Research Article

Flavia Giannetti* and Giorgio Stefani

On the convex components of a set in $\mathbb{R}^n$

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Abstract: We prove a lower bound on the number of the convex components of a compact set with non-empty interior in $\mathbb{R}^n$ for all $n \geq 2$. Our result generalizes and improves the inequalities previously obtained in [M. Carozza, F. Giannetti, F. Leonetti and A. Passarelli di Napoli, Convex components, *Commun. Contemp. Math.* **21** (2019), no. 6, Article ID 1850036] and [M. La Civita and F. Leonetti, Convex components of a set and the measure of its boundary, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **56** (2008/09), 71–78].

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1 Introduction

1.1 Convex components

Let $n \geq 2$. Let us consider a compact set $E \subset \mathbb{R}^n$ with non-empty interior and a decomposition of the form

$$E = \bigcup_{i=1}^{k} E_i, \tag{1.1}$$

where $k \in \mathbb{N}$ and $E_1, \dots, E_k$ are the *convex components* of E, i.e., compact and convex sets with non-empty interior. Since, in general, such a decomposition is obviously not unique, it is interesting to give a lower bound on the minimal number $k_{\min}(E) \in \mathbb{N}$ of the convex components of E. By definition, $k_{\min}(E) = 1$ if and only if E is a convex body. Moreover, we can note that $k_{\min}(E) \geq c(E)$, where $c(E) \in \mathbb{N}$ is the number of connected components of E. Indeed, any convex component of E must lay inside some connected component of E. Therefore, without loss of generality, in the following we will always assume that E is a connected set.

The first lower bound on the minimal number of convex components was given in [9, Theorem 1.1], where the authors proved that

$$k_{\min}(E) \geq \left\lceil \frac{\mathcal{H}^{n-1}(\partial E)}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \right\rceil, \tag{1.2}$$

where $\lceil x \rceil \in \mathbb{Z}$ denotes the upper integer part of $x \in \mathbb{R}$. Here and in the following, for all $s \geq 0$ we let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure (in particular, $\mathcal{H}^0$ is the counting measure). Moreover, we let ∂E be the boundary of E and $\mathrm{co}(E)$ be the convex hull of E. Note that, since E admits at least one decomposition as in (1.1), $\mathcal{H}^{n-1}(\partial E)$ and $\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))$ are two finite and strictly positive real numbers, see [9], so that the right-hand side in (1.2) is well defined.

In the subsequent paper [6], the bound in (1.2) has been improved in the case $n = 2$ in the sense explained in Section 1.3 for a class of compact sets $E \subset \mathbb{R}^2$. In the same spirit, our aim is to provide a refined bound of the

***Corresponding author: Flavia Giannetti,** Dipartimento di Matematica ed Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli “Federico II”, Via Cintia, 80126 Napoli, Italy, e-mail: giannett@unina.it. https://orcid.org/0000-0002-2461-0845
Giorgio Stefani, Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Bonomea 265, 34136 Trieste, Italy, e-mail: giorgio.stefani.math@gmail.com. https://orcid.org/0000-0002-1592-8288

number $k_{\min}(E)$ in any dimension $n \geq 2$. We stress that the estimate we are going to obtain also improves the result in [6].

1.2 Monotonicity of perimeter

The proof of (1.2) is based on the following monotonicity property of the perimeter: if $A \subset B \subset \mathbb{R}^n$ are two convex bodies, then

$$\mathcal{H}^{n-1}(\partial A) \leq \mathcal{H}^{n-1}(\partial B). \tag{1.3}$$

Inequality (1.3) is well known since the ancient Greek (Archimedes himself took it as a postulate in his work on the sphere and the cylinder, see [1, p. 36]) and can be proved in many different ways, for example by exploiting either the Cauchy formula for the area surface or the monotonicity property of mixed volumes, [2, Section 7], by using the Lipschitz property of the projection on a convex closed set, [3, Lemma 2.4], or finally by observing that the perimeter is decreased under intersection with half-spaces, [10, Exercise 15.13]. Actually, a deep inspection of the proof given in [3] shows that the convexity of B is not needed.

Anyway, in [9], a quantitative improvement of formula (1.3) has been obtained if A and B are both convex bodies. Moreover, lower bounds for the perimeter deficit

$$\delta(B, A) := \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}(\partial A)$$

with respect to the Hausdorff distance of A and B have been established for $n = 2$ in [4, 9], for $n = 3$ in [5] and finally for all $n \geq 2$ in [11].

In particular, if $A \subset B$ are two convex bodies in $\mathbb{R}^n$, with $n \geq 2$, then

$$\mathcal{H}^{n-1}(\partial A) + \frac{\omega_{n-1} r^{n-2} h^2}{r + \sqrt{r^2 + h^2}} \leq \mathcal{H}^{n-1}(\partial B), \tag{1.4}$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ denotes the volume of the unit ball in $\mathbb{R}^n$, $h = h(A, B)$ is the Hausdorff distance of A and B and

$$r = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(B \cap \partial H)}{\omega_{n-1}}}, \qquad H = \{x \in \mathbb{R}^n : \langle b - a, x - a \rangle \leq 0\},$$

with $a \in A$ and $b \in B$ such that $|a - b| = h(A, B)$, see [11, Corollary 1.2] and Figure 1.

Actually, the main result of [11] provides a quantitative lower bound for the more general deficit

$$\delta_\Phi(B, A) := P_\Phi(B) - P_\Phi(A),$$

where P_Φ stands for the *anisotropic (Wulff) perimeter* associated to the positively 1-homogeneous convex function $\Phi\colon \mathbb{R}^n \to [0, +\infty)$.

We conclude this subsection by underlying that the quantitative estimates of the perimeter deficit $\delta(B, A)$ obtained in [4, 5, 11] are sharp in the sense that they hold as equalities in some cases, see Figure 1.

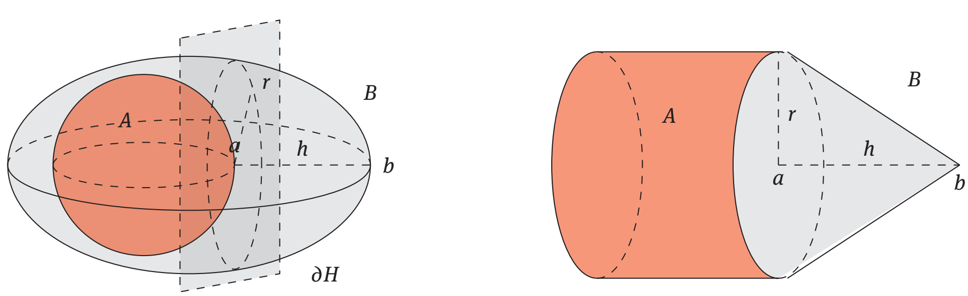

Figure 1: The setting of the estimate (1.4) (on the left) with an example of equality (on the right).

1.3 Improvement of (1.2) in the planar case

Taking advantage of the quantitative estimate (1.4) in the planar case proved in [5], in the more recent paper [6] the authors were able to improve the lower bound (1.2) for $n = 2$ for a class of compact sets $E \subset \mathbb{R}^2$ (see also [7]). Precisely, if for a bounded closed $\emptyset \neq E \subset \mathbb{R}^2$ one can find $q \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $\alpha \in (0, 1)$ such that any decomposition of the form (1.1) admits p convex components $E_{i_1}, \dots, E_{i_p}$ such that

$$h(E_{i_j}, \mathrm{co}(E)) \geq \alpha \, \mathrm{diam}(\mathrm{co}(E)) \quad \text{for all } j = 1, \dots, p \tag{1.5}$$

and

$$q\mathcal{H}^1(\partial(\mathrm{co}(E))) - \mathcal{H}^1(\partial E) < \frac{4\alpha^2 p}{1 + \sqrt{1 + 4\alpha^2}} \, \mathrm{diam}(\mathrm{co}(E)), \tag{1.6}$$

then

$$k_{\min}(E) \geq q + 1. \tag{1.7}$$

Inequality (1.7) is sharp, in the sense that it holds as an equality in some cases. Moreover, it improves the previous lower bound (1.2) in the case $n = 2$. Indeed, in [6] the authors exhibited an example for which (1.2) gives a strict inequality while, on the contrary, (1.7) yields an equality.

The idea behind inequality (1.7) essentially relies on two ingredients. On the one hand, the use of the refined estimate of the deficit obtained in [4] in place of the monotonicity property of the perimeter (1.3). On the other hand, the idea of assuming (1.5) for a finite number p of the components, according to the observation that some planar sets $E \subset \mathbb{R}^2$ have some convex components whose Hausdorff distance from the convex hull $\mathrm{co}(E)$ is comparable to the diameter of $\mathrm{co}(E)$ itself, independently of the chosen decomposition.

By a careful inspection of the proof of (1.7), one realizes that

$$k_{\min}(E) \geq \left\lceil \frac{1}{\mathcal{H}^1(\partial(\mathrm{co}(E)))} \left(\mathcal{H}^1(\partial E) + \sum_{j=1}^{p} \frac{4h(E_{i_j}, \mathrm{co}(E))^2}{\mathrm{diam}(\mathrm{co}(E)) + \sqrt{\mathrm{diam}(\mathrm{co}(E))^2 + 4h(E_{i_j}, \mathrm{co}(E))^2}} \right) \right\rceil$$

and since the function

$$r \mapsto \frac{4h}{r + \sqrt{r^2 + 4h^2}}$$

is monotone for $r > 0$, the assumption (1.5) yields

$$k_{\min}(E) \geq \left\lceil \frac{1}{\mathcal{H}^1(\partial(\mathrm{co}(E)))} \left(\mathcal{H}^1(\partial E) + \frac{4\alpha^2 p}{1 + \sqrt{1 + 4\alpha^2}} \, \mathrm{diam}(\mathrm{co}(E)) \right) \right\rceil, \tag{1.8}$$

which is precisely (1.7), according to the best possible choice of $q \in \mathbb{N}_0$ in (1.6).

1.4 Main result

The aim of the present paper is to improve inequality (1.2) for all $n \geq 2$ exploiting the quantitative monotonicity of the perimeter (1.4) proved in [11], thus generalizing inequality (1.8) to higher dimensions. Before stating our main result, we need to introduce the following notation.

Definition 1.1 (Maximal sectional radius). Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be a compact set with non-empty interior. Given a direction $\nu \in \mathbb{R}^n$, we let

$$\rho_\nu(E) = \sup \left\{ \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(E \cap (t\nu + \partial H_\nu))}{\omega_{n-1}}} : t \in \mathbb{R} \right\}$$

be the *maximal sectional radius of* E *in the direction* ν, where $H_\nu = \{x \in \mathbb{R}^n : \langle x, \nu \rangle \leq 0\}$. Note that, naturally, $\rho_{-\nu}(E) = \rho_\nu(E)$ for all $\nu \in \mathbb{R}^n$.

With the above definition in force, our main result reads as follows.

Theorem 1.2. *Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be a compact set with non-empty interior. Assume that there exist $p \in \mathbb{N}$, $\alpha \in (0,1)$ and $\beta \in [0,1]$ with the following properties. For every family $E_1, \dots, E_k$, with $k \in \mathbb{N}$, of convex bodies with non-empty interior such that $E = \bigcup_{i=1}^k E_i$, we can find a subfamily of p convex bodies $E_{i_1}, \dots, E_{i_p}$ and a family of corresponding p closed half-spaces such that $E_{i_j} \subset H_{i_j}$,*

$$h(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j}) \geq \alpha \operatorname{diam}(\mathrm{co}(E)) \tag{1.9}$$

and

$$\mathcal{H}^{n-1}(\mathrm{co}(E) \cap \partial H_{i_j}) \geq \beta \omega_{n-1} \rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-1} \tag{1.10}$$

for all $j = 1, \dots, p$, where $\nu_{i_j} = a_{i_j} - b_{i_j}$, with $a_{i_j} \in \mathrm{co}(E) \cap H_{i_j}$ and $b_{i_j} \in \mathrm{co}(E)$ such that $h(\mathrm{co}(E) \cap H_{i_j}, \mathrm{co}(E)) = |a_{i_j} - b_{i_j}|$ and $H_{i_j} = \{x \in \mathbb{R}^n : \langle b_{i_j} - a_{i_j}, x - a_{i_j} \rangle \leq 0\}$. Then

$$k_{\min}(E) \geq \left\lceil \frac{1}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \left(\mathcal{H}^{n-1}(\partial E) + \omega_{n-1} \alpha^2 \beta^{\frac{n-2}{n-1}} \sum_{j=1}^{p} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2} \operatorname{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2 \operatorname{diam}(\mathrm{co}(E))^2}} \right) \right\rceil. \tag{1.11}$$

1.5 Comments

First of all, let us remark that inequality (1.11) improves the previous lower bound (1.2). Indeed, inequality (1.11) clearly reduces to the lower bound (1.2) as soon as one drops the additional assumptions on each of all possible decompositions of the form (1.1). Moreover, inequality (1.11) holds as an equality in some cases for which (1.2) gives a strict inequality only. We will give some explicit examples in Section 3 below.

Concerning the statement of Theorem 1.2, it is worth noting that the assumption (1.9) corresponds to (1.5), while the additional assumption (1.10) comes into play for $n \geq 3$ only.

In fact, if we take $n = 2$ in Theorem 1.2, then inequality (1.11) becomes

$$k_{\min}(E) \geq \left\lceil \frac{1}{\mathcal{H}^{1}(\partial(\mathrm{co}(E)))} \left(\mathcal{H}^{1}(\partial E) + 2\alpha^2 \sum_{j=1}^{p} \frac{\operatorname{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2 \operatorname{diam}(\mathrm{co}(E))^2}} \right) \right\rceil \tag{1.12}$$

(as it is customary, we use the convention $0^0 = 1$) and the parameter $\beta \in [0,1]$ provided by (1.10) plays no role in the final estimate (1.12). Consequently, the additional assumption in (1.10) can be dropped and one just needs to choose the closed half-plane $H_{i_j} \subset \mathbb{R}^2$ in such a way that

$$h(\mathrm{co}(E) \cap H_{i_j}, \mathrm{co}(E)) = h(E_{i_j}, \mathrm{co}(E)) \quad \text{for all } j = 1, \dots, p,$$

which is always possible by the definition of the Hausdorff distance and the convexity of each component E_{i_j}.

Concerning the higher-dimensional case $n \geq 3$, a control like the one in (1.10) seems reasonable to be assumed. Indeed, as one may realize by looking at inequality (1.2), the set $E \subset \mathbb{R}^n$ may have a convex component very lengthened in one specific direction $\nu \in \mathbb{S}^{n-1}$ which does not give a substantial contribution to the total perimeter of E but, nevertheless, that strongly affects the total perimeter of the convex hull $\mathrm{co}(E)$.

In addition, we observe that the effectiveness of the lower bound (1.2) drastically changes when passing from the planar case $n = 2$ to the non-planar case $n \geq 3$. Indeed, if $E \subset \mathbb{R}^2$ is a non-convex connected compact set admitting at least one decomposition like (1.1), then

$$\mathcal{H}^1(\partial(\mathrm{co}(E))) < \mathcal{H}^1(\partial E),$$

correctly implying that $k_{\min}(E) \geq 2$. As a matter of fact, in the planar case $n = 2$, the examples given in [6] provide the precise value of $k_{\min}(E)$ for $q \geq 2$, since if $q = 1$ both inequalities (1.2) and (1.7) allow to conclude that $k_{\min}(E) \geq 2$ only. However, as we are going to show with some examples in Section 3 below, there are non-convex connected compact sets $E \subset \mathbb{R}^n$, with $n \geq 3$, such that

$$\mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) \geq \mathcal{H}^{n-1}(\partial E),$$

so that (1.2) only implies that $k_{\min}(E) \geq 1$. Nevertheless, inequality (1.11) given by Theorem 1.2 allows us to recover the correct value of $k_{\min}(E)$ in these examples.

Moreover, let us observe that, in the planar case $n = 2$, one can trivially bound

$$\rho_\nu(\mathrm{co}(E)) \le \frac{\mathrm{diam}(\mathrm{co}(E))}{2} \quad \text{for all } \nu \in \mathbb{S}^1, \tag{1.13}$$

so that inequality (1.12) gives back

$$\begin{aligned} k_{\min}(E) &\ge \left\lceil \frac{1}{\mathcal{H}^1(\partial(\mathrm{co}(E)))}\left(\mathcal{H}^1(\partial E) + \frac{2p\alpha^2\,\mathrm{diam}(\mathrm{co}(E))^2}{\frac{\mathrm{diam}(\mathrm{co}(E))}{2} + \sqrt{\frac{\mathrm{diam}(\mathrm{co}(E))^2}{4} + \alpha^2\,\mathrm{diam}(\mathrm{co}(E))^2}}\right)\right\rceil \\ &= \left\lceil \frac{1}{\mathcal{H}^1(\partial(\mathrm{co}(E)))}\left(\mathcal{H}^1(\partial E) + \frac{4\alpha^2 p}{1+\sqrt{1+4\alpha^2}}\,\mathrm{diam}(\mathrm{co}(E))\right)\right\rceil, \end{aligned}$$

that is the estimate in (1.8). Actually, because of the fact that the upper bound (1.13) can be too rough in general, inequality (1.12) given by our Theorem 1.2 is more precise than the one in (1.8), as we are going to show in Example 3.1 below.

Last but not least, we remark that both the lower bounds provided by estimates (1.2) and (1.11) are not stable under small modifications of the compact set $E \subset \mathbb{R}^n$, $n \ge 2$. In fact, the value of $k_{\min}(E)$ may be changed without substantially altering neither the perimeters of E and of its convex hull $\mathrm{co}(E)$, nor all the other geometrical quantities involved in (1.11), for example by gluing some additional tiny convex components to the original set E.

1.6 Organization of the paper

The rest of the paper is organized as follows.

In Section 2 we detail the proof of our main result, Theorem 1.2. Our approach essentially follows the strategy of [6], up to some minor modifications needed in order to exploit the quantitative estimate (1.4) in conjunction with the notion of maximal radius introduced in Definition 1.1.

In Section 3 we provide some examples proving the effectiveness of our main result with respect to either the general inequality (1.2) or its improvement (1.8) in the planar case, as already observed, due to the fact that $\rho_\nu(\mathrm{co}(E)) \le \frac{\mathrm{diam}(\mathrm{co}(E))}{2}$ for all $\nu \in \mathbb{S}^1$.

2 Proof of Theorem 1.2

We recall that, if $A \subset B$ are two compact sets in $\mathbb{R}^n$, with $n \ge 2$, then the Hausdorff distance $h(A, B)$ between A and B can be written as

$$h(A,B) = \max_{b\in B} \mathrm{dist}(A, b) = \max_{b\in B} \min_{a\in A} |a-b|.$$

As above, given $\emptyset \ne A \subset B$ two convex bodies in $\mathbb{R}^n$, with $n \ge 2$, we let

$$\delta(B, A) := \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}(\partial A) \ge 0$$

be the perimeter deficit between A and B.

Proof of Theorem 1.2. Since E is compact, its convex hull $\mathrm{co}(E)$ is compact too, see [8, Corollary 3.1] for example. As a consequence, $\mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) < +\infty$. Arguing as in [6, Section 2], we can estimate

$$\begin{aligned} \mathcal{H}^{n-1}(\partial E) &\le \mathcal{H}^{n-1}\Big(\bigcup_{i=1}^k \partial E_i\Big) \le \sum_{i=1}^k \mathcal{H}^{n-1}(\partial E_i) = \sum_{j=1}^p \mathcal{H}^{n-1}(\partial E_{i_j}) + \sum_{j=p+1}^k \mathcal{H}^{n-1}(\partial E_{i_j}) \\ &\le \sum_{j=1}^p \big(\mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) - \delta(\mathrm{co}(E), E_{i_j})\big) + \sum_{j=p+1}^k \mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) \\ &= k\mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) - \sum_{j=1}^p \delta(\mathrm{co}(E), E_{i_j}), \end{aligned}$$

so that

$$\left\lceil \frac{\mathcal{H}^{n-1}(\partial E) + \sum_{j=1}^{p} \delta(\mathrm{co}(E), E_{i_j})}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \right\rceil \le k.$$

Now, since $E_{i_j} \subset H_{i_j}$, we observe that

$$\begin{aligned}
\delta(\mathrm{co}(E), E_{i_j}) &= \mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) - \mathcal{H}^{n-1}(\partial E_{i_j}) \\
&= \big(\mathcal{H}^{n-1}(\partial(\mathrm{co}(E))) - \mathcal{H}^{n-1}(\partial(\mathrm{co}(E) \cap H_{i_j}))\big) + \big(\mathcal{H}^{n-1}(\partial(\mathrm{co}(E) \cap H_{i_j})) - \mathcal{H}^{n-1}(\partial E_{i_j})\big) \\
&= \delta(\mathrm{co}(E) \cap H_{i_j}, E_{i_j}) + \delta(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j}) \\
&\ge \delta(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j})
\end{aligned} \tag{2.1}$$

for all $j = 1, \dots, p$. Since $h(\mathrm{co}(E) \cap H_{i_j}, \mathrm{co}(E)) = |a_{i_j} - b_{i_j}|$ with $a_{i_j} \in \mathrm{co}(E) \cap H_{i_j}$ and $b_{i_j} \in \mathrm{co}(E)$ such that

$$H_{i_j} = \{x \in \mathbb{R}^n : \langle b_{i_j} - a_{i_j}, x - a_{i_j} \rangle \le 0\},$$

we can thus apply (1.4) to each couple of convex bodies $\mathrm{co}(E)$ and $\mathrm{co}(E) \cap H_{i_j}$, with $j = 1, \dots, p$, and get

$$\delta(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j}) \ge \frac{\omega_{n-1} r_{i_j}^{n-2} h_{i_j}^2}{r_{i_j} + \sqrt{r_{i_j}^2 + h_{i_j}^2}}, \tag{2.2}$$

where

$$h_{i_j} = h(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j}), \quad r_{i_j} = \sqrt[n-1]{\frac{\mathcal{H}^{n-1}(\mathrm{co}(E) \cap \partial H_{i_j})}{\omega_{n-1}}}.$$

By (1.10), we clearly have

$$\beta^{\frac{1}{n-1}} \rho_{\nu_{i_j}}(\mathrm{co}(E)) \le r_{i_j} \le \rho_{\nu_{i_j}}(\mathrm{co}(E)) \tag{2.3}$$

for all $j = 1, \dots, p$. Inserting (2.3) into (2.2), we immediately obtain that

$$\delta(\mathrm{co}(E), \mathrm{co}(E) \cap H_{i_j}) \ge \frac{\omega_{n-1} \beta^{\frac{n-2}{n-1}} \rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2} h_{i_j}^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + h_{i_j}^2}}$$

for all $j = 1, \dots, p$. Now, for any given $c > 0$, the function

$$s \mapsto \frac{s^2}{c + \sqrt{c + s^2}}$$

is strictly increasing for $s > 0$. Since $h_{i_j} \ge \alpha\, \mathrm{diam}(\mathrm{co}(E))$ for all $j = 1, \dots, p$ by (1.9), thanks to (2.1) we can finally estimate

$$\delta(\mathrm{co}(E), E_{i_j}) \ge \frac{\omega_{n-1} \alpha^2 \beta^{\frac{n-2}{n-1}} \rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2}\, \mathrm{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2\, \mathrm{diam}(\mathrm{co}(E))^2}}$$

for all $j = 1, \dots, p$. In conclusion, we get

$$\begin{aligned}
k &\ge \left\lceil \frac{1}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \left(\mathcal{H}^{n-1}(\partial E) + \sum_{j=1}^{p} \delta(E_{i_j}, \mathrm{co}(E)) \right) \right\rceil \\
&\ge \left\lceil \frac{1}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(E)))} \left(\mathcal{H}^{n-1}(\partial E) + \omega_{n-1} \alpha^2 \beta^{\frac{n-2}{n-1}} \sum_{j=1}^{p} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(E))^{n-2}\, \mathrm{diam}(\mathrm{co}(E))^2}{\rho_{\nu_{i_j}}(\mathrm{co}(E)) + \sqrt{\rho_{\nu_{i_j}}(\mathrm{co}(E))^2 + \alpha^2\, \mathrm{diam}(\mathrm{co}(E))^2}} \right) \right\rceil
\end{aligned}$$

proving (1.11). The proof is thus complete. □

3 Examples

We dedicate the remaining part of the paper to give some explicit examples of compact sets $E \subset \mathbb{R}^n$, $n \ge 2$, for which our main result applies. In each example, we will identify a point $P \in \partial E$ and one convex component E_j of E containing P and we will make a precise choice of parameters in order to satisfy the hypotheses of Theorem 1.2.

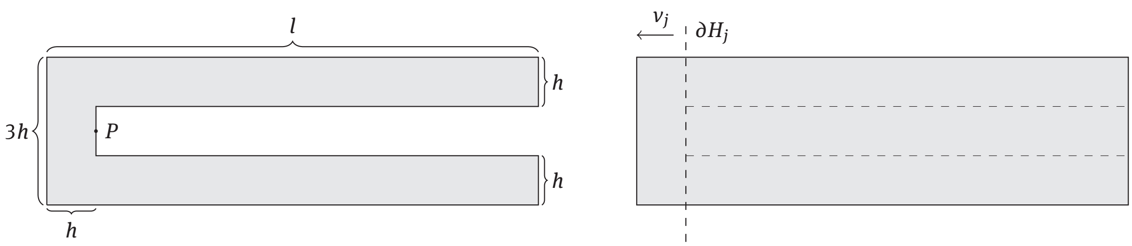

Figure 2: The set $C \subset \mathbb{R}^2$ (on the left) and its convex hull (on the right).

3.1 An example in $\mathbb{R}^2$

We begin with the following example in $\mathbb{R}^2$ showing that our Theorem 1.2 in the planar formulation (1.12), at least in some cases, provides a strictly better estimate than the one in (1.8) previously established in [6]. This example is based on the set $C \subset \mathbb{R}^2$ shown in Figure 2, which was already considered in [9, Example 2.1] and in [6, Example 3.1]. The set C depends on two parameters $l > h > 0$. In [6, Example 3.1], to make the construction work, it was necessary to assume that $h \in (0, \varepsilon)$ for some $\varepsilon \in (0, l)$ sufficiently small. In our situation, thanks to the refined inequality (1.8), our choice of the parameter h is less restrictive, i.e., we are going to choose $h \in (0, \bar{\varepsilon})$ for some $\bar{\varepsilon} \in (\varepsilon, l)$. As matter of fact, when $h \in (\varepsilon, \bar{\varepsilon})$, our inequality (1.8) gives the correct value $k_{\min}(C) = 3$, while inequality (1.7) gives the lower bound $k_{\min}(C) \geq 2$ only.

Example 3.1 (The set $C \subset \mathbb{R}^2$). Let $l > h > 0$ and consider the set $C \subset \mathbb{R}^2$ in Figure 2. We can compute

$$\mathcal{H}^1(\partial C) = 4l + 4h, \quad \mathcal{H}^1(\partial(\operatorname{co}(C))) = 2l + 6h, \quad \operatorname{diam}(\operatorname{co}(C)) = \sqrt{l^2 + 9h^2}.$$

Since C is not convex, we must have that $k_{\min}(C) \geq 2$. After all, it is evident that $k_{\min}(C) = 3$. Our argument will give such right value for a larger class of parameters $l > h > 0$ than the one provided in [6, Example 3.1]. First of all, notice that we do not deduce any further information from the result in [9]. Indeed, inequality (1.2) only yields

$$k_{\min}(C) \geq \left\lceil \frac{\mathcal{H}^1(\partial C)}{\mathcal{H}^1(\partial(\operatorname{co}(C)))} \right\rceil = 2,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^1(\partial C)}{\mathcal{H}^1(\partial(\operatorname{co}(C)))} = \frac{2l + 2h}{l + 3h} \in (1, 2)$$

whenever $l > h > 0$. We now consider the point $P \in \partial C$ as shown in Figure 2. For every decomposition of C into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in C, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of C to which the point P belongs, see Figure 2. Consequently, we must have

$$h(\operatorname{co}(C) \cap H_j, \operatorname{co}(C)) = l - h, \quad \mathcal{H}^1(\operatorname{co}(C) \cap \partial H_j) = 3h, \quad \rho_{\nu_j}(\operatorname{co}(C)) = \frac{3h}{2},$$

where $\nu_j \in \mathbb{S}^1$ is the inner unit normal of the half-space H_j as in Figure 2. Now let $l > 0$ be fixed. In [6], it has been shown that, for any $\alpha \in (0, 1)$, $p = 1$ and $h \ll l$, one has

$$\left\lceil \frac{\mathcal{H}^1(\partial C) + \frac{4\alpha^2}{1+\sqrt{1+4\alpha^2}} \operatorname{diam}(\operatorname{co}(C))}{\mathcal{H}^1(\partial(\operatorname{co}(C)))} \right\rceil = 3.$$

We now apply inequality (1.8) and Theorem 1.2 with

$$p = 1, \quad \alpha = \frac{l - h}{\sqrt{l^2 + 9h^2}}, \quad \beta = 0.$$

We claim that we can choose $h \in (0, l)$ such that

$$\left\lceil \frac{\mathcal{H}^1(\partial C) + \frac{4\alpha^2}{1+\sqrt{1+4\alpha^2}} \operatorname{diam}(\operatorname{co}(C))}{\mathcal{H}^1(\partial(\operatorname{co}(C)))} \right\rceil = 2$$

and

$$\left\lceil \frac{1}{\mathcal{H}^1(\partial(\mathrm{co}(C)))}\left(\mathcal{H}^1(\partial C) + \frac{2\alpha^2 \operatorname{diam}(\mathrm{co}(C))^2}{\rho_{\nu_j}(\mathrm{co}(C)) + \sqrt{\rho_{\nu_j}(\mathrm{co}(C))^2 + \alpha^2 \operatorname{diam}(\mathrm{co}(C))^2}}\right)\right\rceil = 3.$$

In order to have both the claimed inequalities, it is sufficient to find $h \in (0, l)$ such that

$$\frac{\mathcal{H}^1(\partial C) + \frac{4\alpha^2}{1+\sqrt{1+4\alpha^2}} \operatorname{diam}(\mathrm{co}(C))}{\mathcal{H}^1(\partial(\mathrm{co}(C)))} \le 2 < \frac{\mathcal{H}^1(\partial C) + \frac{2\alpha^2 \operatorname{diam}(\mathrm{co}(C))^2}{\rho_{\nu_j}(\mathrm{co}(C))+\sqrt{\rho_{\nu_j}(\mathrm{co}(C))^2+\alpha^2 \operatorname{diam}(\mathrm{co}(C))^2}}}{\mathcal{H}^1(\partial(\mathrm{co}(C)))},$$

that is,

$$\frac{2(l+h) + \frac{2\alpha^2}{1+\sqrt{1+4\alpha^2}}\sqrt{l^2+9h^2}}{l+3h} \le 2 < \frac{2(l+h) + \frac{2\alpha^2(l^2+9h^2)}{3h+\sqrt{9h^2+4\alpha^2(l^2+9h^2)}}}{l+3h}.$$

Up to some elementary algebraic computations, we need to find $h \in (0, l)$ such that

$$\frac{(l-h)^2}{3h+\sqrt{9h^2+4(l-h)^2}} > 2h \ge \frac{(l-h)^2}{\sqrt{l^2+9h^2}+\sqrt{l^2+9h^2+4(l-h)^2}}.$$

If we let $h = tl$ for $t \in (0, 1)$, then we just need to solve

$$\begin{cases} 1-5t^2-2t-2t\sqrt{9t^2+4(1-t)^2} > 0, \\ 2t\sqrt{1+9t^2}+2t\sqrt{1+9t^2+4(1-t)^2}-1-t^2+2t \ge 0, \end{cases}$$

and we let the reader check that the above system of inequalities admits solutions.

3.2 Some examples in $\mathbb{R}^3$

We now give some examples in $\mathbb{R}^3$ showing that for $n = 3$ our Theorem 1.2 provides an improvement of inequality (1.2) established in [9].

Example 3.2 (The set $L \subset \mathbb{R}^3$). Let $l > h > 0$ and consider the set $L \subset \mathbb{R}^3$ in Figure 3. We can compute

$$\mathcal{H}^2(\partial L) = 4hl + 6h^2,$$
$$\mathcal{H}^2(\partial(\mathrm{co}(L))) = 4hl + 5h^2 + h\sqrt{(l-h)^2+h^2},$$
$$\operatorname{diam}(\mathrm{co}(L)) = \sqrt{l^2+5h^2}.$$

Since L is not convex, we must have that $k_{\min}(L) \ge 2$, and a simple geometric argument allows to conclude that $k_{\min}(L) = 2$. From (1.2) we deduce that

$$k_{\min}(L) \ge \left\lceil \frac{\mathcal{H}^2(\partial L)}{\mathcal{H}^2(\partial(\mathrm{co}(L)))}\right\rceil = 1,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^2(\partial L)}{\mathcal{H}^2(\partial(\mathrm{co}(L)))} = \frac{4l+6h}{4l+5h+\sqrt{(l-h)^2+h^2}} \in (0, 1)$$

whenever $l > h > 0$. We now consider the point $P \in \partial L$ as shown in Figure 3. For every decomposition of L into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in L, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of L to which the point P belongs, see Figure 3. Consequently, we must have

$$h(\mathrm{co}(L) \cap H_j, \mathrm{co}(L)) = l - h, \quad \mathcal{H}^2(\mathrm{co}(L) \cap \partial H_j) = 2h^2, \quad \rho_{\nu_j}(\mathrm{co}(L)) = \sqrt{\frac{2h^2}{\pi}},$$

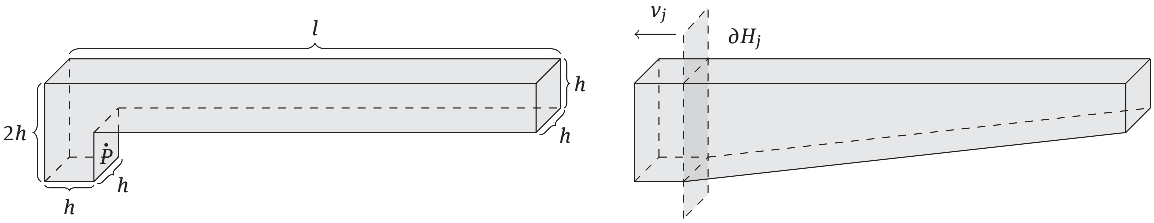

Figure 3: The set $L \subset \mathbb{R}^3$ (on the left) and its convex hull (on the right).

where $\nu_j \in \mathbb{S}^2$ is the inner unit normal of the half-space H_j as in Figure 3. We now let $l > 0$ be fixed. We apply Theorem 1.2 with

$$p = 1, \quad \alpha = \frac{l-h}{\sqrt{l^2+5h^2}}, \quad \beta = 1.$$

Provided that we choose $h \in (0, l)$ sufficiently small, we conclude that

$$\begin{aligned} k_{\min}(L) &\geq \left\lceil \frac{1}{4hl+5h^2+h\sqrt{(l-h)^2+h^2}} \left(4hl+6h^2+\pi\Big(\frac{l-h}{\sqrt{l^2+5h^2}}\Big)^2 \frac{\sqrt{\frac{2h^2}{\pi}}\,(\sqrt{l^2+5h^2})^2}{\sqrt{\frac{2h^2}{\pi}}+\sqrt{\frac{2h^2}{\pi}+(\frac{l-h}{\sqrt{l^2+5h^2}})^2(\sqrt{l^2+5h^2})^2}}\right)\right\rceil \\ &= \left\lceil \frac{1}{4l+5h+\sqrt{(l-h)^2+h^2}} \left(4l+6h+\frac{\sqrt{2\pi}(l-h)^2}{\sqrt{\frac{2h^2}{\pi}}+\sqrt{\frac{2h^2}{\pi}+(l-h)^2}}\right)\right\rceil = 2, \end{aligned}$$

since

$$\lim_{h\to 0^+} \frac{4l+6h+\frac{\sqrt{2\pi}(l-h)^2}{\sqrt{\frac{2h^2}{\pi}}+\sqrt{\frac{2h^2}{\pi}+(l-h)^2}}}{4l+5h+\sqrt{(l-h)^2+h^2}} = \frac{4+\sqrt{2\pi}}{5} \in (1,2).$$

Example 3.3 (The set D in $\mathbb{R}^3$). Let $l > 2h > 0$ and consider the set $D \subset \mathbb{R}^3$ in Figure 4. We can compute

$$\begin{aligned} \mathcal{H}^2(\partial D) &= 12lh + 4h\sqrt{(l-h)^2+h^2} + 4h\sqrt{(l-2h)^2+h^2} + 23h^2, \\ \mathcal{H}^2(\partial(\mathrm{co}(D))) &= 9lh + 4h\sqrt{(l-h)^2+h^2} + 25h^2, \\ \mathrm{diam}(\mathrm{co}(D)) &= \sqrt{l^2+25h^2}. \end{aligned}$$

Since D is not convex, we must have that $k_{\min}(D) \geq 2$, and a simple geometric argument allows to conclude that $k_{\min}(D) = 3$. From (1.2) we deduce that

$$k_{\min}(D) \geq \left\lceil \frac{\mathcal{H}^2(\partial D)}{\mathcal{H}^2(\partial(\mathrm{co}(D)))} \right\rceil = 2,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^2(\partial D)}{\mathcal{H}^2(\partial(\mathrm{co}(D)))} = \frac{12l + 4\sqrt{(l-h)^2+h^2} + 4\sqrt{(l-2h)^2+h^2} + 23h}{9l + 4\sqrt{(l-h)^2+h^2} + 25h} \in (1,2)$$

whenever $l > 2h > 0$. We now consider the point $P \in \partial D$ as shown in Figure 4. For every decomposition of D into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in D, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of D to which the point P belongs, see Figure 4. Consequently, we must have

$$h(\mathrm{co}(D) \cap H_j, \mathrm{co}(D)) = l - h, \quad \mathcal{H}^2(\mathrm{co}(D) \cap \partial H_j) = 12h^2, \quad \rho_{\nu_j}(\mathrm{co}(D)) = \sqrt{\frac{12h^2}{\pi}},$$

where $\nu_j \in \mathbb{S}^2$ is the inner unit normal of the half-space H_j as in Figure 4. We now let $l > 0$ be fixed. We apply Theorem 1.2 with

$$p = 1, \quad \alpha = \frac{l-h}{\sqrt{l^2+25h^2}}, \quad \beta = 1.$$

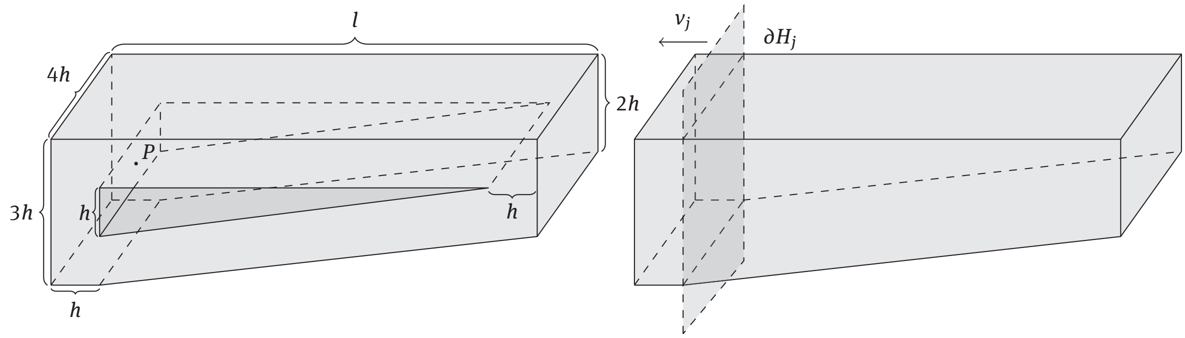

Figure 4: The set $D \subset \mathbb{R}^3$ (on the left) and its convex hull (on the right).

Provided that we choose $h \in \left(0, \frac{l}{2}\right)$ sufficiently small, we conclude that

$$\begin{aligned} k_{\min}(D) &\geq \left\lceil \frac{1}{\mathcal{H}^2(\partial(\mathrm{co}(D)))} \left(\mathcal{H}^2(\partial D) + \pi \left(\frac{l-h}{\sqrt{l^2+25h^2}} \right)^2 \frac{\sqrt{\frac{12h^2}{\pi}} \left(\sqrt{l^2+25h^2}\right)^2}{\sqrt{\frac{12h^2}{\pi}} + \sqrt{\frac{12h^2}{\pi} + \left(\frac{l-h}{\sqrt{l^2+25h^2}}\right)^2 \left(\sqrt{l^2+25h^2}\right)^2}} \right) \right\rceil \\ &= \left\lceil \frac{1}{9l + 4\sqrt{(l-h)^2+h^2} + 25h} \left(12l + 4\sqrt{(l-h)^2+h^2} + 4\sqrt{(l-2h)^2+h^2} + 23h + \frac{\sqrt{12\pi}(l-h)^2}{\sqrt{\frac{12h^2}{\pi}} + \sqrt{\frac{12h^2}{\pi} + (l-h)^2}} \right) \right\rceil \\ &= 3, \end{aligned}$$

since

$$\lim_{h \to 0^+} \frac{12l + 4\sqrt{(l-h)^2+h^2} + 4\sqrt{(l-2h)^2+h^2} + 23h + \frac{\sqrt{12\pi}(l-h)^2}{\sqrt{\frac{12h^2}{\pi}} + \sqrt{\frac{12h^2}{\pi} + (l-h)^2}}}{9l + 4\sqrt{(l-h)^2+h^2} + 25h} = \frac{20 + \sqrt{12\pi}}{13} \in (2, 3).$$

Example 3.4 (The set U in $\mathbb{R}^3$). Let $l > 3h > 0$ and consider the set $U \subset \mathbb{R}^3$ in Figure 5. We can compute

$$\mathcal{H}^2(\partial U) = 4hl + 10h^2, \quad \mathcal{H}^2(\partial(\mathrm{co}(U))) = 6hl + 4h^2, \quad \mathrm{diam}(\mathrm{co}(U)) = \sqrt{l^2 + 5h^2}.$$

Since U is not convex, we must have that $k_{\min}(U) \geq 2$, and a simple geometric argument allows to conclude that $k_{\min}(U) = 3$. From (1.2) we deduce that

$$k_{\min}(U) \geq \left\lceil \frac{\mathcal{H}^2(\partial U)}{\mathcal{H}^2(\partial(\mathrm{co}(U)))} \right\rceil = 1,$$

since an elementary computation shows that

$$\frac{\mathcal{H}^2(\partial U)}{\mathcal{H}^2(\partial(\mathrm{co}(U)))} = \frac{4l + 10h}{6l + 4h} \in (0, 1)$$

whenever $l > 3h > 0$. We now consider the points $P, Q \in \partial U$ as shown in Figure 5. For every decomposition of U into convex bodies, there exists two convex bodies E_j and E_k containing P and Q respectively. Since the segment PQ is not contained in U, it follows that E_j cannot contain Q. Since E_j is convex and contained in U, we must have that $E_j \subset H_j$, where H_j is the half-space such that ∂H_j contains the face of U to which the point P belongs, see Figure 5. Consequently, we must have

$$h(\mathrm{co}(U) \cap H_j, \mathrm{co}(U)) = l - h, \quad \mathcal{H}^2(\mathrm{co}(U) \cap \partial H_j) = 2h^2, \quad \rho_{\nu_j}(\mathrm{co}(U)) = \sqrt{\frac{2h^2}{\pi}},$$

where $\nu_j \in \mathbb{S}^2$ is the inner unit normal of the half-space H_j as in Figure 5. By the symmetry of U, a similar argument can be used for the convex component E_k containing Q. We now let $l > 0$ be fixed. We apply Theorem 1.2 with

$$p = 2, \quad \alpha = \frac{l - h}{\sqrt{l^2 + 5h^2}}, \quad \beta = 1.$$

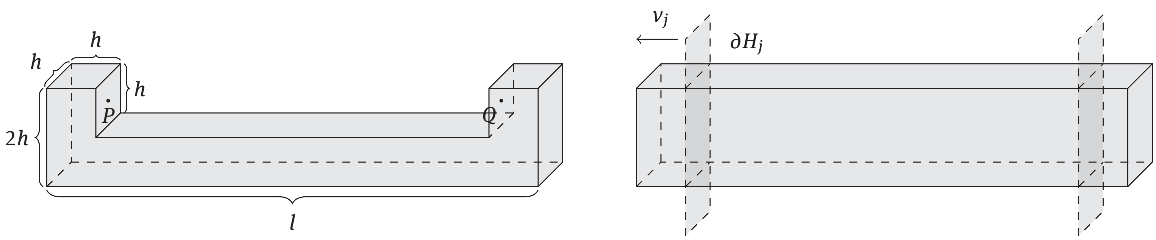

Figure 5: The set $U \subset \mathbb{R}^3$ (on the left) and its convex hull (on the right).

Provided that we choose $h \in (0, \frac{l}{3})$ sufficiently small, we conclude that

$$k_{\min}(U) \geq \left\lceil \frac{1}{6hl + 4h^2}\left(4hl + 10h^2 + 2\pi\left(\frac{l-h}{\sqrt{l^2+5h^2}}\right)^2 \frac{\sqrt{\frac{2h^2}{\pi}}(\sqrt{l^2+5h^2})^2}{\sqrt{\frac{2h^2}{\pi}} + \sqrt{\frac{2h^2}{\pi} + (\frac{l-h}{\sqrt{l^2+5h^2}})^2(\sqrt{l^2+5h^2})^2}}\right)\right\rceil$$
$$= \left\lceil \frac{1}{6l + 4h}\left(4l + 10h + \frac{2\sqrt{2\pi}(l-h)^2}{\sqrt{\frac{2h^2}{\pi}} + \sqrt{\frac{2h^2}{\pi} + (l-h)^2}}\right)\right\rceil = 2,$$

since

$$\lim_{h\to 0^+} \frac{4l + 10h + \frac{2\sqrt{2\pi}(l-h)^2}{\sqrt{\frac{2h^2}{\pi}} + \sqrt{\frac{2h^2}{\pi} + (l-h)^2}}}{6l + 4h} = \frac{4 + 2\sqrt{2\pi}}{6} \in (1, 2).$$

The above computations prove that, in this case, although the lower bound given by (1.11) is strictly better than the one given by (1.2), inequality (1.11) is not sharp.

3.3 An example in $\mathbb{R}^n$

We conclude this section with Example 3.6 below, showing that for all $n \geq 3$ our Theorem 1.2 provides an improvement of inequality (1.2) established in [9]. In Example 3.6 we will need to apply the following result, whose elementary proof is detailed below for the reader's convenience.

Lemma 3.5. *Let $\ell \in (0, +\infty)$ and let $Q \subset \mathbb{R}^2$ be a set with*

$$\mathcal{H}^1(\partial Q) < +\infty \quad \text{and} \quad \mathcal{H}^2(Q) < +\infty.$$

If $E_n = Q \times [0, \ell]^{n-2} \subset \mathbb{R}^n$, then

$$\mathcal{H}^{n-1}(\partial E_n) = \ell^{n-2}\mathcal{H}^1(\partial Q) + 2(n-2)\ell^{n-3}\mathcal{H}^2(Q) \tag{3.1}$$

for all $n \geq 2$.

Proof. By definition, the set $E_n \subset \mathbb{R}^n$ satisfies

$$\mathcal{H}^n(E_n) = \ell^{n-2}\mathcal{H}^2(Q). \tag{3.2}$$

Moreover, since we can recursively write $E_n = E_{n-1} \times [0, \ell]$ and thus

$$\partial E_n = ((\partial E_{n-1}) \times [0, \ell]) \cup (E_{n-1} \times \{0, \ell\}),$$

by the coarea formula we can compute

$$\mathcal{H}^{n-1}(\partial E_n) = 2\mathcal{H}^{n-1}(E_{n-1}) + \ell\mathcal{H}^{n-2}(\partial E_{n-1})$$

for all $n \geq 2$. The validity of (3.1) can thus be checked by induction, thanks to (3.2). □

Example 3.6 (The set $L_n \subset \mathbb{R}^n$ for $n \geq 3$). Let $l > h > 0$ and $\lambda > 1$ and consider the set $L_n = L_2 \times [0, h]^{n-2} \subset \mathbb{R}^n$ for $n \geq 3$, where $L_2 \subset \mathbb{R}^n$ is the set in Figure 6. Note that

$$\mathcal{H}^1(\partial L_2) = 2l + 2\lambda h, \quad \mathcal{H}^2(L_2) = h(l + (\lambda - 1)h)$$

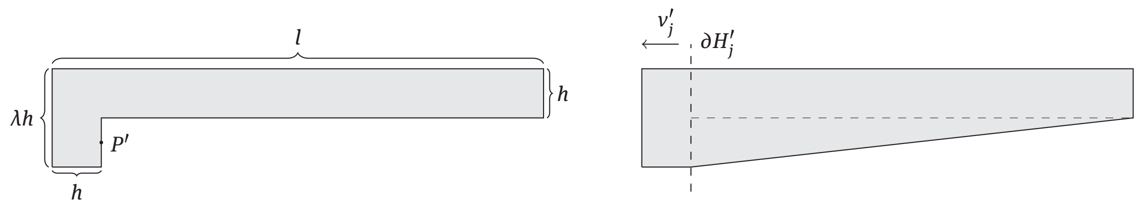

Figure 6: The body $L_2 \subset \mathbb{R}^2$ (on the left) and its convex hull (on the right).

and, similarly,

$$\mathcal{H}^1(\partial(\mathrm{co}(L_2))) = l + \sqrt{(l-h)^2 + (\lambda-1)^2h^2} + (\lambda+2)h,$$
$$\mathcal{H}^2(\mathrm{co}(L_2)) = \frac{h}{2}((\lambda+1)l + (\lambda-1)h).$$

Since $\mathrm{co}(L_n) = \mathrm{co}(L_2) \times [0,h]^{n-2}$, we can apply Lemma 3.5 to compute

$$\mathcal{H}^{n-1}(\partial L_n) = 2h^{n-2}((n-1)l + ((n-1)\lambda - n + 2))h),$$
$$\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n))) = h^{n-2}(((n-2)\lambda + n - 1)l + \sqrt{(l-h)^2 + (\lambda-1)^2h^2} + ((n-1)\lambda - n + 4)h),$$
$$\mathrm{diam}(\mathrm{co}(L_n)) = \sqrt{l^2 + (\lambda^2 + n - 2)h^2}$$

for all $n \geq 3$. Note that L_n is not convex, so we must have that $k_{\min}(L_n) \geq 2$ for all $n \geq 3$. In fact, a simple geometric decomposition proves that $k_{\min}(L_n) = 2$ for all $n \geq 3$. We now consider the point $P = (P', 0) \in L_n$, where $P' \in \partial L_2$ is shown in Figure 6. For every decomposition of L_n into convex bodies, there exists a convex body E_j containing P. Since E_j is convex and contained in L_n, we must have that its projection $E'_j = \mathsf{P}_{\mathbb{R}^2}(E_j)$ is a convex body contained in $L_2 \cap H'_j$, where $\mathsf{P}_{\mathbb{R}^2} : \mathbb{R}^n \to \mathbb{R}^2$ is the canonical projection onto the first two coordinates and H'_j is the half-plane such that $\partial H'_j$ contains the face of L_2 to which the point P belongs, see Figure 6. Therefore, we must have that $E_j \subset H_j$, where H_j is the half-space $H_j = \mathsf{P}^{-1}_{\mathbb{R}^2}(H'_j) \subset \mathbb{R}^n$. Consequently, we must have

$$h(\mathrm{co}(L_n) \cap H_j, \mathrm{co}(L_n)) = l - h, \quad \mathcal{H}^{n-1}(\mathrm{co}(L_n) \cap \partial H_j) = \lambda h^{n-1}, \quad \rho_{\nu_j}(\mathrm{co}(L_n)) = \sqrt[n-1]{\frac{\lambda h^{n-1}}{\omega_{n-1}}},$$

where $\nu_j \in \mathbb{S}^{n-1}$ is the inner unit normal of the half-space H_j (precisely, $\nu_j = (\nu'_j, 0)$, where ν'_j is the inner unit normal of H'_j, see Figure 6). We now let $l > 0$ be fixed. We apply Theorem 1.2 with

$$p = 1, \quad \alpha = \frac{l-h}{\sqrt{l^2 + (\lambda^2 + n - 2)h^2}}, \quad \beta = 1.$$

We are going to choose $\lambda > 1$ as a dimensional constant and $h \in (0, l)$ sufficiently small. Indeed, for any given $\lambda > 1$, we have that

$$\lim_{h\to 0^+} \frac{\mathcal{H}^{n-1}(\partial L_n)}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} = \frac{2n-2}{(n-2)\lambda + n}$$

and, similarly,

$$\lim_{h\to 0^+} \frac{\mathcal{H}^{n-1}(\partial L_n) + \omega_{n-1}\alpha^2\beta^{\frac{n-2}{n-1}} \frac{\rho_{\nu_j}(\mathrm{co}(L_n))^{n-2}\,\mathrm{diam}(\mathrm{co}(L_n))^2}{\rho_{\nu_j}(\mathrm{co}(L_n)) + \sqrt{\rho_{\nu_j}(\mathrm{co}(E))^2 + \alpha^2\,\mathrm{diam}(\mathrm{co}(L_n))^2}}}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} = \frac{2n - 2 + c_n\lambda^{\frac{n-2}{n-1}}}{(n-2)\lambda + n},$$

where $c_n = \omega_{n-1}^{\frac{1}{n-1}} > 0$ is a dimensional constant. Since $\lambda > 1$, we have that

$$\frac{2n-2}{(n-2)\lambda + n} < 1 \quad \text{for all } n \geq 3.$$

On the other hand, we obviously have

$$\frac{2n - 2 + c_n\lambda^{\frac{n-2}{n-1}}}{(n-2)\lambda + n} > 1 \iff \lambda^{\frac{n-2}{n-1}} > \frac{n-2}{c_n}(\lambda - 1)$$

and it is possible to verify that the last inequality admits solutions in the interval $(1, +\infty)$. Consequently, for each $n \geq 3$ we can find $\lambda_n \in (1, +\infty)$ such that

$$\frac{2n - 2 + c_n \lambda_n^{\frac{n-2}{n-1}}}{(n-2)\lambda_n + n} > 1.$$

Therefore, provided that we choose $\lambda = \lambda_n$ as above and $h \in (0, l)$ sufficiently small, we conclude that the set $L_n \subset \mathbb{R}^n$ corresponding to these choices of parameters satisfies

$$\left\lceil \frac{\mathcal{H}^{n-1}(\partial L_n)}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} \right\rceil = 1$$

and

$$\left\lceil \frac{1}{\mathcal{H}^{n-1}(\partial(\mathrm{co}(L_n)))} \left(\mathcal{H}^{n-1}(\partial L_n) + \omega_{n-1} \alpha^2 \beta^{\frac{n-2}{n-1}} \frac{\rho_{\nu_{i_j}}(\mathrm{co}(L_n))^{n-2} \operatorname{diam}(\mathrm{co}(L_n))^2}{\rho_{\nu_j}(\mathrm{co}(L_n)) + \sqrt{\rho_{\nu_j}(\mathrm{co}(E))^2 + \alpha^2 \operatorname{diam}(\mathrm{co}(L_n))^2}} \right) \right\rceil = 2.$$

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References

[1] Archimedes, *The Works of Archimedes. Vol. I*, Cambridge University, Cambridge, 2004.
[2] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, 1987.
[3] G. Buttazzo, V. Ferone and B. Kawohl, Minimum problems over sets of concave functions and related questions, *Math. Nachr.* **173** (1995), 71–89.
[4] M. Carozza, F. Giannetti, F. Leonetti and A. Passarelli di Napoli, A sharp quantitative estimate for the perimeters of convex sets in the plane, *J. Convex Anal.* **22** (2015), no. 3, 853–858.
[5] M. Carozza, F. Giannetti, F. Leonetti and A. Passarelli di Napoli, A sharp quantitative estimate for the surface areas of convex sets in $\mathbb{R}^3$, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 3, 327–333.
[6] M. Carozza, F. Giannetti, F. Leonetti and A. Passarelli di Napoli, Convex components, *Commun. Contemp. Math.* **21** (2019), no. 6, Article ID 1850036.
[7] F. Giannetti, Sharp geometric quantitative estimates, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **28** (2017), no. 1, 1–6.
[8] P. M. Gruber, *Convex and Discrete Geometry*, Grundlehren Math. Wiss. 336, Springer, Berlin, 2007.
[9] M. La Civita and F. Leonetti, Convex components of a set and the measure of its boundary, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* **56** (2008/09), 71–78.
[10] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems*, Cambridge Stud. Adv. Math. 135, Cambridge University, Cambridge, 2012.
[11] G. Stefani, On the monotonicity of perimeter of convex bodies, *J. Convex Anal.* **25** (2018), no. 1, 93–102.