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RECTIFIABILITY OF THE FREE BOUNDARY FOR VARIFOLDS

LUIGI DE MASI

ABSTRACT. We establish a partial rectifiability result for the free boundary of a k-varifold V. Namely, we first refine a theorem of Grüter and Jost by showing that the first variation of a general varifold with free boundary is a Radon measure. Next we show that if the mean curvature H of V is in L^p for some $p \in [1, k]$, then the set of points where the k-density of V does not exist or is infinite has Hausdorff dimension at most $k - p$. We use this result to prove, under suitable assumptions, that the part of the first variation of V with positive and finite $(k-1)$-density is $(k-1)$-rectifiable.

1. INTRODUCTION

1.1. **Motivations.** The main goal of this paper is to study the rectifiability of the free boundary for a k-varifold V in a compact domain $\mathcal{M} \subset \mathbb{R}^n$ with smooth boundary $\partial\mathcal{M}$.

We say that V has free boundary at $\partial\mathcal{M}$ if the following first variation formula holds for every vector field X that is tangent to $\partial\mathcal{M}$ (see next section for more detailed definitions):

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\, \mathrm{d}V(x, S) = -\int_{\mathcal{M}} \langle X, H\rangle\, \mathrm{d}\|V\|, \tag{1.1}$$

where $H \in L^1(\mathcal{M}, \|V\|)$.

If V is the varifold induced by a smooth k-surface Σ with smooth boundary $\partial\Sigma$, H is the mean curvature vector of Σ and (1.1) implies that $\partial\Sigma \subset \partial\mathcal{M}$ and that Σ meets $\partial\mathcal{M}$ orthogonally: that is the unit conormal η to $\partial\Sigma$ coincides with the exterior unit normal vector N to $\partial\mathcal{M}$. So, varifolds with free boundary generalize in a weak sense the idea of surfaces that meet $\partial\mathcal{M}$ orthogonally.

If $\Sigma \subset \mathcal{M}$ is a smooth k-surface with smooth boundary $\partial\Sigma$ and meets $\partial\mathcal{M}$ orthogonally, we can test the first variation formula

$$\int_{\Sigma} \operatorname{div}_{T_x\Sigma} X(x)\, \mathrm{d}\mathcal{H}^k(x) = -\int_{\Sigma} \langle X, H\rangle\, \mathrm{d}\mathcal{H}^k + \int_{\partial\Sigma} \langle X, N\rangle\, \mathrm{d}\mathcal{H}^{k-1} \tag{1.2}$$

(where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure) with a smooth vector field X such that $X(x) = N(x)$ on $\partial\mathcal{M}$, obtaining the estimate

$$\mathcal{H}^{k-1}(\partial\Sigma) \le c\left(\frac{\mathcal{H}^k(\Sigma)}{R(\mathcal{M})} + \int_{\Sigma} |H|\, \mathrm{d}\mathcal{H}^k\right) \tag{1.3}$$

where $c = c(k, \mathcal{M})$ and $R(\mathcal{M})$ is the minimum radius of curvature of $\partial\mathcal{M}$. This bound can be easily localized to any ball $B_r(x)$ where $x \in \partial\mathcal{M}$. (In particular, if $\mathcal{M} = B_1$ is the unit ball with center 0 and $\Sigma \subset B_1$ is a minimal k-surface that meets ∂B_1 orthogonally, choosing $X = x$ we obtain the nice identity $\mathcal{H}^{k-1}(\partial\Sigma) = k\mathcal{H}^k(\Sigma)$.)

The simple proofs of these a-priori bounds strongly rely on the fact that Σ and $\partial\Sigma$ are assumed to be smooth (that is the first variation of Σ is assumed to be bounded) and on the assumption that the conormal η of $\partial\Sigma$ points outside $\mathcal{M}$. It is natural to ask if similar estimates hold also for a general varifold with free boundary V: that is if a varifold V with free boundary at $\partial\mathcal{M}$ has bounded first variation and if its unit conormal on $\partial\mathcal{M}$ is orthogonal to $\partial\mathcal{M}$ and points outside $\mathcal{M}$.

We answer these questions refining a result stated by Grüter and Jost in [9] and by Edelen in [6]: we prove that if V satisfies (1.1), then it has bounded first variation: namely there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ on $\partial\mathcal{M}$ such that, for every smooth

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vector field X on $\mathcal{M}$ we have

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H+\tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V, \tag{1.4}$$

where N is the exterior unit normal vector to $\partial\mathcal{M}$ and $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ is orthogonal to $\partial\mathcal{M}$. Moreover we prove bounds on σ_V similar to (1.3). The measure $\tilde{H}\|V\| + N\sigma_V$ is the orthogonal part of the first variation of V: $\tilde{H}\|V\|$ takes into account the absolutely continuous part (with respect to $\|V\|$), that is the areas where V "lean" on $\partial\mathcal{M}$ tangentially, while σ_V takes into account the boundary part of the first variation, that is where V meets $\partial\mathcal{M}$ transversally.

Indeed, (1.4) is clearly analogous to (1.2): by comparison we have that if V is induced by a smooth surface Σ with smooth boundary $\partial\Sigma$, then $\sigma_V = \mathcal{H}^{k-1} \llcorner \partial\Sigma$. It is then natural to ask also for a general k-varifold with free boundary V, if σ_V is singular with respect to $\|V\|$ or, more precisely, if σ_V is $(k-1)$-rectifiable. As far as we know, this question has not been investigated. Under suitable assumptions, we are able to show a rectifiability result for σ_V.

To prove it, we analyze tangent cones to V at points on $\partial\mathcal{M}$ to get informations about the tangent measures of σ_V; tangent varifolds to V exist if the upper k-density of V is finite. When the mean curvature H of V is in $L^p(\mathcal{M}, \|V\|)$ for some $p > k$, it is well-known that the density of V exists and is finite for every point; whereas if $p \le k$, the finiteness of the upper density is guaranteed just $\|V\|$-a.e. by monotonicity formulae and differentiation theorems, which is not enough to prove any rectifiability result on σ_V (since $\partial\mathcal{M}$ may have $\|V\|$-measure 0).

In order to deal with this case, we prove an estimate of the size of the set where the k-density of the varifold does not exists or is infinite, in terms of Hausdorff measures: if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \in [1,k]$, then this set has Hausdorff dimension at most $k-p$.

1.2. **Background and main results.**

Allard studied the first variation of a varifold in the two seminal papers [1] and [2]. In the former he considered the interior case, while in the latter he studied the behavior of a k-varifold V assuming it has, as a boundary, a smooth $(k-1)$-dimensional submanifold Γ, i.e. he assumes that V has generalized mean curvature with respect to vector fields that vanish on Γ. An extension of the boundary result of Allard can be found in [3].

An ε-regularity theorem similar to the ones by Allard is proved by Grüter and Jost in [9] for varifolds with free boundaries. Moreover they prove in [9, 4.11(ii)] that a varifold with free boundary with $\|V\|(\partial\mathcal{M}) = 0$ has bounded first variation δV; this is also proved by Edelen in [6, Proposition 3.2] removing the hypothesis that $\|V\|(\partial\mathcal{M}) = 0$, but assuming that V is rectifiable.

We refine these boundedness results, extending them to general varifolds and removing the assumption $\|V\|(\partial\mathcal{M}) = 0$. We state it in a slightly more general setting: if V has generalized mean curvature with respect to vector fields that vanish on $\partial\mathcal{M}$, then it has bounded first variation with respect to vector fields that are orthogonal to $\partial\mathcal{M}$ (see next section for precise definitions):

Theorem 1.1. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a k-varifold with generalized mean curvature H with respect to $\mathfrak{X}_0(\mathcal{M})$, with $H \in L^1(\mathcal{M}, \|V\|)$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ on $\partial\mathcal{M}$ such that, for any $X \in \mathfrak{X}_\perp(\mathcal{M})$, it holds*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H+\tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \tag{1.5}$$

where $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$, $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends only on the second fundamental form of $\partial\mathcal{M}$. In particular, V has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$. Moreover, the following global and local estimates hold:

$$\sigma_V(\partial\mathcal{M}) \le c\|V\|(\mathcal{M}) + \int_{\mathcal{M}} |H|\,\mathrm{d}\|V\|; \tag{1.6}$$

$$\sigma_V\big(B_{r/2}(x_0)\big) \le \frac{c}{r}\|V\|\big(B_r(x_0)\big) + \int_{B_r(x_0)} |H|\,\mathrm{d}\|V\| \qquad \forall x_0 \in \partial\mathcal{M}, \forall r \le R(\mathcal{M}), \tag{1.7}$$

where the constant $c = c(\mathcal{M})$ depends only on the second fundamental form of $\partial\mathcal{M}$ and $R(\mathcal{M})$ is such that the signed distance function from $\partial\mathcal{M}$ is of class C^2 in $U_R(\partial\mathcal{M})$.

In other words, this theorem states that the component of the first variation of V on $\partial\mathcal{M}$ that is orthogonal to $\partial\mathcal{M}$ is the sum of two terms:

- An absolutely continuous part with respect to $\|V\|$ given by $\tilde{H}\|V\|$, which takes into account the fact that V can "lean" on $\partial\mathcal{M}$: indeed, if V is induced by a smooth surface Σ with constant multiplicity, then $\tilde{H}$ is the mean curvature of Σ where Σ lean on $\partial\mathcal{M}$. This is orthogonal to $\partial\mathcal{M}$ and depends on $T_x\Sigma$ and of the second fundamental form of $\partial\mathcal{M}$, see (3.12).
- A part given by $N\sigma_V$, which "points outward $\mathcal{M}$" and is bounded. Roughly speaking, we expect that this is the "transversal boundary" of V at $\partial\mathcal{M}$. On the other hand, V can have unbounded first variation on $\partial\mathcal{M}$ only where "V meets $\partial\mathcal{M}$ tangentially".

Looking at the case of varifolds with free boundary, since the tangent part to $\partial\mathcal{M}$ of the first variation of such a varifold is controlled by definition, Theorem 1.1 easily implies that varifolds with free boundary have bounded first variation (Corollary 4.6).

As we said before, Corollary 4.6 was already proved by Grüter and Jost when $\|V\|(\partial\mathcal{M}) = 0$, and Edelen extended the result to $\|V\|(\partial\mathcal{M}) > 0$ but assuming that V is rectifiable. In Edelen's proof, the rectifiability is used to show that for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ the only planes charged by V are those included in $T_x\partial\mathcal{M}$. We are able to remove the rectifiability assumption by the use of Lemma 3.1, which is a form of the Constancy Theorem (see [12, Theorem 41.1]) and asserts that on $\partial\mathcal{M}$, even in the non-rectifiable case, V charges only planes that are included in $T_x\partial\mathcal{M}$.

We have already stated above that, if the varifold is induced by a smooth surface with free boundary at $\partial\mathcal{M}$, then $\sigma_V = \mathcal{H}^{k-1}\llcorner\partial\Sigma$. It is then natural to ask if σ_V is $(k-1)$-rectifiable in a more general case as well.

In order to state precisely our main result concerning σ_V, we define the $(k-1)$*-dimensional part* of σ_V, written σ_V^*, as the restriction of σ_V to those points with strictly positive lower $(k-1)$-density and finite upper $(k-1)$-density :

$$\sigma_V^* = \sigma_V\llcorner E, \qquad E = \{x \mid 0 < \Theta_*^{k-1}(\sigma_V, x) \leq \Theta^{*(k-1)}(\sigma_V, x) < +\infty\}. \tag{1.8}$$

Our main result on σ_V^* is the following:

Theorem 1.2. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$ such that $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > 1$ and $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$. Then σ_V^* is $(k-1)$-rectifiable.*

To prove Theorem 1.2, we make an analysis of the blow-ups of V on $\partial\mathcal{M}$ similar to the one performed in [5]. The study of blow-ups of V allows us to deduce, for σ_V^*-a.e. $x \in \partial\mathcal{M}$, that every $(k-1)$-blow-up of σ_V^* at x is of the form $\beta\mathcal{H}^{k-1}\llcorner S$ for some $(k-1)$-dimensional plane S and $\beta > 0$. The Marstrand-Mattila Rectifiability Criterion (Theorem 5.1), then implies that σ_V^* is $(k-1)$-rectifiable.

As we show at the beginning of section 5, if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the condition $\Theta^{*(k-1)}(\sigma_V, x) < +\infty$ is not restrictive, since it holds for every point x, basically by (1.7) and by the monotonicity formula for $\|V\|$ that we prove in Corollary 4.8. Thus in this case, Theorem 1.2 reads as follows.

Theorem 1.3. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$ such that $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$ and $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$. Then the restriction $\sigma_V\llcorner\{x \mid \Theta_*^{k-1}(\sigma_V, x) > 0\}$ is $(k-1)$-rectifiable.*

To perform the analysis of the blow-ups of V in the proof of Theorem 1.2, we have to distinguish two cases:

- If $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$, then the monotonicity formula (Corollary 4.8) assures the existence of the k-density $\Theta^k(\|V\|, \cdot) < +\infty$ and of tangent cones to V at *every* point in $\partial\mathcal{M}$.
- The situation is more delicate when $p \leq k$, because the Lebesgue-Besicovitch differentation Theorem applied to the classical monotonicity identity for varifolds with bounded first variation guarantees (see e.g. [12, Lemma 40.5]) the existence and finiteness of the k-density and of tangent cones just $\|V\|$-a.e.. Since $\|V\|(\partial\mathcal{M}) = 0$ may hold true, it was in principle

possible that $\Theta^k(\|V\|, x) = +\infty$ for every point in $\partial\mathcal{M} \cap \operatorname{supp}\|V\|$, and this would stop our analysis.

To overcome this difficulty, in subsection 4.6 we study more carefully the set of points where the k-density of V exists and is finite also when $p \leq k$.

The behavior of the density for points in $\mathcal{M}^\circ$ was studied by Menne [11, pp. 2.9–2.11]: in that paper the author was interested in a lower bound for the lower density of the varifold with respect to Hausdorff measures; in particular he shows that, if the density has a lower bound $\|V\|$-a.e. and $H \in L^p(\mathcal{M}, \|V\|)$, then at $\mathcal{H}^{k-p}$-a.e. point, either the lower density still satisfies the lower bound, or it is equal to 0.

Although the existence and finiteness of the density with respect to lower dimensional Hausdorff measures seems to be well-known at least for points in $\mathcal{M}^\circ$, it does not appear in the literature; thus we report the result with its proof, both for points in $\mathcal{M}^\circ$ and on $\partial\mathcal{M}$, since it can be useful for a future reference too.

More precisely, we define the *density set* of V (Definition 4.1), denoted by $\operatorname{Dens}(V)$, and we prove the following result.

Theorem 1.4. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a varifold with free boundary at $\partial\mathcal{M}$ such that $H \in L^p(\mathcal{M}, \|V\|)$ for some $1 \leq p \leq \infty$. Then*

$$\mathcal{H}^s\big(\mathcal{M} \setminus \operatorname{Dens}(V)\big) = 0 \qquad \forall s > k - p \tag{1.9}$$

and, for every $x_0 \in \operatorname{Dens}(V)$ there exists an increasing function $\varphi_{x_0} \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{\|V\|\big(B_r(x_0)\big)}{r^k} \leq \frac{\|V\|\big(B_t(x_0)\big)}{t^k} + \varphi_{x_0}(t) \quad \forall\, 0 < r < t, \qquad \lim_{t\to 0} \varphi_{x_0}(t) = 0. \tag{1.10}$$

In particular, the k-density $\Theta^k(\|V\|, x_0)$ exists and is finite for every $x_0 \in \operatorname{Dens}(V)$; moreover, the restrictions of $\Theta^k(\|V\|, \cdot)$ to $\operatorname{Dens}(V) \cap \mathcal{M}^\circ$ and to $\operatorname{Dens}(V) \cap \partial\mathcal{M}$ are upper semi-continuous.

Thus the set of points where $\Theta^k(\|V\|, \cdot)$ is infinte or does not exist has Hausdorff dimension at most $k - p$. $\operatorname{Dens}(V)$ is the "good set" where we are able to study the blow-ups of $\|V\|$ and σ_V^*, to conclude the proof of Theorem 1.2. We remark that, if $p > k$, then the theorem is the well-known existence of the density at *every* point and is a straightforward consequence of Corollary 4.8.

Besides its application in the proof of Theorem 1.2, Theorem 1.4 is also interesting in itself, since it is a natural counterpart of the similar result for the set of Lebesgue points of a Sobolev function proved by Federer and Ziemer in [8]: if $f \in W^{1,p}(\mathcal{M})$ for $p \in [1, n]$ and if $\operatorname{Leb}(f)$ is the set of Lebesgue points of f, then $\mathcal{M} \setminus \operatorname{Leb}(f)$ has Hausdorff dimension at most $n - p$.

1.3. Outline of the paper.

In section 2 we recall the notations and the definitions used throughout the paper.

In section 3 we first prove Lemma 3.1, which concerns the behavior of V on $\partial\mathcal{M}$; next we move on to the proof of Theorem 1.1.

In section 4 we describe some consequences of Theorem 1.1: we notice that, if $k = n - 1$, $\tilde{H}$ coincides with the mean curvature of $\partial\mathcal{M}$; next we adapt Theorem 1.1 to an other class of varifolds; we move on to varifolds with free boundary, proving that they have bounded first variation, establishing some monotonicity formulae and refining Lemma 3.1 in this framework. Lastly, in subsection 4.6, we prove Theorem 1.4.

In section 5 we prove Theorem 1.2; this is obtained by the study of tangent varifolds to V at points in $\operatorname{Dens}(V)$, to prove that σ_V^* satisfies the hypotheses of the Marstrand-Mattila Rectifiability Criterion.

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2. NOTATIONS

2.1. **Basic notations.** For a fixed orthonormal system of coordinates, we denote by e_i the i-th coordinate unit vector. If $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the closed ball with center x and radius r and by ω_n the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Moreover we set $B_r := B_r(0)$. c and c' are generic positive constants, unless otherwise specified. If $v \in \mathbb{R}^n$ we call L_v the translation

$$L_v : x \mapsto x + v.$$

For each $A \subset \mathbb{R}^n$, we denote by $\mathbf{1}_A$ the indicator function of A, by $\overline{A}$ and A^{o} respectively the closure and the interior of A in the euclidean topology. If $A \subset \mathbb{R}^n$, and $r > 0$ we write $U_r(A)$ for the tubular neighborhood of A, i.e.

$$U_r(A) = \bigcup_{x \in A} B_r(x).$$

We denote by S a generic k-dimensional linear subspace (or k-plane) of $\mathbb{R}^n$ and we write $S^{\perp}$ for the orthogonal complement of S in $\mathbb{R}^n$. We denote by P_S the orthogonal projection on S. If X is a C^1 vector field, we call $\operatorname{div}_S X$ the scalar product $P_S \cdot DX$. If $\tau_1, \dots, \tau_k$ is an orthonormal basis of S, by simple computations one has

$$\operatorname{div}_S X(x) = \sum_{i=1}^{k} D_{\tau_i} \langle X(x), \tau_i \rangle. \tag{2.1}$$

Throughout the paper, $\gamma \in C^{\infty}\big([0,\infty)\big)$ denotes a cut-off function such that

- $\gamma(t) = 1$ for each $t \in [0, \frac{1}{2}]$;
- $\gamma(t) = 0$ for each $t \geq 1$;
- $\gamma'(t) \leq 0$ and $|\gamma'(t)| \leq 3$ for every $t \in \mathbb{R}$.

For each $r > 0$ and $x \in \mathbb{R}^n$ we consider the dilation map

$$\tau_{x,r}(y) = \frac{1}{r}(y - x). \tag{2.2}$$

If Γ is a C^1 k-dimensional sub-manifold in $\mathbb{R}^n$ and $x \in \Gamma$, we write $T_x\Gamma$ for the tangent space to Γ at x. We see $T_x\Gamma$ as an immersed k-plane in $\mathbb{R}^n$; more precisely, we see $T_x\Gamma$ as the blow-up of Γ at the point x. If Γ has non-empty boundary $\partial\Gamma$ of class C^1 and if $x \in \partial\Gamma$, we see $T_x\Gamma$ as containing $T_x\partial\Gamma$, which divide $T_x\Gamma$ into two half-spaces. We call these two parts $T_x^+\Gamma$ and $T_x^-\Gamma$ and we set $T_x^+\Gamma$ to be the blow-up at x of the interior part of Γ.

We work on a compact domain $\mathcal{M} \subset \mathbb{R}^n$ with C^2 boundary $\partial\mathcal{M}$. We write $N(x)$ for the exterior unit normal vector to $\partial\mathcal{M}$ at x. In the following, d denotes the signed distance function from $\partial\mathcal{M}$ such that $d > 0$ in $\mathcal{M}^{\mathrm{o}}$, that is

$$d(x) = \begin{cases} \inf\{|x-y| \mid y \in \partial\mathcal{M}\} & \text{if } x \in \mathcal{M} \\ -\inf\{|x-y| \mid y \in \partial\mathcal{M}\} & \text{if } x \in \mathbb{R}^n \setminus \mathcal{M}. \end{cases} \tag{2.3}$$

Unless otherwise specified, we denote by $R = R(\mathcal{M}) > 0$ a number such that d is C^2 in $\overline{U_R(\partial\mathcal{M})}$. Thus ∇d exists in $\overline{U_R(\partial\mathcal{M})}$ and points inside $\mathcal{M}$.

We work with several classes of vector fields on $\mathcal{M}$, which we denote with the letter $\mathfrak{X}$ with subscripts based on their behavior on $\partial\mathcal{M}$:

$$\begin{gathered} \mathfrak{X}(\mathcal{M}) = C^1(\overline{\mathcal{M}}, \mathbb{R}^{n+1}), \qquad \mathfrak{X}_t(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(x) \in T_x\partial\mathcal{M},\ \forall x \in \partial\mathcal{M}\}, \\ \mathfrak{X}_\perp(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(x) \in (T_x\partial\mathcal{M})^\perp,\ \forall x \in \partial\mathcal{M}\} \\ \mathfrak{X}_0(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid X(x) = 0,\ \forall x \in \partial\mathcal{M}\} \quad \mathfrak{X}_c(\mathcal{M}) = \{X \in \mathfrak{X}(\mathcal{M}) \mid \operatorname{supp} X \subset\subset \mathcal{M}^{\mathrm{o}}\}. \end{gathered} \tag{2.4}$$

If Γ is a C^2 submanifold of $\mathbb{R}^n$, by slight abuse of notation we write $\mathfrak{X}_t(\Gamma)$ (respectively $\mathfrak{X}_0(\Gamma)$) for the set of compactly supported C^1 vector fields on $\mathbb{R}^n$ that are tangent to Γ (respectively that vanish on Γ).

2.2. **Measures, rectifiable sets.** If $A \subset \mathbb{R}^n$, $\mathscr{M}(A, \mathbb{R}^m)$ is the space of $\mathbb{R}^m$-valued Radon measures on A and $\mathscr{M}^+(A)$ is the space of positive Radon measures on A. If $\mu \in \mathscr{M}(A, \mathbb{R}^m)$ we denote by $|\mu|$ the total variation measure of μ. If $B \subset \mathbb{R}^n$ is Borel, we write $\mu \llcorner B$ for the restriction of the measure μ to B. If $A \subset \mathbb{R}^n$ has non-empty interior, we endow $\mathscr{M}(A, \mathbb{R}^m)$ with the weak*-topology: i.e. we say that a sequence of Radon measures $\{\sigma_j\}_j$ converges to μ ($\sigma_j \overset{*}{\rightharpoonup} \mu$) if

$$\lim_{j \to \infty} \int f \, \mathrm{d}\sigma_j = \int f \, \mathrm{d}\mu \qquad \forall f \in C_c(A, \mathbb{R}^m).$$

If $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$, we define the upper and the lower k-densities of μ at x:

$$\Theta^{*k}(\mu, x) = \limsup_{r \to 0} \frac{\mu\big(B_r(x)\big)}{r^k} \qquad \Theta^k_*(\mu, x) = \liminf_{r \to 0} \frac{\mu\big(B_r(x)\big)}{r^k}.$$

If the above limits coincide, then we define the k-density of μ at x as their common value, which we denote by $\Theta^k(\mu, x)$. The k-singular set $\mathrm{Sing}^k(\mu)$ is defined as

$$\mathrm{Sing}^k(\mu) = \{x \in \mathcal{M} \mid \Theta^{*k}(\mu, x) = +\infty\}.$$

If $\mu \in \mathscr{M}(A, \mathbb{R}^m)$ and $f : A \to \mathbb{R}^N$ is proper, we define the push-forward $f_\# \mu$ of μ through f as the Radon measure in $\mathscr{M}(\mathbb{R}^N, \mathbb{R}^m)$ defined by

$$f_\# \mu(B) = \mu\big(f^{-1}(B)\big) \qquad \forall B \subset \mathbb{R}^N \text{ Borel.}$$

If $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, we say that ν is a k-blow-up of μ at x or a k-tangent measure to μ in x if there exists a sequence $r_j \downarrow 0$ such that

$$\mu_j := \frac{1}{r_j^k} \big(\tau_{x, r_j}\big)_\# \mu \overset{*}{\rightharpoonup} \nu. \tag{2.5}$$

We denote by $\mathrm{Tan}^k(\mu, x)$ the (possibly empty) set of k-blow-ups of μ at the point x. If $\Theta^{*k}(\mu, x) < \infty$ then, by Banach-Alaoglu Theorem, $\mathrm{Tan}^k(\mu, x)$ is non-empty. If $\Theta^k_*(\mu, x) > 0$, then every k-blow-up of μ at x is non-trivial; indeed, if $\nu \in \mathrm{Tan}^k(\mu, x)$ and $\mu_j \overset{*}{\rightharpoonup} \nu$ as in (2.5), then

$$\nu(B_1) \geq \limsup_j \mu_j(B_1) = \limsup_k \frac{\mu(B_{r_j}(x))}{r_j^k} \geq \Theta^k_*(\mu, x) > 0. \tag{2.6}$$

For each $s > 0$, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and, if $A \subset \mathbb{R}^N$, $\mathcal{H}_{\dim}(A)$ denotes the Hausdorff dimension of A. We say that a Borel set $M \subset \mathbb{R}^n$ is k-rectifiable if there exist $M_0 \subset \mathbb{R}^n$ with $\mathcal{H}^k(M_0) = 0$ and a countable family of C^1 k-submanifolds $\{M_j\}_{j=1}^\infty$ such that

$$M \subset \bigcup_{i=0}^\infty M_j.$$

We say that a measure $\mu \in \mathscr{M}^+(\mathbb{R}^n)$ is k-rectifiable if there exist a k-rectifiable set M and a positive function $\theta \in L^1_{\mathrm{loc}}(M, \mathcal{H}^k)$ such that $\mu = \theta \mathcal{H}^k \llcorner M$.

2.3. **Varifolds.** If $1 \leq k \leq n$ we call $G(k, n)$ the Grassmannian of the un-oriented k-dimensional linear subspaces (or k-planes) of $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ we denote by $G_k(A) := A \times G(k, n)$ the trivial Grassmannian bundle over A.

A *k-varifold* on A is a positive Radon measure on $G_k(A)$. We denote by $\mathcal{V}_k(A)$ the set of all k-varifolds on A and we endow $\mathcal{V}_k(A)$ with the topology of the weak*-convergence of Radon measures, i.e. we say that $V_j \overset{*}{\rightharpoonup} V$ if

$$\lim_{j \to \infty} \int_{G_k(A)} \varphi(x, S) \, \mathrm{d}V_j(x, S) = \int_{G_k(A)} \varphi(x, S) \, \mathrm{d}V(x, S) \qquad \forall \varphi \in C_c(G_k(A)).$$

A k-rectifiable measure $\mu = \theta \mathcal{H}^k \llcorner M$ in $\mathbb{R}^m$ induces the k-varifold

$$V = \theta \mathcal{H}^k \llcorner M \otimes \delta_{T_x M},$$

where $T_x M$ is the approximate tangent space of M at x. A varifold that is induced by a rectifiable set is called a *rectifiable varifold.* If the multiplicity function assumes only integer values, we say that the varifold is *integer rectifiable.* If V is a k-varifold on Ω, the *mass* $\|V\|$ (or *total variation*) of V is the positive Radon measure defined as

$$\|V\|(A) = V(G_k(A)) \qquad \forall A \subset \Omega \text{ Borel.}$$

If V is the k-varifold induced by the rectifiable measure $\mu = \theta\mathcal{H}^k \llcorner M$, then

$$\|V\|(B) = \int_B \theta(x)\,\mathrm{d}\mathcal{H}^k(x).$$

By slight abuse of notation, we often denote $\operatorname{supp}\|V\|$ by $\operatorname{supp} V$. If $\Omega \subset \mathbb{R}^n$ is a domain, $V \in \mathcal{V}_k(\Omega)$ and if $\psi : \Omega \to \mathbb{R}^n$ is a diffeomorphism, the *push forward* $\psi_\sharp V$ of V through ψ is the varifold in $\mathcal{V}_k\big(\psi(\Omega)\big)$ such that, $\forall \varphi \in C_c\big(G_k(\psi(\Omega))\big)$,

$$\int_{G_k(\psi(\Omega))} \varphi(y,T)\,\mathrm{d}\psi_\sharp V(y,T) = \int_{G_k(\Omega)} J_S\psi(x)\varphi\big(\psi(x), \mathrm{d}\psi_x(S)\big)\,\mathrm{d}V(x,S), \tag{2.7}$$

where $J_S\psi(x)$ is the Jacobian of ψ relative to the k-plane S, i.e.

$$J_S\psi(x) = \sqrt{\det\big((\mathrm{d}\psi_x)^*_{|S} \circ \big((\mathrm{d}\psi_x)_{|S}\big)}.$$

We notice that this *is not* the push forward of measures previous defined (which is denoted by the different symbol $f_\#\mu$). In fact, the push forward of varifolds is defined in this way in order to ensure the validity of the area formula: indeed if V is induced by a rectifiable set M, then $\psi_\sharp V$ is induced by $\psi(M)$. If $V \in \mathcal{V}_k(\mathbb{R}^n)$, we say that $C \in \mathcal{V}_k(\mathbb{R}^n)$ is a *blow-up* of V at x or a *tangent varifold* to V at x if there exists a sequence of radii $r_j \downarrow 0$ such that

$$(\tau_{x,r_j})_\sharp V \overset{*}{\rightharpoonup} C.$$

We write $\operatorname{Tan}(V,x)$ for the set of tangent varifold to V at the point x.

If $V \in \mathcal{V}_k(\mathcal{M})$ and if $X \in \mathfrak{X}(\mathcal{M})$ the *first variation* $\delta V(X)$ of V with respect to X is

$$\delta V(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|(\psi_t)_\sharp V\|(\mathbb{R}^n)\Big)\Big|_{t=0}$$

where ψ_t is the flow map of X at the time t. The following *first variation formula* holds:

$$\delta V(X) = \int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\ \mathrm{d}V(x,S).$$

We now define the class of varifolds with *bounded first variation*:

Definition 2.1. We say that a varifold V has *bounded first variation* in $\mathcal{M}$ if

$$\sup\{|\delta V(X)| \mid X \in \mathfrak{X}(\mathcal{M})\,, \max|X| \le 1\} < +\infty. \tag{2.8}$$

If (2.8) holds with a proper subset of $\mathfrak{X}(\mathcal{M})$(e.g. $\mathfrak{X}_c(\mathcal{M})$, $\mathfrak{X}_0(\mathcal{M})$...) in place of $\mathfrak{X}(\mathcal{M})$, we say that V has bounded first variation with respect to this subset.

Therefore V has bounded first variation if there exists $\delta V \in \mathscr{M}(\mathcal{M}, \mathbb{R}^n)$ such that, for any $X \in \mathfrak{X}(\mathcal{M})$,

$$\delta V(X) = \int_{\mathcal{M}} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{\mathcal{M}} \langle X(x), \zeta(x)\rangle\,\mathrm{d}|\delta V|(x), \tag{2.9}$$

where ζ is the polar vector of δV with respect to $|\delta V|$.

If V has bounded first variation, then by Lebesgue decomposition there exist $|\delta^s V| \in \mathscr{M}^+(\mathcal{M})$, a $|\delta^s V|$-measurable function $\eta : \mathcal{M} \to \mathbb{R}^n$ and a $\|V\|$-measurable function $H : \mathcal{M} \to \mathbb{R}^n$ such that

$$\delta V(X) = -\int_{\mathcal{M}} \langle H, X\rangle\ \mathrm{d}\|V\| + \int_{\mathcal{M}} \langle X, \eta\rangle\ \mathrm{d}|\delta^s V| \qquad \forall X \in \mathfrak{X}(\mathcal{M})$$

where $|\delta^s V|$ is the singular part of $|\delta V|$ with respect to $\|V\|$:

$$|\delta^s V| = |\delta V| \llcorner Z \qquad Z = \Big\{x \in \mathcal{M} \mid \limsup_{r\to 0} \frac{|\delta V|(B_r(x))}{\|V\|(B_r(x))} = +\infty\Big\}.$$

Since the previous formula is similar to the corresponding one for smooth surfaces, we call H the *generalized mean curvature* of V, $|\delta^s V|$ the *boundary measure* of V, the set Z is the *boundary* of V and η is the *unit co-normal* of V.

We now define the classes of varifolds with *generalized mean curvature*:

Definition 2.2. We say that $V \in \mathcal{V}_k(\mathcal{M})$ has *generalized mean curvature* with respect to $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$) if there exists a $\|V\|$-measurable vector field $H \in L^1(\mathcal{M}, \|V\|)$ such that $H(x) = 0$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ and for any $X \in \mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$) the following formula holds:

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x) \, \mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle H, X\rangle \, \mathrm{d}\|V\|. \tag{2.10}$$

Remark 2.1. The assumption that $H(x) = 0$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ is important: without this hypothesis, the generalized mean curvature is not uniquely defined, that is any combination $H + K$ with $\operatorname{supp} K \subset \partial\mathcal{M}$ satisfies (2.10). Without excluding this ambiguity, (1.5) is no longer true because of the extra term $\int_{\partial\mathcal{M}} \langle K, X\rangle \, \mathrm{d}\|V\|$ on the right-hand side. Similarly, if V has bounded variation with respect to $\mathfrak{X}_c(\mathcal{M})$ or $\mathfrak{X}_0(\mathcal{M})$, if not otherwise specified we assume that the polar vector of δV vanishes at $|\delta V|$-a.e. point on $\partial\mathcal{M}$.

Thus V has *generalized mean curvature* with respect to $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$, $\mathfrak{X}_t(\mathcal{M})$) if it has bounded variation with respect to $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$, $\mathfrak{X}_t(\mathcal{M})$) and δV has no singular part with respect to $\|V\|$ when we test with vector fields in $\mathfrak{X}_c(\mathcal{M})$ (respectively $\mathfrak{X}_0(\mathcal{M})$, $\mathfrak{X}_t(\mathcal{M})$).

Definition 2.3 (Varifold with free boundary)**.** We say that $V \in \mathcal{V}_k(\mathcal{M})$ has *free boundary* at $\partial\mathcal{M}$ if there exists a $\|V\|$-measurable vector field $H \in L^1(\mathcal{M}, \|V\|)$ such that $H(x)$ is tangent to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ and such that (2.10) holds for every $X \in \mathfrak{X}_t(\mathcal{M})$.

Remark 2.2. As in Remark 2.1, the assumption that H is tangent to $\partial\mathcal{M}$ $\|V\|$-a.e. is important because otherwise (4.7) is no longer true: every sum $H + K$, with K orthogonal to $\partial\mathcal{M}$, satisfies (2.10) for any $X \in \mathfrak{X}_t(\mathcal{M})$, whereas when we test with non-tangent vector fields the presence of K is relevant. As above, when we say that V has bounded variation with respect to $\mathfrak{X}_t(\mathcal{M})$, if not otherwise specified we assume that the polar vector of δV is tangent to $\partial\mathcal{M}$ at $|\delta V|$-a.e. point on $\partial\mathcal{M}$.

Remark 2.3. As we mentioned in the introduction, a varifold with free bondary meets $\partial\mathcal{M}$ orthogonally in a weak sense: indeed, if V is the varifold induced by a surface Σ with smooth boundary $\partial\Sigma$, (2.10) implies that the conormal η to $\partial\Sigma$ is orthogonal to $\partial\mathcal{M}$.

3. Proof of Theorem 1.1

3.1. **Constancy Lemma.** The proof of the Theorem 1.1 is based on the following lemma, which is a form of the Constancy Theorem [12, Theorem 41.1] with weaker hypotheses (and conclusion) and it is interesting in itself. The result is used in the proof of the Theorem 1.1 to deal with the case $\|V\|(\partial\mathcal{M}) > 0$.

Lemma 3.1. *Let $\Gamma \subset \mathbb{R}^n$ be a C^2-hypersurface without boundary, let $V \in \mathcal{V}_k(\mathbb{R}^n)$ have bounded first variation $\delta_0 V$ with respect to $\mathfrak{X}_0(\Gamma)$ (that is vector fields that vanish on Γ). Then*

$$V\big(\{(x,S) \in G_k(\mathbb{R}^n) \mid x \in \Gamma, S \not\subset T_x\Gamma\}\big) = 0.$$

Proof. By a simple covering argument it is enough to prove the result locally: that is that, for each $x_0 \in \Gamma$, there exists $r = r(x_0) > 0$ and a ball $B_r(x_0)$ such that

$$V\big(\{(x,S) \in G_k(\mathbb{R}^n) \mid x \in \Gamma \cap B_r(x_0), S \not\subset T_x\Gamma\}\big) = 0.$$

We fix $x_0 \in \Gamma$ and without loss of generality we can assume that $x_0 = 0$.

Since Γ is locally-orientable, there exists $r' > 0$ and a ball $B_{r'}$ such that $B_{r'} \setminus \Gamma$ is made of two connected components D^+ and D^- separated by Γ. Only in this proof, d denotes a fixed one of the two signed distance function from Γ in B_ρ, that is

$$d(x) = \begin{cases} \inf\{|x-y| \mid y \in \Gamma\} & \text{if } x \in D^+ \\ -\inf\{|x-y| \mid y \in \Gamma\} & \text{if } x \in D^-. \end{cases}$$

Since Γ is of class C^2, there exists $r \le r'$ such that $d \in C^2\big(\overline{B_r}\big)$. We set $r(x_0) = r$.

We recall that γ is the cut-off function defined in section 2. Since $\nabla d(x)$ is orthogonal to Γ for every $x \in \Gamma$, we have $|P_S \nabla d(x)|^2 = 0$ if and only if $S \subset T_x\Gamma$. Therefore, to get the conclusion, it is enough to prove that

$$\int_{G_k(\Gamma)} \gamma\Big(\frac{|x|}{r}\Big)|P_S\nabla d(x)|^2 \,\mathrm{d}V(x,S) = 0. \tag{3.1}$$

To do so, we test (2.10) with a suitable vector field $X \in \mathfrak{X}_0(\mathcal{M})$. If $\rho < r$, we choose $X(x) = d(x)\gamma\big(\frac{|x|}{r}\big)\gamma\big(\frac{d(x)}{\rho}\big)\nabla d(x)$. We clearly have $X \in \mathfrak{X}_0(\mathcal{M})$ and

$$\begin{aligned} \operatorname{div}_S X(x) =& \gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 + \frac{d(x)}{r}\gamma'\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle \frac{x}{|x|}, P_S\nabla d(x)\rangle \\ &+ \frac{d(x)}{\rho}\gamma\Big(\frac{|x|}{r}\Big)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 + d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S \nabla d(x). \end{aligned}$$

Since V has bounded first variation $\delta_0 V = \zeta|\delta_0 V|$ with respect to $\mathfrak{X}_0(\Gamma)$ (where ζ is the polar vector of $\delta_0 V$ with respect its total variation $|\delta_0 V|$), testing (2.10) with X we obtain

$$\begin{aligned} &\left|\int_{G_k(\mathbb{R}^n)} \gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 \,\mathrm{d}V(x,S)\right| \\ &\le \left|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{r}\gamma'\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle \frac{x}{|x|}, P_S\nabla d(x)\rangle \,\mathrm{d}V(x,S)\right| \\ &\quad + \left|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{\rho}\gamma\Big(\frac{|x|}{r}\Big)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 \,\mathrm{d}V(x,S)\right| \\ &\quad + \left|\int_{G_k(\mathbb{R}^n)} d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S \nabla d(x) \,\mathrm{d}V(x,S)\right| \\ &\quad + \left|\int_{\mathbb{R}^n} \langle X(x), \zeta(x)\rangle \,\mathrm{d}|\delta_0 V|(x)\right| \end{aligned} \tag{3.2}$$

For the left-hand side of the above inequality, by dominated convergence we have

$$\lim_{\rho\to 0}\int_{G_k(\mathbb{R}^n)} \gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 \,\mathrm{d}V(x,S) = \int_{G_k(\Gamma)} \gamma\Big(\frac{|x|}{r}\Big)|P_S\nabla d(x)|^2 \,\mathrm{d}V(x,S).$$

Therefore, to show (3.1), we have to prove that the terms on the right-hand side of (3.2) go to 0 as $\rho \to 0$:

(1) Since γ' is bounded and $d(x) \le \rho$ by cut-off, for the first term we have

$$\lim_{\rho\to 0}\left|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{r}\gamma'\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\langle \frac{x}{|x|}, P_S\nabla d(x)\rangle \,\mathrm{d}V(x,S)\right| \le c\lim_{\rho\to 0}\frac{\rho}{r}\|V\|\big(U_\rho(\Gamma)\cap B_r\big) = 0.$$

(2) Since $\big|\frac{d}{\rho}\big| \le 1$ and $\gamma'(s) \ne 0$ only if $s \in (1/2, 1)$, for the second term we have

$$\begin{aligned} &\lim_{\rho\to 0}\left|\int_{G_k(\mathbb{R}^n)} \frac{d(x)}{\rho}\gamma\Big(\frac{|x|}{r}\Big)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2 \,\mathrm{d}V(x,S)\right| \\ &\qquad \le 3\lim_{\rho\to 0}\|V\|\Big(\big(U_\rho(\Gamma)\setminus U_{\rho/2}(\Gamma)\big)\cap B_r\Big) = 0. \end{aligned}$$

(3) By the choice of r we have that $|\operatorname{div}_S \nabla d(x)| \leq c$ in B_r; thus

$$\lim_{\rho\to 0}\left|\int_{G_k(\mathbb{R}^n)} d(x)\gamma\Big(\frac{|x|}{r}\Big)\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S \nabla d(x)\,\mathrm{d}V(x,S)\right| \leq c\lim_{\rho\to 0}\rho\|V\|\big(U_\rho(\Gamma)\cap B_r\big)=0.$$

(4) For the last term, since $\delta_0 V$ is a Radon measure, it holds

$$\lim_{\rho\to 0}\left|\int_{\mathbb{R}^n}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta_0 V|(x)\right| \leq \lim_{\rho\to 0}\rho|\delta_0 V|\big(U_\rho(\Gamma)\cap B_r\big)=0.$$

This completes the proof. □

3.2. **Proof of Theorem 1.1.** We can now prove Theorem 1.1.

Proof of Theorem 1.1. Let us fix $R>0$ (as in section 2) so that the distance function d from $\partial\mathcal{M}$ defined in (2.3) is of class C^2 in $\overline{U_R(\partial\mathcal{M})}$.

In what follows we are going to repeatedly use the decomposition of a vector field $X\in\mathfrak{X}(\mathcal{M})$ we now present; within $U_R(\partial\mathcal{M})$, we can decompose X in its normal and tangent component: there exists a scalar function $\chi(x)$ such that $X = X^\perp + X^T$ with $X^\perp(x)=\chi(x)\nabla d(x)$ and $\langle X^T(x),\nabla d(x)\rangle = 0$ for all $x\in U_R(\partial\mathcal{M})$.

Step 1: We begin by cut-offing a vector field in its "interior" and "boundary" part. For every $X\in\mathfrak{X}_\perp(\mathcal{M})$ and $\rho<R$ one has

$$X(x)=\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)+\Big(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x).$$

Therefore

$$\begin{aligned}\int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) &= \int_{G_k(\mathcal{M})}\operatorname{div}_S\Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S)\\ &\quad+\int_{G_k(\mathcal{M})}\operatorname{div}_S\Big[\Big(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x)\Big]\,\mathrm{d}V(x,S).\end{aligned}\tag{3.3}$$

Thus we have splitted X in the "interior" and the "boundary part" by cut-offing with $\gamma(d/\rho)$ and the idea is to send $\rho\to 0$.

Step 2: For the interior part, since $\Big(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x)\in\mathfrak{X}_c(\mathcal{M})$ (i.e. it is compactly supported in the interior of $\mathcal{M}$), by (2.10) and dominated convergence we have

$$\begin{aligned}&\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\operatorname{div}_S\Big[\Big(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\Big)X(x)\Big]\,\mathrm{d}V(x,S)\\ &\quad=\lim_{\rho\to 0}\int_{\mathcal{M}}\Big(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\Big)\langle X(x),H(x)\rangle\,\mathrm{d}\|V\|(x)\\ &\quad=-\int_{\mathcal{M}^\circ}\langle X(x),H(x)\rangle\,\mathrm{d}\|V\|(x)\\ &\quad=-\int_{\mathcal{M}}\langle X(x),H(x)\rangle\,\mathrm{d}\|V\|(x),\end{aligned}\tag{3.4}$$

where the last equality follows by the fact that H is assumed to be equal to 0 on $\partial\mathcal{M}$ (see Remark 2.1 for more details).

Step 3: Since the limit in (3.4) exists, also for the "boundary part" of X the limit

$$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\operatorname{div}_S\Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S)\tag{3.5}$$

exists. We want now to compute (3.5) and to show the existence of $\tilde{H}$ and σ_V on $\partial\mathcal{M}$. We begin by writing

$$\begin{aligned}\int \operatorname{div}_S \Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S) &= \int \gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)\\ &\quad+\int \frac{1}{\rho}\gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x), X(x)\rangle\,\mathrm{d}V(x,S).\end{aligned}\tag{3.6}$$

At this point, we want to study separately the limit as $\rho \to 0$ of each term in the right-hand side of the above equation.

- For the first term of the right-hand side in (3.6), we expect that, as $\rho \to 0$, it give a sort of mean curvature of $\partial\mathcal{M}$. This expectation is justified by the fact that, on $\partial\mathcal{M}$, V charges only planes that are tangent to $\partial\mathcal{M}$ by Lemma 3.1 and because X is orthogonal to $\partial\mathcal{M}$ on $\partial\mathcal{M}$. More precisely, we are going to prove that there exists a $\|V\|$-measurable vector field $\tilde{H}$ such that

$$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\partial\mathcal{M}}\langle \tilde{H}(x), X(x)\rangle\,\mathrm{d}\|V\|(x),\tag{3.7}$$

  which is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$. To this aim, we first observe that by dominated convergence we have

$$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\gamma\Big(\frac{d(x)}{\rho}\Big)\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{G_k(\partial\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)\tag{3.8}$$

  We now have to compute the right-hand side of (3.8). To do this, we decompose $\operatorname{div}_S X(x) = \operatorname{div}_S X^{\perp}(x) + \operatorname{div}_S X^T(x)$. For the tangent part, we claim that

$$\int_{G_k(\partial\mathcal{M})}\operatorname{div}_S X^T(x)\,\mathrm{d}V(x,S) = 0.\tag{3.9}$$

  In fact $\operatorname{div}_S X^T(x) = 0$ for any $x \in \partial\mathcal{M}$, and $\forall S \subset T_x\partial\mathcal{M}$. This is true because $X^T \equiv 0$ on $\partial\mathcal{M}$, therefore $D_\tau\langle X(x),\tau\rangle = 0$ for any $\tau \in T_x\partial\mathcal{M}$. Thus (3.9) follows by definition of tangential divergence (2.1) and by Lemma 3.1.
  For the orthogonal component we get

$$\operatorname{div}_S X^{\perp}(x) = \langle P_S\nabla\chi, \nabla d(x)\rangle + \chi(x)\operatorname{div}_S \nabla d(x).$$

  Since $\nabla d(x)$ is orthogonal to $\partial\mathcal{M}$, by Lemma 3.1 again we have $\langle P_S\nabla\chi(x), \nabla d(x)\rangle = 0$ for V-a.e. $(x,S) \in G_k(\partial\mathcal{M})$. Hence

$$\int_{G_k(\partial\mathcal{M})}\langle P_S\nabla\chi(x), \nabla d(x)\rangle\,\mathrm{d}V(x,S) = 0.$$

  Since $\nabla d(x) = -N(x)$ for every $x \in \partial\mathcal{M}$, where $N(x)$ is the unit normal vector to $\partial\mathcal{M}$ at x, we obtain

$$\int_{G_k(\partial\mathcal{M})}\chi(x)\operatorname{div}_S \nabla d(x)\,\mathrm{d}V(x,S) = \int_{G_k(\partial\mathcal{M})}\langle X(x), N(x)\rangle\operatorname{div}_S N(x)\,\mathrm{d}V(x,S)\tag{3.10}$$

  Thus, combining (3.9)-(3.10), we obtain

$$\int_{G_k(\partial\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{G_k(\partial\mathcal{M})}\langle X(x), N(x)\rangle\operatorname{div}_S N(x)\,\mathrm{d}V(x,S).\tag{3.11}$$

  We are now going to define $\tilde{H}$ and write the last integral in terms of it. To do so, by disintegration of V we can write

$$V = \|V\| \otimes \nu_x$$

where for $\|V\|$-a.e. x, ν_x is a probability measure on $G(k,n)$. Hence the right-hand side of (3.11) can be written as

$$\begin{aligned}\int_{G_k(\partial\mathcal{M})} &\langle X(x), N(x)\rangle \operatorname{div}_S N(x)\,\mathrm{d}V(x,S)\\ &= \int_{\partial\mathcal{M}} \langle X(x), N(x)\rangle \left(\int_{G(k,n)} \operatorname{div}_S N(x)\,\mathrm{d}\nu_x(S)\right) \mathrm{d}\|V\|(x).\end{aligned}$$

If we define

$$\tilde{H}(x) := -N(x)\int_{G(k,n)} \operatorname{div}_S N(x)\,\mathrm{d}\nu_x(S) \qquad \text{for } \|V\|\text{-a.e. } x\in\partial\mathcal{M}, \tag{3.12}$$

we can write (3.11) as

$$\int_{G_k(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\partial\mathcal{M}} \langle \tilde{H}(x), X(x)\rangle\,\mathrm{d}\|V\|(x). \tag{3.13}$$

Loosely speaking, $\tilde{H}(x)$ can be interpreted as the "mean curvature of $\partial\mathcal{M}$ weighted according to the planes charged by V at x"; in fact, in co-dimension 1, $\tilde{H}$ turns out to be precisely the mean curvature of $\partial\mathcal{M}$ (see Corollary 4.2). By its definition, it is clear that $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x\in\partial\mathcal{M}$, that $\tilde{H}\in L^\infty(\partial\mathcal{M},\|V\|)$ and that $\|\tilde{H}\|_{L^\infty(\partial\mathcal{M},\|V\|)}$ depends only on the second fundamental form of $\partial\mathcal{M}$. Gathering (3.8) and (3.13) we finally get (3.7).

- We now have to study the second term in the right-hand side of (3.6) Roughly speaking, we can see it as "the mean orthogonal part to $\partial\mathcal{M}$ of V" on the tubular neighborhood $U_\rho(\partial\mathcal{M})$. When $\rho\to 0$, we expect that this term takes into account the "transversal boundary" of V at $\partial\mathcal{M}$, that is the singular part of the first variation of V on $\partial\mathcal{M}$.
  More precisely, we are going to show the existence of a positive Radon measure σ_V (as expressed in the statement of the theorem), such that

$$\lim_{\rho\to 0}\frac{1}{\rho}\int \gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x), X(x)\rangle\,\mathrm{d}V(x,S) = \int_{\partial\mathcal{M}} \langle X(x), N(x)\rangle\,\mathrm{d}\sigma_V(x), \tag{3.14}$$

where $N(x)$ is the exterior unit normal vector to $\partial\mathcal{M}$. We first of all remark that the above limit exists by the existence of limits of the other two terms in (3.6).
To compute the limit in (3.14), we use again the decomposition $X = X^T + X^\perp = X^T + \chi\nabla d$. As $\rho\to 0$, we expect that the contribution of X^T is zero. Indeed, since $X\in\mathfrak{X}_\perp(\mathcal{M})$ and X is of class C^1, there exists a constant $c>0$ such that $|X^T(x)|\le cd(x)$. Therefore, since $\gamma'(s)\neq 0$ only for $s\in(1/2,1)$, we obtain

$$\begin{aligned}\lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})} &\left|\gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x), X^T(x)\rangle\right|\mathrm{d}V(x,S)\\ &\le \lim_{\rho\to 0} 3c\|V\|\big(U_\rho(\partial\mathcal{M})\setminus U_{\rho/2}(\partial\mathcal{M})\big) = 0.\end{aligned} \tag{3.15}$$

This proves the existence of the limit for the orthogonal part

$$\begin{aligned}\lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})} &\gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x), X^\perp(x)\rangle\,\mathrm{d}V(x,S)\\ &= \lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})} \chi(x)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S).\end{aligned} \tag{3.16}$$

(3.15) and (3.16) yield

$$\begin{aligned}\lim_{\rho\to 0}\int &\frac{1}{\rho}\gamma'\Big(\frac{d(x)}{\rho}\Big)\langle P_S\nabla d(x), X(x)\rangle\,\mathrm{d}V(x,S)\\ &= \lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})} \chi(x)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S).\end{aligned} \tag{3.17}$$

We now observe that the map

$$T\colon X\in\mathfrak{X}_\perp(\mathcal{M})\longmapsto \lim_{\rho\to 0}\frac{1}{\rho}\int_{G_k(\mathcal{M})}\chi(x)\gamma'\Big(\frac{d(x)}{\rho}\Big)|P_S\nabla d(x)|^2\,\mathrm{d}V(x,S).$$

is a well-defined distribution and, by its definition, $\operatorname{supp}T\subseteq\partial\mathcal{M}$. Again by definition, if $\chi(x)\le 0$ for every $x\in\partial\mathcal{M}$ (i.e. if X point outward $\partial\mathcal{M}$), then $T(X)\ge 0$. Therefore T is a signed distribution and by Riesz Representation Theorem there exists a positive Radon measure σ_V such that

$$\langle T,X\rangle=\int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V\qquad\forall X\in\mathfrak{X}_\perp(\mathcal{M}).$$

This completes the proof of (3.14).

Step 4: We now gather the previous computations to get (1.5).

By (3.6), (3.7) and (3.14), we can rewrite (3.5) as

$$\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\operatorname{div}_S\Big[\gamma\Big(\frac{d(x)}{\rho}\Big)X(x)\Big]\,\mathrm{d}V(x,S)=-\int_{\partial\mathcal{M}}\langle\tilde H(x),X(x)\rangle\,\mathrm{d}\|V\|(x)+\int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V. \tag{3.18}$$

Going back to (3.3), by (3.4) and (3.18) we finally get

$$\int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)=-\int_{\mathcal{M}}\langle X(x),H(x)+\tilde H(x)\rangle\,\mathrm{d}\|V\|(x)+\int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V.$$

which completes the proof of (1.5).

Step 5: We are left with the proof of (1.6) and (1.7). We begin with (1.7). Let us fix $x_0\in\partial\mathcal{M}$ and $r\le R$. Without loss of generality we can assume that $x_0=0$. To prove the estimate, we test (1.5) with $X(x)=-\gamma\big(\frac{|x|}{r}\big)\nabla d(x)$ which clearly belongs to $\mathfrak{X}_\perp(\mathcal{M})$. We have

$$\operatorname{div}_S X(x)=-\gamma'\Big(\frac{|x|}{r}\Big)\frac{1}{r}\langle P_S\frac{x}{|x|},\nabla d(x)\rangle-\gamma\Big(\frac{|x|}{r}\Big)\operatorname{div}_S\nabla d(x).$$

Therefore

$$\begin{aligned}\sigma_V\big(B_{r/2}(x)\big)&\le\int_{\partial\mathcal{M}}\gamma\Big(\frac{|x|}{r}\Big)\,\mathrm{d}\sigma_V\\&=\int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V\\&=\int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)+\int_{\mathcal{M}}\langle X,H+\tilde H\rangle\,\mathrm{d}\|V\|\\&=\int_{G_k(\mathcal{M})}\Big[-\gamma'\Big(\frac{|x|}{r}\Big)\frac{1}{r}\langle P_S\frac{x}{|x|},\nabla d(x)\rangle-\gamma\Big(\frac{|x|}{r}\Big)\operatorname{div}_S\nabla d(x)\Big]\,\mathrm{d}V(x,S)\\&\quad-\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{r}\Big)\langle\nabla d(x),H(x)+\tilde H(x)\rangle\,\mathrm{d}\|V\|(x)\end{aligned} \tag{3.19}$$

We want to estimate the last member of the above inequality. To do so, we choose a constant $c=c(\partial\mathcal{M},R)$ such that

$$|\operatorname{div}_S\nabla d(x)|\le c\qquad\forall x\in U_R(\partial\mathcal{M}).$$

By (3.12), the choice of c provides also $|\tilde H(x)|\le c$ for $\|V\|$-a.e. $x\in\partial\mathcal{M}$. Thus

$$\begin{aligned}-\int_{G_k(\mathcal{M})}\gamma\Big(\frac{|x|}{r}\Big)\operatorname{div}_S\nabla d(x)\,\mathrm{d}V(x,S)-&\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{r}\Big)\langle\nabla d(x),H(x)+\tilde H(x)\rangle\,\mathrm{d}\|V\|(x)\\&\le\int_{B_r(x)}\big(c+|H(x)|\big)\,\mathrm{d}\|V\|(x).\end{aligned}$$

Substituting in (3.19) and since γ can be taken so that $\|\gamma'\|_\infty\approx 2$, we get

$$\sigma_V\big(B_{r/2}(x)\big)\le\frac{c}{r}\|V\|\big(B_r(x)\big)+\int_{B_r(x)}|H(x)|\,\mathrm{d}\|V\|(x).$$

The proof of (1.6) is similar to the previous one and is in fact easier: it is enough to take a vector field $X \in \mathfrak{X}(\mathcal{M})$ such that $X(x) = \gamma\Big(\frac{d(x)}{\rho}\Big)\nabla d(x)$ for some ρ sufficiently small, so that $X = N$ on $\partial\mathcal{M}$ and use (1.5).

□

4. CONSEQUENCES OF THEOREM 1.1

In this section we clarify some consequences of Theorem 1.1.

- In subsection 4.1 we extend Theorem 1.1 to the case of varifolds with bounded first variation with respect to $\mathfrak{X}_0(\mathcal{M})$;
- In subsection 4.2 we study the codimension 1 case: we prove that if $k = n-1$, then the vector field $\tilde{H}$ given by Theorem 1.1 is the mean curvature vector of $\partial\mathcal{M}$ (Corollary 4.2); next we state a refined version of Lemma 3.1 for varifolds with free boundary: if $k = n-1$, then the restriction of V to $\partial\mathcal{M}$ is $(n-1)$-rectifiable (Corollary 4.3);
- In subsection 4.3 we extend Theorem 1.1 to varifolds with generalized mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$, i.e. vector fields compactly supported in the interior of $\mathcal{M}$ (see (2.1)), assuming that $\|V\|(\partial\mathcal{M}) = 0$ (Corollary 4.4); As a consequence, such varifolds have generalized mean curvature with respect to the larger class of vector fields $\mathfrak{X}_0(\mathcal{M})$;
- In subsection 4.4 we show that varifolds with free boundary have bounded first variation (Corollary 4.6);
- In subsection 4.5 we prove a monotonicity inequality (Lemma 4.7) for points on $\partial\mathcal{M}$; we next use the inequality to obtain monotonicity formulae for points on $\partial\mathcal{M}$ (Corollaries 4.8 and 4.9) without the reflections used by Grüter and Jost in [9].
- In subsection 4.6 we use the monotonicity inequality at the boundary to prove Theorem 1.4.

4.1. Varifolds with bounded variation with respect to $\mathfrak{X}_0(\mathcal{M})$.

The fact that the first variation of V with respect to $\mathfrak{X}_0(\mathcal{M})$ is absolutely continuous with respect to $\|V\|$ is not essential to prove Theorem 1.1. In fact the following slight modification holds true.

Theorem 4.1. *Let $V \in \mathcal{V}_k(\mathcal{M})$ be a k-varifold with bounded first variation $\delta_0 V$ with respect to $\mathfrak{X}_0(\mathcal{M})$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ on $\partial\mathcal{M}$ such that, for any $X \in \mathfrak{X}_\perp(\mathcal{M})$, it holds*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = \int_{\mathcal{M}} \langle X, \zeta\rangle\,\mathrm{d}|\delta_0 V| - \int_{\mathcal{M}} \langle X, \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \tag{4.1}$$

where $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$ for $\|V\|$-a.e. $x \in \partial\mathcal{M}$, $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends on the second fundamental form of $\partial\mathcal{M}$ and ζ is the polar vector of $\delta_0 V$ with respect to $|\delta_0 V|$. In particular, V has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$. Moreover, the following estimates hold:

$$\sigma_V(\partial\mathcal{M}) \le c\|V\|(\mathcal{M}) + |\delta_0 V|(\mathcal{M}); \tag{4.2}$$

$$\sigma_V\big(B_{r/2}(x_0)\big) \le \frac{c}{r}\|V\|\big(B_r(x_0)\big) + |\delta_0 V|\big(B_r(x_0)\big) \qquad \forall x_0 \in \partial\mathcal{M}, \forall r \le R(\mathcal{M}) \tag{4.3}$$

where $R(\mathcal{M})$ is such that the distance function from $\partial\mathcal{M}$ is of class C^2 in $U_R(\partial\mathcal{M})$ and the constant $c = c(\mathcal{M})$ depends on the second fundamental form of $\partial\mathcal{M}$.

Proof. The proof follows the one of Theorem 1.1. The only part of the proof of (1.5) where we use the hypothesis $\delta_0 V \ll \|V\|$ is in (3.4). If V has bounded first variation $\delta_0 V$ with respect to $\mathfrak{X}_0(\mathcal{M})$, then one easily obtain

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{G_k(\mathcal{M})}\operatorname{div}_S\left[\left(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\right)X(x)\right]\mathrm{d}V(x,S)\\
&\quad=\lim_{\rho\to 0}\int_{\mathcal{M}}\left(1-\gamma\Big(\frac{d(x)}{\rho}\Big)\right)\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta_0V|(x)\\
&\quad=-\int_{\mathcal{M}^\circ}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta_0V|(x)\\
&\quad=-\int_{\mathcal{M}}\langle X(x),\zeta(x)\rangle\,\mathrm{d}|\delta_0V|(x)
\end{aligned}\tag{4.4}$$

where ζ is the polar vector of δ_0V with respect to $|\delta_0V|$ and the last equality is due to the assumption $|\delta_0V|(\partial\mathcal{M})=0$ (see Remark 2.1).

The modifications to the proofs of (1.6) and (1.7) to obtain (4.2) and (4.3) are obvious. □

4.2. **The codimension 1 case:** $k=n-1$**.** If $k=n-1$, we can characterize $\tilde{H}$ in a simpler way: if $x\in\partial\mathcal{M}$, then $\tilde{H}(x)$ is the mean curvature vector of $\partial\mathcal{M}$.

Corollary 4.2. *Let $V\in\mathcal{V}_{n-1}(\mathcal{M})$ have generalized mean curvature H with respect to $\mathfrak{X}_0(\mathcal{M})$ with $H\in L^1(\mathcal{M},\|V\|)$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ such that*

$$\int_{G_{n-1}(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)=-\int_{\mathcal{M}}\langle X,H+\tilde{H}\rangle\,\mathrm{d}\|V\|+\int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V\qquad\forall X\in\mathfrak{X}_\perp(\mathcal{M})$$

where $\tilde{H}$ is the mean curvature vector of $\partial\mathcal{M}$, that is $\tilde{H}(x):=-N(x)\big(\operatorname{div}_{T_x\partial\mathcal{M}}N(x)\big)$ for $x\in\partial\mathcal{M}$. Moreover, (1.7) *holds true.*

Proof. If S is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, then $S\subset T_x\partial\mathcal{M}$ if and only if $S=T_x\partial\mathcal{M}$. Therefore, if $k=n-1$ and $V\in\mathcal{V}_{n-1}(\mathcal{M})$ with generalized mean curvature with respect to $\mathfrak{X}_0(\mathcal{M})$, Lemma 3.1 yields

$$V\left(\{(x,S)\in G_{n-1}(\partial\mathcal{M})\mid S\neq T_x\Gamma\}\right)=0.\tag{4.5}$$

Hence, for $X\in\mathfrak{X}_\perp(\mathcal{M})$,

$$\int_{G_{n-1}(\partial\mathcal{M})}X(x)\operatorname{div}_S N(x)\,\mathrm{d}V(x,S)=\int_{\partial\mathcal{M}}X(x)\operatorname{div}_{T_x\partial\mathcal{M}}N(x)\,\mathrm{d}\|V\|,$$

thus (3.11) becomes

$$\int_{G_{n-1}(\partial\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)=\int_{\partial\mathcal{M}}\langle N(x),X(x)\rangle\operatorname{div}_{T_x\partial\mathcal{M}}N(x)\,\mathrm{d}\|V\|(x).$$

Thus, defining

$$\tilde{H}(x):=-N(x)\operatorname{div}_{T_x\partial\mathcal{M}}N(x),$$

we get (3.13). □

If $k=n-1$ and V has bounded first variation with respect to $\mathfrak{X}_t(\mathcal{M})$ instead of $\mathfrak{X}_0(\mathcal{M})$, we can strengthen the conclusion of Lemma 3.1: $V\llcorner G_{n-1}(\partial\mathcal{M})$ is $(n-1)$-rectifiable.

Corollary 4.3. *Let $V\in\mathcal{V}_{n-1}(\mathcal{M})$ with bounded first variation with respect to $\mathfrak{X}_t(\mathcal{M})$. Then $V\llcorner G_{n-1}(\partial\mathcal{M})$ is an $(n-1)$-rectifiable varifold. More precisely, if $\varphi\in C_c\big(G_{n-1}(\mathcal{M})\big)$, then*

$$\int_{G_{n-1}(\mathcal{M})}\varphi(x,S)\,\mathrm{d}V(x,S)=\int_{\partial\mathcal{M}}\varphi(x,T_x\partial\mathcal{M})\theta(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)+\int_{G_{n-1}(\mathcal{M}^\circ)}\varphi(x,S)\,\mathrm{d}V(x,S),$$

where $\theta(x)=(\omega_{n-1})^{-1}\Theta^{n-1}(\|V\|,x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\mathcal{M}$.

Proof. By Lemma 3.1 and in particular by (4.5), for any $\varphi\in C_c\big(G_{n-1}(\mathcal{M})\big)$ we have

$$\int_{G_{n-1}(\mathcal{M})}\varphi(x,S)\,\mathrm{d}V(x,S)=\int_{\partial\mathcal{M}}\varphi(x,T_x\partial\mathcal{M})\,\mathrm{d}\|V\|(x)+\int_{G_{n-1}(\mathcal{M}^\circ)}\varphi(x,S)\,\mathrm{d}V(x,S).$$

By [12, Lemma 40.5]), for $\|V\|$-a.e. $x \in \partial\mathcal{M}$ there exists the density $\Theta^{n-1}(\|V\|, x) < \infty$. Hence, since $\partial\mathcal{M}$ is of class C^2, the quantity

$$\theta(x) := \lim_{\rho\to 0} \frac{\|V\|\big(B_\rho(x)\big)}{\mathcal{H}^{n-1}\llcorner\partial\mathcal{M}\big(B_\rho(x)\big)} = \frac{\Theta^{n-1}(\|V\|, x)}{\omega_{n-1}}$$

exists and is finite for $\|V\|$-a.e. $x \in \partial\mathcal{M}$. By Radon-Nikodym Theorem [12, Theorem 4.7], since the singular set $\{x \in \mathcal{M} \mid \theta(x) = +\infty\}$ of $\|V\|\llcorner\partial\mathcal{M}$ is $\|V\|$-negligible, we have $\|V\|\llcorner\partial\mathcal{M} \ll \mathcal{H}^{n-1}\llcorner\partial\mathcal{M}$ and the conclusion follows. □

4.3. **Varifolds with mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$.** The analogous of Theorem 1.1 holds, adding the extra hypothesis $\|V\|(\partial\mathcal{M}) = 0$, even if V has generalized mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$, i.e. the vector fields with compact support in the interior of $\mathcal{M}$. In fact, if we analyze the proof of Theorem 1.1, we can see that the only point where we used the existence of generalized mean curvature with respect $\mathfrak{X}_0(\mathcal{M})$ is to obtain (3.11), that is to obtain the identity

$$\int_{G_k(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{G_k(\partial\mathcal{M})} \langle X(x), N(x)\rangle \operatorname{div}_S N(x)\,\mathrm{d}V(x,S)$$

by the use of Lemma 3.1, whereas if $\|V\|(\partial\mathcal{M}) = 0$ then obviously we have

$$\int_{G_k(\partial\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = 0.$$

Since the remaining arguments remain valid also if V has mean curvature with respect to $\mathfrak{X}_c(\mathcal{M})$, we have proved the following corollary.

Corollary 4.4. *Let $V \in \mathcal{V}_k(\mathcal{M})$ with generalized mean curvature H with respect to $\mathfrak{X}_c(\mathcal{M})$ with $H \in L^1(\mathcal{M}, \|V\|)$ and $\|V\|(\partial\mathcal{M}) = 0$. Then there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ such that*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \qquad \forall X \in \mathfrak{X}_\perp(\mathcal{M}) \tag{4.6}$$

In particular, V has bounded first variation with respect to $\mathfrak{X}_\perp(\mathcal{M})$ and the estimates (1.6), (1.7) *on σ_V hold true.*

Remark 4.1. If we remove the hypothesis $\|V\|(\partial\mathcal{M}) = 0$ nothing can be said about the behavior of V on $\partial\mathcal{M}$, because any vector field in $\mathfrak{X}_c(\mathcal{M})$ has first derivatives compactly supported in the interior of $\mathcal{M}$. So we have a lack of test vector fields to establish any property of V on $\partial\mathcal{M}$: e.g. take a smooth surface in $\mathcal{M}^\circ$ and add any varifold $W \in \mathcal{V}_k(\partial\mathcal{M})$ with unbounded first variation with respect $\mathfrak{X}_\perp(\mathcal{M})$.

Since $\mathfrak{X}_0(\mathcal{M}) \subset \mathfrak{X}_\perp(\mathcal{M})$, the following result follows.

Corollary 4.5. *Let $V \in \mathcal{V}_k(\mathcal{M})$ with generalized mean curvature H with respect to $\mathfrak{X}_c(\mathcal{M})$, with $H \in L^1(\mathcal{M}, \|V\|)$ and $\|V\|(\partial\mathcal{M}) = 0$. Then V has generalized mean curvature H with respect to $\mathfrak{X}_0(\mathcal{M})$.*

4.4. **Varifolds with free boundaries.** As an immediate corollary of Theorem 1.1, varifolds with free boundaries have bounded first variation.

Corollary 4.6. *Let $V \in \mathcal{V}_k(\mathcal{M})$ have free boundary at $\partial\mathcal{M}$ with $H \in L^1(\mathcal{M}, \|V\|)$. Then V has bounded first variation. More precisely, there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ such that*

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X, H + \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V \qquad \forall X \in \mathfrak{X}(\mathcal{M}), \tag{4.7}$$

where $\tilde{H}$ is defined as in (3.12)*. In particular $\tilde{H}$ is orthogonal to $\partial\mathcal{M}$, $\tilde{H} \in L^\infty(\partial\mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends only on the second fundamental form of $\partial\mathcal{M}$. Moreover,* (1.6) (1.7) *hold true.*

Proof. If $X \in \mathfrak{X}(\mathcal{M})$, there exist $X^T, X^\perp$ such that $X = X^T + X^\perp$, with $X^T \in \mathfrak{X}_t(\mathcal{M})$ and $X^\perp \in \mathfrak{X}_\perp(\mathcal{M})$ (see the decomposition at the beginning of the proof of Theorem 1.1). We have

$$\operatorname{div}_S X(x) = \operatorname{div}_S X^T(x) + \operatorname{div}_S X^\perp(x).$$

Since the mean curvature of V with respect to $\mathfrak{X}_0(\mathcal{M})$ is given by $H\mathbf{1}_{\mathcal{M}^\circ}$, by Theorem 1.1 there exists a positive Radon measure σ_V on $\partial\mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ such that

$$\begin{aligned}\int_{G_k(\mathcal{M})} \operatorname{div}_S X^\perp(x)\,\mathrm{d}V(x,S) &= -\int_{\mathcal{M}} \langle X^\perp, H\mathbf{1}_{\mathcal{M}^\circ} + \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V\\ &= -\int_{\mathcal{M}} \langle X^\perp, H + \tilde{H}\rangle\,\mathrm{d}\|V\| + \int_{\partial\mathcal{M}} \langle X, N\rangle\,\mathrm{d}\sigma_V,\end{aligned}$$

where the last equality follows because H is assumed to be tangent to $\partial\mathcal{M}$ on $\partial\mathcal{M}$ (see Remark 2.2). Moreover we have the same estimates (1.6) and (1.7) on σ_V. For what concerns the tangent part X^T, the definition of varifold with free boundary yields

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X^T(x)\,\mathrm{d}V(x,S) = -\int_{\mathcal{M}} \langle X^T, H\rangle\,\mathrm{d}\|V\|.$$

This shows the conclusion. □

4.5. **Monotonicity formulae.** Grüter and Jost established in [9] several properties of varifolds with free boundaries: monotonicity formulae for $\|V\|$ at the boundary [9, Theorem 3.1], which imply the existence of $\Theta^k(\|V\|, x)$ for every point x if the mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > k$.

The monotonicity results are obtained by reflecting the balls across $\partial\mathcal{M}$, i.e. they have monotonicity of the sum of the masses in the ball and in the reflected ball [9, Theorem 3.1]. Using Corollary 4.6 it is possible to obtain the monotonicity of the mass in $B_r(x)$, without reflecting the balls.

We begin with a monotonicity inequality which is used also in the proof of Theorem 1.4.

Lemma 4.7 (Monotonicity inequality)**.** *Suppose $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$, with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \in [1, +\infty)$. Then there exists a constant $c > 0$ that depends only on n, k, p and on the second fundamental form of $\partial\mathcal{M}$ such that, for all $x_0 \in \partial\mathcal{M}$ and $s \in \mathbb{R}$ the following inequality holds:*

$$\begin{aligned}(1+c\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} \geq &-\rho^{-\frac{k-s}{p}}\Big(\frac{1}{p} + c\rho\Big)\left(\frac{1}{\rho^s}\int_{B_\rho} |H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\\ &- c(1+\rho)\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}\end{aligned} \tag{4.8}$$

Proof. Without loss of generality we can suppose $x_0 = 0$. Since for large ρ the statement is obvious, we have to prove it only for $0 < \rho < R(\mathcal{M})$, where $R(\mathcal{M})$ is defined in section 2. We want to bound from below the following derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1-p}{p}}. \tag{4.9}$$

To do so, we want to bound from below the derivative in the right-hand side to get a differential inequality. We have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) = -\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})} \left(k\gamma\Big(\frac{|x|}{\rho}\Big) + \frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\right)\mathrm{d}V(x,S).$$

Let us choose $X(x) = \gamma\Big(\frac{|x|}{\rho}\Big)x$. Then

$$\operatorname{div}_S X(x) = k\gamma\Big(\frac{|x|}{\rho}\Big) + \frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\left|P_S\frac{x}{|x|}\right|^2.$$

We use Corollary 4.6: by testing (4.7) with X we get the following *monotonicity identity*:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) =& -\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)\\
&-\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\left|P_{S^\perp}\frac{x}{|x|}\right|^2\mathrm{d}V(x,S)\\
=&\frac{1}{\rho^{k+1}}\int_{\mathcal{M}}\langle X,H+\tilde{H}\rangle\,\mathrm{d}\|V\| - \frac{1}{\rho^{k+1}}\int_{\partial\mathcal{M}}\langle X,N\rangle\,\mathrm{d}\sigma_V\\
&-\frac{1}{\rho^{k+1}}\int_{G_k(\mathcal{M})}\frac{|x|}{\rho}\gamma'\Big(\frac{|x|}{\rho}\Big)\left|P_{S^\perp}\frac{x}{|x|}\right|^2\mathrm{d}V(x,S)
\end{aligned}\tag{4.10}$$

We have to estimate from below the last member of the above identity.

Since $\gamma'\leq 0$, we can neglect the last integral and, since $|x|\leq\rho$ by cut-off, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\geq& -\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)|H+\tilde{H}|\,\mathrm{d}\|V\|\\
&-\frac{1}{\rho^k}\int_{\partial\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\Big|\langle\frac{x}{|x|},N\rangle\Big|\,\mathrm{d}\sigma_V.
\end{aligned}\tag{4.11}$$

We have now to bound the two terms in the right-hand side of (4.11).

- For the first one, since $H\in L^p(\mathcal{M},\|V\|)$ and $\tilde{H}\in L^\infty(\partial\mathcal{M},\|V\|)$, also $H+\tilde{H}\in L^p(\mathcal{M},\|V\|)$. Therefore, by Hölder inequality we get

$$\begin{aligned}
\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)|H+\tilde{H}|\,\mathrm{d}\|V\|\leq&\frac{1}{\rho^k}\left(\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\int_{\mathcal{M}}\gamma^{\frac{p}{p-1}}\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}\\
\leq&\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\rho^{-\frac{k-s}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}
\end{aligned}\tag{4.12}$$

- We now move on the estimate of the second integral in the right-hand side of (4.11). Since $\partial\mathcal{M}$ is of class C^2 and since $0\in\partial\mathcal{M}$, there exists a constant c such that

$$|\langle\frac{x}{|x|},N(x)\rangle|\leq c|x|.\tag{4.13}$$

  This yields

$$\frac{1}{\rho^k}\int_{\partial\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\big|\langle\frac{x}{|x|},N\rangle\big|\,\mathrm{d}\sigma_V\leq\frac{c}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V.\tag{4.14}$$

  We have to further estimate the righ-hand side of this inequality. This is done by testing (4.7) with $X(x)=-\gamma\Big(\frac{|x|}{r}\Big)\nabla d(x)$; as in (3.19) we get

$$\begin{aligned}
\frac{1}{\rho^{k-1}}\int\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V=&-\frac{1}{\rho^{k-1}}\int_{G_k(\mathcal{M})}\gamma'\Big(\frac{|x|}{\rho}\Big)\frac{1}{\rho}\langle P_S\frac{x}{|x|},\nabla d(x)\rangle\,\mathrm{d}V(x,S)\\
&-\frac{1}{\rho^{k-1}}\int_{G_k(\mathcal{M})}\gamma\Big(\frac{|x|}{\rho}\Big)\operatorname{div}_S\nabla d(x)\,\mathrm{d}V(x,S)\\
&-\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\langle\nabla d(x),H+\tilde{H}\rangle\,\mathrm{d}\|V\|(x).
\end{aligned}$$

Since $\gamma'(s) = 0$ if $s \in (0,1/2)$, we have $\frac{|x|}{\rho} \geq \frac{1}{2}$; Moreover, using $|\operatorname{div}_S \nabla d| \leq c$ (because $\rho < R$ and d is of class C^2 in $\overline{U_R(\partial\mathcal{M})}$) and (4.12), we obtain

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V \leq & -\frac{2}{\rho^{k-1}}\int_{\mathcal{M}} \gamma'\Big(\frac{|x|}{\rho}\Big)\frac{|x|}{\rho^2}\,\mathrm{d}\|V\|(x) + \frac{c}{\rho^{k-1}}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ & + \rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho} |H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}\\ = & \frac{2}{\rho^{k-1}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) - \frac{2(k-1)}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ & + \frac{2(k-1)+c\rho}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ & + \rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho} |H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}\end{aligned} \tag{4.15}$$

Concerning the last member, we now observe that

$$\begin{aligned}\frac{2}{\rho^{k-1}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right) - \frac{2(k-1)}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ = 2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\end{aligned}$$

Substituting in (4.15) we get

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V \leq & 2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\\ & + \frac{2(k-1)+c\rho}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\\ & + \rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho} |H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}.\end{aligned}$$

Taking into account that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^{k-1}}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\\ &\quad = \frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x) + \rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right),\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{\rho^{k-1}}\int \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\sigma_V \leq & \frac{2k+c\rho}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x) + 2\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)\\ & + \rho^{1-\frac{k-s}{p}}\left(\frac{1}{\rho^s}\int_{B_\rho} |H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{1-\frac{1}{p}}.\end{aligned} \tag{4.16}$$

This complete the estimate of the second term in the right-hand side of (4.11).

Gathering (4.12), (4.14) and (4.16) in (4.11), we can estimate (4.9) as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1}{p}} =&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|\right)^{\frac{1-p}{p}}\\ \geq&-\rho^{-\frac{k-s}{p}}\Big(\frac{1}{p}+c\rho\Big)\left(\frac{1}{\rho^s}\int_{B_\rho}|H+\tilde{H}|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\\ &-c(1+\rho)\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}\\ &-c\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}.\end{aligned}$$

which is the desired inequality. ☐

Corollary 4.8. *Let* $V\in\mathcal{V}_k(\mathcal{M})$ *with free boundary at* $\partial\mathcal{M}$*, with* $H\in L^p(\mathcal{M},\|V\|)$ *for some* $p\in(k,+\infty)$*. Then there exists* $\Lambda=\Lambda(k,p,\mathcal{M},\|H\|_{L^p})>0$ *such that, for all* $x_0\in\partial\mathcal{M}$ *the function*

$$\rho\longmapsto e^{\Lambda\rho}\left(\frac{\|V\|\big(B_\rho(x_0)\big)}{\rho^k}\right)^{\frac{1}{p}}+\Lambda e^{\Lambda\rho}\rho^{1-\frac{k}{p}} \tag{4.17}$$

is monotone increasing.

Proof. Without loss of generality we can assume $x_0=0\in\partial\mathcal{M}$. We first notice that we have to prove the result only for small ρ, since it is clearly true for $\rho>\operatorname{diam}(\mathcal{M})$.

We choose $s=0$ in (4.8); therefore $\frac{k}{p}\in(0,1)$ since $p>k$. Rearranging we obtain the existence of $\Lambda>0$, which depends on the constant c of Lemma 4.7, on $\|H\|_{L^p(\mathcal{M})}$ and on the second fundamental form of $\partial\mathcal{M}$, such that

$$\frac{\mathrm{d}}{\mathrm{d}\rho}e^{\Lambda\rho}\left[\left(\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\right)^{\frac{1}{p}}+\Lambda\rho^{1-\frac{k}{p}}\right]\geq 0.$$

Since the constant c of Lemma 4.7 does not depend on the choice of γ (the estimates are indipendent on the choice of γ unless that $\gamma'(s)=0$ for $s\in(0,1/2)$), letting γ increase to $\mathbf{1}_{[0,1)}$ we have that the function

$$\rho\longmapsto e^{\Lambda\rho}\left[\left(\frac{\|V\|(B_\rho)}{\rho^k}\right)^{\frac{1}{p}}+\Lambda\rho^{1-\frac{k}{p}}\right]$$

is monotone increasing. ☐

If $H\in L^\infty(\mathcal{M},\|V\|)$, in (4.12) we simply estimate

$$\frac{1}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)|H+\tilde{H}|\,\mathrm{d}\|V\|\leq\frac{\|H+\tilde{H}\|_\infty}{\rho^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|$$

By repeating the previous arguments we have the following result:

Corollary 4.9. *Let* $V\in\mathcal{V}_k(\mathcal{M})$ *have free boundary at* $\partial\mathcal{M}$ *with* $H\in L^\infty(\mathcal{M},\|V\|)$*. Then there exists* $\Lambda=\Lambda(k,\mathcal{M},\|H\|_\infty)$ *such that, for all* $x\in\partial\mathcal{M}$ *the function*

$$\rho\longmapsto e^{\Lambda\rho}\frac{\|V\|\big(B_\rho(x)\big)}{\rho^k} \tag{4.18}$$

is monotone increasing.

4.6. **Density set and proof of Theorem 1.4.** In order to prove Theorem 1.2 when $p \in (1,k]$, we have to show that, for every varifold V with free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$, the Hausdorff dimension of the set of points where the k-density $\Theta^k(\|V\|,\cdot)$ does not exist or is infinite is at most $k-p$.

We introduce the definition of *density set* for a k-varifold V.

Definition 4.1 (Density set)**.** Let $V \in V_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ and $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$. Then the density set for V is defined as

$$\mathrm{Dens}(V) = \bigcup_{s\in(k-p,k]} \Big\{x \in \mathcal{M} \mid \limsup_{r\to 0} \frac{1}{r^s}\int_{B_r(x)} |H+\tilde{H}|^p \,\mathrm{d}\|V\| = 0\Big\}.$$

Remark 4.2. We notice that, if $p > k$, then $\mathrm{Dens}(V) = \mathcal{M}$ ($s = 0 \in (k-p,k]$) and Theorem 1.4 is an easy consequence of the monotonicity of density ratios, Corollaries 4.8 and 4.9. Thus Theorem 1.4 is an extension to lower integrability of the mean curvature of the well-known existence of the density at every point when $p > k$.

Proof of Theorem 1.4. We first show that $\mathcal{H}^s\big(M \setminus \mathrm{Dens}(V)\big) = 0$ for all $s > k-p$. We have

$$\mathrm{Dens}(V) = \bigcup_{s\in(k-p,k]} A_s; \qquad A_s = \Big\{x \in \mathcal{M} \mid \limsup_{r\to 0} \frac{1}{r^s}\int_{B_r(x)} |H+\tilde{H}|^p \,\mathrm{d}\|V\| = 0\Big\}.$$

If μ is absolutely continuous with respect to $\|V\|$ and has density $|H+\tilde{H}|^p$, that is $\mu = |H+\tilde{H}|^p\|V\|$, then for any $s \in (k-p,k]$ we have

$$\mathcal{M} \setminus A_s = \bigcup_{i\in\mathbb{N}} \Big\{x \in \mathcal{M} \mid \Theta^{*s}(\mu,x) \geq \frac{1}{i}\Big\}.$$

Then, by [10, Theorem 6.9], for every $i \in \mathbb{N}$

$$\mathcal{H}^s\Big(\Big\{x \mid \Theta^{*s}(\mu,x) \geq \frac{1}{i}\Big\}\Big) \leq i\mu\Big(\Big\{x \mid \Theta^{*s}(\mu,x) \geq \frac{1}{i}\Big\}\Big).$$

We want to show that for every $i \in \mathbb{N}$ the right-hand side is equal to 0, which would prove (1.9). To see this, we first recall that for a varifold with bounded first variation in $\mathcal{M}$ the $\Theta^k(\|V\|,x)$ exists and is finite for $\|V\|$-a.e. $x \in \mathcal{M}$ by [12, Lemma 40.5]; since

$$\lim_{r\to 0} \frac{1}{\|V\|(B_r(x))}\int_{B_r(x)} |H+\tilde{H}|^p \,\mathrm{d}\|V\| < \infty \qquad \text{for } \|V\|\text{-a.e. } x \in \mathcal{M},$$

we have

$$\mu\Big(\Big\{x \mid \Theta^{*s}(\mu,x) \geq \frac{1}{i}\Big\}\Big) = 0.$$

Thus $\mathcal{H}^s\big(\mathcal{M} \setminus \mathrm{Dens}(V)\big) \leq \mathcal{H}^s\big(\mathcal{M} \setminus A_s\big) = 0$. This shows (1.9).

In order to prove (1.10), we fix $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$ and $s \in (k-p,k]$ such that $x_0 \in A_s$. Without loss of generality we can assume $x_0 = 0$. It is enough to show the result only for small radii, because it is clearly valid if $\mathrm{diam}(\mathcal{M}) \leq r < t$. By the choice of s, we have $\frac{k-s}{p} \in [0,1)$. Applying Lemma 4.7 and taking into account $0 \in A_s$, we obtain that exists $\Lambda > 0$ (which depends on the point x_0 chosen) such that

$$\frac{\mathrm{d}}{\mathrm{d}\rho} e^{\Lambda\rho}\left[\Big(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\Big)^{\frac{1}{p}} + \Lambda\rho^{1-\frac{k-s}{p}}\right] \geq 0.$$

Since this is independent on the choice of γ, letting γ to increase to $\mathbf{1}_{[0,1)}$ we obtain (1.10).

If $x \in \mathrm{Dens}(V) \cap M^{\circ}$ the proof of (1.10) is the same, except for the fact that we have to use the classical monotonicity inequality for interior points, which indeed is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}\Big(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|\Big)^{\frac{1}{p}} \geq{}& -\frac{\rho^{-\frac{k-s}{p}}}{p}\Big(\frac{1}{\rho^s}\int_{B_\rho} |H|^p \,\mathrm{d}\|V\|\Big)^{\frac{1}{p}} \\ &- \Big(\frac{1}{\rho^k}\int_{\mathcal{M}} \gamma\Big(\frac{|x-x_0|}{\rho}\Big)\,\mathrm{d}\|V\|(x)\Big)^{\frac{1}{p}}.\end{aligned}$$

The existence of $\Theta^k(\|V\|,\cdot)$ on $\mathrm{Dens}(V)$ and the upper semi-continuity of the restrictions of $\Theta^k(\|V\|,\cdot)$ to $\mathrm{Dens}(V)\cap\mathcal{M}^o$ and to $\mathrm{Dens}(V)\cap\partial\mathcal{M}$ are easy consequences of (1.10). □

5. Proof of Theorem 1.2

In this section we often use the following assumption on V.

Assumption 1. $V\in\mathcal{V}_k(\mathcal{M})$ is a rectifiable k-varifold with free boundary at $\partial\mathcal{M}$, with generalized mean curvature $H\in L^p(\mathcal{M},\|V\|)$ for some $p\geq 1$ and $\Theta^k(\|V\|,x)\geq 1$ for $\|V\|$-a.e. $x\in\mathcal{M}$.

We remark that if V satisfies Assumption 1, Corollary 4.6 establishes the existence of the measure σ_V and that V has bounded first variation.

We first recall the definition of σ_V^*, the $(k-1)$-*dimensional part* of σ_V:

$$\sigma_V^*=\sigma_V\llcorner E,\qquad E=\{x\mid 0<\Theta_*^{k-1}(\sigma_V,x)\leq\Theta^{*(k-1)}(\sigma_V,x)<+\infty\}.$$

As stated in the introduction, we recall that if $H\in L^p(\mathcal{M},\|V\|)$ for some $p>k$, then the condition $\Theta^{*(k-1)}(\sigma_V,x)<+\infty$ is not restrictive, since it holds *for every* point $x\in\partial\mathcal{M}$; indeed (1.7) and Hölder inequality yield

$$\begin{aligned}\frac{\sigma_V(B_{r/2}(x))}{r^{k-1}}\leq& c\frac{\|V\|\big(B_r(x)\big)}{r^k}+\frac{1}{r^{k-1}}\int_{B_r(x)}|H|\,\mathrm{d}\|V\|\\ \leq& c\frac{\|V\|\big(B_r(x)\big)}{r^k}+r^{1-\frac{k}{p}}\bigg(\int_{B_r(x)}|H|^p\,\mathrm{d}\|V\|\bigg)^{\frac1p}\bigg(\frac{\|V\|\big(B_r(x)\big)}{r^k}\bigg)^{1-\frac1p};\end{aligned}$$

By Lemma 4.8 and $p>k$, the last member of the above inequality is bounded as $r\to 0$, therefore it follows $\Theta^{*(k-1)}(\sigma_V,x)<+\infty$. This proves that Theorem 1.3 is a corollary of Theorem 1.2.

Since in the proof of Theorem 1.2 we have to deal with the blow-ups of V at points on $\partial\mathcal{M}$, we introduce a notation for the scalings of V at a point $x_0\in\partial\mathcal{M}$ that we use throughout this section.

Notation (Scalings)**.** If $V\in\mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$, if $x_0\in\partial\mathcal{M}$ is a fixed point and $r_j\downarrow 0$ is a fixed sequence, we use the following notations:

$$\mathcal{M}_j:=\tau_{x_0,r_j}(\mathcal{M})\qquad V_j:=(\tau_{x_0,r_j})_\sharp V\in\mathcal{V}_k(\mathcal{M}_j)\qquad\sigma_j:=\sigma_{V_j};$$

where τ_{x_0,r_j} is the dilation function defined in (2.2) and $f_\sharp V$ denotes the push-forward of V through f defined in (2.7). In Lemma 5.4 we show that each V_j has free boundary at $\partial\mathcal{M}_j$, thus Corollary 4.6 states the existence of σ_{V_j}, which we call σ_j. Moreover, we denote by H_j the generalized mean curvature of V_j and $\tilde{H}_j$ the vector field provided by Corollary 4.6 relative to V_j. If $x\in\partial\mathcal{M}_j$, we denote by $N_j(x)$ the exterior unit normal to $\mathcal{M}_j$ at x. As $j\to\infty$, according to the definition of tangent space given in section 2, we say that $\mathcal{M}_j\to T^+_{x_0}\mathcal{M}$ and that $\partial\mathcal{M}_j\to T_{x_0}\partial\mathcal{M}$.

The idea of the proof is to study the blow-ups of V at points on $\partial\mathcal{M}$, to prove that the at σ_V^*-a.e. point on $\partial\mathcal{M}$, every $(k-1)$-blow-up of σ_V^* is of the form $\alpha\mathcal{H}^{k-1}\llcorner S$ for some $(k-1)$-dimensional linear subspace S. This allows us to apply the Marstrand-Mattila Rectifiability Criterion [4, Theorem 5.1] to σ_V^*. We state explicitly the criterion for the reader convenience:

Theorem 5.1 (Marstrand-Mattila Rectifiability Criterion)**.** *Let $m\leq n$ be a natural number and let μ be a positive Radon measure on $\mathbb{R}^n$ such that, for μ-a.e. $x\in\mathbb{R}^n$, we have*

(1) $0<\Theta_*^m(\mu,x)\leq\Theta^{*m}(\mu,x)<\infty$;
(2) Every m-tangent measure of μ at x is of the form $\beta\mathcal{H}^m\llcorner S$ for some m-dimensional linear subspace of $\mathbb{R}^n$.

Then μ is m-rectifiable.

We begin in subsection 5.1 where we adapt to σ_V^* two well-known facts about measures:

- in Lemma 5.2 we show that $\sigma_V^*\big(\mathcal{M}\setminus\mathrm{Dens}(V)\big)=0$; this allows us to check the conditions of the Marstrand-Mattila criterion just on $\mathrm{Dens}(V)$.
- In Lemma 5.3 we prove that for σ_V^*-a.e. $x\in\partial\mathcal{M}$, σ_V and σ_V^* have the same $(k-1)$-tangent measures.

Next, since by definition

$$0 < \Theta^{k-1}_*(\sigma^*_V, x) \leq \Theta^{*(k-1)}(\sigma^*_V, x) < +\infty \qquad \text{for } \sigma^*_V\text{-a.e. } x \in \partial\mathcal{M},$$

to apply Marstrand-Mattila Rectifiability Criterion it remains to show that for σ^*_V-a.e. $x_0 \in \mathrm{Dens}(V)$, every tangent measure to σ_V is a $(k-1)$-dimensional plane. Subsection 5.2 is devoted to prove this, which is achieved in several steps. The outline of the proof is the following:

(1) In Lemma 5.4 we first prove that at every $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$, tangent varifolds to V are cones in the half-space $T^+_{x_0}\mathcal{M}$ that are stationary with respect to $\mathfrak{X}_t(T^+_{x_0}\mathcal{M})$. Thus, if $C \in \mathrm{Tan}(V, x_0)$ for some $x_0 \in \mathrm{Dens}(V) \cap \partial\mathcal{M}$, Corollary 4.6 provides the existence of the measure σ_C. We also prove that the measures σ_j relative to the scalings of V converge weakly to σ_C; This imply that (Corollary 5.5)

$$\mathrm{Tan}^{k-1}(\sigma^*_V, x_0) = \{\sigma_C \mid C \in \mathrm{Tan}(V, x_0)\} \qquad \text{for } \sigma^*_V\text{-a.e. } x_0 \in \partial\mathcal{M}. \tag{5.1}$$

(2) In Lemma 5.6 we prove that each tangent cone C has an invariant linear subspace D_C that coincides with the set of points where the k-density $\Theta^k(\|C\|, \cdot)$ attains its maximum;

(3) In Lemma 5.7 we show that for σ^*_V-a.e. $x_0 \in \partial\mathcal{M}$, if $C \in \mathrm{Tan}(V, x_0)$, then σ_C is concentrated on D_C;

(4) In Lemma 5.8 we prove that for σ^*_V-a.e. $x_0 \in \partial\mathcal{M}$, if $C \in \mathrm{Tan}(V, x_0)$, then D_C is $(k-1)$-dimensional and $\sigma_C = \alpha\mathcal{H}^{k-1} \llcorner D_C$ for some constant $\alpha = \alpha(x_0, C)$;

(5) In subsection 5.3 we summarize all these facts to conclude the proof of Theorem 1.2: by Lemma 5.8 and (5.1) it follows that, for σ^*_V-a.e. $x_0 \in \partial\mathcal{M}$, every $(k-1)$-tangent measure to σ_V at x_0 is a $(k-1)$-plane.

5.1. **Summary of well-known facts.** We begin with a measure-theoretic lemma applied to σ^*_V.

Lemma 5.2. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$. Then $\sigma^*_V \ll \mathcal{H}^{k-1}$. In particular, if $p > 1$, we have*

$$\sigma^*_V\big(\partial\mathcal{M} \setminus \mathrm{Dens}(V)\big) = 0 \tag{5.2}$$

Proof. The proof is an easy consequence of [10, Theorem 6.9] which states that if $\mu \in \mathscr{M}^+(\mathbb{R}^n)$ is a positive Radon measure and if

$$A \subset \{x \in \mathbb{R}^n \mid \Theta^{*(k-1)}(\mu, x) \leq \lambda\},$$

then $\mu(A) \leq 2^{k-1}\lambda\mathcal{H}^{k-1}(A)$. Let us assume $\mathcal{H}^{k-1}(A) = 0$; since by definition $\Theta^{*(k-1)}(\sigma^*_V, x) < \infty$ for σ^*_V-a.e. $x \in \mathcal{M}$ we have that

$$\sigma^*_V(A) = \sigma^*_V\Big(\bigcup_{i\in\mathbb{N}} A \cap \{\Theta^{*(k-1)}(\sigma^*_V, x) < i\}\Big) \leq \sum_{i\in\mathbb{N}} i2^{k-1}\mathcal{H}^{k-1}(A) = 0.$$

(5.2) clearly follows by $\sigma^*_V \ll \mathcal{H}^{k-1}$ and (1.9). □

Since σ^*_V is concentrated on E, we expect that $\mathrm{Tan}^{k-1}(\sigma^*_V, x) = \mathrm{Tan}^{k-1}(\sigma_V, x)$ for every point of E that has density 1 with respect to σ_V. This is a well-known fact, see e.g. [4, Remark 3.13], but we recall the proof for the reader convenience.

Lemma 5.3. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$, let σ_V be the measure provided by Corollary 4.6 and let E be defined as in* (1.8). *Then*

$$\mathrm{Tan}^{k-1}(\sigma^*_V, x_0) = \mathrm{Tan}^{k-1}(\sigma_V, x_0) \qquad \forall x_0 \in E \text{ such that } \lim_{r\to 0} \frac{\sigma_V\big(E \cap B_r(x)\big)}{\sigma_V\big(B_r(x)\big)} = 1. \tag{5.3}$$

In particular, the first equality in (5.3) *holds σ^*_V-a.e..*

Proof. Let us fix $x_0 \in \partial\mathcal{M}$ such that

$$\lim_{r\to 0} \frac{\sigma_V\big(E \cap B_r(x)\big)}{\sigma_V\big(B_r(x)\big)} = \lim_{r\to 0} \frac{\sigma^*_V\big(B_r(x)\big)}{\sigma_V\big(B_r(x)\big)} = 1,$$

let $\nu \in \operatorname{Tan}^{k-1}(\sigma_V, x_0)$ and $r_j \downarrow 0$ such that

$$\sigma_j := \frac{1}{r_j^{k-1}} \big(\tau_{x_0,r_j}\big)_{\#} \sigma_V \overset{*}{\rightharpoonup} \nu. \tag{5.4}$$

We are going to prove that

$$\sigma_j^* := \frac{1}{r_j^{k-1}} \big(\tau_{x_0,r_j}\big)_{\#} \sigma_V^* \overset{*}{\rightharpoonup} \nu. \tag{5.5}$$

Without loss of generality we can assume $x_0 = 0$. Let us consider $f \in C_c(\mathbb{R}^n)$. We have

$$\begin{aligned}\int f(x)\,\mathrm{d}\nu(x) &= \lim_{j\to\infty} \frac{1}{r_j^{k-1}} \int f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V(x)\\ &= \lim_{j\to\infty} \frac{1}{r_j^{k-1}} \left(\int_E f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V(x) + \int_{\mathbb{R}^n\setminus E} f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V(x)\right).\end{aligned}$$

For the last term there exists $c > 0$ such that

$$\begin{aligned}\frac{1}{r_j^{k-1}} \int_{\mathbb{R}^n\setminus E} \left| f\Big(\frac{x}{r_j}\Big)\right| \mathrm{d}\sigma_V(x) &\le \frac{c}{r_j^{k-1}} \sigma_V\big(B_{cr_j} \setminus E\big)\\ &\le c \frac{\sigma_V\big(B_{cr_j}\big)}{r_j^{k-1}} \frac{\sigma_V\big(B_{cr_j}\setminus E\big)}{\sigma_V\big(B_{cr_j}\big)} \xrightarrow[j\to\infty]{} 0\end{aligned}$$

since E has density 1 with respect to σ_V and $\sigma_V(B_{cr_j})/r_j^{k-1}$ remains bounded by $\sigma_j \overset{*}{\rightharpoonup} \nu$. Hence

$$\int f(x)\,\mathrm{d}\nu(x) = \lim_{j\to\infty} \frac{1}{r_j^{k-1}} \int f\Big(\frac{x}{r_j}\Big)\,\mathrm{d}\sigma_V^*(x),$$

thus

$$\sigma_j^* = \frac{1}{r_j^{k-1}} \big(\tau_{0,r_j}\big)_{\#} \sigma_V^* \overset{*}{\rightharpoonup} \nu.$$

This proves $\operatorname{Tan}^{k-1}(\sigma_V, x_0) \subset \operatorname{Tan}^{k-1}(\sigma_V^*, x_0)$; the reverse inclusion can be proved in a similar way. ☐

5.2. **Proof of the second condition of the Marstrand-Mattila Criterion.** We begin by studying tangent varifolds to V in a point $x \in \partial\mathcal{M}$.

Lemma 5.4. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \ge 1$. Then, for every $x_0 \in \operatorname{Dens}(V) \cap \partial\mathcal{M}$ the following statements hold: let $r_j \downarrow 0$ be fixed and let us use the notations for the scalings defined on p. 22; then*

1. *every V_j has free boundary at $\partial\mathcal{M}_j$ with $\tilde{H}_j(x) = r_j \tilde{H}(r_j x)$ for every $x \in \partial\mathcal{M}_j$ and*

$$\sigma_{V_j} = \frac{1}{r_j^{k-1}} (\tau_{0,r_j})_{\#} \sigma_V; \tag{5.6}$$

2. *there exist a subsequence of r_j, not relabeled, and a k-varifold C with $\operatorname{supp} C \subset T_0^+\mathcal{M}$ such that*

$$V_j \underset{j\to\infty}{\overset{*}{\rightharpoonup}} C.$$

   In particular, $\operatorname{Tan}(V, x_0) \neq \emptyset$.
3. *C is stationary with respect to $\mathfrak{X}_t(T_{x_0}^+\mathcal{M})$;*
4. *if σ_C is the measure given by Corollary 4.6 relative to C, we have*

$$\sigma_j := \sigma_{V_j} \underset{j\to\infty}{\overset{*}{\rightharpoonup}} \sigma_C; \tag{5.7}$$

5. *If in addition $\Theta^k(\|V\|, x) \ge 1$ for $\|V\|$-a.e. $x \in \mathcal{M}$ (that is if V satisfies Assumption 1), then C is a rectifiable cone and σ_C is scaling invariant, that is*

$$\frac{1}{r^{k-1}} (\tau_{0,r})_{\#} \sigma_C = \sigma_C \qquad \forall r > 0.$$

Proof. To simplify the notations, we assume without loss of generality that $x_0 = 0 \in \partial\mathcal{M} \cap \mathrm{Dens}(V)$ and $T_0^+\mathcal{M} = \{x_n \geq 0\}$.

Step 1: We first prove that every V_j has free boundary at $\partial\mathcal{M}_j$. For every $X \in \mathfrak{X}_t(\mathcal{M}_j)$ the vector field $x \mapsto X(\frac{x}{r_j})$ belongs to $\mathfrak{X}_t(\mathcal{M})$, thus we can apply (1.1) to V and get

$$\begin{aligned}\int_{G_k(\mathcal{M}_j)} \mathrm{div}_S\, X(x)\, \mathrm{d}V_j(x,S) =& \frac{1}{r_j^{k-1}} \int_{G_k(\mathcal{M})} \mathrm{div}_S\, X\big(\frac{x}{r_j}\big)\, \mathrm{d}V(x,S) \\ =& -\frac{1}{r_j^{k-1}} \int_{\mathcal{M}} \langle X\big(\frac{x}{r_j}\big), H(x)\rangle\, \mathrm{d}\|V\|(x).\end{aligned}$$

Thus, if we define $H_j(x) = r_j H(r_j x)$ for all $x \in \mathcal{M}_j$, by changing again variables we get

$$\int_{G_k(\mathcal{M}_j)} \mathrm{div}_S\, X(x)\, \mathrm{d}V_j(x,S) = -\int_{\mathcal{M}_j} \langle X(x), H_j(x)\rangle\, \mathrm{d}\|V_j\|(x),$$

that is V_j has free boundary at $\partial\mathcal{M}_j$, with $H_j(x) = r_j H(r_j x)$. By Corollary 4.6, V_j has bounded first variation and there exists $\sigma_j = \sigma_{V_j}$ and $\tilde{H}_j$. By a similar argument and by definition of $\tilde{H}$ it easily seen that

$$\tilde{H}_j(x) = r_j \tilde{H}(r_j x) \qquad \text{for } \|V_j\|\text{-a.e. } x \in \partial\mathcal{M}_j.$$

If $X \in \mathfrak{X}(\mathcal{M}_j)$, then

$$\begin{aligned}\int_{G_k(\mathcal{M}_j)} \mathrm{div}_S\, X(x)\, \mathrm{d}V_j(x,S) =& \frac{1}{r_j^{k-1}} \int_{G_k(\mathcal{M})} \mathrm{div}_S\, X\big(\frac{x}{r_j}\big)\, \mathrm{d}V(x,S) \\ =& -\frac{1}{r_j^{k-1}} \int_{\mathcal{M}} \langle X\big(\frac{x}{r_j}\big), H(x) + \tilde{H}(x)\rangle\, \mathrm{d}\|V\|(x) \\ &+ \frac{1}{r_j^{k-1}} \int_{\partial\mathcal{M}} \langle X\big(\frac{x}{r_j}\big), N(x)\rangle\, \mathrm{d}\sigma_V(x) \\ =& -\int_{\mathcal{M}_j} \langle X, H_j + \tilde{H}_j\rangle\, \mathrm{d}\|V_j\| + \frac{1}{r_j^{k-1}} \int_{\partial\mathcal{M}_j} \langle X, N_j\rangle\, \mathrm{d}(\tau_{0,r_j})_\# \sigma_V\end{aligned}$$

If we compare this equality with (4.7) for V_j tested with X, we get

$$\sigma_j := \sigma_{V_j} = \frac{1}{r_j^{k-1}} (\tau_{0,r_j})_\# \sigma_V,$$

that is σ_j is obtained by scaling σ_V.

Step 2: We now want to study the limit of the sequence V_j. Since $0 \in \mathrm{Dens}(V)$, by Theorem 1.4 there exists a constant $c > 0$ such that

$$\|V_j\|(B_1) = \|(\tau_{0,r_j})_\sharp V\|(B_1) = \frac{\|V\|(B_{r_j})}{r_j^k} \leq c \qquad \forall j \in \mathbb{N}. \tag{5.8}$$

Thus by compactness of Radon measures, there exist $C \in \mathcal{V}_k(\mathbb{R}^n)$ and a subsequence of r_j, not relabeled, such that

$$V_j \underset{j\to\infty}{\overset{*}{\rightharpoonup}} C. \tag{5.9}$$

Step 3: Clearly $\operatorname{supp} C \subset T_0^+\mathcal{M}$. To show that C is stationary with respect to $\mathfrak{X}_t(T_0^+\mathcal{M})$, we test with a vector field $X \in \mathfrak{X}_t(T_0^+\mathcal{M})$. We first need the following estimate on H_j (exactly the same relation holds for $\tilde{H}_j$):

$$\|H_j\|_{L^p(B_1)} = \Big(\int_{B_1} |H_j|^p\, \mathrm{d}\|V_j\|\Big)^{\frac{1}{p}} = \Big(\frac{1}{r_j^{k-p}} \int_{B_{r_j}} |H|^p\, \mathrm{d}\|V\|\Big)^{\frac{1}{p}} \xrightarrow[j\to\infty]{} 0. \tag{5.10}$$

where in the second equality we have used $H_j(x) = r_j H(r_j(x))$ and the limit follows by $0 \in \mathrm{Dens}(V)$ and $|H| \leq |H + \tilde{H}|$ because $H(x) \in T_x\partial\mathcal{M}$ (Remark 2.2) while $\tilde{H} \in (T_x\partial\mathcal{M})^\perp$.

To prove that C is stationary, let us pick $X \in \mathfrak{X}_t(T_0^+\mathcal{M})$ with compact support; there exists a sequence of vector fields $X_j \in \mathfrak{X}_t(\mathcal{M}_j)$ with compact support such that $X_j \to X$ in the C^1 topology. Hence, using (5.9) and (5.10) we get

$$\begin{aligned}\int_{G_k(T_0^+\mathcal{M})} \operatorname{div}_S X(x)\,\mathrm{d}C(x,S) &= \lim_{j\to\infty}\int_{G_k(\mathcal{M}_j)} \operatorname{div}_S X_j(x)\,\mathrm{d}V_j(x,S)\\ &= -\lim_{j\to\infty}\int_{\mathcal{M}_j}\langle X_j(x), H_j(x)\rangle\,\mathrm{d}\|V_j\|(x)\\ &= 0\end{aligned}$$

that is, by definition, C is stationary with respect to $\mathfrak{X}_t(T_0^+\mathcal{M})$.

Step 4: Corollary 4.6 provides that C has bounded first variation and the existence of σ_C. We want to prove that

$$\sigma_j \xrightharpoonup[j\to\infty]{*} \sigma_C. \tag{5.11}$$

To this aim, we first remark that $\operatorname{supp}\sigma_C \subset T_0\partial\mathcal{M}$ and, since $T_0\partial\mathcal{M}$ is flat, we have that $\tilde{H}_C = 0$. We next need an uniform bound on $\sigma_j(B_1)$ and we use (1.7): since the second fundamental forms of $\partial\mathcal{M}_j$ go to 0 as $r_j \to 0$, when we apply (1.7) to V_j in B_1, the constant c in (1.7) is bounded uniformly in j. This implies that there exists an uniform constant c such that for each $j \in \mathbb{N}$

$$\begin{aligned}\sigma_j(B_1) =&\frac{\sigma_V(B_{r_j})}{r_j^{k-1}}\\ \le&\frac{c\|V\|(B_{2r_j})}{r_j^k} + \frac{1}{r_j^{k-1}}\int_{B_{2r_j}}|H|\,\mathrm{d}\|V\|.\\ \le&\frac{c\|V\|(B_{2r_j})}{r_j^k} + r_j^{1-\frac{k-s}{p}}\left(\frac{1}{r_j^s}\int_{B_{2r_j}}|H|^p\,\mathrm{d}\|V\|\right)^{\frac{1}{p}}\left(\frac{\|V\|(B_{2r_j})}{r_j^k}\right)^{1-\frac{1}{p}}\end{aligned} \tag{5.12}$$

where $s \in (k-p, k]$ is such that 0 satisfies the s-density condition in the definition of $\operatorname{Dens}(V)$. The last member of (5.12) is uniformly bounded in j since $0 \in \operatorname{Dens}(V)$, Theorem 1.4 and by $\frac{k-s}{p} \le 1$. This proves the uniform bound on $\sigma_j(B_1)$.

To complete the proof of (5.11), let $X \in \mathfrak{X}(\mathbb{R}^n)$ be a vector field with compact support. If e_n is the n-th coordinate unit vector (that is the interior unit normal vector to $\partial T_0^+\mathcal{M}$), we have

$$\begin{aligned}-\int_{T_0\partial\mathcal{M}}\langle X, e_n\rangle\,\mathrm{d}\sigma_C &= \int_{G_k(T_0^+\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}C(x,S)\\ &= \lim_{j\to\infty}\int_{G_k(\mathcal{M}_j)}\operatorname{div}_S X(x)\,\mathrm{d}V_j(x,S)\\ &= -\lim_{j\to\infty}\left(\int_{\mathcal{M}_j}\langle X(x), H_j(x)+\tilde{H}_j(x)\rangle\,\mathrm{d}\|V_j\|(x) + \int_{\partial\mathcal{M}_j}\langle X(x), N_j(x)\rangle\,\mathrm{d}\sigma_j(x)\right)\\ &= \lim_{j\to\infty}\int_{\partial\mathcal{M}_j}\langle X(x), N_j(x)\rangle\,\mathrm{d}\sigma_j(x),\end{aligned} \tag{5.13}$$

where the first identity follows by $H_C = \tilde{H}_C = 0$ and the last one follows by (5.10). Thus $N_j\sigma_j \xrightharpoonup{*} -e_n\sigma_C$ is proved, since $\mathfrak{X}(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n,\mathbb{R}^n)$, by (5.13) and the uniform bound on $\sigma_j(B_1)$ (5.12). To prove (5.11), it is enough to observe that, if $f \in C_c(\mathbb{R}^n,\mathbb{R})$, then

$$\int f\,\mathrm{d}\sigma_C = \int\langle fe_n, e_n\rangle\,\mathrm{d}\sigma_C = -\lim_{j\to\infty}\int\langle fe_n, N_j\rangle\,\mathrm{d}\sigma_j = \lim_{j\to\infty}\int f\,\mathrm{d}\sigma_j,$$

by the uniform bound on $\sigma_j(B_1)$ and since $\partial\mathcal{M}_j \to \partial T_0^+\mathcal{M}$ in the C^1 topology.

Step 5: We are left to prove that, if $\Theta^k(\|V\|,x)\geq 1$ for $\|V\|$-a.e. $x\in\mathcal{M}$, then C is a rectifiable cone. The scaling invariance of σ_C will be an easy consequence of this. We first claim that C is rectifiable. In fact, we have:

- $\Theta^k(\|V_j\|,x)\geq 1$ for $\|V_j\|$-a.e. $x\in\mathcal{M}_j$, since $\Theta^k(\|V\|,x)\geq 1$ for $\|V\|$-a.e. $x\in\mathcal{M}$;
- $\sup_{j\in\mathbb{N}}\|V_j\|(B_1)<\infty$ by (5.8);
- the V_j's have locally uniformly bounded first variations because $0\in\mathrm{Dens}(V)$, by (5.10) and the uniform bound on (5.12).

Thus we can apply to the sequence V_j the compactnes theorem for rectifiable varifolds [12, Theorem 42.7], which proves that C is rectifiable and that $\Theta^k(\|C\|,x)\geq 1$ for $\|C\|$-a.e. $x\in\mathcal{M}$. In particular, there exists a k-rectifiable set $\Gamma\subset T_0^+\mathcal{M}$ such that

$$C=\theta(x)\mathcal{H}^k\llcorner\Gamma$$

with $\theta(x)=\Theta^k(\|C\|,x)\geq 1$ for $\mathcal{H}^k$-a.e. $x\in\Gamma$.

It remains to show that C is a cone. The argument is exactly the same as [12, Theorem 19.3], but we recall it for the reader convenience. Since C is rectifiable, to prove

$$(\tau_{0,\lambda})_\sharp C=C\qquad\forall\lambda>0,\tag{5.14}$$

(that is the fact that C is a cone), it is enough to show that

$$\Theta^k(\|C\|,\lambda x)=\Theta^k(\|C\|,x)\qquad\forall x\in\mathbb{R}^n,\forall\lambda>0,\tag{5.15}$$

that is θ is homogeneous of degree 0. This is clearly implied by

$$\|C\|(\lambda A)=\lambda^k\|C\|(A)\qquad\forall A\subset\mathbb{R}^n\text{ Borel},\forall\lambda>0.\tag{5.16}$$

By approximation, it is enough to prove that, for every 0-homogeneous function $h\in C^1(\mathbb{R}^n)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\lambda^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\lambda}\Big)h(x)\,\mathrm{d}\|V\|(x)\right)=0\qquad\forall\lambda>0,\tag{5.17}$$

where γ is the cut-off function defined in Section 2. To this aim, if $\lambda>0$ is such that $\|C\|(\partial B_\lambda)=0$, we observe that

$$\frac{\|C\|(B_\lambda)}{\lambda^k}=\lim_{j\to\infty}\frac{\|V_j\|(B_\lambda)}{\lambda^k}=\lim_{j\to\infty}\frac{\|V\|(B_{\lambda r_j})}{\lambda^k r_j^k}=\Theta^k(\|V\|,0).$$

Since there exists at most a countable number of radii λ_j such that $\|C\|(\partial B_{\lambda_j})>0$, by approximation it follows that

$$\frac{\|C\|(B_\lambda)}{\lambda^k}=\Theta^k(\|V\|,0)\qquad\forall\lambda>0.\tag{5.18}$$

If we write the monotonicity identity (4.10) for C, since $H_C=\tilde{H_C}=0$ and $\partial(T_0^+\mathcal{M})$ is flat, integrating between σ,λ we obtain

$$\begin{aligned}\frac{1}{\lambda^k}\int_W\gamma\Big(\frac{|x|}{\lambda}\Big)\,\mathrm{d}\|C\|(x)&-\frac{1}{\sigma^k}\int_W\gamma\Big(\frac{|x|}{\sigma}\Big)\,\mathrm{d}\|C\|(x)\\&=\frac{1}{\lambda^k}\int_{G_k(W)}\gamma\Big(\frac{|x|}{\lambda}\Big)\big|P_{S^\perp}\nabla|x|\big|^2\,\mathrm{d}C(x,S)\\&\quad-\frac{1}{\sigma^k}\int_{G_k(W)}\gamma\Big(\frac{|x|}{\sigma}\Big)\big|P_{S^\perp}\nabla|x|\big|^2\,\mathrm{d}C(x,S)\\&\quad+\int_{G_k(W)}\big|P_{S^\perp}\nabla|x|\big|^2\left(\int_\sigma^\lambda\frac{k}{\rho^{k+1}}\gamma\Big(\frac{|x|}{\rho}\Big)\,\mathrm{d}\rho\right)\mathrm{d}C(x,S)\end{aligned}\tag{5.19}$$

Letting γ increase to $\mathbf{1}_{[0,1)}$ in (5.19), by dominated convergence and (5.18) we get

$$0=\frac{\|C\|(B_\lambda)}{\lambda^k}-\frac{\|C\|(B_\sigma)}{\sigma^k}=\int_{G_k(B_\lambda\setminus B_\sigma)}\frac{\big|P_{S^\perp}\nabla|x|\big|^2}{|x|^k}\,\mathrm{d}C(x,S).\tag{5.20}$$

Since the last term is non-negative, we have

$$\big|P_{S^\perp}\nabla|x|\big|^2 = \Big|P_{S^\perp}\frac{x}{|x|}\Big|^2 = 0 \qquad \text{for } C\text{-a.e. } (x,S) \in G_k(T_0^+\mathcal{M}), \tag{5.21}$$

that is

$$C\big((x,S) \mid P_S(x) \neq x\big) = C\big((x,S) \mid x \notin S\big) = 0. \tag{5.22}$$

If h is 0-homogeneous, then $\langle \nabla h(x), x\rangle = 0$ and (5.22) implies

$$\langle \nabla h(x), P_S\, x\rangle = \langle \nabla h(x), x\rangle = 0 \qquad \text{for } C\text{-a.e. } (x,S) \in G_k(T_0^+\mathcal{M}). \tag{5.23}$$

Since C has free boundary at $\partial T_0^+\mathcal{M}$, by testing (2.10) for C with $X(x) = h(x)\gamma\big(\frac{|x|}{\lambda}\big)x \in \mathfrak{X}_t(T_0^+\mathcal{M})$, one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{\lambda^k}\int_{\mathcal{M}}\gamma\Big(\frac{|x|}{\lambda}\Big)h(x)\,\mathrm{d}\|V\|(x)\right) &= -\frac{1}{\lambda^{k+1}}\int_{G_k(T_0^+\mathcal{M})}\operatorname{div}_S X(x)\,\mathrm{d}V(x,S)\\ &\quad + \frac{1}{\lambda^{k+1}}\int_{G_k(T_0^+\mathcal{M})}\gamma\Big(\frac{|x|}{\lambda}\Big)\langle\nabla h(x), P_S\,x\rangle\,\mathrm{d}V(x,S)\\ &= 0,\end{aligned}$$

where the last equality follows by $H_C = 0$ and (5.23). This proves (5.17), thus C is a cone. The scaling invariance of σ_C is a trivial consequence of (4.7) applied to C and (5.14).

□

Before of going on, we highlight an easy consequence of Lemma 5.2, Lemma 5.3 and Lemma 5.4.

Corollary 5.5. *Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial\mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p > 1$ and let σ_V be the measure provided by Corollary 4.6. Then*

$$\operatorname{Tan}^{k-1}(\sigma_V^*, x_0) = \{\sigma_C \mid C \in \operatorname{Tan}(V, x_0)\} \qquad \text{for } \sigma_V^*\text{-a.e. } x_0 \in \partial\mathcal{M}. \tag{5.24}$$

Proof. By Lemma 5.4 it clearly follows that for each $x_0 \in \operatorname{Dens}(V) \cap \partial\mathcal{M}$,

$$\operatorname{Tan}^{k-1}(\sigma_V, x_0) = \{\sigma_C \mid C \in \operatorname{Tan}(V, x_0)\}.$$

By Lemma 5.2 and Lemma 5.3 we have the conclusion. □

It is well-known that, for a cone, points with maximal density form a linear subspace and that the cone is invariant by translation with respect these points (see e.g. [13, Section 3.3] and [14, Theorem 3.1, Example (4) of Section 4]). We report here the simple proof of this fact for the sake of completeness.

Lemma 5.6. *Let V satisfy Assumption 1, let $x_0 \in \operatorname{Dens}(V) \cap \partial\mathcal{M}$ be fixed and let C be a tangent cone to V at x_0. Then the set*

$$D_C = \{y \in T_{x_0}\partial\mathcal{M} \mid \Theta^k(\|C\|, y) = \Theta^k(\|C\|, 0)\}$$

is a linear subspace of $\mathbb{R}^n$. Moreover the translated cone $(L_y)_\sharp C$ coincides with C for all $y \in D_C$. In particular, if σ_C is the measure given by Corollary 4.6 relative to C, then $(L_y)_\#\sigma_C = \sigma_C$.

Definition 5.1. We say that D_C is the *invariant subspace* of the cone C.

Proof of Lemma 5.6. Without loss of generality we can assume that $x_0 = 0$. Let us call $\theta_0 = \Theta^k(\|V\|, 0)$. By Lemma 5.4 C is rectifiable and it holds $\Theta^k(\|V\|, 0) = \Theta^k(\|C\|, 0)$. Since C is a cone, for $y \in T_x\partial\mathcal{M}$ we have

(5.25)

$$\theta_0 \overset{(5.18)}{=} \lim_{r\to+\infty}\frac{\|C\|\big(B_r\big)}{r^k} \geq \lim_{r\to+\infty}\frac{\|C\|\big(B_{r-|y|}(y)\big)}{(r-|y|)^k}\frac{(r-|y|)^k}{r^k} = \lim_{r\to+\infty}\frac{\|C\|\big(B_r(y)\big)}{r^k} \geq \Theta^k(\|C\|, y).$$

The last inequality is given by the monotonicity identity for C: in the last member of (4.10), the first two integrals disapper (since $H + \tilde{H} = 0$ and $\langle N(x), x\rangle = 0$) and the last term is non-negative. (5.25) shows that

$$\Theta^k(\|C\|, 0) \geq \Theta^k(\|C\|, y) \qquad \forall y \in T_0\partial\mathcal{M}. \tag{5.26}$$

If $y \in D_C$, then (5.25) yields

$$\frac{\|C\|\big(B_r(y)\big)}{r^k} = \theta_0 \qquad \forall r > 0.$$

By the same arguments of Lemma 5.4, it follows that C is a rectifiable cone also with respect to y. By rectifiability of C, in order to show that $(L_y)_\sharp C = C$, it is enough to prove that

$$\Theta^k(C, z) = \Theta^k(C, y + z) \qquad \forall z \in T_0^+\mathcal{M}. \tag{5.27}$$

To this aim, let $z \in T_0^+\mathcal{M}$ be an arbitrary point. Since C is a cone with respect to y, we have that

$$\Theta^k(C, z) = \Theta^k\Big(C, y + \frac{1}{2}(z - y)\Big) = \Theta^k\Big(C, \frac{1}{2}(y + z)\Big).$$

On the other hand, since C is a cone we have

$$\Theta^k(\|C\|, y + z) = \Theta^k\Big(C, \frac{1}{2}(y + z)\Big).$$

This shows (5.27) and hence $(L_y)_\sharp C = C$. The translation invariance of σ_C is a trivial consequence of (4.7) applied to C and of $(L_y)_\sharp C = C$.

It remains to show that D_C is a linear subspace of $\mathbb{R}^n$. Since C is a cone, if $y \in D_C$, then $\lambda y \in D_C$ for each $\lambda > 0$. Since C is a cone also with respect to y, then $\lambda y \in D_C$ also if $\lambda < 0$. If $y, z \in D_C$, it follows from the previous discussion that also $y + z \in D_C$ and this proves that D_C is a linear subspace. ☐

Before going on, we first recall the definition of *approximate continuity*:

Definition 5.2 (Approximate continuity)**.** If μ is a positive Radon measure and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Borel function, we say that f is approximate continuous at $x \in \mathbb{R}^n$ with respect to μ if

$$\lim_{r \to 0} \frac{\mu\big(\{z \in B_r(x) \mid |f(z) - f(x)| > \varepsilon\}\big)}{\mu(B_r(x))} = 0 \qquad \forall \varepsilon > 0.$$

Remark 5.1. It is well-known that, if μ is a Radon measure, then every μ-measurable function is approximate continuous at μ-a.e. point (see e.g. [7, Theorem 1.37] where the proof is done for the Lebesgue measure, but the same arguments can be applied to any Radon measure).

The following lemma states that there exists a set F of full σ_V-measure with respect to $\mathrm{Dens}(V)$ such that for every $x \in F$ the invariant subspace D_C of any cone $C \in \mathrm{Tan}(V, x)$ coincides with $\operatorname{supp}\sigma_C$.

Lemma 5.7. *Let V satisfy Assumption 1. Then there exists a set $F \subset \mathrm{Dens}(V) \cap \partial\mathcal{M}$ that satisfies*

$$\sigma_V(\mathrm{Dens}(V) \setminus F) = 0$$

and has the following property: for every $x_0 \in F$ and for every $C \in \mathrm{Tan}(V, x_0)$ we have either $\sigma_C = 0$ or $\sigma_C \neq 0$ and

$$\Theta^k(\|C\|, y) = \Theta^k(\|C\|, 0) \qquad \forall y \in \operatorname{supp}\sigma_C.$$

In particular, either $\sigma_C = 0$ or $\operatorname{supp}\sigma_C = D_C$.

Proof. The idea of the proof is the following: we first define the "good" set F of full σ_V-measure with respect to $\mathrm{Dens}(V)$ where $\Theta^k(\|V\|, \cdot)$ exists and is approximate continuous with respect to σ_V. Next, we fix $x_0 \in F$, $r_j \to 0$ and, using the notations for the scalings,

$$V_j \overset{*}{\rightharpoonup} C \in \mathrm{Tan}(V, x_0).$$

We fix $y \in \operatorname{supp}\sigma_C$ and we assume by contradiction that the statement is false. We find a tiny ball $B_r(y)$ where $\Theta^k(\|V_j\|, \cdot)$ is close to $\Theta^k(\|C\|, y)$ for j sufficiently large. Since $\Theta^k(\|C\|, y)$ is close to $\|C\|(B_\rho(y))/\rho^k$ for small ρ, this is achieved by weak convergence $V_j \overset{*}{\rightharpoonup} C$ and using the properties of φ_{x_0} stated in Theorem 1.4. Since $\sigma_j(B_r(y)) > \beta > 0$ for large j, this contradicts the approximate continuity of $\Theta^k(\|V\|, \cdot)$ at x_0 with respect to σ_V.

Step 1: We are going first to define the set F of full σ_V-measure with respect to $\mathrm{Dens}(V)$ and next we will prove that the conclusion of the theorem holds for every $x \in F$.

We call A the set of points $x \in \partial\mathcal{M}$ that satisfy all the following conditions:

(1) $x \in \mathrm{Dens}(V) \cap \operatorname{supp} \sigma_V$ and is of density 1 for $\mathrm{Dens}(V)$ with respect to σ_V, that is

$$\lim_{\rho \to 0} \frac{\sigma_V\big(\mathrm{Dens}(V) \cap B_\rho(x)\big)}{\sigma_V\big(B_\rho(x)\big)} = 1; \tag{5.28}$$

(2) x is a point of approximate continuity for $\Theta^k(\|V\|, \cdot)$ with respect to σ_V (which is a well-defined Borel function in $\mathrm{Dens}(V)$ and can be extended, for instance, to be 0 outside $\mathrm{Dens}(V)$; the chosen extension does not influence the approximate continuity, because by (5.28) x has density 1 in $\mathrm{Dens}(V)$, where $\Theta^k(\|V\|, \cdot)$ is well-defined).

By Lebesgue differentiation theorem and by Remark 5.1, we have

$$\sigma_V(\mathrm{Dens}(V) \setminus A) = 0. \tag{5.29}$$

In addition, by Theorem 1.4, for every $x \in A$ the map $\rho \mapsto \varphi_x(\rho)$ is monotone increasing and converge pointwise to 0 as $\rho \downarrow 0$, that is

$$\lim_{\rho \to 0} \varphi_x(\rho) = 0 \qquad \forall x \in A.$$

Thus, by Egoroff's Theorem, for every $h \in \mathbb{N}$ there exists a set $F_h \subset A$ such that

$$\sigma_V(A \setminus F_h) \leq 1/h, \qquad \varphi_x(\rho) \underset{\rho \to 0}{\overset{\text{unif}}{\longrightarrow}} 0 \text{ on } F_h. \tag{5.30}$$

Up to removing sets of σ_V-measure 0, we can assume that every $x \in F_h$ is a point of density 1 with respect to σ_V, that is

$$\lim_{\rho \to 0} \frac{\sigma_V\big(F_h \cap B_\rho(x)\big)}{\sigma_V\big(B_\rho(x)\big)} = 1 \qquad \forall x \in F_h. \tag{5.31}$$

We now define

$$F = A \cap \Big(\bigcup_{h \in \mathbb{N}} F_h\Big).$$

By (5.29) and (5.30) it follows

$$\sigma_V^*(\mathrm{Dens}(V) \setminus F) = 0.$$

Let us fix $x_0 \in F$ and let us consider $h \in \mathbb{N}$ such that $x_0 \in F_h$. Without loss of generality we can assume $x_0 = 0$.

Now let us fix $r_j \downarrow 0$. Using the notations for the scalings, since $F \subset \mathrm{Dens}(V)$, by Lemma 5.4 there exists a subsequence, not relabeled, such that $V_j \overset{*}{\rightharpoonup} C$ with $C \in \mathrm{Tan}(V, 0)$ and C is a rectifiable cone with $\Theta^k(\|C\|, y) \geq 1$ for $\|C\|$-a.e. $y \in \mathbb{R}^n$.

We now need a technical remark that is useful in the rest of the proof: recalling that $H_j(y) = r_j H(r_j y)$, by a simple change of variables one obtains

$$r_j x \in \mathrm{Dens}(V) \quad \Leftrightarrow \quad x \in \mathrm{Dens}(V_j).$$

More precisely, if we call φ_x^j the function for V_j defined in Theorem 1.4, we get

$$\varphi_x^j(\rho) = \varphi_{r_j x}(r_j \rho). \tag{5.32}$$

Step 2: We can now begin with the proof. If $\sigma_C = 0$ there is nothing to prove. Thus we can assume $\sigma_C \neq 0$ and $\emptyset \neq \operatorname{supp} \sigma_C \subset \partial T_0^+ \mathcal{M}$. Since C is a cone, By (5.26) we have

$$\Theta^k(\|C\|, y) \leq \Theta^k(\|C\|, 0) \qquad \forall y \in \partial T_0^+ \mathcal{M}.$$

By contradiction, let us assume that there exists $y \in \operatorname{supp} \sigma_C$ and $\varepsilon > 0$ such that

$$\Theta^k(\|C\|, y) < \Theta^k(\|C\|, 0) - \varepsilon. \tag{5.33}$$

Since σ_C is scaling invariant by Lemma 5.4, without loss of generality we can assume that $y \in B_{1/2}$. By the uniform convergence (5.30) and by definition of density, there exists $\rho \in (0, 1/2)$ such that

$$\frac{\|C\|\big(B_\rho(y)\big)}{\rho^k} + \varphi_z(\rho) \leq \Theta^k(\|C\|, y) + \frac{\varepsilon}{8} \qquad \forall z \in F_h. \tag{5.34}$$

Since $V_j \overset{*}{\rightharpoonup} C$, without loss of generality we can choose ρ so that there exists $J \in \mathbb{N}$ and a small $0 < r < \rho$ for which

$$\left| \frac{\|C\|\big(B_\rho(y)\big)}{\rho^k} - \frac{\|V_j\|\big(B_\rho(y)\big)}{(\rho - r)^k} \right| < \frac{\varepsilon}{8} \qquad \forall j > J. \tag{5.35}$$

Let us choose $j > J$; for every $z \in B_r(y)$ such that $r_j z \in F_h$ we have

$$\begin{aligned}
\Theta^k(\|V_j\|, z) &\leq \frac{\|V_j\|\big(B_{\rho-r}(z)\big)}{(\rho - r)^k} + \varphi^j_z(\rho - r) \\
&\leq \frac{\|V_j\|\big(B_\rho(y)\big)}{(\rho - r)^k} + \varphi_{r_j z}(r_j \rho) \\
&\overset{(5.35)}{\leq} \frac{\|C\|\big(B_\rho(y)\big)}{\rho^k} + \varphi_{r_j z}(r_j \rho) + \frac{\varepsilon}{8} \\
&\overset{(5.34)}{\leq} \Theta^k(\|C\|, y) + \frac{\varepsilon}{4} \\
&\overset{(5.33)}{\leq} \Theta^k(\|C\|, 0) - \frac{3\varepsilon}{4} \\
&= \Theta^k(\|V_j\|, 0) - \frac{3\varepsilon}{4}.
\end{aligned}$$

where we used the fact that every φ_x is increasing and (5.32). This shows that, for $j > J$,

$$B_r(y) \cap \frac{1}{r_j} F_h \subseteq \left\{ z \in B_1 \mid \left| \Theta^k(\|V_j\|, z) - \Theta^k(\|V_j\|, 0) \right| > \frac{\varepsilon}{2} \right\}.$$

Step 3: We now want to estimate from below the measure of this set to get a contradiction with the approximate continuity of $\Theta^k(\|V\|, \cdot)$ in 0.

By approximate continuity of the $\Theta^k(\|V\|, \cdot)$ in 0 with respect to σ_V we have

$$\begin{aligned}
0 &= \limsup_{j \to \infty} \frac{\sigma_V\left(\left\{ z \in B_{r_j} \mid \left|\Theta^k(\|V\|, z) - \Theta^k(\|V\|, 0)\right| > \frac{\varepsilon}{2} \right\}\right)}{\sigma_V(B_{r_j})} \\
&\geq \limsup_{j \to \infty} \frac{\sigma_V\big(B_{r r_j}(r_j y) \cap F_h\big)}{\sigma_V(B_{r_j})} \\
&= \limsup_{j \to \infty} \frac{\sigma_V\big(B_{r r_j}(r_j y)\big)}{\sigma_V(B_{r_j})} \\
&= \limsup_{j \to \infty} \frac{\sigma_j\big(B_r(y)\big)}{\sigma_j(B_1)},
\end{aligned} \tag{5.36}$$

where the second identity is consequence of (5.31) and the last one follows by (5.6). We want to estimate from below the last term to get a contradiction. To do so, let us notice that, by $\sigma_j \overset{*}{\rightharpoonup} \sigma_C$ and $y \in \operatorname{supp} \sigma_C$, there exist two constants $c, \beta > 0$ such that, for j sufficiently large,

$$\sigma_j\big(B_1\big) \leq c, \qquad \sigma_j\big(B_r(y)\big) \geq \beta,$$

which contradicts (5.36).

This also prove the inclusion $\operatorname{supp} \sigma_C \subset D_C$. To prove the other inclusion, let us notice that the scaling invariance of σ_C and $\operatorname{supp} \sigma_C \neq \emptyset$ imply $0 \in \operatorname{supp} \sigma_C$. Since σ_C is invariant by translations along D_C by Lemma 5.6, we have the opposite inclusion and $\operatorname{supp} \sigma_C = D_C$.

□

We next prove that, for every $x \in F$ (where F is the set defined in the previous Lemma) such that $\Theta^{k-1}_*(\sigma_V, x) > 0$ and for every $C \in \operatorname{Tan}(V, x)$, σ_C is the surface measure of a $(k-1)$-plane, which coincides with D_C.

Lemma 5.8. *Let V satisfy Assumption 1 and let F be the set defined in Lemma 5.7. For every $x_0 \in F$ such that $\Theta_*^{k-1}(\sigma_V, x) > 0$ and for every $C \in \operatorname{Tan}(V, x_0)$, the invariant subspace D_C of C is $(k-1)$-dimensional; moreover there exists $\alpha_0 > 0$ such that*

$$\sigma_C = \alpha_0 \mathcal{H}^{k-1} \llcorner D_C.$$

Proof. Let $x_0 \in F$ be a fixed point and let us assume $C \in \operatorname{Tan}(V, x_0)$. Without loss of generality we can assume that $x_0 = 0$ and that $T_0 \partial \mathcal{M}$ is the subspace $\{x_n = 0\}$. Since $C \in \operatorname{Tan}(V, 0)$ there exists $r_j \downarrow 0$ such that, using the notations for the scalings, $V_j \overset{*}{\rightharpoonup} C$. Lemma 5.4 asserts C is a rectifiable cone and that that $\sigma_j \overset{*}{\rightharpoonup} \sigma_C$, where σ_C is the measure relative to C given by Corollary 4.6.

We first recall that the condition

$$\Theta_*^{k-1}(\sigma_V, 0) = \liminf_{r \to 0} \frac{\sigma_V\big(B_r(x_0)\big)}{r^{k-1}} > 0, \tag{5.37}$$

together with $\sigma_j \overset{*}{\rightharpoonup} \sigma_C$, implies that $\sigma_C \neq 0$. Thus, Lemma 5.7 provides $\operatorname{supp} \sigma_C = D_C$, where D_C is the invariant subspace D_C of C, given by Lemma 5.6. D_C is a linear subspace of $\mathbb{R}^n$ and throughout this proof we call $m = \dim D_C$ its dimension. By definition of D_C, we clearly have $D_C \subset \{x_n = 0\}$. Thus $m \leq n-1$. After a suitable change of coordinates, we can assume that $D_C = \{x_{m+1} = \cdots = x_n = 0\}$.

For any $y \in D_C$ and any $r > 0$, we denote by $Q_{D_C}(y, r)$ the closed cube included in D_C with center y, side of length r and faces parallel to the coordinate vectors $e_1, \ldots, e_m$. If we set

$$\alpha_0 = \liminf_{j \to \infty} \frac{\sigma_V\big(B_{r_j}\big)}{\omega_{k-1} r_j^{k-1}}. \tag{5.38}$$

we have $\alpha_0 > 0$, by (5.37). Since σ_C is invariant by scalings (by Lemma 5.4) and by translations in D_C (by Lemma 5.6), there exists a fixed $\beta_0 > 0$ such that

$$\sigma_C\big(Q_{D_C}(y, r)\big) = \beta_0 r^{k-1} \qquad \forall y \in D_C \quad \forall r > 0. \tag{5.39}$$

We are going to show that (5.39) implies $m = k-1$. We argue by contradiction and by cases:

- Let us assume, by contradiction, that $m < k-1$. For each $l \in \mathbb{N}$, there exists a covering $\{Q_i^l\}_{i=1}^{2^{lm}}$ of $Q_{D_C}(0,1)$ such that each Q_i^l is a cube included in D_C and of side length 2^{-l}. Therefore

  $$\sigma_C\big(Q_{D_C}(0,1)\big) \leq \sum_{i=1}^{2^{lm}} \sigma_C\big(Q_i^l\big) = 2^{lm} \beta_0 2^{-l(k-1)} \leq \beta_0 2^{-l} \xrightarrow[l \to \infty]{} 0.$$

  Thus $\sigma_C\big(Q_{D_C}(0,1)\big) = 0$. By translation invariance of σ_C, it follows that $\sigma_C = 0$, which is a contradiction.
- Let us assume now that $m \geq k$. We first observe that by approximation, (5.39) holds also for cubes that are open in D_C. Hence, taking for every $l \in \mathbb{N}$ a covering $\{Q_i^l\}_{i=1}^{2^{lm}}$ of $Q_{D_C}(0,1)$ of cubes included in D_C with disjoint interiors and of side length 2^{-l}, we have

  $$\sigma_C\big(Q_{D_C}(0,1)\big) \geq \sum_{i=1}^{2^{lm}} \sigma_C\Big(\big(Q_i^l\big)^{\mathrm{o}}\Big) = 2^{lm} \beta_0 2^{-l(k-1)} \geq \beta_0 2^{l} \xrightarrow[l \to \infty]{} +\infty,$$

  (where $\big(Q_i^l\big)^{\mathrm{o}}$ is intended in the topology of D_C) which is a contradiction.

This shows that $\dim D_C = k-1$. Since σ_C is invariant by scalings and translations in D_C, we have

$$\frac{\sigma_C\big(B(r(y))\big)}{\mathcal{H}^{k-1}\big(D_C \cap B_r(y)\big)} = \alpha_0 \qquad \forall y \in D_C, \, \forall r > 0.$$

By Radon-Nikodym Theorem [12, Theorem 4.7], we obtain that $\sigma_C = \alpha_0 \mathcal{H}^{k-1} \llcorner D_C$. □

5.3. **Proof of Theorem 1.2.**

We can now prove Theorem 1.2.

Proof of Theorem 1.2. By definition of σ_V^* we have

$$0 < \Theta_*^{k-1}(\sigma_V^*, x) \leq \Theta^{*(k-1)}(\sigma_V^*, x) < +\infty \qquad \text{for } \sigma_V^*\text{-a.e. } x \in \partial\mathcal{M}, \tag{5.40}$$

thus σ_V^* satisfies the first condition of the Marstrand-Mattila Rectifiability Criterion (Theorem 5.1).

To check the second condition of the criterion, let us notice that Corollary 5.5 yield

$$\operatorname{Tan}^{k-1}(\sigma_V^*, x) = \{\sigma_C \mid C \in \operatorname{Tan}(V, x)\} \qquad \text{for } \sigma_V^*\text{-a.e. } x \in \partial\mathcal{M}. \tag{5.41}$$

Since $\Theta_*^{k-1}(\sigma_V, x) > 0$ for every $x \in E$ where E is defined in (1.8), Lemma 5.7 and Lemma 5.8 yields

$$\{\sigma_C \mid C \in \operatorname{Tan}(V, x)\} \subset \{\alpha\mathcal{H}^{k-1} \llcorner S \mid S\, (k-1)\text{-dimensional plane}, \alpha > 0\} \qquad \forall x \in F \cap E, \tag{5.42}$$

where F is the set defined in Lemma 5.7. Moreover

$$\begin{aligned} \sigma_V^*\big(\partial\mathcal{M} \setminus (F \cap E)\big) &\leq \sigma_V^*\big(\partial\mathcal{M} \setminus \operatorname{Dens}(V)\big) + \sigma_V^*\big(\operatorname{Dens}(V) \setminus F\big) + \sigma_V^*\big(\partial\mathcal{M} \setminus E\big) \\ &= 0. \end{aligned} \tag{5.43}$$

Every set in the right-hand side is σ_V^*-negligible because: $\sigma_V^*\big(\partial\mathcal{M} \setminus \operatorname{Dens}(V)\big) = 0$ by Lemma 5.2 and $p > 1$; $\sigma_V^*\big(\operatorname{Dens}(V) \setminus F\big) = 0$ by Lemma 5.7 and $\sigma_V^* \ll \sigma_V$; $\sigma_V^*\big(\partial\mathcal{M} \setminus E\big) = 0$ by definition of σ_V^*.

Summarizing (5.41), (5.42) and (5.43), we obtain

$$\operatorname{Tan}^{k-1}(\sigma_V^*, x) \subset \{\alpha\mathcal{H}^{k-1} \llcorner S \mid \alpha > 0,\ S \text{ is a } (k-1)\text{-dimensional plane}\} \qquad \text{for } \sigma_V^*\text{-a.e. } x \in \partial\mathcal{M}.$$

Since this is the second condition for the Marstrand-Mattila Rectifiability Criterion, we have that σ_V^* is $(k-1)$-rectifiable. ☐

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