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Derived Equivalences for the Flops of Type C_2 and A_4^G via Mutation of Semiorthogonal Decomposition

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Abstract
We give a new proof of the derived equivalence of a pair of varieties connected by the flop of type C_2 in the list of Kanemitsu (2018), which is originally due to Segal (Bull. Lond. Math. Soc., **48** (3) 533–538, 2016). We also prove the derived equivalence of a pair of varieties connected by the flop of type A_4^G in the same list. The latter proof follows that of the derived equivalence of Calabi–Yau 3-folds in Grassmannians Gr(2, 5) and Gr(3, 5) by Kapustka and Rampazzo (Commun. Num. Theor. Phys., **13** (4) 725–761 2019) closely.

Keywords Calabi–Yau manifolds · Flops and derived categories · Mutation of semiorthogonal decomposition

Mathematics Subject Classification (2010) 14J32

1 Introduction

Let G be a semisimple Lie group and B a Borel subgroup of G. For distinct maximal parabolic subgroups P and Q of G containing B, three homogeneous spaces G/P, G/Q, and $G/(P \cap Q)$ form the following diagram:

$$\begin{array}{ccc} & \mathbf{F} := G/(P \cap Q) & \\ \swarrow^{\varpi_-} & & \searrow^{\varpi_+} \\ \mathbf{P} := G/P & & \mathbf{Q} := G/Q \end{array}$$

We write the hyperplane classes of $\mathbf{P}$ and $\mathbf{Q}$ as h and H respectively. By abuse of notation, the pull-back to $\mathbf{F}$ of the hyperplane classes h and H will be denoted by the same symbol.

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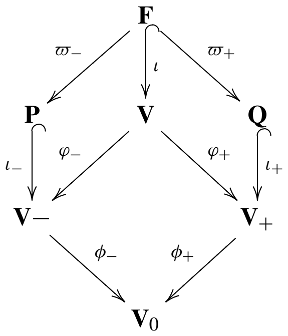
The morphisms ϖ_- and ϖ_+ are projective morphisms whose relative $\mathcal{O}(1)$ are $\mathcal{O}(H)$ and $\mathcal{O}(h)$ respectively. We consider the diagram



(1.1)

where

- $\mathbf{V}_-$ is the total space of $((\varpi_-)_*\mathcal{O}(h+H))^\vee$ over $\mathbf{P}$,
- $\mathbf{V}_+$ is the total space of$((\varpi_+)_*\mathcal{O}(h+H))^\vee$ over $\mathbf{Q}$,
- $\mathbf{V}$ is the total space of $\mathcal{O}(-h-H)$ over $\mathbf{F}$,
- ι_-, ι_+, and ι are the zero-sections,
- φ_- and φ_+ are blow-ups of the zero sections, and
- ϕ_- and ϕ_+ are the affinizations which contract the zero sections.

If $\mathbf{V}_-$ and $\mathbf{V}_+$ have the trivial canonical bundles, then one expects from [4, Conjecture 4.4] or [16, Conjecture 1.2] that $\mathbf{V}_-$ and $\mathbf{V}_+$ are derived-equivalent.

When G is the simple Lie group of type G_2, Ueda [24] used sequence of mutations of semiorthogonal decompositions of $D^b(\mathbf{V})$ obtained by applying Orlov's theorem [20] to the diagram Eq. 1.1 to prove the derived equivalence of $\mathbf{V}_-$ and $\mathbf{V}_+$. This sequence of mutations in turn follows that of Kuznetsov [18] closely.

In this paper, by using the same method, we give a new proof to the following theorem, which is originally due to Segal [22], where the flop was attributed to Abuaf:

Theorem 1.1 *Varieties connected by the flop of type C_2 are derived-equivalent.*

The term *the flop of type C_2* was introduced in [13], where simple K-equivalent maps in dimension at most 8 were classified. There are several ways to prove Theorem 1.1. In [22], Segal showed the derived equivalence by using tilting vector bundles. Hara [8] constructed alternative tilting vector bundles and studied the relation between functors defined by him and Segal.

The flop of type A^G_{2r-2} is also in the list of Kanemitsu [13]. It connects $\mathbf{V}_-$ and $\mathbf{V}_+$ for $\mathbf{P} = \mathrm{Gr}(r-1, 2r-1)$ and $\mathbf{Q} = \mathrm{Gr}(r, 2r-1)$. Similarly, we prove the following theorem:

Theorem 1.2 *Varieties connected by the flop of type A^G_4 are derived-equivalent.*

Although the proof of Theorem 1.2 is parallel to that of the derived equivalence of Calabi–Yau complete intersections in $\mathbf{P} = \mathrm{Gr}(2, 5)$ and $\mathbf{Q} = \mathrm{Gr}(3, 5)$ defined by global sections of the equivariant vector bundles dual to $\mathbf{V}_-$ and $\mathbf{V}_+$ in [15, Theorem 5.7], we write down a full detail for clarity. As explained in [24], the derived equivalence obtained in [15] in turn follows from Theorem 1.2 using matrix factorizations.

We also give a similar proof of derived equivalences for a Mukai flop and a standard flop. For a Mukai flop, Kawamata [16] and Namikawa [19] independently showed the derived equivalence by using the pull-back and the push-forward along the fiber product $\mathbf{V}_- \times_{\mathbf{V}_0} \mathbf{V}_+$. Addington, Donovan, and Meachan [1] introduced a generalization of the functor of Kawamata and Namikawa parametrized by an integer, and discovered that certain compositions of these functors give the $\mathbb{P}$-twist in the sense of Huybrechts and Thomas [11]. They also considered the case of a standard flop, where the derived equivalence is originally proved by Bondal and Orlov [5]. Our proof is obtained by proceeding the mutation performed in [5] and [1] a little further in a straightforward way. Hara [7] also studied a Mukai flop in terms of non-commutative crepant resolutions.

For a standard flop, Segal [21] showed the derived equivalence by using the grade restriction rule for variation of geometric invariant theory quotients (VGIT) originally introduced by Hori, Herbst, and Page [10]. VGIT method was subsequently developed by Halpern-Leistner [6] and Ballard, Favero, and Katzarkov [2]. It is an interesting problem to develop this method further to prove the derived equivalence for the flop of type C_2 and A_4^G, and a Mukai flop.

Notations and conventions *We work over an algebraically closed field* $\mathbf{k}$ *of characteristic 0 throughout this paper. All pull-back and push-forward are derived unless otherwise specified. The complexes underlying* $\mathrm{Ext}^\bullet(-,-)$ *and* $\mathrm{H}^\bullet(-)$ *will be denoted by* $\mathbf{hom}(-,-)$ *and* $\mathbf{h}(-)$ *respectively.*

2 Flop of Type C_2

Let P and Q be the parabolic subgroups of the simple Lie group G of type C_2 associated with the crossed Dynkin diagrams $\times\!\Leftarrow\!\bullet$ and $\bullet\!\Leftarrow\!\times$. The corresponding homogeneous spaces are the projective space $\mathbf{P} = \mathbb{P}(V)$, the Lagrangian Grassmannian $\mathbf{Q} = \mathrm{LGr}(V)$, and the isotropic flag variety $\mathbf{F} = \mathbb{P}_{\mathbf{P}}\left(\mathscr{L}_{\mathbf{P}}^{\perp}/\mathscr{L}_{\mathbf{P}}\right) = \mathbb{P}_{\mathbf{Q}}\left(\mathscr{S}_{\mathbf{Q}}\right)$. Here V is a 4-dimensional symplectic vector space, $\mathscr{L}_{\mathbf{P}}^{\perp}$ is the rank 3 vector bundle given as the symplectic orthogonal to the tautological line bundle $\mathscr{L}_{\mathbf{P}} \cong \mathscr{O}_{\mathbf{P}}(-h)$ on $\mathbf{P}$, and $\mathscr{S}_{\mathbf{Q}}$ is the tautological rank 2 bundle on $\mathbf{Q}$. Note that $\mathbf{Q}$ is also a quadric hypersurface in $\mathbb{P}^4$. Tautological sequences on $\mathbf{Q} = \mathrm{LGr}(V)$ and $\mathbf{F} \cong \mathbb{P}_{\mathbf{Q}}\left(\mathscr{S}_{\mathbf{Q}}\right)$ give

$$0 \to \mathscr{S}_{\mathbf{Q}} \to \mathscr{O}_{\mathbf{Q}} \otimes V \to \mathscr{S}_{\mathbf{Q}}^{\vee} \to 0 \tag{2.1}$$

and

$$0 \to \mathscr{O}_{\mathbf{F}}(-h+H) \to \mathscr{S}_{\mathbf{F}}^{\vee} \to \mathscr{O}_{\mathbf{F}}(h) \to 0, \tag{2.2}$$

where $\mathscr{S}_{\mathbf{F}} := \varpi_+^* \mathscr{S}_{\mathbf{Q}}$. We have

$$(\varpi_-)_* \left(\mathscr{O}_{\mathbf{F}}(H)\right) \cong \left(\left(\mathscr{L}_{\mathbf{P}}^{\perp}/\mathscr{L}_{\mathbf{P}}\right) \otimes \mathscr{L}_{\mathbf{P}}\right)^{\vee}$$

and

$$(\varpi_+)_* \left(\mathscr{O}_{\mathbf{F}}(h)\right) \cong \mathscr{S}_{\mathbf{Q}}^{\vee},$$

whose determinants are given by $\mathscr{O}_{\mathbf{P}}(2h)$ and $\mathscr{O}_{\mathbf{Q}}(H)$ respectively. Since $\omega_{\mathbf{P}} \cong \mathscr{O}_{\mathbf{P}}(-4h)$, $\omega_{\mathbf{Q}} \cong \mathscr{O}_{\mathbf{Q}}(-3H)$, and $\omega_{\mathbf{F}} \cong \mathscr{O}_{\mathbf{F}}(-2h-2H)$, we have $\omega_{\mathbf{V}_-} \cong \mathscr{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathscr{O}_{\mathbf{V}_+}$, and $\omega_{\mathbf{V}} \cong \mathscr{O}_{\mathbf{V}}(-h-H)$.

Recall from [3] that

$$D^b(\mathbf{P}) = \langle \mathcal{O}_{\mathbf{P}}(-2h), \mathcal{O}_{\mathbf{P}}(-h), \mathcal{O}_{\mathbf{P}}, \mathcal{O}_{\mathbf{P}}(h) \rangle, \tag{2.3}$$

and from [17] (cf. also [14]) that

$$D^b(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-H), \mathscr{S}_{\mathbf{Q}}^{\vee}(-H), \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(H) \rangle.$$

Since $\varphi_{\pm}$ are blow-ups along the zero-sections, it follows from [20] that

$$D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \Phi_-(D^b(\mathbf{V}_-)) \rangle \tag{2.4}$$

and

$$D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \Phi_+(D^b(\mathbf{V}_+)) \rangle, \tag{2.5}$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_{\mathbf{V}}(H)) \circ \varphi_-^* \colon D^b(\mathbf{V}_-) \to D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_{\mathbf{V}}(h)) \circ \varphi_+^* \colon D^b(\mathbf{V}_+) \to D^b(\mathbf{V}).$$

By abuse of notation, we use the same symbol for an object of $D^b(\mathbf{F})$ and its image in $D^b(\mathbf{V})$ by the push-forward ι_*. Equations 2.3 and 2.4 give

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-2h), \mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \Phi_-(D^b(\mathbf{V}_-)) \rangle.$$

Since $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-h-H)$, by mutating the first term to the far right, and then $\Phi_-(D^b(\mathbf{V}_-))$ one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(-h+H), \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1 := R_{\langle \mathcal{O}_{\mathbf{F}}(-h+H) \rangle} \circ \Phi_-.$$

In the sequel, we will use the following fact.

Lemma 2.1 *Given two vector bundles $\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}$ on $\mathbf{F}$, if $\mathbf{h}\left(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}}(-h-H)\right) \simeq 0$, then we have $\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}) \simeq \mathbf{h}\left(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}}\right)$.*

Proof We have

$$\begin{aligned} \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}) &\simeq \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\{\mathcal{E}_{\mathbf{V}}(h+H) \to \mathcal{E}_{\mathbf{V}}\}, \mathcal{F}_{\mathbf{F}}) \\ &\simeq \mathbf{h}\left(\left\{\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}} \to \mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}}(-h-H)\right\}\right) \\ &\simeq \mathbf{h}\left(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}}\right). \end{aligned}$$

□

Note that the canonical extension of $\mathcal{O}_{\mathbf{F}}(h)$ by $\mathcal{O}_{\mathbf{F}}(-h+H)$ associated with

$$\begin{aligned} \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(-h+H)) &\simeq \mathbf{h}(\mathcal{O}_{\mathbf{F}}(-2h+H)) \\ &\simeq \mathbf{h}\left((\varpi_+)_* \mathcal{O}_{\mathbf{F}}(-2h) \otimes \mathcal{O}_{\mathbf{Q}}(H)\right) \\ &\simeq \mathbf{h}\left(\mathcal{O}_{\mathbf{Q}}[-1]\right) \\ &\simeq \mathbf{k}[-1] \end{aligned}$$

is given by the short exact sequence Eq. 2.2. By mutating $\mathcal{O}_{\mathbf{F}}(-h+H)$ one step to the left, $\mathcal{O}_{\mathbf{F}}(-h)$ to the far right, and then $\Phi_1(D^b(\mathbf{V}_-))$ one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}, \mathscr{S}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(H), \Phi_2(D^b(\mathbf{V}_-))\rangle,$$

where

$$\Phi_2 := R_{\langle \mathcal{O}_{\mathbf{F}}(H)\rangle} \circ \Phi_1.$$

One can easily see that $\mathcal{O}_{\mathbf{F}}(h)$ and $\mathcal{O}_{\mathbf{F}}(H)$ are orthogonal, so that

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}, \mathscr{S}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}(H), \mathcal{O}_{\mathbf{F}}(h), \Phi_2(D^b(\mathbf{V}_-))\rangle. \tag{2.6}$$

By mutating $\Phi_2(D^b(\mathbf{V}_-))$ one step to the left, and then $\mathcal{O}_{\mathbf{F}}(h)$ to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-H), \mathcal{O}_{\mathbf{F}}, \mathscr{S}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}(H), \Phi_3(D^b(\mathbf{V}_-))\rangle,$$

where

$$\Phi_3 := L_{\langle \mathcal{O}_{\mathbf{F}}(h)\rangle} \circ \Phi_2.$$

We have

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathcal{O}_{\mathbf{F}}, \mathscr{S}_{\mathbf{F}}^{\vee}\right) \simeq \mathbf{h}\left(\mathscr{S}_{\mathbf{F}}^{\vee}\right) \simeq V^{\vee},$$

and the dual of Eq. 2.1 shows that the kernel of the evaluation map $\mathcal{O}_{\mathbf{F}} \otimes V^{\vee} \to \mathscr{S}_{\mathbf{F}}^{\vee}$ is $\mathscr{S}_{\mathbf{F}} \cong \mathscr{S}_{\mathbf{F}}^{\vee}(-H)$. By mutating $\mathscr{S}_{\mathbf{F}}^{\vee}$ one step to the left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-H), \mathscr{S}_{\mathbf{F}}^{\vee}(-H), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(H), \Phi_3(D^b(\mathbf{V}_-))\rangle. \tag{2.7}$$

By comparing Eq. 2.7 with Eq. 2.5, we obtain a derived equivalence

$$\Phi := \Phi_+^! \circ \Phi_3 \colon D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+),$$

where

$$\Phi_+^!(-) := (\varphi_+)_* \circ ((-) \otimes \mathcal{O}_{\mathbf{V}}(-h)) : D^b(\mathbf{V}) \to D^b(\mathbf{V}_+)$$

is the left adjoint functor of Φ_+.

3 Flop of Type A_4^G

Let P and Q be the parabolic subgroups of the simple Lie group G of type A_4 associated with the crossed Dynkin diagrams •-×-•-• and •-•-×-•. The corresponding homogeneous spaces are the Grassmannians $\mathbf{P} = \mathrm{Gr}(2, V)$, $\mathbf{Q} = \mathrm{Gr}(3, V)$, and the partial flag variety $\mathbf{F} = \mathbb{P}_{\mathbf{P}}\left(\wedge^2 \mathscr{Q}_{\mathbf{P}}^{\vee}\right) = \mathbb{P}_{\mathbf{Q}}\left(\wedge^2 \mathscr{S}_{\mathbf{Q}}\right)$. Here V is a 5-dimensional vector space, $\mathscr{Q}_{\mathbf{P}}^{\vee}$ is the dual of the universal quotient bundle on $\mathbf{P}$, and $\mathscr{S}_{\mathbf{Q}}$ is the tautological rank 3 bundle on $\mathbf{Q}$. We have

$$(\varpi_-)_*\left(\mathcal{O}_{\mathbf{F}}(H)\right) \cong \wedge^2 \mathscr{Q}_{\mathbf{P}}$$

and

$$(\varpi_+)_*\left(\mathcal{O}_{\mathbf{F}}(h)\right) \cong \wedge^2 \mathscr{S}_{\mathbf{Q}}^{\vee},$$

whose determinants are given by $\mathcal{O}_{\mathbf{P}}(2h)$ and $\mathcal{O}_{\mathbf{Q}}(2H)$ respectively. Since $\omega_{\mathbf{P}} \cong \mathcal{O}_{\mathbf{P}}(-5h)$, $\omega_{\mathbf{Q}} \cong \mathcal{O}_{\mathbf{Q}}(-5H)$, and $\omega_{\mathbf{F}} \cong \mathcal{O}_{\mathbf{F}}(-3h-3H)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$ and $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-2h-2H)$.

First, we adapt several lemmas in [15] to our situation. To distinguish vector bundles which are obtained as a pull-back to $\mathbf{F}$ from $\mathbf{P}$ or $\mathbf{Q}$, we put tilde on the pull-back from

Q. By abuse of notation, we use the same symbol for an object of $D^b(\mathbf{F})$ and its image in $D^b(\mathbf{V})$ by the push-forward ι_*.

Lemma 3.1 $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\widetilde{\mathscr{Q}}_\mathbf{F}, \mathcal{O}_\mathbf{F}(h + aH)\right) \simeq 0$ *for integers* $-4 \le a \le -2$.

Proof We have

$$\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\widetilde{\mathscr{Q}}_\mathbf{F}, \mathcal{O}_\mathbf{F}(h + aH)\right) \simeq \mathbf{h}\left(\widetilde{\mathscr{Q}}_\mathbf{F}^\vee(h + aH)\right) \simeq 0,$$

where the first and the second isomorphisms follow from Lemma 2.1, Borel-Bott-Weil theorem and [15, Lemma 5.1] respectively. □

Similarly, one can deduce Lemmas 3.2 and 3.3 below from [15, Lemma 5.2, Lemma 5.3] by checking that $\mathcal{O}_\mathbf{F}((a-1)H)$, $\mathcal{E}_\mathbf{F}^\vee \otimes \mathcal{E}'_\mathbf{F}((a-1)h - 2H)$, and $\widetilde{\mathcal{F}}_\mathbf{F}^\vee \otimes \widetilde{\mathcal{F}}'_\mathbf{F}(-2h + (a-1)H)$ are acyclic as an object of $D^b(\mathbf{F})$.

Lemma 3.2 $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}(\mathcal{O}_\mathbf{F}, \mathcal{O}_\mathbf{F}(h + aH)) \simeq 0$ *for integers* $-3 \le a \le -1$.

Lemma 3.3 *Let* $\mathcal{E}_\mathbf{F}, \mathcal{E}'_\mathbf{F}$ *be the pull-back to* **F** *of vector bundles* $\mathcal{E}, \mathcal{E}'$ *on* **P**, *and let* $\widetilde{\mathcal{F}}_\mathbf{F}, \widetilde{\mathcal{F}}'_\mathbf{F}$ *be the pull-back to* **F** *of vector bundles* $\mathcal{F}, \mathcal{F}'$ *on* **Q**. *Then we have* $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\mathcal{E}_\mathbf{F}, \mathcal{E}'_\mathbf{F}(ah - H)\right) \simeq 0$ *and* $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\widetilde{\mathcal{F}}_\mathbf{F}, \widetilde{\mathcal{F}}'_\mathbf{F}(-h + aH)\right) \simeq 0$ *for all integers* a.

The parallel result to the following lemma was tacitly used in [15].

Lemma 3.4 *As an object of* $D^b(\mathbf{V})$, $\mathcal{O}_\mathbf{F}, \widetilde{\mathscr{Q}}_\mathbf{F}, \mathscr{S}_\mathbf{F}$, *and* $\mathscr{S}_\mathbf{F}^\vee$ *are left orthogonal to* $\widetilde{\mathscr{S}}_\mathbf{F}^\vee(h - 2H)$, $\widetilde{\mathscr{S}}_\mathbf{F}^\vee(h - 2H)$, $\mathcal{O}_\mathbf{F}(2h - 2H)$, *and* $\mathscr{Q}_\mathbf{F}$ *respectively.*

Lemma 3.5 below and the tautological sequence show that $R_{\mathcal{O}_\mathbf{F}}\widetilde{\mathscr{Q}}_\mathbf{F}^\vee \simeq \widetilde{\mathscr{S}}_\mathbf{F}^\vee$ and $R_{\mathcal{O}_\mathbf{F}}\mathscr{S}_\mathbf{F} \simeq \mathscr{Q}_\mathbf{F}$ in $D^b(\mathbf{V})$.

Lemma 3.5 $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\widetilde{\mathscr{Q}}_\mathbf{F}^\vee, \mathcal{O}_\mathbf{F}\right) \simeq V$ *and* $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\mathscr{S}_\mathbf{F}, \mathcal{O}_\mathbf{F}\right) \simeq V$.

Again, both Lemmas 3.4 and 3.5 follow from Lemma 2.1 and Borel-Bott-Weil theorem. Lemma 3.6 below and the exact sequences

$$0 \to \mathcal{O}_\mathbf{F}(h - H) \to \mathscr{Q}_\mathbf{F} \to \widetilde{\mathscr{Q}}_\mathbf{F} \to 0$$

and

$$0 \to \mathscr{S}_\mathbf{F} \to \widetilde{\mathscr{S}}_\mathbf{F} \to \mathcal{O}_\mathbf{F}(h - H) \to 0$$

obtained in [15] show that $R_{\mathcal{O}_\mathbf{F}(h-H)}\widetilde{\mathscr{Q}}_\mathbf{F} \simeq \mathscr{Q}_\mathbf{F}[1]$ and $L_{\mathcal{O}_\mathbf{F}(-h+H)}\widetilde{\mathscr{S}}_\mathbf{F}^\vee \simeq \mathscr{S}_\mathbf{F}^\vee$ in $D^b(\mathbf{V})$.

Lemma 3.6 $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\widetilde{\mathscr{Q}}_\mathbf{F}, \mathcal{O}_\mathbf{F}(h - H)\right) \simeq \mathbf{k}[-1]$ *and* $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\mathcal{O}_\mathbf{F}(-h + H), \widetilde{\mathscr{S}}_\mathbf{F}^\vee\right) \simeq \mathbf{k}$.

Proof We have

$$\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\widetilde{\mathscr{Q}}_\mathbf{F}, \mathcal{O}_\mathbf{F}(h - H)\right) \simeq \mathbf{h}\left(\widetilde{\mathscr{Q}}_\mathbf{F}^\vee(h - H)\right) \simeq \mathbf{k}[-1],$$

where the isomorphisms follow from Lemma 2.1 and Borel-Bott-Weil theorem. Similarly, we have

$$\mathbf{hom}_{\mathcal{O}_\mathbf{V}}\left(\mathcal{O}_\mathbf{F}(-h+H), \widetilde{\mathscr{S}}_\mathbf{F}^\vee\right) \simeq \mathbf{h}\left(\widetilde{\mathscr{S}}_\mathbf{F}^\vee(h-H)\right) \simeq \mathbf{k}. \qquad \square$$

Recall from [17] (cf. also [14])

$$D^b(\mathbf{P}) = \langle \mathscr{S}_\mathbf{P}(-2h), \mathcal{O}_\mathbf{P}(-2h), \mathscr{S}_\mathbf{P}(-h), \mathcal{O}_\mathbf{P}(-h), \cdots, \mathscr{S}_\mathbf{P}(2h), \mathcal{O}_\mathbf{P}(2h)\rangle,$$

and

$$D^b(\mathbf{Q}) = \langle \mathcal{O}_\mathbf{Q}, \mathscr{Q}_\mathbf{Q}, \mathcal{O}_\mathbf{Q}(H), \mathscr{Q}_\mathbf{Q}(H), \cdots, \mathcal{O}_\mathbf{Q}(4H), \mathscr{Q}_\mathbf{Q}(4H)\rangle. \tag{3.1}$$

Since $\varphi_\pm$ are blow-ups along the zero-sections, it follows from [20] that

$$D^b(\mathbf{V}) = \langle \iota_*\varpi_-^* D^b(\mathbf{P}), \iota_*\varpi_-^* D^b(\mathbf{P})(h+H), \Phi_-(D^b(\mathbf{V}_-))\rangle \tag{3.2}$$

and

$$D^b(\mathbf{V}) = \langle \iota_*\varpi_+^* D^b(\mathbf{Q}), \iota_*\varpi_+^* D^b(\mathbf{Q})(h+H), \Phi_+(D^b(\mathbf{V}_+))\rangle, \tag{3.3}$$

where

$$\Phi_- := ((-)\otimes \mathcal{O}_\mathbf{V}(2H))\circ \varphi_-^* \colon D^b(\mathbf{V}_-) \to D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-)\otimes \mathcal{O}_\mathbf{V}(2h))\circ \varphi_+^* \colon D^b(\mathbf{V}_+) \to D^b(\mathbf{V}).$$

We write $\mathcal{O}_{i,j} := \mathcal{O}_\mathbf{F}(ih+jH)$. Equations 3.1 and 3.3 give a semiorthogonal decomposition of the form

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathcal{O}_{0,0}, \widetilde{\mathscr{Q}}_{0,0}, \mathcal{O}_{0,1}, \widetilde{\mathscr{Q}}_{0,1}, \mathcal{O}_{0,2}, \widetilde{\mathscr{Q}}_{0,2}, \mathcal{O}_{0,3}, \widetilde{\mathscr{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathscr{Q}}_{0,4} \\ &\mathcal{O}_{1,1}, \widetilde{\mathscr{Q}}_{1,1}, \mathcal{O}_{1,2}, \widetilde{\mathscr{Q}}_{1,2}, \mathcal{O}_{1,3}, \widetilde{\mathscr{Q}}_{1,3}, \mathcal{O}_{1,4}, \widetilde{\mathscr{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathscr{Q}}_{1,5}, \Phi_+(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

Since $\omega_\mathbf{V} \cong \mathcal{O}_\mathbf{V}(-2h-2H)$, by mutating the first five terms to the far right, and then $\Phi_+(D^b(\mathbf{V}_+))$ five steps to the right, we obtain

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\widetilde{\mathscr{Q}}_{0,2}, \mathcal{O}_{0,3}, \widetilde{\mathscr{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathscr{Q}}_{0,4}, \mathcal{O}_{1,1}, \widetilde{\mathscr{Q}}_{1,1}, \mathcal{O}_{1,2}, \widetilde{\mathscr{Q}}_{1,2}, \mathcal{O}_{1,3} \\ &\widetilde{\mathscr{Q}}_{1,3}, \mathcal{O}_{1,4}, \widetilde{\mathscr{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathscr{Q}}_{1,5}, \mathcal{O}_{2,2}, \widetilde{\mathscr{Q}}_{2,2}, \mathcal{O}_{2,3}, \widetilde{\mathscr{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+))\rangle, \end{aligned}$$

where

$$\Phi_1 := R_{\langle \mathcal{O}_{2,2}, \widetilde{\mathscr{Q}}_{2,2}, \mathcal{O}_{2,3}, \widetilde{\mathscr{Q}}_{2,3}, \mathcal{O}_{2,4}\rangle} \circ \Phi_+.$$

One can easily see that $\mathcal{O}_{1,1}$ is orthogonal to $\mathcal{O}_{0,3}$, $\widetilde{\mathscr{Q}}_{0,3}$, $\mathcal{O}_{0,4}$, and $\widetilde{\mathscr{Q}}_{0,4}$ by Lemmas 3.1 and 3.2, so that

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\widetilde{\mathscr{Q}}_{0,2}, \mathcal{O}_{1,1}, \mathcal{O}_{0,3}, \widetilde{\mathscr{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathscr{Q}}_{0,4}, \widetilde{\mathscr{Q}}_{1,1}, \mathcal{O}_{1,2}, \widetilde{\mathscr{Q}}_{1,2}, \mathcal{O}_{1,3} \\ &\widetilde{\mathscr{Q}}_{1,3}, \mathcal{O}_{2,2}, \mathcal{O}_{1,4}, \widetilde{\mathscr{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathscr{Q}}_{1,5}, \widetilde{\mathscr{Q}}_{2,2}, \mathcal{O}_{2,3}, \widetilde{\mathscr{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

By mutating $\widetilde{\mathscr{Q}}_{0,2}$, $\widetilde{\mathscr{Q}}_{1,3}$, $\widetilde{\mathscr{Q}}_{1,1}$, and $\widetilde{\mathscr{Q}}_{2,2}$ one step to the right, we obtain by $\widetilde{\mathscr{Q}}_{1,1} \cong \widetilde{\mathscr{Q}}_{1,2}^\vee$, Lemmas 3.5, and 3.6

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathcal{O}_{1,1}, \mathscr{Q}_{0,2}, \mathcal{O}_{0,3}, \widetilde{\mathscr{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathscr{Q}}_{0,4}, \mathcal{O}_{1,2}, \widetilde{\mathscr{S}}_{1,2}^\vee, \widetilde{\mathscr{Q}}_{1,2}, \mathcal{O}_{1,3} \\ &\mathcal{O}_{2,2}, \mathscr{Q}_{1,3}, \mathcal{O}_{1,4}, \widetilde{\mathscr{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathscr{Q}}_{1,5}, \mathcal{O}_{2,3}, \widetilde{\mathscr{S}}_{2,3}^\vee, \widetilde{\mathscr{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

By mutating $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,3}$ four steps to the left, we obtain by Lemmas 3.1, 3.2, and 3.6

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathcal{O}_{1,1}, \mathscr{Q}_{0,2}, \mathcal{O}_{1,2}, \mathcal{O}_{0,3}, \mathscr{Q}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathscr{Q}}_{0,4}, \widetilde{\mathscr{S}}_{1,2}^\vee, \widetilde{\mathscr{Q}}_{1,2}, \mathcal{O}_{1,3} \\ &\mathcal{O}_{2,2}, \mathscr{Q}_{1,3}, \mathcal{O}_{2,3}, \mathcal{O}_{1,4}, \mathscr{Q}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathscr{Q}}_{1,5}, \widetilde{\mathscr{S}}_{2,3}^\vee, \widetilde{\mathscr{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

One can easily see that $\widetilde{\mathscr{F}}^{\vee}_{1,2}$ is orthogonal to $\mathcal{O}_{0,4}$ and $\widetilde{\mathscr{Q}}_{0,4}$ by Lemmas 3.4, so that

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathcal{O}_{1,1}, \mathscr{Q}_{0,2}, \mathcal{O}_{1,2}, \mathcal{O}_{0,3}, \mathscr{Q}_{0,3}, \widetilde{\mathscr{F}}^{\vee}_{1,2}, \mathcal{O}_{0,4}, \widetilde{\mathscr{Q}}_{0,4}, \widetilde{\mathscr{Q}}_{1,2}, \mathcal{O}_{1,3} \\ &\mathcal{O}_{2,2}, \mathscr{Q}_{1,3}, \mathcal{O}_{2,3}, \mathcal{O}_{1,4}, \mathscr{Q}_{1,4}, \widetilde{\mathscr{F}}^{\vee}_{2,3}, \mathcal{O}_{1,5}, \widetilde{\mathscr{Q}}_{1,5}, \widetilde{\mathscr{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

By mutating $\mathcal{O}_{0,3}$ and $\mathcal{O}_{1,4}$ two steps to the right, $\mathcal{O}_{1,3}$ and $\mathcal{O}_{2,4}$ three steps to the left, and then $\mathcal{O}_{0,4}$ and $\mathcal{O}_{1,5}$ two steps to the right, we obtain by Lemmas 3.5 and 3.6

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathcal{O}_{1,1}, \mathscr{Q}_{0,2}, \mathcal{O}_{1,2}, \mathscr{S}_{0,3}, \mathscr{S}^{\vee}_{1,2}, \mathcal{O}_{0,3}, \mathcal{O}_{1,3}, \mathscr{S}_{0,4}, \mathscr{S}^{\vee}_{1,3}, \mathcal{O}_{0,4} \\ &\mathcal{O}_{2,2}, \mathscr{Q}_{1,3}, \mathcal{O}_{2,3}, \mathscr{S}_{1,4}, \mathscr{S}^{\vee}_{2,3}, \mathcal{O}_{1,4}, \mathcal{O}_{2,4}, \mathscr{S}_{1,5}, \mathscr{S}^{\vee}_{2,4}, \mathcal{O}_{1,5}, \Phi_1(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

By mutating $\mathcal{O}_{1,1}$ to the far right, and then $\Phi_1(D^b(\mathbf{V}_+))$ one step to the right, we obtain

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathscr{Q}_{0,2}, \mathcal{O}_{1,2}, \mathscr{S}_{0,3}, \mathscr{S}^{\vee}_{1,2}, \mathcal{O}_{0,3}, \mathcal{O}_{1,3}, \mathscr{S}_{0,4}, \mathscr{S}^{\vee}_{1,3}, \mathcal{O}_{0,4}, \mathcal{O}_{2,2} \\ &\mathscr{Q}_{1,3}, \mathcal{O}_{2,3}, \mathscr{S}_{1,4}, \mathscr{S}^{\vee}_{2,3}, \mathcal{O}_{1,4}, \mathcal{O}_{2,4}, \mathscr{S}_{1,5}, \mathscr{S}^{\vee}_{2,4}, \mathcal{O}_{1,5}, \mathcal{O}_{3,3}, \Phi_2(D^b(\mathbf{V}_+))\rangle, \end{aligned}$$

where

$$\Phi_2 := R_{\langle \mathcal{O}_{3,3}\rangle} \circ \Phi_1.$$

By Lemmas 3.2, 3.3, and 3.4, we obtain

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathscr{Q}_{0,2}, \mathcal{O}_{1,2}, \mathscr{S}^{\vee}_{1,2}, \mathcal{O}_{2,2}, \mathscr{S}_{0,3}, \mathcal{O}_{0,3}, \mathcal{O}_{1,3}, \mathscr{S}^{\vee}_{1,3}, \mathscr{Q}_{1,3}, \mathcal{O}_{2,3} \\ &\mathscr{S}^{\vee}_{2,3}, \mathcal{O}_{3,3}, \mathscr{S}_{0,4}, \mathcal{O}_{0,4}, \mathscr{S}_{1,4}, \mathcal{O}_{1,4}, \mathcal{O}_{2,4}, \mathscr{S}^{\vee}_{2,4}, \mathscr{S}_{1,5}, \mathcal{O}_{1,5}, \Phi_2(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

By mutating $\Phi_2(D^b(\mathbf{V}_+))$ ten steps to the left, and then last ten terms to the far left, we obtain

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathscr{S}^{\vee}_{0,1}, \mathcal{O}_{1,1}, \mathscr{S}_{-2,2}, \mathcal{O}_{-2,2}, \mathscr{S}_{-1,2}, \mathcal{O}_{-1,2}, \mathcal{O}_{0,2}, \mathscr{S}^{\vee}_{0,2}, \mathscr{S}_{-1,3}, \mathcal{O}_{-1,3} \\ &\mathscr{Q}_{0,2}, \mathcal{O}_{1,2}, \mathscr{S}^{\vee}_{1,2}, \mathcal{O}_{2,2}, \mathscr{S}_{0,3}, \mathcal{O}_{0,3}, \mathcal{O}_{1,3}, \mathscr{S}^{\vee}_{1,3}, \mathscr{Q}_{1,3}, \mathcal{O}_{2,3}, \Phi_3(D^b(\mathbf{V}_+))\rangle, \end{aligned}$$

where

$$\Phi_3 := L_{\langle \mathscr{S}^{\vee}_{2,3}, \mathcal{O}_{3,3}, \mathscr{S}_{0,4}, \mathcal{O}_{0,4}, \mathscr{S}_{1,4}, \mathcal{O}_{1,4}, \mathcal{O}_{2,4}, \mathscr{S}^{\vee}_{2,4}, \mathscr{S}_{1,5}, \mathcal{O}_{1,5}\rangle} \circ \Phi_2.$$

By Lemma 3.3, we obtain

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathscr{S}^{\vee}_{0,1}, \mathcal{O}_{1,1}, \mathscr{S}_{-2,2}, \mathcal{O}_{-2,2}, \mathscr{S}_{-1,2}, \mathcal{O}_{-1,2}, \mathcal{O}_{0,2}, \mathscr{S}^{\vee}_{0,2}, \mathscr{Q}_{0,2}, \mathcal{O}_{1,2} \\ &\mathscr{S}^{\vee}_{1,2}, \mathcal{O}_{2,2}, \mathscr{S}_{-1,3}, \mathcal{O}_{-1,3}, \mathscr{S}_{0,3}, \mathcal{O}_{0,3}, \mathcal{O}_{1,3}, \mathscr{S}^{\vee}_{1,3}, \mathscr{Q}_{1,3}, \mathcal{O}_{2,3}, \Phi_3(D^b(\mathbf{V}_+))\rangle. \end{aligned}$$

By mutating $\mathscr{Q}_{0,2}$ and $\mathscr{Q}_{1,3}$ two steps to the left, the first two terms to the far right, and then $\Phi_3(D^b(\mathbf{V}_+))$ two steps to the right, we obtain by $\mathscr{S}^{\vee}_{0,0} \simeq \mathscr{S}_{1,0}$, Lemmas 3.4, and 3.6

$$\begin{aligned} D^b(\mathbf{V}) = \langle &\mathscr{S}_{-2,2}, \mathcal{O}_{-2,2}, \mathscr{S}_{-1,2}, \mathcal{O}_{-1,2}, \mathscr{S}_{0,2}, \mathcal{O}_{0,2}, \mathscr{S}_{1,2}, \mathcal{O}_{1,2}, \mathscr{S}_{2,2}, \mathcal{O}_{2,2} \\ &\mathscr{S}_{-1,3}, \mathcal{O}_{-1,3}, \mathscr{S}_{0,3}, \mathcal{O}_{0,3}, \mathscr{S}_{1,3}, \mathcal{O}_{1,3}, \mathscr{S}_{2,3}, \mathcal{O}_{2,3}, \mathscr{S}_{3,3}, \mathcal{O}_{3,3}, \Phi_4(D^b(\mathbf{V}_+))\rangle, \end{aligned} \tag{3.4}$$

where

$$\Phi_4 := R_{\langle \mathscr{S}^{\vee}_{2,3}, \mathcal{O}_{3,3}\rangle} \circ \Phi_3.$$

By comparing Eq. 3.4 with Eq. 3.2, we obtain a derived equivalence

$$\Phi := \Phi^{!}_{-} \circ \Phi_4 \colon D^b(\mathbf{V}_+) \xrightarrow{\sim} D^b(\mathbf{V}_-),$$

where

$$\Phi^{!}_{-}(-) := (\varphi_-)_* \circ ((-) \otimes \mathcal{O}_{\mathbf{V}}(-2H)) : D^b(\mathbf{V}) \to D^b(\mathbf{V}_-)$$

is the left adjoint functor of Φ_-.

4 Mukai Flop

For $n \geq 2$, let P and Q be the maximal parabolic subgroups of the simple Lie group of type A_n associated with the crossed Dynkin diagrams ×—•⋯—• and •—⋯•—×. The corresponding homogeneous spaces are the projective spaces $\mathbf{P} = \mathbb{P}V$, $\mathbf{Q} = \mathbb{P}V^\vee$, and the partial flag variety $\mathbf{F} = \mathrm{F}(1, n; V)$, where V is an $(n+1)$-dimensional vector space. Since $\omega_\mathbf{P} \cong \mathcal{O}(-(n+1)h)$, $\omega_\mathbf{Q} \cong \mathcal{O}(-(n+1)H)$, and $\omega_\mathbf{F} \cong \mathcal{O}(-nh - nH)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$, and $\omega_\mathbf{V} \cong \mathcal{O}(-(n-1)h - (n-1)H)$.

Lemma 4.1 *$\mathcal{O}_\mathbf{F}(-ih + jH)$ and $\mathcal{O}_\mathbf{F}(-(i+1)h + (j-1)H)$ are acyclic for $1 \leq j \leq n-1$ and $1 \leq i \leq n-j$.*

Proof Since $j - n \leq -i \leq -1$ and $j - n - 1 \leq -i - 1 \leq -2$, the derived push-foward of $\mathcal{O}_\mathbf{F}(-ih + jH)$ and $\mathcal{O}_\mathbf{F}(-(i+1)h + (j-1)H)$ vanish by [9, Exercise III.8.4] unless $i = n-1$ and $j = 1$, in which case the acyclicity of $\mathcal{O}_\mathbf{F}(-nh)$ is obvious. □

Lemma 4.2 $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}(\mathcal{O}_\mathbf{F}(ih - jH), \mathcal{O}_\mathbf{F}) \simeq 0$ *for $1 \leq j \leq n-1$ and $1 \leq i \leq n-j$.*

Proof We have

$$\mathbf{hom}_{\mathcal{O}_\mathbf{V}}(\mathcal{O}_\mathbf{F}(ih - jH), \mathcal{O}_\mathbf{F}) \simeq \mathbf{h}\left(\{\mathcal{O}_\mathbf{F}(-ih + jH) \to \mathcal{O}_\mathbf{F}(-(i+1)h + (j-1)H)\}\right),$$

which vanishes by Lemma 4.1. □

Recall from [3] that

$$D^b(\mathbf{P}) = \langle \mathcal{O}_\mathbf{P}, \mathcal{O}_\mathbf{P}(h), \cdots, \mathcal{O}_\mathbf{P}(nh)\rangle \tag{4.1}$$

and

$$D^b(\mathbf{Q}) = \langle \mathcal{O}_\mathbf{Q}, \mathcal{O}_\mathbf{Q}(H), \cdots, \mathcal{O}_\mathbf{Q}(nH)\rangle. \tag{4.2}$$

Since $\varphi_\pm$ are blow-ups along the zero-sections, it follows from [20] that

$$D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \cdots, \iota_* \varpi_-^* D^b(\mathbf{P}) \otimes \mathcal{O}_\mathbf{V}((n-2)H), \Phi_-(D^b(\mathbf{V}_-))\rangle \tag{4.3}$$

and

$$D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \cdots, \iota_* \varpi_+^* D^b(\mathbf{Q}) \otimes \mathcal{O}_\mathbf{V}((n-2)h), \Phi_+(D^b(\mathbf{V}_+))\rangle, \tag{4.4}$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_\mathbf{V}((n-1)H)) \circ \varphi_-^* \colon D^b(\mathbf{V}_-) \to D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_\mathbf{V}((n-1)h)) \circ \varphi_+^* \colon D^b(\mathbf{V}_+) \to D^b(\mathbf{V}).$$

We write $\mathcal{O}_{i,j} := \mathcal{O}_\mathbf{F}(ih + jH)$. Equations 4.1 and 4.3 give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{A}_0, \Phi_-(D^b(\mathbf{V}_-))\rangle$$

where $\mathcal{A}_0$ is given by

$$\begin{array}{cccccccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} & \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} & \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ & & & \mathcal{O}_{n-2,n-2} & \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots\ \mathcal{O}_{2n-2,n-2}. \end{array} \tag{4.5}$$

Note from Lemma 4.2 that there are no morphisms from right to left in Eq. 4.5. Since $\omega_{\mathbf{V}} \cong \mathcal{O}_{-(n-1),-(n-1)}$, by mutating first

$$\begin{array}{cccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} \\ & & \ddots & \vdots \\ & & & \mathcal{O}_{n-2,n-2} \end{array}$$

to the far right, and then $\Phi_-(D^b(\mathbf{V}_-))$ to the far right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_1, \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1(D^b(\mathbf{V}_-)) \coloneqq R_{\langle \mathcal{O}_{n-1,n-1}, \cdots, \mathcal{O}_{2n-3,2n-3} \rangle} \circ \Phi_-$$

and $\mathcal{A}_1$ is given by

$$\begin{array}{cccccc} \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & & \\ \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-3,n-2} & \mathcal{O}_{2n-2,n-2} \\ \mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-3,n-1} & \\ & \mathcal{O}_{n,n} & \mathcal{O}_{n+1,n} & \cdots & \mathcal{O}_{2n-3,n} & \\ & & \mathcal{O}_{n+1,n+1} & \cdots & \mathcal{O}_{2n-3,n+1} & \\ & & & \ddots & \vdots & \\ & & & & \mathcal{O}_{2n-3,2n-3}. & \end{array}$$

By mutating $\Phi_1(D^b(\mathbf{V}_-))$ one step to the left, and then $\mathcal{O}_{2n-2,n-2}$ to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_2, \Phi_2(D^b(\mathbf{V}_-)) \rangle, \tag{4.6}$$

where

$$\Phi_2(D^b(\mathbf{V}_-)) \coloneqq L_{\mathcal{O}_{2n-2,n-2}} \circ \Phi_1$$

and $\mathcal{A}_2$ is given by

$$\begin{array}{ccccc}
\mathcal{O}_{n-1,-1} & & & & \\
\mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & \\
\mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & \\
\vdots & \vdots & \vdots & \ddots & \\
\mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-3,n-2} \\
\mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-3,n-1} \\
 & \mathcal{O}_{n,n} & \mathcal{O}_{n+1,n} & \cdots & \mathcal{O}_{2n-3,n} \\
 & & \mathcal{O}_{n+1,n+1} & \cdots & \mathcal{O}_{2n-3,n+1} \\
 & & & \ddots & \vdots \\
 & & & & \mathcal{O}_{2n-3,2n-3}.
\end{array}$$

By comparing Eq. 4.6 with Eqs. 4.2 and 4.4, we obtain a derived equivalence

$$\Phi := (\varphi_+)_* \circ ((-) \otimes \mathcal{O}_{-(2n-2),0}) \circ \Phi_2 \colon D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+).$$

5 Standard Flop

For $n \geq 1$, let P and Q be the maximal parabolic subgroups of the semisimple Lie group $G = \mathrm{SL}(V) \times \mathrm{SL}(V^\vee)$ associated with the crossed Dynkin diagram ×—•⋯•—• ⊕ •—•⋯•—• and •—•⋯•—• ⊕ •—•⋯•—×. The corresponding homogeneous spaces are the projective spaces $\mathbf{P} = \mathbb{P}V$, $\mathbf{Q} = \mathbb{P}V^\vee$, and their product $\mathbf{F} = \mathbb{P}V \times \mathbb{P}V^\vee$. Since $\omega_\mathbf{P} \cong \mathcal{O}(-(n+1)h)$, $\omega_\mathbf{Q} \cong \mathcal{O}(-(n+1)H)$, and $\omega_\mathbf{F} \cong \mathcal{O}(-(n+1)h-(n+1)H)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$, and $\omega_\mathbf{V} \cong \mathcal{O}(-nh-nH)$.

Lemma 5.1 $\mathbf{hom}_{\mathcal{O}_\mathbf{V}}(\mathcal{O}_\mathbf{F}(ih-jH), \mathcal{O}_\mathbf{F}) \simeq 0$ *for* $1 \leq j \leq n-1$ *and* $1 \leq i \leq n-j$.

Proof We have

$$\mathbf{hom}_{\mathcal{O}_\mathbf{V}}(\mathcal{O}_\mathbf{F}(ih-jH), \mathcal{O}_\mathbf{F}) \simeq \mathbf{h}\left(\{\mathcal{O}_\mathbf{F}(-ih+jH) \to \mathcal{O}_\mathbf{F}(-(i+1)h+(j-1)H)\}\right),$$

which vanishes for $1 \leq i \leq n-j \leq n-1$. □

It follows from [20] that

$$D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \cdots, \iota_* \varpi_-^* D^b(\mathbf{P}) \otimes \mathcal{O}((n-1)(h+H)), \Phi_-(D^b(\mathbf{V}_-)) \rangle \quad (5.1)$$

and

$$D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \cdots, \iota_* \varpi_+^* D^b(\mathbf{Q}) \otimes \mathcal{O}((n-1)(h+H)), \Phi_+(D^b(\mathbf{V}_+)) \rangle, \quad (5.2)$$

where

$$\Phi_- := (-) \otimes \mathcal{O}_\mathbf{V}(n(h+H)) \circ \varphi_-^* \colon D^b(\mathbf{V}_-) \to D^b(\mathbf{V})$$

and

$$\Phi_+ := (-) \otimes \mathcal{O}_\mathbf{V}(n(h+H)) \circ \varphi_+^* \colon D^b(\mathbf{V}_+) \to D^b(\mathbf{V}).$$

We write $\mathcal{O}_{i,j} := \mathcal{O}_\mathbf{F}(ih+jH)$. Equations 4.1 and 5.1 give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{A}_0, \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

where $\mathcal{A}_0$ is given by

$$\begin{array}{cccccccccc}
\mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} & \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & & \\
 & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} & \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & & \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
 & & & \mathcal{O}_{n-2,n-2} & \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-2,n-2} & \\
 & & & & \mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-2,n-1} & \mathcal{O}_{2n-1,n-1}.
\end{array} \tag{5.3}$$

Note from Lemma 5.1 that there are no morphisms from right to left in Eq. 5.3. Since $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-nh - nH)$, by mutating first

$$\begin{array}{cccc}
\mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} \\
 & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} \\
 & & \ddots & \vdots \\
 & & & \mathcal{O}_{n-2,n-2}
\end{array}$$

to the far right, and then $\Phi_-(D^b(\mathbf{V}_-))$ to the far right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_1, \Phi_1(D^b(\mathbf{V}_-))\rangle,$$

where

$$\Phi_1(D^b(\mathbf{V}_-)) \coloneqq R_{\langle \mathcal{O}_{n,n}, \cdots, \mathcal{O}_{2n-2,2n-2}\rangle} \circ \Phi_-$$

and $\mathcal{A}_1$ is given by

$$\begin{array}{cccccc}
\mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & & \\
\mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
\mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-2,n-1} & \mathcal{O}_{2n-1,n-1} \\
 & \mathcal{O}_{n,n} & \mathcal{O}_{n+1,n} & \cdots & \mathcal{O}_{2n-2,n} & \\
 & & \mathcal{O}_{n+1,n+1} & \cdots & \mathcal{O}_{2n-2,n+1} & \\
 & & & \ddots & \vdots & \\
 & & & & \mathcal{O}_{2n-2,2n-2}. &
\end{array}$$

By mutating $\Phi_1(D^b(\mathbf{V}_-))$ one step to the left, and then $\mathcal{O}_{2n-1,n-1}$ to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_2, \Phi_2(D^b(\mathbf{V}_-))\rangle, \tag{5.4}$$

where

$$\Phi_2(D^b(\mathbf{V}_-)) \coloneqq L_{\mathcal{O}_{2n-1,n-1}} \circ \Phi_1$$

and $\mathcal{A}_2$ is given by

$$\begin{array}{ccccc}
\mathcal{O}_{n-1,-1} & & & & \\
\mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & \\
\mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & \\
\vdots & \vdots & \vdots & \ddots & \\
\mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-2,n-1} \\
 & \mathcal{O}_{n,n} & \mathcal{O}_{n+1,n} & \cdots & \mathcal{O}_{2n-2,n} \\
 & & \mathcal{O}_{n+1,n+1} & \cdots & \mathcal{O}_{2n-2,n+1} \\
 & & & \ddots & \vdots \\
 & & & & \mathcal{O}_{2n-2,2n-2}.
\end{array}$$

By comparing Eq. 5.4 with Eqs. 4.2 and 5.2, we obtain a derived equivalence

$$\Phi := (\varphi_+)_* \circ ((-) \otimes \mathcal{O}_{-(2n-1),0}) \circ \Phi_2 \colon D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+).$$

Remark 1 The way of presenting our proof in Section 4 and 5 is called chess game by some authors [12, 23].

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