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Mathematical Analysis, Modelling, and Applications

Depicting instabilities in linear dispersive PDEs:
a dynamical systems approach

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Abstract

In the present thesis, we study instabilities in different classes of linear dispersive PDEs, in the form of weak turbulence formation and energy cascades. We use the growth in time of positive Sobolev norms of solutions as an indicator of instability in terms of transfer of energy towards higher frequencies.

We identify a resonant mechanism responsible for instability formation in different models. More precisely, we base our procedure on a combination of resonant normal forms, pseudodifferential techniques and positive commutator arguments, and we reduce the instability problem to the study of the underlying classical dynamics. We obtain three different results.

The first concerns time dependently perturbed quantum harmonic oscillators in $\mathbb{R}^2$, where the perturbation is a selfadjoint pseudodifferential operator of degree zero, 2π -periodic in time. We identify sufficient conditions on the principal symbol of the potential that ensure existence of solutions exhibiting unbounded growth in time of their positive Sobolev norms and we show that the class of symbols satisfying such conditions is generic in the Fréchet space of classical 2π -time periodic symbols of order zero. To prove our result we apply the abstract Theorem of [69]: the main difficulty is to find a conjugate operator A for the resonant average of the perturbation. We construct explicitly the symbol of the conjugate operator A , called escape function, combining techniques from microlocal analysis, dynamical systems and contact topology.

In the second result, we study linear and time-dependent perturbations of periodic transport equations on the two-dimensional torus. For generic perturbations, we prove

the existence of a large class of initial data whose Sobolev norms diverge exponentially fast. In higher dimensions, the same conclusion holds under a Morse-Smale assumption on the resonant part of the perturbation. In both cases, the proof is based on a normal form procedure and on the study of Sobolev instabilities for time-dependent perturbations of Morse-Smale transport equations. The latter are analyzed on general compact manifolds using techniques from microlocal analysis and hyperbolic dynamics.

Finally, we consider the one-dimensional instance of the previous model: we study linear, time-dependent and skewadjoint perturbations of periodic transport equations on the one-dimensional torus. In this low-dimensional setting, we are able to describe the long-time behavior of solutions for all non-degenerate perturbations in resonant regime, proving that either there exist solutions whose Sobolev norms grow exponentially fast, provoking an energy transfer towards higher frequencies, or all solutions remain stable over arbitrarily long time scales. In this case as well, the proof combines pseudodifferential tools with dynamical systems techniques: a resonant normal form procedure reduces the problem to the analysis of the classical dynamics associated with the resonant equation. The main difficulty lies in the instability mechanism, for which we explicitly construct an escape function adapted to the dynamics and use the associated operator for a positive commutator estimate.

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Chapter 1

Introduction

The study of instabilities in linear and nonlinear PDEs is a problem of broad mathematical interest and an active area of current research, that has been approached from different viewpoints. The phenomena considered in the present thesis are weak turbulence formation and energy cascades in dispersive equations. In this framework, the time evolution of the Sobolev norms of solutions is a widely used tool to depict unstable paths: an unbounded growth in time of the positive Sobolev norms of solutions is an indicator of instability, in the form of transfer of energy from low to high modes in the frequency space.

The present thesis provides new results in this direction: we analyze resonant time-dependent perturbations of certain classes of linear dispersive equations, and we identify models admitting *generic* sets of perturbations that trigger instability phenomena.

In the next paragraphs, we will describe precisely the models we study. Nonetheless, we anticipate here some common features they share, to motivate our results. A first common feature is the conservation of a low norm (in our case the L^2 -norm) of all solutions along the time evolution. This is due to the selfadjointness of the operators governing the dynamics. We use this conservation law to describe the common principle underlying all our instability results: the formation of instabilities that we describe is entirely caused by a redistribution of energy across frequencies (and not by other mechanisms). In this sense, an unbounded growth in time of some positive Sobolev norms is an indicator of weakly turbulent dynamics.

A second common feature is that the underlying mechanism behind the instability phenomena is purely resonant. All the models considered in the present thesis are obtained as time-periodic perturbations of principal operators whose solutions are all periodic in time and have the same period. Equivalently, this property can be read at the spectral level by noting that the unperturbed operator has discrete spectrum with constant spectral gaps. For this reason, the time-frequency of the perturbation is tuned to be resonant with the spectral gaps of the unperturbed operator. Consequently, the philosophy underlying the results presented is to identify the resonant component of the perturbation, and use it to determine sufficient conditions to enhance resonant interactions between Fourier modes, giving rise to the instability phenomena and, ultimately, to unbounded trajectories.

Finally, for all the equations we consider we have at our disposal global upper bounds for the growth in time of the Sobolev norms of all solutions. More precisely, in the work [72] by Maspero and Robert the authors consider an abstract framework encompassing all the models studied in this thesis and obtain global upper bounds for the Sobolev norms of all solutions. In all the cases considered, we identify large classes of perturbations for which these upper bounds are saturated: the growth rates we obtain are thus the optimal ones.

In order to contextualize our results, we briefly review the main developments that laid the groundwork for this line of research, referring to Section 1.3 for a complete overview of the literature.

The problem of growth of Sobolev norms for linear, Schrödinger-type equations on the torus has been first issued by Bourgain in [18] in 1999 and, since then, the research in this direction has been very active. In recent years, many authors have produced examples of time-dependent perturbations provoking unbounded growth in time of some positive Sobolev norms of some solutions of the corresponding equations. Here we only mention that, after the pioneering work of Bourgain in [18], Delort in [29] is the first to construct unbounded trajectories for quantum harmonic oscillators. We postpone to Section 1.3 a wider overview of the literature for results of this kind.

A new paradigm in this direction is developed by Maspero in [69]: the author studies time-periodic perturbations of operators with constant spectral gaps and introduces

a new approach to detect instability. The main idea is to isolate the time-independent, resonant component of the perturbation via a resonant normal form procedure, and then establish a Mourre positive commutator estimate for the resulting operator. As shown in [69], such Mourre estimate is a sufficient condition for the existence of unbounded trajectories.

The challenge in implementing this strategy is the verification of the Mourre estimate: this in turn requires to construct a *conjugate operator* for the resonant average of the perturbation or, in the pseudodifferential setting, an *escape function* for its principal symbol.

The first work in which this goal is achieved is [70], where the author studies time-periodic perturbations of quantum harmonic oscillators on $\mathbb{R}$. As a result, it is proven that the set of perturbations generating solutions with unbounded growth in time of their positive Sobolev norms is *generic* among time-periodic, classical pseudodifferential operators of order zero. The argument used in [70] crucially relies on the low dimensionality of the problem, where it turns out that the escape function can be constructed explicitly. However, it is not possible to extend such construction in dimension greater or equal than two: different techniques and ideas need to be introduced to tackle the higher dimensional setup.

The present thesis aims at proceeding in this direction by developing a general strategy to construct escape functions in higher-dimensional settings. To this aim, we combine techniques of dynamical systems, contact topology and microlocal analysis, thereby extending the applicability of the instability criterion of [69] to a broad class of higher-dimensional models. All our results provide explicit sufficient conditions on the perturbation to provoke growth of Sobolev norms, and show that the sets of perturbations verifying them are very large. Indeed, we emphasize once more that the main novelty of these results is to identify *generic* (in a sense to be specified later in this introduction) sets of resonant, time-periodic perturbations of first-order operators that generate unstable dynamics. This is the main distinction with the previously mentioned results where isolated instances of such perturbations were constructed. The ideas briefly introduced in this paragraph will be better explained in Sections 1.2.1-1.2.3.

We conclude by presenting informally the results that we prove in this thesis, and providing reference to the articles where these problems are studied:

1. Quantum harmonic oscillators in $\mathbb{R}^2$: we consider

$$i\partial_t u = H_0 u + V(t, x, D)u, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where

$$H_0 := \frac{1}{2}(-\partial_{x_1}^2 - \partial_{x_2}^2 + x_1^2 + x_2^2), \quad D := (D_1, D_2) = -(i\partial_{x_1}, i\partial_{x_2}),$$

and $V(t, x, D)$ is a pseudodifferential operator of degree zero, selfadjoint for every t and 2π -periodic in time. We prove existence of a generic set of perturbations, among classical pseudodifferential perturbations V of order zero, provoking growth of Sobolev norms.

This is object of study of the work "*Growth of Sobolev norms for completely resonant quantum harmonic oscillators on $\mathbb{R}^2$* " ([61]), joint work with B. Langella and A. Maspero, published in *Journal of Differential Equations*.

2. Unstable perturbations of periodic transport equations: we study the following class of transport equations on n -dimensional tori $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$

$$\partial_t u_\varepsilon = (\nu + \varepsilon V(t, x)) \cdot \nabla_x u_\varepsilon + \frac{\varepsilon}{2} \operatorname{div}_x(V(t, x))u_\varepsilon, \quad x \in \mathbb{T}^n, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad (1.0.1)$$

where $\nu \in \mathbb{N}^n$, $V(t, x) \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ is a time-dependent vector field, $\operatorname{div}(\cdot)$ denotes the divergence with respect to the standard Euclidean volume on $\mathbb{T}^n$ and $\varepsilon > 0$ plays the role of a parameter. We prove that, in the case $n = 2$ and for generic potentials $V \in C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ there exist solutions whose positive Sobolev norms grow exponentially.

In fact, we also prove our result in the more general framework of linear, skew-adjoint transport equations on arbitrary compact manifolds. We identify an open set of transport coefficients leading to instability, and we prove that it is also dense in the set of smooth vector fields on compact surfaces (i.e., in dimension two).

This is object of study of the work "*Sobolev instability for perturbations of periodic transport equations*" ([79]), joint work with G. Rivière, submitted.

3. Long-time behavior of resonant transport equations in dimension one:

we study the particular case of equation (1.0.1) when $n = 1$ and we prove that, depending on the choice of the perturbation, only two possibilities occur: either all Sobolev norms of all solutions are uniformly bounded for long time scales, or instability phenomena arise. This provides a complete characterization of both the sets of stable and unstable potentials in this one-dimensional setting.

This is object of study of the work "*Long-time behavior of resonant time-dependent perturbations of periodic transport equations on $\mathbb{R}$* " ([80]), published in *Nonlinearity*.

1.1 Main results

In this section, we state our main results in a precise form. For each model, we introduce the notation required to formulate the theorems and we complete the statements with some comments.

1.1.1 Resonant quantum harmonic oscillators in $\mathbb{R}^2$

We consider

$$i\partial_t u = H_0 u + V(t, x, D)u, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (1.1.1)$$

where

$$H_0 := \frac{1}{2}(-\partial_{x_1}^2 - \partial_{x_2}^2 + x_1^2 + x_2^2), \quad D := (D_1, D_2) = -(\mathrm{i}\partial_{x_1}, \mathrm{i}\partial_{x_2}),$$

and $V(t, x, D)$ is a pseudodifferential operator of degree zero (see Definition 1.1.3), selfadjoint for every t and 2π -periodic in time, and we prove existence of global in time solutions fulfilling

$$\lim_{t \rightarrow +\infty} \|u(t)\|_s = +\infty, \quad s > 0;$$

here the norm $\|\cdot\|_s$ is the one of the Sobolev space

$$\mathcal{H}_{H_0}^s(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2; \mathbb{C}) : \|u\|_s := \|H_0^s u\|_{L^2} < +\infty\}, \quad s \in \mathbb{R}. \quad (1.1.2)$$

Note immediately that, when $V \equiv 0$, every solution of (1.1.1) has constant $\mathcal{H}_{H_0}^s$ -norm for every s and every time. Hence, as already alluded, the existence of solutions

with unbounded paths depends on the ability of the perturbation $V(t, x, D)$ to couple together sufficiently many different Fourier modes to provoke energy cascades towards higher and higher frequencies.

We now define the set of functions and then the class of symbols that we will use:

Definition 1.1.1 (Homogeneous function). A smooth function $f: \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{C}$ is said to be *positively homogeneous of degree m* if

$$f(\lambda x, \lambda \xi) = \lambda^m f(x, \xi), \quad \forall \lambda > 0, \quad \forall (x, \xi) \neq 0, \quad (1.1.3)$$

and *definitely homogeneous* if it is smooth at zero and (1.1.3) holds only for $\lambda \geq 1$ and $|(x, \xi)| \geq 1$.

Definition 1.1.2 (Symbols). We define the class of symbols of order $\rho \in \mathbb{R}$ as

$$\begin{aligned} \mathbf{S}_{H_0}^\rho := & \left\{ f \in C^\infty(\mathbb{R}^4, \mathbb{C}) : \forall \alpha, \beta \in \mathbb{N}_0^2, \exists C_{\alpha, \beta} > 0 \right. \\ & \left. \text{such that } |\partial_x^\alpha \partial_\xi^\beta f(x, \xi)| \leq C_{\alpha, \beta} (1 + |x|^2 + |\xi|^2)^{\rho - \frac{|\alpha| + |\beta|}{2}} \right\}. \end{aligned}$$

We say that a symbol $f \in \mathbf{S}_{H_0}^\rho$ is *classical* of order ρ if there exist f_ρ definitely homogeneous of degree 2ρ and $\mu < \rho$ such that

$$f - f_\rho := f_\mu \in \mathbf{S}_{H_0}^\mu.$$

We call f_ρ the *principal symbol* of f , and denote by $\mathbf{S}_{cl}^\rho$ the set of classical symbols of order ρ . Note that the inclusion $\mathbf{S}_{cl}^0 \subset \mathbf{S}_{H_0}^0$ holds.

The space $\mathbf{S}_{H_0}^\rho$, endowed with the family of seminorms

$$\wp_j^\rho(f) := \sum_{|\alpha| + |\beta| \leq j} \sup_{x, \xi \in \mathbb{R}^2} \frac{|\partial_x^\alpha \partial_\xi^\beta f(x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{\rho - \frac{|\alpha| + |\beta|}{2}}}, \quad \forall j \in \mathbb{N},$$

is a Fréchet space with distance defined as

$$\mathbf{d}^\rho(f, g) := \sum_{j \geq 0} \frac{1}{2^j} \frac{\wp_j^\rho(f - g)}{1 + \wp_j^\rho(f - g)}.$$

Analogously, we also define the Fréchet space $C^0(\mathbb{T}; \mathbf{S}_{H_0}^\rho)$ of continuous in time maps $f: t \rightarrow f(t, \cdot) \in \mathbf{S}_{H_0}^\rho$, with finite seminorms:

$$\wp_{\mathbb{T}, j}^\rho(f) := \sum_{|\alpha| + |\beta| \leq j} \sup_{\substack{x, \xi \in \mathbb{R}^2 \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta f(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{\rho - \frac{|\alpha| + |\beta|}{2}}}, \quad \forall j \in \mathbb{N}.$$

The distance between two functions $f, g \in C^0(\mathbb{T}; \mathbf{S}_{H_0}^\rho)$ is hence given by:

$$\mathbf{d}_{\mathbb{T}}^\rho(f, g) := \sum_{j \geq 0} \frac{1}{2^j} \frac{\wp_{\mathbb{T}, j}^\rho(f - g)}{1 + \wp_{\mathbb{T}, j}^\rho(f - g)}. \quad (1.1.4)$$

We now define pseudodifferential operators of order ρ :

Definition 1.1.3 (Pseudodifferential operators). We say that the operator F belongs to $\mathcal{S}_{H_0}^\rho$, the class of pseudodifferential operators of order ρ , if there exists a symbol $f \in \mathbf{S}_{H_0}^\rho$ such that $F = \text{Op}^w(f)$, where

$$\text{Op}^w(f)[\psi](x) := \frac{1}{2\pi} \iint_{y, \xi \in \mathbb{R}^2} e^{i(x-y)\xi} f\left(\frac{x+y}{2}, \xi\right) \psi(y) dy d\xi \quad (1.1.5)$$

is the Weyl quantization of the symbol f .

Finally, we define the resonant average of a symbol v and the average of an operator V : we denote by $\phi_{h_0}^t(x_1, x_2, \xi_1, \xi_2)$ the classical flow of the harmonic oscillator in $\mathbb{R}^4$ with initial datum (x_1, x_2, ξ_1, ξ_2) , where h_0 is defined as in (2.1.2):

$$\phi_{h_0}^t(x, \xi) = (x_1 \cos t + \xi_1 \sin t, x_2 \cos t + \xi_2 \sin t, -x_1 \sin t + \xi_1 \cos t, -x_2 \sin t + \xi_2 \cos t). \quad (1.1.6)$$

One easily checks the following result:

Lemma 1.1.4. *Let $a(t, x, \xi) \in C^0(\mathbb{T}; \mathbf{S}_{H_0}^0)$, then $a(t, \phi_{h_0}^t(x, \xi)) \in C^0(\mathbb{T}; \mathbf{S}_{H_0}^0)$.*

Definition 1.1.5 (Resonant average). Given a symbol $v \in C^0(\mathbb{T}; \mathbf{S}_{H_0}^0)$, we define the resonant average $\langle v \rangle_{h_0}$ with respect to the flow (1.1.6) of the harmonic oscillator as:

$$\langle v \rangle_{h_0}(x, \xi) := \frac{1}{2\pi} \int_0^{2\pi} v(t, \phi_{h_0}^t(x, \xi)) dt, \quad (1.1.7)$$

which is a time independent symbol of order zero. In addition if $v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$, then $\langle v \rangle_{h_0} \in \mathbf{S}_{cl}^0$. We define the average of the pseudodifferential operator $V \in C^0(\mathbb{T}, \mathcal{S}_{H_0}^0)$, $V(t, x, D) = \text{Op}^w(v(t, x, \xi))$, as:

$$\langle V \rangle_{H_0} := \text{Op}^w(\langle v \rangle_{h_0}), \quad (1.1.8)$$

which is a time independent operator, $\langle V \rangle_{H_0} \in \mathcal{S}_{H_0}^0$.

We are finally ready to state our theorem:

Theorem 1.1.6. *Denote by*

$$\mathcal{V} := \left\{ v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0) : \exists I \subset \text{Ran}(\langle v_0 \rangle_{h_0}), \exists a \in \mathbf{S}_{cl}^1 \text{ and } \exists \delta > 0 \right. \\ \left. \text{such that } \{ \langle v_0 \rangle_{h_0}, a \} \geq \delta \quad \forall (x, \xi) \in \langle v_0 \rangle_{h_0}^{-1}(I) \cap \{ |(x, \xi)| \geq 1 \} \right\}, \quad (1.1.9)$$

where v_0 denotes the principal symbol of v . Then

(i) **Genericity:** *The set $\mathcal{V}$ is generic in $C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$; precisely it is open and dense with respect to the metric $\mathbf{d}_{\mathbb{T}}^0$ in (1.1.4).*

(ii) **Instability:** *for every $v \in \mathcal{V}$, for every $s > 0$, there exist a function $u_0 \in \mathcal{H}_{H_0}^s(\mathbb{R}^2)$ and a time $T > 0$ such that the solution $u(t, x)$ of equation (1.1.1) with $V(t, x, D) = \text{Op}^w(v)$ and initial datum u_0 satisfies*

$$\|u(t)\|_s \geq C_{s, u_0} \langle t \rangle^s, \quad \forall t > T, \quad (1.1.10)$$

for some positive constant C_{s, u_0} .

The proof of Theorem 1.1.6 is the content of Chapter 2. We conclude this section with some comments about Theorem 1.1.6:

- First of all, we remark that the lower bound in (1.1.10) is sharp, indeed in [72] the authors prove that, for all $s > 0$ and for all $u_0 \in \mathcal{H}_{H_0}^s$, there exists a constant $\tilde{C}_s > 0$ such that $\|u(t)\|_s \leq \tilde{C}_s \langle t \rangle^s \|u_0\|_s$, showing that our unstable solutions grow with the optimal speed.
- We also point out that, in the recent paper [70], the author proves that, in one-spatial dimension, generic 2π -periodic pseudodifferential perturbations $V(t, x_1, D_1)$ of the harmonic oscillator provoke growth of Sobolev norms. However, such one-dimensional perturbations, when considered as perturbations on $\mathbb{R}^2$, are neither pseudodifferential, nor generic in the higher dimensional setting. Moreover, as we will detail in Section 1.2.1, in the one-dimensional setting of [70], both the construction of the escape function and the proof that (1.1.10) holds for generic symbols can be done in a rather direct and explicit way. This is not possible in the present higher dimensional setting and has further motivated the study of this problem with new different techniques.

We refer to Section 1.2.1 for a description of the main ideas used in the proof.

1.1.2 Unstable perturbations of periodic transport equations

In [79], in collaboration with Gabriel Rivière, we identify a class of equations for which the previous instability result (described in Section 1.1.1) can be generalized to arbitrary compact and smooth manifolds M . In particular, we are able to extend the study of the resonant mechanism leading to instability formation to this more general topological setting. Nevertheless, due to the peculiar properties of the torus, we are able to achieve two different but closely related goals in the cases $M = \mathbb{T}^n$, and M arbitrary. We now state and comment separately both results.

We first study a class of transport equations on n -dimensional tori. More precisely, for any $n \geq 1$, we denote by $\mathbb{T}^{n+1}$ (with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$) the Euclidean $(n+1)$ -dimensional torus. Given $\nu \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$, we consider the following equations

$$\partial_t u_\varepsilon = (\nu + \varepsilon V(\omega t, x)) \cdot \nabla_x u_\varepsilon + \frac{\varepsilon}{2} \operatorname{div}_x (V(\omega t, x)) u_\varepsilon, \quad x \in \mathbb{T}^n, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad (1.1.11)$$

where $V(t, x) \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ is a time-dependent vector field, $\operatorname{div}(\cdot)$ denotes the divergence with respect to the standard Euclidean volume on $\mathbb{T}^n$ and $\varepsilon > 0$ plays the role of a parameter. Note immediately that, in analogy with (1.1.1), the transport operator on the right-hand side of the equation is formally skew-adjoint on $L^2(\mathbb{T}^n)$ for every $t \in \mathbb{R}$. Thus one has

$$\|u_\varepsilon(t)\|_{L^2} = \|u_\varepsilon(0)\|_{L^2}, \quad \forall t \in \mathbb{R}, \quad \forall \varepsilon \geq 0.$$

Moreover, it is worth noting that, for $\varepsilon = 0$, all the Sobolev norms are preserved, i.e.

$$\|u_0(t)\|_\sigma = \|u_0(0)\|_\sigma, \quad \forall t \in \mathbb{R}, \quad \forall \sigma \in \mathbb{R}.$$

Here we denote by

$$\mathcal{H}^\sigma(\mathbb{T}^n) \equiv \mathcal{H}_{\mathbb{T}^n}^\sigma := \{u \in L^2(\mathbb{T}^n; \mathbb{C}) : \|u\|_\sigma^2 := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2\sigma} |u_k|^2 < +\infty\}, \quad (1.1.12)$$

(where we use the notation $\langle k \rangle := \sqrt{1 + |k|^2}$ for the Japanese brackets and we denote by $\{u_k\}_{k \in \mathbb{Z}^n}$ the Fourier coefficients of the function u). Our first main result is the following theorem:

Theorem 1.1.7. *Let $n = 2$, $\omega = 1$ and $\nu \in \mathbb{N}^2 \setminus \{0\}$. Then there exists an open and dense subset $\mathcal{O}_\nu$ of ¹ $C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ such that, for every $V \in \mathcal{O}_\nu$, the following holds.*

There exists $\delta_0 > 0$ such that, for every $0 \leq \sigma \leq \delta_0$ and for every $0 \leq \varepsilon \leq \delta_0 \sigma^2$, one can find $v_\varepsilon \in \mathcal{H}^\sigma(\mathbb{T}^2)$ and $r := r(\varepsilon, \sigma, v_\varepsilon) > 0$ such that

$$\|u_\varepsilon(0) - v_\varepsilon\|_\sigma \leq r \quad \implies \quad \|u_\varepsilon(t)\|_\sigma \geq \delta_0 e^{\delta_0 \sigma t \varepsilon} \|u_\varepsilon(0)\|_\sigma, \quad \text{for any } t > 0,$$

where $u_\varepsilon(t)$ is the solution to equation (1.1.11) with initial datum $u_\varepsilon(0)$.

The proof of Theorem 1.1.7 is presented in Chapter 3. Here follow some comments on the result:

- We immediately remark that, for every potential $V \in C^\infty(\mathbb{T} \times \mathbb{T}^2; \mathbb{R}^2)$, the Sobolev norms of all solutions to equation (1.1.11) grow at most exponentially fast with the L^2 -norm being preserved – see [72, Th. 1.2]. Thus, as anticipated, we identify a large class of time dependent transport equations for which this exponential growth is indeed sharp. Yet, we emphasize that the growth from this lower bound is effective for times $t \gg \varepsilon^{-1}$.
- Note that, in Theorem 1.1.7, we consider a resonant regime, i.e., the potential V has the same time-period as the solution to the unperturbed equation, since this may create resonance phenomena. This choice, which is in clear continuity with the previously outlined case of the harmonic oscillator, is nevertheless further motivated: in the previous work [10, Th. 2.4] (see also [39]), the authors study long-time dynamics of equation (1.1.11) in a complementary *non-resonant* regime. They prove that, for generic choices of parameters (ω, ν) in $[1, 2]^{n+1}$ satisfying some Diophantine conditions, equation (1.1.11) is *reducible*. Roughly speaking, it means that, up to a bounded transformation on the Sobolev space $\mathcal{H}^\sigma(\mathbb{T}^n)$, one can conjugate the equation to a diagonal one in the Fourier basis (up to smoothing remainders). A direct consequence of this result is that either there exist solutions whose positive Sobolev norms grow exponentially (in the case where the diagonal operator has eigenvalues with nonzero real parts) or the Sobolev norms $\|u_\varepsilon(t)\|_\sigma$ remain bounded (for instance, if V is even). See [10, Cor. 2.5] for details;

¹The set is endowed with its standard Fréchet topology.

- In order to produce instabilities we exploit the resonant part of the perturbation $V(t, x)$. Namely, we rely on the dynamical properties of the *resonant* vector field

$$\langle V \rangle_\nu(x) := \frac{1}{2\pi} \int_0^{2\pi} V(\tau, x - \tau\nu) d\tau, \quad x \in \mathbb{T}^n. \quad (1.1.13)$$

Indeed, after a normal form procedure (see Section 3.4), instabilities for (1.1.11) can be obtained from unstable solutions to time dependent perturbations of transport equations generated by $\langle V \rangle_\nu$.

Note that here we call *resonant* vector field (associated with the perturbation V) the vector field $\langle V \rangle_\nu$ defined in (1.1.13) and in Section 1.1.1 we define instead the *resonant average* of the perturbation as in Definition 1.1.5. Clearly the two definitions do not coincide, and this is a slight abuse of terminology. However, this is heuristically due to the fact that both are obtained performing a time average of the perturbation along the flow of the unperturbed equation.

- As we will explain later on (see Section 1.2.2), the set $\mathcal{O}_\nu$ consists of all perturbations $V \in C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ such that the resonant average (1.1.13) satisfies some dynamical properties (see Definition 1.1.8). We refer to Section 1.2.2 for a description of this class of vector fields, and we only anticipate here that, in dimension 2 these form a generic set among smooth vector fields (see [78]). This property is, however, peculiar of dimension 2 and we are not aware of analogous generic sets of vector fields in higher dimension. Hence, in higher dimensions, we will only be able to produce unstable solutions for an open set of perturbations.

We now state our second main result concerning transport equations on arbitrary compact manifolds M . In order to motivate the statement, we anticipate here that the proof of Theorem 1.1.7 proceeds in two steps: first, we perform a *resonant normal form* reduction, to conjugate equation (1.1.11) to a new transport equation having as leading term the resonant average $\langle V \rangle_\nu$ of V , up to small-in-size, time dependent perturbations. Next, we obtain an instability result for this latter class of equations. We postpone to Section 1.2.2 a more detailed explanation of our proof, but we anticipate here that the main technical accomplishment of the work is the study of instabilities for Morse-Smale transport equations (see Definition 1.1.8) on arbitrary smooth, compact

and Riemannian manifolds of arbitrary dimension. This is the content of the second theorem that we will prove in Chapter 3, and that we now state below.

We consider M a smooth (C^∞), compact and oriented manifold of dimension $n \geq 1$, without boundary. We endow the manifold M with a Riemannian metric that we denote by g and we let v_g be the corresponding Riemannian volume on M . We denote by $\mathfrak{X}(M)$ the set of smooth vector fields on M and we consider the following class of transport equations:

$$\partial_t u_\varepsilon = \mathcal{L}_{V + \varepsilon \mathcal{P}(t)} u_\varepsilon + \frac{1}{2} \operatorname{div}_{v_g}(V + \varepsilon \mathcal{P}(t)) u_\varepsilon, \quad x \in M, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad (1.1.14)$$

where $V \in \mathfrak{X}(M)$, $\mathcal{P}(t) \in C_b^0(\mathbb{R}; \mathfrak{X}(M))$ and div_{v_g} is the divergence with respect to the Riemannian volume form. Here, $\mathcal{L}$ denotes the usual Lie derivative, i.e., the order one differential operator defined by $\mathcal{L}_Y u := du[Y]$, for every $Y \in \mathfrak{X}(M)$, and $C_b^0(\mathbb{R}; \mathfrak{X}(M))$ denotes the set of time dependent vector fields on M , which are continuous and bounded in time. Note that, as in equation (1.1.11), the addition of the divergence term in equation (1.1.14) ensures that the operator on the right-hand side is skew-adjoint for every time $t \in \mathbb{R}$ so that

$$\|u_\varepsilon(t)\|_{L^2(M, v_g)} = \|u_\varepsilon(0)\|_{L^2(M, v_g)}, \quad \forall t \in \mathbb{R}, \quad \forall \varepsilon > 0.$$

Our main goal is to prove existence of unstable solutions for equation (1.1.14), where we again identify unstable solutions with the ones that exhibit an unbounded growth in time of their positive Sobolev norms. To this aim, recall that the Sobolev spaces $\mathcal{H}^\sigma(M)$ are defined as

$$\mathcal{H}^\sigma(M) := \left\{ u \in L^2(M; \mathbb{C}) : \|(1 - \Delta_g)^{\frac{\sigma}{2}} u\|_{L^2(M, v_g)} < +\infty \right\}, \quad \sigma \in \mathbb{R},$$

where Δ_g is the Laplace-Beltrami operator on M . The corresponding Sobolev norm will be denoted by $\|\cdot\|_\sigma$. We emphasize that, also in this general case, one has only the a priori bound for every $\sigma > 0$ [72, Th. 1.2]

$$\|u_\varepsilon(t)\|_\sigma \leq C_\sigma e^{C_\sigma |t|} \|u_\varepsilon(0)\|_\sigma, \quad \forall t \in \mathbb{R}, \quad \text{for some } C_\sigma > 0.$$

Our main theorem shows that this exponential growth is in fact sharp for the class of Morse-Smale vector fields in any dimension. Originally introduced by Smale in [82] as a

generalization of gradient flows induced by Morse functions, Morse-Smale vector fields on compact manifolds are characterized by a non-wandering set consisting exclusively of a finite number of hyperbolic critical elements (both fixed points and periodic orbits), with no saddle-to-saddle connections. Precisely, they are defined as follows:

Definition 1.1.8 (Morse-Smale vector fields). We say that a smooth vector field $V \in \mathfrak{X}(M)$ is Morse-Smale if

1. V has a finite number of critical elements (i.e. critical points and closed orbits), which are all hyperbolic (see Definition 3.5.1);
2. for any pair Λ_1, Λ_2 of critical elements, the stable manifold $W^s(\Lambda_1)$ and the unstable one $W^u(\Lambda_2)$ (see Definition 3.5.5) intersect transversally;
3. the non-wandering set (see Definition 3.5.3) $NW(V)$ coincides with the union of the critical elements of V .

We denote by $\mathfrak{X}^{MS}(M) \subset \mathfrak{X}(M)$ the set of Morse-Smale vector fields on M .

We are now in position to state our second main result for transport equations:

Theorem 1.1.9. *Let $V \in \mathfrak{X}(M)$ and $\mathcal{P} \in C_b^0(\mathbb{R}, \mathfrak{X}(M))$. Suppose that V is a Morse-Smale vector field (see Definition 1.1.8) and, if $n = 1$, suppose in addition that it has at least one critical point.*

Then, there exists $\delta_0 > 0$ such that, for every $0 \leq \sigma \leq \delta_0$ and for every $0 \leq \varepsilon \leq \delta_0 \sigma^2$, one can find $v_\varepsilon \in \mathcal{H}^\sigma(M)$ and $r := r(\varepsilon, \sigma, v_\varepsilon) > 0$ such that

$$\|u_\varepsilon(0) - v_\varepsilon\|_\sigma \leq r \quad \implies \quad \|u_\varepsilon(t)\|_\sigma \geq \delta_0 e^{\delta_0 \sigma t} \|u_\varepsilon(0)\|_\sigma, \quad \text{for any } t > 0, \quad (1.1.15)$$

where $u_\varepsilon(t)$ is the solution to equation (1.1.14) with initial datum $u_\varepsilon(0)$.

The proof of Theorem 1.1.9 is given in Chapter 3, and we refer to Section 1.2.2 for an outline of the main ideas used.

1.1.3 Long-time behavior of resonant transport equations in dimension one

In this section, we state our main result concerning one-dimensional transport equations in resonant regime: in this low dimensional setting, we are indeed able to completely classify the long-time dynamics of solutions, depending on the perturbation (and not only to identify the ones generating instability). Precisely, we consider

$$\partial_t u = (\nu + \varepsilon V(t, x)) \partial_x u + \frac{\varepsilon}{2} \partial_x V(t, x) u, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R}, \varepsilon > 0, \quad (1.1.16)$$

where $\nu \in \mathbb{N}^+$, ε is a positive parameter, $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$, and we use the definition in (1.1.12) with $n = 1$ for the scale of Sobolev spaces on $\mathbb{T}$. Roughly speaking we prove that, depending on the choice of the perturbation V , only two possibilities occur (up to a nowhere dense set of degenerate perturbations): either all Sobolev norms of all solutions are uniformly bounded for finite but arbitrarily long time scales, or instability phenomena arise. In other words, exploiting the simpler one-dimensional Hamiltonian dynamics associated with the problem, we completely characterize both the sets of stable and unstable perturbations. We now state precisely our result. First, we report here an equivalent definition of (1.1.13).

Definition 1.1.10 (Resonant average). Let $V \in C^\infty(\mathbb{T}^2; \mathbb{C})$. For $k, \ell \in \mathbb{Z}$ denote by $\{v_{k, \ell}\}_{k, \ell \in \mathbb{Z}}$ the Fourier coefficients of V , i.e., $V(t, x) = \sum_{k, \ell \in \mathbb{Z}} v_{k, \ell} e^{i(kx + \ell t)}$, for all t, x in $\mathbb{T}$. We define the *resonant average* of V with respect to the frequency $\nu \in \mathbb{N}^+$ as

$$\langle V \rangle_\nu(x) := \sum_{k \in \mathbb{Z}} v_{k, \nu k} e^{ikx}. \quad (1.1.17)$$

We say that a function $V(t, x)$ is *completely resonant* with respect to the frequency $\nu \in \mathbb{N}^+$ if $v_{k, \ell} = 0$ for all $\ell \neq \nu k$, or equivalently if $V(t, x) \equiv \langle V \rangle_\nu(x + \nu t)$.

Definition 1.1.11. We say that a function $W \in C^\infty(\mathbb{T}; \mathbb{C})$ has a degenerate zero in $x_0 \in \mathbb{T}$ if $W(x_0) = 0$ and $W'(x_0) = 0$. We say that a function is non degenerate if it does not have any degenerate zero.

We also introduce the following terminology:

Definition 1.1.12. We say that a function $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$ is

- *resonantly stable* with respect to the frequency $\nu \in \mathbb{N}^+$ (or simply resonantly stable) if its resonant average $\langle V \rangle_\nu$ (see Definition 1.1.10) does not have any zero;
- *resonantly unstable* with respect to the frequency $\nu \in \mathbb{N}^+$ (or simply resonantly unstable) if its resonant average $\langle V \rangle_\nu$ (see Definition 1.1.10) has at least one zero, but all its zeroes are non degenerate.

Our main result is the following theorem.

Theorem 1.1.13. *For any $\nu \in \mathbb{N}^+$ let*

$$\mathcal{V}_{\nu,s} := \{V \in C^\infty(\mathbb{T}^2; \mathbb{R}) : V \text{ is resonantly stable} \}, \quad (1.1.18)$$

$$\mathcal{V}_{\nu,u} := \{V \in C^\infty(\mathbb{T}^2; \mathbb{R}) : V \text{ is resonantly unstable} \}, \quad (1.1.19)$$

$$\mathcal{V}_{\nu,d} := \{V \in C^\infty(\mathbb{T}^2; \mathbb{R}) : \langle V \rangle_\nu \text{ has degenerate zeroes} \}, \quad (1.1.20)$$

where we recall Definitions 1.1.10, 1.1.11 and 1.1.12. The following holds:

1. *Long-time stability: for any $V \in \mathcal{V}_{\nu,s}$, for any $N > 0$ there exists $\varepsilon_N(V) > 0$ such that, for any $0 < \varepsilon < \varepsilon_N(V)$ and for any $s > 0$, there exists $C_1 > 0$ (independent of ε) such that, for any $u_0(x) \in \mathcal{H}^s(\mathbb{T})$, the corresponding solution $u(t, x)$ of equation (1.1.16) (i.e. $u(0, x) = u_0(x)$) satisfies*

$$\|u(t)\|_s \leq C_1 \|u_0\|_s \quad \forall |t| \leq C_V \varepsilon^{-(N+1)}, \quad \text{where } C_1 \equiv C_1(s, V)$$

for some $C_V > 0$.

2. *Instability: for any $V \in \mathcal{V}_{\nu,u}$ there exist $\varepsilon(V) > 0$, $\delta_2 > 0$ such that, for any $0 < \varepsilon < \varepsilon(V)$ and for any $s \geq 1$ there exists $u_0 \in \mathcal{H}^s(\mathbb{T})$, such that the corresponding solution $u(t, x)$ of equation (1.1.16) satisfies:*

$$\|u(t)\|_s \geq e^{\varepsilon \delta_2 t} \delta_1(u_0), \quad \forall t > 0, \quad (1.1.21)$$

for some $\delta_1(u_0) > 0$.

3. *Genericity: The set $\mathcal{V}_{\nu,u} \cup \mathcal{V}_{\nu,s}$ is open and dense in $C^\infty(\mathbb{T}^2; \mathbb{R})$ endowed with its Fréchet topology.*

The proof of Theorem 1.1.13 is the content of Chapter 4. We conclude this section with some comments about Theorem 1.1.13:

- First remark that

$$C^\infty(\mathbb{T}^2; \mathbb{R}) = \mathcal{V}_{\nu,s} \cup \mathcal{V}_{\nu,u} \cup \mathcal{V}_{\nu,d}$$

(see (1.1.18), (1.1.19), (1.1.20)). Thus, from the genericity property in point 3, Theorem 1.1.13 answers the question of stability of equation (1.1.16) for the vast majority of time-dependent, 2π -periodic perturbations V .

- We note again that the lower bound in (1.1.21) is sharp: in [72] the authors prove that all positive Sobolev norms of all solutions of (1.1.16) grow at most exponentially fast. We remark that, in analogy with Theorem 1.1.7, also in this case our bound (1.1.21) is effective for $t \gtrsim \varepsilon^{-1}$.
- We now motivate the definitions of the sets (1.1.18), (1.1.19), and (1.1.20). The main difficulty encountered in the proofs of the previous results is the identification of sufficient conditions on the dynamics generated by the resonant averages of the perturbation for the formation of instability phenomena. This procedure involves the study of hyperbolic vector fields on high-dimensional manifolds, together with the analysis of their dynamics in nontrivial topologies. As expected, this strategy becomes considerably simpler in one-dimensional settings: in the one-dimensional case, this property follows directly from the structure of the function $\langle V \rangle_\nu$ (see Section 4.3). In particular, a necessary condition for instability is that the resonant average admits at least one non degenerate zero (see Definition 1.1.11). As we will explain in detail in Chapter 4, this condition is closely related to the dynamics of the system $\dot{x} = \langle V \rangle_\nu(x)$, since such a zero corresponds to a hyperbolic equilibrium point for this one-dimensional ODE. On the other hand, the absence of zeroes for the resonant average is a sufficient condition for the long time stability.

We refer to Section 1.2.3 for an idea of the techniques used in the proof of Theorem 1.1.13.

1.2 Strategies of the proofs

In this section, we outline the central elements used in the proofs of our main results.

1.2.1 Ideas of the proof of Theorem 1.1.6

In order to prove Theorem 1.1.6, two distinct objectives must be achieved. First, given a symbol v in the set $\mathcal{V}$ (see (1.1.9)), one has to prove that the corresponding operator $V = \text{Op}^w(v)$ (see (1.1.5)) generates instability in the sense of (1.1.10). Next, we need to establish the genericity of the set $\mathcal{V}$. We now briefly describe the ideas underlying these two steps.

Instability: The instability result relies on the abstract framework developed in [69]. In Section 2.1.1, we present the abstract result in detail together with a minor adaptation required in our setting. In particular, Theorem 2.1.6 states that instability follows as soon as three classes of assumptions are verified by the operators H_0 and $V(t, x, D)$ (see I, II, III). Therefore, our main task for this part of the proof is to show that these assumptions hold for every perturbation $V \in \mathcal{V}$ (see (1.1.9)).

The assumptions in I are purely algebraic and concern the operator algebras where both the principal operator and the perturbation must lie. In our framework, these are automatically satisfied since we work in the pseudodifferential class. The assumptions in II concern the spectral properties of the unperturbed operator, which is required to have purely discrete spectrum and constant spectral gaps; again, these properties hold for the harmonic oscillator H_0 .

The core of the analysis is therefore the verification of the Mourre estimate in III. The fundamental intuition behind the abstract result of [69] is that the formation of instability is driven by the time-average of the perturbation along the flow generated by the unperturbed dynamics. At the operator level, this mechanism is encoded in a positive commutator estimate for the averaged perturbation, called Mourre estimate.

For this reason, in Section 2.4 we prove that every operator $V(t) = \text{Op}^w(v(t))$ with symbol $v(t) \in \mathcal{V}$ (recall the definition of the set $\mathcal{V}$ in (2.0.3)) has resonant average of its principal symbol $\langle v_0 \rangle_{h_0}$ satisfying a Mourre estimate (see (1.2.1)) on the interval

I appearing in the definition of $\mathcal{V}$. More precisely, we show that there exist $A \in C^\infty(\mathbb{T}; \mathcal{S}_{H_0}^1)$, $g_I \in C_c^\infty(\mathbb{R}; \mathbb{R}_{\geq 0})$ with $g_I \equiv 1$ on I , and $\theta > 0$ such that

$$g_I(\langle V \rangle_{H_0})i[\langle V \rangle_{H_0}, A]g_I(\langle V \rangle_{H_0}) \geq \theta g_I(\langle V \rangle_{H_0})^2 + K, \quad (1.2.1)$$

where K is a compact remainder. The key difficulty in establishing (1.2.1) is the construction of the conjugate operator A . This is precisely the motivation behind the definition of $\mathcal{V}$ in (1.1.9). Indeed, in Section 2.4 we prove that, if one defines $A := \text{Op}^w(a)$, where a is the *escape function* appearing in the definition of $\mathcal{V}$ in (1.1.9), then A is a valid conjugate operator for the averaged perturbation. The passage from the symbolic condition

$$\{\langle v_0 \rangle_{h_0}, a\} \geq \delta, \quad \text{on } I \subset \text{Ran}(\langle v_0 \rangle_{h_0})$$

to the operator estimate (1.2.1) is achieved through pseudodifferential calculus and, in particular, Gårding inequality. This correspondence between dynamical properties at the symbolic level and positive commutator estimates at the operator level constitutes the core of the instability proof. We refer to Section 2.4 for a detailed description.

Genericity: We now turn to the proof of the genericity statement. This is the most delicate part of the proof, where we introduce the main novelties of this approach. Given the characterization of the set $\mathcal{V}$, the main issue is to show that a generic classical symbol v (see Definition 1.1.2) has resonant average $\langle v_0 \rangle_{h_0}$ of its principal symbol (see Definition 1.1.5) which admits an escape function. We accomplish this in two steps.

At a first stage, in place of the definitively homogeneous function $\langle v_0 \rangle_{h_0}$ we consider an arbitrary function $h \in C^\infty(\mathbb{R}^4 \setminus \{0\}; \mathbb{R})$ positively homogeneous of degree zero. In Section 2.2 we establish sufficient conditions for the existence of an escape function a (see Definition 2.2.2) associated with h at a regular value $\mathbf{e}_0$ of h . This result is obtained adapting to our framework a construction introduced by Colin de Verdière in [26] for symbols on the torus (see also [27]). The crucial point is that such sufficient conditions are entirely of dynamical nature. More precisely, they depend on the properties of the vector field $\tilde{X}_h$ obtained as follows: first, one restricts the Hamiltonian vector field X_h to the energy surface $h = \mathbf{e}_0$, which has a conic structure due to the homogeneity of h ;

then one projects the resulting dynamics onto the unit sphere $\mathbb{S}^3$ (see (2.2.6)). The key dynamical assumption that we make is that the projected vector field $\tilde{X}_h$ is of Morse-Smale type on the surface $Z_{e_0}^h := h^{-1}(e_0) \cap \mathbb{S}^3$: we prove that this property is sufficient to construct the escape function. Even though a more general dynamical condition is sufficient (see Proposition 2.2.9), we focus here on the Morse-Smale framework to provide the geometric intuition for our main argument. Before outlining the second step of our proof, we now briefly explain the properties of Morse-Smale vector fields that are relevant for our construction.

To this aim, recall the general definition 1.1.8 of Morse-Smale vector fields on compact manifolds. However, the following is an equivalent definition for the particular case of Morse-Smale vector fields on surfaces (this is the case of the manifold $Z_{e_0}^h$):

Definition 1.2.1 (Morse-Smale vector fields). A vector field X on a closed, orientable surface M is said to be of *Morse-Smale* type if it has the following properties:

1. There are finitely many singularities and closed orbits, all of which are non degenerate;
2. There are no trajectories connecting saddle points;
3. The α and ω -limit set of each flow line is either a singular point or a closed orbit.

Broadly speaking, the motivation for choosing Morse-Smale vector fields comes from both the possibility to provide a global dynamical picture of the associated flow, and their structural stability. We now explain both such properties.

- The hyperbolicity of the flow allows to identify two disjoint, compact, and invariant subsets of M that act as a global attractor and repeller respectively, for the dynamics; in particular, the entire manifold is covered by their basins of attraction. This provides a global picture of the flow: roughly speaking, all trajectories start from the repelling set and move towards the attractor, to which they eventually converge. As first pointed out by Colin de Verdière in [27], this global structure of the trajectories is sufficient to construct an escape function. More precisely, this is done progressively following the trajectories: first, one defines a local escape function near the attractor and repeller, and then extends it to the

whole basins of attraction of the two sets. We refer to Section 2.2 for the details of this construction in our framework.

- The second crucial information about Morse-Smale vector fields, is that they form an open and dense set among smooth vector fields on closed and oriented surfaces: this is a famous result by Peixoto in [78](see Theorem 2.1.9). This genericity property ensures structural stability of the vector field $\tilde{X}_h$, going in the direction of the genericity result achieved in Theorem 1.1.6.

We now turn to the second step in our proof of the genericity result. At this point, our problem reduces to proving that, for the *generic* symbol $v(t, x, \xi)$, the resonant average $\langle v_0 \rangle_{h_0}$ of its principal symbol (see (1.1.7)) generates a projected vector field $\tilde{X}_{\langle v_0 \rangle_{h_0}}$ which is Morse-Smale. Even though we already emphasized that Morse-Smale vector fields are generic among smooth vector fields on closed oriented surfaces, this is not enough to immediately achieve our goal: the vector fields $\tilde{X}_{\langle v_0 \rangle_{h_0}}$ form a rather specific set, so that Peixoto's theorem cannot be applied directly. The key point is therefore to exploit the additional geometric structure of projected vector fields of the form $\tilde{X}_h$, with h homogeneous of degree zero. To this end, we regard $Z_{e_0}^h$ as a submanifold of the contact manifold $\mathbb{S}^3$ (see Section 2.1.3 for the main definitions and results of contact geometry used), endowed with the standard contact structure inherited from the symplectic structure of $\mathbb{R}^4$. This contact structure naturally induces a foliation on $Z_{e_0}^h$, called the *characteristic foliation* (Definition 2.1.17). We prove that the vector field $\tilde{X}_h$ is precisely the vector field generating such characteristic foliation (Proposition 2.3.5). This geometric interpretation allows us to exploit tools from contact geometry. In particular, by applying a local version of Gray's theorem (the contact version of Darboux's theorem in symplectic geometry), see Theorem 2.1.15, we show that one can always slightly deform $Z_{e_0}^h$ into a nearby submanifold $Z_{e_0}^{h_\varepsilon}$, where h_ε is an arbitrarily small perturbation of h , in such a way that the associated vector field $\tilde{X}_{h_\varepsilon}$ is Morse-Smale (Proposition 2.3.6). This ultimately yields the desired genericity result. We refer to Section 2.3 for the details of this construction.

1.2.2 Ideas of the proof of Theorems 1.1.7 and 1.1.9

We now outline the proof of Theorems 1.1.7 and 1.1.9. While Theorem 1.1.7 comes as an application of Theorem 1.1.9, the proof of the latter is where we introduce the main novelties of our approach. For this reason, we first anticipate the strategy of the proof of Theorem 1.1.9.

Proof of Theorem 1.1.9 In analogy with the harmonic oscillator problem described in Section 1.2.1, the core of the proof of the instability result in Theorem 1.1.9 relies on a positive commutator estimate. However, due to the different nature of the problem at hand, in this case we are able to establish a stronger commutator estimate than the Mourre estimate obtained in (1.2.1). This is essentially due to the fact that the perturbation considered in (1.1.14) is of the same order as the principal operator. This feature, in turn, yields the exponential growth in (1.1.15). Note that this growth is stronger than the polynomial one in (1.1.10) derived via the Mourre estimate.

We now explain this mechanism in more detail. For convenience, throughout this section we write equation (1.1.14) in the compact form $\partial_t u_\varepsilon = H u_\varepsilon$.

The energy estimate. Given a Morse-Smale vector field $V \in \mathfrak{X}(M)$ and for a sufficiently small $\varepsilon > 0$, we look for a *conjugate operator* $A \in \mathcal{S}^\sigma$, with $\sigma > 0$, and a positive constant $\tilde{\alpha} > 0$ such that, up to bounded remainders, we have

$$[A, H] \geq \tilde{\alpha} \sigma A, \quad \forall t \in \mathbb{R}. \quad (1.2.2)$$

Then, defining for every u_ε solution of (1.1.14)

$$\mathcal{A}(t) := \langle A u_\varepsilon(t, x), u_\varepsilon(t, x) \rangle_{L^2(M)}, \quad \forall t \in \mathbb{R},$$

and computing the time derivative, we get

$$\frac{d}{dt} \mathcal{A}(t) \geq \tilde{\alpha} \sigma \mathcal{A}(t) - \beta_\sigma \|u_\varepsilon(0)\|_{L^2(M)}, \quad \forall t \in \mathbb{R}, \quad (1.2.3)$$

where $\beta_\sigma > 0$ takes into account the contributions given by the bounded remainders in (1.2.2). Inequality (1.2.3) follows from skew-adjointness of the operator H , and is the content of Lemma 3.3.14, to which we refer for a precise statement of our energy

estimate. We now show how to obtain the instability result from (1.2.3): integrating (1.2.3) we obtain

$$\mathcal{A}(t) \geq e^{\tilde{\alpha}\sigma t} \underbrace{\left(\mathcal{A}(0) - \frac{\beta_\sigma}{\tilde{\alpha}\sigma} \|u_\varepsilon(0)\|_{L^2(M)} \right)}_{=: \mathcal{F}(u_\varepsilon(0))}, \quad \forall t \geq 0. \quad (1.2.4)$$

Finally, recalling that $A \in \mathcal{S}^\sigma$ with $\sigma > 0$, a Cauchy-Schwarz inequality yields

$$\mathcal{A}(t) := \langle Au_\varepsilon(t, x), u_\varepsilon(t, x) \rangle_{L^2(M)} \leq C_{A_\sigma} \|u_\varepsilon(t)\|_\sigma \|u_\varepsilon(0)\|_{L^2(M)}. \quad (1.2.5)$$

Putting together (1.2.5) and (1.2.4), we obtain the growth result for all solutions $u_\varepsilon(t, x)$ with initial condition $u_\varepsilon(0)$ satisfying $\mathcal{F}(u_\varepsilon(0)) > 0$, and for all norms $\|u(t)\|_s$, with $s \geq \sigma$. Thus, given $V \in \mathfrak{X}(M)$ Morse-Smale vector field, two goals must be achieved:

- find a conjugate operator A satisfying (1.2.2);
- find initial conditions satisfying $\mathcal{F}(u_\varepsilon(0)) > 0$ (see (1.2.4)).

These results are the core of the argument, and establishing them requires the main new ideas of this part of the work. They are obtained respectively in Sections 3.2 and in Section 3.3.5. We now describe the general ideas behind each of the two steps.

Construction of the conjugate operator. In order to construct the conjugate operator A , we first use results of symbolic calculus and translate the problem at the level of the corresponding symbols (see Section 3.3.1 for an introduction to pseudodifferential operators on compact manifolds and related properties). Recalling that the symbol associated with the commutator $[A, H]$ coincides, at leading order, with the Poisson brackets between the symbols of A and H respectively, this translates condition (1.2.2) in searching for a function $a \in C^\infty(T^*M \setminus \{0\})$, positively homogeneous of order $\sigma > 0$, such that

$$\{a, h\} \geq \tilde{\alpha}\sigma a, \quad \forall t \in \mathbb{R},$$

where $h := \xi \cdot V(x)$ (see (1.1.14)).

This reduces our problem to studying the Hamiltonian dynamics generated by $h := \xi \cdot V(x)$ on T^*M . Using the dynamical ingredients introduced in [24], we develop a

general and robust *construction of an escape function* a associated with the symbol h , whenever the vector field V is Morse–Smale. This setting differs substantially from the previously discussed case of the harmonic oscillator (see Section 1.2.1), making it impossible to adapt the construction of Colin de Verdière [26] to this case. This motivates the introduction of new ideas and techniques.

Heuristically, the main difference is that, even though in both situations the Hamiltonian dynamics for which we seek an escape function is associated with an underlying Morse–Smale dynamics (in the present case, the dynamics of V on M , whereas in the harmonic oscillator case it is the dynamics of $\tilde{X}_h$ on $Z_{e_0}^h$, see Section 1.2.1 for a description of these objects), the resulting Hamiltonian flows have significantly different structures. More precisely, in the harmonic oscillator case, one can construct global, disjoint, conic attractors and repellers for the Hamiltonian flow starting from the attractors and repellers of the underlying Morse–Smale vector field. In the present setting, this is no longer true: although global attractors and repellers can still be identified, they are not necessarily disjoint (see Proposition 3.1.10 and [24] for a detailed description of the dynamics). This introduces the main obstacle to constructing the escape function and motivates the introduction of a different approach. More precisely, we construct an escape function $a \in C^\infty(T^*M \setminus \{0\})$ of the form

$$a(x, \xi) := m(x, \xi)f(x, \xi),$$

where $m \in C^\infty(T^*M \setminus \{0\})$ is a positively homogeneous function of degree zero, and $f(x, \xi)$ is positively homogeneous of order $\sigma > 0$, where σ depends on the vector field V .

To perform such construction, we follow the microlocal point of view introduced in [37] to study the correlation (or Ruelle) spectrum of dynamical systems with hyperbolic behavior, which was further developed in [32, 31, 24, 52, 38, 73], among others. Roughly speaking, the main difference with these references is that the function f that we consider is of order $\sigma > 0$, whilst they use a function of order $\log(1 + \|\xi\|_x^2)$.

This is due to the different nature of the problems tackled, and the quite strong nature of the time-dependent perturbations that we consider: we need to adapt the dynamical results from these references carefully, and we can not directly apply them. For this reason, we also revisit the dynamical construction from [24] which deals with

the Morse-Smale case.

We refer to Section 3.2 for a more detailed description of our construction and an overview of the related literature.

The choice of the initial condition. As one can see from (1.2.4), the exponential growth of the solution is conditioned to the proper choice of the initial condition $u_\varepsilon(0)$: to this aim, we provide an explicit microlocal criterium (3.3.32). Roughly speaking, initial data for which the Sobolev norms will grow exponentially will be highly oscillating in the conormal direction to the stable manifold for the Hamiltonian flow. Somehow equivalently, we will pick initial data whose semiclassical wavefront set [86, Section 8.4] intersects this direction in a nontrivial way.

We conclude by anticipating here that Theorem 1.1.9 holds true for more general time dependent perturbations of equation (1.1.14) and we refer to equation (3.3.9) for a more general class of equations (including unbounded pseudodifferential perturbations of (1.1.14)) to which Theorem 1.1.9 applies – see Theorem 3.3.13.

Proof of Theorem 1.1.7 For simplicity in the exposition, we rewrite here equation (1.1.11) in its resonant form and with $n = 2$, as considered by Theorem 1.1.7: for $\nu \in \mathbb{N}^2 \setminus \{0\}$, $\varepsilon > 0$ we study

$$\partial_t u_\varepsilon = (\nu + \varepsilon V(t, x)) \cdot \nabla_x u_\varepsilon + \frac{\varepsilon}{2} \operatorname{div}_x(V(t, x)) u_\varepsilon, \quad x \in \mathbb{T}^2, t \in \mathbb{R}, \varepsilon > 0. \quad (1.2.6)$$

The proof of Theorem 1.1.7 consists of two main steps. First, we perform a resonant normal form reduction to reduce the transport coefficient to a time-independent one: this reduces our problem to the same setting of Theorem 1.1.9, that we apply to conclude. We describe each of these steps separately. As anticipated above, the goal of the normal form procedure is to conjugate equation (1.2.6) to a new transport equation whose leading transport coefficient is given by the *resonant average* of the perturbation V (see (1.1.13)). More precisely, we prove that for $\varepsilon > 0$ sufficiently small there exist a 2π -periodic in time, invertible linear map $\Phi_\varepsilon(t)$, bounded on $\mathcal{H}^\sigma$ for every $\sigma \in \mathbb{R}$, invertible with bounded inverse, and a smooth vector field $W_{\nu, \varepsilon} \in C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ such that, for any solution u_ε of equation (1.2.6), defining v_ε as $u_\varepsilon = \Phi_\varepsilon(t)v_\varepsilon$, then v_ε solves

$$\partial_t v_\varepsilon = \operatorname{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W_{\nu, \varepsilon}(t, x))) v_\varepsilon. \quad (1.2.7)$$

We refer to Proposition 3.4.3 for a precise statement of our conjugation result. Note immediately that the conjugated operator in (1.2.7) is still skew-adjoint, thus in particular the conservation of the L^2 -norm of solutions is preserved for equation (1.2.7). We also emphasize that, since both Φ_ε and Φ_ε^{-1} are bounded on $\mathcal{H}^\sigma$, for any $\sigma \in \mathbb{R}$, we reduce our initial problem to the study of the growth of Sobolev norms for equation (1.2.7) (see Corollary 3.4.4). We now describe in detail the conjugation procedure, anticipating first that it is inspired by the ideas of [10]. This procedure holds in any dimension $n \geq 1$ (and not only for $n = 2$), thus we report it here in its full generality. We consider a class of diffeomorphisms of the torus of the form

$$\varphi(t) : \mathbb{T}^n \rightarrow \mathbb{T}^n, \quad x \mapsto x + \varepsilon\beta(t, x),$$

where $\beta \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ has to be determined, and we define the associated transformation as

$$\tilde{\Phi}_\varepsilon(t) := \det(\text{Id} + \varepsilon \nabla_x \beta(t, x))^{\frac{1}{2}} u \circ \varphi(t, x), \quad \forall u \in C^\infty(\mathbb{T}; \mathcal{H}^\sigma(\mathbb{T}^n)), \forall t \in \mathbb{R}, \sigma \in \mathbb{R}. \quad (1.2.8)$$

Both φ and $\tilde{\Phi}_\varepsilon$ are invertible for ε sufficiently small, moreover $\tilde{\Phi}_\varepsilon$ is a unitary transformation. We refer to Section 3.4.1 for details on these properties. Remark that, writing equation (1.2.6) in compact form as $\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon$, and defining $\tilde{v}_\varepsilon$ as $\tilde{v}_\varepsilon := \tilde{\Phi}_\varepsilon(t) u_\varepsilon$, then $\tilde{v}_\varepsilon$ solves

$$\partial_t v_\varepsilon = \tilde{\Phi}_\varepsilon(t)_* H_\varepsilon v_\varepsilon, \quad \text{where } \tilde{\Phi}_\varepsilon(t)_* H_\varepsilon := \tilde{\Phi}_\varepsilon(t)^{-1} [H_\varepsilon \tilde{\Phi}_\varepsilon(t) - \partial_t \tilde{\Phi}_\varepsilon(t)]. \quad (1.2.9)$$

The core of this first part thus reduces to computing the pushforward in (1.2.9) with $\tilde{\Phi}_\varepsilon$ in (1.2.8). This is the content of Lemma 3.4.6, to which we refer for details. We emphasize here that, due to the simple expression of our transport operator in (1.2.6), and the properties of the map $\tilde{\Phi}_\varepsilon$, this computation can be done explicitly and both terms of the pushforward are pseudodifferential operators and we can express explicitly their symbols (see (3.4.15), (3.4.16)).

We only report here the resulting operator: for $\varepsilon > 0$ small enough we get

$$\tilde{\Phi}_\varepsilon(t)_* H_\varepsilon = \text{Op}^w(i\xi \cdot (\nu + \varepsilon(V(t, x) + \nu \cdot \nabla_x \beta - \partial_t \beta) + \mathcal{O}_{C^\infty}(\varepsilon^2))).$$

Thus, we look for $\beta \in C^\infty(\mathbb{T}^{n+1}; \mathbb{R}^n)$ such that the following homological equation is satisfied:

$$V(t, x) + \nu \cdot \nabla_x \beta - \partial_t \beta = \langle V \rangle_\nu(x + \nu t), \quad \forall (t, x) \in \mathbb{T} \times \mathbb{T}^n. \quad (1.2.10)$$

This can be achieved passing to the Fourier expansion of (1.2.10). This is the point in which the resonant nature of our problem crucially enters: due to the choice of the resonant parameters (ω, ν) in (1.1.11), one can not hope to reduce the left hand side of (1.2.10) to constant coefficients, but only to eliminate the Fourier modes of the perturbation which are non-resonant with the constant coefficient ν , collecting in the resonant average $\langle V \rangle_\nu$ the remaining resonant modes. In other words, the solution of (1.2.10) is given by $\beta(t, x) = \sum_{k, \ell} \beta_{k, \ell} e^{i(kx + \ell t)}$, with

$$\beta_{k, \ell} := \begin{cases} \frac{v_{k, \ell}}{i(\ell - k \cdot \nu)} & \text{if } \ell \neq k \cdot \nu, \\ 0 & \text{otherwise,} \end{cases}$$

where $V(t, x) = \sum_{k, \ell} v_{k, \ell} e^{i(kx + \ell t)}$ is the Fourier series of the perturbation V . Again, from the expression of the Fourier coefficients of β , it is clear that one can eliminate all and only the non-resonant modes of the function $V(t, x) + \nu \cdot \nabla_x \beta - \partial_t \beta$, due to the presence of the denominator $\ell - k \cdot \nu$, which has to be different from zero. Finally, one performs a time translation of (1.2.10) to recover the operator in (1.2.7).

In the second part of the proof, we note that equation (1.2.7) is a particular instance of (1.1.14). Thus, applying Theorem 1.1.9 we obtain the instability result for all vector fields $V \in C^\infty(\mathbb{T}^{n+1}; \mathbb{R}^n)$ which have Morse-Smale resonant average. This is the content of Corollary 3.4.4.

Finally, we are left to prove that, when $n = 2$, such perturbations form a generic set in $C^\infty(\mathbb{T}^3; \mathbb{R}^2)$. Recall that, in the case of the harmonic oscillator, this point is one of the key difficulties of the proof, since the projected vector field generating the Morse-Smale dynamics relates in a non-trivial way with the Hamiltonian vector field induced by the resonant average (see Section 1.2.1). In this case instead, the proof of the genericity result is much more direct: the projected dynamics studied coincides exactly with the one generated by the ODE $\dot{x} = \langle V \rangle_\nu(x)$ on $\mathbb{T}^2$. Thus, genericity follows in a rather direct way by the fact that Morse-Smale sets are generic among smooth vector fields on surfaces, and by continuity of the map

$$C^\infty(\mathbb{T}^3; \mathbb{R}^2) \ni V(t, x) \rightarrow \langle V \rangle_\nu(x) \in C^\infty(\mathbb{T}^2; \mathbb{R}^2).$$

We refer to Section 3.4.2 for details on this last part of the proof.

1.2.3 Ideas of the proof of Theorem 1.1.13

In this section, we describe the main tools needed to prove Theorem 1.1.13.

The proof of Theorem 1.1.13 is naturally divided in two parts, corresponding to the stability and instability results described in the statement. Even though the two strategies are quite different and lead to radically different dynamical behaviors, both analyses originate from the same normal form procedure, which constitutes the first technical step of both arguments. Given any smooth function $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$ we are able to eliminate, through an iterative conjugation scheme, the non-resonant components of the perturbation, reducing, at the n -th iteration, to a new transport equation having as leading coefficient the resonant average $\langle V \rangle_\nu(x)$ of the perturbation. More precisely we prove that, for $\varepsilon > 0$ sufficiently small and for any $N \in \mathbb{N}$, there exists a 2π -periodic in time linear map $\Psi_{N,\varepsilon}$, bounded on $\mathcal{H}^s$ for every $s \in \mathbb{R}$, invertible with bounded inverse, such that, given any $u(t, x)$ solution of (1.1.16) and defining u_N as $u(t, x) = \Psi_{N,\varepsilon} u_N(t, x)$, then u_N solves

$$\partial_t u_N = (\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 Z(x) + \varepsilon^{N+1} \tilde{W}_N(t, x)) \partial_x u_N + \frac{\varepsilon}{2} \operatorname{div} \left(\langle V \rangle_\nu(x) + \varepsilon Z(x) + \varepsilon^N \tilde{W}_N(t, x) \right) u_N, \quad (1.2.11)$$

where $Z \in C^\infty(\mathbb{T}; \mathbb{R})$ is a time independent smooth function, and $\tilde{W}_N \in C^\infty(\mathbb{T}^2; \mathbb{R})$. We refer to Lemma 4.2.2 for a precise statement of our conjugation result. Remark that, in order to construct the map $\Psi_{N,\varepsilon}$, we define at each step a close to the identity map of the following form: at the n -th step we consider a class of diffeomorphisms of the torus of the form

$$\varphi(t) : \mathbb{T} \rightarrow \mathbb{T}, \quad x \mapsto x + \varepsilon^n \beta(t, x),$$

where $\beta \in C^\infty(\mathbb{T} \times \mathbb{T}; \mathbb{R})$ has to be determined, and we define the associated transformation as

$$\tilde{\Phi}_\varepsilon(t) := \sqrt{1 + \varepsilon^n \partial_x \beta(t, x)} u \circ \varphi(t, x), \quad \forall u \in C^\infty(\mathbb{T}; \mathcal{H}^s(\mathbb{T})), \forall t \in \mathbb{R}, s \in \mathbb{R}.$$

This first step is analogous to the one used to prove Theorem 1.1.7. For this reason we do not detail further this construction referring to Section 1.2.2 for an idea of the technique and to Section 4.1.2 for the precise conjugation procedure.

At this stage, the original time-dependent problem has been reduced to the study of an autonomous effective transport equation whose leading coefficient is the resonant

average $\langle V \rangle_\nu$. Consequently, the qualitative behavior of the solutions is encoded in the geometric properties of this function. In particular, the presence or absence of zeros of $\langle V \rangle_\nu$ (see (1.1.18), (1.1.19)) will determine whether the reduced dynamics is stable or exhibits hyperbolic mechanisms leading to growth. The second part of the proof therefore splits into two different procedures tailored for the stability and instability result respectively, and it is in particular for the instability result that we need to introduce new constructions.

The stable regime. In the stable regime, we assume that V is *resonantly stable* (see (1.1.18) and Definition 1.1.12), or, in other words $\langle V \rangle_\nu(x) \neq 0$, for all $x \in \mathbb{T}$. This allows one to straighten the vector field by means of a time-independent change of variables on the torus, transforming the principal transport operator into one with constant velocity, up to arbitrarily small-in-size remainders: in other words, the reduced equation (1.2.11) can be further conjugated to a constant-coefficient transport equation, up to smaller size remainders. More precisely, in Proposition 4.2.1 we prove that, if $V \in \mathcal{V}_{\nu,s}$ we can reduce the transport coefficient of equation (1.2.11) to $\hat{m}_N + \varepsilon^N W(t, x)$, where $\hat{m}_N \in \mathbb{R} \setminus \{0\}$ and $W(t, x) \in C^\infty(\mathbb{T}^2; \mathbb{R})$. This is done performing a change of variable with a similar structure as the previous one, but with a time independent diffeomorphism of the torus. We only write here the explicit expression of the constant coefficient

$$\hat{m}_N := 2\pi \left(\int_0^{2\pi} (\langle V \rangle_\nu(x) + \varepsilon Z_N(x))^{-1} dx \right)^{-1}.$$

From this formula we explicitly see that the fact that $\langle V \rangle_\nu(x)$ never vanishes is crucial to properly define $\hat{m}_N$, and thus to successfully conclude the conjugation argument.

Finally, since constant-coefficient transport equations always preserve all Sobolev norms of all solutions, in order to conclude the proof of the stability result the problem reduces to estimating the contributions given by the smoothing terms. Note that, after N steps of normal forms, the remainder is of size ε^N . Thus one can show, using standard energy estimates, that its contribution remains negligible on polynomial time scales and therefore does not affect the leading-order conservation of Sobolev norms induced by the constant-coefficient transport dynamics, up to timescales $|t| \lesssim \varepsilon^{-(N+1)}$. We refer to Proposition 4.2.1 for details on the proof.

The unstable regime. We now describe the proof of the instability result. In this case, we assume that V is *resonantly unstable* (see (1.1.19) and Definition 1.1.12): we require that $\langle V \rangle_\nu$ has at least one non degenerate zero, and that it has no degenerate zeroes. In analogy with the proof of Theorem 1.1.7, also in this case the core of the proof is a positive commutator estimate analogous to (1.2.2). Once this is established, the proof of the growth result proceeds exactly as in the proof of Theorem 1.1.7. Thus, we refer to Section 1.2.2 for details on this mechanism.

Here we focus instead on the strategy used to obtain the commutator estimate, which in this context, is significantly different with respect to the higher dimensional context of Theorem 1.1.7, and constitutes the core novelty of this part. Recall that the goal is to construct a *conjugate operator* A such that, writing in compact form equation (1.1.16) as $\partial_t u = Hu$, A satisfies

$$[A, H] \geq cA, \quad \forall t \in \mathbb{R},$$

for some $c > 0$. Using again pseudodifferential techniques, we consider the associated problem at the level of symbols, and we provide a general and explicit construction of an *escape function* a for every Hamiltonian $h := \xi \langle V \rangle_\nu \in C^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R})$, when $\langle V \rangle_\nu$ has nondegenerate zeroes. Unlike the higher-dimensional one described above, the one-dimensional setting allows for a different construction, which is more direct, explicit, and simpler.

In this case indeed, the presence of non-degenerate critical points for the ODE $\dot{x} = \langle V \rangle_\nu(x)$ automatically gives rise to a hyperbolic structure in phase space. In the higher dimensional setting this corresponds to requiring Morse-Smale structure of the flow induced by $\langle V \rangle_\nu$. Exploiting this simpler dynamical setting, we are able to identify global, disjoint attractors and repellers for the Hamiltonian flow, thereby recovering an attractor–repellor structure similar to the one used by Colin de Verdière in [27] (and which we also exploit in the harmonic oscillator case; see Section 1.2.1).

Finally, we anticipate that, due to the low dimensionality of the problem, we are able to find an escape function a such that

$$\{h, a\} \geq \tilde{c}|\xi|, \quad \forall t \in \mathbb{R},$$

for some $\tilde{c} > 0$. In other words, the *escape function* a is *positively homogeneous of order one* and *strictly increasing* along the trajectories of the Hamiltonian flow induced by h , differently from the one constructed in the higher-dimensional setting (see Section 1.2.2). Both these results were not achievable in the higher-dimensional case: depending on the perturbation we were able to construct an escape function of order $\sigma > 0$, but typically $\sigma < 1$. Moreover, we were not able to achieve strict growth of the escape function along the trajectories but only a bound of the form $\{h, a\} \geq a$.

The final step consists in quantizing the escape function and transferring the positivity of the Poisson bracket to the quantum level recovering the positive commutator estimate.

1.3 Previous results

The problem of the growth of Sobolev norms for solutions to Hamiltonian dispersive PDEs emerged in the 1990s and has, since then, become a valuable framework for the study of asymptotic phenomena such as long-time stability, weak turbulence, and energy transfer to high frequencies. Although related questions had already appeared in earlier works, the formulation that attracted the greatest attention is due to Bourgain in 2000 in his survey [1] *Problems in Hamiltonian PDE's*. There, Bourgain considers the cubic defocusing nonlinear Schrödinger equation on the two-dimensional torus and raises the question of whether there exist solutions whose higher Sobolev norms grow unbounded as time tends to infinity. He identifies this problem as one of the central open questions in the theory of Hamiltonian partial differential equations.

Starting from this pioneering work, substantial progress has been made in the study of growth of Sobolev norms for nonlinear dispersive PDEs. We mention in this direction the seminal contributions of Kuksin [58, 59], the breakthrough work of Colliander, Keel, Staffilani, Takaoka, and Tao [23], and the refinements and extensions obtained in [50, 53, 56, 49, 48, 45] that exhibit strong forms of norm inflation. However, truly unbounded trajectories for nonlinear Hamiltonian PDEs are known only in a handful of cases; see [42, 54, 43, 14, 46]. We also emphasize the recent result of [20], concerning

time-dependent perturbations of nonlinear harmonic oscillators on $\mathbb{R}^2$.

Motivated by the scarcity of genuinely unbounded nonlinear examples and by the intrinsic difficulty of constructing unstable trajectories in nonlinear settings, in the present thesis we focus on linear models, for which instability mechanisms can be investigated in a more transparent framework.

In particular, in this section we discuss the present literature concerning the study of Sobolev instabilities for time dependent perturbations of abstract, linear Schrödinger-type equations

$$i\partial_t u = \mathcal{L}u + Q(t)u, \quad t \in \mathbb{R}, \quad (1.3.1)$$

where $\mathcal{L}$ is a linear operator and $Q(t)$ is smooth and real valued. From the perspective of quantum mechanics, this is a classical problem addressed already by Bellissard in the '80s in [15, Section I.2] (in this setting $\mathcal{L} \equiv -\Delta$). In this context, the growth of Sobolev norms of solutions can be viewed as an indicator of instabilities at the quantum level. Indeed, a growth for the Sobolev norms of solutions to (1.3.1) amounts to say that the quantum state does not stay in a bounded region of phase space. As explained in [15], this can be understood as the quantum analogue that an Hamiltonian flow has unbounded trajectories.

For what concerns linear, time-dependent Schrödinger equations on the torus, the problem has been first issued by Bourgain in [18] where the author constructs a solution with growing positive Sobolev norms when $Q(t)$ is quasiperiodic in time and multiplicative operator.

In the last years a lot of efforts have been done in the construction of time dependent perturbations $Q(t)$ giving rise to solutions with unbounded trajectories. In particular, concerning the case of harmonic oscillators (ie., $\mathcal{L} \equiv H_0$ with H_0 as in (1.1.1)), the one-dimensional case has been first investigated by Delort in [29], where the author constructs the first example of bounded, pseudodifferential V admitting solutions whose $\mathcal{H}^s$ -norm grows as t^s . Then, Maspero in [68] constructs an example of V , still bounded and pseudodifferential, provoking t^s growth of Sobolev norms for *all* the solutions. In [5], Bambusi, Grebert, Maspero and Robert construct an example of unbounded V generating solutions with $\mathcal{H}^s$ -norm growing as t^{2s} , while examples of

multiplication operators provoking a logarithmic growth are exhibited in [35] by Faou and Raphael.

Then Maspero in [69] uses for the first time Mourre's positive commutator theory to perform the key energy estimate for the growth result and exhibits a wider class of such perturbations which in [70] are proven to be generic among pseudodifferential operators of order zero.

In the peculiar case of pseudodifferential perturbations which are quadratic polynomials both in x and D , the growth rate at infinity of $\mathcal{H}^s$ -norms of the solutions in the one-dimensional setting has been completely described in the works [65, 67, 66] by Liang, Luo, Zhao and Zhou, and recently extended in [64] to higher dimensions.

Considering different models, we also mention [7] for perturbations of the Landau Hamiltonian, [55] for anharmonic oscillators on $\mathbb{R}^d$, [84] for perturbations causing infinite growth of solutions in the Bargmann-Fock space, and [21] for a recent result in the same formulation as [18], for a smooth perturbation on the two-dimensional torus, decaying in time along with all its time derivatives and admitting growing solutions with logarithmic speed.

On the other hand, the closely related question of exhibiting a priori upper bounds on the growth of Sobolev norms for all solutions of linear time-dependent systems has been widely investigated. A first important class of results in this direction, concerns t^ε upper bounds, with $\varepsilon > 0$ arbitrarily small. We mention the pioneering result [17] by Bourgain, in the quasi-periodic case, and more recent results for systems with increasing spectral gaps (such as the one-dimensional anharmonic oscillator or the Laplace-Beltrami operator on Zoll manifolds) in [76, 30, 72, 6, 85, 34], and for perturbations of the Laplacian on d -dimensional flat tori in [17, 28, 16, 11]. In this last case, it is important to exploit the specific cluster structure of the spectrum of the Laplacian on tori. A generalization of such structural property has recently been identified in [12, 9, 8] and used to extend the t^ε upper bound to a wider class of Schrödinger equations, such as perturbations of the Laplace-Beltrami operator on Lie groups and of the anharmonic oscillators on $\mathbb{R}^2$.

We separately mention the work by Maspero and Robert [72], where upper bounds for different abstract classes of Schrödinger equations in arbitrary spatial dimension

are obtained. Note that both the case of the harmonic oscillator equation (1.1.1) and the transport equation (1.1.11) considered in this thesis are particular instances of the equations considered in [72], and our results saturate the corresponding upper bounds (see Theorems 2.0.1 and 1.1.7). We also refer to [29] for a proof of a t^s upper bound for the growth of the $\mathcal{H}^s$ -Sobolev norm of all solutions of harmonic oscillators in spatial dimension one.

We mention for completeness that there are different classes of results that study the complementary question of reducibility for non-resonant, Schrödinger-type equations. We do not provide a complete overview of these results here since these involve completely different methods and models with respect to the ones considered in this thesis. We only mention the ones strictly related with the models considered in the present thesis: first the result by Bambusi, Grebert, Maspero and Robert, and the one by Grebert and Paturel [5, 47], where reducibility for small-in-size perturbations of harmonic oscillators is obtained, implying uniform $\mathcal{H}^s$ -boundedness in time of the solutions, and the ones by Bambusi [4] [3] and Bambusi, Montalto [13] where reducibility is achieved for different instances of one-dimensional Schrödinger operators with unbounded, quasiperiodic perturbations. Concerning transport equations we refer to the results by Bambusi, Langella Montalto [10] and Feola, Giuliani, Montalto and Procesi [39] where, as already mentioned, reducibility for non-resonant transport equations as (1.1.11) is obtained.

Finally we point out that the techniques developed in this thesis have recently proven to apply also to nonlinear models. First we mention the work by Maspero and Murgante [71], in which the method of the escape function is successfully extended to a one-dimensional fractional quasilinear Schrödinger equation, allowing to construct solutions undergoing growth of Sobolev norms. Next, the one by Langella, Maspero, Murgante and Terracina [60], in which the authors study two-dimensional periodic gravity water waves with constant nonzero vorticity in infinite depth and with periodic boundary conditions. Under suitable non resonance conditions on the wave number, they prove that the system admits smooth small-amplitude solutions exhibiting a transfer of energy toward high frequencies. This yields the first rigorous construction of weakly turbulent solutions for a quasilinear hydrodynamic wave system. Finally, the

work by Chaubet, Han-Kwan and Rivière [22], in which strategies similar to the one used to prove Theorem 1.1.9 are used to study the exponential stability of nonlinear Vlasov equations on negatively curved manifolds.

Notational remark

In the rest of this thesis, each chapter is devoted to the proof of one of the main results stated in this introduction and is independent from the others. Precisely, in Chapter 2 we prove Theorem 1.1.6, in Chapter 3 we prove Theorem 1.1.7 and Theorem 1.1.9, whilst in Chapter 4 we prove Theorem 1.1.13. As a consequence, the notation is local to each chapter and should be understood as valid only within that chapter.

For this reason, to simplify the notation, in the rest of the thesis we will always denote by $\mathcal{H}^s$ (or $\mathcal{H}^s(\mathbb{R}^d)$, $\mathcal{H}^s(\mathbb{T}^d)$) the Sobolev spaces of order s , omitting the subscripts since no ambiguity arises. More precisely, in Chapter 2 $\mathcal{H}^s$ will always refer to $\mathcal{H}_{H_0}^s$ defined in (1.1.2), whilst in the remaining chapters it will always denote the standard Sobolev spaces of the torus defined in (1.1.12).

Analogously, we will denote by $\mathbf{S}^\rho$ the set of symbols of order ρ , omitting the subscripts: in Chapter 2 $\mathbf{S}^\rho$ will always refer to $\mathbf{S}_{H_0}^\rho$ (see Definition 1.1.2), and in the other chapters $\mathbf{S}^\rho$ will always refer to the space of symbols of order ρ on the torus. The same convention will hold for the space $\mathcal{S}^\rho$ of pseudodifferential operators of order ρ .

Chapter 2

Resonant quantum harmonic oscillators in $\mathbb{R}^2$

In this chapter, we consider

$$i\partial_t u = H_0 u + V(t, x, D)u, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (2.0.1)$$

where

$$H_0 := \frac{1}{2}(-\partial_{x_1}^2 - \partial_{x_2}^2 + x_1^2 + x_2^2), \quad D := (D_1, D_2) = -(\mathrm{i}\partial_{x_1}, \mathrm{i}\partial_{x_2}),$$

and $V(t, x, D)$ is a pseudodifferential operator of degree zero (see Definition 1.1.3), selfadjoint for every t and 2π -periodic in time. As anticipated, we prove existence of global in time solutions undergoing unbounded growth in time of their positive Sobolev norms, where we define the norm $\|\cdot\|_s$ as the one of the Sobolev space

$$\mathcal{H}^s(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2; \mathbb{C}) : \|u\|_s := \|H_0^s u\|_{L^2} < +\infty\}, \quad s \in \mathbb{R}. \quad (2.0.2)$$

For simplicity in the exposition, we now state the main result of this chapter, referring to previous chapter for the preliminary definitions and notations used, and for a broader introduction to our result.

Theorem 2.0.1. *Denote by*

$$\begin{aligned} \mathcal{V} := \Big\{ v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0) : \exists I \subset \mathrm{Ran}(\langle v_0 \rangle_{h_0}), \exists a \in \mathbf{S}_{cl}^1 \text{ and } \exists \delta > 0 \\ \text{such that } \{\langle v_0 \rangle_{h_0}, a\} \geq \delta \quad \forall (x, \xi) \in \langle v_0 \rangle_{h_0}^{-1}(I) \cap \{|(x, \xi)| \geq 1\} \Big\}, \end{aligned} \quad (2.0.3)$$

where v_0 denotes the principal symbol of v . Then

(i) **Genericity:** The set $\mathcal{V}$ is generic in $C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$; precisely it is open and dense with respect to the metric $d_{\mathbb{T}}^0$ in (1.1.4).

(ii) **Instability:** for every $v \in \mathcal{V}$, for every $s > 0$, there exist a function $u_0 \in \mathcal{H}_{H_0}^s(\mathbb{R}^2)$ and a time $T > 0$ such that the solution $u(t, x)$ of equation (2.0.1) with $V(t, x, D) = \text{Op}^w(v)$ and initial datum u_0 satisfies

$$\|u(t)\|_s \geq C_{s,u_0} \langle t \rangle^s, \quad \forall t > T,$$

for some positive constant C_{s,u_0} .

We refer to Section 1.1.1 for a detailed explanation of our result, and to Section 1.2.1 for an overview of the techniques used in the proof. The rest of the chapter is devoted to the proof of Theorem 2.0.1.

Organization of the Chapter. The central element, as already discussed, is the construction of the escape function a for generic symbols $v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$, for which we proceed as follows. In Section 2.2 we consider an arbitrary function h , positively homogeneous of degree zero, a regular value e_0 for h and the projection $\tilde{X}_h$ of the Hamiltonian vector field X_h on the manifold $Z_{e_0}^h := h^{-1}(e_0) \cap \mathbb{S}^3$. Following Colin de Verdière [26], we prove that if the flow of $\tilde{X}_h$ has a weakly hyperbolic simple structure (see Definitions 2.2.8, 2.2.7) then there exists an escape function a for h at e_0 , i.e. $\{h, a\} \geq \delta > 0$ for any (x, ξ) on $h^{-1}(e_0)$ (Proposition 2.2.9).

In Section 2.3 we prove genericity of the set $\mathcal{V}$. The main difficulty is to prove its density. This is achieved in several steps. First we prove that $\tilde{X}_h$ directs the characteristic foliation of $Z_{e_0}^h$ (Proposition 2.3.5). Then, via a local Gray's theorem, we slightly perturb h to a new positively homogeneous function h_ε whose projected vector field $\tilde{X}_{h_\varepsilon}$ is Morse-Smale (Proposition 2.3.6), and actually has even weakly hyperbolic simple structure (Proposition 2.3.9). This implies the genericity result in $C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ (Proposition 2.3.10).

In Section 2.4 we prove instability using the abstract result of [69], reported in Theorem 2.1.6. We verify its assumptions by proving that any operator V with symbol $v \in \mathcal{V}$ fulfills the Mourre estimate (2.1.7).

Notational remark: we recall that in this Chapter we denote by $\mathcal{H}^s$, $s \in \mathbb{R}$, the Sobolev spaces defined in 2.0.2, by $\mathbf{S}^m$, $m \in \mathbb{R}$, the class of symbols of order m (see Definition 1.1.2) and by $\mathcal{S}^m$ the class of pseudodifferential operators of degree m (see Definition 1.1.3). For simplicity in the notation we will also denote simply by $\langle \cdot \rangle$ the average of a symbol (or an operator), omitting the subscript $\langle \cdot \rangle_{h_0}$ since no ambiguity arises.

2.1 Preliminaries and abstract results

We start by giving some preliminary definitions and results that we will need in the rest of the chapter.

2.1.1 Symbolic calculus and the abstract result of [69]

In this section we state some results of symbolic calculus and next we report the abstract result of [69] (which in particular also applies to our pseudodifferential setting), that we use to prove point (ii) of our main Theorem 2.0.1 (see Section 2.4).

Symbolic calculus. We refer for example to [81] for the results of symbolic calculus.

Proposition 2.1.1. *Let $a \in \mathbf{S}^m$, $b \in \mathbf{S}^n$, $m, n \in \mathbb{R}$. Then:*

1. *For any $s \in \mathbb{R}$, there exist $C, M > 0$ such that: $\|\text{Op}^w(a)u\|_{s-m} \leq C \wp_M^m(a) \|u\|_s$;*
2. *$\text{Op}^w(a)^* = \text{Op}^w(\bar{a})$, (here $*$ denotes the adjoint operator);*
3. *There exists $c \in \mathbf{S}^{m+n}$ such that $\text{Op}^w(a) \circ \text{Op}^w(b) = \text{Op}^w(c)$. Furthermore $c - ab \in \mathbf{S}^{m+n-1}$, and*

$$\forall k \in \mathbb{N}, \exists N, C > 0 : \wp_k^{m+n-1}(c - ab) \leq C \wp_N^m(a) \wp_N^n(b); \quad (2.1.1)$$

4. *There exists $d \in \mathbf{S}^{m+n-1}$ such that $i[\text{Op}^w(a), \text{Op}^w(b)] = \text{Op}^w(d)$. Moreover $d - \{a, b\} \in \mathbf{S}^{m+n-3}$, and*

$$\forall k \in \mathbb{N}, \exists N, C > 0 : \wp_k^{m+n-3}(d - \{a, b\}) \leq C \wp_N^m(a) \wp_N^n(b);$$

5. If $m < 0$, then $\text{Op}^w(a)$ is compact as an operator from $L^2(\mathbb{R}^2)$ to itself.

Remark 2.1.2. (i) The symbol

$$h_0(x, \xi) := \frac{|x|^2 + |\xi|^2}{2}, \quad (x, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad (2.1.2)$$

is definitely homogeneous of degree two, therefore it belongs to $\mathbf{S}_{cl}^1$.

(ii) If $v(x, \xi)$ is a positively homogeneous function of degree 0, then

$$dv[\nabla h_0] = \langle \nabla v(x, \xi), \nabla h_0(x, \xi) \rangle = 0. \quad (2.1.3)$$

Theorem 2.1.3 (Exact Egorov Theorem). *Let $a \in \mathbf{S}^\rho$, $\rho \in \mathbb{R}$. Then $e^{itH_0} \text{Op}^w(a) e^{-itH_0} = \text{Op}^w(a \circ \phi_{h_0}^t)$ for any $t \in \mathbb{R}$, where $\phi_{h_0}^t$ is the Hamiltonian flow generated by h_0 (written explicitly in (1.1.6)).*

Next we state a strong Gårding inequality [81], referring to [70] for a proof of this statement.

Theorem 2.1.4 (Strong Gårding inequality). *Let $a \in \mathbf{S}^0$ be a symbol of order zero, and suppose that there exists $R > 0$ such that*

$$a(x, \xi) \geq 0, \quad \forall |(x, \xi)| \geq R.$$

Then there exists a positive constant C such that

$$\langle \text{Op}^w(a)u, u \rangle \geq -C \|u\|_{-\frac{1}{2}}^2, \quad \forall u \in L^2(\mathbb{R}^2).$$

We conclude by stating the following result about composition of symbols with compactly supported functions. It is an application of pseudodifferential functional calculus, for which we refer to [25] and [69, Appendix B].

Lemma 2.1.5. *Consider a smooth compactly supported real function $g \in C_c^\infty(\mathbb{R}; \mathbb{R})$.*

(i) *Let $a \in \mathbf{S}^0$ be real valued. Then the operator $g(\text{Op}^w(a))$, defined via functional calculus, is a pseudodifferential operator of order zero: $g(\text{Op}^w(a)) \in \mathbf{S}^0$. Moreover $g(\text{Op}^w(a)) - \text{Op}^w(g(a)) \in \mathbf{S}^{-1}$.*

(ii) *Let A, B be bounded selfadjoint operators in an Hilbert space. If $A - B$ is compact, so is $g(A) - g(B)$.*

The abstract result of [69]. We now report the abstract result proven in [69], in a small variant suited for our setting. Given an Hilbert space $\mathcal{H}$, endowed with the scalar product $\langle \cdot, \cdot \rangle$, let K_0 be a selfadjoint, positive operator, with compact resolvent. Define a scale of Hilbert spaces by $\mathcal{H}^r := D(K_0^r)$ (the domain of the operator K_0^r) if $r \geq 0$, and $\mathcal{H}^r = (\mathcal{H}^{-r})'$ (the dual space) if $r < 0$, and denote by $\mathcal{H}^{-\infty} = \bigcup_{r \in \mathbb{R}} \mathcal{H}^r$ and $\mathcal{H}^{+\infty} = \bigcap_{r \in \mathbb{R}} \mathcal{H}^r$. We endow $\mathcal{H}^r$ with the norm $\|\psi\|_r := \|K_0^r \psi\|_0$, where $\|\cdot\|_0$ is the norm of $\mathcal{H}^0 \equiv \mathcal{H}$.

On $\mathcal{H}^r$ consider the abstract Schrödinger equation

$$i\partial_t \psi = K_0 \psi + V(t)\psi, \quad (2.1.4)$$

with $V(t)$ a time dependent selfadjoint operator.

Next assume to have a graded algebra $\mathcal{A}$ of operators which mimic pseudodifferential operators. For $m \in \mathbb{R}$ let $\mathcal{A}_m$ be a linear subspace of $\bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ (where $\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ denotes the space of linear and bounded operators from $\mathcal{H}^s$ to $\mathcal{H}^{s-m}$) and define $\mathcal{A} := \bigcup_{m \in \mathbb{R}} \mathcal{A}_m$. We notice that the space $\bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$, is a Fréchet space equipped with the seminorms: $\|A\|_{m,s} := \|A\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})}$. We say that A is of *order* m if $A \in \mathcal{A}_m$.

We shall also use the following notations. For $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{F}$ a Fréchet space, we denote by $C_b^0(\Omega; \mathcal{F})$ the space of C^m maps $f : \Omega \ni x \mapsto f(x) \in \mathcal{F}$ such that, for every seminorm $\|\cdot\|_j$ of $\mathcal{F}$, one has

$$\sup_{x \in \Omega} \|\partial_x^\alpha f(x)\|_j < +\infty, \quad \forall \alpha \in \mathbb{N}^d : |\alpha| \leq m. \quad (2.1.5)$$

If (2.1.5) is true for any m , we say $f \in C_b^\infty(\Omega; \mathcal{F})$. Similarly we denote by $C^\infty(\mathbb{T}; \mathcal{F})$ the space of smooth maps from the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ to the Fréchet space $\mathcal{F}$. Given two operators $A, B \in \mathcal{L}(\mathcal{H})$, we write $A \geq B$ with the meaning $\langle A\varphi, \varphi \rangle \geq \langle B\varphi, \varphi \rangle \quad \forall \varphi \in \mathcal{H}$.

We now state a list of assumptions on $\mathcal{A}, K_0, V(t)$ required to apply the abstract theorem in [69].

I. Pseudodifferential algebra

- (i) For each $m \in \mathbb{R}$, $K_0^m \in \mathcal{A}_m$; in particular K_0 is an operator of order one.

- (ii) For each $m \in \mathbb{R}$, $\mathcal{A}_m$ is a Fréchet space for a family of filtering semi-norms $\{\wp_j^m\}_{j \geq 0}$ such that the embedding $\mathcal{A}_m \hookrightarrow \bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ is continuous¹. If $m' \leq m$ then $\mathcal{A}_{m'} \subseteq \mathcal{A}_m$ with a continuous embedding.
- (iii) For all $m, n \in \mathbb{R}$, if $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then $AB \in \mathcal{A}_{m+n}$ and the map $(A, B) \mapsto AB$ is continuous from $\mathcal{A}_m \times \mathcal{A}_n$ into $\mathcal{A}_{m+n}$.
- (iv) If $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$, then the commutator $[A, B] \in \mathcal{A}_{m+n-1}$ and the map $(A, B) \mapsto [A, B]$ is continuous from $\mathcal{A}_m \times \mathcal{A}_n$ into $\mathcal{A}_{m+n-1}$.
- (v) $\mathcal{A}$ is closed under perturbation by smoothing operators in the following sense: let A be a linear map: $\mathcal{H}^{+\infty} \rightarrow \mathcal{H}^{-\infty}$. If there exists $m \in \mathbb{R}$ such that for every $N > 0$ we have a decomposition $A = A^{(N)} + S^{(N)}$, with $A^{(N)} \in \mathcal{A}_m$ and $S^{(N)}$ is N -smoothing², then $A \in \mathcal{A}_m$.
- (vi) If $A \in \mathcal{A}_m$ then also the adjoint operator $A^* \in \mathcal{A}_m$. The duality here is defined by the scalar product $\langle \cdot, \cdot \rangle$ of $\mathcal{H} = \mathcal{H}^0$. The adjoint A^* is defined by $\langle u, Av \rangle = \langle A^*u, v \rangle$ for $u, v \in \mathcal{H}^\infty$ and extended by continuity.

II. Properties of K_0

- (i) The operator K_0 has purely discrete spectrum fulfilling, for some $\lambda \geq 0$, $\sigma(K_0) \subseteq \mathbb{N} + \lambda$.
- (ii) For any $m \in \mathbb{R}$ and $A \in \mathcal{A}_m$, the map defined on $\mathbb{R}$ by $\tau \mapsto A(\tau) := e^{i\tau K_0} A e^{-i\tau K_0}$ belongs to $C_b^\infty(\mathbb{R}, \mathcal{A}_m)$ and one has:

$$\forall j \quad \exists N \text{ s.t. } \sup_{\tau \in \mathbb{R}} \wp_j^m(A(\tau)) \leq C \wp_N^m(A),$$

for some positive constant $C(m, j)$.

III. Properties of the potential $V(t)$

The operator $V \in C^0(\mathbb{T}, \mathcal{A}_0)$, $V(t)$ is selfadjoint for any $t \in \mathbb{T}$, and its resonant

¹A family of seminorms $\{\wp_j^m\}_{j \geq 0}$ is called filtering if for any $j_1, j_2 \geq 0$ there exist $k \geq 0$ and $c_1, c_2 > 0$ such that the two inequalities $\wp_{j_1}^m(A) \leq c_1 \wp_k^m(A)$ and $\wp_{j_2}^m(A) \leq c_2 \wp_k^m(A)$ hold for any $A \in \mathcal{A}_m$.

²Namely, if $\forall s \in \mathbb{R}$, it can be extended to an operator in $\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+N})$. When this is true for every $N \geq 0$, we call it a smoothing operator.

average

$$\langle\langle V \rangle\rangle := \frac{1}{2\pi} \int_0^{2\pi} e^{isK_0} V(s) e^{-isK_0} ds \quad (2.1.6)$$

fulfills:

- (i) There exists an interval $I_0 \subset \mathbb{R}$ such that $|\sigma(\langle\langle V \rangle\rangle) \cap I_0| > 0$; here $|\cdot|$ denotes the Lebesgue measure.
- (ii) *Mourre estimate* over I_0 : there exist a selfadjoint operator $A \in \mathcal{A}_1$ and a function $g_{I_0} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $g_{I_0} \equiv 1$ on I_0 such that

$$g_{I_0}(\langle\langle V \rangle\rangle) i[\langle\langle V \rangle\rangle, A] g_{I_0}(\langle\langle V \rangle\rangle) \geq \theta g_{I_0}(\langle\langle V \rangle\rangle)^2 + K, \quad (2.1.7)$$

for some $\theta > 0$ and K a selfadjoint compact operator.

The following is a small variant of the abstract result in [69]:

Theorem 2.1.6. *Assume that $\mathcal{A}$ is a graded algebra as in Assumption I, and that K_0 and $V(t) \in C^0(\mathbb{T}; \mathcal{A}_0)$ satisfy Assumptions II and III. Then for any $r > 0$ there exist a solution $\psi(t)$ of (2.1.4) in $\mathcal{H}^r$ and constants $C, T > 0$ such that*

$$\|\psi(t)\|_r \geq C \langle t \rangle^r, \quad \forall t \geq T.$$

Proof. The proof goes through exactly as the one of Theorem 2.7 of [69], which is proven in Section 3 therein. The only difference is that here we only suppose $V \in C^0(\mathbb{T}; \mathcal{A}^0)$, instead of requiring the stronger hypothesis $V \in C^\infty(\mathbb{T}; \mathcal{A}^0)$, assumed in [69]. In particular, all the results of Section 3 of [69] remain unchanged, with the only exception of Lemma 3.3, which is the only point where this hypothesis might enter. Given a time periodic family $W \in C^\infty(\mathbb{T}; \mathcal{A}^m)$ for some m , such lemma ensures the existence of a solution X of the homological equation

$$\partial_t X + i[K_0, X(t)] = W(t) - \int_0^{2\pi} e^{-isK_0} W(t+s) e^{isK_0} ds.$$

However, for our purpose it is sufficient to note that X gains regularity in time w.r.t. W , namely $X \in C^1(\mathbb{T}; \mathcal{A}^m)$ if $W \in C^0(\mathbb{T}; \mathcal{A}^m)$. $\square$

2.1.2 Morse-Smale vector fields

In this section we recall the definition of Morse-Smale vector fields and state a genericity result by Peixoto, that we shall use in Section 2.3.

Given M a smooth manifold and X a smooth vector field on M , we denote by $\phi_X^t(x)$ the flow of X that passes through x at time $t = 0$. We refer to [77] for most of the definitions related to dynamical systems.

Definition 2.1.7 (Attractor and repeller). Let M be a smooth closed manifold and X a smooth vector field on M . A compact, invariant subset $K^+ \subset M$ [respectively K^-] is called an *attractor* [respectively a *repeller*] if there exists an open neighbourhood U^+ of K^+ [respectively U^- of K^-] in M such that

$$K^+ = \bigcap_{t \geq 0} \phi_X^t(U^+) \quad [\text{respectively} \quad K^- = \bigcap_{t \leq 0} \phi_X^t(U^-)] . \quad (2.1.8)$$

If K^+ is an attractor, the *basin* of K^+ is the set of points x such that $\phi_X^t(x) \rightarrow K^+$ as $t \rightarrow +\infty$. One defines analogously the basin of K^- .

We recall the definition of Morse-Smale vector field on a surface, referring to [41, Definition 4.6.8]:

Definition 2.1.8 (Morse-Smale vector field). A vector field X on a closed, orientable surface M is said to be of *Morse-Smale* type if it has the following properties:

1. There are finitely many singularities and closed orbits, all of which are non-degenerate³;
2. There are no trajectories connecting saddle points;
3. The α and ω -limit set of each flow line is either a singular point or a closed orbit.

An important property of Morse-Smale vector fields is that, on any given closed and orientable surface, they are dense in the set of smooth vector fields, as it was proven by Peixoto [78, Theorem 2].

³Recall that a fixed point p is non-degenerate if any eigenvalue of the linearized vector field at p has non-vanishing real part. A closed orbit γ is called non-degenerate if given $p \in \gamma$, p is a non-degenerate fixed point of the Poincaré return map.

Theorem 2.1.9 (Peixoto). *Let S be a smooth closed orientable surface. The set of smooth Morse-Smale vector fields is open and dense in the set of smooth vector fields on S , with respect to the C^∞ -topology.*

We also recall that, given compact smooth manifolds M and N , the $C^k(M; N)$ norm of a function $f : M \rightarrow N$ is defined once finite atlases (M_i, ϕ_i) , (N_i, ψ_i) and a partition of the unity $\{\chi_i\}_i$ subordinated to $\{M_i\}_i$ are fixed on M and N . Then

$$\|f\|_{C^k(M; N)} := \max_{i,j} \sup_{\substack{x \in M_i \\ 0 \leq |\alpha| \leq k}} |\partial_x^\alpha (\psi_j \circ ((\chi_i f) \circ \phi_i^{-1}))|_{M_j}|.$$

Furthermore, given a vector field X on a compact smooth manifold M of class C^k , we define its norm as

$$\|X\|_{C^k} := \max_i \|X_i\|_{C^k(M; \mathbb{R})} \quad (2.1.9)$$

where X_i are the components of X . In order to define rigorously $\|X\|_{C^\infty(M; N)}$ we first recall that, given a Fréchet space $\mathcal{F}$ equipped with a sequence of seminorms $\{\wp_j\}_{j \in \mathbb{N}}$, we define on $\mathcal{F}$ the natural distance

$$d_{\mathcal{F}}(f, g) := \sum_{j \geq 0} \frac{1}{2^j} \frac{\wp_j(f - g)}{1 + \wp_j(f - g)}. \quad (2.1.10)$$

Then the C^∞ norm of a vector field X is defined as in (2.1.10) with $\wp_k(X) := \|X\|_{C^k}$.

2.1.3 Contact geometry

In this section we recall some basic concepts of contact geometry, referring to [41, 44].

Definition 2.1.10 (Contact structure). Let M be a odd dimensional smooth manifold. We say that a smooth field of hyperplanes $\kappa \subset TM$ is a *contact structure* on M if there exists a 1-form α such that $\kappa = \ker(\alpha)$ and $d\alpha|_{\ker(\alpha)}$ is non-degenerate. We say that α is a *contact form* on M and we say that the pair (M, κ) is a *contact manifold*.

Remark 2.1.11. The contact form is not uniquely determined by the contact structure: indeed if α is a contact form, then for every $f : M \rightarrow \mathbb{R}$ never vanishing, $f\alpha$ is also a contact form for (M, κ) .

Remark 2.1.12. Let α be a 1-form on M , $\dim M = 2n + 1$, then $d\alpha|_{\ker(\alpha)}$ is non-degenerate if and only if $\alpha \wedge (d\alpha)^n \neq 0$, (where by $(d\alpha)^n$ we denote the wedge product of $d\alpha$ repeated n times) see e.g. [44, Section I.1.A].

The following lemma is well known (see [41, Lemma 1.4.5]), but we sketch its proof for completeness.

Lemma 2.1.13. *Let (V, Ω) be an exact symplectic manifold of dimension $2n$ and let X be a Liouville vector field on V , namely a vector field fulfilling $\mathcal{L}_X \Omega = \Omega$. Every hypersurface $E \subset V$ of dimension $2n - 1$ and transverse to X (i.e. $X(p) \notin T_p E$, $\forall p \in E$) is a contact hypersurface with contact form $\alpha := i_X \Omega|_E$ and contact structure $\kappa_p = \ker(\alpha_p)$.*

Proof. We show the equivalent condition $\alpha \wedge (d\alpha)^{(n-1)} \neq 0$. Using Cartan's formula and that X is a Liouville vector field we have

$$d\alpha = d(i_X \Omega) = \mathcal{L}_X \Omega = \Omega \implies \alpha \wedge (d\alpha)^{n-1} = i_X \Omega \wedge \Omega^{n-1} = \frac{i_X(\Omega^n)}{n}.$$

We now claim that $i_X(\Omega^n) \neq 0$ as $2n - 1$ form on E , thus proving the lemma. Indeed, since E is transversal to X , it is possible to pick, for every point $p \in E \subset V$, a basis for $T_p V$ of the form $\mathcal{B} = \{e_1, \dots, e_{2n-1}, X(p)\}$, where $\{e_j\}_{j=1}^{2n-1}$ is a basis of $T_p E$. Evaluating the volume form Ω^n at this basis we have that $0 \neq \Omega^n(X(p), e_1, \dots, e_{2n-1}) = i_X(\Omega^n)(e_1, \dots, e_{2n-1})$, proving that $i_X(\Omega^n) \neq 0$. $\square$

As a consequence we have:

Corollary 2.1.14. *The sphere $\mathbb{S}^3 \equiv \{(x, \xi) \in \mathbb{R}^4 : |(x, \xi)| = 1\}$ is a contact manifold, with contact form locally given by $\alpha := \frac{1}{2}(-\xi_1 dx_1 - \xi_2 dx_2 + x_1 d\xi_1 + x_2 d\xi_2)|_{\mathbb{S}^3}$.*

Proof. We regard $\mathbb{S}^3$ as a submanifold of the exact symplectic manifold $(\mathbb{R}^4, \Omega)$, with Ω the standard symplectic form. A Liouville vector field is given by $\frac{1}{2}\nabla h_0$, with h_0 as in (2.1.2), which is transverse to $\mathbb{S}^3$. Lemma 2.1.13 yields the result, with $\alpha := i_{\frac{\nabla h_0}{2}} \Omega|_{\mathbb{S}^3}$ having the claimed form. $\square$

The following result is a local reformulation of the result in [41, Theorem 2.2.2].

Proposition 2.1.15 (Local Gray Theorem). *Let M be a smooth closed contact manifold and $S \subset M$ a codimension one smooth compact submanifold. Let $\delta > 0$. Suppose that there exist $\mathcal{U}(S) \subset M$ tubular neighbourhood of S , and $\{\alpha_t\}_{t \in [0,1]}$ smooth family of contact forms defined on $\mathcal{U}(S)$ such that⁴*

$$\|\dot{\alpha}_t\|_{C^\infty(\mathcal{U}(S))} < \delta'. \quad (2.1.11)$$

Then there exists $\delta'_0 > 0$ such that, if $\delta' < \delta'_0$, there exist a neighborhood $\mathcal{U}'(S) \subset \mathcal{U}(S)$ of S , and two smooth families of maps $\{\psi_t\}_{t \in [0,1]}$, $\psi_t : \mathcal{U}'(S) \rightarrow \mathcal{U}(S)$, $\{\lambda_t\}_{t \in [0,1]}$, $\lambda_t : \mathcal{U}(S) \rightarrow \mathbb{R}^+$, such that, for any t , ψ_t is a diffeomorphism on $\psi_t(\mathcal{U}'(S))$,

$$\|\psi_t - \mathbf{1}\|_{C^\infty(\mathcal{U}'(S))} < \delta, \quad (2.1.12)$$

$$\psi_t^* \alpha_t = \lambda_t \alpha_0 \quad (2.1.13)$$

as 1-forms on $\mathcal{U}'(S)$.

Proof. The idea is to look for ψ_t as the flow at time t of a suitable vector field $Y_t \in \kappa_t := \ker(\alpha_t)$, defined in $\mathcal{U}(S)$. More precisely, suppose that

$$\|Y_t\|_{C^\infty(\mathcal{U}(S))} < \delta'' \quad (2.1.14)$$

for some positive δ'' . Take local coordinates (z, ζ) around S such that S can be identified with the surface $\{z = 0\}$ and, given $r > 0$, let $\mathcal{U}_r(S)$ be the neighborhood of S corresponding to $|z| < r$. If r and δ'' are small enough, then $\mathcal{U}_{r+\delta''}(S) \subset \mathcal{U}(S)$. Then the Cauchy problem $\dot{\psi}_t = Y_t(\psi_t)$, $\psi_0 = \mathbf{1}$, is well defined on $\mathcal{U}'(S) := \mathcal{U}_r(S)$, $\psi_t(z) \in \mathcal{U}(S)$ for any $z \in \mathcal{U}'(S)$ and $t \in [0, 1]$. Furthermore, by (2.1.14) one deduces that $\forall k \in \mathbb{N}$ there exists $D_k > 0$ such that $\|\psi_t - \mathbf{1}\|_{C^k(\mathcal{U}'(S))} \leq \delta'' t D_k$ thus, recalling (2.1.10) with $\wp_k(\cdot) := \|\cdot\|_{C^k(\mathcal{U}'(S))}$, and eventually shrinking δ'' , also $\|\psi_t - \mathbf{1}\|_{C^\infty(\mathcal{U}'(S))} \leq \delta t$. Thus, to prove the result it is sufficient to find Y_t such that (2.1.13), (2.1.14) hold.

Now, derive (2.1.13) with respect to time. Since $\frac{d}{dt} \psi_t^*(\alpha_t) = \psi_t^*(\dot{\alpha}_t + \mathcal{L}_{Y_t}(\alpha_t))$ (see for example [41, Lemma 2.2.1]), using Cartan's formula, and the fact that $Y_t \in \kappa_t$, we

⁴Here to define $\|\alpha\|_{C^\infty(\mathcal{U}(S))}$ we pick a finite open covering $\{U_i\}$ of $\overline{\mathcal{U}(S)}$ on which local coordinates $y = (y_1, y_2, y_3)$ are defined, and we write locally $\alpha = \sum_{i=1}^3 \alpha_i(y) dy_i$ in such coordinates. Then, for every $k \in \mathbb{N}$, the C^k norm of α is $\|\alpha\|_{C^k} := \max_i \sum_{j=1}^3 \sup_{\substack{y \in U_i \\ 0 \leq |\beta| \leq k}} |\partial_y^\beta \alpha_j(y)|$, and the C^∞ norm is defined via the Fréchet distance (2.1.10).

obtain

$$\frac{d}{dt}\psi_t^*(\alpha_t) = \psi_t^*(\dot{\alpha}_t + i_{Y_t}d\alpha_t) = \dot{\lambda}_t\alpha_0. \quad (2.1.15)$$

Since

$$\frac{\dot{\lambda}_t}{\lambda_t}\alpha_0 = \frac{\dot{\lambda}_t}{\lambda_t}\psi_t^*(\alpha_t) = \psi_t^*(\mu_t\alpha_t), \quad \text{with } \mu_t := \partial_t(\log(\lambda_t)) \circ \psi_t^{-1},$$

equation (2.1.15) reads:

$$\psi_t^*(\dot{\alpha}_t + i_{Y_t}d\alpha_t) = \psi_t^*(\mu_t\alpha_t) \iff \dot{\alpha}_t + i_{Y_t}d\alpha_t = \mu_t\alpha_t. \quad (2.1.16)$$

Let now R_{α_t} be the Reeb vector field associated to α_t , i.e. the vector field on $\mathcal{U}(S)$ uniquely defined by $\alpha_t(R_{\alpha_t}) = 1$, $d\alpha_t(R_{\alpha_t}, \cdot) = 0$. Evaluating (2.1.16) in R_{α_t} gives $\mu_t = \dot{\alpha}_t(R_{\alpha_t})$, which defines μ_t . Hence, by (2.1.16), we obtain

$$d\alpha_t(Y_t, \cdot)|_{\kappa_t} = (\mu_t\alpha_t - \dot{\alpha}_t)|_{\kappa_t} = -\dot{\alpha}_t|_{\kappa_t},$$

which uniquely determines Y_t due to the non degeneracy of $d\alpha_t|_{\kappa_t}$. Finally, the smallness condition (2.1.14) with $\delta'' = C\delta'$ for some positive C , depending only on the coefficients of $d\alpha_t$, follows from (2.1.11). $\square$

Lemma 2.1.16. *Let $\{\psi_t\}_{t \in [0,1]}$ as in Proposition 2.1.15. For any $t \in [0,1]$ there exists $\delta_0 > 0$ such that, if $\delta < \delta_0$, then $S \subset \psi_t(\mathcal{U}'(S))$.*

Proof. Take a tubular neighborhood $\mathcal{U}''(S)$ of S such that $\overline{\mathcal{U}''(S)} \subset \mathcal{U}'(S)$ and for any $\zeta \in S$ consider the map $F_{t,\zeta} : \overline{\mathcal{U}''(S)} \rightarrow \mathcal{U}(S)$ defined by $F_{t,\zeta}(\eta) := \zeta - (\psi_t(\eta) - \eta)$. Then by (2.1.12), if $\delta < \delta_0$ is small enough, $F_{t,\zeta}$ maps $\overline{\mathcal{U}''(S)}$ into itself and it is a contraction. Moreover, compactness of S ensures that δ_0 can be taken independently of ζ . Then the map $F_{t,\zeta}$ admits a fixed point η_0 , namely a point such that $\psi_t(\eta_0) = \zeta$. $\square$

Characteristic foliation. First recall that a 1-dimensional oriented foliation $\mathcal{F}$ on a manifold S is the equivalence class of vector fields on S positively proportional to a given vector field X on S , which we say to *direct* the foliation. Note that $\mathcal{F}$ is a singular foliation since X might vanish.

Consider now a hypersurface S embedded in a contact manifold (M, κ) . We define its *characteristic foliation* [44, 41]:

Definition 2.1.17. Let S be a oriented hypersurface in a contact 3-dimensional manifold (M, κ) and fix μ a volume form on S (compatible with the given orientation). The characteristic foliation $\mathcal{F}$ is the foliation directed by the vector field X such that

$$i_X \mu = \alpha|_S, \quad (2.1.17)$$

where α is a contact form for (M, κ) .

Note that different choices of volume form μ and of contact form α differ by a positive function (Remark 2.1.11), so the vector field X in (2.1.17) is defined only up to a positive function, however the induced foliation is the same.

2.2 Construction of the escape function

In this section we consider a smooth function $h: \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ positively homogeneous of degree zero (see Definition 1.1.1) and prove that, under suitable dynamical conditions on its flow, it is possible to construct an escape function for it (see Definition 2.2.2). Our construction extends to our setting the one done by Colin de Verdière in [26] for pseudodifferential operators on the two-dimensional torus.

Notation: We denote by X_h the Hamiltonian vector field of the function $h(x, \xi)$, i.e.

$$X_h := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \nabla_x h \\ \nabla_\xi h \end{pmatrix} = J \nabla_{x, \xi} h$$

and the Poisson bracket

$$\{h, f\} := \nabla_\xi h \cdot \nabla_x f - \nabla_x h \cdot \nabla_\xi f = \mathrm{d}f[X_h]. \quad (2.2.1)$$

Remark 2.2.1. If h is a positively homogeneous function of degree 0, its Hamiltonian vector field X_h is homogeneous of degree -1 ; consequently the flow $\Phi_{X_h}^t(x, \xi)$ satisfies

$$\Phi_{X_h}^t(\lambda x, \lambda \xi) = \lambda \Phi_{X_h}^{t/\lambda^2}(x, \xi), \quad \forall \lambda > 0, t \in \mathbb{R}.$$

2.2.1 Setting and main result

We first give the definition of escape function.

Definition 2.2.2 (Escape function). Let $h : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ be a positively homogeneous function of degree zero (see Definition 1.1.1), and let $e_0 \in \mathbb{R}$ be a regular value for h (i.e. $dh(e_0) \neq 0$). Denoting by

$$\Sigma_{e_0}^h := h^{-1}(e_0) \quad (2.2.2)$$

the energy shell associated to e_0 , we say that a function $a : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$, positively homogeneous of degree two, is an *escape function* for h at e_0 if the Poisson brackets $\{h, a\}$ (see (2.2.1)) satisfy:

$$\{h, a\} \geq \delta > 0 \quad \text{on } \Sigma_{e_0}^h. \quad (2.2.3)$$

Let π be the projection of $\mathbb{R}^4 \setminus \{0\}$ on $\mathbb{S}^3$, i.e.,

$$\pi : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{S}^3, \quad \pi(x, \xi) = \left(\frac{x}{\rho}, \frac{\xi}{\rho} \right) =: \zeta, \quad \rho := \sqrt{|x|^2 + |\xi|^2} \quad (2.2.4)$$

and denote by $Z_{e_0}^h$ the set

$$Z_{e_0}^h := \pi(\Sigma_{e_0}^h) = \Sigma_{e_0}^h \cap \mathbb{S}^3. \quad (2.2.5)$$

Note that $Z_{e_0}^h$ is a smooth manifold of dimension 2 since, due to the homogeneity property of h , e_0 is still a regular value of $h|_{\mathbb{S}^3}$ (this follows from (2.1.3) with $v \rightsquigarrow h$), and that, due to the homogeneity of h , the energy shell $\Sigma_{e_0}^h$ is a cone.

Remark 2.2.3. Note that $Z_{e_0}^h$ is compact, thus it only has a finite number of connected components⁵. Moreover, if $Z_{e_0}^h$ has n connected components, then $\Sigma_{e_0}^h$ is the disjoint union of n open connected cones. Therefore in the present section we are going to assume without loss of generality that $Z_{e_0}^h$ and $\Sigma_{e_0}^h$ are connected, since if they are not, the construction of the escape function presented in this section can be repeated separately on each one of the finitely many open cones forming $\Sigma_{e_0}^h$.

⁵To see this, recall that the union of all the connected components of $Z_{e_0}^h$ is an open covering of $Z_{e_0}^h$, see for instance Proposition 1.8(d) of [62]; then the finiteness of the covering follows by compactness.

Due to its cone structure, we can parametrize $\Sigma_{e_0}^h$ with “polar” coordinates of the form $(\rho, \zeta) \in \mathbb{R}^+ \times Z_{e_0}^h$. We now express the Hamiltonian vector field X_h in such coordinates.

Lemma 2.2.4. *Let h be a positively homogeneous function of degree 0, and let X_h be its Hamiltonian vector field. There exists a vector field $\tilde{X}_h$ on $\Sigma_{e_0}^h$, homogeneous of order 0, such that $\tilde{X}_h|_{Z_{e_0}^h} \in TZ_{e_0}^h$ and*

$$X_h = \frac{\{h, h_0\}}{\rho^2} \nabla h_0 + \frac{1}{\rho} \tilde{X}_h, \quad (2.2.6)$$

where h_0 is defined in (2.1.2).

Proof. First note that X_h is homogeneous of degree -1 . Hence the vector field

$$\tilde{X}_h := \rho \left(X_h - \frac{\{h, h_0\}}{\rho^2} \nabla h_0 \right) \quad (2.2.7)$$

is homogeneous of order zero. We show that $\tilde{X}_h|_{Z_{e_0}^h} \in TZ_{e_0}^h$. In view of (2.2.5), it is sufficient to show that, at each point of $Z_{e_0}^h$, $\tilde{X}_h$ is tangent to both $\Sigma_{e_0}^h$ and $\mathbb{S}^3$. It is tangent to $\Sigma_{e_0}^h$ since

$$dh[\tilde{X}_h] \stackrel{(2.2.7)}{=} \rho dh[X_h] - \frac{\{h, h_0\}}{\rho} dh[\nabla h_0] \stackrel{(2.1.3)}{=} 0.$$

Next we show that the vector field $\tilde{X}_h$ is tangent to $\mathbb{S}^3 \equiv \{(x, \xi) : h_0(x, \xi) = \frac{1}{2}\}$. Indeed, as $\rho^2 = 2h_0$, we also have

$$dh_0[\tilde{X}_h] \stackrel{(2.2.7)}{=} \rho \left(dh_0[X_h] - \underbrace{\frac{\{h, h_0\}}{\rho^2} dh_0[\nabla h_0]}_{=2h_0} \right) = \rho (dh_0[X_h] - \{h, h_0\}) = 0,$$

proving (2.2.6). □

Remark 2.2.5. Similarly it follows that $d\rho[\tilde{X}_h] = 0$ and $d\rho[X_h] = \frac{\{h, h_0\}}{\rho}$.

It is convenient to decompose also the Liouville measure on the energy level $\Sigma_{e_0}^h$ in a “radial part” and a component supported on $Z_{e_0}^h$. So let us denote by ν_0 the Liouville measure induced on the energy level $\Sigma_{e_0}^h$ (see (2.2.2)) by the standard Liouville measure ν on $\mathbb{R}^4$; it is given explicitly by

$$\nu_0 := i_V \nu|_{\Sigma_{e_0}^h}, \quad \text{where} \quad V := \frac{\nabla h}{|\nabla h|} \quad (2.2.8)$$

and i is the contraction operator on differential forms. We have

Lemma 2.2.6. *The Liouville measure ν_0 in (2.2.8) can be written as*

$$\nu_0 = \rho^3 d\rho \wedge \tilde{\mu}, \quad (2.2.9)$$

where $\tilde{\mu}$ is the two dimensional volume form on $Z_{e_0}^h$ defined by

$$\tilde{\mu} := -i_V \nu_{\mathbb{S}^3}, \quad (2.2.10)$$

with V in (2.2.8) and $\nu_{\mathbb{S}^3}$ the spherical measure on $\mathbb{S}^3$. One has

$$i_{\nabla h_0} \tilde{\mu} = 0. \quad (2.2.11)$$

Proof. In spherical coordinates $\nu = \rho^3 d\rho \wedge \nu_{\mathbb{S}^3}$. Then using (2.2.8) and $d\rho[V] = \frac{1}{|\nabla h|} d\rho[\nabla h] = 0$ (which follows from (2.1.3)), we obtain (2.2.9) with $\tilde{\mu}$ in (2.2.10).

We now show (2.2.11). From the definition of $\tilde{\mu}$ and since $i_X i_Y = -i_Y i_X$, we have $i_{\nabla h_0} \tilde{\mu} = i_V i_{\nabla h_0} \nu_{\mathbb{S}^3} = 0$, concluding the proof. $\square$

We denote by $\mathcal{F}^h$ the 1-dimensional, oriented, singular foliation generated by $\tilde{X}_h$ on $Z_{e_0}^h$, namely

$$\mathcal{F}^h := \{X \in TZ_{e_0}^h \mid \exists \lambda \in C^\infty(Z_{e_0}^h; \mathbb{R}^+) \text{ s.t. } X = \lambda \tilde{X}_h\}.$$

It turns out that the existence of an escape function (according to Definition 2.2.2) is guaranteed by certain dynamical conditions on the foliation $\mathcal{F}^h$ that we now describe. Following [26], we now recall the definitions of simple structure and of weak hyperbolicity (recall also Definitions 2.1.7 of attractor, repeller, and of their basins of attraction).

Definition 2.2.7 (Simple structure). An oriented one-dimensional singular foliation $\mathcal{F}^h$ of the compact manifold $Z_{e_0}^h$ admits a simple structure (K^+, K^-) if $Z_{e_0}^h = K^+ \cup K^- \cup G$ as a disjoint union where:

- K^+ is a compact attractor of the oriented foliation $\mathcal{F}^h$;
- K^- is a compact repeller of the oriented foliation $\mathcal{F}^h$;
- all leaves of points in G converge to K^+ for positive times and to K^- for negative times. In particular the basin of attraction of K^+ is $G \cup K^+$ and the basin of attraction of K^- for the reversed orientation of $\mathcal{F}^h$ is $G \cup K^-$.

We introduce a further property of an invariant set, namely weak hyperbolicity. In order to do this, we recall that, given a volume form μ on a manifold M and a vector field X on TM , the divergence $\operatorname{div}_\mu(X)$ is the function defined by

$$\operatorname{div}_\mu(X)\mu = d(i_X\mu) . \quad (2.2.12)$$

We also recall the following standard formulas: if f is a smooth function on M ,

$$\operatorname{div}_\mu(fX) = df(X) + f\operatorname{div}_\mu(X) , \quad \operatorname{div}_{f\mu}(X) = \frac{df(X)}{f} + \operatorname{div}_\mu(X) . \quad (2.2.13)$$

Definition 2.2.8 (Weakly hyperbolic sets). Let (K^+, K^-) be a simple structure of a one-dimensional singular foliation $\mathcal{F}^h$ on $Z_{e_0}^h$. The invariant sets $K^\pm$ are called weakly hyperbolic if there exist neighbourhoods $U^\pm \supset K^\pm$ and a smooth density μ , absolutely continuous with respect to $\tilde{\mu}$ (i.e. a density of the form $\tilde{f}\tilde{\mu}$, with $\tilde{f}$ smooth and strictly positive function defined on $Z_{e_0}^h$), such that $\operatorname{div}_\mu(\tilde{X}_h) < 0$ on U^+ respectively $\operatorname{div}_\mu(\tilde{X}_h) > 0$ on U^- . Here $\tilde{X}_h$ is the vector field in (2.2.6) and $\tilde{\mu}$ the measure on $Z_{e_0}^h$ in (2.2.10).

We are now able to state the main result of this section.

Proposition 2.2.9. *Let $h : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ be a positively homogeneous function of degree 0 (according to Definition 1.1.1). If $e_0 \in \mathbb{R}$ is a regular value for h such that the foliation $\mathcal{F}^h$ induced by $\tilde{X}_h$ on $Z_{e_0}^h$ (see (2.2.5) and (2.2.6)) has simple structure (K^+, K^-) with K^+ and K^- weakly hyperbolic, then there exists an escape function for h at e_0 (see Definition 2.2.2).*

Let us give now an example of a definitely homogeneous function having a weakly hyperbolic simple structure; see Section 2.5 for the proof.

Lemma 2.2.10. *The definitely homogeneous function $h_\star : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ defined by*

$$h_\star(x, \xi) := \frac{1}{2} \frac{x_1^2}{h_0(x, \xi)}$$

induces on the surface $Z_{\frac{1}{2}}^{h_\star}$ a foliation $\mathcal{F}^{h_\star}$ with weakly hyperbolic simple structure.

The rest of the section is devoted to the proof of Proposition 2.2.9. The construction is rather involved and follows the following steps:

1. we construct local escape functions for h in small neighbourhoods of $\Gamma^+ := \pi^{-1}(K^+)$ and $\Gamma^- := \pi^{-1}(K^-)$;
2. we extend the local escape functions to the basins of attraction $B(\Gamma^\pm)$ of $\Gamma^\pm$ (see (2.2.21)), and we show that their union covers the entire $\Sigma_{e_0}^h$;
3. we glue together the local escape functions defined on $B(\Gamma^\pm)$ obtaining a global escape function on $\Sigma_{e_0}^h$, which we finally extend to the whole $\mathbb{R}^4 \setminus \{0\}$.

Since the function h is fixed, from now on we shall simply denote $\Sigma_{e_0} := \Sigma_{e_0}^h$, $Z_{e_0} := Z_{e_0}^h$ and $\mathcal{F} := \mathcal{F}^h$.

2.2.2 Construction of a local escape function

The goal of this section is to construct a local escape function for h in a neighbourhood of

$$\Gamma^+ := \pi^{-1}(K^+) \quad \text{and} \quad \Gamma^- := \pi^{-1}(K^-).$$

In all this section, we shall denote by $U^\pm$ the open sets defined in Definition 2.2.8 of weak hyperbolicity. First we write the Hamilton equations of h in the "radial" variables $(\rho, \zeta) \in \mathbb{R}^+ \times Z_{e_0}$:

Lemma 2.2.11. *The variables $(\rho, \zeta) \in \mathbb{R}^+ \times Z_{e_0}$ defined in (2.2.4) satisfy*

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = X_h(x, \xi) \quad \Leftrightarrow \quad \begin{cases} \dot{\rho} = \frac{\{h, h_0\}}{\rho} \\ \dot{\zeta} = \frac{1}{\rho^2} \tilde{X}_h(\zeta). \end{cases}$$

Proof. We compute the time derivatives of ρ and ζ :

$$\dot{\rho} = \frac{1}{\rho} \langle \nabla h_0, (\dot{x}, \dot{\xi}) \rangle = \frac{1}{\rho} \mathrm{d}h_0[X_h] = \frac{1}{\rho} \{h, h_0\}. \quad (2.2.14)$$

Similarly

$$\dot{\zeta} = \frac{\rho(\dot{x}, \dot{\xi}) - \dot{\rho}(x, \xi)}{\rho^2} \stackrel{(2.2.14)}{=} \frac{1}{\rho^2} \left(\rho X_h - \frac{\{h, h_0\}}{\rho} \nabla h_0 \right) \stackrel{(2.2.7)}{=} \frac{1}{\rho^2} \tilde{X}_h.$$

□

Next we also prove the following identity.

Lemma 2.2.12. *One has*

$$2\{h, h_0\} + \rho \operatorname{div}_{\tilde{\mu}}(\tilde{X}_h) = 0 \quad \text{on } \Sigma_{e_0}. \quad (2.2.15)$$

Proof. We start from the identity $\operatorname{div}_{\nu_0} X_h = 0$ on Σ_{e_0} , which follows from the fact that the Hamiltonian flow of h preserves the Liouville measure. We have

$$\operatorname{div}_{\nu_0}(X_h) \nu_0 \stackrel{(2.2.12)}{=} d(i_{X_h} \nu_0) = d(\rho^2 \{h, h_0\} \tilde{\mu} - \rho^2 d\rho \wedge i_{\tilde{X}_h} \tilde{\mu}),$$

where in the last equality we used (2.2.9), Remark 2.2.5 and

$$i_{X_h} \tilde{\mu} \stackrel{(2.2.6)}{=} \frac{\{h, h_0\}}{\rho^2} i_{\nabla h_0} \tilde{\mu} + \frac{1}{\rho} i_{\tilde{X}_h} \tilde{\mu} \stackrel{(2.2.11)}{=} \frac{1}{\rho} i_{\tilde{X}_h} \tilde{\mu}.$$

Finally, taking the external differential and observing that, since $\{h, h_0\}$ is an homogeneous function of degree zero and $\tilde{\mu}$ is a volume form on $Z_{e_0}^h$, then, by (2.1.3), $d\{h, h_0\} \wedge \tilde{\mu} = 0$, we obtain

$$\operatorname{div}_{\nu_0}(X_h) \nu_0 = 2\rho \{h, h_0\} d\rho \wedge \tilde{\mu} + \rho^2 d\rho \wedge d(i_{\tilde{X}_h} \tilde{\mu}) \stackrel{(2.2.9)}{=} \stackrel{(2.2.12)}{=} (2\{h, h_0\} \rho^{-2} + \rho^{-1} \operatorname{div}_{\tilde{\mu}}(\tilde{X}_h)) \nu_0.$$

As $\operatorname{div}_{\nu_0}(X_h) = 0$, we get (2.2.15). Remark that $\operatorname{div}_{\tilde{\mu}}(\tilde{X}_h)$ is a function homogeneous of order -1 . $\square$

Remark 2.2.13. Identity (2.2.15) on Z_{e_0} reads $2\{h, h_0\} + \operatorname{div}_{\tilde{\mu}}(\tilde{X}_h) = 0$.

We begin now the construction of the global escape function for h . The first step is to construct two functions $k^\pm$, which are local escape functions in neighbourhoods of $\Gamma^\pm$.

Proposition 2.2.14. *Assume that the foliation $\mathcal{F}$ has a simple structure (K^+, K^-) with $K^\pm$ weakly hyperbolic and let $\tilde{f} : Z_{e_0} \rightarrow \mathbb{R}$ be the strictly positive, smooth function given by Definition 2.2.8. Define the functions*

$$k^\pm : \Sigma_{e_0} \rightarrow \mathbb{R}, \quad k^\pm := \pm \frac{h_0}{\tilde{f}}, \quad (2.2.16)$$

then there exists $\delta > 0$ such that

$$\{h, k^\pm\} \geq \delta > 0 \quad \text{on} \quad \pi^{-1}(U^\pm). \quad (2.2.17)$$

Proof. We prove the statement with “+”. From weak hyperbolicity of K^+ one has

$$\operatorname{div}_{\tilde{f}\tilde{\mu}}(\tilde{X}_h) < 0 \text{ in } U^+ . \quad (2.2.18)$$

We compute:

$$\{h, k^+\} = dk^+[X_h] = d\left(\frac{h_0}{\tilde{f}}\right)[X_h] = \frac{1}{\tilde{f}}dh_0[X_h] - \frac{h_0}{\tilde{f}^2}d\tilde{f}[X_h]. \quad (2.2.19)$$

Evaluating separately the term $d\tilde{f}[X_h]$ we have:

$$d\tilde{f}[X_h] \stackrel{(2.2.6)}{=} \frac{\{h, h_0\}}{\rho^2}d\tilde{f}[\nabla h_0] + \frac{1}{\rho}d\tilde{f}[\tilde{X}_h] = \frac{1}{\rho}d\tilde{f}[\tilde{X}_h], \quad (2.2.20)$$

where the last inequality holds by (2.1.3), recalling that $\tilde{f}$ is homogeneous of degree 0. Inserting this last identity in (2.2.19) and using $h_0 = \frac{1}{2}\rho^2$ gives

$$\{h, k^+\} \stackrel{(2.2.20)}{=} \frac{1}{\tilde{f}}\{h, h_0\} - \frac{h_0}{\rho} \frac{d\tilde{f}[\tilde{X}_h]}{\tilde{f}^2} = \frac{1}{\tilde{f}}\left[\{h, h_0\} - \frac{\rho}{2} \frac{d\tilde{f}[\tilde{X}_h]}{\tilde{f}}\right].$$

By identity (2.2.15), $\{h, h_0\} = -\frac{\rho}{2}\operatorname{div}_{\tilde{\mu}}(\tilde{X}_h)$ so we obtain

$$\{h, k^+\} = -\frac{\rho}{2\tilde{f}}\left[\frac{d\tilde{f}[\tilde{X}_h]}{\tilde{f}} + \operatorname{div}_{\tilde{\mu}}\tilde{X}_h\right] \stackrel{(2.2.13)}{=} -\frac{\rho}{2\tilde{f}}\operatorname{div}_{\tilde{f}\tilde{\mu}}(\tilde{X}_h).$$

Then, by (2.2.18) and since the function $\frac{\rho}{2\tilde{f}}\operatorname{div}_{\tilde{f}\tilde{\mu}}(\tilde{X}_h)$ is homogeneous of degree 0, there exists a strictly positive δ such that

$$\{h, k^+\} = -\frac{\rho}{2\tilde{f}}\operatorname{div}_{\tilde{f}\tilde{\mu}}(\tilde{X}_h) \geq \delta > 0 \quad \text{on } \pi^{-1}(U^+) ,$$

proving (2.2.17). The statement with “−” is analogous. $\square$

2.2.3 Extension to the basins of attraction of $\Gamma^\pm$

The next goal is to extend the local escape functions of Proposition 2.2.14 to the whole *basins of attraction* $B(\Gamma^\pm)$ of $\Gamma^\pm = \pi^{-1}(K^\pm)$ defined as

$$B(\Gamma^\pm) := \{z = (x, \xi) \in \Sigma_{e_0} : \operatorname{dist}(\Phi_{X_h}^t(z), \Gamma^\pm) \rightarrow 0 \text{ as } t \rightarrow \pm\infty\} . \quad (2.2.21)$$

We first characterize $B(\Gamma^\pm)$ and prove several properties of these sets.

Proposition 2.2.15. *One has $B(\Gamma^+) \cup B(\Gamma^-) = \Sigma_{e_0}$. Precisely:*

(i) *There exist open sets $U^\pm \subset Z_{e_0}$, $U^\pm \supset K^\pm$, such that for any $z_0 \in \pi^{-1}(U^\pm)$ the flow $\Phi_{X_h}^t(z_0)$ is globally defined for $\pm t \geq 0$ and $\Phi_{X_h}^t(\pi^{-1}(U^\pm)) \subseteq \pi^{-1}(U^\pm)$ for $\pm t \geq 0$.*

(ii) *If $z_0 \in \Gamma^+$, the flow $\Phi_{X_h}^t(z_0)$ exists globally positively in time and $\Phi_{X_h}^t(z_0) \subseteq \Gamma^+$ for any $t \geq 0$, whereas there exists a time $T^+(z_0) < 0$ such that*

$$\Phi_{X_h}^t(z_0) \xrightarrow[t \searrow T^+(z_0)]{} 0;$$

(iii) *If $z_0 \in \Gamma^-$, the flow $\Phi_{X_h}^t(z_0)$ exists globally negatively in time and $\Phi_{X_h}^t(z_0) \subseteq \Gamma^-$ for any $t \leq 0$, whereas there exists a time $T^-(z_0) > 0$ such that*

$$\Phi_{X_h}^t(z_0) \xrightarrow[t \nearrow T^-(z_0)]{} 0.$$

(iv) *If $z_0 \in \Sigma_{e_0} \setminus (\Gamma^+ \cup \Gamma^-)$, then the flow $\Phi_{X_h}^t(z_0)$ exists globally both positively and negatively in time, and one has*

$$\Phi_{X_h}^t(z_0) \xrightarrow[t \rightarrow +\infty]{} \Gamma^+, \quad \Phi_{X_h}^t(z_0) \xrightarrow[t \rightarrow -\infty]{} \Gamma^-;$$

in particular $z_0 \in B(\Gamma^+) \cap B(\Gamma^-)$.

The long proof of this Proposition is postponed to Section 2.5.2.

We shall also use the following lemma.

Lemma 2.2.16. *There exists a positively homogeneous function $\mathbf{m}$ of degree zero, defined on Σ_{e_0} , such that*

$$\mathbf{m} = \{h, k^\pm\} \text{ on } \pi^{-1}(U^\pm), \quad \text{and} \quad \mathbf{m}(z) \geq \frac{\delta}{2} \quad \forall z \in \Sigma_{e_0}, \quad (2.2.22)$$

where δ is the positive value fixed by (2.2.17) and $U^\pm$ are as in Proposition 2.2.15.

Proof. Take open sets $V^\pm$ with $V^\pm \supset U^\pm$, $V^\pm \cap U^\mp = \emptyset$ and complete $\{V^+, V^-\}$ to a covering of Z_{e_0} via finitely many open sets $\{V_i\}_i$. Up to restricting $V^\pm$ one has $\{h, k^\pm\} \geq \frac{\delta}{2}$ on $V^\pm$, with δ in (2.2.17). Consider a smooth, non-negative, partition

of the identity $\varphi^+, \varphi^-, \{\varphi_i\}_i$ with the properties $\text{supp}(\varphi^\pm) \subset V^\pm$ and $\varphi^\pm \equiv 1$ on $U^\pm$, $\text{supp}(\varphi_i) \subset V_i$. Define the smooth function

$$\tilde{\mathbf{m}} : Z_{e_0} \rightarrow \mathbb{R}, \quad \tilde{\mathbf{m}} := \{h, k^+\}\varphi^+ + \{h, k^-\}\varphi^- + \delta \sum_i \varphi_i.$$

One has $\tilde{\mathbf{m}} \equiv \{h, k^\pm\} \geq \frac{\delta}{2}$ on $V^\pm$ and thus $\tilde{\mathbf{m}} \geq \frac{\delta}{2}$ on Z_{e_0} . Then extend $\tilde{\mathbf{m}}$ by homogeneity to a function $\mathbf{m}$ on Σ_{e_0} , namely put $\mathbf{m}(z) := \tilde{\mathbf{m}}(\pi(z))$. $\square$

We are finally ready to extend the local escape functions $k^\pm$ in (2.2.14) to escape functions on the whole basins of attraction $B(\Gamma^\pm)$.

Lemma 2.2.17. *For $z \in B(\Gamma^\pm)$, define*

$$\ell^\pm(z) := \lim_{t \rightarrow \pm\infty} \left(k^\pm(\Phi_{X_h}^t(z)) - \int_0^t \mathbf{m}(\Phi_{X_h}^s(z)) ds \right). \quad (2.2.23)$$

Then

(i) *the functions $\ell^\pm$ are well defined on $B(\Gamma^\pm)$, smooth and positively homogeneous of degree two;*

(ii) *$\ell^\pm$ are escape functions on $B(\Gamma^\pm)$, namely there exists $\delta > 0$ such that*

$$\{h, \ell^\pm\} \geq \delta > 0 \quad \text{in } B(\Gamma^\pm). \quad (2.2.24)$$

Proof. We prove the statement with “+”, the other being analogous.

(i) Well defined: Let $z \in B(\Gamma^+)$ and let $t_0(z) > 0$ be such that $\Phi_{X_h}^{t_0(z)}(z) \in \pi^{-1}(U^+)$, with U^+ as in Proposition 2.2.15. By definition of $B(\Gamma^+)$, $t_0(z)$ is finite. We show that for any $t > t_0(z)$, the argument of the limit in (2.2.23) is constant. Since, by Proposition 2.2.15 (i), the set $\pi^{-1}(U^+)$ is positively invariant, by (2.2.22) we have $\mathbf{m}(\Phi_{X_h}^t(z)) = \{h, k^+\}(\Phi_{X_h}^t(z))$ for any $t \geq t_0(z)$. Therefore for $t \geq t_0(z)$ one has

$$\begin{aligned} & k^+(\Phi_{X_h}^t(z)) - \int_0^t \mathbf{m}(\Phi_{X_h}^s(z)) ds \\ &= k^+(\Phi_{X_h}^t(z)) - \int_0^{t_0(z)} \mathbf{m}(\Phi_{X_h}^s(z)) ds - \int_{t_0(z)}^t \{h, k^+\}(\Phi_{X_h}^s(z)) ds \\ &= k^+(\Phi_{X_h}^t(z)) - \int_0^{t_0(z)} \mathbf{m}(\Phi_{X_h}^s(z)) ds - k^+(\Phi_{X_h}^{t_0(z)}(z)) + k^+(\Phi_{X_h}^{t_0(z)}(z)) \\ &= - \int_0^{t_0(z)} \mathbf{m}(\Phi_{X_h}^s(z)) ds + k^+(\Phi_{X_h}^{t_0(z)}(z)), \end{aligned} \quad (2.2.25)$$

and thus the argument of the limit is definitely constant, proving that ℓ^+ is well defined.

Smoothness: fix $z_0 \in B(\Gamma^+)$ and let $V_{z_0} \subset B(\Gamma^+)$ be a neighbourhood of z_0 . We observe that, again by Proposition 2.2.15, there exists $t_1(V_{z_0}) > 0$, eventually larger than $t_0(z_0)$, such that $\Phi_{X_h}^t(z) \in \pi^{-1}(U^+)$ for all $z \in V_{z_0}$ and $t \geq t_1(V_{z_0})$. Then, by (2.2.25), on the set V_{z_0} the function ℓ^+ reduces to

$$\ell^+(z) = - \int_0^{t_1(V_{z_0})} \mathbf{m}(\Phi_{X_h}^s(z)) ds + k^+(\Phi_{X_h}^{t_1(V_{z_0})}(z)), \quad \forall z \in V_{z_0}, \quad (2.2.26)$$

which is smooth.

Homogeneity: It follows using the homogeneities of k^+ and $\mathbf{m}$ and Remark 2.2.1.

(ii) Given $z_0 \in B(\Gamma^+)$ and $V_{z_0} \subset B(\Gamma^+)$ a small neighbourhood of z_0 , let $t_1(V_{z_0})$ as above. We have

$$\begin{aligned} \{h, \ell^+\}(z) &\stackrel{(2.2.26)}{=} \{h, k^+ \circ \Phi_{X_h}^{t_1(V_{z_0})}\}(z) - \int_0^{t_1(V_{z_0})} \{h, \mathbf{m} \circ \Phi_{X_h}^s(z)\} ds \\ &= \{h, k^+\}(\Phi_{X_h}^{t_1(V_{z_0})}(z)) - \frac{d}{dt} \Big|_{t=0} \int_0^{t_1(V_{z_0})} \mathbf{m}(\Phi_{X_h}^{t+s}(z)) ds \\ &= \{h, k^+\}(\Phi_{X_h}^{t_1(V_{z_0})}(z)) - \mathbf{m}(\Phi_{X_h}^{t_1(V_{z_0})}(z)) + \mathbf{m}(z) \\ &= \mathbf{m}(z) \geq \frac{\delta}{2}, \end{aligned}$$

where in the last two passages we used (2.2.22). Finally rename $\delta \rightsquigarrow 2\delta$. $\square$

2.2.4 Construction of the global escape function

In this section we glue together the local escape functions ℓ^+ and ℓ^- of Lemma 2.2.17, defined in $B(\Gamma^+)$ and $B(\Gamma^-)$ respectively, to obtain a global escape function on Σ_{e_0} . We start by proving the following result.

Proposition 2.2.18. *Given $\varepsilon > 0$ sufficiently small, there exists a smooth function $\eta_\varepsilon: \Sigma_{e_0} \rightarrow \mathbb{R}$ such that*

- (i) η_ε is homogeneous of degree 0;
- (ii) $\eta_\varepsilon(\Gamma^+) \equiv 1$, $\eta_\varepsilon(\Gamma^-) \equiv 0$;
- (iii) $|\mathrm{d}\eta_\varepsilon(\zeta)[\tilde{X}_h(\zeta)]| \leq \varepsilon$ for any $\zeta \in Z_{e_0}$.

To prove this proposition we shall use the following result, whose proof is postponed to Section 2.5.3.

Lemma 2.2.19. *There exist a smooth function $\tilde{\mathcal{L}} : Z_{e_0} \rightarrow \mathbb{R}$ and an interval $(\alpha, +\infty) \subset \mathbb{R}$ such that the open set $V^- := \tilde{\mathcal{L}}^{-1}((\alpha, +\infty))$ has the following properties:*

- (i) $V^- \supset K^-$ is invariant by the flow of $\tilde{X}_h$ for negative times, and $Z_{e_0} \setminus V^- \supset K^+$ is invariant by the flow of $\tilde{X}_h$ for positive times;
- (ii) ∂V^- is transverse to the flow of $\tilde{X}_h$, in particular $\tilde{X}_h[\tilde{\mathcal{L}}] < 0$ on ∂V^- .

We use this result to define the gluing function. Take $\zeta \in Z_{e_0} \setminus (K^+ \cup K^-)$. Since $\Phi_{\tilde{X}_h}^t(\zeta) \rightarrow K^\pm$ when $t \rightarrow \pm\infty$, by Lemma 2.2.19 there will be a unique time $t_\zeta \in \mathbb{R}$ such that $\Phi_{\tilde{X}_h}^{-t_\zeta}(\zeta)$ intersects ∂V^- ; we then put

$$\tilde{t}(\zeta) := \begin{cases} t_\zeta, & \zeta \in Z_{e_0} \setminus (K^+ \cup K^-) \\ +\infty, & \zeta \in K^+ \\ -\infty, & \zeta \in K^-. \end{cases} \quad (2.2.27)$$

Next, given $\varepsilon > 0$ sufficiently small, take a smooth non-decreasing function $\phi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\phi_\varepsilon(\tau) := \begin{cases} 0, & \tau \leq -\varepsilon \\ \varepsilon\tau, & \varepsilon < \tau \leq \varepsilon^{-1} - \varepsilon \\ 1, & \tau > \varepsilon^{-1} + \varepsilon \end{cases} \quad \text{and} \quad \sup_{\tau \in \mathbb{R}} |\phi'_\varepsilon(\tau)| \leq 10\varepsilon. \quad (2.2.28)$$

The gluing function of Proposition 2.2.18 is

$$\eta_\varepsilon(z) := \phi_{\frac{\varepsilon}{10}}(\tilde{t}(\boldsymbol{\pi}(z))) , \quad z \in \Sigma_{e_0} , \quad (2.2.29)$$

where $\tilde{t}$ is defined in (2.2.27) and ϕ_ε in (2.2.28).

Proof of Proposition 2.2.18. It is clear from its definition that η_ε is homogeneous of degree 0. Moreover from (2.2.27), (2.2.28) we have $\eta_\varepsilon(z) := \phi_{\frac{\varepsilon}{10}}(\tilde{t}(\boldsymbol{\pi}(z))) = 1$ for any $z \in \Gamma^+$; similarly $\eta_\varepsilon(\Gamma^-) \equiv 0$. This proves items (i) and (ii).

We prove now item (iii). Let $\zeta \in Z_{e_0}$ and note that, by (2.2.27), for any $\tau \in \mathbb{R}$ one has

$$\tilde{t}(\Phi_{\tilde{X}_h}^\tau(\zeta)) = \tilde{t}(\zeta) + \tau ; \quad (2.2.30)$$

hence we compute

$$d\eta_\varepsilon(\zeta)[\tilde{X}_h(\zeta)] = \frac{d}{d\tau} \Big|_{\tau=0} \eta_\varepsilon \left(\Phi_{\tilde{X}_h}^\tau(\zeta) \right) \stackrel{(2.2.30)}{=} \frac{d}{d\tau} \Big|_{\tau=0} \phi_{\frac{\varepsilon}{10}}(\tilde{t}(\zeta) + \tau) = \phi'_{\frac{\varepsilon}{10}}(\tilde{t}(\zeta)) ,$$

and (iii) follows from (2.2.28).

We are left to prove that η_ε is smooth. Remark that, due to the homogeneity of η_ε , we can assume $z =: \zeta \in Z_{e_0}$. We are going to prove that η_ε is constant on sufficiently small neighbourhoods of $K^\pm$, whereas outside such neighbourhoods we will prove smoothness of $\tilde{t}$ (and thus of η_ε) by implicit function theorem. Take a point $\underline{\zeta} \in Z_{e_0} \setminus (K^+ \cup K^-)$ and consider the corresponding point $\underline{y} = \Phi_{\tilde{X}_h}^{-\tilde{t}(\underline{\zeta})}(\underline{\zeta}) \in \partial V^-$, with V^- the set of Lemma 2.2.19. Recall that $\partial V^- = \tilde{\mathcal{L}}^{-1}(\alpha)$. Then, let $U_{\underline{\zeta}}$ be a small neighbourhood of $\underline{\zeta}$ with $U_{\underline{\zeta}} \cap (K^+ \cup K^-) = \emptyset$: the function

$$F : U_{\underline{\zeta}} \times \mathbb{R} \rightarrow \mathbb{R} , \quad F(\underline{\zeta}, t) := \tilde{\mathcal{L}}(\Phi_{\tilde{X}_h}^{-t}(\underline{\zeta})) - \alpha$$

is smooth, $F(\underline{\zeta}, \tilde{t}(\underline{\zeta})) = 0$ by the very definition of $\tilde{t}$ and, being ∂V^- transverse to $\tilde{X}_h$ by Lemma 2.2.19 (iii),

$$\partial_t F(\underline{\zeta}, \tilde{t}(\underline{\zeta})) = -d\tilde{\mathcal{L}}(\underline{y})[\tilde{X}_h(\underline{y})] > 0 .$$

Then, by implicit function theorem, in $U_{\underline{\zeta}}$ the function $\zeta \mapsto \tilde{t}(\zeta)$ is smooth. We conclude that η_ε is smooth in every point of $Z_{e_0} \setminus (K^+ \cup K^-)$.

The final claim is that there exist $W^\pm$ neighbourhoods of $K^\pm$ such that $\eta_\varepsilon \equiv 0$ in W^- and $\eta_\varepsilon \equiv 1$ in W^+ . We prove this claim with "+". To this aim it suffices to define

$$W^+ := \Phi_{\tilde{X}_h}^{\varepsilon^{-1}+1}(Z_{e_0} \setminus V^-) .$$

First remark that W^+ is a neighbourhood of K^+ since $K^+ \subset Z_{e_0} \setminus V^-$, and since $K^- \not\subset Z_{e_0} \setminus V^-$, one has $Z_{e_0} \setminus V^-$ is contained in the basin of attraction of K^+ ; then using (2.1.8)

$$K^+ = \bigcap_{t>0} \Phi_{\tilde{X}_h}^t(Z_{e_0} \setminus V^-) \subset \Phi_{\tilde{X}_h}^{\varepsilon^{-1}+1}(Z_{e_0} \setminus V^-) = W^+ .$$

Next, for any $\zeta \in W^+$, there is $v \in Z_{e_0} \setminus V^-$ such that $\zeta = \Phi_{\tilde{X}_h}^{\varepsilon^{-1}+1}(v)$ and with $\tilde{t}(v) \geq 0$, hence $\tilde{t}(\zeta) \stackrel{(2.2.30)}{=} \varepsilon^{-1} + 1 + \tilde{t}(v) > \varepsilon^{-1} + \varepsilon$; therefore $\eta_\varepsilon(\zeta) = \phi_{\frac{\varepsilon}{10}}(\tilde{t}(\zeta)) = 1$ by the properties of ϕ_ε in (2.2.28).

The same argument can be repeated for K^- taking $W^- := \Phi_{\tilde{X}_h}^{-1}(V^-)$, obtaining that $\eta_\varepsilon \equiv 0$ in W^- . $\square$

We are now in position to construct a global escape function. First we construct it on Σ_{e_0} and then extend it to the whole $\mathbb{R}^4 \setminus \{0\}$. Using all previous notation for the local escapes and the gluing functions, we prove the following result:

Proposition 2.2.20. *There exists a function $a : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ positively homogeneous of degree 2, such that a is an escape function for h at e_0 in the sense of Definition 2.2.2.*

Proof. Step 1: First we construct a smooth function $\tilde{a} : \Sigma_{e_0} \rightarrow \mathbb{R}$, positively homogeneous of degree 2, that satisfies (2.2.3) for some $\delta > 0$. To this goal, it suffices to define, for ε sufficiently small,

$$\tilde{a}(z) := \eta_\varepsilon(z)\ell^+(z) + (1 - \eta_\varepsilon(z))\ell^-(z) ,$$

with η_ε defined in (2.2.29) and $\ell^\pm$ in (2.2.23). Remark that $\tilde{a}$ is well defined: indeed, from Lemma 2.2.19, $\eta_\varepsilon \equiv 0$ in Γ^- where ℓ^+ is not defined, whereas $1 - \eta_\varepsilon \equiv 0$ in Γ^+ where ℓ^- is not defined.

Next note that $\tilde{a}$ is smooth, since all terms in its expression are smooth and $\tilde{a}$ is positively homogeneous of degree 2, since both ℓ^+ and ℓ^- are (see Lemma 2.2.17) and η_ε is positively homogeneous of degree zero (see Proposition 2.2.18). We now show that $\tilde{a}$ satisfies (2.2.3). One has:

$$\{h, \tilde{a}\} = \{h, \eta_\varepsilon\}\ell^+ + \{h, \ell^+\}\eta_\varepsilon + \{h, \ell^-\}(1 - \eta_\varepsilon) + \{h, (1 - \eta_\varepsilon)\}\ell^- \stackrel{(2.2.24)}{\geq} \{h, \eta_\varepsilon\}(\ell^+ - \ell^-) + \delta . \quad (2.2.31)$$

Next we remark that, by (2.2.6) we have

$$\{h, \eta_\varepsilon\} = d\eta_\varepsilon[X_h] = \frac{\{h, h_0\}}{\rho^2} d\eta_\varepsilon[\nabla h_0] + \frac{1}{\rho} d\eta_\varepsilon[\tilde{X}_h] = \frac{1}{\rho} d\eta_\varepsilon[\tilde{X}_h],$$

where the last equality is obtained using (2.1.3) and the homogeneity of η_ε . Since $d\eta_\varepsilon[\tilde{X}_h]$ is homogeneous of degree -1 , using Proposition 2.2.18 (iii) and writing $z \equiv \rho\pi(z) \in \Sigma_{e_0}$, we get

$$|\{h, \eta_\varepsilon\}(z)| = \left| \frac{1}{\rho} d\eta_\varepsilon[\tilde{X}_h](z) \right| = \left| \frac{1}{\rho^2} d\eta_\varepsilon[\tilde{X}_h](\pi(z)) \right| \leq \frac{\varepsilon}{\rho^2}.$$

In conclusion, using also that $\ell^\pm$ are positively homogeneous functions of degree 2, we have the following bound:

$$|\{h, \eta_\varepsilon\}(z)(\ell^+(z) - \ell^-(z))| \leq \frac{\varepsilon}{\rho^2} \sup_{z \in \Sigma_{e_0}} |\ell^+(z) - \ell^-(z)| \leq \varepsilon \sup_{\zeta \in Z_{e_0}} |\ell^+(\zeta) - \ell^-(\zeta)| \leq C\varepsilon.$$

Hence, shrinking ε , we obtain from (2.2.31) the lower bound

$$\{h, \tilde{a}\} > \delta - C\varepsilon > \frac{\delta}{2} \quad \text{on } \Sigma_{e_0}. \quad (2.2.32)$$

Step 2: We extend $\tilde{a}$ to the whole $\mathbb{R}^4 \setminus \{0\}$. To this aim consider first the restriction $\tilde{a}|_{Z_{e_0}} : Z_{e_0} \rightarrow \mathbb{R}$. It is a smooth function defined on the closed submanifold Z_{e_0} of $\mathbb{S}^3$. By classical extension Lemma (see e.g. [62, Lemma 2.26]), given an open neighbourhood $\mathcal{U}$ of Z_{e_0} in $\mathbb{S}^3$, there exists a smooth function $\hat{a}$,

$$\hat{a} : \mathbb{S}^3 \rightarrow \mathbb{R}, \text{ such that } \hat{a} = \tilde{a} \text{ on } Z_{e_0} \text{ and } \text{supp}(\hat{a}) \subset \mathcal{U}.$$

Then we define a as the extension of $\hat{a}$ by homogeneity on the whole $\mathbb{R}^4 \setminus \{0\}$. Due to the cone structure of Σ_{e_0} and the fact that $\tilde{a}$ and a are both homogeneous of degree two, $\tilde{a} = a$ on the whole Σ_{e_0} , since they coincide on Z_{e_0} . In particular $\{h, a\} = \{h, \tilde{a}\} \geq \frac{\delta}{2}$ on Σ_{e_0} . $\square$

Remark 2.2.21. If Σ_{e_0} is not connected, one proves Step 1 of the proof of Proposition 2.2.20 separately for each one of its n connected components Σ_j , obtaining n functions $\tilde{a}_j : \Sigma_j \rightarrow \mathbb{R}$, homogeneous of degree 2, such that $\{h, \tilde{a}_j\} > \frac{\delta_j}{2}$ for all j and for some positive $\delta_1, \dots, \delta_n$. Then one defines $\tilde{a}$ on the whole Σ_{e_0} as $\tilde{a} = \tilde{a}_j$ on Σ_j ; (2.2.32) holds with $\delta = \min\{\delta_j\} > 0$, and Step 2 follows unchanged.

We conclude this section proving that if a is an escape function for h at e_0 in the sense of Definition 2.2.2, then a is also an escape function for h at ω , for ω sufficiently close to e_0 . Precisely we have:

Corollary 2.2.22. *Let $\mathbf{e}_0$ be a regular value of h . If a is an escape function for h at $\mathbf{e}_0$ (see Definition 2.2.2), there exists a neighbourhood I of $\mathbf{e}_0$, contained in the image of h , such that a is still an escape function for h at all values $\omega \in I$.*

Proof. Let $\delta > 0$ be such that (2.2.3) is satisfied on $\Sigma_{\mathbf{e}_0}$; we start by showing that there exists a sufficiently small interval $I \ni \mathbf{e}_0$ such that

$$\{h, a\} \geq \frac{\delta}{2}, \quad \forall (x, \xi) \in \bigcup_{\omega \in I} \Sigma_\omega. \quad (2.2.33)$$

Since the function $\{h, a\}$ is homogeneous of degree zero, it is sufficient to prove (2.2.33) for any $\zeta \in \bigcup_{\omega \in I} Z_\omega$, with $Z_\omega := h^{-1}(\omega) \cap \mathbb{S}^3$. From continuity of $\{h, a\}|_{\mathbb{S}^3}$ and (2.2.3), there exists an open neighbourhood $\mathcal{U}_{Z_{\mathbf{e}_0}} \subset \mathbb{S}^3$ of $Z_{\mathbf{e}_0}$ such that

$$\{h, a\} \geq \frac{\delta}{2} \quad \forall \zeta \in \mathcal{U}_{Z_{\mathbf{e}_0}}.$$

We show that, choosing $\varepsilon > 0$ small enough, the interval $I := (\mathbf{e}_0 - \varepsilon, \mathbf{e}_0 + \varepsilon)$ fulfills $h^{-1}(I) \subset \mathcal{U}_{Z_{\mathbf{e}_0}}$, thus proving (2.2.33). Suppose by contradiction that $\exists N \in \mathbb{N}$, such that, defining

$$I_n := \left(\mathbf{e}_0 - \frac{1}{n}, \mathbf{e}_0 + \frac{1}{n} \right), \quad (2.2.34)$$

one has

$$\{h^{-1}(I_n) \cap \mathbb{S}^3\} \not\subset \mathcal{U}_{Z_{\mathbf{e}_0}}, \quad \forall n > N.$$

Take now a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in \{h^{-1}(I_n) \cap \mathbb{S}^3\} \setminus \mathcal{U}_{Z_{\mathbf{e}_0}}$. By compactness, up to a subsequence it converges to a point $x_\infty \in \mathbb{S}^3 \setminus \mathcal{U}_{Z_{\mathbf{e}_0}}$. From continuity of h we get $h(x_n) \rightarrow h(x_\infty)$, but $I_n \ni h(x_n) \rightarrow \mathbf{e}_0$ from the definition (2.2.34) of I_n , whilst $x_\infty \notin Z_{\mathbf{e}_0}$, hence $h(x_\infty) \neq \mathbf{e}_0$, obtaining a contradiction.

Finally remark that, since $\mathbf{e}_0$ is a regular value of the smooth function h , up to shrinking ε we can assume that I is contained in the image of h and only consists of regular values, concluding the proof. $\square$

2.3 Genericity of the set $\mathcal{V}$

The goal of this section is to show that the set $\mathcal{V}$ defined in (2.0.3) is generic (i.e., open and dense) in $C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ (see Definition 1.1.2) with respect to the distance $d_{\mathbb{T}}^0$ defined in (1.1.4), proving Theorem 2.0.1 (i).

2.3.1 The set $\mathcal{V}$ is open

Proposition 2.3.1. *The set $\mathcal{V}$ defined in (2.0.3) is open with respect to the distance $d_{\mathbb{T}}^0$ defined in (1.1.4).*

We separately state the following preliminary lemma.

Lemma 2.3.2. *Let $v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ and let $\langle v \rangle$ be its average along the flow of h_0 (see Definition 1.1.5), then*

$$\sum_{|\alpha|+|\beta| \leq 1} \sup_{|(x,\xi)| \geq 1} \left| \partial_x^\alpha \partial_\xi^\beta \langle v \rangle(x, \xi) \right| \leq 4 \sum_{|\alpha|+|\beta| \leq 1} \sup_{\substack{|(x,\xi)| \geq 1 \\ t \in \mathbb{T}}} \left| \partial_x^\alpha \partial_\xi^\beta v(t, x, \xi) \right|.$$

Proof. We only show the bound for the case $\alpha = (1, 0)$ and $\beta = (0, 0)$, all the others being analogous. Using the explicit form (1.1.6) of the classical flow $(x(t), \xi(t)) := \phi_{h_0}^t(x, \xi)$ of the harmonic oscillator, and the definition (1.1.7), we have

$$\begin{aligned} \sup_{|(x,\xi)| \geq 1} |\partial_{x_1} \langle v \rangle(x, \xi)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{\substack{|(x,\xi)| \geq 1 \\ t \in \mathbb{T}}} (|\partial_{x_1} v(t, x(t), \xi(t))| + |\partial_{\xi_1} v(t, x(t), \xi(t))|) dt \\ &\leq \sup_{\substack{|(x,\xi)| \geq 1 \\ t \in \mathbb{T}}} (|\partial_{x_1} v(t, x, \xi)| + |\partial_{\xi_1} v(t, x, \xi)|). \end{aligned}$$

Analogously one bounds the other terms with $|\alpha| + |\beta| \leq 1$, obtaining the claim. $\square$

Proof of Proposition 2.3.1. Take $v \in \mathcal{V}$. We show the existence of $\varepsilon > 0$ sufficiently small such that

$$d_{\mathbb{T}}^0(v, v + w) < \varepsilon \quad \Rightarrow \quad v + w \in \mathcal{V} \quad (2.3.1)$$

for any $w \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$. Decompose $v = v_0 + v_{-\mu}$, and $w = w_0 + w_{-\mu'}$ following Definition 1.1.2, where v_0 and w_0 are the principal symbols of v and w respectively, and $v_{-\mu} \in C^0(\mathbb{T}; \mathbf{S}^{-\mu})$ and $w_{-\mu'} \in C^0(\mathbb{T}; \mathbf{S}^{-\mu'})$, with $\mu, \mu' > 0$.

By the very definition of $\mathcal{V}$ in (2.0.3), since $v \in \mathcal{V}$ there exist $a \in \mathbf{S}_{cl}^1$, $I \subset \text{Ran}(\langle v_0 \rangle)$ and $\delta > 0$ such that

$$\{\langle v_0 \rangle, a\} \geq \delta, \quad \forall (x, \xi) \in \langle v_0 \rangle^{-1}(I) \cap \{|(x, \xi)| \geq 1\}.$$

We are going to show that there exist $J \subset \text{Ran}(\langle v_0 + w_0 \rangle)$ and $\delta' > 0$ such that

$$\{\langle v_0 + w_0 \rangle, a\} \geq \delta', \quad \forall (x, \xi) \in \langle v_0 + w_0 \rangle^{-1}(J) \cap \{|(x, \xi)| \geq 1\}. \quad (2.3.2)$$

To this aim, we claim that

$$\sum_{|\alpha|+|\beta|\leq 1} \sup_{\substack{|(x,\xi)|\geq 1 \\ t\in\mathbb{T}}} \left| \partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi) \right| \leq 10\varepsilon. \quad (2.3.3)$$

Assuming this estimate, we continue with the proof of (2.3.2). We decompose

$$\{\langle v_0 + w_0 \rangle, a\} = \{\langle v_0 \rangle, a\} + \{\langle w_0 \rangle, a\},$$

and we observe that, by Lemma 2.3.2,

$$\sup_{|(x,\xi)|\geq 1} |\{\langle w_0 \rangle, a\}| \leq 2\wp_1^1(a) \sum_{|\alpha|+|\beta|\leq 1} \sup_{|(x,\xi)|\geq 1} \left| \partial_x^\alpha \partial_\xi^\beta \langle w_0 \rangle(x, \xi) \right| \stackrel{(2.3.3)}{\leq} 80\wp_1^1(a)\varepsilon.$$

Thus, choosing $\varepsilon \leq \frac{\delta}{160\wp_1^1(a)}$, estimate (2.3.2) follows with $\delta' = \frac{\delta}{2}$ in $\langle v_0 \rangle^{-1}(I) \cap \{|(x, \xi)| \geq 1\}$. In order to conclude the proof of (2.3.2) we show that there exists $J \subset I$ such that, for all w satisfying the bound in (2.3.1),

$$\langle v_0 + w_0 \rangle^{-1}(J) \cap \{|(x, \xi)| \geq 1\} \subset \langle v_0 \rangle^{-1}(I) \cap \{|(x, \xi)| \geq 1\}. \quad (2.3.4)$$

From equation (2.3.3) and Lemma 2.3.2 we have that

$$\sup_{|(x,\xi)|\geq 1} |\langle w_0 \rangle(x, \xi)| \leq 4 \sup_{\substack{|(x,\xi)|\geq 1 \\ t\in\mathbb{T}}} |w_0(t, x, \xi)| \leq 40\varepsilon. \quad (2.3.5)$$

Hence, denoting by $I = [a, b]$, we choose $J := [a + 40\varepsilon, b - 40\varepsilon]$. Up to shrinking ε we have $\emptyset \neq J \subset I$, and by (2.3.5) J satisfies (2.3.4). Thus (2.3.2) is proven.

It remains to prove (2.3.3). We first notice that, from the definition (1.1.4) of the distance, if $\mathbf{d}_{\mathbb{T}}^0(v, v + w) < \varepsilon$, up to shrinking ε we have

$$\wp_{\mathbb{T},1}^0(w) = \sum_{|\alpha|+|\beta|\leq 1} \sup_{\substack{(x,\xi)\in\mathbb{R}^4 \\ t\in\mathbb{T}}} \frac{\left| \partial_x^\alpha \partial_\xi^\beta w(t, x, \xi) \right|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} \leq 4\varepsilon. \quad (2.3.6)$$

Recall the decomposition $w = w_0 + w_{-\mu'}$; since $w_{-\mu'} \in C^0(\mathbb{T}; \mathbf{S}^{-\mu'})$ (see Definition 1.1.2), we have

$$\frac{\left| \partial_x^\alpha \partial_\xi^\beta w_{-\mu'}(t, x, \xi) \right|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} \leq \frac{C_{\alpha,\beta}(w)}{(1 + |x|^2 + |\xi|^2)^{\mu'}}, \quad \forall \alpha, \beta \in \mathbb{N}^2.$$

Fix now $\lambda_\varepsilon = \lambda_\varepsilon(w) \geq 1$ sufficiently large such that

$$\sum_{|\alpha|+|\beta| \leq 1} \frac{C_{\alpha,\beta}(w)}{(1 + \lambda_\varepsilon^2)^{\mu'}} < \varepsilon. \quad (2.3.7)$$

We have

$$\begin{aligned} \sum_{|\alpha|+|\beta| \leq 1} \sup_{\substack{|(x,\xi)| \geq \lambda_\varepsilon \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} &\leq \sum_{|\alpha|+|\beta| \leq 1} \sup_{\substack{|(x,\xi)| \geq \lambda_\varepsilon \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_{-\mu'}(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} + \wp_{\mathbb{T},1}^0(w) \\ &\stackrel{(2.3.6), (2.3.7)}{\leq} \varepsilon + 4\varepsilon = 5\varepsilon. \end{aligned} \quad (2.3.8)$$

Finally remark that, for all $|(x, \xi)| \geq 1$,

$$\frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} \leq 2 \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(|x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} \leq 2 \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}}, \quad (2.3.9)$$

and that, since w_0 is definitely homogeneous of degree zero (see Definition 1.1.1), then

$$\sup_{\substack{|(x,\xi)| \geq \lambda_\varepsilon \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(|x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} = \sup_{\substack{|(x,\xi)| \geq 1 \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(|x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}}, \quad (2.3.10)$$

hence

$$\begin{aligned} \sum_{|\alpha|+|\beta| \leq 1} \sup_{\substack{|(x,\xi)| \geq 1 \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} &\stackrel{(2.3.9)}{\leq} 2 \sum_{|\alpha|+|\beta| \leq 1} \sup_{\substack{|(x,\xi)| \geq 1 \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(|x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} \\ &\stackrel{(2.3.10)}{\leq} 2 \sum_{|\alpha|+|\beta| \leq 1} \sup_{\substack{|(x,\xi)| \geq \lambda_\varepsilon \\ t \in \mathbb{T}}} \frac{|\partial_x^\alpha \partial_\xi^\beta w_0(t, x, \xi)|}{(1 + |x|^2 + |\xi|^2)^{-\frac{|\alpha|+|\beta|}{2}}} \stackrel{(2.3.8)}{\leq} 10\varepsilon, \end{aligned}$$

proving (2.3.3). $\square$

2.3.2 The set $\mathcal{V}$ is dense

The starting point to prove density of $\mathcal{V}$ is to use Proposition 2.2.9 to characterize a subset $\mathcal{A} \subset \mathcal{V}$, that we will prove to be dense in $C^0(\mathbb{T}; \mathbb{S}_{cl}^0)$.

We recall the definitions of classical symbol 1.1.2, resonant average of a symbol 1.1.5

and that of Z_{e_0} (2.2.4).

Throughout this section we will profusely use the results of Section 2.2, where the functions are positively homogeneous and not definitely homogeneous as in the definition of classical symbols. In particular we need to transform the definitely homogeneous symbol $\langle v_0 \rangle$ (averaged with respect to time), into a positively homogeneous function; we do this defining

$$[v_0] := \pi^* \langle v_0 \rangle = \langle v_0 \rangle \circ \pi, \quad \pi \text{ in (2.2.4)}. \quad (2.3.11)$$

Remark 2.3.3. Note that $[v_0] = \langle v_0 \rangle$ for $|(x, \xi)| \geq 1$.

We shall use repeatedly the notations and definitions of Section 2.2.1. The main result of this section is the following:

Proposition 2.3.4. *With the notation in (2.3.11), denote by*

$$\begin{aligned} \mathcal{A} := & \{v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0) : \exists v_0 \text{ principal symbol of } v \text{ such that the vector field } \tilde{X}_{[v_0]} \\ & \text{induces a foliation on } Z_{e_0}^{[v_0]} \text{ with weakly hyperbolic simple structure,} \\ & \text{for some regular value } e_0 \in \text{Ran}([v_0])\} . \end{aligned} \quad (2.3.12)$$

Then $\mathcal{A} \subseteq \mathcal{V}$, with $\mathcal{V}$ in (2.0.3), and $\mathcal{A}$ is dense in $C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ with respect to the distance $d_{\mathbb{T}}^0$ (see (1.1.4)).

In the rest of this section we prove Proposition 2.3.4, that, together with Proposition 2.3.1, proves Theorem 2.0.1 point (i). To prove Proposition 2.3.4 we shall follow the following scheme:

1. given any positively homogeneous function h of degree zero and a regular value e_0 for h , the foliation $\mathcal{F}^h$ induced by $\tilde{X}_h$ on $Z_{e_0}^h$ is the characteristic foliation of $Z_{e_0}^h$ regarded as a submanifold of a contact manifold;
2. given h as in point 1, there exists h_ε , positively homogeneous of degree zero and arbitrarily close to h (when restricted to $\mathbb{S}^3$), such that the foliation induced by $\tilde{X}_{h_\varepsilon}$ is Morse-Smale;
3. given h_ε as in point 2, the induced characteristic foliation $\mathcal{F}^{h_\varepsilon}$ has actually weakly hyperbolic simple structure;

4. given $v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$, let $h = [v_0]$, and h_ε as in point 3, there exists a definitely homogeneous symbol $v_\varepsilon \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ arbitrarily close to v_0 and such that $[v_\varepsilon] = h_\varepsilon$. This concludes the proof of density of the set $\mathcal{A}$.
5. Prove that $\mathcal{A} \subset \mathcal{V}$.

We now prove each step of the scheme.

The induced foliation is a characteristic foliation

To prove the first step we are going to identify $Z_{e_0}^h$ as a submanifold of $\mathbb{S}^3$, which we regard as a contact manifold with contact structure induced by the symplectic structure of $(\mathbb{R}^4, \Omega)$, where Ω is the standard symplectic form, $\Omega := dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$.

Proposition 2.3.5. *Let $h : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ be a positively homogeneous function of degree zero, fix a regular value e_0 for h and let $\tilde{X}_h$ be the vector field on $Z_{e_0}^h$ (see (2.2.5)) defined in (2.2.6). Let $\tilde{\mu}$ be the volume form on $Z_{e_0}^h$ defined in (2.2.10). Then $\tilde{X}_h$ directs the characteristic foliation of $Z_{e_0}^h$ induced by the contact manifold $(\mathbb{S}^3, \kappa)$ with $\kappa = \ker(\alpha)$ and α defined by Corollary 2.1.14. In particular,*

$$i_{\tilde{X}_h} \tilde{\mu} = (2|\nabla h| \alpha)|_{Z_{e_0}^h}. \quad (2.3.13)$$

Proof. We compute $i_{\tilde{X}_h} \tilde{\mu}$. Using the definition (2.2.7) of $\tilde{X}_h$ and identity (2.2.11), we have $i_{\tilde{X}_h} \tilde{\mu} = i_{\rho X_h} \tilde{\mu}$. Next, from the definition (2.2.10),

$$\tilde{\mu} = -i_{\frac{\nabla h}{|\nabla h|}} i_{\nabla h_0} \nu = i_{\frac{\nabla h}{|\nabla h|}} i_{\nabla h_0} \frac{\Omega \wedge \Omega}{2}$$

with Ω the standard symplectic form in $\mathbb{R}^4$. Recalling the expression of α in Corollary 2.1.14, the equality (2.3.13) reads

$$i_{\rho X_h} i_{\frac{\nabla h}{|\nabla h|}} i_{\frac{\nabla h_0}{2}} \Omega \wedge \Omega = (2|\nabla h| i_{\frac{\nabla h_0}{2}} \Omega)|_{Z_{e_0}^h}.$$

We now verify it. Recalling that $\Omega\left(\frac{\nabla h}{|\nabla h|}, X_h\right) = |\nabla h|$, a direct computation yields

$$i_{X_h} i_{\frac{\nabla h}{|\nabla h|}} i_{\frac{\nabla h_0}{2}} (\Omega \wedge \Omega) = 2|\nabla h| i_{\frac{\nabla h_0}{2}} \Omega - \Omega\left(\frac{\nabla h}{|\nabla h|}, \nabla h_0\right) i_{X_h} \Omega + \Omega\left(X_h, \nabla h_0\right) i_{\frac{\nabla h}{|\nabla h|}} \Omega.$$

Now note that the coefficient $\Omega(X_h, \nabla h_0) = 0$ due to (2.1.3), whereas the one form $i_{X_h} \Omega|_{Z_{e_0}^h} = dh|_{Z_{e_0}^h}$ vanishes on $TZ_{e_0}^h$ since $Z_{e_0}^h$ is inside the energy manifold $h^{-1}(e_0)$. $\square$

Density of functions h such that $\tilde{X}_h$ has Morse-Smale foliation

The main result of this section is the following:

Proposition 2.3.6. *For any $h : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ positively homogeneous of degree zero, for any regular value e_0 of h and for any $\varepsilon > 0$, there exists h_ε , positively homogeneous of degree zero, such that $\tilde{X}_{h_\varepsilon}$ directs a Morse-Smale characteristic foliation on $Z_{e_0}^{h_\varepsilon}$, and $d_{C^\infty(\mathbb{S}^3)}(h, h_\varepsilon) < \varepsilon$.*

We need a preliminary result.

Lemma 2.3.7. *Let M, N be smooth compact manifolds of dimension two and let $f : M \rightarrow N$ be a smooth diffeomorphism. If X is a Morse-Smale vector field on N (see Definition 2.1.8), then also $f^*(X)$ is a Morse-Smale vector field on M .*

Proof. Recalling the definition of the vector field $f^*(X)$ as

$$f^*(X)(p) = df^{-1}(f(p))[X(f(p))],$$

one starts with observing that the number of critical elements of $f^*(X)$ coincides with the number of critical elements of X , hence it is finite. Furthermore, since $f^{-1} \circ \phi_X^t \circ f = \phi_{f^*(X)}^t$, one checks that if every orbit of X has one and only one critical element as α limit and one and only one as ω limit, then the same holds for every orbit of $f^*(X)$. Finally, since the Jacobian matrix of X is similar to that of $f^*(X)$, the hyperbolicity of the critical points is preserved. If there are no saddle connections for the flow of X , the same holds for the flow of $f^*(X)$. $\square$

We say that a smooth family of maps $\psi_t : M \rightarrow M$, $t \in [0, 1]$, has size $\varepsilon > 0$ if $d_{C^\infty}(\psi_t, \mathbb{1}) \leq \varepsilon$ for all $t \in [0, 1]$. In particular it is possible to write

$$\psi_1 = \mathbb{1} + \varepsilon \phi_1, \quad d_{C^\infty}(\phi_1, 0) \leq 1. \quad (2.3.14)$$

Proof of Proposition 2.3.6. Fix $h : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$ positively homogeneous function of degree zero, e_0 regular value for h and $\varepsilon > 0$. Let $\tilde{X}_h$ be the vector field on $Z_{e_0}^h$ defined in (2.2.6). Using the genericity result of Theorem 2.1.9 by Peixoto, for any $\varepsilon_0 > 0$

there exists a Morse-Smale vector field X_1 on $Z_{e_0}^h$, such that, recalling the definition of distance between vector fields in (2.1.9),

$$d_{C^\infty(Z_{e_0}^h)}(\tilde{X}_h, X_1) < \varepsilon_0. \quad (2.3.15)$$

Note that X_1 does not necessarily direct the characteristic foliation on $Z_{e_0}^h$ (hence we cannot guarantee that it is a vector field induced by a positively homogeneous function). However, we shall prove that it is possible to slightly deform $Z_{e_0}^h$ to a new submanifold $Z_{e_0}^{h_\varepsilon}$ whose characteristic foliation (which is directed by $\tilde{X}_{h_\varepsilon}$) is actually also directed by the pullback of X_1 , and so it is Morse-Smale.

Precisely, we are going to show that for any $\varepsilon > 0$, provided ε_0 is small enough, there exist a positively homogeneous function of order zero $h_\varepsilon : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}$, a diffeomorphism $\psi : Z_{e_0}^{h_\varepsilon} \rightarrow Z_{e_0}^h$ and a smooth strictly positive function f on $Z_{e_0}^{h_\varepsilon}$ such that

$$\psi^*(X_1) = f\tilde{X}_{h_\varepsilon} \text{ as vector fields on } Z_{e_0}^{h_\varepsilon} \quad \text{and} \quad d_{C^\infty(\mathbb{S}^3)}(h, h_\varepsilon) < \varepsilon. \quad (2.3.16)$$

This concludes the proof since, from Lemma 2.3.7, the flow of $\psi^*(X_1)$ on $Z_{e_0}^{h_\varepsilon}$ is still Morse-Smale, and $f\tilde{X}_{h_\varepsilon}$ directs the same foliation of $\tilde{X}_{h_\varepsilon}$.

We now prove (2.3.16). The key idea is to regard $Z_{e_0}^h$ as a submanifold of the contact manifold $\mathbb{S}^3$, and use the local Gray theorem 2.1.15 to deform it to a new submanifold whose characteristic foliation is Morse-Smale. So define $\alpha_0 := 2|\nabla h|\alpha$ and α the Liouville 1-form as in Corollary 2.1.14. Note that α and α_0 define the same contact structure in a neighbourhood $\mathcal{U}(Z_{e_0}^h) \subset \mathbb{S}^3$ since they differ for function never-vanishing on $\mathcal{U}(Z_{e_0}^h)$ (provided the neighbourhood is chosen sufficiently small, since $|\nabla h|_{|Z_{e_0}^h}$ is non-vanishing).

So define on $Z_{e_0}^h$ the 1-form given by $\beta := i_{X_1 - \tilde{X}_h}\tilde{\mu}$. By (2.3.15) on a sufficiently small neighborhood $\mathcal{U}(Z_{e_0}^h)$ it is possible to extend β to a 1-form, which we still denote by β , such that $\|\beta\|_{C^\infty(\mathcal{U}(Z_{e_0}^h))} \leq 2\varepsilon_0$.

Thus for any $t \in [0, 1]$ the 1-form

$$\alpha_t := \alpha_0 + t\beta$$

is well defined on $\mathcal{U}(Z_{e_0}^h)$, and it is still a contact form, since

$$\alpha_t \wedge d\alpha_t = \alpha_0 \wedge d\alpha_0 + t\alpha_0 \wedge d\beta + t\beta \wedge d\alpha_0 + t^2\beta \wedge d\beta$$

is non-degenerate on $\mathcal{U}(Z_{e_0}^h)$, due to the fact that α_0 is positively proportional on $\mathcal{U}(Z_{e_0}^h)$ to the contact form α and to (2.3.15). Moreover on $Z_{e_0}^h$, in view of (2.3.13) and the very definition of β ,

$$i_{X_1}\tilde{\mu} = \alpha_1|_{Z_{e_0}^h}. \quad (2.3.17)$$

Since α_t is a family of contact forms in $\mathcal{U}(Z_{e_0}^h)$ with $\|\dot{\alpha}_t\|_{C^\infty(\mathcal{U}(Z_{e_0}^h))} = \|\beta\|_{C^\infty(\mathcal{U}(Z_{e_0}^h))} \leq 2\varepsilon_0$, we can now use Gray's Theorem 2.1.15: for any $\varepsilon_1 > 0$ (which will be fixed later), if ε_0 is small enough, there exists a family of smooth diffeomorphisms $\{\psi_t\}_{t \in [0,1]}$ on $\mathcal{U}(Z_{e_0}^h)$ and a family of strictly positive smooth functions λ_t on $\mathcal{U}(Z_{e_0}^h)$ such that $\psi_t^*(\alpha_t) = \lambda_t\alpha_0$ and (2.3.14) holds, with ε replaced by ε_1 . Define now the positively homogeneous function

$$h_\varepsilon := h \circ \psi_1 \circ \pi;$$

clearly $Z_{e_0}^{h_\varepsilon} = \psi_1^{-1}(Z_{e_0}^h)$. Now we show that the diffeomorphism in equation (2.3.16) can be taken as $\psi = \psi_1$. We first evaluate $\psi_1^*(\alpha_1)$ on $Z_{e_0}^{h_\varepsilon}$:

$$\psi_1^*(\alpha_1) \stackrel{(2.3.17)}{=} \psi_1^*(i_{X_1}\tilde{\mu}) = i_{\psi_1^*(X_1)}\psi_1^*(\tilde{\mu}).$$

Combining this identity with (2.1.13) and using the definition of α_0 , we have:

$$i_{\psi_1^*(X_1)}\psi_1^*(\tilde{\mu}) = \lambda_1\alpha_0 = 2|\nabla h|\lambda_1\alpha \quad \text{on} \quad Z_{e_0}^{h_\varepsilon}. \quad (2.3.18)$$

We define on $Z_{e_0}^{h_\varepsilon}$ the volume form $\tilde{\mu}_\varepsilon := -i_{\frac{\nabla h_\varepsilon}{|\nabla h_\varepsilon|}}\nu_{\mathbb{S}^3}$, with $\nu_{\mathbb{S}^3}$ as in Lemma 2.2.6; by Proposition 2.3.5 with h replaced by h_ε and $\tilde{\mu}$ replaced by $\tilde{\mu}_\varepsilon$, we have

$$i_{\tilde{X}_{h_\varepsilon}}\tilde{\mu}_\varepsilon = 2|\nabla h_\varepsilon|\alpha \quad \text{on} \quad Z_{e_0}^{h_\varepsilon}. \quad (2.3.19)$$

Moreover, we observe that both $\tilde{\mu}_\varepsilon$ and $\psi_1^*(\tilde{\mu})$ are volume forms on $Z_{e_0}^{h_\varepsilon}$, thus there exists a smooth function $g : Z_{e_0}^{h_\varepsilon} \rightarrow \mathbb{R}^+$ such that $\psi_1^*(\tilde{\mu}) = g\tilde{\mu}_\varepsilon$. Then comparing (2.3.19) and (2.3.18) one gets that on $Z_{e_0}^{h_\varepsilon}$

$$i_{\psi_1^*(X_1)}\tilde{\mu}_\varepsilon = g^{-1}i_{\psi_1^*(X_1)}\psi_1^*(\tilde{\mu}) = 2g^{-1}|\nabla h|\lambda_1\alpha = g^{-1}\lambda_1|\nabla h||\nabla h_\varepsilon|^{-1}i_{\tilde{X}_{h_\varepsilon}}\tilde{\mu}_\varepsilon,$$

thus, the two vector fields $\psi_1^*(X_1)$ and $\tilde{X}_{h_\varepsilon}$ are proportional.

To prove (2.3.16), it only remains to show that $d_{C^\infty}(h, h_\varepsilon) < \varepsilon$. First choose $N \in \mathbb{N}$ large enough such that $\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \frac{\varepsilon}{2}$, so that:

$$d_{C^\infty}(h, h_\varepsilon) \leq \sum_{n=0}^N \frac{1}{2^n} \frac{\|h - h_\varepsilon\|_{C^n(\mathbb{S}^3)}}{1 + \|h - h_\varepsilon\|_{C^n(\mathbb{S}^3)}} + \frac{\varepsilon}{2}. \quad (2.3.20)$$

Note that it is sufficient to prove that $\forall n = 0, \dots, N$ there exists $C_n > 0$ such that

$$\|h - h_\varepsilon\|_{C^n(\mathbb{S}^3)} \leq C_n \varepsilon_1, \quad (2.3.21)$$

since then (2.3.20) and (2.3.21) give $\mathbf{d}_{C^\infty(\mathbb{S}^3)}(h, h_\varepsilon) \leq \sum_{n=0}^N C_n \varepsilon_1 + \frac{\varepsilon}{2} < \varepsilon$, up to choosing $\varepsilon_1 < \varepsilon(2 \sum_{n=0}^N C_n)^{-1}$. To deduce (2.3.21), with a direct computation one inductively checks that, $\forall n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^2$ with $|\alpha| + |\beta| = n$, one has

$$\partial_x^\alpha \partial_\xi^\beta (h - h_\varepsilon)(x, \xi) = f_{\varepsilon_1, \alpha, \beta}^{(1)}(x, \xi) - f_{\varepsilon_1, \alpha, \beta}^{(1)}((x, \xi) + \varepsilon_1 \phi_1(x, \xi)) + \varepsilon_1 f_{\varepsilon_1, \alpha, \beta}^{(2)}(x, \xi), \quad (2.3.22)$$

with $f_{\varepsilon_1, \alpha, \beta}^{(j)}$ smooth functions, whose $C^k(\mathbb{S}^3)$ norms are finite and have uniform upper bound in ε_1 for any k , and ϕ_1 as in (2.3.14). Then (2.3.22) gives

$$\sup_{\mathbb{S}^3} |\partial_x^\alpha \partial_\xi^\beta (h - h_\varepsilon)(x, \xi)| \leq \varepsilon_1 \left(\sup_{\mathbb{S}^3} |\nabla f^{(1)}| \sup_{\mathbb{S}^3} |\phi_1| + \sup_{\mathbb{S}^3} |f_{\varepsilon_1, \alpha, \beta}^{(2)}| \right) \leq \varepsilon_1 C_{\alpha, \beta},$$

for some positive $C_{\alpha, \beta}$, and taking the sup over all α, β with $|\alpha| + |\beta| \leq n$ (2.3.21) follows, concluding the proof. $\square$

If $\mathcal{F}$ is Morse-Smale, it has weakly hyperbolic simple structure

In this section we show that if the foliation $\mathcal{F}^h$ induced by $\tilde{X}_h$ on $Z_{e_0}^h$ is Morse-Smale, then it has weakly hyperbolic simple structure (see Definition 2.2.8). To state precisely the result we need the following preliminary definition:

Definition 2.3.8. Let ζ_0 be a hyperbolic critical point of the foliation $\mathcal{F}^h$. Then ζ_0 is called weakly stable if there exists a strictly positive function $f : Z_{e_0}^h \rightarrow \mathbb{R}$ such that $\operatorname{div}_{\tilde{\mu}}(f \tilde{X}_h)(\zeta_0) < 0$. Similarly, ζ_0 is called weakly unstable if there exists a positive function $f : Z_{e_0}^h \rightarrow \mathbb{R}$ such that $\operatorname{div}_{\tilde{\mu}}(f \tilde{X}_h)(\zeta_0) > 0$.

The following result is a small variant of Theorem 6.2 of [26]:

Proposition 2.3.9. *Let $\mathcal{F}^h$ be the foliation induced by $\tilde{X}_h$ on $Z_{e_0}^h$. If $\mathcal{F}^h$ is Morse-Smale, then it admits a weakly hyperbolic simple structure (see Definitions 2.2.7 and 2.2.8). In particular K^+ is the finite union of the stable critical points and closed orbits, and of the unstable manifolds of the weakly stable saddle points, whereas K^- is the finite union of the unstable critical points and closed orbits, and of the stable manifolds of the weakly unstable saddle points.*

Proof. The proof follows the same construction of Theorem 6.2 in [26]. The only difference is that, due to the difference in the homogeneity classes of the involved functions, we need to show in a different way that all critical points of $\tilde{X}_h$ are either weakly stable or weakly unstable, according to Definition 2.3.8. In order to prove it, we proceed as follows: suppose by contradiction that there exists $\zeta_0 \in Z_{e_0}^h$ such that, for any smooth function f defined in a neighborhood of ζ_0 and such that $f(\zeta_0) > 0$, $\operatorname{div}_{\tilde{\mu}}(f\tilde{X}_h)(\zeta_0) = 0$. Then, choosing f such that $f(\zeta_0) = 1$ and using (2.2.15), we get $\{h, h_0\}(\zeta_0) = 0$. But then, since ζ_0 is a critical point of $\tilde{X}_h$, from (2.2.6) we deduce $X_h(\zeta_0) = 0$, contradicting that e_0 is a regular value for h . $\square$

Density of symbols whose characteristic foliation is Morse-Smale

In this section we put together the results obtained so far, and we show that, given any symbol $v \in C^0(\mathbb{T}, \mathbf{S}_{cl}^0)$, it is possible to slightly perturb it into a new symbol v_ε whose projected vector field directs a foliation with weakly hyperbolic simple structure. Precisely we have (recall the notation (2.3.11) and (2.2.5)):

Proposition 2.3.10. *Let $v \in C^0(\mathbb{T}, \mathbf{S}_{cl}^0)$. Given any $\varepsilon > 0$, there exists $v_\varepsilon \in C^0(\mathbb{T}, \mathbf{S}_{cl}^0)$ of the form $v_\varepsilon = v_{\varepsilon 0} + v_{-\mu}$, with $v_{\varepsilon 0}$ positively homogeneous and $v_{-\mu} \in C^0(\mathbb{T}, \mathbf{S}^{-\mu})$, $\mu > 0$, and e_0 regular value of $[v_{\varepsilon 0}]$, such that $\mathbf{d}_{\mathbb{T}}^0(v, v_\varepsilon) < \varepsilon$, and the foliation induced by $\tilde{X}_{[v_{\varepsilon 0}]}$ on $Z_{e_0}^{[v_{\varepsilon 0}]}$ has weakly hyperbolic simple structure.*

Proof. First we observe that, without loss of generality, we can assume that the definitely homogeneous symbol v admits the decomposition $v = v_0 + v_{-\mu}$, with

$$v_0(x, \xi) = v_0(\lambda x, \lambda \xi) \quad \forall \lambda \geq 1, \quad |(x, \xi)| \geq \frac{1}{4}, \quad v_0 \in C^0(\mathbb{T}; \mathbf{S}^0), \quad v_{-\mu} \in C^0(\mathbb{T}; \mathbf{S}^{-\mu}), \quad \mu > 0. \quad (2.3.23)$$

Note that the homogeneity is here required to hold in $|(x, \xi)| \geq \frac{1}{4}$ and not only in $|(x, \xi)| \geq 1$ as in Definition 1.1.1 of definitely homogeneous symbol. However, we can always reduce to (2.3.23) by a different definition of the smoothing component. Precisely, since $v \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$, we have $v = \tilde{v}_0 + \tilde{v}_{-\mu}$, with $\tilde{v}_j \in C^0(\mathbb{T}; \mathbf{S}_{cl}^j)$, $j = 0, -\mu$, and $\tilde{v}_0$ definitely homogeneous according to Definition 1.1.1. Then to obtain (2.3.23) it is sufficient to choose

$$v_0 := (1 - \eta)(\tilde{v}_0 \circ \boldsymbol{\pi}), \quad v_{-\mu} := \tilde{v}_0 - (1 - \eta)(\tilde{v}_0 \circ \boldsymbol{\pi}) + \tilde{v}_{-\mu},$$

with $\eta \in C^\infty(\mathbb{R}^4; \mathbb{R})$ a non-negative radial cut-off function satisfying

$$\eta(x, \xi) = \begin{cases} 1 & |(x, \xi)| \leq \frac{1}{8}, \\ 0 & |(x, \xi)| \geq \frac{1}{4}. \end{cases}$$

Note that $v_0(t, \cdot)$ is a smooth function, since the function $\tilde{v}_0 \circ \pi$ has a singularity only at $(x, \xi) = 0$, where $1 - \eta$ identically vanishes; furthermore v_0 coincides with $\tilde{v}_0 \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ outside the ball $B_1(0) := \{(x, \xi) \in \mathbb{R}^4 \mid |(x, \xi)| \leq 1\}$, thus one also has $v_0 \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$, and v_0 satisfies the homogeneity property in (2.3.23) by construction. Similarly, we observe that the function $\tilde{v}_0 - (1 - \eta)(\tilde{v}_0 \circ \pi)$ has no singularities and identically vanishes outside $B_1(0)$, thus we deduce that $\tilde{v}_0 - (1 - \eta)(\tilde{v}_0 \circ \pi) \in C^0(\mathbb{T}; \mathbf{S}_{cl}^{-\infty})$ and $v_{-\mu} \in C^0(\mathbb{T}; \mathbf{S}_{cl}^{-\mu})$. This proves (2.3.23).

Assume now that v_0 is as in (2.3.23). We fix a smooth, non-negative, radial cut-off function χ fulfilling

$$\chi \in C_c^\infty(\mathbb{R}^4, [0, 1]), \quad \chi(x, \xi) = \begin{cases} 1 & |(x, \xi)| \leq \frac{1}{4} \\ 0 & |(x, \xi)| \geq 1 \end{cases}, \quad (2.3.24)$$

and we are going to construct v_ε as follows:

$$v_\varepsilon := v_{\varepsilon 0} + v_{-\mu},$$

where

$$v_{\varepsilon 0}(t, \cdot) := v_0 + ((1 - \chi)(h_\varepsilon - [v_0])) \circ \phi_{h_0}^{-t},$$

and $h_\varepsilon \in C^\infty(\mathbb{R}^4 \setminus \{0\}; \mathbb{R})$ is a suitable smooth, positively homogeneous function of degree 0, that we now are going to exhibit. We shall require that h_ε also satisfies the following properties:

- (i) $\mathbf{d}_{C^\infty(\mathbb{S}^3)}([v_0], h_\varepsilon) < \varepsilon$;
- (ii) there exists $\mathbf{e}_0 \in \text{Ran}(h_\varepsilon)$, regular value for h_ε , such that the foliation induced by $\tilde{X}_{h_\varepsilon}$ on $Z_{\mathbf{e}_0}^{h_\varepsilon}$ has weakly hyperbolic simple structure.

If h_ε is as above, then observing that, on the support of $1 - \chi$, both h_ε and $[v_0]$ are smooth and positively homogeneous functions of degree 0, and using Lemma 1.1.4, it

is clear that $v_{\varepsilon 0}$ is a smooth, definitely homogeneous function of degree 0. Thus we have $v_{\varepsilon} = v_{\varepsilon 0} + v_{-\mu} \in C^0(\mathbb{T}; \mathbf{S}_{cl}^0)$ and, by (i),

$$\mathbf{d}_{\mathbb{T}}^0(v, v_{\varepsilon}) \leq \mathbf{d}^0([v_0](1 - \chi), h_{\varepsilon}(1 - \chi)) \leq \varepsilon.$$

Furthermore, by construction we have that $\langle v_{\varepsilon 0} \rangle = \langle v_0 \rangle + (1 - \chi)(h_{\varepsilon} - [v_0])$ (see Definition 1.1.5), and using that $\chi \equiv 0$ on $|(x, \xi)| \geq 1$, we also have $[v_{\varepsilon 0}] = h_{\varepsilon}$, thus by (ii) there exists $\mathbf{e}_0 \in \text{Ran}[v_{\varepsilon 0}]$ such that the foliation induced by $\tilde{X}_{[v_{\varepsilon 0}]}$ on $Z_{\mathbf{e}_0}^{[v_{\varepsilon 0}]}$ has weakly hyperbolic simple structure.

Then it remains to exhibit h_{ε} with the above properties in order to prove the proposition. To this aim, consider the positively homogeneous function $[v_0]$. Two cases are given: either $[v_0]$ is identically constant, or not. In the first case, we define

$$h_{\varepsilon} := [v_0] + \varepsilon h_{\star},$$

with $h_{\star}$ as in Lemma 2.2.10 and we take $\mathbf{e}_0 = [v_0] + \frac{\varepsilon}{2}$, and items (i) and (ii) follow from Lemma 2.2.10 and the arbitrariness of ε .

In the second case, let $\mathbf{e}_0$ be a regular value of $[v_0]$; the existence of a smooth, definitely homogeneous, function h_{ε} such that item (i) holds and the foliation induced by $\tilde{X}_{h_{\varepsilon}}$ on $Z_{\mathbf{e}_0}^{h_{\varepsilon}}$ is Morse-Smale is guaranteed by Proposition 2.3.6. Finally, item (ii) holds by Proposition 2.3.9. $\square$

Proof of Proposition 2.3.4

First we show that $\mathcal{A} \subseteq \mathcal{V}$. Take $v \in \mathcal{A}$; by the very definition (2.3.12), the vector field $\tilde{X}_{[v_0]}$ induces a foliation on $Z_{\mathbf{e}_0}^{[v_0]}$ with a weakly hyperbolic simple structure, with $\mathbf{e}_0$ a regular value of $[v_0]$. By Proposition 2.2.9 and Corollary 2.2.22, there exist a positively homogeneous function $\tilde{a}$ of degree two and an interval $I \ni \mathbf{e}_0$ and contained in $\text{Ran}([v_0])$, such that $\tilde{a}$ is an escape function for $[v_0]$ at any $\omega \in I$ and for any $(x, \xi) \in [v_0]^{-1}(I) \cap \{|(x, \xi)| \geq 1\}$.

Since $\tilde{a}$ is only positively homogeneous of degree 2, we regularize it at 0 with the cut-off in (2.3.24), i.e. we put $a := (1 - \chi)\tilde{a}$. Then

$$\{\langle v_0 \rangle, a\} = \{[v_0], \tilde{a}\} \geq \delta, \quad \forall (x, \xi) \in \langle v_0 \rangle^{-1}(I) \cap \{|(x, \xi)| \geq 1\}.$$

Since, by Remark 2.3.3, $\langle v_0 \rangle^{-1}(I) \cap \{|(x, \xi)| \geq 1\} = [v_0]^{-1}(I) \cap \{|(x, \xi)| \geq 1\}$, we have thus showed that $v \in \mathcal{V}$.

The density of $\mathcal{A}$ in $C^0(\mathbb{T}, \mathbf{S}_{cl}^0)$ with respect to the distance $d_{\mathbb{T}}^0$ is Proposition 2.3.10. $\square$

2.4 Growth of Sobolev norms for potentials in $\mathcal{V}$

In this last section we prove Theorem 2.0.1 point (ii), by applying the abstract Theorem 2.1.6 with $K_0 := H_0$ and $V(t) = \text{Op}^w(v(t))$, $v \in \mathcal{V}$. We verify that assumptions **I**, **II**, **III** are met.

Verification of Assumption I: it holds trivially by the symbolic calculus of pseudodifferential operators (see Proposition 2.1.1).

Verification of Assumption II: (i) holds since $\sigma(H_0) = \mathbb{N} + 1$, whereas (ii) by the exact Egorov Theorem (see Proposition 2.1.3).

Verification of Assumption III: First note that, applying again Theorem 2.1.3, if $K_0 = H_0$, then the two operators $\langle\langle V \rangle\rangle$ and $\langle V \rangle$ respectively defined in (2.1.6) and (1.1.8) coincide.

(i) Decompose as usual $v = v_0 + v_{-\mu}$, $\mu > 0$, then by (2.1.6), (1.1.7) and the exact Egorov theorem 2.1.3, one has

$$\langle V \rangle = \langle V_0 \rangle + \langle V_{-\mu} \rangle, \quad \langle V_0 \rangle := \text{Op}^w(\langle v_0 \rangle) \in \mathcal{S}^0,$$

and $\langle V_{-\mu} \rangle = \text{Op}^w(\langle v_{-\mu} \rangle) \in \mathcal{S}^{-\mu}$ a compact operator. In particular $\sigma_{\text{ess}}(\langle V \rangle) = \sigma_{\text{ess}}(\langle V_0 \rangle)$ and the last spectrum is known to be

$$\sigma_{\text{ess}}(\text{Op}^w(\langle v_0 \rangle)) = [\min \langle v_0 \rangle|_{\mathbb{S}^3}, \max \langle v_0 \rangle|_{\mathbb{S}^3}],$$

see e.g [70, Theorem 2.7] or [26, Theorem 2.1]. Since for symbols in $\mathcal{V}$ the function $\langle v_0 \rangle$ is not constant on $\mathbb{S}^3$ (otherwise $\{\langle v_0 \rangle, a\} \equiv 0$ in $\{|(x, \xi)| \geq 1\}$, Assumption **III** (i) holds taking an interval $I_0 \subset \text{Ran}(\langle v_0 \rangle|_{\mathbb{S}^3})$.

(ii) Put $A := \text{Op}^w(a)$, where $a \in \mathbf{S}_{cl}^1$ and $I \subset I_0$ are the escape function and the interval in (2.0.3). Let $g \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ be a smooth and compactly supported function such that $\text{supp}(g) \subset I$.

We verify (2.1.7) for some $\theta > 0$ and K compact and selfadjoint operator in $L^2(\mathbb{R}^2)$, and eventually shrinking I . It is sufficient to show (2.1.7) for V_0 , i.e.

$$g_I(\langle V_0 \rangle) i[\langle V_0 \rangle, A] g_I(\langle V_0 \rangle) \geq \theta g_I^2(\langle V_0 \rangle) + K, \quad (2.4.1)$$

since then by Lemma 2.1.5 and the fact that $[\langle V - V_0 \rangle, A]$ is a compact operator

$$g_I(\langle V \rangle) i[\langle V \rangle, A] g_I(\langle V \rangle) = g_I(\langle V_0 \rangle) i[\langle V_0 \rangle, A] g_I(\langle V_0 \rangle) + K,$$

with K compact, proving (2.1.7).

To show (2.4.1) we use first symbolic and functional calculus (see Section 2.1.1) to write

$$g_I(\langle V_0 \rangle) i[\langle V_0 \rangle, A] g_I(\langle V_0 \rangle) = \text{Op}^w(g_I^2(\langle v_0 \rangle) \{ \langle v_0 \rangle, a \}) + K \quad (2.4.2)$$

with K a pseudodifferential operator of strictly negative order, hence compact.

Next we remark that, from the definition of the set $\mathcal{V}$ in (2.0.3) and the support properties of g_I :

$$g_I^2(\langle v_0 \rangle) (\{ \langle v_0 \rangle, a \} - \delta) \geq 0, \quad \forall |(x, \xi)| \geq 1,$$

so the strong Gårding inequality 2.1.4 yields

$$\langle \text{Op}^w(g_I^2(\langle v_0 \rangle) (\{ \langle v_0 \rangle, a \} - \delta)) u, u \rangle \geq -C \|u\|_{-\frac{1}{2}}^2. \quad (2.4.3)$$

Hence

$$\begin{aligned} \langle \text{Op}^w(g_I^2(\langle v_0 \rangle) \{ \langle v_0 \rangle, a \}) u, u \rangle &\stackrel{(2.4.3)}{\geq} \delta \langle \text{Op}^w(g_I^2(\langle v_0 \rangle)) u, u \rangle - C \|u\|_{-\frac{1}{2}}^2 \\ &\stackrel{(2.0.2)}{=} \delta \langle [\text{Op}^w(g_I^2(\langle v_0 \rangle)) - CH_0^{-1}] u, u \rangle, \end{aligned} \quad (2.4.4)$$

and since from (2.1.1)

$$\text{Op}^w(g_I^2(\langle v_0 \rangle)) = \text{Op}^w(g_I(\langle v_0 \rangle))^2 + K_1, \quad \text{with } K_1 \in \mathcal{S}^{-1}, \quad (2.4.5)$$

we obtain

$$\text{Op}^w(g_I^2(\langle v_0 \rangle) \{ \langle v_0 \rangle, a \}) \stackrel{(2.4.4), (2.4.5)}{\geq} \delta \text{Op}^w(g_I(\langle v_0 \rangle))^2 + K = \delta g_I^2(\langle V_0 \rangle) + K, \quad (2.4.6)$$

where again the compact contribution K changes at each equality.

Plugging (2.4.6) in (2.4.2) we obtain (2.4.1) with $\theta = \delta$, concluding the proof.

2.5 Proof of technical results

2.5.1 Proof of Lemma 2.2.10

First note that the energy level $\Sigma_{\frac{1}{2}}^{h_\star} = \{(x, \xi) \in \mathbb{R}^4 : x_1^2 = \xi_1^2 + x_2^2 + \xi_2^2\}$. Then, by (2.2.5),

$$Z_{\frac{1}{2}}^{h_\star} = \{x_1 = \pm \frac{1}{\sqrt{2}}\} \cup \{\xi_1^2 + x_2^2 + \xi_2^2 = \frac{1}{2}\}$$

which is topologically $\mathbb{S}^2 \cup \mathbb{S}^2$. We now compute the induced vector field $\tilde{X}_{h_\star}$ on one of the spheres, say the one corresponding to $x_1 = \frac{1}{\sqrt{2}}$. Such sphere is described, in the coordinates ζ of (2.2.4), by

$$\zeta_2^2 + \zeta_3^2 + \zeta_4^2 = \frac{1}{2}, \quad (2.5.1)$$

where we have identified $\left(\frac{x_1}{\rho}, \frac{x_2}{\rho}, \frac{\xi_1}{\rho}, \frac{\xi_2}{\rho}\right) \equiv (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$. We now compute $\tilde{X}_{h_\star}(\zeta)$ via its definition (2.2.7), getting

$$\tilde{X}_{h_\star}(\zeta) = 2\frac{x_1}{\rho^3} \begin{pmatrix} 0 \\ \xi_1 x_2 - x_1 \xi_2 \\ \xi_1^2 + x_1^2 - 2h_0(x, \xi) \\ x_1 x_2 + \xi_1 \xi_2 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ \zeta_3 \zeta_2 - \frac{1}{\sqrt{2}} \zeta_4 \\ \zeta_3^2 - \frac{1}{2} \\ \zeta_3 \zeta_4 + \frac{1}{\sqrt{2}} \zeta_2 \end{pmatrix}.$$

By (2.5.1), each variable ζ_j is defined in the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$. Consider first the equation for ζ_3 , given by $\dot{\zeta}_3 = \sqrt{2}(\zeta_3^2 - \frac{1}{2})$. The dynamics is clear: the fixed point $-\frac{1}{\sqrt{2}}$ is a global attractor and the fixed point $\frac{1}{\sqrt{2}}$ a global repeller. Then, in view of (2.5.1), $\zeta^\pm := (\frac{1}{\sqrt{2}}, 0, \mp \frac{1}{\sqrt{2}}, 0)$ are fixed points for the dynamics of $\tilde{X}_{h_\star}$. Moreover, since $\zeta_3(t) \rightarrow \mp \frac{1}{\sqrt{2}}$ for $t \rightarrow \pm\infty$ whenever $\zeta_3(0) \neq \pm \frac{1}{\sqrt{2}}$, still from (2.5.1) we deduce that $(\frac{1}{\sqrt{2}}, \zeta_2(t), \zeta_3(t), \zeta_4(t)) \rightarrow \zeta^\pm$ when $t \rightarrow \pm\infty$. This, putting $K^\pm := \{\zeta^\pm\}$, shows that the foliation has a simple structure (Definition 2.2.7). Note that the flow of $\tilde{X}_{h_\star}$ is the simplest Morse-Smale flow on $\mathbb{S}^2$! See Figure 2.1.

To show that the foliation is also weakly hyperbolic, it is easier to use formula (2.2.15), that reads

$$\operatorname{div}_{\tilde{\mu}}(\tilde{X}_{h_\star}) = -2\{h_\star, h_0\} = 4x_1 \xi_1|_{Z_{\frac{1}{2}}^{h_\star}},$$

with $\tilde{\mu}$ as in (2.2.10). Then, at the point ζ^+ , corresponding to K^+ , the right hand side is strictly negative, hence by continuity the l.h.s is strictly negative also in a

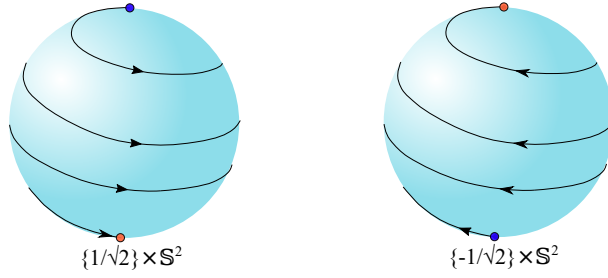


Figure 2.1: The flow of the vector field $\tilde{X}_{h_*}$. The repellor K^- consists in the two blue dots, whereas the attractor K^+ is formed by the orange dots.

neighbourhood of K^+ . Analogously, in a neighbourhood of K^- , the l.h.s. is strictly positive, proving the weak hyperbolicity (Definition 2.2.8).

As a final comment, an escape function for h_* can be directly computed: put $a(x, \xi) := -x_1 \xi_1$ to get

$$\{h_*, a\} = \frac{\xi_1^2 x_1^2}{2h_0^2} + \frac{x_1^2}{h_0} \left(1 - \frac{x_1^2}{2h_0}\right) \geq 2h_*(1 - h_*)$$

which is strictly positive on any energy level $\Sigma_{e_0}^{h_*}$ with $e_0 \in (0, 1)$.

2.5.2 Proof of Proposition 2.2.15

Before starting the proof, we recall the following topological properties of attractors [57, Lemma 1.6].

Lemma 2.5.1. *Let K^+ be an attractor and V a neighbourhood of K^+ . Then there exists an open neighbourhood U of K^+ such that $K^+ \subset U \subset V$, $\bigcap_{t \geq 0} \phi_X^t(\overline{U}) = K^+$ and $\phi_X^t(U) \subset U$ for all positive times.*

We divide the proof of Proposition 2.2.15 in several steps. A key point will be to observe that, by Lemma 2.2.11, if $z(t)$ solves the Cauchy problem $\dot{z} = X_h(z)$, $z(t_0) = z_0$ for some $t_0 \in \mathbb{R}$, then its components $\rho(t)$, $\zeta(t)$ solve the Cauchy problem

$$\begin{cases} \dot{\zeta} = \rho^{-2} \tilde{X}_h(\zeta) \\ \dot{\rho} = \rho^{-1} \{h, h_0\} \end{cases}, \quad \begin{cases} \zeta(t_0) = \zeta_0 := \pi(z_0) \\ \rho(t_0) = \rho_0 := \|z_0\| \end{cases}.$$

For this reason, we will often reparametrize time as

$$\tau_{t_0}(t) := \int_{t_0}^t \rho^{-2}(s) \, ds \quad (2.5.2)$$

and exploit the fact that, for any time interval $I \ni t_0$ such that $\rho(t) > 0 \, \forall t \in I$, one has

$$\zeta(t) = w(\tau_{t_0}(t)) \quad \forall t \in I, \quad (2.5.3)$$

where w solves the Cauchy problem

$$\frac{dw}{d\tau}(\tau) = \tilde{X}_h(w(\tau)), \quad w(0) := \zeta(t_0). \quad (2.5.4)$$

Note that the flow of (2.5.4) is globally defined in time, since Z_{e_0} is a compact manifold. Moreover, let U^+ be the open set given by the definition of weak hyperbolicity 2.2.8. Since K^+ is an attractor, Lemma 2.5.1 guarantees the existence of an open neighbourhood of K^+ , $\tilde{U}^+ \subset U^+$ such that, if $w(0) \in \tilde{U}^+$,

$$w(t) \in \tilde{U}^+ \quad \forall t \geq 0 \quad \text{and} \quad w(t) \rightarrow K^+ \text{ as } t \rightarrow +\infty.$$

Up to shrinking U^+ we are going to suppose $\tilde{U}^+ = U^+$.

Step 1: *Item (i) holds, namely there exist open neighborhoods $U^\pm \subset Z_{e_0}$ of $K^\pm$ such that for any $z_0 \in \pi^{-1}(U^\pm)$, the flow $\Phi_{X_h}^t(z_0)$ is globally defined for $\pm t \geq 0$ and $\Phi_{X_h}^t(\pi^{-1}(U^\pm)) \subseteq \pi^{-1}(U^\pm)$ for $\pm t \geq 0$.*

Proof of Step 1. We prove the statement with “+”. The idea is to exploit that the projected flow $\zeta(t)$ stays in U^+ for all times; the difficulty is that the radial component $\rho(t)$ can either blow up or go to 0 in finite time. While the first possibility is ruled out by homogeneity of the vector field, the second one would contradict the existence of the local escape function k^+ .

Let $z_0 = (\rho_0, \zeta_0) \in \pi^{-1}(U^+)$ (recall (2.2.4)) with $\rho_0 \neq 0$, and denote $\Phi_{X_h}^t(\rho_0, \zeta_0) = (\rho(t), \zeta(t))$. We need to show that for all positive times $\rho(t) \in (0, +\infty)$ and $\zeta(t) \in U^+$. Denote by $T_{\max} \in (0, +\infty]$ the maximal time such that $\rho(t) \in (0, +\infty)$. First remark that $\rho(t)$ can’t blow up in finite time, since $\rho = \sqrt{2h_0}$ and one has

$$|\dot{h}_0| = |\{h, h_0\}| \leq \max_{Z_{e_0}} |\{h, h_0\}| =: \nu \quad (2.5.5)$$

that leads to the upper bound

$$|h_0(t) - h_0(0)| \leq \nu t, \quad \forall t \geq 0. \quad (2.5.6)$$

Hence the only possibility to have $T_{\max} < +\infty$ is that $\rho(t) \rightarrow 0$ as $t \rightarrow T_{\max}^-$. This is not possible as long as $\zeta(t)$ belongs to U^+ , as we now show. Denote by t_* the escape time from $\pi^{-1}(U^+)$, namely

$$t_* := \inf\{t \geq 0 : \zeta(t) \notin U^+\}.$$

As $\zeta(0) \in U^+$, $t_* > 0$. Now, by (2.2.17), the escape function $k^+(t)$ is strictly increasing as long as $\zeta(t)$ belongs to U^+ and, by (2.2.16), it is equivalent to h_0 :

$$c_2 h_0(t) \geq k^+(t) \geq k^+(0) \geq c_1 h_0(0) > 0, \quad \forall t \in [0, t_*]. \quad (2.5.7)$$

This gives also that $T_{\max} > t_*$.

We are now going to prove that $t_* = +\infty$, implying both that $\Phi_{X_h}^t(\pi^{-1}(U^+))$ is defined for all $t \geq 0$, and that $\pi^{-1}(U^+)$ is invariant along $\Phi_{X_h}^t$ for $t \geq 0$.

Assume by contradiction that $t_* < +\infty$. Then, recalling $t_* < T_{\max}$, take $\varepsilon > 0$ so small that $t_* + \varepsilon < T_{\max}$, so that we have

$$\rho(t) \geq \bar{\rho} > 0, \quad \forall t \in [t_* - \varepsilon, t_* + \varepsilon]. \quad (2.5.8)$$

Then the time reparametrization $\tau_{t_* - \varepsilon}(t)$ in (2.5.2) is well defined in the time interval of (2.5.8). On this time interval, the identity (2.5.3) holds with $t_0 := t_* - \varepsilon$ and $w(0) = \zeta(t_* - \varepsilon) \in U^+$. Since $w(\tau)$ is globally defined in time and $w(\tau) \in U^+$ for any $\tau \geq 0$, we get $\zeta(t_* + \varepsilon) = w(\tau_{t_* - \varepsilon}(t_* + \varepsilon)) \in U^+$ contradicting the maximality of t_* . $\square$

Step 2: *The set $\Gamma^\pm$ is invariant for all $\pm t \geq 0$.*

Proof of Step 2. We prove the statement with “+”. Denote again by $\Phi_{X_h}^t(\rho_0, \zeta_0) = (\rho(t), \zeta(t))$ the flow of X_h with initial datum $z_0 = (\rho_0, \zeta_0)$, written in the coordinates (2.2.4). We point out that $\rho(t)$ and $\zeta(t)$ are well defined for all $t \geq 0$ from Step 1 and

$$\rho(s) \geq \bar{\rho}_t > 0, \quad \forall s \in [0, t].$$

In particular the function $\tau_0(t)$ in (2.5.2) is well defined for all $t \geq 0$.

We must show that whenever $\rho_0 > 0$ and $\zeta_0 \in K^+$, $\zeta(t) \in K^+$ for any $t > 0$. We have that $\zeta(t) = w(\tau_0(t))$ and since $w(0) = \zeta(0) \in K^+$ and K^+ is invariant for all $t > 0$ under the flow of $\tilde{X}_h$, $\zeta(t) \in K^+$ for all $t \geq 0$, which proves Step 2. $\square$

Step 3: *If $z_0 \in \Gamma^\pm$, there exists a finite time $\pm T^\pm(z_0) < \infty$ such that*

$$\Phi_{X_h}^t(z_0) \in \Gamma^\pm \quad \forall t \quad \text{s.t.} \quad \pm T^\pm(z_0) < \pm t, \quad (2.5.9)$$

$$\Phi_{X_h}^t(z_0) \rightarrow 0 \quad \text{as} \quad t \rightarrow T^\pm(z_0)^\pm. \quad (2.5.10)$$

In particular, the maximal time of existence of $\Phi_{X_h}^t(z_0)$ for $z_0 \in \Gamma^+$ is $(T^+(z_0), +\infty)$, and it is $(-\infty, T^-(z_0))$ for $z_0 \in \Gamma^-$.

Proof of Step 3. We prove the statement with “+”. Let $T^+(z_0) < \infty$ be the smallest time such that $\Phi_{X_h}^t(z_0)$ exists. The fact that the invariance property (2.5.9) holds for all $t > T^+(z_0)$ can be shown exactly as in Step 2.

We now prove that (2.5.10) holds for a finite time $T^+(z_0)$. Pick $z_0 \in \Gamma^+$, and consider the escape function k^+ defined in (2.2.16). The evolution of k^+ along the flow is given by the equation $\dot{k}^+(t) = \{h, k^+\}$ for all $t \in (T^+(z_0), \infty)$. From (2.2.17), in Γ^+ we have a strictly positive bound from below on $\{h, k^+\}$, and since Γ^+ is invariant for the flow $\Phi_{X_h}^t$ as proved in (2.5.9), we have:

$$\dot{k}^+(t) \geq \delta, \quad \forall t \in (T^+(z_0), \infty).$$

Integrating both sides between $t < 0$ and 0 we have:

$$\int_t^0 \dot{k}^+(s) ds \geq -\delta t \implies k^+(t) \leq \delta t + k^+(0). \quad (2.5.11)$$

Then, combining (2.5.7) with (2.5.11), we deduce the existence of positive constants $c_1, c_2 > 0$ such that

$$h_0(t) \leq c_1 \delta t + c_2 h_0(0).$$

The right hand side above tends to 0 as $t \rightarrow -\frac{c_2 h_0(0)}{c_1 \delta} < 0$ (recall $h_0(0) > 0$). Thus there exists a negative time $T^+(z_0) \in \left[-\frac{c_2 h_0(0)}{c_1 \delta}, 0\right)$ such that $\rho(t) = \sqrt{2h_0(t)} \rightarrow 0$ as $t \rightarrow T^+(z_0)$, proving (2.5.10). $\square$

Steps 2 and 3 prove Items (ii) and (iii) of Proposition 2.2.15.

Step 4: If $z_0 \in \Sigma_{e_0}$ and $\Phi_{X_h}^t(z_0)$ is defined for all $t \geq 0$, resp. $t \leq 0$, then $z_0 \in B(\Gamma^+)$, resp. $B(\Gamma^-)$.

Proof of Step 4. We prove the statement with “+”. It suffices to show that

$$\text{dist}(\zeta(t), K^+) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (2.5.12)$$

By Step 2, this holds for $z_0 \in \Gamma^+$ and we restrict to the case $z_0 \notin \Gamma^+$. To show (2.5.12), we note that, since $\rho^2 = 2h_0$ and $\Phi_{X_h}^t(z_0)$ is defined for all times,

$$\tau_0(t) = \int_0^t \rho^{-2}(s) ds \stackrel{(2.5.6)}{\geq} \int_0^t \frac{1}{2(h_0(0) + \nu s)} ds \rightarrow \infty \text{ as } t \rightarrow \infty.$$

Hence, since $w(\tau) \rightarrow K^+$ as $\tau \rightarrow +\infty$, so does $\zeta(t) = w(\tau_0(t))$.

□

Step 5: If $z_0 \in \Sigma_{e_0} \setminus (\Gamma^+ \cup \Gamma^-)$, then $\Phi_{X_h}^t(z_0)$ exists for all times $t \in \mathbb{R}$.

Proof of Step 5. We show that if $z_0 \in \Sigma_{e_0} \setminus \Gamma^-$ then $\Phi_{X_h}^t(z_0)$ exists for any $t \geq 0$. Analogously one can prove that if $z_0 \in \Sigma_{e_0} \setminus \Gamma^+$ then $\Phi_{X_h}^t(z_0)$ exists for any $t \leq 0$ concluding the proof.

To this aim, we prove that there exists a sequence of sets $\{V_k\}_{k \in \mathbb{N}_0} \subset Z_{e_0}$ satisfying

$$\Sigma_{e_0} \setminus \Gamma^- = \bigcup_{k \in \mathbb{N}_0} \pi^{-1}(V_k), \quad (2.5.13)$$

and we prove, inductively on k , that

$$\forall k \in \mathbb{N}_0, \quad \Phi_{X_h}^t(\pi^{-1}(V_k)) \text{ is well defined } \forall t \geq 0. \quad (2.5.14)$$

The set V_0 . To ensure (2.5.14) for $k = 0$, it is sufficient to choose $V_0 := U^+$, with U^+ the set of Step 1.

The set V_k . For $k \geq 1$, we recursively define

$$V_k := \Phi_{\tilde{X}_h}^{-\underline{t}}(V_{k-1}), \quad \text{where } \underline{t} := \frac{1}{4\tilde{\nu}} \ln \left(\frac{3}{2} \right), \quad \tilde{\nu} := \max\{1, \nu\},$$

with ν defined in (2.5.5). Suppose now that (2.5.14) holds for $k-1$. We are going to prove that (2.5.14) holds for k . Let $z_0 \in \pi^{-1}(V_k)$ and suppose by contradiction that there exists $0 < T_{\max} := T_{\max}(z_0) < +\infty$ such that

$$\rho(t) \rightarrow 0, \quad \text{as } t \rightarrow T_{\max}^-.$$

Then for all $t \in [0, T_{\max}/2]$, $\Phi_{X_h}^t(z_0)$ is well defined and $\rho(t) \geq \underline{\rho} > 0$ for some $\underline{\rho} > 0$, thus on such time interval one also has $\zeta(t) = w(\tau_0(t))$. We now claim that

$$\tau_0\left(\frac{T_{\max}}{2}\right) \geq 2\underline{t}. \quad (2.5.15)$$

Indeed, passing to the limit $t \rightarrow T_{\max}^-$ in (2.5.6), one has

$$T_{\max} \geq \frac{h_0(z_0)}{\tilde{\nu}}. \quad (2.5.16)$$

This, recalling that, again by (2.5.6), $\rho^2(t) = 2h_0(z(t)) \leq 2(h_0(z_0) + \tilde{\nu}t)$, gives

$$\begin{aligned} \tau_0\left(\frac{T_{\max}}{2}\right) &= \int_0^{\frac{T_{\max}}{2}} \frac{1}{\rho^2(s)} \, ds \stackrel{(2.5.16)}{\geq} \int_0^{\frac{h_0(z_0)}{2\tilde{\nu}}} \frac{1}{\rho^2(s)} \, ds \\ &\geq \int_0^{\frac{h_0(z_0)}{2\tilde{\nu}}} \frac{1}{2(h_0(z_0) + \tilde{\nu}s)} \, ds \\ &= \frac{1}{2\tilde{\nu}} \int_0^{\frac{1}{2}} \frac{1}{1+s} \, ds = \frac{1}{2\tilde{\nu}} \ln\left(\frac{3}{2}\right) = 2\underline{t}, \end{aligned}$$

which proves the claim in (2.5.15). Now, one observes that τ_0 is an increasing continuous function on $[0, T_{\max}/2]$ satisfying $\tau_0(0) = 0$ and $\tau_0\left(\frac{T_{\max}}{2}\right) > \underline{t}$, thus $\exists! t_\star \in (0, T_{\max}/2)$ such that $\tau_0(t_\star) = \underline{t}$. But then, recalling $\zeta(0) = w(0) \in V_k := \Phi_{\tilde{X}_h}^{-\underline{t}}(V_{k-1})$, one has

$$\zeta(t_\star) = w(\tau_0(t_\star)) = w(\underline{t}) \in V_{k-1},$$

namely $\Phi_{X_h}^{t_\star}(z_0) \in \pi^{-1}(V_{k-1})$. Then, for any $t \geq t_\star$, one has

$$\Phi_{X_h}^t(z_0) = \Phi_{X_h}^{t-t_\star}(\Phi_{X_h}^{t_\star}(z_0)) \in \Phi_{X_h}^{t-t_\star}(\pi^{-1}(V_{k-1})),$$

which is well defined by the inductive hypothesis. This then implies that $T_{\max}(z_0) = +\infty$, which gives the contradiction.

It remains to show that (2.5.13) holds; but this is a consequence of the fact that K^+ is a global attractor, thus for any $z_0 \in \Sigma_{e_0} \setminus \Gamma^-$, there exists $k_0 \in \mathbb{N}_0$ such that $\Phi_{\tilde{X}_h}^{k_0 \underline{t}}(\zeta_0) \in U^+$, namely $z_0 \in \pi^{-1}(V_{k_0})$. $\square$

Steps 4 and 5 prove Item (iv) of Proposition 2.2.15.

2.5.3 Proof of Lemma 2.2.19

The idea is to construct V^- as the sublevel of a smooth Lyapunov function defined on Z_{e_0} for the flow of $\tilde{X}_h$. The first step is to construct a continuous Lyapunov function, for which we invoke the following classical result, see e.g. [40, Proposition 1.4.21].

Proposition 2.5.2. *There exists a continuous Lyapunov function $\mathcal{L} : Z_{e_0} \rightarrow [0, 1]$ associated to the flow of $\tilde{X}_h$, such that $\mathcal{L}^{-1}(\{0\}) = K^+$, $\mathcal{L}^{-1}(\{1\}) = K^-$, and $\mathcal{L}$ is strictly decreasing along orbits lying outside $K^+ \cup K^-$.*

The next step is to regularize the Lyapunov function. We shall exploit a result by Fathi-Pageault [36] that adapts to our case. First we need the following definitions:

Definition 2.5.3. Given a Lyapunov function $\mathcal{L} : M \rightarrow \mathbb{R}$ for a flow ϕ^t on a compact manifold M , the *neutral set* $N(\mathcal{L})$ is defined as

$$N(\mathcal{L}) := \{x \in M : \exists t > 0 : \mathcal{L}(\phi^t(x)) = \mathcal{L}(x)\}.$$

Theorem 2.5.4. [36, Theorem 2.1] *Let M be a smooth manifold, and let ϕ^t be a continuous flow on M . Assume that*

- (i) *the vector field $X(x) := \frac{d}{dt}\phi^t(x)|_{t=0}$ exists everywhere, is continuous and is a Lipschitz vector field;*
- (ii) *the flow ϕ^t admits a continuous Lyapunov function $\mathcal{L} : M \rightarrow \mathbb{R}$ such that $\mathcal{L}(N(\mathcal{L}))$ is contained in a closed subset of $\mathbb{R}$ with empty interior.*

Then we can approximate $\mathcal{L}$ in the compact open topology of $C^0(M, \mathbb{R})$ by a C^∞ Lyapunov function $\tilde{\mathcal{L}} : M \rightarrow \mathbb{R}$, arbitrarily close to $\mathcal{L}$ and such that $N(\tilde{\mathcal{L}}) = \text{Crit}(\tilde{\mathcal{L}})$, the set of critical points of $\tilde{\mathcal{L}}$, and $X[\tilde{\mathcal{L}}] < 0$ on $M \setminus N(\tilde{\mathcal{L}})$.

We apply this result to the continuous Lyapunov function $\mathcal{L} : Z_{e_0} \rightarrow \mathbb{R}$ constructed in Proposition 2.5.2. Clearly item (i) of Theorem 2.5.4 is fulfilled since the vector field $\tilde{X}_h$ is smooth. To check (ii), note that $N(\mathcal{L}) = K^+ \cup K^-$, and again by Proposition 2.5.2 we have $\mathcal{L}(N(\mathcal{L})) = \{0, 1\}$ which is a closed subset of $\mathbb{R}$ with empty interior. Hence Theorem 2.5.4 yields a smooth Lyapunov function $\tilde{\mathcal{L}} : Z_{e_0} \rightarrow \mathbb{R}$ such that

$$\|\tilde{\mathcal{L}} - \mathcal{L}\|_{C^0} < \frac{1}{10}. \quad (2.5.17)$$

Note that $\tilde{\mathcal{L}}(Z_{e_0}) \subseteq [-\frac{1}{10}, \frac{11}{10}]$.

Proof of Lemma 2.2.19. Since $\mathcal{L}^{-1}(1) = K^-$, we deduce from (2.5.17) that $K^- \subset \tilde{\mathcal{L}}^{-1}([\frac{9}{10}, \frac{11}{10}])$, and analogously $K^+ \subset \tilde{\mathcal{L}}^{-1}([-\frac{1}{10}, \frac{1}{10}])$. Then choose $\varepsilon_1 \in (\frac{1}{10}, \frac{9}{10})$ which is not a critical value of $\tilde{\mathcal{L}}$; such number exists by Sard's theorem being $\tilde{\mathcal{L}}$ non-constant. Put $V^- := \tilde{\mathcal{L}}^{-1}((\varepsilon_1, \frac{11}{10}])$; then V^- contains K^- and $(V^-)^c$ contains K^+ , V^- is invariant for negative times being the superlevel of a Lyapunov function, and its boundary $\partial V^- = \tilde{\mathcal{L}}^{-1}(\varepsilon_1)$ is transverse to $\tilde{X}_h$ since, by Theorem 2.5.4, $\tilde{X}_h[\tilde{\mathcal{L}}] < 0$. $\square$

Chapter 3

Unstable perturbations of periodic transport equations

In this Chapter, we study instabilities in linear transport equations with time-dependent perturbations on compact manifolds. As explained in the introduction, we consider two different but closely related models: first, we study time-dependent perturbations of linear, skew-adjoint transport equations with constant coefficients on tori, and then we turn to the more general case of time-dependent perturbations of Morse-Smale transport equations on compact manifolds.

We now state the two results separately, referring to Section 1.1.2 for a broader introduction to the models, and to Section 1.2.2 for an outline of the techniques used in the proofs.

For any $n \geq 1$, we denote by $\mathbb{T}^{n+1}$ (with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$) the Euclidean $(n+1)$ -dimensional torus. Given $\nu \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$, we consider the following equations

$$\partial_t u_\varepsilon = (\nu + \varepsilon V(\omega t, x)) \cdot \nabla_x u_\varepsilon + \frac{\varepsilon}{2} \operatorname{div}_x(V(\omega t, x)) u_\varepsilon, \quad x \in \mathbb{T}^n, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad (3.0.1)$$

where $V(t, x) \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ is a time-dependent vector field, $\operatorname{div}(\cdot)$ denotes the divergence with respect to the standard Euclidean volume on $\mathbb{T}^n$ and $\varepsilon > 0$ plays the role of a parameter. Here we denote by

$$\mathcal{H}^\sigma(\mathbb{T}^n) \equiv \mathcal{H}^\sigma := \{u \in L^2(\mathbb{T}^n; \mathbb{C}) : \|u\|_\sigma^2 := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2\sigma} |u_k|^2 < +\infty\},$$

(where we use the notation $\langle k \rangle := \sqrt{1 + |k|^2}$ for the Japanese brackets and we denote by $\{u_k\}_{k \in \mathbb{Z}^n}$ the Fourier coefficients of the function u). Our first main result is the following theorem:

Theorem 3.0.1. *Let $n = 2$, $\omega = 1$ and $\nu \in \mathbb{N}^2 \setminus \{0\}$. Then there exists an open and dense subset $\mathcal{O}_\nu$ of ¹ $C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ such that, for every $V \in \mathcal{O}_\nu$, the following holds.*

There exists $\delta_0 > 0$ such that, for every $0 \leq \sigma \leq \delta_0$ and for every $0 \leq \varepsilon \leq \delta_0 \sigma^2$, one can find $v_\varepsilon \in \mathcal{H}^\sigma(\mathbb{T}^2)$ and $r := r(\varepsilon, \sigma, v_\varepsilon) > 0$ such that

$$\|u_\varepsilon(0) - v_\varepsilon\|_\sigma \leq r \quad \implies \quad \|u_\varepsilon(t)\|_\sigma \geq \delta_0 e^{\delta_0 \sigma t \varepsilon} \|u_\varepsilon(0)\|_\sigma, \quad \text{for any } t > 0,$$

where $u_\varepsilon(t)$ is the solution to equation (3.0.1) with initial datum $u_\varepsilon(0)$.

As we explained in Section 1.2.2, the set $\mathcal{O}_\nu$ consists of all perturbations $V \in C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ such that the resonant average (1.1.13) is Morse-Smale (see Definition 1.1.8). For this reason, our second main result concerns the study of instabilities for Morse-Smale transport equations on arbitrary smooth, compact, and Riemannian manifolds of arbitrary dimension.

We consider M a smooth (C^∞), compact and oriented manifold of dimension $n \geq 1$, without boundary. We endow the manifold M with a Riemannian metric that we denote by g and we let v_g be the corresponding Riemannian volume on M . We denote by $\mathfrak{X}(M)$ the set of smooth vector fields on M and we consider the following class of transport equations:

$$\partial_t u_\varepsilon = \mathcal{L}_{V + \varepsilon \mathcal{P}(t)} u_\varepsilon + \frac{1}{2} \operatorname{div}_{v_g}(V + \varepsilon \mathcal{P}(t)) u_\varepsilon, \quad x \in M, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad (3.0.2)$$

where $V \in \mathfrak{X}(M)$, $\mathcal{P}(t) \in C_b^0(\mathbb{R}; \mathfrak{X}(M))$ and div_{v_g} is the divergence with respect to the Riemannian volume form. Here, $\mathcal{L}$ denotes the usual Lie derivative, i.e., the order one differential operator defined by $\mathcal{L}_Y u := du[Y]$, for every $Y \in \mathfrak{X}(M)$, and $C_b^0(\mathbb{R}; \mathfrak{X}(M))$ denotes the set of time dependent vector fields on M , which are continuous and bounded in time. The Sobolev spaces $\mathcal{H}^\sigma(M)$ are defined as

$$\mathcal{H}^\sigma(M) := \left\{ u \in L^2(M; \mathbb{C}) : \|(1 - \Delta_g)^{\frac{\sigma}{2}} u\|_{L^2(M, v_g)} < +\infty \right\}, \quad \sigma \in \mathbb{R}, \quad (3.0.3)$$

¹The set is endowed with its standard Fréchet topology.

where Δ_g is the Laplace-Beltrami operator on M . The corresponding Sobolev norm will be denoted by $\|\cdot\|_\sigma$.

Theorem 3.0.2. *Let $V \in \mathfrak{X}(M)$ and $\mathcal{P} \in C_b^0(\mathbb{R}, \mathfrak{X}(M))$. Suppose that V is a Morse-Smale vector field (see Definition 1.1.8) and, if $n = 1$, suppose in addition that it has at least one critical point.*

Then, there exists $\delta_0 > 0$ such that, for every $0 \leq \sigma \leq \delta_0$ and for every $0 \leq \varepsilon \leq \delta_0 \sigma^2$, one can find $v_\varepsilon \in \mathcal{H}^\sigma(M)$ and $r := r(\varepsilon, \sigma, v_\varepsilon) > 0$ such that

$$\|u_\varepsilon(0) - v_\varepsilon\|_\sigma \leq r \quad \implies \quad \|u_\varepsilon(t)\|_\sigma \geq \delta_0 e^{\delta_0 \sigma t} \|u_\varepsilon(0)\|_\sigma, \quad \text{for any } t > 0,$$

where $u_\varepsilon(t)$ is the solution to equation (3.0.2) with initial datum $u_\varepsilon(0)$.

This Chapter is devoted to the proof of Theorems 3.0.1 and 3.0.2.

Organization of the Chapter. The core of our proof is the construction of an escape function for the transport operator in the Morse-Smale case. This requires a detailed description of the Hamiltonian flow induced by its principal symbol. Thus we devote Section 3.1 to a brief reminder of Morse-Smale dynamics and their symplectic lifts and then Section 3.2 to the construction of the corresponding escape function. In Section 3.3.1, we review tools from microlocal analysis and apply them to the proof of (a slightly stronger version of) Theorem 3.0.2 by a positive commutator argument making use of the escape function from Section 3.2. Once this is done, we prove Theorem 3.0.1 in Section 3.4 by combining Theorem 3.0.2 with a normal form procedure. Finally, the auxiliary section 3.5.1 reviews a few definitions from dynamical systems theory that are used all along the Chapter (especially in Sections 3.1 and 3.2) and in Section 3.5.2 we prove global well posedness for the class of transport equations that we use in Section 3.3.2.

3.1 Review of dynamical setting

In this section, we recall the properties of the class of Morse-Smale vector fields V on a smooth compact manifold M (see Definition 1.1.8) and of their symplectic lift to the cotangent bundle T^*M . Along the way, we collect some of their properties that will be

used in the following. More precisely, in Paragraph 3.1.1, we review the definition of such vector fields. After that, we briefly recall in Paragraph 3.1.2 how to lift a vector field in a symplectic manner to T^*M and we review in Paragraph 3.1.3 some material from [24] on the dynamics of symplectic lifts for Morse-Smale vector fields.

Remark 3.1.1. We refer to [77, Chapter 4] for a detailed introduction to Morse-Smale vector fields and flows on smooth manifolds and to [24] for a detailed study of the lifted dynamics on the cotangent bundle. We collect in Section 3.5.1 standard definitions from the theory of dynamical systems that are used in this section.

3.1.1 Morse-Smale flows

In this section we will always refer to Definition 1.1.8 of Morse-Smale vector fields on arbitrary compact manifolds (confront with Definition 1.2.1, for an equivalent formulation on surfaces). We also recall the notation $\mathfrak{X}^{MS}(M) \subset \mathfrak{X}(M)$ for the set of Morse-Smale vector fields on M . Given a vector field $V \in \mathfrak{X}(M)$, we denote by $\varphi_V^t(x)$ its flow on M with starting point $x \in M$:

$$\frac{d}{dt}\varphi_V^t(x) = V(\varphi_V^t(x)), \quad \forall t \in \mathbb{R}. \quad (3.1.1)$$

We recall that, from compactness of the manifold M , the flow $\varphi_V^t(x)$ is complete, i.e., for each initial datum $x \in M$, the flow line $\varphi_V^t(x)$ exists globally in time – see e.g. [62, Corollary 9.17]. We denote by

$$\text{Crit}(V) := \{\Lambda_1, \dots, \Lambda_K\} \quad (3.1.2)$$

the set of critical elements (basic sets) of V , which are either critical points or closed orbits. It will be important for the final steps² of our analysis that $n \geq 2$ or that $\text{Crit}(V) \neq M$ if $n = 1$.

From the Definition 1.1.8 of Morse-Smale vector fields, one can prove the following properties:

Lemma 3.1.2. *Let $V \in \mathfrak{X}^{MS}(M)$.*

²More precisely, this is used when we pick the initial data generating unstable solutions.

(i) For every $x \in M$ there exists a unique pair $(i, j)_x \in \{1, \dots, K\}^2$ such that

$$x \in W^s(\Lambda_i) \cap W^u(\Lambda_j),$$

(recall notation in (3.1.2) and Definition 3.5.5).

(ii) The unstable manifolds of the critical elements of V form a partition of M , precisely:

$$M = \bigcup_{i=1}^K W^u(\Lambda_i), \quad \text{and} \quad W^u(\Lambda_i) \cap W^u(\Lambda_j) = \emptyset, \quad \forall i \neq j.$$

We refer for instance to [24, Lemmas 3.3, 3.4] for the proof of these statements.

Energy function. We now report a result by Meyer [74], that proves the existence of a function which is non-decreasing along the flow that we call *energy function* following the literature. It will be a fundamental element for our construction in Section 3.2.

Lemma 3.1.3 ([74] Energy function). *Let $V \in \mathfrak{X}^{MS}(M)$. Then there exists a function $\mathcal{E} \in C^\infty(M)$ such that*

$$\mathcal{L}_V \mathcal{E} \geq 0, \quad \text{everywhere on } M, \text{ and} \quad \mathcal{L}_V \mathcal{E} > 0 \quad \text{on } M \setminus \text{Crit}(V). \quad (3.1.3)$$

Remark 3.1.4. In the case where V is a gradient vector field associated with a Morse function f , such a function is given by f itself.

Morse-Smale vector fields on surfaces. The case of Morse-Smale vector fields on compact surfaces (i.e. $n = 2$) is peculiar and further motivates the choice of such class of vector fields. Indeed, as we already emphasized (see Definition 1.2.1), in this case, the definition of Morse-Smale vector fields can be simplified.

Remark 3.1.5. We recall that the set $\mathfrak{X}^{MS}(M)$ is generic in $\mathfrak{X}(M)$ (i.e., it is open and dense with respect to the C^∞ topology), for M smooth, compact manifold of dimension two. This important property of Morse-Smale vector fields has been proved by Peixoto in [78, Theorem 2]. As anticipated, we will use this result in Section 3.4 in order to prove Theorem 3.0.1. In higher dimension, it is still an open set which is contained in the larger class of Axiom A flows (with appropriate transversality properties). However, Axiom A flows are not generic in higher dimensions even if they are structurally stable.

3.1.2 Symplectic lift on the cotangent bundle

From now on, we will use the notation $T^*M \setminus \{0\} := \{(x, \xi) \in T^*M : \xi \neq 0\}$ where T^*M is the cotangent bundle of M . The main dynamical object considered in this Chapter is the Hamiltonian flow of

$$h : T^*M \rightarrow \mathbb{R}, \quad h(x, \xi) := \xi(V(x)). \quad (3.1.4)$$

It turns out that the dynamics of such flow is linked to that of the flow of $V(x)$ on M and, in this paragraph, we make this connection precise.

We denote by $\omega(\cdot, \cdot)$ the canonical symplectic form on T^*M and by X_h be the Hamiltonian vector field defined by h (see (3.1.4)) through the relation

$$dh(Y) = \omega(X_h, Y), \quad \forall Y \in \mathfrak{X}(M). \quad (3.1.5)$$

The upcoming definitions and results in this paragraph hold for any vector field, not necessarily Morse-Smale and we denote by $\pi : (x, \xi) \in T^*M \mapsto x \in M$ the canonical projection.

Definition 3.1.6 (Symplectic lift). Let $V \in \mathfrak{X}(M)$ and let $\varphi^t(x)$ be a flow on M .

I) The symplectic lift of V to T^*M is the vector field $Y \in \mathfrak{X}(T^*M)$ such that

- a) $d\pi(Y) = V$;
- b) the vector field Y satisfies $\mathcal{L}_Y \omega = 0$.

II) The symplectic lift of a flow φ^t on M is the flow Φ^t on T^*M such that

- a) $\pi \circ \Phi^t = \varphi^t \circ \pi$, for all $t \in \mathbb{R}$, as maps on T^*M ;
- b) $(\Phi^t)^* \omega = \omega$, $\forall t \in \mathbb{R}$,

where $*$ denotes the pullback of ω with respect to Φ^t .

In order to write down explicitly the relation between a flow on M and its symplectic lift on T^*M , we recall the following definition.

Definition 3.1.7 (Transposed operator). Let $\Phi : TM \rightarrow TM$ and set $\pi_0 : (x, v) \in TM \mapsto x \in M$. We define the transposed map $\Phi^\top$ as

$$\Phi^\top : T^*M \rightarrow T^*M, \quad \Phi^\top(\xi)(w) := \xi(\Phi(w)), \quad \forall \xi \in T_{\pi_0 \circ \Phi(w)}^*M, \quad w \in TM.$$

The following Lemma explicits the relation between a vector field $V(x)$ and the Hamiltonian vector field X_h on T^*M , as defined in (3.1.4) and (3.1.5). We only state the next result, since the proof is standard and follows directly applying Definitions 3.1.6 and 3.1.7.

Lemma 3.1.8. *For every $V \in \mathfrak{X}(M)$, the vector field X_h defined by (3.1.5) is the symplectic lift of V on T^*M . Moreover, for every flow $\varphi^t(x)$ on M , its symplectic lift is given by*

$$\Phi^t(x, \xi) = \left(\varphi^t(x), [d_x \varphi^t(x)]^{-\top} \xi \right), \quad (3.1.6)$$

where $[d_x \varphi^t(x)]^{-\top}$ is the inverse of the transposed operator of $d_x \varphi^t(x)$, as in Definition 3.1.7.

In particular remark that, as an immediate consequence of Lemma 3.1.8, the flow of the Hamiltonian vector field X_h on T^*M , with h in (3.1.4), coincides with the lift of the flow φ_V^t . Thus we will use the notation $\Phi_V^t(x, \xi)$ for such flow, whose explicit expression is the one in (3.1.6).

3.1.3 Dynamics of Morse-Smale symplectic lifts

In this paragraph, we take $V \in \mathfrak{X}^{MS}(M)$ and we describe the dynamics of the Hamiltonian flow induced by the Hamiltonian vector field X_h , with $h : T^*M \rightarrow \mathbb{R}$ defined in (3.1.4). In particular, we describe global attractors and repellers for the lifted flow on T^*M and some of their properties, using that V is Morse-Smale. We closely follow [24], to which we refer for the proof of most results.

Let S^*M denote the unit cotangent bundle of M :

$$S^*M := \{(x, \xi) \in T^*M : \|\xi\|_x = 1\},$$

where $\|\cdot\|_x$ is the norm on the cotangent space at the point x , T_x^*M , induced by the metric g_x . The flow Φ_V^t of X_h on T^*M , which is explicitly given in (3.1.6), induces a

projected flow on S^*M :

$$\tilde{\Phi}_V^t(x, \xi) := \left(\varphi_V^t(x), \frac{[d_x \varphi_V^t(x)]^{-\top} \xi}{\|[d_x \varphi_V^t(x)]^{-\top} \xi\|_x} \right), \quad (3.1.7)$$

and we denote by $\tilde{X}_h$ the induced smooth vector field on S^*M , which is the generator of $\tilde{\Phi}_V^t$.

We now identify global attractors and repellers for such flows on S^*M . To this aim, we start with the general definition of *conormal bundle* of a submanifold. Let $S \subset M$ be a smooth submanifold of M . We define the submanifold N^*S of T^*M as

$$N^*S := \{(x, \xi) \in T^*M \text{ such that } x \in S, \xi \neq 0, \xi(w) = 0, \forall w \in T_x S\},$$

and we call it the conormal bundle of S . We define the following sets:

$$R_p := \bigcup_{i=1}^K N^*(W^s(\Lambda_i)) \cap S^*M, \quad A_p := \bigcup_{i=1}^K N^*(W^u(\Lambda_i)) \cap S^*M. \quad (3.1.8)$$

Next we recall that every stable and unstable manifold is foliated by a family of submanifolds (see Definition 3.5.6):

$$W^u(\Lambda_i) = \bigcup_{x \in \Lambda_i} W^{uu}(x), \quad W^s(\Lambda_i) = \bigcup_{x \in \Lambda_i} W^{ss}(x), \quad \forall 1 \leq i \leq K.$$

On the one hand, this foliation is trivial for critical points. On the other hand, for every $x \in \Lambda_i$ with Λ_i closed orbit, $W^{ss}(x)$ has codimension one in $W^s(\Lambda_i)$ and analogously $W^{uu}(x)$ in $W^u(\Lambda_i)$. As we shall see, they also turn out to be relevant for the dynamics. Hence, we define

$$R_p^r := \bigcup_{i=1}^K \bigcup_{x \in \Lambda_i} N^*(W^{ss}(x)) \cap S^*M, \quad A_p^a := \bigcup_{i=1}^K \bigcup_{x \in \Lambda_i} N^*(W^{uu}(x)) \cap S^*M. \quad (3.1.9)$$

Remark 3.1.9. The transversality assumption in Definition 1.1.8 can be written explicitly as follows: for any $(i, j) \in \{1, \dots, K\}^2$, for any $x \in W^s(\Lambda_i) \cap W^u(\Lambda_j)$ we have

$$T_x M = T_x W^s(\Lambda_i) + T_x W^u(\Lambda_j).$$

Using this decomposition of $T_x M$, one can verify that transversality implies

$$R_p \cap A_p = \emptyset, \quad R_p \cap A_p^a = \emptyset, \quad R_p^r \cap A_p = \emptyset. \quad (3.1.10)$$

We are now in position to describe the dynamics of $\tilde{\Phi}_V^t(x, \xi)$ in (3.1.7) on S^*M .

Proposition 3.1.10 ([24], Lemma 5.1, Theorem 5.2). *Let $V \in \mathfrak{X}^{MS}(M)$. Then, the sets R_p , A_p , R_p^r , A_p^a in (3.1.8) and (3.1.9) are compact subsets of S^*M . Moreover, denoting by $\mathbf{d}$ the induced Riemannian distance on S^*M , the following holds:*

$$\text{for every } (x, \xi) \in S^*M \setminus A_p \quad \text{we have} \quad \lim_{t \rightarrow -\infty} \mathbf{d}(\tilde{\Phi}_V^t(x, \xi), R_p^r) = 0,$$

$$\text{for every } (x, \xi) \in S^*M \setminus R_p^r \quad \text{we have} \quad \lim_{t \rightarrow +\infty} \mathbf{d}(\tilde{\Phi}_V^t(x, \xi), A_p) = 0,$$

and

$$\text{for every } (x, \xi) \in S^*M \setminus R_p \quad \text{we have} \quad \lim_{t \rightarrow +\infty} \mathbf{d}(\tilde{\Phi}_V^t(x, \xi), A_p^a) = 0,$$

$$\text{for every } (x, \xi) \in S^*M \setminus A_p^a \quad \text{we have} \quad \lim_{t \rightarrow -\infty} \mathbf{d}(\tilde{\Phi}_V^t(x, \xi), R_p) = 0.$$

We refer to [24] for the proof of this statement, which involves a careful study of the Morse-Smale structure of the flow induced by V on M . Strictly speaking, the proof in that reference is given under the extra assumption that the vector field is $\mathcal{C}^1$ -linearizable near critical elements. Yet, this hypothesis is removed by Meddane in [73, Lemma 3.2, Prop. 3.3] where he deals with general Axiom A flows satisfying the corresponding transversality property. We emphasize that one of the main technical issue in proving this statement is the compactness property. Finally, for later purpose, we define:

$$A := \bigcup_{i=1}^K N^*(W^s(\Lambda_i)), \quad R := \bigcup_{i=1}^K N^*(W^u(\Lambda_i)).$$

Analogously we define A^a and R^r as subsets of $T^*M \setminus \{0\}$, obtained extending by homogeneity the projective ones in (3.1.9).

3.2 Construction of the escape function

We are now in position to construct the main dynamical ingredient for the proof of Theorem 3.0.2: an *escape (or Lyapunov) function* for the Hamiltonian $h(x, \xi)$ defined in (3.1.4). Roughly speaking, we are looking for a function that increases along the flow lines of Φ_V^t in view of applying a positive commutators argument in our analysis of the transport equation. Before stating the main dynamical result we are aiming at, we start by giving a general definition of escape functions in our context.

Definition 3.2.1 (Positively homogeneous function). We say that a smooth function $f : T^*M \setminus \{0\} \rightarrow \mathbb{C}$ is a positively homogeneous function of degree $\sigma \geq 0$ if

$$f(x, \lambda\xi) = \lambda^\sigma f(x, \xi), \quad \forall \lambda > 0, (x, \xi) \neq 0.$$

Definition 3.2.2 (Escape function). Let $h : T^*M \rightarrow \mathbb{R}$ be the Hamiltonian function defined in (3.1.4). We say that a smooth function $a : T^*M \setminus \{0\} \rightarrow \mathbb{R}$ is an escape function of order $\sigma > 0$ for h if a is positively homogeneous of degree $\sigma > 0$ and if there exist $\delta > 0$ and a closed conic set $E \subset T^*M \setminus \{0\}$ such that

$$X_h(a) \geq \delta \|\xi\|_x^\sigma, \quad \text{on } \{(x, \xi) \in T^*M \setminus E\}, \quad (3.2.1)$$

$$X_h(a) \geq 0 \text{ and } a(x, \xi) \geq \delta \|\xi\|_x^\sigma, \quad \text{on } \{(x, \xi) \in E\}, \quad (3.2.2)$$

where we recall the notation X_h in (3.1.5) for the Hamiltonian vector field defined by h .

Remark 3.2.3. We remark that, if h is a positively homogeneous function of degree σ_1 and a is a positively homogeneous function of degree σ_2 , then $X_h(a)$ is a positively homogeneous function of degree $\sigma_1 + \sigma_2 - 1$. In particular this means that, in Definition 3.2.2, both $X_h(a)$ and a are homogeneous of degree σ .

The main result of this section is the following Proposition:

Proposition 3.2.4. *Let $V \in \mathfrak{X}^{MS}(M)$ be a Morse-Smale vector field on M and let $h(x, \xi) := \xi(V(x))$. Then there exist a neighborhood $\mathcal{U}$ of $\text{NW}(V)$ (see 3.5.3), $c_0 > 0$ and $0 < \sigma_0 < 1$, depending on V such that, for every $0 < \sigma < \sigma_0$, h admits an escape function a of order σ in the sense of Definition 3.2.2, with E being a closed conic neighborhood of $A^a \cap R^r \cap T^*\mathcal{U}$ and with $\delta = c_0\sigma$. Moreover, if $n \geq 2$ or if V has at least one critical point, there exists a nonempty open conic subset C satisfying*

$$a(x, \xi) \leq -\frac{c_0}{2} \|\xi\|_x^\sigma \text{ on } \{(x, \xi) \in C\}. \quad (3.2.3)$$

Remark 3.2.5. Observe that when V has no closed orbits, $A^a \cap R^r$ is empty and our proof will in fact show that E is empty.

In order to prove Proposition 3.2.4, we will use all the dynamical ingredients introduced in Section 3.1. We will construct an escape function $a \in C^\infty(T^*M \setminus \{0\})$ of the form

$$a(x, \xi) := m(x, \xi)f(x, \xi), \quad \forall (x, \xi) \in T^*M \setminus \{0\}, \quad (3.2.4)$$

where $m \in C^\infty(T^*M \setminus \{0\})$ is a positively homogeneous function of degree zero (recall Definition 3.2.1) that we call *order function* following the literature (see e.g. [37]) and $f(x, \xi)$ is a positively homogeneous function of degree $\sigma > 0$, with σ to be determined depending on the vector field. Our construction is close to the one appearing in the microlocal approach to Ruelle resonances in dynamical systems and we refer for instance to [37, 24, 73] for presentations close to ours. We also mention [63] for a recent comprehensive description of microlocal techniques and applications to dynamical systems. The main difference with these references is that we pick f to be σ -homogeneous while these earlier works use symbols of order σ for any $\sigma > 0$. Roughly speaking, they use $\log(1 + \|\xi\|_x^2)$ while we use $\|\xi\|_x^\sigma$. This is due to the different nature of the problems at hand: growth of Sobolev norms for time dependent vector fields in our case compared with the spectral study that is considered in these previous contributions.

We will first construct the function m that will strictly decay away from the repellers and attractors of the lifted flow. A similar construction has been already achieved in [24] in the case of Morse-Smale flows. In view of our application, the only thing we need to pay attention to, compared with this reference, is to specify the values of m near the various subsets A_p , A_p^a , R_p and R_p^r . This is the content of paragraph 3.2.1 and more specifically of Lemma 3.2.8. Once this is done, we will construct in paragraph 3.2.2 the function f and show that it has the expected properties (up to signs issues) near the attractors and repellers. We will also verify that its Poisson bracket with h behaves nicely away from these sets. Once this is done, we will gather these two functions as in (3.2.4) and prove Proposition 3.2.4 in paragraph 3.2.3.

3.2.1 Construction of the order function

In this section, we construct the first component of the escape function, the *order function* $m \in C^\infty(T^*M \setminus \{0\})$ in (3.2.4). Recall the definition of the sets in (3.1.8) and (3.1.9). Our first ingredient is the following lemma, which is an adaptation to our

context of [37, Lemma 2.1] (in which the authors deal with Anosov flows instead of Morse-Smale ones).

Lemma 3.2.6. *Let $V \in \mathfrak{X}^{MS}(M)$ and let $\tilde{X}_h$ be the induced vector field on S^*M (see 3.1.7). For every $\varepsilon > 0$ and for all neighborhoods $V(R_p)$ and $V(A_p^a)$ (of R_p and A_p^a respectively), there exist $\eta_1 > 0$, neighborhoods $W(R_p) \subset V(R_p)$, $W(A_p^a) \subset V(A_p^a)$ (of R_p and A_p^a respectively) and a function $m_1 \in C^\infty(S^*M; [0, 1])$ such that*

$$\begin{aligned} \tilde{X}_h(m_1) &\geq 0 && \text{on the whole } S^*M, \\ \tilde{X}_h(m_1) &\geq \eta_1 && \text{on } S^*M \setminus (W(R_p) \cup W(A_p^a)). \end{aligned} \quad (3.2.5)$$

Moreover $m_1 < \varepsilon$ on $W(R_p)$ and $m_1 > 1 - \varepsilon$ on $W(A_p^a)$.

The proof of this Lemma was given in [24, Lemma 8.2] using the exact same argument as that of [37, Lemma 2.1]. Note that, in order to follow this argument, one needs compactness of the sets A_p^a and R_p and existence of invariant neighborhoods of these sets for the flow – see [24, Th. 5.4 and 7.1]. While this is obvious in the Anosov case, these two properties are much more subtle in the Morse-Smale case. Again, we emphasize that [24] made the assumption that the flow is $\mathcal{C}^1$ -linearizable and this restriction was removed by Meddane who performed this construction in the general case of Axiom A flows without any linearization assumption [73, Sect. 8].

Remark 3.2.7. By applying the flow in backward time, the analog of Lemma 3.2.6 holds for R_p^r and A_p . For every $\varepsilon > 0$ and for every $V(R_p^r)$ and $V(A_p)$ (neighborhoods of R_p^r and A_p respectively), there exist $\eta_2 > 0$, $W(R_p^r) \subset V(R_p^r)$ and $W(A_p) \subset V(A_p)$ (neighborhoods of R_p^r and A_p respectively) and $m_2 \in C^\infty(S^*M; [0, 1])$ satisfying the analog of properties (3.2.5) and such that $m_2 < \varepsilon$ in $W(R_p^r)$ and $m_2 > 1 - \varepsilon$ in $W(A_p)$.

Notations. We denote by $W(R)$, $W(R^r)$, $W(A)$, $W(A^a)$ the extensions by homogeneity to the whole $T^*M \setminus \{0\}$ of the sets in Lemma 3.2.6 and Remark 3.2.7. We define

$$\mathbf{B} := (W(R) \cap W(R^r)) \cup (W(A) \cap W(A^a)) \cup (W(R^r) \cap W(A^a)). \quad (3.2.6)$$

Remark that, up to shrinking the W -neighborhoods, we can suppose that all the other intersections among such neighborhoods are empty (see (3.1.10)). Given a neighborhood $\mathcal{U}$ of $\text{NW}(V)$ (see 3.5.3), we also define

$$\mathbf{B}_{\mathcal{U}} := \mathbf{B} \cap T^*\mathcal{U}. \quad (3.2.7)$$

Recall that, from Definition 1.1.8, $\text{NW}(V)$ coincides with the set of critical elements of V .

We can now state the main result of this paragraph.

Lemma 3.2.8. *Let $V \in \mathfrak{X}^{MS}(M)$, let X_h be the induced Hamiltonian vector field on $T^*M \setminus \{0\}$ (see (3.1.4)), let $\mathbf{B}$ as in (3.2.6) and let $\mathcal{U}$ be a neighborhood of $\text{NW}(V)$. There exists a function $m \in C^\infty(T^*M \setminus \{0\}; [-4, 4])$, positively homogeneous of degree zero, such that*

$$X_h(m) \geq 0, \quad \text{on the whole } T^*M \setminus \{0\}; \quad (3.2.8)$$

and

$$X_h(m) \geq \eta > 0, \quad \text{on } T^*M \setminus \mathbf{B}_{\mathcal{U}}. \quad (3.2.9)$$

Moreover,

- (i) $m < -\frac{1}{2}$ on $W(R^r) \cap W(R)$;
- (ii) $m > \frac{1}{2}$ on $W(A) \cap W(A^a)$;
- (iii) $m > \frac{1}{4}$ on $W(A^a) \cap W(R^r)$.

Proof. In order to prove the Lemma, we construct a function $\tilde{m} \in C^\infty(S^*M; [-4, 4])$ such that conditions (3.2.8), (3.2.9) and items (i), (ii) and (iii) hold for $\tilde{m}$ and $\tilde{X}_h$ on S^*M . Then, defining by homogeneity

$$m(x, \xi) := \tilde{m}\left(x, \frac{\xi}{\|\xi\|_x}\right), \quad \forall (x, \xi) \in T^*M \setminus \{0\}, \quad (3.2.10)$$

we have $m \in C^\infty(T^*M \setminus \{0\}; \mathbb{R})$ and $X_h(m) = \tilde{X}_h(\tilde{m})$. Thus we now turn to defining the function $\tilde{m}$ following the lines of [24]. To this aim we recall the definition of the

energy function $\mathcal{E}(x)$ in Lemma 3.1.3 and we claim that, choosing $\varepsilon > 0$ small enough in Lemma 3.2.6, we can put

$$\tilde{m}\left(x, \frac{\xi}{\|\xi\|_x}\right) := \mathcal{E}(x) - 1 + \frac{3}{2}m_1\left(x, \frac{\xi}{\|\xi\|_x}\right) + \frac{1}{2}m_2\left(x, \frac{\xi}{\|\xi\|_x}\right), \quad (3.2.11)$$

(we recall that m_1 and m_2 defined in Lemma 3.2.6 and Remark 3.2.7 depend on ε). First of all we remark that $\tilde{m}$ is smooth since both $\mathcal{E}$, m_1 and m_2 are. Next we immediately see that $\tilde{X}_h(\tilde{m}) \geq 0$ since

$$\begin{aligned} \tilde{X}_h(\tilde{m}) &\stackrel{(3.2.11)}{=} \underbrace{\tilde{X}_h(\mathcal{E})}_{\stackrel{(3.1.7)}{=} \mathcal{L}_V(\mathcal{E})} + \frac{3}{2}\tilde{X}_h(m_1) + \frac{1}{2}\tilde{X}_h(m_2) \stackrel{(3.2.5)}{\geq} 0, \\ &\stackrel{(3.1.3)}{\geq} 0 \end{aligned}$$

proving the first condition. Moreover, outside $\mathbf{B}_{\mathcal{U}}$, we have

$$\tilde{X}_h(\tilde{m}) \stackrel{(3.1.3)}{=} \tilde{X}_h(E) + \frac{3}{2}\tilde{X}_h(m_1) + \frac{1}{2}\tilde{X}_h(m_2) \geq \min \left\{ \min_{M \setminus \mathcal{U}} \mathcal{L}_V(E), \frac{3}{2}\eta_1, \frac{\eta_2}{2} \right\} =: \eta > 0,$$

where we recall the definition of $\mathbf{B}_{\mathcal{U}}$ in (3.2.7) and we use, for the last inequality, condition (3.2.5), the analog in Remark 3.2.7 for the function m_2 and Lemma 3.1.3 for $\mathcal{E}$. Thanks to (3.2.10), this concludes the proof of (3.2.8) and (3.2.9). We are left to prove items (i), (ii) and (iii). To this aim we make the assumption

$$\sup_M |\mathcal{E}| < \frac{1}{8}, \quad (3.2.12)$$

that we will use along the proof of the three items. Notice that this is not in contradiction with previous part of the proof nor restrictive: indeed the function $\mathcal{E}$ is clearly bounded and, up to now, we have only used that $\mathcal{L}_V(\mathcal{E}) > 0$, which is still true if we rescale $\mathcal{E}$ by a positive factor and impose (3.2.12).

Item (i). For this item, we have to look at the set $W(R_p^r) \cap W(R_p)$, where both $m_1 < \varepsilon$ and $m_2 < \varepsilon$. Thus we have

$$\tilde{m} = \mathcal{E}(x) - 1 + \frac{3}{2}m_1 + \frac{1}{2}m_2 \stackrel{(3.2.12)}{\leq} -\frac{7}{8} + 2\varepsilon < -\frac{1}{2},$$

choosing $0 < \varepsilon < \frac{3}{16}$. From (3.2.10), this concludes the proof of the first item.

Item (ii). For this item, we have to look at the set $W(A_p) \cap W(A_p^a)$, where both $m_1 > 1 - \varepsilon$ and $m_2 > 1 - \varepsilon$. Thus we have

$$\tilde{m} = \mathcal{E}(x) - 1 + \frac{3}{2}m_1 + \frac{1}{2}m_2 \stackrel{(3.2.12)}{\geq} -\frac{9}{8} + 2(1 - \varepsilon) = \frac{7}{8} - 2\varepsilon > \frac{1}{2},$$

choosing again $0 < \varepsilon < \frac{3}{16}$. From (3.2.10), this concludes the proof of the second item.

Item (iii). For this item, we have to look at the set $W(A_p^a) \cap W(R_p^r)$, in which $m_1 > 1 - \varepsilon$. We also use that $m_2 \geq 0$. We have:

$$\tilde{m} = \mathcal{E}(x) - 1 + \frac{3}{2}m_1 + \frac{1}{2}m_2 \stackrel{(3.2.12)}{\geq} -\frac{9}{8} + \frac{3}{2}(1 - \varepsilon) = \frac{3}{8} - \frac{3}{2}\varepsilon > \frac{1}{4},$$

choosing $0 < \varepsilon < \frac{1}{12}$.

In conclusion choosing any $0 < \varepsilon < \frac{1}{12}$ and the corresponding functions m_1 and m_2 , all three items are satisfied, concluding the proof. $\square$

3.2.2 Construction of the function f

In this section we construct the second component of the escape function, i.e. the function denoted by f in (3.2.4). Precisely, we have the following Lemma.

Lemma 3.2.9. *There exist conical neighborhoods $V(A)$, $V(A^a)$, $V(R)$ and $V(R^r)$ (respectively of A , A^a , R and R^r) and a neighborhood $\mathcal{U}$ of $\text{NW}(V)$ such that the following holds.*

For any $\sigma > 0$ one can find a function $f_\sigma : T^*M \setminus \{0\} \rightarrow \mathbb{R}$, smooth, positively homogeneous of degree σ , with restriction to S^*M positive, which satisfies the following properties:

- (i) $X_h(f_\sigma) \leq -\gamma_1 \sigma f_\sigma$, in $V(R) \cap V(R^r) \cap T^*\mathcal{U}$;
- (ii) $X_h(f_\sigma) \geq \gamma_2 \sigma f_\sigma$, in $V(A) \cap V(A^a) \cap T^*\mathcal{U}$;
- (iii) $f_\sigma(x, \xi) = |h(x, \xi)|^\sigma$, in $V(R^r) \cap V(A^a) \cap T^*\mathcal{U}$;

for some $\gamma_1, \gamma_2 > 0$ independent of σ . Moreover, there exists a constant $c_0 > 0$ such that, for every $0 < \sigma \leq 1$,

$$c_0^{-1} \|\xi\|_x^\sigma \leq f_\sigma(x, \xi) \leq c_0 \|\xi\|_x^\sigma \text{ and } |X_h(f_\sigma)(x, \xi)| \leq c_0 \sigma \|\xi\|_x^\sigma, \quad \forall (x, \xi) \in T^*M \setminus \{0\}. \quad (3.2.13)$$

In order to prove this Lemma, we need the following result, which follows from the hyperbolic properties of the non-wandering set (see Definition 1.1.8).

Lemma 3.2.10. *Let $V \in \mathfrak{X}^{MS}(M)$ and consider the associated Hamiltonian vector field X_h with $h(x, \xi) := \xi(V(x))$.*

Then, there exist $\theta, C_a, C_r > 0$ such that, for every $(x, \xi) \in T^*M$ verifying $x \in \text{NW}(V)$,

$$\|[\mathrm{d}\varphi_V^{-t}(x)]^{-\top} \xi\|_{\varphi_V^{-t}(x)} \leq C_a e^{-\theta t} \|\xi\|_x, \quad \text{in } A \cap A^a, \text{ and } \forall t > 0, \quad (3.2.14)$$

and

$$\|[\mathrm{d}\varphi_V^t(x)]^{-\top} \xi\|_{\varphi_V^t(x)} \leq C_r e^{-\theta t} \|\xi\|_x, \quad \text{in } R \cap R^r \text{ and } \forall t > 0. \quad (3.2.15)$$

We are now ready to prove Lemma 3.2.9.

Proof of Lemma 3.2.9. Mimicking the proof of [52, Lemma 2.4], we first define the two functions f_1^r and f_1^a as

$$f_1^{a/r}(x, \xi) := e^{I^{a/r}(x, \xi)}, \quad (3.2.16)$$

with

$$I^{a/r}(x, \xi) := \begin{cases} T_1^{-1} \int_0^{T_1} \ln \|[\mathrm{d}\varphi_V^{-t}(x)]^{-\top} \xi\|_x dt, & \text{in } V(A) \cap V(A^a) \\ T_1^{-1} \int_0^{T_1} \ln \|[\mathrm{d}\varphi_V^t(x)]^{-\top} \xi\|_x dt, & \text{in } V(R) \cap V(R^r), \end{cases}$$

for $T_1 > 0$ to be chosen big enough and for small neighborhoods $V(\cdot)$. First of all we remark that the functions $f_1^{a/r}$ are positively homogeneous of degree 1 in their domains. Indeed, via a direct computation one obtains

$$I^{a/r}(x, \lambda \xi) = \ln \lambda + I^{a/r}(x, \xi) \implies f_1^{a/r}(x, \lambda \xi) = \lambda f_1^{a/r}(x, \xi), \quad \forall \lambda > 0, (x, \xi) \neq 0. \quad (3.2.17)$$

Next we claim that, up to considering slightly smaller neighborhoods, the functions $f_\sigma^{a/r} := (f_1^{a/r})^\sigma$ verify the inequalities of items (i) and (ii) of the Lemma. Let us postpone this verification and conclude the proof extending $f_1^{a/r}$ to the whole $T^*M \setminus \{0\}$. To this aim, we exploit homogeneity (3.2.17) and construct a function $\tilde{f}_1 \in C^\infty(S^*M)$ such that

$$\tilde{f}_1 = \begin{cases} f_1^a & \text{in } V(A_p) \cap V(A_p^a) \\ f_1^r & \text{in } V(R_p) \cap V(R_p^r) \\ \left| h\left(x, \frac{\xi}{\|\xi\|_x}\right) \right| & \text{in } V(R_p^r) \cap V(A_p^a). \end{cases}$$

Recall that, from (3.1.10), the three sets $V(R) \cap V(R^r)$, $V(A) \cap V(A^a)$ and $V(A^a) \cap V(R^r)$ do not intersect (up to restricting the neighborhoods if needed). Moreover the function $\tilde{f}_1$ is positive in the closed set $\bar{\mathbf{B}}$ defined by 3.2.6. It is then possible to extend $\tilde{f}_1$ to a positive smooth function on the whole compact manifold S^*M (see for example [62, 2.26]). Note that, by compactness, the extended function and its derivatives are uniformly bounded. Finally we extend $\tilde{f}_1$ by homogeneity to the whole $T^*M \setminus \{0\}$ and obtain f_1 positively homogeneous of degree 1. We define then $f_\sigma := (f_1)^\sigma$ so that (3.2.13) is automatically satisfied using the fact that f_1 is 1-homogeneous.

We are thus left to prove the initial claim, i.e., that f_1^a and f_1^r in (3.2.16) verify items (i) and (ii) of the Lemma (the bound for $\sigma > 0$ follows again directly by construction). We show only item (i) since the two calculations are exactly the same. We first prove the claim on $R \cap R^r \cap \text{NW}(V)$. To this aim, we first we evaluate $X_h(f_1^r)$:

$$X_h(f_1^r) = \frac{d}{ds} \Big|_{s=0} f_1^r(\Phi_V^s(x, \xi)) \stackrel{(3.2.16)}{=} f_1^r(x, \xi) \frac{d}{ds} \Big|_{s=0} I^r(\Phi_V^s(x, \xi)), \quad (3.2.18)$$

and, using Lemma 3.2.10, we have, on $R \cap R^r \cap \text{NW}(V)$:

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} I^r(\Phi_V^s(x, \xi)) &= \frac{1}{T_1} \int_0^{T_1} \frac{d}{ds} \ln(\|[\text{d}\varphi_V^{t+s}(x)]^{-\top} \xi\|) \Big|_{s=0} dt \\ &= \frac{1}{T_1} \int_0^{T_1} \frac{d}{dt} (\ln(\|[\text{d}\varphi_V^t(x)]^{-\top} \xi\|)) dt \\ &= \frac{1}{T_1} (\ln(\|[\text{d}\varphi_V^{T_1}(x)]^{-\top} \xi\|) - \ln \|\xi\|) \\ &\stackrel{(3.2.15)}{\leq} \frac{1}{T_1} (\ln(C_r) - \theta T_1). \end{aligned} \quad (3.2.19)$$

Thus, choosing T_1 big enough so that $\ln(C_r) - \theta T_1 < 0$ and plugging (3.2.19) in (3.2.18) we have

$$X_h(f_1^r) \leq f_1^r \left(\frac{\ln(C_r)}{T_1} - \theta \right) < -f_1^r \gamma_1,$$

with $\gamma_1 = -\frac{\ln(C_r)}{T_1} + \theta > 0$. This proves item (i) for f_1^r on $R \cap R^r \cap \text{NW}(V)$. Then we can extend by continuity the inequality to a sufficiently small conical neighborhood $\mathbf{C}$ of $R \cap R^r \cap \text{NW}(V)$. Recall that we aim at showing such an upper bound inside $V(R) \cap V(R^r) \cap T^*\mathcal{U}$. Up to shrinking the size of the neighborhood $\mathcal{U}$ of $\text{NW}(V)$ and of the conical neighborhoods $V(R) \cap V(R^r)$ and recalling from Proposition 3.1.7 that R_p^r and R_p are both compact subsets, one finds that $\mathbf{C}$ contains $V(R) \cap V(R^r) \cap T^*\mathcal{U}$ in its interior, proving item (i). One analogously proves item (ii) using (3.2.14). $\square$

3.2.3 Proof of Proposition 3.2.4

We now put together Lemmas 3.2.8 and 3.2.9 to prove Proposition 3.2.4. Let $m \in C^\infty(T^*M \setminus \{0\})$ and $\{f_\sigma \in C^\infty(T^*M \setminus \{0\})\}_{0 < \sigma < 1}$ be respectively the order function given by Lemma 3.2.8 and the family of functions given by Lemma 3.2.9, for the vector field $V \in \mathfrak{X}^{MS}(M)$. We note that the neighborhoods $\mathcal{U}$ and $W(S)$ in Lemma 3.2.8 are chosen in terms of the neighborhoods $\mathcal{U}$ and $V(S)$ given by Lemma 3.2.9.

We will prove that it is possible to choose $\sigma > 0$ small enough, such that the function

$$a(x, \xi) := m(x, \xi) f_\sigma(x, \xi), \quad \forall (x, \xi) \in T^*M \setminus \{0\}, \quad (3.2.20)$$

satisfies Definition 3.2.2 for some $\delta > 0$ and E a closed conical neighborhood containing $A^a \cap R^r$ in its interior (where we recall the definition of the sets A^a and R^r in (3.1.9)). Note that (3.2.3) is satisfied in $W(R) \cap W(R^r)$ thanks to item (i) in Lemma 3.2.8 and to (3.2.13). We now use the explicit expression of $X_h(a)$ to show that a satisfies the requirements of Definition 3.2.2:

$$X_h(a) \stackrel{(3.2.20)}{=} X_h(m f_\sigma) = X_h(m) f_\sigma + X_h(f_\sigma) m. \quad (3.2.21)$$

Next, recalling the notation in (3.2.7), we split the cotangent space in

$$T^*M = (T^*M \setminus \mathbf{B}_\mathcal{U}) \cup \mathbf{B}_\mathcal{U},$$

and treat the two regions separately. Remark that $W(A^a) \cap W(R^r) \cap T^*\mathcal{U} \subset \mathbf{B}_\mathcal{U}$ is one of the three components of the set $\mathbf{B}_\mathcal{U}$. Let us start with $T^*M \setminus \mathbf{B}_\mathcal{U}$, for which we have to prove (3.2.1). Using non-negativity of f_σ and equations (3.2.9) and (3.2.13), we have:

$$X_h(a) = X_h(m) f_\sigma + X_h(f_\sigma) m \stackrel{(3.2.9)}{\geq} \eta f_\sigma + X_h(f_\sigma) m \stackrel{(3.2.13)}{\geq} (\eta c_0^{-1} - \sigma c_0) \|\xi\|_x^\sigma. \quad (3.2.22)$$

Thus choosing σ small enough (precisely $0 < \sigma < \eta/(2c_0^2)$), we can conclude that

$$X_h(a) \stackrel{(3.2.22)}{\geq} \frac{\eta}{2c_0} \|\xi\|_x^\sigma, \quad \text{on } T^*M \setminus \mathbf{B}_\mathcal{U}, \quad (3.2.23)$$

proving (3.2.1) in this set. Remark that it is not restrictive to suppose that $\sigma < 1$, as in the statement of Proposition 3.2.4.

We now consider the three components of the region $\mathbf{B}_\mathcal{U}$ and use the properties of m and f_σ in each of them (see Lemma 3.2.9 and Lemma 3.2.8).

- In the set $W(R)_- \cap W(R^r) \cap T^*\mathcal{U}$, (3.2.1) is given by:

$$X_h(a) = X_h(m)f_\sigma + X_h(f_\sigma)m \stackrel{(3.2.8)}{\geq} X_h(f_\sigma)m \geq \frac{\gamma_1\sigma}{2}f_\sigma,$$

where for the last inequality we have used point (i) of Lemma 3.2.8 and point (i) of Lemma 3.2.9. This concludes, for this set, the proof of (3.2.1). Using (3.2.13), we have

$$\frac{\gamma_1\sigma}{2}f_\sigma \geq \frac{\gamma_1\sigma}{2c_0} \|\xi\|_x^\sigma, \quad \implies \quad X_h(a) \geq \frac{\gamma_1\sigma}{2c_0} \|\xi\|_x^\sigma, \quad \forall (x, \xi) \in W(R)_- \cap W(R^r) \cap T^*\mathcal{U}. \quad (3.2.24)$$

Remark that in this set we have not used any smallness of σ and that this bound is valid for any $0 < \sigma \leq 1$.

- In the set $W(A)_- \cap W(A^a) \cap T^*\mathcal{U}$ analogously, (3.2.1) is obtained from:

$$X_h(a) = X_h(m)f_\sigma + X_h(f_\sigma)m \stackrel{(3.2.8)}{\geq} X_h(f_\sigma)m \geq \frac{\gamma_2\sigma}{2}f_\sigma,$$

where for the last inequality we have used point (ii) of Lemma 3.2.8 and point (ii) of Lemma 3.2.9. This concludes the proof as in previous point using (3.2.13):

$$X_h(a) \geq \frac{\gamma_1\sigma}{2}f_\sigma \geq \frac{\gamma_2\sigma}{2c_0} \|\xi\|_x^\sigma. \quad (3.2.25)$$

- In the set $W(R^r)_- \cap W(A^a) \cap T^*\mathcal{U}$, we instead have to prove (3.2.2). First of all remark that, from point (iii) of Lemma 3.2.9, we have that, for every $\sigma > 0$, $f_\sigma \equiv |h(x, \xi)|^\sigma$, thus in particular

$$X_h(f_\sigma) = 0. \quad (3.2.26)$$

Plugging this expression in (3.2.21) we obtain

$$X_h(a) \stackrel{(3.2.21)}{=} X_h(m)f_\sigma + X_h(f_\sigma)m \stackrel{(3.2.26)}{=} X_h(m)f_\sigma \geq 0,$$

from (3.2.8) and the fact that f_σ is non negative. This concludes the proof, since from point (iii) of Lemma 3.2.8 $m(x, \xi) > \frac{1}{4}$ in this region, thus, using (3.2.13) one more time

$$a \stackrel{(3.2.20)}{\geq} \frac{1}{4}f_\sigma \geq \frac{1}{4c_0} \|\xi\|_x^\sigma. \quad (3.2.27)$$

In conclusion, we obtain, for every $\sigma > 0$ small enough, (3.2.1) and (3.2.2) with $\delta := \frac{\sigma}{2c_0} \min\{\eta, \gamma_1, \gamma_2, 1/2\}$ (see (3.2.23), (3.2.24), (3.2.25) and (3.2.27)).

3.3 Instability and growth of Sobolev norms

We are now in position to prove Theorem 3.0.2. The two main ingredients we use are the existence of an escape function for the Hamiltonian $h(x, \xi) = \xi(V(x))$ with V a Morse-Smale vector field on M (proven in Proposition 3.2.4) and some results of symbolic calculus on M . For this reason, before proving Theorem 3.0.2, we will start this section by reviewing some preliminary material on pseudodifferential operators in paragraph 3.3.1. We refer to [86, Ch. 14] and [33, App. E] for an introduction to pseudodifferential calculus on compact manifolds. See also [22, Appendix A2, A3] for a brief presentation close to ours.

Once these preliminary tools are settled, we introduce a generalized transport equation (3.3.9) in paragraph 3.3.2 for which we will prove existence of solutions whose positive Sobolev norms grow exponentially. This is achieved in two steps. We introduce first in paragraph 3.3.3 an energy type functional that is bounded from above by the Sobolev norm of the solution and we prove a growth property (3.3.13) for this functional. Then, in paragraph 3.3.5, we introduce appropriate initial data such that this functional indeed blows up exponentially in time along the corresponding trajectories. This concludes the proof of Theorem 3.0.2 in the case of the generalized transport equation (3.3.9).

3.3.1 Pseudodifferential operators

Some of the definitions and results presented in this section are analogous to those introduced in Section 2.1.1. However, the different topological setting considered in this chapter requires a number of minor modifications. For the reader's convenience, we therefore present here all the material needed in the sequel, making this section self-contained rather than referring back to Section 2.1.1.

Pseudodifferential operators in $\mathbb{R}^n$. We start by giving here some results of symbolic calculus in the case of pseudodifferential operators on $\mathbb{R}^n$, as the case of manifolds will be locally modeled on it. First, we recall the definition of the set of symbols of

order $\rho \in \mathbb{R}$ over $\mathbb{R}^n$:

$$\mathbf{S}^\rho(\mathbb{R}^{2n}) := \{a \in C^\infty(\mathbb{R}^{2n}) : \forall \alpha, \beta \in \mathbb{N}^n \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{\rho - |\beta|}\}$$

(where we denote by $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$), and the definition of the Weyl quantization of symbols $a \in \mathbf{S}^\rho(\mathbb{R}^{2n})$:

$$\text{Op}^w(a)[u] := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (3.3.1)$$

This last definition makes sense apriori only for a and u belonging to the Schwartz class but one can show that this still makes sense for $a \in \mathbf{S}^\rho(\mathbb{R}^{2n})$ and u in the Schwartz class. In this setting, we say that an operator F belongs to $\mathcal{S}^\rho(\mathbb{R}^n)$, the class of pseudodifferential operators of order ρ , if there exists a symbol $f \in \mathbf{S}^\rho(\mathbb{R}^{2n})$ such that $F = \text{Op}^w(f)$. The following general results for pseudodifferential calculus over $\mathbb{R}^n$ hold. We refer for example to [81, 86] for proofs.

Lemma 3.3.1. *Let $f \in \mathbf{S}^{\rho_1}(\mathbb{R}^{2n})$, $g \in \mathbf{S}^{\rho_2}(\mathbb{R}^{2n})$, $\rho_1, \rho_2 \in \mathbb{R}$. Then:*

1. *For any $s \in \mathbb{R}$, there exist $C_{f,s} > 0$ such that:*

$$\|\text{Op}^w(f)u\|_{s-\rho_1} \leq C_{f,s} \|u\|_s, \quad (3.3.2)$$

where we recall the definition of Sobolev norms in (3.0.3) (with $M \rightsquigarrow \mathbb{R}^n$);

2. *$\text{Op}^w(f)^* = \text{Op}^w(\bar{f})$ (here $*$ denotes the adjoint operator) and in particular skew-adjoint operators have purely imaginary symbols;*
3. *There exists $h \in \mathbf{S}^{\rho_1+\rho_2}(\mathbb{R}^{2n})$ such that $\text{Op}^w(f) \circ \text{Op}^w(g) = \text{Op}^w(h)$. Furthermore,*

$$h := fg - \frac{i}{2} \{f, g\} \mod \mathbf{S}^{\rho_1+\rho_2-2}(\mathbb{R}^{2n});$$

4. *There exists $\ell \in \mathbf{S}^{\rho_1+\rho_2-1}(\mathbb{R}^{2n})$ such that $i[\text{Op}^w(f), \text{Op}^w(g)] = \text{Op}^w(\ell)$. Moreover*

$$\ell := \{f, g\} \mod \mathbf{S}^{\rho_1+\rho_2-2}(\mathbb{R}^{2n});$$

5. *If $\rho_1 < 0$, then $\text{Op}^w(f)$ is compact as an operator from $L^2(\mathbb{R}^n)$ to itself.*

We conclude this introductory section with the following observation, that is a crucial ingredient to consider pseudodifferential operators on manifolds. Given a smooth diffeomorphism $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with bounded derivatives and such that also γ^{-1} is bounded with bounded derivatives, one can define its symplectic lift as

$$\tilde{\gamma} : T^*\mathbb{R}^n \simeq \mathbb{R}^{2n} \rightarrow T^*\mathbb{R}^n, (x, \xi) \rightarrow (\gamma^{-1}(x), d\gamma(x)^\top \xi). \quad (3.3.3)$$

It is possible to prove that if $a \in \mathbf{S}^\rho(\mathbb{R}^{2n})$, then $a \circ \tilde{\gamma} \in \mathbf{S}^\rho(\mathbb{R}^{2n})$, see for example [86, Theorem 9.4].

Pseudodifferential operators on a compact manifold M . Following [86], we first fix a finite atlas for M ,

$$\{(\mathcal{U}_i, \gamma_i)\}_{i=1}^N, \quad \bigcup_{i=1}^N \mathcal{U}_i = M, \quad (3.3.4)$$

where $\{\mathcal{U}_i\}_{i=1}^N$ are coordinate patches and $\gamma_i : \mathcal{U}_i \rightarrow V_i \subset \mathbb{R}^n$ are smooth homeomorphisms onto open subsets $\{V_i\}_{i=1}^N$ of $\mathbb{R}^n$. We are now in position to introduce symbols on the compact manifold M . This definition is given for example in [86, Sections 14.2.2, 14.2.3], and the idea is that a function $a \in C^\infty(T^*M)$ is a symbol over M if, in local coordinates, it corresponds to a symbol on each open subset $\gamma_i(\mathcal{U}_i)$ over $\mathbb{R}^n$.

Definition 3.3.2 (Symbols on M). We say that a smooth function $a \in C^\infty(T^*M)$ is a symbol of order ρ on M if, for any $i = 1, \dots, N$, the pullback of a under the identification (3.3.3)

$$V_i \times \mathbb{R}^n \rightarrow T^*\mathcal{U}_i$$

belongs to $\mathbf{S}^\rho(T^*V_i)$. Equivalently we can define the set of symbols of order $\rho \in \mathbb{R}$ on M as:

$$\mathbf{S}^\rho(T^*M) := \{a \in C^\infty(T^*M) : \forall \alpha, \beta \in \mathbb{N}^n \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < C_{\alpha, \beta} \langle \xi \rangle^{\rho - |\beta|}\}, \quad (3.3.5)$$

where the derivatives are again understood in local coordinate charts.

We remark that Definition 3.3.2 does not depend on the choice of the atlas in (3.3.4). We record the following property:

Lemma 3.3.3. *Let $a \in C^\infty(T^*M)$ be a compactly supported function, then $a \in \mathbf{S}^\rho(T^*M)$ for any $\rho < 0$.*

Following [86], we now set

Definition 3.3.4 (Pseudodifferential operators of order ρ). We say that a linear operator $A : C^\infty(M) \rightarrow C^\infty(M)$ is a pseudodifferential operator of order ρ on M if the following two conditions hold:

1. for every coordinate patch $\mathcal{U}_i$ there exists a symbol $a_i \in \mathbf{S}^\rho(\mathbb{R}^{2n})$ such that

$$\varphi A(\psi u) = \varphi \gamma_i^* \text{Op}^w(a_i)(\gamma_i^{-1})^*(\psi u), \quad \forall \varphi, \psi \in C_c^\infty(\mathcal{U}_i), \quad \forall u \in C^\infty(M),$$

where γ_i is the homomorphism associated to $\mathcal{U}_i$ (see (3.3.4)), $*$ denotes the pull-back operator and $\text{Op}^w(a_i)$ is defined as in (3.3.1);

2. for every $\varphi_1, \varphi_2 \in C^\infty(M)$ such that $\text{supp}(\varphi_1) \cap \text{supp}(\varphi_2) = \emptyset$, the operator $\varphi_1 A \varphi_2$ is bounded from $\mathcal{H}^{-k}(M)$ to $\mathcal{H}^k(M)$ for every $k \in \mathbb{N}$ (recall (3.0.3)).

We denote by $\Psi^\rho(M)$ the class of pseudodifferential operators of order ρ on M .

Remark 3.3.5. We will use the notation $\Psi^\rho(M)$, $\rho \in \mathbb{R}$, for the class of operators of order ρ on a generic Riemannian manifold M . For coherence with the rest of the thesis, we will instead denote by $\mathcal{S}^\rho(\mathbb{T}^n)$, $\rho \in \mathbb{R}$, $n \in \mathbb{N}$, the set of pseudodifferential operators of order ρ on $\mathbb{T}^n$.

It is non-trivial to associate to each symbol in $\mathbf{S}^\rho(T^*M)$ an operator in $\Psi^\rho(M)$ and vice versa (see [86, Theorem 14.1]). Following [51, Sect. 4] (see also [22, App. A]) and in view of simplifying some aspects of the discussion, we now assume that the atlas in (3.3.4) is an isochore atlas in the sense of the following definition:

Definition 3.3.6 (Isochore atlas). We say that the atlas $\{(\mathcal{U}_i, \gamma_i)\}_{i=1}^N$ is *isochore* if

$$\gamma_i^*(d\mu_g) = \text{Leb}_{\mathbb{R}^n}, \quad \forall i = 1, \dots, N,$$

where $d\mu_g$ is the Riemmanian volume form on M and Leb is the Lebesgue measure.

Roughly speaking, Definition 3.3.6 can be rephrased saying that an atlas is isochore if, for every change of coordinates between two patches $\mathcal{U}_i$ and $\mathcal{U}_j$, the determinant of the Jacobian matrix related to such transformation is equal to one. It is a general result that there always exists an isochore atlas on a compact smooth and oriented Riemannian manifold (see [75]). Next, we fix a partition of unity $\{\chi_i\}_{i=1}^N$ subordinated to the atlas, i.e.,

$$\sum_{i=1}^N \chi_i^2(x) = 1, \quad \forall x \in M, \quad \text{and} \quad \chi_i \in C_c^\infty(\mathcal{U}_i; [0, 1]), \quad \forall i = 1, \dots, N.$$

Finally, following [86, Ch. 14], we can define the quantization of a symbol $a \in \mathbf{S}^\rho(T^*M)$:

$$\text{Op}^M(a) := \sum_{i=1}^N \chi_i \gamma_i^* \text{Op}^w((\tilde{\gamma}_i^{-1})^*(\tilde{\chi}_i a))(\gamma_i^{-1})^* \chi_i, \quad (3.3.6)$$

where we recall the notation in (3.3.3) for $\tilde{\gamma}_i$ and where $\tilde{\chi}_i \in C_c^\infty(\mathcal{U}_i; [0, 1])$ is such that $\tilde{\chi}_i \chi_i = \chi_i$.

Remark 3.3.7. Let us briefly motivate the expression in (3.3.6). Given a symbol $a \in \mathbf{S}^\rho(T^*M)$, in order to define its quantization we need to use its expression in local charts to rewrite it as sum of symbols over $\mathbb{R}^n$. Indeed, each $\{(\tilde{\gamma}_i^{-1})^*(\chi_i a)\}_{i=1}^N$ is a symbol over $\mathbb{R}^{2n}$. Thus we can quantize it using the Weyl quantization over $\mathbb{R}^n$ (see (3.3.1)), and then go back to the manifold localizing in the desired chart, thanks to the composition with $(\gamma_i^{-1})^* \chi_i$ and its inverse.

We now state the main result related to this choice of the quantization and of an isochore atlas: roughly speaking, the quantization of a symbol is independent on the choice of charts, up to lower order remainders. Precisely, we have the following result (see [51] or [22, Remark A.2], to which we refer for a proof):

Lemma 3.3.8. *Let $a \in \mathbf{S}^\rho(T^*M)$ be a symbol of order ρ over M and let $\{(\mathcal{U}_i, \gamma_i)\}_{i=1}^N$ be an isochore atlas in the sense of Definition 3.3.6. Let $\text{Op}^M(a)$ be the quantization of a in the sense of (3.3.6). Then the principal symbol of $\text{Op}^M(a)$ is well defined and independent of the chart, up to symbols in $\mathbf{S}^{\rho-2}(M)$.*

We also record the following useful properties (see [22, Remark A.3] for a proof of the second point):

Lemma 3.3.9. 1. Let $f \in C^\infty(M)$ be a smooth function on M . Then $f \in \mathbf{S}^0(T^*M)$ and $\text{Op}^M(f) \in \Psi^0(M)$ is a multiplicative operator with factor f , i.e.,

$$\text{Op}^M(f)u(x) := f(x)u(x), \quad \forall x \in M, \forall u \in C^\infty(M).$$

2. Let $X \in \mathfrak{X}(M)$ be a smooth vector field on M , then

$$\mathcal{L}_X u(x) = \text{Op}^M(i\xi(X(x)) + r(x))u(x), \quad \forall u \in C^\infty(M),$$

where $r \in \mathbf{S}^0(T^*M)$ is a smooth function on M .

We now state the main properties of pseudodifferential operators that we will need in our proof. We proceed in analogy with the ones of Lemma 3.3.1 on $\mathbb{R}^n$.

Lemma 3.3.10. Let $a \in \mathbf{S}^{\rho_1}(T^*M)$ and $b \in \mathbf{S}^{\rho_2}(T^*M)$, with $\rho_1, \rho_2 \in \mathbb{R}$. Then

1. For any $s \in \mathbb{R}$ there exists $C_{a,s} > 0$ such that

$$\|\text{Op}^M(a)u\|_{s-\rho_1} \leq C_{a,s} \|u\|_s; \quad (3.3.7)$$

2. $\text{Op}^M(a)^* = \text{Op}^M(\bar{a})$ (here $*$ denotes the adjoint operator) and in particular purely imaginary symbols correspond to skew-adjoint operators;

3. There exists a symbol $c \in \mathbf{S}^{\rho_1+\rho_2}(T^*M)$ such that

$$\text{Op}^M(c) = \text{Op}^M(a) \circ \text{Op}^M(b) \mod \Psi^{-\infty}(M).$$

Moreover, the operator $\text{Op}^M(c)$ is given by

$$\text{Op}^M(c) = \text{Op}^M\left(ab + \frac{1}{2i}\{a, b\}\right) \mod \Psi^{\rho_1+\rho_2-2}(M);$$

4. The symbol for the commutators between $\text{Op}^M(a)$ and $\text{Op}^M(b)$ is given by

$$[\text{Op}^M(a), \text{Op}^M(b)] = -i \text{Op}^M(\{a, b\}) \mod \Psi^{\rho_1+\rho_2-2}(M); \quad (3.3.8)$$

5. if $\rho_1 < 0$, then $\text{Op}^M(a)$ is a compact operator.

Observe that item 2 follows from our choice of isochore charts. Finally, we also need the following Gårding inequality:

Lemma 3.3.11. Let $a \in \mathbf{S}^\rho(T^*M)$, $a \geq 0$. Then, for every $u \in C^\infty(M)$, we have

$$\langle \text{Op}^M(a)u, u \rangle_{L^2(M)} \geq -C_{\rho,a} \|u\|_{\mathcal{H}^{\frac{\rho-1}{2}}(M)}^2.$$

We refer to [86, Theorem 9.11] for a proof in $\mathbb{R}^n$ from which the case on M follows.

3.3.2 Generalized transport equations

In the rest of this section, we prove our main result, namely Theorem 3.0.2. The starting point is to rewrite the transport equation (3.0.2) in terms of the symbols associated to pseudodifferential operators in view of using the above microlocal tools. Precisely, using Lemma 3.3.9, we can in fact consider more generally the transport like equation:

$$\partial_t u_\varepsilon = \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t, x)))u_\varepsilon + \text{Op}^M(ib_\varepsilon(t, x, \xi))u_\varepsilon, \quad x \in M, \quad t \in \mathbb{R}, \quad \varepsilon > 0, \quad (3.3.9)$$

where

- V belongs to $\mathfrak{X}^{MS}(M)$,
- $\mathcal{P}_\varepsilon(t, x)$ is a smooth time dependent vector field belonging to $\mathcal{C}_b^0(\mathbb{R}; \mathfrak{X}(M))$ with all semi-norms uniformly bounded in terms of $0 \leq \varepsilon \leq 1$,
- there exists $\rho < 1$ such that $b_{\varepsilon,1}(t, x, \xi) := \text{Re } b_\varepsilon(t, x, \xi) \in C_b^0(\mathbb{R}; \mathbf{S}^\rho(T^*M))$ with all seminorms uniformly bounded in terms of $0 \leq \varepsilon \leq 1$,
- $b_{\varepsilon,2}(t, x) := \varepsilon^{-1} \text{Im } b_\varepsilon(t, x) \in C_b^0(\mathbb{R}; \mathbf{S}^0(T^*M))$ with all seminorms uniformly bounded in terms of $0 \leq \varepsilon \leq 1$.

Note that the function $b_\varepsilon(t, x, \xi)$ appearing in Theorem 3.0.2 would not depend on ξ (thanks to Lemma 3.3.9) and that it would be of order 0 (and proportional to ε). Yet, as our proof allows to handle this more general case, we consider such a general b_ε .

Remark 3.3.12. According to [72, Th. 1.2], one has that, for any $\sigma \geq 0$ and for any $u_0 \in \mathcal{H}^\sigma(M)$, the solution $u(t, x)$ exists globally in time and belongs to $\mathcal{C}^0(\mathbb{R}; \mathcal{H}^\sigma(M))$ when $b_{\varepsilon,2} \equiv 0$. Here, we want to allow this extra selfadjoint term and we postpone the existence of the solution to Section 3.5.2 following the results from this reference and classical perturbative arguments from ordinary differential equations.

Our main result reads

Theorem 3.3.13. *Suppose that the above assumptions on V , $\mathcal{P}_\varepsilon$ and b_ε hold and, if $n = 1$, suppose in addition that V has at least one critical point.*

There exists $\delta_0 > 0$ such that, for every $0 \leq \sigma \leq \delta_0$ and for every $0 \leq \varepsilon \leq \delta_0 \sigma^2$, one can find $v_\varepsilon \in \mathcal{H}^\sigma(M)$ and $r := r(\varepsilon, \sigma, v_\varepsilon) > 0$ such that

$$\|u_\varepsilon(0) - v_\varepsilon\|_\sigma \leq r \implies \|u_\varepsilon(t)\|_\sigma \geq \delta_0 e^{\delta_0 \sigma t} \|u_\varepsilon(0)\|_\sigma, \quad \text{for any } t > 0,$$

where $u_\varepsilon(t)$ is the solution to equation (3.3.9) with initial datum $u_\varepsilon(0)$.

Observe that this Theorem implies Theorem 3.0.2. Yet the equation considered in (3.3.9) is slightly more general and it encompasses the case where the right-hand side of the equation is not necessarily skew-adjoint. We emphasize that the selfadjoint perturbation in (3.3.9) is small both in terms of ε and of the order. As anticipated, the key ingredient of our proof is the existence of an escape function for $h(x, \xi) = \xi(V(x))$, proven in Proposition 3.2.4. Thus, we start associating a symbol to the escape function.

Let f be any positively homogeneous function of order $\sigma > 0$, in the sense of Definition 3.2.1. Let $\chi \in C_c^\infty(T^*M; [0, 1])$ be a bump function on T^*M , such that

$$\text{supp}(\chi) \subseteq \{(x, \xi) : x \in M, \|\xi\|_x \leq 2\}, \quad \chi(x, \xi) \equiv 1, \quad \forall \|\xi\|_x \leq 1. \quad (3.3.10)$$

Then the function $\tilde{f}(x, \xi) := f(x, \xi)(1 - \chi(x, \xi))$ is a symbol of order σ , i.e., $\tilde{f} \in \mathbf{S}^\sigma(T^*M)$. Indeed, it is immediate to see from its definition that $\tilde{f}$ is a smooth function on the whole T^*M . Moreover using the characterization in (3.3.5) of $\mathbf{S}^\sigma(T^*M)$, one obtains the decay property of symbols from homogeneity of f .

In particular, for every choice of $0 < \sigma < \sigma_0$ (where the threshold σ_0 is given by Proposition 3.2.4), the escape function a for $h(x, \xi) = \xi(V(x))$ is a positively homogeneous function of order σ , where σ depends on the vector field V (recall the Definition 3.2.2 of escape function). Thus we denote by

$$\tilde{a}(x, \xi) := a(x, \xi)(1 - \chi(x, \xi)) \in \mathbf{S}^\sigma(T^*M), \quad (3.3.11)$$

a smooth symbol associated with the escape function. We remark that, choosing χ as in (3.3.10), we have $\tilde{a} \equiv a$ in $\{(x, \xi) \in T^*M : \|\xi\|_x \geq 2\}$, so that, in this set, conditions (3.2.1) and (3.2.2) hold for $\tilde{a}$ as well.

3.3.3 Energy type estimate

Let $u_\varepsilon(t, x)$ be a solution of equation (3.3.9) and define

$$\mathcal{A}(t) := \langle \text{Op}^M(-\tilde{a})u_\varepsilon(t, x), u_\varepsilon(t, x) \rangle_{L^2(M)}, \quad \forall t \in \mathbb{R}, \quad (3.3.12)$$

where $\tilde{a}$ is given in (3.3.11). The main result of this paragraph is the following Lemma.

Lemma 3.3.14. *There exist $\varepsilon_0 > 0$, $0 < \sigma_0 < 1$ and $\tilde{\alpha} > 0$ such that, for any $0 < \sigma \leq \sigma_0$, one can find $\beta_\sigma > 0$ so that, for every $0 < \varepsilon \leq \sigma^2 \varepsilon_0$, the function $\mathcal{A}(t)$ in (3.3.12) satisfies*

$$\frac{d}{dt} \mathcal{A}(t) \geq \tilde{\alpha} \sigma \mathcal{A}(t) - \beta_\sigma \|u_\varepsilon(t)\|_{L^2}^2, \quad \forall t \in \mathbb{R}, \quad (3.3.13)$$

where $u_\varepsilon(t, x)$ is the solution of (3.0.2) with initial condition $u_\varepsilon(0, x)$.

For simplicity of notations, from now on we drop the index ε and just write $u_\varepsilon = u$. Before proving the Lemma, we remark that, from the properties of the operator $\mathcal{B}(t) := \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}(t, x)) + b_\varepsilon(t, x, \xi))$ and the definitions of the functions b_{ε_1} , b_{ε_2} in (3.3.9), we have

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 = 2\varepsilon \langle \text{Op}^M(b_{\varepsilon,2})u(t), u(t) \rangle \leq 2C_b \varepsilon \|u(t)\|_{L^2}^2,$$

where we used the Calderón-Vaillancourt property (3.3.2) in the last inequality and where the involved constant $C_b > 0$ is uniform for $0 \leq \varepsilon \leq 1$. In particular,

$$\|u(t)\|_{L^2} \leq e^{C_b \varepsilon t} \|u(0)\|_{L^2}, \quad \forall t \geq 0, \quad (3.3.14)$$

and equation (3.3.13) also reads, for any $c > 0$,

$$\frac{d}{dt} (\mathcal{A}(t) - c \|u(t)\|_{L^2}^2) \geq \tilde{\alpha} \sigma \left(\mathcal{A}(t) - \frac{\beta_\sigma + 2cC_b \varepsilon}{\tilde{\alpha} \sigma} \|u(t)\|_{L^2}^2 \right), \quad \forall t \in \mathbb{R}.$$

Hence, if we set $c = \beta_{\sigma,\varepsilon} := \frac{\beta_\sigma}{\tilde{\alpha} \sigma - 2C_b \varepsilon}$ for $0 \leq \varepsilon \leq \tilde{\alpha} \sigma / 4C_b$ (recall that $0 \leq \varepsilon \leq \varepsilon_0 \sigma^2$), we find that

$$\frac{d}{dt} (\mathcal{A}(t) - \beta_{\sigma,\varepsilon} \|u(t)\|_{L^2}^2) \geq \tilde{\alpha} \sigma (\mathcal{A}(t) - \beta_{\sigma,\varepsilon} \|u(t)\|_{L^2}^2), \quad \forall t \in \mathbb{R}. \quad (3.3.15)$$

In paragraph 3.3.5, we will explain how one can derive Theorem 3.3.13 from this energy estimate in (3.3.15).

3.3.4 Proof of Lemma 3.3.14

First of all, we remark that, using (3.3.9) and the properties of the operator $\mathcal{B}(t)$, we can write:

$$\begin{aligned} \frac{d}{dt} \mathcal{A}(t) &= \langle [\text{Op}^M(-\tilde{a}), \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t, x)) + ib_{\varepsilon,1}(t, x, \xi))]u, u \rangle \\ &\quad + 2\varepsilon \text{Re} \langle \text{Op}^M(\tilde{a}) \text{Op}^M(b_{\varepsilon,2})u, u \rangle. \end{aligned}$$

Using the Gårding inequality from Lemma 3.3.11, one can find a constant C_b (uniform for $t \in \mathbb{R}$ and $0 \leq \varepsilon \leq 1$) such that this equality implies

$$\frac{d}{dt} \mathcal{A}(t) \geq \langle [\text{Op}^M(-\tilde{a}), \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t, x)) + ib_{\varepsilon,1}(t, x, \xi))]u, u \rangle - C_b \varepsilon \langle \text{Op}^M(\langle \xi \rangle^\sigma)u, u \rangle, \quad (3.3.16)$$

where $\langle \xi \rangle^\sigma := (1 + \|\xi\|_x^2)^{\frac{\sigma}{2}}$.

Hence, we are left with studying the operator $[\text{Op}^M(-\tilde{a}), \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t, x)) + ib_{\varepsilon,1})]$. We first use Lemma 3.3.10 to recover its symbol:

$$\begin{aligned} & [\text{Op}^M(-\tilde{a}), \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t, x)) + ib_{\varepsilon,1})] \\ & \stackrel{(3.3.8)}{=} \text{Op}^M(-i\{-\tilde{a}, i\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t)) + ib_{\varepsilon,1}\}) \mod \Psi^{\sigma-1}(M) \\ & = \text{Op}^M(\{\xi(V(x) + \varepsilon \mathcal{P}_\varepsilon(t)) + b_{\varepsilon,1}, \tilde{a}\}) \mod \Psi^{\sigma-1}(M) \\ & = \text{Op}^M(\{\xi(V(x)), \tilde{a}\}) + \text{Op}^M(\varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t)) \mod \Psi^{\sigma-1}(M) \end{aligned} \quad (3.3.17)$$

where the remainder in $\Psi^{\sigma-1}(M)$ results from the symbolic calculus operations (see (3.3.8)) with all the seminorms of the symbols uniformly bounded in terms of $t \in \mathbb{R}$, $0 < \sigma \leq 1$ and $0 < \varepsilon < 1$. Here, we have denoted by $\mathcal{P}_{\tilde{a}}(t) := \{\xi(\mathcal{P}_\varepsilon(t, x)), \tilde{a}\}$ and $\mathcal{P}_{b,\varepsilon}(t) := \{b_{\varepsilon,1}, \tilde{a}\}$, just to shorten the notation. Remark that $\mathcal{P}_{\tilde{a}}(t) \in C_b^0(\mathbb{R}; \mathbf{S}^\sigma(M))$, that it depends implicitly on ε and that it is a homogeneous function of degree σ on $\{(x, \xi) : \|\xi\|_x \geq 2\}$ for all $t \in \mathbb{R}$. Similarly, $\mathcal{P}_{b,\varepsilon}(t)$ belongs to $\mathcal{C}^0(\mathbb{R}; \mathbf{S}^{\sigma+\rho-1}(M))$. Consider now a cutoff function $\chi_3 \in C_c^\infty(T^*M; [0, 1])$, such that

$$\text{supp}(\chi_3) \subseteq \{(x, \xi) : \|\xi\|_x \leq R_0 + 1\}, \quad \text{and} \quad \chi_3(x, \xi) \equiv 1 \text{ in } \{(x, \xi) : \|\xi\|_x \leq R_0\}, \quad (3.3.18)$$

where $R_0 \geq 2$ will be determined later on. We can rewrite

$$\begin{aligned} & \{\xi(V(x)), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma = (\{\xi(V(x)), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma)(1 - \chi_3) \\ & + (\{\xi(V(x)), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma) \chi_3. \end{aligned} \quad (3.3.19)$$

Since the cutoff function is compactly supported, we have from Lemma 3.3.3 that

$$\text{Op}^M((\{\xi(V(x)), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma) \chi_3) \in C_b^0(\mathbb{R}; \mathcal{S}^{-\nu}), \quad \forall \nu > 0, \quad (3.3.20)$$

with the involved constants in the semi-norms being uniform for $t \in \mathbb{R}$, $0 \leq \varepsilon \leq 1$, $0 \leq \sigma \leq 1$ and R_0 varying in a compact interval of $(0, \infty)$. Thus the right-hand side of

(3.3.16) reads:

$$\begin{aligned} & [\text{Op}^M(-\tilde{a}), \text{Op}^M(i\xi(V(x) + \varepsilon\mathcal{P}_\varepsilon(t, x)) + ib_{\varepsilon_1})] - C_b\varepsilon \text{Op}^M(\langle \xi \rangle^\sigma) \\ & \stackrel{(3.3.19)}{=} \stackrel{(3.3.20)}{=} \text{Op}^M((\{\xi(V(x)), \tilde{a}\} + \varepsilon\mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b\varepsilon\langle \xi \rangle^\sigma)(1 - \chi_3)) \mod \Psi^{\sigma-1}(M), \end{aligned} \quad (3.3.21)$$

where the remainder takes into account both the remainder in (3.3.17) and the compact operator in (3.3.20). One more time, all the seminorms of the symbols in this remainder are uniformly bounded in time and in terms of $0 < \varepsilon < 1$ and $0 < \sigma \leq 1$. Yet it is worth noting that the constants depend on $R_0 \geq 2$.

We now claim that, from (3.2.1) and (3.2.2), there exist $\varepsilon_0, \sigma_0 > 0$ small enough and a positive constant C_+ such that, for every $0 < \sigma \leq \sigma_0$, one can find $R_0 \geq 2$ so that, for every $0 \leq \varepsilon \leq \varepsilon_0\sigma^2$,

$$(\{\xi V(x), \tilde{a}\} + \varepsilon\mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b\varepsilon\langle \xi \rangle^\sigma)(1 - \chi_3) > -C_+\sigma\tilde{a}(1 - \chi_3), \quad \forall (x, \xi) \in T^*M, \forall t \in \mathbb{R}. \quad (3.3.22)$$

Let us postpone the proof of (3.3.22) and conclude the proof of (3.3.13). Applying Gårding inequality (see Lemma 3.3.11) with $0 < \sigma < 1$, we indeed obtain from (3.3.22) that, for every $0 < \varepsilon < \varepsilon_0\sigma^2$:

$$\text{Op}^M((\{\xi V(x), \tilde{a}\} + \varepsilon\mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b\varepsilon\langle \xi \rangle^\sigma)(1 - \chi_3)) \geq C_+\sigma \text{Op}^M(-\tilde{a}) - C_0\text{Id}, \quad (3.3.23)$$

where $C_0 > 0$ is the constant resulting from Gårding inequality and the remainder in $\Psi^{\sigma-1}(M)$. It is again uniformly bounded for $t \in \mathbb{R}$ but now depends on $0 < \sigma \leq 1$ (through its dependence in R_0). Plugging (3.3.21) and (3.3.23) in (3.3.16) we obtain

$$\frac{d}{dt}\mathcal{A}(t) \geq C_+\sigma \langle \text{Op}^M(-\tilde{a})u, u \rangle - (C_K + C_0)\|u(t)\|_{L^2}^2, \quad (3.3.24)$$

with $C_K > 0$ taking into account the compact remainder from (3.3.20). Observe that again this remainder depends implicitly on $R_0 \geq 2$ and thus on $0 \leq \sigma \leq \sigma_0$. See Remark 3.3.15 for discussion on this dependence. This concludes the proof, since (3.3.24) is (3.3.13) with $\tilde{\alpha} := C_+$ and $\tilde{\beta}_\sigma := C_K + C_0 > 0$.

We are left to prove (3.3.22). To this aim, we first recall that, from the properties of χ_3 in (3.3.18), we only have to verify (3.3.22) in $\{(x, \xi) \in T^*M : \|\xi\|_x \geq R_0\}$, since

$(1 - \chi_3) \equiv 0$ otherwise. We thus claim that

$$\{\xi V(x), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma > -C_+ \sigma \tilde{a}, \quad \text{in } \{(x, \xi) \in T^*M : \|\xi\|_x \geq R_0\}, \quad (3.3.25)$$

for some $C_+ > 0$ (depending only V) and $0 < \varepsilon < \varepsilon_0 \sigma^2$ with ε_0 to be determined. Clearly (3.3.25) implies (3.3.22) since $(1 - \chi_3) \geq 0$ everywhere. We remark that in this remaining region $\tilde{a} \equiv a$, where a is the homogeneous escape function given by Proposition 3.2.4. To prove (3.3.25) we split the domain as

$$\{(x, \xi) \in T^*M : \|\xi\|_x \geq R_0\} = \underbrace{\{(x, \xi) \in E : \|\xi\|_x \geq R_0\}}_{=: \mathcal{E}} \cup \underbrace{\{(x, \xi) \in T^*M \setminus E : \|\xi\|_x \geq R_0\}}_{=: \mathcal{I}}, \quad (3.3.26)$$

where E is the set appearing in the Definition 3.2.2 of the escape function. First, we look at the set $\mathcal{I}$. Using (3.2.1) (with $\delta = c_0 \sigma$ given by Proposition 3.2.4), we have that

$$\{\xi V(x), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma \geq (c_0 \sigma - \varepsilon(M_2 + 2C_b) - \tilde{C}_b R_0^{\rho-1}) \|\xi\|_x^\sigma, \quad (3.3.27)$$

where $\tilde{C}_b > 0$ is a constant (uniform in $0 \leq \varepsilon \leq 1$ and $0 \leq \sigma \leq 1$) that depends on the seminorms of b_ε and where $M_2 := \max_{t \in \mathbb{R}, (x, \xi) \in \mathcal{I}} \left| \mathcal{P}_{\tilde{a}} \left(t, x, \frac{\xi}{\|\xi\|} \right) \right| > 0$ is again uniform for $0 < \sigma \leq 1$. Moreover, using again homogeneity of a , one has $a(x, \xi) \geq -C_a \|\xi\|_x^\sigma$ for some $C_a > 0$ (that is uniform for $0 \leq \sigma \leq 1$) and for all $(x, \xi) \in \mathcal{I}$. Thus choosing $\varepsilon_{\mathcal{I}} := \frac{c_0}{4(M_2 + 2C_b)} > 0$ and $R_0^{\rho-1} \leq \frac{c_0 \sigma}{4\tilde{C}_b}$, we obtain from (3.3.27) that, for all $0 < \varepsilon < \varepsilon_{\mathcal{I}} \sigma$ we have:

$$\{\xi(V(x)), \tilde{a}\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma \geq \frac{c_0 \sigma}{4} \|\xi\|_x^\sigma \geq -\frac{c_0}{4C_a} \sigma a.$$

Finally, from condition (3.2.2) (with $\delta = c_0 \sigma$ from Proposition 3.2.4), we see that, in $\mathcal{E}$ (see (3.3.26)):

$$\tilde{a} \equiv a \geq c_0 \sigma \|\xi\|_x^\sigma$$

and

$$\{\xi V(x), a\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma \stackrel{(3.2.2)}{\geq} -(\varepsilon(M_1 + C_b) + \tilde{C}_b R_0^{\rho-1}) \|\xi\|_x^\sigma,$$

where $M_1 := \max_{t \in \mathbb{R}, (x, \xi) \in \mathcal{E}} \left| \mathcal{P}_{\tilde{a}} \left(t, x, \frac{\xi}{\|\xi\|_x} \right) \right| \geq 0$ (since $\mathcal{P}_{\tilde{a}}(t)$ is positively homogeneous of degree σ in $\mathcal{E}$) is uniformly bounded with respect to the parameter $0 < \sigma \leq 1$. Thus

choosing $\varepsilon_{\mathcal{E}} := \frac{c_0^2}{8(M_1+C_b)C_a}$ and $R_0^{\rho-1} \leq \frac{c_0^2\sigma^2}{8C_bC_a}$, we obtain that, for all $0 < \varepsilon < \varepsilon_{\mathcal{E}}\sigma^2$,

$$\{\xi V(x), a\} + \varepsilon \mathcal{P}_{\tilde{a}}(t) + \mathcal{P}_{b,\varepsilon}(t) - C_b \varepsilon \langle \xi \rangle^\sigma \geq -\frac{c_0^2\sigma^2}{4C_a} \|\xi\|_x^\sigma \geq -\frac{c_0\sigma}{4C_a} a.$$

This proves (3.3.25) with $C_+ := \frac{c_0}{4C_a} > 0$ and $\varepsilon_0 := \min\{\varepsilon_{\mathcal{I}}, \varepsilon_{\mathcal{E}}\} > 0$, concluding the proof of the energy estimate (3.3.13).

Remark 3.3.15. Observe that, the smaller σ is, the larger $R_0 \geq 2$ has to be. If $b_{\varepsilon,1}$ was of the form $\varepsilon \tilde{b}_{\varepsilon,1}$ with $\tilde{b}_{\varepsilon,1} \in \mathcal{C}_b^0(\mathbb{R}; \mathbf{S}^\rho(M))$ having all its seminorms uniformly bounded in terms of $0 \leq \varepsilon \leq 1$, one could in fact pick $R_0 = 2$ up to shrinking the value of ε_0 . In particular, $\tilde{\beta}_\sigma$ could be chosen uniformly in terms of $0 \leq \sigma \leq \sigma_0$. Finally, note that the choice of the parameter ε_0 depends not only on V and $\mathcal{P}_\varepsilon$ but also on $b_{\varepsilon,2}$.

3.3.5 Initial data and end of the proof of Theorem 3.3.13

We are now ready to prove Theorem 3.3.13. According to Equation (3.3.15), one has that, for any $0 < \sigma \leq \sigma_0$, for any $0 < \varepsilon \leq \sigma^2 \varepsilon_0$ and for any initial datum $u_0 \in L^2$,

$$\mathcal{A}(t) \geq e^{\tilde{\alpha}\sigma t} (\mathcal{A}(0) - \beta_{\sigma,\varepsilon} \|u_0\|_{L^2}^2) + \beta_{\sigma,\varepsilon} \|u(t)\|_{L^2}^2, \quad \forall t \geq 0. \quad (3.3.28)$$

Cauchy-Schwarz inequality together with (3.3.14) now gives:

$$\mathcal{A}(t) \stackrel{(3.3.12)}{=} \langle \text{Op}^M(-\tilde{a})u(t, x), u(t, x) \rangle_{L^2(M)} \leq \|\text{Op}^M(-\tilde{a})u\|_{L^2(M)} e^{C_b \varepsilon t} \|u_0\|_{L^2}. \quad (3.3.29)$$

Since $\tilde{a} \in \mathbf{S}^\sigma(T^*M)$, Lemma 3.3.10 (inequality (3.3.7)) gives:

$$\|\text{Op}^M(\tilde{a})u\|_{L^2(M)} \leq C_{\tilde{a}} \|u\|_{\mathcal{H}^\sigma}, \quad \forall u \in C^\infty(M). \quad (3.3.30)$$

Thus, plugging (3.3.30) and (3.3.29) in (3.3.28), we have, for every $0 < \sigma \leq \sigma_0 < 1$,

$$C_{\tilde{a}} \|u\|_{\mathcal{H}^\sigma} \|u_0\|_{L^2} \geq e^{(\tilde{\alpha}\sigma - C_b \varepsilon)t} (\mathcal{A}(0) - \beta_{\sigma,\varepsilon} \|u_0\|_{L^2}^2) \geq e^{(\tilde{\alpha} - C_b \varepsilon_0 \sigma) \sigma t} (\mathcal{A}(0) - \beta_{\sigma,\varepsilon} \|u_0\|_{L^2}^2). \quad (3.3.31)$$

Hence, we will be done with the proof of Theorem 3.3.13 if we are able to find initial datum u_0 such that

$$F(u_0) := -(\mathcal{A}(0) - \beta_{\sigma,\varepsilon} \|u_0\|_{L^2}^2) \equiv \langle \text{Op}^M(\tilde{a})u_0, u_0 \rangle_{L^2(M)} + \beta_{\sigma,\varepsilon} \|u_0\|_{L^2}^2 < 0. \quad (3.3.32)$$

Remark 3.3.16. Note that the proof is valid for any $0 < \sigma < \sigma_0$ with σ_0 given by Proposition 3.2.4. Yet, since for every $s \geq \sigma$, we have $\|u\|_s \geq \|u\|_\sigma$, exponential blow up of the Sobolev norm of the solutions to (3.3.9) holds for any $s \geq \sigma$.

Let us now construct u_0 verifying (3.3.32). To that aim, we recall property (3.2.3) of the escape function a , namely that it is $\leq -c_0\|\xi\|^\sigma/2$ in some non empty³ closed conic subset C . Hence, we will construct initial data which are microlocalized in this region where $a < 0$. To that aim, we fix $\chi \in \mathcal{C}^\infty(M; [0, 1])$ such that the support of χ intersects C and such that the support of χ is contained in an isochore local chart $\gamma : \mathcal{U}_j \subset M \rightarrow B(0, 1) := \{x \in \mathbb{R}^n : \|x\| < 1\}$. In order to work with the Weyl quantization on $\mathbb{R}^n$, we will consider initial data of the form $u_0 = \chi v_0 \circ \gamma$ with $v_0 \in \mathcal{C}_c^\infty(B(0, 1))$. This allows us to write

$$\langle \text{Op}^M(\tilde{a})u_0, u_0 \rangle_{L^2(M)} = \langle \gamma^{-1*} \chi \text{Op}^M(\tilde{a}) \chi \gamma^* v_0, v_0 \rangle_{L^2(\mathbb{R}^n)}.$$

Thanks to the rule for the change of coordinates for pseudodifferential operators [86, Th. 9.9], one finds that

$$\langle \text{Op}^M(\tilde{a})u_0, u_0 \rangle_{L^2(M)} \leq \langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma})v_0, v_0 \rangle_{L^2(\mathbb{R}^n)} + C\|v_0\|_{L^2(\mathbb{R}^n)}^2, \quad (3.3.33)$$

for some constant C that depends only on the coordinate chart, on $\tilde{a}$ and on the choice of the quantization procedure Op^M and with $\tilde{\gamma}$ defined in (3.3.3). We now fix some $\xi_0 \neq 0$ such that ξ_0 lies in the region where $(\chi^2 \tilde{a}) \circ \tilde{\gamma} \leq -c_\gamma \|\xi\|^\sigma$ and we set

$$\tilde{v}_h(x) = \chi_0(x) e^{\frac{i\xi_0 \cdot x}{h}}, \quad 0 < h \leq 1,$$

with $\chi_0 \in \mathcal{C}_c^\infty(B(0, 1))$ which is not identically 0. With this definition at hand, one has

Lemma 3.3.17. *There exists a constant $C_0 > 0$ (depending on ξ_0 and χ_0) such that, for every $0 < h \leq 1$ and for every $0 \leq \sigma \leq 1$, one has*

$$C_0^{-1} \leq \|\tilde{v}_h \circ \gamma\|_{L^2} \leq C_0, \text{ and } \|\tilde{v}_h \circ \gamma\|_\sigma \leq C_0 h^{-\sigma}. \quad (3.3.34)$$

Proof. Regarding the L^2 -norm, one has

$$\|\tilde{v}_h\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\chi_0(x)|^2 dx,$$

³This is exactly for this reason that we need to impose that V has a critical point when $n = 1$.

from which we deduce the expected equality as we have chosen an isochore chart. Regarding the Sobolev norm, one has

$$\|\tilde{v}_h\|_\sigma^2 = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^\sigma \left| \int_{\mathbb{R}^n} \chi_0(x) e^{ix \cdot (\frac{\xi_0}{h} - \xi)} dx \right|^2 d\xi,$$

where $\|\cdot\|_\sigma$ denotes here the Sobolev norm in $\mathbb{R}^n$ (endowed with the standard Euclidean metric). Hence, denoting by $\hat{\chi}_0$ the Fourier transform of χ_0 , one finds

$$\|\tilde{v}_h\|_\sigma^2 = \int_{\mathbb{R}^n} \left(1 + \left\| \xi + \frac{\xi_0}{h} \right\|^2 \right)^\sigma |\hat{\chi}_0(\xi)|^2 d\xi \leq 2^\sigma \left(1 + 2 \frac{\|\xi_0\|^2}{h^2} \right)^\sigma \|\chi_0\|_\sigma.$$

The conclusion follows then from the fact that the Sobolev norm induced by the Riemannian metric g on M and the one induced by the Euclidean metric in the local chart are uniformly equivalent (with constants that can be made uniform in terms of $0 \leq \sigma \leq 1$). $\square$

Remark 3.3.18. Recall that $\tilde{a}$ is σ homogeneous for $\|\xi\|_x \geq 2$ with $0 < \sigma < 1$. Hence, its pullback to $T^*\mathbb{R}^n$ is also σ -homogeneous for $\|\xi\|$ large enough, say R_0 (where $\|\cdot\|$ now denotes the Euclidean norm in $\mathbb{R}^n$). Hence, we pick $\|\xi_0\| > 2R_0$ in order to have this covector in the region where the pullback symbol $(\chi^2 \tilde{a}) \circ \tilde{\gamma}$ is σ -homogeneous.

We now study the behavior of

$$\langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)}, \text{ as } h \rightarrow 0^+. \quad (3.3.35)$$

Observing that the L^2 norm of $\tilde{v}_h$ is uniformly bounded in terms of $0 < h \leq 1$, we will be done with the proof of (3.3.32) if we can prove that (3.3.35) tends to $-\infty$ as $h \rightarrow 0^+$. Indeed, from Cauchy-Schwarz inequality and (3.3.33), we will obtain (3.3.32). This is the content of the next Lemma.

Lemma 3.3.19. *With the above conventions, there exist $0 < c_1, h_1 < 1$ such that, for every $0 < h < h_1$ and for every $0 \leq \sigma \leq 1$,*

$$\langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} \leq -c_1 h^{-\sigma} \leq -c_1 C_0^{-2} \|\tilde{v}_h\|_\sigma \|\tilde{v}_h\|_{L^2}.$$

Note that the last inequality follows from (3.3.34) (thus the constant C_0 depends on ξ_0 and χ_0). The above discussion combined with this Lemma yields that, for $h > 0$

small enough (depending on $0 < \sigma \leq 1$),

$$F(\tilde{v}_h \circ \gamma) \leq -\frac{c_1 C_0^{-2}}{2} \|\tilde{v}_h \circ \gamma\|_\sigma \|\tilde{v}_h \circ \gamma\|_{L^2}. \quad (3.3.36)$$

As already explained, this yields the proof of Theorem 3.3.13 with $v_\varepsilon = \tilde{v}_h \circ \gamma$. More precisely, we have shown the existence of a constant $\delta_1 > 0$ such that, for every $0 < \sigma \leq \delta_1$ and for every $0 < \varepsilon \leq \delta_1 \sigma$, one can find $v_\varepsilon(0) \in \mathcal{H}^\sigma \setminus \{0\}$ (even $\mathcal{C}^\infty$) such that

$$\forall t > 0, \quad \|v_\varepsilon(t)\|_\sigma \geq \delta_1 (e^{\sigma \delta_1 t} \|v_\varepsilon(0)\|_\sigma + \|v_\varepsilon(0)\|_{L^2}), \quad (3.3.37)$$

where we have picked $\delta_1 := \min\{\tilde{\alpha}, \tilde{\beta}/(C_{\tilde{a}} \tilde{\alpha}), c_1 C_0^{-2}/(2C_{\tilde{a}})\}$ and where $v_\varepsilon(t)$ is the solution to (3.3.9) with initial datum v_ε .

We will explain below why this remains true in a small neighborhood of $v_\varepsilon(0) = \tilde{v}_h \circ \gamma$ and thus we will be done with the proof of Theorem 3.3.13.

Proof of Lemma 3.3.19. We will make use of semiclassical pseudodifferential calculus as in [86]. First, we introduce ψ in $\mathcal{C}_c^\infty((-2, 2); [0, 1])$ which is identically equal to 1 on $[-1, 1]$ and we set $\psi_R(\xi) := \psi(\|\xi\|/R)$. Using non stationary phase Lemma [86] and that $\|\xi_0\| > 2R_0$, one can prove that $\|\text{Op}^w(\psi_{R_0 h^{-1}}) \tilde{v}_h\|_{L^2} = \mathcal{O}(h)$. Now using Calderon-Vaillancourt property 3.3.2 with $s = \rho_1 = \sigma$ and Lemma 3.3.17, one finds that

$$\|\text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h\|_{L^2} \leq \tilde{C}_a \|\tilde{v}_h\|_\sigma = \mathcal{O}(h^{-\sigma}).$$

Hence, gathering these two properties with the Cauchy-Schwarz inequality, one finds that

$$\langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} = \langle \text{Op}^w(1 - \psi_{R_0 h^{-1}}) \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^{1-\sigma}).$$

Using the composition rule for pseudodifferential operators together with (3.3.2) and the fact that $\|\tilde{v}_h\|_{L^2} = \mathcal{O}(1)$, we get

$$\langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} = \langle \text{Op}^w((1 - \psi_{R_0 h^{-1}})(\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} + \mathcal{O}(1).$$

Thanks to the introduction of the cutoff $(1 - \psi_{R_0 h^{-1}})$, the function $(\chi^2 \tilde{a}) \circ \tilde{\gamma}$ is now σ -homogeneous in the support of our cutoff function. Hence, using the semiclassical quantization Op_h^w as in [86, Ch.4], one finds

$$\langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} = h^{-\sigma} \langle \text{Op}_h^w((1 - \psi_{R_0})(\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} + \mathcal{O}(1).$$

Let now $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ which is equal to 1 near ξ_0 . Another application of the non stationary phase lemma yields that

$$\langle \text{Op}^w((\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} = h^{-\sigma} \langle \text{Op}_h^w(\tilde{\psi}(1 - \psi_{R_0})(\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} + \mathcal{O}(1).$$

The conclusion follows then from an application of the stationary phase Lemma (see [86, Ch. 5, Ex. 2] for the exact same integral) which gives

$$\langle \text{Op}_h^w(\tilde{\psi}(1 - \psi_{R_0})(\chi^2 \tilde{a}) \circ \tilde{\gamma}) \tilde{v}_h, \tilde{v}_h \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^d} (\chi^2 \tilde{a}) \circ \tilde{\gamma}(x, \xi_0) \chi_0^2(x) dx + o(1).$$

By construction, this quantity is negative thanks to (3.2.3) as $h \rightarrow 0^+$. $\square$

We can now conclude the proof of Theorem 3.3.13.

Proof of Theorem 3.3.13. Thanks to (3.3.37), we have already the candidate for the center of the ball $v_\varepsilon(0) \neq 0$ in $\mathcal{H}^\sigma$ and it remains to prove that exponential growth is true in a small neighborhood of $v_\varepsilon(0)$. We let $\|u_\varepsilon(0) - v_\varepsilon(0)\|_\sigma \leq r$ with $r > 0$ to be determined. Thanks to (3.3.32), one has exponential growth if $F(u_\varepsilon(0)) < 0$. By continuity of the map F on $\mathcal{H}^\sigma$ and thanks to (3.3.36), one has that, for $r > 0$ small enough,

$$F(u_\varepsilon(0)) \leq \frac{F(v_\varepsilon(0))}{2} \leq -\frac{c_1 C_0^{-2}}{4} \|v_\varepsilon(0)\|_\sigma \|v_\varepsilon(0)\|_{L^2}.$$

Hence, using (3.3.31), one finds that, for $t \geq 0$,

$$C_{\tilde{a}} \|u_\varepsilon(t)\|_\sigma \|u_\varepsilon(0)\|_{L^2} \geq e^{\tilde{\alpha}\sigma t} \frac{c_1 C_0^{-2}}{4} \|v_\varepsilon(0)\|_\sigma \|v_\varepsilon(0)\|_{L^2}.$$

Choosing $r \leq \|v_\varepsilon(0)\|_{L^2}/2$, one has, for every $0 \leq \sigma \leq 1$,

$$\|u_\varepsilon(0)\|_\sigma \leq r + \|v_\varepsilon(0)\|_\sigma \leq \frac{3}{2} \|v_\varepsilon(0)\|_\sigma,$$

from which we infer the expected result

$$C_{\tilde{a}} \|u_\varepsilon(t)\|_\sigma \geq e^{\tilde{\alpha}\sigma t} \frac{c_1 C_0^{-2}}{9} \|u_\varepsilon(0)\|_\sigma.$$

$\square$

Remark 3.3.20. Observe that our proof dealt with the general equation (3.3.9) whose principal symbol is given by a (time-dependent) transport perturbation term. In fact, instead of $\varepsilon \xi(\mathcal{P}(t, x))$, we could also have considered perturbations of the form $\varepsilon p_\varepsilon(t, x, \xi)$ with $p_\varepsilon \in \mathcal{C}_b^0(\mathbb{R}; \mathbf{S}^1(M))$ with p_ε real valued and with all the seminorms of p_ε uniformly bounded in terms of $0 \leq \varepsilon \leq 1$.

3.4 Application to periodic transport equations

In this last section we prove Theorem 3.0.1 and its extension in dimension $n > 2$. Let us recall the equation we consider (see equation (3.0.1)):

$$\partial_t u = (\nu + \varepsilon V(t, x)) \cdot \nabla_x u + \frac{\varepsilon}{2} \operatorname{div}(V(t, x))u, \quad x \in \mathbb{T}^n, t \in \mathbb{R}, \quad (3.4.1)$$

where $\nu \in \mathbb{N}_+^n$, $V(t, x) \in C^\infty(\mathbb{T} \times \mathbb{T}^n)$ and $\operatorname{div}(\cdot)$ denotes the divergence with respect to the Euclidean volume on $\mathbb{T}^n$. In this section, we often drop the index ε for the solution of equation (3.4.1) and we simply use $u(t, x)$ even if the solution depends on the parameter $\varepsilon > 0$ (which requires some attention at certain steps).

Let us start by recalling the scheme of the proof (see Section 1.2.2 for more details), in order to explain why this result comes as an application of Theorem 3.0.2. We first perform a *resonant normal form reduction* adapted from [10], conjugating equation (3.4.1) to an equation with the same structure as (3.3.9), but with leading transport term which is completely resonant with respect to the frequency ν (see Definition (3.4.1)). This reduces the study of instabilities of equation (3.4.1), to that of an equation analogous to the one previously studied in this Chapter. Indeed, after the normal form procedure, the transport term $\nu + \varepsilon V(t, x)$ in (3.4.1) is reduced to a time independent one (up to small-in-size remainders), given by the resonant average

$$\langle V \rangle_\nu(x) := \frac{1}{2\pi} \int_0^{2\pi} V(\tau, x - \tau\nu) d\tau \in C^\infty(\mathbb{T}^n)$$

of V . Equivalently, $\langle V \rangle_\nu$ is obtained selecting only the resonant Fourier modes of V (see (3.4.3)) with respect to the periodic flow $\varphi^\tau(x) = x + \tau\nu$, $x \in \mathbb{T}^n$. Loosely speaking, these Fourier modes are the ones provoking the unstable dynamics.

Next, using Theorem 3.0.2, we can identify the set of perturbations V such that their resonant average $\langle V \rangle_\nu$ will provoke unstable dynamics. Finally, we show that, for $n = 2$, such set of potentials is generic. Let us remark that it is for this last part of the proof that we need to ask that $M = \mathbb{T}^2$. We are indeed able to perform the resonant normal form on all tori, i.e. $M = \mathbb{T}^n$, for all $n \in \mathbb{N}$, but in order to achieve genericity, we need to use the fact that Morse-Smale vector fields are generic in compact manifolds of dimension two (see Peixoto [78] and Remark 3.1.5). This is the reason why the only

possible choice (through our approach) to state Theorem 3.0.1 is to pick $M = \mathbb{T}^2$ if we aim at proving genericity among smooth perturbations.

3.4.1 The resonant normal form

In this section, we perform the normal form reduction. We use the ideas from [10] and we adapt them to our context. First of all, given a C^∞ function $V : \mathbb{T} \times \mathbb{T}^n \rightarrow \mathbb{R}^n$, with components $V(t, x) := (V_1(t, x), \dots, V_n(t, x))$, we use the standard compact notation for the Fourier coefficients of V :

$$V(t, x) := \sum_{\substack{k \in \mathbb{Z}^n \\ \ell \in \mathbb{Z}}} \underbrace{v_{k, \ell}}_{\in \mathbb{C}^n} e^{i(k \cdot x + \ell t)}, \quad \forall (t, x) \in \mathbb{T} \times \mathbb{T}^n, \quad (3.4.2)$$

where $\{v_{k, \ell}\}_{k, \ell}$ are the Fourier coefficients of V . We recall the definition of resonant average of a function V .

Definition 3.4.1 (Resonant average). Let $V(t, x) \in C^\infty(\mathbb{T}^{n+1}; \mathbb{R}^n)$ and recall the notation in (3.4.2) for the Fourier coefficients of V . For any $\nu \in \mathbb{N}_+^n$, we define the *resonant average* of V with respect to the frequency vector ν as

$$\langle V \rangle_\nu(x) := \sum_{k \in \mathbb{Z}^n} v_{k, \nu \cdot k} e^{ik \cdot x}, \quad x \in \mathbb{T}^n. \quad (3.4.3)$$

Remark 3.4.2. One can verify that this definition coincides with the earlier convention,

$$\langle V \rangle_\nu(x) := \frac{1}{2\pi} \int_0^{2\pi} V(t, x - \nu t) dt.$$

The main result of this section is the following Proposition:

Proposition 3.4.3. *There exists $\varepsilon_0 > 0$ sufficiently small such that, for every $0 \leq \varepsilon \leq \varepsilon_0$, there exists an invertible linear map*

$$\Phi_\varepsilon(t) : \mathcal{D}'(\mathbb{T}^n) \rightarrow \mathcal{D}'(\mathbb{T}^n),$$

and a vector field $W_{\nu, \varepsilon} \in C^\infty(\mathbb{T}^{n+1}; \mathbb{R}^n)$, with all its seminorms uniformly bounded in terms of $\varepsilon \in [0, \varepsilon_0]$ such that the following properties hold:

1. $\Phi_\varepsilon(t)$ is 2π -periodic,

2. for all $\sigma \in \mathbb{R}_+$, there exists $C_\sigma > 0$ such that, for all $0 \leq \varepsilon \leq \varepsilon_0$ and for all $t \in \mathbb{R}$

$$\|\Phi_\varepsilon(t)\|_{\mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma} + \|\Phi_\varepsilon(t)^{-1}\|_{\mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma} \leq C_\sigma,$$

3. the constant C_σ is uniformly bounded when σ lies in a compact interval,

4. for any solution $u_\varepsilon(t, x)$ to equation (3.4.1), the function $v_\varepsilon(t, x)$ defined by $u_\varepsilon(t, x) := \Phi_\varepsilon(t)v_\varepsilon(t, x)$ satisfies

$$\partial_t v_\varepsilon(t, x) = \text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W_{\nu, \varepsilon}(t, x)))v_\varepsilon(t, x). \quad (3.4.4)$$

Remark that equation (3.4.4) is a particular case of (3.3.9) with $b_\varepsilon = 0$. Before proving this proposition, let us write the following corollary that motivates the choice of the normal form and that allows to derive unstable solutions to (3.0.1) from the unstable solutions arising in Theorem 3.0.2 (as soon as $\langle V \rangle_\nu$ is Morse-Smale).

Corollary 3.4.4. *Let $u_\varepsilon(t, x)$ be a solution of (3.4.1) and let $v_\varepsilon(t, x)$ be the corresponding solution of (3.4.4) (i.e. $u_\varepsilon = \Phi_\varepsilon(t)v_\varepsilon$ with $\Phi_\varepsilon(t)$ given in Proposition 3.4.3). If*

$$\|v_\varepsilon(t)\|_\sigma \geq \tilde{\delta}_0 e^{\tilde{\delta}_0 \sigma \varepsilon t} \|v_\varepsilon(0)\|_\sigma, \quad \text{for some } \tilde{\delta}_0 > 0, \sigma \geq 0, \quad (3.4.5)$$

then one has

$$\|u_\varepsilon(t)\|_\sigma \geq C_\sigma^{-2} \tilde{\delta}_0 e^{\tilde{\delta}_0 \sigma \varepsilon t} \|u_\varepsilon(0)\|_\sigma, \quad \text{for some } \tilde{\delta}_0 > 0, \sigma \geq 0,$$

where C_σ is the constant from Proposition 3.4.3.

Equivalently, the behavior of the Sobolev norms of solutions to (3.4.4) is the same as that to (3.4.1), up to constants. Recall that the existence of solutions to (3.4.4) verifying (3.4.5) follows from Theorem 3.3.13 as soon as $\langle V \rangle_\nu$ has the Morse-Smale property:

Theorem 3.4.5. *Let $\omega = 1$ and $\nu \in \mathbb{N}^2 \setminus \{0\}$. Suppose that $\langle V \rangle_\nu$ satisfies the Morse-Smale property and, if $n = 1$, suppose in addition that $\langle V \rangle_\nu$ has at least one critical point.*

Then, there exists $\delta_0 > 0$ such that, for every $0 \leq \sigma \leq \delta_0$ and for every $0 \leq \varepsilon \leq \delta_0 \sigma^2$, one can find $v_\varepsilon \in \mathcal{H}^\sigma(\mathbb{T}^2)$ and $r := r(\varepsilon, \sigma, v_\varepsilon) > 0$ such that

$$\|u_\varepsilon(0) - v_\varepsilon\|_\sigma \leq r \implies \|u_\varepsilon(t)\|_\sigma \geq \delta_0 e^{\delta_0 \sigma t \varepsilon} \|u_\varepsilon(0)\|_\sigma, \quad \text{for any } t > 0,$$

where $u_\varepsilon(t)$ is the solution to equation (3.0.1) with initial datum $u_\varepsilon(0)$.

We now turn to the proof of Proposition 3.4.3.

Preliminary constructions

As already explained, we drop the index ε at several places (e.g. u, v, Φ instead of $u_\varepsilon, v_\varepsilon, \Phi_\varepsilon$) in view of alleviating the notations. We now report some preliminary remarks and notation that we introduce following [10, Section 3.1]. Some of these definitions were already presented in Section 1.2.2. For convenience of the reader we collect here all the material needed to avoid any confusion. First of all, recalling the definition in (3.0.3) of the Sobolev spaces $\mathcal{H}^\sigma$, $\sigma \in \mathbb{R}$, let $A(t) : \mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma$ be any linear invertible transformation, periodic in time of period 2π . For any differential equation of the form $\partial_t u = Hu$, defining the function $v(t, x)$ as $u = A(t)v$, one can check that v solves $\dot{v} = A(t)_* H v$, where the pushforward $A(t)_* H$ is given by

$$A(t)_* H := A(t)^{-1} [H A(t) - \partial_t A(t)]. \quad (3.4.6)$$

We remark, to avoid any confusion in the notation, that $\partial_t A(t)$ is the derivative in time of the linear transformation (and not the composition $\partial_t \circ A(t)$). The core of the proof of Proposition 3.4.3 consists in choosing the proper transformation $A(t)$ and computing the corresponding pushforward in (3.4.6). To this aim, we consider a family of diffeomorphisms of the torus of the form

$$\varphi(t) : \mathbb{T}^n \rightarrow \mathbb{T}^n, \quad x \mapsto x + \varepsilon \beta(t, x), \quad (3.4.7)$$

where $\beta \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ is a function to be determined. It has been proved (see for example [2, Lemma B.4]) that, for $\varepsilon > 0$ small enough, this diffeomorphism is invertible and its inverse has the form

$$\varphi(t)^{-1} : \mathbb{T}^n \rightarrow \mathbb{T}^n, \quad y \mapsto y + \varepsilon \tilde{\beta}_\varepsilon(t, y), \quad (3.4.8)$$

for $\tilde{\beta}_\varepsilon \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$. Moreover, $\tilde{\beta}_\varepsilon$ has all its derivatives uniformly bounded in terms of $\varepsilon > 0$ in the admissible range of ε . Note that β is chosen independently of ε in our construction and that φ depends implicitly on ε .

We define the transformation associated with (3.4.7) as:

$$\tilde{\Phi}(t)u(t, x) := \det(\text{Id} + \varepsilon \nabla_x \beta(t, x))^{\frac{1}{2}} u \circ \varphi(t, x), \quad \forall u \in C^\infty(\mathbb{T}; \mathcal{H}^\sigma(\mathbb{T}^n)) \quad \forall t \in \mathbb{T}, \forall \sigma \in \mathbb{R}, \quad (3.4.9)$$

with inverse

$$\tilde{\Phi}(t)^{-1}u(t, x) := \det(\text{Id} + \varepsilon \nabla_x \tilde{\beta}_\varepsilon(t, x))^{\frac{1}{2}} u \circ \varphi^{-1}(t, x), \quad \forall u \in C^\infty(\mathbb{T}; \mathcal{H}^\sigma(\mathbb{T}^n)) \quad \forall t \in \mathbb{T}, \forall \sigma \in \mathbb{R}. \quad (3.4.10)$$

Remark that

$$\det(\text{Id} + \varepsilon \nabla_y \tilde{\beta}_\varepsilon(t, y)) = \frac{1}{\det(\text{Id} + \varepsilon \nabla_x \beta(t, x))|_{x=y+\varepsilon \tilde{\beta}(t, y)}}, \quad \forall y \in \mathbb{T}^n. \quad (3.4.11)$$

Using (3.4.11) and via a direct computation one can check that

$$\tilde{\Phi}(t)^* = \tilde{\Phi}(t)^{-1}, \quad \forall t \in \mathbb{T}, \quad (3.4.12)$$

where $*$ denotes the adjoint with respect to the standard inner product on L^2 . Finally, defining

$$H(t) := (\nu + \varepsilon V(t, x)) \cdot \nabla_x + \frac{\varepsilon}{2} \text{div}(V(t, x)), \quad (3.4.13)$$

where ν and V are as in (3.4.1), (so that equation (3.4.1) reads $\dot{u} = H(t)u$), and using formula (3.4.6), we see that the pushforward of $H(t)$ in (3.4.13) with respect to $\tilde{\Phi}(t)$ in (3.4.9) is given by

$$\tilde{\Phi}(t)_* H = \tilde{\Phi}(t)^{-1} H \tilde{\Phi}(t) - \tilde{\Phi}(t)^{-1} \partial_t \tilde{\Phi}(t). \quad (3.4.14)$$

The following Lemma computes explicitly this pushforward.

Lemma 3.4.6. *Let H as in (3.4.13) and $\tilde{\Phi}$ as in (3.4.9). Then the order one differential operator $\tilde{\Phi}(t)^{-1} H \tilde{\Phi}(t)$ is skew-adjoint and the following identities hold:*

$$\tilde{\Phi}(t)^{-1} H \tilde{\Phi}(t) = \text{Op}^w(i\xi(\nu + \varepsilon V(t, x) + \varepsilon \nu \cdot \nabla_x \beta + \varepsilon^2 \mathcal{R}_\varepsilon^1(t, x))), \quad (3.4.15)$$

and

$$\tilde{\Phi}(t)^{-1} \partial_t \tilde{\Phi}(t) = \text{Op}^w(i\xi(\varepsilon \partial_t \beta + \varepsilon^2 \mathcal{R}_\varepsilon^2(t, x))), \quad (3.4.16)$$

with $\mathcal{R}_\varepsilon^{1/2}(t, x) \in C^\infty(\mathbb{T}^{n+1}; \mathbb{R})$ having all their seminorms uniformly bounded in terms of $\varepsilon \in [0, \varepsilon_0]$.

Here, we use the formalism of pseudodifferential operators (see section 3.3.1) in order to identify the symbols of $\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t)$ and $\tilde{\Phi}(t)^{-1}\partial_t\tilde{\Phi}(t)$, that we will need to prove Proposition 3.4.3. We remark that on the torus it is possible to use the Weyl quantization exactly as over $\mathbb{R}^n$ and that all properties in Lemma 3.3.1 hold. See [86, Ch. 5] for details.

Proof. First of all remark that, since $H(t)$ defined as in (3.4.13) is skew-adjoint, using (3.4.12) we have

$$[\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t)]^* = \tilde{\Phi}(t)^*H^*(\tilde{\Phi}(t)^{-1})^* \stackrel{(3.4.12)}{=} -\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t),$$

which proves skewadjointness of $\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t)$. Moreover, from its definition, $\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t)$ is a real valued differential operator of order one. Thus, from point 2 of Lemma 3.3.1, we have

$$\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t) = \text{Op}^w(i\xi(f_\varepsilon(t, x))), \quad (3.4.17)$$

for some $f_\varepsilon \in C^\infty(\mathbb{T}^{n+1})$, real valued. We now compute such a function f_ε in order to prove (3.4.15). For every $u \in C^\infty(\mathbb{T}^{n+1})$ we have

$$\begin{aligned} \tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t)u(t, x) &= \tilde{\Phi}(t)^{-1}[(\nu + \varepsilon V(t, x)) \cdot \nabla_x (\det(\text{Id} + \varepsilon \nabla_x \beta(t, x))^{\frac{1}{2}} u(t, x + \varepsilon \beta(t, x)))] \mod \mathcal{S}^0(\mathbb{T}^{n+1}) \\ &= \tilde{\Phi}(t)^{-1}[(\nu + \varepsilon V(t, x)) \cdot \det(\text{Id} + \varepsilon \nabla_x \beta(t, x))^{\frac{1}{2}} (\text{Id} + \varepsilon \nabla_x \beta(t, x)) \cdot \nabla_x u(t, x + \varepsilon \beta(t, x))] \mod \mathcal{S}^0(\mathbb{T}^{n+1}) \\ &\stackrel{(3.4.10)}{=} \\ &\stackrel{(3.4.11)}{=} (\nu + \varepsilon V(t, x + \varepsilon \tilde{\beta}(t, x))) (\text{Id} + \varepsilon \nabla_x \beta(t, x + \varepsilon \tilde{\beta}(t, x))) \nabla_x u(t, x) \mod \mathcal{S}^0(\mathbb{T}^{n+1}). \end{aligned}$$

Since we are interested in finding the first term of the expansion in ε of f_ε in (3.4.17), we now perform a Taylor expansion in a neighborhood of $\varepsilon = 0$ and we obtain:

$$\tilde{\Phi}(t)^{-1}H\tilde{\Phi}(t)u(t, x) = [(\nu + \varepsilon V(t, x))(\text{Id} + \varepsilon \nabla_x \beta(t, x)) + O_{C^\infty}(\varepsilon^2)] \nabla_x u(t, x) \mod \mathcal{S}^0(\mathbb{T}^{n+1}),$$

which gives the expansion in (3.4.15).

We now prove (3.4.16). We first show that there exists $g_\varepsilon \in C^\infty(\mathbb{T}^{n+1})$ such that

$$\tilde{\Phi}(t)^{-1}\partial_t\tilde{\Phi}(t) = \text{Op}^w(i\xi g_\varepsilon(t, x)). \quad (3.4.18)$$

This proves that $\tilde{\Phi}(t)^{-1}\partial_t\tilde{\Phi}(t)$ is a skew-adjoint operator (see Lemma 3.3.1). Next, we derive the explicit expression in (3.4.16). In order to prove (3.4.18), we first remark that

$$\tilde{\Phi}(t)^{-1}\partial_t\tilde{\Phi}(t)[u](t, x) = \tilde{\Phi}(t)^{-1}\partial_t[\tilde{\Phi}(t)u](t, x) - \partial_t u(t, x), \quad \forall u \in C^\infty(\mathbb{T}^{n+1}). \quad (3.4.19)$$

Next we compute $\tilde{\Phi}(t)^{-1}\partial_t[\tilde{\Phi}(t)u](t, x)$:

$$\begin{aligned} \tilde{\Phi}(t)^{-1}\partial_t[\tilde{\Phi}(t)u](t, x) &\stackrel{(3.4.9)}{=} \tilde{\Phi}(t)^{-1}\partial_t[\det(\text{Id} + \varepsilon\nabla_x\beta(t, x))^{\frac{1}{2}}u(t, x + \varepsilon\beta(t, x))] \\ &= \tilde{\Phi}(t)^{-1}\left[\left(\partial_t \det(\text{Id} + \varepsilon\nabla_x\beta(t, x))^{\frac{1}{2}}\right)u(t, x + \varepsilon\beta(t, x))\right] \\ &\quad + \tilde{\Phi}(t)^{-1}\left[\det(\text{Id} + \varepsilon\nabla_x\beta(t, x))^{\frac{1}{2}}(\partial_t u(t, x + \varepsilon\beta(t, x)))\right] \\ &\quad + \tilde{\Phi}(t)^{-1}\left[\det(\text{Id} + \varepsilon\nabla_x\beta(t, x))^{\frac{1}{2}}\varepsilon\partial_t\beta(t, x) \cdot \nabla_x u(t, x + \varepsilon\beta(t, x))\right] \\ &\stackrel{(3.4.10)}{=} \left(\partial_t \det(\text{Id} + \varepsilon\nabla_x\beta(t, x))^{\frac{1}{2}}\right)|_{y=x+\varepsilon\tilde{\beta}_\varepsilon} \det(\text{Id} + \varepsilon\nabla_x\tilde{\beta}_\varepsilon(t, x))^{\frac{1}{2}}u(t, x) \\ &\quad + \partial_t u(t, x) + \varepsilon\partial_t\beta(t, x + \varepsilon\tilde{\beta}_\varepsilon(t, x)) \cdot \nabla_x u(t, x). \end{aligned} \quad (3.4.20)$$

Thus, plugging (3.4.20) in (3.4.19), we obtain that

$$\begin{aligned} \tilde{\Phi}(t)^{-1}\partial_t\tilde{\Phi}(t)[u](t, x) &= \underbrace{\left(\partial_t \det(\text{Id} + \varepsilon\nabla_x\beta(t, x))^{\frac{1}{2}}\right)|_{y=x+\varepsilon\tilde{\beta}_\varepsilon} \det(\text{Id} + \varepsilon\nabla_x\tilde{\beta}_\varepsilon(t, x))^{\frac{1}{2}}u(t, x)}_{g_{\varepsilon,1}(t, x)} \\ &\quad + \underbrace{\varepsilon\partial_t\beta(t, x + \varepsilon\tilde{\beta}_\varepsilon(t, x)) \cdot \nabla_x u(t, x)}_{g_{\varepsilon,2}(t, x)}. \end{aligned} \quad (3.4.21)$$

Recalling now that, from the composition rule for pseudodifferential operators (see Lemma 3.3.1), for any $h(t, x) \in C^\infty(\mathbb{T}^{n+1})$ we have

$$h(t, x) \cdot \nabla_x + \frac{\text{div}_x h(t, x)}{2} = \text{Op}^w(i\xi \cdot h(t, x)),$$

we are done with the proof of (3.4.18) if we show that

$$\frac{\text{div}_{g_{\varepsilon,2}}(t, x)}{2} = g_{\varepsilon,1}(t, x), \quad (3.4.22)$$

with $g_{\varepsilon,1}$ and $g_{\varepsilon,2}$ given in (3.4.21). Indeed, plugging (3.4.22) in (3.4.21) we obtain (3.4.18) with $g_\varepsilon(t, x) := \varepsilon \partial_t \beta(t, x + \varepsilon \tilde{\beta}_\varepsilon(t, x))$. We show that (3.4.22) holds with a direct computation:

$$\begin{aligned}
g_{\varepsilon,1}(t, x) &\stackrel{(3.4.21)}{=} \left(\partial_t \det(\text{Id} + \varepsilon \nabla_x \beta(t, x))^{\frac{1}{2}} \right) \Big|_{y=x+\varepsilon \tilde{\beta}_\varepsilon} \det(\text{Id} + \varepsilon \nabla_x \tilde{\beta}_\varepsilon(t, x))^{\frac{1}{2}} \\
&\stackrel{(3.4.11)}{=} \frac{1}{2} \partial_t \det(\text{Id} + \varepsilon \nabla_x \beta(t, x + \varepsilon \tilde{\beta}_\varepsilon(t, x))) \det(\text{Id} + \varepsilon \nabla_x \tilde{\beta}_\varepsilon(t, x)) \\
&= \frac{1}{2} \text{tr} \left((\text{Id} + \varepsilon \nabla_x \beta)^{-1} \partial_t (\text{Id} + \varepsilon \nabla_x \beta) \right) \\
&\stackrel{(3.4.11)}{=} \frac{1}{2} \text{tr} \left((\text{Id} + \varepsilon \nabla_x \tilde{\beta}_\varepsilon) \varepsilon \partial_t \nabla_x \beta \right)
\end{aligned} \tag{3.4.23}$$

where for the unlabeled equality we used Jacobi identity. On the other hand we also have:

$$\begin{aligned}
\text{div} g_{\varepsilon,2}(t, x) &\stackrel{(3.4.21)}{=} \text{div}(\varepsilon \partial_t \beta(t, x + \varepsilon \tilde{\beta}_\varepsilon(t, x))) \\
&= \text{tr} \left(\varepsilon \partial_t \nabla_y \beta(t, y) \Big|_{y=x+\varepsilon \tilde{\beta}_\varepsilon(t, x)} (\text{Id} + \varepsilon \nabla_x \tilde{\beta}_\varepsilon(t, x)) \right).
\end{aligned} \tag{3.4.24}$$

Putting together (3.4.23) and (3.4.24), we obtain (3.4.22). Thus we have shown that

$$\tilde{\Phi}(t)^{-1} \partial_t \tilde{\Phi}(t)[u](t, x) = \text{Op}^w(i\xi \cdot \varepsilon \partial_t \beta(t, x + \varepsilon \tilde{\beta}_\varepsilon(t, x))). \tag{3.4.25}$$

Finally, we obtain (3.4.16) performing a Taylor expansion of (3.4.25) in a neighborhood of $\varepsilon = 0$, concluding the proof of the lemma. $\square$

Proof of Proposition 3.4.3

Let $\tilde{\Phi}(t)$ be a transformation of the form in (3.4.9), with $\beta \in C^\infty(\mathbb{T}^{n+1})$ to be determined and recall the expression for its inverse in (3.4.10). Take $u(t, x)$ solution of (3.4.1), that we rewrite as $\partial_t u = H(t)u$ with $H(t)$ in (3.4.13), and recall from previous discussion that defining $v(t, x)$ via $u(t, x) = \tilde{\Phi}(t)v(t, x)$ then $v(t, x)$ solves

$$\partial_t v(t, x) = \tilde{\Phi}(t)_* H(t) v(t, x). \tag{3.4.26}$$

See (3.4.6). We claim that it is possible to choose $\varepsilon_0 > 0$ small enough and $\beta \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ such that, for every $0 < \varepsilon < \varepsilon_0$,

$$\tilde{\Phi}(t)_* H(t) = \text{Op}^w(i\xi(\nu + \varepsilon V_{R_\nu}(t, x) + \varepsilon^2 \tilde{W}_{\nu, \varepsilon}(t, x))) \quad \forall t \in \mathbb{T}, \tag{3.4.27}$$

where $\tilde{W}_{\nu,\varepsilon}(t, x) \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$ with semi-norms uniformly bounded in terms of ε and

$$V_{R_\nu}(t, x) := \sum_{k \in \mathbb{Z}^n} v_{k,\nu \cdot k} e^{ik \cdot (x + \nu t)} \equiv \langle V \rangle_\nu(x + \nu t), \quad (3.4.28)$$

where $v_{k,\nu \cdot k}$ are the resonant Fourier coefficients of V (see (3.4.2) and (3.4.3)). Let us postpone the proof of this claim, and conclude the proof of the proposition assuming (3.4.27). Defining indeed the time translation $\mathcal{U}(t) : \mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma$, which acts as $\mathcal{U}(t)w(t, x) := w(t, x - \nu t)$, for all $t \in \mathbb{R}$ and for all $w \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$, we consider

$$u_1(t, x) := \mathcal{U}(t)v(t, x) = v(t, x - \nu t), \quad (3.4.29)$$

where $v(t, x)$ solves equation (3.4.26). First we compute the equation solved by $u_1(t, x)$:

$$\begin{aligned} \partial_t u_1(t, x) &= \partial_t v(t, x - \nu t) - \nu \cdot \nabla_x v(t, x - \nu t) \\ &= \partial_t v(t, x - \nu t) - \text{Op}^w(i\nu \cdot \xi)v(t, x - \nu t) \\ &\stackrel{(3.4.26)}{=} \stackrel{(3.4.27)}{=} \text{Op}^w(i\xi \cdot (\varepsilon V_{R_\nu}(t, x - \nu t) + \varepsilon^2 \tilde{W}_{\nu,\varepsilon}(t, x - \nu t)))v(t, x - \nu t) \\ &\stackrel{(3.4.28)}{=} \stackrel{(3.4.29)}{=} \text{Op}^w(i\xi \cdot (\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W_{\nu,\varepsilon}(t, x)))u_1(t, x), \end{aligned}$$

where $W_{\nu,\varepsilon} := \tilde{W}_{\nu,\varepsilon}(t, x - \nu t)$. Thus $u_1(t, x)$ defined as in (3.4.29) solves (3.4.4). Moreover the transformation $\Phi(t) := \mathcal{U}(t)\tilde{\Phi}(t)^{-1} : \mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma$ is linear, 2π -periodic and bounded for every $\sigma \in \mathbb{R}$, since both $\mathcal{U}(t)$ and $\tilde{\Phi}(t)^{-1}$ are. Furthermore, the continuity constants are uniform in terms of $0 \leq \varepsilon \leq \varepsilon_0$ (recall that $\Phi(t)$ depends implicitly on ε). This concludes the proof of the proposition since $u(t, x) = \Phi(t)u_1(t, x)$ by construction.

Thus we are left with finding $\varepsilon_0 > 0$ and $\beta \in C^\infty(\mathbb{T} \times \mathbb{T}^n; \mathbb{R}^n)$, so that (3.4.27) holds. First of all, we plug the expansions (3.4.15) and (3.4.16) in (3.4.14), obtaining

$$\Phi_*(t)H(t) = \text{Op}^w(i\xi \cdot (\nu + \varepsilon V(t, x) + \varepsilon \nu \cdot \nabla_x \beta - \varepsilon \partial_t \beta + O_{C^\infty}(\varepsilon^2))), \quad (3.4.30)$$

in a neighborhood of $\varepsilon = 0$. Next we consider the Fourier expansion of the term $T(t, x) := \nu + \varepsilon V + \varepsilon \nu \cdot \nabla_x \beta - \varepsilon \partial_t \beta$:

$$T(t, x) = \nu + \varepsilon V + \varepsilon \nu \cdot \nabla_x \beta - \varepsilon \partial_t \beta = \nu + \varepsilon \sum_{(k,\ell) \in \mathbb{Z}^{n+1}} (v_{k,\ell} + i(k \cdot \nu - \ell)\beta_{k,\ell}) e^{ikx + i\ell t}, \quad (3.4.31)$$

where $V(t, x) = \sum_{k, \ell} v_{k, \ell} e^{i(kx + \ell t)}$ and $\beta(t, x) = \sum_{k, \ell \in \mathbb{Z}} \beta_{k, \ell} e^{i(kx + \ell t)}$. Define

$$\beta_{k, \ell} := \begin{cases} \frac{v_{k, \ell}}{i(\ell - k \cdot \nu)} & \text{if } \ell \neq k \cdot \nu, \\ 0 & \text{otherwise.} \end{cases} \quad (3.4.32)$$

The corresponding β is a well defined and smooth function over $\mathbb{T}^n$. Substituting the Fourier coefficients of β (see (3.4.32)), in the expression of $T(t, x)$ in (3.4.31), we get:

$$T(t, x) = \nu + \varepsilon \sum_{\substack{\ell + \nu \cdot k = 0 \\ k \in \mathbb{Z}}} v_{k, \nu \cdot k} e^{ik(x + \nu t)} = \nu + \varepsilon V_{R_\nu}(t, x), \quad (3.4.33)$$

where we recall the definition of V_{R_ν} in (3.4.28). Thus recognizing that $\Phi_*(t)H(t) = \text{Op}^w(i\xi(T(t, x) + O_{C^\infty}(\varepsilon^2)))$ (see (3.4.31) and (3.4.30)) and plugging (3.4.33) in (3.4.30), we obtain

$$\Phi_*(t)H(t) = \text{Op}^w(i\xi(\nu + \varepsilon V_{R_\nu}(t, x) + \tilde{W}_{\nu, \varepsilon})),$$

where $\tilde{W}_{\nu, \varepsilon}$ is the remainder in the order one operator. This gives (3.4.27). Remark that ε_0 is chosen so that both (3.4.8) and the Taylor expansions hold for all $0 < \varepsilon < \varepsilon_0$.

3.4.2 Genericity in dimension 2 and proof of Theorem 3.0.1

In this last section we conclude the proof of Theorem 3.0.1. Fix any $\nu \in \mathbb{N}_+^2$ and define

$$\mathcal{A}_\nu := \{V \in C^\infty(\mathbb{T}^3; \mathbb{R}^2) : \langle V \rangle_\nu(x) \in \mathfrak{X}^{MS}(\mathbb{T}^2)\}.$$

Recall that C^∞ is endowed with its natural Fréchet topology, i.e. the one induced by the seminorms $(\|\cdot\|_{C^k})_{k \geq 0}$. One can verify that the map

$$V \in C^\infty(\mathbb{T}^3; \mathbb{R}^2) \mapsto \langle V \rangle_\nu \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$$

is linear and continuous with respect to the Fréchet topology on each space. In particular, $\mathfrak{X}^{MS}(\mathbb{T}^2)$ being an open subset of $C^\infty(\mathbb{T}^2; \mathbb{R}^2)$, one finds that $\mathcal{A}_\nu$ is open for the Fréchet topology on C^∞ . Regarding density, we let $V_0 \in C^\infty(\mathbb{T}^3; \mathbb{R}^2)$ and we decompose it as

$$V_0(t, x) = \langle V_0 \rangle_\nu(x + t\nu) + (V_0(t, x) - \langle V_0 \rangle_\nu(x + t\nu)).$$

The second term on the right hand side has all its Fourier coefficients in $\mathbb{Z}^3 \setminus \{(k \cdot \nu, k) : k \in \mathbb{Z}^2\}$. In particular, its average $\langle \cdot \rangle_\nu$ is identically zero. Now recalling that $\mathfrak{X}^{MS}(\mathbb{T}^2)$

is a dense subset of $C^\infty(\mathbb{T}^2; \mathbb{R}^2)$, one can find W in $\mathfrak{X}^{MS}(\mathbb{T}^2)$ arbitrarily close to $\langle V_0 \rangle_\nu$. Letting

$$V(t, x) := W(x + t\nu) + (V_0(t, x) - \langle V_0 \rangle_\nu(x + t\nu)),$$

one finds that V can be made arbitrarily close to V_0 . This shows that $\mathcal{A}_\nu$ is dense and concludes the proof of Theorem 3.0.1 when combined with Theorem 3.4.5.

3.5 Preliminary and auxiliary results

3.5.1 Definitions in dynamical systems

In this section, we review the definitions of dynamical systems theory that we use in Sections 3.1 and 3.2. We refer to [77] for Morse-Smale theory and to [19, 40] for a complete review of the theory related to hyperbolic dynamical systems. Let V be a smooth vector field on a smooth manifold M of dimension $n \geq 1$. We denote by $\varphi_V^t(x)$ its flow at time $t \in \mathbb{R}$ (see (3.1.1)).

Definition 3.5.1. A point $\Lambda \in M$ is a *critical point* of V if $V(\Lambda) = 0$. We say that a critical point is *hyperbolic* if $dV(\Lambda) : T_\Lambda M \rightarrow T_\Lambda M$ has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ such that $\operatorname{Re}(\lambda_i) \neq 0$, for all $i = 1, \dots, n$.

Definition 3.5.2. A point $x_0 \in M$ is a *periodic point* if $V(x_0) \neq 0$, and there exists $T_0 > 0$ such that $\varphi_V^{T_0}(x_0) = x_0$. Moreover, a periodic point x_0 is *hyperbolic* if $d\varphi_V^{T_0}(x_0)$ has 1 as a simple eigenvalue and no other eigenvalues of modulus 1. The set $\Lambda = \{\varphi_V^t(x_0) : 0 \leq t \leq T_0\}$ is then called a *hyperbolic closed orbit*.

Definition 3.5.3 (Non-wandering set). We say that a point $x \in M$ is *wandering* if there exist some open neighborhood $\mathcal{O}$ of x and a time $T_x > 0$ such that

$$\mathcal{O} \cap \left(\bigcup_{|t| > T_x} \varphi_V^t(\mathcal{O}) \right) = \emptyset.$$

We denote by $\operatorname{NW}(V) = \operatorname{NW}(\varphi_V^t)$ the *non-wandering set* of V , i.e., the union of all points $y \in M$ which are not wandering.

Definition 3.5.4 (α and ω limit sets). Let $x \in M$. We define

$$\alpha(x) := \bigcap_{T \leq 0} \overline{\{\varphi_V^t(x) : t \leq T\}}, \quad \text{and} \quad \omega(x) := \bigcap_{T \geq 0} \overline{\{\varphi_V^t(x) : t \geq T\}}.$$

We remark that, if $x \in \Lambda_i$, with $\Lambda_i \in \text{Crit}(V)$ (see (3.1.2)), then $\alpha(x) = \omega(x) = \Lambda_i$.

Definition 3.5.5 (Stable and unstable manifolds). Let K be a closed invariant subset of M . We define the stable and unstable manifolds associated with K as

$$W^s(K) := \{x \in M : \omega(x) \subset K\} \quad \text{and} \quad W^u(K) := \{x \in M : \alpha(x) \subset K\}.$$

Definition 3.5.6. Let $\Lambda \in \text{Crit}(V)$ be a hyperbolic closed orbit of period T_0 . The stable and unstable manifolds $W^{s/u}(\Lambda)$ are foliated by the family of smooth submanifolds of codimension one $\{W^{ss/uu}(x_0)\}_{x_0 \in \Lambda}$, where

$$W^{ss}(x_0) := \{x \in M : \lim_{n \rightarrow +\infty} \varphi^{nT_0}(x) = x_0\},$$

and

$$W^{uu}(x_0) := \{x \in M : \lim_{n \rightarrow +\infty} \varphi^{-nT_0}(x) = x_0\}.$$

We remark that the same foliation can be trivially defined also in the case of critical points, for which $W^{ss/uu}(x_0) = W^{s/u}(x_0)$.

3.5.2 Existence of solutions to transport equations

In this section, we discuss the existence of solutions to (3.3.9). Recall that the main difference with [72] is the presence of a non skew-adjoint term $\varepsilon \text{Op}^M(b_{\varepsilon,2})$ in the right-hand side of the equation with $b_{\varepsilon,2}(t)$ being real valued, lying in $\mathcal{C}_b^0(\mathbb{R}, \mathbf{S}^0)$ and having all its seminorms uniformly bounded in terms of $0 \leq \varepsilon \leq 1$ and $t \in \mathbb{R}$. To deal with this issue, we will apply a perturbative argument compared with the results from this reference. For the sake of simplicity, we set

$$\mathcal{B}_0(t) = \text{Op}^M(i\xi(V(x) + \varepsilon \mathcal{P}(t, x)) + ib_{\varepsilon,1}), \quad \text{and} \quad \mathcal{R}_0(t) = \varepsilon \text{Op}^M(b_{\varepsilon,2}).$$

We denote by $\mathcal{U}_0(t, s) \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}; \mathcal{L}(\mathcal{H}^\sigma))$ the flow associated with $\mathcal{B}_0(t)$. Recall from [72, Th. 1.2] that it satisfies

$$\|\mathcal{U}_0(t, s)\|_{\mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma} \leq C_\sigma e^{C_\sigma |t-s|} \quad (3.5.1)$$

for every $\sigma \geq 0$ and $t, s \in \mathbb{R}$ and that it is unitary in L^2 (i.e., for $\sigma = 0$). Thanks to Duhamel's principle, the equation we are interested in can be rewritten under an integral form, for every t in $\mathbb{R}$,

$$u_\varepsilon(t) = \mathcal{U}_0(t, 0)u_\varepsilon(0) + \int_0^t \mathcal{U}_0(t, \tau)\mathcal{R}_0(\tau)u_\varepsilon(\tau)d\tau.$$

Hence, we fix $u_0 \in \mathcal{H}^\sigma$, $T > 0$ and we consider the functional

$$F : u \in \mathcal{C}^0([-T, T]; \mathcal{H}^\sigma) \mapsto \mathcal{U}_0(t, 0)u_0 + \int_0^t \mathcal{U}_0(t, \tau)\mathcal{R}_0(\tau)u(\tau)d\tau \in \mathcal{C}^0([-T, T]; \mathcal{H}^\sigma).$$

Our goal is to prove the existence of a fixed point $F(v) = v$ in $\mathcal{H}^\sigma$. To do that, we proceed by induction and we set

$$v_0 = u_0, \quad v_{n+1} = F(v_n).$$

One has

$$\|v_{n+1}(t) - v_n(t)\|_\sigma \leq C_{\sigma, T} \int_0^t \|v_n(\tau) - v_{n-1}(\tau)\|_\sigma d\tau, \quad \forall -T \leq t \leq T, \quad \forall n \in \mathbb{N},$$

where we used (3.5.1) and the property that $\mathcal{R}_0(\tau) : \mathcal{H}^\sigma \rightarrow \mathcal{H}^\sigma$ is a bounded operator thanks to (3.3.2). We remark that $C_{\sigma, T} > 0$ depends only on σ , T and the norms of $\mathcal{B}_0$ and $\mathcal{R}_0$. By induction, one finds

$$\|v_{n+1}(t) - v_n(t)\|_\sigma \leq \frac{(C_{\sigma, T}t)^n}{n!} \sup_{\tau \in [-T, T]} \|v_1(\tau) - v_0(\tau)\|_\sigma,$$

from which we can infer that $(v_n)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{C}^0([-T, T], \mathcal{H}^\sigma)$. By continuity of the functional F , we find that the limit v_∞ satisfies $F(v_\infty) = v_\infty$. The same argument shows that the solution is unique. As T can be chosen arbitrarily, we find that there exists an unique solution to (3.3.9) that is defined on $\mathbb{R}$.

Chapter 4

Long-time behavior of resonant transport equations in dimension one

This chapter complements the study carried out in the previous one: we consider the same class of transport equations, but we restrict ourselves to the one-dimensional case. In this setting, we are able to completely classify the long-time dynamics of solutions for any non-degenerate perturbation (and not only to identify the potentials generating instability). Precisely, we consider

$$\partial_t u = (\nu + \varepsilon V(t, x)) \partial_x u + \frac{\varepsilon}{2} \partial_x V(t, x) u, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R}, \varepsilon > 0, \quad (4.0.1)$$

where $\nu \in \mathbb{N}^+$, ε is a positive parameter, $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$, and we use the definition in (1.1.12) with $n = 1$ for the scale of Sobolev spaces on $\mathbb{T}$. As anticipated in the introduction, we prove that, depending on the choice of the perturbation V , only two possibilities occur (up to a nowhere dense set of degenerate perturbations): either all Sobolev norms of all solutions are uniformly bounded for finite but arbitrarily long time scales, or instability phenomena arise. We recall Definition 1.1.10 of resonant average of a given perturbation and of completely resonant function, and Definitions 1.1.11-1.1.12 of resonantly stable and unstable functions. Our main result is the following theorem.

Theorem 4.0.1. *For any $\nu \in \mathbb{N}^+$ let*

$$\mathcal{V}_{\nu,s} := \{V \in C^\infty(\mathbb{T}^2; \mathbb{R}) : V \text{ is resonantly stable} \}, \quad (4.0.2)$$

$$\mathcal{V}_{\nu,u} := \{V \in C^\infty(\mathbb{T}^2; \mathbb{R}) : V \text{ is resonantly unstable} \}, \quad (4.0.3)$$

$$\mathcal{V}_{\nu,d} := \{V \in C^\infty(\mathbb{T}^2; \mathbb{R}) : \langle V \rangle_\nu \text{ has degenerate zeroes} \},$$

where we recall Definitions 1.1.10, 1.1.11 and 1.1.12. The following holds:

1. *Long-time stability:* for any $V \in \mathcal{V}_{\nu,s}$, for any $N > 0$ there exists $\varepsilon_N(V) > 0$ such that, for any $0 < \varepsilon < \varepsilon_N(V)$ and for any $s > 0$, there exists $C_1 > 0$ (independent of ε) such that, for any $u_0(x) \in \mathcal{H}^s(\mathbb{T})$, the corresponding solution $u(t, x)$ of equation (4.0.1) (i.e. $u(0, x) = u_0(x)$) satisfies

$$\|u(t)\|_s \leq C_1 \|u_0\|_s \quad \forall |t| \leq C_V \varepsilon^{-(N+1)}, \quad \text{where } C_1 \equiv C_1(s, V) \quad (4.0.4)$$

for some $C_V > 0$.

2. *Instability:* for any $V \in \mathcal{V}_{\nu,u}$ there exist $\varepsilon(V) > 0$, $\delta_2 > 0$ such that, for any $0 < \varepsilon < \varepsilon(V)$ and for any $s \geq 1$ there exists $u_0 \in \mathcal{H}^s(\mathbb{T})$, such that the corresponding solution $u(t, x)$ of equation (4.0.1) satisfies:

$$\|u(t)\|_s \geq e^{\varepsilon \delta_2 t} \delta_1(u_0), \quad \forall t > 0, \quad (4.0.5)$$

for some $\delta_1(u_0) > 0$.

3. *Genericity:* The set $\mathcal{V}_{\nu,u} \cup \mathcal{V}_{\nu,s}$ is open and dense in $C^\infty(\mathbb{T}^2; \mathbb{R})$ endowed with its Fréchet topology.

We refer to Section 1.1.3 for a broader introduction to this result and to Section 1.2.3 for an idea of the main tools used in the proof. The rest of the Chapter is devoted to the proof of Theorem 4.0.1.

Organization of the chapter. In Section 4.1 we collect preliminary results needed along the chapter. In particular, in Section 4.1.1 we introduce the class of pseudodifferential operators along with some of their properties, whilst in Section 4.1.2 we define the class of transformations that we will use all along the chapter. In Section 4.2 we prove the first point of Theorem 4.0.1, using first a normal form procedure to reduce equation (4.0.1) to constant coefficients up to small-in-size remainders, and then proving estimate (4.0.4).

In order to prove the second point of Theorem 4.0.1, we need to construct an escape function for the Hamiltonian lift of the flow generated by the resonant average of V on the torus. Thus in Section 4.3.1 we construct the escape function and in Section 4.3.2 we use it to prove estimate (4.0.5). Finally, in Section 4.4 we conclude the proof of Theorem 4.0.1 studying the structure of the sets $\mathcal{V}_{\nu,s}, \mathcal{V}_{\nu,u}$.

4.1 Preliminary results

In this section we collect some preliminary results that we will use in the rest of the chapter. Precisely, in Section 4.1.1 we collect some definitions and results of symbolic calculus and in Section 4.1.2 we define a class of linear transformations (that we will need to prove both the reducibility and the instability results in Theorem 4.0.1) and we study their action on transport operators. Some of the material of this section was already presented in the previous chapter, but for the reader's convenience (and in order to make the chapter selfcontained) we report here all the preliminary material we will use in the chapter.

4.1.1 Pseudodifferential operators

We define the class of symbols of order $\rho \in \mathbb{R}$ as

$$\mathbf{S}^\rho(\mathbb{T}) := \{a \in C^\infty(\mathbb{T} \times \mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}^n \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{\rho-|\beta|}\}, \quad \forall \rho \in \mathbb{R}. \quad (4.1.1)$$

We also denote by $C^\infty(\mathbb{T}; \mathbf{S}^\rho)$ the space of smooth maps $t \mapsto a(t, \cdot) \in \mathbf{S}^\rho(\mathbb{T})$, for every $\rho \in \mathbb{R}$. For every $a \in \mathbf{S}^\rho(\mathbb{T})$ we define the Weyl quantization $\text{Op}^w(a)$ of a as

$$\text{Op}^w(a)[u](x) := \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{R}} e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

We say that an operator A is pseudodifferential of order ρ if there exists $a \in C^\infty(\mathbb{T} \times \mathbb{R})$ such that $A = \text{Op}^w(a)$. We write $A \in \mathcal{S}^\rho$ and we denote by $C^\infty(\mathbb{T}; \mathcal{S}^\rho)$ the set of smooth maps $t \mapsto A(t) \in \mathcal{S}^\rho$, for every $\rho \in \mathbb{R}$. We now collect in the following lemma the main results of symbolic calculus that we will need in the Chapter. We refer for example to [81], [86] for the proofs.

Lemma 4.1.1. *Let $f \in \mathbf{S}^{\rho_1}(\mathbb{T})$, $g \in \mathbf{S}^{\rho_2}(\mathbb{T})$, $\rho_1, \rho_2 \in \mathbb{R}$. Then:*

1. *For any $s \in \mathbb{R}$, there exist $C_{f,s} > 0$ such that:*

$$\|\mathrm{Op}^w(f)u\|_{s-\rho_1} \leq C_{f,s}\|u\|_s, \quad \forall u \in \mathcal{H}^s, \quad (4.1.2)$$

where we recall the definition of Sobolev norms in (1.1.12);

2. *$\mathrm{Op}^w(f)^* = \mathrm{Op}^w(\bar{f})$ (here $*$ denotes the adjoint operator) and in particular skew-adjoint operators have purely imaginary symbols;*

3. *There exists $h \in \mathbf{S}^{\rho_1+\rho_2}(\mathbb{T})$ such that $\mathrm{Op}^w(f) \circ \mathrm{Op}^w(g) = \mathrm{Op}^w(h)$. Furthermore,*

$$h := fg - \frac{i}{2}\{f, g\} \mod \mathbf{S}^{\rho_1+\rho_2-2}(\mathbb{T}); \quad (4.1.3)$$

4. *There exists $\ell \in \mathbf{S}^{\rho_1+\rho_2-1}(\mathbb{T})$ such that $i[\mathrm{Op}^w(f), \mathrm{Op}^w(g)] = \mathrm{Op}^w(\ell)$. Moreover*

$$\ell := \{f, g\} \mod \mathbf{S}^{\rho_1+\rho_2-2}(\mathbb{T}); \quad (4.1.4)$$

5. *If $\rho_1 < 0$, then $\mathrm{Op}^w(f)$ is compact as an operator from $L^2(\mathbb{T})$ to itself.*

6. *If $a \in C^\infty(\mathbb{T} \times \mathbb{R})$ is a compactly supported function, then $a \in \mathbf{S}^\rho$ for any $\rho < 0$.*

Remark 4.1.2. Let $p \in C^\infty(\mathbb{T})$. Then $p \in \mathbf{S}^0$ and $\mathrm{Op}^w(p)[u](x) = p(x)u(x)$. Moreover

$$\mathrm{Op}^w(i\xi p(x)) = p(x)\partial_x + \frac{1}{2}p'(x).$$

Finally, we state the following Gårding inequality, referring to [86, Theorem 9.11] for a proof.

Lemma 4.1.3. *Let $a \in \mathbf{S}^\rho(\mathbb{T})$, $a \geq 0$. Then for every $u \in C^\infty(\mathbb{T})$ we have*

$$\langle \mathrm{Op}^w(a)u, u \rangle_{L^2(\mathbb{T})} \geq -C_{\rho,a} \|u\|_{\frac{\rho-1}{2}}^2, \quad (4.1.5)$$

for some $C_{\rho,a} > 0$.

4.1.2 Conjugation results

In this section we define a class of linear, invertible transformations and study their action on transport equations. This conjugation technique is already used in [10] and [39]. The same transformation as the one presented in this section is studied in [79] (and described in Section 3.4.1 of the present thesis) for tori of any dimension. We report it here since we slightly modify the discussion to adapt it also to the proof of the stability result (see Section 4.2).

First of all we remark that a linear, invertible transformation $u = A(t)u_1$, with $A(t)$ periodic in time of period 2π , transforms equation $\dot{u} = Hu$ into $\dot{u}_1 = A(t)_*Hu_1$, where the pushforward $A(t)_*H$ is given by

$$A(t)_*H := A(t)^{-1}[HA(t) - \partial_t A(t)]. \quad (4.1.6)$$

We remark, to avoid any confusion in the notation, that $\partial_t A(t)$ is the derivative in time of the linear transformation (and not the composition $\partial_t \circ A(t)$). Next, we consider a family of time-dependent diffeomorphisms of the torus of the form

$$\varphi_k(t) : \mathbb{T} \rightarrow \mathbb{T}, \quad x \mapsto x + \varepsilon^k \beta(t, x), \quad k \in \mathbb{N}^+, \varepsilon > 0, \beta \in C^\infty(\mathbb{T}^2; \mathbb{R}). \quad (4.1.7)$$

It has been proved (see for example [2, Lemma B.4]) that for $\varepsilon > 0$ small enough $\varphi_k(t)$ is an invertible diffeomorphism and its inverse has the form

$$\varphi_k^{-1}(t) : \mathbb{T} \rightarrow \mathbb{T}, \quad y \mapsto y + \varepsilon^k \tilde{\beta}(t, y), \quad k \in \mathbb{N}^+, \varepsilon > 0, \quad (4.1.8)$$

for some $\tilde{\beta} \in C^\infty(\mathbb{T}^2; \mathbb{R})$. We define the transformation associated with (4.1.7) as:

$$\tilde{\Phi}_k(t) : u(x) \mapsto (1 + \varepsilon^k \partial_x \beta)^{\frac{1}{2}}(u \circ \varphi_k)(x), \quad \forall t \in \mathbb{T}, k \in \mathbb{N}^+, \quad (4.1.9)$$

with inverse given by

$$\tilde{\Phi}_k(t)^{-1} : u(y) \mapsto (1 + \varepsilon^k \partial_x \tilde{\beta})^{\frac{1}{2}}(u \circ \varphi_k^{-1})(y), \quad \forall t \in \mathbb{T}, k \in \mathbb{N}^+. \quad (4.1.10)$$

We stress here that $\tilde{\Phi}_k(t) \equiv \tilde{\Phi}_{k,\varepsilon,\beta}(t)$, i.e., the transformation depends both on β and ε , but we will always write as in (4.1.9), omitting such labels to simplify the notation. Remark that, for every $k \in \mathbb{N}^+$, both $\tilde{\Phi}_k(t)$ and $\tilde{\Phi}_k(t)^{-1}$ are 2π -periodic in time. We

now show that, for every $k \in \mathbb{N}^+$, the operator $\tilde{\Phi}_k(t)$ in (4.1.9) is unitary: substituting the explicit expressions of (4.1.7) and (4.1.8) in the identity

$$\varphi_k|_{x=y+\varepsilon^k\tilde{\beta}(t,y)} \circ \varphi_k^{-1}(y) = y \quad \forall t \in \mathbb{R}, y \in \mathbb{T}, k \in \mathbb{N}^+, \quad (4.1.11)$$

and deriving (4.1.11) with respect to y one obtains that, for every $\varepsilon > 0$ and $\beta \in C^\infty(\mathbb{T}^2; \mathbb{R})$,

$$1 + \varepsilon^k \partial_y \tilde{\beta}(t, y) = \frac{1}{1 + \varepsilon^k \partial_x \beta(t, x)|_{x=y+\varepsilon^k\tilde{\beta}(t,y)}}, \quad \forall y \in \mathbb{T}, k \in \mathbb{N}^+. \quad (4.1.12)$$

This in turn gives, via a direct computation, unitarity of $\tilde{\Phi}_k(t)$, namely

$$\tilde{\Phi}_k(t)^* = \tilde{\Phi}_k(t)^{-1}, \quad \forall t \in \mathbb{T}, k \in \mathbb{N}^+, \quad (4.1.13)$$

where $\tilde{\Phi}_k(t)^*$ denotes the adjoint operator of $\tilde{\Phi}_k(t)$. We now turn to studying the action of $\tilde{\Phi}_k(t)$ on transport operators.

Lemma 4.1.4. *Take any $f, g \in C^\infty(\mathbb{T}^2; \mathbb{R})$, $n \in \mathbb{N}^+$ and $\varepsilon > 0$. Define*

$$H_n(t) := \text{Op}^w(i\xi(f(t, x) + \varepsilon^n g(t, x))). \quad (4.1.14)$$

For any $\beta \in C^\infty(\mathbb{T}^2; \mathbb{R})$ there exists $\varepsilon_\beta > 0$ such that, for every $0 < \varepsilon < \varepsilon_\beta$, the associated transformation $\tilde{\Phi}_n(t)$ defined as in (4.1.9) satisfies

$$\tilde{\Phi}_n(t)^{-1} H_n(t) \tilde{\Phi}_n(t) = \text{Op}^w(i\xi(f(t, x) + \varepsilon^n(g(t, x) + f(t, x)\partial_x \beta(t, x)) + \varepsilon^{n+1} \mathcal{R}_{n,\varepsilon}^1(t, x))) \quad (4.1.15)$$

and

$$\tilde{\Phi}_n(t)^{-1} \partial_t \tilde{\Phi}_n(t) = \text{Op}^w(i\xi(\varepsilon^n \partial_t \beta(t, x) + \varepsilon^{n+1} \mathcal{R}_{n,\varepsilon}^2(t, x))), \quad (4.1.16)$$

where $\mathcal{R}_{n,\varepsilon}^1, \mathcal{R}_{n,\varepsilon}^2 \in C^\infty(\mathbb{T}^2; \mathbb{R})$ have all seminorms uniformly bounded in terms of $\varepsilon \in [0, \varepsilon_\beta]$.

Proof. Given $\beta \in C^\infty(\mathbb{T}^2; \mathbb{R})$, let $\tilde{\Phi}_n(t)$ be the associated transformation in (4.1.9), with $k \rightsquigarrow n$ and recall the expression for its inverse in (4.1.10). Fix $\varepsilon_\beta > 0$ such that, for every $0 < \varepsilon < \varepsilon_\beta$, $\tilde{\Phi}_n(t)$ is a bounded and invertible transformation (recall previous discussion in (4.1.7) and (4.1.8) and see [2, Lemma B.4]). We first prove (4.1.15). To

this aim, first remark that, since $H_n(t)$ in (4.1.14) is skew-adjoint (see Lemma 4.1.1), using (4.1.13) we have

$$[\tilde{\Phi}_n(t)^{-1}H_n(t)\tilde{\Phi}_n(t)]^* \stackrel{(4.1.13)}{=} -\tilde{\Phi}_n(t)^{-1}H_n(t)\tilde{\Phi}_n(t).$$

Next, using Lemma 4.1.1, we can rewrite $H_n(t)$ in (4.1.14) as

$$H_n(t) = (f(t, x) + \varepsilon^n g(t, x))\partial_x + \frac{1}{2}\partial_x(f(t, x) + \varepsilon^n g(t, x)). \quad (4.1.17)$$

Thus, using (4.1.17), (4.1.9), (4.1.10), (4.1.12) and performing a direct computation one obtains

$$\begin{aligned} \tilde{\Phi}_n(t)^{-1}H_n(t)\tilde{\Phi}_n(t) &= (f + \varepsilon^n g)(t, \varphi_n^{-1}(t)[x])(1 + \varepsilon^n \partial_x \beta(t, \varphi_n^{-1}(t)[x]))\partial_x \\ &\quad + \frac{1}{2}\partial_x((f + \varepsilon^n g)(t, \varphi_n^{-1}(t)[x])(1 + \varepsilon^n \partial_x \beta(t, \varphi_n^{-1}(t)[x]))) \\ &\stackrel{(4.1.3)}{=} \text{Op}^w(i\xi((f + \varepsilon^n g)(t, \varphi_n^{-1}(t)[x])(1 + \varepsilon^n \partial_x \beta(t, \varphi_n^{-1}(t)[x])))). \end{aligned} \quad (4.1.18)$$

Analogously, the expression for $\tilde{\Phi}_n(t)^{-1}\partial_t\tilde{\Phi}_n(t)$ can be explicitly computed and is given by

$$\begin{aligned} \tilde{\Phi}_n(t)^{-1}\partial_t\tilde{\Phi}_n(t) &= \varepsilon^n \partial_t \beta(t, \varphi_n^{-1}(t)[x])\partial_x + \frac{\varepsilon^n}{2}\partial_x(\partial_t \beta(t, \varphi_n^{-1}(t)[x])) \\ &\stackrel{(4.1.3)}{=} \text{Op}^w(i\xi \partial_t \beta(t, \varphi_n^{-1}(t)[x])). \end{aligned} \quad (4.1.19)$$

Finally, performing a Taylor expansion of both (4.1.18) and (4.1.19) in a neighborhood of $\varepsilon = 0$ we obtain respectively (4.1.15) and (4.1.16) concluding the proof. $\square$

We conclude this preliminary section with the following Lemma, which contains a computation that we will use repeatedly along the Chapter.

Lemma 4.1.5. *Let $\nu \in \mathbb{N}^+$ and $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$. There exists a smooth function $\beta_{\nu, V} \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that*

$$V(t, x) + \nu \partial_x \beta_{\nu, V}(t, x) - \partial_t \beta_{\nu, V}(t, x) = \langle V \rangle_\nu(x + \nu t) \quad (4.1.20)$$

(recall the definition (1.1.17) of resonant average of V with respect to ν).

Proof. Denote by $T(t, x) := V + \nu \partial_x \beta_{\nu, V} - \partial_t \beta_{\nu, V}$, with $\beta_{\nu, V} \in C^\infty(\mathbb{T}^2; \mathbb{R})$ to be determined and write the Fourier expansion of T :

$$T(t, x) = \sum_{k, \ell \in \mathbb{Z}} (v_{k, \ell} + i(k\nu - \ell)\beta_{\nu, V_{k, \ell}}) e^{ikx + i\ell t}, \quad (4.1.21)$$

where $V(t, x) = \sum_{k, \ell} v_{k, \ell} e^{i(kx + \ell t)}$ and $\beta_{\nu, V}(t, x) = \sum_{k, \ell \in \mathbb{Z}} \beta_{\nu, V_{k, \ell}} e^{i(kx + \ell t)}$. Define

$$\beta_{\nu, V_{k, \ell}} := \begin{cases} \frac{v_{k, \ell}}{i(\ell - k\nu)} & \text{if } \ell \neq k\nu, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1.22)$$

The corresponding function $\beta_{\nu, V}(t, x) := \sum \beta_{\nu, V_{k, \ell}} e^{i(kx + \ell t)}$ is a well defined smooth function over $\mathbb{T}^2$. This can be directly checked using that V is smooth and that, since $\nu \in \mathbb{N}^+$, the sequence $\{\ell - k\nu\}_{\ell \neq k\nu}$ in (4.1.22) never accumulates to zero. Substituting the Fourier coefficients of $\beta_{\nu, V}$ (see (4.1.22)), in the expression of $T(t, x)$ in (4.1.21), one obtains (4.1.20), concluding the proof. $\square$

4.2 Long-time stability

In this section we prove the first point of Theorem (4.0.1). We recall here the idea of the proof (see Section 1.2.2 for a wider presentation of the strategy): first we take any $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$ and, using the diffeomorphisms of the torus introduced in Section 4.1.2, we conjugate equation (4.0.1) to a new transport equation having as leading transport term the resonant average $\langle V \rangle_\nu$ of V (see (4.2.2) and recall Definition 1.1.10 of resonant average). This is done in Lemma 4.2.2.

Next, we exploit the characterization of the set $\mathcal{V}_{\nu, s}$ to reduce such equation to constant coefficients (up to arbitrarily small in size remainders) and obtain the estimate in (4.0.4). Precisely, the main result of this section is the following Proposition:

Proposition 4.2.1. *For any $V \in \mathcal{V}_{\nu, s}$ in (4.0.2), for any $N \in \mathbb{N}^+$ there exists $\varepsilon_N(V) > 0$ such that, for any $0 < \varepsilon < \varepsilon_N(V)$, there exists a 2π -periodic in time, invertible linear map $\Phi_{N, \varepsilon}(t)$, $\hat{m}_N \in \mathbb{R} \setminus \{0\}$ and a smooth function $W_N \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that*

1. *for all $s \in \mathbb{R}$, there exists $C_{N, s} > 0$ such that, for all $0 \leq \varepsilon \leq \varepsilon_N(V)$ and for all $t \in \mathbb{R}$*

$$\|\Phi_{N, \varepsilon}(t)\|_{\mathcal{H}^s \rightarrow \mathcal{H}^s} + \|\Phi_{N, \varepsilon}(t)^{-1}\|_{\mathcal{H}^s \rightarrow \mathcal{H}^s} \leq C_{N, s}; \quad (4.2.1)$$

2. for any solution u of equation (4.0.1), defining $v_N(t, x)$ as $u = \Phi_{N,\varepsilon}(t)v_N$, then v_N solves

$$\partial_t v_N = \text{Op}^w(i\xi\varepsilon(\hat{m}_N + \varepsilon^N W_N(t, x)))v_N, \quad \forall t \in \mathbb{R}. \quad (4.2.2)$$

In the rest of this Section we will prove Proposition 4.2.1, but we first use it to conclude the proof of point 1 of Theorem 4.0.1.

Proof of Theorem 4.0.1, point 1. Fix $V \in \mathcal{V}_{\nu,s}$, $N > 0$, let $\varepsilon_N(V) > 0$ be the threshold given by Proposition 4.2.1 and, for every $0 < \varepsilon < \varepsilon_N(V)$ let $\Phi_{N,\varepsilon}$ be the corresponding transformation given by Proposition 4.2.1. Recall that, defining $v_N(t, x)$ as $u = \Phi_{N,\varepsilon}(t)v_N$, then v_N solves (4.2.2). Since $\Phi_{N,\varepsilon}(t)$ satisfies condition (4.2.1) it is enough to prove that, for every $s \in \mathbb{R}$ there exists $C_{V,s} > 0$ such that, for any $v_{N_0} \in C^\infty(\mathbb{T})$,

$$\|v_N(t)\|_s \leq 2\|v_{N_0}\|_s \quad \forall |t| \leq C_{V,s}\varepsilon^{-(N+1)}, \quad (4.2.3)$$

where $v_N(t, x)$ is the solution of (4.2.2) satisfying $v_N(0, x) = v_{N_0}(x)$. Indeed, from the definition of v_N , this gives (4.0.4) with $C_1 := 2C_{\Phi_{N,s}}^2$ (recall (4.2.1)). In order to prove (4.2.3), we claim that there exists $C_{W_N,s} > 0$ such that

$$\frac{d}{dt} \|v_N\|_s^2 \leq 2\varepsilon^{N+1}C_{W_N,s} \|v_N\|_s^2, \quad \forall t, s \in \mathbb{R}. \quad (4.2.4)$$

This immediately implies (4.2.3) since integrating (4.2.4) one obtains

$$\|v_N\|_s \leq e^{C_{W_N,s}\varepsilon^{N+1}t} \|v_{N_0}\|_s, \quad \forall t, s \in \mathbb{R},$$

which is (4.2.3) with $C_{V,s} := \log(2)C_{W_N,s}^{-1} > 0$. Thus we are left to show (4.2.4). To this aim, we introduce the compact notation $H_N(t) := \text{Op}^w(i\xi\varepsilon(\hat{m}_N + \varepsilon^N W_N(t, x)))$, so that equation (4.2.2) reads $\partial_t v_N = H_N(t)v_N$. We obtain via a direct calculation, using skewadjointness of H_N

$$\frac{d}{dt} \|v_N\|_s^2 = 2\text{Re}(\langle [\text{Op}^w(\langle \xi \rangle^s), H_N]v_N, \text{Op}^w(\langle \xi \rangle^s)v_N \rangle). \quad (4.2.5)$$

Next, using the explicit expression of $H_N(t)$ we have

$$[\text{Op}^w(\langle \xi \rangle^s), H_N] = \varepsilon^{N+1}[\text{Op}^w(\langle \xi \rangle^s), \text{Op}^w(i\xi W_N(t, x))] \in C^\infty(\mathbb{T}; \mathcal{S}^s). \quad (4.2.6)$$

Thus, via Cauchy-Schwarz inequality we get

$$\frac{d}{dt} \|v_N\|_s^2 \stackrel{(4.2.5)}{\leq} 2\varepsilon^{N+1} \|[\text{Op}^w(\langle \xi \rangle^s), \text{Op}^w(i\xi W_N(t, x))]v_N\|_{L^2} \|v_N\|_s \leq 2\varepsilon^{N+1} C_{W_N, s} \|v_N\|_s^2,$$

with $C_{W_N, s} := \|[\text{Op}^w(\langle \xi \rangle^s), \text{Op}^w(i\xi W_N(t, x))]\|_{\mathcal{H}^s \rightarrow L^2}$, proving inequality (4.2.4) and concluding the proof. $\square$

We now turn to the proof of Proposition 4.2.1. We first prove the following Lemma, to reduce the transport operator of equation (4.0.1) to a new one, having as leading term the resonant average of the potential. Precisely, we have the following result:

Lemma 4.2.2. *Let $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$ and take $u(t, x)$ any solution of equation (4.0.1). For every $N \in \mathbb{N}^+$ there exists $\varepsilon_N(V) > 0$ such that, for every $0 < \varepsilon < \varepsilon_N(V)$ there exists a 2π -periodic, linear, invertible transformation $\Psi_{N, \varepsilon}(t) : \mathcal{H}^s \rightarrow \mathcal{H}^s$, bounded for every $s \in \mathbb{R}$ and with bounded inverse, such that, defining $u_N(t, x)$ as $u(t, x) = \Psi_{N, \varepsilon}(t)u_N(t, x)$, then $u_N(t, x)$ solves*

$$\partial_t u_N = \text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 Z_N(x) + \varepsilon^{N+1} \tilde{W}_N(t, x)))u_N, \quad (4.2.7)$$

where $Z_N \equiv Z_N(x, \varepsilon) \in C^\infty(\mathbb{T}; \mathbb{R})$, $\tilde{W}_N \equiv W_N(x, \varepsilon) \in C^\infty(\mathbb{T}^2; \mathbb{R})$ have all seminorms uniformly bounded in terms of $\varepsilon \in [0, \varepsilon_N(V)]$.

Remark 4.2.3. We remark that this first step can be performed for a generic $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$ and we still do not use the characterization of $\mathcal{V}_{\nu, s}$ given in (4.0.2).

Proof. Take $V \in C^\infty(\mathbb{T}^2; \mathbb{R})$, $u(t, x)$ solution of (4.0.1) and fix any $N \in \mathbb{N}^+$. We claim that there exists $\varepsilon_N(V) > 0$ such that, for every $0 < \varepsilon < \varepsilon_N(V)$ there exists a sequence of 2π -periodic, linear and invertible transformations $\tilde{\Phi}_1(t), \dots, \tilde{\Phi}_N(t)$ such that, defining $u_N^{(1)}$ as $u(t, x) = \tilde{\Phi}_1(t) \circ \dots \circ \tilde{\Phi}_N(t)u_N^{(1)}(t, x)$, then $u_N^{(1)}(t, x)$ solves

$$\begin{aligned} \partial_t u_N^{(1)} &= H_N(t)u_N^{(1)}, \\ H_N(t) &:= \text{Op}^w(i\xi(\nu + \varepsilon \langle V \rangle_\nu(x + \nu t) + \varepsilon^2 \tilde{Z}_N(t, x) + \varepsilon^{N+1} \hat{W}_N(t, x))), \end{aligned} \quad (4.2.8)$$

where $\tilde{Z}_N(t, x)$ is real valued and completely resonant with respect to ν (see Definition 1.1.10), $\hat{W}_N \in C^\infty(\mathbb{T}^2; \mathbb{R})$. We stress that also in this case $\tilde{\Phi}_i(t) \equiv \tilde{\Phi}_{i, \varepsilon}(t)$, for all $i =$

$1, \dots, N$, but we omit the dependence on ε to simplify the notation. Let us postpone the proof of (4.2.8) and show first how to conclude the proof of the proposition: denote by $\mathcal{T}_\nu$ the following translation in space

$$[\mathcal{T}_\nu u](t, x) := u(t, x - \nu t), \quad \forall u \in C^\infty(\mathbb{T}^2),$$

and define the function u_N as $u_N := \mathcal{T}_\nu u_N^{(1)}$, where $u_N^{(1)}$ solves (4.2.8). From (4.2.8) we see that u_N solves

$$\partial_t u_N = \text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 \tilde{Z}_N(t, x - \nu t) + \varepsilon^{N+1} \hat{W}_N(t, x - \nu t)))u_N.$$

Remark that, since $\tilde{Z}_N$ is completely resonant with respect to ν (see Definition 1.1.10), then $\tilde{Z}_N(t, x - \nu t) = \tilde{Z}_N(0, x)$. Thus defining $Z_N(x) := \tilde{Z}_N(0, x)$, and $\tilde{W}_N(t, x) := \hat{W}_N(t, x - \nu t)$, we obtain that u_N solves (4.2.7), concluding the proof. Remark that $u = \Psi_{N,\varepsilon} u_N$, with

$$\Psi_{N,\varepsilon}(t) := \tilde{\Phi}_1(t) \circ \dots \circ \tilde{\Phi}_N(t) \circ \mathcal{T}_\nu^{-1}.$$

We now turn to the proof of (4.2.8). To this aim, we use Lemma 4.1.4 and proceed inductively. First consider the case $N = 1$: denote by

$$H(t) := \text{Op}^w(i\xi(\nu + \varepsilon V(t, x))),$$

(so that equation (4.0.1) reads $\partial_t u = H(t)u$) and apply Lemma (4.1.4) with $n = 1$, $f = \nu$ and $g = V$: let $\beta \in C^\infty(\mathbb{T}^2; \mathbb{R})$ to be fixed later and consider the associated transformation $\tilde{\Phi}_1(t)$ defined in (4.1.9) and define $u_1^{(1)}(t, x)$ as $u = \tilde{\Phi}_1(t)u_1^{(1)}$. Recall from previous discussion that $u_1^{(1)}(t, x)$ satisfies $\partial_t u_1^{(1)} = \tilde{\Phi}_1(t)_* H(t)u_1^{(1)}$. Using Lemma 4.1.4 and the expression of the pushforward in (4.1.6), we have

$$\tilde{\Phi}_1(t)_* H(t) = \text{Op}^w(i\xi(\nu + \varepsilon(V(t, x) + \nu \partial_x \beta(t, x) - \partial_t \beta(t, x)) + \varepsilon^2 \mathcal{R}_{1,\varepsilon}(t, x))),$$

where $\mathcal{R}_{1,\varepsilon} = \mathcal{R}_{1,\varepsilon}^1 + \mathcal{R}_{1,\varepsilon}^2$ is the sum of the remainders given by (4.1.15) and (4.1.16). We now apply Lemma 4.1.5 to fix β : there exists $\beta_{\nu,V}^1 \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that

$$V(t, x) + \nu \partial_x \beta_{\nu,V}^1(t, x) - \partial_t \beta_{\nu,V}^1(t, x) = \langle V \rangle_\nu(x + \nu t), \quad (4.2.9)$$

where we recall the definition of resonant average in (1.1.17). Thus $u_1^{(1)}(t, x)$ solves (4.2.8) with $N \rightsquigarrow 1$, $\tilde{Z}_1 = 0$ and $\hat{W}_1(t, x) = \mathcal{R}_{1,\varepsilon}(t, x)$. This concludes the proof of

(4.2.8) when $N = 1$. Let us remark that we fix $\tilde{\varepsilon}_1(V) := \varepsilon_{\beta_{\nu,V}^1} > 0$, where $\varepsilon_{\beta_{\nu,V}^1}$ is given by Lemma 4.1.4 and is in turn fixed by the choice of $\beta_{\nu,V}^1$ in (4.2.9).

Finally, we take any $0 < n < N$ and consider equation (4.2.8) with $N \rightsquigarrow n$. Remark that $H_n(t)$ in (4.2.8) is a particular instance of (4.1.14) with $f = \nu + \varepsilon \langle V \rangle_\nu (x + \nu t) + \varepsilon^2 \tilde{Z}_n(t, x)$ and $g(t, x) = \varepsilon \hat{W}_n(t, x)$. Thus we can apply Lemma 4.1.4: define $u_{n+1}^{(1)}(t, x)$ as $u_n^{(1)} := \tilde{\Phi}_{n+1}(t) u_{n+1}^{(1)}(t, x)$, where $\tilde{\Phi}_{n+1}(t)$ is given in (4.1.9), (with $\beta(t, x)$ to be determined). Again $u_{n+1}^{(1)}$ solves $\partial_t u_{n+1}^{(1)} = \tilde{\Phi}_{n+1}(t)_* H_n(t) u_{n+1}^{(1)}$. From (4.1.15) and (4.1.16) we have

$$\begin{aligned} \tilde{\Phi}_{n+1}(t)_* H_n(t) u_{n+1}^{(1)} &= \text{Op}^w(i\xi(\nu + \varepsilon \langle V \rangle_\nu (x + \nu t) + \varepsilon^2 \tilde{Z}_n(t, x) \\ &\quad + \varepsilon^{n+1}(\hat{W}_n(t, x) + \nu \partial_x \beta(t, x) - \partial_t \beta(t, x)) \\ &\quad + \varepsilon^{n+2} \mathcal{R}_{n+1,\varepsilon}(t, x))), \end{aligned} \quad (4.2.10)$$

where $\mathcal{R}_{n+1,\varepsilon} = \mathcal{R}_{n+1,\varepsilon}^1 + \mathcal{R}_{n+1,\varepsilon}^2$ is the sum of the remainders given by (4.1.15) and (4.1.16). We can now conclude as in the previous step: applying Lemma 4.1.5, we can find $\beta_{\nu,V}^n(t, x) \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that

$$\hat{W}_n(t, x) + \nu \partial_x \beta_{\nu,V}^n(t, x) - \partial_t \beta_{\nu,V}^n(t, x) = \left\langle \hat{W}_n \right\rangle_\nu (x + \nu t), \quad (4.2.11)$$

(recall (1.1.17)). Substituting (4.2.11) in (4.2.10) we obtain that $u_{n+1}^{(1)}(t, x)$ solves (4.2.8) with $N \rightsquigarrow n+1$, $\tilde{Z}_{n+1} := \tilde{Z}_n + \varepsilon^{n+1} \left\langle \hat{W}_n \right\rangle_\nu (x + \nu t)$, which is still completely resonant with ν (see Definition 1.1.10) and $\hat{W}_{n+1}(t, x) := \mathcal{R}_{n+1,\varepsilon}$. Finally define $\tilde{\varepsilon}_n(V) := \varepsilon_{\beta_{\nu,V}^n} > 0$ given by Lemma 4.1.4, in turn fixed by $\beta_{\nu,V}^n$. This concludes the proof of the Lemma, choosing $\varepsilon_N(V) := \min\{\tilde{\varepsilon}_1(V), \dots, \tilde{\varepsilon}_N(V)\} > 0$. $\square$

We are now in position to prove Proposition 4.2.1.

Proof of Proposition 4.2.1. Fix $V \in \mathcal{V}_{\nu,s}$, $N \in \mathbb{N}^+$ and let $\varepsilon_N(V) > 0$ and $\Psi_{N,\varepsilon}(t)$ be respectively the threshold and the transformation given by Lemma 4.2.2. Recall that, defining u_N as $u(t, x) = [\Psi_{N,\varepsilon}(t) u_N](t, x)$, then u_N solves equation (4.2.7). Let now

$$[\Lambda u](t, x) := (1 + \lambda'(x))^{\frac{1}{2}} u(t, x + \lambda(x)), \quad \forall u \in C^\infty(\mathbb{T}^2) \quad (4.2.12)$$

where $\lambda \in C^\infty(\mathbb{T}; \mathbb{R})$ has to be determined. We claim that it is possible to choose $\lambda \in C^\infty(\mathbb{T}; \mathbb{R})$ such that, considering the corresponding transformation Λ in (4.2.12)

and defining $u_N = \Lambda v_N$, where u_N solves (4.2.7), then v_N solves (4.2.2). Moreover such transformation Λ is bounded and invertible with inverse bounded in $\mathcal{H}^s$ for every $s \in \mathbb{R}$. This would conclude the proof with $\Phi_{N,\varepsilon}(t) := \Psi_{N,\varepsilon}(t) \circ \Lambda$.

To prove our claim, we first remark that if $1 + \lambda'(x) \neq 0$ for any $x \in \mathbb{T}$, then Λ in (4.2.12) is invertible with inverse of the form

$$[\Lambda^{-1}u](t, y) := (1 + \tilde{\lambda}'(y))^{\frac{1}{2}} u(t, y + \tilde{\lambda}(y)), \quad \forall u \in C^\infty(\mathbb{T}^2), \quad (t, x) \in \mathbb{T} \times \mathbb{T}, \quad (4.2.13)$$

for some $\tilde{\lambda} \in C^\infty(\mathbb{T}; \mathbb{R})$. We now proceed to identify the claimed function $\lambda \in C^\infty(\mathbb{T}; \mathbb{R})$ assuming that $1 + \lambda'(x) \neq 0$ and thus formally writing Λ^{-1} as in (4.2.13). We will check in the end that the condition $1 + \lambda'(x) \neq 0$ is satisfied. Denote by

$$\omega(t, x) := \varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 Z_N(x) + \varepsilon^{N+1} \tilde{W}_N(t, x), \quad (4.2.14)$$

where Z_N and $\tilde{W}_N$ are given by Lemma 4.2.2, so that equation (4.2.7) reads $\partial_t u_N = \text{Op}^w(i\xi\omega(t, x))u_N$. Then from (4.1.6) v_N defined as $u_N = \Lambda v_N$ solves

$$\partial_t v_N = \Lambda^{-1} \text{Op}^w(i\xi\omega(t, x))\Lambda v_N$$

(recall (4.2.14) and (4.2.12) and notice that the transformation Λ is time independent). Noticing that

$$\text{Op}^w(i\xi\omega(t, x)) = \omega(t, x)\partial_x + \frac{1}{2}\partial_x(\omega(t, x))$$

and performing a direct computation one can show that

$$\begin{aligned} \Lambda^{-1} \text{Op}^w(i\xi\omega(t, x))\Lambda &= \omega(t, y)(1 + \lambda'(y))|_{y=x+\tilde{\lambda}(x)}\partial_x + \frac{1}{2}\partial_x \left(\omega(t, y)(1 + \lambda'(y))|_{y=x+\tilde{\lambda}(x)} \right) \\ &\stackrel{(4.1.3)}{=} \text{Op}^w(i\xi\omega(t, y)(1 + \lambda'(y))|_{y=x+\tilde{\lambda}(x)}). \end{aligned} \quad (4.2.15)$$

Recalling now the explicit expression of $\omega(t, x)$ in (4.2.14), we can rewrite

$$\begin{aligned} \Lambda^{-1} \text{Op}^w(i\xi\omega(t, x))\Lambda &\stackrel{(4.2.15)}{=} \text{Op}^w \left(i\xi(\varepsilon \langle V \rangle_\nu(y) + \varepsilon^2 Z_N(y))(1 + \lambda'(y))|_{y=x+\tilde{\lambda}(x)} \right) \\ &\quad + \varepsilon^{N+1} \text{Op}^w \left(\tilde{W}_N(t, y)(1 + \lambda'(y))|_{y=x+\tilde{\lambda}(x)} \right). \end{aligned} \quad (4.2.16)$$

We now find $\lambda \in C^\infty(\mathbb{T})$ such that

$$(1 + \lambda'(x))(\langle V \rangle_\nu(x) + \varepsilon Z_N(x)) = \hat{m}_N, \quad (4.2.17)$$

for some $\hat{m}_N \in \mathbb{R}$. Notice that, since $V \in \mathcal{V}_{\nu,s}$, up to shrinking $\varepsilon_N(V) > 0$ we can suppose that, for every $0 < \varepsilon < \varepsilon_N(V)$, the function $\langle V \rangle_\nu(x) + \varepsilon Z_N(x)$ does not have any zero (recall (4.0.2)). Thus (4.2.17) immediately implies $1 + \lambda'(x) \neq 0$ proving invertibility of Λ in (4.2.12). Plugging (4.2.17) in (4.2.16) we obtain (4.2.2) with $W_N(t, x) := \tilde{W}_N(t, y)(1 + \lambda'(y))|_{y=x+\tilde{\lambda}(x)}$ concluding the proof.

To find λ satisfying (4.2.17), we first rewrite it as

$$\lambda'(x) = \left(\frac{\hat{m}_N}{\langle V \rangle_\nu(x) + \varepsilon Z_N(x)} - 1 \right), \quad (4.2.18)$$

using again that $\langle V \rangle_\nu + \varepsilon Z_N(x)$ has no zeroes. Next, requiring $\lambda \in C^\infty(\mathbb{T})$ and integrating equation (4.2.18) on $\mathbb{T}$ one finds

$$\hat{m}_N = 2\pi \left(\int_0^{2\pi} (\langle V \rangle_\nu(x) + \varepsilon Z_N(x))^{-1} dx \right)^{-1} \in \mathbb{R}.$$

Remark that, since $V \in \mathcal{V}_{\nu,s}$ and since $\langle V \rangle_\nu + \varepsilon Z_N(x)$ has no zeroes, then the function $(\langle V \rangle_\nu(x) + \varepsilon Z_N(x))^{-1}$ has constant sign, and thus $\hat{m}_N$ can not be zero.

Finally, in order to find $\lambda(x)$, write its Fourier series as $\lambda(x) = \sum_{k \in \mathbb{Z}} \lambda_k e^{ikx}$, for all $x \in \mathbb{T}$. Next, let $f(x) := \left(\frac{\hat{m}_N}{\langle V \rangle_\nu(x) + \varepsilon Z_N(x)} - 1 \right)$ and write its Fourier series as $f(x) = \sum_{k \in \mathbb{Z}} f_k e^{ikx}$. Remark that, from (4.2.18), $f(x)$ has zero average, i.e., $f_0 = 0$. Thus (4.2.18) reads

$$\lambda_k = \frac{f_k}{ik} \quad \forall k \neq 0, \quad \lambda_0 = 0. \quad (4.2.19)$$

One can check that, since $f \in C^\infty(\mathbb{T}; \mathbb{R})$, then $\lambda(x) := \sum_{k \in \mathbb{Z}} \lambda_k e^{ikx}$, with $\{\lambda_k\}_{k \in \mathbb{Z}}$ given by (4.2.19) is a well defined, real valued and smooth function as well. By construction λ verifies (4.2.17), concluding the proof. $\square$

4.3 Instability

In this section we prove the second point of Theorem 4.0.1. As anticipated, the core of the proof is the construction of an escape function for $h(x, \xi) = \xi \langle V \rangle_\nu(x)$, with $V \in \mathcal{V}_{\nu,u}$ (see (4.0.3)). This is done in Section 4.3.1, adapting the construction of [26] (see also [61]). Next, in section 4.3.2 we prove existence of unstable solutions performing first a normal form reduction as the one in Lemma 4.2.2 and then a positive commutator estimate in which we exploit the properties of the escape function.

4.3.1 Construction of the escape function

In this section we consider a smooth function

$$h(x, \xi) := \xi X(x), \quad (x, \xi) \in T^*\mathbb{T} \simeq \mathbb{T} \times \mathbb{R}, \quad X \in C^\infty(\mathbb{T}), \quad (4.3.1)$$

we assume that that X is non degenerate and has at least one zero (see Definition 1.1.11) and we construct an *escape function* for h . We denote by $\mathcal{X}_h$ the Hamiltonian vector field of h on $T^*\mathbb{T}$, that in coordinates reads

$$\mathcal{X}_h(x, \xi) = (X(x), -\xi \partial_x X(x)), \quad \forall (x, \xi) \in T^*\mathbb{T}, \quad (4.3.2)$$

and we denote by $\Phi_{\mathcal{X}_h}^t(x, \xi)$ the flow of $\mathcal{X}_h$ on $T^*\mathbb{T}$. Finally, for every $f \in C^\infty(T^*\mathbb{T})$ we write the Poisson brackets between h and f as

$$\{h, f\} := \partial_\xi h \partial_x f - \partial_x h \partial_\xi f.$$

We now give the definition of escape function. For all this section we will use the notation $T^*\mathbb{T} \setminus \{0\} = \{(x, \xi) \in T^*\mathbb{T} : \xi \neq 0\}$.

Definition 4.3.1. We say that a smooth function $f : T^*\mathbb{T} \setminus \{0\} \rightarrow \mathbb{R}$ is positively homogeneous of degree ρ if

$$f(x, \lambda \xi) = \lambda^\rho f(x, \xi), \quad \forall (x, \xi) \neq 0, \quad \forall \lambda > 0.$$

Definition 4.3.2 (Escape function). Let $h : T^*\mathbb{T} \rightarrow \mathbb{R}$ be as in (4.3.1). We say that a function $a \in C^\infty(T^*\mathbb{T} \setminus \{0\})$, positively homogeneous of degree one (see Definition 4.3.1) is an *escape function* for h if there exists some positive δ such that

$$\{h, a\} \geq \delta |\xi|, \quad \forall (x, \xi) \in T^*\mathbb{T} \setminus \{0\}. \quad (4.3.3)$$

The main results of this section is the following proposition.

Proposition 4.3.3. Let $X \in C^\infty(\mathbb{T})$ and suppose that X has at least one zero and that all the zeroes are non degenerate (see Definition 1.1.11). Then the function h defined in (4.3.1) admits an escape function in the sense of Definition 4.3.2. Moreover, there exists an open subset $\mathcal{W} \subset \mathbb{T}$ such that

$$a(x, \xi) \leq -\frac{|\xi|}{2}, \quad \forall x \in \mathcal{W}. \quad (4.3.4)$$

Dynamical properties. In order to prove Proposition 4.3.3 we need some dynamical properties of the Hamiltonian flow induced by $h(x, \xi)$ in (4.3.1), that we list in this section. First of all, we recall (see, for example, [62, Chapter 9]) that a smooth vector field is *complete* if it generates a global flow, or equivalently if, for every initial datum, the associated flow exists for all in times. We have the following result, for whose proof we refer to [83, Section 5.2].

Lemma 4.3.4. *Let h as in (4.3.1). Then the Hamiltonian vector field $\mathcal{X}_h$ (see (4.3.2)) is complete.*

Remark 4.3.5. Denoting by $\Phi_{\mathcal{X}_h}^t(x, \xi) = (X^t(x, \xi), \Xi^t(x, \xi))$ the components of the flow of $\mathcal{X}_h$ in (4.3.2) with initial datum $(x, \xi) \in T^*\mathbb{T}$, the following homogeneity property holds:

$$(X^t(x, \lambda\xi), \Xi^t(x, \lambda\xi)) = (X^t(x, \xi), \lambda\Xi^t(x, \xi)), \quad \forall \lambda \in \mathbb{R}, \quad \forall (x, \xi) \in T^*\mathbb{T}, \quad \forall t \geq 0.$$

We now identify global attractors and repellers for the flow. In particular, we show that the dynamics of $\mathcal{X}_h$ on $T^*\mathbb{T}$ is completely determined once the zeroes of X are known. To this aim let us first introduce the following terminology:

Definition 4.3.6 (Attractor-Repellor structure). We say that the flow $\Phi_{\mathcal{X}_h}^t$ on $T^*\mathbb{T}$ has a *global attractor-repellor* structure if there exist A, R closed invariant subsets of $T^*\mathbb{T}$ such that:

$$\begin{aligned} &\text{for every } (x, \xi) \in T^*\mathbb{T} \setminus R, \quad \Phi_{\mathcal{X}_h}^t(x, \xi) \rightarrow A \quad \text{as } t \rightarrow +\infty, \\ &\text{for every } (x, \xi) \in T^*\mathbb{T} \setminus A \quad \Phi_{\mathcal{X}_h}^t(x, \xi) \rightarrow R \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

We denote the basins of attraction of A and R respectively as $B(A) := T^*\mathbb{T} \setminus (R)$ and $B(R) := T^*\mathbb{T} \setminus (A)$.

Lemma 4.3.7. *Let $h(x, \xi) = \xi X(x)$ and suppose that X has at least one zero, but does not have any degenerate zero (see Definition 1.1.11). Denote by*

$$K^+ := \{x \in \mathbb{T} : X(x) = 0, \quad X'(x) < 0\} \quad \text{and} \quad K^- := \{x \in \mathbb{T} : X(x) = 0, \quad X'(x) > 0\}, \quad (4.3.5)$$

and by

$$\Gamma^+ := \{(x, \xi) : x \in K^+\} \simeq K^+ \times \mathbb{R} \quad \text{and} \quad \Gamma^- := \{(x, \xi) : x \in K^-\} \simeq K^- \times \mathbb{R}. \quad (4.3.6)$$

Then the sets Γ^+ , Γ^- and $\mathcal{T} := \{\xi = 0\}$ are invariant for the flow of $\mathcal{X}_h$ in (4.3.2). Moreover the flow $\Phi_{\mathcal{X}_h}^t$ has a global attractor-repellor structure (see Definition 4.3.6) with attractor Γ^+ and repellor Γ^- . In particular, for all $(x_0, \xi_0) \in T^*\mathbb{T} \setminus (\Gamma^+ \cup \Gamma^-)$ we have

$$\Phi_{\mathcal{X}_h}^t(x_0, \xi_0) \rightarrow \Gamma^+, \quad \text{as } t \rightarrow +\infty, \quad \text{and} \quad \Phi_{\mathcal{X}_h}^t(x_0, \xi_0) \rightarrow \Gamma^-, \quad \text{as } t \rightarrow -\infty. \quad (4.3.7)$$

Proof. First of all remark that, since X has no degenerate zeroes, then

$$X^{-1}(0) = K^+ \cup K^-.$$

Moreover, using the explicit expression of $\mathcal{X}_h$ in (4.3.2), one can check that $\mathcal{T}$, Γ^+ and Γ^- are the only invariant sets for the flow. We now prove (4.3.7): let $(x_0, \xi_0) \in T^*\mathbb{T} \setminus (\Gamma^+ \cup \Gamma^-)$ and recall that, from Lemma 4.3.4, the flow $\Phi_{\mathcal{X}_h}^t(x_0, \xi_0)$ exists for all $t \in \mathbb{R}$. Thus, in order to prove (4.3.7) we have to show that $x(t) \rightarrow K^+$ as $t \rightarrow +\infty$ and $x(t) \rightarrow K^-$ as $t \rightarrow -\infty$ (see (4.3.5) and (4.3.6)). This follows immediately from the fact that the dynamics of $x(t)$ is determined by the one-dimensional system $\dot{x} = X(x)$ (see (4.3.2)) and from the definitions of K^+ and K^- in (4.3.5), thus we omit the details of this last part of the proof. $\square$

Remark 4.3.8. From the structure of the flow described in Lemma 4.3.7 one can check that any conical neighborhood $\mathcal{U}^+$ of Γ^+ is positively invariant for the flow (i.e., $\Phi_{\mathcal{X}_h}^t(\mathcal{U}^+) \subset \mathcal{U}^+$, for all $t > 0$) and analogously any conical neighborhood of Γ^- is invariant for the flow at negative times.

Construction of the escape function

In this section we use the structure of the flow of $\mathcal{X}_h$ (see (4.3.2)) described in Lemma 4.3.7 and the sets defined therein ((4.3.5), (4.3.6)) to construct an escape function for $h(x, \xi)$ in the sense of Definition 4.3.2. We first construct a local escape function close to the zeroes of X and then we progressively extend it to the whole space.

Lemma 4.3.9. *Let $h(x, \xi) = \xi X(x)$ and suppose that $X \in C^\infty(\mathbb{T})$ has at least one zero and is non degenerate (see Definition 1.1.11). There exist conical neighborhoods $\mathcal{U}^\pm$ of $\Gamma^\pm$ (see (4.3.6)), $\delta > 0$ and a smooth function $k \in C^\infty(T^*\mathbb{T} \setminus \{0\})$ such that*

$$k(x, \xi) = \pm|\xi| \text{ on } \Gamma^\pm, \quad \text{and} \quad \{h, k\} \geq \delta|\xi|, \quad \forall (x, \xi) \in \mathcal{U}^+ \cup \mathcal{U}^-. \quad (4.3.8)$$

Moreover the function k is positively homogeneous of degree one (see Definition 4.3.1) on its domain.

Proof. Denote by $\{x_1, \dots, x_n\}$ the zeroes of $X(x)$ and, using non-degeneracy of X , let $\mu := \min_i |X'(x_i)| > 0$. Let $\tilde{\mathcal{U}}^\pm \subseteq \mathbb{T}$ be neighborhoods of $K^\pm$ in $\mathbb{T}$ (see (4.3.5)) to be fixed later, and let $\tilde{k} \in C^\infty(\mathbb{T})$ be such that

$$\tilde{k}(x) = \begin{cases} 1 & \text{if } x \in \tilde{\mathcal{U}}^+ \\ -1 & \text{if } x \in \tilde{\mathcal{U}}^-. \end{cases} \quad (4.3.9)$$

Set $k(x, \xi) := |\xi| \tilde{k}(x)$. Clearly k is positively homogeneous of degree one and $k = \pm|\xi|$ on $\Gamma^\pm$. Moreover, computing $\{h, k\}(x_i)$ for any $i \in \{1, \dots, n\}$ we also have

$$\{h, k\}(x_i, \xi) = -|\xi| X'(x_i) \tilde{k}(x_i) = |\xi| |X'(x_i)| \geq |\xi| \mu, \quad \forall i = 1, \dots, n,$$

where for the last equality we have used (4.3.9) and the definition of $K^\pm$ in (4.3.5). Thus, up to choosing $\tilde{\mathcal{U}}^\pm$ small enough, we have by continuity $\{h, k\} \geq \frac{\mu}{2}|\xi|$ on $\mathcal{U}^+ \cup \mathcal{U}^-$ (where $\mathcal{U}^\pm := \tilde{\mathcal{U}}^\pm \times \mathbb{R}$), concluding the proof of (4.3.8) with $\delta := \frac{\mu}{2}$. $\square$

The next goal is to extend the local escape function k to the basins of attraction $B(\Gamma^\pm) := T^*\mathbb{T} \setminus \Gamma^\mp$ (see Definition 4.3.6). To this aim we first need the following auxiliary function, which extends the Poisson brackets $\{h, k\}$, defined on $\mathcal{U}^\pm$, to a positive function defined on the whole $T^*\mathbb{T}$.

Lemma 4.3.10. *There exists a smooth function $\mathfrak{m} : T^*\mathbb{T} \setminus \{0\} \rightarrow \mathbb{R}$, positively homogeneous of degree one, such that*

$$\mathfrak{m}(x, \xi) = \{h, k\}(x, \xi), \quad \forall (x, \xi) \in \mathcal{U}^\pm, \quad \text{and} \quad \mathfrak{m}(x, \xi) \geq \frac{\delta}{2}|\xi|, \quad \forall (x, \xi) \in T^*\mathbb{T}, \quad (4.3.10)$$

where k , δ and $\mathcal{U}^\pm$ are as in Lemma 4.3.9.

Proof. Let $\mathcal{U}^\pm$ be the conical neighborhoods of $\Gamma^\pm$ given by Lemma 4.3.9. Denote by $\tilde{\mathcal{U}}^\pm \subset \mathbb{T}$ the neighborhoods of $K^\pm$ obtained projecting $\mathcal{U}^\pm$, i.e. $\tilde{\mathcal{U}}^\pm := \{x \in \mathbb{T} : (x, \xi) \in \mathcal{U}^\pm, \forall \xi \in \mathbb{R}\}$. Then it is possible to define $\tilde{\mathfrak{m}} : \mathbb{T} \rightarrow \mathbb{R}$, a smooth function satisfying

$$\tilde{\mathfrak{m}}(x) = \{h, k\}(x, 1) \quad \forall x \in (\overline{\mathcal{U}^+} \cup \overline{\mathcal{U}^-}) \quad \text{and} \quad \tilde{\mathfrak{m}} \geq \frac{\delta}{2} \quad \forall x \in \mathbb{T}. \quad (4.3.11)$$

This can be done since, from (4.3.8), it is possible to extend $\{h, k\}(x, 1)$ in a smooth way outside $\overline{\tilde{\mathcal{U}}^\pm}$, preserving positivity as in (4.3.11). Finally, we define $\mathfrak{m} : T^*\mathbb{T} \setminus \{0\} \rightarrow \mathbb{R}$ extending $\tilde{\mathfrak{m}}$ by homogeneity, i.e. $\mathfrak{m}(x, \xi) := \tilde{\mathfrak{m}}(x)|\xi|$, for all $(x, \xi) \in T^*\mathbb{T}$, obtaining that $\mathfrak{m}$ is a positively homogeneous function of degree one and satisfies (4.3.10). $\square$

We can now extend the local escape functions to the sets $B(\Gamma^\pm) := T^*\mathbb{T} \setminus \Gamma^\mp$ (see Definition 4.3.6). In the next Lemma we write $z := (x, \xi) \in T^*\mathbb{T}$ for simplicity in the notation.

Lemma 4.3.11. *Let $z = (x, \xi) \in B(\Gamma^\pm)$ and define*

$$\ell^\pm(z) := \lim_{t \rightarrow \pm\infty} \left[k(\Phi_{\mathcal{X}_h}^t(z)) - \int_0^t \mathfrak{m}(\Phi_{\mathcal{X}_h}^s(z)) ds \right], \quad (4.3.12)$$

where k is given by Lemma 4.3.9 and $\mathfrak{m}$ by Lemma 4.3.10. Then

(i) $\ell^\pm$ are well defined on $B(\Gamma^\pm)$, smooth, positively homogeneous of degree one and $\ell^\pm = k$ on $\Gamma^\pm$;

(ii) $\ell^\pm$ are escape functions on $B(\Gamma^\pm)$. In particular, let $\delta > 0$ be given by Lemma 4.3.9, then

$$\{h, \ell^\pm\} \geq \frac{\delta}{2} |\xi|, \quad \forall z \in B(\Gamma^\pm). \quad (4.3.13)$$

Proof. We prove the statement with $+$.

Let $z \in B(\Gamma^+)$ and let $t_0(z) \geq 0$ be such that $\Phi_{\mathcal{X}_h}^{t_0(z)}(z) \in \mathcal{U}^+$, with $\mathcal{U}^+$ the neighborhood of Γ^+ defined in Lemma 4.3.9. By Definition 4.3.6 of $B(\Gamma^+)$, $t_0(z)$ is finite. We show that for any $t > t_0$, the argument of the limit in (4.3.12) is constant. Since the set $\mathcal{U}^+$ is positively invariant for the flow of $\mathcal{X}_h$ (see Remark 4.3.8), we have $\mathfrak{m}(\Phi_{\mathcal{X}_h}^t(z)) =$

$\{h, k\}(\Phi_{\mathcal{X}_h}^t(z))$ for any $t \geq t_0(z)$ (recall (4.3.10)). Therefore, for $t \geq t_0(z)$ one has

$$\begin{aligned}
 k(\Phi_{\mathcal{X}_h}^t(z)) - \int_0^t \mathfrak{m}(\Phi_{\mathcal{X}_h}^s(z)) ds &= k(\Phi_{\mathcal{X}_h}^t(z)) - \int_0^{t_0(z)} \mathfrak{m}(\Phi_{\mathcal{X}_h}^s(z)) ds - \int_{t_0(z)}^t \{h, k\}(\Phi_{\mathcal{X}_h}^s(z)) ds \\
 &= k(\Phi_{\mathcal{X}_h}^t(z)) - \int_0^{t_0(z)} \mathfrak{m}(\Phi_{\mathcal{X}_h}^s(z)) ds - k(\Phi_{\mathcal{X}_h}^t(z)) + k(\Phi_{\mathcal{X}_h}^{t_0(z)}(z)) \\
 &= - \int_0^{t_0(z)} \mathfrak{m}(\Phi_{\mathcal{X}_h}^s(z)) ds + k(\Phi_{\mathcal{X}_h}^{t_0(z)}(z)), \tag{4.3.14}
 \end{aligned}$$

and thus the argument of the limit is definitely constant, proving that ℓ^+ is well defined.

Moreover, if $z \in \Gamma^+$, then $t_0(z) = 0$ and (4.3.14) reads $\ell^+(z) = k(z)$.

Smoothness: fix $z_0 \in B(\Gamma^+)$ and let $V_{z_0} \subset B(\Gamma^+)$ be a neighborhood of z_0 . We observe that, from the global picture of the flow of $\mathcal{X}_h$ in Lemma 4.3.7 and Remark 4.3.8, there exists $t_1(V_{z_0}) > 0$, possibly larger than $t_0(z_0)$, such that $\Phi_{\mathcal{X}_h}^t(z) \in \mathcal{U}^+$ for all $z \in V_{z_0}$ and $t \geq t_1(V_{z_0})$. Then, by (4.3.14), on the set V_{z_0} the function ℓ^+ reduces to

$$\ell^+(z) = - \int_0^{t_1(V_{z_0})} \mathfrak{m}(\Phi_{\mathcal{X}_h}^s(z)) ds + k(\Phi_{\mathcal{X}_h}^{t_1(V_{z_0})}(z)), \quad \forall z \in V_{z_0}, \tag{4.3.15}$$

which is smooth.

Homogeneity: Using remark 4.3.5 and recalling the homogeneity properties of $\mathfrak{m}$ and k given in Lemma 4.3.10 and 4.3.9 respectively, then one can directly check from the definition of ℓ^+ in (4.3.12) that $\ell^+(x, \lambda\xi) = \lambda\ell^+(x, \xi)$, for all $\lambda > 0$, proving that ℓ^+ is positively homogeneous of degree one.

(ii) Given $z_0 \in B(\Gamma^+)$ and $V_{z_0} \subset B(\Gamma^+)$ a small neighborhood of z_0 , let $t_1(V_{z_0})$ as above. We have

$$\begin{aligned}
 \{h, \ell^+\}(z) &\stackrel{(4.3.15)}{=} \{h, k \circ \Phi_{\mathcal{X}_h}^{t_1(V_{z_0})}\}(z) - \int_0^{t_1(V_{z_0})} \{h, \mathfrak{m} \circ \Phi_{\mathcal{X}_h}^s(z)\} ds \\
 &= \{h, k\}(\Phi_{\mathcal{X}_h}^{t_1(V_{z_0})}(z)) - \frac{d}{dt} \Big|_{t=0} \int_0^{t_1(V_{z_0})} \mathfrak{m}(\Phi_{\mathcal{X}_h}^{t+s}(z)) ds \\
 &= \{h, k\}(\Phi_{\mathcal{X}_h}^{t_1(V_{z_0})}(z)) - \mathfrak{m}(\Phi_{\mathcal{X}_h}^{t_1(V_{z_0})}(z)) + \mathfrak{m}(z) \\
 &= \mathfrak{m}(z) \stackrel{(4.3.10)}{\geq} \frac{\delta}{2} |\xi|,
 \end{aligned}$$

concluding the proof. $\square$

Finally we conclude the construction gluing together the functions ℓ^+ and ℓ^- of Lemma 4.3.11.

Lemma 4.3.12. *Given $\sigma > 0$ sufficiently small, there exists a smooth function $\eta_\sigma : \mathbb{T} \rightarrow \mathbb{R}$, satisfying $\eta_\sigma(K^+) \equiv 1$, $\eta_\sigma(K^-) \equiv 0$, and such that*

$$\left| \frac{d}{dt} \Big|_{t=0} \eta_\sigma[X^t(x, \xi)] \right| < \sigma, \quad \forall (x, \xi) \in \mathbb{T} \times \mathbb{R}. \quad (4.3.16)$$

where we recall the notation in Remark 4.3.5 for the components of the flow of $\mathcal{X}_h$.

Proof. Recalling the definition of K^- in (4.3.5), let $\tilde{\mathcal{U}}^- \subseteq \mathbb{T}$ be an open neighborhood of K^- . Using again the dynamics of $X^t(x, \xi)$ (see (4.3.2)), we deduce that for any $x \notin K^+ \cup K^-$ there exists a unique $t_x \in \mathbb{R}$ such that $X^{-t_x}(x, \xi) \in \partial\tilde{\mathcal{U}}^-$ (see Remark 4.3.8). Define then

$$\tilde{t}(x) := \begin{cases} t_x, & x \in \mathbb{T} \setminus (K^+ \cup K^-) \\ +\infty, & x \in K^+ \\ -\infty, & x \in K^-. \end{cases} \quad (4.3.17)$$

Remark that for any $t \in \mathbb{R}$ one has

$$\tilde{t}(X^t(x, \xi)) = \tilde{t}(x) + t. \quad (4.3.18)$$

Next take a smooth non-decreasing function $\phi_\sigma : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\phi_\sigma(\tau) := \begin{cases} 0, & \tau \leq -\sigma \\ \sigma\tau, & \sigma < \tau \leq \sigma^{-1} - \sigma \\ 1, & \tau > \sigma^{-1} + \sigma, \end{cases} \quad \text{and} \quad \sup_{\tau \in \mathbb{R}} |\phi'_\sigma(\tau)| \leq 10\sigma. \quad (4.3.19)$$

We now prove that the function η_σ claimed by the lemma is $\eta_\sigma(x) := \phi_{\frac{\sigma}{10}}(\tilde{t}(x))$, for all $x \in \mathbb{T}$, with $\tilde{t}$ defined in (4.3.17) and $\phi_{\frac{\sigma}{10}}$ in (4.3.19). First of all notice that, from the definition of $\tilde{t}$ in (4.3.17), if $x \in K^+$ then $\eta_\sigma(x) = 1$ and analogously if $x \in K^-$, then $\eta_\sigma(x) = 0$. Next notice that, for any $(x, \xi) \in \mathbb{T} \times \mathbb{R}$ we have:

$$\begin{aligned} \left| \frac{d}{dt} \Big|_{t=0} \eta_\sigma[X^t(x, \xi)] \right| &= \left| \frac{d}{dt} \Big|_{t=0} \phi_{\frac{\sigma}{10}}(\tilde{t}(X^t(x, \xi))) \right| \\ &\stackrel{(4.3.18)}{=} \left| \frac{d}{dt} \Big|_{t=0} \phi_{\frac{\sigma}{10}}(\tilde{t}(x) + t) \right| = \left| \phi'_{\frac{\sigma}{10}}(\tilde{t}(x)) \right| \stackrel{(4.3.19)}{<} \sigma. \end{aligned}$$

This proves point (ii) and concludes the proof of the Lemma. $\square$

We can now conclude the proof of Proposition 4.3.3.

Proof of Proposition 4.3.3. Define, for $\sigma > 0$ sufficiently small and to be fixed later,

$$a(x, \xi) := \eta_\sigma(x)\ell^+(x, \xi) + (1 - \eta_\sigma)\ell^-(x, \xi),$$

with η_σ defined in Lemma 4.3.12 and ℓ^+ and ℓ^- in Lemma 4.3.11. First of all notice that a is well defined, since $\eta_\sigma(x) = 0$ for all $(x, \xi) \in \Gamma^-$, where ℓ^+ is not defined and analogously $(1 - \eta_\sigma)(x) = 0$ for all $(x, \xi) \in \Gamma^+$ where ℓ^- is not defined. Next remark that a is smooth in $T^*\mathbb{T} \setminus \{0\}$, since both $\ell^\pm$ and η_σ are and, from the homogeneity properties of η_σ and $\ell^\pm$, we infer that a is positively homogeneous of degree one.

Next we show property (4.3.4). We remark that $a(x, \xi) = -|\xi|$ on Γ^- : indeed, $\eta_\sigma = 0$ on K^- from Lemma 4.3.12 and putting together (4.3.8) and point (i) of Lemma 4.3.11, we get $\ell^-(x, \xi) = k(x, \xi) = -|\xi|$ on Γ^- . Thus from continuity of a there exists $\mathcal{W}$ neighborhood of K^- such that (4.3.4) holds for all $x \in \mathcal{W}$.

We are left to prove (4.3.3). We have

$$\{h, a\} = \{h, \eta_\sigma\}\ell^+ + \{h, \ell^+\}\eta_\sigma + \{h, \ell^-\}(1 - \eta_\sigma) - \{h, \eta_\sigma\}\ell^- \stackrel{(4.3.13)}{\geq} \frac{\delta}{2}|\xi| + \{h, \eta_\sigma\}(\ell^+ - \ell^-). \quad (4.3.20)$$

Finally, remark that

$$\{h, \eta_\sigma\}(x, \xi) = X(x)\eta'_\sigma(x) = \frac{d}{dt}\bigg|_{t=0}\eta_\sigma(X^t(x, \xi)),$$

thus using (4.3.16) we have $|\{h, \eta_\sigma\}| < \sigma$ and this gives

$$\{h, a\} \stackrel{(4.3.20)}{\geq} \frac{\delta}{2}|\xi| - |\{h, \eta_\sigma\}||\xi| \underbrace{\max_{\mathbb{T}} \left| (\ell^+ - \ell^-) \left(x, \frac{\xi}{|\xi|} \right) \right|}_{=: C} = |\xi| \left(\frac{\delta}{2} - C\sigma \right),$$

where for this last inequality we have used homogeneity of $\ell^\pm$. In conclusion, choosing $\sigma < \frac{\delta}{4C}$ we obtain (4.3.3) with $\delta \rightsquigarrow \frac{\delta}{4}$. $\square$

4.3.2 Energy estimate

We are now in position to prove the second point of Theorem 4.0.1. We need to first transform equation (4.0.1) in a new transport equation, having $\langle V \rangle_\nu(x)$ as leading

transport term (recall Definition 1.1.10). Precisely, given any $V \in \mathcal{V}_{\nu,u}$, we apply Lemma 4.2.2 with $N = 1$ (see Remark 4.2.3): there exists $\varepsilon_1(V) > 0$ and for every $0 < \varepsilon < \varepsilon_1(V)$ a 2π -periodic, linear transformation $\Psi_{1,\varepsilon}(t)$, such that, given $u(t, x)$ solution of equation (4.0.1) and defining $u_1(t, x)$ as $u(t, x) = \Psi_{1,\varepsilon}(t)u_1(t, x)$, then $u_1(t, x)$ solves

$$\partial_t u_1 = \text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W(t, x)))u_1(t, x), \quad (4.3.21)$$

for some $W \in C^\infty(\mathbb{T}^2; \mathbb{R})$. Moreover, the transformation $\Psi_{1,\varepsilon}(t) : \mathcal{H}^s \rightarrow \mathcal{H}^s$ is bounded for any $s \in \mathbb{R}$ with bounded inverse. We now show that there exists $u_{1_0} \in C^\infty(\mathbb{T})$ such that, if $u_1(t, x)$ is the solution of (4.3.21) satisfying $u_1(0, x) = u_{1_0}(x)$, then for every $s > 1$ we have

$$\|u_1(t)\|_s \geq \tilde{\delta}_1(u_0)e^{\varepsilon\delta_2 t}, \quad \forall t \in \mathbb{R}, \quad (4.3.22)$$

for some $\tilde{\delta}_1 \equiv \tilde{\delta}_1(u_0) > 0$, $\delta_2 > 0$. Remark that, from boundedness of the transformation $\Psi_{1,\varepsilon}$ and of its inverse, this implies inequality (4.0.5), concluding the proof. Thus in the rest of this section we prove (4.3.22). Note that we will perform an energy estimate analogous to the one presented in Section 3.3.3, with the due modifications.

Energy Estimate. In order to prove (4.3.22) first recall that, since $V \in \mathcal{V}_{\nu,u}$, we can apply Proposition 4.3.3 to $h(x, \xi) = \xi \langle V \rangle_\nu(x)$: denote by $a(x, \xi) \in C^\infty(T^*\mathbb{T} \setminus \{0\})$ the escape function for $h(x, \xi)$ constructed in Proposition 4.3.3. We start by associating a symbol $\tilde{a} \in \mathbf{S}^1(\mathbb{T})$ (see (4.1.1)) to the escape function: take $\chi \in C_c^\infty(T^*\mathbb{T}; [0, 1])$, satisfying $\chi(x, \xi) = 1$ for all $|\xi| \leq \frac{1}{2}$ and $\chi(x, \xi) = 0$ for all $|\xi| \geq 1$ and define

$$\tilde{a}(x, \xi) = (1 - \chi)(x, \xi)a(x, \xi), \quad \forall (x, \xi) \in \mathbb{T} \times \mathbb{R}. \quad (4.3.23)$$

Clearly $\tilde{a} \in C^\infty(T^*\mathbb{T})$ and $\tilde{a} = a$ on $|\xi| \geq 1$. Using the homogeneity of a one can check that $\tilde{a} \in \mathbf{S}^1$. We now use a virial argument like in [71]: given any $u_1(t, x)$ solution of (4.3.21), we define

$$\mathcal{A}(t) := \langle \text{Op}^w(-\tilde{a})u_1(t), u_1(t) \rangle_{L^2(\mathbb{T})} \quad \forall t \in \mathbb{R}. \quad (4.3.24)$$

We have the following result:

Lemma 4.3.13. *Let $u_1(t, x)$ be a solution of equation (4.3.21) with initial condition $u_1(0, x) = u_{1_0}(x)$. There exist constants $\delta_2, \beta_{\varepsilon, W, \tilde{a}} > 0$ such that*

$$\dot{\mathcal{A}}(t) \geq \varepsilon \left(\delta_2 \mathcal{A}(t) - \beta_{\varepsilon, W, \tilde{a}} \|u_{1_0}\|_{L^2(\mathbb{T})}^2 \right), \quad \forall t \in \mathbb{R}, \quad (4.3.25)$$

where $\mathcal{A}(t)$ is defined in (4.3.24).

Before proving Lemma 4.3.13 let us remark that, integrating inequality (4.3.25) one obtains

$$\mathcal{A}(t) \geq e^{\varepsilon \delta_2 t} \left(\mathcal{A}(0) - \frac{\beta_{\varepsilon, W, \tilde{a}}}{\delta_2} \|u_{1_0}\|_{L^2(\mathbb{T})}^2 \right). \quad (4.3.26)$$

On the other hand, using Cauchy-Schwarz inequality and property (4.1.2), we find $C_{\tilde{a}} > 0$ such that

$$\mathcal{A}(t) \stackrel{(4.3.24)}{\leq} \|\text{Op}^w(-\tilde{a})u_1(t)\|_{L^2(\mathbb{T})} \|u_1(t)\|_{L^2(\mathbb{T})} \stackrel{(4.1.2)}{\leq} C_{\tilde{a}} \|u_1(t)\|_{\mathcal{H}^1} \|u_{1_0}\|_{L^2(\mathbb{T})}, \quad (4.3.27)$$

where in the last inequality we used that, for any solution $u_1(t, x)$ of (4.3.21) we have conservation of the L^2 -norm of the solution, i.e., $\|u_1(t)\|_{L^2(\mathbb{T})} = \|u_{1_0}\|_{L^2(\mathbb{T})}$, for every $t \in \mathbb{R}$, due to skewadjointness of the operator $\text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W(t, x)))$ (see (4.3.21)). Thus putting together (4.3.26) and (4.3.27) we have

$$\|u_1(t)\|_{\mathcal{H}^1} \geq \left(C_{\tilde{a}} \|u_{1_0}\|_{L^2(\mathbb{T})} \right)^{-1} e^{\varepsilon \delta_2 t} \left(\mathcal{A}(0) - \frac{\beta_{\varepsilon, W, \tilde{a}}}{\delta_2} \|u_{1_0}\|_{L^2(\mathbb{T})}^2 \right). \quad (4.3.28)$$

In the next paragraph we will show how to find $u_{1_0} \neq 0$ such that

$$\left(\mathcal{A}(0) - \frac{\beta_{\varepsilon, W, \tilde{a}}}{\delta_2} \|u_{1_0}\|_{L^2(\mathbb{T})}^2 \right) =: c_0 > 0. \quad (4.3.29)$$

This would conclude the proof of (4.3.22) for $s = 1$ since using (4.3.29), then (4.3.28) gives (4.3.22), with $\tilde{\delta}_1 := c_0 (C_{\tilde{a}} \|u_{1_0}\|_{L^2(\mathbb{T})})^{-1} > 0$. Finally remark that $\|u\|_s \geq \|u\|_1$ for every $s \geq 1$, thus exponential blow up of the Sobolev norm of the solutions to (4.3.21) holds for any $s \geq 1$. We now prove Lemma 4.3.13.

Proof of Lemma 4.3.13. First of all remark that, computing the time derivative of $\mathcal{A}(t)$ in (4.3.24) and using skewadjointness of $\text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W(t, x)))$ in (4.3.21), we have

$$\dot{\mathcal{A}}(t) = \langle [\text{Op}^w(-\tilde{a}), \text{Op}^w(i\xi(\varepsilon \langle V \rangle_\nu(x) + \varepsilon^2 W(t, x)))] u_1, u_1 \rangle_{L^2(\mathbb{T})}. \quad (4.3.30)$$

Next, using results of symbolic calculus we have

$$\begin{aligned}
& [\text{Op}^w(-\tilde{a}), \text{Op}^w(i\xi(\varepsilon\langle V \rangle_\nu(x) + \varepsilon^2 W(t, x)))] \\
& \stackrel{(4.1.4)}{=} \text{Op}^w(\{-\tilde{a}, \xi(\varepsilon\langle V \rangle_\nu(x) + \varepsilon^2 W(t, x))\}) + \varepsilon \mathcal{B}(t) \\
& = \varepsilon \text{Op}^w(\{\xi\langle V \rangle_\nu(x), \tilde{a}\}) + \varepsilon^2 \text{Op}^w(\{\xi W(t, x), \tilde{a}\}) + \varepsilon \mathcal{B}(t),
\end{aligned} \tag{4.3.31}$$

with $\mathcal{B}(t) \in C^\infty(\mathbb{T}, \mathcal{S}^0)$ bounded remainder resulting from (4.1.4). Since $\tilde{a} \in \mathbf{S}^1$, then $\{\tilde{a}, \xi W(t, x)\} \in C^\infty(\mathbb{T}, \mathbf{S}^1)$ (see (4.1.4)). In particular, denoting by $M_W := \max_{|\xi|=1} |\{\xi W(x), \tilde{a}(x, \xi)\}| \geq 0$ one has

$$\text{Op}^w(\{\xi W(t, x), a\}) \geq -M_W \text{Op}^w(|\xi|). \tag{4.3.32}$$

Moreover, from boundedness of $\mathcal{B}(t)$ there exists $C_{\mathcal{B}} > 0$ such that

$$\langle \mathcal{B}(t)u_1, u_1 \rangle_{L^2(\mathbb{T})} \geq -C_{\mathcal{B}} \|u_1\|_{L^2(\mathbb{T})}^2, \quad \forall t \in \mathbb{R}. \tag{4.3.33}$$

Recalling again that $\|u_1(t)\|_{L^2(\mathbb{T})} = \|u_{10}\|_{L^2(\mathbb{T})}$ and plugging (4.3.33), (4.3.32) and (4.3.31) in (4.3.30) we get

$$\dot{\mathcal{A}}(t) \geq \varepsilon \langle \text{Op}^w(\{\xi\langle V \rangle_\nu(x), \tilde{a}\} - \varepsilon M_W |\xi|)u_1, u_1 \rangle_{L^2(\mathbb{T})} - \varepsilon C_{\mathcal{B}} \|u_{10}\|_{L^2(\mathbb{T})}^2, \quad \forall t \in \mathbb{R}. \tag{4.3.34}$$

Thus we are left to show that there exists $\delta_2, \tilde{C}_{\tilde{a}} > 0$ such that

$$\langle \text{Op}^w(\{\xi\langle V \rangle_m(x), \tilde{a}\} - \varepsilon M_W |\xi|)u_1, u_1 \rangle_{L^2(\mathbb{T})} \geq \delta_2 \mathcal{A}(t) - \tilde{C}_{\tilde{a}} \|u_{10}\|_{L^2(\mathbb{T})}^2, \quad \forall t \in \mathbb{R} \tag{4.3.35}$$

Indeed, putting together (4.3.35) and (4.3.34), we get (4.3.25) with $\beta_{\varepsilon, W, \tilde{a}} := C_{\mathcal{B}} + \tilde{C}_{\tilde{a}}$. To prove (4.3.35), let $\chi_1 \in C_c^\infty(\mathbb{T} \times \mathbb{R}; [0, 1])$ be such that $\chi_1 = 1$ if $|\xi| \leq 1$ and $\chi_1 = 0$ if $|\xi| \geq 2$ and write

$$\begin{aligned}
\text{Op}^w(\{\xi\langle V \rangle_\nu(x), \tilde{a}\} - \varepsilon M_W |\xi|) &= \text{Op}^w((\{\xi\langle V \rangle_\nu(x), \tilde{a}\} - \varepsilon M_W |\xi|)(1 - \chi_1)) \\
&\quad + \text{Op}^w((\{\xi\langle V \rangle_\nu(x), \tilde{a}\} - \varepsilon M_W |\xi|)\chi_1).
\end{aligned}$$

Remark that, by (4.3.23), $\{\xi\langle V \rangle_\nu(x), \tilde{a}\}(1 - \chi_1) = \{\xi\langle V \rangle_\nu(x), a\}(1 - \chi_1)$. Moreover, from point 6 of Lemma 4.1.1 we have $\text{Op}^w((\{\xi\langle V \rangle_\nu(x), \tilde{a}\} - \varepsilon M_W |\xi|)\chi_1) \in \mathcal{S}^\rho$, for any

$\rho < 0$. In particular there exists $C_{\chi_1} > 0$ such that

$$\begin{aligned} \langle \text{Op}^w(\{\xi \langle V \rangle_\nu(x), \tilde{a}\} - \varepsilon M_W |\xi|) u_1, u_1 \rangle &\geq \langle \text{Op}^w((\{\xi \langle V \rangle_\nu(x), a\} - \varepsilon M_W |\xi|)(1 - \chi_1)) u_1, u_1 \rangle \\ &\quad - C_{\chi_1} \|u_1\|_{L^2(\mathbb{T})}^2. \end{aligned} \quad (4.3.36)$$

We exploit now (4.3.3) and we apply Gårding inequality:

$$\begin{aligned} \langle \text{Op}^w((\{\xi \langle V \rangle_\nu(x), a\} - \varepsilon M_W |\xi|)(1 - \chi_1)) u_1, u_1 \rangle &\stackrel{(4.3.3)}{\stackrel{(4.1.5)}{\geq}} (\delta - \varepsilon M_W) \langle \text{Op}^w((1 - \chi_1)|\xi|) u_1, u_1 \rangle \\ &\quad - C_{G,1} \|u_1\|_{L^2(\mathbb{T})}^2, \end{aligned} \quad (4.3.37)$$

where $C_{G,1} > 0$ is the constant resulting from Gårding inequality and $\delta > 0$ is given by (4.3.3). Recalling now the definition of $\tilde{a}$ in (4.3.23) and using the fact that a is positively homogeneous of degree one, there exists $C_{2,a} > 0$ such that $a(x, \xi) \geq -C_{2,a}|\xi|$ for all $(x, \xi) \in \mathbb{T} \times \mathbb{R}$. Thus, applying again Gårding inequality, we obtain:

$$\begin{aligned} \langle \text{Op}^w((1 - \chi_1)|\xi|) u_1, u_1 \rangle &\geq -\frac{1}{C_{2,a}} \langle \text{Op}^w((1 - \chi_1)a) u_1, u_1 \rangle - C_{G,2} \|u_1\|_{L^2(\mathbb{T})}^2 \\ &\geq -\frac{1}{C_{2,a}} \langle \text{Op}^w(\tilde{a}) u_1, u_1 \rangle - (C_{G,2} + C_K) \|u_1\|_{L^2(\mathbb{T})}^2, \end{aligned} \quad (4.3.38)$$

where $C_{G,2} > 0$ is the constant resulting from Gårding inequality and $C_K > 0$ results from the compact correction to pass from $\text{Op}^w((1 - \chi_1)a)$ to $\text{Op}^w(\tilde{a})$.

Finally, plugging (4.3.38) and (4.3.37) in (4.3.36) and choosing $0 < \varepsilon < \frac{\delta}{2M_W}$ we obtain (4.3.35) with $\tilde{C}_{\tilde{a}} = C_{G,1} + C_{G,2} + C_K + C_{\chi_1} > 0$ and $\delta_2 := \frac{\delta}{2C_{2,a}} > 0$ and concluding the proof. $\square$

Initial datum. In this section we prove that there exists $u_{1_0} \in C^\infty(\mathbb{T})$, such that $\|u_{1_0}\|_{L^2(\mathbb{T})} = 1$ and

$$\langle \text{Op}^w(-\tilde{a}) u_{1_0}, u_{1_0} \rangle_{L^2(\mathbb{T})} - \frac{\beta_{\varepsilon, W, \tilde{a}}}{\delta_2} > 0, \quad (4.3.39)$$

where the constants $\delta_2, \beta_{\varepsilon, W, \tilde{a}}$ are given by Lemma 4.3.13. This gives (4.3.29) and, as discussed above, concludes the proof of the second point of Theorem 4.0.1. To prove (4.3.39) we recall the property (4.3.4) of the escape function, and we define

$\chi_3 \in C_c^\infty(\mathbb{T}; [0, 1])$, such that $\text{supp}(\chi_3) \subset \mathcal{W}$, with $\mathcal{W}$ given in (4.3.4). Next we define the family of functions

$$u_{\xi_0}(x) := \frac{\chi_3(x)}{\|\chi_3\|_{L^2(\mathbb{T})}} e^{i\xi_0 x}, \quad \forall x \in \mathbb{T}.$$

We claim that there exist constants $C_\chi, C_r > 0$ such that

$$\langle \text{Op}^w(-\tilde{a})u_{\xi_0}, u_{\xi_0} \rangle_{L^2(\mathbb{T})} \geq C_\chi |\xi_0| - C_r. \quad (4.3.40)$$

Thus choosing ξ_0 big enough, (4.3.40) proves (4.3.39) with $u_{1_0} = u_{\xi_0}$. To prove (4.3.40) first remark that, using symbolic calculus results in Lemma 4.1.1 we have

$$\langle \text{Op}^w(-\tilde{a})u_{\xi_0}, u_{\xi_0} \rangle_{L^2(\mathbb{T})} \stackrel{(4.1.3)}{=} \frac{1}{\|\chi_3\|_{L^2(\mathbb{T})}^2} \langle \text{Op}^w(-\tilde{a}\chi_3^2)e^{i\xi_0 x}, e^{i\xi_0 x} \rangle_{L^2(\mathbb{T})} + \langle \text{Op}^w(r_0)u_{\xi_0}, u_{\xi_0} \rangle_{L^2(\mathbb{T})}, \quad (4.3.41)$$

where $r_0 \in \mathbf{S}^0(\mathbb{T})$ results from the remainders in the pseudodifferential operations. Next, recalling the definition of $\tilde{a}$ in (4.3.23) and using (4.3.4) we obtain

$$\begin{aligned} \langle \text{Op}^w(-\tilde{a}\chi_3^2)e^{i\xi_0 x}, e^{i\xi_0 x} \rangle_{L^2(\mathbb{T})} &\geq \left\langle \text{Op}^w\left(\frac{|\xi|}{2}(1-\chi)\chi_3^2\right)e^{i\xi_0 x}, e^{i\xi_0 x} \right\rangle_{L^2(\mathbb{T})} - C_g \\ &= \left\langle \text{Op}^w\left(\frac{|\xi|}{2}(1-\chi)\right)\chi_3 e^{i\xi_0 x}, \chi_3 e^{i\xi_0 x} \right\rangle_{L^2(\mathbb{T})} - C_g \\ &\quad + \langle \text{Op}^w(r_1)e^{i\xi_0 x}, e^{i\xi_0 x} \rangle_{L^2(\mathbb{T})}, \end{aligned} \quad (4.3.42)$$

where $C_g > 0$ is the constant resulting from the Gårding inequality and where for the last equality we used again Lemma 4.1.1 and thus $r_1 \in \mathbf{S}^0(\mathbb{T})$ results from the remainders in the pseudodifferential operations. Finally, using the definition of Weyl quantization and performing a direct computation one obtains

$$\left\langle \text{Op}^w\left(\frac{|\xi|}{2}(1-\chi)\right)\chi_3 e^{i\xi_0 x}, \chi_3 e^{i\xi_0 x} \right\rangle_{L^2(\mathbb{T})} \stackrel{(4.3.23)}{\geq} \int_{|\xi| \geq 1} |\xi| \left(\int_{\mathbb{T}^2} \chi_3(y) \overline{\chi_3(x)} e^{i(\xi - \xi_0)(x-y)} dy dx \right) d\xi. \quad (4.3.43)$$

Using the Fourier series of χ_3 given by $\chi_3(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}$, one can check that

$$\int_{\mathbb{T}^2} \chi_3(y) \overline{\chi_3(x)} e^{i(\xi - \xi_0)(x-y)} dy dx = (2\pi)^2 \sum_{k \in \mathbb{Z}} |c_k|^2 \delta(\xi - \xi_0 - k), \quad (4.3.44)$$

where $\delta(\cdot)$ denotes the dirac-delta. Thus plugging (4.3.44) in (4.3.43), we obtain

$$\begin{aligned}
\left\langle \text{Op}^w \left(\frac{|\xi|}{2} (1 - \chi) \right) \chi_3 e^{i\xi_0 x}, \chi_3 e^{i\xi_0 x} \right\rangle_{L^2(\mathbb{T})} &\geq (2\pi)^2 \sum_k |c_k|^2 \int_{|\xi| \geq 1} |\xi| \delta(\xi - \xi_0 - k) d\xi \\
&= (2\pi)^2 \sum_k |c_k|^2 |\xi_0 + k| \\
&\geq (2\pi)^2 |\xi_0| \|\chi_3\|_{L^2}^2 - C_0 \|\chi_3\|_{\frac{1}{2}}^2.
\end{aligned} \tag{4.3.45}$$

Thus putting (4.3.45) and (4.3.42) in (4.3.41) we obtain (4.3.40) with $C_\chi = (2\pi)^2$ and C_r resulting from the sum of the constants C_g and $C_0 \|\chi_3\|_{\frac{1}{2}}$ and the remainders r_0 and r_1 (see (4.3.42) and (4.3.41)). This concludes the proof.

4.4 Genericity and conclusion of the proof of Theorem 4.0.1

We now conclude proving Point 3 of Theorem 4.0.1. To this aim we first define the set

$$\mathcal{R} := \{f \in C^\infty(\mathbb{T}) : f(x) = 0 \implies f'(x) \neq 0\} \subset C^\infty(\mathbb{T}), \tag{4.4.1}$$

of smooth non degenerate functions on $\mathbb{T}$ (see Definition 1.1.11). We have the following property:

Lemma 4.4.1. *The set $\mathcal{R}$ is open and dense in $C^\infty(\mathbb{T})$ endowed with its Fréchet topology (i.e., the one induced by the family of seminorms $(\|\cdot\|_{C^k})_{k \geq 0}$).*

Let us postpone the proof of this fact and first use it to conclude the proof. To this aim, consider the application

$$\mathcal{F} : C^\infty(\mathbb{T}^2) \rightarrow C^\infty(\mathbb{T}), \quad \mathcal{F}(V(t, x)) = \langle V \rangle_\nu(x).$$

One can verify that $\mathcal{F}$ is linear and continuous. Moreover $\mathcal{V}_{\nu,s} \cup \mathcal{V}_{\nu,u} = \mathcal{F}^{-1}(\mathcal{R})$. This immediately implies that $\mathcal{V}_{\nu,s} \cup \mathcal{V}_{\nu,u}$ is open. We now prove density of $\mathcal{V}_{\nu,s} \cup \mathcal{V}_{\nu,u}$. To this aim, take any $W \in C^\infty(\mathbb{T}^2)$, and consider its resonant average $\langle W \rangle_\nu(x)$ (recall

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Definition 1.1.10). From density of the set $\mathcal{R}$ in (4.4.1), for every $\varepsilon > 0$ there exists $V_0 \in \mathcal{R}$ such that $\mathbf{d}_{C^\infty(\mathbb{T})}(V_0, \langle W \rangle_\nu) < \varepsilon$. Thus, considering the function

$$W_0(t, x) := V_0(x + \nu t) + (W(t, x) - \langle W \rangle_\nu(x + \nu t))$$

one finds that $\langle W_0 \rangle_\nu(x) = V_0(x) \in \mathcal{R}$ by construction. Moreover,

$$\mathbf{d}_{C^\infty(\mathbb{T}^2)}(W_0, W) = \mathbf{d}_{C^\infty(\mathbb{T})}(V_0, \langle W \rangle_\nu) < \varepsilon,$$

concluding the proof.

We now sketch the proof Lemma 4.4.1.

Proof of Lemma 4.4.1. We omit the proof of the fact that the set $\mathcal{R}$ is open, since it is a direct application of Sard's Lemma and of the properties of the Fréchet topology (which ensures a control both on the C^0 and C^1 norms). We now prove that $\mathcal{R}$ is dense in $C^\infty(\mathbb{T})$: take any function $g \in C^\infty(\mathbb{T}) \setminus \mathcal{R}$ (this means that g has some degenerate zeroes). Using Sard's Lemma, for any $\varepsilon > 0$, there exists $\varepsilon_0 \in [-\varepsilon, \varepsilon]$, regular value for g . Thus defining $h(x) := g(x) - \varepsilon_0$, we obtain that $\mathbf{d}_{C^\infty(\mathbb{T})}(h, g) = \varepsilon_0 < \varepsilon$. Moreover, the zeroes of $h(x)$ are the points such that $g(x) = \varepsilon_0$, thus $h'(x) = g'(x) \neq 0$, for all such $x \in \mathbb{T}$ since ε_0 is a regular value for g . Thus $h \in \mathcal{R}$ concluding the proof. $\square$

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