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# INTEGRABILITY OF THE SUB-RIEMANNIAN MEAN CURVATURE AT DEGENERATE CHARACTERISTIC POINTS IN THE HEISENBERG GROUP

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ABSTRACT. We address the problem of integrability of the sub-Riemannian mean curvature of an embedded hypersurface around isolated characteristic points. The main contribution of this note is the introduction of a concept of mildly degenerate characteristic point for a smooth surface of the Heisenberg group, in a neighborhood of which the sub-Riemannian mean curvature is integrable (with respect to the perimeter measure induced by the Euclidean structure). As a consequence we partially answer to a question posed by Danielli-Garofalo-Nhieu in [DGN12], proving that the mean curvature of a real-analytic surface with discrete characteristic set is locally integrable.

## 1. INTRODUCTION AND STATEMENTS

Let  $M$  be a sub-Riemannian manifold, and  $\Sigma \subset M$  be an embedded hypersurface. The horizontal mean curvature  $\mathcal{H} : \Sigma \rightarrow \mathbb{R}$  is a geometrical invariant which arises naturally in different areas of geometric analysis. It appears in the theory of minimal surfaces [DGN07, HP08, Mon15, Pau04, CHMY05], in the study of the heat content asymptotics in sub-Riemannian manifolds [TW18, RR20], and in Steiner-type formulas for the volume of tubes around hypersurfaces [BFF<sup>+</sup>15].

Of particular relevance in all aforementioned applications is the (local) integrability of  $\mathcal{H}$ , either with respect to the horizontal perimeter measure or the Riemannian one (cf. Section 2 for precise definitions). An important fact is that, even for smooth hypersurfaces  $\Sigma$ , the horizontal mean curvature blows-up at the so-called characteristic points, where the subspace of horizontal directions is tangent to  $\Sigma$ , making the local integrability problem a non-trivial one.

For what concerns the sub-Riemannian perimeter measure  $\sigma_H$ , as remarked first in [DGN12] for the Heisenberg group, the blow-up of  $\mathcal{H}$  is compensated by the degeneration of  $\sigma_H$ , and thus  $\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_H)$ .

The aforementioned compensation fails if one replaces the sub-Riemannian perimeter measure  $\sigma_H$  on  $\Sigma$  with the Riemannian one  $\sigma_R$ , and in general  $\mathcal{H} \notin L^1_{\text{loc}}(\Sigma, \sigma_R)$ . In all known examples, however, either  $\Sigma$  is not very smooth, or the set of characteristic points has positive dimension, cf. [DGN12]. On the other hand, if  $\Sigma$  is at least  $C^2$  and the characteristic set is discrete, no counter-examples to local integrability are known. Furthermore, a thorough analysis of many specific cases led the authors in [DGN12] to conjecture that, under these assumptions,  $\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_R)$ .

In this note we address the question of local integrability with respect to the Riemannian perimeter measure in the Heisenberg group and in the more general context of

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three-dimensional contact manifolds. The case of non-degenerate and isolated characteristic points is elementary, and we can state the following result (proved in a setting that includes the Heisenberg group).

**Theorem 1.1.** *Let  $M$  be a three-dimensional contact sub-Riemannian manifold, equipped with a smooth measure  $\mu$  and let  $\Sigma \subset M$  be a  $C^2$  embedded surface. Assume that all characteristic points of  $\Sigma$  are isolated and non-degenerate. Then,*

$$\mathcal{H} \in L_{\text{loc}}^1(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure by  $\mu$  on  $\Sigma$ .

The case of *degenerate* characteristic points is less understood, even for the case of smooth surfaces in the Heisenberg group  $\mathbb{H}$  (to which we restrict to for the rest of this introduction). The main result of this note is the definition of a concept of *mildly degenerate* characteristic point for surfaces in the Heisenberg group, for which we are able to prove local integrability. In the following,  $\mathbb{H}$  is equipped with the Lebesgue measure, which induces the Riemannian measure  $\sigma_{\mathbb{R}}$  on a smooth embedded hypersurface  $\Sigma$ .

**Theorem 1.2.** *Let  $\Sigma \subset \mathbb{H}$  be a smooth embedded surface. Assume that the all characteristic points of  $\Sigma$  are isolated and mildly degenerate. Then*

$$\mathcal{H} \in L_{\text{loc}}^1(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ .

The concept of *mild degeneration* is based on a finite-order condition along an intrinsic curve  $\mathcal{C} \subset \Sigma$  emanating from degenerate characteristic points, which to our best knowledge does not appear in previous literature (cf. Definitions 4.1 and 4.4). In particular, if  $\Sigma$  is *real-analytic*, all degenerate characteristic points are mildly degenerate. As a consequence, we have the following corollary, which answers affirmatively to the conjecture in [DGN12], at least for real-analytic surfaces.

**Theorem 1.3.** *Let  $\Sigma \subset \mathbb{H}$  be a real-analytic embedded surface. Assume that the all characteristic points of  $\Sigma$  are isolated. Then,*

$$\mathcal{H} \in L_{\text{loc}}^1(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ .

Our results include and unify several previous examples of local integrability of the horizontal mean curvature present in literature. In particular, one can compare our results with Propositions 3.1-3.4 in [DGN12]: Propositions 3.1, 3.2 are included in Theorem 1.1, while Proposition 3.3 is covered by Theorem 1.3.

*Remark 1.4.* Theorems 1.1, 1.2 and 1.3 are proved below in a slightly stronger form, cf. Theorems 3.2, 4.5 and 4.7, respectively. Namely, in each case, we prove the local integrability of  $\frac{1}{W}$ , where  $W$  denotes the norm of horizontal projection of the Riemannian horizontal normal to  $\Sigma$ . For this stronger result, the mild degeneration assumption is sharp and cannot be improved, cf. Remark 4.6. In particular, it implies the local integrability of the horizontal mean curvature, but also the local integrability of the intrinsic horizontal Gaussian curvature as defined in [BTV17], yielding Gauss-Bonnet-type theorems for surfaces with isolated and mildly degenerate (or non-degenerate) characteristic points. We refer to [BTV17, Thm. 1.1] for details.

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## 2. PRELIMINARIES

Let  $M$  be a smooth, connected  $m$ -dimensional manifold. For our purposes, a sub-Riemannian structure on  $M$  is defined by a subbundle of the tangent bundle  $\mathcal{D} \subset TM$ , which we call *distribution*, and a metric on it, namely a positive, symmetric  $(0, 2)$ -tensor on  $\mathcal{D}$ , denoted by  $g$ .

If the distribution has rank  $k$ , then, locally in an open set  $U$ , we may describe it via a local orthonormal frame, namely a family of  $k$  vector fields such that

$$(1) \quad \mathcal{D}_p = \text{span}_p\{X_1, \dots, X_k\} \subset T_pM, \quad \forall p \in U.$$

We assume that the distribution is *bracket-generating*, cf. [ABB20] for details.

**Divergence and horizontal gradient.** Let  $\mu$  be a smooth measure on  $M$ , defined by a positive tensor density. The *divergence* of a smooth vector field is defined by

$$\text{div}_\mu(X)\mu = \mathcal{L}_X\mu, \quad \forall X \in \Gamma(TM),$$

where  $\mathcal{L}_X$  denotes the Lie derivative in the direction of  $X$ . The *horizontal gradient* of a function  $f \in C^1(M)$ , denoted by  $\nabla f$ , is defined as the horizontal vector field (i.e. tangent to the distribution at each point), such that

$$g(\nabla f, V) = Vf, \quad \forall V \in \Gamma(\mathcal{D}),$$

where  $V$  acts as a derivation. In terms of a local orthonormal frame as in (1), one has

$$\nabla f = \sum_{i=1}^k (X_i f) X_i, \quad \forall f \in C^1(M).$$

**Characteristic points.** Let  $\Sigma \subset M$  be a  $C^1$  embedded hypersurface. We say that  $p \in \Sigma$  is a *characteristic point* if

$$\mathcal{D}_p \subseteq T_p\Sigma.$$

We denote by  $C(\Sigma)$  the set of characteristic points. Notice that  $C(\Sigma) \subset \Sigma$  is a closed set, and it has zero measure if  $\Sigma$  is at least  $C^2$ . We refer to [Bal03, Der72] for fine results about the size of  $C(\Sigma)$  under suitable assumptions on the regularity of  $\Sigma$ .

The hypersurface  $\Sigma$  can be locally described as follows: at  $p \in M$ , there exists a neighborhood  $U \subset M$  of  $p$  and  $u \in C^1(U)$  such that

$$\Sigma \cap U = \{u = 0\}, \quad du|_{\Sigma \cap U} \neq 0.$$

When  $\Sigma$  is locally given as the zero-locus of  $u$ , then  $p \in C(\Sigma)$  if and only if

$$X_i u(p) = 0, \quad \forall i = 1, \dots, k.$$

**Horizontal Hessian.** We introduce the horizontal Hessian for classifying characteristic points (cf. also [BBCH20]). Fix an affine connection  $\tilde{\nabla}$  on the distribution  $\mathcal{D}$ . Then, the horizontal Hessian of  $u \in C^2(M)$  is the  $(0, 2)$ -tensor on  $\mathcal{D}$ , defined as

$$\text{Hess}_{\mathbb{H}}(u)(V, W) = g(\tilde{\nabla}_V(\nabla u), W), \quad \forall V, W \in \Gamma(\mathcal{D}).$$

While in general the definition of horizontal Hessian depends on the choice of the connection, it is intrinsic at characteristic points.

**Lemma 2.1.** *Let  $M$  be a sub-Riemannian manifold and let  $\Sigma = \{u = 0\} \subset M$ , where  $u: M \rightarrow \mathbb{R}$  is a  $C^2$  submersion on  $\Sigma$ . If  $p \in C(\Sigma)$ , then  $\text{Hess}_{\mathbb{H}}(u)|_p$  does not depend on the choice of the connection and thus is a well-defined bilinear map on  $\mathcal{D}_p$ .*

*Proof.* Let  $\{X_1, \dots, X_k\}$  be a local orthonormal frame around  $p$ , then, by definition of horizontal Hessian, for  $V, W \in \Gamma(\mathcal{D})$ , we have

$$\begin{aligned} \text{Hess}_{\mathbb{H}}(u)(V, W) &= \sum_{i,j=1}^k V^i W^j g\left(\tilde{\nabla}_{X_i} \nabla u, X_j\right) \\ &= \sum_{i,j=1}^k V^i W^j g\left(\tilde{\nabla}_{X_i} \left(\sum_{\ell=1}^k (X_\ell u) X_\ell\right), X_j\right) \\ &= \sum_{i,j,\ell=1}^k V^i W^j g\left((X_i X_\ell u) X_\ell + (X_\ell u) \tilde{\nabla}_{X_i} X_\ell, X_j\right), \end{aligned}$$

using the linearity and the Leibniz formula for the connection. Finally, since  $p \in C(\Sigma)$ ,  $X_\ell u(p) = 0$  for any  $\ell = 1, \dots, k$ , therefore we conclude that

$$\text{Hess}_{\mathbb{H}}(u)(V, W)|_p = \sum_{i,j=1}^k V^i W^j X_i X_j u(p),$$

and the right-hand side does not depend on the choice of the connection  $\tilde{\nabla}$ .  $\square$

**Definition 2.2.** Let  $M$  be a sub-Riemannian manifold and  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface in  $M$ . We say that  $p \in C(\Sigma)$  is a *non-degenerate characteristic point* if

$$\det(\text{Hess}_{\mathbb{H}}(u)|_p) \neq 0,$$

where  $u \in C^2$  is a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Notice that this property does not depend on the choice of  $u$ .

**Horizontal mean curvature.** The *horizontal mean curvature* at  $p \in \Sigma$  is defined as

$$\mathcal{H}(p) = -\text{div}_\mu \left( \frac{\nabla u}{\|\nabla u\|} \right) \Big|_p,$$

where  $u \in C^2$  is a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Notice that the value of  $\mathcal{H}$  does not depend on the choice of  $u$ .

**(Sub-)Riemannian induced measure.** Let  $\nu$  be the horizontal unit normal to  $\Sigma$ , then the sub-Riemannian induced measure  $\sigma_{\mathbb{H}}$  on  $\Sigma$  is the positive smooth measure with density  $|i_\nu \mu|$ . If  $u$  is a local defining function for  $\Sigma$ ,  $\nu$  is given by

$$(2) \quad \nu = \frac{\nabla u}{\|\nabla u\|}.$$

Analogously, to define the Riemannian induced measure  $\sigma_{\mathbb{R}}$ , consider any Riemannian extension of the sub-Riemannian structure, then replace  $\nu$  with the Riemannian unit normal, which is given by (2) with the Riemannian gradient. Notice that  $\sigma_{\mathbb{R}}$  coincides with the  $n - 1$  dimensional Hausdorff measure on  $\Sigma$  induced by the Riemannian structure. Moreover, it depends on the choice of a Riemannian extension, but this choice does not play any role concerning the integrability of the horizontal mean curvature.

**2.1. A general estimate for horizontal mean curvature.** We provide here a general estimate for the horizontal mean curvature. Let  $M$  be a sub-Riemannian manifold and  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface. Without loss of generality, assume that  $u \in C^2(M)$  is a global defining function for  $\Sigma$ , that is  $u: M \rightarrow \mathbb{R}$  is a submersion on  $\Sigma$  and  $\Sigma = \{u = 0\}$ . Having fixed a local orthonormal frame for the sub-Riemannian structure at a point  $p$ , say  $\{X_1, \dots, X_k\}$ , recall that

$$(3) \quad \nabla u = \sum_{i=1}^k (X_i u) X_i,$$

and its norm, which we denote by  $W$ , is given by

$$(4) \quad W^2 = \|\nabla u\|^2 = g(\nabla u, \nabla u) = \sum_{i=1}^k (X_i u)^2.$$

Therefore, we can write the horizontal mean curvature explicitly in terms of  $\nabla u$  and  $W$

$$\mathcal{H} = -\operatorname{div}_{\mu} \left( \frac{\nabla u}{W} \right) = -\frac{1}{W} \Delta u + \frac{1}{W^2} g(\nabla u, W).$$

Using formula (3) and (4), we obtain

$$\mathcal{H} = -\frac{1}{W} \Delta u + \frac{1}{W^3} \sum_{i,j=1}^k (X_i u)(X_j u)(X_i X_j u),$$

which gives the estimate

$$(5) \quad |\mathcal{H}| \leq \frac{1}{W} \left( \|\Delta u\|_{L^\infty(U)} + \sum_{i,j=1}^k \|X_i X_j u\|_{L^\infty(U)} \right) \leq \frac{C_0}{W},$$

for a suitable constant  $C_0 > 0$ , where we have used the inequality:  $|X_i u| \leq W$ , for any  $i = 1, \dots, k$ . Here  $\|\cdot\|_{L^\infty(U)}$  denotes the supremum norm and  $U$  is a relatively compact neighborhood of  $p$ .

We recover the well-known integrability result for the horizontal mean curvature with respect to the sub-Riemannian perimeter measure (see also [DGN12, Prop. 3.5]).

**Lemma 2.3.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\mu$  and let  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface in  $M$ . Then*

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{H}}).$$

Here  $\sigma_{\mathbb{H}}$  denotes the sub-Riemannian induced measure on  $\Sigma$ .

*Proof.* Let  $U$  be a neighborhood of  $p$  and consider  $u \in C^2$ , a local defining function for  $\Sigma$  on  $U$ . Then, denoting by  $W = \|\nabla u\|$ , we have  $\sigma_{\mathbb{H}} \propto W \sigma_{\mathbb{R}}$ , up to a smooth never-vanishing function. Therefore, using (5), we obtain

$$\left| \int_{\Sigma \cap U} \mathcal{H} d\sigma_{\mathbb{H}} \right| \leq C \int_{\Sigma \cap U} |\mathcal{H}| W d\sigma_{\mathbb{R}} \leq C \int_{\Sigma \cap U} \frac{C_0}{W} W d\sigma_{\mathbb{R}} < +\infty. \quad \square$$

### 3. INTEGRABILITY FOR NON-DEGENERATE CHARACTERISTIC POINTS IN 3D CONTACT SUB-RIEMANNIAN MANIFOLDS

Let  $M$  be a smooth manifold of dimension 3. Let  $\omega$  be a contact one-form, that is such that  $\omega \wedge d\omega \neq 0$ . Then, the contact distribution is

$$\mathcal{D}_p = \ker(\omega_p) \subset T_p M, \quad \forall p \in M.$$

By the non-degeneracy assumption on  $d\omega$ ,  $\mathcal{D}$  is a subbundle of rank 2 and is bracket-generating. Any metric on  $\mathcal{D}$  defines a sub-Riemannian structure on  $M$ . We will refer to  $M$  as *contact sub-Riemannian manifold*. Recall the following normal form for an orthonormal frame (see [ACEAG98, AG99]).

**Theorem 3.1.** *Let  $M$  be a 3D contact sub-Riemannian manifold, with contact 1-form  $\omega$ , and  $\{X_1, X_2\}$  be a local orthonormal frame for  $\mathcal{D} = \ker(\omega)$ . There exists a smooth coordinate system  $(x, y, z)$  such that*

$$\begin{aligned} X_1 &= \partial_x - \frac{y}{2}\partial_z + \beta y(y\partial_x - x\partial_y) + \gamma y\partial_z, \\ X_2 &= \partial_y + \frac{x}{2}\partial_z - \beta x(y\partial_x - x\partial_y) + \gamma x\partial_z, \end{aligned}$$

where  $\beta = \beta(x, y, z)$  and  $\gamma = \gamma(x, y, z)$  are smooth functions satisfying

$$\beta(0, 0, z) = \gamma(0, 0, z) = \partial_x \gamma(0, 0, z) = \partial_y \gamma(0, 0, z) = 0.$$

We now prove the first integrability result for isolated non-degenerate characteristic points on general contact manifolds.

**Theorem 3.2.** *Let  $\Sigma \subset M$  be a  $C^2$  embedded surface, let  $p$  be an isolated non-degenerate characteristic point and let  $u \in C^2$  be a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Denote with  $W$  the norm of the horizontal gradient of  $u$ . Then*

$$(6) \quad \frac{1}{W} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ . In particular

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}).$$

*Proof.* Introducing the normal form given by Theorem 3.1, we may assume that the characteristic point is at the origin and that  $\Sigma$  is locally a graph around the origin. Indeed, recall that  $\Sigma \cap U = \{u = 0\}$  and  $du \neq 0$  on  $\Sigma \cap U$ . However, since  $\mathbf{0} \in C(\Sigma)$

$$(7) \quad 0 = d_p u(X_1) = \partial_x u(\mathbf{0}) \quad \text{and} \quad 0 = d_p u(X_2) = \partial_y u(\mathbf{0}),$$

therefore  $\partial_z u(\mathbf{0}) \neq 0$ , which implies that, up to restricting  $U$ ,  $\Sigma \cap U = \{z = g(x, y)\}$ , for some  $C^2$  function  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ . Moreover, in this coordinates, relations (7) give first-order conditions on  $g$

$$g(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0.$$

Recall that we want to discuss the finiteness of the following integral

$$(8) \quad \int_{\Sigma \cap U} \frac{1}{W} d\sigma_{\mathbb{R}} = \int_V \frac{1}{W(x, y, g(x, y))} f(x, y) dx dy,$$

where  $V$  is a neighborhood of  $(0, 0)$  and  $f$  is the Riemannian density in coordinates. Since the characteristic point is non-degenerate, up to restricting  $V$ , the map

$$\varphi: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad \text{such that} \quad \begin{cases} \tilde{x} = X_1 u(x, y, g(x, y)) \\ \tilde{y} = X_2 u(x, y, g(x, y)) \end{cases}$$

defines a smooth change coordinate on  $V$ . Indeed, its Jacobian at  $(0,0)$  equals the determinant of the horizontal Hessian at the origin

$$\det(\mathcal{J}\varphi)|_{(0,0)} = \det \begin{pmatrix} \partial_x X_1 u + g_x \partial_z X_1 u & \partial_y X_1 u + g_y \partial_z X_1 u \\ \partial_x X_2 u + g_x \partial_z X_2 u & \partial_y X_2 u + g_y \partial_z X_2 u \end{pmatrix} \Big|_{(0,0)} = \det(\text{Hess}_{\mathbb{H}}(u)|_{\mathbf{0}}),$$

and by the non-degeneracy assumption is non-zero. Thus, after the change of variables, the integral (8) becomes

$$\int_V \frac{1}{W} f(x, y) dx dy = \int_{\varphi(V)} \frac{1}{|\det(\mathcal{J}\varphi)| \sqrt{\tilde{x}^2 + \tilde{y}^2}} \tilde{f}(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} < +\infty.$$

The claim (6) follows. Using estimate (5), we obtain the local integrability of  $\mathcal{H}$ .  $\square$

*Remark 3.3.* A non-degenerate characteristic point need not to be isolated, however, this situation is quite pathological. For example, in the Heisenberg group  $\mathbb{H}$  (cf. Section 4), the only situation in which this can occur is when we have a sequence of characteristic points  $(x_n, y_n)$  accumulating at the origin, and not contained in any absolutely continuous curve. Indeed, consider  $\Sigma = \{z = g(x, y)\}$  in  $\mathbb{H}$  and assume we have an absolutely continuous curve  $\gamma: (-\varepsilon, \varepsilon) \rightarrow \Sigma$  of characteristic points with  $\gamma(0) = \mathbf{0}$ , and such that  $\dot{\gamma}(0)$  exists. Then, the origin is a degenerate characteristic point, indeed for each  $t \in (-\varepsilon, \varepsilon)$ , we have

$$\begin{cases} g_x(\gamma_1(t), \gamma_2(t)) + \frac{\gamma_2(t)}{2} = 0, \\ g_y(\gamma_1(t), \gamma_2(t)) - \frac{\gamma_1(t)}{2} = 0. \end{cases}$$

Differentiating both equations with respect to  $t$ , and evaluating at  $t = 0$ , we have

$$\begin{cases} \dot{\gamma}_1(0)g_{xx}(0,0) + \dot{\gamma}_2(0)g_{xy}(0,0) + \frac{\dot{\gamma}_2(0)}{2} = 0, \\ \dot{\gamma}_1(0)g_{xy}(0,0) + \dot{\gamma}_2(0)g_{yy}(0,0) - \frac{\dot{\gamma}_1(0)}{2} = 0, \end{cases}$$

thus,  $\begin{pmatrix} \dot{\gamma}_1(0) \\ \dot{\gamma}_2(0) \end{pmatrix} \in \ker(\text{Hess}_{\mathbb{H}}(u)|_{\mathbf{0}})$ , implying that  $\mathbf{0}$  is a degenerate characteristic point.

#### 4. INTEGRABILITY FOR MILDLY DEGENERATE CHARACTERISTIC POINTS IN $\mathbb{H}$

The *Heisenberg group* is the 3D contact structure on  $\mathbb{R}^3$ , defined by the 1-form

$$\omega = dz - \frac{1}{2}(xdy - ydx).$$

A global frame for the contact distribution is given by  $\{X, Y\}$ , where

$$(9) \quad X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z.$$

Setting  $\{X, Y\}$  to be an orthonormal frame, the resulting sub-Riemannian manifold is the well-known first Heisenberg group,  $\mathbb{H}$ . We equip it with the Lebesgue measure.

Let us consider in  $\mathbb{H}$  a surface  $\Sigma = \{u = 0\}$ , where  $u \in C^\infty(\mathbb{R}^3)$  with  $du \neq 0$  on  $\Sigma$ . Assume that  $p \in C(\Sigma)$  is a degenerate characteristic point, meaning that the horizontal Hessian of  $u$  has zero determinant at  $p$ . Notice that, in  $\mathbb{H}$ , the horizontal Hessian at  $p \in C(\Sigma)$  coincides with the one introduced in [DGN03], and, in terms of the orthonormal basis  $\{X, Y\}$ , takes the form:

$$\text{Hess}_{\mathbb{H}}(u)|_p = \begin{pmatrix} XXu(p) & XYu(p) \\ YXu(p) & YYu(p) \end{pmatrix}.$$



By the bracket-generating assumption, one of the entries of the Hessian must be non-zero at  $p$ , thus, it has a 1-dimensional kernel at  $p$ , spanned by some unitary vector, say  $N_p \in \mathcal{D}_p$ , which is unique, up to a sign. We extend  $N_p$  to a left-invariant vector field  $N \in \Gamma(\mathcal{D})$ . Taking an orthogonal vector field to  $N$  in  $\Gamma(\mathcal{D})$ , we obtain an orthonormal frame  $\{N, T\}$  for the distribution, which, up to changing sign, we assume to be co-oriented with the standard one (9).

**Definition 4.1.** Let  $\Sigma \subset \mathbb{H}$  be a smooth embedded surface and let  $p \in C(\Sigma)$  be degenerate. The critical curve of  $p$  is defined as the set of points in  $\Sigma$  where  $N$  is tangent to  $\Sigma$ , i.e.

$$(10) \quad \mathcal{C} = \{q \in \Sigma \mid N(q) \in T_q \Sigma\}.$$

We prove now that Definition 4.1 is well-posed.

**Lemma 4.2.** *Let  $\Sigma \subset \mathbb{H}$  be a smooth embedded surface and let  $p \in C(\Sigma)$  be degenerate. Then, in a neighborhood of  $p$ , the set  $\mathcal{C}$  as in (10), is a smooth curve in  $\Sigma$ , through  $p$ .*

*Proof.* Consider for  $\Sigma$  a local defining function  $u \in C^\infty$ . Then,  $N(q) \in T_q \Sigma$  if and only if  $d_q u(N) = 0$ , thus

$$\mathcal{C} = \{u = 0\} \cap \{Nu = 0\}.$$

In the orthonormal frame  $\{N, T\}$ ,  $Nu(p) = Tu(p) = 0$  since  $p \in C(\Sigma)$ . However,  $d_p u \neq 0$ , therefore, by the bracket-generating assumption

$$[N, T]u(p) = NTu(p) - TNu(p) \neq 0.$$

We are going to show that  $NTu(p) = 0$ . First of all, since the frame  $\{N, T\}$  is co-oriented with  $\{X, Y\}$ , there exists  $R \in \text{SO}(2)$ , such that

$$(11) \quad \begin{pmatrix} N \\ T \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

where we used the shorthand  $a = \cos(\theta)$ ,  $b = \sin(\theta)$ , for some  $\theta \in [0, 2\pi)$ . Hence,

$$\begin{aligned} NTu(p) &= (aX + bY)(-bX + aY)u(p) \\ &= -abXXu(p) + a^2XYu(p) - b^2YXu(p) + abYYu(p). \end{aligned}$$

Second of all, by definition of  $N$ ,  $N_p \in \ker(\text{Hess}_{\mathbb{H}}(u)|_p)$ , thus we obtain

$$0 = \begin{pmatrix} -b & a \end{pmatrix} \begin{pmatrix} XXu(p) & YXu(p) \\ XYu(p) & YYu(p) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = NTu(p).$$

Finally,  $TNu(p) \neq 0$  so the differential of the map  $\phi = (u, Nu)$  has maximal rank at  $p$ , implying that  $\mathcal{C}$  is a smooth curve in  $\Sigma$ , in a neighborhood of  $p$ .  $\square$

*Remark 4.3.* Notice that, in general, the critical curve  $\mathcal{C}$  is not necessarily horizontal, and it is not related to the characteristic foliation induced on  $\Sigma$  by the contact structure.

**Definition 4.4.** Let  $\Sigma \subset \mathbb{H}$  be an embedded smooth surface, let  $p \in C(\Sigma)$  be a degenerate characteristic point and let  $u$  be a local defining function for  $\Sigma$  around  $p$ . Let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{C}$  be a regular parametrization of  $\mathcal{C}$ , with  $\gamma(0) = p$ . We say that  $p$  is mildly degenerate if the function

$$s \mapsto Tu(\gamma(s)),$$

has a finite order zero at  $s = 0$ . Notice that this definition does not depend on the choice of  $u$  and the regular parametrization.

**Theorem 4.5.** *Let  $\Sigma \subset \mathbb{H}$  be an embedded surface, let  $p$  be an isolated mildly degenerate characteristic point and let  $u \in C^\infty$  be a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Denote with  $W$  the norm of the horizontal gradient of  $u$ . Then*

$$\frac{1}{W} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ . In particular

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}).$$

*Proof.* Without loss of generality, we may assume that  $u(x, y, z) = z - g(x, y)$ , i.e.  $\Sigma = \{z = g(x, y)\}$ , where  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth, and that  $C(\Sigma) = \{\mathbf{0}\}$ . This implies

$$g(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0.$$

Notice that the local integrability of  $W$  is preserved by the action of isometries of  $\mathbb{H}$ . We can exploit this fact to reduce  $g$  to a normal form. Recall first that Heisenberg isometries (preserving the origin and the orientation of the  $z$ -axis) are given by the standard action of  $\text{SO}(2)$  on the  $xy$ -plane. Consider then the isometry

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \left( \begin{array}{c|c} R & \\ \hline & 1 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where  $R \in \text{SO}(2)$  is defined in (11), in this way the frame  $\{N, T\}$  is sent to the standard frame  $\{X, Y\}$ . In particular, since  $N_{\mathbf{0}} \in \ker(\text{Hess}_{\mathbb{H}}(u)|_{\mathbf{0}})$ ,  $N$  is given by

$$N = aX + bY = \frac{-YXu(p)X + XXu(p)Y}{\sqrt{YXu(p)^2 + XXu(p)^2}} = \frac{\left(g_{11} + \frac{1}{2}\right)X - g_{20}Y}{\sqrt{\left(g_{11} + \frac{1}{2}\right)^2 + g_{20}^2}},$$

where  $g_{ij} = \partial_x^i \partial_y^j g(0, 0)$ . Hence, the change of coordinates is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{W_0} \begin{pmatrix} g_{11} + \frac{1}{2} & g_{20} & \\ -g_{20} & g_{11} + \frac{1}{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \frac{1}{W_0} \begin{pmatrix} \left(g_{11} + \frac{1}{2}\right)\tilde{x} + g_{20}\tilde{y} \\ -g_{20}\tilde{x} + \left(g_{11} + \frac{1}{2}\right)\tilde{y} \\ \tilde{z} \end{pmatrix},$$

where  $W_0 = \sqrt{\left(g_{11} + \frac{1}{2}\right)^2 + g_{20}^2}$ . Therefore, expanding  $u$ , we have

$$\begin{aligned} u(x, y, z) &= z - \frac{g_{20}}{2}x^2 - \frac{g_{02}}{2}y^2 - g_{11}xy + O(r^3) \\ &= \tilde{z} - \frac{g_{20}}{2W_0^2} \left( \left(g_{11} + \frac{1}{2}\right)\tilde{x} + g_{20}\tilde{y} \right)^2 - \frac{g_{02}}{2W_0^2} \left( -g_{20}\tilde{x} + \left(g_{11} + \frac{1}{2}\right)\tilde{y} \right)^2 \\ &\quad - \frac{g_{11}}{W_0^2} \left( \left(g_{11} + \frac{1}{2}\right)\tilde{x} + g_{20}\tilde{y} \right) \left( -g_{20}\tilde{x} + \left(g_{11} + \frac{1}{2}\right)\tilde{y} \right) + O(r^3), \end{aligned}$$

where  $r = \sqrt{x^2 + y^2}$ , and the coefficients of the second-order terms become

$$\tilde{x}^2 \rightsquigarrow \frac{1}{W_0^2} \left( -\frac{g_{20}}{2} \left( g_{11} + \frac{1}{2} \right)^2 + g_{11} \left( g_{11} + \frac{1}{2} \right) g_{20} - \frac{g_{02}}{2} g_{20}^2 \right) = 0,$$

$$\tilde{x}\tilde{y} \rightsquigarrow \frac{1}{W_0^2} \left( -g_{20} \left( g_{11} + \frac{1}{2} \right) g_{20} - g_{11} \left( \left( g_{11} + \frac{1}{2} \right)^2 - g_{20}^2 \right) + g_{02} g_{20} \left( g_{11} + \frac{1}{2} \right) \right) = -\frac{1}{2},$$

having used the fact that the characteristic point is degenerate, which gives the condition  $g_{20}g_{02} = g_{11}^2 - \frac{1}{4}$ . Hence, the function  $g$  simplifies to

$$(12) \quad g(\tilde{x}, \tilde{y}) = \frac{1}{2}\tilde{x}\tilde{y} + \frac{\alpha}{2}\tilde{y}^2 + h(\tilde{x}, \tilde{y}),$$

where  $\alpha \in \mathbb{R}$  and  $h \in C^\infty(\mathbb{R}^2)$  with order  $\geq 3$ . Notice that the specific value of  $\alpha$  won't play any role in the integrability of  $\mathcal{H}$ .

We can then assume that  $u(x, y, z) = z - g(x, y)$  where  $g$  has the normal form (12), and that  $N = X$ ,  $T = Y$ . For such a function  $u$ , the norm of the horizontal gradient is

$$W^2 = (\alpha y + h_y)^2 + (y + h_x)^2,$$

so, since in these coordinates  $d\sigma_R = f(x, y)dxdy$ , where  $f$  is a strictly positive and smooth function, we focus on

$$(13) \quad \int_V \frac{1}{W} f(x, y) dx dy,$$

where  $V$  is a neighborhood of  $(0, 0)$ . We may set  $f \equiv 1$ , since its explicit expression plays no role in the integrability. From Lemma 4.2,  $\mathcal{C} = \{Nu = 0\} \cap \Sigma$  is a smooth curve, whose expression in coordinates is  $\{y + h_x = 0\} \cap \Sigma$ . Thus, we introduce the following smooth change of variables around the origin, rectifying  $\mathcal{C}$

$$(14) \quad \varphi: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ t \end{pmatrix} \quad \text{such that} \quad \begin{cases} x = x \\ t = y + h_x(x, y) \end{cases}$$

and the integral (13) becomes

$$(15) \quad \int_{V'} \frac{1}{((\alpha t + (h_y - \alpha h_x))^2 + t^2)^{1/2} |1 + 2h_{xy}|} dt dx,$$

where the integrand is evaluated in  $\varphi^{-1}(x, t)$ . Here  $V' = \varphi(V)$ . We expand in Taylor series the function  $h_y - \alpha h_x$ , with respect to the  $t$ -variable at the point  $(x, 0)$ , obtaining

$$h_y(\varphi^{-1}(x, t)) - \alpha h_x(\varphi^{-1}(x, t)) = \xi(x) + tR(x, t),$$

where  $\xi, R$  are smooth functions of order  $\geq 2$  and  $\geq 1$  respectively, since  $h$  was of order at least 3 and the notion of order in  $t$  is preserved by  $\varphi$ . But now, parametrizing the critical curve by  $x \mapsto \gamma(x) = (x, y(x), g(x, y(x)))$  where  $y(x) + h_x(x, y(x)) = 0$ , we have that  $\varphi^{-1}(x, 0) = (x, y(x))$  and

$$\xi(x) = -\alpha h_x(\varphi^{-1}(x, 0)) + h_y(\varphi^{-1}(x, 0)) = -\alpha h_x(x, y(x)) + h_y(x, y(x)) = Tu(\gamma(x)).$$

Thus, by assumption of mildly degenerate characteristic point,  $\xi$  has a zero of finite order at  $x = 0$ . So, we may write

$$\xi(x) = c_0 x^k (1 + r(x)),$$

where  $k$  is an integer  $\geq 2$ , and  $r$  is a smooth function of order  $\geq 1$ . Thus, we introduce the following weighted polar coordinates in the plane

$$\psi: \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} \rho \\ \theta \end{pmatrix} \quad \text{such that} \quad \begin{cases} c_0 x^k = \rho \cos(\theta) \\ (\alpha^2 + 1)^{1/2} t = \rho \sin(\theta) \end{cases}$$

whose Jacobian is  $\frac{1}{(\alpha^2 + 1)^{1/2} k c_0} \rho^{1/k} |\cos(\theta)|^{1/k-1}$ . In these new coordinates, the function  $W$  becomes

$$W^2 = (\alpha t + \xi(x) + tR(x, t))^2 + t^2 = \rho^2 \left( 1 + \frac{\alpha \sin(2\theta)}{(1 + \alpha^2)^{1/2}} + R_{\text{pol}}(\rho, \theta) \right),$$

where  $R_{\text{pol}}(\rho, \theta)$  is a remainder term vanishing at  $\rho = 0$ . Therefore, the integral (15) is controlled by

$$\int_{V''} \frac{|\rho \cos(\theta)|^{1/k-1}}{\sqrt{1 + \frac{\alpha \sin(2\theta)}{(1 + \alpha^2)^{1/2}} + R_{\text{pol}}(\rho, \theta)}} d\rho d\theta,$$

where  $V'' = \psi(\varphi(V))$ . But now this integral is finite, since

$$1 + \frac{\alpha \sin(2\theta)}{(\alpha^2 + 1)^{1/2}} > 1 - \frac{|\alpha|}{(\alpha^2 + 1)^{1/2}} > 0$$

and thus the denominator, up to restricting  $V''$ , is never-vanishing.  $\square$

*Remark 4.6.* The mild degeneration assumption is sharp for the local integrability of  $W^{-1}$ . Consider the example taken from [DGN12, Prop. 3.4] where  $\Sigma = \{z = g(x, y)\}$ , with

$$g(x, y) = \frac{1}{2}xy + \frac{1}{2}y^2 + \int_0^x e^{-\tau^{-2}} d\tau.$$

Here  $N = X$  and  $T = Y$ , being  $g$  in the normal form (12). Then, the critical curve of  $\mathbf{0}$  is  $\mathcal{C} = \{y + e^{-x^{-2}} = 0\} \cap \Sigma$ , which can be parametrized by

$$\gamma(x) = \left( x, -e^{-x^{-2}}, g\left(x, e^{-x^{-2}}\right) \right).$$

Thus,  $Tu(\gamma(x)) = -e^{-x^{-2}}$ , which has infinite order at  $x = 0$ . Therefore,  $\mathbf{0}$  is not a mildly degenerate characteristic point and one can check that  $W^{-1}$  is not locally integrable.

Notice, however, that in the previous example,  $\mathcal{H}$  is locally integrable. Thus, in general, to prove the integrability of  $\mathcal{H}$ , one should take into account also its numerator, which vanishes at characteristic points.

**Theorem 4.7.** *Let  $\Sigma \subset \mathbb{H}$  be a real-analytic embedded surface, let  $p$  be an isolated characteristic point and let  $u \in C^\omega$  be a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Denote with  $W$  the norm of the horizontal gradient of  $u$ . Then*

$$\frac{1}{W} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ . In particular

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}).$$

*Proof.* If  $C(\Sigma)$  consists of non-degenerate characteristic points, the result follows from Theorem 3.2. If  $p \in C(\Sigma)$  is degenerate, we show that  $p$  is actually mildly degenerate. We can assume that  $\Sigma = \{z - g(x, y) = 0\}$ , where  $g \in C^\omega(\mathbb{R}^2)$  has the normal form (12), and  $\mathbf{0} \in C(\Sigma)$  is degenerate. In this case, in coordinates  $(x, t) = \varphi(x, y)$  defined in (14), the critical curve of  $\mathbf{0}$  is  $\mathcal{C} = \{t = 0\}$  and  $p \in C(\Sigma)$  if and only if

$$\begin{cases} \xi(x) = 0, \\ t = 0. \end{cases}$$

Since  $\mathbf{0}$  is an isolated characteristic point,  $\xi$  is not identically zero. Thus, since  $\xi$  is real-analytic it has finite order at  $x = 0$ .  $\square$

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