

SISSA

Scuola
Internazionale
Superiore di
Studi Avanzati

Mathematics Area - PhD course in
Mathematical Analysis, Modelling, and Applications

Geometric features of Fourier transform and applications to discrepancy theory

Candidate:
Thomas Beretti

Advisor:
Luca Brandolini
Co-advisor:
Massimiliano Berti

Academic Year 2025-26



Acknowledgements

I thank the referees and anyone who reads (at least one chapter of) this thesis. I wrote this work for you first and foremost, and I sincerely hope every page reads smoothly.

Di seguito, una serie di sentiti ringraziamenti a chi, matematicamente o moralmente, ha contribuito al mio percorso di dottorato. Ringrazio la mia famiglia allargata di relatori, Giancarlo Travaglini, Leonardo Colzani, Luca Brandolini, e Giacomo Gigante, per avermi accolto e supportato durante questi tre anni. In particolare, ringrazio Luca Brandolini, il relatore ufficiale di questa tesi, per essere stato paziente e disponibile nei miei confronti. Devo un riconoscimento speciale a Giancarlo Travaglini, che più di una volta è intervenuto nel mio percorso accademico, e a cui sarò sempre grato. Seguirebbe una lunga lista di colleghi ed amici della SISSA con cui ho condiviso risate, pensieri, e teoremi, ma – poiché vengo dalla strada – non farò i nomi. Sappiate solo che vi ringrazio di cuore e che tiferò sempre per voi! Una menzione è dovuta a Luca Gennaioli, che ha contribuito al contenuto matematico di questa tesi e che, ad oggi, resta l'unico coautore ad avermi sopportato. Infine, come da prassi, ringrazio i miei familiari, che sono la cosa più importante che ho.

Contents

Acknowledgements	iii
1 Introduction	1
1.1 The optimal discrepancy problem	1
1.2 Fourier transform and geometry	3
1.3 Common ground and structure of the thesis	5
2 The Fourier transform of sets	11
2.1 Introduction	11
2.2 Averages over dilations	15
2.3 Averages over dilations and intervals of rotations	19
3 Discrepancy over intervals of rotations	27
3.1 Introduction	27
3.1.1 Brandolini-Travaglini's argument	28
3.2 Lower and upper bounds	31
3.3 Intermediate orders	37
3.3.1 Geometric construction	37
3.3.2 Lower and upper bounds	44
4 Polynomial-order oscillations	49
4.1 Introduction	49
4.2 The first method: a geometric approach	52
4.3 The second method: a Fourier analytic approach	55
4.3.1 Geometric construction	55
4.3.2 Lower bound	62
4.3.3 Upper bound	64
5 The Fourier transform of BV functions	69
5.1 Preliminary results on BV functions	69
5.2 Introduction	72

5.3	Detecting jumps via the Fourier transform	75
5.3.1	A weighted Plancherel identity for BV functions	75
5.3.2	An “isoperimetric inequality” for BV functions	83
5.4	Application to discrepancy: sharpening a result of Beck	85
5.5	A question in between harmonic analysis and GMT	89
6	Set-theoretic derivative and Fourier transform	93
6.1	Introduction	93
6.2	Sharpening a result of Herz	94
	Bibliography	97

Chapter 1

Introduction

This thesis deals with the interplay between harmonic analysis and geometry. Although the topics we consider are disparate in character, recurring themes and techniques provide them with a unifying framework. The central theme of our work is the theory of irregularities of distribution, often referred to as discrepancy theory, which quantifies how point distributions in a space deviate from uniformity. The following papers constitute the backbone of the present thesis.

- [Ber26a] *The Fourier transform of planar convex bodies and discrepancy over intervals of rotations*, Advances in Mathematics **503** (2026), Paper No. 111220,
- [Ber26b] *Polynomial-order oscillations in geometric discrepancy*, Mathematika **72** (2026), Paper No. e70121,
- [BG26] *Fourier transform of BV functions and applications* (with Luca Gennaioli), Journal of Functional Analysis (2026), Paper No. 111700.

While [Ber26b] deals only with discrepancy theory, both [BG26] and [Ber26a] may be split into two different efforts, one concerning the interplay between the Fourier transform and hypersurface geometry, and the other concerning applications to discrepancy theory. In the two sections that follow, we outline the historical backgrounds that led up to our questions.

1.1 The optimal discrepancy problem

Consider the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A one-dimensional sequence $\{p_j\}_{j=1}^\infty \subset \mathbb{T}$ is said to be uniformly distributed if, for every interval $I \subseteq \mathbb{T}$, it holds

$$\lim_{N \rightarrow +\infty} N^{-1} \sum_{j=1}^N \mathbf{1}_I(p_j) = |I|,$$

where $\mathbb{1}_I$ denotes the indicator function of I . Uniformly distributed sequences are easy to construct; a classical example is the result of Weyl [Wey16] that the Kronecker sequence $\{n\alpha\}_{n=1}^\infty$ is uniformly distributed whenever α is irrational. The concept of discrepancy is introduced as a quantitative counterpart to uniform distribution. Let N be a positive integer and consider a sequence $\mathcal{P} = \{p_j\}_{j=1}^\infty \subset \mathbb{T}$, then define the star-discrepancy of $\mathcal{P}$ at N as

$$D(\mathcal{P}, N) = \sup_{0 < x < 1} \left| \sum_{j=1}^N \mathbb{1}_{[0,x)}(p_j) - Nx \right|.$$

It was van der Corput [van35] who conjectured that, for any sequence $\mathcal{P} \subset \mathbb{T}$, the quantity $D(\mathcal{P}, N)$ cannot stay bounded with respect to N , and the conjecture was proved to be true by van Aardenne-Ehrenfest [van45, van49] who first obtained a lower bound. Finding the optimal order of growth for the star-discrepancy stood as an unsolved problem for several years, until Schmidt [Sch72] showed that there exists a positive constant c such that, for every sequence $\mathcal{P} \subset \mathbb{T}$, it holds

$$\limsup_{N \rightarrow +\infty} \frac{D(\mathcal{P}, N)}{\log N} \geq c.$$

Indeed, if one considers the Kronecker sequence $\{n\alpha\}_{n=1}^\infty \subset \mathbb{T}$, with α being a quadratic irrational number, then its star-discrepancy is of order $\log N$.

Before Schmidt's result, Roth [Rot54] already made significant improvements on van Aardenne-Ehrenfest's lower bound, and interestingly, these followed from a work carried out in a two-dimensional setting. Consider a set $\Omega \subset \mathbb{T}^2$ and a set of N points $\mathcal{P}_N \subset \mathbb{T}^2$, then define the discrepancy of $\mathcal{P}_N$ with respect to Ω as

$$\mathcal{D}(\Omega; \mathcal{P}_N) = \sum_{\mathbf{p} \in \mathcal{P}_N} \mathbb{1}_\Omega(\mathbf{p}) - N|\Omega|.$$

We state Roth's result on the optimal quadratic discrepancy with respect to rectangles.

Theorem 1.1.1 (Roth). *There exists $c > 0$ such that, for every positive integer N , it holds*

$$\inf_{\#\mathcal{P}=N} \int_0^1 \int_0^1 |\mathcal{D}([0,x) \times [0,y); \mathcal{P})|^2 dx dy \geq c \log N.$$

The latter is a turning point in discrepancy theory; we refer to [CV17] for more historical details and to [Bill1] for an extensive survey on the impact of Roth's result. In particular, the proof employs the classic Haar basis and introduces an original geometric perspective. Later, Davenport [Dav56] showed that Roth's lower bound can not be improved, proving its sharpness.

Roth's result quantifies how a two-dimensional point distribution $\mathcal{P}$ deviates from uniformity by testing it over a family of rectangles. It is natural to replace the rectangles in

Roth's theorem with more general sets; this is the starting point of geometric discrepancy. Notably, within the family of convex sets, the lower bound for the optimal discrepancy can be much higher than the logarithm, and indeed, Schmidt [Sch69] showed that the optimal worst-case discrepancy with respect to discs has a polynomial lower bound.

Theorem 1.1.2 (Schmidt). *Let $\varepsilon > 0$, then there exists $c_\varepsilon > 0$ such that, for every positive integer N , it holds*

$$\inf_{\#\mathcal{P}=N} \sup_{x,r} |\mathcal{D}(B(x,r);\mathcal{P})| \geq c_\varepsilon N^{\frac{1}{4}-\varepsilon},$$

where $B(x,r) \subset \mathbb{T}^2$ denotes the Euclidean disc centered at x of radius r , and the supremum is taken over all $x \in \mathbb{T}^2$ and $r \in [0, 1/2)$.

Hence, a general question arises: which geometric properties dictate the magnitude of the optimal discrepancy? In this thesis, we address this question by considering quadratic averages of the discrepancy over families of sets generated by the similarity group.

We employ harmonic analysis methods tailored to quadratic discrepancy and, in particular, exploit its convolutional structure through Parseval's identity. It turns out that, for a configuration of points $\mathcal{P}_N \subset \mathbb{T}^d$, the quadratic discrepancy with respect to translated copies of a set C can be expressed as a weighted sum over the frequencies $|\widehat{\mathbf{1}}_C(n)|^2$ with $n \in \mathbb{Z}^d \setminus \{0\}$. As a consequence, the decay of the latter quantity dictates the optimal order of discrepancy. Notably, the decay of $\widehat{\mathbf{1}}_C$ is much intertwined with the geometric and analytic properties of ∂C ; this is the subject of the following section.

Lastly, we mention that discrepancy theory has applications to numerical integration, quasi-Monte Carlo methods, and number theory, among other areas; we refer the reader to the monographs [Nie92, DT97, Cha00, Mat10, CST14, Dic14, Tra14, BDP20].

1.2 Fourier transform and geometry

For $\xi \in \mathbb{R}^d$ and a function $f \in L^1(\mathbb{R}^d)$, we use the convention

$$\mathcal{F}\{f\}(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Consider a bounded domain $\Omega \subset \mathbb{R}^d$ whose boundary $\partial\Omega$ is piecewise smooth. By an integration by parts step followed by the divergence theorem, one obtains

$$\widehat{\mathbf{1}}_\Omega(\xi) = -\frac{1}{2\pi i |\xi|^2} \int_\Omega \operatorname{div}_x(\xi e^{-2\pi i x \cdot \xi}) dx = -\frac{1}{2\pi i |\xi|} \int_{\partial\Omega} \nu(s) \cdot \frac{\xi}{|\xi|} e^{-2\pi i s \cdot \xi} d\sigma(s), \quad (1.1)$$

where $\nu(s)$ denotes the outward pointing unit normal at $s \in \partial\Omega$ and σ denotes the surface measure over $\partial\Omega$. Viewing the boundary term in (1.1) as an oscillatory integral, one may deduce that the decay of $\widehat{\mathbf{1}}_\Omega(\xi)$ as $|\xi| \rightarrow +\infty$ reflects the amount of cancellation produced

by the phase $s \mapsto s \cdot \xi$ across $\partial\Omega$. It is natural to ask which geometric properties of the boundary force cancellations and at what rate.

We briefly recall a classical property of the Fourier transform that relates the regularity of f with the decay of $\widehat{f}$. For a positive integer k , consider a function with compact support $f \in \mathcal{C}^k(\mathbb{R}^d)$, then, for all multi-indices α such that $|\alpha| = k$, it holds

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi),$$

and since for every $\xi \in \mathbb{R}^d$ it also holds

$$|\widehat{\partial^\alpha f}(\xi)| \leq \|\partial^\alpha f\|_{L^1},$$

then one obtains the estimate

$$|\widehat{f}(\xi)| = \mathcal{O}(|\xi|^{-k}).$$

Since indicator functions are not smooth, their Fourier transform is expected to decay more slowly. Remarkably, it still retains information on how rapidly discontinuities occur; this, in turn, reflects the local “flatness” of $\partial\Omega$.

For example, let B^d be the unit d -dimensional ball. Then, by using classic estimates on Bessel functions of the first kind, one finds the sharp estimate

$$|\widehat{\mathbf{1}}_{B^d}(\xi)| = \mathcal{O}(|\xi|^{-(d+1)/2}).$$

The exponent $-(d+1)/2$ may actually be deduced from (1.1) by the following heuristic. For $|\xi| \geq 1$, the phase $s \mapsto s \cdot \xi$ on $\mathbb{S}^{d-1}$ is stationary at the two antipodal points $\xi/|\xi|$ and $-\xi/|\xi|$, and the level sets of the phase are hyperplanes orthogonal to ξ . The wavelength of this phase is proportional to $|\xi|^{-1}$, and consequently, the phase is “approximately stationary” on a spherical cap formed by intersecting $\mathbb{S}^{d-1}$ with a $|\xi|^{-1}$ -neighborhood of the tangent hyperplane at one of the stationary points. Thus, such a spherical cap will have a diameter of $2|\xi|^{-1/2}$ in the directions orthogonal to ξ – related to the unit Gaussian curvature of $\mathbb{S}^{d-1}$ at the stationary points – and so it will have a surface measure proportional to $|\xi|^{-(d-1)/2}$. Combining with the external factor $|\xi|^{-1}$ in (1.1) gives the claimed estimate.

A second useful example is the Fourier transform of the d -dimensional unit cube $Q^d = [-1/2, 1/2]^d$, that is easily calculated as

$$\widehat{\mathbf{1}}_{Q^d}(\xi) = \prod_{j=1}^d \frac{\sin(\pi \xi_j)}{\pi \xi_j}.$$

Notice that the latter is $\mathcal{O}(|\xi|^{-1})$ when ξ is parallel to a coordinate axis, while it is $\mathcal{O}(|\xi|^{-d})$ when no component of ξ vanishes (for instance, when ξ points along a vertex direction of Q^d). Thus, decay can be strongly anisotropic.

These two examples suggest that the local geometry of $\partial\Omega$, in the frequency direction, governs the decay of $\widehat{\mathbf{1}}_\Omega$. Indeed, one could deduce that a higher “flatness” at $s \in \partial\Omega$

implies a slower decay of $|\widehat{\mathbf{1}}_{\Omega}(t\nu(s))|$ as $t \rightarrow +\infty$. This perspective naturally leads to the theory of oscillatory integrals, a classic topic in harmonic analysis.

Hlawka's work [Hla50] on oscillatory integrals over convex bodies is an early investigation into how convexity alone influences decay, anticipating successive stationary-phase approaches. Later, Herz's seminal paper [Her62] established a foundational treatment for Fourier transforms related to convex sets, including bounds that encode geometric information, such as convexity, boundary regularity, and associated geometric measures. Lastly, the modern stationary-phase method and its incorporation into PDEs is treated systematically in Hörmander's monograph [Hör63].

The interplay between the Fourier transform and hypersurface geometry has been central to several problems in harmonic analysis. For example, we mention Strichartz's seminal work [Str77] on wave equations, where the Fourier transform of certain measures supported on hypersurfaces is shown to yield integrability properties of the solutions. This phenomenon underscores the relevance of the Fourier restriction problem, which asks for which exponents p the Fourier transform of an L^p function admits a meaningful restriction to a hypersurface; for a comprehensive picture, we refer the reader to [Dem20].

Throughout this thesis, we present novel relationships between the Fourier transform and the geometric properties of sets; in Chapters 5–6, this extends to the broader class of functions of bounded variation.

1.3 Common ground and structure of the thesis

Besides this introduction, the thesis is organised into five chapters, each corresponding to a specific set of results from the three papers listed above. Before delving into the structure of the thesis, we discuss some notations and definitions that recur throughout the thesis.

Each of the papers listed above deals with different subsets of the similarity group — that is, translations, dilations, and rotations. We establish suitable notation for the action of the similarity group on sets and functions.

Consider a set $\Omega \subset \mathbb{R}^d$ and a real function $u \in L^1(\mathbb{R}^d)$. Let $\tau \in \mathbb{R}^d$ be a translation factor, let $\delta > 0$ be a dilation factor, and let $\rho: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a rotation; in particular, we identify $\text{SO}(d)$ with the set of all rotations. We define the action of the similarity group on Ω and u , respectively, as

$$[\tau, \delta, \rho]\Omega = \tau + \delta\rho\Omega \quad \text{and} \quad [\tau, \delta, \rho]u(x) = u(\rho^{-1}(x - \tau)/\delta),$$

with the convention that if a transformation is null, we omit its writing in the square brackets. We remark that, by employing this notation, the action of the similarity group “commutes” with characteristic functions of sets; that is, it holds

$$[\tau, \delta, \rho]\mathbf{1}_{\Omega} = \mathbf{1}_{[\tau, \delta, \rho]\Omega}.$$

Now, we introduce the periodization functional $\mathfrak{P}: L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{T}^d)$ defined in the sense that

$$\mathfrak{P}\{u\}(x) = \sum_{n \in \mathbb{Z}^d} u(x + n).$$

Hence, we are able to extend the previous notion of discrepancy.

Definition 1.3.1. Let $u \in L^1(\mathbb{R}^d)$ and let $\mathcal{P}_N \subset \mathbb{T}^d$ be a configuration of N points (that is, a collection of N not necessarily distinct points counted with multiplicity). We define the discrepancy of $\mathcal{P}_N$ with respect to u as

$$\mathcal{D}(u; \mathcal{P}_N) = \sum_{p \in \mathcal{P}_N} \mathfrak{P}\{u\}(p) - N\hat{u}(\mathbf{0}), \quad (1.2)$$

with the convention that

$$\mathcal{D}(\Omega; \mathcal{P}_N) = \mathcal{D}(\mathbf{1}_\Omega; \mathcal{P}_N).$$

When averaging the square of (1.2) over translations of u , we may exploit the convolutional structure of discrepancy via the Plancherel identity. We proceed to show such a relation explicitly. Let μ_L be the Lebesgue measure on $\mathbb{T}^d$, and for $p \in \mathbb{T}^d$, let $\mu_D(p)$ be the Dirac delta centered at p . By setting

$$\tilde{\mu} = \sum_{p \in \mathcal{P}_N} \mu_D(-p) - N\mu_L,$$

we may rewrite (1.2) as

$$\mathcal{D}(u; \mathcal{P}_N) = \int_{\mathbb{T}^d} \mathfrak{P}\{u\}(x) d\tilde{\mu}(-x) = (\mathfrak{P}\{u\} * \tilde{\mu})(\mathbf{0}).$$

For a finite measure (or a summable function) $g \in \mathcal{M}(\mathbb{T}^d)$, we let $\tilde{\mathcal{F}}\{g\}: \mathbb{Z}^d \rightarrow \mathbb{C}$ be the function of the Fourier coefficients of g . Then, it is not difficult to see that for $u \in L^1(\mathbb{R}^d)$, and for every $n \in \mathbb{Z}^d$, it holds

$$\tilde{\mathcal{F}} \circ \mathfrak{P}\{u\}(n) = \mathcal{F}\{u\}(n).$$

Moreover, by standard properties of the Fourier transform, it also holds

$$\mathcal{F}\{[\delta, \rho]u\} = \delta^d [\delta^{-1}, \rho] \mathcal{F}\{u\}. \quad (1.3)$$

Therefore, by applying Plancherel identity on $\mathbb{T}^d$ (in particular, we identify $[0, 1)^d$ with $\mathbb{T}^d$ via the quotient map $\boldsymbol{\tau} \mapsto \boldsymbol{\tau} + \mathbb{Z}^d$), we get

$$\begin{aligned} \int_{\mathbb{T}^d} |\mathcal{D}([\tau, \delta, \rho]u; \mathcal{P}_N)|^2 d\tau &= \int_{\mathbb{T}^d} |(\mathfrak{P}\{[\delta, \rho]u\} * \tilde{\mu})|^2(\tau) d\tau \\ &= \sum_{n \in \mathbb{Z}^d} \left| \tilde{\mathcal{F}}\{\tilde{\mu}\}(n) \right|^2 \left| \tilde{\mathcal{F}} \circ \mathfrak{P}\{[\delta, \rho]u\}(n) \right|^2 \\ &= \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 |\mathcal{F}\{[\delta, \rho]u\}(n)|^2, \end{aligned}$$

where $d\tau$ is the normalized Lebesgue measure on $\mathbb{T}^d$ and, for the sake of notation, we have set $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Thus, our bridge between discrepancy and harmonic analysis is the equality

$$\int_{\mathbb{T}^d} |\mathcal{D}([\tau, \delta, \rho]u; \mathcal{P}_N)|^2 d\tau = \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 |\mathcal{F}\{\delta, \rho\}u(n)|^2. \quad (1.4)$$

The three papers that make up this thesis consider three different kinds of discrepancy, but share a common structure. Indeed, let $u \in L^1(\mathbb{R}^d)$ and let $\mathcal{P}_N \subset \mathbb{T}^d$ be a configuration of N points. Then, given a uniform distribution η on a subset of rotations $\mathcal{R} \subset \text{SO}(d)$, we are concerned with the general-form quadratic discrepancy

$$\mathcal{D}_2^{\mathcal{R}}(u; \mathcal{P}_N) = \int_{\mathcal{R}} \int_0^1 \int_{\mathbb{T}^d} |\mathcal{D}([\tau, \delta, \rho]u; \mathcal{P}_N)|^2 d\tau d\delta d\eta, \quad (1.5)$$

where $d\delta$ is the Lebesgue measure on $[0, 1]$. As in Definition 1.3.1, we use the convention

$$\mathcal{D}_2^{\mathcal{R}}(\Omega; \mathcal{P}_N) = \mathcal{D}_2^{\mathcal{R}}(\mathbf{1}_{\Omega}; \mathcal{P}_N).$$

Finally, our results concern the general-form optimal discrepancy

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\mathcal{R}}(u; \mathcal{P})$$

for three different choices of $\mathcal{R}$; namely, we consider $\mathcal{R}$ to be the null rotation $\text{id} \in \text{SO}(2)$, an interval of rotations $I \subset \text{SO}(2)$, and the full set of rotations $\text{SO}(d)$.

While obtaining an upper bound on the latter quantity requires constructing a suitable configuration of points, the lower bound is much subtler. The original approach introduced by Montgomery [Mon94, Ch. 6] serves as the common ground for the lower bounds presented in this thesis. Namely, he exploited (1.4) and adapted an argument of Cassels [Cas56] for estimating exponential sums; for the sake of completeness, we state his consequent lower bound on the discrepancy over squares (Drmota [Drm96] proved Montgomery's estimate to be sharp, owing to its substantial equivalence to Theorem 1.1.1).

Theorem 1.3.2 (Montgomery). *There exists a positive value c such that, for every positive integer N , it holds*

$$\inf_{\#\mathcal{P}=N} \int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}(\tau + [0, \delta]^2; \mathcal{P})|^2 d\tau d\delta \geq c \log N.$$

We state and prove the argument of Cassels [Cas56] for estimating exponential sums from below, since it is fundamental for obtaining all our lower bounds on the quadratic discrepancy. We also point out that an analogous result holds on manifolds, as recently presented in [BGG21] and [GG22].

Lemma 1.3.3 (Cassels-Montgomery). *Let $U \subset \mathbb{R}^d$ be a neighbourhood of the origin. For every origin-symmetric convex body $\Omega \subset \mathbb{R}^d$ and for every finite configuration of points $\{p_j\}_{j=1}^N \subset \mathbb{T}^d$, it holds*

$$\sum_{n \in (\Omega \setminus U) \cap \mathbb{Z}^d} \left| \sum_{j=1}^N e^{2\pi i n \cdot p_j} \right|^2 \geq 2^{-d} |\Omega| N - \#\{U \cap \mathbb{Z}^d\} N^2.$$

Proof. Consider the auxiliary sets $A_\Omega(x) \subset \mathbb{Z}^d$ defined by

$$A_\Omega(x) = (x + \Omega/2) \cap \mathbb{Z}^d.$$

Notice that

$$\int_{\mathbb{T}^d} \#A_\Omega(x) \, dx = \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} \mathbf{1}_{\Omega/2}(n - x) \, dx = \int_{\mathbb{R}^d} \mathbf{1}_{\Omega/2}(x) \, dx = 2^{-d} |\Omega|,$$

and therefore, we find a point

$$\tilde{x} \in [0, 1)^d \quad \text{such that} \quad \#A_\Omega(\tilde{x}) \geq 2^{-d} |\Omega|.$$

Now, consider the non-negative trigonometric polynomial

$$T(y) = \frac{1}{\#A_\Omega(\tilde{x})} \left| \sum_{n \in A_\Omega(\tilde{x})} e^{2\pi i n \cdot y} \right|^2 = \frac{1}{\#A_\Omega(\tilde{x})} \sum_{n, m \in A_\Omega(\tilde{x})} e^{2\pi i (n-m) \cdot y},$$

and notice that the function of its Fourier coefficients $\widehat{T}: \mathbb{Z}^d \rightarrow \mathbb{R}$ is non-negative as well and, since $n, m \in A_\Omega(\tilde{x})$ imply $(n - m) \in \Omega$, then its support is contained in Ω . Further, observe that we have

$$T(0) = \#A_\Omega(\tilde{x}) \geq 2^{-d} |\Omega|,$$

moreover, it is not difficult to notice that for all $n \in \mathbb{Z}^d$, it holds

$$0 \leq \widehat{T}(n) \leq \widehat{T}(\mathbf{0}) = \int_{\mathbb{T}^d} T(x) \, dx = 1.$$

Finally, it follows that

$$\begin{aligned}
\sum_{n \in (\Omega \setminus U) \cap \mathbb{Z}^d} \left| \sum_{j=1}^N e^{2\pi i n \cdot p_j} \right|^2 &\geq \sum_{n \in (\Omega \setminus U) \cap \mathbb{Z}^d} \widehat{T}(n) \left| \sum_{j=1}^N e^{2\pi i n \cdot p_j} \right|^2 \\
&= \sum_{j=1}^N \sum_{\ell=1}^N \sum_{n \in (\Omega \setminus U) \cap \mathbb{Z}^d} \widehat{T}(n) e^{2\pi i n \cdot (p_j - p_\ell)} \\
&\geq \sum_{j=1}^N \sum_{\ell=1}^N \sum_{n \in \Omega \cap \mathbb{Z}^d} \widehat{T}(n) e^{2\pi i n \cdot (p_j - p_\ell)} - \#\{U \cap \mathbb{Z}^d\} N^2 \\
&= \sum_{j=1}^N \sum_{\ell=1}^N T(p_j - p_\ell) - \#\{U \cap \mathbb{Z}^d\} N^2 \\
&\geq 2^{-d} |\Omega| N - \#\{U \cap \mathbb{Z}^d\} N^2.
\end{aligned}$$

□

Having gathered the previous useful results, we now move on to presenting the content of each chapter.

Chapters 2–3 present the content of [Ber26a]. Chapter 2 concerns harmonic analysis and planar convex bodies. Namely, for a planar convex body C , we establish sharp two-sided estimates for averages of the Fourier transform $\widehat{\mathbf{1}}_C$ over dilations and partial rotations. The key geometric quantity is the chord function $\gamma_C(\theta, \lambda)$, which records the length of the chord in C in directions $\{\theta, \theta + \pi\}$ and at distance λ from ∂C . These estimates are central to the discrepancy results that follow, but are also of independent interest. Then, given a planar convex body C and interval of rotations $I \subset \text{SO}(2)$, Chapter 3 makes use of the content of Chapter 2 to obtain sharp bounds on the optimal *partial-similar* quadratic discrepancy

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^I(C; \mathcal{P}).$$

The answer is governed entirely by the relation between $|I|$ and the symmetric angular threshold ψ_C of C (in the simpler case of C having a center of symmetry, ψ_C is just π minus the smallest interior angle of ∂C). The optimal order is $N^{1/2}$ when $|I| > \psi_C$ and $N^{2/5}$ when $|I| < \psi_C$. In the borderline case $|I| = \psi_C$, we construct suitable convex bodies to show that every intermediate polynomial order between $N^{2/5}$ and $N^{1/2}$ is attained. In particular, for each $\alpha > 1$, we achieve the order $N^{2\alpha/(1+4\alpha)}$ by designing a convex body with curvature exponent α near its angular points.

Chapter 4 presents the results of [Ber26b], again concerning the discrepancy of planar convex bodies. Here, the focus shifts to the optimal *homothetic* quadratic discrepancy

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}),$$

where only translations and dilations are averaged over. The central question is whether this quantity must converge to a single order of growth as $N \rightarrow +\infty$, and the answer is negative. Two construction methods yield planar convex bodies whose optimal *homothetic* quadratic discrepancy oscillates between prescribed orders. The first, relying on a purely geometric argument, allows oscillations between $\log N$ and $N^{1/2}$. The second draws on the Fourier-analytic estimates of Chapter 2 and produces prescribed polynomial-order oscillations of N^α with α in the range $(2/5, 1/2)$. As a byproduct of the first method, the set of planar convex bodies whose optimal discrepancy converges to a single order is shown to be meagre (that is, a countable union of nowhere-dense sets) in the space of planar convex bodies equipped with the Hausdorff metric.

Chapters 5–6 present the results of [BG26], and the setting is now $\mathbb{R}^d$ for arbitrary $d \geq 1$. In Chapter 5, the role of characteristic functions of sets is taken over by functions of bounded variation. We establish a weighted Plancherel identity for functions of bounded variation, expressed by the L^2 -jump product $\mathcal{J}(u, v)$, that is, a bilinear form that compares the jump part of the distributional gradients of u and v . Such an identity is then applied to derive sharp asymptotic bounds of order $N^{(d-1)/d}$ (attained when $\mathcal{J}(u) \neq 0$) on the optimal *similar* quadratic discrepancy

$$\inf_{\#P=N} \mathcal{D}_2^{\text{SO}}(u; \mathcal{P}),$$

averaged over the full similarity group $\text{SO}(d) \times [0, 1] \times \mathbb{T}^d$. The latter extends the classical results of Beck [Bec87] and Montgomery [Mon94, Ch. 6] to the class of bounded BV functions with bounded support. We conclude Chapter 5 with a speculative note: can one construct a space of functions that is defined in terms of the Fourier transform only and whose discontinuity set is rectifiable in codimension one? An answer to the latter question may provide a natural setting for studying hyperbolic equations in which discontinuities are expected. Last, Chapter 6 is of a geometric measure-theoretic character, and in particular, it deals with the outer Minkowski content of sets. Here, we present a characterisation of the set-theoretic derivative of the Fourier transform of sets of finite perimeter.

The orders listed for the chapters above all refer to the squared quantity $\mathcal{D}_2^{\mathcal{R}}$; no square root is ever taken. An exponent β here therefore corresponds to the exponent $\beta/2$ in the sources that normalise by the square root. Schmidt’s bound $N^{1/4-\varepsilon}$ for discs, stated above for an unsquared supremum, sits at the same scale as our $1/2$. Read this way, the three families line up. The exponent $1/2$ is reached in each of them: for $\mathcal{C}^2$ convex bodies with no rotation averaging, by a known result recalled in Chapter 4; for every convex body once $|I| > \psi_C$, in Chapter 3; and, when $d = 2$, in the *similar* case of Chapter 5.

More details, historical context, and further references for each of the above topics may be found at the start of the respective chapters.

Chapter 2

The Fourier transform of sets

This chapter presents the harmonic analysis content of [Ber26a]. We establish sharp two-sided estimates on the Fourier transform of the indicator function of planar convex bodies, averaged over dilations and rotations; the main results are Lemma 2.1.2 and Lemma 2.1.7. The key geometric objects are introduced in Section 2.1 and collected below. The estimates of this chapter are fundamental for the discrepancy results of Chapter 3, and Lemma 2.1.2 is used again in Chapter 4.

Index of geometric notation

η_p at (2.1)	P_C at Definition 2.1.6
K_C at Definition 2.1.1	s_C^- and s_C^+ at Definition 2.3.1
γ_C at Definition 2.1.1	$s_C^o, s_C^{o-}, s_C^{o+}$ at Definition 2.3.1
L_C and S_C at Definition 2.1.1	K_C^- and K_C^+ at Definition 2.3.1
Γ_C at Definition 2.1.4	$\mathcal{T}_C$ at Definition 2.3.6
ν_C, ν_C^-, ν_C^+ at Definition 2.1.4	ψ_C at Definition 2.3.6

Index of comparison notation

$\preceq, \succ, \asymp$ at (2.2)	$\lesssim, \gtrsim, \approx$ in Chapter 4
$\approx$ in Section 3.3	$\sim$ in Chapter 5

2.1 Introduction

For a real positive number p , we define the one-dimensional torus with period p and the unit two-dimensional torus, respectively, as

$$\mathbb{T}_p = \mathbb{R}/p\mathbb{Z} \quad \text{and} \quad \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2.$$

Then, we define the *ordered-distance function*

$$\eta_p: \mathbb{T}_p \times \mathbb{T}_p \rightarrow [0, p) \quad (2.1)$$

in such a way that

$$\eta_p(x_1, x_2) = y \quad \text{if and only if} \quad x_1 + y \equiv x_2 \pmod{p}.$$

Moreover, for an angle $\theta \in \mathbb{T}_{2\pi}$, we let $\sigma_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the counterclockwise rotation by θ , and we set

$$\mathbf{u}(\theta) = (\cos \theta, \sin \theta)$$

to be the unit vector in $\mathbb{R}^2$ that makes an angle θ with the x -axis.

We introduce convenient notation on limit behaviours. Consider an unbounded set $U \subseteq [0, +\infty)$ and let f and g be two positive functions defined on U , then we say that it holds

$$f(x) \preccurlyeq g(x) \quad (2.2)$$

to intend that there exists a positive value c such that

$$\limsup_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \leq c.$$

Moreover, in the case where $f_y(\cdot)$ and $g_y(\cdot)$ further depend on a variable $y \in V \subseteq \mathbb{R}$, then we say that (2.2) holds uniformly for every $y \in V$ to intend that the involved value c does not depend on y . Last, if both $f(x) \preccurlyeq g(x)$ and $f(x) \succcurlyeq g(x)$ hold, then we say that it holds

$$f(x) \asymp g(x).$$

In order to state our first main result, we need the following notions of chords and diameters of a body.

Definition 2.1.1. Let $C \subset \mathbb{R}^2$ be a convex body. For an angle $\theta \in \mathbb{T}_{2\pi}$ and value $\lambda > 0$, we define (as shown in Figure 2.1) the *chord* of C in direction $\mathbf{u}(\theta)$ at distance λ as

$$K_C(\theta, \lambda) = \left\{ \mathbf{x} \in C : \mathbf{x} \cdot \mathbf{u}(\theta) = \inf_{\mathbf{y} \in C} (\mathbf{y} \cdot \mathbf{u}(\theta)) + \lambda \right\}.$$

Further, we consider its length $|K_C(\theta, \lambda)|$, and we define the quantity

$$\gamma_C(\theta, \lambda) = \max\{|K_C(\theta, \lambda)|, |K_C(\theta + \pi, \lambda)|\}.$$

Last, we define the *longest directional diameter* (or classic diameter) of C as

$$L_C = \max_{\mathbf{x}, \mathbf{y} \in C} |\mathbf{x} - \mathbf{y}|,$$

and we define the *shortest directional diameter* of C as

$$S_C = \min_{\theta \in \mathbb{T}_{2\pi}} \max_{\lambda \geq 0} |K_C(\theta, \lambda)|.$$

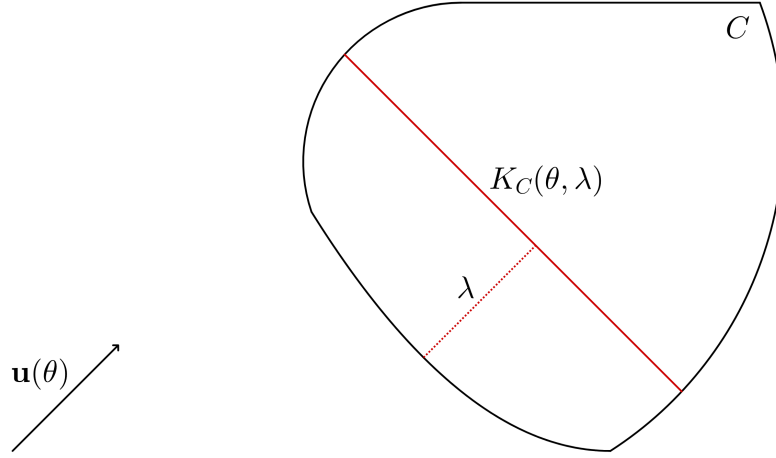


Figure 2.1: The chord in Definition 2.1.1.

Then, we state our result on the Fourier transform of the characteristic function of C averaged over dilations.

Lemma 2.1.2. *There exist positive absolute constants κ_3 and κ_4 such that, for every convex body $C \subset \mathbb{R}^2$, for every angle $\theta \in \mathbb{T}_{2\pi}$ and for every value $\rho \geq \kappa_3 L_C^6 / S_C^7$, it holds*

$$\kappa_4 \rho^{-2} \gamma_C^2(\theta, \rho^{-1}) \leq \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta \leq 2 \rho^{-2} \gamma_C^2(\theta, \rho^{-1}).$$

Remark 2.1.3. Notice that if C has a $\mathcal{C}^2$ boundary with curvature that is uniformly bounded away from zero and infinity, then the order in the previous inequality would be of ρ^{-3} uniformly in θ . On the other hand, if C is an axis-aligned square, then one would find an order of ρ^{-4} at direction $\theta = \pi/4$ and an order of ρ^{-2} at direction $\theta = 0$. Last, with some work, one may construct C in such a way that chords in the same direction $\mathbf{u}(\theta)$ display different polynomial decays at different magnitudes of ρ ; therefore, one would get that the asymptotic behaviour of $\widehat{\mathbf{1}}_C(\rho \mathbf{u}(\theta))$ oscillates between different polynomial orders at different magnitudes (actually, the latter is the *leitmotiv* of Chapter 4).

Before showing our second main result, we provide some preliminary definitions concerning points at the boundary.

Definition 2.1.4. Let $C \subset \mathbb{R}^2$ be a convex body. We set

$$\Gamma_C: \mathbb{T}_{|\partial C|} \rightarrow \mathbb{R}^2$$

to be the counterclockwise arc-length parameterisation of the boundary ∂C . For $s \in \mathbb{T}_{|\partial C|}$, we define the *set of normals* at s as

$$\nu_C(s) = [\nu_C^-(s), \nu_C^+(s)] = \left\{ \theta \in \mathbb{T}_{2\pi} : \min_{\mathbf{a} \in C} (\mathbf{a} \cdot \mathbf{u}(\theta)) = \Gamma_C(s) \cdot \mathbf{u}(\theta) \right\},$$

with the convention that if $\nu_C(s)$ is a single angle, then we simply consider

$$\nu_C^-(s) = \nu_C^+(s) = \nu_C(s).$$

In particular, we say that $s \in \mathbb{T}_{|\partial C|}$ is an angular point if $\nu_C^-(s) \neq \nu_C^+(s)$.

Remark 2.1.5. Notice that if C has a $\mathcal{C}^1$ boundary, then it has no angular point. Moreover, if ∂C has strictly positive curvature, then ν_C is a bijection between $\mathbb{T}_{|\partial C|}$ and $\mathbb{T}_{2\pi}$. Last, if we choose C to be an axis-aligned square, then its vertices identify angular points whose sets of normals are $[(n-1)\pi/2, n\pi/2]$, with n being an integer such that $1 \leq n \leq 4$.

Secondly, we introduce an object that relates directions and perimeter.

Definition 2.1.6. Let $C \subset \mathbb{R}^2$ be a convex body. For an interval of angles $I \subset \mathbb{T}_{2\pi}$, we define the *portion of perimeter* of C with respect to I as

$$P_C(I) = |\{s \in \mathbb{T}_{|\partial C|} : \nu_C(s) \cap I \neq \emptyset\}|.$$

Finally, we state our results on the Fourier transform of the characteristic function of C averaged over dilations and an interval of rotations.

Lemma 2.1.7. *Uniformly for every convex body $C \subset \mathbb{R}^2$, and uniformly for every closed interval $I \subset \mathbb{T}_{2\pi}$, it holds*

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}_{[\delta]C}}(\rho \mathbf{u}(\theta)) \right|^2 d\delta d\theta \asymp \rho^{-3} (P_C(I) + P_C(I + \pi)),$$

with the convention that if $P_C(I) + P_C(I + \pi) = 0$, then it holds

$$\lim_{\rho \rightarrow +\infty} \rho^3 \int_I \int_0^1 \left| \widehat{\mathbf{1}_{[\delta]C}}(\rho \mathbf{u}(\theta)) \right|^2 d\delta d\theta = 0.$$

In particular, our methods do not involve the Gaussian curvature, as they do not make use of the method of stationary phase for oscillatory integrals.

2.2 Averages over dilations

We study the asymptotic behaviour of $\widehat{\mathbb{1}}_C$. Namely, letting $\theta \in \mathbb{T}_{2\pi}$ be an angle and considering ρ to be a real positive number, we are concerned with the decay of

$$\widehat{\mathbb{1}}_C(\rho \mathbf{u}(\theta)) \quad \text{as } \rho \rightarrow +\infty.$$

First notice that, since $\mathbb{1}_C$ is a real function, it holds

$$\left| \widehat{\mathbb{1}}_C(\rho \mathbf{u}(\theta)) \right| = \left| \widehat{\mathbb{1}}_C(\rho \mathbf{u}(\theta + \pi)) \right|.$$

Without loss of generality assume $\theta = 0$, so that

$$\widehat{\mathbb{1}}_C((\rho, 0)) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_C(x_1, x_2) e^{-2\pi i \rho x_1} dx_1 dx_2 = \int_{\mathbb{R}} g(x_1) e^{-2\pi i \rho x_1} dx_1 = \widehat{g}(\rho),$$

where we have set

$$g(t) = \int_{\mathbb{R}} \mathbb{1}_C(t, x_2) dx_2. \quad (2.3)$$

Since C is convex, the non-negative function g is supported and concave on an interval $[a, b] \subset \mathbb{R}$. Therefore, we are led to study the Fourier transform of such a one-dimensional function, and to proceed, we define an auxiliary tool.

Definition 2.2.1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function supported and concave on $[a, b]$, then for every $\lambda \in [0, \frac{b-a}{2}]$ we define the height of g at distance λ from the support as

$$\zeta_g(\lambda) = \max \{g(a + \lambda), g(b - \lambda)\}.$$

We remark on the duality between the latter quantity and the chord in Definition 2.1.1, which is strongly related to the decay of the Fourier transform of $\mathbb{1}_C$. It holds the following estimate, obtained through a simple geometric argument. In particular, notice that the threshold and the values involved depend solely on the diameters of C .

Proposition 2.2.2. Let $C \subset \mathbb{R}^2$ be a convex body. For every $\theta \in \mathbb{T}_{2\pi}$ and for every $\rho \geq 2/S_C$, it holds

$$\gamma_C(\theta, \rho^{-1}) \geq \frac{S_C}{L_C} \rho^{-1}.$$

Proof. Without loss of generality, suppose $\theta = 0$ and define g as in (2.3). In particular, notice that

$$\zeta_g(\rho^{-1}) = \gamma_C(0, \rho^{-1}),$$

so that it is enough to estimate g . It is not difficult to see that

$$S_C \leq \max_{x \in \mathbb{R}} g(x) \leq L_C \quad \text{and} \quad S_C \leq |\text{supp}(g)| \leq L_C, \quad (2.4)$$

and by the concavity of g on its support, it follows from some easy geometric observations that, for every $\rho \geq 2/S_C$, it holds

$$g(\rho^{-1}) \geq \frac{\max_{x \in \mathbb{R}} g(x)}{|\text{supp}(g)|} \rho^{-1} \geq \frac{S_C}{L_C} \rho^{-1}.$$

□

We state a classic upper bound on such one-dimensional functions due to Podkorytov [Pod91]. For more results in this direction, we refer the interested reader to [Tra14, Ch. 8].

Lemma 2.2.3 (Podkorytov). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative continuous function supported and concave on $[-1, 1]$, then for every value $s \geq 1$ it holds*

$$|\widehat{f}(s)| \leq s^{-1} \zeta_f(s^{-1}).$$

Let us show how the latter lemma evolves into estimates on the decay of the Fourier transform of $\mathbb{1}_C$. Consider a non-negative function $g : \mathbb{R} \rightarrow \mathbb{R}$ supported and concave on a bounded interval $[a, b] \subset \mathbb{R}$, and apply the affine change of variable

$$f(s) = g\left(\frac{b+a}{2} + s\frac{b-a}{2}\right), \quad (2.5)$$

hence obtaining

$$|\widehat{f}(s)| = \frac{2}{b-a} \left| \widehat{g}\left(\frac{2s}{b-a}\right) \right|. \quad (2.6)$$

Further, notice that it holds

$$f(\pm(1-\lambda)) = g\left(\frac{b+a}{2} \pm (1-\lambda)\frac{b-a}{2}\right),$$

and therefore, for every $\lambda \in [0, \frac{b-a}{2}]$, we get

$$\zeta_f(\lambda) = \zeta_g\left(\lambda \frac{b-a}{2}\right).$$

Hence, by applying the latter lemma to f and by translating into terms of g , we have that, for every $s \geq 1$, it holds

$$\frac{2}{b-a} \left| \widehat{g}\left(\frac{2s}{b-a}\right) \right| \leq s^{-1} \zeta_g\left(\frac{b-a}{2s}\right),$$

so that by the change of variable

$$\rho = 2s/(b-a),$$

we get that, for every $\rho \geq 2/(b-a)$, it holds

$$|\widehat{g}(\rho)| \leq \rho^{-1} \zeta_g(\rho^{-1}).$$

In particular, we remark that $|b-a|$ is bounded from below by S_C independently of the choice of θ , and therefore, by turning into terms of the convex body C , we get the following formulation.

Lemma 2.2.4. *Let $C \subset \mathbb{R}^2$ be a convex body. For every $\theta \in \mathbb{T}_{2\pi}$ and for every $\rho \geq 2/S_C$, it holds*

$$\left| \widehat{\mathbf{1}}_C(\rho \mathbf{u}(\theta)) \right| \leq \rho^{-1} \gamma_C(\theta, \rho^{-1}).$$

We now state an essential result that establishes both a lower and an upper bound on the Fourier transform of one-dimensional functions as the one in (2.3).

Lemma 2.2.5 (Brandolini-Travaglini). *There exist positive absolute constants $\kappa_1 < 1 < \kappa_2$ such that, uniformly for every non-negative continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ supported and concave on $[-1, 1]$, it holds*

$$\int_{\kappa_1}^{\kappa_2} \left| \widehat{f}(\delta s) \right|^2 d\delta \asymp s^{-2} \zeta_f^2(s^{-1}).$$

Actually, it was Podkorytov who first achieved the latter estimate and then showed it to Travaglini during a personal communication in 2001, but the original proof has never been published. The authors in [BT22, Lem. 23] give an original proof by relating the Fourier transform of such f with its moduli of smoothness (see [DL93, Ch. 2]). Now, with Proposition 2.2.2 and Lemma 2.2.4 in mind, we proceed to prove Lemma 2.1.2.

Proof of Lemma 2.1.2. Let us start by proving the upper bound. First, we set $\rho_0 = 2/S_C$ and consider $\rho \geq \rho_0$. Then, it is useful to split the integral as

$$\begin{aligned} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta &= \\ &= \int_0^{\rho_0/\rho} \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta + \int_{\rho_0/\rho}^1 \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta. \end{aligned} \quad (2.7)$$

By basic properties of the Fourier transform and the fact that $|C| \leq L_C^2$ (this easily follows by (2.4)), we obtain

$$\left\| \widehat{\mathbf{1}}_{[\delta]C} \right\|_{L^\infty(\mathbb{R}^2)} \leq \left\| \mathbf{1}_{[\delta]C} \right\|_{L^1(\mathbb{R}^2)} = \delta^2 |C| \leq \delta^2 L_C^2,$$

so that, for the first integral in the right-hand term of (2.7), we get

$$\int_0^{\rho_0/\rho} \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta \leq L_C^4 \int_0^{\rho_0/\rho} \delta^4 d\delta = \frac{32L_C^4}{5S_C^5} \rho^{-5}. \quad (2.8)$$

Now, notice that by the concavity of $|K_C(\theta, \cdot)|$ on its support, we have that, for every angle $\theta \in \mathbb{T}_{2\pi}$, for every $\rho > 0$, and for every $\delta \in (0, 1]$, it holds

$$\gamma_C(\theta, \delta^{-1}\rho^{-1}) \leq \delta^{-1} \gamma_C(\theta, \rho^{-1}).$$

Therefore, by the latter observation, and by (1.3) and Lemma 2.2.4, for the second integral in the right-hand term of (2.7) we get

$$\begin{aligned} \int_{\rho_0/\rho}^1 \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta &= \int_{\rho_0/\rho}^1 \delta^4 \left| \widehat{\mathbf{1}}_C(\delta \rho \mathbf{u}(\theta)) \right|^2 d\delta \\ &\leq \int_{\rho_0/\rho}^1 \delta^4 \left| \delta^{-1} \rho^{-1} \gamma_C(\theta, \delta^{-1} \rho^{-1}) \right|^2 d\delta \\ &\leq \int_{\rho_0/\rho}^1 \delta^2 \left| \rho^{-1} \delta^{-1} \gamma_C(\theta, \rho^{-1}) \right|^2 d\delta \leq \rho^{-2} \gamma_C^2(\theta, \rho^{-1}). \end{aligned}$$

By Proposition 2.2.2, it holds

$$\rho^{-2} \gamma_C^2(\theta, \rho^{-1}) \geq \frac{S_C^2}{L_C^2} \rho^{-4},$$

so that, by defining

$$\rho_1 = \frac{32L_C^6}{5S_C^7},$$

one can deduce from (2.8) that, for every $\rho \geq \rho_1$, it holds

$$\int_0^{\rho_0/\rho} \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta \leq \frac{S_C^2}{L_C^2} \rho^{-4} \leq \rho^{-2} \gamma_C^2(\theta, \rho^{-1}).$$

Finally, by combining the latter observations into (2.7), we obtain that, for every $\rho \geq \max\{\rho_0, \rho_1\}$, it holds

$$\begin{aligned} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta &\leq \int_0^{\rho_0/\rho} \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta + \rho^{-2} \gamma_C^2(\theta, \rho^{-1}) \\ &\leq 2\rho^{-2} \gamma_C^2(\theta, \rho^{-1}). \end{aligned}$$

Let us now proceed to prove the lower bound. As before, and without loss of generality, we assume $\theta = 0$, and we define g as in (2.3). Hence, we define f by the same affine change of variable as in (2.5), so that its support is the interval $(-1, 1)$. By the latter lemma, it follows that there exist positive absolute constants $\tilde{s} > 1$ and $\tilde{c} > 0$ such that, uniformly for every such f and for every $s \geq \tilde{s}$, it holds

$$s^{-2} \zeta_f^2(s^{-1}) \leq \tilde{c} \int_{\kappa_1}^{\kappa_2} \left| \widehat{f}(\delta s) \right|^2 d\delta.$$

By the concavity of f on its support, it follows that, for every s_1 and s_2 such that $0 \leq s_1 < s_2 \leq 1$, it holds

$$f(-1 + s_1) \leq 2f(-1 + s_2) \quad \text{and} \quad f(1 - s_1) \leq 2f(1 - s_2).$$

Hence, since $\kappa_2 > 1$ and $\tilde{s} > 1$, then for every $s \geq \tilde{s}$ it holds

$$s^{-2} \zeta_f^2(\kappa_2^{-1} s^{-1}) \leq 4\tilde{c} \int_{\kappa_1}^{\kappa_2} \left| \widehat{f}(\delta s) \right|^2 d\delta \leq \frac{4\tilde{c}}{\kappa_1^4} \int_{\kappa_1}^{\kappa_2} \delta^4 \left| \widehat{f}(\delta s) \right|^2 d\delta.$$

Turning into terms of g , and by (2.6) and the change of variable, $\rho = 2s\kappa_2/(b-a)$, we get that, for every ρ such that $\rho \geq 2\tilde{s}\kappa_2/(b-a)$, it holds

$$\kappa_2^2 \rho^{-2} \zeta_g^2(\rho^{-1}) \leq \frac{4\tilde{c}}{\kappa_1^4} \int_{\kappa_1}^{\kappa_2} \delta^4 \left| \widehat{g}(\delta \kappa_2^{-1} \rho) \right|^2 d\delta.$$

Independently of the choice of θ , it holds $|b-a| \geq S_C$, and then we set

$$\rho_2 = 2\tilde{s}\kappa_2/S_C.$$

Hence, by rewriting the last inequality in terms of C , and by the change of variable $\delta = \kappa_2 \ell$, we get that for every $\rho \geq \rho_2$ it holds

$$\rho^{-2} \gamma_C^2(0, \rho^{-1}) \leq \frac{4\tilde{c}\kappa_2^3}{\kappa_1^4} \int_0^1 \ell^4 \left| \widehat{\mathbb{1}}_C(\ell \rho, 0) \right|^2 d\ell = \frac{4\tilde{c}\kappa_2^3}{\kappa_1^4} \int_0^1 \left| \widehat{\mathbb{1}}_{[\ell]C}(\rho, 0) \right|^2 d\ell.$$

Last, we set $\kappa_4 = \kappa_1^4/(4\tilde{c}\kappa_2^3)$, and the conclusion follows once we acknowledge that there exists a positive absolute constant κ_3 , independent of C , such that it holds

$$\max\{\rho_0, \rho_1, \rho_2\} \leq \kappa_3 L_C^6 / S_C^7.$$

□

Remark 2.2.6. Notice that the estimates in the latter lemma are uniform for a class of planar convex bodies whose longest and shortest directional diameters are uniformly bounded.

2.3 Averages over dilations and intervals of rotations

Once Lemma 2.1.2 is taken into account, the proof of Lemma 2.1.7 relies on Proposition 2.3.2, which finds an exact relation between averages over semi-chords of a planar convex body and portions of its perimeter. It is a pivotal point of this chapter, but in order to state such a result, we need to expand on our previous notion of chord.

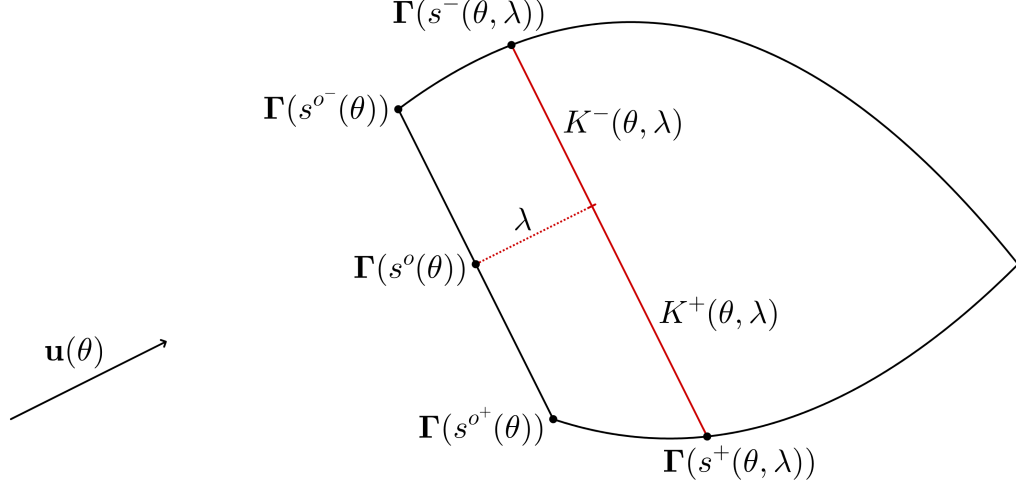


Figure 2.2: The objects in Definition 2.3.1. For simplicity, we omit writing C .

Definition 2.3.1. Let $K_C(\theta, \lambda)$ be as in Definition 2.1.1. We set (as shown in Figure 2.2)

$$s_C^-(\theta, \lambda) \quad \text{and} \quad s_C^+(\theta, \lambda)$$

to be the parameterization by Γ_C of the extreme points of $K_C(\theta, \lambda)$, with the convention that

$$\Gamma_C(s_C^-(\theta, \lambda)) - \Gamma_C(s_C^+(\theta, \lambda)) = |K_C(\theta, \lambda)| \mathbf{u}'(\theta).$$

Further, we define

$$s_C^{o-}(\theta) = \lim_{\lambda \rightarrow 0} s_C^-(\theta, \lambda) \quad \text{and} \quad s_C^{o+}(\theta) = \lim_{\lambda \rightarrow 0} s_C^+(\theta, \lambda),$$

and we set

$$s_C^o(\theta) = s_C^{o-}(\theta) + \frac{\eta_{|\partial C|}(s_C^{o-}(\theta), s_C^{o+}(\theta))}{2}.$$

Hence, we define the *right semi-chord* $K_C^+(\theta, \lambda)$ to be the projection of

$$\Gamma_C([s_C^o(\theta), s_C^+(\theta, \lambda)]) \quad \text{in direction} \quad \mathbf{u}(\theta) \quad \text{on} \quad K_C(\theta, \lambda),$$

and we define $K_C^-(\theta, \lambda)$ analogously.

Having gathered all the previous definitions, we are able to state our technical result; in particular, we state it in the case of right semi-chords.

Proposition 2.3.2. *Let $C \subset \mathbb{R}^2$ be a convex body, and let $I = (\alpha, \beta] \subset \mathbb{T}_{2\pi}$ be a left semi-open interval. It holds*

$$\lim_{\lambda \rightarrow 0} \frac{1}{2\lambda} \int_I |K_C^+(\theta, \lambda)|^2 d\theta = P_C(I). \quad (2.9)$$

Remark 2.3.3. Suppose there exists $s \in \mathbb{T}_{|\partial C|}$ such that

$$\nu_C(s) = [\alpha_1, \beta_1] \subset \mathbb{T}_{2\pi} \quad \text{with} \quad \alpha_1 \neq \beta_1.$$

By some basic geometry, we get that for every $\theta \in (\alpha_1, \beta_1)$ it holds

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} |K_C(\theta, \lambda)| = \cot(\eta_{2\pi}(\alpha_1, \theta)) + \cot(\eta_{2\pi}(\theta, \beta_1)).$$

Hence, if we consider an interval $[\alpha_2, \beta_2] \subset (\alpha_1, \beta_1)$ and set $\rho = 1/\lambda$, we get

$$\int_{\alpha_2}^{\beta_2} |K_C(\theta, \rho^{-1})|^2 d\theta \asymp \rho^{-2}.$$

We proceed with the proof of Proposition 2.3.2.

Proof of Proposition 2.3.2. For the sake of simplicity, we omit the subscript C under the geometric objects. With the help of Figure 2.3, observe that

$$|K^+(\theta, \lambda)| = - \int_{s^o(\theta)}^{s^+(\theta, \lambda)} \mathbf{u}'(t) \cdot \mathbf{\Gamma}'(t) dt, \quad (2.10)$$

and

$$\int_{s^o(\theta)}^{s^+(\theta, \lambda)} \mathbf{u}(\theta) \cdot \mathbf{\Gamma}'(t) dt = \lambda. \quad (2.11)$$

Since C is a convex body, it is not difficult to deduce that the set of angular points of C is at most countable. In turn, this implies that the derivatives

$$\frac{\partial}{\partial \lambda} s^+, \quad \frac{\partial}{\partial \theta} s^+, \quad \text{and} \quad \mathbf{\Gamma}', \quad \text{exist almost everywhere.}$$

For fixed $\lambda > 0$, the map $\theta \mapsto s^+(\theta, \lambda)$ is non-decreasing and absolutely continuous, so that $\int_I \left| \frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right| d\theta \leq |\partial C|$, uniformly in λ . Hence, by taking the distributional derivative with respect to λ of both sides of (2.11), we get

$$\left(\frac{\partial}{\partial \lambda} s^+(\theta, \lambda) \right) \mathbf{u}(\theta) \cdot \mathbf{\Gamma}'(s^+(\theta, \lambda)) = 1. \quad (2.12)$$

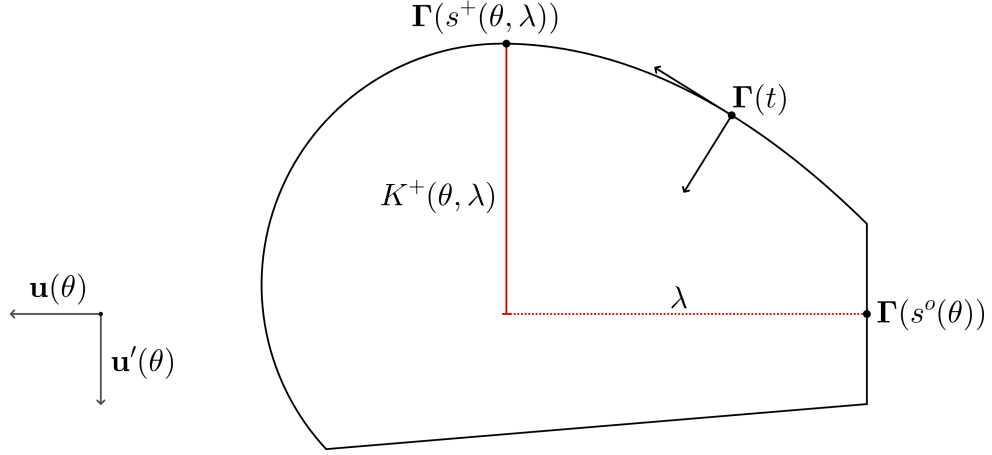


Figure 2.3: An auxiliary image for the proof of Proposition 2.3.2. For simplicity, we omit to write C .

Also, by taking the distributional derivative with respect to θ of both sides of (2.11) and by applying Leibniz integral rule, we obtain

$$\begin{aligned} & \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}(\theta) \cdot \boldsymbol{\Gamma}'(s^+(\theta, \lambda)) + \int_{s^o(\theta)}^{s^+(\theta, \lambda)} \mathbf{u}'(\theta) \cdot \boldsymbol{\Gamma}'(t) \, dt = \\ & = \left(\frac{\partial}{\partial \theta} s^o(\theta) \right) \mathbf{u}(\theta) \cdot \boldsymbol{\Gamma}'(s^o(\theta)). \end{aligned} \quad (2.13)$$

It is simple to notice that, for every $\theta \in \mathcal{T}$, it holds

$$\frac{\partial}{\partial \theta} s^o(\theta) = 0.$$

On the other hand, for every $\theta \in \mathcal{T}^c$, it holds

$$\mathbf{u}(\theta) \cdot \boldsymbol{\Gamma}'(s^o(\theta)) = 0.$$

Therefore, for every angle $\theta \in \mathbb{T}_{2\pi}$, it holds

$$\left(\frac{\partial}{\partial \theta} s^o(\theta) \right) \mathbf{u}(\theta) \cdot \boldsymbol{\Gamma}'(s^o(\theta)) = 0. \quad (2.14)$$

Hence, by (2.10), (2.13), and (2.14), it follows that

$$|K^+(\theta, \lambda)| = \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}(\theta) \cdot \boldsymbol{\Gamma}'(s^+(\theta, \lambda)). \quad (2.15)$$

Also, by taking the distributional derivative with respect to λ of both sides of (2.10), we get

$$\frac{\partial}{\partial \lambda} |K^+(\theta, \lambda)| = - \left(\frac{\partial}{\partial \lambda} s^+(\theta, \lambda) \right) \mathbf{u}'(\theta) \cdot \mathbf{\Gamma}'(s^+(\theta, \lambda)). \quad (2.16)$$

Then, since we can apply the dominated convergence theorem to the integral at the left-hand side of (2.9), and by (2.12), (2.15), and (2.16), it follows that

$$\begin{aligned} & \frac{\partial}{\partial \lambda} \int_I |K^+(\theta, \lambda)|^2 d\theta = \\ &= 2 \int_I |K^+(\theta, \lambda)| \left(\frac{\partial}{\partial \lambda} |K^+(\theta, \lambda)| \right) d\theta \\ &= -2 \int_I \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}(\theta) \cdot \mathbf{\Gamma}'(s^+(\theta, \lambda)) \left(\frac{\partial}{\partial \lambda} s^+(\theta, \lambda) \right) \mathbf{u}'(\theta) \cdot \mathbf{\Gamma}'(s^+(\theta, \lambda)) d\theta \\ &= -2 \int_I \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}'(\theta) \cdot \mathbf{\Gamma}'(s^+(\theta, \lambda)) d\theta. \end{aligned}$$

Now, notice that $s^{o+}(\theta) = s^o(\theta)$ if and only if

$$\left\{ \mathbf{b} \in C : \mathbf{b} \cdot \mathbf{u}(\theta) = \min_{\mathbf{a} \in C} \mathbf{a} \cdot \mathbf{u}(\theta) \right\} \quad \text{is a single point.}$$

Also, it is not difficult to see that, for every fixed $\theta \in \mathbb{T}_{2\pi}$, it holds

$$\lim_{\lambda \rightarrow 0} \mathbf{\Gamma}'(s^+(\theta, \lambda)) = -\mathbf{u}'(\nu^+(s^{o+}(\theta))),$$

and therefore, by the dominated convergence theorem, for every small $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that, for every $\lambda \in \mathbb{R}$ with $0 < \lambda \leq \lambda_\varepsilon$, it holds

$$\int_{A_\varepsilon^\lambda} \left| \frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right| d\theta < \varepsilon,$$

where we have set

$$A_\varepsilon^\lambda = \left\{ \theta \in I : \left| \mathbf{\Gamma}'(s^+(\theta, \lambda)) + \mathbf{u}'(\nu^+(s^{o+}(\theta))) \right| \geq \varepsilon \right\}.$$

Now, consider the set

$$E_\varepsilon = \left\{ \theta \in I : \eta_{2\pi}(\theta, \nu^+(s^{o+}(\theta))) \geq \varepsilon \right\},$$

and let $[\alpha_j, \beta_j]$ be one of its connected components; in particular, notice that these are at most $2\pi/\varepsilon$. Hence, we set

$$\ell(\theta, \lambda) = \eta_{|\partial C|}(s^{o+}(\theta), s^+(\theta, \lambda)),$$

and since $s^{o+}(\alpha_j) = s^{o+}(\beta_j)$, and by some basic geometry, we get that

$$\begin{aligned} \left| \int_{\alpha_j}^{\beta_j} \frac{\partial}{\partial \theta} s^+(\theta, \lambda) d\theta \right| &= |\ell(\beta_j, \lambda) - \ell(\alpha_j, \lambda)| \\ &\leq \frac{\lambda}{\sin(\varepsilon)} = \lambda \mathcal{O}(\varepsilon^{-1}). \end{aligned}$$

Moreover, notice that, for every $\theta \in I \setminus E_\varepsilon$, it holds

$$\mathbf{u}'(\theta) \cdot \mathbf{u}'(\nu^+(s^{o+}(\theta))) = \cos(\theta - \nu^+(s^{o+}(\theta))) \geq \cos(\varepsilon) = 1 + \mathcal{O}(\varepsilon^2).$$

By the latter observations, for every λ such that $0 < \lambda \leq \lambda_\varepsilon$, it follows that

$$\begin{aligned} & - \int_I \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}'(\theta) \cdot \boldsymbol{\Gamma}'(s^+(\theta, \lambda)) d\theta = \\ &= \int_I \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}'(\theta) \cdot \mathbf{u}'(\nu^+(s^{o+}(\theta))) d\theta + \mathcal{O}(\varepsilon) \\ &= \int_{I \setminus E_\varepsilon} \left(\frac{\partial}{\partial \theta} s^+(\theta, \lambda) \right) \mathbf{u}'(\theta) \cdot \mathbf{u}'(\nu^+(s^{o+}(\theta))) d\theta + \lambda \mathcal{O}(\varepsilon^{-2}) + \mathcal{O}(\varepsilon) \\ &= \int_{I \setminus E_\varepsilon} \frac{\partial}{\partial \theta} s^+(\theta, \lambda) d\theta + \lambda \mathcal{O}(\varepsilon^{-2}) + \mathcal{O}(\varepsilon) \\ &= \int_I \frac{\partial}{\partial \theta} s^+(\theta, \lambda) d\theta + \lambda \mathcal{O}(\varepsilon^{-2}) + \mathcal{O}(\varepsilon) \\ &= \eta_{|\partial C|}(s^{o+}(\alpha), s^{o+}(\beta)) + \ell(\beta, \lambda) - \ell(\alpha, \lambda) + \lambda \mathcal{O}(\varepsilon^{-2}) + \mathcal{O}(\varepsilon). \end{aligned}$$

Finally, $\ell(\theta, \lambda)$ vanishes as $\lambda \rightarrow 0$ for every fixed θ , and it holds

$$\eta_{|\partial C|}(s^{o+}(\alpha), s^{o+}(\beta)) = P_C((\alpha, \beta]).$$

We stress that the identity above is where the half-open convention of the statement comes from. If ∂C contains a segment, then all of its interior points s share the same set of normals $\nu_C(s) = \{\theta_0\}$, so the segment contributes its whole length to $P_C(I)$ as soon as $\theta_0 \in I$. Now, $s^{o+}(\theta)$ is the endpoint of the contact set at θ that is reached last in the counterclockwise orientation of ∂C . The arc from $s^{o+}(\alpha)$ to $s^{o+}(\beta)$ therefore contains the whole segment associated with β , while the one associated with α lies entirely before its starting point. This is exactly $P_C((\alpha, \beta])$. Hence, by choosing $\lambda = \min(\lambda_\varepsilon, \varepsilon^3)$ and letting $\varepsilon \rightarrow 0$, we obtain

$$\lim_{\lambda \rightarrow 0} \frac{\partial}{\partial \lambda} \int_I |K^+(\theta, \lambda)|^2 d\theta = 2P_C((\alpha, \beta]).$$

Last, the claim follows at once by applying L'Hospital's rule. $\square$

By an analogous proof, the same result for K_C^- and right semi-open intervals $I = [a, b)$ holds. As for entire chords K_C , by the fact that for every $a, b \geq 0$ it holds

$$\frac{a^2 + b^2}{2} \leq \max(a^2, b^2) \leq a^2 + b^2 \quad \text{and} \quad a^2 + b^2 \leq (a + b)^2 \leq 2a^2 + 2b^2,$$

it easily follows a handy result.

Corollary 2.3.4. *Let $C \subset \mathbb{R}^2$ be a convex body, and let $I \subset \mathbb{T}_{2\pi}$ be a closed interval. It holds*

$$\liminf_{\rho \rightarrow +\infty} \rho \int_I \gamma_C^2(\theta, \rho^{-1}) d\theta \geq P_C(I) + P_C(I + \pi)$$

and

$$\limsup_{\rho \rightarrow +\infty} \rho \int_I \gamma_C^2(\theta, \rho^{-1}) d\theta \leq 8P_C(I) + 8P_C(I + \pi).$$

Pairing the latter result with Lemma 2.2.4 and Lemma 2.1.2, we immediately get Lemma 2.1.7. By analogous reasoning, we further obtain the following spherical estimates for the Fourier transform.

Lemma 2.3.5. *Uniformly for every convex body $C \subset \mathbb{R}^2$, and uniformly for every closed interval $I \subset \mathbb{T}_{2\pi}$, it holds*

$$\int_I \left| \widehat{\mathbf{1}}_C(\rho \mathbf{u}(\theta)) \right|^2 d\theta \preceq \rho^{-3} (P_C(I) + P_C(I + \pi)),$$

with the convention that if $P_C(I) + P_C(I + \pi) = 0$, then it holds

$$\lim_{\rho \rightarrow +\infty} \rho^3 \int_I \left| \widehat{\mathbf{1}}_C(\rho \mathbf{u}(\theta)) \right|^2 d\theta = 0.$$

In particular, considering complete rotations in the latter result, we recover an asymptotic version of the spherical estimates in [Pod91]. Also, we point out that in arbitrary dimensions and for more general sets, the authors in [BHI03] obtain an analogous decay; it seems plausible that the best constant should (as well) be the perimeter of the set considered, but this is an open matter.

Last, it further follows a useful result for the next chapter. To state it, we introduce one last geometric quantity: the angular trace. For a generic set A , we write $\text{int}(A)$ to denote its interior.

Definition 2.3.6. Let $C \subset \mathbb{R}^2$ be a convex body. We define the *angular trace* of C as

$$\mathcal{T}_C = \bigcup_{s \in \mathbb{T}_{|\partial C|}} \text{int}(\nu_C(s)),$$

and further, we define the *symmetric angular threshold* of C as

$$\psi_C = \max \{ |J| : J \text{ is a connected component of } \mathcal{T}_C \cap (\mathcal{T}_C + \pi) \}.$$

Remark 2.3.7. Notice that if C has a centre of symmetry, then it holds

$$\psi_C = \max_{s \in \mathbb{T}_{|\partial C|}} |\nu_C(s)|.$$

Moreover, given an integer $n > 1$, if C is the regular polygon with $2n$ sides, then it follows that $\psi_C = \pi/n$. On the other hand, if C has a $\mathcal{C}^1$ boundary (that is, it has no angular points), then it follows that $\mathcal{T}_C = \emptyset$ and $\psi_C = 0$. Last, notice that it always holds $0 \leq \psi_C < \pi$.

We present the last result of this chapter.

Lemma 2.3.8. *Let $C \subset \mathbb{R}^2$ be a convex body, and let $I \subseteq \mathbb{T}_{2\pi}$ be an interval of angles such that $\psi_C < |I| \leq 2\pi$. Uniformly for every $\omega \in \mathbb{T}_{2\pi}$, it holds*

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \asymp \rho^{-3}.$$

Proof. First, we prove that there exists a positive value c such that for every $\omega \in \mathbb{T}_{2\pi}$ it holds

$$\liminf_{\rho \rightarrow +\infty} \rho \int_{\omega+I} \gamma_C^2(\theta, \rho^{-1}) d\theta \geq c.$$

If this were not the case, then, by the latter corollary, we would have a sequence of $\{\omega_j\}_{j \in \mathbb{N}} \subset \mathbb{T}_{2\pi}$ such that

$$\lim_{j \rightarrow +\infty} (P_C(\omega_j + I) + P_C(\omega_j + I + \pi)) = 0.$$

Hence, by the compactness of $\mathbb{T}_{2\pi}$, we would get the existence of a $\tilde{\omega} \in \mathbb{T}_{2\pi}$ such that

$$P_C(\tilde{\omega} + I) = 0 = P_C(\tilde{\omega} + I + \pi),$$

but this is a contradiction since it implies that

$$(\tilde{\omega} + I) \cup (\tilde{\omega} + I + \pi) \subset \mathcal{T}_C,$$

and consequently, it would hold $\psi_C \geq |I|$.

Finally, by Lemma 2.1.2 and by the compactness of $\mathbb{T}_{2\pi}$, it follows that, uniformly for every $\omega \in \mathbb{T}_{2\pi}$, it holds

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta]C}(\rho \mathbf{u}(\omega - \theta)) \right|^2 d\delta d\theta \asymp \rho^{-2} \int_{\omega+I} \gamma_C^2(\theta, \rho^{-1}) d\theta \asymp \rho^{-3}.$$

□

Chapter 3

Discrepancy over intervals of rotations

This chapter presents the discrepancy content of [Ber26a]. It applies the Fourier-analytic estimates of Chapter 2 to obtain sharp bounds on the optimal *partial-similar* quadratic discrepancy; the main results are Theorems 3.1.2, 3.1.3, and 3.3.6. We employ the same notation and geometric objects as in Chapter 2.

3.1 Introduction

We recall the notion of *partial-similar* quadratic discrepancy, in the planar case.

Definition 3.1.1. Let $\Omega \subset \mathbb{R}^2$, let $\mathcal{P}_N \subset \mathbb{T}^2$ be a configuration of N points, and let $I \subseteq \mathbb{T}_{2\pi}$ be an interval of angles. We define the *partial-similar* quadratic discrepancy of $\mathcal{P}_N$ with respect to Ω and I as

$$\mathcal{D}_2^I(\Omega; \mathcal{P}_N) = \int_I \int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}([\boldsymbol{\tau}, \delta, \theta] \Omega; \mathcal{P}_N)|^2 \, d\boldsymbol{\tau} \, d\delta \, d\theta, \quad (3.1)$$

where $d\boldsymbol{\tau}$ is the normalized Haar measure on $\mathbb{T}^2$, $d\delta$ is the Lebesgue measure on $[0, 1]$, and $d\theta$ is the arc-length measure on $\mathbb{T}_{2\pi}$ (of total mass 2π). In particular, we will always assume that $|I| > 0$.

The *partial-similar* quadratic discrepancy with respect to intervals of rotations has been recently investigated by Bilyk and Mastrianni [BM23]. The authors obtained partial results for the case where Ω is a square, and the questions raised thereafter motivated the current work. Indeed, we disprove the expectations stated at the end of their paper, which suggested that the *partial-similar* quadratic discrepancy behaves independently of the interval considered.

We also mention that the authors in [BMPS11, BMPS16] investigated the discrepancy of rectangles averaged over sets of (possibly uncountably many) rotations with empty interiors, and interestingly, the results depend heavily on Diophantine approximation properties.

We state our two main results on the optimal *partial-similar* quadratic discrepancy. The first one shows that, for averages over sufficiently large intervals of rotations, we essentially obtain the same asymptotic order as for a complete rotation (see Theorem 5.2.6 from Chapter 5).

Theorem 3.1.2. *Let $C \subset \mathbb{R}^2$ be a convex body, and let $I \subseteq \mathbb{T}_{2\pi}$ be an interval of angles such that $\psi_C < |I| \leq 2\pi$. Then, it holds*

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^I(C; \mathcal{P}) \asymp N^{1/2}.$$

Studying the asymptotic behaviour of the Fourier transform of $\mathbf{1}_C$ is a fundamental step of the proof, and this was the purpose of Lemma 2.1.7. Then, the proof of the lower bound in Theorem 3.1.2 requires Lemma 1.3.3 and an argument of Montgomery [Mon94, Ch. 6]. On the other hand, the upper bound is straightforward, as it simply requires unions of uniform lattices.

Our second main result concerns the complementary case of averages over small enough intervals of rotations.

Theorem 3.1.3. *Let $C \subset \mathbb{R}^2$ be a convex body, and let the interval $I \subset \mathbb{T}_{2\pi}$ be such that $0 < |I| < \psi_C$. It holds*

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^I(C; \mathcal{P}) \asymp N^{2/5}.$$

Again, Lemma 2.1.7 is the starting point for the proof. Then, the proof of the lower bound in Theorem 3.1.3 relies on an argument of Brandolini and Travaglini [BT22] that we present under a general form in Theorem 3.1.4. Finally, the proof of the upper bound is more involved than the one in Theorem 3.1.2 and requires unions of special configurations of points that are lattices under a certain action of the similarity group.

Finally, in Section 3.3, we study the intermediate case of $|I| = \psi_C$. Namely, we show that in such circumstances, the optimal *partial-similar* quadratic discrepancy can achieve any polynomial order in between $N^{1/2}$ and $N^{2/5}$ (notably, this is the same range as in Theorem 4.1.4 from Chapter 4). First, we construct suitable planar convex bodies and establish subtle geometric estimates for their Fourier transform. Then, the main result of the last section, Theorem 3.3.6, follows from adjusting the arguments in Subsection 3.3.2.

3.1.1 Brandolini-Travaglini's argument

We show a general result that allows to obtain lower bounds for the quadratic discrepancy. The original argument is in [BT22], and we present an integral version of the proof. As further notation, we consider the argument function

$$\arg: \mathbb{R}^2 \setminus \{\mathbf{0}\} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{defined as} \quad \arg(x_1, x_2) = \arctan \frac{x_2}{x_1}.$$

Theorem 3.1.4. *Let $C \subset \mathbb{R}^2$ be a convex body. Let Ξ be a measure space of transformations of C and let $h \in [0, 1]$. If there exist an interval of angles $I \subset \mathbb{T}_{2\pi}$ and values $\tilde{\rho}, \tilde{c} > 0$ such that for every $\rho \geq \tilde{\rho}$ it holds*

$$\int_{\Xi} \left| \widehat{\mathbf{1}}_{[\xi]C}(\rho \mathbf{u}(\omega)) \right|^2 d\xi \geq \tilde{c} \begin{cases} \rho^{-3} & \text{if } \omega \in I \cup (I + \pi) \\ \rho^{-3-h} & \text{else} \end{cases},$$

then it holds

$$\inf_{\#\mathcal{P}_N=N} \int_{\Xi} \int_{\mathbb{T}^2} |\mathcal{D}([\tau, \xi]C; \mathcal{P}_N)|^2 d\tau d\xi \gtrsim N^{\frac{2}{4+h}}.$$

Proof. Since the hypothesis only involves $I \cup (I + \pi)$, we may assume $|I| \leq \pi$. Let $N \in \mathbb{N} \setminus \{0\}$. Consider a rectangle $R \subset \mathbb{R}^2$ such that it is symmetric with respect to the axes and has a vertex in (X, Y) where $X = X(N)$ and $Y = Y(N)$ are positive parameters of N to be chosen later. As for now, we set them in such a way that

$$|R| = 4XY = \kappa N,$$

where κ is a positive value to be chosen later. Now, we define the function $\Phi: \mathbb{Z}^2 \rightarrow \mathbb{R}^+$ as

$$\Phi(\mathbf{m}) = \int_I \mathbf{1}_{[\theta]R}(\mathbf{m}) d\theta$$

and then aim to find a parameter $Z = Z(N)$ such that for every $\mathbf{m} \in \mathbb{Z}^2$ it holds

$$Z\Phi(\mathbf{m}) \leq \begin{cases} |\mathbf{m}|^{-3} & \text{if } \arg(\mathbf{m}) \in I \cup (I + \pi) \\ |\mathbf{m}|^{-3-h} & \text{else} \end{cases}.$$

Notice that, if $\mathbf{m} \in [\theta]R$, then $|\mathbf{m}| \leq \sqrt{X^2 + Y^2}$ and the angle between $\mathbf{m}$ and the line of direction θ is at most $\arcsin(Y/|\mathbf{m}|)$. Hence, if we integrate over the middle half of I instead of I in the definition of Φ , then $\Phi(\mathbf{m}) = 0$ whenever $|\mathbf{m}| \geq 2X$, and whenever $|\mathbf{m}| \geq Y/\sin(|I|/4)$ and $\arg(\mathbf{m}) \notin I \cup (I + \pi)$. For the first property, we use that $X \geq Y$, which holds for the choices below, since equalizing the two terms in the minimum gives $X = Y^{1+h/2}$, and $Y \geq 1$. For the sake of readability, we keep Φ as defined above and use the thresholds X and Y in place of $2X$ and $Y/\sin(|I|/4)$. Restoring them only multiplies Z by $\min\{1/4, \sin^{3+h}(|I|/4)\}$ and $|I|$ by $1/2$ in the last step, so it changes the constants c_3 and c_4 below, but not the exponents.

First, we consider all $\mathbf{m} \in \mathbb{Z}^2$ such that $|\mathbf{m}| \geq Y$. By some basic geometry, we find that $\Phi(\mathbf{m}) \leq 2\alpha$ where α is such that $|\mathbf{m}|\sin \alpha = Y$, and therefore we obtain

$$\Phi(\mathbf{m}) \leq \pi \frac{Y}{|\mathbf{m}|}.$$

Also, by the discussion above, $\Phi(\mathbf{m}) = 0$ for every $\mathbf{m} \in \mathbb{Z}^2$ such that $\arg(\mathbf{m}) \notin I \cup (I + \pi)$. Recall that in the sector $\arg(\mathbf{m}) \in I \cup (I + \pi)$ we aim for $Z\Phi(\mathbf{m}) \leq |\mathbf{m}|^{-3}$. Since $\Phi(\mathbf{m}) = 0$ for every $\mathbf{m}$ such that $|\mathbf{m}| \geq X$, we are therefore led to choose

$$Z \leq \frac{1}{\pi Y X^2}.$$

On the other hand, we consider all $\mathbf{m} \in \mathbb{Z}^2$ such that $|\mathbf{m}| \leq Y$. We have the trivial estimate $\Phi(\mathbf{m}) \leq |I| \leq \pi$. It is enough to aim for $Z\Phi(\mathbf{m}) \leq |\mathbf{m}|^{-3-h}$, and therefore we are led to choose

$$Z \leq \frac{1}{\pi Y^{3+h}}.$$

Thus, the choice

$$Z \leq \min\left(\pi^{-1}Y^{-1}X^{-2}, \pi^{-1}Y^{-3-h}\right)$$

will suit us overall. By equalizing the two terms in the minimum, while keeping in mind the constraint $4XY = \kappa N$, we finally get

$$X = c_1 N^{\frac{2+h}{4+h}}, \quad Y = c_2 N^{\frac{2}{4+h}}, \quad \text{and} \quad Z = c_3 N^{-\frac{6+2h}{4+h}},$$

where c_i are positive values that eventually depend on κ and h .

Finally, for any configuration of N points $\mathcal{P}_N = \{\mathbf{p}_j\}_{j=1}^N \subset \mathbb{T}^2$, by Parseval's identity and by Lemma 1.3.3, we get

$$\begin{aligned} \int_{\Xi} \int_{\mathbb{T}^2} |\mathcal{D}([\tau, \xi]C; \mathcal{P}_N)|^2 d\tau d\xi &= \sum_{\mathbf{m} \in \mathbb{Z}^2 \setminus \{0\}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{p}_j} \right|^2 \int_{\Xi} |\widehat{\mathbf{1}}_{\xi C}(\mathbf{m})|^2 d\xi \\ &\geq \tilde{c} \sum_{|\mathbf{m}| \geq \tilde{\rho}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{p}_j} \right|^2 Z\Phi(\mathbf{m}) \\ &= \tilde{c} Z \int_I \sum_{|\mathbf{m}| \geq \tilde{\rho} \text{ and } \mathbf{m} \in [\theta]R} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{p}_j} \right|^2 d\theta \\ &\geq \tilde{c} Z |I| (\kappa N^2/4 - c_{\tilde{\rho}} N^2), \end{aligned}$$

where we have set $c_{\tilde{\rho}} = \#\{B_{\tilde{\rho}} \cap \mathbb{Z}^2\}$ (here $B_{\tilde{\rho}}$ denotes the ball of radius $\tilde{\rho}$ centered at the origin). By choosing $\kappa = 8c_{\tilde{\rho}}$ in the last line, we obtain

$$\int_{\Xi} \int_{\mathbb{T}^2} |\mathcal{D}([\tau, \xi]C; \mathcal{P}_N)|^2 d\tau d\xi \geq c_4 N^{-\frac{6+2h}{4+h}} N^2 = c_4 N^{\frac{2}{4+h}},$$

where c_4 is a positive value that eventually depends on h , $\tilde{\rho}$, $\tilde{c}$ and $|I|$. □

3.2 Lower and upper bounds

We now turn to the proof of our main results on the optimal *partial-similar* quadratic discrepancy of planar convex bodies.

Proof of Theorem 3.1.2. By Lemma 2.3.8, we have that, uniformly for every $\omega \in \mathbb{T}_{2\pi}$, it holds

$$\int_I \int_0^1 \left| \widehat{\mathbb{1}}_{[\delta, \theta]C}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \asymp \rho^{-3}. \quad (3.2)$$

Hence, by Theorem 3.1.4 in the case of $h = 0$, we get the lower bound

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^I(C; \mathcal{P}) \gtrsim N^{1/2}.$$

In order to show the upper bound, we aim to find a suitable sampling for every N . To proceed, we first show it in the case of N being a square, and then the general upper bound will follow from Lagrange's four-square theorem and the fact that, for $a_1, \dots, a_4 \geq 0$, it holds

$$\left(\sum_{j=1}^4 a_j \right)^2 \leq 4 \sum_{j=1}^4 a_j^2.$$

Hence, let N be a square and consider

$$\mathcal{P}_N = \{\mathbf{p}_{h,j}\}_{h,j=1}^{N^{1/2}} = \left\{ \left(\frac{h}{N^{1/2}}, \frac{j}{N^{1/2}} \right) \right\}_{h,j=1}^{N^{1/2}} \subset \mathbb{T}^2.$$

By Parseval's identity, we get

$$\mathcal{D}_2^I(C; \mathcal{P}_N) = \sum_{\mathbf{m} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} \left| \sum_{h=1}^{N^{1/2}} \sum_{j=1}^{N^{1/2}} e^{2\pi i \mathbf{m} \cdot \mathbf{p}_{h,j}} \right|^2 \int_I \int_0^1 \left| \widehat{\mathbb{1}}_{[\delta, \theta]C}(\mathbf{m}) \right|^2 d\delta d\theta,$$

and in particular, notice that

$$\sum_{h=1}^{N^{1/2}} \sum_{j=1}^{N^{1/2}} e^{2\pi i \mathbf{m} \cdot \mathbf{p}_{h,j}} = \begin{cases} N & \text{if } m_1 = n_1 N^{1/2} \text{ and } m_2 = n_2 N^{1/2} \text{ for some } \mathbf{n} \in \mathbb{Z}^2 \\ 0 & \text{else} \end{cases}.$$

Finally, by (3.2), we get

$$\begin{aligned} \mathcal{D}_2^I(C; \mathcal{P}_N) &= \sum_{\mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} N^2 \int_I \int_0^1 \left| \widehat{\mathbb{1}}_{[\delta, \theta]C}(\mathbf{n} N^{1/2}) \right|^2 d\delta d\theta \\ &\preccurlyeq N^2 \sum_{\mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} |\mathbf{n}|^{-3} N^{-3/2} \preccurlyeq N^{1/2}. \end{aligned}$$

□

The proof of Theorem 3.1.3 requires more attention. The first step to prove both lower and upper bound will be to identify two sectors of $\mathbb{T}_{2\pi}$ where the averaged Fourier transform of C has different magnitudes of decay.

Proof of Theorem 3.1.3. First, we prove the lower bound. By Lemma 2.1.2 and Corollary 2.3.4, and by accounting for the fact that not all points on the boundary of a planar convex body can be angular points, it follows that there exists an interval $I_1 \subset \mathbb{T}_{2\pi}$ such that, uniformly for every $\omega \in I_1 \cup (I_1 + \pi)$, it holds

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \asymp \rho^{-3}.$$

Moreover, by the results in Lemma 2.1.2 and Proposition 2.2.2, we obtain that, uniformly for every $\omega \in (I_1 \cup (I_1 + \pi))^c$, it holds

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \succcurlyeq \rho^{-2} \int_{\omega+I} \gamma_C^2(\theta, \rho^{-1}) d\theta \succcurlyeq \rho^{-4}.$$

Therefore, by applying Theorem 3.1.4 in the case of $h = 1$, we get the lower bound

$$\inf_{\#P=N} \mathcal{D}_2^I(C; \mathcal{P}) \succcurlyeq N^{2/5}.$$

Now, we proceed to show the upper bound. Since $|I| < \psi_C$, and by Lemma 2.1.2 and Remark 2.3.3, we get the existence of an open interval I_2 such that, uniformly for every $\omega \in I_2 \cup (I_2 + \pi)$, it holds

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \asymp \rho^{-2} \int_{\omega+I} \gamma_C^2(\theta, \rho^{-1}) d\theta \asymp \rho^{-4}. \quad (3.3)$$

Hence, by the results in Lemma 2.1.2 and Corollary 2.3.4, we have that, uniformly for every $\omega \in (I_2 \cup (I_2 + \pi))^c$, it holds

$$\int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \preccurlyeq \rho^{-2} \int_{\omega+I} \gamma_C^2(\theta, \rho^{-1}) d\theta \preccurlyeq \rho^{-3}. \quad (3.4)$$

We proceed to show an explicit construction of suitable samplings. First, let us do it for a number N of points such that

$$N = \lfloor n^{3/5} \rfloor \lfloor n^{2/5} \rfloor \quad \text{for some } n \in \mathbb{N}.$$

Hence, set

$$G = \lfloor n^{3/5} \rfloor, \quad L = \lfloor n^{2/5} \rfloor, \quad J_G = [0, G-1] \cap \mathbb{N} \quad \text{and} \quad J_L = [0, L-1] \cap \mathbb{N}.$$

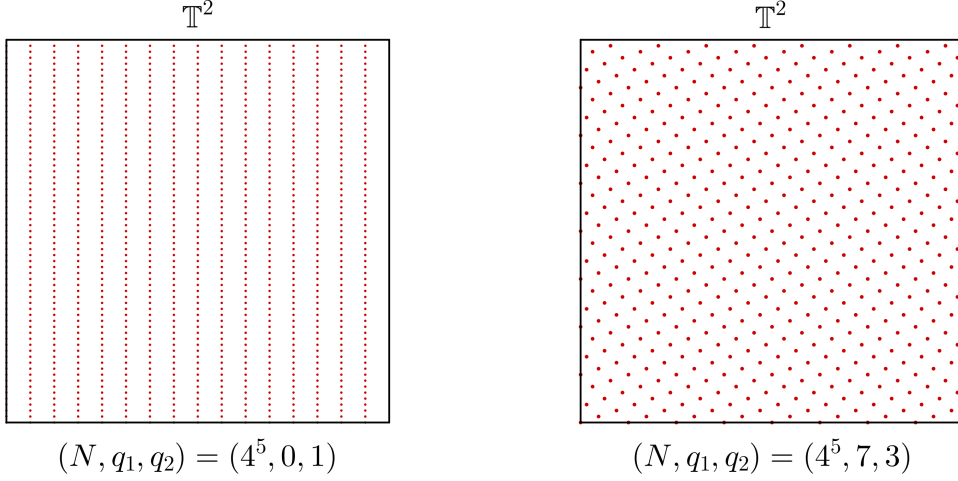


Figure 3.1: Two different dispositions of the points in $\mathcal{P}_N$ for $N = 4^5$. The one on the left side corresponds to the starting lattice. The points on the right side are to be counted with multiplicity 2.

Now, take

$$\frac{q_1}{q_2} \in \mathbb{Q} \quad \text{such that} \quad \gcd(q_1, q_2) = 1 \quad \text{and} \quad \arctan \frac{q_1}{q_2} \in I_2 \cup (I_2 + \pi),$$

so that the line $q_2 y = q_1 x$ makes an angle in $I_2 \cup (I_2 + \pi)$ with the x -axis. For the sake of simplicity, we set

$$\tilde{\omega} = \arctan \frac{q_1}{q_2}.$$

To glimpse the idea behind the coming construction, notice that

$$\sigma_{\tilde{\omega}}(x, y) = (q_1^2 + q_2^2)^{-1/2} (q_2 x - q_1 y, q_1 x + q_2 y).$$

Hence, consider the configuration of points $\mathcal{P}_N \subset \mathbb{T}^2$ defined by

$$\mathcal{P}_N = \{\mathbf{p}_j\}_{j=1}^N = \{\mathbf{p}_{\ell, g}\}_{\ell \in J_L, g \in J_G} \quad \text{with} \quad \mathbf{p}_{\ell, g} = \left(q_2 \frac{\ell}{L} - q_1 \frac{g}{G}, q_1 \frac{\ell}{L} + q_2 \frac{g}{G} \right),$$

where the coordinates of $\mathbf{p}_{\ell, g}$ are taken modulo 1 (in particular, repetition of points in $\mathcal{P}_N$ is admitted, and we refer to Figure 3.1 for an example). Further, one may notice that $\mathbf{p}_{\ell, g}$ is the representative in $\mathbb{T}^2$ of the point $(\ell/L, g/G) \in \mathbb{R}^2$ after a counterclockwise rotation by the angle $\tilde{\omega}$ and a dilation by the factor $(q_1^2 + q_2^2)^{1/2}$. Again, by Parseval's identity, we

obtain

$$\int_{\mathbb{T}^2} |\mathcal{D}([\boldsymbol{\tau}, \delta, \theta]C; \mathcal{P}_N)|^2 d\boldsymbol{\tau} = \sum_{\mathbf{m} \neq (0,0)} \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\mathbf{m}) \right|^2 \left| \sum_{g \in J_G} \sum_{\ell \in J_L} e^{2\pi i \mathbf{m} \cdot \mathbf{p}_{\ell, g}} \right|^2.$$

Observe that

$$\begin{aligned} & \sum_{g \in J_G} \sum_{\ell \in J_L} e^{2\pi i \left(\frac{\ell}{L} (q_2 m_1 + q_1 m_2) + \frac{g}{G} (q_2 m_2 - q_1 m_1) \right)} = \\ & = \begin{cases} GL & \text{if } q_2 m_1 + q_1 m_2 = n_1 L \text{ and } q_2 m_2 - q_1 m_1 = n_2 G \text{ for some } \mathbf{n} \in \mathbb{Z}^2, \\ 0 & \text{else} \end{cases}, \end{aligned}$$

hence we are looking for all non-zero $\mathbf{m} \in \mathbb{Z}^2$ for which there exist some $\mathbf{n} \in \mathbb{Z}^2$ such that

$$\begin{cases} m_1 = \frac{1}{q_1^2 + q_2^2} (q_2 n_1 L - q_1 n_2 G) \\ m_2 = \frac{1}{q_1^2 + q_2^2} (q_1 n_1 L + q_2 n_2 G) \end{cases}.$$

We label as $\mathcal{R}$ the set of all the non-zero $\mathbf{m} \in \mathbb{Z}^2$ that happen to be solutions to the latter system. Furthermore, we consider the auxiliary set

$$\tilde{\mathcal{R}} = (q_1^2 + q_2^2)^{-1/2} \{ (n_1 L, n_2 G) : \mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \},$$

and in particular, we notice that $[-\tilde{\omega}] \mathcal{R} \subset \tilde{\mathcal{R}}$.

Again, since the Fourier transform and rotations commute, we get

$$\begin{aligned} \sum_{\mathbf{m} \in \mathcal{R}} \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\mathbf{m}) \right|^2 &= \sum_{\mathbf{m} \in [-\tilde{\omega}] \mathcal{R}} \left| \widehat{\mathbf{1}}_{[\delta, \theta]C}(\sigma_{\tilde{\omega}} \mathbf{m}) \right|^2 \\ &= \sum_{\mathbf{m} \in [-\tilde{\omega}] \mathcal{R}} \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2 \leq \sum_{\mathbf{m} \in \tilde{\mathcal{R}}} \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2. \end{aligned} \tag{3.5}$$

In order to estimate the latter quantity, we distinguish between two different regions of $\tilde{\mathcal{R}}$. First, we let

$$\alpha \in \left(0, \frac{\pi}{2}\right) \quad \text{be such that} \quad [-\alpha, \alpha] \subset (I_2 \cup (I_2 + \pi) - \tilde{\omega}),$$

and then we split $\tilde{\mathcal{R}}$ in the region

$$V = \left\{ \mathbf{m} \in \tilde{\mathcal{R}} : \arg \mathbf{m} \in [-\alpha, \alpha] \right\}$$

and its complementary V^c . In particular, the condition $\arg \mathbf{m} \in [-\alpha, \alpha]$ in the definition of V translates into the requirement

$$|n_2|G \leq |n_1|L \tan \alpha.$$

Hence, by (3.5), we have

$$\begin{aligned}
\mathcal{D}_2^I(C; \mathcal{P}_N) &\leq \\
&\leq \int_I \int_0^1 G^2 L^2 \sum_{\mathbf{m} \in \tilde{\mathcal{R}}} \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2 d\delta d\theta \\
&= G^2 L^2 \sum_{\mathbf{m} \in V} \int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2 d\delta d\theta + \\
&\quad + G^2 L^2 \sum_{\mathbf{m} \in V^c} \int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2 d\delta d\theta.
\end{aligned}$$

For the first sum in the last term, by (3.3), we get

$$\begin{aligned}
&G^2 L^2 \sum_{\mathbf{m} \in V} \int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2 d\delta d\theta \preccurlyeq \\
&\preccurlyeq G^2 L^2 \sum_{\mathbf{m} \in V} |\mathbf{m}|^{-4} \\
&\preccurlyeq G^2 L^2 (q_1^2 + q_2^2)^2 (1 + \tan \alpha)^{-4} \sum_{n_1=1}^{+\infty} \sum_{n_2=0}^{n_1 \frac{L}{G} \tan \alpha} (n_1 L)^{-4} \\
&\preccurlyeq G^2 L^{-2} \sum_{n_1=1}^{+\infty} n_1^{-4} \left(n_1 \frac{L}{G} \tan \alpha + 1 \right) \preccurlyeq G^2 L^{-2} \preccurlyeq N^{2/5}.
\end{aligned}$$

On the other hand, for the second sum, we get

$$\begin{aligned}
&G^2 L^2 \sum_{\mathbf{m} \in V^c} \int_I \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta - \tilde{\omega}]C}(\mathbf{m}) \right|^2 d\delta d\theta \preccurlyeq \\
&\preccurlyeq G^2 L^2 \sum_{\mathbf{m} \in V^c} |\mathbf{m}|^{-3} \\
&\preccurlyeq G^2 L^2 (q_1^2 + q_2^2)^{3/2} (1 + \cot \alpha)^{-3} \sum_{n_2=1}^{+\infty} \sum_{n_1=0}^{n_2 \frac{G}{L} \cot \alpha} (n_2 G)^{-3} \\
&\preccurlyeq G^{-1} L^2 \sum_{n_2=1}^{+\infty} n_2^{-3} \left(n_2 \frac{G}{L} \cot \alpha + 1 \right) \preccurlyeq L \preccurlyeq N^{2/5},
\end{aligned}$$

and we can conclude that the initial claim holds for all N of the form $N = \lfloor n^{3/5} \rfloor \lfloor n^{2/5} \rfloor$.

In order to prove that there is a suitable choice of points for every positive integer N ,

consider the following recursive definition

$$n_j = \max \left\{ n \in \mathbb{N} : \lfloor n^{3/5} \rfloor \lfloor n^{2/5} \rfloor \leq N - \sum_{i=1}^{j-1} \lfloor n_i^{3/5} \rfloor \lfloor n_i^{2/5} \rfloor \right\} \quad \text{for } j \in \mathbb{N} \setminus \{0\},$$

where improper sums are conventionally considered as zeros. Now, notice that the latter definition implies that

$$\begin{aligned} N - \sum_{i=1}^{j-1} \lfloor n_i^{3/5} \rfloor \lfloor n_i^{2/5} \rfloor &\leq \lfloor (n_j + 1)^{3/5} \rfloor \lfloor (n_j + 1)^{2/5} \rfloor \\ &\leq \lfloor n_j^{3/5} \rfloor \lfloor n_j^{2/5} \rfloor + \lfloor n_j^{3/5} \rfloor + \lfloor n_j^{2/5} \rfloor + 1, \end{aligned}$$

and therefore, it follows that

$$N - \sum_{i=1}^j \lfloor n_i^{3/5} \rfloor \lfloor n_i^{2/5} \rfloor \leq 2n_j^{3/5}.$$

Again, by the definition of n_j , it is easy to see that

$$\frac{n_1}{2} \leq N \quad \text{and} \quad \frac{n_{j+1}}{2} \leq N - \sum_{i=1}^j \lfloor n_i^{3/5} \rfloor \lfloor n_i^{2/5} \rfloor,$$

so that by induction, we get

$$N - \sum_{i=1}^j \lfloor n_i^{3/5} \rfloor \lfloor n_i^{2/5} \rfloor \leq 2n_j^{3/5} \leq 2^4 n_{j-1}^{9/25} \leq 2^{2j} N^{(3/5)^j}.$$

In particular, notice that

$$N - \sum_{j=1}^4 \lfloor n_j^{3/5} \rfloor \lfloor n_j^{2/5} \rfloor \leq 2^8 N^{(3/5)^4} = o(N^{1/5}).$$

Finally, we associate a choice of points as in the previous construction to every $N_j = \lfloor n_j^{3/5} \rfloor \lfloor n_j^{2/5} \rfloor$ with $1 \leq j \leq 4$, and we ignore the remaining points since their contribution is $o(N^{1/5})$. The conclusion follows at once since, for $a_1, \dots, a_5 \geq 0$, it holds

$$\left(\sum_{j=1}^5 a_j \right)^2 \leq 5 \sum_{j=1}^5 a_j^2.$$

□

3.3 Intermediate orders

We proceed to show that the optimal *partial-similar* quadratic discrepancy can achieve any polynomial order in between $N^{1/2}$ and $N^{2/5}$. First, we carry on the explicit geometric construction of our convex planar bodies, and then, we move onto the discrepancy estimates.

3.3.1 Geometric construction

We prove that, for an interval of angles

$$I(\phi) = \left[-\frac{\phi}{2}, \frac{\phi}{2}\right] \subset \mathbb{T}_{2\pi} \quad \text{with} \quad \phi \in (0, \pi), \quad \text{and for} \quad \alpha \in (1, +\infty), \quad (3.6)$$

there exists a planar convex body $C(\phi, \alpha)$ with piecewise- $\mathcal{C}^\infty$ boundary such that it holds

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{I(\phi)}(C(\phi, \alpha); \mathcal{P}) \asymp N^{\frac{2\alpha}{4\alpha+1}}.$$

For the sake of notation, the letter ε will stand for a generic positive small value throughout this section. Moreover, for an interval $U \subseteq [0, +\infty)$ and two positive functions f and g defined on U , we say that for $x \in U$ it holds

$$f(x) \approx g(x)$$

to intend that there exist positive values c_1 and c_2 (which eventually depend on α and ϕ) such that, for every $x \in U$, it holds

$$c_1 g(x) \leq f(x) \leq c_2 g(x).$$

The key to obtaining these intermediate orders is to build such a convex body in a way that $\psi_{C(\phi, \alpha)} = \phi$. For the sake of construction, first, consider a planar convex body $H(\phi, \alpha)$ such that it has a centre of symmetry and such that it is symmetric with respect to the line

$$y = x \tan\left(\frac{\pi}{2} - \frac{\phi}{2}\right).$$

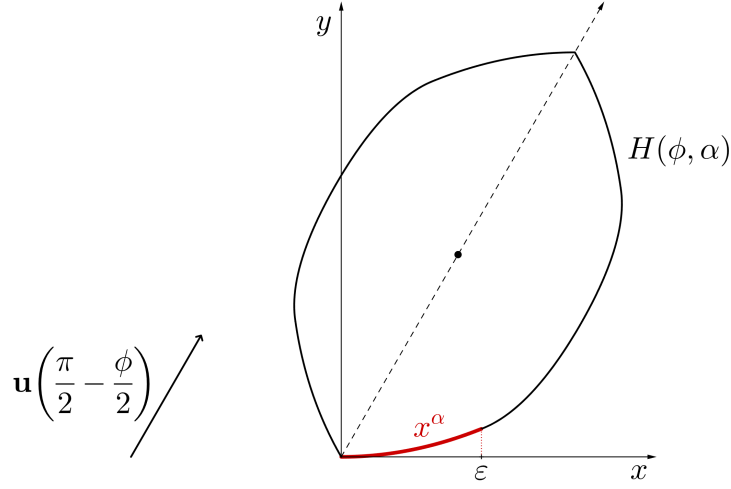
Moreover, build it in such a way that

$$\{(x, x^\alpha) \in \mathbb{R}^2 : x \in [0, \varepsilon]\} \subset \partial H(\phi, \alpha).$$

Last, construct $H(\phi, \alpha)$ in such a way that its boundary is $\mathcal{C}^\infty$ except at the origin and at its symmetric counterpart (see Figure 3.2). Hence, in order to evaluate its optimal *partial-similar* quadratic discrepancy, it is sufficient to get estimates for the chords of $H(\phi, \alpha)$ about the origin. By symmetry, we can restrict ourselves to studying the directions

$$\mathbf{u}(\theta) \quad \text{for} \quad \theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2} + \varepsilon\right].$$

First, we present an auxiliary technical result.

Figure 3.2: A depiction of $H(\phi, \alpha)$.

Lemma 3.3.1. *Let α and β be positive numbers, and let $g: [0, +\infty) \rightarrow [0, +\infty)$ be such that*

$$g(x) \approx \begin{cases} x^\alpha & \text{if } 0 \leq x < 1 \\ x^\beta & \text{if } x \geq 1 \end{cases}.$$

If x_y is such that $g(x_y) = y$, then it holds

$$x_y \approx \begin{cases} y^{1/\alpha} & \text{if } 0 \leq y < 1 \\ y^{1/\beta} & \text{if } y \geq 1 \end{cases}.$$

Proof. By hypothesis, there exist two positive values c_1 and c_2 such that it holds

$$\begin{cases} c_1 x^\alpha \leq g(x) \leq c_2 x^\alpha & \text{if } 0 \leq x < 1 \\ c_1 x^\beta \leq g(x) \leq c_2 x^\beta & \text{if } x \geq 1 \end{cases}.$$

If $y < c_1$ then we necessarily have $0 \leq x_y \leq 1$, and therefore

$$c_1 x_y^\alpha \leq y \leq c_2 x_y^\alpha.$$

Rearranging, one gets that

$$c_1^{1/\alpha} x_y \leq y^{1/\alpha} \leq c_2^{1/\alpha} x_y \quad \text{for } y \in [0, c_1).$$

On the other hand, if $y > c_2$ then we necessarily have $x_y \geq 1$, and therefore

$$c_1 x_y^\beta \leq y \leq c_2 x_y^\beta.$$

Rearranging, one gets that

$$c_1^{1/\beta} x_y \leq y^{1/\beta} \leq c_2^{1/\beta} x_y \quad \text{for } y \in (c_2, +\infty).$$

The claim follows since, for every $y \in [c_1, c_2]$, we have that x_y is bounded away from 0 or $+\infty$. $\square$

Let us first study the case when $s_{H(\phi, \alpha)}^o(\theta)$ is the origin, or in other words, when $\theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2}\right]$.

Proposition 3.3.2. *Let $H(\phi, \alpha)$ be as above. Uniformly for every $\theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2}\right]$, it holds*

$$|K_{H(\phi, \alpha)}(\theta, \rho^{-1})| \asymp \begin{cases} \rho^{-1/\alpha} & \text{if } 0 \leq \frac{\pi}{2} - \theta < \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-1}(\frac{\pi}{2} - \theta)^{-1} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} \leq \frac{\pi}{2} - \theta \leq \frac{\phi}{2} \end{cases}.$$

Proof. By symmetry, there exists $\rho_0 > 0$ such that, for every $\theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2}\right]$ and for every $\rho \geq \rho_0$, we have that the part of the chord $K_{H(\phi, \alpha)}(\theta, \rho^{-1})$ at the right of $y = x \tan(\frac{\pi}{2} - \frac{\phi}{2})$ is longer than the part at the left. Hence, by considering the auxiliary shape

$$F(\alpha) = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq x^\alpha\},$$

it is not difficult to see that, uniformly for every $\theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2}\right]$, it holds

$$|K_{H(\phi, \alpha)}(\theta, \rho^{-1})| \asymp |K_{F(\alpha)}(\theta, \rho^{-1})|.$$

Therefore, we can restrict ourselves to studying the chords of $F(\alpha)$. Now, for the sake of notation, we let

$$x_+ = x_{F(\alpha)}^+(\theta, \rho^{-1}) \quad \text{be the abscissa of } \Gamma_{F(\alpha)}(s_{F(\alpha)}^+(\theta, \rho^{-1})),$$

and define x_- analogously. It is immediate to see that, for every $\theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2}\right]$, we have $x_- = 0$, and it also holds

$$|K_{F(\alpha)}(\theta, \rho^{-1})| = \frac{x_+ - x_-}{\sin \theta}.$$

On the other hand, x_+ is the abscissa of the intersection in $x \geq 0$ between the curve $y = x^\alpha$ and the straight line

$$y - \rho^{-1} \sin \theta = -\frac{1}{\tan \theta}(x - \rho^{-1} \cos \theta).$$

Rearranging, we have that x_+ is a solution of

$$x(x^{\alpha-1} \sin \theta + \cos \theta) = \rho^{-1},$$

and by the normalization

$$z = x^{\alpha-1} \tan \theta,$$

we get the equation

$$f(z) = z^{\frac{1}{\alpha-1}} (z+1) = \frac{(\tan \theta)^{\frac{\alpha}{\alpha-1}}}{\rho \sin \theta}.$$

Notice that it holds

$$f(z) \approx \begin{cases} z^{\frac{1}{\alpha-1}} & \text{if } 0 \leq z < 1 \\ z^{\frac{\alpha}{\alpha-1}} & \text{if } z \geq 1 \end{cases},$$

and by applying Lemma 3.3.1, and the fact that for $\theta \in \left[\frac{\pi}{2} - \frac{\phi}{2}, \frac{\pi}{2}\right]$ it holds

$$\sin \theta \approx 1 \quad \text{and} \quad \cot \theta \approx \frac{\pi}{2} - \theta,$$

it follows that

$$x_+^{\alpha-1} \left(\frac{\pi}{2} - \theta\right)^{-1} \approx \begin{cases} \rho^{\frac{1-\alpha}{\alpha}} \left(\frac{\pi}{2} - \theta\right)^{-1} & \text{if } 0 \leq \frac{\pi}{2} - \theta < \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{1-\alpha} \left(\frac{\pi}{2} - \theta\right)^{-\alpha} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} \leq \frac{\pi}{2} - \theta \leq \frac{\phi}{2} \end{cases}.$$

By a last rearrangement, we get

$$x_+ \approx \begin{cases} \rho^{-1/\alpha} & \text{if } 0 \leq \frac{\pi}{2} - \theta < \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-1} \left(\frac{\pi}{2} - \theta\right)^{-1} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} \leq \frac{\pi}{2} - \theta \leq \frac{\phi}{2} \end{cases}.$$

□

We now turn to estimating $|K_{H(\phi, \alpha)}(\theta, \rho^{-1})|$ in the case of $\theta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right]$. Again, we make use of an auxiliary shape. Namely, consider

$$G(\alpha) = \{(x, y) \in \mathbb{R}^2 : y \geq |x|^\alpha\},$$

and as before, notice that, uniformly for every $\theta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right]$, it holds

$$|K_{H(\phi, \alpha)}(\theta, \rho^{-1})| \asymp |K_{G(\alpha)}(\theta, \rho^{-1})|.$$

First, we need a technical observation on the chords of $G(\alpha)$.

Proposition 3.3.3. *Let $G(\alpha)$ be as above. There exists a positive value c_α such that, for every $\theta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right]$ and for every $\rho \geq 1$, it holds*

$$|K_{G(\alpha)}^-(\theta, \rho^{-1})| \leq c_\alpha |K_{G(\alpha)}^+(\theta, \rho^{-1})|.$$

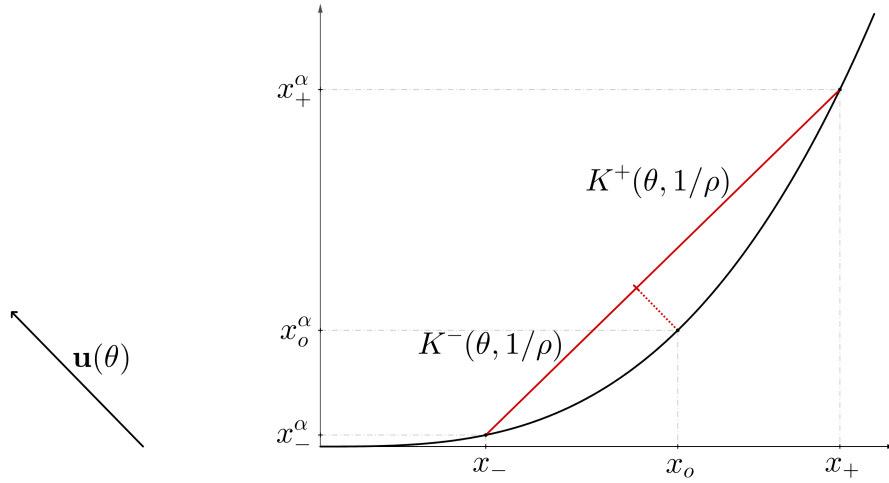


Figure 3.3: An auxiliary image for the proof of Proposition 3.3.3. For simplicity, we omit to write $G(\alpha)$.

Proof. For the sake of notation, let

$$x_o = x_{G(\alpha)}^o(\theta) \quad \text{be the abscissa of } \Gamma_{G(\alpha)}(s_{G(\alpha)}^o(\theta)).$$

Moreover, we let

$$x_+ = x_{G(\alpha)}^+(\theta, \rho^{-1}) \quad \text{be the abscissa of } \Gamma_{G(\alpha)}(s_{G(\alpha)}^+(\theta, \rho^{-1})),$$

and define x_- analogously. With the help of Figure 3.3, notice that, for every $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$, it holds

$$\left| K_{G(\alpha)}^-(\theta, \rho^{-1}) \right| \sin \theta \leq x_o - x_- \quad \text{and} \quad \left| K_{G(\alpha)}^+(\theta, \rho^{-1}) \right| \sin \theta \geq x_+ - x_o,$$

and therefore, it is enough to show that

$$x_o - x_- \leq c_\alpha(x_+ - x_o).$$

Indeed, for every $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$, we have that x_- and x_+ are the abscissas of the intersections of the curve $y = |x|^\alpha$ with the straight line

$$y = (x - x_o)\alpha x_o^{\alpha-1} + x_o^\alpha + \frac{1}{\rho \sin \theta}.$$

Equalizing, and with the normalization $z = \frac{x-x_o}{x_o}$, we get to the equation

$$f(z) = |z+1|^\alpha - z\alpha - 1 = \frac{1}{x_o^\alpha \rho \sin \theta}, \quad (3.7)$$

and we also remark that, for every $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$, both x_o and $\sin \theta$ are non-negative. Hence, the conclusion follows once we show that

$$f(z) \leq f(-c_\alpha z) \quad \text{for every } z \geq 0,$$

since this would imply

$$\frac{x_+ - x_o}{x_o} \geq -\frac{1}{c_\alpha} \cdot \frac{x_- - x_o}{x_o}.$$

Last, it is not difficult to see that by choosing $c_\alpha = 2^\alpha$ then, for every $z \geq 0$, it holds

$$f'(z) = \alpha (|z+1|^{\alpha-1} - 1) \leq \alpha 2^\alpha (|2^\alpha z - 1|^{\alpha-1} + 1) = \frac{\partial}{\partial z} f(-2^\alpha z),$$

and indeed, one has

$$z+1 \leq 2 \quad \text{for } 0 \leq z < 1, \quad \text{and} \quad z+1 \leq 2(2z-1) \quad \text{for } z \geq 1.$$

□

Now, we proceed to estimate the chords in the case of $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$.

Proposition 3.3.4. *Let $H(\phi, \alpha)$ be as above. Uniformly for every $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$, it holds*

$$|K_{H(\phi, \alpha)}(\theta, \rho^{-1})| \asymp \begin{cases} \rho^{-1/\alpha} & \text{if } 0 \leq \theta - \frac{\pi}{2} < \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-1/2} \left(\theta - \frac{\pi}{2}\right)^{\frac{2-\alpha}{2(\alpha-1)}} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} \leq \theta - \frac{\pi}{2} \leq \varepsilon \end{cases}.$$

Proof. We have already noted that we can equivalently study the chords of the auxiliary shape $G(\alpha)$, and therefore, we define x_- , x_o , x_+ , and f , as in the Proposition 3.3.3. Since

$$|K_{G(\alpha)}(\theta, \rho^{-1})| \sin \theta = (x_+ - x_-),$$

then, by the arguments in Proposition 3.3.3, it is enough to estimate $(x_+ - x_o)$. As before, x_+ is a solution of

$$|x|^\alpha = (x - x_o)\alpha x_o^{\alpha-1} + x_o^\alpha + \frac{1}{\rho \sin \theta},$$

and again by the normalization $z = \frac{x-x_o}{x_o}$, we get (3.7). In particular, we remark that the solution x_+ corresponds to the range $z \geq 0$. Now, by applying Taylor's formula with integral remainder to f , we get

$$f(z) = \alpha(\alpha-1) \int_0^z (1+t)^{\alpha-2} (z-t) dt.$$

Notice that for $z \in [0, 1)$ it holds

$$\int_0^z (1+t)^{\alpha-2} (z-t) dt \approx \int_0^z (z-t) dt = \frac{z^2}{2}.$$

On the other hand, for $z \in [1, +\infty)$ it holds

$$\begin{aligned} \int_0^z (1+t)^{\alpha-2} (z-t) dt &= \int_0^{z/2} (1+t)^{\alpha-2} (z-t) dt + \int_{z/2}^z (1+t)^{\alpha-2} (z-t) dt \\ &\approx z \int_0^{z/2} (1+t)^{\alpha-2} dt + z^{\alpha-2} \int_{z/2}^z (z-t) dt \\ &= \frac{z}{\alpha-1} \left((1+z/2)^{\alpha-1} - 1 \right) + z^{\alpha-2} \frac{z^2}{8} \approx z^\alpha. \end{aligned}$$

Hence, we get

$$f(z) \approx \begin{cases} z^2 & \text{if } 0 \leq z < 1 \\ z^\alpha & \text{if } z \geq 1 \end{cases},$$

and if we consider (3.7), by applying Lemma 3.3.1, and by the fact that for $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$ it holds $\sin \theta \approx 1$, then it follows that

$$\frac{x_+ - x_o}{x_o} \approx \begin{cases} \rho^{-1/2} x_o^{-\alpha/2} & \text{if } 0 \leq \rho^{-1} x_o^{-\alpha} < 1 \\ \rho^{-1/\alpha} x_o^{-1} & \text{if } \rho^{-1} x_o^{-\alpha} \geq 1 \end{cases}. \quad (3.8)$$

Last, by the definition of x_o , we have

$$\alpha x_o^{\alpha-1} = \left. \frac{d}{dx} x^\alpha \right|_{x=x_o} = \tan\left(\theta - \frac{\pi}{2}\right),$$

and therefore, we get that for $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon]$ it holds

$$x_o \approx \left(\theta - \frac{\pi}{2}\right)^{\frac{1}{\alpha-1}}.$$

The conclusion hence follows by a simple rearrangement of the terms in (3.8). $\square$

Now, we are able to estimate the Fourier transform.

Proposition 3.3.5. *Let $I(\phi)$ and $H(\phi, \alpha)$ be as above, and let $\tilde{\phi} = \frac{\pi}{2} - \frac{\phi}{2}$. Uniformly for every $\omega \in [-\varepsilon, \varepsilon]$, it holds*

$$\int_{I(\phi)} \int_0^1 \left| \widehat{\mathbb{1}}_{[\delta, \theta]H(\phi, \alpha)} \left(\rho \mathbf{u}(\tilde{\phi} + \omega) \right) \right|^2 d\delta d\theta \asymp \begin{cases} \rho^{-3-\frac{1}{\alpha}} & \text{if } |\omega| \leq \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-3} \omega^{\frac{1}{\alpha-1}} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} < |\omega| \leq \varepsilon \end{cases}.$$

Proof. By symmetry, we can restrict ourselves to studying the case of $\omega \in [0, \varepsilon]$. Indeed, by Lemma 2.1.2, we have that, uniformly for every $\omega \in [0, \varepsilon]$, it holds

$$\begin{aligned} \int_{I(\phi)} \int_0^1 \left| \widehat{\mathbb{1}}_{[\delta, \theta]H(\phi, \alpha)} \left(\rho \mathbf{u}(\tilde{\phi} + \omega) \right) \right|^2 d\delta d\theta &\asymp \rho^{-2} \int_{-\phi/2}^{\phi/2} \gamma_{[\theta]H(\phi, \alpha)}^2 \left(\tilde{\phi} + \omega, \rho^{-1} \right) d\theta \\ &= \rho^{-2} \int_{\omega-\phi}^{\omega} \gamma_{H(\phi, \alpha)}^2 \left(\frac{\pi}{2} + \theta, \rho^{-1} \right) d\theta \\ &\asymp \rho^{-2} \int_{-\phi/2}^{\omega} \gamma_{H(\phi, \alpha)}^2 \left(\frac{\pi}{2} + \theta, \rho^{-1} \right) d\theta, \end{aligned}$$

where the last approximation follows from the symmetries of $H(\phi, \alpha)$. By Proposition 3.3.2 and by Proposition 3.3.4, we get that

$$\gamma_{H(\phi, \alpha)} \left(\frac{\pi}{2} + \theta, \rho^{-1} \right) \asymp \begin{cases} -\rho^{-1}\theta^{-1} & \text{if } -\frac{\phi}{2} \leq \theta < -\rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-1/\alpha} & \text{if } |\theta| \leq \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-1/2}\theta^{\frac{2-\alpha}{2(\alpha-1)}} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} < \theta \leq \varepsilon \end{cases}.$$

Therefore, uniformly for every $\omega \in [0, \rho^{\frac{1-\alpha}{\alpha}}]$, we have

$$\begin{aligned} \int_{-\frac{\phi}{2}}^{\omega} \gamma_{H(\phi, \alpha)}^2 \left(\frac{\pi}{2} + \theta, \rho^{-1} \right) d\theta &\asymp \int_{-\frac{\phi}{2}}^{-\rho^{\frac{1-\alpha}{\alpha}}} \rho^{-2} |\theta|^{-2} d\theta + \int_{-\rho^{\frac{1-\alpha}{\alpha}}}^{\omega} \rho^{-2/\alpha} d\theta \\ &= \rho^{-2} \left(\rho^{\frac{\alpha-1}{\alpha}} - 2\phi^{-1} \right) + \rho^{-2/\alpha} \left(\omega + \rho^{\frac{1-\alpha}{\alpha}} \right) \asymp \rho^{-\frac{\alpha+1}{\alpha}}. \end{aligned}$$

On the other hand, in the case of $\omega \in \left(\rho^{\frac{1-\alpha}{\alpha}}, \varepsilon \right]$, we must take into account the additional term

$$\begin{aligned} \int_{\rho^{\frac{1-\alpha}{\alpha}}}^{\omega} \gamma_{H(\phi, \alpha)}^2 \left(\frac{\pi}{2} + \theta, \rho^{-1} \right) d\theta &\asymp \int_{\rho^{\frac{1-\alpha}{\alpha}}}^{\omega} \rho^{-1} \theta^{\frac{2-\alpha}{\alpha-1}} d\theta \\ &= \rho^{-1}(\alpha-1) \left(\omega^{\frac{1}{\alpha-1}} - \rho^{-1/\alpha} \right), \end{aligned}$$

and the initial claim easily follows. $\square$

3.3.2 Lower and upper bounds

We have gathered the necessary estimates to prove the main result of this section, which we now state.

Theorem 3.3.6. *Let $I(\phi)$ and α be as in (3.6). There exists a convex body $C(\phi, \alpha)$ with piecewise- $\mathcal{C}^\infty$ boundary such that it holds*

$$\inf_{\#P=N} \mathcal{D}_2^{I(\phi)}(C(\phi, \alpha); P) \asymp N^{\frac{2\alpha}{1+4\alpha}}.$$

Proof. Let $H(\phi, \alpha)$ be as above, and consider

$$C(\phi, \alpha) = \left[\frac{\phi}{2} - \frac{\pi}{2} \right] H(\phi, \alpha).$$

In particular, notice that $C(\phi, \alpha)$ is symmetric with respect to the x -axis. Further, by Proposition 3.3.5, we have that, uniformly for every $\omega \in (-\varepsilon, \varepsilon)$, it holds

$$\int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\rho \mathbf{u}(\omega)) \right|^2 d\delta d\theta \asymp \begin{cases} \rho^{-3-\frac{1}{\alpha}} & \text{if } |\omega| \leq \rho^{\frac{1-\alpha}{\alpha}} \\ \rho^{-3}\omega^{\frac{1}{\alpha-1}} & \text{if } \rho^{\frac{1-\alpha}{\alpha}} < |\omega| \leq \varepsilon \end{cases}, \quad (3.9)$$

and, by symmetry, analogous estimates hold in the case of $\omega \in (\pi - \varepsilon, \pi + \varepsilon)$. On the other hand, since by construction $\partial C(\phi, \alpha)$ is $\mathcal{C}^\infty$ everywhere except at the origin and at its symmetric counterpart, by Lemma 2.1.2 and Corollary 2.3.4, we have that, uniformly for every

$$\omega \in [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon],$$

it holds

$$\int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta]C(\phi, \alpha)}(\rho \mathbf{u}(\omega - \theta)) \right|^2 d\delta d\theta \asymp \rho^{-3}. \quad (3.10)$$

In particular, notice that the hypotheses of Theorem 3.1.4 are satisfied, and we may apply it in the case of $h = 1/\alpha$. Consequently, we get the lower bound

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{I(\phi)}(C(\phi, \alpha); \mathcal{P}) \gtrsim N^{\frac{2\alpha}{1+4\alpha}}.$$

Now, we turn our attention to the upper bound. We show this by employing a suitable sampling scheme, namely the one in [BT22, Thm. 8]. First, let us do it for a number of N points such that

$$N = \lfloor n^{\frac{1+2\alpha}{1+4\alpha}} \rfloor \lfloor n^{\frac{2\alpha}{1+4\alpha}} \rfloor \quad \text{for some } n \in \mathbb{N}.$$

Hence, set

$$G = \lfloor n^{\frac{1+2\alpha}{1+4\alpha}} \rfloor, \quad L = \lfloor n^{\frac{2\alpha}{1+4\alpha}} \rfloor, \quad J_G = [0, G-1] \cap \mathbb{N}, \quad \text{and} \quad J_L = [0, L-1] \cap \mathbb{N}.$$

Consider the set of points $\mathcal{P}_N \subset \mathbb{T}^2$ defined by

$$\mathcal{P}_N = \{\mathbf{p}_j\}_{j=1}^N = \{\mathbf{p}_{\ell, g}\}_{\ell \in J_L, g \in J_G} \quad \text{with} \quad \mathbf{p}_{\ell, g} = \left(\frac{\ell}{L}, \frac{g}{G} \right),$$

where the coordinates of $\mathbf{p}_{\ell, g}$ are taken modulo 1. Again, by Parseval's identity, we get

$$\int_{\mathbb{T}^2} |\mathcal{D}([\boldsymbol{\tau}, \delta, \theta]C(\phi, \alpha); \mathcal{P}_N)|^2 d\boldsymbol{\tau} = \sum_{\mathbf{m} \neq (0,0)} \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 \left| \sum_{g \in J_G} \sum_{\ell \in J_L} e^{2\pi i \mathbf{m} \cdot \mathbf{p}_{\ell, g}} \right|^2,$$

and we observe that

$$\sum_{g \in J_G} \sum_{\ell \in J_L} e^{2\pi i(m_1 \frac{\ell}{L} + m_2 \frac{g}{G})} = \begin{cases} GL & \text{if } m_1 \in L\mathbb{Z} \text{ and } m_2 \in G\mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Hence, we can consider

$$\mathbf{m} = (Ln_1, Gn_2) \quad \text{with} \quad \mathbf{n} \in \mathbb{Z}^2,$$

and split the set

$$\mathcal{R} = (L\mathbb{Z} \times G\mathbb{Z}) \setminus \{\mathbf{0}\}$$

into the regions

$$\begin{aligned} V_1 &= \{\mathbf{m} \in \mathcal{R} : |m_2|^\alpha \leq |m_1|\}, \\ V_2 &= \{\mathbf{m} \in \mathcal{R} : |m_2| \leq |m_1| < |m_2|^\alpha\}, \\ V_3 &= \{\mathbf{m} \in \mathcal{R} : |m_1| < |m_2|\}. \end{aligned}$$

Then, we write

$$\begin{aligned} \mathcal{D}_2^{I(\phi)}(C(\phi, \alpha); \mathcal{P}_N) &= \\ &= \int_{I(\phi)} \int_0^1 G^2 L^2 \sum_{\mathbf{m} \in \mathcal{R}} \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 d\delta d\theta \\ &= G^2 L^2 \left(\sum_{\mathbf{m} \in V_1} + \sum_{\mathbf{m} \in V_2} + \sum_{\mathbf{m} \in V_3} \right) \int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 d\delta d\theta. \end{aligned} \tag{3.11}$$

We exploit (3.9) and (3.10) in order to study the three sums in the latter equation. In this case, we must consider

$$\rho = |\mathbf{m}| \quad \text{and} \quad \tan \omega = \frac{m_2}{m_1}.$$

We notice that for $\omega \in [-1, 1]$ it holds $\tan \omega \approx \omega$, and consequently, with a bit of rearrangement, we can rewrite the estimates in (3.9) and (3.10) as

$$\int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 d\delta d\theta \asymp \begin{cases} |m_1|^{-3-\frac{1}{\alpha}} & \text{if } |m_2|^\alpha \leq |m_1| \\ |m_1|^{\frac{2-3\alpha}{\alpha-1}} |m_2|^{\frac{1}{\alpha-1}} & \text{if } |m_2| \leq |m_1| < |m_2|^\alpha \\ |m_2|^{-3} & \text{if } |m_1| < |m_2| \end{cases}. \tag{3.12}$$

By the latter, for the first sum in the last term of (3.11), we get

$$G^2 L^2 \sum_{\mathbf{m} \in V_1} \int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 d\delta d\theta$$

$$\begin{aligned}
&\preceq G^2 L^2 \sum_{\mathbf{m} \in V_1} |m_1|^{-3-\frac{1}{\alpha}} \\
&\preceq G^2 L^2 \sum_{n_1=1}^{+\infty} n_1^{1/\alpha} L^{1/\alpha} G^{-1} (Ln_1)^{-3-\frac{1}{\alpha}} \\
&\preceq G^2 L^{-\frac{1+\alpha}{\alpha}} \sum_{n_1=1}^{+\infty} n_1^{-3-\frac{1}{\alpha}} \left(1 + n_1^{1/\alpha} L^{1/\alpha} G^{-1}\right) \\
&\preceq \left(G^2 L^{-\frac{1+\alpha}{\alpha}} + GL^{-1}\right) \preceq N^{\frac{2\alpha}{1+4\alpha}}.
\end{aligned}$$

For the second sum in the last term of (3.11), by applying (3.12), we get

$$\begin{aligned}
&G^2 L^2 \sum_{\mathbf{m} \in V_2} \int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 d\delta d\theta \\
&\preceq G^2 L^2 \sum_{\mathbf{m} \in V_2} |m_1|^{\frac{2-3\alpha}{\alpha-1}} |m_2|^{\frac{1}{\alpha-1}} \\
&\preceq G^2 L^2 \sum_{n_2=1}^{+\infty} \sum_{n_1=n_2 GL^{-1}}^{n_2^\alpha G^\alpha L^{-1}} (Ln_1)^{\frac{2-3\alpha}{\alpha-1}} (Gn_2)^{\frac{1}{\alpha-1}} \\
&\preceq G^{\frac{2\alpha-1}{\alpha-1}} L^{\frac{-\alpha}{\alpha-1}} \sum_{n_2=1}^{+\infty} n_2^{\frac{1}{\alpha-1}} \sum_{n_1=n_2 GL^{-1}}^{+\infty} n_1^{\frac{2-3\alpha}{\alpha-1}} \\
&\preceq G^{\frac{2\alpha-1}{\alpha-1}} L^{\frac{-\alpha}{\alpha-1}} \sum_{n_2=1}^{+\infty} n_2^{\frac{1}{\alpha-1}} (n_2 GL^{-1})^{\frac{1-2\alpha}{\alpha-1}} \\
&\preceq L \sum_{n_2=1}^{+\infty} n_2^{-2} \preceq L \preceq N^{\frac{2\alpha}{1+4\alpha}}.
\end{aligned}$$

Finally, for the last sum in the last term in (3.11), again by applying (3.12), we get

$$\begin{aligned}
&G^2 L^2 \sum_{\mathbf{m} \in V_3} \int_{I(\phi)} \int_0^1 \left| \widehat{\mathbf{1}}_{[\delta, \theta]C(\phi, \alpha)}(\mathbf{m}) \right|^2 d\delta d\theta \\
&\preceq G^2 L^2 \sum_{\mathbf{m} \in V_3} |m_2|^{-3} \\
&\preceq G^2 L^2 \sum_{n_2=1}^{+\infty} \sum_{n_1=0}^{n_2 GL^{-1}} (Gn_2)^{-3} \\
&\preceq G^{-1} L^2 \sum_{n_2=1}^{+\infty} n_2^{-2} GL^{-1} \preceq L \preceq N^{\frac{2\alpha}{1+4\alpha}}.
\end{aligned}$$

Hence, we conclude that the upper bound holds for all N of the form $N = \lfloor n^{\frac{1+2\alpha}{1+4\alpha}} \rfloor \lfloor n^{\frac{2\alpha}{1+4\alpha}} \rfloor$.

Last, in order to prove the initial claim holds for every $N \in \mathbb{N}$, it is enough to repeat the argument at the end of the proof of Theorem 3.1.3 with adjusted exponents. $\square$

Chapter 4

Polynomial-order oscillations in geometric discrepancy

This chapter presents the content of [Ber26b]. We study the optimal *homothetic* quadratic discrepancy and show, by two independent methods, that it need not exhibit a single order of growth; the main results are Theorems 4.1.5 and 4.1.6. The second method draws on Lemma 2.1.2 from Chapter 2, connecting this chapter's results to those of Chapters 2–3.

4.1 Introduction

We recall the notion of *homothetic* quadratic discrepancy, in the planar case.

Definition 4.1.1. Let $\Omega \subset \mathbb{R}^2$ and let $\mathcal{P}_N \subset \mathbb{T}^2$ be a configuration of N points. We define the *homothetic* quadratic discrepancy of $\mathcal{P}_N$ with respect to Ω as

$$\mathcal{D}_2^{\text{id}}(\Omega; \mathcal{P}_N) = \int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}([\tau, \delta]\Omega; \mathcal{P}_N)|^2 \, d\tau \, d\delta, \quad (4.1)$$

where $d\tau$ is the normalized Haar measure on $\mathbb{T}^2$ and $d\delta$ is the Lebesgue measure on $[0, 1]$.

It is a classical question in the theory of irregularities of distribution to determine the optimal behaviour of the *homothetic* quadratic discrepancy as N (the number of samples per unit square) goes to infinity. In other words, we are concerned with the quantity

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \quad \text{as } N \rightarrow +\infty.$$

Surprisingly, the qualitative behaviour of this quantity depends strongly on the geometry of C : for instance, when C is the unit square one obtains a logarithmic order of growth, while when C is the unit disk one obtains a polynomial order. Before showing known

estimates for more general bodies, we introduce suitable notation for orders of growth (in particular, this differs from the notation for limit behaviours we used in Chapters 2–3).

Let f and g be two functions defined on the same domain, and let q be an object; we write $f \lesssim_q g$ to indicate that there exists a positive value c , that may depend on q , such that $f \leq cg$ in the whole domain. If we omit q in the subscript, then c is a positive constant. If the $\lesssim_q$ (or $\lesssim$) holds in both senses, then we write $f \approx_q g$ (or $f \approx g$). We also write $f \gtrsim_q g$ (or $f \gtrsim g$) for $g \lesssim_q f$ (or $g \lesssim f$). We stress that, in contrast with Section 3.3, a bare $\approx$ here involves an absolute constant.

First, we state a result on the optimal *homothetic* quadratic discrepancy with respect to polygons. Throughout this chapter, we assume that N is an integer greater than 1.

Theorem 4.1.2 (Beck-Chen). *Let C be a convex polygon. Then, it holds*

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \approx_C \log N.$$

In particular, the lower bound is due to a classical work of Beck [Bec88], while the upper bound follows from a subsequent work of Beck and Chen [BC97].

In contrast with the latter result, one has the following result on the optimal *homothetic* quadratic discrepancy for convex bodies with sufficiently regular boundaries.

Theorem 4.1.3 (Brandolini-Travaglini). *Let C be a planar convex body. If the boundary of C is $\mathcal{C}^2$ (regardless of curvature), then it holds*

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \approx_C N^{1/2}.$$

The upper bound is well-known and can be easily shown via a probabilistic argument employing a randomly shifted lattice (for example, see [BCC⁺19]). On the other hand, the lower bound is due to a recent work of Brandolini and Travaglini [BT22] and requires subtle Fourier-analytic techniques; these provide a starting point for our work. In the same paper, the following result is shown, notably recovering a polynomial order of growth across an interval of exponents.

Theorem 4.1.4 (Brandolini-Travaglini). *Let $\alpha \in [2/5, 1/2)$. Then, there exists a planar convex body C_α such that*

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C_\alpha; \mathcal{P}) \approx_\alpha N^\alpha.$$

In this chapter, we aim to show that, in general, the optimal *homothetic* quadratic discrepancy does not need to exhibit a single order of growth, and prescribed stationary orders of growth can be achieved. For this purpose, we present two general methods for constructing special sets whose optimal *homothetic* quadratic discrepancy continuously switch between different orders of growth (logarithmic or polynomial).

Our first method employs a simple geometric argument. This allows prescribed oscillations from a logarithmic order of $\log N$ to a polynomial order of $N^{1/2}$, and vice versa. We state our result as follows.

Theorem 4.1.5. *Let $\{\alpha_i\}_{i \in \mathbb{N}}$ be in $\{0, 1/2\}$, let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be in $(0, \varepsilon]$ for some small positive ε , and let $\{q_i\}_{i \in \mathbb{N}}$ be positive integers. Then there exists a planar convex body C and an increasing sequence $\{N_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ such that, for every $i \in \mathbb{N}$ such that $\alpha_i = 0$, it holds*

$$\log^{1-\varepsilon_i} N \leq \inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \leq \log^{1+\varepsilon_i} N \quad \text{for every } N \in [N_i, N_i + q_i],$$

while, for every $i \in \mathbb{N}$ such that $\alpha_i = 1/2$, it holds

$$N^{1/2-\varepsilon_i} \leq \inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \leq N^{1/2+\varepsilon_i} \quad \text{for every } N \in [N_i, N_i + q_i].$$

This first method obtains the planar convex body C as the limit of an increasing sequence of planar convex bodies; thus, C is constructed implicitly. The simplicity of our first method comes at the cost of limited knowledge on the geometry of the boundary of C . In particular, we rely on Lemma 4.2.1, showing that two bodies with a small symmetric difference have close discrepancies. Further, by pairing the just-mentioned lemma with a classical topological argument, we show that the set of planar convex bodies whose optimal *homothetic* quadratic discrepancy has a single order of growth is meagre in the space of planar convex bodies endowed with the Hausdorff metric.

On the other hand, our second method provides a direct construction. This employs harmonic analysis techniques and subtle geometric estimates on the Fourier transform of sets. We obtain prescribed polynomial-order oscillations in the range N^α with $\alpha \in (2/5, 1/2)$, and state our second result as follows.

Theorem 4.1.6. *Let $\{\alpha_i\}_{i \in \mathbb{N}}$ be in $(2/5, 1/2)$, let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be in $(0, \varepsilon]$ for some small positive ε , and let $\{q_i\}_{i \in \mathbb{N}}$ be positive integers. Then there exists a planar convex body C and an increasing sequence $\{N_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ such that, for every $i \in \mathbb{N}$, it holds*

$$N^{\alpha_i-\varepsilon_i} \leq \inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \leq N^{\alpha_i+\varepsilon_i} \quad \text{for every } N \in [N_i, N_i + q_i].$$

The second method involves constructing the boundary of C “by hand”; as we will see, it suffices to design the boundary only in a small neighbourhood of a point and its centrally symmetric counterpart, while leaving the remaining part as $\mathcal{C}^2$. Moreover, we note that we can alternate exponents within the same range as in Theorem 4.1.4.

To the best of our knowledge, these two results settle the first examples of sets for which the optimal *homothetic* quadratic discrepancy has prescribed stationary orders of growth.

In the pages that follow, Section 4.2 contains the implicit geometric construction that leads to Theorem 4.1.5, while Section 4.3 contains the direct construction that proves Theorem 4.1.6. In particular, the proof of Theorem 4.1.6 requires different techniques to establish lower and upper bounds, and therefore, these are treated in distinct subsections.

4.2 The first method: a geometric approach

Recalling the hypotheses of Theorem 4.1.5, we have that $\{\alpha_i\}_{i \in \mathbb{N}}$ is a sequence of exponents with values in $\{0, 1/2\}$, $\{\varepsilon_i\}_{i \in \mathbb{N}}$ is a (possibly decreasing) sequence in $(0, \varepsilon]$ for some small positive ε , and $\{q_i\}_{i \in \mathbb{N}}$ is a (possibly increasing) sequence of positive integers. We remark that α_i prescribes the target order of growth, while ε_i and q_i quantify, respectively, how closely and for how long we approximate it.

We begin by presenting a simple but essential auxiliary result.

Lemma 4.2.1. *Let $\mathcal{P}$ be a configuration of N points in $\mathbb{T}^2$, and let A and B be two planar sets such that $\text{diam}(A) \leq 1$, $A \supset B$, and $|A \setminus B| \leq \eta \leq 1$. Then, it holds*

$$\left| \mathcal{D}_2^{\text{id}}(A; \mathcal{P}) - \mathcal{D}_2^{\text{id}}(B; \mathcal{P}) \right| \leq 4N^2\eta^{1/2}.$$

Proof. By the reverse triangle inequality, it holds

$$\left| \mathcal{D}_2^{\text{id}}(A; \mathcal{P}) - \mathcal{D}_2^{\text{id}}(B; \mathcal{P}) \right|^2 \leq 4N^2 \int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}(\tau + \delta A; \mathcal{P}) - \mathcal{D}(\tau + \delta B; \mathcal{P})|^2 d\tau d\delta. \quad (4.2)$$

Now, since $A \supset B$, it holds

$$\mathcal{D}(\tau + \delta A; \mathcal{P}) - \mathcal{D}(\tau + \delta B; \mathcal{P}) = \mathcal{D}(\tau + \delta(A \setminus B); \mathcal{P}). \quad (4.3)$$

Since $\text{diam}(A) \leq 1$, then, for any $\mathbf{p} \in \mathbb{T}^2$, the set of τ 's such that

$$\mathbb{1}_{\tau + \delta(A \setminus B)}(\mathbf{p}) \neq 0$$

has measure $\delta^2|A \setminus B| \leq \delta^2\eta$. Therefore, it holds

$$\int_{\mathbb{T}^2} |\mathbb{1}_{\tau + \delta(A \setminus B)}(\mathbf{p})|^2 d\tau \leq \delta^2\eta.$$

Now, by applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}(\tau + \delta(A \setminus B); \mathcal{P})|^2 d\tau d\delta &\leq 2 \int_0^1 \int_{\mathbb{T}^2} \left| \sum_{\mathbf{p} \in \mathcal{P}} \mathbb{1}_{\tau + \delta(A \setminus B)}(\mathbf{p}) \right|^2 d\tau d\delta + 2N^2\eta^2 \\ &\leq 2N \sum_{\mathbf{p} \in \mathcal{P}} \int_0^1 \int_{\mathbb{T}^2} |\mathbb{1}_{\tau + \delta(A \setminus B)}(\mathbf{p})|^2 d\tau d\delta + 2N^2\eta^2 \\ &\leq 2N^2\eta + 2N^2\eta^2 \leq 4N^2\eta. \end{aligned}$$

Combining the latter with (4.2) and (4.3) yields the claim. $\square$

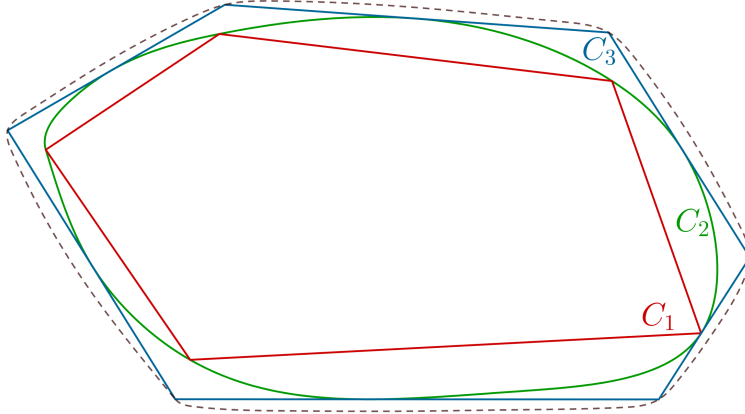


Figure 4.1: The construction in the proof of Theorem 4.1.5.

The latter result is all we need to proceed with the proof of Theorem 4.1.5. We start by considering a sequence of convex planar bodies $\{C_i\}_{i \in \mathbb{N}}$ to be chosen later (see Figure 4.1), but such that each of their diameters is less than 1 and

$$\partial C_i \text{ is } \begin{cases} \text{a polygon} & \text{if } \alpha_i = 0 \\ \text{a } \mathcal{C}^2 \text{ curve} & \text{if } \alpha_i = 1/2 \end{cases}.$$

Now, let $\{\eta_i\}_{i \in \mathbb{N}}$ be a (decreasing) sequence with values in $(0, 1]$, but to be chosen later as well. Further, for every $i \in \mathbb{N}$, we further impose that $C_i \subset C_{i+1}$ and that $|C_{i+1} \setminus C_i| \leq \eta_{i+1}$.

For the sake of notation, consider the function

$$H_i(N) = \begin{cases} \log N & \text{if } \alpha_i = 0 \\ N^{1/2} & \text{if } \alpha_i = 1/2 \end{cases}.$$

We recall that, by Theorem 4.1.2 and Theorem 4.1.3, for every $i \in \mathbb{N}$, it holds

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C_i; \mathcal{P}) \approx_{C_i} H_i(N). \quad (4.4)$$

We now construct the sequence $\{\eta_i\}_{i \in \mathbb{N}}$ recursively. Suppose $\{\eta_j\}_{j=1}^i$ and $\{C_j\}_{j=1}^i$ have already been chosen. Then, let N_i be such that

$$H_i^{1-\varepsilon_i}(N) + 1 \leq \inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C_i; \mathcal{P}) \leq H_i^{1+\varepsilon_i}(N) - 1 \quad \text{for every } N \in [N_i, N_i + q_i]. \quad (4.5)$$

By (4.4), and since $H_i(N) \rightarrow +\infty$, this holds for all sufficiently large N_i . In particular, we may take $\{N_i\}_{i \in \mathbb{N}}$ increasing. Then, choose η_{i+1} in such a way that $\eta_{i+1} \leq \eta_i/2$ and

$$\eta_{i+1} < (N_i + q_i)^{-4}/32.$$

Next, we choose C_{i+1} with the properties imposed above. This is possible, since C_i can be approximated from outside, in the Hausdorff metric, both by convex polygons and by convex bodies with $\mathcal{C}^2$ boundary. Thus, the choices are made in the order $\eta_1, C_1, N_1, \eta_2, C_2, N_2, \dots$, with η_1 and C_1 arbitrary. Every later body C_j , with $j > i$, satisfies $|C_j \setminus C_i| \leq 2\eta_{i+1}$. Since η_{i+1} is chosen after N_i , this changes the discrepancy in (4.5) by less than 1, as shown below. The terms ± 1 in (4.5) absorb this change.

By choosing the whole sequence $\{\eta_i\}_{i \in \mathbb{N}}$ in such a fashion, it follows that the nested sequence of the planar convex bodies $\{C_i\}_{i \in \mathbb{N}}$ converges in the Hausdorff metric to a planar convex body C such that, for every $i \in \mathbb{N}$, it holds

$$|C \setminus C_i| \leq \sum_{j=i+1}^{\infty} \eta_j \leq 2\eta_{i+1}.$$

Therefore, by applying Lemma 4.2.1, we obtain that, for every $i \in \mathbb{N}$, for every $N \in [N_i, N_i + q_i]$, and for every configuration $\mathcal{P}$ of N points in $\mathbb{T}^2$, it holds

$$|\mathcal{D}_2^{\text{id}}(C; \mathcal{P}) - \mathcal{D}_2^{\text{id}}(C_i; \mathcal{P})| \leq 4N^2(2\eta_{i+1})^{1/2} < 1.$$

Finally, by pairing the latter inequality with (4.5), the claim of Theorem 4.1.5 follows.

What follows is inspired by [BRT98, Sec. 4], where the authors applied an analogous argument to study the average decay of the Fourier transform of convex sets. We let $\mathfrak{C}$ be the space of convex bodies in $\mathbb{R}^2$ endowed with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y| \right\} \quad \text{for } A, B \in \mathfrak{C}.$$

By the Blaschke selection theorem, the metric space $(\mathfrak{C}, d_H)$ is locally compact, and by the Baire category theorem, it follows that $(\mathfrak{C}, d_H)$ is a Baire space and, in particular, it is nonmeagre.

In [SW65] it is shown that the Hausdorff metric and the symmetric-difference metric $d_\Delta(A, B) = |A \Delta B|$ induce the same topology on $\mathfrak{C}$. It is not difficult to adapt the proof of Lemma 4.2.1 and show that, for a fixed $N \in \mathbb{N}$, the functional

$$\mathcal{D}_2^{\text{id}}(\cdot; \mathcal{P}): (\mathfrak{C}, d_\Delta) \rightarrow (0, +\infty)$$

is (locally) uniformly continuous in its argument with respect to all configurations $\mathcal{P}$ of N points. Indeed, it is enough to pair the equality $A \Delta B = (A \setminus B) \sqcup (B \setminus A)$ with the fact that $\mathcal{D}(\cdot; \mathcal{P})$ is additive on disjoint sets, and note that the condition on the diameter in the hypothesis of Lemma 4.2.1 is (locally) not restrictive. Hence, we infer that, for a fixed $N \in \mathbb{N}$, the functional

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(\cdot; \mathcal{P}): (\mathfrak{C}, d_\Delta) \rightarrow (0, +\infty)$$

is continuous. It is time to state a technical result of Gruber [Gru83].

Lemma 4.2.2 (Gruber). *Let $\{a_j\}_{j \in \mathbb{N}} \subset (0, +\infty)$, let T be a nonmeagre topological space, and let $\{\phi_j\}_{j \in \mathbb{N}}$ be a sequence of continuous functionals with $\phi_j : T \rightarrow (0, +\infty)$. Define*

$$\begin{aligned} \mathcal{A} &= \{x \in T : \phi_j(x) = o(a_j) \text{ as } j \rightarrow +\infty\}, \\ \mathcal{B} &= \{x \in T : a_j = o(\phi_j(x)) \text{ as } j \rightarrow +\infty\}. \end{aligned}$$

Then, the following holds:

- i) *If $\mathcal{A}$ is dense in T , then, for every $x \in T$ except for a meagre subset, the inequality $\phi_j(x) < a_j$ holds for infinitely many values of j .*
- ii) *If $\mathcal{B}$ is dense in T , then, for every $x \in T$ except for a meagre subset, the inequality $\phi_j(x) > a_j$ holds for infinitely many values of j .*

Finally, by pairing Theorem 4.1.2 and Theorem 4.1.3 with the latter lemma and the fact that $\inf_{\#P=N} \mathcal{D}_2^{\text{id}}(\cdot; \mathcal{P})$ is continuous in $(\mathfrak{C}, d_\Delta)$, we obtain the following result.

Theorem 4.2.3. *Let $\varepsilon \in (0, 1]$. Then, for every planar convex body $C \in \mathfrak{C}$ except for a meagre subset, there exist two infinite subsets $\mathcal{N}_C^-, \mathcal{N}_C^+ \subset \mathbb{N}$ such that*

$$\inf_{\#P=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \leq \log^{1+\varepsilon} N \quad \forall N \in \mathcal{N}_C^- \quad \text{and} \quad \inf_{\#P=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \geq N^{1/2-\varepsilon} \quad \forall N \in \mathcal{N}_C^+.$$

4.3 The second method: a Fourier analytic approach

The notation we are to employ in this section is the same as that in Chapters 2–3.

We recall that for a convex body $C \subset \mathbb{R}^2$, by Lemma 2.1.2 (established in Chapter 2), there exists a positive value λ_C such that, for every angle $\theta \in [0, 2\pi)$ and every $\lambda \in (0, \lambda_C]$, it holds

$$\int_0^1 \left| \widehat{\mathbf{1}}_{\delta C}(\lambda^{-1} \mathbf{u}(\theta)) \right|^2 d\delta \approx \lambda^2 \gamma_C^2(\theta, \lambda).$$

4.3.1 Geometric construction

Recalling the hypotheses of Theorem 4.1.6, we have that $\{\alpha_i\}_{i \in \mathbb{N}}$ is a sequence of exponents with values in $(2/5, 1/2)$, $\{\varepsilon_i\}_{i \in \mathbb{N}}$ is a (possibly decreasing) sequence in $(0, \varepsilon]$ for some small positive ε , and $\{q_i\}_{i \in \mathbb{N}}$ is a (possibly increasing) sequence of positive integers. We remark that α_i prescribes the target order of growth, while ε_i and q_i quantify, respectively, how closely and for how long we approximate it.

We start our construction. Consider a planar convex body C such that it has a centre of symmetry (this ensures $\gamma_C = |K_C|$) and is symmetric with respect to the y -axis. Then, suppose that $\mathbf{0} \in \partial C$ and that its inner normal is the vector $(0, 1)$. By symmetry, we may

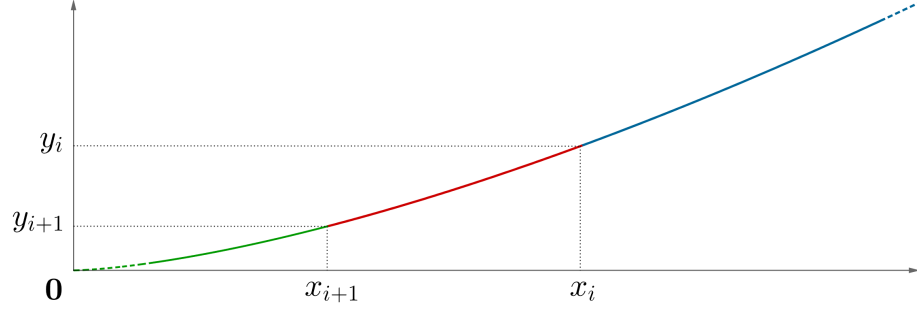


Figure 4.2: A close-up look at P , in which different auxiliary monomial curve segments are displayed in different colours.

construct ∂C in the first quadrant $\mathbb{R}_+^2$. For this purpose, consider the graphs in $\mathbb{R}_+^2$ of the monomials

$$\{P_i\}_{i \in \mathbb{N}} \quad \text{with} \quad P_i(x) = x^{\beta_i} \quad \text{and} \quad \beta_i = \frac{2\alpha_i}{2 - 3\alpha_i} \in (1, 2),$$

and notice that their curvatures

$$\kappa_i(x) = \beta_i(\beta_i - 1) \frac{x^{\beta_i - 2}}{(1 + \beta_i^2 x^{2\beta_i - 2})^{3/2}} \quad (4.6)$$

are strictly decreasing on $(0, +\infty)$, and that $\kappa_i(x) \rightarrow +\infty$ as $x \rightarrow 0^+$.

Now, let $\{k_i\}_{i \in \mathbb{N}}$ be a sequence to be chosen later, but such that it is contained in $[10, +\infty)$ and it holds $k_{i+1} > k_i$ with $k_i \rightarrow +\infty$. By induction, we construct a curve segment of finite length as follows. First, consider the auxiliary curve segment

$$\{(x, y) \in \mathbb{R}_+^2 : P_1(x) = y, a_1 < x < b_1, \kappa_1(a_1) = k_1, \kappa_1(b_1) = 10\},$$

and then glue at the endpoint $(a_1, P_1(a_1))$ a tangentially-aligned copy of the auxiliary curve segment

$$\{(x, y) \in \mathbb{R}_+^2 : P_2(x) = y, a_2 < x < b_2, \kappa_2(a_2) = k_2, \kappa_2(b_2) = k_1\}.$$

In particular, notice that we obtain a $\mathcal{C}^2$ link since the signed curvatures at the endpoints $(a_1, P_1(a_1))$ and $(b_2, P_2(b_2))$ coincide. Inductively, and in such a $\mathcal{C}^2$ fashion, we glue all the auxiliary curve segments

$$\{(x, y) \in \mathbb{R}_+^2 : P_i(x) = y, a_i < x < b_i, \kappa_i(a_i) = k_i, \kappa_i(b_i) = k_{i-1}\},$$

and in the end, we obtain a curve segment of finite length. By a final rigid motion, we may place this curve segment so that it is tangent to the x -axis and its left endpoint is the

origin; see Figure 4.2 for an illustration of the resulting curve segment P . By construction, P has strictly decreasing continuous curvature (and this goes to infinity as one approaches the origin). Finally, we let ∂C coincide with P for its length, and then we design the rest of ∂C , apart from the symmetric copies of P , in such a way that it is $\mathcal{C}^2$ with signed curvature bounded from below by $\kappa_1(1)$ and from above by, say, 10. Hence, ∂C is $\mathcal{C}^2$ except at the origin and at its symmetric counterpart. At these two points, ∂C is only $\mathcal{C}^1$. Indeed, P and its reflection across the y -axis are both tangent to the x -axis at the origin, but the curvature is unbounded there.

The purpose of the latter construction is the following: ∂C has to look like the graphs of different P_i 's at increasingly smaller neighbourhoods of the origin. Ultimately, this is achieved by ensuring that the sequence $\{k_i\}_{i \in \mathbb{N}}$ increases very rapidly. Indeed, for every $i \in \mathbb{N}$, mark by (x_i, y_i) the left endpoint of P_i after being assembled into P , as in Figure 4.2. First, the fact that, for $i \in \mathbb{N}$ and $x \in (0, 1]$, it holds

$$\kappa_i(x) \leq 2x^{-1} \quad \text{and} \quad P'_i(x) \leq 2,$$

ensures that we may get any (x_i, y_i) arbitrarily close to the origin by a large enough choice of k_i . Secondly, since the elements in $\{k_j\}_{j \geq i}$ may be chosen arbitrarily large, we may get the elements in $\{P'_{j+1}(b_{j+1})\}_{j \geq i}$ arbitrarily close to 0, and this, in turn, ensures that we may get the inner normal at (x_i, y_i) arbitrarily close to the vector $(0, 1)$; in particular, this prevents the curve segment P from becoming a spiral.

In order to apply Lemma 2.1.2 to C , we are interested in how the chords $K_C(\theta + \pi/2, \lambda)$ behave for small values of θ . For this purpose, let $\beta \in (1, 2)$, and consider a second planar convex body $A(\beta)$ whose boundary coincides with the graph of $y = |x|^\beta$ in $[-1, 1]^2$. Moreover, suppose that $A(\beta)$ has a centre of symmetry and is symmetric with respect to the y -axis. We will show the following estimates on the chords of $A(\beta)$.

Proposition 4.3.1. *Let $A(\beta)$ be as above. Then, there exists a positive small constant c (independent of β) such that it holds*

$$|K_{A(\beta)}(\theta + \pi/2, \lambda)| \approx_\beta \begin{cases} \lambda^{1/\beta} & \text{if } 0 \leq |\theta| < \lambda^{\frac{\beta-1}{\beta}} < c \\ \lambda^{1/2} |\theta|^{\frac{2-\beta}{2(\beta-1)}} & \text{if } \lambda^{\frac{\beta-1}{\beta}} \leq |\theta| < c \end{cases}.$$

Recall that, by a large enough choice of the elements in $\{k_j\}_{j \geq i}$, we may get (x_i, y_i) arbitrarily close to the origin, as we may get the inner normal at (x_i, y_i) to be arbitrarily close to the vector $(0, 1)$. Ultimately, this guarantees that, for every $i \in \mathbb{N}$, we may obtain an interval $[\lambda_{i,1}, \lambda_{i,2}]$ and a positive value θ_i such that, for every θ such that $|\theta| \leq \theta_i$ and every λ such that $\lambda \in [\lambda_{i,1}, \lambda_{i,2}]$, it holds

$$|K_C(\theta + \pi/2, \lambda)| \approx |K_{A(\beta_i)}(\theta + \pi/2, \lambda)|.$$

Notice that, as θ_i and $\lambda_{i,2}$ depend on $\{\beta_j\}_{j=1}^{i-1}$ and $\{k_j\}_{j=1}^{i-1}$, the choice of $\lambda_{i,1}$ can be made arbitrarily small at every i -th step; this will be key for the estimates in the following subsection. More precisely, at the i -th step, we first fix $\lambda_{i,1}$, and then we choose k_i large enough. The later choices only change the part of P between the origin and (x_i, y_i) , and move the rest of P by a rigid motion. Since the comparison above only involves chords at depth at least $\lambda_{i,1}$, it persists as long as this part has length and total turning of the tangent much smaller than $\lambda_{i,1}$. Since $\kappa_{j+1}(b_{j+1}) = k_j$, the bounds above give that, whatever k_{j+1} is, the auxiliary curve segment built from P_{j+1} has length at most $2\sqrt{5}/k_j$ and turns the tangent by at most $2(2/k_j)^{\beta_{j+1}-1}$. Hence, all the comparisons persist if each k_j is so large that both bounds are at most 2^{-j} times a small fraction of $\min_{l \leq j} \lambda_{l,1}$.

Finally, Proposition 4.3.1 and Lemma 2.1.2 may be applied to C to infer the following result. From now on, we write ρ^{-1} in place of λ , to prioritise the argument of the Fourier transform.

Lemma 4.3.2. *Let C be as above. Then, one may choose the sequence $\{k_i\}_{i \in \mathbb{N}}$ in such a way that, for every $i \in \mathbb{N}$, there exists an interval $[\rho_{i,1}, \rho_{i,2}] \subset [1, +\infty)$ (with $\rho_{i+1,1} > \rho_{i,2}$ and $\lim_{i \rightarrow +\infty} \rho_{i,1} = +\infty$) and a positive value θ_i (with $\theta_{i+1} < \theta_i$ and $\lim_{i \rightarrow +\infty} \theta_i = 0$) such that, for $\rho \in [\rho_{i,1}, \rho_{i,2}]$ and $|\theta| \leq \theta_i$, it holds*

$$\int_0^1 \left| \widehat{\mathbb{1}}_{\delta C}(\rho \mathbf{u}(\theta + \pi/2)) \right|^2 d\delta \approx_{\beta_i} \begin{cases} \rho^{-2-2/\beta_i} & \text{if } |\theta| \leq \rho^{\frac{1-\beta_i}{\beta_i}} \\ \rho^{-3} |\theta|^{\frac{2-\beta_i}{\beta_i-1}} & \text{if } \rho^{\frac{1-\beta_i}{\beta_i}} < |\theta| \leq \theta_i \end{cases}.$$

In particular, although $\rho_{i,1}$ and θ_i are determined by the elements in $\{k_j\}_{j=1}^{i-1}$, the endpoint $\rho_{i,2}$ can be made arbitrarily large upon a larger choice of k_i .

Starting from the fact that we have designed P so that its curvature is strictly decreasing, we obtain the following auxiliary result.

Lemma 4.3.3. *Let C be as above. Then, there exist $\lambda_0 > 0$ and $\theta_0 > \pi/2$ such that, for $\lambda \in [0, \lambda_0)$ and for $\pi/2 \leq \theta_1 < \theta_2 < \theta_0$, it holds*

$$|K_C(\theta_1, \lambda)| \leq 2|K_C(\theta_2, \lambda)|.$$

Proof. We need some auxiliary geometric tools defined in Chapters 2–3, and for the sake of notation, we omit C in the subscript. Without loss of generality, we let $\mathbf{\Gamma}: [-1, 1) \rightarrow \mathbb{R}^2$ be the counter-clockwise arc-length parametrisation of ∂C such that $\mathbf{\Gamma}(0) = \mathbf{0}$, and we write $\mathbf{\Gamma}'$ as its velocity (in particular, $|\mathbf{\Gamma}'| \equiv 1$). For a chord $K(\theta, \lambda)$, we define $a(\theta, \lambda)$ and $b(\theta, \lambda)$ to be the parametrisation by $\mathbf{\Gamma}$ of the extreme points of $K(\theta, \lambda)$, with the convention that

$$\mathbf{\Gamma}(a(\theta, \lambda)) - \mathbf{\Gamma}(b(\theta, \lambda)) = |K(\theta, \lambda)| \mathbf{u}'(\theta).$$

Moreover, we set $p(\theta)$ to be the parametrisation by $\mathbf{\Gamma}$ of the point at ∂C whose inner normal is $\mathbf{u}(\theta)$. Finally, we split the chord $K(\theta, \lambda)$ into two semi-chords $K^+(\theta, \lambda)$ and $K^-(\theta, \lambda)$ as

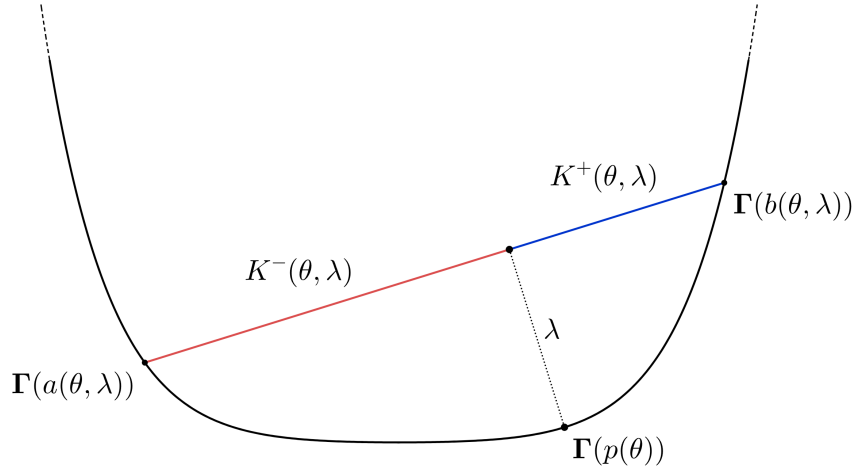


Figure 4.3: The auxiliary tools we constructed.

displayed in Figure 4.3. Now, choose θ_0 and λ_0 so that, for $\lambda \in [0, \lambda_0)$ and $\theta \in [\pi/2, \theta_0)$, it holds $\Gamma'_2(b(\theta, \lambda)) \leq 1/10$. The latter requirement bounds the endpoints of the chords $K(\theta, \lambda)$ to the curve segment constructed earlier. Thus, we may conclude once we prove that, for $\lambda \in [0, \lambda_0)$ and $\pi/2 \leq \theta_1 < \theta_2 < \theta_0$, it holds

$$|K^+(\theta_2, \lambda)| \geq |K^+(\theta_1, \lambda)| \geq |K^-(\theta_1, \lambda)|.$$

We only show the inequality $|K^+(\theta_2, \lambda)| \geq |K^+(\theta_1, \lambda)|$, since the argument for the other one is analogous. We begin by noting that, for $i \in \{1, 2\}$, it holds

$$\lambda = \int_0^{b(\theta_i, \lambda) - p(\theta_i)} \mathbf{u}(\theta_i) \cdot \Gamma'(p(\theta_i) + s) \, ds \quad (4.7)$$

and

$$|K^+(\theta_i, \lambda)| = \int_0^{b(\theta_i, \lambda) - p(\theta_i)} \mathbf{u}(\theta_i - \pi/2) \cdot \Gamma'(p(\theta_i) + s) \, ds. \quad (4.8)$$

Since the derivative of the argument of Γ' is exactly the signed curvature κ , then for $i \in \{1, 2\}$ we obtain

$$\Gamma'(p(\theta_i)) = \mathbf{u}(\theta_i - \pi/2) \quad \text{and} \quad \Gamma'(p(\theta_i) + s) = \mathbf{u} \left(\theta_i - \pi/2 + \int_0^s \kappa(p(\theta_i) + t) \, dt \right),$$

so that, by some basic trigonometry, we may rewrite (4.7) and (4.8) as

$$\lambda = \int_0^{b(\theta_i, \lambda) - p(\theta_i)} \sin \left(\int_0^s \kappa(p(\theta_i) + t) \, dt \right) \, ds \quad (4.9)$$

and

$$|K^+(\theta_i, \lambda)| = \int_0^{b(\theta_i, \lambda) - p(\theta_i)} \cos \left(\int_0^s \kappa(p(\theta_i) + t) dt \right) ds. \quad (4.10)$$

Notice that, by our construction of the curve segment P , it holds the inequality

$$\kappa(p(\theta_1) + t) \geq \kappa(p(\theta_2) + t) \quad \text{for } t \in [0, b(\theta_0, \lambda_0) - p(\theta_2)],$$

but this, combined with (4.9) and the monotonicity of the sine function in $[0, \pi/2)$, implies that

$$b(\theta_1, \lambda) - p(\theta_1) \leq b(\theta_2, \lambda) - p(\theta_2),$$

and in turn, combining the latter with (4.10) and the monotonicity of the cosine function in $[0, \pi/2)$, implies

$$|K^+(\theta_2, \lambda)| \geq |K^+(\theta_1, \lambda)|.$$

□

By pairing Lemma 2.1.2 with Lemma 4.3.3, we get that, for every $\rho > \max\{\lambda_C^{-1}, \lambda_0^{-1}\}$ and every angle θ_1 and θ_2 such that $\pi/2 \leq \theta_1 < \theta_2 < \theta_0$, it holds

$$\int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\rho \mathbf{u}(\theta_1)) \right|^2 d\delta \lesssim \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\rho \mathbf{u}(\theta_2)) \right|^2 d\delta. \quad (4.11)$$

We proceed with the proof of Proposition 4.3.1, and by symmetry, we may restrict ourselves to the case $\theta \geq 0$. Since the curvature of the graph of $y = |x|^\beta$ decreases as one moves away from the origin, then, by arguing as in the proof of Lemma 4.3.3, we obtain that there exist positive small values $\tilde{\lambda}$ and $\tilde{\theta}$ such that, for every $\lambda \in [0, \tilde{\lambda})$ and $\theta \in [0, \tilde{\theta})$, it holds

$$\left| K_{A(\beta)}^+(\theta + \pi/2, \lambda) \right| \geq \left| K_{A(\beta)}^-(\theta + \pi/2, \lambda) \right|.$$

In this same range, the endpoints of the chord $K(\theta + \pi/2, \lambda)$ are bounded to the part of $\partial A(\beta)$ that coincides with the graph of $y = |x|^\beta$. Thus, it is enough to establish the estimates in Proposition 4.3.1 for $K_{A(\beta)}^+(\theta + \pi/2, \lambda)$ with $\lambda \in [0, \tilde{\lambda})$ and $\theta \in [0, \tilde{\theta})$.

Proof of Proposition 4.3.1. Without loss of generality, let $\mathbf{\Gamma}_{A(\beta)}: [-1, 1) \rightarrow \mathbb{R}^2$ be the counter-clockwise arc-length parametrisation of $\partial A(\beta)$ such that $\mathbf{\Gamma}_{A(\beta)}(0) = \mathbf{0}$, and let b and p be defined analogously as in the proof of Lemma 4.3.3. For the sake of notation, we set $x_o(\theta)$ to be the abscissa of $\mathbf{\Gamma}_{A(\beta)}(p(\theta))$ and $x_+(\theta, \lambda)$ to be the abscissa of $\mathbf{\Gamma}_{A(\beta)}(b(\theta, \lambda))$. First, by some basic geometric calculations, we note that, for $\lambda \in [0, \tilde{\lambda})$ and for $\theta \in [0, \tilde{\theta})$, it holds

$$\left| K_{A(\beta)}^+(\theta + \pi/2, \lambda) \right| \approx x_+(\theta + \pi/2, \lambda) - x_o(\theta + \pi/2).$$

For the sake of brevity, we will write $x_o = x_o(\theta + \pi/2)$ and $x_+ = x_+(\theta + \pi/2, \lambda)$. Now, notice that x_+ is the abscissa of the (right) intersection of the curve $y = |x|^\beta$ with the straight line

$$y = (x - x_o)\beta x_o^{\beta-1} + x_o^\beta + \frac{\lambda}{\cos \theta}.$$

Equating, and with the normalisation $z = \frac{x-x_o}{x_o}$, we get to the equation

$$f(z) = |z + 1|^\beta - z^\beta - 1 = \frac{\lambda}{x_o^\beta \cos \theta}, \quad (4.12)$$

and we note that, for $\theta \in [0, \tilde{\theta})$, both x_o and $\cos \theta$ are non-negative. By applying Taylor's formula with integral remainder to f , we obtain

$$f(z) = \beta(\beta - 1) \int_0^z (1+t)^{\beta-2} (z-t) dt.$$

Notice that for $z \in [0, 1)$ it holds

$$\int_0^z (1+t)^{\beta-2} (z-t) dt \approx \int_0^z (z-t) dt \approx z^2.$$

On the other hand, for $z \in [1, +\infty)$ it holds

$$\begin{aligned} \int_0^z (1+t)^{\beta-2} (z-t) dt &= \int_0^{z/2} (1+t)^{\beta-2} (z-t) dt + \int_{z/2}^z (1+t)^{\beta-2} (z-t) dt \\ &\approx z \int_0^{z/2} (1+t)^{\beta-2} dt + z^{\beta-2} \int_{z/2}^z (z-t) dt \\ &= \frac{z}{\beta-1} \left((1+z/2)^{\beta-1} - 1 \right) + z^{\beta-2} \frac{z^2}{8} \approx_\beta z^\beta. \end{aligned}$$

Hence, we get

$$f(z) \approx_\beta \begin{cases} z^2 & \text{if } 0 \leq z < 1 \\ z^\beta & \text{if } z \geq 1 \end{cases},$$

and if we consider (4.12), by applying Lemma 3.3.1, and by the fact that for $\theta \in [0, \tilde{\theta})$ it holds $\cos \theta \approx 1$, it follows that

$$\frac{x_+ - x_o}{x_o} \approx_\beta \begin{cases} \lambda^{1/2} x_o^{-\beta/2} & \text{if } 0 \leq \lambda x_o^{-\beta} < 1 \\ \lambda^{1/\beta} x_o^{-1} & \text{if } \lambda x_o^{-\beta} \geq 1 \end{cases}. \quad (4.13)$$

Finally, by the definition of x_o , we have

$$\beta x_o^{\beta-1} = \left. \frac{d}{dx} x^\beta \right|_{x=x_o} = \tan \theta,$$

and therefore, we get that for $\theta \in [0, \tilde{\theta})$ it holds

$$x_o \approx \theta^{\frac{1}{\beta-1}}.$$

The conclusion follows from a rearrangement of the terms in (4.13). $\square$

4.3.2 Lower bound

We show how we may construct the sequence $\{k_i\}_{i \in \mathbb{N}}$ in such a way that, for every $i \in \mathbb{N}$, it holds the lower bound

$$N^{\alpha_i - \varepsilon_i} \leq \inf_{\# \mathcal{P} = N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \quad \text{for } N \in [N_i, N_i + q_i].$$

Similarly as in Chapter 3, we proceed with the geometric argument that pairs with Lemma 1.3.3. For the sake of notation, we write

$$\|\theta\|_\pi = \min_{j \in \mathbb{Z}} |\theta - j\pi|.$$

First, notice that, since we have constructed ∂C in such a way that its curvature is uniformly bounded from below and above when excluding a neighbourhood of the origin (and its symmetric counterpart), then for every $\lambda \in [0, \lambda_0)$ and for every θ such that $\|\theta\|_\pi > 1/10$, it holds

$$|K_C(\theta + \pi/2, \lambda)| \approx_C \lambda^{1/2}. \quad (4.14)$$

Now, fix an $i \in \mathbb{N}$. Then, by the latter consideration paired with Lemma 2.1.2, and by Lemma 4.3.2 and (4.11), we get the lower bound

$$\int_0^1 \left| \widehat{\mathbf{1}}_{\delta C}(\rho \mathbf{u}(\theta + \pi/2)) \right|^2 d\delta \gtrsim_{\beta_i} \begin{cases} \rho^{-2-2/\beta_i} & \text{if } \rho \in [\rho_{i,1}, \rho_{i,2}] \wedge \|\theta\|_\pi \leq 1/10 \\ \rho^{-3} & \text{if } \rho \in [\rho_{i,1}, \rho_{i,2}] \wedge \|\theta\|_\pi > 1/10 \\ 0 & \text{else} \end{cases}.$$

Let $\tilde{\varepsilon}_i \in (0, 1]$ be a value to be chosen later. Moreover, let N , X , and Y be positive parameters to be chosen later, but such that $\rho_{i,1} \ll Y \ll X \ll \rho_{i,2}$ and $XY = N^{1+\tilde{\varepsilon}_i}$. Then, consider a rectangle $R \subset \mathbb{R}^2$ such that it is symmetric with respect to the axes and has a vertex in $(X/2, Y/2)$; consequently, it also holds $|R| = N^{1+\tilde{\varepsilon}_i}$. Denote by $R(\omega)$ the same rectangle rotated counterclockwise by an angle ω , and define the function $\Phi: \mathbb{Z}^2 \rightarrow [0, +\infty)$ by

$$\Phi(\mathbf{m}) = \int_{-1/10}^{1/10} \mathbf{1}_{R(\omega) \setminus B(\rho_{i,1})}(\mathbf{m}) d\omega,$$

Now, in order to apply Lemma 1.3.3 effectively, we need to find a positive parameter $Z = Z(N)$ such that in the annulus $|\mathbf{m}| \in [\rho_{i,1}, \rho_{i,2}]$ it holds

$$Z\Phi(\mathbf{m}) \leq \begin{cases} |\mathbf{m}|^{-2-2/\beta_i} & \text{if } |\mathbf{m}| \in [\rho_{i,1}, \rho_{i,2}] \wedge \|\arg(\mathbf{m}) + \pi/2\|_\pi \leq 1/10 \\ |\mathbf{m}|^{-3} & \text{if } |\mathbf{m}| \in [\rho_{i,1}, \rho_{i,2}] \wedge \|\arg(\mathbf{m}) + \pi/2\|_\pi > 1/10 \\ 0 & \text{else} \end{cases}.$$

First, we describe two regions where Φ vanishes. By construction, for all $\mathbf{m} \in \mathbb{Z}^2$ such that $|\mathbf{m}| \in [\rho_{i,1}, X]^c$, it holds $\Phi(\mathbf{m}) = 0$. Secondly, for all $\mathbf{m} \in \mathbb{Z}^2$ such that $|\mathbf{m}| \geq Y$ and $\|\arg(\mathbf{m}) + \pi/2\|_\pi \leq 1/10$, it holds $\Phi(\mathbf{m}) = 0$ as well.

Now, for all $\mathbf{m} \in \mathbb{Z}^2$ such that $|\mathbf{m}| \leq Y$, we have the trivial estimate $\Phi(\mathbf{m}) \leq 1/5$. Since in this region we are aiming for $Z\Phi(\mathbf{m}) \leq |\mathbf{m}|^{-2-2/\beta_i}$, then we settle for

$$Z \lesssim Y^{-2-2/\beta_i}.$$

Lastly, we consider the remaining region defined by $|\mathbf{m}| \in [Y, X]$ and $\|\arg(\mathbf{m}) + \pi/2\|_\pi > 1/10$. By some elementary geometry, we find that $\Phi(\mathbf{m}) \leq 2\psi$ where ψ is such that $|\mathbf{m}| \sin \psi = Y/2$. Hence, it holds $\Phi(\mathbf{m}) \leq \pi Y/(2|\mathbf{m}|)$, and since in this region we are aiming for $Z\Phi(\mathbf{m}) \leq |\mathbf{m}|^{-3}$, then we settle for

$$Z \lesssim Y^{-1}X^{-2}.$$

Having considered all the possible regions, we are left with the requirement

$$Z \lesssim \min \left(Y^{-2-2/\beta_i}, Y^{-1}X^{-2} \right),$$

and if, by an optimisation argument, we equalise the two terms in the minimum and use the constraint $XY = N^{1+\tilde{\varepsilon}_i}$, then we obtain

$$X \approx N^{(1+\tilde{\varepsilon}_i)(\beta_i+2)/(3\beta_i+2)}, \quad Y \approx N^{(1+\tilde{\varepsilon}_i)2\beta_i/(3\beta_i+2)}, \quad \text{and} \quad Z \approx N^{-(1+\tilde{\varepsilon}_i)(4\beta_i+4)/(3\beta_i+2)}.$$

Starting from (1.4) and applying Lemma 1.3.3, we get that, for any configuration of N points $\mathcal{P} = \{\mathbf{p}_j\}_{j=1}^N$ in $\mathbb{T}^2$, it holds

$$\begin{aligned} \int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}(\boldsymbol{\tau} + \delta C; \mathcal{P})|^2 \, d\boldsymbol{\tau} \, d\delta &= \sum_{\mathbf{m} \in \mathbb{Z}^2 \setminus \{0\}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{p}_j} \right|^2 \int_0^1 |\widehat{\mathbb{1}}_{\delta C}(\mathbf{m})|^2 \, d\delta \\ &\gtrsim_{\beta_i} \sum_{|\mathbf{m}| \geq \rho_{i,1}} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{p}_j} \right|^2 Z\Phi(\mathbf{m}) \\ &= Z \int_{-1/10}^{1/10} \sum_{\mathbf{m} \in R(\omega) \setminus B(\rho_{i,1})} \left| \sum_{j=1}^N e^{2\pi i \mathbf{m} \cdot \mathbf{p}_j} \right|^2 \, d\omega \\ &\gtrsim Z \left(N^{2+\tilde{\varepsilon}_i} - 2\pi(\rho_{i,1}^2 + 1)N^2 \right). \end{aligned}$$

By construction, $\rho_{i,2}$ can be chosen arbitrarily large, and this also applies to N since our only constraint is $X \ll \rho_{i,2}$. Hence, by taking $\tilde{\varepsilon}_i$ in such a way that $\tilde{\varepsilon}_i(\beta_i+2)/(3\beta_i+2) < \varepsilon_i$, we find an N_i such that

$$N_i^{\tilde{\varepsilon}_i} \geq 4\pi(\rho_{i,1}^2 + 1) \quad \text{and} \quad (N_i + q_i)^{(1+\tilde{\varepsilon}_i)(\beta_i+2)/(3\beta_i+2)} \ll \rho_{i,2}.$$

Therefore, it follows that, for every $N \in [N_i, N_i + q_i]$, it holds

$$\int_0^1 \int_{\mathbb{T}^2} |\mathcal{D}(\boldsymbol{\tau} + \delta C; \mathcal{P})|^2 d\boldsymbol{\tau} d\delta \gtrsim_{\beta_i} N^{-(1+\tilde{\varepsilon}_i)(4\beta_i+4)/(3\beta_i+2)} N^{2+\tilde{\varepsilon}_i} = N^{\frac{2\beta_i}{3\beta_i+2} - \tilde{\varepsilon}_i \frac{\beta_i+2}{3\beta_i+2}} > N^{\alpha_i - \varepsilon_i}.$$

Notice that, since $N^{-\varepsilon_i}$ can be made arbitrarily small, we can ultimately get the implicit constant in front of $N^{\alpha_i - \varepsilon_i}$ to be 1.

4.3.3 Upper bound

For every $i \in \mathbb{N}$, we proceed to show the upper bound

$$\inf_{\#\mathcal{P}=N} \mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \lesssim N^{\alpha_i + \varepsilon_i} \quad \text{for } N \in [N_i, N_i + q_i].$$

We show this by employing a suitable sampling scheme, namely the one in [BT22, Thm. 8].

Fix $i \in \mathbb{N}$, and again, we recall that at every i -th step of our construction, $\rho_{i,2}$ serves as a free parameter that may be chosen arbitrarily large. By pairing Lemma 2.1.2 with (4.14), and by Lemma 4.3.2 and (4.11), in the annulus $\rho \in [\rho_{i,1}, \rho_{i,2}]$ it holds the upper bound

$$\int_0^1 \left| \widehat{\mathbf{1}}_{\delta C}(\rho \mathbf{u}(\theta + \pi/2)) \right|^2 d\delta \lesssim_{\beta_i} \begin{cases} \rho^{-2-2/\beta_i} & \text{if } \|\theta\|_\pi \leq \rho^{\frac{1-\beta_i}{\beta_i}} \\ \rho^{-3} \|\theta\|_\pi^{\frac{2-\beta_i}{\beta_i-1}} & \text{if } \rho^{\frac{1-\beta_i}{\beta_i}} < \|\theta\|_\pi \leq \theta_i \\ \rho^{-3} & \text{if } \theta_i < \|\theta\|_\pi \end{cases}. \quad (4.15)$$

By Lemma 2.1.2, and since we have constructed ∂C in such a way that its curvature is uniformly bounded from below, then, for every $\rho > \lambda_C^{-1}$ and any angle θ , it holds

$$\int_0^1 \left| \widehat{\mathbf{1}}_{\delta C}(\rho \mathbf{u}(\theta)) \right|^2 d\delta \lesssim \rho^{-3}. \quad (4.16)$$

Again, let $\tilde{N} \in \mathbb{N}$ be a free large parameter to be chosen later, and set

$$G = \lfloor \tilde{N}^{\frac{2+\beta_i}{2+3\beta_i}} \rfloor, \quad L = \lfloor \tilde{N}^{\frac{2\beta_i}{2+3\beta_i}} \rfloor, \quad J_G = [0, G-1] \cap \mathbb{N}, \quad \text{and} \quad J_L = [0, L-1] \cap \mathbb{N}.$$

Set $N = GL$ (trivially, $L < G$ and $N/2 < \tilde{N} < 2N$), and consider the configuration of N points $\mathcal{P}$ in $\mathbb{T}^2$ defined by

$$\mathcal{P} = \{\mathbf{p}_j\}_{j=1}^N = \{\mathbf{p}_{\ell,g}\}_{g \in J_G, \ell \in J_L} \quad \text{with} \quad \mathbf{p}_{\ell,g} = \left(\frac{g}{G}, \frac{\ell}{L} \right).$$

Recall that, by Parseval's identity, we get

$$\int_{\mathbb{T}^2} |\mathcal{D}(\boldsymbol{\tau} + \delta C; \mathcal{P})|^2 d\boldsymbol{\tau} = \sum_{\mathbf{m} \neq (0,0)} \left| \widehat{\mathbf{1}}_{\delta C}(\mathbf{m}) \right|^2 \left| \sum_{g \in J_G} \sum_{\ell \in J_L} e^{2\pi i \mathbf{m} \cdot \mathbf{p}_{\ell,g}} \right|^2,$$

and in particular, it holds the identity

$$\sum_{g \in J_G} \sum_{\ell \in J_L} e^{2\pi i(m_1 \frac{g}{G} + m_2 \frac{\ell}{L})} = \begin{cases} GL & \text{if } m_1 \in G\mathbb{Z} \text{ and } m_2 \in L\mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Hence, consider the set $\mathcal{R} = (G\mathbb{Z} \times L\mathbb{Z}) \setminus \{\mathbf{0}\}$, and split it into the regions

$$\begin{aligned} V_1 &= \left\{ \mathbf{m} \in \mathcal{R} : |m_1| \leq |m_2|^{1/\beta_i} \right\}, \\ V_2 &= \left\{ \mathbf{m} \in \mathcal{R} : |m_2|^{1/\beta_i} < |m_1| \leq \theta_i |m_2| \right\}, \\ V_3 &= \left\{ \mathbf{m} \in \mathcal{R} : \theta_i |m_2| < |m_1| \right\}, \end{aligned}$$

so that we may write

$$\mathcal{D}_2^{\text{id}}(C; \mathcal{P}) = G^2 L^2 \left(\sum_{\mathbf{m} \in V_1} + \sum_{\mathbf{m} \in V_2} + \sum_{\mathbf{m} \in V_3} \right) \int_0^1 \left| \widehat{\mathbf{1}}_{\delta C}(\mathbf{m}) \right|^2 d\delta. \quad (4.17)$$

Now, we constrain $\tilde{N}$ to be so large that $L > \rho_{i,1}$, and as a consequence, it follows that $\mathcal{R} \cap B(\rho_{i,1}) = \emptyset$. Moreover, we constrain $\rho_{i,2}$ to be so large that

$$\mathcal{Q}_i = \sum_{\mathbf{m} \in \mathcal{R} \cap B^c(\rho_{i,2})} |\mathbf{m}|^{-3} < N^{-2}.$$

We exploit (4.15) to study the three sums in (4.17). In this case, we must consider

$$\rho = |\mathbf{m}| \quad \text{and} \quad \tan \theta = -\frac{m_1}{m_2}.$$

For the sake of notation, we write the annulus $\mathcal{A} = \{\mathbf{m} \in \mathcal{R} : |\mathbf{m}| \in [\rho_{i,1}, \rho_{i,2}]\}$. Notice that for small θ it holds $\tan \theta \approx \theta$, and consequently, with a bit of rearrangement, we can rewrite the estimates in (4.15) as: for every $\mathbf{m} \in \mathcal{A}$ it holds

$$\int_0^1 \left| \widehat{\mathbf{1}}_{\delta C}(\mathbf{m}) \right|^2 d\delta \lesssim_{\beta_i} \begin{cases} |m_2|^{-2-2/\beta_i} & \text{if } |m_1| \leq |m_2|^{1/\beta_i} \\ |m_1|^{\frac{2-\beta_i}{\beta_i-1}} |m_2|^{\frac{1-2\beta_i}{\beta_i-1}} & \text{if } |m_2|^{1/\beta_i} \leq |m_1| < \theta_i |m_2| \\ |m_1|^{-3} & \text{if } \theta_i |m_2| < |m_1| \end{cases}. \quad (4.18)$$

For $j \in \{1, 2, 3\}$, notice that by the constraints we have imposed earlier, it holds $V_j \cap \mathcal{A} =$

$V_j \cap B(\rho_{i,2})$ and $G^2 L^2 \mathcal{Q}_i = N^2 \mathcal{Q}_i < 1$, and by further applying (4.16), we obtain

$$\begin{aligned}
G^2 L^2 \sum_{\mathbf{m} \in V_j} \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta &= G^2 L^2 \left(\sum_{\mathbf{m} \in V_j \cap \mathcal{A}} + \sum_{\mathbf{m} \in V_j \cap B^c(\rho_{i,2})} \right) \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta \\
&\lesssim G^2 L^2 \sum_{\mathbf{m} \in V_j \cap \mathcal{A}} \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta + G^2 L^2 \mathcal{Q}_i \\
&\lesssim G^2 L^2 \sum_{\mathbf{m} \in V_j \cap \mathcal{A}} \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta.
\end{aligned} \tag{4.19}$$

Then, by applying (4.18) and (4.19), for the first sum in the last term of (4.17) we obtain

$$\begin{aligned}
\sum_{\mathbf{m} \in V_1} \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta &\lesssim_{\beta_i} \sum_{\mathbf{m} \in V_1} |m_2|^{-2-2/\beta_i} \lesssim \sum_{n_2=1}^{+\infty} \sum_{n_1=0}^{(Ln_2)^{1/\beta_i}/G} (Ln_2)^{-2-2/\beta_i} \\
&\lesssim L^{-2-2/\beta_i} \sum_{n_2=1}^{+\infty} n_2^{-2-2/\beta_i} \left(1 + (Ln_2)^{1/\beta_i}/G \right) \\
&\lesssim L^{-2-2/\beta_i}.
\end{aligned}$$

Similarly, for the second sum in the last term of (4.17), we get

$$\begin{aligned}
\sum_{\mathbf{m} \in V_2} \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta &\lesssim_{\beta_i} \sum_{\mathbf{m} \in V_2} |m_1|^{\frac{2-\beta_i}{\beta_i-1}} |m_2|^{\frac{1-2\beta_i}{\beta_i-1}} \\
&\lesssim \sum_{n_1=1}^{+\infty} \sum_{n_2=Gn_1/(\theta_i L)}^{(Gn_1)^{\beta_i}/L} (Gn_1)^{\frac{2-\beta_i}{\beta_i-1}} (Ln_2)^{\frac{1-2\beta_i}{\beta_i-1}} \\
&\lesssim G^{\frac{2-\beta_i}{\beta_i-1}} L^{\frac{1-2\beta_i}{\beta_i-1}} \sum_{n_1=1}^{+\infty} n_1^{\frac{2-\beta_i}{\beta_i-1}} \sum_{n_2=Gn_1/(\theta_i L)}^{+\infty} n_2^{\frac{1-2\beta_i}{\beta_i-1}} \\
&\lesssim G^{\frac{2-\beta_i}{\beta_i-1}} L^{\frac{1-2\beta_i}{\beta_i-1}} \sum_{n_1=1}^{+\infty} n_1^{\frac{2-\beta_i}{\beta_i-1}} (Gn_1/(\theta_i L))^{\frac{-\beta_i}{\beta_i-1}} \\
&\lesssim G^{-2} L^{-1} \sum_{n_1=1}^{+\infty} n_1^{-2} \lesssim G^{-2} L^{-1}.
\end{aligned}$$

For the last sum in the last term in (4.17), we obtain

$$\sum_{\mathbf{m} \in V_3} \int_0^1 \left| \widehat{\mathbf{l}}_{\delta C}(\mathbf{m}) \right|^2 d\delta \lesssim_{\beta_i} \sum_{\mathbf{m} \in V_3} |m_1|^{-3} \lesssim \sum_{n_1=1}^{+\infty} \sum_{n_2=0}^{Gn_1/(\theta_i L)} (Gn_1)^{-3}$$

$$\lesssim G^{-3} \sum_{n_1=1}^{+\infty} n_1^{-2} G/(\theta_i L) \lesssim G^{-2} L^{-1} \theta_i^{-1}.$$

Finally, since $G^2 L^{-2/\beta_i} = L \approx N^{\frac{2\beta_i}{2+3\beta_i}} = N^{\alpha_i}$, and by comparing the previous three estimates with (4.17), we obtain

$$\mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \lesssim_{\beta_i} G^2 L^2 \left(L^{-2-2/\beta_i} + G^{-2} L^{-1} / \theta_i \right) \lesssim G^2 L^{-2/\beta_i} + L / \theta_i \lesssim N^{\alpha_i} \theta_i^{-1}$$

Hence, since we may choose $\rho_{i,2}$ and $\tilde{N}$ arbitrarily large, then we can ultimately obtain N such that $N^{\varepsilon_i} \geq \theta_i^{-1}$, and for the same reason, we may ultimately write

$$\mathcal{D}_2^{\text{id}}(C; \mathcal{P}) \leq N^{\alpha_i + \varepsilon_i}.$$

At this stage, by the constraints we have imposed, we may only conclude that the upper bound holds for all the integers N of the form

$$N = \lfloor n^{\frac{2+\beta_i}{2+3\beta_i}} \rfloor \lfloor n^{\frac{2\beta_i}{2+3\beta_i}} \rfloor, \quad \text{with } n^{\alpha_i} > \rho_{i,1} \quad \text{and } n^2 < \rho_{i,2}.$$

In order to prove that there is a suitable choice of points for some q_i consecutive integers in the interval $(\rho_{i,1}^{1/\alpha_i}, \rho_{i,2}^{1/2})$, we need to adapt the argument at the end of the proof of Theorem 3.1.3. Consider the following recursive definition

$$n_j = \max \left\{ n \in \mathbb{N} : \lfloor n^{1-\alpha_i} \rfloor \lfloor n^{\alpha_i} \rfloor \leq N - \sum_{k=1}^{j-1} \lfloor n_k^{1-\alpha_i} \rfloor \lfloor n_k^{\alpha_i} \rfloor \right\} \quad \text{for } j \in \mathbb{N} \setminus \{0\},$$

where improper sums are conventionally considered as zeros. Now, notice that the latter definition implies that

$$\begin{aligned} N - \sum_{k=1}^{j-1} \lfloor n_k^{1-\alpha_i} \rfloor \lfloor n_k^{\alpha_i} \rfloor &\leq \lfloor (n_j + 1)^{1-\alpha_i} \rfloor \lfloor (n_j + 1)^{\alpha_i} \rfloor \\ &\leq \lfloor n_j^{1-\alpha_i} \rfloor \lfloor n_j^{\alpha_i} \rfloor + \lfloor n_j^{1-\alpha_i} \rfloor + \lfloor n_j^{\alpha_i} \rfloor + 1, \end{aligned}$$

and therefore, it follows that

$$N - \sum_{k=1}^j \lfloor n_k^{1-\alpha_i} \rfloor \lfloor n_k^{\alpha_i} \rfloor \leq 2n_j^{1-\alpha_i}.$$

Again, by the definition of n_j , we point out that

$$\frac{n_1}{2} \leq N \quad \text{and} \quad \frac{n_{j+1}}{2} \leq N - \sum_{k=1}^j \lfloor n_k^{1-\alpha_i} \rfloor \lfloor n_k^{\alpha_i} \rfloor,$$

and therefore, by induction, we obtain

$$N - \sum_{k=1}^j \lfloor n_k^{1-\alpha_i} \rfloor \lfloor n_k^{\alpha_i} \rfloor \leq 2n_j^{1-\alpha_i} \leq 2^4 n_{j-1}^{(1-\alpha_i)^2} \leq 2^{2j} N^{(1-\alpha_i)^j}.$$

In particular, notice that

$$N - \sum_{k=1}^5 \lfloor n_k^{1-\alpha_i} \rfloor \lfloor n_k^{\alpha_i} \rfloor \leq 2^{10} N^{(1-\alpha_i)^5} \ll (N^{\alpha_i/2}).$$

Then, if we further impose that $N^{\alpha_i^2/2} > \rho_{i,1}$, we may associate a choice of points as in the previous construction to every $N_j = \lfloor n_j^{1-\alpha_i} \rfloor \lfloor n_j^{\alpha_i} \rfloor$ such that $N_j \gtrsim N^{\alpha_i/2}$, and we neglect the contribution of the remaining points since, by the latter equation, it is certainly much smaller than N^{α_i} . In particular, it is not restrictive to admit repetition of points in the sampling sequence $\mathcal{P}_N$. Finally, it is harmless to sum the contribution of our (at most) five point-configurations and the remainder points, since, for $a_1, \dots, a_6 \geq 0$, it holds

$$\left(\sum_{j=1}^6 a_j \right)^2 \leq 6 \sum_{j=1}^6 a_j^2.$$

By the argument we have just carried over, we may find an integer $N_i \gg q_i$ for which the upper bound in Theorem 4.1.6 holds, and by choosing $\rho_{i,2}$ large enough, we ensure that the same construction may be carried over for the next consecutive q_i integers after N_i .

Remark 4.3.4. Notice that all the constraints needed in the proof of the lower bound and in the proof of the upper bound are ultimately a matter of choosing $\rho_{i,2}$ large enough; this guarantees that we may get the N_i obtained at the end of the two proofs to be the same integer. Namely, at the i -th step, $\rho_{i,1}$ and θ_i are given by the previous steps. We first choose N_i large in terms of them, and of β_i , ε_i and q_i . Then, we choose $\rho_{i,2}$ large in terms of N_i . Finally, we choose k_i accordingly, as in Lemma 4.3.2.

Chapter 5

The Fourier transform of BV functions

This chapter presents the main content of [BG26]. Section 5.1 gathers the introductory material for BV functions. The subsequent sections then establish, respectively, the weighted Plancherel identity (Theorem 5.2.2, proved in Section 5.3) and the resulting sharp discrepancy bounds (Theorems 5.2.7 and 5.2.8, proved in Section 5.4). The L^2 -jump product $\mathcal{J}(u, v)$ is the central quantity of this chapter; when $u = v = \mathbb{1}_\Omega$, it reduces to the perimeter $\text{Per}(\Omega)$.

5.1 Preliminary results on BV functions

In what follows, d stands for a positive integer. Consider a vector-valued measure $\mu = (\mu_1, \dots, \mu_k)$ of $\mathbb{R}^d$. We define the total variation of μ as the measure $|\mu|$ which, to any Borel set $E \subset \mathbb{R}^d$, assigns the value

$$|\mu|(E) = \sup \left\{ \sum_n \|\mu(E_n)\|_{\mathbb{R}^k} : (E_n)_n \text{ is a Borel partition of } E \right\}.$$

We say that a vector-valued measure μ is finite if

$$|\mu|(\mathbb{R}^d) < +\infty,$$

and we shall denote the vector space of finite measures of $\mathbb{R}^d$ with values in $\mathbb{R}^k$ with $\mathcal{M}(\mathbb{R}^d; \mathbb{R}^k)$, with the convention that $\mathcal{M}(\mathbb{R}^d) = \mathcal{M}(\mathbb{R}^d; \mathbb{R}^1)$. Equipping such spaces with the norm

$$\|\mu\|_{\text{TV}} = |\mu|(\mathbb{R}^d)$$

makes them Banach spaces. We also remark that it makes sense to compute the Fourier transform of a measure, and indeed, the Fourier transform

$$\mathcal{F} : \mathcal{M}(\mathbb{R}^d) \rightarrow \mathcal{C}_b(\mathbb{R}^d)$$

is a linear and continuous operator with norm equal to 1, and the same holds component-wise if we consider the operator

$$\mathcal{F} : \mathcal{M}(\mathbb{R}^d; \mathbb{R}^k) \rightarrow \mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^k).$$

Now, we proceed with defining the class of functions of bounded variation.

Definition 5.1.1 (BV functions). We say that $u \in L^1(\mathbb{R}^d)$ is a function of bounded variation if $Du \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$, where Du is the distributional gradient of u . We denote the space of such functions with $\text{BV}(\mathbb{R}^d)$.

For a positive value R , we write B_R to denote the ball of radius R centered at the origin. For a Borel set E and a Borel measure μ , we shall write $\mu|_E$ to denote the Borel measure such that $\mu|_E(A) = \mu(E \cap A)$ for any Borel set A . We shall recall the decomposition of the distributional gradient of a BV function as in [AFP00, Sec. 3.7]. Indeed, if $u \in \text{BV}(\mathbb{R}^d)$, then Du can be written as

$$Du = \nabla u \mathcal{L}^d + (u^\vee - u^\wedge) \nu_{S_u} \mathcal{H}_{|S_u}^{d-1} + D^c u. \quad (5.1)$$

In the previous decomposition, ∇u is the density of the absolutely continuous part with respect to $\mathcal{L}^d$, and $D^c u$ is the Cantor part of the distributional gradient. The second term (often denoted with $D^j u$) is the jump part of the distributional gradient, and we briefly explain the notation used for it. Here, $\mathcal{H}^d$ stands for the d -dimensional Hausdorff measure, and S_u denotes the approximate discontinuity set of u , while ν_{S_u} stands for its measure-theoretic normal. Namely, we have that $x \in S_u^c$ if there exists $a \in \mathbb{R}$ such that

$$\lim_{\rho \rightarrow 0^+} \int_{B_\rho(x)} |u(y) - a| dy = 0, \quad (5.2)$$

and on the other hand, for $x \in S_u$, we have

$$\begin{aligned} u^\vee(x) &= \inf \{t \in [-\infty, +\infty] : \lim_{\rho \rightarrow 0^+} \rho^{-d} |\{u > t\} \cap B_\rho(x)| = 0\}, \\ u^\wedge(x) &= \sup \{t \in [-\infty, +\infty] : \lim_{\rho \rightarrow 0^+} \rho^{-d} |\{u < t\} \cap B_\rho(x)| = 0\}. \end{aligned}$$

We shall use the fact that there exists a $\mathcal{H}^{d-1}$ -rectifiable Borel set $J_u \subseteq S_u$, usually referred to as the jump set of u , such that $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$ and that allows us to rewrite (5.1) as

$$Du = \nabla u \mathcal{L}^d + (u^+ - u^-) \nu_{J_u} \mathcal{H}_{|J_u}^{d-1} + D^c u. \quad (5.3)$$

Here, u^+ and u^- are the one-sided approximate limits of u , defined in such a way that

$$\lim_{\rho \rightarrow 0^+} \int_{B_\rho^+(x)} |u(y) - u^+(x)| dy = \lim_{\rho \rightarrow 0^+} \int_{B_\rho^-(x)} |u(y) - u^-(x)| dy = 0,$$

where, denoting by $\langle \cdot, \cdot \rangle$ the inner product of $\mathbb{R}^d$, we have

$$\begin{aligned} B_\rho^+(x) &= \{y \in B_\rho(x) : \langle y - x, \nu_{J_u} \rangle > 0\}, \\ B_\rho^-(x) &= \{y \in B_\rho(x) : \langle y - x, \nu_{J_u} \rangle < 0\}. \end{aligned}$$

We proceed with the definition of sets of finite perimeter.

Definition 5.1.2 (Finite perimeter set). For a Borel set $\Omega \subset \mathbb{R}^d$, we define the perimeter of Ω as

$$\text{Per}(\Omega) = |D\mathbf{1}_\Omega|(\mathbb{R}^d),$$

and we say that Ω is of finite perimeter if $\mathbf{1}_\Omega \in \text{BV}(\mathbb{R}^d)$.

Due to the isoperimetric inequality, if Ω is of finite perimeter, then either Ω or Ω^c is of finite measure. In the sequel, we shall assume that our sets of finite perimeter come with finite measure.

For a set of finite perimeter, it is possible to define the notion of reduced boundary, which identifies the more “regular” points of the boundary. In particular, this coincides with the jump set of the characteristic function of the set itself.

Definition 5.1.3 (Reduced boundary). For a set of finite perimeter $\Omega \subset \mathbb{R}^d$, we define its reduced boundary $\partial^*\Omega$ as the Borel set of all the points $x \in \text{supp}|D\mathbf{1}_\Omega|$ such that the limit

$$\nu_\Omega(x) = \lim_{r \rightarrow 0^+} \frac{D\mathbf{1}_\Omega(B_r(x))}{|D\mathbf{1}_\Omega|(B_r(x))}$$

exists and has norm equal to 1.

Remark 5.1.4. Observe that $\partial^*\Omega$ is always a subset of the topological boundary, and it may happen that the two do not coincide. Trivially, the reduced boundary of a square consists of its topological boundary except for the 4 points at the corners.

We are now in the position to recall part of the celebrated structural result for sets of finite perimeter, namely De Giorgi theorem (see [AFP00, Thm. 3.59]).

Theorem 5.1.5 (De Giorgi). *Let $\Omega \subset \mathbb{R}^d$ be a set of finite perimeter, then $\partial^*\Omega$ is countably $\mathcal{H}^{d-1}$ -rectifiable and $|D\mathbf{1}_\Omega| = \mathcal{H}^{d-1}_{|\partial^*\Omega|}$.*

Remark 5.1.6. We do not delve into the definition of rectifiability in the above sense, and again, we refer the reader to [AFP00] or [Mag12]. However, one can think of it as a measure-theoretic notion of being $(d-1)$ -dimensional.

5.2 Introduction

We move onto the original content of this chapter. In Section 5.3, we show that, for functions of bounded variation, the problem of estimating the short-time behaviour of the (relative) heat content is dual to determining the limit behaviour of averages of the Fourier transform. Since the former retrieves several quantities of geometric interest and is strictly related to perimeters, the latter will naturally inherit such a property.

We turn to the main result of Section 5.3. Before proceeding with its statement, we introduce a special geometric quantity.

Definition 5.2.1 (L^2 -jump product). Let $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then we define the L^2 -jump product of u and v as

$$\mathcal{J}(u, v) = \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} d\mathcal{H}^{d-1},$$

with the convention that $\mathcal{J}(u) = \mathcal{J}(u, u)$.

We proceed with stating what we interpret as a weighted Plancherel identity.

Theorem 5.2.2. *Let $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then it holds*

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 \hat{u}(\xi) \bar{\hat{v}}(\xi) d\xi = \mathcal{J}(u, v). \quad (5.4)$$

In particular, if $u = v$, we get

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) d\xi = \mathcal{J}(u). \quad (5.5)$$

We make a few comments on the latter theorem. By classic properties of the Fourier transform, we can rewrite (5.5) as

$$\lim_{R \rightarrow +\infty} \frac{1}{2R} \int_{B_R} |\mathcal{F}\{Du\}|^2(\xi) d\xi = \mathcal{J}(u), \quad (5.6)$$

where Du stands for the distributional gradient of u . As a well-known property, the distributional gradient of a BV function decomposes in an absolutely continuous part, a Cantor part, and a jump part. In (5.6), the jump part alone plays a role in the limit, and we give the following heuristic interpretation. The Fourier transform tells how influential a certain oscillation is, and when looking at increasingly high frequencies, one expects to detect faster oscillations. Since the oscillation frequency is “infinity” at the jump set of a function, then the jump part of the distributional gradient is the only one caught up in the limit.

We also mention that we recover Wiener's lemma by considering $d = 1$ in (5.6). Indeed, a finite measure μ on $\mathbb{R}$ is the distributional derivative of its distribution function $F(x) = \mu((-\infty, x])$, which is bounded and has finite total variation, but need not be integrable on $\mathbb{R}$. Nevertheless, let $\chi_L \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfy $0 \leq \chi_L \leq 1$ and $\chi_L = 1$ on $[-L, L]$. Then $\chi_L F \in \text{BV}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $\mu - D(\chi_L F) = (1 - \chi_L)\mu - F\chi'_L$ is the sum of a measure of total variation at most $|\mu|(\mathbb{R} \setminus [-L, L])$ and of a function in $L^2(\mathbb{R})$. Hence, Wiener's lemma for μ follows by applying (5.6) to $\chi_L F$ and letting $L \rightarrow +\infty$.

Last, it is worth comparing Theorem 5.2.2 with the results in [Str90], where analogous estimates for real-valued measures are obtained under different hypotheses. Namely, for a non-negative integer k , the author considers measures $\mathcal{H}_{|E}^k$ that are the restriction of the k -dimensional Hausdorff measure to a k -regular set E , and that are locally uniformly k -dimensional, that is, there exists a positive constant c such that

$$\mathcal{H}_{|E}^k(B_r(x)) \leq cr^k \quad \forall x \in \mathbb{R}^d \quad \forall r \in (0, 1].$$

We remark that a k -regular set E , as defined in [Fal86], is a set whose upper and lower k -density with respect to $\mathcal{H}^k$ are jointly equal to 1, and by the theorem of Marstrand [Mar64], it follows that the condition of k being a non-negative integer is necessary. Moreover, the techniques employed by the author, such as maximal operators and covering theorems, differ from ours in nature. We address his question of removing the regularity assumptions in the case of measures that are also distributional gradients, and this is done by employing tools from the theory of BV functions, exploiting the decomposition of the distributional gradient and the good behaviour of its jump part. For completeness, we also mention the preceding study in [AH76] concerning measures supported on $\mathcal{C}^1$ submanifolds of $\mathbb{R}^d$, where the authors obtain analogous estimates.

Integrating by parts (5.5), we immediately deduce this first corollary to Theorem 5.2.2.

Corollary 5.2.3. *Let $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then*

$$\lim_{R \rightarrow +\infty} 2\pi^2 R \int_{B_R^c} |\hat{u}|^2(\xi) \, d\xi = \mathcal{J}(u). \quad (5.7)$$

The latter equality is significant for at least two reasons, the first being that it quantifies the error committed by cutting off the higher frequencies in Plancherel identity. Namely, we obtain

$$\int_{\mathbb{R}^d} |u|^2(x) \, dx = \int_{B_R} |\hat{u}|^2(\xi) \, d\xi + \frac{\mathcal{J}(u)}{2\pi^2 R} + o(R^{-1}) \quad \text{as } R \rightarrow +\infty.$$

Moreover, by letting $\Omega \subset \mathbb{R}^d$ be a set of finite measure and perimeter, and by setting $u = \mathbb{1}_\Omega$ in equation (5.7), we get

$$\lim_{R \rightarrow +\infty} 2\pi^2 R \int_{B_R^c} |\widehat{\mathbb{1}_\Omega}|^2(\xi) \, d\xi = \text{Per}(\Omega),$$

which extends the estimates in [Mon94, Ch. 6] to any dimension and for any set of finite perimeter.

Dealing with characteristic functions of sets, we further obtain this second corollary to Theorem 5.2.2. Again, we refer to the preliminary part of Section 5.3 for the notion of reduced boundary.

Corollary 5.2.4. *Let Ω_1 and Ω_2 be sets of finite measure and perimeter of $\mathbb{R}^d$, then it holds*

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 \widehat{\mathbb{1}_{\Omega_1}}(\xi) \widehat{\mathbb{1}_{\Omega_2}}(\xi) d\xi = \int_{\partial^* \Omega_1 \cap \partial^* \Omega_2} \nu_{\partial^* \Omega_1} \cdot \nu_{\partial^* \Omega_2} d\mathcal{H}^{d-1}. \quad (5.8)$$

In particular, if $\Omega_1 = \Omega_2 = \Omega$, we get

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 |\widehat{\mathbb{1}_{\Omega}}|^2(\xi) d\xi = \text{Per}(\Omega). \quad (5.9)$$

Finally, if for a set $\Omega \subset \mathbb{R}^d$ of finite measure it holds

$$\limsup_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 |\widehat{\mathbb{1}_{\Omega}}|^2(\xi) d\xi < +\infty, \quad (5.10)$$

then Ω is of finite perimeter.

The latter is a newfound characterization of sets of finite perimeter that does not involve distributional calculus and derivatives. Additionally, (5.9) improves on the estimates in [Her62, Thm. 2], where an upper bound for convex sets is obtained.

In the conclusion of Subsection 5.3.2, we adapt an argument of [Her62] and obtain a functional inequality that retrieves a non-sharp version of the celebrated isoperimetric inequality (when specialized to sets).

In Section 5.4, we show how the asymptotic relation in (5.5) is key for improving on a major result in the theory of irregularities of distribution. In what follows, we assume the dimension d to be greater than or equal to 2.

We recall the notion of *similar* quadratic discrepancy.

Definition 5.2.5. Let $u \in L^1(\mathbb{R}^d)$ and let $\mathcal{P}_N \subset \mathbb{T}^d$ be a configuration of N points. We define the *similar* quadratic discrepancy of $\mathcal{P}_N$ with respect to u

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) = \int_{\text{SO}(d)} \int_0^1 \int_{\mathbb{T}^d} |\mathcal{D}([\tau, \delta, \rho]u; \mathcal{P}_N)|^2 d\tau d\delta d\rho, \quad (5.11)$$

where $d\tau$ is the normalized Lebesgue measure on $\mathbb{T}^d$, $d\delta$ is the Lebesgue measure on $[0, 1]$, and $d\rho$ is the Haar probability measure on $\text{SO}(d)$.

In a classic work, Beck [Bec87] got the following major result on the optimal *similar* quadratic discrepancy in the case of characteristic functions of convex compact sets (with non-empty interior) of $\mathbb{R}^d$. In particular, the author employed probabilistic methods.

Theorem 5.2.6 (Beck). *Uniformly for every convex body $\Omega \subset \mathbb{R}^d$, it holds*

$$\liminf_{N \rightarrow +\infty} N^{(1-d)/d} \inf_{\mathcal{P}_N \subset \mathbb{T}^d} \mathcal{D}_2^{\text{SO}}(\Omega; \mathcal{P}_N) \geq C_d \text{Per}(\Omega).$$

In the same setting, the study in [BC90] proved the order $N^{(1-d)/d}$ to be sharp, although a generic set-dependent constant replaces the dependence on the perimeter. Moreover, in an independent work, Montgomery [Mon94, Ch. 6] showed that the same lower bound holds for every compact set (with non-empty interior) of $\mathbb{R}^2$ with piecewise- $\mathcal{C}^1$ boundary, hence trading convexity for a requirement of regularity. In particular, his work employed Fourier series and Lemma 1.3.3, therefore inspiring our results.

It is time to state the sharp discrepancy bounds that follow from Theorem 5.2.2, starting from the upper one.

Theorem 5.2.7. *There exists a positive dimensional constant C_d such that for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it holds*

$$\limsup_{N \rightarrow +\infty} N^{(1-d)/d} \inf_{\mathcal{P}_N \subset \mathbb{T}^d} \mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) \leq C_d \mathcal{J}(u).$$

As the following result shows, the latter estimate is the best achievable with respect to the order of N and the dependence on u , under the additional hypothesis of u having compact support.

Theorem 5.2.8. *There exists a positive dimensional constant C_d such that for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with compact support, it holds*

$$\liminf_{N \rightarrow +\infty} N^{(1-d)/d} \inf_{\mathcal{P}_N \subset \mathbb{T}^d} \mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) \geq C_d \mathcal{J}(u).$$

5.3 Detecting jumps via the Fourier transform

We start by treating the geometric measure-theoretic content of the chapter, and then move on to the consequential results in discrepancy theory.

5.3.1 A weighted Plancherel identity for BV functions

Following the notation in [MPPP07], for a function $u \in L^1$, we denote by $T(t)u$ the unique solution to the differential problem

$$\begin{cases} \partial_t v = \Delta_x v & \text{in } \mathbb{R}^d \times (0, +\infty) \\ v(0, x) = u & \text{on } \mathbb{R}^d \end{cases}, \quad (5.12)$$

evaluated at time $t > 0$. Moreover, we recall the explicit representation via convolution

$$(T(t)u)(x) = \int_{\mathbb{R}^d} G_d(x - y, t) u(y) dy,$$

where G_d stands for the classic Gaussian heat kernel

$$G_d(z, t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|z|^2}{4t}}. \quad (5.13)$$

Now, we recall two results on the (relative) heat content. First, we state [MPPP07, Thm. 4.3].

Theorem 5.3.1. *Let $u, v \in \text{BV}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then it holds*

$$\lim_{t \rightarrow 0^+} \frac{\sqrt{\pi}}{\sqrt{t}} \langle u - T(t)u, v \rangle = \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} d\mathcal{H}^{d-1}. \quad (5.14)$$

Remark 5.3.2. For our purposes, given $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we consider the map

$$H(t) = \int_{\mathbb{R}^d} (T(t^2)u)(x)v(x) dx. \quad (5.15)$$

By the latter result, we can only deduce that H is (right) differentiable at $t = 0$. In order to prove Theorem 5.2.2, we actually need

$$\lim_{t \rightarrow 0^+} H'(t) \quad \text{to exist,}$$

and in this case, the limit would clearly coincide with the right-hand term of (5.14). We address such a matter with Proposition 5.3.4.

Secondly, we state [MPPP07, Thm 3.4]. The following is a sufficient condition for a set to be of finite perimeter in terms of its (relative) heat content, and we use it in the proof of Corollary 5.2.4.

Theorem 5.3.3. *Let $\Omega \subset \mathbb{R}^d$ be such that either $|\Omega| < +\infty$ or $|\Omega^c| < +\infty$. If*

$$\liminf_{t \rightarrow 0^+} \frac{1}{t} \int_{\Omega^c} T(t^2) \mathbb{1}_\Omega(x) dx < +\infty,$$

then Ω has finite perimeter.

We proceed with the proof of the main technical result of the section.

Proposition 5.3.4. *Let $u, v \in \text{BV}(\mathbb{R}^d)$, with $u \in L^\infty(\mathbb{R}^d)$ or $v \in L^\infty(\mathbb{R}^d)$, then*

$$\lim_{t \rightarrow 0^+} H'(t) = -\frac{\mathcal{J}(u, v)}{\sqrt{\pi}}. \quad (5.16)$$

Proof. Without loss of generality, let us assume $u \in L^\infty(\mathbb{R}^d)$. Moreover, let H be defined as in (5.15). Clearly $H \in \mathcal{C}^1((0, +\infty))$, and by (5.12), for every $t \in (0, +\infty)$, we are allowed to write

$$H'(t) = 2t \int_{\mathbb{R}^d} \Delta_x(T(t^2)u)(x)v(x) \, dx = -2t \int_{\mathbb{R}^d} \nabla_x(T(t^2)u)(x) \, dDv(x). \quad (5.17)$$

We shall now compute the term $\nabla_x(T(t^2)u)(x)$ explicitly. Namely, we have

$$\begin{aligned} \nabla_x(T(t^2)u)(x) &= \int_{\mathbb{R}^d} \nabla_x G_d(x-y, t^2)u(y) \, dy \\ &= -\frac{1}{2t^2} \int_{\mathbb{R}^d} (x-y) \frac{e^{-\frac{|x-y|^2}{4t^2}}}{(4\pi t^2)^{d/2}} u(y) \, dy \\ &= -\frac{1}{(4\pi)^{d/2} 2t} \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} u(x-tz) \, dz. \end{aligned}$$

Therefore, (5.17) becomes

$$H'(t) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} u(x-tz) \, dz \, dDv(x). \quad (5.18)$$

Now, let us consider

$$g(x, t) = \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} u(x-tz) \, dz, \quad (5.19)$$

and split the integral in (5.18) as

$$H'(t) = G_1(t) + G_2(t),$$

with

$$\begin{aligned} G_1(t) &:= \frac{1}{(4\pi)^{d/2}} \int_{J_u} g(x, t) \, dDv(x), \\ G_2(t) &:= \frac{1}{(4\pi)^{d/2}} \int_{S_u^c} g(x, t) \, dDv(x). \end{aligned}$$

Here, as in (5.1), S_u and J_u stand respectively for the approximate discontinuity set and jump set of u (as in (5.3)). In particular, we are allowed to write the latter decomposition since $\mathcal{H}^{d-1}(S_w \setminus J_w) = 0$ for $w \in \{u, v, u+v\}$. Moreover, observe that $g(\cdot, t)$ is a continuous function such that

$$\sup_{x \in \mathbb{R}^d} |g(x, t)| \leq \|u\|_{L^\infty} \int_{\mathbb{R}^d} |z| e^{-\frac{|z|^2}{4}} \, dz = \frac{2^{d+1} \pi^{d/2}}{\Gamma(d/2)} \Gamma((d+1)/2) \|u\|_{L^\infty}. \quad (5.20)$$

Claim: $\lim_{t \rightarrow 0^+} G_2(t) = 0$.

Let us prove the claim. Observe that, by (5.2), for every $x \in S_u^c$ we are allowed to write

$$g(x, t) = \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} (u(x - tz) - \tilde{u}(x)) \, dz,$$

where $\tilde{u}(x)$ stands for the approximate limit of u at x . Now, fix a $R > 0$ and estimate g as

$$\begin{aligned} |g(x, t)| &\leq \int_{B_R} |z| e^{-\frac{|z|^2}{4}} |u(x - tz) - \tilde{u}(x)| \, dz \\ &\quad + \int_{B_R^c} |z| e^{-\frac{|z|^2}{4}} |u(x - tz) - \tilde{u}(x)| \, dz \\ &\leq \int_{B_R} |z| e^{-\frac{|z|^2}{4}} |u(x - tz) - \tilde{u}(x)| \, dz + 2\|u\|_{L^\infty} \int_{B_R^c} |z| e^{-\frac{|z|^2}{4}} \, dz. \end{aligned}$$

We shall now take the $\limsup$ as $t \rightarrow 0^+$ of the previous expression, so that we get

$$\limsup_{t \rightarrow 0^+} |g(x, t)| \leq 2\|u\|_{L^\infty} \int_{B_R^c} |z| e^{-\frac{|z|^2}{4}} \, dz,$$

since x is a point of approximate continuity. Finally, by taking the $\limsup$ as $R \rightarrow +\infty$, we obtain that

$$\lim_{t \rightarrow 0^+} g(x, t) = 0 \quad \forall x \in S_u^c.$$

Using the fact that $|Dv|(\mathbb{R}^d) < +\infty$ and the bound in (5.20), we obtain the claim by a dominated convergence argument.

We are thus left to study what happens in the set J_u . First, we have that J_u is countably $\mathcal{H}^{d-1}$ -rectifiable, hence we can apply [AFP00, Prop. 3.92] to deduce that $|D^c v|(J_u) = 0$. Moreover, the set J_u has Lebesgue measure zero, and therefore, it only remains to deal with

$$G_1(t) = \frac{1}{(4\pi)^{d/2}} \int_{J_u \cap J_v} g(x, t) \cdot \nu_{J_v}(v^+ - v^-) \, d\mathcal{H}^{d-1}(x). \quad (5.21)$$

Now, define

$$f(x, t) := \langle g(x, t), \nu_{J_v}(x) \rangle,$$

and consider

$$\begin{aligned} H_{\nu_{J_u}(x)}^+ &= \{z \in \mathbb{R}^d : \langle z, \nu_{J_u}(x) \rangle \leq 0\}, \\ H_{\nu_{J_u}(x)}^- &= \{z \in \mathbb{R}^d : \langle z, \nu_{J_u}(x) \rangle \geq 0\}. \end{aligned}$$

By exploiting the symmetries of the integrand in (5.21), we get

$$\begin{aligned}
f(x, t) &= \int_{H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^+(x)) \, dz \\
&\quad + u^+(x) \int_{H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} \, dz \\
&\quad + \int_{H_{\nu_{J_u}}^-} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^-(x)) \, dz \\
&\quad + u^-(x) \int_{H_{\nu_{J_u}}^-} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} \, dz \\
&=: A + B + C + D.
\end{aligned}$$

Claim: $\lim_{t \rightarrow 0^+} A = \lim_{t \rightarrow 0^+} C = 0$ for every $x \in J_u \cap J_v$.

First, let us consider the term A ; in particular, the term C can be treated analogously. We have

$$\begin{aligned}
A &= \int_{B_R \cap H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^+(x)) \, dz \\
&\quad + \int_{B_R^c \cap H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^+(x)) \, dz.
\end{aligned}$$

By the definition of $u^+(x)$, the first integral on the right-hand side goes to zero as $t \rightarrow 0^+$. For the second integral, as in (5.20), we have that for every $t \in (0, +\infty)$ it holds

$$\int_{B_R^c \cap H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^+(x)) \, dz \leq \|u\|_{L^\infty} \int_{B_R^c} |z| e^{-|z|^2/4} \, dz,$$

and therefore, it converges to 0 as $R \rightarrow +\infty$. This proves the claim.

Now observe that J_u and J_v are $\mathcal{H}^{d-1}$ -rectifiable, so that their approximate tangent planes coincide for $\mathcal{H}^{d-1}$ -a.e. $x \in J_u \cap J_v$. Hence, since for such x it holds

$$\nu_{J_v}(x) = \nu_{J_u}(x) \langle \nu_{J_u}(x), \nu_{J_v}(x) \rangle,$$

then we get

$$\begin{aligned}
B &= -\frac{(4\pi)^{d/2}}{\sqrt{\pi}} u^+(x) \langle \nu_{J_u}(x), \nu_{J_v}(x) \rangle, \\
D &= \frac{(4\pi)^{d/2}}{\sqrt{\pi}} u^-(x) \langle \nu_{J_u}(x), \nu_{J_v}(x) \rangle.
\end{aligned}$$

A straightforward application of dominated convergence finally gives

$$\lim_{t \rightarrow 0^+} G_1(t) = -\frac{1}{\sqrt{\pi}} \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} d\mathcal{H}^{d-1},$$

which is the thesis. $\square$

Before proceeding with the proof of Theorem 5.2.2, we recall the integral Abelian-Tauberian theorem. The reader may find related proofs in the appendix of [AHT18].

Theorem 5.3.5 (Abelian-Tauberian). *Let ν be a σ -finite positive Borel measure and let $\gamma \in [0, +\infty)$, then it holds*

$$\lim_{t \rightarrow 0^+} t^\gamma \int_0^{+\infty} e^{-t\lambda} d\nu(\lambda) = C \quad \Longleftrightarrow \quad \lim_{a \rightarrow +\infty} \frac{\nu([0, a])}{a^\gamma} = \frac{C}{\Gamma(\gamma + 1)}.$$

In the proof of Corollary 5.2.4, we use a special one-sided version of the latter theorem, in which the existence of the limit is not required. Again, we refer to the appendix of [AHT18].

Theorem 5.3.6. *Let ν be a σ -finite positive Borel measure and let $\gamma \in [0, +\infty)$. If*

$$\limsup_{a \rightarrow \infty} \frac{\nu([0, a])}{a^\gamma} \leq C < +\infty$$

for some $C \geq 0$, then

$$\limsup_{t \rightarrow 0^+} t^\gamma \int_0^\infty e^{-t\lambda} d\nu(\lambda) \leq C\Gamma(\gamma + 1).$$

Finally, we have gathered all the tools to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. First, we shall prove (5.5). Let u be as in the assumption, and similarly to (5.15), consider the quantity

$$H(t) = \int_{\mathbb{R}^d} (T(t^2)u)(x)u(x) dx.$$

By Plancherel identity and by the explicit expression of the heat kernel in (5.13), we get

$$H(t) = \int_{\mathbb{R}^d} \mathcal{F}\{T(t^2)u\}(\xi) \overline{\mathcal{F}\{u\}}(\xi) d\xi = \int_{\mathbb{R}^d} e^{-4\pi^2|\xi|^2 t^2} |\hat{u}|^2(\xi) d\xi.$$

Observe that $t \mapsto H(t)$ is continuous in $[0, 1]$, and by Plancherel identity and dominated convergence, it follows that $H(0) = \|u\|_{L^2}^2$. Again, by dominated convergence, we obtain

$$H'(t) = -8\pi^2 t \int_{\mathbb{R}^d} |\xi|^2 e^{-4\pi^2|\xi|^2 t^2} |\hat{u}|^2(\xi) d\xi,$$

and in particular, H' is continuous on $(0, 1]$. Therefore, thanks to Proposition 5.3.4, we get

$$t \int_{\mathbb{R}^d} |\xi|^2 e^{-4\pi^2 |\xi|^2 t^2} |\hat{u}|^2(\xi) d\xi \rightarrow \frac{\mathcal{J}(u)}{8\pi^{\frac{5}{2}}}.$$

Now, let us convert to polar coordinates and obtain

$$t \int_0^{+\infty} e^{-4\pi^2 r^2 t^2} r^{d+1} g(r) dr \rightarrow \frac{\mathcal{J}(u)}{8\pi^{\frac{5}{2}}},$$

where we have defined

$$g(r) := \int_{\mathbb{S}^{d-1}} |\hat{u}|^2(rv) d\sigma(v).$$

Let us write $\delta := 4\pi^2 t^2$, and then, by the change of variable $r^2 = s$, we get

$$\int_0^{+\infty} e^{-\delta s} s^{\frac{d}{2}} g(\sqrt{s}) ds \sim \frac{\mathcal{J}(u)}{2\pi^{\frac{3}{2}} \sqrt{\delta}} \quad \text{as } \delta \rightarrow 0^+, \quad (5.22)$$

where “ $\sim$ ” denotes asymptotic equivalence, that is, the ratio of the two sides tends to 1. Here and in the rest of the chapter, we assume $\mathcal{J}(u) \neq 0$ whenever $\sim$ is used; if $\mathcal{J}(u) = 0$, the same computations show that the corresponding limits vanish. Now, consider

$$\Phi(s) := \int_0^s z^{\frac{d}{2}} g(\sqrt{z}) dz = 2 \int_0^{\sqrt{s}} y^{d+1} g(y) dy = 2 \int_{B_{\sqrt{s}}} |\xi|^2 |\hat{u}|^2(\xi) d\xi.$$

With the previous definition in mind, we can write (5.22) as

$$\int_0^{+\infty} e^{-\delta s} d\Phi(s) \sim \frac{\mathcal{J}(u)}{2\pi^{\frac{3}{2}} \sqrt{\delta}} \quad \text{as } \delta \rightarrow 0^+.$$

Finally, we apply Theorem 5.3.5 and obtain

$$\Phi(R) \sim \sqrt{R} \frac{\mathcal{J}(u)}{\pi^2} \quad \text{as } R \rightarrow +\infty,$$

which (by replacing R with R^2) means that

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) d\xi = \frac{\mathcal{J}(u)}{2\pi^2}. \quad (5.23)$$

In order to prove (5.4), we shall proceed by polarization. Again, let u and v satisfy our hypotheses. We exploit the fact that

$$\Re\{\hat{u}(\xi) \overline{\hat{v}(\xi)}\} = \frac{|\widehat{(u+v)}(\xi)|^2}{2} - \frac{|\hat{u}(\xi)|^2}{2} - \frac{|\hat{v}(\xi)|^2}{2}.$$

Indeed, by the latter equation and by (5.5), we obtain

$$\begin{aligned} \lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 \hat{u}(\xi) \bar{\hat{v}}(\xi) d\xi &= \frac{1}{4\pi^2} \left(\int_{J_{u+v}} ((u+v)^+ - (u+v)^-)^2 d\mathcal{H}^{d-1} \right. \\ &\quad - \int_{J_u} (u^+ - u^-)^2 d\mathcal{H}^{d-1} \\ &\quad \left. - \int_{J_v} (v^+ - v^-)^2 d\mathcal{H}^{d-1} \right). \end{aligned} \quad (5.24)$$

Now, we fix an orientation ν on $J_u \cup J_v$, with $\nu = \nu_{J_u}$ on J_u and $\nu = \nu_{J_v}$ on $J_v \setminus J_u$, and we take all one-sided approximate limits with respect to ν ; this is possible since $\nu_{J_u} \cdot \nu_{J_v} = \pm 1$ for $\mathcal{H}^{d-1}$ -a.e. $x \in J_u \cap J_v$ (see the proof of Proposition 5.3.4). Since $\mathcal{H}^{d-1}(S_w \setminus J_w) = 0$ for $w \in \{u, v, u+v\}$, for $\mathcal{H}^{d-1}$ -a.e. $x \in J_u \cup J_v$ it holds

$$(u+v)^\pm(x) = u^\pm(x) + v^\pm(x),$$

where $u^+ = u^-$ on $J_v \setminus J_u$ and $v^+ = v^-$ on $J_u \setminus J_v$. Moreover, $J_{u+v} \subseteq J_u \cup J_v$ up to an $\mathcal{H}^{d-1}$ -null set, and the inclusion may be strict, since jumps may cancel (for instance, if $v = -u$): on $(J_u \cup J_v) \setminus J_{u+v}$, we have $(u+v)^+ = (u+v)^-$. Hence, the first integral in (5.24) may be taken over $J_u \cup J_v$, and expanding the square gives

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 \hat{u}(\xi) \bar{\hat{v}}(\xi) d\xi = \frac{1}{2\pi^2} \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} d\mathcal{H}^{d-1},$$

where on the right-hand side the limits of v are taken again with respect to ν_{J_v} : the factor $\nu_{J_u} \cdot \nu_{J_v}$ accounts for this change of orientation. This is (5.4). $\square$

Now, we proceed with the proof of the first corollary to Theorem 5.2.2.

Proof of Corollary 5.2.3. Before performing the limit in (5.5), multiply by R^{-3} both sides and then integrate with respect to R from $L > 0$ to $+\infty$, so to get

$$\int_L^\infty R^{-3} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) d\xi dR = \frac{\mathcal{J}(u)}{2\pi^2 L} + o(L^{-1}).$$

Now, we integrate by parts (justified by (5.5)) the left-hand side above and obtain

$$\frac{1}{2L^2} \int_{B_L} |\xi|^2 |\hat{u}|^2(\xi) d\xi + \frac{1}{2} \int_{B_L^\complement} |\hat{u}|^2(\xi) d\xi = \frac{\mathcal{J}(u)}{2\pi^2 L} + o(L^{-1}).$$

Finally, by applying (5.5) a second time, we end up with

$$\int_{B_L^\complement} |\hat{u}|^2(\xi) d\xi = \frac{\mathcal{J}(u)}{2\pi^2 L} + o(L^{-1}),$$

which is indeed (5.7). $\square$

We proceed with the proof of the second and last corollary, in which we present a newfound characterization of sets of finite perimeter.

Proof of Corollary 5.2.4. The relations (5.8) and (5.9) easily follow from Theorem 5.2.2 and the definition of reduced boundary. Hence, let us suppose that (5.10) holds. By Theorem 5.3.6, we have

$$\limsup_{\delta \rightarrow 0^+} \sqrt{\delta} \int_0^{+\infty} e^{-\delta s} d\Phi(s) < +\infty,$$

with (we shall neglect the constants)

$$\Phi(s) = \int_0^s z^{\frac{d}{2}} g(\sqrt{z}) dz \quad \text{and} \quad g(r) = \int_{\mathbb{S}^{d-1}} |\widehat{\mathbf{1}}_\Omega|^2(rv) dv.$$

By a change of variables, we obtain

$$\limsup_{t \rightarrow 0^+} t \int_{\mathbb{R}^d} e^{-|\xi|^2 t^2} |\xi|^2 |\widehat{\mathbf{1}}_\Omega|^2(\xi) d\xi < +\infty.$$

By Plancherel identity, the latter inequality implies that

$$\limsup_{t \rightarrow 0^+} |H'(t)| < +\infty,$$

where we intend H' as in (5.17) with $u = v = \mathbf{1}_\Omega$. However, by the mean value theorem and by the monotonicity properties of H , it follows that

$$0 \leq \limsup_{t \rightarrow 0} \frac{H(0) - H(t)}{t} < +\infty. \quad (5.25)$$

Exploiting the definition of H , we turn (5.25) into

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} \int_{\Omega^c} (T(t^2) \mathbf{1}_\Omega)(x) dx < +\infty,$$

and we can conclude by Theorem 5.3.3. $\square$

5.3.2 An “isoperimetric inequality” for BV functions

For simplicity, in what follows we denote $\|\cdot\|_{L^p}$ with $\|\cdot\|_p$.

Lemma 5.3.7. *There exists a positive absolute constant C such that, for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and for every $v \in \mathbb{S}^{d-1}$, it holds*

$$\frac{1}{R} \int_{B_R} |\langle \xi, v \rangle|^2 |\hat{u}|^2(\xi) d\xi \leq C \|u\|_\infty |D_v u|(\mathbb{R}^d) \quad \forall R > 0. \quad (5.26)$$

In particular, by integrating over $\mathbb{S}^{d-1}$, we get

$$\frac{1}{R} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) d\xi \leq C \|u\|_\infty |Du|(\mathbb{R}^d) \quad \forall R > 0. \quad (5.27)$$

Proof. First, we fix a vector $v \in \mathbb{S}^{d-1}$ and a $h > 0$. By standard properties of the Fourier transform and by Plancherel identity, we have

$$\int_{\mathbb{R}^d} \frac{|u(x + hv) - u(x)|^2}{h} dx = \frac{2}{h} \int_{\mathbb{R}^d} (1 - \cos(2\pi h \langle \xi, v \rangle)) |\hat{u}|^2(\xi) d\xi.$$

The left-hand side is bounded from above by $2\|u\|_\infty |D_v u|(\mathbb{R}^d)$, where $|D_v u|$ stands for the directional total variation. Now, we exploit the fact that if $\xi \in B_{1/(2\pi h)}$, then there exists a positive absolute constant c such that

$$ch^2 |\langle \xi, v \rangle|^2 \leq 2(1 - \cos(2\pi h \langle \xi, v \rangle)).$$

Therefore, we get

$$h \int_{B_{1/(2\pi h)}} |\langle \xi, v \rangle|^2 |\hat{u}|^2(\xi) d\xi \leq C \|u\|_\infty |D_v u|(\mathbb{R}^d),$$

which is (5.26). Finally, by setting $R = 1/(2\pi h)$ and by integrating over $\mathbb{S}^{d-1}$, we obtain (5.27). $\square$

Remark 5.3.8. By integrating by parts, we can reformulate (5.27) as

$$\int_{B_R^c} |\hat{u}|^2(\xi) d\xi \leq C \frac{\|u\|_\infty |Du|(\mathbb{R}^d)}{R}.$$

Last, we proceed with showing an interesting inequality. In particular, we obtain a non-sharp isoperimetric inequality by substituting $u = \mathbf{1}_\Omega$ in the following proposition.

Proposition 5.3.9. *There exists a positive dimensional constant C_d such that, for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it holds*

$$\|u\|_2^2 \leq C_d \|u\|_1^{\frac{2}{d+1}} \|u\|_\infty^{\frac{d}{d+1}} \left(|Du|(\mathbb{R}^d) \right)^{\frac{d}{d+1}}.$$

Proof. First, we use Plancherel identity to write

$$\int_{\mathbb{R}^d} |u|^2(x) dx = \int_{B_R} |\hat{u}|^2(\xi) d\xi + \int_{B_R^c} |\hat{u}|^2(\xi) d\xi.$$

By the fact that the Fourier transform is 1-Lipschitz between L^1 and L^∞ , and by (5.27), we obtain

$$\|u\|_2^2 \leq \omega_d R^d \|u\|_1^2 + C \frac{\|u\|_\infty |Du|(\mathbb{R}^d)}{R}.$$

By optimizing over $R > 0$, we get the optimal value

$$R = \left(\frac{C \|u\|_\infty |Du|(\mathbb{R}^d)}{d\omega_d \|u\|_1^2} \right)^{\frac{1}{d+1}}.$$

Therefore, it follows that

$$\|u\|_2^2 \leq C_d \|u\|_1^{\frac{2}{d+1}} \|u\|_\infty^{\frac{d}{d+1}} \left(|Du|(\mathbb{R}^d) \right)^{\frac{d}{d+1}},$$

and this is the thesis. $\square$

5.4 Application to discrepancy: sharpening a result of Beck

We now turn our attention to the quadratic discrepancy and prove Theorem 5.2.7 and Theorem 5.2.8. By (1.3) and the symmetry of $\text{SO}(d)$, we may rewrite (5.11) as

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) = \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 \int_{\text{SO}(d)} \int_0^1 \delta^{2d} |[\mathbf{0}, \delta^{-1}, \rho] \mathcal{F}\{u\}(n)|^2 d\delta d\rho,$$

and further, by a change of variable, we get

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) = \frac{1}{|\mathbb{S}^{d-1}|} \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi. \quad (5.28)$$

Here, $|\mathbb{S}^{d-1}|$ is the surface measure of the unit sphere, which appears since $d\rho$ is a probability measure. As it depends only on d , we omit it in the rest of the section, that is, we work throughout with $|\mathbb{S}^{d-1}| \mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N)$; this merely rescales the constants C_d in Theorems 5.2.7 and 5.2.8. The rest of the section relies on the following asymptotic relation resulting from integrating by parts in (5.5). Namely, it holds

$$\int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi \sim \frac{|n|^d \mathcal{J}(u)}{2d\pi^2} \quad \text{as } |n| \rightarrow \infty. \quad (5.29)$$

Hence, let us start by proving the upper bound in Theorem 5.2.7.

Proof of Theorem 5.2.7. We write $c_i = c_i(d)$ with $i \in \mathbb{N}$ to denote generic dimensional constants throughout the proof. Assume $\mathcal{J}(u) > 0$. By (5.29), there exists $n_0 \in \mathbb{N}$ such that for every n with $|n| \geq n_0$, it holds

$$\int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi \leq \frac{|n|^d \mathcal{J}(u)}{d\pi^2}.$$

If instead $\mathcal{J}(u) = 0$, we fix $\eta > 0$ and choose n_0 so that the left-hand side is at most $\eta|n|^d$ for $|n| \geq n_0$; the argument below then gives the claim with $\mathcal{J}(u)$ replaced by $d\pi^2\eta$, and we conclude by letting $\eta \rightarrow 0$. First, we prove the upper bound for all positive integers N such that $N^{1/d} \geq n_0$ and $N^{1/d} \in \mathbb{N}$. Consider the configuration of N points $\mathcal{P}_N \subset \mathbb{T}^d$ to be the periodization of

$$\frac{[0, N^{1/d})^d \cap \mathbb{Z}^d}{N^{1/d}} \subset [0, 1)^d. \quad (5.30)$$

Hence, it is not difficult to see that

$$\sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} = \begin{cases} N & \text{if } n \in N^{1/d} \mathbb{Z}^d \\ 0 & \text{else} \end{cases},$$

and therefore, by (5.28) and the assumptions on N , we get

$$\begin{aligned} \mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) &= \sum_{m \in \mathbb{Z}_*^d} N^2 |N^{1/d} m|^{-2d-1} \int_{|\xi| \leq N^{1/d} |m|} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi \\ &\leq \frac{1}{d\pi^2} \mathcal{J}(u) N^2 \sum_{m \in \mathbb{Z}_*^d} |N^{1/d} m|^{-d-1} \\ &\leq c_1 \mathcal{J}(u) N^{(d-1)/d}. \end{aligned}$$

Now, we prove the upper bound for every positive integer N . Again, we adapt the argument at the end of the proof of Theorem 3.1.3. As k goes through the positive integers, we consider the recursive decomposition of N defined as

$$N_k = \max \left\{ n \in \mathbb{N} : n^{1/d} \in \mathbb{N}, n \leq N - \sum_{j=1}^{k-1} N_j \right\},$$

where improper sums are zeros. By the latter definition, we get that

$$N - \sum_{j=1}^{k-1} N_j \leq (N_k^{1/d} + 1)^d \leq N_k + c_2 N_k^{(d-1)/d},$$

and therefore, for every integer $k \geq 1$, it holds

$$N_{k+1} \leq N - \sum_{j=1}^k N_j \leq c_2 N_k^{(d-1)/d}. \quad (5.31)$$

Hence, since by definition $N_1 \leq N$, then we recursively get

$$N_{k+1}^{1/d} \leq c_2^{1/d} N_k^{(d-1)/d^2} \leq c_2^{(k-1)/d} N^{(d-1)^{k-1}/d^k}. \quad (5.32)$$

Now, set $K = K(d)$ to be the smallest integer such that

$$\frac{(d-1)^K}{d^K} \leq \frac{d-1}{4d},$$

and in particular, notice that by (5.32), we have

$$N_{K+1}^{(d-1)/d} \leq c_3 N^{(d-1)/(4d)}. \quad (5.33)$$

Now, consider the remainder

$$N_0 = N - \sum_{j=1}^{K+1} N_j,$$

and by (5.31) and (5.33), it is immediate to see that

$$N_0 \leq c_2 c_3 N^{(d-1)/(4d)}. \quad (5.34)$$

Last, consider the constant depending on u defined as

$$M(u) = \max_{|n| \geq 1} |n|^{-d} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi,$$

and in particular, notice that this quantity is indeed finite. Finally, for $1 \leq j \leq K+1$, we assign a uniform lattice $\mathcal{P}_{N_j}$ to every N_j as in (5.30), and by proceeding as in the first part of the proof, we get the estimate

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_{N_j}) \leq c_1 \mathcal{J}(u) N_j^{(d-1)/d} + c_4 M(u) n_0^{d-1}, \quad (5.35)$$

where the first term bounds the case $N_j^{1/d} \geq n_0$, and the second one the case $N_j^{1/d} < n_0$. On the other hand, for every configuration of N_0 points $\mathcal{P}_{N_0} \subset \mathbb{T}^d$, by the inequality in (5.34), we get

$$\begin{aligned} \mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_{N_0}) &= \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_{N_0}} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\ &\leq c_5 M(u) N^{(d-1)/(2d)}. \end{aligned} \quad (5.36)$$

Therefore, by (5.35) and (5.36), we get

$$\begin{aligned}
\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) &= \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{j=0}^{K+1} \sum_{p \in \mathcal{P}_{N_j}} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\
&\leq (K+2) \sum_{j=0}^{K+1} \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_{N_j}} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\
&\leq (K+2) \mathbf{M}(u) \left(c_4(K+1)n_0^{d-1} + c_5 N^{(d-1)/(2d)} \right) + \\
&\quad + c_1(K+2) \mathcal{J}(u) \sum_{j=1}^{K+1} N_j^{(d-1)/d} \\
&\leq c_5 \mathbf{M}(u) \left(n_0^{d-1} + N^{(d-1)/(2d)} \right) + C_d \mathcal{J}(u) N^{(d-1)/d},
\end{aligned}$$

so that, dividing by $N^{(d-1)/d}$ and by taking the limsup as $N \rightarrow +\infty$, the conclusion follows. $\square$

Again, to prove the lower bound in Theorem 5.2.8, we use Lemma 1.3.3.

Proof of Theorem 5.2.8. We write $c_i = c_i(d)$ with $i \in \mathbb{N}$ to denote generic dimensional constants throughout the proof. If $\mathcal{J}(u) = 0$, there is nothing to prove, so we assume $\mathcal{J}(u) > 0$. By (5.29), there exists $n_0 \in \mathbb{N}$ such that for every n with $|n| \geq n_0$, it holds

$$\int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \geq \frac{|n|^d \mathcal{J}(u)}{4d\pi^2}.$$

Hence, consider

$$\mathbf{m}(u) = \min_{1 \leq |n| \leq n_0} |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi.$$

Since (by hypothesis) u is of compact support, we have that $\hat{u}$ is of exponential type, and therefore $\hat{u} \equiv 0$ in B_1 would imply $u \equiv 0$. Hence, we may assume that $\mathbf{m}(u) > 0$. We prove the lower bound for all the positive integers N such that

$$\frac{\mathcal{J}(u)}{4d\pi^2} N^{-(d+1)/d} \leq \mathbf{m}(u),$$

and for every configuration of N points $\mathcal{P}_N \subset \mathbb{T}^d$. Indeed, within this assumption, for all $n \in B_{N^{1/d}}$ it holds

$$|n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \geq \frac{\mathcal{J}(u)}{4d\pi^2} N^{-(d+1)/d}.$$

Therefore, by applying the latter inequality in (5.28), we get that for a dimensional constant $\kappa \geq 1$ to be chosen later, it holds

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) \geq \sum_{n \in \mathbb{Z}_*^d \cap B_{(\kappa N)^{1/d}}} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 \frac{\mathcal{J}(u)}{4d\pi^2} (\kappa N)^{-(d+1)/d}.$$

Finally, we can apply Lemma 1.3.3 with $U = B_{1/2}$, and obtain

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) \geq c_1 \mathcal{J}(u) (\kappa N)^{-(d+1)/d} \left(2^{-d} \omega_d \kappa N^2 - N^2 \right),$$

and by setting $\kappa = 2^{d+1} \omega_d^{-1}$, it follows that

$$\mathcal{D}_2^{\text{SO}}(u; \mathcal{P}_N) \geq c_2 \mathcal{J}(u) N^{(d-1)/d}.$$

□

5.5 A question in between harmonic analysis and GMT

We close by recording a question that the results of this chapter raise but do not settle. Throughout, the Fourier transform, the heat semigroup $T(t)$ and the Gaussian kernel G_d are as in Section 5.3, and the geometric notation $(S_u, J_u, \text{the one-sided traces } u^\pm)$ is that of Section 5.1.

Recall from Theorem 5.2.2 that, for $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it holds

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 \hat{u}(\xi) \bar{\hat{v}}(\xi) \, d\xi = \mathcal{J}(u, v), \quad (5.37)$$

where $\mathcal{J}(u, v)$ is the L^2 -jump product of Section 5.3. As shown there, (5.37) is obtained by studying the short-time behaviour of the derivative of the relative heat content

$$H'(t; u, v) = 2t \int_{\mathbb{R}^d} \Delta(T(t^2)u)(x) v(x) \, dx,$$

whose limit as $t \rightarrow 0^+$ is identified in Proposition 5.3.4.

We begin with a remark on the natural function-space setting of the latter functional. Consider the auxiliary functional $\mathcal{G}_t$ defined by

$$\mathcal{G}_t(u) = \sqrt{2t} \nabla T(t^2/2)u.$$

Then

$$H'(t; u, v) = -\langle \mathcal{G}_t(u), \mathcal{G}_t(v) \rangle_{L^2},$$

and it is not difficult to prove that

$$\|\mathcal{G}_t(u)\|_{L^1} \leq \sqrt{2t} \|Du\|_{\text{TV}},$$

where $\|Du\|_{\text{TV}}$ is the total variation of the distributional gradient of u , while it is immediate that

$$\|\mathcal{G}_t(u)\|_{L^\infty} \leq ct^{-1/2} \|u\|_{L^\infty},$$

where, here and in what follows, c denotes an absolute generic constant. Hence, by real-interpolation arguments, and by setting

$$\mathcal{K} := (\text{BV}, L^\infty)_{1/2,2} \supset \text{BV} \cap L^\infty,$$

it follows that

$$|H'(t; u, v)| \leq c \|u\|_{\mathcal{K}} \|v\|_{\mathcal{K}} \quad \forall t > 0.$$

A question arises: does $H'(t; u, v)$ converge as $t \rightarrow 0^+$ for $u, v \in \mathcal{K}$?

This is one instance of a broader question. Luc Tartar's [Tar07, Ch. 38] is devoted to some remarks (and intriguing ideas) regarding a possible interpolation space that recovers some of the good geometric properties of BV. At the end of his considerations, Tartar finds the Besov space

$$\mathcal{B}_{2,\infty}^{1/2} = (W^{1,2}, L^2)_{1/2,\infty}$$

as a candidate; namely, he writes “*Could the space $\mathcal{B}_{2,\infty}^{1/2}$ be used for quasi-linear hyperbolic equations where one expects discontinuities?*”. By using moduli of continuity, this space can be defined as

$$\mathcal{B}_{2,\infty}^{1/2} := \left\{ u \in L^2(\mathbb{R}^d) \quad : \quad \exists b > 0 \quad \text{such that} \quad \|\tau_h u - u\|_{L^2} \leq b|h|^{1/2} \quad \forall h \in \mathbb{R}^d \right\}.$$

Alternatively, by the classic definition in [Tri78, Page 169], one has the following characterisation on the frequency domain: setting

$$M_0 = \left\{ x \in \mathbb{R}^d \quad : \quad |x| \leq 2 \right\} \quad \text{and} \quad M_j = \left\{ x \in \mathbb{R}^d \quad : \quad 2^{j-1} \leq |x| \leq 2^{j+1} \right\},$$

then

$$\mathcal{B}_{2,\infty}^{1/2} := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \quad : \quad u = \sum_{j=0}^{\infty} a_j, \text{ Supp}(\hat{a}_j) \subset M_j, \sup_j 2^{j/2} \|a_j\|_{L^2} < +\infty \right\}.$$

Amongst other things, Tartar shows that $\mathcal{K} \subset \mathcal{B}_{2,\infty}^{1/2}$. This is somehow encouraging, since, as noted above, the functions in $\mathcal{K}$ may enjoy nice geometric properties. Nonetheless, it

is not difficult to show that the limit in (5.37) does not exist (in general) for functions in $\mathcal{B}_{2,\infty}^{1/2}$. Indeed, by integrating by parts, one may write

$$\mathcal{B}_{2,\infty}^{1/2} = \left\{ u \in L^2(\mathbb{R}^d) : \limsup_{R \rightarrow \infty} \frac{1}{R} \int_{B_R} |\xi|^2 |\hat{u}(\xi)|^2 d\xi < +\infty \right\}.$$

Moreover, the discontinuity set need not be well behaved on $\mathcal{B}_{2,\infty}^{1/2}$, as shown by Hashash and Poliakovsky [HP24]. Indeed, even though the jump set J_u of any locally integrable function is $(d-1)$ -rectifiable (see [DN21]), they proved that the good Federer–Vol’pert structure of BV functions (recalled in Section 5.1, where $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$) is lost at this regularity; namely, there exists $u \in W^{1/2,2} \subset \mathcal{B}_{2,\infty}^{1/2}$ for which $\mathcal{H}^{d-1}(S_u) > 0$ but $\mathcal{H}^{d-1}(J_u) = 0$. What survives, in general, is only that a suitable restricted discontinuity set is $\mathcal{H}^{d-1}$ σ -finite [HP24, Theorem 1.5]. Thus membership in $\mathcal{B}_{2,\infty}^{1/2}$ alone cannot recover BV-type geometry on S_u , and any candidate space must be strictly smaller.

One may finally observe that even the stronger requirement that the limit in the left-hand side of (5.37) exist may not be sufficient to recover any geometric property on S_u : heuristically, such a limit may exist even if $\hat{u}$ has masses that concentrate in sectors with different directions for increasing dyadic annuli, thereby leaving little hope for any regularity. This sharpens the question: does the space conjectured by Tartar exist, and how does one find it? A reasonable first guess would be to consider a directional form of (5.37), and this would be

$$\lim_{R \rightarrow \infty} R^{-1} \int_0^R |s|^{d+1} |\hat{u}(s\nu)|^2 ds < +\infty \quad \forall \nu \in \mathbb{S}^{d-1}.$$

The latter condition turns out to be too strong; for example, if u is the characteristic function of the unit square, then the previous limit is 0 for all directions ν except for the 4 cardinal directions, where it diverges. One may also take into consideration more complicated constructions, such as spaces of the form

$$\left\{ u \in L^2(\mathbb{R}^2) : f_u(\nu) := \lim_{r \rightarrow 0^+} r^{-1/2} \|u(\cdot - r\nu) - u\|_{L^2}, f_u \text{ has some regularity} \right\}.$$

An established way to study geometric sets with codimension 1 by harmonic analysis methods comes from restriction theory and the wave packet decomposition techniques. Indeed, the use of Gaussians to recover geometric properties has been widely employed for the latter problems, and as an example, we refer to [BCT06]. In particular, we note that, in restriction theory, the curvature (or rather, the transversality between surfaces) plays a crucial role, whereas for our purposes it should not.

Now, we present a remark (for which I am grateful to Hannah Cairo) that gives a simple heuristic explanation of (5.37) (starting from the case $u = v$). If η_R is a smooth cutoff function with support in B_R , then, by Plancherel,

$$R^{-1} \|\xi |\eta_R \hat{u}|\|_{L^2}^2 = cR^{-1} \|\nabla(\tilde{\eta}_R * u)\|_{L^2}^2.$$

Since $\check{\eta}_R$ is morally $R^d \mathbf{1}_{B_{R^{-1}}}$, convolution with it is a blur at scale R^{-1} . Hence we can represent $\check{\eta}_R * u$ as a function that is constant on a grid of hypercubes of side R^{-1} , that is, as a discretisation. The distributional gradient can then be interpreted as a discrete gradient; accordingly, let g_R/R^{-1} be the result of the discrete gradient applied to the discretised function (g_R morally represents the grid of the differences). It is not difficult to show that

$$R\|g_R\|_{L^1} \leq c\|Du\|_{\text{TV}},$$

and the bound $\|g_R\|_{L^\infty} \leq 2\|u\|_{L^\infty}$ is trivial. Then, again by interpolation, one could conclude that

$$R^{1/2}\|g_R\|_{L^2} \leq \|u\|_{\mathcal{K}}.$$

Moreover, one can verify that

$$\lim_{R \rightarrow +\infty} R|g_R|^{-1}((\varepsilon, +\infty]) = \mathcal{J}_\varepsilon(u),$$

where $\mathcal{J}_\varepsilon(u)$ is the part of the jump set of u on which $|u^+ - u^-| > \varepsilon$. Furthermore, if $Q = (\mathcal{J}_\varepsilon(u))_{R^{-1}}$ denotes the R^{-1} neighbourhood of the previous set, then $\|g_R\|_{L^\infty(Q^c)} \leq \varepsilon$, so that interpolating again gives

$$R^{1/2}\|g_R\|_{L^2(Q^c)} \leq \varepsilon^{1/2}\|Du\|_{\text{TV}}^{1/2},$$

so that only the contribution of Q plays a role in the limit. It seems that one should be able to proceed with a covering argument on the jump set and recover (5.37), but we stop here.

On the other hand, if one takes η_R supported on a tube (as in classical arguments of restriction theory) and applies the same heuristic, then one recovers some directional geometric properties, albeit with complications. In particular, one could impose a directional condition on the decay of the Fourier transform and then exploit it via wave packet decomposition techniques; this path is yet to be explored, but one could start by examining the following set. For $\nu \in \mathbb{S}^{d-1}$ and $R > 0$ large, consider the tube

$$T_\nu^R = \left\{ \xi \in \mathbb{R}^d : |\xi \cdot \nu| \leq R, |\xi - (\xi \cdot \nu)\nu| \leq R^{1/2} \right\},$$

and, for $u \in L^2(\mathbb{R}^d)$, the finite positive measure on the sphere

$$d\mu_{u,R}(\nu) := \left(R^{-\frac{d+1}{2}} \int_{T_\nu^R} |\xi|^{d+1} |\hat{u}(\xi)|^2 d\xi \right) d\mathcal{H}^{d-1}(\nu).$$

A candidate space could then be

$$\left\{ u \in L^2(\mathbb{R}^d) : \mu_{u,R} \xrightarrow{*} \mu_u \text{ for some } \mu_u \in \mathcal{M}(\mathbb{S}^{d-1}) \text{ as } R \rightarrow \infty \right\},$$

where $\xrightarrow{*}$ denotes weak-* convergence in $\mathcal{M}(\mathbb{S}^{d-1}) = (C(\mathbb{S}^{d-1}))^*$. Establishing the utility of such a space (whether $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$, and so on) would require further work, which we leave to future investigation.

Chapter 6

Set-theoretic derivative and Fourier transform

This chapter presents the content of [BG26] concerning the set-theoretic derivative of the Fourier transform. The main result, Theorem 6.1.3, characterizes when the difference quotient of the Fourier transform under Minkowski expansion recovers the Fourier transform of the perimeter measure. The framework and the notation used in this chapter are developed in detail in Section 5.1 of Chapter 5.

6.1 Introduction

The central hypothesis of Theorem 6.1.3 is the equality between the outer Minkowski content and the perimeter of a set. We introduce these notions, along with the relevant weak convergence of measures. Throughout this chapter, Ω denotes a Borel set, and not its equivalence class modulo $\mathcal{L}^d$ -null sets. Indeed, the perimeter depends only on the equivalence class, whereas the outer parallel sets Ω_r below, and hence the outer Minkowski content, depend on the representative. For a set $\Omega \subset \mathbb{R}^d$ of finite measure and perimeter, we consider the distance function

$$d_\Omega(x) = \inf\{|x - y| : y \in \Omega\},$$

and write Ω_r to denote the set

$$\Omega_r = \{x \in \mathbb{R}^d : d_\Omega(x) \leq r\},$$

which coincides with the Minkowski sum $\overline{\Omega} + \overline{B_r}$, and hence with $\Omega + B_r$ up to the $\mathcal{L}^d$ -negligible set $\{d_\Omega = r\}$. We have the following definition.

Definition 6.1.1 (outer Minkowski content). We say that a Borel set $\Omega \subseteq \mathbb{R}^d$ has a finite outer Minkowski content if the limit

$$\mathcal{SM}(\Omega) = \lim_{r \rightarrow 0^+} \frac{|\Omega_r \setminus \Omega|}{r}$$

exists and is finite.

Moreover, we give the following notion of convergence for sequences of measures.

Definition 6.1.2. We say that a sequence of measures $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$ weakly converges to a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, if for every $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ it holds

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi \, d\mu_n = \int_{\mathbb{R}^d} \varphi \, d\mu.$$

In this chapter, we characterize the set-theoretic derivative of the Fourier transform of sets of finite perimeter. The following result improves on [Her62, Thm. 1], in which the author only considered convex sets.

Theorem 6.1.3. *Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure and perimeter, with $|\Omega + h_0 B_1| < +\infty$ for some $h_0 > 0$, then*

$$\lim_{h \rightarrow 0^+} \frac{\widehat{\mathbf{1}}_{\Omega + h B_1}(\xi) - \widehat{\mathbf{1}}_{\Omega}(\xi)}{h} = \mathcal{F}\{|D\mathbf{1}_{\Omega}|\}(\xi) \quad \forall \xi \in \mathbb{R}^d \quad (6.1)$$

if and only if $\text{Per}(\Omega) = \mathcal{SM}(\Omega)$.

The hypotheses under which we obtain (6.1) are the weakest possible, and as an example, any set with Lipschitz boundary satisfies our hypotheses, as the reader may verify in [ACV08]. In addition, our proof appears more straightforward than the one in [Her62, Thm. 1] since the former relies on classic functional analysis techniques and geometric measure theory.

6.2 Sharpening a result of Herz

The following result is essential to obtain Theorem 6.1.3, and it is a refinement of [ACV08, Prop. 2].

Proposition 6.2.1. *Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure and perimeter. Then $\mathcal{SM}(\Omega) = \text{Per}(\Omega)$ if and only if the measures*

$$\mu_h := \frac{\mathbf{1}_{\Omega + h B_1} - \mathbf{1}_{\Omega}}{h} \quad (6.2)$$

weakly converge to $|D\mathbf{1}_{\Omega}|$.

Proof. First, let us assume that

$$\mathcal{SM}(\Omega) = \text{Per}(\Omega) < +\infty.$$

Since $\overline{\Omega} \subseteq \Omega_r$ for every $r > 0$, we have $|\overline{\Omega} \setminus \Omega| \leq |\Omega_r \setminus \Omega| = \mathcal{O}(r)$ as $r \rightarrow 0^+$, because $\mathcal{SM}(\Omega) < +\infty$. Hence, $|\overline{\Omega} \setminus \Omega| = 0$, so that $\{d_\Omega > 0\} = \mathbb{R}^d \setminus \overline{\Omega}$ coincides with Ω^c up to a null set: we use this identification below. By Portmanteau theorem (see [Bog07, Thm. 8.2.3]), and since the assumption implies that

$$\lim_{h \rightarrow 0^+} \mu_h(\mathbb{R}^d) = \text{Per}(\Omega),$$

then it is enough to show that

$$\liminf_{h \rightarrow 0^+} \mu_h(U) \geq |D\mathbf{1}_\Omega|(U) \quad \text{for every open set } U.$$

It is clear that the distance function d_Ω is Lipschitz continuous, and as in [AD00, p. 11], we have that

$$|\nabla d_\Omega| = 1 \quad \mathcal{L}^d\text{-almost everywhere in } \Omega^c.$$

Given the previous observations, we apply Coarea formula to obtain

$$\begin{aligned} \mu_h(U) &= \frac{1}{h} \int_{\{0 < d_\Omega < h\}} \mathbf{1}_U |\nabla d_\Omega| \, d\mathcal{H}^d = \frac{1}{h} \int_0^h \text{Per}(\{d_\Omega > t\}; U) \, dt \\ &= \int_0^1 \text{Per}(\{d_\Omega > hr\}; U) \, dr. \end{aligned}$$

Since the sets $\{d_\Omega > hr\}$ converge locally in measure to $\{d_\Omega > 0\} = \Omega^c$ as $h \rightarrow 0^+$, then, by the lower semicontinuity of the perimeter (see [AFP00, Prop. 3.38]) and by Fatou lemma, we get

$$\liminf_{h \rightarrow 0^+} \mu_h(U) \geq \text{Per}(\Omega^c; U) = \text{Per}(\Omega; U) = |D\mathbf{1}_\Omega|(U).$$

This proves the first part of the result.

Conversely, let μ_h be weakly convergent to $|D\mathbf{1}_\Omega|$. Then this implies the boundedness of the sequence, and moreover, we have that

$$\mu_h(\mathbb{R}^d) \rightarrow |D\mathbf{1}_\Omega|(\mathbb{R}^d) = \text{Per}(\Omega),$$

which gives the thesis. $\square$

In light of the latter proposition, we give a short proof of Theorem 6.1.3.

Proof of Theorem 6.1.3. Notice that the function

$$x \mapsto e^{-2\pi i \xi \cdot x}$$

is bounded and continuous for all $\xi \in \mathbb{R}^d$. If the measures μ_h (defined as in (6.2)) converge in duality with $\mathcal{C}_b(\mathbb{R}^d)$ to $|D\mathbf{1}_\Omega|$, then we have

$$\lim_{h \rightarrow 0^+} \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d\mu_h(x) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d|D\mathbf{1}_\Omega|(x),$$

which is equivalent to (6.1). The converse implication is trivial since one can test the pointwise convergence of the Fourier transforms at $\xi = 0$, easily implying the thesis. $\square$

Bibliography

- [ACV08] Luigi Ambrosio, Andrea Colesanti, and Elena Villa. Outer Minkowski content for some classes of closed sets. *Math. Ann.*, 342(4):727–748, 2008.
- [AD00] L. Ambrosio and N. Dancer. *Calculus of variations and partial differential equations*. Springer-Verlag, Berlin, 2000. Topics on geometrical evolution problems and degree theory, Papers from the Summer School held in Pisa, September 1996.
- [AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [AH76] S. Agmon and L. Hörmander. Asymptotic properties of solutions of differential equations with simple characteristics. *J. Analyse Math.*, 30:1–38, 1976.
- [AHT18] Luigi Ambrosio, Shouhei Honda, and David Tewodrose. Short-time behavior of the heat kernel and Weyl’s law on $\mathrm{RCD}^*(K, N)$ spaces. *Ann. Global Anal. Geom.*, 53(1):97–119, 2018.
- [BC90] J. Beck and W. W. L. Chen. Note on irregularities of distribution. II. *Proc. London Math. Soc. (3)*, 61(2):251–272, 1990.
- [BC97] J. Beck and W. W. L. Chen. Irregularities of point distribution relative to convex polygons. III. *J. London Math. Soc. (2)*, 56(2):222–230, 1997.
- [BCC⁺19] Luca Brandolini, William W. L. Chen, Leonardo Colzani, Giacomo Gigante, and Giancarlo Travaglini. Discrepancy and numerical integration on metric measure spaces. *J. Geom. Anal.*, 29(1):328–369, 2019.
- [BCT06] Jonathan Bennett, Anthony Carbery, and Terence Tao. On the multilinear restriction and Kakeya conjectures. *Acta Math.*, 196(2):261–302, 2006.
- [BDP20] Dmitriy Bilyk, Josef Dick, and Friedrich Pillichshammer, editors. *Discrepancy theory*. De Gruyter, Berlin, 2020.

- [Bec87] József Beck. Irregularities of distribution. I. *Acta Math.*, 159(1-2):1–49, 1987.
- [Bec88] József Beck. Irregularities of distribution. II. *Proc. London Math. Soc.* (3), 56(1):1–50, 1988.
- [Ber26a] Thomas Beretti. The Fourier transform of planar convex bodies and discrepancy over intervals of rotations. *Adv. Math.*, 503:Paper No. 111220, 45, 2026.
- [Ber26b] Thomas Beretti. Polynomial-order oscillations in geometric discrepancy. *Mathematika*, 72(4):Paper No. e70121, 2026.
- [BG26] Thomas Beretti and Luca Gennaioli. Fourier transform of BV functions and applications. *Journal of Functional Analysis*, page 111700, 2026.
- [BG21] Luca Brandolini, Bianca Gariboldi, and Giacomo Gigante. On a sharp lemma of Cassels and Montgomery on manifolds. *Math. Ann.*, 379(3-4):1807–1834, 2021.
- [BHI03] L. Brandolini, S. Hofmann, and A. Iosevich. Sharp rate of average decay of the Fourier transform of a bounded set. *Geom. Funct. Anal.*, 13(4):671–680, 2003.
- [Bil11] Dmitriy Bilyk. On Roth’s orthogonal function method in discrepancy theory. *Unif. Distrib. Theory*, 6(1):143–184, 2011.
- [BM23] Dmitriy Bilyk and Michelle Mastrianni. Lower bounds for the directional discrepancy with respect to an interval of rotations. *J. Fourier Anal. Appl.*, 29(3):Paper No. 25, 2023.
- [BMPS11] Dmitriy Bilyk, Xiaomin Ma, Jill Pipher, and Craig Spencer. Directional discrepancy in two dimensions. *Bull. Lond. Math. Soc.*, 43(6):1151–1166, 2011.
- [BMPS16] Dmitriy Bilyk, Xiaomin Ma, Jill Pipher, and Craig Spencer. Diophantine approximations and directional discrepancy of rotated lattices. *Trans. Amer. Math. Soc.*, 368(6):3871–3897, 2016.
- [Bog07] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.
- [BRT98] Luca Brandolini, Marco Rigoli, and Giancarlo Travaglini. Average decay of Fourier transforms and geometry of convex sets. *Rev. Mat. Iberoamericana*, 14(3):519–560, 1998.
- [BT22] Luca Brandolini and Giancarlo Travaglini. Irregularities of distribution and geometry of planar convex sets. *Adv. Math.*, 396:Paper No. 108162, 2022.
- [Cas56] J. W. S. Cassels. On the sums of powers of complex numbers. *Acta Math. Acad. Sci. Hungar.*, 7:283–289, 1956.

- [Cha00] Bernard Chazelle. *The discrepancy method*. Cambridge University Press, Cambridge, 2000.
- [CST14] William Chen, Anand Srivastav, and Giancarlo Travaglini, editors. *A panorama of discrepancy theory*. Springer, Cham, 2014.
- [CV17] William W. L. Chen and Robert C. Vaughan. In memoriam Klaus Friedrich Roth 1925–2015. *Mathematika*, 63(3):711–712, 2017.
- [Dav56] H. Davenport. Note on irregularities of distribution. *Mathematika*, 3:131–135, 1956.
- [Dem20] Ciprian Demeter. *Fourier restriction, decoupling, and applications*, volume 184 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.
- [Dic14] Josef Dick. Applications of geometric discrepancy in numerical analysis and statistics. In *Applied algebra and number theory*, pages 39–57. Cambridge Univ. Press, Cambridge, 2014.
- [DL93] Ronald A. DeVore and George G. Lorentz. *Constructive approximation*. Springer-Verlag, Berlin, 1993.
- [DN21] Giacomo Del Nin. Rectifiability of the jump set of locally integrable functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 22(3):1233–1240, 2021.
- [Drm96] Michael Drmota. Irregularities of distributions with respect to polytopes. *Mathematika*, 43(1):108–119, 1996.
- [DT97] Michael Drmota and Robert F. Tichy. *Sequences, discrepancies and applications*. Springer-Verlag, Berlin, 1997.
- [Fal86] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.
- [GG22] Bianca Gariboldi and Giacomo Gigante. Almost positive kernels on compact Riemannian manifolds. *Math. Z.*, 302(2):783–801, 2022.
- [Gru83] Peter M. Gruber. In most cases approximation is irregular. *Rend. Sem. Mat. Univ. Politec. Torino*, 41(1):19–33, 1983.
- [Her62] C. S. Herz. Fourier transforms related to convex sets. *Ann. of Math. (2)*, 75:81–92, 1962.
- [Hla50] Edmund Hlawka. Über Integrale auf konvexen Körpern. I. *Monatsh. Math.*, 54:1–36, 1950.

- [Hör63] Lars Hörmander. *Linear partial differential operators*, volume Band 116 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Göttingen-Heidelberg; Academic Press, Inc., Publishers, New York, 1963.
- [HP24] Paz Hashash and Arkady Poliakovsky. Jumps in Besov spaces and fine properties of Besov and fractional Sobolev functions. *Calc. Var. Partial Differential Equations*, 63(2):Paper No. 28, 49, 2024.
- [Mag12] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.
- [Mar64] John M. Marstrand. The (φ, s) regular subsets of n -space. *Trans. Amer. Math. Soc.*, 113:369–392, 1964.
- [Mat10] Jiří Matoušek. *Geometric discrepancy*. Springer-Verlag, Berlin, 2010.
- [Mon94] Hugh L. Montgomery. *Ten lectures on the interface between analytic number theory and harmonic analysis*. American Mathematical Society, Providence, 1994.
- [MPPP07] Michele Miranda, Jr., Diego Pallara, Fabio Paronetto, and Marc Preunkert. Short-time heat flow and functions of bounded variation in $\mathbb{R}^N$. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(1):125–145, 2007.
- [Nie92] Harald Niederreiter. *Random number generation and quasi-Monte Carlo methods*, volume 63 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
- [Pod91] A. N. Podkorytov. On the asymptotics of the Fourier transform on a convex curve. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 2:50–57, 125, 1991. Translation in *Vestnik Leningrad Univ. Math.* **24** (1991), no. 2, 57–65.
- [Rot54] K. F. Roth. On irregularities of distribution. *Mathematika*, 1:73–79, 1954.
- [Sch69] Wolfgang M. Schmidt. Irregularities of distribution. IV. *Invent. Math.*, 7:55–82, 1969.
- [Sch72] Wolfgang M. Schmidt. Irregularities of distribution. VII. *Acta Arith.*, 21:45–50, 1972.
- [Str77] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

- [Str90] Robert S. Strichartz. Fourier asymptotics of fractal measures. *J. Funct. Anal.*, 89(1):154–187, 1990.
- [SW65] G. C. Shephard and R. J. Webster. Metrics for sets of convex bodies. *Mathematika*, 12:73–88, 1965.
- [Tar07] Luc Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin; UMI, Bologna, 2007.
- [Tra14] Giancarlo Travaglini. *Number theory, Fourier analysis and geometric discrepancy*. Cambridge University Press, Cambridge, 2014.
- [Tri78] H. Triebel. *Interpolation theory, function spaces, differential operators*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
- [van35] J. G. van der Corput. Verteilungsfunktionen. I. *Proc. Akad. Wet. Amsterdam*, 38:813–821, 1935.
- [van45] T. van Aardenne-Ehrenfest. Proof of the impossibility of a just distribution of an infinite sequence of points over an interval. *Nederl. Akad. Wetensch., Proc.*, 48:71–76, 1945.
- [van49] T. van Aardenne-Ehrenfest. On the impossibility of a just distribution. *Nederl. Akad. Wetensch., Proc.*, 52:734–739, 1949.
- [Wey16] Hermann Weyl. Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.*, 77(3):313–352, 1916.