_k^{-\delta,1} - v_k^{\delta,1})[\varphi\partial_s u_k^{-\delta,2} + (1-\varphi)\partial_s v_k^{\delta,2}] + \varphi'(u_k^{-\delta,2} - v_k^{\delta,2})[\varphi\partial_s u_k^{-\delta,1} + (1-\varphi)\partial_s v_k^{\delta,1}].$$

Unfortunately, for any $\delta = \delta_k \to 0$, the properties of strict convergence on the slices $\{t = \pm\delta_k\}$ are not enough to control $\int_S |Jw_k|$ with an $o(\delta_k)$. The core issue is to figure out how to use at the level of slices that $u \in \mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,; \Omega))$. Another issue would be to remove the assumption $u \in W^{1,1}$: we do not have directly uniform convergence on slices, but only at the level of reparametrizations, so the question is how to glue the reparametrizations of u_k and v_k on slices with the "true" sequences u_k and v_k. For the moment, we have no clue about how to proceed.

Another possibility, unless it does not lead to the same issues, is to consider a sort of *countably subadditive envelope* of $\overline{\mathcal{A}}_{BV}(u;\cdot)$, namely, the set function defined by

$$\overline{\overline{\mathcal{A}}}_{BV}(u; A) := \inf\left\{\sum_{i=1}^{\infty} \overline{\mathcal{A}}_{BV}(u; A_i);\ A_i \text{ open}, A = \bigcup_{i=1}^{\infty} A_i\right\} \qquad \forall A \subset \Omega \text{ open}. \tag{6.1.1}$$

The idea of this "double relaxation" goes back to the groundbreaking lecture by De Giorgi in [20], where he defines it for the L^1-relaxed area, in order to replace it with a measure. Indeed, the set function defined in (6.1.1) is clearly a measure, and so the goal is to prove that $\overline{\mathcal{A}}_{BV}(u; A) = \overline{\overline{\mathcal{A}}}_{BV}(u; A)$ for every open set $A \subset \Omega$.

Of course, the most ambitious strategy remains to prove directly an integral representation formula for $\overline{\mathcal{A}}_{BV}(u; \Omega)$ for a generic $u \in \mathrm{Dom}(\overline{\mathcal{A}}_{BV}(\cdot\,; \Omega))$. In order to apply what we have obtained so far, one can begin with proving some kind of density result in BV for the class of piecewise Lipschitz maps of Chapter 5 with respect to the strict convergence (we are thinking in the direction of [34], for instance). If one were able to pass to the limit in (5.2.1) and ends up again with an integral formula, then this should provide an upper bound for $\overline{\mathcal{A}}_{BV}(u; \Omega)$, that one could conjecture to be optimal.

Concerning the case of higher dimension and codimension, we have less explicit examples and so, we are not able to guess whether the BV-relaxed area could be subadditive. We have also less arguments in favour, since in higher dimension we loose the inheritance of strict convergence on 2-dimensional slices, while in higher codimension we do not have uniqueness of the minimal lifting current (see [40]).

After all, we can say that the idea of studying the functional $\overline{\mathcal{A}}_{BV}$ significantly simplified the analysis and enabled us to compute it for relevant classes of singular maps, but, on the other hand, the story is far from being to an end, and many efforts are still required to completely understand it.

6.2 Some extension to higher dimension and codimension

In Theorem 4.3.15 of Chapter 4 we computed the explicit expression of $\overline{\mathcal{A}}_{BV}$ for homogeneous maps valued in $\mathbb{R}^2$. We believe that a similar result holds true also for homogeneous maps valued in $\mathbb{R}^m$. More explicitely, if $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^m)$ and u is its homogeneous extension on B_ℓ, then we conjecture that

$$\overline{\mathcal{A}}_{BV}(u;B_\ell) = \int_{B_\ell} \sqrt{1+|\nabla u|^2}dx + |D^s u|(B_\ell) + \overline{P}_m(\gamma),$$

where $\overline{P}_m(\gamma)$ is the relaxation of the (singular) Plateau problem in $\mathbb{R}^m$ defined as

$$P_m(\varphi) = \inf\left\{\int_{B_1} |\partial_{x_1} v \wedge \partial_{x_2} v| dx; v \in \mathrm{Lip}(B_1;\mathbb{R}^m) : v_{|\partial B_1} = \varphi\right\}$$

for $\varphi \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^m)$. Indeed, $P_m(\cdot)$ should have the same fundamental features as $P(\cdot)$, namely the invariance by rescaling and the continuity property with respect to the strict convergence. Moreover, one can define also in this case the "completed" curve $\widetilde{\gamma}$ associated to γ and should be able to prove that $\overline{P}_m(\gamma) = P_m(\widetilde{\gamma})$.
Another possible extension can be consider for Theorem 2.3.6 in the case of maps $u \in W^{1,1}(\Omega;\mathbb{S}^1)$, where Ω is an open bounded set of $\mathbb{R}^n$, $n \geq 3$. Indeed, for maps with finite relaxed energy, we expect the singularities to be detected by the distributional Jacobian determinant, that lives in a set of codimension two. These issues are contained in some work in progress.

6.3 On the L^1-relaxed area

Although the focus of this thesis is the BV-relaxed area, it is worth to briefly mention some challenging problems concerning the L^1-relaxed area, on which we started to work. First, we recall that one of the main motivations to study the functional $\overline{\mathcal{A}}_{L^1}(\cdot\,;\Omega)$ is the approach to the Plateau problem in codimension 2. A possible formulation, with Dirichlet boundary conditions, can be the following: Let $\Omega \subset \mathbb{R}^2$ an open bounded set with C^1-boundary and $\varphi \in L^1(\Omega;\mathbb{R}^2)$, then consider (compare [28])

$$\inf\left\{\overline{\mathcal{A}}_{L^1}(u;\Omega); u \in BV(\Omega;\mathbb{R}^2) : u = \varphi \text{ on } \partial\Omega\right\}. \tag{6.3.1}$$

Of course, the analysis of the L^1-relaxed area is preliminary to address the problem (6.3.1).

6.3.1 Perturbated vortex

Let $\varphi \in C_c^\infty(B_\ell \setminus \{(0,0)\})$ and let $u : B_\ell :\to \mathbb{S}^1$ be a vortex with perturbation φ, i.e. in complex coordinates $u(\rho,\theta) = e^{i(\theta+\varphi(\rho,\theta))}$ for $\rho \in (0,\ell]$, $\theta \in [0,2\pi)$. Following some ideas in [5], we conjecture that the singular contribution of $\overline{\mathcal{A}}_{L^1}(u; B_\ell)$ is the result of an area-minimizing problem among all catenoids having a curve as a constraint, and among all curves connecting the origin to ∂B_ℓ. The choice of a minimizing path must highly depend on φ. Moreover, its existence and regularity properties are not clear, in general. Typically, the lack of symmetry of this problem should represent a relevant issue.

6.3.2 Double vortex

Let $u : B_\ell \to \mathbb{S}^1$ be the double vortex map, i.e. a vortex with degree and multiplicity equal to 2. In complex coordinates, $u(\rho, \theta) = e^{2i\theta}$ for $\rho \in (0, \ell]$, $\theta \in [0, 2\pi)$. We conjecture that the singular contribution of $\overline{\mathcal{A}}_{L^1}(u; B_\ell)$ is the solution of a non-standard Plateau problem, whose minimal profile looks like a double (half) catenoid departing from the circular hole in the graph upon the origin and attaching to the boundary of the cartesian domain (compare [6]). The bigger catenoid has a constrained segment connecting the origin to the boundary of the domain, and partial free boundary, while the smaller one seems to be a standard catenoid, which should coincide with the bigger one on part of the free boundary. More explicitely, let[1] $R_{2\ell} = (0, 2\ell) \times (-1, 1)$ and consider

$$\mathcal{H} = \{h : [0, 2\ell] \to [-1, 1] \quad \text{convex, } h(0) = h(2\ell) = 1\},$$
$$\mathcal{K} = \{k : [0, 2\ell] \to [-1, 1] \quad \text{concave, } k(0) = k(2\ell) = -1\}.$$

Notice that $\mathcal{H}$ and $\mathcal{K}$ could contain discontinuous maps: for instance the map $h = -1$ on $(0, 2\ell)$, with $h = 1$ at 0 and 2ℓ, belongs to $\mathcal{H}$. For a map $v : [0, 2\ell] \to \mathbb{R}$, denote by $UG_v = \{(x, y) \in R_{2\ell} : y > v(x)\}$ and $SG_v = \{(x, y) \in R_{2\ell} : y < v(x)\}$. Define the spaces

$$\mathcal{F}_h = \{f \in BV(R_{2\ell}) : \quad f = 0 \text{ on } UG_h\},$$
$$\mathcal{G}_{h,k} = \{g \in BV(R_{2\ell}) : \quad f = 0 \text{ on } UG_h \cup SG_k\}.$$

Now we want to minimize the functional

$$W(f, g, h, k) = \overline{\mathcal{A}}_{L^1}(f, R_{2\ell}) + \overline{\mathcal{A}}_{L^1}(g, R_{2\ell}) + \int_{\partial R_{2\ell}} |f - \varphi| d\mathcal{H}^1 + \int_{\partial R_{2\ell}} |g - \varphi| d\mathcal{H}^1 - |UG_h| - |SG_k|$$

where $\varphi(x, y) = \sqrt{1 - y^2}$ for $(x, y) \in R_{2\ell}$, among all functions $f \in \mathcal{F}_h$, $g \in \mathcal{G}_{h,k}$, $h \in \mathcal{H}$, $k \in \mathcal{K}$, with the condition $h \geq k$ in $[0, 2\ell]$. Then we conjecture that

$$\overline{\mathcal{A}}_{L^1}(u; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u|^2} dx + \inf_{f,g,h,k} W. \tag{6.3.2}$$

For large values of ℓ, this double (half) catenoid must degenerate in four half unit disks, recovering the result in Theorem 2.2.3, which is compatible with the vortex case.
The proof of the lower bound in (6.3.2) is probably very complicated, but at least we can exploit the symmetries of the problem.
Moreover, we believe that the same can be conjectured for a vortex with generic degree d, by constructing a d-ple catenoid, with the small inner catenoid covered $(d - 1)$ times.

[1] We are doubling the length of the radius and cutting the surface in half, so that the area does not change (see [6]).

6.3.3 Multipole I

Let $u : \mathbb{R}^2 \to \mathbb{S}^1$ be a multipole map, i.e. $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^2; \mathbb{S}^1)$ with a finite number of singular points x_i, $i = 1, \dots, N$ (as in (2.1.9), Chapter 2). Assume also that $\deg(u)$ has constant sign around each x_i (in other words, if d_i is the degree of u around x_i, then either $d_i > 0$ for every i or $d_i < 0$ for every i.) We conjecture that if ℓ is large enough, then

$$\overline{\mathcal{A}}_{L^1}(u; B_\ell) = \int_{B_\ell} \sqrt{1 + |\nabla u|^2} dx + \pi \sum_{i=1}^{N} |d_i|. \tag{6.3.3}$$

Notice that the right hand side coincides with $\overline{\mathcal{A}}_{BV}(u; B_\ell)$, by Theorem 2.3.6. Roughly, the fact that ℓ is large should prevent the interaction of each pole with the boundary of Ω (compare [1, Lemma 5.2]), so that one cannot construct a d_i-ple catenoid like in Subsection 6.3.2. Moreover, since the degree has the same sign at each x_i, we do not expect any interaction between the poles either. Therefore, the relaxed area functional should "localize" around x_i and the only way to fill the hole generated by the cavitation must be the trivial one, with a unit disk of multiplicity $|d_i|$.
We believe that a similar formula holds true also in higher dimension, i.e. for $u : \mathbb{R}^n \to \mathbb{S}^{n-1}$, $u \in W^{1,n-1}_{\text{loc}}(\mathbb{R}^n; \mathbb{S}^{n-1})$ with the same properties as before, motivated also by the fact that the proof of [1, Lemma 5.2] is valid in every dimension.
Of course, since the configuration of the poles is arbitrary, we cannot expect to have any symmetry property at our disposal.

6.3.4 Multipole II

We expect the same behaviour as in (6.3.3) also if we relax the hypotheses on the degrees, but we add the condition of "well separated" poles: Let $u : \mathbb{R}^2 \to \mathbb{S}^1$ be a multipole map with a finite number of singular points $\{x_i\}_{i=1,\dots,N}$. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set containing x_i for every i. Then we conjecture that there exists $r_0 > 0$ such that, if $\min_{i,j=1,\dots,N} \text{dist}(x_i, x_j) \geq r_0$ and $\text{dist}(x_i, \partial\Omega) \geq r_0$, then

$$\overline{\mathcal{A}}_{L^1}(u; \Omega) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx + \pi \sum_{i=1}^{N} |d_i|.$$

Again, the argument should generalize in every dimension.

6.3.5 Multipole III

If we omit both the hypotheses on well separation between the poles and between each pole and the boundary of the domain, and we do not assume anything on the degree at each singularity, then the poles are free to interact together and with the boundary of Ω. In this case, the situation is more involved, but we think that an optimal profile of the minimal surfaces filling the holes should always be made by catenoids like in Subsection 6.3.2, with the corresponding degree, which are constrained to the *minimal connection* path between the x_i's. This path could connect x_i also to $\partial\Omega$, generating a "virtual" pole of opposite degree at $\partial\Omega$ (see [12]).

6.3.6 Symmetric quadruple point

Let $u : B_\ell \to \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the symmetric quadruple point map, as in Remark 2.4.3 for $n = 4$. We expect a result similar to [8] in the expression of the L^1-relaxed area, but it is not clear which is the boundary datum of the 4 Plateau problems entagled at the target plane. In fact, we have at least 2 possibilities: one is the path made by the two diagonals of the square $P_{\alpha_1\alpha_2\alpha_3\alpha_4}$ (that is rotationally symmetric), the second path one of the two Steiner graphs, which have of course minimal length (but it is not rotationally symmetric). We do not know which is the most convenient datum in terms of area surface in the corresponding Plateau problem, but we believe that at least if ℓ is small enough, the Steiner graph is the best candidate.

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