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ORIGINAL PAPER

Existence results of $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordisms

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Abstract
In this note, we study necessary and sufficient conditions for the existence of a $\mathrm{Spin}(n+1)$-dimensional cobordism that supports a non-singular and non-degenerate pseudo-Riemannian metric of signature $(2, n-1)$, which restricts to a non-singular time-orientable Lorentzian metric on its boundary. The corresponding cobordism groups are computed.

Keywords Distributions · Spin structures · Cobordisms · Indefinite metrics

Mathematics Subject Classification Primary 53C50 · 57M50 · 57R25; Secondary 53B30

Contents

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1 Introduction and main results

Reinhart [26] and Sorkin [28] determined necessary and sufficient topological conditions for a compact $(n+1)$-manifold W to admit a non-singular time-orientable Lorentzian metric g^L inducing a Riemannian metric on its boundary $\partial W = M_1 \sqcup M_2$. In particular, a pair $((W; M_1, M_2), g^L)$ satisfying such properties is known as an $\mathrm{SO}(1,n)_0$-Lorentzian cobordism (Definition 41) and the existence of the Lorentzian metric g^L on W has been completely characterized by Reinhart and Sorkin by means of the Euler characteristics of W and of the closed n-manifolds M_1 and M_2. These objects have been studied extensively throughout the years by several mathematicians and mathematical physicists, including Chamblin [3], Geroch [9] Gibbons–Hawking [10], Reinhart [26], Sorkin [28] and Yodzis [40].

An interesting question is whether two given topological and geometric properties can co-exist on a given cobordism $(W; M_1, M_2)$. Let us think for a second about the third-dimensional case to fix ideas. Any two closed oriented 3-manifolds are the boundary of some compact Spin 4-manifold since the third Spin-cobordism group is trivial [23]. The results of Reinhart and Sorkin that were just mentioned also guarantee the existence of a $\mathrm{SO}(1,3)_0$-Lorentzian cobordism between any two closed oriented 3-manifolds. The latter cobordism, however, need not support a Spin-structure. Gibbons–Hawking [10] showed that the Kervaire semi-characteristic (see Definition 9) of $\partial W = M_1 \sqcup M_2$ is the only topological obstruction for a $\mathrm{SO}(1,3)_0$-Lorentzian cobordism to admit a compatible structure of a Spin-cobordism. Smirnov–Torres [27] generalized their results to arbitrary dimensions and computed the corresponding $\mathrm{Spin}(1,n)_0$-Lorentzian cobordism groups.

In this paper, we extend these results further into the pseudo-Riemannian realm and occupy ourselves with the study of the following objects.

Definition 1 An $\mathrm{SO}(2, n-1)_0$-pseudo-Riemannian cobordism between closed smooth oriented n-manifolds M_1 and M_2 is a pair

$$((W; M_1, M_2), g) \tag{2}$$

that consists of

(A) a cobordism $(W; M_1, M_2)$,

(B.1) a non-singular indefinite metric (W, g) of signature $(2, n-1)$ such that

(C.1) its restriction to the boundary $\partial W = M_1 \sqcup M_2$ gives rise to non-singular time-orientable Lorentzian metrics $(M_1, g^L_{M_1})$ and $(M_2, g^L_{M_2})$; please see Sect. 3 for an explanation on our notation.

In the sequel, we focus on the case where the cobordism of Item (A) is a Spin-cobordism and require for the Spin-structures on the boundary components to be induced by the Spin-structure on the cobordism. In thise case, we call (2) a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism between M_1 and M_2 and say that M_1 is $\mathrm{Spin}(2, n-1)_0$-cobordant to M_2.

A prototype example of a $\mathrm{Spin}(2, n-2)_0$-pseudo-Riemannian cobordism is as follows.

Example A Let (X, g) be a closed Riemannian $(n-2)$-manifold of dimension at least two, take the 2-disk D^2 with polar coordinates $(r, \theta) \in D^2$, and consider the compact pseudo-Riemannian n-manifold

$$(D^2, -dr^2 - r^2 d\theta^2) \times (X, g) \tag{3}$$

with the indefinite product metric of signature $(2, n-2)$. Furthermore, if X admits a Spin-structure, then (3) is a $\mathrm{Spin}(2, n-2)_0$-pseudo-Riemannian cobordism with boundary the

Lorentzian $(n-1)$-manifold

$$(S^1, -d\theta^2) \times (X, g). \tag{4}$$

While some partial results on $\mathrm{Spin}(2,2)_0$-pseudo-Riemannian cobordisms have been obtained by Alty–Chamblin in [1], there is no systematic study of these objects available in the literature. The goal of this paper is to fill such gap. In our first main result, we provide an almost complete topological characterization for the existence of a $\mathrm{Spin}(2,n-1)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$; see Remark 55. The Euler characteristic of a manifold X is denoted by $\chi(X)$ and the Kervaire semi-characteristic of an odd-dimensional manifold Y is denoted by $\hat{\chi}_{\mathbb{Z}/2}(Y)$; see Definition 9.

Theorem B *Let $\{M_1, M_2\}$ be closed smooth* Spin*-cobordant n-manifolds.*

• *Suppose $n \not\equiv 1, 5, 7 \mod 8$. There exists a* $\mathrm{Spin}(2,n-1)_0$*-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ if and only if*

(1) *The Euler characteristic of every connected component of M_1 and M_2 is trivial for n even*
(2) *$\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$ for n odd.*

• *Suppose $n \equiv 1 \mod 4$. If $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$, there exists a* $\mathrm{Spin}(2,n-1)_0$*-pseudo-Riemannian cobordism.*

• *Suppose $n \equiv 7 \mod 8$. There is a* $\mathrm{Spin}(2,n-1)_0$*-pseudo-Riemannian cobordism without any further assumptions.*

The main ingredients of our proof of Theorem B are results of Atiyah [2], Frank [5], Hirzebruch–Hopf [12], Matsushita [22], Thomas [33–39] on the existence of 2-distributions of the tangent bundle of an oriented smooth manifold along with work of Gibbons–Hawking [10], Kervaire [15], Kervaire–Milnor [16], Lusztig–Milnor–Peterson [20] and Smirnov–Torres [27] relating the Euler characteristic of a Spin-cobordism with the Kervaire characteristic of its boundary.

In Sect. 4, we define the corresponding $\mathrm{Spin}(2,n-1)_0$-cobordism groups, which we denote by $\Omega^{\mathrm{Spin}}_{2,n-1}$. We build on Milnor's computations of Spin-groups in low dimensions [23] in order to determine them. Along with Theorem B, the task yields the following depiction of the cobordisms of Definition 1 in terms of simple topological invariants. We also compute the cobordism groups.

Theorem C *Let $\{M_1, M_2\}$ be closed smooth* Spin *n-manifolds.*

• *If $n = 3$, there is a* $\mathrm{Spin}(2,2)_0$*-pseudo-Riemannian cobordism $((W; M_1, M_2))$ if and only if $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$. There is a group isomorphism*

$$\Omega^{\mathrm{Spin}_0}_{2,2} \to \mathbb{Z}/2. \tag{5}$$

• *If $n = 4$, there is a* $\mathrm{Spin}(2,3)_0$*-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ if and only if all the connected components of M_1 and M_2 have trivial Euler characteristic and they have the same signature $\sigma(M_1) = \sigma(M_2)$. There is a group isomorphism*

$$\Omega^{\mathrm{Spin}_0}_{2,3} \to \mathbb{Z}. \tag{6}$$

• *If $n = 6$, there is a* $\mathrm{Spin}(2,5)_0$*-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ if and only if all the connected components of M_1 and M_2 have trivial Euler characteristic. There is a group isomorphism*

$$\Omega^{\mathrm{Spin}_0}_{2,5} \to \{0\}. \tag{7}$$

• *If $n = 7$, there is a* $\mathrm{Spin}(2,6)_0$*-pseudo-Riemannian cobordism* $((W; M_1, M_2), g)$ *without any further assumptions. There is a group isomorphism*

$$\Omega_{2,6}^{\mathrm{Spin}_0} \to \{0\}. \tag{8}$$

The last result to be presented in this introduction contains a myriad of examples of the cobordisms of Definition 1; cf. [27, Corollary G].

Theorem D *Let M be a closed connected smooth n-manifold that Spin-bounds and whose Euler characteristic is zero, let $n \geq 5$. For any finitely presented group G, there exists a closed smooth* Spin *n-manifold $M(G)$ such that $\pi_1 M(G) = G$ and a* $\mathrm{Spin}(2, n-1)_0$*-pseudo-Riemannian cobordism* $((W; M, M(G)), g)$.

The proof of Theorem D follows from Theorem B and a well-known argument to construct closed high-dimensional stably-parallelizable n-manifolds with prescribed fundamental group [13, Theorem A].

We have organized the paper as follows. In Sect. 2 we present some background material for the convenience of the reader. It includes a description of the topological constructions that are used in the paper as well as background existence results on indefinite metrics. A discussion on the co-existence of Spin-structures and indefinite metrics on our cobordisms can be found in Sect. 3. The cobordism groups are defined in Sect. 4. Section 5 contains a comparison between the cobordisms of Definition 1 and Lorentzian cobordisms. Section 6 contains a discussion and open questions on expressing our main results in the language of tangential structures of Galatius–Madsen–Tillmann–Weiss [6] as Ebert did in [4] for Reinhart's work on Lorentzian cobordisms [26]. This section arose from a suggestion of Oscar Randal–Williams. The proofs of our main results are given in Sect. 7. For background results, the reader is directed to Atiyah [2], Chamblin [3], Gibbons–Hawking [10], Milnor [23], O'Neill [25], Reinhart [26], Smirnov-Torres [27], Steenrod [29], Stong [30], Thom [32], Thomas [34–36, 38, 39].

All manifolds in this paper are assumed to be C^∞-smooth and Hausdorff. All pseudo-Riemannian metrics in this paper are assumed to be time-orientable and non-degenerate.

2 Background results

2.1 Kervaire semi-characteristic and spin-structures

The following fundamental invariant of odd-dimensional manifolds was introduced by Kervaire [15].

Definition 9 The Kervaire semi-characteristic of a closed $(2k+1)$-manifold M is

$$\hat{\chi}_{\mathbb{Z}/2}(M) = \sum_{i=0}^{k} b_i(M; \mathbb{Z}/2) \mod 2 \tag{10}$$

where $b_i(M; \mathbb{Z}/2)$ denotes the i^{th} Betti number of M with $\mathbb{Z}/2$-coefficients.

There is a relation between the Euler characteristic of an even-dimensional manifold with boundary and the Kervaire semi-characteristic of its boundary in the presence of a Spin-structure as observed by Geiges [8], Gibbons–Hawking [10], Kervaire [15], Kervaire-Milnor [16], Lusztig–Milnor–Peterson [20] and Smirnov-Torres [27].

Theorem 11 *Let W be a compact even-dimensional manifold with non-empty boundary $\partial W \neq \emptyset$. The identity*

$$\chi(W) + \hat{\chi}_{\mathbb{Z}/2}(\partial W) \equiv 0 \mod 2 \tag{12}$$

holds provided either

- *W is stably-parallelizable [15] or*
- *W is $2q$-dimensional for $q \not\equiv 0 \mod 4$ and admits a* Spin-*structure [8, Lemma 8.2.13], [27, Lemma 6].*

The value of the Kervaire semi-characteristic is independent of the choice of field of coefficients for manifolds that admit a Spin-structure [20]; see [2, Remark p. 16].

2.2 Double of a compact manifold and its Euler characteristic and Kervaire semi-characteristic

Let W be a compact oriented n-manifold with non-empty boundary $\partial W = M$. The double $2W$ of W is the closed smooth oriented n-manifold

$$2W = W \cup_M \overline{W}, \tag{13}$$

where corners have been smoothed out. It can also be described as the boundary

$$2W = \partial(W \times [0, 1]). \tag{14}$$

The computation of the Euler characteristic of (13) and its signature, whenever it is defined, are immediate and we record them.

Lemma 15 *The Euler characteristic of the double (13) is*

$$\chi(2W) = 2\chi(W) - \chi(\partial W). \tag{16}$$

Suppose W is $4q$-dimensional. The signature of $2W$ satisfies

$$\sigma(2W) = 0. \tag{17}$$

A result of Zadeh [41] is useful for the computation of the Kervaire semi-characteristic of (14).

Proposition 18 *Let W be a* Spin *compact $2q-1$ oriented manifold with $q \not\equiv 0 \mod 4$. The Kervaire semi-characteristic of the double $2W$ satisfies*

$$\hat{\chi}_{\mathbb{Z}/2}(2W) = 0. \tag{19}$$

The proof of Proposition 18 follows from Theorem 11 and the fact that $2W$ is the boundary of the Spin-manifold $W \times [0, 1]$.

2.3 Existence of 2-distributions

Let us now discuss existence results of subbundles of the tangent bundle of a manifold that are directly related to the existence of pseudo-Riemannian metrics.

Definition 20 Let W be an oriented n-manifold. A 2-distribution V is a non-singular field of tangent 2-planes, i.e. an oriented 2-plane sub-bundle of TW.

Remark 21 An oriented rank 2 vector bundle over a manifold is determined up to isomorphism by its Euler class [29]. It follows that a 2-distribution on W with trivial Euler class satisfies $\mathrm{Span}(W) \geq 2$, where $\mathrm{Span}(W)$ is the maximum number of everywhere linearly independent and nowhere vanishing vector fields on W.

The following result collects several foundational theorems on the existence of 2-distributions on oriented and Spin-manifolds due to Atiyah [2], Frank [5], Hirzebruch–Hopf [12], Matsushita [22] and Thomas [33–39].

Theorem 22 *Let X be a closed connected oriented n-manifold.*

• *[5], [12, 4.5], [21, Theorem 2], [2, Theorem 3.1]. Suppose $n \equiv 0 \mod 4$ and that the signature satisfies $\sigma(X) = 0$. There is a 2-distribution on X if and only if*

$$\chi(X) \equiv 0 \mod 4.$$

• *[36, Theorem 1.3]. Suppose $n \equiv 1 \mod 4$ and that X admits a* Spin*-structure. There is a 2-distribution on X with Euler class $2v$ for each class $v \in H^2(X;\mathbb{Z})$ if and only if*

$$w_{n-1}(X) = 0 \ \text{and} \ \hat{\chi}_{\mathbb{Z}/2}(X) = 0,$$

where $w_{n-1}(X)$ is the $(n-1)$nth Stiefel–Whitney class of X. Moreover, if X admits a 2-distribution, then $\hat{\chi}_{\mathbb{Z}/2}(X) \equiv 0 \mod 2$ [2, Theorem 4.1].

• *[36, Theorem 1.1, Corollary 1.2]. If $n \equiv 3 \mod 4$, then there is a 2-distribution on X with Euler class $2v$ for each class $v \in H^2(X;\mathbb{Z})$.*

Theorem 22 is a key ingredient in the proofs of our main results. The following technical lemma is used to build a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism whenever a given Spin-cobordism $(W; M_1, M_2)$ satisfies $\chi(W) = \chi(M_1) = \chi(M_2) = 0$ and $\mathrm{Span}(W) \geq 2$.

Lemma 23 *Let $(W; M_1, M_2)$ be a cobordism between closed orientable n-manifolds such that the Euler characteristic of every connected component of W, M_1 and M_2 is trivial. Suppose furthermore that there exist two everywhere linearly independent vector fields $X, Y \in \mathfrak{X}(W)$. Then we can find two everywhere linearly independent vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(W)$ such that $\tilde{X}|_{\partial W}$ is the exterior normal to the boundary ∂W.*

Proof Without loss of generality, we assume the cobordism W to be connected. Indeed, if this is not the case we can just build the vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(W)$ from the statement on each connected component. We start by proving the lemma in the case in which both M_1 and M_2 are connected.

Fix a Riemannian metric g on W. Let $X, Y \in \mathfrak{X}(W)$ be as above and let ν_i be a tubular neighbourhood of M_i inside W for $i = 1, 2$ respectively. In the following, we will make use of the identification $M_i \times [0, 1] \cong \nu_i$, with $M_i \times \{1\}$ corresponding to the submanifold $M_i \subset \nu_i$. Let $\bar{X} \in \mathfrak{X}(W)$ be a vector field satisfying the following conditions:

(1) $\bar{X}|_{W \setminus (\nu_1 \sqcup \nu_2)} = X|_{W \setminus (\nu_1 \sqcup \nu_2)}$, i.e. $\bar{X}$ coincides with X outside the disjoint union of the collar neighbourhoods of the boundary components of W;
(2) $\bar{X}|_{\partial W}$ is the outward-pointing normal vector field;
(3) $\bar{X}$ has finitely many singular points in $\nu_1 \sqcup \nu_2$.

Notice that such a vector field can always be built out of X. We can assume without loss of generality that all the singular points of $\bar{X}$ are contained in the disjoint union of two small closed balls $B_i \subset \nu_i$ for $i = 1, 2$ (see [24, Chapter 4]).

Our first aim is building out of Y a new vector field $\bar{Y} \in \mathfrak{X}(W \setminus (B_1 \cup B_2))$ with the following properties:

(1) $\bar{Y}|_{W\setminus(\nu_1\sqcup\nu_2)} = Y|_{W\setminus(\nu_1\sqcup\nu_2)}$, i.e. $\bar{Y}$ coincides with Y outside the disjoint union of the collar neighbourhoods of the boundary components of W;
(2) $\bar{X}, \bar{Y}$ are everywhere linearly independent on $W \setminus (B_1 \sqcup B_2)$.

Let us build $\bar{Y}$ by first extending the restriction of Y to $W \setminus (\nu_1 \sqcup \nu_2)$ to the complement of a small ball B_1 inside the tubular neighbourhood ν_1, the procedure in ν_2 will be analogous. Without loss of generality, we can suppose $\bar{X}$ to be unitary outside $W\setminus(B_1 \sqcup B_2)$ with respect to the metric g. The vector field $\bar{X}$ defines an isotopy between the linear span of $\bar{X}|_{M_1\times\{0\}}$ inside $TW|_{M_1\times\{0\}}$ and the one of $\bar{X}|_{M_1\times\{t\}}$ inside $TW|_{M_1\times\{t\}} \cong TW|_{M_1\times\{0\}}$ for every t sufficiently small. Such isotopy can be extended to an ambient isotopy of the total space of the bundle $TW|_{M_1\times\{0\}}$ consisting of vector bundle isomorphisms (by adapting the proof of Hirsch's Isotopy extension theorem in [11]). This allows us to define $\bar{Y}$ on $(M_1 \times [0, \bar{t})) \setminus B_1$, where $\bar{t}$ is the maximum $t \in [0, 1]$ such that $B_1 \cap (M_1 \times \{t\}) \neq \emptyset$. In particular, we can suppose without loss of generality that $B_1 \cap (M_1 \times \{\bar{t}\})$ consists of one single point $\{pt\}$. Since $\bar{X}|_{M_1\times\{\bar{t}\}}$ is isotopic to the outward-pointing normal and linearly independent with respect to $\bar{Y}$ on $(M_1 \times \{\bar{t}\})\setminus\{pt\}$, up to a slight perturbation we can regard $\bar{Y}|_{(M_1\times\{\bar{t}\})\setminus\{pt\}}$ as a tangent vector field on $M_1 \setminus \{pt\}$. Since $\chi(M_1) = 0$, Poincaré-Hopf's theorem (see [24, Chapter 6]) allows then to define $\bar{Y}$ globally on $\nu_1\setminus B_1$. Indeed, $\bar{Y}|_{(M_1\times\{\bar{t}\})\setminus\{pt\}}$ can be extended to a tangent vector field to $M_1 \times \{\bar{t}\}$ with an isolated zero at $\{pt\}$ of trivial index, up to scaling the vector field in a neighbourhood of $\{pt\}$ by using a smooth function which is constantly equal to one in a collar of such neighbourhood and vanishes exactly at the point $\{pt\} \in M_1 \times \{\bar{t}\}$. Hence, up to homotopy, we can assume that $\bar{Y}|_{(M_1\times\{\bar{t}\})\setminus\{pt\}}$ is constant in a small neighbourhood of $\{pt\}$ inside $M_1 \times \{\bar{t}\}$ and hence we can extend $\bar{Y}$ first to a tangent vector field to the slice $M_1 \times \{\bar{t}\}$ and then to the whole $\nu_1 \setminus B_1$ as done before in the collar $M_1 \times [0, \bar{t})$.

Once we have built such $\bar{Y} \in \mathfrak{X}(W\setminus(B_1 \cup B_2))$, we can assume (up to composing with a self-diffeomorphism of the cobordism W, which we may assume to be connected) that $\bar{X}, \bar{Y}$ are linearly independent outside a little ball $B \subset \nu_1$. In particular, we can choose such diffeomorphism to fix a small collar of the boundary ∂W and to send both B_1 and B_2 into B. Moreover, we can assume without loss of generality that the restrictions of $\bar{X}, \bar{Y}$ to the complement of such ball B are orthonormal with respect to the fixed Riemannian metric. Poincaré–Hopf's theorem together with the vanishing of the Euler characteristic of W implies that the map

$$\partial B \to S^n$$

sending each point $x \in \partial B$ to $\bar{X}(x)$ is of zero degree and is therefore null-homotopic. Hence, we can suppose that the restriction $\bar{X}|_{\partial B}$ is the constant vector field $\frac{\partial}{\partial t}$ under the identification $\nu_1 \cong M_1 \times [0, 1]$, where t parametrizes the unit interval. In this way, $\bar{X}$ can be extended to the whole interior of B by setting it to be constantly equal to $\frac{\partial}{\partial t}$, defining a global vector field $\tilde{X} \in \mathfrak{X}(W)$ with the desired properties. In particular, $\tilde{X}|_{\partial W} = \bar{X}|_{\partial W}$ coincides with the outward-pointing normal by construction. Moreover, we can assume without loss of generality that the projection on the unit interval $[0, 1]$ parametrizing the collar ν_1 is a height function with just one maximum t_M and one minimum t_m for the sphere ∂B. Up to isotopy, $\bar{Y}|_{M_1\times\{t\}}$ can be seen as a tangent vector field to $M_1 \times \{t\}$ for any $t \in [t_m, t_M]$. The fact that $\chi(M_1) = 0$ will then allow us to extend $\bar{Y}$ to a vector field $\tilde{Y} \in \mathfrak{X}(W)$ by using Poincaré-Hopf's theorem, as it was already done in the construction of $\bar{Y}$ inside $\nu_1\setminus B_1$. This finishes the proof of Lemma 23 in the case where M_1 and M_2 are connected.

In the case in which M_1 and M_2 are not necessarily connected, the vector field $\bar{X} \in \mathfrak{X}(W)$ is built in the same way as in the case where these manifolds are connected. Denote by

$\{M_{i,j} : j = 1, 2, \ldots, k_i\}$ the connected components of M_i for $i = 1, 2$. Without loss of generality, the singular points of $\bar{X} \in \mathfrak{X}(W)$ are now contained in the union of closed balls $\{B_{i,j} \subset \nu(M_{i,j}) : j = 1, \ldots, k_i\}$ for $i = 1, 2$. For the purpose of defining the vector field $\bar{Y}$ on the complement of these balls, it is enough to apply Poincaré–Hopf's theorem to each connected component $M_{i,j}$ using our hypothesis that $\chi(M_{i,j}) = 0$. In order to build the vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(W)$, we collect all points at which $\bar{X}, \bar{Y}$ are not linearly independent inside a closed ball $B \subset \nu(M_{1,1})$ and proceed as it was done in the connected case. This concludes the proof of Lemma 23. □

2.4 Existence of indefinite metrics

We now recall some basic definitions and existence results of pseudo-Riemannian metrics with non-trivial signature on manifolds. It is well known that the existence of a Lorentzian metric on a manifold is equivalent to the existence of a nowhere vanishing vector field, i.e. a nowhere vanishing section of its tangent bundle [25], [27, Lemma 1]. Sub-bundles of the tangent bundle of a manifold yield other indefinite metrics [25, 29], and the role in this note of the existence results on 2-distributions that were described in Sect. 2.3 is explained in the following lemma; cf. [29].

Lemma 24 *Let W be a n-manifold. There is a pseudo-Riemannian metric (W, g) of signature (p, q) if and only if there is a decomposition of the tangent bundle*

$$TW = \xi \oplus \eta, \tag{25}$$

where ξ and η are vector sub-bundles of rank p and q, respectively. Moreover, these sub-bundles can be chosen such that ξ is time-like and η is space-like.

In particular, there is a non-singular indefinite metric of signature $(2, n-2)$ on a n-manifold W if and only if there is a 2-distribution $V \subset TW$.

The vector sub-bundle ξ from the decomposition (25) is called a time-like sub-bundle of maximal rank. Recall that a pseudo-Riemannian manifold (W, g) of signature (p, q) is called time-orientable if there exists an orientable time-like sub-bundle of maximal rank.

The following result will be used to construct a non-singular pseudo-Riemannian metric of signature $(2, n-2)$ on the double $2W$ whenever there is such a metric on W, which restricts to a Lorentzian metric on ∂W.

Theorem 26 *Let W be a n-manifold with non-empty boundary ∂W. The following statements are equivalent.*

(1) *There is an indefinite metric (W, g) of signature $(2, n-2)$ such that its restriction to the boundary $g|_{\partial W}$ is a Lorentzian metric;*
(2) *There is a rank 2 sub-bundle $\xi \subset TW$ and a line bundle $\eta' \subset \xi|_{\partial W}$ such that η' is transversal to the zero section ∂W of $\xi|_{\partial W}$;*
(3) *There is a rank 2 sub-bundle $\xi \subset TW$ and a nowhere vanishing section $\nu : \partial W \to \xi|_{\partial W}$ that is everywhere outward pointing.*

Proof We begin by showing that the implication (1)$\Rightarrow$ (2) holds. Lemma 24 implies the existence of a decomposition

$$TW = \xi \oplus \eta$$

with $\xi, \eta \subset TW$ sub-bundles of rank 2 and $n-2$ (respectively), and such that g is negative-definite on ξ and positive-definite on η. Since $g' := g\big|_{\partial W}$ is a Lorentzian metric by hypothesis, Lemma 24 implies the existence of a further decomposition

$$T\big(\partial W\big) = \tilde{\xi} \oplus \tilde{\eta}$$

where $\tilde{\xi}$ is a time-like rank 1 vector sub-bundle of $T\big(\partial W\big)$, while $\tilde{\eta}$ can be chosen to be space-like.

We will prove that the map

$$p' : \tilde{\xi} \to \xi\big|_{\partial W}$$

given by the composing the inclusion of the bundle $\tilde{\xi}$ in TW with the projection

$$p : TW = \xi \oplus \eta \to \xi.$$

is a bundle monomorphism. Let $x \in \partial W$ and consider a non-zero vector $v \in \tilde{\xi}_x$. We will show that $w := p'(v) \in \xi_x$ is non-zero as well. Indeed, suppose that $w \in \xi_x$ is the zero vector. Then (after composing with the inclusion $\tilde{\xi}_x \subset TW$) we would have $v \in \eta_x$ and thus $g|_{T_x W}(v, v) > 0$. This is a contradiction.

Then, the image of $\tilde{\xi}$ under the bundle map $p' : \tilde{\xi} \to \xi|_{\partial W}$ defines a line sub-bundle $\xi_1 \subset \xi\big|_{\partial W}$, yielding a further decomposition

$$\xi\big|_{\partial W} = \xi_1 \oplus \xi_2.$$

In order to finish the proof of the implication (1)⇒(2), it will be enough to prove the transversality condition $\xi_2 \pitchfork \partial W$. We proceed by contradiction and suppose that there is a point $x \in \partial W$ such that $(\xi_2)_x \subset T_x \partial W$. This implies that $(\xi_2)_x = \langle v \rangle$ is generated by a non-zero vector

$$v = v_1 + v_2 \in \tilde{\xi}_x \oplus \tilde{\eta}_x$$

where $v_1 \in \tilde{\xi}_x$ and $v_2 \in \tilde{\eta}_x$. Since p' is an isomorphism and $v \notin (\xi_1)_x$, we have that $v_1 = 0$, while the fact of v being time-like implies that also v_2 is trivial, a contradiction. This concludes the proof that the implication (1)⇒ (2) holds.

We now prove that the implication (2)⇒ (3) holds. Let $\hat{n} : \partial W \to TW\big|_{\partial W}$ be an outward-pointing vector field. For all $x \in \partial W$, there is a decomposition

$$T_x W = T_x\big(\partial W\big) \oplus \mathrm{Span}\big(\hat{n}(x)\big).$$

Moreover, from the decomposition $T_x W = T_x(\partial W) \oplus \eta'_x$ we have that

$$\hat{n}(x) = v_x + w_x,$$

where $v_x \in T_x\big(\partial W\big)$ and $w_x \in \eta'_x$. Note that the condition of $\hat{n}$ being outward pointing trivially implies that $w_x \neq 0$ for all $x \in \partial W$. In particular, the map

$$n : x \mapsto w_x$$

defines a nowhere vanishing section of $\xi\big|_{\partial W}$. Since $w_x \notin T_x\big(\partial W\big)$ at every point by construction, we have that n is either everywhere outward pointing or everywhere inward pointing. In the latter case, we can simply consider $-n$ as the desired section. This concludes the proof that the implication (2)⇒ (3) holds.

To finish the proof of Theorem 26, we proceed to show that the implication (3) ⇒ (1) holds. We can use the given 2-distribution ξ to construct a pseudo-Riemannian metric g on

Table 1 Summary on Span X

$n = \dim X$	Necessary and sufficient conditions for $\mathrm{Span}(X) \geq 2$
$n \equiv 0 \mod 4$	$\sigma(X) \equiv 0 \mod 4$, $\chi(X) = 0$
$n \equiv 1 \mod 4$	$w_{n-1}(X) = 0$, $\hat{\chi}_{\mathbb{R}}(X) = 0$
$n \equiv 2 \mod 4$	$\chi(X) = 0$
$n \equiv 3 \mod 4$	Span $X \geq 2$ is always true

W of signature $(2, n-2)$ with $\xi \subset TW$ being time-like. We consider also a rank 2 space-like sub-bundle $\eta \subset TW$ complementary to ξ. The everywhere outward-pointing vector field $n : \partial W \to \xi\big|_{\partial W}$ spans a trivial sub-bundle $\xi_1 \subset \xi\big|_{\partial W}$. Hence, we have a decomposition $\xi\big|_{\partial W} = \xi_1 \oplus \xi_2$. Clearly, we may assume that $\xi_2 \subset T\big(\partial W\big)$. We thus have a decomposition

$$T\big(\partial W\big) = \xi_2 \oplus \eta\big|_{\partial W};$$

with ξ_2 time-like and $\eta|_{\partial W}$ space-like. Such a decomposition allows us to get the desired Lorentzian metric on ∂W of Item (3). $\square$

At this point, we are able to formulate the following consequence of Lemma 23 and Theorem 26 that will be useful for our purposes.

Corollary 27 *Let $(W; M_1, M_2)$ be a* Spin*-cobordism between closed* Spin *n-manifolds. The following statements are equivalent.*

- *There is an indefinite metric g such that $((W; M_1, M_2), g)$ is a* $\mathrm{Spin}(2, n-1)_0$*-pseudo-Riemannian cobordism and g is induced by a 2-distribution of trivial Euler class;*
- *There exist everywhere linearly independent vector fields $X, Y \in \mathfrak{X}(W)$ and the Euler characteristic of every connected component of W, M_1 and M_2 is trivial.*

We will make use of the necessary and sufficient conditions for a closed n-manifold X to admit two everywhere linearly independent vector fields, with the purpose of building a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism out of a Spin-cobordism as in Corollary 27. These conditions have been obtained by Thomas [39, Table 2, pp. 652] and are summarized in Table 1.

Proposition 28 *Let W be an orientable n-manifold with non-empty boundary $\partial W = M_1 \sqcup M_2$, where the Euler characteristic of every connected component of M_1 and M_2 is trivial.*

- *Suppose that $n \not\equiv 3 \mod 4$ and that the Euler characteristic of every connected component of W is trivial. If $n \equiv 1 \mod 4$, assume further that W admits a* Spin*-structure. There are non-singular Lorentzian metrics $(2W, g^L)$ and (W, g^L_W) such that the latter restricts to a Riemannian metric on ∂W. Moreover, there are indefinite metrics $(2W, g)$ and (W, g_W) of signature $(2, n-2)$ that arise from a 2-distribution with trivial Euler class and such that the restriction of g_W to ∂W yields a non-singular Lorentzian metric.*
- *If $n \equiv 3 \mod 4$, such pseudo-Riemannian metrics exist on $2W$ and W without any other assumptions.*

The proof of Proposition 28 is a straight-forward application of results of Reinhart [26, Theorem 1], Lemma 23 and Table 1.

3 Spin structures on pseudo-Riemannian cobordisms and their structure groups

Let us justify now the notation $\mathrm{Spin}(2, n-1)_0$ in our definition. Let X be a orientable compact n-manifold. If X admits a pseudo-Riemannian metric (X, g) of signature (p, q) with $p+q=n$, then it is well known that the structure group of its tangent bundle TX can be reduced to $O(p, q)$ [29], where $O(p, q)$ is the indefinite orthogonal group of signature (p, q)

$$O(p, q) = \{A \in \mathbb{R}^{n\times n} \mid A^T I_{p,q} A = I_{p,q}\}, \tag{29}$$

and $I_{p,q}$ is the diagonal matrix with the first p diagonal entries equal -1 and the last q diagonal entries equal $+1$. It is easy to see that if $A \in O(p, q)$, then $\det A = \pm 1$. In particular, if (X, g) is orientable, the structure group can be further reduced to $SO(p, q)$, namely to the group of indefinite orthogonal matrices of signature (p, q) with positive determinant. Such group has two connected components [7]; the connected component of the identity is denoted by $SO(p, q)_0$. One can prove that the structure group of the tangent bundle TX can be reduced to $SO(p, q)_0$ if (X, g) is time-orientable [7]. As for the definite case, we have a double cover of $SO(p, q)_0$ by the $\mathrm{Spin}(p, q)_0$ group (see [19] for the definitions) which satisfies a short exact sequence of groups

$$0 \to \mathbb{Z}/2 \to \mathrm{Spin}(p, q)_0 \xrightarrow{\mathrm{Ad}} SO(p, q)_0 \to 0. \tag{30}$$

Definition 31 Let X be a orientable compact n-manifold with a time-oriented pseudo-Riemannian metric g of signature (p, q), where $p+q=n$. Let $\pi_{SO} : F(X) \to X$ be the principal $SO(p, q)_0$-bundle of oriented orthonormal frames of (X, g). A $\mathrm{Spin}(p, q)_0$-structure on X is a principal $\mathrm{Spin}(p, q)_0$-bundle $\pi_{\mathrm{Spin}} : \tilde{F}(X) \to X$ with a 2-fold cover $\Lambda : \tilde{F}(X) \to F(X)$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Spin}(p, q)_0 & \longrightarrow & \tilde{F}(X) & \longrightarrow & X \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \mathrm{Id}} \\ SO(p, q)_0 & \longrightarrow & F(X) & \xrightarrow{\pi_{SO}} & X \end{array} \tag{32}$$

commutes.

The existence of a $\mathrm{Spin}(p, q)_0$-structure under these hypotheses does not depend on the metric. The obstruction for it is merely topological, as the following result exhibits.

Lemma 33 *Let X be a compact orientable n-manifold X with a time-orientable pseudo-Riemannian metric of signature (p, q), with $p+q=n$. Then X admits a* $\mathrm{Spin}(p, q)_0$*-structure if and only if it admits a* $\mathrm{Spin}(n)$*-structure.*

Proof In the presence of a pseudo-Riemannian metric of signature (p, q) on X, there is a splitting of its tangent bundle $TX = \xi \oplus \eta$. A result of Karoubi [14, Proposition 1.1.26] says that the principal $SO(p, q)_0$-bundle $F(X)$ defined above admits a lift to $\tilde{F}(X)$ as in (32) if and only if

$$w_1(\xi) + w_1(\eta) = 0 \qquad \text{and} \qquad w_2(\xi) + w_2(\eta) = 0. \tag{34}$$

The claim follows from the properties of Stiefel–Whitney classes. □

Corollary 35 *Let $((W; M_1, M_2), g)$ be an $SO(2, n-1)_0$-pseudo-Riemannian cobordism. Then $((W; M_1, M_2), g)$ is a* $\mathrm{Spin}(2, n-1)_0$*-pseudo-Riemannian cobordism if and only if W admits a* $\mathrm{Spin}(2, n-1)_0$*-structure.*

4 Spin pseudo-Riemannian cobordism groups

We now discuss the definition of the $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism groups that appear in the statement of Theorem C. For each integer $n \geq 1$, let $\mathcal{A}_n$ be the set of diffeomorphism classes of closed Spin n-manifolds with the property that all of their connected components have vanishing Euler characteristic. Define the following relation in $\mathcal{A}_n$.

Definition 36

$$M_1 \sim M_2 \qquad \text{if and only if} \qquad \{M_1, M_2\} \text{ are } \mathrm{Spin}(2, n-1)_0\text{-cobordant.}$$

Proposition 37 *The relation in Definition 36 defines an equivalence relation on $\mathcal{A}_n$.*

Lemma 24 is a key ingredient in the proof of Proposition 37, as it gives us tools to glue different $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordisms with mutual boundary connected components.

Definition 38 The $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism group is the set of the equivalence classes of the relation in Definition 36 equipped with the disjoint union as group product and it is denoted by $\Omega^{\mathrm{Spin}_0}_{2,n-1}$.

The reader might have already noticed that the group operation in Definition 38 cannot be the connected sum of M_1 and M_2 as it is the case in cobordisms of other flavors. Notice that if M_1 and M_2 are even-dimensional Lorentzian manifolds, their connected sum $M_1 \# M_2$ does not admit a non-singular Lorentzian metric.

The main result of this section is the following theorem.

Theorem 39 *The* $\mathrm{Spin}(2, n-1)_0$*-pseudo-Riemannian groups $\Omega^{\mathrm{Spin}_0}_{2,n-1}$ are abelian groups. Moreover, the Cartesian product of manifolds yields a graded ring structure*

$$\Omega^{\mathrm{Spin}_0}_{2,*-1} = \bigoplus_{n=1}^{\infty} \Omega^{\mathrm{Spin}_0}_{2,n-1}. \tag{40}$$

The proof of Theorem 39 is a straightforward adaptation of well-known arguments in cobordism theory [17, 23, 26, 30, 40]. We refer to the ring (40) as the $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism ring.

5 Comparison of pseudo-Riemannian cobordisms

In this section, we draw a comparison between the pseudo-Riemannian cobordisms of Definition 1 and $\mathrm{Spin}(1, n-1)_0$-Lorentzian cobordisms. The contrast between these objects sheds light on the topological restrictions imposed by the coexistence of the Spin-structure with the pseudo-Riemannian structure of the cobordism. We first recall the definition of a $\mathrm{Spin}(1, n-1)_0$-Lorentzian cobordism.

Definition 41 A Lorentzian cobordism between closed n-manifolds M_1 and M_2 is a pair

$$((\widehat{W}; M_1, M_2), g^L) \tag{42}$$

that consists of

(A) a cobordism $(\widehat{W}; M_1, M_2)$,

(B.2) a non-singular Lorentzian metric $(\widehat{W}, g^L)$ with a time-like line field V,

(C.2) and the boundary $\partial \widehat{W} = M_1 \sqcup M_2$ is space-like, i.e. $(M_1, g^L|_{M_1})$ and $(M_2, g^L|_{M_2})$ are Riemannian manifolds, where $g^L|_{M_i}$ is the restriction of g^L to M_i.

If the cobordism $(\widehat{W}; M_1, M_2)$ of Item (A) is a Spin-cobordism, we say that the cobordism (42) is a $\mathrm{Spin}(1, n-1)_0$-Lorentzian cobordism.

We keep the discussion at a three-dimensional level for the sake of brevity, although similar comparisons apply to any dimension. More precisely, we pivot the comparison and the discussion of the topological restrictions on the following result.

Theorem 43 *Gibbons–Hawking [10], Smirnov–Torres [27]. Let $\{M_1, M_2\}$ be closed oriented 3-manifolds. The following conditions are equivalent.*

(1) There exists a $\mathrm{Spin}(1, 3)_0$*-Lorentzian cobordism*

$$((\widehat{W}; M_1, M_2), g),$$

where $(\widehat{W}; M_1, M_2)$ is a Spin*-cobordism.*

(2) Their Kervaire semi-characteristics satisfy $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$.

(3) There exists a cobordism $(\widehat{W}'; M_1, M_2)$, where $\widehat{W}'$ is a parallelizable manifold with trivial Euler characteristic.

There is a group isomorphism

$$\Omega_{1,3}^{\mathrm{Spin}_0} \to \mathbb{Z}/2. \tag{44}$$

While the existence of a Spin-cobordism imposes no restrictions on the boundary 3-manifold since the third cobordism group Ω_3^{Spin} is trivial [23], the presence of the required Lorentzian metric on $\widehat{W}$ forces its Euler characteristic to be $\chi(\widehat{W}) = 0$ [26]. For the latter structure to co-exist with the Spin-cobordism, the Kervaire semi-characteristic of the boundary 3-manifolds must coincide as indicated by Theorem 11. This invariant gives us the isomorphism (44).

The corresponding statement for $\mathrm{Spin}(2, 2)_0$-pseudo-Riemannian cobordisms is the following.

Theorem 45 *Let $\{M_1, M_2\}$ be closed oriented 3-manifolds. The following conditions are equivalent*

(1) *There exists a* $\mathrm{Spin}(2, 2)_0$*-pseudo-Riemannian cobordism*

$$((W; M_1, M_2), g).$$

(2) *Their Kervaire semi-characteristics satisfy $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$.*

(3) *There exists a cobordism $(W'; M_1, M_2)$, where W' is a parallelizable manifold.*

There is a group isomorphism

$$\Omega_{2,2}^{\mathrm{Spin}_0} \to \mathbb{Z}/2. \tag{46}$$

The proof of Theorem 45 is given in Sect. 7.1. The reader will notice that while both 4-manifolds $\widehat{W}$ and W in Theorems 43 and 45 are parallelizable, the Euler characteristic of the cobordism W need not be zero. The following example is illustrative of the situation.

Example 47 The product of a 2-disk with the round 2-sphere admits a Spin-structure as well as an indefinite metric of signature (2, 2)

$$(D^2, -dr^2 - r^2 d\theta^2) \times (S^2, g_{S^2}) \tag{48}$$

that restrict to a Spin-structure and a Lorentzian metric on the boundary

$$(S^1, -d\theta^2) \times (S^2, g_{S^2}) \tag{49}$$

as indicated in Example A. The Euler characteristic of (48) is $\chi(D^2 \times S^2) = 2$ and it does not admit a Lorentzian metric that restricts to a Riemannian metric on (49). The connected sum

$$\widehat{W} = (D^2 \times S^2)\#(S^1 \times S^3)$$

on the other hand, does support both kinds of non-singular pseudo-Riemannian metrics as well as Spin-structures that restrict to (49).

The phenomenon displayed in Example 47 occurs for all oriented 3-manifolds.

Corollary 50 *Let M_1 and M_2 be closed oriented 3-manifolds. There is a* $\mathrm{Spin}(1,3)_0$*-Lorentzian cobordism if and only if there is a* $\mathrm{Spin}(2,2)_0$*-pseudo-Riemannian cobordism.*

We now elucidate on the reason behind the difference in the obstructions. The existence of a 2-distribution is not equivalent to the existence of a pair of linearly independent and nowhere vanishing vector fields as shown in the work of Atiyah [2], Frank [5], Hirzebruch-Hopf [12], Matsushita [22], Svane [31], and Thomas [33–39] among many others. The existence of a pair of linearly independent vector fields on a manifold requires for its Euler characteristic to vanish in the even-dimensional case and for its Kervaire semi-characteristic to vanish in the odd-dimensional case.

We finish this section with an amendment to [27, Corollary E]. The correct statement is as follows.

Corollary 51 *Let M_1 and M_2 be closed* Spin *4-manifolds. There is a* $\mathrm{Spin}(1,4)_0$*-Lorentzian cobordism* $((W; M_1, M_2), g^L)$ *if and only if* $\chi(M_1) = \chi(M_2)$ *and* $\sigma(M_1) = \sigma(M_2)$.

6 Tangential structures and spectrum

This section arose from a suggestion of Oscar Randal–Williams to the first named author of this note. We record and elaborate on a plausible extension of results of Ebert for future work. Ebert [4, Appendix A] expressed Reinhart's work on Lorentzian cobordisms in the terms of tangential structures in the sense of Galatius–Madsen–Tillmann–Weiss [6, Section 5]. In particular, Ebert [4, Theorem A1] has shown that the Reinhart Lorentzian cobordism groups [26] are isomorphic to the cobordism group $\pi_0(MTSO(n))$.

Once a comparison with the Spin Lorentzian cobordisms of Gibbons–Hawking [10] and Smirnov–Torres [27] is drawn, the following question arises immediately.

Question 1 *Can the* $\mathrm{Spin}(1,n)_0$*-Lorentzian cobordism studied by Gibbons–Hawking and Smirnov-Torres be phrased in terms of tangential structures and, if so, are the corresponding Spin Lorentzian groups* $\Omega_{1,n}^{Spin}$ *isomorphic to the connected component of the Spin version of the spectrum* $\pi_0(MTSpin(n+1))$*?*

An explanation on the construction of the Spin version of the Thom spectrum $MSpin(n)$ can be found in work of Svane [31, Remark 1.15]. A definition of the Madsen–Tillmann–Weiss spectrum $TMSO(n)$ is found in [4, Section 2.4]. Given the topic of this paper, we state the following version of Question 1 due to Randal-Williams.

Question 2 *Randal–Williams. Is it possible to express the existence results on* $\mathrm{Spin}(2,n-1)_0$*-pseudo-Riemannian cobordism obtained in this note in the language of tangential structures?*

An answer to Question 2 in the affirmative would suggest further study on the corresponding cobordism groups. The proposed extensions of Ebert's results provide answers to Question 1 and Question 2.

7 Proofs

The proofs of the results that are mentioned in the introduction have the following structure.

7.1 Proof of Theorem 45

We first show the equivalence (1) $\Leftrightarrow$ (2) and begin with (1) $\Leftarrow$ (2). Suppose M_1 and M_2 are two closed oriented 3-manifolds whose Kervaire semi-characteristics satisfy $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2)$. As it was mentioned in the previous section, the third Spin-cobordism group is $\Omega_3^{\mathrm{Spin}} = \{0\}$ and there is a (connected) Spin-cobordism $(\widehat{W}; M_1, M_2)$. Theorem 11 then implies that

$$\chi(\widehat{W}) + \hat{\chi}_{\mathbb{Z}/2}(\partial\widehat{W}) \equiv 0 \mod 2. \tag{52}$$

Thus, we have that

$$\chi(\widehat{W}) \equiv 0 \mod 2. \tag{53}$$

Take connected sums of $\widehat{W}$ with copies of $S^1 \times S^3$ and $S^2 \times S^2$ and obtain a manifold

$$W := \widehat{W} \# k_1(S^1 \times S^3) \# k_2(S^2 \times S^2)$$

which has zero Euler characteristic, by choosing k_1 and k_2 appropriately. Proposition 28 allows us to conclude the proof of the implication. To show that the implication (1) $\Rightarrow$ (2) holds, we proceed as follows. Assume that there is a $\mathrm{Spin}(2,2)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ and consider the closed 4-manifold with the indefinite metric $(2W, g_{2W})$ given as the double of (W, g). Applying Theorem 22 to $2W$ we get that $\chi(W) = 0$ mod 2 and the Kervaire semi-characteristics of M_1 and M_2 coincide by (52). We conclude that the equivalence (1) $\Leftrightarrow$ (2) holds. The implication (1) $\Rightarrow$ (3) follows from W being an almost parallelizable manifold with non-empty boundary. Such a manifold is stably-parallelizable, and hence parallelizable [16], [18, §7, §8]. The implication (3) $\Rightarrow$ (2) follows from Theorem 11.

The Kervaire semi-characteristic yields an isomorphism

$$\hat{\chi}_{\mathbb{Z}/2} : \Omega_{2,2}^{\mathrm{Spin}} \to \mathbb{Z}/2 \tag{54}$$

and its generator is S^3; see [27, Theorem C]. □

7.2 Proof of Theorem B

Let us begin by studying the case $n \equiv 0 \mod 2$ and let $(W; M_1, M_2)$ be a Spin-cobordism (which we can assume to be connected), where M_1 and M_2 are such that all their connected components have trivial Euler characteristic. Since

$$2\chi(W) = \chi(2W) + \chi(\partial W) = 0$$

by Lemma 15, Proposition 28 implies that there is a metric g on W for which $((W; M_1, M_2), g)$ is a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism. Conversely, if $((W; M_1, M_2), g)$ is a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism, then the pseudo-Riemannian metric g restricts to a Lorentzian metric on $\partial W = M_1 \sqcup M_2$ and we have that all the connected components of M_1 and M_2 have trivial Euler characteristic.

We now address the cases $n \equiv 1, 3, 5 \mod 8$. If $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2) \mod 2$, there is a $\mathrm{Spin}(1, n)_0$-Lorentzian cobordism $((W; M_1, M_2), g^L)$ by [27, Theorem D], where W has connected com ponents $W_1, \ldots, W_k$. In particular, we have that for all $i = 1, \ldots, k$

$$\chi(2W_i) = 2\chi(W_i) = 0.$$

Being M_1 and M_2 odd-dimensional manifolds, we also have that all their connected components have trivial Euler characteristic and we can apply Proposition 28 to obtain a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$. Suppose now that $((W; M_1, M_2), g)$ is a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism and let us restrict to the case $n \equiv 3 \mod 8$. We have that $\chi(W) \equiv 0 \mod 2$ by Theorem 22. Theorem 11 allows us to conclude that $\hat{\chi}_{\mathbb{Z}/2}(M_1) = \hat{\chi}_{\mathbb{Z}/2}(M_2) \mod 2$.

Consider the case $n \equiv 7 \mod 8$. By [27, Theorem D] there is a $\mathrm{Spin}(1, n)_0$-Lorentzian cobordism $((W; M_1, M_2), g^L)$. A result of Reinhart [26] implies that each connected component of W must have trivial Euler characteristic. Hence, Proposition 28 implies the existence of a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$. □

Remark 55 The reason for the absence of a complete characterization in the case $n \equiv 1 \mod 4$ of Theorem B is essentially due to the lack of existence results in literature of distributions of tangent 2-planes on closed $4q + 2$-dimensional manifolds. In particular, it is not clear whether the existence of a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism implies the existence of a $\mathrm{Spin}(1, n)_0$-Lorentzian cobordism. Thus, the missing implication. By inspecting the proof of Theorem B, one realizes that solving this issue is equivalent to giving an answer to the following open question.

Question *Is it possible to find two closed n-manifolds M_1, M_2 which are* $\mathrm{Spin}(2, n-1)_0$*-pseudo-Riemannian cobordant and such that the Euler class of the* 2*-distribution inducing the pseudo-Riemannian metric on the cobordism is necessarily non-trivial?*

On the other hand, from the proof of Theorem B we get that we may ask for $\mathrm{Span}(W) \geq 2$ for a $\mathrm{Spin}(2, n-1)_0$-pseudo-Riemannian cobordism whenever $n \not\equiv 1 \mod 4$. This yields the following corollary.

Corollary E *Let n be a positive integer with $n \not\equiv 1 \mod 4$. Let $\{M_1, M_2\}$ be two closed connected* $\mathrm{Spin}(2, n-1)_0$*-pseudo-Riemannian cobordant n-manifolds. There exists a* $\mathrm{Spin}(n+1)$*-manifold W and two everywhere linearly independent vector fields $X, Y \in \mathfrak{X}(W)$ such that*

- *$(W; M_1, M_2)$ is a* Spin*-cobordism;*
- *X is interior normal to M_1 and exterior normal to M_2.*

In particular, the existence of the Spin-cobordism $(W; M_1, M_2)$ and a nowhere vanishing vector field $X \in \mathfrak{X}(W)$ as above is equivalent to the existence of a $\mathrm{Spin}(1, n)_0$-Lorentzian cobordism $((W; M_1, M_2), g)$ (see [27]).

7.3 Proof of Theorem C

The three-dimensional case has been addressed in Theorem 45. We argue the four-dimensional case first; cf. [27, Proof of Corollary E]. Suppose $((W; M_1, M_2), g)$ is a $\mathrm{Spin}(2, 3)_0$-pseudo-Riemannian cobordism. Since $(W; M_1, M_2)$ is a Spin-cobordism, it is in particular an oriented cobordism and therefore we have the condition $\sigma(M_1) = \sigma(M_2)$. Moreover, the existence of a Lorentzian metric on M_i for $i = 1, 2$ implies that the Euler characteristic of each connected component of M_i is trivial. To prove the converse, we argue as follows. If M_1 and M_2 are closed Spin 4-manifolds with the same signature, there is a Spin-cobordism $(W; M_1, M_2)$ [32], [17, Chapter VIII]. The existence of an indefinite $(2, 3)$-metric on W restricting to a Lorentzian one on the boundary follows from the vanishing of the Euler characteristics of the connected components of M_1 and M_2 and Theorem B. The group isomorphism

$$\Omega^{\mathrm{Spin}}_{2,3} \to \mathbb{Z}$$

is given by the signature. In particular, the map $[M] \mapsto \sigma(M)$ is well defined, being the signature an oriented cobordism invariant.

Let us address now the six-dimensional case. The existence of a $\mathrm{Spin}(2, 5)_0$-pseudo-Riemannian cobordism $((W; M_1, M_2), g)$ implies that the Euler characteristic of all the connected components of M_1 and M_2 is trivial [32]. The converse follows from results of Milnor and Thomas. Since the sixth cobordism group is $\Omega^{\mathrm{Spin}}_6 = \{0\}$ [23], we know that there is a Spin-cobordism $(W; M_1, M_2)$. The conclusion follows again from Theorem B. The group $\Omega^{\mathrm{Spin}_0}_{2,5}$ is hence trivial.

In the seven-dimensional case, Milnor observed that $\Omega^{\mathrm{Spin}}_7 = \{0\}$ [23, p. 201] and any two closed Spin 7-manifolds bound a Spin 8-manifold. The existence of a $\mathrm{Spin}(2, 6)_0$-pseudo-Riemannian cobordism follows from Theorem B, and the group $\Omega^{\mathrm{Spin}_0}_{2,6}$ is hence trivial. □

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