Lyapunov Exponents of Linear Switched Systems

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Abstract

We explicitly compute the maximal Lyapunov exponent for a switched system on $\mathrm{SL}_2(\mathbb{R})$ and the corresponding switching function which realizes the maximal exponent. This computation is reduced to the characterization of optimal trajectories for an optimal control problem on the Lie group.

Keywords. Control Theory, Optimal Control, Switched system, Dynamical Systems, Lyapunov Exponents

Contents

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1 Introduction

Let

$$\dot{y} = A(t)y, \quad y \in \mathbb{R}^n, \tag{1.1}$$

be a linear system of ordinary differential equations; here $t \mapsto A(t),\ t \geq 0$, is a measurable bounded family of $n \times n$-matrices. Let $X(t)$ be the fundamental matrix of system (1.1), the solutions of (1.1) have a form: $y(t) = X(t)y(0)$.

Principal Lyapunov exponent of system (1.1) is defined as follows:

$$\ell(A(\cdot)) = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)\|. \tag{1.2}$$

This quantity does not depend on the choice of norm in the space of matrices and is a natural measure of instability of system (1.1). Indeed, system (1.1) is asymptotically stable with an exponential convergence rate if and only if $\ell(A(\cdot)) < 0$.

Principal Lyapunov exponents play a key role in the general theory of Dynamical Systems, see [11], [2].

If $A(t) \equiv A$ is a constant matrix, then $\ell(A)$ is maximum of the real parts of the eigenvalues of A. Moreover, for any $A(\cdot)$ and any $c \in \mathbb{R}$, we have $\ell(A(\cdot) + cI) = \ell(A(\cdot)) + c$.

In what follows, we use standard notations: $\mathfrak{gl}_n(\mathbb{R})$ is Lie algebra of all real $n \times n$-matrices and $\mathrm{GL}_n(\mathbb{R})$ is Lie group of all nondegenerate real $n \times n$-matrices. Similarly $\mathfrak{sl}_n(\mathbb{R})$ is Lie algebra of all real $n \times n$-matrices with zero trace and $\mathrm{SL}_n(\mathbb{R})$ is Lie group of all real $n \times n$-matrices whose determinant equals 1.

Let $S \subset \mathfrak{sl}_n(\mathbb{R})$ be a compact subset. A switched system induced by S is the set of systems of ordinary differential equations

$$\dot{y} = A(t)y, \quad A(t) \in S,\ 0 \leq t < \infty,$$

where $t \mapsto A(t),\ t \geq 0$, is any measurable map with values in S.

Let $\mathcal{A}_S \subset L_\infty([0,\infty); S)$ be the set of all such maps. Principal Lyapunov exponent of the switched system is defined as follows:

$$\ell(S) = \sup_{A(\cdot)\in\mathcal{A}_S} \ell(A(\cdot)).$$

Clearly, $\ell(S) \geq \sup\limits_{A\in S} \ell(A)$.

We say that the switched system is exponentially stable if $\ell(S) < 0$. Stability of switched dynamical systems is really important for applications and there is a big literature devoted to this subject, see for instance [1, 3, 5, 8, 7, 9, 10] and references there.

A more natural mathematical problem is to compute or at least to estimate principal Lyapunov exponent of any switched system; many results about stability can be naturally generalized in this direction. This paper is devoted to the detailed analysis of the most simple but fundamental case of two matrices generating Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. We give a simple explicit formula of the maximal Lyapunov exponent as a function of the matrices and explicitly describe switching functions which realize maximal Lyapunov exponents.

This work is mainly motivated by paper [1]. The paper [1] is about stability but if you analyse proofs of the main results you will find the characterization of Lie subalgebras $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ such that

$$\ell(S) = \sup_{A\in S} \ell(A), \quad \forall\, S \subset \mathfrak{g}. \tag{1.3}$$

Namely, a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ has property (1.3) if and only if $\mathfrak{g}$ does not contain a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

In the current paper, we explicitly compute $\ell(\{A, B\})$[1] for any pair of matrices A, B which generates Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Such a formula automatically provides us with an explicit expression of $\ell(A, B)$ for matrices A, B of any size if these matrices generate a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

Indeed, let $\Psi : \mathfrak{sl}_2(\mathbb{R}) \to Lie(A, B)$ be such isomorphism; then Ψ is a representation of $\mathfrak{sl}_2(\mathbb{R})$. The representation Ψ is a direct sum of irreducible representations, $\Psi = \bigoplus_{i=1}^{k} \Psi_i$, where $\Psi_i : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_{n_i}(\mathbb{R})$, $i = 1, \dots, k$, $n_1 \geq \cdots \geq n_k \geq 2$. Then

$$\ell(A, B) = (n_1 - 1)\ell(\Psi^{-1}(A), \Psi^{-1}(B)).$$

Indeed, for any $n \geq 2$ there is a unique up-to an isomorphism irreducible n-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$. The representation of the group $\mathrm{SL}_2(\mathbb{R})$ generated by this representation of $\mathfrak{sl}_2(\mathbb{R})$ is the $(n-1)$th symmetric power of the basic two-dimensional representation. If $X \in \mathrm{SL}_2(\mathbb{R})$ has eigenvalues λ, λ^{-1}, then the $(n-1)$th symmetric power of X has the eigenvalues $\lambda^{n-1}, \lambda^{n-3}, \dots \lambda^{1-n}$. We hope that our computation will serve as a building block for eventual estimates of the Lyapunov exponents in much more general cases.

Our final result can be stated as follows. To simplify the notation, set $a = \mathrm{tr}(A^2), b = \mathrm{tr}(B^2), c = \mathrm{tr}(AB)$.

Theorem 1.1 (Main Theorem)**.** *Suppose that $A, B \in \mathfrak{sl}_2(\mathbb{R})$, with $a \geq b$.*

1. *If $a \geq 0$ and $c > a$ or $a < 0$ and $c \geq \sqrt{ab}$, then*

$$\ell(A, B) = \frac{1}{2}\sqrt{\frac{c^2 - ab}{c - \frac{a+b}{2}}};$$

2. *If $a \geq 0$ and $c \leq a$, then*

$$\ell(A, B) = \sqrt{\frac{a}{2}};$$

3. *If $a < 0$ and $c \leq -\sqrt{ab}$, then*

$$\ell(A, B) = \frac{2}{\pi}\frac{1}{\sqrt{\frac{-2}{a}} + 3\sqrt{\frac{-2}{b}}}\operatorname{arcosh}\left(\frac{-c}{\sqrt{ab}}\right).$$

We point out that for every $A, B \in \mathfrak{sl}_2(\mathbb{R})$ such that $a, b \leq 0$, then $c^2 \geq ab$ (see Lemma 6.1).

As we will show in the proof of this result, we also find the trajectories realizing these exponents (see also Table 1). More specifically, in point 1. the value of the Lyapunov exponent is not attained exactly, but it is the supremum among a family of periodic piecewise constant right hand side switching faster and faster. We compute exactly this trajectories and its limit as the switching time goes to zero. In point 2. the Lyapunov exponent is exactly attained by the trajectory with a constant right hand side. In point 3. the Lyapunov exponent is attained by a fixed periodic piecewise constant right hand side. The whole paper is devoted to prove our Main Theorem and it is structured as follows: in Section 2 we

[1]In what follows we use shortened notation $\ell(A, B) \doteq \ell(\{A, B\})$

show how computing the Principal Lyapunov exponent $\ell(A,B)$ can be reduced to an optimal control problem and we apply Pontryagin Maximum Principle (PMP) to this problem. In Section 3 we deduce from PMP some general properties of Pontryagin extremals. In particular, we divide them into two categories: *bang-bang* extremals and singular extremals. In Section 4 and 5 we study *bang-bang* and singular extremals respectively. In Section 6 we show which extremal is the optimal one, depending on $\mathrm{tr}(A^2), \mathrm{tr}(B^2)$ and $\mathrm{tr}(AB)$.

2 Reduction to OC Problem and application of PMP

As explained in the Introduction, we will study the switched system induced by two matrices $A, B \in \mathfrak{sl}_2(\mathbb{R})$:

$$\begin{cases} \dot{X}(t) = X(t)\big(u(t)A + (1-u(t))B\big), \\ X(0) = \mathrm{Id}, \end{cases} \tag{2.1}$$

where $X : [0,T] \to \mathrm{SL}_2(\mathbb{R})$ and $u \in \mathcal{U} := \{v : [0,T] \to \{0,1\} \mid v \text{ measurable}\}$. We suppose moreover that $A, B, [A,B]$ are linearly independent and, without loss of generality, $\mathrm{tr}(A^2) \geq \mathrm{tr}(B^2)$.
We remark that taking $A, B, [A,B]$ linearly independent is not restrictive, because otherwise $\ell(A,B)$ can be computed as in [1].
Notice that in (2.1) we write $\dot{X} = XA(t)$, whereas in the Introduction, the expression was $\dot{X} = A(t)X$. Using the Introduction's notation, system (2.1) should appear as $\dot{X}^T = X^T A(t)^T$, where the superscript "T" represents the transpose. For simplification, we will omit this superscript. We prefer to solve the system with A and X in reversed order because system (2.1) is left-invariant.

Now, our aim is to find the principal Lyapunov exponent $\ell(A,B)$. Fixed $u \in \mathcal{U}$, we will denote by $X(\cdot; u)$ the solution of (2.1) with control equal to u.
A straightforward way to relax our problem (2.1) is to let $u : [0,T] \to [0,1]$ be a measurable function rather than restricting to $u(t) \in \{0,1\}$. This choice is quite natural, since the trajectories with $u(t) \in [0,1]$ can be uniformly approximated by trajectories with controls $u(t) \in \{0,1\}$ (see, for instance, Theorem 8.2 in [12]).
This relaxation will allow us to apply classical results from Control Theory, such as Filippov theorem, which requires the convexity of the set of admissible velocities.
Given $X \in \mathrm{SL}_2(\mathbb{R})$, we denote by $\lambda(X)$ the eigenvalue of X bigger than 1 if both eigenvalues are real. If X has complex eigenvalues, we denote by $\lambda(X)$ the real part of these eigenvalues.
Another crucial observation is that if the norm in (1.2) is the norm induced by the Euclidean metric on $\mathbb{R}^2$, then its value coincides with the absolute value of $\lambda(X(T))$. Hence, if for every $T > 0$ we can compute

$$c(T) := \sup_{u \in \mathcal{U}} \lambda(X(T;u)),$$

then, we obtain that

$$\ell(A,B) = \limsup_{T \to +\infty} \frac{1}{T} \ln c(T). \tag{2.2}$$

Strictly speaking, the previous equality should be a "less than or equal to" a priori, since the optimal strategy to maximize λ up to time T may depend on T itself. However, a posterori, we found that, once A and B are fixed, for any $T_1 < T_2$ the optimal control for the problem with final time T_1 is the restriction to $[0, T_1]$ of the optimal control for the problem with final time T_2. For this reason, we obtain the equality in (2.2).

In terms of Control Theory, the optimal strategy will be one of the following: 1. Constant singular control (case 1 of the Main Theorem). 2. Constant control equal to 1 (case 2). 3. Bang-bang periodic control (case 3). These concepts will be defined later.

Now, instead of finding $\sup_u \lambda(X(T;u))$, one can find $\sup_u f\big(\lambda(X(T;u))\big)$, for a suitable f monotone in λ. A convenient choice of f is the trace $\mathrm{tr}(X(T;u))$: besides being a monotone function of λ, it is a linear function of X and it is invariant after a change of basis.

Summing up all these observations, we can reduce the original problem of computing $\ell(A,B)$ into the following Optimal Control problem:

Problem 1. *Let $A, B \in \mathfrak{sl}_2(\mathbb{R})$ such that $A, B, [A,B]$ are linearly independent and fix $T \geq 0$. Let $\mathcal{U} = \{u : [0,T] \to [0,1] \mid u \text{ measurable}\}$ and for each $u \in \mathcal{U}$ denote by $X(\cdot;u)$ the solution of the following Cauchy problem on $\mathrm{SL}_2(\mathbb{R})$:*

$$\begin{cases} \dot{X}(t) = X(t)(u(t)A + (1-u(t))B), \\ X(0) = \mathrm{Id} \in \mathrm{SL}_2(\mathbb{R}), \end{cases} \qquad t \in [0,T]. \tag{2.3}$$

We want to find $\tilde{u} \in \mathcal{U}$ which maximizes $\mathrm{tr}X(T)$:

$$\mathrm{tr}X(T;\tilde{u}) = \max_{u \in \mathcal{U}} \mathrm{tr}X(T;u). \tag{2.4}$$

If we are able to solve this problem for each fixed T, then from the estimate of $\mathrm{tr}X(T;u)$ we can deduce an estimate for $\|X(T;u)\|$ which is uniform with respect to $u \in \mathcal{U}$ and depends only on T. Then, the desired value of the principal Lyapunov exponent follows.

Existence of maximum in (2.4) follows by Filippov Theorem (see, for instance, Theorem 10.1 in [12]). So, we can apply Pontryagin Maximum Principle (PMP) to our problem. For a general reference, see Theorem 12.3 in [12]. We recall here the precise statement of PMP suited to our setting.

2.1 Pontryagin Maximum Principle

Let M be a smooth manifold of dimension n, $U \subset \mathbb{R}^m$, and $(f_u)_{u \in U}$, a family of smooth vector field on M. Suppose furthermore that $M \times \overline{U} \ni (q,u) \mapsto f_u(q)$ and $M \times \overline{U} \ni (q,u) \mapsto \frac{\partial f_u}{\partial q}(q)$ are continuous. Fix $T > 0$ and define $\mathcal{U} = \{u : [0,T] \to U, u \in L^\infty_{loc}([0,T], U)\}$ the space of admissible controls. Fix $q_0 \in M$ and for all $u \in \mathcal{U}$ denote by $q_u : [0,T] \to M$ the solution to the Cauchy Problem

$$\begin{cases} \dot{q}(t) = f(q(t), u(t)), \\ q(0) = q_0. \end{cases} \tag{2.5}$$

Existence and uniqueness of solution to (2.5) are guaranteed by Carathéodory Theorem, see, for instance, [4].

Let $a : M \to \mathbb{R}$ be a smooth function. We define the cost functional

$$J(u) = a(q(T;u)), \quad u \in \mathcal{U},$$

where $q(\cdot\,;u)$ denotes the solution to (2.5) corresponding to the choice of the control $u \in \mathcal{U}$. In the following, given any $h \in T^*M$, we will denote by $\vec{h}$ the hamiltonian vector field of h, that is $dh = \sigma(\cdot, \vec{h})$, where σ is the standard symplectic form of T^*M.

Theorem 2.1 (Pontryagin Maximum Principle)**.** *Let $\tilde{u} \in \mathcal{U}$ be a local maximum point for J. Define $h_u(\lambda) = \langle \lambda, f(q,u) \rangle$, for $q \in M$, $u \in U$ and $\lambda \in T_q^*M$. Then, there exists a Lipschitz curve $\lambda : [0,T] \to T^*M$ such that*

$$\lambda(t) \neq 0, \tag{2.6}$$

$$\dot{\lambda}(t) = \vec{h}_{\tilde{u}(t)}(\lambda(t)), \tag{2.7}$$

$$h_{\tilde{u}(t)}(\lambda(t)) = \max_{u \in U} h_u(\lambda_t), \tag{2.8}$$

for almost all $t \in [0,T]$, and moreover

$$\lambda_T = \mathrm{d}_{q_{\tilde{u}}(T)} a. \tag{2.9}$$

We will call (2.7) the Hamiltonian system associated to (2.5), (2.8) is the maximal condition and (2.9) is the transversality condition. We will call *extremal* any curve λ satisfying all the necessary condition given by PMP.

We recall now some results about left-invariant Hamiltonian systems on Lie groups. For more detailed reference see Chapter 18 in [12] and Section 7.6 in [6].

Let $M \subset GL(N)$ be a Lie Group and $\mathcal{M}$ its Lie algebra. Given $q \in M$, define $L_q(p) = qp$, $p \in M$, the multiplication on the left by q.

Take a Hamiltonian function $h \in C^\infty(T^*M)$ and define

$$\mathcal{H}(\xi, q) = h(L_{q^{-1}}^* \xi, q), \quad \xi \in T_q^*M, q \in M.$$

We will call $\mathcal{H}$ the trivialized Hamiltonian function of h. We say that h is left-invariant if $\mathcal{H}$ does not depend on q. In this case, the function $\mathcal{H}$ is smooth on $\mathcal{M}^*$.

Recall the definition of the adjoint representation: given $v \in \mathcal{M}$, $\mathrm{ad}v(w) = [v,w]$, where $w \in \mathcal{M}$. The Killing form K of $\mathcal{M}$ is defined by $K(v_1, v_2) \coloneqq \mathrm{tr}(\mathrm{ad}v_1 \circ \mathrm{ad}v_2)$, for $v_1, v_2 \in \mathcal{M}$.

Proposition 2.2. *Let M be a Lie group and take $h \in C^\infty(T^*M)$. Let $\mathcal{M}$ be the Lie algebra of M and K the Killing form of $\mathcal{M}$. Suppose that K is non-degenerate.*
If h is left-invariant, then its associated Hamiltonian system can be rewritten as

$$\begin{cases} \dot{q} = L_{q*}\mathrm{d}\mathcal{H}, \\ \dot{\xi} = (\mathrm{ad}\ \mathrm{d}\mathcal{H})^*\xi, \end{cases} \qquad \text{or} \qquad \begin{cases} \dot{q} = L_{q*}\mathrm{d}\mathcal{H}, \\ \dot{\eta} = [\eta, \mathrm{d}\mathcal{H}], \end{cases} \tag{2.10}$$

where $q : [0,T] \to M$, $\xi : [0,T] \to \mathcal{M}^*$, $\eta : [0,T] \to \mathcal{M}$ and ξ and η are dual with respect to K:

$$K(\eta, v) = \langle \xi, v \rangle \quad \forall v \in \mathcal{M}. \tag{2.11}$$

Now, we go back to the discussion about Problem 1.

Corollary 2.3 (PMP applied to Problem 1)**.** *Let $\tilde{u}$ be an optimal solution to Problem 1. Define*

$$H(\eta, u) = \mathrm{tr}\big(\eta(uA + (1-u)B)\big) \quad \eta \in \mathfrak{sl}_2(\mathbb{R}), u \in [0,1]. \tag{2.12}$$

Then there is $\eta : [0,T] \to \mathfrak{sl}_2(\mathbb{R})$ such that:

1. *there is some* $\eta_0 \in \mathfrak{sl}_2(\mathbb{R})$ *such that* η *satisfies the equation*

$$\begin{cases} \dot{\eta}(t) = \big[\eta(t),\ \tilde{u}(t)A + \big(1-\tilde{u}(t)\big)B\big], \\ \eta(0) = \eta_0, \end{cases} \tag{2.13}$$

   for a.e. $t \in [0,T]$.

2. *the following two conditions hold*

$$H(\eta(t), \tilde{u}(t)) = \max_{u\in[0,1]} H(\eta(t), u), \quad \textit{for a.e. } t \in [0,T]. \tag{2.14}$$

$$\eta(T) = X_{\tilde{u}}(T) - \frac{\mathrm{tr}(X_{\tilde{u}}(T))}{2} I. \tag{2.15}$$

We will call (2.12) the *adjoint equation* and its solution η *adjoint trajectory.*

Proof. We apply PMP as formulated in Theorem 2.1 to Problem 1: (2.12) is the trivialized Hamiltonian function corresponding to the one in the statement of PMP, identifying tangent and cotangent space of $\mathrm{SL}_2(\mathbb{R})$ through Killing form of $\mathfrak{sl}_2(\mathbb{R})$, which is non-degenerate since $\mathfrak{sl}_2(\mathbb{R})$ is simple. Recall that Killing form K on $\mathfrak{sl}_2(\mathbb{R})$ can be computed as

$$K(M_1, M_2) = 4\mathrm{tr}(M_1 M_2), \quad M_1, M_2 \in \mathfrak{sl}_2(\mathbb{R}).$$

The system (2.13) is simply the equation in the fibers of (2.7) (see (2.10)).
Condition (2.14) is a straightforward consequence of (2.8) and (2.12).
It remains to prove (2.15). We have to apply (2.9). Let $\mathrm{tr} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ be the trace operator on $\mathrm{SL}_2(\mathbb{R})$ and set $X = X_{\tilde{u}}(T)$. In our setting (2.9) reads

$$\mathrm{tr}(\eta(T)U) = \langle \mathrm{d}_X \mathrm{tr}, XU \rangle, \quad \forall U \in \mathfrak{sl}_2(\mathbb{R}). \tag{2.16}$$

On the other hand, we have

$$\langle \mathrm{d}_X \mathrm{tr}, XU \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{tr}(Xe^{tU})\big)_{|t=0} = \mathrm{tr}(XU). \tag{2.17}$$

Moreover $\mathrm{tr}(XU) = \mathrm{tr}((X - \frac{\mathrm{tr}(X)}{2}I)U)$ and $X - \frac{\mathrm{tr}(X)}{2}I \in \mathfrak{sl}_2(\mathbb{R})$. So, we obtained $K(\eta(T), U) = K((X - \frac{\mathrm{tr}(X)}{2}I), U)$ for every $U \in \mathfrak{sl}_2(\mathbb{R})$. Since the Killing form of $\mathfrak{sl}_2(\mathbb{R})$ is a non-degenerate bilinear form, (2.15) follows. □

In the remaining part of the paper, we are going to find *all* the possible extremal trajectories that satisfy Corollary 2.3. We will see that there are four possibilities: constant control $u = 1$, two different kinds of periodic piecewise constant controls and singular controls (which will be defined later, Definition 3.5). Depending on the values of $\mathrm{tr}(A^2), \mathrm{tr}(B^2), \mathrm{tr}(AB)$, some of them will be extremals and some of them will not.

	$a > 0$ and $c \geq a$	$a \geq 0$ and $b \leq c < a$	$a \geq 0$ and $c < b$	$a \leq 0$ and $c \geq a$	$a \leq 0$ and $c < b$
Regular constant control $u = 1$	Yes	Yes	Yes	No	No
Periodic control of 1^{st} kind	No	No	No	Yes	Yes
Periodic control of 2^{nd} kind	Yes	No	Yes	Yes	Yes
Singular constant controls	Yes	No	Yes	Yes	Yes

Table 1: Classification of the extremals, depending on the values of $a = \mathrm{tr}(A^2), b = \mathrm{tr}(B^2), c = \mathrm{tr}(AB)$. In this table, for all the possible instances of a, b, c we wrote which are the possible kind of extremals among all the families that we found. The green cells are the optimal one.

3 General properties of extremals

In this Section we discuss some general properties of solutions of (2.13). In particular, we are going to show that this Equation admits two first integrals. So, we can reduce the number of the degrees of freedom from 3 to 1. So, we will determine geometrically the trajectories of the solutions, without solving explicitly the equations.
In the following, if it is not important to specify the control we will denote by X the solution of (2.3) instead of $X(\cdot; u)$.

Proposition 3.1. *Let η be a solution of* (2.13). *Then*

$$\eta(t) = X(t)^{-1}\eta_0 X(t). \tag{3.1}$$

This is an immediate consequence of

$$\frac{d}{dt}[X(t)^{-1}] = -X(t)^{-1}\dot{X}(t)X(t)^{-1}.$$

In particular, we obtain the following Corollary.

Corollary 3.2. *Let η be a solution of* (2.13). *Then $\det \eta(t) = \eta_0 = const.$ for every $t \geq 0$.*

For $M \in \mathfrak{sl}_2(\mathbb{R})$, we have the formula $2\det(M) = -\mathrm{tr}(M^2)$. In what follows, it will be more convenient to use the expression $\mathrm{tr}(M^2)$ instead of $\det M$.

Proposition 3.3. *Let $\tilde{u}$ be an optimal solution of (OCP) and let (X, η) be an extremal corresponding to $\tilde{u}$. Then, if $T > 0$ is large enough, it holds*

$$\mathrm{tr}(\eta(t)^2) > 0 \quad \forall t \in [0, T]. \tag{3.2}$$

In particular, $\eta(t)$ has real eigenvalues for any $t \in [0, T]$.

Proof. From transversality condition (2.15), it follows that

$$\mathrm{tr}(\eta(T)^2) = \frac{1}{2}\big(\mathrm{tr}X(T)\big)^2 - 2.$$

So, since we want to maximize $\mathrm{tr}(X(T))$, for large T we will have $\mathrm{tr}(X(T))^2 > 4$. Moreover, since $\mathrm{tr}(\eta^2)$ is constant, we can deduce that $\mathrm{tr}(\eta(t)^2) > 0$ for all $t \in [0, T]$. □

3.1 Invariant Hyperboloid

We now want to give a geometric interpretation of Corollary 3.2. Equation (2.13) is invariant after a dilation, in the sense that for any $c \in \mathbb{R}$, if $\eta(\cdot)$ is a solution with initial value $\eta(0) = \eta_0$, then $\tilde{\eta}(t) = c\eta(t)$ is a solution with initial value $\tilde{\eta}(0) = c\eta_0$.
So, it is not restrictive to suppose that $\mathrm{tr}(\eta(t)^2) = \mathrm{tr}(\eta_0^2) = 1$. Hence, the trajectory η is contained in the set

$$H := \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \mathrm{tr}(M^2) = 1\},$$

which is a (connected) hyperboloid in the three-dimensional vector space $\mathfrak{sl}_2(\mathbb{R})$. To see this, just take

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix}.$$

Then

$$\mathrm{tr}(M^2) = 2(m_1^2 + m_2 m_3) = 2\left(m_1^2 + \left(\frac{m_2 + m_3}{2}\right)^2 - \left(\frac{m_2 - m_3}{2}\right)^2\right).$$

Since $A, B, [A, B]$ are linearly independent, we can write

$$M = xA + yB + z[A, B],$$

and the equation defining H reads

$$x^2\mathrm{tr}(A^2) + y^2\mathrm{tr}(B^2) + z^2\mathrm{tr}([A, B]^2) + 2xy\mathrm{tr}(AB) = 1.$$

We recall an identity that we will use many times in this paper.

Lemma 3.4. *Let $A, B \in \mathfrak{sl}_2(\mathbb{R})$. Then*

$$\mathrm{tr}([A, B]^2) = 2\mathrm{tr}(AB)^2 - 2\mathrm{tr}(A^2)\mathrm{tr}(B^2). \tag{3.3}$$

The proof is an elementary exercise of Linear Algebra and we omit it.

3.2 Switching function, *bang-bang* and singular extremals

Now we want to analyze the condition on the Hamiltonian (2.14). It can be rewritten more explicitly as

$$\mathrm{tr}(\eta(t)B) + \tilde{u}(t)\mathrm{tr}(\eta(t)(A - B)) = \max_{u \in [0,1]} \left[\mathrm{tr}(\eta(t)B) + u\mathrm{tr}(\eta(t)(A - B))\right] \tag{3.4}$$

$$= \mathrm{tr}(\eta(t)B) + \max_{u \in [0,1]} u\mathrm{tr}(\eta(t)(A - B)), \tag{3.5}$$

for almost every $t \in [0, T]$. Hence, we are lead to define $\varphi : [0, T] \to \mathbb{R}$,

$$\varphi(t) = \mathrm{tr}(\eta(t)(A - B)). \tag{3.6}$$

We will call φ *switching function*. Notice that φ is at least Lipschitz. If φ has only isolated zeros, then a control u, in order to satisfy Pontryagin Maximum Principle, must be of the form

$$u(t) = \begin{cases} 1 & \text{if } \varphi(t) > 0, \\ 0 & \text{if } \varphi(t) < 0, \end{cases} \tag{3.7}$$

that is, the isolated zeros of φ are exactly the times when there is a *switch* in the dynamics.
If $\varphi(t_0) = 0$, we will call t_0 a *switching time* and the corresponding $\eta(t_0)$ *switching point*.
If φ has any nonisolated zero, then Pontryagin Maximum Principle does not determine the control directly, but, as we will see in Section 5, in our case it is still possible to bypass this obstacle and determine the control.

Definition 3.5 (*bang-bang* control, singular control)**.** We will call *bang-bang* a control u and its corresponding extremal if φ has only isolated zeros.
If instead $\varphi \equiv 0$, we will call *singular* the corresponding control u and its extremal.

A control u which takes only extremal values, i.e. $u(t) \in \{0, 1\}$ for a.e. $t \in [0, T]$, is called *bang-bang*. We will first focus on *bang-bang* extremals and then we will discuss separately the singular case.

4 *Bang-bang* extremals

4.1 Properties of *bang-bang* extremals

The goal of this Section is to prove that extremal controls must be either constant or periodic piecewise constant.

Theorem 4.1. *Any admissible control corresponding to a* bang-bang *extremal must belong to one of the following two families of controls:*

- *constant controls, that is either $u(t) \equiv 1$ or $u(t) \equiv 0$ for all $t \in [0, T]$;*
- bang-bang *periodic controls, that is controls in the form*

$$u(t) = \begin{cases} 1 & \text{if } t \in [0, \tau), \\ 0 & \text{if } t \in [\tau, T/k), \end{cases} \tag{4.1}$$

  for $k \in \mathbb{N}$ and $\tau \in (0, T/k)$ and then extended by periodicity on the whole interval $[0, T]$.

We explain briefly the idea behind this result. We have two first integrals for equation (2.13): the quantity $\det \eta(t)$ and the Hamiltonian $H(\eta(t), u(t))$. In the 3D linear space $\mathfrak{sl}_2(\mathbb{R})$, the set $\{\det \eta = \text{const}\}$ is a connected hyperboloid, while $\{H(\eta, 0) = \text{const}, \operatorname{tr}(\eta(A - B) < 0\}$ and $\{H(\eta, 1) = \text{const}, \operatorname{tr}(\eta(A - B) > 0\}$ form half-planes. Thus, any trajectory η solving (2.13) must be in the intersections of these surfaces, resulting in a concateation of conic paths, which can be either unbounded or a loop. If unbounded, after a finite number of switches, the control remains constant. If instead η is contained in a loop, then the solution η and the switching function are periodic, making the control periodic as well.

Before proving Theorem 4.1, we prove the following useful remark: the solution η of (2.13) satisfies transversality condition (2.15) if and only if it is periodic.

Proposition 4.2. *A couple (X, η) solution of* (2.3) *and* (2.13) *respectively satisfies transversality condition* (2.15) *at time $T > 0$ if and only if η is periodic of period T.*

Proof. If (2.15) is satisfied, then

$$X(T)^{-1}\eta(0)X(T) = \eta(T) = X(T) - \frac{\operatorname{tr}(X(T))}{2}\text{Id},$$

and simply multiplying both sides by $X(T)$ on the left and by $X(T)^{-1}$ on the left one obtains that also

$$\eta(0) = X(T) - \frac{\operatorname{tr}(X(T))}{2}\mathrm{Id} = \eta(T),$$

that is, η is periodic of period T.
Viceversa, if $\eta(0) = \eta(T)$, then, from $\eta(T) = X(T)^{-1}\eta(0)X(T)$, it follows that

$$X(T)\eta(0) = \eta(0)X(T),$$

that is, $\eta(0), \eta(T)$ and $X(T)$ commutes. But then also $X(T) - \frac{\operatorname{tr}(X(T))}{2}\mathrm{Id}$ and $\eta(T)$ commutes, and since they are both matrices in $\mathfrak{sl}_2(\mathbb{R})$, which is semisimple, they must be proportional, i.e. there is $c \in \mathbb{R}$ such that :

$$c\eta(T) = X(T) - \frac{\operatorname{tr}(X(T))}{2}\mathrm{Id}.$$

Hence, since we can always rescale η, transversality condition (2.15) is satisfied. $\square$

The remaining part of this section and Subection 4.2 are devoted to the proof of Theorem 4.1.

In order to find the switches of a trajectory η, we define the plane of switching points, that is the subset of $\mathfrak{sl}_2(\mathbb{R})$ where the switching function ϕ is zero:

$$\Pi = \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \operatorname{tr}(MA) = \operatorname{tr}(MB)\}. \tag{4.2}$$

Since the matrices $A, B, [A, B]$ form a basis of $\mathfrak{sl}_2(\mathbb{R})$ by hypothesis, we can write the equation $\operatorname{tr}(MA) = \operatorname{tr}(MB)$ as

$$xa + yc = xc + yb,$$

where, according to the notation introduced in the Main Theorem, we set $a = \operatorname{tr}(A^2)$, $b = \operatorname{tr}(B^2)$, $c = \operatorname{tr}(AB)$. So, if $c \neq b$, we have

$$y = \frac{c-a}{c-b}x. \tag{4.3}$$

So, if we impose $\operatorname{tr}(M^2) = 1$, that is if we intersect the hyperboloid H with the plane Π we obtain the equation of a conic

$$\frac{(2c-a-b)(c^2-ab)}{\left(c-b\right)^2}x^2 + 2(c^2-ab)z^2 = 1. \tag{4.4}$$

Definition 4.3 (Switching curve)**.** We will call *switching curve* S the intersection curve $\Pi \cap H$, whose equation is given by (4.3), (4.4).

So, the coefficient in front of x is positive if and only if

$$\big(2c - a - b\big)(c^2 - ab) > 0.$$

Hence, if $c^2 - ab > 0$, then the switching curve is an ellipse if $c > \frac{a+b}{2}$, and is an hyperbola if $c^2 - ab > 0$ and $c < \frac{a+b}{2}$.
If instead $c^2 - ab < 0$, the switching curve is an hyperbola if $c < \frac{a+b}{2}$. Notice that $c > \frac{a+b}{2}$ implies $c^2 - ab > 0$, so we can neglect the case $c > \frac{a+b}{2}$ and $c^2 - ab < 0$.

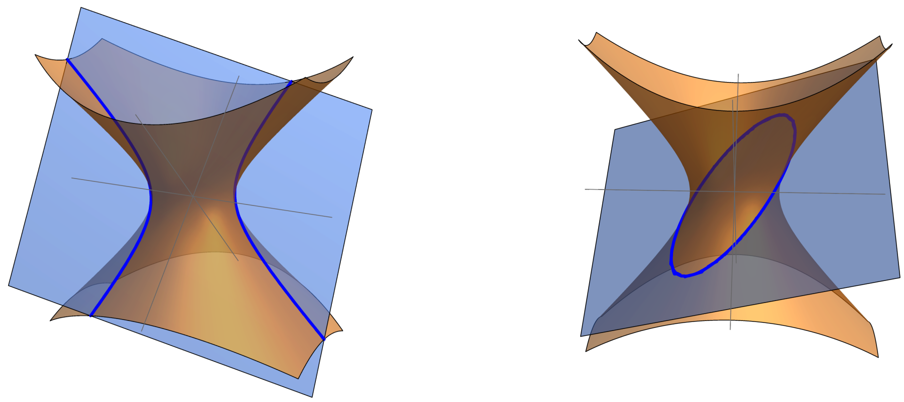

Figure 1: Examples of switching curves. In both pictures you can see the orange hyperboloid H, the blue plane Π and their intersection, the switching curve S.

4.2 Hyperboloid foliation

Let us now fix an initial point on the hyperboloid:

$$\eta_0 \in H.$$

Define the half spaces and the half hyperboloids

$$\begin{aligned} \Pi^+ &= \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \operatorname{tr}(MA) > \operatorname{tr}(MB)\}, \qquad & H^+ = H \cap \Pi^+, \\ \Pi^- &= \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \operatorname{tr}(MA) < \operatorname{tr}(MB)\}, \qquad & H^- = H \cap \Pi^-. \end{aligned}$$

From condition (3.7), we can see that if $\eta_0 \in H^+$, the control u, in order to satisfy PMP, must be equal to 1 until the trajectory η meets the switching curve S. Notice that, as far as $u(t) = 1$, the Hamiltonian function of the problem is $H(\eta(t), 1) = \operatorname{tr}(\eta(t)A)$, hence the quantity $\operatorname{tr}(\eta(t)A)$ is constant.
Similarly, if $\eta(t) \in H^-$ for $t \in [\tau_1, \tau_2]$, for some $0 < \tau_1 < \tau_2$, then $H(\eta(t), 0) = \operatorname{tr}(\eta(t)B)$ is constant.

Define the planes

$$\begin{aligned} \Pi_{A,\gamma} &= \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \operatorname{tr}(MA) = \gamma\} \\ \Pi_{B,\gamma} &= \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \operatorname{tr}(MB) = \gamma\}, \end{aligned}$$

for $\gamma \in \mathbb{R}$.
So, if $\eta_0 \in H^+$ is such that $\operatorname{tr}(\eta_0 A) = \gamma$, then the trajectory η is contained in the curve $H^+ \cap \Pi_{A,\gamma}$ (see Figure 3). In particular, we can suppose that geometrically the trajectory η coincides with the whole curve $H^+ \cap \Pi_{A,\gamma}$. Indeed, if η accumulates at some point, then the control $u(t)$ is identically equal to 1 or to 0 for every t sufficiently large and, as we will see, the corresponding Lyapunov exponents are easy to compute.

Now, we want to study all the possible configurations of H, S and the planes $\Pi_{A,\gamma}, \Pi_{B,\gamma}$. Consider the straight line given by the intersection of the three planes $\Pi, \Pi_{A,\gamma}, \Pi_{B,\gamma}$:

$$r_\gamma : \begin{cases} \operatorname{tr}(MA) = \gamma, \\ \operatorname{tr}(MB) = \gamma, \\ \operatorname{tr}(MA) = \operatorname{tr}(MB). \end{cases} \tag{4.5}$$

There are three cases: either this line does not intersect the hyperboloid H, either it is tangent to the hyperboloid, or it has two distinct intersection with the hyperboloid (See Figure 2).
We will see later that if r_γ is tangent, then the corresponding trajectory η is constant and in particular it is a singular trajectory. So, we will consider first the non-tangent cases and we postpone the tangent case to the Section of singular extremals.

If r_γ does not intersect the hyperboloid, then the trajectory η starting from η_0 does not meet the switching curve S. This means that η is confined either in the region H^+ or H^-, so the control u is identically equal to 1 or 0 for all times. If instead there are two distinct intersection between r_γ and

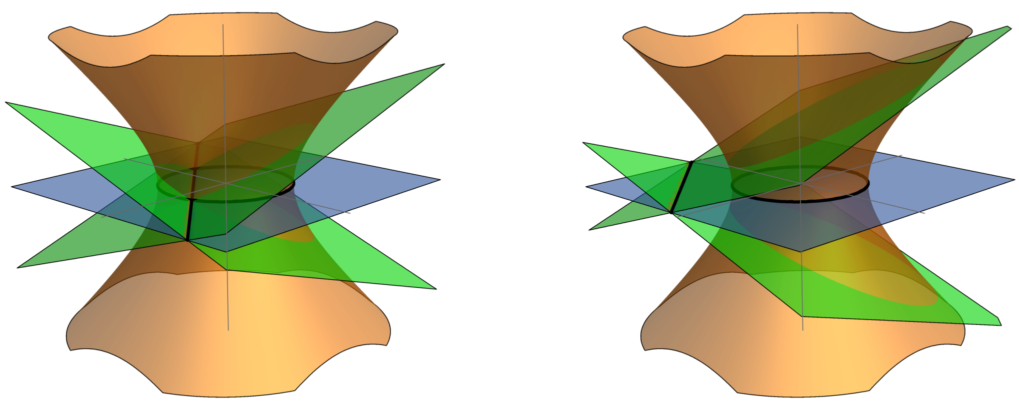

Figure 2: Example of possible position of the line r_γ.

H, then there are three possibilities (see Figure 3):

1. there is exactly one switch;
2. there are two switches and after the second switch the adjoint trajectory does not meet anymore the switching curve;
3. the adjoint trajectory is periodic with infinite switches.

Since the trajectories with no switches give the same Lyapunov exponents as the trajectories with a finite number of switches, we will not consider them. This ends the proof of Theorem 4.1.

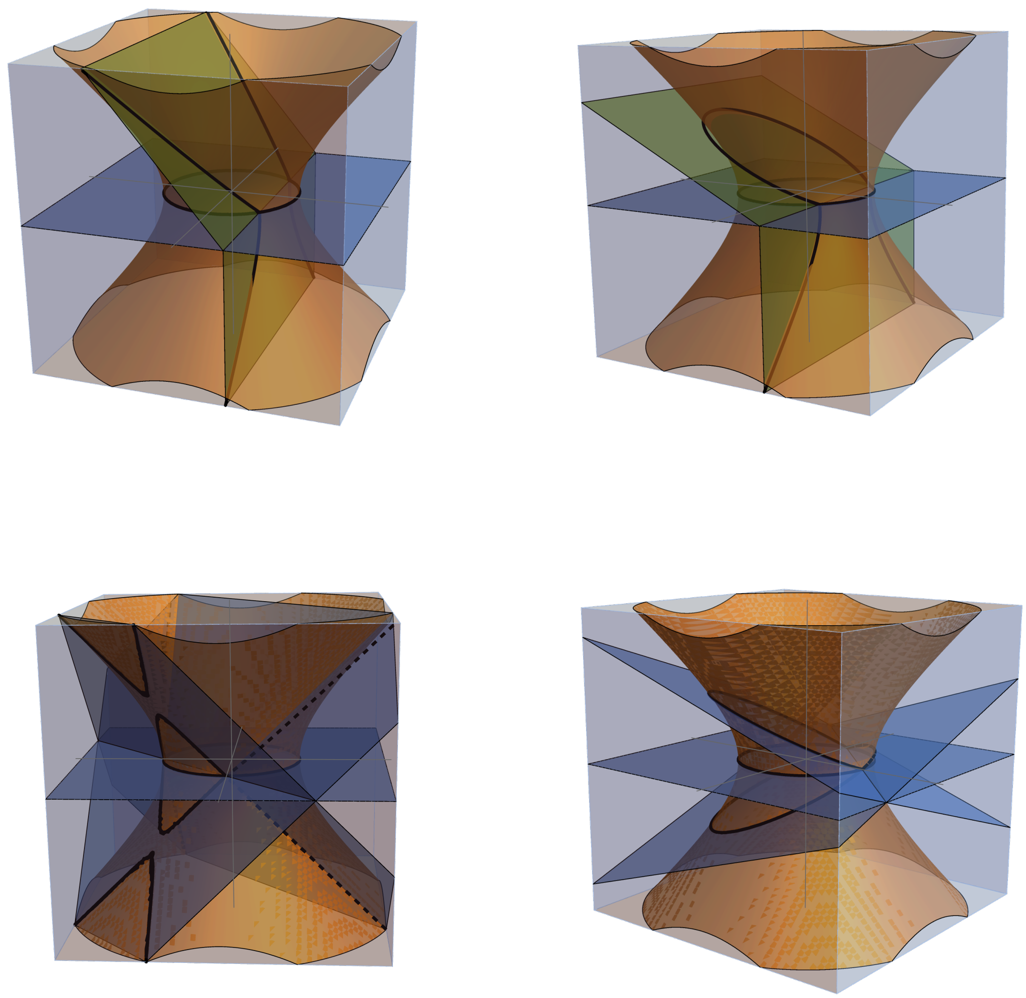

Figure 3: The figure illustrates the possible cases when the switching curve is a horizontal circle: in the first figure the intersection between the hyperboloid H and the planes $\Pi_{A,\gamma}$ and $\Pi_{B,\gamma}$ are two hyperbolas and the corresponding adjoint trajectory has only one switch. In the second figure the intersection with one plane is an ellipse and the other is a hyperbola, and the resulting adjoint trajectory has two switch. In the third case, the two intersections between the hyperboloid and the planes are hyperbolas, but in this case there are three connected components and one of them is closed, which corresponds to a periodic adjoint trajectory. The last figure is similar to the third one but with ellipses instead of hyperbolas.

Before ending this Subsection, we provide explicit formulas for the coordinates of the switching points. Using the same notation introduced before, we can write $\eta_0 = xA+yB+z[A,B]$. Then, from (4.3),(4.5) we obtain

$$x = \frac{b-c}{c^2-ab}\gamma, \quad y = \frac{a-c}{c^2-ab}\gamma, \quad z = \pm\frac{\sqrt{\big(a+b-2c\big)\gamma^2+c^2-ab}}{\sqrt{2}|c^2-ab|}. \tag{4.6}$$

Thus, there are two distinct intersection if γ is such that

$$\big(a+b-2c\big)\gamma^2 + (c^2-ab) > 0.$$

If $a+b-2c>0$ and $c^2>ab$ then the previous condition holds for every $\gamma\in\mathbb{R}$. If instead $a+b-2c>0$ and $c^2<ab$ this condition corresponds to

$$\gamma > \frac{1}{\sqrt{2}}\sqrt{\frac{2(c^2-ab)}{2c-a-b}} \quad \text{or} \quad \gamma < -\frac{1}{\sqrt{2}}\sqrt{\frac{2(c^2-ab)}{2c-a-b}}.$$

If instead $a+b-2c<0$, this implies $c^2>ab$, hence γ can assume the values

$$-\frac{1}{\sqrt{2}}\sqrt{\frac{2(c^2-ab)}{2c-a-b}} < \gamma < \frac{1}{\sqrt{2}}\sqrt{\frac{2(c^2-ab)}{2c-a-b}}.$$

Notice that if $\gamma=0$, then the line r_γ is tangent to the hyperboloid.

As already mentioned in the Introduction, if the control u in equation (2.3) is constant equal to 1, then the corresponding principal Lyapunov exponent is equal to the positive eigenvalue of A, i.e.

$$\ell_{u=1}(A,B) = \sqrt{\frac{a}{2}}.$$

Analogously, if the control u is constant equal to 0, then

$$\ell_{u=0}(A,B) = \sqrt{\frac{b}{2}}.$$

Clearly, since we are assuming $a\geq b$, we have $\ell_{u=1}\geq \ell_{u=0}$.
Concerning of a periodic periodic control, the principal Lyapunov exponent can be computed as follows: let T be the period of the control, then

$$\begin{aligned}\ell_{\mathrm{per}}(T) &= \limsup_{n\to+\infty}\frac{1}{nT}\log\|X(nT)\| = \\ &= \limsup_{n\to+\infty}\frac{1}{nT}\log\|X(T)^n\| = \\ &= \limsup_{n\to+\infty}\frac{1}{nT}\log\lambda(T)^n = \frac{1}{T}\log\lambda(T),\end{aligned} \tag{4.7}$$

where $\lambda(T)$ is the greatest eigenvalue of $X(T)$. We can further simplify the expression for $\ell_{\mathrm{per}}(T)$. Indeed, the functional of our Optimal Control Problem does not appear explicitly in the formula for $\ell_{\mathrm{per}}(T)$. However, we can deduce the value the eigenvalue from the trace solving

$$\lambda(T)^2 - \mathrm{tr}(X(T))\lambda(T) + 1 = 0,$$

which gives

$$\lambda(T) = \frac{\mathrm{tr}(X(T)) + \sqrt{\mathrm{tr}(X(T))^2 - 4}}{2}.$$

Using the identity $\mathrm{arcosh} x = \log(x + \sqrt{x^2 - 1})$, we obtain

$$\ell_{\mathrm{per}}(T) = \frac{\mathrm{arcosh}\left(\frac{\mathrm{tr}(X(T))}{2}\right)}{T}.$$

4.3 ***Bang-bang*** periodic controls

Up to this point, we have demonstrated that adjoint trajectories can be classified into two distinct categories: those without switches and those that are piecewise constant and periodic.

For periodic adjoint trajectories, the switching function (3.6) is also periodic, thereby the associated control is periodic too. Nevertheless, in order to satisfy PMP, we have to choose carefully the switching time τ in (4.1). In this Section, we will see how we can determine this τ.

From (3.1), finding all periodic solution to equation adjoint equation (2.13) amounts to finding $X(T)$ in the form

$$X(T) = e^{\tau A} e^{\sigma B},$$

with $\tau + \sigma = T$ and $0, \tau, T$ switching times.

We will proceed as follows: for every t we will find $\sigma = \sigma(\tau)$ such that $0, \tau, T = \tau + \sigma(\tau)$ are switching points. The following result simplifies this task.

Proposition 4.4. *Define the bang-bang control*

$$u(t) = \begin{cases} 1 & \text{if} \quad t \in [0, \tau], \\ 0 & \text{if} \quad t \in [\tau, T]. \end{cases}$$

Let X be the corresponding solution of (2.3) *and η be the solution of* (2.13)*. Suppose that η satisfies the condition of transversality* (2.15) *at time T. If τ is a switching time, then also* 0 *and T are switching times.*

So, if we know that η satisfies transversality condition at T and τ is a switching time, we know that automatically T is a switching time, so we do not have to check that the switching condition is satisfied also at time T.

Proof. We have to check that $\mathrm{tr}(\eta(T)A) = \mathrm{tr}(\eta(T)B)$. From transversality condition (2.15), this is equivalent to $\mathrm{tr}(X(T)A) = \mathrm{tr}(X(T)B)$.
transversality condition also implies, by Proposition 4.2, that η is periodic of period T, so it must be

$$\eta_0 = X(T) - \frac{\mathrm{tr}\big(X(T)\big)}{2}\mathrm{Id}.$$

So, putting together this identity with the formula (3.1), we obtain that

$$\eta(\tau) = e^{\sigma B} e^{\tau A} - \frac{\mathrm{tr}\big(X(T)\big)}{2}\mathrm{Id}.$$

We are assuming that τ to be a switching time, so it must satisfy

$$\mathrm{tr}(\eta(\tau)A) = \mathrm{tr}(\eta(\tau)B).$$

Using the formula for η that we have just obtained, we get

$$\mathrm{tr}(e^{\sigma B}e^{\tau A}A) = \mathrm{tr}(e^{\sigma B}e^{\tau A}B). \tag{4.8}$$

The left hand side can be rewritten as

$$\mathrm{tr}(e^{\sigma B}e^{\tau A}A) = \mathrm{tr}(e^{\sigma B}Ae^{\tau A}) = \mathrm{tr}(e^{\tau A}e^{\sigma B}A) = \mathrm{tr}(X(T)A).$$

Similarly, the right hand side of (4.8) is equal to $\mathrm{tr}(X(T)B)$. Hence, it follows that also

$$\mathrm{tr}(X(T)A) = \mathrm{tr}(X(T)B), \tag{4.9}$$

which implies that T (and thus, since η is T-periodic, also 0) is a switching time. □

Remark 4.5. If we consider a solution of (2.3) with one switch, the final point is in the form

$$X(T) = X(\tau, \sigma) = e^{\tau A}e^{\sigma B}$$

where $T = \tau + \sigma$. So, we can see the cost functional as a function of t and s:

$$\Phi(\tau, \sigma) = \mathrm{tr}(e^{\tau A}e^{\sigma B}).$$

With this notation, if τ is a switching point, from the previous proof follows that $X(T)$ must satisfy Equation (4.9), which can be rewritten as

$$\frac{\partial \Phi}{\partial \tau}(\tau, \sigma) = \frac{\partial \Phi}{\partial \sigma}(\tau, \sigma). \tag{4.10}$$

To summarize, given any $\tau > 0$, equation (4.10) give us a condition to find $\sigma > 0$ as a function of τ such that the control

$$u(t) = \begin{cases} 1 & \text{if} \quad t \in [0, \tau], \\ 0 & \text{if} \quad t \in [\tau, \tau + \sigma], \end{cases}$$

satisfies Pontryagin Maximum Principle on the time interval $[0, \tau + \sigma]$. The corresponding solution to the adjoint system η satisfies transversality condition on the same time interval, so one can extend this control by periodicity on the whole line $[0, +\infty)$.

4.4 Principal Lyapunov exponent for periodic controls

Now, we are going to compute the upper bound for Principal Lyapunov exponent of periodic controls. As we saw previously, in order to find the extremals of our problem we have to study the functional $\Phi : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$

$$\Phi(\tau, \sigma) = \mathrm{tr}(e^{\tau A}e^{\sigma B}).$$

Since equations (2.3) and (2.13) are invariant after a change of basis, if A and B are not nilpotent (see Remark 4.8 for the nilpotent case), without loss of generality we can assume $\mathrm{spec}(A) = \{\lambda, -\lambda\}, \mathrm{spec}(B) = \{\mu, -\mu\}$, with $\lambda, \mu \in \mathbb{R} \cup i\mathbb{R}$. Up to exchange the role of A and B, we can assume $\mathrm{tr}(A^2) \geq \mathrm{tr}(B^2)$.

Proposition 4.6. *With this notations, the functional in* (4.4) *is*

$$\Phi(\tau,\sigma) = 2\cosh(\lambda\tau)\cosh(\mu\sigma) + \frac{\mathrm{tr}(AB)}{\lambda\mu}\sinh(\lambda\tau)\sinh(\mu\sigma). \tag{4.11}$$

This formula is a simple consequence of the following Lemma.

Lemma 4.7. *If $M \in \mathfrak{sl}_2(\mathbb{R})$, with positive eigenvalue $\alpha > 0$, then*

$$e^{tM} = \cosh(\alpha t)\mathrm{Id} + \frac{\sinh(\alpha t)}{\alpha}M. \tag{4.12}$$

Proof. From the characteristic polynomial of M, we know that $M^2 = \alpha^2\mathrm{Id}$, and so $M^{2k} = \alpha^{2k}\mathrm{Id}$, $M^{2k+1} = \alpha^{2k}M$. Using these facts in the series expansion of e^{tM} one obtains the formula above. $\square$

Remark 4.8. If $M \in \mathfrak{sl}_2(\mathbb{R})$ is nilpotent, i.e. $M^2 = 0$, then

$$e^{tM} = \mathrm{Id} + tM. \tag{4.13}$$

So, if either A or B is nilpotent, one can just replace formula (4.11) with a formula for Φ obtained using (4.13).

4.4.1 Imaginary eigenvalues

We are now going to examine a special case. Suppose that $\mathrm{tr}(A^2), \mathrm{tr}(B^2) < 0$, i.e. A, B have imaginary eigenvalues.

Proposition 4.9. *If both A, B have imaginary eigenvalues, the functional Φ is*

$$\Phi(\tau,\sigma) = \mathrm{tr}\big(\exp(\tau A)\exp(\sigma B)\big) = 2\cos(\lambda\tau)\cos(\mu\sigma) - 2\gamma\sin(\lambda\tau)\sin(\mu\sigma),$$

Where $\gamma = \frac{|\mathrm{tr}(AB)|}{\sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}}$.

This is simply a consequence of formula (4.11) and $\cosh(i\lambda\tau) = \cos(\lambda\tau)$, $\sinh(i\mu\sigma) = i\sin(\mu\sigma)$. With this formula, the switching equation (4.10) reads

$$(\gamma\lambda - \mu)\cos(\lambda\tau)\sin(\mu\sigma) = (\gamma\mu - \lambda)\cos(\mu\sigma)\sin(\lambda\tau),$$

from which we obtain immediately that there are always solutions for $\cos(\lambda\tau), \cos(\mu\sigma) = 0$, that is

$$(\tau,\sigma) \in \left\{\left(\frac{\pi}{2\lambda}, \frac{3\pi}{2\mu}\right), \left(\frac{\pi}{2\lambda}, \frac{\pi}{2\mu}\right), \left(\frac{3\pi}{2\lambda}, \frac{\pi}{2\mu}\right)\right\}.$$

From these choices the value of this functional is $\Phi(\tau,\sigma) = \pm 2\gamma$. Using formula (4.2) one obtains

$$\ell_{\mathrm{per,im}}(A,B) = \frac{1}{\tau+\sigma}\mathrm{arcosh}(\gamma).$$

If $\mathrm{tr}(AB) < 0$, then we have

$$\ell_{\mathrm{per,im}}(A,B) = \frac{2}{\pi\left(\sqrt{\frac{-2}{\mathrm{tr}(A^2)}} + 3\sqrt{\frac{-2}{\mathrm{tr}(B^2)}}\right)}\mathrm{arcosh}\left(\frac{-\mathrm{tr}(AB)}{\sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}}\right). \tag{4.14}$$

Notice that we have chosen the couple (τ, σ) so to maximize $\frac{1}{\tau+\sigma}$ under the hypothesis $\mathrm{tr}(B^2) < \mathrm{tr}(A^2) < 0$.
If instead $\mathrm{tr}(AB) > 0$, then

$$\ell_{\mathrm{per,im}}(A,B) = \frac{2}{\pi\left(\sqrt{\frac{-2}{\mathrm{tr}(A^2)}} + \sqrt{\frac{-2}{\mathrm{tr}(B^2)}}\right)} \operatorname{arcosh}\left(\frac{\mathrm{tr}(AB)}{\sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}}\right). \tag{4.15}$$

We will see in the next subsection how to deal with the extremals with $\cos(\lambda\tau), \cos(\mu\sigma) \neq 0$.

4.4.2 General case

We suppose now $\cosh(\lambda\tau), \cosh(\mu\sigma) \neq 0$ (we already saw the case $\cosh(\lambda\tau), \cosh(\mu\sigma) = 0$, which can happen if $\lambda, \mu \in i\mathbb{R}$). With this assumption, equation (4.10) becomes

$$\tanh(\mu\sigma) = \frac{\mu}{\lambda} \frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)} \tanh(\lambda\tau). \tag{4.16}$$

If $\frac{\mathrm{tr}(AB)-\mathrm{tr}(A^2)}{\mathrm{tr}(AB)-\mathrm{tr}(B^2)} > 0$, then, for every $\tau > 0$ we can find $\sigma = \sigma(\tau)$ for which (4.16) holds. In particular

$$\sigma(\tau) = \frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)}\tau + o(\tau) \quad \text{as } \tau \to 0. \tag{4.17}$$

So, for every $\tau > 0$ we obtained an $\sigma(\tau)$ such that the *bang-bang* control defined by

$$u(t) = \begin{cases} 1 & \text{if } \tau \in [0, \tau), \\ 0 & \text{if } \tau \in [\tau, \tau + \sigma(\tau)), \end{cases}$$

and extended by periodicity for $t \in [\tau + \sigma(\tau), +\infty)$, satisfies all conditions of Pontryagin Maximum Principle.
Now, it remains just to determine for which τ the Principal Lyapunov exponent of the system is maximal. Define the family of controls

$$u_{\tau,\sigma}(t) = \begin{cases} 1 & \text{if } t \in [0, \tau), \\ 0 & \text{if } t \in [\tau, \tau + \sigma), \end{cases} \qquad \tau, \sigma > 0,$$

and denote by $X_{\tau,\sigma}$ the solution of (2.3) with control equal to $u_{\tau,\sigma}$. Let $\alpha(\tau,\sigma)$ be the greatest eigenvalue of $X_{\tau,\sigma}(\tau+\sigma)$, so that the Principal Lyapunov exponent corresponding to the control $u_{\tau,\sigma}$ is

$$\ell(\tau,\sigma) = \frac{1}{\tau+\sigma} \log\big(\alpha(\tau,\sigma)\big).$$

The following result helps us in this task.

Lemma 4.10. *With the notation introduced above:*

1. *if* $\mathrm{tr}\big([A,B]^2\big) > 0$, *then* $\ell(\frac{\tau}{2}, \frac{\sigma}{2}) > \ell(\tau,\sigma)$;
2. *if* $\mathrm{tr}\big([A,B]^2\big) < 0$, *then* $\ell(\frac{\tau}{2}, \frac{\sigma}{2}) < \ell(\tau,\sigma)$;

We first show how to obtain the estimate for the Principal Lyapunov exponent, then we prove the Lemma.

Theorem 4.11. *If* $\mathrm{tr}\big([A,B]^2\big) > 0$, *then for every* $\tau > 0$ *it holds*

$$\ell(\tau, \sigma(\tau)) < \lim_{\tau_1 \to 0} \ell(\tau_1, \sigma(\tau_1)) = \frac{1}{2}\sqrt{\frac{\mathrm{tr}\big([A,B]^2\big)}{2\mathrm{tr}(AB) - \mathrm{tr}(A^2) - \mathrm{tr}(B^2)}} \tag{4.18}$$

If instead $\mathrm{tr}\big([A,B]^2\big) < 0$, *then control with switches are not optimal.*

Proof. First, suppose that $\mathrm{tr}\big([A,B]^2\big) > 0$. One can check that this condition implies that the coefficient in (4.16) is smaller than 1:

$$\frac{\mu}{\lambda}\frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)} < 1.$$

So, for every $\tau > 0$ it makes sense to take the inverse hyperbolic tangent in (4.16) on both sides. Moreover, from point (1) of Lemma 4.10, for any $\tau > 0$ we have

$$\ell\left(\frac{\tau}{2}, \frac{\sigma(\tau)}{2}\right) > \ell(\tau, \sigma(\tau))$$

In particular, we can find $\tau_1 < \tau$ such that $\tau_1 + s(\tau_1) = \frac{1}{2}(\tau + \sigma(\tau))$, so that

$$\ell(\tau_1, s(\tau_1)) > \ell\left(\frac{\tau}{2}, \frac{\sigma(\tau)}{2}\right) > \ell(\tau, \sigma(\tau)).$$

So, the inequality in (4.18) follows. To evaluate the limit, one can use Taylor expansions for $\sigma(\tau)$ as $\tau \to 0$.

Suppose now that $\mathrm{tr}\big([A,B]^2\big) < 0$. Notice that this can happen only if $\mathrm{tr}(A^2), \mathrm{tr}(B^2) > 0$. Similarly to the previous case, this inequality implies

$$\frac{\mu}{\lambda}\frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)} > 1.$$

So, we can find $\sigma = \sigma(\tau)$ solving (4.16) only for $\tau < \bar{\tau}$, where

$$\bar{\tau} = \frac{1}{\lambda}\mathrm{arctanh}\left(\frac{\lambda}{\mu}\frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)}\right).$$

Moreover, $\lim_{\tau \to \bar{\tau}} \sigma(\tau) = +\infty$. Then, from point (2) of Lemma 4.10, we have $\ell(2\tau, 2\sigma(\tau)) > \ell(\tau, \sigma(\tau))$. So, similarly to the previous case, this implies that we can find τ_1 such that

$$\ell(\tau_1, \sigma(\tau_1)) > \ell(\tau, \sigma(\tau)).$$

So, we obtain that

$$\ell(\tau, \sigma(\tau)) < \lim_{\tau_1 \to \bar{\tau}} \ell(\tau_1, \sigma(\tau_1)).$$

But as $\tau \to \bar{\tau}$, the trajectory spends more and more time with control equal to 0. So, this strategy is for sure worse than spending all the time with control equal to 1, which is always admissible if $\mathrm{tr}\big([A,B]^2\big) < 0$. Hence, in this case switches are not optimal. $\square$

We now prove Lemma 4.10.

Proof of Lemma 4.10. Recall that

$$l(\tau,\sigma) = \frac{1}{\tau+\sigma}\log\big(\alpha(\tau,\sigma)\big).$$

Hence, we can write

$$\ell\left(\frac{\tau}{2},\frac{\sigma}{2}\right) - \ell(\tau,\sigma) = \frac{2}{\tau+\sigma}\log\left(\alpha\left(\frac{\tau}{2},\frac{\sigma}{2}\right)\right) - \frac{1}{\tau+\sigma}\log\big(\alpha(\tau,\sigma)\big) = \frac{1}{\tau+\sigma}\log\left(\frac{\alpha\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2}{\alpha(\tau,\sigma)}\right).$$

So, we can see that this difference is positive if and only if

$$\frac{\alpha\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2}{\alpha(\tau,\sigma)} > 1.$$

Since $\alpha(\tau,\sigma) > 1$ for any $\tau,\sigma > 0$, we have that

$$\alpha\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2 > \alpha(\tau,\sigma) \quad \text{iff} \quad \alpha\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2 + \frac{1}{\alpha\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2} > \alpha(\tau,\sigma) + \frac{1}{\alpha(\tau,\sigma)},$$

which is the same as

$$\operatorname{tr}\left(X\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2\right) > \operatorname{tr}\big(X(\tau,\sigma)\big).$$

The term in the left hand side can be rewritten as

$$X\left(\frac{\tau}{2},\frac{\sigma}{2}\right)^2 = X(\tau,\sigma) + e^{\frac{\tau}{2}A}\big[e^{\frac{\sigma}{2}B}, e^{\frac{\tau}{2}A}\big]e^{\frac{\sigma}{2}B}.$$

So, in the end, it amounts just to compute $\operatorname{tr}\left(e^{\frac{\tau}{2}A}\big[e^{\frac{\sigma}{2}B}, e^{\frac{\tau}{2}A}\big]e^{\frac{\sigma}{2}B}\right)$, which can be done using formula (4.12). One obtains

$$\operatorname{tr}\left(e^{\frac{\tau}{2}A}\big[e^{\frac{\sigma}{2}B}, e^{\frac{\tau}{2}A}\big]e^{\frac{\sigma}{2}B}\right) = \frac{\sinh(\lambda\frac{\tau}{2})^2\sinh(\mu\frac{\sigma}{2})^2}{2\lambda^2\mu^2}\operatorname{tr}\big([A,B]^2\big).$$

So, if $\operatorname{tr}\big([A,B]^2\big) > 0$, then we obtained point (1) of the Lemma. If instead $\operatorname{tr}\big([A,B]^2\big) < 0$, then all inequalities are reversed and we obtain point (2). $\square$

Remark 4.12. Notice that this is a completely general result about product of matrix exponentials. Indeed, it holds for a generic control $u_{\tau,\sigma}$, even if it does not correspond to a Pontryagin extremal.

4.4.3 Limit trajectory as switching time tends to zero

In the previous Subsection we saw that if $\operatorname{tr}([A,B]^2) > 0$, then the upper bound for the principal Lyapunov is obtained as the switching time tends to zero. Now, we want to discuss some more details about the limit trajectory as the switching time goes to zero.

Proposition 4.13. *The limit trajectory as the switching time goes to zero is the singular trajectory.*

Proof. First, recall that from (4.17)

$$\sigma(\tau) = \frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)}\tau + o(\tau) \quad \text{as } \tau \to 0.$$

So, our trajectory spends τ time with control equal to 1, which corresponds to matrix A, and approximately $\frac{\mathrm{tr}(AB)-\mathrm{tr}(A^2)}{\mathrm{tr}(AB)-\mathrm{tr}(B^2)}\tau$ time with control equal to 0, corresponding to matrix B. For simplicity, let us call

$$c = \frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)},$$

the coefficient appearing in the expansion of $\sigma(\tau)$.
So, by convex approximation (see Theorem 8.2 in [12]), we know that as the number of switches goes to infinity (i.e., τ goes to zero), the flow tends uniformly (in the $C^\infty(\mathrm{SL}_2(\mathbb{R}))$-topology) to

$$T \mapsto \exp\big(T(vA + (1-v)B)\big),$$

where

$$v = \frac{1}{1+c} = \frac{\mathrm{tr}(AB) - \mathrm{tr}(B^2)}{2\mathrm{tr}(AB) - \mathrm{tr}(A^2) - \mathrm{tr}(B^2)},$$

which corresponds to the value of the singular control (see Section 5). □

5 Singular extremals

In this Section we are going to deal with the case of singular extremals (see Definition 3.5). Recall the definition of the Hamiltonian function H and the switching function φ

$$H(\eta, u) = \mathrm{tr}\big(\eta(uA + (1-u)B)\big), \quad \varphi(t) = \mathrm{tr}\big(\eta(t)(A-B)\big).$$

Recall also that the singular extremals correspond to the case of $\varphi(t) = 0$ for $t \in [\tau_1, \tau_2]$, and $\tau_1, \tau_2 > 0$. In this case, since φ is Lipschitz, we can differentiate it:

$$0 = \dot\varphi(t) = \mathrm{tr}\big(\dot\eta(t)(A-B)\big) = \mathrm{tr}\Big(\big[\eta(t), uA + (1-u)B\big](A-B)\Big) = \mathrm{tr}\big(\eta(t)[A,B]\big),$$

where in the last equality we used the identity $\mathrm{tr}([M_1, M_2]M_3) = \mathrm{tr}(M_1[M_2, M_3])$. Notice that $\dot\varphi$ does not depend on u and in particular it is again a Lipschitz function. So, if η_* is a point of a singular trajectory, it must satisfy the two following conditions:

$$\begin{cases} \mathrm{tr}\big(\eta_*(A-B)\big) = 0, \\ \mathrm{tr}\big(\eta_*[A,B]\big) = 0 \end{cases}. \tag{5.1}$$

These two conditions are linear in η_*, so, since A, B and $[A,B]$ are linearly independent, they determine a straight line in $\mathfrak{sl}_2(\mathbb{R})$.
If we write $\eta_* = xA + yB + z[A,B]$, then the second equation implies $z = 0$. Hence, from (4.6), we see that if we take $\gamma_* = \mathrm{tr}(\eta_* A)$, the line r_{γ_*} is tangent to the hyperboloid H (see discussion in Subsection 4.2).

There are two intersection between the line (5.1) and the hyperboloid H described in Section 2 if and only if

$$\big(2\mathrm{tr}(AB) - \mathrm{tr}(A^2) - \mathrm{tr}(B^2)\big)\mathrm{tr}([A,B]^2) > 0. \tag{5.2}$$

If there are no intersection, then there are no singular extremals. So, from now on we will assume that this inequality hold.
We can compute explicitly these two intersection: let us call η_* one of the two intersection point. Then, from (4.3) and $\gamma = 0$ we obtain

$$\eta_* = \alpha_* A + \alpha_* \frac{\mathrm{tr}(AB) - \mathrm{tr}(A^2)}{\mathrm{tr}(AB) - \mathrm{tr}(B^2)} B,$$

where $\alpha_* \in \mathbb{R}$ is chosen such that $\mathrm{tr}(\eta_*^2) = 1$. Since the intersection between (5.1) and H is a discrete set and since the solution η_* is Lipschitz, the only possibility for η_* is to be constant. Hence

$$0 = \dot{\eta}_*(t) = [\eta_*, u_* A + (1 - u_*)B],$$

from which we can determine the value of the singular control u_*

$$u_* = \frac{\mathrm{tr}(AB) - \mathrm{tr}(B^2)}{2\mathrm{tr}(AB) - \mathrm{tr}(A^2) - \mathrm{tr}(B^2)}. \tag{5.3}$$

In order to be an admissible control, it must hold $u_* \in [0,1]$.

Proposition 5.1. *The value u_* is an admissible control if and only if*

$$\mathrm{tr}(AB) \geq \mathrm{tr}(A^2) \quad \text{or} \quad \mathrm{tr}(AB) \leq \mathrm{tr}(B^2).$$

So, if we take the constant singular control equal to u_*, then the associated Lyapunov exponent is simply the greatest eigenvalue of $u_* A + (1 - u_*)B$, which can be easily computed:

$$l = \frac{1}{2}\sqrt{\frac{\mathrm{tr}\big([A,B]^2\big)}{2\mathrm{tr}(AB) - \mathrm{tr}(A^2) - \mathrm{tr}(B^2)}}. \tag{5.4}$$

A direct computation shows that

$$\ell_{\mathrm{sing}} > \ell_{u=1} \iff \mathrm{tr}\Big(\big(u_* A + (1-u_*)B\big)^2\Big) \geq \mathrm{tr}(A^2) \qquad \iff \qquad \mathrm{tr}(AB) \geq \frac{\mathrm{tr}(A^2) + \mathrm{tr}(B^2)}{2}. \tag{5.5}$$

So, taking into account that singular control is admissible if $\mathrm{tr}(AB) \geq \mathrm{tr}(A^2)$, we obtain that singular constant control are better than constant control if $\mathrm{tr}(AB) \geq \mathrm{tr}(A^2)$. On the other hand, if $\mathrm{tr}(AB) \leq \mathrm{tr}(B^2)$ then constant control is better than constant singular control.
The following result ends the discussion about the singular case.

Proposition 5.2. *There are no solutions to the adjoint system* (2.13) *containing a* bang-bang *piece and a singular piece.*

Proof. We already noticed that for η_* we have $\gamma = 0$ in (4.6) and the intersection line between the planes Π_{A,c_*} and Π_{B,c_*} is tangent to the hyperboloid H, where $c_* = \mathrm{tr}(A\eta_*)$. Recall from Section 4 that as the switching time in a *bang-bang* trajectory in $\mathrm{SL}_2(\mathbb{R})$ tends to zero, the resulting trajectory tends

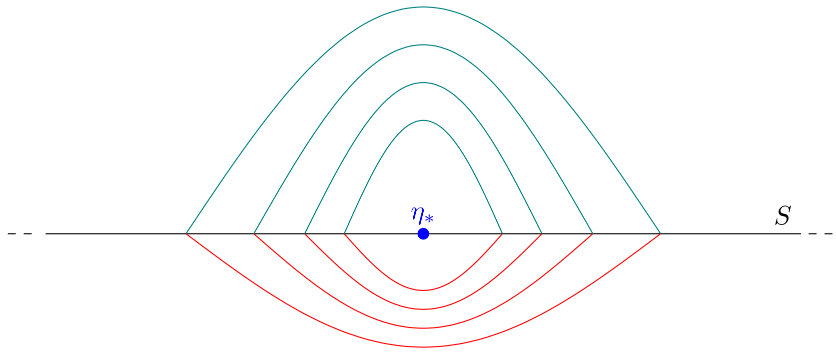

Figure 4: Phase portrait on the hyperboloid H for the adjoint system (2.13) near η_*. The black line S is the switching curve, η_* is the singular point. Above and below S we have the two *bang* pieces, and as the switching time goes to zero the adjoint trajectory tends to η_*.

to a singular trajectory. The convergence in $\mathrm{SL}_2(\mathbb{R})$ implies that also the sequence of the corresponding adjoint trajectories tends to the singular adjoint trajectory, which is constant. Thus, in a neighborhood of η_* sufficiently small and contained in the hyperboloid H, the phase portrait is as in Figure 4.
Thus, any possible trajectory starting from η_* and escaping the switching curve S would inevitably cross some of the periodic trajectories tending to η_*, contradicting the local uniqueness of the solution of an ODE. $\square$

So, to resume, we found that if an extremal contains a singular piece, then the whole extremal is singular, and the corresponding Lyapunov exponent is given by (5.4). This is the optimal solution if $\mathrm{tr}(AB) \geq \mathrm{tr}(A^2)$.

6 Final remarks

Since the nature of the extremals changes a lot with the values of $\mathrm{tr}(A^2)$,$\mathrm{tr}(B^2)$,$\mathrm{tr}(AB)$, we end the discussion with some final remarks about which is the optimal strategy depending on these values. As before, we always assume $\mathrm{tr}(A^2) \geq \mathrm{tr}(B^2)$.
So far, we have obtained that all possible extremals satisfying PMP are in one of the following form:

- constant control;
- constant singular control;
- if $\mathrm{tr}(A^2) < 0$, *bang-bang* periodic control as in Subsection 4.4.1.

As pointed out in Subsection 4.4.2, there is also this other kind of extremals accumulating at singular extremals. However, we saw that their associated principal Lyapunov exponent is at most strictly less than the one associated to singular extremal, so we can omit them from the discussion.
First, assume that $\mathrm{tr}(A^2) \geq 0$. In this case we already saw in Section 5 that if $\mathrm{tr}(AB) \geq \mathrm{tr}(A^2)$, then the value of the Principal Lyapunov exponent given by the constant singular control is bigger than the one given by the constant control. Viceversa, if $\mathrm{tr}(AB) \leq \mathrm{tr}(A^2)$, then the optimal solution is obtained for $u = 1$ constant. So, the first two point of the Main Theorem are proved.
We turn now to the $\mathrm{tr}(A^2) < 0$ case. We have the following restriction on the possible values of $\mathrm{tr}(AB)$.

Lemma 6.1. *If* $\mathrm{tr}(B^2) \leq \mathrm{tr}(A^2) < 0$*, then* $|\mathrm{tr}(AB)| \geq \sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}$*. In particular* $\mathrm{tr}([A,B]^2) \geq 0$*.*

Proof. Up to a change of base, we can suppose

$$A = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \quad B = P \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} P^{-1}, \quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad \lambda, \mu \in \mathbb{R}.$$

Since A in this form commutes with rotations, that is matrix of the form

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi],$$

we can use another change of base given by a rotation matrix to kill one element of P. That is, we can further assume that P is in the form

$$P = \begin{pmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha \neq 0.$$

With A, B in this form, we can compute directly $\mathrm{tr}(AB)$:

$$\mathrm{tr}(AB) = -\lambda\mu \left(\frac{1}{\alpha^2} + \alpha^2 + \beta^2 \right).$$

Since $\frac{1}{\alpha^2} + \alpha^2 \geq 2$, we obtain

$$|\mathrm{tr}(AB)| \geq 2|\lambda\mu| = \sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}.$$

□

Remark 6.2. From the previous proof, we can give a geometric interpretation of the sign of $\mathrm{tr}(AB)$ in the case $\mathrm{tr}(B^2) \leq \mathrm{tr}(A^2) < 0$. Define the cone of imaginary matrices in $\mathfrak{sl}_2(\mathbb{R})$:

$$\mathcal{C} = \{M \in \mathfrak{sl}_2(\mathbb{R}) \mid \mathrm{tr}(M^2) < 0\}.$$

To see that $\mathcal{C}$ is a cone, one can argue as at the beginning of Subsection 3.1. Using the form for A, B introduced in the previous proof, if $\mathrm{tr}(AB) < 0$, then $\lambda\mu > 0$, that is A, B are in the same connected component. If instead $\mathrm{tr}(AB) > 0$, then $\lambda\mu < 0$, hence A, B are in two different connected component.

In particular, if $\mathrm{tr}(AB) < 0$, then the whole segment of velocities

$$\{uA + (1-u)B \mid u \in [0,1]\}$$

is inside the cone $\mathcal{C}$. In this case, if the singular control is admissible, then it has imaginary eigenvalues, hence singular constant control cannot be optimal (the resulting trajectory in $\mathrm{SL}_2(\mathbb{R})$ would be periodic, hence bounded). So, in this case the best admissible strategy is the one given in Subsection 4.4.1, and we obtain the third point of Main Theorem.
If instead $\mathrm{tr}(AB) > 0$, then part of the segment of velocities is outside the cone, and in particular singular control is admissible since $\mathrm{tr}(A^2) < 0$.
The following Lemma shows that in this case constant singular control are again optimal.

Lemma 6.3. *If* $\mathrm{tr}(B^2) \leq \mathrm{tr}(A^2) < 0$ *and* $\mathrm{tr}(AB) \geq \sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}$*, then*

$$\frac{2}{\pi} \frac{1}{\sqrt{\frac{-2}{\mathrm{tr}(A^2)}} + \sqrt{\frac{-2}{\mathrm{tr}(B^2)}}} \mathrm{arcosh}\left(\frac{\mathrm{tr}(AB)}{\sqrt{\mathrm{tr}(A^2)\mathrm{tr}(B^2)}} \right) \leq \sqrt{\frac{1}{2} \frac{\mathrm{tr}(AB)^2 - \mathrm{tr}(A^2)\mathrm{tr}(B^2)}{2\mathrm{tr}(AB) - \mathrm{tr}(A^2) - \mathrm{tr}(B^2)}}.$$

The proof of this Lemma is an elementary but rather long computation, so we omit it.
This concludes the proof of the fourth case of Main Theorem.

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