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SYMPLECTIC MAPPING CLASS GROUPS OF K3 SURFACES AND SEIBERG–WITTEN INVARIANTS

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Abstract. The purpose of this note is to prove that the symplectic mapping class groups of many K3 surfaces are infinitely generated. Our proof makes no use of any Floer-theoretic machinery but instead follows the approach of Kronheimer and uses invariants derived from the Seiberg–Witten equations.

1 Main Result

Let (X, ω) be a symplectic manifold, $\mathrm{Symp}(X, \omega)$ the symplectomorphism group of (X, ω), and $\mathrm{Diff}(X)$ the diffeomorphism group of X. Define

$$K(X, \omega) = \ker \left[\pi_0 \mathrm{Symp}(X, \omega) \to \pi_0 \mathrm{Diff}(X) \right].$$

In his thesis [Sei08], Seidel found examples where $K(X, \omega)$ is non-trivial: If (X, ω) is a complete intersection that is neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then there exists a symplectomorphism $\tau \colon (X, \omega) \to (X, \omega)$ called the four-dimensional Dehn twist such that τ^2 is smoothly isotopic to the identity but not symplectically so. Seidel also proved [Sei00] that for certain symplectic K3 surfaces (X, ω) the group $K(X, \omega)$ is infinite. Results of Tonkonog [Ton15] show that $K(X, \omega)$ is infinite for most hypersurfaces in Grassmannians. Until recently, however, it was unknown whether $K(X, \omega)$ can be infinitely generated. The question has been answered in the positive by Sheridan and Smith [SS20], who gave examples of algebraic K3 surfaces (X, ω) with $K(X, \omega)$ infinitely generated. The present paper aims to extend their result to a large class of K3 surfaces, including some non-algebraic K3 surfaces.

Let (X, ω) be a Kähler K3 surface, and let $\kappa = [\omega] \in H^{1,1}(X; \mathbb{R})$ be the corresponding Kähler class. We set

$$\Delta_\kappa = \left\{ \delta \in H^2(X; \mathbb{Z}) \mid \langle \kappa, \delta \rangle = 0,\ \langle \delta, \delta \rangle = -2 \right\},$$

where $\langle , \rangle$ denotes the cup product pairing.

Our goal in this note is to prove the following statement:

Theorem 1. *If Δ_κ is infinite, then $K(X, \omega)$ is infinitely generated.*

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The plan of the proof is as follows: We start from the results of [Kro97] and construct a homomorphism

$$q\colon K(X,\omega) \to \prod_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z}_2, \quad \text{where } \overline{\Delta}_\kappa \text{ is defined as } \Delta_\kappa/\sim \text{ with } \delta \sim (-\delta).$$

We then consider the moduli space B of marked (κ-)polarized K3 surfaces. This moduli space is a smooth manifold and has the following properties:

(1) B is a fine moduli space, meaning it carries a universal family of K3 surfaces $\{X_t\}_{t\in B}$ together with a family of fiberwise cohomologous Kähler forms $\{\omega_t\}_{t\in B}$.
(2) $H_1(B;\mathbb{Z}_2) = \bigoplus_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z}_2$[1].

Fix a basepoint $t_0 \in B$. Identify (X,ω) with (X_t,ω_{t_0}). Provided by Moser's theorem, there is a monodromy homomorphism

$$\pi_1(B,t_0) \to \pi_0\mathrm{Symp}(X,\omega).$$

We shall prove that the image of this homomorphism is contained in $K(X,\omega)$ and that the composite homomorphism

$$\pi_1(B,t_0) \to \pi_0\mathrm{Symp}(X,\omega) \xrightarrow{q} \prod_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z}_2$$

surjects onto $\bigoplus_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z}_2 \subset \prod_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z}_2$.

Remark 1. Theorem 1 has a natural generalization, with practically identical proof: There is a homomorphism

$$q : K(X,\omega) \to \prod_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z},$$

such that the subgroup $\bigoplus_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z} \subset \prod_{\delta\in\overline{\Delta}_\kappa} \mathbb{Z}$ is in the image of q. This stronger version of Theorem 1 can be proved by using Seiberg–Witten invariants taking values in $\mathbb{Z}$.

2 Family Seiberg–Witten Invariants

Here, we briefly recall the definition of the Seiberg–Witten invariants in the family setting. The given exposition is extremely brief, meant mainly to fix notations. We refer the reader to [Nic00, Mor96] for a comprehensive introduction to four-dimensional gauge theory. The Seiberg–Witten equations for families of smooth 4-manifolds have been studied in various works including [Kro97, Rub98, Rub01, LL01, Nak03, BK20, Bar19].

[1] By definition, an infinite sum of groups $\bigoplus_{i\in\mathbb{Z}} G_i$ is the subgroup of $\prod_{i\in\mathbb{Z}} G_i$ consisting of sequences $(g_1, g_2, \ldots)$ such that all g_i are zero but a finite number.

Let X be a closed oriented *simply-connected* 4-manifold, B a closed n-manifold, $\mathcal{X} \to B$ a fiber bundle with fiber X. Choose a family of fiberwise metrics $\{g_b\}_{b\in B}$. Pick a spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on the vertical tangent bundle $T_{\mathcal{X}/B}$ of $\mathcal{X}$. By restricting $\mathfrak{s}$ to a fiber X_b at $b \in B$, we get a spin$^{\mathbb{C}}$ structure $\mathfrak{s}_b$ on X_b. Hereafter, for any object on the total space $\mathcal{X}$, the object with subscript b stands for the restriction of the object to the fiber X_b. Conversely: Suppose we are given a spin$^{\mathbb{C}}$ structure $\mathfrak{s}_b$ on X_b. When can we find a spin$^{\mathbb{C}}$ structure on $T_{\mathcal{X}/B}$ whose restriction to X_b is $\mathfrak{s}_b$? The following is a sufficient condition: B is a homotopy S^2. (This is the only case we will be considering in the sequel.) Let us briefly sketch why this is sufficient. Chapter 3 in [Mor96] presents necessary preliminaries on spin$^{\mathbb{C}}$ structures.

Lemma 1. *Let $\mathcal{X} \to B$ be a fiber bundle whose fiber X_b is a closed simply-connected 4-manifold, and whose base B is a homotopy S^2. Suppose we are given a spin$^{\mathbb{C}}$ structure $\mathfrak{s}_b$ on X_b. Then there exists a spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on $T_{\mathcal{X}/B}$ extending the spin$^{\mathbb{C}}$ structure $\mathfrak{s}_b$ on X_b.*

Proof. We begin with a general result on spin$^{\mathbb{C}}$ structures. Let Y be an orientable manifold, which does not need to be four-dimensional nor closed. Let $V \to Y$ be a real oriented rank 4 vector bundle over Y. Endow V with a positive-definite inner product so that the structure group of V is $\boldsymbol{SO}(4)$. Suppose that its Stiefel-Whitney class $w_2(V) \in H^2(Y;\mathbb{Z}_2)$ can be lifted to an integral class $c_1(\mathcal{L})$, for some complex line bundle $\mathcal{L} \to Y$. Then there is a spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ whose determinant line bundle is $\mathcal{L}$; that is, we have

$$c_1(\mathfrak{s}) = c_1(\mathcal{L}).$$

On the other hand, if a bundle carries one spin$^{\mathbb{C}}$ structure, it carries many; they are parameterized by the elements in $H^2(Y;\mathbb{Z})$. In particular, if $H^2(Y;\mathbb{Z})$ has no 2-torsion, then the Chern class $c_1(\mathfrak{s})$ determines uniquely the spin$^{\mathbb{C}}$ structure $\mathfrak{s}$.

Specialize to the case of $Y = \mathcal{X}$. What remains is to show that $w_2(T_{\mathcal{X}/B}) \in H^2(\mathcal{X};\mathbb{Z}_2)$ lifts to a class $a \in H^2(\mathcal{X};\mathbb{Z})$ whose restriction to X_b is equal to $c_1(\mathfrak{s}_b)$. Since X is simply-connected, the group $H^2(Y;\mathbb{Z})$ has no 2-torsion. Thus, we may choose $\mathfrak{s}$ such that $c_1(\mathfrak{s}) = a \in H^2(\mathcal{X};\mathbb{Z})$, and the extension is done.

Using a Mayer-Vietoris argument, we obtain the following exact sequence:

$$0 \to H^2(B;\mathbb{Z}) \to H^2(\mathcal{X};\mathbb{Z}) \to H^2(X_b;\mathbb{Z}) \to 0.$$

Here the first arrow comes from the projection $\mathcal{X} \to B$, whereas the latter arrow is induced by the inclusion $X_b \to \mathcal{X}$. This exact sequence provides a lift of $c_1(\mathfrak{s}_b) \in H^2(X_b;\mathbb{Z})$ to a class $a \in H^2(\mathcal{X};\mathbb{Z})$. Such a lift is not unique; however, letting e denote the generator of $H^2(B;\mathbb{Z})$, one writes all other lifts as translates $a + k\,e$ by k's from $\mathbb{Z}$. It is clear that either a or $a + e$ has to be an integral lift of $w_2(T_{\mathcal{X}/B})$.□

Fix a spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on $T_{\mathcal{X}/B}$. Associated to $\mathfrak{s}$, there are spinor bundles $W^{\pm} \to B$ and determinant line bundle $\mathcal{L}$, which we regard as families of bundles

$$W^{\pm} = \bigcup_{b\in B} W_b^{\pm}, \quad \mathcal{L} = \bigcup_{b\in B} \mathcal{L}_b.$$

Let $\mathcal{A}_b$ be the space of $\boldsymbol{U}(1)$-connections on $\mathcal{L}_b$, $\mathcal{U}_b$ the gauge groups acting on $(W_b^{\pm}, \mathcal{A}_b)$ as follows:

$$\text{for } u_b = e^{-if_b} \in \mathcal{U}_b \text{ and } (\varphi_b, A_b) \in W_b^+ \times \mathcal{A}_b, \quad u_b \cdot (\varphi_b, A_b) = (e^{-if_b}\varphi_b, A_b + 2i d\, f_b).$$

Given $b \in B$, let Π_b be the space of g_b-self-dual forms on X_b, $\Pi_b^* \subset \Pi_b$ be the subset of Π_b given by

$$\langle \eta_b \rangle_{g_b} + \langle 2\pi c_1(\mathcal{L}_b)\rangle_{g_b} \neq 0, \tag{2.1}$$

where $\langle \eta_b \rangle_{g_b}$ is the harmonic part of η_b and $\langle 2\pi c_1(\mathcal{L}_b)\rangle_{g_b}$ is the self-dual part of the harmonic representative of the class $2\pi[c_1(\mathcal{L}_b)] \in H^2(X_b;\mathbb{R})$. For the family of metrics $\{g_b\}_{b\in B}$, let Π^* be the set of all pairs (g_b, η_b) where $\eta_b \in \Pi_b^*$ and g_b varies with $b \in B$. Π^* may be thought of as the fiber bundle over B whose fiber over $b \in B$ is the space Π_b^*.

Given a family of fiberwise self-dual 2-forms $\{\eta_b\}_{b\in B}$ satisfying (2.1), the Seiberg–Witten equations with perturbing terms $\{\eta_b\}_{b\in B}$ are equations for a family $\{(\varphi_b, A_b)\}$. The equations are:

$$\begin{cases} \mathcal{D}_{A_b}\varphi_b = 0, \\ F_{A_b}^+ = \sigma(\varphi_b) + i\,\eta_b, \end{cases} \tag{2.2}$$

where $\mathcal{D}_{A_b} \colon \Gamma(W_b^+) \to \Gamma(W_b^-)$ is the Dirac operator, $\sigma(\varphi)$ is the squaring map, and $F_{A_b}^+$ is the self-dual part of the curvature of A_b. Letting

$$\begin{aligned} \mathscr{M}(g_b, \eta_b) &= \big\{(\varphi_b, A_b) \in \Gamma(W_b^+) \times \mathcal{A}_b \mid (\varphi_b, A_b) \text{ is a solution to (2.2)}\big\} / \sim, \\ (\varphi_b, A_b) &\sim (\varphi_b', A_b') \text{ if } u_b \cdot (\varphi_b', A_b') = (\varphi_b, A_b) \text{ for some } u_b \in \mathcal{U}_b, \end{aligned} \tag{2.3}$$

we define the parametrized moduli space as:

$$\mathfrak{M}^{\mathfrak{s}} = \bigcup_{b\in B,\, \eta_b \in \Pi_b^*} \mathscr{M}(g_b, \eta_b).$$

We let $\pi_{\mathfrak{s}} \colon \mathfrak{M}^{\mathfrak{s}} \to \Pi^*$ be the projection whose fiber over $(g, \eta) \in \Pi^*$ is $\mathscr{M}(g, \eta)$. It is shown in [KM94] that $\pi_{\mathfrak{s}}$ is a smooth and proper Fredholm map. The index of $\pi_{\mathfrak{s}}$ is given by:

$$\operatorname{ind} \pi_{\mathfrak{s}} = \frac{1}{4}(c_1^2(\mathfrak{s}_b) - 3\sigma(X) - 2\chi(X)),$$

where $c_1(\mathfrak{s}_b) = c_1(\mathcal{L}_b)$ is the Chern class of $\mathfrak{s}_b$.

Fix a family of fiberwise self-dual 2-forms $\{\eta_b\}_{b\in B}$ satisfying (2.1), and consider it as a section of Π^*. If $\{\eta_b\}_{b\in B}$ is chosen generic, then the moduli space

$$\mathfrak{M}^{\mathfrak{s}}_{(g_b,\eta_b)} = \bigcup_{b\in B} \pi_{\mathfrak{s}}^{-1}(g_b, \eta_b)$$

is either empty or a compact manifold of dimension

$$d(\mathfrak{s}, B) = \frac{1}{4}(c_1^2(\mathfrak{s}_b) - 3\sigma(X) - 2\chi(X)) + n.$$

Now suppose that $d(\mathfrak{s}, B) = 0$. Then $\mathfrak{M}^{\mathfrak{s}}_{(g_b,\eta_b)}$ is zero-dimensional, and thus consists of finitely-many points. We call

$$\mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}) = \#\left\{\text{points of } \mathfrak{M}^{\mathfrak{s}}_{(g_b,\eta_b)}\right\} \bmod 2 \tag{2.4}$$

the family ($\mathbb{Z}_2$-)Seiberg–Witten invariant for the spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ with respect to the family $\{(g_b, \eta_b)\}_{b\in B}$. The following properties of family invariants are well-known:

(1) There is a "charge conjugation" involution $\mathfrak{s} \to -\mathfrak{s}$ on the set of spin$^{\mathbb{C}}$ structures that changes the sign of $c_1(\mathfrak{s})$. This involution provides us with a canonical isomorphism between

$$\mathfrak{M}^{\mathfrak{s}}_{(g_b,\eta_b)} \quad \text{and} \quad \mathfrak{M}^{-\mathfrak{s}}_{(g_b,-\eta_b)}.$$

Hence,

$$\mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}) = \mathsf{FSW}_{(g_b,-\eta_b)}(-\mathfrak{s}). \tag{2.5}$$

See, e.g., Proposition 2.2.22 in [Nic00]. The corresponding $\mathbb{Z}$-valued Seiberg–Witten invariants are also equal to each other, but only up to sign. See Proposition 2.2.26 in [Nic00] for the precise statement.

(2) If $\mathfrak{s}$, $\mathfrak{s}'$ are two spin$^{\mathbb{C}}$ structures on $T_{\mathcal{X}/B}$ that are isomorphic on X_b for each $b \in B$, then

$$\mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}) = \mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}'),$$

in fact, the corresponding moduli spaces $\mathfrak{M}^{\mathfrak{s}}_{(g_b,\eta_b)}$ and $\mathfrak{M}^{\mathfrak{s}'}_{(g_b,\eta_b)}$ are canonically diffeomorphic. See [Bar19, § 2.2] for details.

(3) Suppose we have two families $\{\eta_b\}_{b\in B}$, $\{\eta'_b\}_{b\in B}$ of g_b-self-dual 2-forms satisfying (2.1). Suppose further that they are homotopic, when considered as sections of Π^*; then

$$\mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}) = \mathsf{FSW}_{(g_b,\eta'_b)}(\mathfrak{s}).$$

This is proved by applying the Sard-Smale theorem. See [LL01, § 2] for details. More generally, the family Seiberg–Witten invariants are unchanged under the homotopies of $\{(g_b, \eta_b)\}_{b\in B}$ that satisfy (2.1).

3 Unwinding Families

Let $\mathcal{X}$ be a fiber bundle over B with fiber X. From now on, we assume that B is the 2-sphere S^2 and X is the K3 surface. Pick a family $\{g_b\}_{b\in B}$ of fiberwise metrics on the fibers of $\mathcal{X}$. Let $\mathfrak{s}_b$ be a spin$^{\mathbb{C}}$ structure on a fiber X_b, and let $\mathfrak{s}$ be a spin$^{\mathbb{C}}$ structure on $T_{\mathcal{X}/B}$ extending $\mathfrak{s}_b$.
The group $H_2(X;\mathbb{Z})$ is a free abelian group of rank 22 which, when endowed with the bilinear form coming from the cup product, becomes a unimodular lattice of signature $(3,19)$. Let us fix (once and for all) an abstract lattice Λ which isometric to $H^2(X;\mathbb{Z})$ and an isometry $\alpha\colon H^2(X_b;\mathbb{Z})\to\Lambda$, where $b\in B$ is some fixed base-point. Since B is simply-connected, the groups $\left\{H^2(X_b;\mathbb{Z})\right\}_{b\in B}$ are all canonically isomorphic to each other, and hence they are isomorphic to Λ through the isometry α. Let $\mathbf{K}\subset\Lambda\otimes\mathbb{R}$ be the (open) positive cone:

$$\mathbf{K}=\left\{\kappa\in\Lambda\otimes\mathbb{R}\,|\,\kappa^2>0\right\},$$

which is homotopy-equivalent to S^2.

Let H_b be the space of g_b-self-dual harmonic forms on X_b, and let $\mathcal{H}\to B$ be the vector bundle whose fiber over $b\in B$ is H_b. Pick a family $\{\eta_b\}_{b\in B}$ of g_b-self-dual forms. Suppose that (g_b,η_b) satisfies

$$\langle\eta_b\rangle_{g_b}\neq 0\quad\text{for each } b\in B,$$

so that the correspondence $b\to\langle\eta_b\rangle_{g_b}$ yields a non-vanishing section of $\mathcal{H}$. Then, associated to such a section, there is a map:

$$B\to\mathbf{K}-\{0\}\,,\quad b\to[\langle\eta_b\rangle_{g_b}],$$

where the brackets $[\,]$ signify the cohomology class of $\langle\eta_b\rangle$. Since both B and $\mathbf{K}$ are homotopy S^2, this map has a degree, called the winding number of the family (g_b,η_b).

Lemma 2. *Suppose that the winding number of (g_b,η_b) vanishes. Then*

$$\mathsf{FSW}_{(g_b,\lambda\eta_b)}(\mathfrak{s})=\mathsf{FSW}_{(g_b,-\lambda\eta_b)}(\mathfrak{s})\tag{3.1}$$

for λ sufficiently large.

Proof. By choosing λ large enough, we can make

$$\lambda^2\,\mathsf{min}_{b\in B}\int_{X_b}\langle\eta_b\rangle^2_{g_b}>4\,\pi^2\mathsf{max}_{b\in B}\int_{X_b}\langle c_1(\mathfrak{s}_b)\rangle^2_{g_b},\tag{3.2}$$

so that both $(g_b,\lambda\eta_b)$ and $(g_b,-\lambda\eta_b)$ satisfies (2.1) for λ large enough, and both sides of (3.1) are well defined. Let us show that there exists a homotopy between $\{(g_b,\lambda\eta_b)\}_{b\in B}$ and $\{(g_b,-\lambda\eta_b)\}_{b\in B}$ that satisfies (2.1).
To begin with, we can assume that $\eta_b=\langle\eta_b\rangle_{g_b}$ for each $b\in B$. This can be assumed because:

$$\text{If } \eta_b \text{ satisfies (2.1), then so does } \eta_b + \operatorname{Image} \mathrm{d}^+.$$

If (3.2) holds, then the range of both maps

$$b \to \lambda[\eta_b], \quad b \to -\lambda[\eta_b] \tag{3.3}$$

lies in the complement of the ball $O \subset \mathbf{K}$,

$$O = \left\{ \kappa \in \mathbf{K} \mid \kappa^2 < 4\,\pi^2 \max_{b \in B} \langle c_1(\mathfrak{s}_b) \rangle_{g_b}^2 \right\}. \tag{3.4}$$

For every map $\chi \colon B \to \mathbf{K}$, there exists a unique section $\tilde{\chi} \colon B \to \mathcal{H}$ such that the diagram

$$\begin{array}{ccc} \mathcal{H} & & \\ {\scriptstyle \tilde{\chi}} \uparrow & \searrow {\scriptstyle [\,]} & \\ B & \xrightarrow{\ \chi\ } & \mathbf{K} \end{array}$$

is commutative. If the range of χ is contained in $\mathbf{K} - O$, then $\tilde{\chi}(b)$ satisfies (2.1) for each $b \in B$. To conclude the proof, it suffices to show that the maps (3.3) are homotopic as maps from B to $\mathbf{K} - O$. Since $\mathbf{K} - O$ is a homotopy S^2, the maps (3.3) are homotopic iff their degrees are equal to each other. This is the case, as the winding number of $(g_b, \pm\lambda\eta_b)$ is equal to that of $(g_b, \pm\eta_b)$, and the latter is zero. □

Combining (3.1) and (2.5), we obtain

$$\mathsf{FSW}_{(g_b, \lambda\eta_b)}(-\mathfrak{s}) = \mathsf{FSW}_{(g_b, \lambda\eta_b)}(\mathfrak{s}) \quad \text{for } \lambda \text{ sufficiently large.} \tag{3.5}$$

4 Seiberg–Witten for Symplectic Manifolds

The following material is well-known; see, e.g., [Nic00, § 3.3], [Mor96, Ch. 7] for details. On a symplectic 4-manifold (X, ω) endowed with a compatible almost-complex structure J and the associated Hermitian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, each spin$^{\mathbb{C}}$ structure has the following form:

$$W^+ = L_\varepsilon \oplus \left(\Lambda^{0,2} \otimes L_\varepsilon \right), \quad W^- = \Lambda^{0,1} \otimes L_\varepsilon, \tag{4.1}$$

where L_ε is a line bundle on X with $c_1(L_\varepsilon) = \varepsilon \in H^2(X;\mathbb{Z})$. K_X^* denotes the anticanonical bundle of X. We parameterize all connections on $\mathcal{L} = K_X^* \otimes L_\varepsilon^2$ as $A = A_0 + 2\,B$, where B is a $\boldsymbol{U}(1)$-connection on L_ε and A_0 is the Chern connection on K_X^*. We also write $\varphi = (\ell, \beta)$ for $\varphi \in W^+$. Following Taubes, we choose the perturbation

$$i\,\eta = F_{A_0}^+ - i\,\rho\,\omega. \tag{4.2}$$

Note that ω is g-self-dual and of type $(1,1)$ with respect to J. The Seiberg–Witten equations are:

$$\begin{cases} \bar{\partial}_B \ell + \bar{\partial}_B^* \beta = 0, \\ F_{A_0}^{0,2} + 2\, F_B^{0,2} = \dfrac{\ell^* \beta}{2} + i\eta^{0,2}, \\ (F_{A_0}^+)^{1,1} + 2(F_B^+)^{1,1} = \dfrac{i}{4}\,(|\ell|^2 - |\beta|^2)\omega + i\eta^{1,1}\,. \end{cases} \tag{4.3}$$

Theorem 2 (Taubes [Tau95]). *Suppose that*

$$\varepsilon \neq 0 \quad \textit{and} \quad \int_X \varepsilon \cup \omega \leqslant 0.$$

Then the equations (4.3) *with the perturbing term* (4.2) *have no solutions for* ρ *positive sufficiently large.*

Proof. See Theorem 3.3.29 in [Nic00]. □

When (X, ω) is Kähler we have the following result: Set

$$\rho_0 = 4\pi \left(\int_X \varepsilon \cup \omega \right) \left(\int_X \omega \cup \omega \right)^{-1}.$$

Theorem 3. *Let* η *be as in* (4.2). *If* $\varepsilon \notin H^{1,1}(X;\mathbb{R})$, *then the equations* (4.3) *have no solutions. If* $\varepsilon \in H^{1,1}(X;\mathbb{R})$ *and* $\rho > \rho_0$, *then solutions to* (4.3) *are irreducible and, modulo gauge transformations, are in one-to-one correspondence with the set of effective divisors in the class* ε.

Proof. See [Mor96, Ch. 7]. □

5 The Homomorphism q

Consider the following fibration, introduced in [Kro97] and studied in [McD01]:

$$\mathrm{Symp}(X,\omega) \to \mathrm{Diff}(X) \xrightarrow{\psi \to (\psi^{-1})^* \omega} S_{[\omega]}, \tag{5.1}$$

where $\mathrm{Symp}(X,\omega)$ is the symplectomorphism group of (X,ω), $\mathrm{Diff}(X)$ the diffeomorphism group of X, and $S_{[\omega]}$ is the space of those symplectic forms which can be joined with ω through a path of cohomologous symplectic forms. We first recall the construction of Kronheimer's homomorphism [Kro97]:

$$Q\colon \pi_1(S_{[\omega]}) \to \mathbb{Z}_2,$$

and then define the homomorphism q afterwards. Kronheimer's original construction restricts to the case of $b_2^+(X) > 3$, and a mild refinement of his argument is given here in order to deal with $b_2^+(X) = 3$.

Let $\{\omega_t\}_{t\in S^1}$ be a loop in $S_{[\omega]}$. $\{\omega_t\}_{t\in S^1}$ can always be equipped with a family of ω_t-compatible almost-complex structures $\{J_t\}_{t\in S^1}$ on X. This follows from the fact that the space of compatible almost-complex structures is non-empty and contractible; see, e.g., [MS17, Prop. 4.1.1]. We let $\{g_t\}_{t\in S^1}$ be the associated family of Hermitian metrics on X.
Let $\mathcal{X}$ be a trivial bundle over the 2-disc D with fiber X. Let $\{g_b\}_{b\in D}$ be a family of fiberwise metrics on $\mathcal{X}$, providing a nullhomotopy of the family $\{g_t\}_{t\in S^1}$ in the space of all Riemannian metrics on X. Pick a class $\varepsilon \in H^2(X;\mathbb{Z})$ that satisfies:

$$\int_X \varepsilon \cup \omega = 0, \quad \int_X \varepsilon \cup \varepsilon = -2. \tag{5.2}$$

These include, for examples, those classes represented by smooth Lagrangian spheres in (X,ω). Let $\mathfrak{s}_\varepsilon$ be the spin$^{\mathbb{C}}$ structure on X given by (4.1). We have $c_1(\mathfrak{s}_\varepsilon) = c_1(X) + 2\,\varepsilon$. Choose a spin$^{\mathbb{C}}$ structure on $T_{\mathcal{X}/D}$ extending $\mathfrak{s}_\varepsilon$. We shall use $\mathfrak{s}_\varepsilon$ to denote this spin$^{\mathbb{C}}$ structure also.
As in (3.4), set:

$$O = \left\{\kappa \in \mathbf{K} \mid \kappa^2 < 4\,\pi^2 \,\mathsf{max}_{t\in S^1} \langle c_1(\mathfrak{s}_\varepsilon)\rangle_{g_t}^2 \right\}.$$

Let A_{0t} denote the Chern connection on K_X^* determined by g_t. As in (4.2), set:

$$\eta_t = -iF^+_{A_{0t}} - \rho\,\omega_t. \tag{5.3}$$

Choosing ρ large enough, we can assume that

$$[\langle \eta_t\rangle_{g_t}] \in \mathbf{K} - O \quad \text{for each } t \in S^1.$$

Note that $\mathbf{K}-O$ has the homotopy type of the sphere $S^{b_2^+(X)-1}$; hence, $\pi_i(\mathbf{K}-O) = 0$ for $i < b_2^+(X) - 1$.

Let $\{\eta_b\}_{b\in D}$ be a family of fiberwise g_b-self-dual forms on $\mathcal{X}$ that agree with η_t on ∂D. We call $\{\eta_b\}_{b\in D}$ an admissible extension of $\{\eta_t\}_{t\in S^1}$ if

$$[\langle \eta_b\rangle_{g_b}] \in \mathbf{K} - O \quad \text{for each } b \in D. \tag{5.4}$$

If $b_2^+(X) > 2$, then $\pi_1(\mathbf{K} - O) = 0$ and an admissible extension always exists. Moreover, if $b_2^+(X) > 3$, an admissible extension is essentially unique: Suppose we are given another admissible extension $\{\eta'_b\}_{b\in D}$ of $\{\eta_t\}_{t\in S^1}$. Using the fact that $\pi_2(\mathbf{K} - O) = 0$ and then arguing as in the proof of Lemma 2, one shows that there exists a homotopy $\{\eta^s_b\}_{b\in D}$ from $\{\eta_b\}_{b\in D}$ to $\{\eta'_b\}_{b\in D}$ that agrees with $\{\eta_t\}_{t\in S^1}$ at each stage and such that $[\langle \eta^s_b\rangle_{g_b}] \in \mathbf{K} - O$.

Fix an admissible extension $\{\eta_b\}_{b\in D}$ of $\{\eta_t\}_{t\in S^1}$. By (5.4),

$$\langle \eta_b\rangle_{g_b} + 2\pi\langle c_1(\mathfrak{s}_\varepsilon)\rangle_{g_b} \neq 0 \quad \text{for each } b \in D. \tag{5.5}$$

Now we consider the Seiberg–Witten equations parametrized by the family $\{(g_b,\eta_b)\}_{b\in D}$. By (5.5), for all $b \in B$, these equations have no reducible solutions. By Theorem 2, for ρ large enough, it is true that

$$\pi^{-1}_{\mathfrak{s}_\varepsilon}(g_t,\eta_t) = \varnothing \quad \text{for all } t \in S^1.$$

Here, following the notation of § 2, we let $\pi_{\mathfrak{s}_\varepsilon}^{-1}(g_t, \eta_t)$ stand for the moduli space of solutions of the Seiberg–Witten equations parameterized by (g_t, η_t).

Now the relative version of Sard-Smale theorem is applied: By perturbing $\{\eta_b\}_{b\in D}$, we can assume that the moduli space $\mathfrak{M}^{\mathfrak{s}_\varepsilon}_{(g_b,\eta_b)}$, lying over D, is a manifold of dimension $d(\mathfrak{s}_\varepsilon, D) = 0$. Now set:

$$Q_\varepsilon(\{\omega_t\}_{t\in S^1}) = \#\left\{\text{points of } \mathfrak{M}^{\mathfrak{s}_\varepsilon}_{(g_b,\eta_b)}\right\} \bmod 2.$$

This gives an element of $\mathbb{Z}_2$ depending only on the homotopy class of $\{\omega_t\}_{t\in S^1}$ but not on our choice of an admissible extension. Thus, Q_ε gives a group homomorphism $\pi_1(S_{[\omega]}) \to \mathbb{Z}_2$.

One can extend the above definition of Q to the case of $b_2^+(X) = 3$. Letting

$$N_\omega = \left\{\kappa \in \mathbf{K} \mid \int_X \kappa \cup \omega = 0\right\},$$

the complement of N_ω in $\mathbf{K}$ has two connected components $\mathbf{K}^\pm$, each being contractible; the component $\mathbf{K}^+$ is specified by the condition $[\omega] \in \mathbf{K}^+$. With η_t as in (5.3), we choose ρ large enough so that $[\langle\eta_t\rangle_{g_t}] \in \mathbf{K} - O$ for each $t \in S^1$. Observe that $\langle -iF^+_{A_{0t}}\rangle_{g_t} = 0$ because K_X^* is topologically trivial. Thus $\langle\eta_t\rangle_{g_t} = -\rho\langle\omega_t\rangle_{g_t}$, and we have the inequality:

$$\int_X \langle\eta_t\rangle_{g_t} \wedge \omega_t < 0, \quad \text{and thus} \quad [\langle\eta_t\rangle_{g_t}] \in \mathbf{K}^- - O \quad \text{for each } t \in S^1.$$

An admissible extension of $\{\eta_t\}_{t\in S^1}$ is now defined as follows: An extension $\{\eta_b\}_{b\in D}$ is admissible if it satisfies

$$[\langle\eta_b\rangle_{g_b}] \in \mathbf{K}^- - O \quad \text{for each } b \in D.$$

Since $\mathbf{K}^- - O$ is contractible, an admissible extension exists and it is unique up to homotopy. The rest of the definition of Q goes just as before.

Note that if ε satisfies (5.2), then so does $(-\varepsilon)$. Define $q_\varepsilon\colon \pi_1(S_{[\omega]}) \to \mathbb{Z}_2$ as:

$$q_\varepsilon = Q_\varepsilon - Q_{-\varepsilon}. \tag{5.6}$$

Lemma 3. *The composite homomorphism*

$$\pi_1\mathrm{Diff}(X) \to \pi_1(S_{[\omega]}) \xrightarrow{q_\varepsilon} \mathbb{Z}_2$$

is a nullhomomorphism.

Proof. Assume that there is a family of symplectomorphisms

$$f_t\colon (X, \omega_t) \to (X, \omega) \quad \text{for } t \in \partial D.$$

Via the clutching construction, the family $\{f_t\}_{t\in\partial D}$ corresponds to the quotient space:

$$\mathcal{Y} = \mathcal{X} \cup X/\sim, \quad \text{where } (t, x) \sim f_t(x) \text{ for each } t \in \partial D \text{ and } x \in X,$$

which is a fiber bundle over the 2-sphere $B = D/\partial D$. Pick an ω-compatible almost-complex structure J on X. Let g be the associated Hermitian metric. Now let $J_t = (f_t^{-1})_* \circ J \circ (f_t)_*$, $g_t = g \circ (f_t)_*$. Then, there is a g-self-dual form η on X such that:

$$(f_t^{-1})^* \eta_t = \eta \quad \text{for each } t \in \partial D.$$

Let $\{g_b\}_{b \in D}$ be a family of Riemannian metrics on X that agree with $\{g_t\}_{t \in \partial D}$ at each $t \in \partial D$. We repeat the above construction of the family $\{\eta_b\}_{b \in D}$, and observe that we get a family $\{(g_b, \eta_b)\}_{b \in B}$ on $\mathfrak{Y}$. By definition, we have

$$q_\varepsilon(\{\omega_t\}_{t \in S^1}) = \mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}_\varepsilon) - \mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}_{-\varepsilon}).$$

The Chern classes $c_1(\mathfrak{s}_{-\varepsilon})$ and $c_1(-\mathfrak{s}_\varepsilon)$ are equal to each other, when restricted to X_b, and hence:

$$q_\varepsilon(\{\omega_t\}_{t \in S^1}) = \mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}_\varepsilon) - \mathsf{FSW}_{(g_b,\eta_b)}(-\mathfrak{s}_\varepsilon).$$

Recall that η_b satisfies (5.5), and so does $\lambda\eta_b$ for all $\lambda > 1$, and hence:

$$\mathsf{FSW}_{(g_b,\eta_b)}(\mathfrak{s}_\varepsilon) = \mathsf{FSW}_{(g_b,\lambda\eta_b)}(\mathfrak{s}_\varepsilon) \quad \text{for } \lambda \text{ positive arbitrary large,}$$

and likewise for $-\mathfrak{s}_\varepsilon$. Since $[\langle \eta_b \rangle] \in \mathbf{K}^-$ for each $b \in B$, it follows that the winding number of $\{(g_b, \eta_b)\}_{b \in B}$ vanishes. The lemma now follows by (3.5). □

Let $\Delta_{[\omega]}$ be the (possibly infinite) set of classes satisfying (5.2), and let $\overline{\Delta}_{[\omega]}$ be defined as: $\overline{\Delta}_{[\omega]} = \Delta_{[\omega]}/\sim$, where $\varepsilon \sim -\varepsilon$. Set: $\mathbb{Z}_2^\infty = \prod_{\varepsilon \in \overline{\Delta}_{[\omega]}} \mathbb{Z}_2$. For $\varepsilon_k \in \overline{\Delta}_{[\omega]}$, let q_{ε_k} be the homomorphism defined by (5.6) above. Extending q_{ε_k} as

$$\pi_1(S_{[\omega]}) \to \mathbb{Z}_2 \xrightarrow{I_{\varepsilon_k}} \mathbb{Z}_2^\infty,$$

where $I_{\varepsilon_k} : \mathbb{Z}_2 \to \mathbb{Z}_2^\infty$ is the inclusion homomorphism, we define $q : \pi_1(S_{[\omega]}) \to \mathbb{Z}_2^\infty$ as the (infinite) sum:

$$q = \oplus_{\varepsilon_k \in \overline{\Delta}_{[\omega]}} q_{\varepsilon_k}.$$

The fibration (5.1) leads to the following long exact sequence:

$$\cdots \to \pi_1 \mathrm{Diff}(X) \to \pi_1(S_{[\omega]}, \omega) \to \pi_0 \mathrm{Symp}(X, \omega) \to \pi_0 \mathrm{Diff}(X) \to \cdots.$$

It follows from Lemma 3 that q gives a homomorphism:

$$q : \pi_1(S_{[\omega]}, \omega)/\pi_1 \mathrm{Diff}(X) \cong K(X, \omega) \to \mathbb{Z}_2^\infty.$$

6 Period Domains for K3 Surfaces

The following material is well-known; see, e.g., [Huy16, LP80, BR75]. A K3 surface is a simply-connected compact complex surface X that has trivial canonical bundle. By a theorem of Siu [Siu83] every K3 surface X admits a Kähler form. Fix an even unimodular lattice $(\Lambda, \langle , \rangle)$ of signature $(3, 19)$. (All such lattices are isometric: see [MH73]). Set: $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ and $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$. Given a K3 surface X, there are isometries $\alpha\colon H^2(X;\mathbb{Z}) \cong \Lambda$; a choice of such an isometry is called a marking of X. The isometry α determines the subspace $H^{2,0}(X) \subset H^2(X;\mathbb{C}) \cong \Lambda_{\mathbb{C}}$. If $\varphi_X \in H^{2,0}(X)$ is a generator, then $\langle \varphi_X, \varphi_X \rangle = 0$ and $\langle \varphi_X, \bar{\varphi}_X \rangle > 0$. The period map associates to a marked K3 surface (X, α) a point in the period domain

$$\Phi = \{\varphi \in \Lambda_{\mathbb{C}} \mid \langle \varphi_X, \varphi_X \rangle = 0,\ \langle \varphi_X, \bar{\varphi}_X \rangle > 0\} / \mathbb{C}^* \subset \mathbb{P}^{21},$$

which is a complex manifold of dimension 20. Every point $\varphi \in \Phi$ determines the Hodge structure on $\Lambda_{\mathbb{C}}$ as follows:

$$H^{2,0} = \mathbb{C}\varphi, \quad H^{0,2} = \mathbb{C}\bar{\varphi}, \quad H^{1,1} = \left(H^{2,0} \oplus H^{0,2}\right)^{\perp}.$$

Define $\overline{M}$ as:

$$\overline{M} = \{(\varphi, \kappa) \in \Phi \times \Lambda_{\mathbb{R}} \mid \langle \varphi, \kappa \rangle = 0,\ \langle \kappa, \kappa \rangle > 0\}.$$

We set $\Delta = \{\delta \in \Lambda \mid \langle \delta, \delta \rangle = -2\}$. Define $M \subset \overline{M}$ as:

$$M = \left\{(\varphi, \kappa) \in \overline{M} \mid \text{for all } \delta \in \Delta \text{ if } \langle \varphi, \delta \rangle = 0 \text{ then } \langle \kappa, \delta \rangle \neq 0\right\}.$$

Letting

$$\mathrm{pr}\colon M \to \Phi, \quad \mathrm{pr}(\varphi, \kappa) = \varphi,$$

we define an equivalence relation on M as follows: $(\varphi, \kappa) \sim (\varphi, \kappa')$ iff κ and κ' are in the same connected component of the fiber $\mathrm{pr}^{-1}(\varphi) \subset M$. We call

$$\widetilde{\Phi} = M/\sim$$

the Burns-Rapoport period domain. In [BR75] Burns and Rapoport prove that $\widetilde{\Phi}$ is a (non-Hausdorff) complex-analytic space. A point $(\varphi, \kappa) \in \widetilde{\Phi}$ gives rise to:

(1) the Hodge structure on $\Lambda_{\mathbb{C}}$ determined by φ,
(2) a choice $V^+(\varphi)$ of one of the two connected components of

$$V(\varphi) = \left\{\kappa \in H^{1,1} \cap \Lambda_{\mathbb{R}} \mid \langle \kappa, \kappa \rangle > 0\right\}, \tag{6.1}$$

(3) a partition of $\Delta(\varphi) = \Delta \cap H^{1,1}$ into $P = \Delta^+(\varphi) \cup \Delta^-(\varphi)$ such that:
   (a) if $\delta_1, \ldots, \delta_k \in \Delta^+(\varphi)$ and $\delta = \sum n_i \delta_i \in \Delta(\varphi)$ with $n_i \geqslant 0$, then $\delta \in \Delta^+(\varphi)$, and
   (b) $V_P^+(\varphi) = \{\kappa \in V^+(\varphi) \mid \langle \kappa, \delta \rangle > 0 \ \text{ for all } \delta \in \Delta^+(\varphi)\}$ is not empty.

The Burns-Rapoport period map associates to a marked K3 surface (X, α) the point of $(\varphi, \kappa) \in \widetilde{\Phi}$ determined by

(1) the Hodge structure of $H^2(X;\mathbb{C})$,
(2) the component $V^+(X)$ of $V(X) = \{\kappa \in H^{1,1}(X;\mathbb{R}) \mid \langle \kappa, \kappa \rangle > 0\}$ containing the cohomology class of any Kähler form on X,
(3) the partition of
$$\Delta(X) = \{\delta \in H^{1,1}(X;\mathbb{R}) \cap H^2(X;\mathbb{Z}) \mid \langle \delta, \delta \rangle = -2\}$$
into $P = \Delta^+(X) \cup \Delta^-(X)$, where
$$\begin{aligned}\Delta^+(X) &= \{\delta \in \Delta(X) \mid \delta \text{ is an effective divisor}\}, \\ \Delta^-(X) &= \{\delta \in \Delta(X) \mid -\delta \in \Delta^+(X)\}. \end{aligned} \tag{6.2}$$

It follows from the Riemann-Roch formula that either δ or $-\delta$ is effective for each $\delta \in \Delta(X)$, hence (6.2) is indeed a partition. Finally, we set:
$$V^+_P(X) = \{\kappa \in V^+(X) \mid \langle \kappa, \delta \rangle > 0 \text{ for all } \delta \in \Delta^+(X)\}.$$
An element $\kappa \in V^+_P(X)$ is called a Kähler polarization on X. If X is given a Kähler form, then the cohomology class of this form gives a polarization. Conversely, every class $\kappa \in V^+_P(X)$ is a cohomology class of some Kähler form on X. We call X polarized if the choice of $\kappa \in V^+_P(X)$ has been specified. A classical result (see, e.g., [Siu81]) is that every point $(\varphi, \kappa) \in M$ is a period of some marked κ-polarized K3 surface. Two smooth marked K3 surfaces with the same Burns-Rapoport periods are isomorphic. In other words, we have:

Theorem 4 (Burns-Rapoport [BR75]). *Let X and X' be two non-singular K3 surfaces. If $\theta\colon H^2(X;\mathbb{Z}) \to H^2(X';\mathbb{Z})$ is an isometry which preserves the Hodge structures, maps $V^+(X)$ to $V^+(X')$ and $\Delta^+(X)$ to $\Delta^+(X')$, then there is a unique isomorphism $\Theta\colon X' \to X$ with $\Theta^* = \theta$.*

More generally, we have:

Theorem 5 (Burns-Rapoport [BR75]). *Let S be a complex-analytic manifold, and let $p\colon \mathcal{X} \to S$ and $p'\colon \mathcal{X}' \to S$ be two families of non-singular K3 surfaces. If*
$$\theta\colon \mathcal{R}^2 p_*(\mathbb{Z}) \to \mathcal{R}^2 p'_*(\mathbb{Z})$$
is an isomorphism of second cohomology lattices which preserves the Hodge structures, maps $V^+(X_s)$ to $V^+(X'_s)$ and $\Delta^+(X_s)$ to $\Delta^+(X'_s)$, then there is a unique family isomorphism $\Theta\colon \mathcal{X}' \to \mathcal{X}$, with $\Theta^* = \theta$, such that the following diagram is commutative:
$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\Theta} & \mathcal{X} \\ & \searrow \quad \swarrow & \\ & S. & \end{array} \tag{6.3}$$

Let us show how this theorem is used to construct a fine moduli space of polarized K3 surfaces.

7 Universal Family of Marked Polarized K3's

Let $p\colon \mathcal{X} \to S$ be a complex-analytic family of K3 surfaces. Regarding Λ as a group, let $\overline{\Lambda}_S$ be a locally-constant sheaf on S taking values in Λ. If $\mathcal{R}^2p(\mathbb{Z})$ is globally-constant, then there are isomorphisms $\alpha\colon \mathcal{R}^2p(\mathbb{Z}) \to \overline{\Lambda}_S$. A choice of an isomorphism $\alpha\colon \mathcal{R}^2p(\mathbb{Z}) \to \overline{\Lambda}_S$ is called a marking of $\mathcal{X}$. A marked family of K3 surfaces $(\mathcal{X}, \alpha)$ carries a holomorphic map $\mathrm{T}_{(\mathcal{X},\alpha)}\colon S \to \Phi$ which associates to each marked fiber X_s the corresponding point of φ. This map is called the period map for the family $\mathcal{X}$. A polarization of $\mathcal{X}$ is a section $\kappa \in \Gamma(S, \overline{\Lambda}_S \otimes \mathbb{R})$ such that $\kappa|_s \in V_P^+(X_s)$ for each $s \in S$. The period map $\mathrm{T}_{(\mathcal{X},\alpha)}$ together with κ gives a map $S \to \Phi \times \Lambda_{\mathbb{R}}$, whose image is contained in M; the composite map

$$S \xrightarrow{(\mathrm{T}_{(\mathcal{X},\alpha)},\kappa)} M \xrightarrow{/\sim} \widetilde{\Phi}.$$

is called the polarized period map for the family $\mathcal{X}$. This map is independent of the choice of κ, because $V_P^+(X_s)$ is connected. We can restate Theorem 5 as follows: Let $(\mathcal{X}, \alpha)$ and $(\mathcal{X}', \alpha')$ be two marked families of K3 surfaces over a complex-analytic manifold S. Suppose that their polarized period maps agree on S. Then there exists a unique family isomorphism $\Theta\colon \mathcal{X}' \to \mathcal{X}$, with $\alpha' \circ \Theta^* = \alpha$, such that diagram (6.3) is commutative.

Fix $\kappa \in \Lambda_{\mathbb{R}}$ with $\kappa^2 > 0$. Letting

$$\Delta_\kappa = \{\delta \in \Delta \mid \langle \kappa, \delta\rangle = 0\},$$

we define two complex manifolds $M_\kappa \subset \overline{M}_\kappa$ as:

$$\overline{M}_\kappa = \{\varphi \in \Phi \mid \langle \varphi, \kappa\rangle = 0\}, \quad M_\kappa = \{\varphi \in \Phi \mid \langle \varphi, \kappa\rangle = 0, \text{ and } \langle \varphi, \delta\rangle \neq 0 \text{ for all } \delta \in \Delta_\kappa\}.$$

Setting $H_\delta = \left\{\varphi \in \overline{M}_\kappa \mid \langle \delta, \varphi\rangle = 0\right\}$, where $\delta \in \Delta_\kappa$, we have $M_\kappa = \overline{M}_\kappa - \cup_{\Delta_\kappa} H_\delta$.

LEMMA 4. ([BR75]) *Let $\kappa_0 \in \Lambda_{\mathbb{R}}$, and assume $\kappa_0^2 > 0$. Let $\varphi_0 \in \overline{M}_{\kappa_0}$. Then there is a neighbourhood U of φ_0 in $\overline{M}_\kappa$ and a neighbourhood K of κ_0 in $\Lambda_{\mathbb{R}}$ such that for all $(\varphi, \kappa) \in U \times K$,*

$$\text{if } \delta \in \Delta \text{ satisfies } \langle \delta, \kappa\rangle = \langle \delta, \varphi\rangle = 0, \text{ then } \langle \delta, \kappa_0\rangle = \langle \delta, \varphi_0\rangle = 0.$$

Proof. See Proposition 2.3 in [BR75] and also see the proof of Lemma 5 below. □

In particular, we have:

LEMMA 5. *Every $\varphi \in \overline{M}_\kappa$ has neighbourhood U such that $H_\delta \cap U = \varnothing$ for all but finitely many $\delta \in \Delta_\kappa$. Hence, in particular, M_κ is an open submanifold of $\overline{M}_\kappa$.*

Proof. We let $x \in \Lambda_{\mathbb{C}}$ be the vector corresponding to the point $\varphi \in \Phi$. Letting $x = x_1 + ix_2$, $x_i \in \Lambda_{\mathbb{R}}$, we obtain three pairwise orthogonal vectors (κ, x_1, x_2) in $\Lambda_{\mathbb{R}}$ such that

$$\kappa^2 > 0, \quad x_1^2 > 0, \quad x_2^2 > 0.$$

Fix some euclidean norm $||\,||$ on $\Lambda_\mathbb{R}$. It is clear that any ball (with respect to the norm $||\,||$) contains only finitely many elements of Δ_κ. Suppose, contrary to our claim, that there is an unbounded sequence $\{\delta_i\}_{k=1}^{\infty}$ such that:

$$||\delta_i|| \to \infty \text{ and } (\delta_i, x_1), (\delta_i, x_2), (\delta_i, \kappa) \to 0 \text{ as } i \to \infty.$$

Assuming, as we may, that $\{\{\delta_i\}/||\delta_i||\}_{i=1}^{\infty} \to \delta_\infty \in \Lambda_\mathbb{R}$ as $i \to \infty$, we obtain four pairwise orthogonal non-zero vectors $(\delta_\infty, \kappa, x_1, x_2)$ such that

$$\delta_\infty^2 = 0 \quad \text{and} \quad \kappa^2 > 0, \quad x_1^2 > 0, \quad x_2^2 > 0.$$

Such a configuration of vectors, however, is not realizable in the space of signature $(3, 19)$. □

For a point $\varphi \in M_\kappa$, let (X, α) be a marked K3 surface whose Burns-Rapoport period is (κ, φ). Let $p\colon (\mathcal{S}, X) \to (S, *)$ be its Kuranishi family. By restricting to smaller neighbourhoods of $*$, we may assume that S is contractible. Then the family $\mathcal{S}$ has a natural marking $\alpha\colon \mathcal{R}^2 p_*(\mathbb{Z}) \to H^2(X;\mathbb{Z})$, uniquely determined by the marking of X. The corresponding period map $\mathrm{T}_{(\mathcal{S},\alpha)}\colon S \to \Phi$ is a local isomorphism at $*$ (the local Torelli theorem). Thus, M_κ admits an open cover $\{U_i\}$ such that: for each U_i, there is a marked family $\mathcal{X}_i \to U_i$ with $\mathrm{T}_{(\mathcal{X}_i,\alpha_i)} = \mathrm{id}$. Each $(\mathcal{X}_i, \alpha_i)$ is polarized by the constant section $\kappa \in \Gamma(U_i, \overline{\Lambda}_{U_i} \otimes \mathbb{R})$. Applying the Burns-Rapoport theorem for families, one can construct a global marked family $\mathcal{X} \to M_\kappa$ by gluing all the $\mathcal{X}_i$'s; namely, the families $\mathcal{X}_i$ and $\mathcal{X}_j$ can be uniquely identified over $U_i \cap U_j$ by a morphism $\Theta_{ij}\colon \mathcal{X}_j \to \mathcal{X}_i$ such that $\Theta_{ij}^* \circ \alpha_j = \alpha_i$ and such that Θ_{ij} fits into the diagram:

$$\begin{array}{ccc} \mathcal{X}_j & \xrightarrow{\Theta_{ij}} & \mathcal{X}_i \\ & \searrow \quad \swarrow & \\ & U_i \cap U_j & \end{array}$$

We call the family $\mathcal{X} \to M_\kappa$ the universal family of marked (κ-)polarized K3's.

8 Proof of Theorem 1

Given $\kappa \in \Lambda_\mathbb{R}$, with $\kappa^2 > 0$, the space $\overline{M}_\kappa$ consists of two connected components $\overline{M}_\kappa^{\pm}$, each being contractible; they are interchanged by the mapping $\varphi \to \bar{\varphi}$. M_κ also consists of two connected components $M_\kappa^{\pm}$, which, however, are not contractible.

Lemma 6. *$H_1(M_\kappa^+;\mathbb{Z}) = \bigoplus_{\delta \in \overline{\Delta}_\kappa} \mathbb{Z}$, and likewise for M_κ^-. $\overline{\Delta}_\kappa$ denotes the quotient space obtained by identifying the elements $\delta \in \Delta_\kappa$ and $(-\delta) \in \Delta_\kappa$.*

Proof. Let $\gamma\colon [0,1] \to M_\kappa^+$ be a loop. Since $\overline{M}_\kappa^+$ is contractible, it follows that γ is nullhomotopic in $\overline{M}_\kappa^+$. Let $\psi\colon D \to \overline{M}_\kappa^+$, where D is a 2-disc, be a nullhomotopy of γ in $\overline{M}_\kappa^+$. Since $\overline{M}_\kappa^+$ is contractible, it follows that such a map ψ is unique up to homotopies that agree with ψ on ∂D. By Lemma 5, there are but finitely many $\delta \in \Delta_\kappa$ such that $\psi(D) \cap H_\delta$ is not empty. Each H_δ is a smooth codimension-2 subvariety of $\overline{M}^+_\kappa$. Hence, we may perturb ψ so as it is transverse to each H_δ. Setting

$$\ell_\delta(\gamma) = \#\left\{\text{points of } \psi^{-1}(H_\delta)\right\} \bmod 2,$$

we associate to γ a sequence $\{\ell_\delta(\gamma)\}_{\delta\in\overline{\Delta}_\kappa}$, which is an element of $\bigoplus_{\delta\in\overline{\Delta}_\kappa}\mathbb{Z}$. It is clear that $\ell_\delta(\gamma)$ depends only on the homology class of γ, so the correspondence

$$\ell\colon \gamma \to \{\ell_\delta(\gamma)\}_{\delta\in\overline{\Delta}_\kappa} \tag{8.1}$$

gives a group homomorphism. It is easy to show that (8.1) is an isomorphism. □

Fix a basepoint $b_0 \in M^+_\kappa$. We now specify "generators" for $\pi_1(M^+_\kappa, b_0)$. For each H_δ we pick a loop γ_δ such that there exists a nullhomotopy of γ_δ in $M^+_\kappa \cup H_\delta$ that intersects H_δ transversally at a single point.

Lemma 7. *$\pi_1(M^+_\kappa, b_0)$ is normally-generated by the set $\{\gamma_\delta\}_{\delta\in\overline{\Delta}_\kappa}$.*

Proof. Throughout the proof, all loops are based at b_0. Let μ be a loop in M^+_κ such that there exists H_{δ_0} and a nullhomotopy of μ in $M^+_\kappa \cup H_{\delta_0}$ that intersects H_{δ_0} transversally at a single point. Such a μ is called a meridian. Since H_{δ_0} is connected, it follows that μ and γ_{δ_0} are conjugate in $\pi_1(M^+_\kappa, b_0)$. Let γ be an arbitrary loop in M^+_κ. Since M^+_κ is contractible, the loop γ bounds a disc. We may assume that this disc is transverse to each H_δ, $\delta \in \Delta_\kappa$. It clear now that γ is a product of a bunch of meridians, each being a conjugate of some γ_δ. □

Fix $\kappa_0 \in \Lambda_\mathbb{R}$ with $\langle\kappa_0, \kappa_0\rangle$ ¿ 0. From now on, we write B (resp. $\overline{B}$) for $M^+_{\kappa_0}$ (resp. $\overline{M}^+_{\kappa_0}$). Let $\mathfrak{X} \to B$ the universal family of polarized K3 surfaces, defined in § 7. Each fiber X_b admits a Kähler form in the class $\kappa_0 \in V^+_P(X_b)$. Since the space of Kähler forms representing a given Kähler class is convex and therefore contractible, we may assume given a family of fiberwise Kähler forms $\{\omega_b\}_{b\in B}$ which varies smoothly with b ([KS60]). Thus, there is a monodromy map

$$\pi_1(B, b_0) \to \pi_0\mathrm{Symp}(X_{b_0}, \omega_{b_0}). \tag{8.2}$$

We shall prove:

(a) $\pi_1(B, b_0) \xrightarrow{(8.2)} \pi_0\mathrm{Symp}(X_{b_0}, \omega_{b_0}) \to \pi_0\mathrm{Diff}(X_{b_0})$ is a nullhomomorphism.
(b) The following diagram is commutative:

$$\begin{array}{ccc} \pi_1(B, b_0) & \longrightarrow & \pi_0\mathrm{Symp}(X_{b_0}, \omega_{b_0}) \\ {\scriptstyle \pi_1/[\pi_1,\pi_1]}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ H_1(B, b_0) & \xrightarrow{\ \ell\ } & \oplus_{\delta\in\overline{\Delta}_{\kappa_0}}\mathbb{Z}_2\,, \end{array}$$

where ℓ is the homomorphism defined in Lemma 6.

Before proving (a) we make a definition: Given $\delta_0 \in \Delta_{\kappa_0}$ and a point $\varphi \in \overline{B}$, with $\langle \varphi, \delta_0 \rangle = 0$, we say that φ is **good** if $\langle \varphi, \delta \rangle \neq 0$ for all $\delta \in \Delta_{\kappa_0} - \{\delta_0\}$. The subset of H_{δ_0} consisting of good points is the complement of a collection of proper analytic subvarieties, and hence it is open and dense.

To prove (a), it suffices by Lemma 7 to show that the restriction of $\mathcal{X}$ to each γ_δ is C^∞-trivial. Fix $\delta_0 \in \Delta_{\kappa_0}$. Considering γ_{δ_0} as a free loop we find a homotopy of γ_{δ_0} into a loop so small that it becomes the boundary of a holomorphic disc D transverse to H_{δ_0}. By perturbing D, we may arrange that it intersects H_{δ_0} at a good point; that is, setting $\varphi_0 = D \cap H_{\delta_0}$, we get:

$$\langle \varphi_0, \delta \rangle \neq 0 \quad \text{for each } \delta \in \Delta_{\kappa_0} - \{\delta_0\}\,.$$

By Lemma 5, D can be chosen small enough so that:

$$D \cap H_{\delta_0} = \{\varphi_0\} \quad \text{and} \quad D \cap H_\delta = \varnothing \text{ for each } \delta \in \Delta_{\kappa_0} - \{\delta_0\}\,. \tag{8.3}$$

Choose a coordinate t on D such that φ_0 is given by $t = 0$. Let $D^* = D - \{0\}$. Let $\mathcal{Y} = \mathcal{X}|_{D^*}$ be the restriction of $\mathcal{X}$ to D^*, and let $p\colon \mathcal{Y} \to D^*$ be the projection. The family $\mathcal{X}$ carries a canonical marking. So does $\mathcal{Y}$, being a subfamily of $\mathcal{X}$; call this marking $\alpha\colon \mathcal{R}^2 p_*(\mathbb{Z}) \to \overline{\Lambda}_{D^*}$. We shall prove that there is a marked family of non-singular K3 surfaces $\mathcal{Y}' \to D$ whose restriction to D^* coincides with $\mathcal{Y}$. Let (Y_0', α') be a marked K3 surface whose Burns-Rapoport period is given by

$$(\varphi_0, \kappa_0 - \hbar\, \delta_0) \quad \text{for } \hbar \text{ positive small enough.}$$

Let S be a sufficiently small neighbourhood of φ_0 in Φ. Let $p'\colon (\mathcal{Y}', Y_0') \to (S, \varphi_0)$ be the Kuranishi family of (Y_0', α'), endowed with a natural marking $\mathcal{R}^2 p'_*(\mathbb{Z}) \to H^2(Y_0'; \mathbb{Z})$. We assume (by further shrinking D toward $t = 0$) that $D \subset S$. Now consider the restriction $\mathcal{Y}'|_D$. We shall use $\mathcal{Y}'$ to denote this family, also.

Lemma 8. *There is a neighbourhood U of $0 \in D$ such that for each $t \in U - \{0\}$, the class κ_0 gives a polarization on Y_t'.*

Proof. It is enough to prove that there is a neighbourhood U of $0 \in D$ such that for each $t \in U - \{0\}$,

$$\langle \kappa_0, \delta \rangle > 0 \quad \text{for all } \delta \in \Delta^+(Y_t'). \tag{8.4}$$

We first show that there is a neighbourhood U of $0 \in D$ and $\hbar^* > 0$ such that for each $t \in U - \{0\}$,

$$\text{if } \delta \in \Delta^+(Y_t') \text{ and } \langle \kappa_0 - \hbar\, \delta_0, \delta \rangle = 0, \text{ then } |\hbar| \geqslant \hbar^*. \tag{8.5}$$

Suppose, contrary to our claim, that there is a sequence

$$(t_k, \delta_k, \hbar_k), t_k \in D - \{0\}\,, \delta_k \in \Delta^+(Y_{t_k}') \quad \text{with } t_k \to 0, \hbar_k \to 0,$$

where $\hbar_k$ is the unique solution to the following equation:

$$\langle \kappa_0 - \hbar_k\, \delta_0, \delta_k \rangle = 0.$$

We have, by (8.3), that

$$\langle \kappa_0, \delta_k \rangle \neq 0 \quad \text{for all } (t_k, \delta_k).$$

Thus, $\hbar_k$, for all (t_k, δ_k), is non-zero. Observe that if $\hbar_k \neq 0$, then our sequence contains infinitely many pairwise distinct values of $\hbar_k$, and hence it contains infinitely many pairwise distinct classes $\delta_k \in \Delta$. The rest of the proof is similar to that of Lemma 5. Fix some euclidean norm $|| \, ||$ on $\Lambda_{\mathbb{R}}$. The set of numbers $||\delta_k||$ is unbounded for otherwise we would have only finitely many δ_k in our sequence; thus $\delta_k/||\delta_k||$ converges to a class δ_∞ such that:

$$\delta_\infty^2 = 0, \quad \langle \delta_\infty, \kappa_0 \rangle = 0, \quad \langle \delta_\infty, \varphi_0 \rangle = 0.$$

But this is impossible. Therefore we may choose a small enough U so that (8.5) holds true.

Now choose $\hbar > 0$ so small that $\hbar < \hbar^*$ and also that the class $\kappa_0 - \hbar\, \delta_0$ is a polarization on Y_0'; the latter is needed to claim that for each Y_t', sufficiently close to Y_0', we have that

$$\langle \kappa_0 - \hbar\, \delta_0, \delta \rangle > 0 \quad \text{for each } \delta \in \Delta^+(Y_t'). \tag{8.6}$$

Make U small enough so that (8.6) holds for each $t \in U$. Then (8.4) holds for each $t \in U - \{0\}$, because $\hbar < \hbar^*$. This completes the proof. □

We make D still smaller so that the neighbourhood U may be chosen to cover the whole of D. Then both $\mathcal{Y}$ and $\mathcal{Y}'$ are polarized by the constant section $\kappa_0 \in \Gamma(D^*, \overline{\Lambda}_{D^*} \otimes \mathbb{R})$; therefore, their polarized period maps agree over D^*. Then there exists a canonical family isomorphism $\Theta\colon \mathcal{Y}' \to \mathcal{Y}$, with $\alpha' \circ \Theta^* = \alpha$, that fits into the diagram:

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\Theta} & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & D^*. & \end{array} \tag{8.7}$$

In other words, the family $\mathcal{Y}$, which is defined over $D - \{0\}$, extends to a family of non-singular surfaces defined for all $t \in D$. Conclusion: the fiber bundle $\mathcal{Y} \to D - \{0\}$ is C^∞-trivial, and (a) follows.

Abusing notation, we write $\mathcal{Y}$ for the extension of $\mathcal{Y} \to D^*$ to the whole disc D. We write Y_0 instead of Y_0' for the central fiber of this extension. Let $\{\omega_t\}_{t \in \partial D}$ be a family of cohomologous Kähler forms, with $[\omega_t] = \kappa_0$, on the fibers Y_t over ∂D. To prove (b), it suffice to show that

$$q_\delta(\{\omega_t\}_{t \in \partial D}) = \begin{cases} 1 \text{ for } \delta = \delta_0, \\ 0 \text{ for all } \delta = \overline{\Delta}_{\kappa_0} - \{\delta_0\}\,. \end{cases}$$

To begin with, we choose an extension of $\{\omega_t\}_{t \in \partial D}$ to a family of non-cohomologous Kähler forms $\{\omega_t\}_{t \in D}$ over the whole of D. Such an extension always exists, and may be defined by using partitions of unity in local charts on D.

We claim that for each $t \in D$,

$$\int_{Y_t} [\omega_t] \cup \kappa_0 > 0. \tag{8.8}$$

To see this, recall that for each $t \in D$, we have,

$$[\omega_t]^2 > 0,\ \kappa_0^2 > 0 \quad \text{and} \quad [\omega_t], \kappa_0 \in H^{1,1}(Y_t; \mathbb{R}).$$

Then for each $t \in D - \{0\}$, and hence, by continuity, for $t = 0$, we have,

$$[\omega_t], \kappa_0 \in V^+(Y_t).$$

Since neither $[\omega_t]$ nor κ_0 is isotropic, it follows that their cup product must be positive.

Observe that

$$\langle -\delta_0, \kappa_0 - \hbar\,\delta_0 \rangle = -2\,\hbar < 0,$$

hence δ_0 lies in $\Delta^+(Y_0)$ and $(-\delta_0)$ does not. It follows then that

$$\Delta^+(Y_0) \cap \Delta_{\kappa_0} = \{\delta_0\}\,.$$

Recall that, by (8.3), we get:

$$\Delta^+(Y_t) \cap \Delta_{\kappa_0} = \varnothing \quad \text{for each } t \in D^*.$$

Let $\{g_t\}_{t\in D}$ be the family of fiberwise Hermitian metrics on $\mathfrak{Y}$ associated to $\{\omega_t\}_{t\in D}$. Pick a spin$^{\mathbb{C}}$ structure $\mathfrak{s}_\delta$ on $T_{\mathfrak{Y}/D}$ which, when restricted to Y_0, satisfies:

$$c_1(\mathfrak{s}_\delta) = c_1(Y_0)(= 0) + 2\,\delta. \tag{8.9}$$

Note that (8.9) specifies $\mathfrak{s}_\delta$ uniquely. As in (4.2), set:

$$\eta_t = -iF^+_{A_{0\,t}} - \rho\,\omega_t. \tag{8.10}$$

By (8.8), there is ρ so large that $\{\eta_t\}_{t\in D}$ becomes an admissible extension of $\{\eta_t\}_{t\in\partial D}$. Let us consider the Seiberg–Witten equations parametrized by the family $\{(g_t, \eta_t)\}_{t\in D}$. To describe their solutions, we use Theorem 3. Let Π^*, $\mathfrak{M}^{\mathfrak{s}_\delta}$, and $\pi_{\mathfrak{s}_\delta} \colon \mathfrak{M}^{\mathfrak{s}_\delta} \to \Pi^*$ be as in § 2. We embed D into Π^* by the map

$$t \to (g_t, \eta_t), \quad \text{where } \eta_t \text{ is given by (8.10)}.$$

If $\delta \neq \pm\delta_0$ and $\delta \in \Delta_{\kappa_0}$, then $\delta \notin H^{1,1}(Y_t; \mathbb{R})$ for all $t \in D$, and we have (by Theorem 3)

$$\bigcup_{t\in D} \pi^{-1}_{\mathfrak{s}_\delta}(g_t, \eta_t) = \varnothing \quad \text{for all } \delta \in \Delta_{\kappa_0} - \{\pm\delta_0\}\,. \tag{8.11}$$

Hence,

$$Q_\delta(\{\omega_t\}_{t\in\partial D}) = 0 \quad \text{for all } \delta \in \Delta_{\kappa_0} - \{\pm\delta_0\}\,, \tag{8.12}$$

and $q_\delta(\{\omega_t\}_{t\in\partial D}) = 0$ for all $\delta \in \overline{\Delta}_\kappa - \{\delta_0\}$.

Now let $\delta = \pm\delta_0$. Making ρ so large that

$$\rho > \rho_0 = 4\pi \left(\int_X \delta_0 \cup [\omega_t]\right)\left(\int_X [\omega_t] \cup [\omega_t]\right)^{-1},$$

we insure that the corresponding Seiberg–Witten equations have no reducible solutions. Since for all $t \in D$, $(-\delta_0) \notin \Delta^+(Y_t)$, it follows that (8.11) still holds for $\delta = -\delta_0$. Hence,

$$Q_{(-\delta_0)}(\{\omega_t\}_{t\in\partial D}) = 0.$$

$\delta_0 \notin \Delta^+(Y_t)$ unless $t = 0$. Let C be a divisor in Y_0 representing δ_0. The divisor C is irreducible. Moreover, C is a smooth rational curve. This follows upon applying the adjunction formula to C. If C' is another effective divisor in the class δ_0, then $C' = C$. This is proved by observing that C is irreducible and has negative self-intersection number. Thus, if we abbreviate $\mathfrak{s}_{\delta_0}$ to $\mathfrak{s}_0$, we have

$$\pi^{-1}_{\mathfrak{s}_0}(g_0, \eta_0) = \mathrm{pt}, \quad \pi^{-1}_{\mathfrak{s}_0}(g_t, \eta_t) = \varnothing \quad \text{for all } t \in D^*.$$

In order to prove that $Q_{\delta_0}(\{\omega_t\}_{t\in\partial D}) = 1$ it suffice to show that $\pi_{\mathfrak{s}_0}$ is transverse to D. Identifying the groups $\left\{H^2(Y_t;\mathbb{C})\right\}_{t\in D}$, we consider the infinitesimal variation of Hodge structures ([Gri68]):

$$\Omega_*\colon T_D \to \mathrm{Hom}\,(H^{1,1}, H^{0,2}), \quad \text{where } H^{p,q} = H^{p,q}(Y_0;\mathbb{C}).$$

It was shown in [Smi21, § 6] that $\pi_{\mathfrak{s}_0}$ is transverse to D, provided

$$\delta_0 \notin \ker \Omega_*(\partial_t), \quad \text{where } \partial_t \text{ is a generator for } T_D. \tag{8.13}$$

This last condition is equivalent to the condition that the period map

$$\mathrm{T}_{(\mathcal{Y},\alpha)}\colon D \to \Phi$$

is transverse to the divisor H_{δ_0}. This is the case by our choice of D.

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