Physics Area - PhD course in Theoretical Particle Physics

SISSA, Trieste

TOPOLOGICAL ASPECTS OF QUANTUM FIELD THEORY

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ACADEMIC YEAR 2023/2024

”We will fix it soon”

Acknowledgments

When I moved to Trieste to start the PhD I was very excited. I consider myself a person with a vivid imagination. However, I could not even remotely imagine how incredible these four years would have been. I am happy to be able to use this space to acknowledge all the people who contributed to the cause. It will be quite long; hence the reader interested only in the scientific part of this thesis is encouraged to skip this section and go directly to the Introduction.

Tenderò ad usare l'italiano per rigraziare coloro con cui normalmente interagisco nella mia lingua madre. Quando ho iniziato mi sentivo uno studente, oggi mi sento un ricercatore. Questo step altamente non banale è avvenuto soprattutto grazie al mio supervisor Francesco Benini. Non credo che potrò mai completamente sdebitarmi per la fiducia che mi ha dato in questi anni, e che mi ha permesso di credere in me stesso e diventare indipendente. Il nostro rapporto ha subito un evoluzione drastica ed inspettata durante gli anni, sia dal punto di vista umano che professionale, diventando per me oltre che un supervisor, un amico e un collega. Questo mi ha permesso di usufruire sempre di piu' della sua immensa conoscenza e arguzia nel capire ad analizzare i problemi. Ad oggi sento di avere una visione della fisica molto simile alla sua, cosa che ho sviluppato durante le infinite ore di discussione in cui niente veniva mai dato per scontato. Potrei continuare per ore ma per ragioni di spazio vado avanti.

E potrei parlare per anni degli altri due *Sissa boys*, Giovanni Rizi e Giovanni Galati (Giuà)[1]. Chiaramente il mio PhD non sarebbe nemmeno esistito senza di loro. Modestamente, abbiamo realizzato uno dei migliori esempi nella storia degli esempi di come essere amici e colleghi allo stesso tempo. I due aspetti si sono influenzati a vicenda in una maniera tremenda. La quantità di cose che tutti noi abbiamo imparato dalle continue discussioni quotidiane (anche mentre eravamo fuori a bere con gli altri, mi dispiace per loro che non erano mai troppo contenti) è superiore alla quantità di cose che immaginavo avrei imparato nella vita.

Grazie mille a Christian Copetti per essere stato prima il mio Postdoc di riferimento, poi una sorta di Sissa Boy aggiunto, un amico, e tra poco un mio collega come Postdoc. La sua peculiare maniera di vivere e di fare ricerca mi ha offerto continuamente un punto di vista diverso che mi ha fatto crescere.

Grazie a Luigi Tizzano per essere man mano diventato una sorta di guru spirituale che ha dispensato consigli e suggeriementi sull'accademia, la fisica, e la vita.

Grazie a Marco Serone, dapprima professore di QFT e mentore, poi collaboratore e saggio consigliere. Le sue molte domande su come capire in maniera piu' fisica le cose che facevo, mi hanno in effetti fatto capire tanta fisica.

Non ho mai capito se in inglese c'è un significato diverso tra "supervisor" e "advisor", ma se c'è, Matteo Bertolini è senz'altro stato il mio unofficial advisor, nel senso di persona che consiglia. E' stato una sorta di "padre" e "fratello maggiore" a Trieste, e l'ambiente in SISSA non sarebbe stato così

[1] Per ulteriori aspetti sul rapporto tra i Sissa boys il lettore è invatato a consulatare gli aknowledgments nella tesi di Giuà [1].

divertente senza di lui. E il numero infinito di domande che mi ha fatto su generalized symmetries (e sul suo libro, compratelo) mi hanno migliorato sotto tanti punti di vista.

Thanks to Bobby Acharya for being the academic grandfather that everybody would love to have. Thanks for all the lessions about life, music, drinks (the only Gin that deserves that name is Tanqueray), and physics.

Ringrazio poi tutti gli altri professori di TPP e della "Trieste area". Lorenzo di Pietro per essere stato un punto di riferimento nonchè grande compagno di serate, Sergio Benvenuti per la sua visione unica del mondo, Giulio Bonelli per non averlo mai fixato dandoci l'impressione che questo PhD non sarebbe mai finito, Laura Donnay for bringing a fresh female perspective to TPP, Agnese Bissi per l'infinita ironia e la risata contagiosa.

Thanks to all TPP Postdocs. In particular, to Andreas Blommaert for the many advices, to Jesse van Muiden for the fun, to Ohad Mamroud, Tomas Reis and Eduardo Garcia Valdecasas for many interesting discussions, and to Victor Godet for being so similar to Christian. Thanks also the TPP students, in particlare a Beniamino Valsesia, pace all'anima sua, per aver sopportato i miei soprusi.

I am thankful to Zohar Komargodski for having allowed me to spend three fantastic months at Simons Center, and for all the physics discussions in which he shared with me his very deep point of view. And thanks to all the people who contributed to make great my period at SCGP: Diego Delmastro, Pietro Ferrero, Justin Kulp, Michelangelo Preti, Brandon Rayhaun, Adar Sharon, Marieke Van Beest. In particular I want to thank Yunquin Zheng for the time he spent discussing, from which I learned so much.

I have been incredibly lucky to have the opportunity to travel a lot, participate in conferences and workshops, and meet many researchers from whom I learned most of the things that allowed me to do what is contained in this thesis. In particular I want to mention Jeremias Aguilera Damia, Fabio Apruzzi, Riccardo Argurio, Pietro Benetti Genolini, Lakshya Bhardwaj, Lea Bottini, Daniel Brennan, Matthew Bullimore, Yichul Choi, Clay Córdova, Michele Del Zotto, Iñaki García Etxebarria, Andrea Grigoletto, Julius Grimminger, Salvo Mancani, Pierluigi Niro, Pavel Putrov, Rajath Radhakrishnan, Matteo Sacchi, Sakura Schäfer-Nameki, Shu-Heng Shao, Tin Sulejmanpasic, Matthew Yu, Yifan Wang.

Il grazie piu' grande, però, non può che andare a tutti gli amici di Trieste durante questi anni. Grazie per tutte le fantastiche serate, le gite, le sciate, le vacanze, e grazie per aver sopportato il mio essere estremo su (quasi) tutto. Si, finalmente l'ho ammesso.
Non ricordo chi di loro (forse Fabiana?) una volta disse che il nostro gruppo è stato un miracolo. Penso sia vero. E senz'altro in nostro gruppo non sarebbe esistito senza Max. Non posso immaginare questi anni senza il "Gruppo Buongiorno" (che includendo anche Giovanni Rizi interseca per due terzi i Sissa Boys), casa sua, le sue canzoni (se un giorno inciderà un disco, compratelo), le barzellette in salentino, i taralli, la ricotta forte, etc...
Come avremmo fatto senza Stefano Baggioni (all'anagrafe Stephane Bajot), il suo italiano sgrammaticato, le sue massime, il suo punto di vista a volte un po' esagerato anche per me, la sua ospitalità in Francia (grazie anche a tutta la sua famiglia!), e tutte le birre fresche insieme?
Grazie poi ai nostri compagni di viaggio di TPP, nonchè vicini di ufficio, Giulio Barni, Fragarosi, e Fabiana (se aprirà uno studio di armocromia, andateci).
Grazie a Delpo per essere sempre così delpico e un po' sfrastico (e scusa per quella volta al bar H...).
Grazie a Lucia per la sua confusione e irrequietezza ordinata, a cui sono stato felice di aver imparato ad adattarmi, e per la pazienza che ha avuto con me in certi periodi.
E a Francesca che è inaspettatamente risucita a diventare, da persona che non sopportavo, una delle migliori amiche mai avute, e avermi poi dovuto sopportare lei durante i miei scleri.

Grazie a tutti gli altri compagni di serate e compagni di viaggio di questi anni, Angelo Nestola, Alice, Cristiano, Cristoforo, Costi, Daniel, Daniele Perri, Davidone, Diego 1, Diego 2, Elisa (ex-)manager, Giacomo, Jacopo, Luca mi cuggino, Mavi, Morgana, Saman, Stracchino, Vale, Valeria, Vania.

Grazie poi a tutte le bariste della SISSA, a Patrizia, Lucio, e ai gestori/proprietrari dei locali di questi anni: Savo e Ilaria del dlf, Chen, il Barmacista, il Baritono, e tanti altri che mi sto dimenticando, ma che non leggeranno mai questa tesi e quindi non avranno nulla da ridire.

Last but non least, grazie infinite ai miei genitori per tutto il supporto, l'amore e l'orgoglio dimostratomi in tutti i modi possibili. Questa tesi esiste grazie a voi.

Abstract

Quantum Field Theories (QFTs) have a universal protected sector provided by topological operators. They generalize the ordinary notion of symmetry in various directions, allowing to transport many familiar concepts (symmetry breaking, anomalies, gauging, etc...) in a much more general framework. This thesis is devoted to the study of several aspects of these topological properties, and their interplay with the dynamics. A central tool is provided by a Topological Quantum Field Theory (TQFT) living in one higher dimension, that encodes all the properties of the topological sector in an elegant way, and allows to extract, from topology, dynamical constraints that would be inaccessible otherwise. Some of the applications that we will find include the holographic dual of the so-called *categorical* symmetries, constraints on the infrared imposed by generalized symmetries, the discovery of new topological properties of certain gapless phases, a new class of exotic TQFTs, and the derivation of the holographic dual of any Goldstone theory describing spontaneous symmetry breaking.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following published works:

- A. Antinucci, F. Benini, C. Copetti, G. Galati, and G. Rizi, “The holography of non-invertible self-duality symmetries” [2].
- A. Antinucci, F. Benini, C. Copetti, G. Galati, and G. Rizi, “Anomalies of non-invertible self-duality symmetries: fractionalization and gauging” [3].
- A. Antinucci, C. Copetti and S. Schafer-Nameki, ”SymTFT for (3+1)d Gapless SPTs and Obstructions to Confinement” [4].
- A. Antinucci and F. Benini, “Anomalies and gauging of U(1) symmetries” [5].
- A. Antinucci, F. Benini, and G. Rizi, “Holographic duals of symmetry broken phases” [6].

During the PhD, I also published the following papers, that I did not included in the thesis to preserve a liner logical flow in the presentation:

- A. Antinucci, M. Bianchi, S. Mancani, and F. Riccioni, ”Suspended fixed points” [7].
- A. Antinucci, G. Galati, and G. Rizi, “On continuous 2-category symmetries and Yang-Mills theory” [8].
- A. Antinucci, C. Copetti, G. Galati, and G. Rizi, “”Zoology” of non-invertible duality defects: the view from class $\mathcal{S}$” [9].
- A. Antinucci, G. Galati, G. Rizi, and M. Serone ”Symmetries and topological operators, on average” [10].

Contents

Introduction

One exciting fact in physics is that, often, completely different phenomena can be described by the same formalism. The reason for this is not unique. In some cases, the underlying deep fact is that the phenomena were not really distinct but merely two aspects of the same thing. That is the case for falling objects on Earth and the motion of planets in the solar system. In other cases, this fascinating fact can be understood by recognizing that the formalism with which we are working is extremely broad and powerful. The main example in contemporary physics is Quantum Field Theory (QFT). This is the language we use to describe a surprisingly large class of systems, ranging from high-energy physics, including both particle physics and gravity, to critical phenomena in statistical mechanics, passing through condensed matter systems, quantum computation, and many others. As a by-product QFT also provided deep insights for basically all branches of mathematics, creating a lot of connections to these fields.

Probably the main reason why QFT has so diverse applications is that it is very hard, has many aspects, and encompasses numerous facets. It is so hard that it is not even clear how to define it rigorously. The only instances in which QFTs have mathematically rigorous definitions are Topological Quantum Field Theories (TQFT) and two-dimensional Conformal Field Theories (CFT). In all other cases, physicists think about QFTs in a heuristic way, defined by some *path integral*, which is notoriously ill-defined mathematically. Moreover, even if we accept this lack of a rigorous definition, a path integral formulation requires a Lagrangian, or equivalently the existence of a classical system that one quantizes, and this is a luxury that we are often not granted.

Despite lacking a first principle definition, several quantities and the underlying physics have been computed and analyzed in various cases. The common factor in these successes is that some QFT properties are so *robust* that they persist regardless of the precise mathematical definition of the theory. An example is conformal field theories, where the conformal algebra is so constraining that some correlation functions are determined by it. An other successful instance is supersymmetry, especially in the extended case, where several quantities are protected against quantum corrections and can be determined exactly. While robust properties may be seen merely as a sort of trick to analyze the system, they can actually be interpreted as instrumental toward giving an actual definition of the theory. Indeed, the reason why TQFTs and 2d CFTs have mathematically rigorous definitions is because they are completely determined by their robust properties. In the TQFT case these are symmetries, localities, and so on, while for 2d CFT is the infinite-dimensional Virasoro algebra.

While supersymmetric and conformal field theories are very good laboratories for our understanding of QFT, it is equally important to understand how much the knowledge we acquired in these toy models can be applied to more general theories. Analyzing the dynamics of non-supersymmetric theories requires looking for some other robust properties. In recent years there has been a huge activity in understanding universal robust properties related with the *topological sector* of QFTs. This is the set of operators (or defects), whose support is generically extended, with the property that any corre-

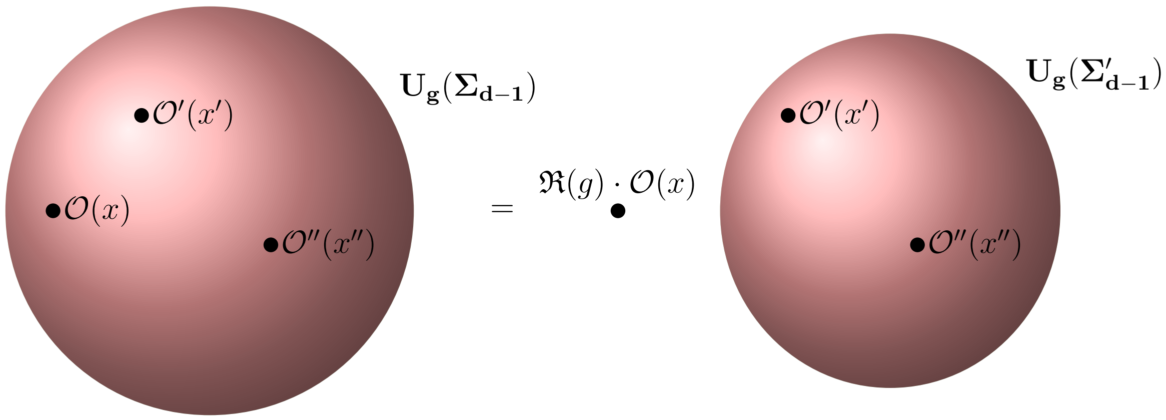


Figure 1: The action of a topological codimension one operator when it crosses a charged local operator.

lation function with those operators inserted is independent of the deformation of the supports that do not change their topological class. The topological class is modified when the operator's support moves in a way such that it intersects with another operator. In such scenarios, the adjustment of the correlation functions can be non-trivial and defines the *data* of the topological sector. These data are precisely the robust properties that can be analyzed reliably.

The most basic example of our discussion is provided by conventional global symmetries represented by a Lie group G. This is associated with some conserved current $J_\mu(x) = J^a_\mu(x)T^a$ leading to Ward identities and, as we will review in Section 1 following [11], we can construct a *topological operator*

$$U_g(\Sigma_{d-1}) = \exp\left(i\epsilon^a \int_{\Sigma_{d-1}} *J^a\right) .$$

Here $g = e^{i\epsilon^a T^a} \in G$ is a group element labelling the operator, while $J^a = J^a_\mu dx^\mu$ is a 1-form so that its Hodge dual $*J^a$ can be integrated on any $(d-1)-$dimensional submanifold Σ_{d-1}, the support of the operator $U_g(\Sigma_{d-1})$. The Ward identities imply that we can reshape the submanifold Σ_{d-1} without altering the value of the correlation function, as long as we do not intersect a point where a local operator $\mathcal{O}(x)$ is placed. If $\mathcal{O}(x)$ transforms in a non-trivial representation $\mathfrak{R}$ of G, the topological defect acts on the operator when it is intersected (fig. 1). This property is robust: starting from a weakly coupled theory and deforming by RG flow or additional interactions, the conclusions hold as long as symmetry is preserved. Thus, whenever a QFT has a G global symmetry, the operator algebra and the Hilbert space remain organized in representations of G even at strong coupling. Although this is very standard, reformulating it as a consequence of a topological operator $U_g(\Sigma_{d-1})$ is crucial, as it will allow us to generalize it to less standard situations.

The interplay between the topological operator $U_g(\Sigma_{d-1})$ and the charged local operators $\mathcal{O}(x)$ is not the only important information. When two or more topological operators intersect in *junctions*, modifying their structure can introduce a phase in the correlation functions. This is the manifestation of a 't Hooft anomaly for the symmetry G, which is an important and robust datum. Indeed, there are (at least) two highly non-trivial facts related to 't Hooft anomalies. First, they are invariant under continuous deformations of the theory, being a motion in a conformal manifold, an RG flow, or any continuous deformation. Second, a non-trivial 't Hooft anomaly is incompatible with a trivial theory. These two facts have a strong consequence: a weakly coupled theory with a 't Hooft anomaly, if deformed and driven to strong coupling, cannot flow to a trivial IR. Moreover, certain anomalies are incompatible with certain IR theories, for instance, perturbative anomalies of continuous groups require a gapless sector. Hence in such cases, the theory cannot develop a mass-gap. This highly non-

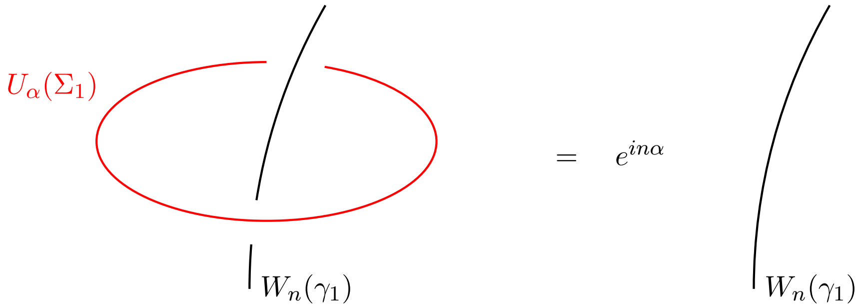


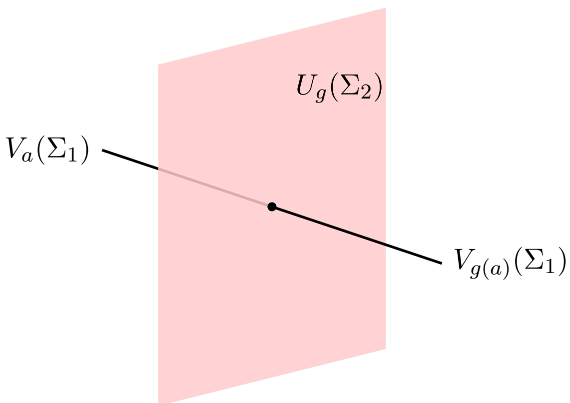
Figure 2: The action of a $U(1)$ 1-form symmetry in three-dimensions. The topological operator is labelled by $\alpha \sim \alpha + 2\pi$ and supported on a 1-dimensional manifold Σ_1. It can act on a 1-dimensional line operator $W_n(\gamma_1)$ of charge $n \in \mathbb{Z}$, multiplying it by a phase.

trivial result is a modern form of 't Hooft anomaly matching [12], and follows from the topological nature of the operator $U_g(\Sigma_{d-1})$.

While ordinary global symmetries lead to topological operators, the topological sector of QFT is all but exhausted by them. Starting from [11], the intense activity of the last years focused on identifying other types of topological operators in QFT, understanding the structure of the topological sector, and extracting the pieces of information that lead to consequences for the dynamics. These more general topological operators are commonly called *generalized global symmetries*, even if the word *symmetry* is more appropriate for some of them than for others[2].

The generalizations are of various levels of complication. The simpler one concerns topological operators still labeled by some group G, but whose support is of higher co-dimension $p+1$. These are called higher-form symmetries of degree p, or $p-$form symmetries. Ordinary symmetries are then $0-$form symmetries. If $p > 0$, a p-form symmetry cannot act on local operators, but on operators of dimensionality at least p. The action of a p-form symmetry defects on a $p-$dimensional operator is by *linking* (see fig. 2). In fact, in a $d-$dimensional manifold, submanifolds whose dimensions sum up to $d-1$ can be geometrically linked. Moreover, it turns out that higher-form symmetries must be Abelian.

If a theory has various higher-form symmetries of different degrees, the corresponding topological operators can have a non-trivial interplay among themselves. For instance, the topological operators of a 1-form symmetry can be permuted when they cross a co-dimension one topological defect of a 0-form symmetry:



[2] *All the symmetries are the same, but some are more symmetric than others.*

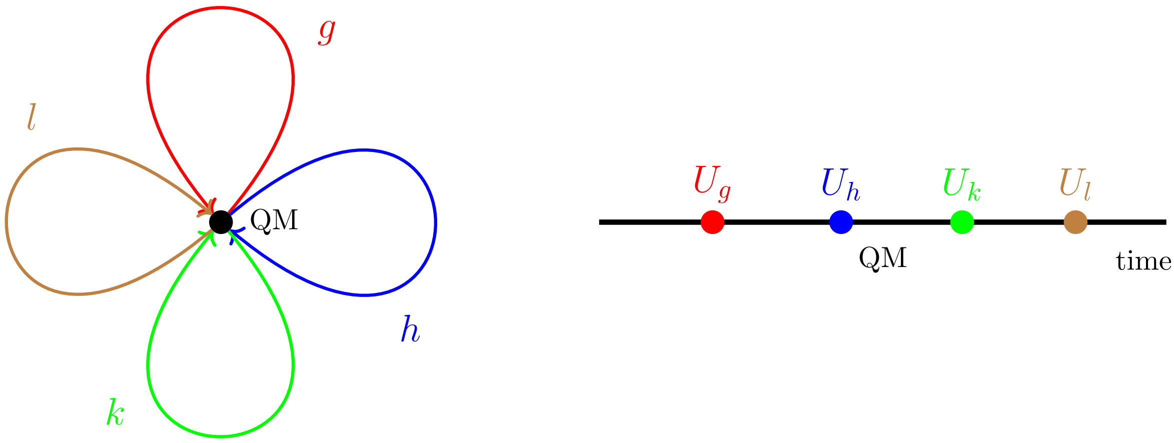


Figure 3: Left: categorification of a group G in terms of a category with a unique object (that can be thought of as a quantum mechanical system) and endomorphisms labelled by group elements. Right: the realization of this structure on a (0+1)-dimensional theory, namely a quantum mechanical system, where the endomorphisms of the category are topological local operators.

There can be more intricate interplay arising from the co-existence of higher-form symmetries of various degrees, forming a structure called *higher-group* [13–16], that is well known in mathematics [17].

This is the simplest non-trivial appearance of *higher-categorical* structures which will play an important role. Higher categories are algebraic structures that abstract ordinary notions of groups, algebras, and so on, allowing very interesting generalizations. Roughly speaking a category $\mathcal{C}$ is a collection of *objects* $a \in \mathrm{Ob}(\mathcal{C})$ together with maps among them $f : a \mapsto b$, called *morphisms*, that can be composed. A group G can be understood as a category with a unique object, with endomorphisms labelled by $g \in G$, and whose composition is governed by the group law of G. In particular all morphisms are invertible, namely are isomorphisms. This is how one may like to think group-symmetries in Quantum Mechanics (QM): the unique object is the QM system itself, and the group elements act on it. In QFT language a QM is a (0+1)-dimensional theory, the symmetry operators are topological local operator, hence being points they do not have higher structures (see fig. 3 for an illustration). That is the reason why they are described by category theory, as opposite to higher-category theory.

In a $(D+1)-$dimensional QFT the topological operators are extended, so they can have higher-codimension topological operators placed on it, or more generally there can be topological interfaces between two difference symmetry group operators. They can be understood as morphisms between morphisms, or *2-morphisms*. These are the building blocks of a higher-category: an algebraic structure with objects whose morphisms form itself a category with 2-morphsims among them, which again can form a category with 3-morphisms among them and so on. An $n-$category is such that there are morphisms up to degree n. A 0-form group symmetry G in a d-dimensional QFT can be thought of as a d-category with a unique object (the QFT itself), and 1-morphisms labeled by $g \in G$ that compose according to the group law.

This structure has room to accommodate 1-form symmetry defects that are of dimension $d-2$. These are 2-(endo)morphisms of the identity 1-morphism, and are composed according to the group law of an Abelian group $\mathbb{A}$. We can then go higher by introducing 2-form symmetry operators as 3-endomorphisms, and so on up to $(d-1)$-form symmetries that are given by topological local operators. The 2-endomorphisms of the identity 1-morphism must induce 2-endomorphisms of any other 1-morphism: codimension two defects can be pushed on top of codimension one defects.

A mathematical structure putting together various higher-form symmetries is called a higher-group, whenever all morphisms are invertible, hence each set of (higher)endomorphism of the identity form some group. The possible non-trivial interplay among the various layers arise from the various associativity conditions. For instance taking three 0-form symmetry defects U_g, U_h, U_k, we may compose $(U_g U_h)U_k$ or $U_g(U_h U_k)$ and they could not coincide, but being related a non-trivial 2-morphism, namely a 1-form symmetry defect $\beta(g, h, k) \in \mathbb{A}$. If this is non-trivial we have an interesting interplay between the 0-form and the 1-form symmetry.

Let us pause for a moment to clarify some terminology. As we discussed, the higher-form symmetries of a d-dimensional QFT combine in d-category with a unique object. These categories are called *monoidal*. The unique object is the QFT itself. This is a trivial information that does not play a role. For this reason it is customary to just look at the $(d-1)$-category of endomorphisms, whose objects are the 0-form symmetry defects. Since these objects started their life as morphisms, they inherit an additional structure from the composition, which is called a *tensor product*: monoidal d-categories are often called *tensor* $(d-1)$-categories, and it is often stated that that *symmetries of a d-dimensional QFT are given by a* $(d-1)$*-category*. Mathematicians have mostly studied higher-categories with finitness properties, called *fusion categories* and are suitable for describing finite symmetries in QFT. The terminology can be somewhat confusing for physicists, since the world *fusion* does not really have to do with the possibility of fusing the objects, which remains true even for continuous symmetries.

At this point, one may ask if there is some reason why the topological sector of QFTs must be described by these special types of higher categories governed by various layers of group symmetries. Indeed, there is no such reason, and we can look for general tensor $(d-1)$-categories where some morphism does not have an inverse. After all, we are looking for the most general robust properties, and these only depend on having topological defects that can be fused and combined in various ways. These more general structures are often called *non-invertible symmetries* by physicists, even if the word *symmetry* is probably not appropriate here. *Symmetry* is a word used for some action on some system that does not change its fundamental properties (e.g. a rotated ball is still a ball, not an ellipse) and is always reversible. This property does not apply to non-invertible symmetries whose action is not reversible and often modify some very fundamental properties. For instance, spontaneously broken non-invertible symmetries can map one vacuum to a physically distinguishable one. Non-invertible symmetries are really associated with some conservation law (namely a topological operator) that does not come from any symmetry of the theory. For this reason, it would be much better to call them *non-invertible topological operators*, or *non-symmetries*. Nevertheless, we will use the terminology *non-invertible symmetries* since it is now standard in the physics community.

Non-invertible symmetries described by fusion (1-)categories are well known to exist in two-dimensional Rational Conformal Field Theories (RCFT) since many decades and are provided by Verlinde lines [18]. However, only in recent years have they been recognized to fit within the framework of generalized symmetries [19–21]. Many people started to study their structure, 't Hooft anomaly, gauging, spontaneous breaking, and all that. This program proved to be extremely successful even for non-conformal theories obtained by deforming some RCFT with a symmetry preserving relevant operator, resulting in striking dynamical predictions [22].

The developments in higher dimensions are much more recent, starting with [23–29], constituting a very active area of current research. One side of the activity puts a lot of effort into understanding the general structure of the topological sector of QFTs. The broad question in this business is how to classify and compute all possible robust data associated with a generalized symmetry structure. An other side of the activity concerns deriving physically relevant consequences from these structures. Of course, the two analyses are all but unrelated.

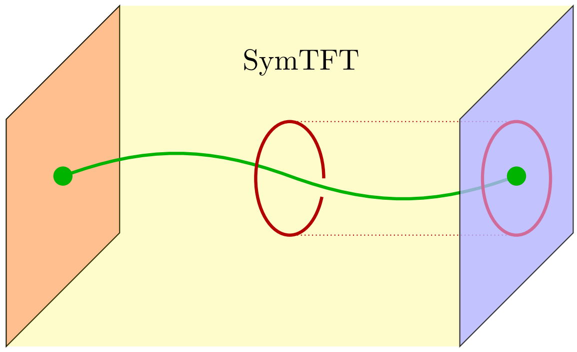
A fundamental role in this story is played by Topological Quantum Field Theories (TQFTs). Roughly speaking, a TQFT is a theory with a vanishing stress-energy tensor. Since the stress-energy tensor generates all space-time symmetries, in a TQFT all operators are topological. In a sense, a TQFT is completely determined by its symmetries and for this reason can be rigorously defined. Hence, the study of TQFTs is an important and active area of mathematics, which is helpful for physics, since we can use many results that mathematicians proved for us. For instance, in sufficiently low dimensions, TQFTs have been essentially classified. There are many reasons why TQFTs are relevant for non-topological theories. First, they describe the infrared limit of gapped phases. Second, a QFT with finite symmetry $\mathcal{C}$ can be coupled with a TQFT with the same symmetry by stacking the two theories and gauging the diagonal symmetry [30]. This operation modifies the global properties of the original theory, for instance changing the number of vacua on non-trivial space-manifolds. Similarly, in a d-dimensional QFT with a higher-form finite symmetry we can construct further topological defects by staking on some submanifold a lower-dimensional TQFT with the same finite symmetry, whose degree is such that the dimensionality of the topological operators match, and again performing a diagonal gauging on the submanifold [26, 31].

However, there is an other application, much more subtle but also much more general, of TQFTs to non-topological theories. This has really to do with the fact that symmetries are identified with topological operators, which intuitively are those that do not *interact* with the stress-energy tensor. One may then be tempted to conjecture that the full set of topological operators of any given d-dimensional QFT produces a consistent d-dimensional TQFT that captures all the properties of the symmetries. However, this is not true for two distinct reasons. First, standard TQFTs as rigorously defined and studied by mathematicians can only capture finite symmetries, while general QFTs also have continuous symmetries. We will see how a new type of TQFTs (Appendix E) can also capture continuous symmetries in Chapter 8, reviewing [5], but for the moment we restrict to finite symmetries. However, the temptation above is not valid even with this restriction, for the very basic reason that among the important robust data relevant for the topological sector, is the interplay of topological operators with non-topological charged operators.

It turns out that a TQFT capturing completely the data associated with the topological sector of d-dimensional QFT exists, but it lives in $d+1$ dimensions. In the high-energy community it is called the *Symmetry Topological Field Theory* (SymTFT) [11, 32–36], while the same concept appears both in mathematics and in condensed matter theory with other names[3]. Roughly speaking, the d-dimension QFT we want to study lives at the boundary of the SymTFT. An intuitive reason why we need to go one dimension higher is the following. Both the topological defects of a $p-$form symmetry, and the charged operators, arise from topological operators of the TQFT. But while the topological ones are just $d-p-1$ dimensional operators pushed at the boundary, for the charged operators we need

[3] The first construction was given for a 3d TQFT associated with a fusion category in mathematics by Turaev and Viro [37, 38], and successively extended by Barrett and Westbury [39, 40], hence it known as Turaev-Viro-Barrett-Westbury theory. Later on the same technique was used to construct TQFTs in various dimensions from the datum of a fusion category in one dimension less, and these construction are generically known as *state-sums*. In the condensed matter community, on the other hand, the SymTFT is often called *bulk-boundary correspondence*, or *symmetry-topological order correspondence*, generalizing one original construction by Levin and Wen [41].

some mechanism to make them non-topological. The mechanism is that the $p-$dimensional charged operator is the end-surface of a $p+1$ dimensional topological defect of the TQFT. We then notice that a $d-p-1$ and a $p+1$ dimensional operators cannot link in $d-$dimensions, but they can in $d+1$. More precisely the set-up is the following:



The TQFT is placed on a slab with two diffeomorphic boundaries. One supports the physical QFT we are interested in but is coupled to the TQFT in the bulk. A common terminology is to call this QFT a *relative theory*, since once we couple it to a higher-dimensional TQFT it no longer exists on its own (is not *absolute*) but it is relative to the TQFT. The precise statement is that it does not have a unique partition function, but a vector of them. To get a unique partition function and recover an absolute theory, one needs to project the partition vector onto another given vector. This is realized by adding the other boundary, which is a topological boundary condition for the TQFT.

Given the original higher category $\mathcal{C}$ that we want to describe, we get an additional structure on the TQFT. We may think of this additional structure as a *presentation* of the TQFT (a duality frame). More precisely, the additional structure is the datum of a *canonical* topological boundary condition, which projects the partition vector on the partition function of the theory we started with. The topological boundary condition is equivalent to the prescription of which bulk defects can end on it. These defects are trivialized if we push them parallel to the topological boundary. All the other defects can be pushed on the boundary remaining non-trivial but are identified modulo those that can end. On the other hand, all bulk defects can end consistently at the physical boundary, but their end points are generically non-topological.

The slab is topological, and we can compactify it with the cost of (at most) a normalization factor. This operation reproduces the absolute theory, where the topological sector is provided by the bulk defects that are not trivialized by the topological boundary, while the charged operators are obtained from the bulk defects that end on both boundaries, as in the picture above.

Then we can start playing with the topological boundary conditions, modifying them away from the canonical one. This operation reshuffles which bulk defects can end and which stay non-trivial after the slab compactification. The absolute theories we get will be different versions of the original one, sharing the same local dynamics but different global properties [34]. They are obtained one from the other by one of the operations we described above: coupling some of original theory with a TQFT by the diagonal gauging of some finite symmetry. These are often called *topological manipulations*, and these different versions are usually called *global variants* [42]. They have different symmetries related by some type of generalized gauging, but they all share the SymTFT. The latter is indeed the full invariant of this equivalence class of symmetries, called *Morita class*.

The SymTFT is very powerful since it contains not only the full information on the topological sector of a QFT, but also those related by topological manipulations. This observation is the starting point that allows to detect 't Hooft anomalies of all possible generalized symmetries (including the non-invertible ones) using the SymTFT. In fact, anomalies are, roughly speaking, the obstruction in performing certain topological manipulations, so they can be computed by studying the topological boundary conditions of the SymTFT [3, 5, 43–45]. This is a very important step, since it allows one to extract the robust data relevant for constraining the dynamics, from a part of the topological sector that does not come from ordinary symmetries, and hence it would not be accessible by other (known) methods.[4]

The SymTFT allows us to completely disentangle the topological sector from the dynamics, and concretely provides a framework where one can use tools from topological quantum field theories in non-topological theories. In a sense, it replaces the complicated study of higher-categories and their action on QFTs, with the simpler problem of analyzing TQFTs. Moreover some non-invertible symmetries that cannot be related with invertible symmetries, can be actually understood in terms of invertible symmetries of the SymTFT, simplifying drastically the analysis [2, 3, 46]. Least but not last, the SymTFT provides a classification of the possible non-invertible symmetries of d-dimensional QFTs in terms of classification of d+1 dimensional TQFTs. For instance it provides an explanation of the richness of topological defect lines of 2d RCFT in terms Wilson lines of 3d Chern-Simons theory, through the 2d/3d correspondence [47–50], while it rules out finite 0-form non-invertible symmetries in 4d other than the known ones [45] by using a classification result for 5d TQFTs [51].

This thesis is organized as follows. Chapter 1 is a review of some basic concepts in generalized global symmetries, not including non-invertible topological operators. Before introducing them in Chapter 3, it is necessary to give an overview on ideas and techniques of TQFTs in Chapter 2. The background material concludes with Chapter 4 that introduces the SymTFT construction.

The original work of the author is presented in the Chapters 5 6 7 8 9. In Chapter 5, based on [2], we study how non-invertible self-duality defects arise in theories with a holographic dual. One important observation is that the supergravity theory contains a topological sector that coincides with the SymTFT. We will use this to deduce the SymTFT for certain non-invertible defects (confirming also the finding of [52]), and use it to derive many properties holographically, and in a controlled setup.

In Chapter 6, which reviews [3], we use the SymTFT derived holographically in Chapter 5 to study anomalies of non-invertible symmetries. We do this both in 2d, where some results were already available and we could compare with them, and in 4d where the results are new.

The study of anomalies can be viewed as understanding which symmetries are compatible with a trivial gapped phase, technically a Symmetry Protected Topological (SPT) phase. Recently, it was realized that, at least in two space-time dimensions, these phases can also exist in gapless systems, leading to the notion of gapless SPT (gSPT) phases. Some of them do not have an analog in gapped systems, hence are *intrinsically gapless* (igSPT). Chapter 7 is based on the recent work [4], and shows that we can use the SymTFT approach to understand these (intrinsically) gapless topological phases in a more general way, which is not tight to two dimensions. Hence, we generalize them to three and four dimensions. In the last case, in particular, igSPT phases acquire a beautiful physical interpretation as obstruction to confinement in 4d gauge theories.

In Chapter 8 we review [5] where we extend the formalism of the SymTFT, that traditionally is restricted to finite symmetries, to include continuous symmetries. The SymTFT describes the structure of the symmetry, its anomalies, and possible topological manipulations. One should notice that the bulk theories used here are beyond the standard realm of TQFTs as introduced in Chapter 2, and

[4]In 2d, thanks to the well-developed theory of fusion categories, alternative methods are often available. However they are hard to generalize to higher dimensions.

we provide a hint toward their mathematical definition in appendix E. We also propose an operation that produces the SymTFT for the theory obtained by dynamically gauging the $U(1)$ symmetry, and discuss many examples. Finally we propose that the various SymTFTs of theories related by dynamical gauging can be realized as different boundary conditions of a unique $d+2$ dimensional TQFT, that is a dynamical version of the anomaly polynomial.

In the final Chapter 9, that is based on [6], we propose a different interpretation of the TQFTs usually employed in the SymTFT business. Focusing on continuous symmetries, we propose that these TQFTs can be viewed as theories of gravity, holographically dual to the universal effective field theory (EFT) that describe spontaneous symmetry breaking on the boundary. This provides a concrete and controlled model for holography, where the boundary dynamics arises from edge modes of the bulk, and can be viewed as a generalization of the CS/WZW correspondence to higher dimensions and higher symmetries. We also comment on some expectations of how to incorporate this framework as the limit of standard holography with dynamical gravity.

Chapter 1

Topological operators in quantum field theory

In this chapter, we explain that symmetries in QFT give rise to topological operators [11]. We argue that the latter can be taken as an intrinsic definition and can be used to generalize the notion of symmetry. The only generalization discussed in detail in this chapter is *higher-form symmetries*, which act on extended operators, leaving more drastic generalizations for the next chapters. We introduce several basic concepts such as anomalies, inflow, gauging, and so on, that are common to this generalized notion of symmetry, and we discuss several examples.

1.1 Symmetries as topological operators

Consider a $d-$dimensional local Euclidean QFT. The theory comes with a family of Euclidean operators (or defects) $\mathcal{O}(x), L(\gamma_1), S(\Sigma_2), ...$ labeled by submanifolds of various dimensionality. The theory can be put into an arbitrary Riemannian manifold X_d (we denote by $\text{vol}(X_d)$ its volume form) and produces Euclidean correlation functions, which can include local and extended operators. Suppose that the theory has a connected Lie group global symmetry G with conserved currents $J^a(x) = J^a_\mu(x)dx^\mu$ (a is a Lie algebra index, with generators T_a). The currents satisfy Ward identities

$$\Big\langle d * J^a(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}(x_n)\Big\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\,\text{vol}(X_d)\,\Big\langle\mathcal{O}_1(x_1)\cdots\delta_a\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\Big\rangle \qquad (1.1.1)$$

where $\delta_a\mathcal{O}_i$ is the action of the Lie algebra vector T_a on $\mathcal{O}_i$. Assuming $\mathcal{O}_i$ is in an irreducible representation $\mathfrak{R}_i$, $\delta_a\mathcal{O}_i = i\mathfrak{R}_i(T_a)\cdot\mathcal{O}_i$.

The Ward identities can be rewritten in a finite form as follows [11]. We take an open region $D_d \subset X_d$ whose boundary is a compact, homologically trivial submanifold Σ_{d-1} of codimension one, and we integrate (1.1.1) on D_d. Defining the extended codimension one charge operator

$$Q^a(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} *J^a(x) \qquad (1.1.2)$$

the result is

$$\Big\langle Q^a(\Sigma_{d-1})\mathcal{O}_1(x_1)\cdots\mathcal{O}(x_n)\Big\rangle = \sum_{i=1}^{n}\text{Lk}(\Sigma_{d-1},x_i)\,\Big\langle\mathcal{O}_1(x_1)\cdots\mathfrak{R}_i(T_a)\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\Big\rangle\,. \qquad (1.1.3)$$

We introduced the *linking number*

$$\text{Lk}(\Sigma_{d-1},x) = \begin{cases} 1 & \text{if } x\in D_d \\ 0 & \text{if } x\notin D_d \end{cases} \qquad (1.1.4)$$

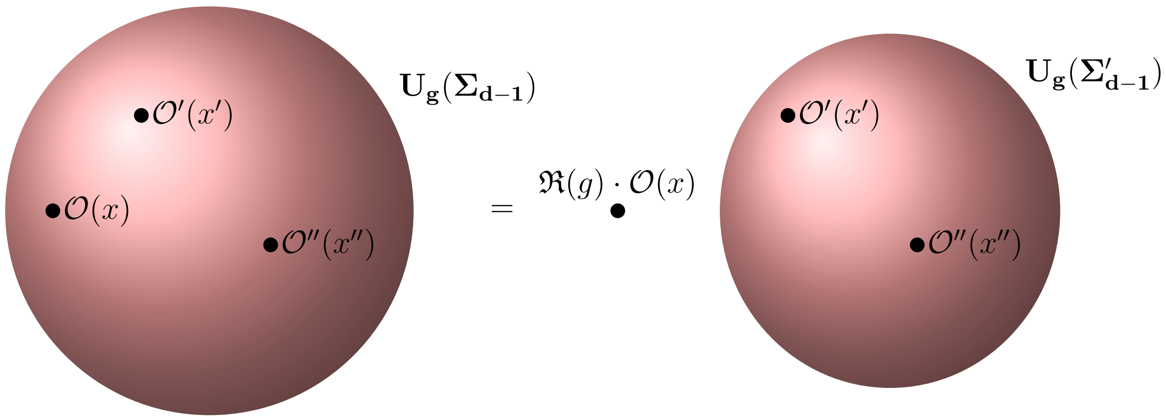


Figure 1.1: Graphical representation of the Ward identity.

In general, we can consider $\mathrm{Lk}(\Sigma_p, \Sigma_q)$ for any homologically trivial submanifolds, where:

$$p + q = d - 1 \ . \tag{1.1.5}$$

It is defined as the number of intersections of Σ_q with a $p+1$ dimensional filling of Σ_p, or vice versa[1]. For $q = 0$, the result depends on the orientation of Σ_{d-1}.

The linking number is topological: it is invariant under any deformation of one of the submanifolds, provided that we do not allow them to intersect. In the case $p = d-1, q = 0$ considered here, this means that we can deform Σ_{d-1} in any way, as long as the point x stays inside. Then (1.1.3) tells us that $Q^a(\Sigma_{d-1})$ is a *topological operator*: correlation functions with its insertions depend only topologically on the support, but are invariant under small deformations of it. When we move the support from Σ_{d-1} to Σ'_{d-1} crossing exactly one point, say x_1, we have

$$\Big\langle Q^a(\Sigma'_{d-1})\mathcal{O}_1(x_1)\cdots\mathcal{O}(x_n)\Big\rangle - \Big\langle Q^a(\Sigma_{d-1})\mathcal{O}_1(x_1)\cdots\mathcal{O}(x_n)\Big\rangle = \Big\langle \mathfrak{R}_1(T_a)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\Big\rangle \ . \tag{1.1.6}$$

There is one more step we can take. Given any group element $g = e^{i\epsilon_a T_a} \in G$, we construct

$$U_g(\Sigma_{d-1}) = \exp\left(i\epsilon_a Q^a(\Sigma_{d-1})\right) \ . \tag{1.1.7}$$

By expanding it, and using repeatedly (1.1.3), we get

$$\Big\langle U_g(\Sigma_{d-1})\mathcal{O}_1(x_1)\cdots\mathcal{O}(x_n)\Big\rangle = \Big\langle \mathfrak{R}_1\left(g^{\mathrm{Lk}(\Sigma_{d-1},x)}\right)\cdot\mathcal{O}_1(x_1)\cdots\mathfrak{R}_n\left(g^{\mathrm{Lk}(\Sigma_{d-1},x)}\right)\cdot\mathcal{O}_n(x_n)\Big\rangle \ . \tag{1.1.8}$$

This relation, that is the finite-group form of the Ward identity, tells that the insertion of the topological operator $U_g(\Sigma_{d-1})$ can be removed at the cost of transforming all the local operators inside Σ_{d-1}. More locally, we may express this as the property that, if we deform the support passing through one local operator, we act on it with g, as summarized in figure 1.1. In formulae we write

$$U_g(\Sigma_{d-1})\ \mathcal{O}(x) = \mathfrak{R}_{\mathcal{O}}(g)\cdot\mathcal{O}(x)\ U_g(\Sigma'_{d-1}) \ . \tag{1.1.9}$$

The reason why formulating the Ward identities in terms of $U_g(\Sigma_{d-1})$ instead of $J^a(x)$ or $Q^a(\Sigma_{d-1})$ is preferable, is that this makes sense also for discrete symmetries. In that case, there is no current; hence $U_g(\Sigma_{d-1})$ will not always have an explicit expression, but the Ward identity formulated as (1.1.8) can be taken as an intrinsic definition of a discrete symmetry action in the QFT. The punchline is that

- Any global group symmetry in QFT leads to codimension-one topological operators labeled by $g \in G$.

[1] The result is independent of this choice

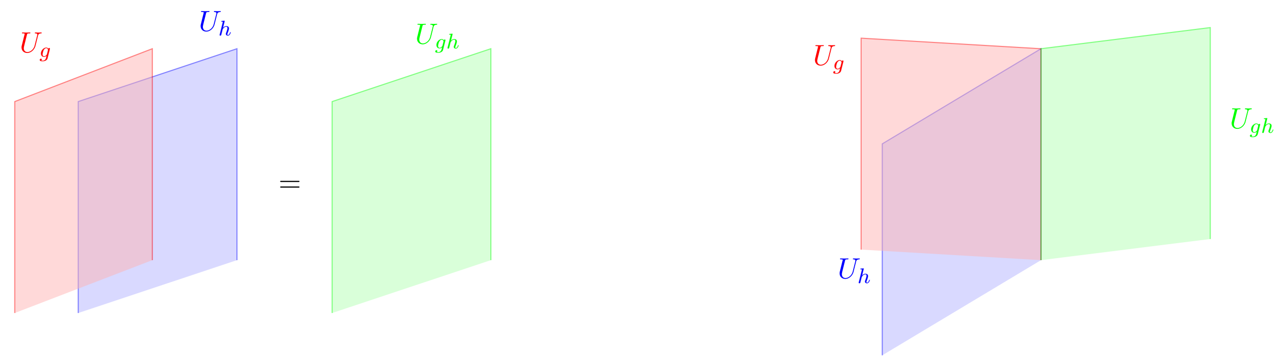


Figure 1.2: Left: parallel fusion of two topological operators labelled by g and h, producing a topological operator labelled by gh. Right: triple junction among topological operators, obtained by topologically deforming the parallel configuration on the left and using the fusion property.

A crucial property of topological operators is that they are not subject to short-distance divergences. Therefore, there is a perfectly well defined notion of products among them: two or more topological operators can be fused. From (1.1.8) it follows that

$$U_g(\Sigma_{d-1})U_h(\Sigma_{d-1}) = U_{gh}(\Sigma_{d-1}) \ . \tag{1.1.10}$$

This corresponds to the possibility of performing parallel fusion, as in the left picture in figure 1.2. For this picture to make sense we need to choose an orientation of the codimension one surface inside the space-time.

The configuration in the left picture of 1.2 can be topologically deformed to the one on the right of the figure 1.2, by performing the parallel fusion only in half of the surface. This configuration has a *junction* of codimension two where two topological defects meet and become a third topological defect obtained by fusion. This operation, however, is all but trivial, since we are now placing our defects on singular manifolds. What happens on the singular locus, the junction, is not completely specified by what happens outside of it; hence, we should specify further data to completely characterize the symmetry. For invertible symmetries, this leads to the notion of *symmetry fractionalization* [15, 53, 54] (see Section 1.5 for a review), or even more drastically to higher-groups [15] while it will result in a richer structure for more general non-invertible categorical symmetries. For the moment the only important thing to notice is that the right picture in figure 1.2 contains more information than the left picture.

1.2 Anomalies and inflow

A natural operation with global symmetries is to couple them to background gauge fields: if the system has a global symmetry G we can introduce a principle $G-$bundle with connection A, which is coupled with the QFT. For a continuous symmetry we have conserved currents J^a and the coupling is achieved by adding a term in the action

$$S[A] = S + i\int_{X_d} A_a \wedge *J^a \ . \tag{1.2.1}$$

Consider the $U(1)$ case for simplicity. The gauge field is subject to gauge invariance $A \mapsto A + d\lambda$. For flat backgrounds, the coupling $A \wedge *J$ can be interpreted as the insertion of a topological defect configuration. Indeed, by Poincare' duality a closed 1-form can be written as $A = \alpha\delta^{(d-1)}(\Sigma_{d-1})$, with

$\alpha \sim \alpha + 2\pi$ representing the holonomy on a closed curve that crosses Σ_{d-1} once. Hence the coupling with the current is equivalent to the insertion of

$$U_\alpha(\Sigma_{d-1}) = \exp\left(i\alpha \int_{\Sigma_{d-1}} *J\right) \tag{1.2.2}$$

in the path integral, with α determined by the holonomy of the gauge field. A gauge transformation $A \mapsto A + d\lambda$ modifies Σ_{d-1} into a homologous support, and this does not change the correlation functions because the operator is topological. Equivalently, gauge invariance is a consequence of the conservation $d * J = 0$.

There are interesting situations where, while this conservation equation holds in the absence of background fields (hence the theory has the symmetry), it is broken as soon as the symmetry is coupled to the background, and there is no way to rescue this. Hence, even if the theory has a symmetry, it is impossible to couple it to a background field that preserves the invariance of the partition function under background gauge transformations: the theory has an *'t Hooft anomaly* [12]. A familiar textbook example is the theory of a 4d free massless Weyl fermion. The 0-form $U(1)$ symmetry rotating the fermion is associated with a current J_μ, and famously the conservation equation is violated in the three-point function of the current by a contact term, which in momentum space is

$$p_1^\mu \Big\langle J_\mu(p_1)\, J_\nu(p_2)\, J_\rho(p_3)\Big\rangle = \frac{i}{16\pi^3}\, \epsilon_{\nu\rho\xi\eta}\, p_2^\xi\, p_3^\eta \;. \tag{1.2.3}$$

Equivalently, when the current is coupled with a background field A, since the latter is a source for the current, the conservation equation is violated by an A dependent term:

$$d * J = \frac{i}{8\pi^2} F \wedge F \;. \tag{1.2.4}$$

This means that the partition function coupled with A is not gauge invariant under $A \mapsto A^\lambda = A + d\lambda$, but it takes a phase expressed as a local integral

$$Z[A] \mapsto Z[A^\lambda] = \exp\left(\frac{i}{8\pi^2}\int_{X_4} \lambda F \wedge F\right) Z[A] \;. \tag{1.2.5}$$

We may try to fix this by modifying the action with a local functional of the background field $S_\text{ct} = \int_{X_4} \mathcal{L}_\text{ct}(A)$, namely a *local counterterm*, such that $\delta_\lambda S_\text{ct} = -\frac{i}{8\pi^2}\int_{X_4} \lambda F \wedge F$. It is easy to see that there is no such local counterterm. However, if we view X_4 as the boundary of a 5d manifold X_5, extending A to the bulk, we can write a local functional

$$S_\text{inflow}(A) = \frac{i}{24\pi^2}\int_{X_5} A \wedge dA \wedge dA \;, \tag{1.2.6}$$

whose gauge variation is a boundary term exactly cancelling the anomalous variation. This mechanism is called *anomaly inflow* [55, 56].

The exponentiated action $Z_\text{inflow}[A] = e^{-S_\text{inflow}}$ is a phase that can be thought of as the partition function of a 5d TQFT on X_5, coupled to a background A for a $U(1)$ symmetry. Since this is a phase, the TQFT is trivial: the Hilbert space on any 4-manifold is one-dimensional. Technically we say that the TQFT is *invertible* [57, 58], but it has a global $U(1)$ symmetry. In the condensed matter literature this is called a Symmetry Protected Topological (SPT) phase [59]. The combined 4d/5d system consisting of a 4d Weyl fermion coupled with the 5d SPT is perfectly gauge invariant when we couple the $U(1)$ symmetry to A.

The fact that anomalies of global symmetries can be canceled by coupling the system to a higher-dimensional trivial TQFT by inflow is very deep and powerful. In fact, TQFTs are rigid: their space is discrete, and in particular invertible TQFTs are classified, in the bosonic case by group cohomology [59], and more generally by cobordism [60, 61]. Hence, deformations of the d-dimensional QFT that preserve the symmetry cannot change the value of the anomaly, simply because the deformations are continuous and the anomaly cannot jump. Moreover, TQFTs are clearly renormalization group (RG) invariant; hence 't Hooft anomalies are also RG invariant. This is essentially a modern version of the 't Hooft argument about anomaly matching [12]. This is one of the facts that makes the study of global symmetries incredibly powerful for studying the dynamics of QFTs: if we compute the anomalies of a UV theory, any symmetry-preserving relevant deformation generates an RG flow whose IR fixed point must reproduce the same anomaly. This puts severe constraints on possible strongly coupled RG flows. Moreover, reformulating anomaly matching in terms of anomaly inflow is a huge step forward. As we shall see shortly, the anomaly inflow paradigm holds also for the generalizations of global symmetries we are going to consider, hence it allows us to derive additional constraints.

There are two general disbeliefs about anomalies. One is that they only exist in fermionic systems, and the other is that they necessarily have to do with some subtle issue concerning the regularization of the path integral measure. While it is true that anomalies of fermionic systems often come from the path integral measure, anomalies can very well exist in bosonic systems and they can show up in much more elementary fashion. The simplest example is a 2d compact boson

$$S = \frac{R^2}{4\pi} \int_{X_2} d\Phi \wedge * d\Phi \ , \qquad \Phi \sim \Phi + 2\pi \ . \tag{1.2.7}$$

This theory has two $U(1)$ symmetries, denoted respectively by $U(1)_M$ and $U(1)_W$. The momentum (or shift) symmetry $\Phi \mapsto \Phi + \alpha$ and the winding (or topological) symmetry that measures how many times Φ winds around a 1-cycle. The currents are respectively

$$J_M = \frac{iR^2}{2\pi} d\Phi \ , \qquad J_W = * \frac{d\Phi}{2\pi} \ . \tag{1.2.8}$$

While J_M is conserved because of the equations of motion, J_W is conserved off-shell. For this reason, the winding symmetry is sometimes called a *topological symmetry*: it follows from the topology of the field space.

From the currents we can construct the topological operators

$$U_\alpha(\Sigma_1) = \exp\left(-\frac{R^2 \alpha}{2\pi} \int_{\Sigma_1} * d\Phi \right) \ , \qquad V_\beta(\Sigma_1) = \exp\left(i\beta \int_{\Sigma_1} \frac{d\Phi}{2\pi} \right) \ . \tag{1.2.9}$$

They act respectively on the vertex operators $\mathcal{O}_n(x) = e^{in\Phi(x)}$ and on vortices $H_w(x)$, whose insertion in a correlation function is obtained by path integrating over singular field configurations that wind $w \in \mathbb{Z}$ times around x.

We can follow the standard prescription to couple these symmetries to background fields. Let us start with $U(1)_M$. By adding the current-gauge field coupling $iA \wedge * J_M$ we modify the action to $\frac{R^2}{4\pi} \int (d\Phi \wedge * d\Phi - 2A \wedge * d\Phi)$. However, this is not gauge invariant since J_M itself changes under gauge transformations $J_M \mapsto J_M + \frac{iR^2}{2\pi} d\alpha$. This issue does not represent an anomaly, since it can be fixed by adding a local counterterm $\frac{R^2}{4\pi} A \wedge *A$ obtaining a new action[2]

$$S[A] = \frac{R^2}{4\pi} \int (d\Phi - A) \wedge *(d\Phi - A) \ . \tag{1.2.12}$$

Coupling the compact boson to a background B for $U(1)_W$ is smoother: we simply have the action

$$S[B] = \frac{R^2}{4\pi} \int d\Phi \wedge *d\Phi + i \int \frac{d\Phi}{2\pi} \wedge B \tag{1.2.13}$$

and this is gauge invariant for $B \mapsto B + d\beta$. The troubles come if we try to couple the full $U(1)_M \times U(1)_W$ to backgrounds A and B. Indeed the action

$$S[A, B] = \frac{R^2}{4\pi} \int (d\Phi - A) \wedge *(d\Phi - A) + \frac{i}{2\pi} \int d\Phi \wedge B \tag{1.2.14}$$

fails to be gauge invariant for $\Phi \mapsto \Phi + \alpha$, $A \mapsto A + d\alpha$ because of the $B-$coupling. We can try to fix this by adding a counterterm proportional to $A \wedge B$:

$$S'[A, B] = \frac{R^2}{4\pi} \int (d\Phi - A) \wedge *(d\Phi - A) + \frac{i}{2\pi} \int (d\Phi - A) \wedge B \ . \tag{1.2.15}$$

This restores gauge invariance for the momentum symmetry, but is not gauge invariant for $B \mapsto B + \mathrm{d}\beta$.

The final result is the same as in the massless Weyl fermion: the theory has a $U(1)_M \times U(1)_W$ symmetry but there is no way to couple it to background fields without breaking the symmetry. This is again a 't Hooft anomaly. This time is a mixed anomaly because it involves two symmetry, each of them by its own being non-anomalous. The origin of the phenomenon appears to be different with respect to the massless Weyl case. There are no fermions here, and there is no issue with the regularization of the path integral measure, but the phenomenon is definitely the same.

Also in this case the only way to turn on consistently both backgrounds is to add an invertible TQFT in one dimension higher that cancels the anomaly by inflow. If we define the coupling as in (1.2.14) we are lacking gauge invariance for $U(1)_M$ as $\delta S[A, B] = \frac{i}{2\pi} \int d\alpha \wedge B$, and this is cancelled by the following inflow action in 3d:

$$S_{\text{inflow}} = \frac{i}{2\pi} \int_{X_3} A \wedge dB \ . \tag{1.2.16}$$

This action is gauge invariant on closed manifold, but in presence of a boundary fails to be gauge invariant for $A \mapsto A + d\alpha$ by a boundary term that exactly cancel the one of the 2d theory. If we define the coupling as in (1.2.15) we have $\delta S'[A, B] = -\frac{i}{2\pi} \int A \wedge d\beta$ that is cancelled by the inflow action

$$S'_{\text{inflow}} = \frac{i}{2\pi} \int_{X_3} dA \wedge B \ . \tag{1.2.17}$$

We see that S_{inflow} and S'_{inflow} only differ by a boundary term (they are obtained one from the other integrating by parts), that is precisely the local counterterm $A \wedge B$ we added in the 2d theory.

[2]Equivalently this can be understood as a modification of the definition of the current when the background is turned on. Indeed the current can be in general defined through the functional derivative of the action with respect to the background as

$$*J = -i \frac{\delta S[A]}{\delta A} \ . \tag{1.2.10}$$

Hence the new current is

$$J'_M = \frac{iR^2}{2\pi} (d\Phi - A) \tag{1.2.11}$$

and this is gauge invariant.

This is a general fact: anomalies can appear in different ways by modifying local counterterms in terms of the background fields, and the corresponding inflow actions will be related by boundary terms. The non-triviality of the anomaly, namely the impossibility of fixing the lack of gauge invariance by adding counterterms (as we did for instance in the case of $U(1)_M$ alone), is measured by the non-triviality of the inlow action. The anomaly inflow paradigm is the idea of classifying all possible anomalies in terms of invertible TQFTs in one dimension higher. As we shall see, this principle is valid also for finite symmetries, and more generally to the higher-form symmetries that we are going to introduce. With some modification, this paradigm is valid for any type of generalized symmetry.

1.3 Higher-form symmetries

So far we have arrived at the intrinsic notion of ordinary symmetries as topological codimension one operators satisfying Ward identities (1.1.8). The simplest generalization considers topological operators on submanifolds with codimension greater than one. Clearly if the codimension is $p+1$, with $p>0$, the surface cannot link with a point operator, but can with an operator supported on a $p-$dimensional submanifold (see figure 2). This means that such $d-p-1$ dimensional topological operators cannot lead to non-trivial Ward identities in correlation functions of local operators, but something interesting can arise if the correlation function contains some extended $p-$dimensional operator. If this happens, these new objects are called *higher-form symmetries* of degree p, or *$p-$form symmetries*. The existence of these symmetries has been clearly stated in [11, 30] for the first time. They have properties very similar to ordinary symmetries, which in this context are referred to as 0-form symmetries. Before discussing these properties in general, let us consider a few concrete examples.

1.3.1 Abelian gauge theories

Maxwell theory One of the simplest examples is pure $U(1)$ gauge theory, namely Maxwell theory

$$S=\frac{1}{4e^2}\int_{X_d} F\wedge *F \ . \tag{1.3.1}$$

$F=dA$ is the curvature of a $U(1)$ connection A, and satisfies Dirac quantization condition $\int_{\Sigma_2}\frac{F}{2\pi}\in\mathbb{Z}$, with $\Sigma_2\subset X_d$ any compact 2-manifold. The path integral sums over all topological classes of $U(1)$ bundles, meaning that all possible integer values of the fluxes are included, and for each given bundle we sum over all possible connections A modulo gauge transformations

$$A\mapsto A^\lambda = A+\lambda \ . \tag{1.3.2}$$

λ is a globally defined closed 1-form, with quantized periods $\int_{\gamma_1}\lambda=2\pi n_{\gamma_1}$, $n_{\gamma_1}\in\mathbb{Z}$. For $n_{\gamma_1}=0$ on all γ_1, $\lambda=d\theta$ is exact and this is a *small* gauge transformation. If this is not the case, $A\mapsto A^\lambda$ is a large gauge transformation. In both cases these are redundancies of the path integrals, so no operator can transform non-trivially under them, and they do not correspond to a global symmetry of any type. In particular this implies that Wilson line operators

$$W_n(\gamma_1)=e^{in\int_{\gamma_1}A} \tag{1.3.3}$$

must have integer charge $n\in\mathbb{Z}$.

The main observation is that, shifting $A \mapsto A+\xi$ by a closed 1-form ξ with non-quantized periods, we still leave the action invariant. But the Wilson lines transform non-trivially

$$A \mapsto A+\xi \implies W_n(\gamma_1) \mapsto e^{i\alpha} W_n(\gamma_1) \ , \qquad \alpha = \int_{\gamma_1} \xi \in \mathbb{R}/2\pi\mathbb{Z} \cong U(1) \ , \tag{1.3.4}$$

hence this is not a redundancy, but a true global symmetry of the type described above, more precisely a *1-form symmetry*. Here $\alpha \sim \alpha + 2\pi$, because the 2π periods belong to large gauge transformations.

To translate this observation into a robust quantum statement, we need a Ward identity for a codimension-two topological operator. We derive it by using the same trick that is usually employed: we change variable in the path integral, applying a symmetry transformation with a non-constant parameter. For 1-form symmetries, this amounts to changing A into $A+\xi$ where ξ is a *non* closed 1-form. The action changes by

$$\delta S = \frac{1}{2e^2} \int_{X_d} \xi \wedge d * F \ . \tag{1.3.5}$$

Therefore we get

$$\left\langle \frac{1}{2e^2} d*F(x) W_{n_1}\left(\gamma_1^{(n_1)}\right) \cdots W_{n_N}\left(\gamma_1^{(n_N)}\right) \right\rangle = \sum_{i=1}^{N} n_i \delta^{d-1}\left(\gamma_1^{(n_i)}\right) \left\langle W_{n_1}\left(\gamma_1^{(n_1)}\right) \cdots W_{n_N}\left(\gamma_1^{(n_N)}\right) \right\rangle \ . \tag{1.3.6}$$

Here $\delta^{d-1}\left(\gamma_1^{(n_i)}\right)$ is the Poincare' dual of $\gamma_1^{(n_i)}$. It is a $d-1$ form with the property that for any 1-form η we have

$$\int_{X_d} \eta \wedge \delta^{d-1}\left(\gamma_1^{(n_i)}\right) = \int_{\gamma_1^{(n_i)}} \eta \ . \tag{1.3.7}$$

Consider a $d-2$ dimensional closed, but homologically trivial submanifold Σ_{d-2}. It is the boundary of an open $d-1$ dimensional manifold $D_{d-1} \subset X_{d-1}$. The integral of $\delta^{d-1}\left(\gamma_1^{(n_i)}\right)$ on D_{d-1} gives the number of intersections of $\gamma_1^{(n_i)}$ with D_{d-1}, that is the linking number $\mathrm{Lk}\left(\Sigma_{d-2}, \gamma_1^{(n_i)}\right)$. Then integrating (1.3.6) on D_{d-1} we get

$$\left\langle Q(\Sigma_{d-2}) W_{n_1}\left(\gamma_1^{(n_1)}\right) \cdots W_{n_N}\left(\gamma_1^{(n_N)}\right) \right\rangle = \sum_{i=1}^{N} n_i \mathrm{Lk}\left(\Sigma_{d-2}, \gamma_1^{(n_i)}\right) \left\langle W_{n_1}\left(\gamma_1^{(n_1)}\right) \cdots W_{n_N}\left(\gamma_1^{(n_N)}\right) \right\rangle \ , \tag{1.3.8}$$

where $Q(\Sigma_{d-2}) = \frac{1}{2e^2} \int_{\Sigma_{d-2}} *F$. By exponentiating we conclude that

$$U_\alpha(\Sigma_{d-2}) = \exp\left(\frac{\alpha}{2e^2} \int_{\Sigma_{d-2}} *F \right) \tag{1.3.9}$$

is a topological codimension two operator labelled by an angle $\alpha \sim \alpha + 2\pi$ and satisfy a relation similar to (1.1.9), but involving a Wilson line (see figure 1.3):

$$U_\alpha(\Sigma_{d-2}) \cdot W_n(\gamma_1) = e^{in\alpha} \ W_n(\gamma_1) \ U_\alpha(\Sigma'_{d-2}) \ . \tag{1.3.10}$$

This 1-form symmetry is called *electric* 1-form symmetry: it acts on Wilson lines that can be thought of as world-lines electrically charged particles. Its underlying group is $U(1)$, which means that the defects $U_\alpha(\Sigma_{d-2})$ fuse according to the $U(1)$ group law.

Maxwell theory has another $U(1)$ higher-form symmetry of degree $d-3$, called *magnetic* symmetry: it acts on 't Hooft operators $H_n(\gamma_{d-3})$. They can be defined as *disorder operators*. Inserting $H_n(\gamma_{d-3})$ in a correlator amounts to modifying the path integral domain, including singular connections on γ_{d-3} such that

$$\int_{S^2} \frac{F}{2\pi} = n \ , \qquad \mathrm{Lk}(S^2, \gamma_{d-3}) = 1 \ . \tag{1.3.11}$$

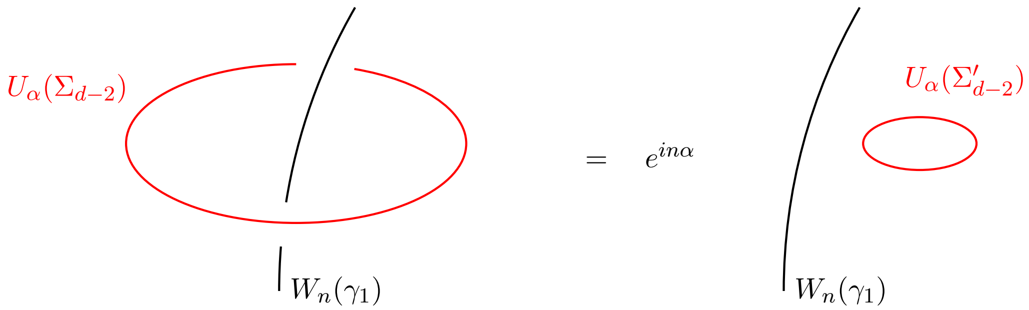


Figure 1.3: The action of the codimension two topological defect on a Wilson line.

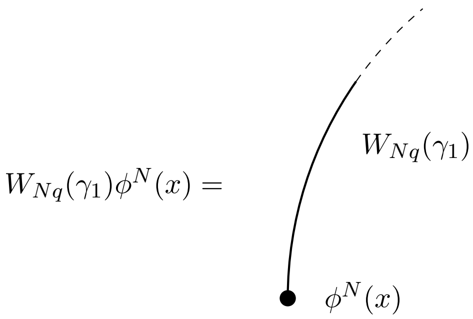
Tautologically, the two-dimensional topological operators are

$$V_\alpha(\Sigma_2) = \exp\left(i\alpha \int_{\Sigma_2} \frac{F}{2\pi}\right) . \tag{1.3.12}$$

In $d = 3$ the $H_n(x)$ is a local operator, the monopole operator, and the magnetic symmetry is a 0-form $U(1)$ symmetry. For $d = 4$ it is 1-form symmetry, and $H_n(\gamma_1)$ are 't Hooft lines.

Inclusion of matter Consider Abelian gauge theories with matter (QED-like theories), bosonic or fermionic. Suppose that there is a matter field $\phi(x)$ of charge $q \in \mathbb{Z}$, which means the covariant derivative is $D = d - iqA$. A gauge transformation $A \mapsto A + \lambda$ acts on the matter field as $\phi(x) \mapsto e^{iq\theta(x)}\phi(x)$, where $\theta(x)$ is a compact scalar such that locally $\lambda = d\theta$.[3]

While in pure Maxwell theory the Wilson lines must be closed by gauge invariance, now some of them can be opened, ending on the matter fields: a Wilson line $W_n(\gamma_1)$ of charge $n = Nq$ can end on the local non gauge-invariant operator $\phi(x)^N$ producing a gauge invariant line operator:



This implies that most of the electric 1-form symmetry defects $U_\alpha(\Sigma_{d-2}) = e^{\frac{\alpha}{2e^2}\int_{\Sigma_{d-2}} *F}$ can no longer be topological. In fact, suppose that $U_\alpha(\Sigma_{d-2})$ is topological. Putting it in a configuration where it links with an open Wilson line $W_q(\gamma_1)\phi(x)$, close to the line. We can unlink it in two ways: directly, or first sliding it below the end point $\phi(x)$ and then closing the surface Σ_{d-2}. In the first way, we get a phase $e^{i\alpha q}$, while nothing happens in the second case. By consistency

$$\alpha = \frac{2\pi a}{q} \, , \qquad a = 0, ..., q-1 \in \mathbb{Z}_q \, . \tag{1.3.13}$$

All the other operators must be non-topological.

This argument alone does not prove that $U_{\frac{2\pi a}{q}}$ is still topological, but we will show in the next example – discussing a more general context – that this *endability argument* works in both directions:

[3]Even though strictly speaking $\lambda = d\theta$ is only valid locally, the condition $\int_{\gamma_1} \lambda \in 2\pi\mathbb{Z}$ allows to extend the validity of $\lambda = d\theta$ at the global leave, at the prize of making θ multi-valued, with 2π-periods. Hence, the transformation $\phi(x) \mapsto e^{iq\theta(x)}\phi(x)$ makes perfect sense.

a topological operator of the pure gauge theory remains topological if and only if all endable lines are uncharged under it. Hence, the theory has a discrete electric 1-form symmetry in $\mathbb{Z}_q$.

The magnetic symmetry is still there even in presence of matter. In $d = 3$ it is a 0-form symmetry, and it can be broken by adding the monopole operator to the action. For $d = 4$ it is a 1-form symmetry, and to break it one would need to add magnetically charged matter. For $d > 4$ the magnetic symmetry has degree $p > 1$ and is not broken by any natural operation.

1.3.2 Non-Abelian gauge theories

Consider pure G-Yang-Mills theory, with connection A and field strength $F = dA + iA \wedge A$:

$$S = \frac{1}{2g^2} \int_{X_d} \mathrm{Tr}(F \wedge *F) \tag{1.3.14}$$

This discussion generalizes the Abelian case and clarifies previously overlooked subtleties. The path integral sums over all G bundles and their connections, modulo gauge transformations. It is convenient to describe a G bundle by decomposing space-time into patches $\mathcal{U}_i$ and assigning transition functions $g_{ij} : \mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j \to G$, satisfying the cocycle condition on triple intersections:

$$g_{ij}\; g_{jk}\; g_{ki} = 1 \; . \tag{1.3.15}$$

The connection A is given by local 1-forms $A_i \in \Omega^1(\mathcal{U}_i, \mathfrak{g})$ glued on double intersections:

$$A_j = g_{ij}^{-1}\, A_i\, g_{ij} + g_{ij}^{-1} dg_{ij} \; . \tag{1.3.16}$$

This is not to be confused with gauge transformations: it's just the connection's defining property. Gauge transformations are redundancies of the path integral that act on both A_i and g_{ij}

$$A_i \mapsto U_i^{-1}\, A_i\, U_i + U_i^{-1}\, dU_i \; , \qquad g_{ij} \mapsto U_i\, g_{ij}\, U_j^{-1} \; , \tag{1.3.17}$$

with $U_i : \mathcal{U}_i \to G$ a family of locally defined functions. All operators, including the Wilson lines

$$W_{\mathfrak{R}}(\gamma) = \mathrm{Tr}_{\mathfrak{R}}\, P \exp\left(i \int_\gamma A \right) \; , \tag{1.3.18}$$

are invariant under (1.3.17).

To define the path-ordering exponential (holonomy) on an arbitrary curve γ, we cut small arcs $\gamma_i \subset \mathcal{U}_i$ and use A_i to compute its holonomy $\mathrm{hol}_{\gamma_i}(A_i) = P \exp\left(i \int_{\gamma_i} A_i \right)$. Under a gauge transformation, this transforms into $U_i(x_1) \mathrm{hol}_{\gamma_i}(A_i) U_i(x_2)^{-1}$ (x_1, x_2 are the initial and final points). The full holonomy is constructed choosing a base-point $x \in \gamma_{i_1}$:

$$P \exp\left(i \int_\gamma A \right) = \prod_{k=1,\ldots} \mathrm{hol}_{\gamma_{i_k}}(A_{i_k}) g_{i_k\, i_{k+1}} \; . \tag{1.3.19}$$

This transforms under conjugation by $U_{i_1}(x)$, hence the Wilson loops are gauge invariant.

Electric 1-form symmetry. The pure Yang-Mills theory has a 1-form *center* symmetry $\mathcal{Z}(G)$ under which the Wilson loops are charged. Under a transformation by $g \in \mathcal{Z}(G)$, a Wilson loop in representation $\mathfrak{R}$ picks a phase[4]

$$\chi_{\mathfrak{R}}(g) = \frac{\mathrm{Tr}_{\mathfrak{R}}(g)}{\mathrm{Tr}_{\mathfrak{R}}(1)} \in U(1) \tag{1.3.20}$$

[4]The center is an Abelian group (it is either $\mathbb{Z}_N$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$), and every representation of G descents to a representation of $\mathcal{Z}(G)$, hence $\chi_{\mathfrak{R}}(g) \in U(1)$ it is just its character.

To see where this 1-form symmetry comes from, let us first return to the case $G = U(1)$, where a 1-form symmetry transformation is $A \mapsto A + \lambda$, with λ a globally defined closed 1-form. In every patch, we can write $\lambda\big|_{\mathcal{U}_i} = d\eta_i$, and completely eliminate its action on the connection by local gauge transformations with $U_i = e^{-i\eta_i}$. The prize is that we modify the transition functions as

$$g_{ij} \mapsto g_{ij}t_{ij} \ , \qquad t_{ij} = e^{-i(\eta_i - \eta_j)} \ . \tag{1.3.21}$$

This reproduces the action on the Wilson lines, carefully defined using (1.3.19).

This reformulation has an obvious generalization to the non-Abelian case. We define the action of the 1-form symmetry directly on transition functions

$$g_{ij} \mapsto g_{ij}t_{ij} \ . \tag{1.3.22}$$

To preserve the cocycle condition, all the t_{ij} must commute with any possible g_{ij}, hence $t_{ij} \in \mathcal{Z}(G)$. On top of this we must impose

$$t_{ij}\ t_{jk}\ t_{ki} = 1 \ . \tag{1.3.23}$$

A Wilson line $W_{\mathfrak{R}}(\gamma)$ transforms as

$$W_{\mathfrak{R}}(\gamma) \mapsto \chi_{\mathfrak{R}}(g) W_{\mathfrak{R}}(\gamma) \ , \qquad g = \prod_{\gamma \cap \mathcal{U}_{ij} \neq \emptyset} t_{ij} \in \mathcal{Z}(G) \ . \tag{1.3.24}$$

For $G = \mathrm{SU}(N)$ the center is $\mathbb{Z}_N$, and the 1-form symmetry $a \in \mathbb{Z}_N$ on $W_{\mathfrak{R}}$ is a multiplication by $e^{\frac{2\pi i q a}{N}}$, where $q = 0, ..., N-1$ is the $N-$ality of the representation.

The codimension-two topological operators $U_g(\Sigma_{d-2})$ do not have a very explicit expression. They can be defined as *disorder* operators: their insertion in a correlation function is specified by modifying the path integral summing over connections which are singular on Σ_{d-2}, such that

$$\mathrm{hol}_{S^1}(A) = g \in \mathcal{Z}(G) \ , \qquad \mathrm{Lk}(\Sigma_{d-1}, S^1) = 1 \ . \tag{1.3.25}$$

They are usually called Gukov-Witten (GW) operators, since they are similar to those introduced in [62] in 4d $\mathcal{N} = 4$ SYM.

Magnetic $(d-3)-$form symmetry. Like in the Abelian case, $G-$gauge theories have a $d-3$ degree magnetic symmetry with the group $\pi_1(G)^\vee$. The charged objects are 't Hooft operators: monopoles in 3d, 't Hooft lines in 4d and so on. Contrary to the electric symmetry, the charged objects do not have an explicit representation, while the topological operators do. We use that $G = \widetilde{G}/\Gamma$, with $\widetilde{G}$ the universal cover and $\Gamma \cong \pi_1(G) \subset \mathcal{Z}(\widetilde{G})$. There is a characteristic class $[w] \in H^2(X_d, \pi_1(G))$ of $G--$ bundles —-sometimes called the Stiefel-Whitney (or Breuer) class– measuring the obstruction to lift them to $\widetilde{G}-$bundles, and it is defined as follows. Pick a lift of the transition functions $g_{ij} \in G$ to $\widetilde{g}_{ij} \in \widetilde{G}$. The cocycle condition is modified:

$$\widetilde{g}_{ij}\ \widetilde{g}_{jk}\ \widetilde{g}_{ki} = w_{ijk} \in \pi_1(G) \subset \mathcal{Z}(\widetilde{G}) \ . \tag{1.3.26}$$

On quadruple intersections $\mathcal{U}_{ijkl}$ we have

$$w_{ijk}\ w_{jkl}\ w_{kli}\ w_{lij} = 1 \ . \tag{1.3.27}$$

Hence w_{ijk} defines $[w] \in H^2(X_d, \pi_1(G))$. In the G gauge theory we sum over all possible G bundles, hence $[w]$ is dynamical and we can construct 2-dimensional topological operators

$$V_\chi(\Sigma_2) = \chi\left(e^{i\int_{\Sigma_2} w}\right) \ , \qquad \chi \in \pi_1(G)^\vee \tag{1.3.28}$$

Gauge theories with matter. Let us add matter in representation $\mathfrak{R}$ of G. This will not affect the magnetic symmetry, but the electric 1-form symmetry will be generally reduced to a subgroup. Indeed, the Wilson lines in representation $\mathfrak{R}$ can end on the matter field. For the same argument as in the Abelian case, the operators U_g labeled by central elements $g \in \mathcal{Z}(G)$ that are non-trivial in the representation $\mathfrak{R}$, can no longer be topological. We remain with the subgroup

$$\Gamma_{\mathfrak{R}} = \left\{ g \in \mathcal{Z}(G) \;\middle|\; \chi_{\mathfrak{R}}(g) = 1 \right\} \subset \mathcal{Z}(G) \ . \tag{1.3.29}$$

Let us argue now that we indeed have this 1-form symmetry. A matter field ϕ is given by a family of locally defined ϕ_i on each $\mathcal{U}_i$, valued in the $\mathfrak{R}$, and glued on the double intersections as[5]

$$\phi_j = \mathfrak{R}(g_{ij}) \cdot \phi_i \tag{1.3.30}$$

A 1-form symmetry transformation $g_{ij} \mapsto g_{ij} t_{ij}$ affects the matter field modifying the gluing condition, unless $t_{ij} \in \Gamma_{\mathfrak{R}}$. If this condition is satisfied, the matter field does not see the action of the 1-form symmetry transformation, hence it remains a good symmetry.

1.3.3 General properties of higher-form symmetries

The first important observation is that higher-form symmetries are always Abelian: when we fuse two topological operators of codimension greater than one, there is no concept of ordering and the multiplication of the labels must be commutative.

Also, higher-form symmetries cannot act on local operators and therefore cannot be explicitly broken by adding interactions, in sharp contrast with ordinary 0-form symmetries. As discussed, 1-form symmetries realized as center symmetries of some gauge group G can be broken adding matter. For this reason, these symmetries are often emergent: they arise in the IR of some theories with massive matter charged under the center of the gauge group. However, in contrast with emergent 0-form symmetries, emergent 1-form symmetries are not violated by any irrelevant operator.

Background fields and anomalies. As for ordinary symmetries, $p-$form symmetries can be coupled with background gauge fields, which are $p+1$ forms. A $U(1)$ $p-$form symmetry has a $(p+1)-$form conserved current $d * J_{p+1} = 0$. The topological operators are $\exp\left(i\alpha \int_{\Sigma_{d-p-1}} *J_{p+1}\right)$. We can couple the symmetry to a background B_{p+1} as

$$S[B] = S + i \int_{X_d} B_{p+1} \wedge *J_{p+1} \ . \tag{1.3.31}$$

The background field gauge transformation is $B_{p+1} \mapsto B_{p+1} + d\lambda_p$, with λ_p a $p-$form gauge field.

If the background field is flat $dB_{p+1} = 0$, (1.3.31) can be understood as the insertion of a topological defect. Indeed, by Poincare' duality, a closed $p+1$ form gauge field can be realized as $B_{p+1} = \alpha\delta^{(p+1)}(\Sigma_{d-p-1})$ for a closed submanifold Σ_{d-p-1} and a certain $\alpha \sim \alpha + 2\pi$ (its holonomy). Then

$$\int_{X_d} B_{p+1} \wedge *J_{p+1} \quad \Longleftrightarrow \quad \text{insertion of } \exp\left(i\alpha \int_{\Sigma_{d-p-1}} *J_{p+1} \right) \ . \tag{1.3.32}$$

A gauge transformation $B_{d+1} \mapsto B_{p+1} + d\lambda_p$ deforms Σ_{d-p-1}, and this does not modify anything since the operator is topological. Equivalently, gauge invariance is a consequence of $d * J_{p+1} = 0$.

As for 0-form symmetries, higher-form symmetries can have anomalies: it might be impossible to couple the symmetry to a background field in a gauge-invariant way, unless we add a $d+1$ invertible

[5]Mathematically ϕ is a section of an associated vector bundle.

TQFT. We will discuss the discrete case in Section 1.4. In the continuous world, the simplest example is Maxwell theory, say in 4d. The currents for the electric and magnetic 1-form symmetries are respectively

$$J_e = \frac{i}{2e^2} F \ , \qquad J_m = \frac{*F}{2\pi} \tag{1.3.33}$$

They have a mixed anomaly similarly to 2d compact boson: coupling $U(1)_e$ to a background B_e amounts to shifting $F \mapsto F - B_e$ in the action[6]. On the other hand, $U(1)_m$ is coupled to a background B_m by adding a coupling

$$\frac{i}{2\pi} \int_{X_4} F \wedge B_m \tag{1.3.34}$$

that is not gauge invariant for $U(1)_e$. It might be made so by adding a counterterm $-\frac{i}{2\pi} B_e \wedge B_m$, which spoils gauge invariance for $U(1)_m$. This clash represents an anomaly with a 5d inflow action

$$S_{\text{inflow}} = -\frac{i}{2\pi} \int_{X_5} B_e \wedge dB_m \tag{1.3.35}$$

Selection rules. In compact space-times a $p-$form symmetry implies selection rules for correlation functions of $p-$dimensional operators: a correlator containing extended operators wrapping non-trivial homologous $p-$cycles $\gamma_{p,1}, \gamma_{p,2}, ..., \gamma_{p,n}$ can be non-zero only if the $p-$form symmetry charges of the operators sum up to zero. Moreover, this selection rule is valid independently for each homology class of cycles.

The derivation is straightforward. In the correlator, we nucleate a topological operator for the symmetry element $e^{i\alpha}$, whose support is not linked with any operator. Then we enlarge the support, crossing various operators until we shrink the topological operator back to a point. For each crossing, we get a phase $e^{\pm i\alpha q}$, with q the charge of the operator. This process can be done in topologically distinct ways, each crossing only the operators on a given cycle. The final result, for each distinct process, is that the correlation function gets multiplied by a phase

$$\exp\left(i\alpha \sum_{i=1}^{n} q_i\right) \tag{1.3.36}$$

which must be one to allow a non-zero correlator. For a $U(1)$ symmetry, this implies $\sum_i q_i = 0$, while for a $\mathbb{Z}_N$ symmetry, this implies $\sum_i q_i = 0 \text{ mod}(N)$.

Spontaneous breaking and phases of gauge theories. If the space-time is non-compact this argument may fail. After the topological operator crosses all the extended operators, we remain with a very big topological operator in the exterior of the region containing all operator insertions. Hence, to make the previous argument work, we need the extra assumption that this big operator is still the identity. This is the same as requiring that the vacuum state is invariant, namely that the symmetry is not spontaneously broken. If this is not the case, then the selection rules are not valid.

To possibly have a violation of the selection rule due to spontaneous symmetry breaking, we need to place charged extended operators on homologically non-trivial cycles. Consider a 4d SU(N) gauge theory with adjoint matter (having 1-form symmetry $\mathbb{Z}_N$) on $\mathbb{R}^3 \times S^1$. Place a Wilson loop in the fundamental representation on S^1 and examine its VEV. If non-zero, the 1-form symmetry is spontaneously broken. In the decompactification limit of S^1, the theory is on $\mathbb{R}^4$ with the Wilson loop

[6] As in the 2d compact boson, this includes a counterterm $B_e \wedge *B_e$ that is necessary for gauge invariance

on a straight line closed at infinity. If the theory confines, the Wilson loop obeys area law, and the VEV of a large Wilson loop vanishes:

$$\text{confinement:} \qquad \left\langle W(S^1) \right\rangle \sim e^{-\sigma A(D_2)} \longrightarrow 0 \ . \qquad . \tag{1.3.37}$$

If the theory does not confine, this VEV is either a constant (Coulomb phase) or follows perimeter law. In the second case we can modify the definition of the operator by a counterterm localized on the line (its line element) to make the VEV a non-zero constant. Hence in both cases we have

$$\text{deconfinement:} \qquad \left\langle W(S^1_\infty) \right\rangle \longrightarrow c \neq 0 \ . \qquad . \tag{1.3.38}$$

This observation puts confinement and deconfinement phases in the realm of Landau paradigm, which identifies phases of QFTs in terms of their symmetries and how they are realized on the vacuum [11].

The confinement criterion we discussed, *Wilson criterion* [63], uses electric probes (Wilson lines), and cannot be applied if the gauge group does not have a center. However, a second criterion due to 't Hooft [64] can be used when $\pi_1(G) \neq 1$, and hence there is a magnetic 1-form symmetry. The idea is that confinement is dually described in terms of monopole condensation, and hence in the presence of a magnetic symmetry it is detected by spontaneous breaking of it. On the other hand, deconfinement takes place if 't Hooft loops have area law.

Spontaneous breaking of finite symmetries: topological order. For ordinary 0-form symmetries, spontaneous breaking implies degeneracy of vacua. One might expect the same to be true for higher-form symmetries. There is, however, a very interesting difference: the number of such ground states depends on the topology of space. This phenomenon is known as *topological order* [65, 66].

Consider a gapped theory with spontaneously broken discrete $p-$form symmetry. Below the gap we do not have local excitations, hence we can only find a discrete set of vacua. These vacua are mapped to each other by the action of the topological operators that generate the symmetry. However, if the space is S^{d-1} and $p > 0$, there is no $p-$ cycle in space, and the symmetry defect acting on the vacuum is necessarily trivial: even if the symmetry is broken the number of *local vacua*, namely the vacua on the sphere, is one. To obtain multiple vacua, we need the space to contain $p-$cycles. For example, if $X_{d-1} = S^{d-p-1} \times S^p$ (we assume $d \neq 2p+1$), then we can wrap the $d-p-1$ dimensional topological symmetry defects on S^{d-p-1}, and by acting on the vacuum we generate other states. If we have a symmetry breaking pattern $\mathbb{A} \to \mathbb{B} \subset \mathbb{A}$, we will have $\left|\mathbb{A}/\mathbb{B}\right|$ vacua. More generally, the number of vacua on a spatial manifold X_{d-1} is the cardinality of

$$H_{d-p-1}(X_{d-1}, \mathbb{A}/\mathbb{B}) \ . \tag{1.3.39}$$

We can reinterpret this fact in terms of an effective low-energy theory below the gap. Since there are no local excitations, this theory must be topological. Consider again the example of 4d gauge theories with 1-form center symmetry $\mathbb{Z}_N$, and assume it to be in a deconfined gapped phase. All Wilson lines have perimeter law; hence, they have zero tension at low energy and become topological. Moreover, we have N topological surfaces $U_{a=0,\ldots,N-1}(\gamma_2)$ for the 1-form symmetry, which produce a braiding phase $e^{2\pi i \frac{ab}{N}}$ when passing a Wilson line of $N-$ality $b = 0, \ldots, n-1$. There is a simple topological theory with N lines and N surfaces with this braiding: a $\mathbb{Z}_N$ gauge theory. It can be presented in the continuum [30] in terms of two $U(1)$ gauge fields, a 1-form A and a 2-form B:

$$S = \frac{iN}{2\pi} \int_{X_4} A \wedge dB \ . \tag{1.3.40}$$

This is called a BF theory, and we will analyze these simple TQFTs in more detail in Chapter 2. For the moment let us just notice that it has lines $W_a(\gamma_1) = e^{ia\int_{\gamma_1} A}$ and surfaces $U_b(\gamma_2) = e^{ib\int_{\gamma_2} B}$ with braiding phase $e^{2\pi\frac{ab}{N}}$. Gauge invariance requires $a, b, \in \mathbb{Z}$, while the braiding phase is periodic for shifts of a and b by N, hence $a, b \in \mathbb{Z}_N$. From the viewpoint of the UV gauge theory, the surfaces $U_b(\gamma_2)$ are identified with the topological operators of the center symmetry, while the lines $W_a(\gamma_1)$ are identified with the IR limit of the Wilson lines.

The theory (1.3.40) nicely encodes the vacuum structure. Let us take the spatial manifold to be $S^2 \times S^1$, and consider the operators $W_a(S^1)$ and $U_b(S^2)$ acting on the Hilbert space. Due to the braiding phase they do not commute, but

$$W_a(S^1)\; U_b(S^2) = e^{2\pi i\frac{ab}{N}}\; U_b(S^2)\; W_a(S^1)\;. \tag{1.3.41}$$

This is called *Heisenberg algebra*. The states can be labeled in terms of their eigenvalues for $W_a(S^1)$:

$$W_a(S^1)\,|b\rangle = e^{\frac{2\pi i ab}{N}}\,|b\rangle\;. \tag{1.3.42}$$

By using the algebra we realize that

$$U_b(S^2)\,|b'\rangle = |b + b'\rangle\;. \tag{1.3.43}$$

We conclude that there are N ground states mapped among each other by the action of the 1-form symmetry defect wrapped on S^2, and distinguished by the phase of the VEV of the Wilson lines.

Spontaneous breaking of continuous higher-form symmetries. While spontaneous breaking of finite higher-form symmetries in gapped phases implies topological order, breaking continuous higher-form symmetries produces Goldstone bosons and requires a gapless theory. For a $p-$form $U(1)$ symmetry the Goldstone bosons are $p-$form gauge fields A_p with a Maxwell action [11]:

$$S = \frac{1}{4e^2}\int_{X_d} dA_p \wedge *dA_p\;. \tag{1.3.44}$$

In the modern perspective the photon is interpreted as the Goldstone boson of the spontaneously broken magnetic $U(1)$ 1-form symmetry. Indeed, while in a 4d QED-like theories[7] the electic 1-form symmetry is generically explicitly broken by the presence of charged matter, the magnetic symmetry is robust but spontaneously broken by the VEV of 't Hooft line operators. Notice that the IR will contain a decoupled Maxwell sector[8], hence also an electric $U(1)$ 1-form symmetry. This is an emergent symmetry: the matter must be integrated out below the energy scale set by the masses, and hence the Wilson lines can no longer be cut open at those scales.

Another example is 4d non-Abelian gauge theories in the Coulomb phase. This can be realized with adjoint scalars that condense and generically break the gauge group as $G \mapsto U(1)^r$, with r the rank of G. In the UV there might be no 1-form symmetry at all. However, below the scale of the condensation, the theory is effectively Abelian and there is an emergent magnetic 1-form symmetry. This is also spontaneously broken and implies the existence of r photons in the low-energy spectrum. How do we understand the explicit breaking of this magnetic symmetry at high energy? The breaking pattern $G \mapsto U(1)^r$ implies the presence of dynamical monopoles labeled $\pi_2\left(G/U(1)^r\right)$, with a mass of the order VEVs. The 't Hooft lines of the low-energy Abelian gauge theory are world-lines of these massive monopoles that ere effectively probes in the IR. Hence, going up in energy, at scales where the monopoles become dynamical, the 't Hooft lines become endable on the monopoles, and the 1-form magnetic symmetry is broken.

[7]We assume all the matter fields to be massive, in order to avoid IR divergences that would make the theory ill-defined.
[8]The electric charge is determined by the smallest mass. Indeed, the running coupling stops at that scale.

1.4 Gauging finite symmetries

1.4.1 Background fields and anomalies

As we have seen, coupling a continuous symmetry to a flat background is equivalent to inserting a topological operator. For discrete symmetries we do not have currents, but we extrapolate this principle and *define* the coupling with background fields as a network of topological defects. By construction, a background field for a discrete symmetry $\mathbb{A}$ is necessarily flat. The defects of the network intersect in *junctions*, as in the right picture in Figure 1.2. For a $p-$form symmetry the generic junction is not in codimension two, but in codimension $p+2$, and it is made by $p+3$ defect $U_{a_1}, ..., U_{a_{p+3}}$. For consistency

$$a_1 \cdot a_2 \cdots a_{p+3} = 1 \ . \tag{1.4.1}$$

Using this *cocycle condition* repeatedly, it is easy to convince himself that, if we go around a contractible cycle, the product of all symmetry transformations is 1. Only if we go around a non-contractible cycle we can have a non-trivial transformation. This path must be *orthogonal* to the planes of the network, hence the cycle needs to be of complementary dimension to the defects: it is $(p+1)-$cycle.

We can make this description more systematic by using a *simplicial decomposition*, or triangulation, S of X_d. A $d-$dimensional simplex is an open subset bounded by $(d+1)-$many codimension-one planes, and we cover X_d with simplices, joining along the planes. S has planes of all possible dimensions: each plane of dimension n is bounded by $n+1$ planes of dimension $n-1$. The topological defects are placed on the *dual triangulation* S^*: we place a point at the center of each simplex and join the various points through edges orthogonal to the codimension one faces of the simplex. We then have two-dimensional planes orthogonal to codimension-two planes of S, and so on. This means that $m-$dimensional planes of S^* meet in $(d-m+2)-$valent junctions[9]. The defects of a $p-$form symmetry are of dimension $m = d-p-1$, so their junctions are $(p+3)-$valent, as we wanted.

It is natural to introduce a function that, to any $(p+1)$ dimensional plane Δ_{p+1} of S^*, associates the symmetry element $A(\Delta_{p+1}) \in \mathbb{A}$ of the defect placed on the $(d-p-1)-$dimensional orthogonal simplex Σ_{d-p-1}. It is convenient to label the vertices of S^* with $i = 1, 2, ...$, so that Δ_{p+1} is identified by the $p+2$ vertices delimiting it, and the function $A(\Delta_{p+1})$ is represented in components:

$$A_{i_0, i_1, ..., i_{p+1}} \in \mathbb{A} \tag{1.4.2}$$

This is called a *co-chain* $A \in C^{p+1}(X_d, \mathbb{A})$. The cocycle condition (1.4.1) becomes (using additive notation for $\mathbb{A}$)

$$(\delta A)_{i_0, i_1, ..., i_{p+2}} := A_{i_1, i_2, ..., i_{p+2}} - A_{i_0, i_2, ..., i_{p+2}} + \cdots + (-1)^{p+2} A_{i_0, i_1, ..., i_{p+1}} = 0 \tag{1.4.3}$$

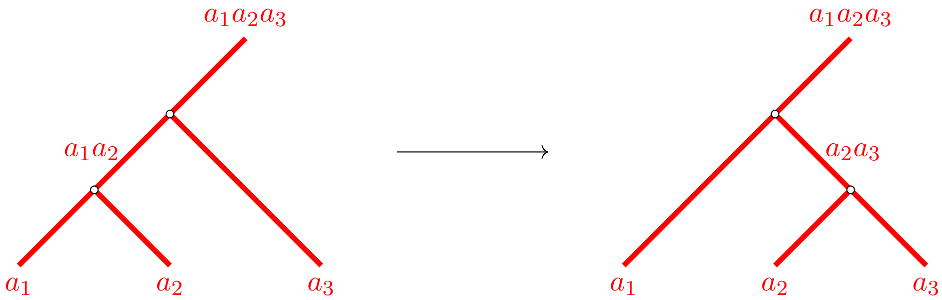
This means that A is a cocycle $A \in Z^{p+1}(X_d, \mathbb{A})$ in singular cohomology. This is the precise mathematical concept for a discrete gauge field, and makes much more manageable the definition in terms of network of topological defects. One way to think about $A \in Z^{p+1}(X_d, \mathbb{A})$ is as a map that assigns an holonomy to each $(p+1)-$dimensional cycle

$$\text{hol}(\gamma_{p+1}) = \int_{\gamma_{p+1}} A \in \mathbb{A} \tag{1.4.4}$$

[9]Indeed $m-$dimensional planes are orthogonal to $(d-m)-$dimensional simplices of S, and $(d-m+2)$ of them bound a $(d-m+1)-$dimensional simplex of S.

where the integral is obtained by decomposing γ_{p+1} in $(p+1)-$simplices, and summing over the value of A on all of them.

To gauge the discrete $p-$form symmetry $\mathbb{A}$, we need to compute the partition function $Z[A]$ as a function of the background A, and sum over it. Notice that the choice of the precise location of the simplices is arbitrary and the independence of $Z[A]$ on it is ensured by the topological nature of the defects. There is one more redundancy that we need to mod-out though. The network of defects can be modified *locally* near junctions, using the so-called *Pachner moves*. For instance, for line defects in 2d we can perform



This does not change the holonomy hol(γ_{p+1}), and the two configurations should be physically indistinguishable. Indeed, one can easily convince himself that they are obtained one from the other by *nucleating* a topological defect from a point, and enlarging to adhere to other defects of the network. In this specific example, it is U_{a_2}. From the point of view of the cocycle $A \in Z^{p+1}(X_d, \mathbb{A})$, this operation adds a *coboundary*

$$A \mapsto A^{\lambda} = A + \delta\lambda \ , \qquad \lambda \in C^p(X_d, \mathbb{A}) \ , \tag{1.4.5}$$

and it is interpreted as a gauge transformation for A.

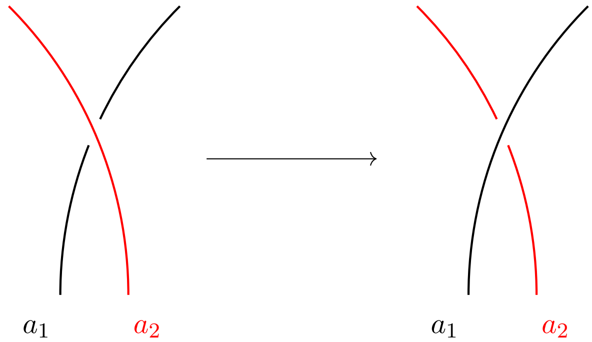
As we emphasized at the end of Section 1.1, introducing junctions requires specifying further data there. There is no a priori reason why the local modification of the junction must leads to a strict equality, and might in principle introduce a phase. In the 2d example above it is $\alpha(a_1, a_2, a_3) \in U(1)$. In terms of the gauge field A, we may generically have

$$Z[A] \mapsto Z[A^{\lambda}] = \alpha(A;\lambda) Z[A] \tag{1.4.6}$$

with $\alpha(A;\lambda) \in U(1)$ that should be expressed as a local function of A and λ. If this phase cannot be eliminated by redefining the coupling of the theory with A, this represents an inconsistency in coupling with background: it is a 't Hooft anomaly. Indeed, even if the symmetry is there and produces topological operators, as soon as we couple the symmetry with a background, the topological nature of the defects is broken: nucleating a bubble of the defects does not lead to an equivalent configuration.

A natural example of a discrete 't Hooft anomaly of this kind arises in 4d $\mathcal{N} = 1$ SYM theory with gauge group SU(N). The theory has a classical $U(1)_R$ R-symmetry, but it suffers from an ABJ anomaly that reduce the true quantum-machanical symmetry to $\mathbb{Z}_{2N} \subset U(1)_R$. This inherits the standard cubic anomaly of the classical $U(1)_R$, hence $\mathbb{Z}_{2N}$ has an anomaly (only the $\mathbb{Z}_2 = (-1)^F$ subgroup is anomaly free), which can be captured by a Pachner move similar to the one above, with the only difference that it involves five 3-dimensional defects joining to create a sixth one.

If the symmetry is of higher-form, $p > 0$, there are other types of local modifications of the network that we should take care of. The prototypical example is for $d = 3$, $p = 1$ (with a straightforward generalization for any d odd and $p = \frac{d-1}{2}$). The topological operators are lines, and they can link non-trivially inside the network. Thus, a possible move is



and in general we may get a phase $\alpha(a_1, a_2)$. If this is non-trivial, we have an 't Hooft anomaly. This phase is detected with the action by linking of a symmetry generator on the other: an 't Hooft anomaly of this kind arises if the symmetry generator are charged among themselves. The typical example of this kind of anomaly is 3d $U(1)_N$ Chern-Simons theory (see Section 2.3 for details)

$$S = \frac{iN}{4\pi} \int_{X_3} A \wedge dA \ . \tag{1.4.7}$$

There is a 1-form symmetry $\mathbb{Z}_N$ generated by the lines $W_a = e^{ia \int A}$, $a = 0, ..., N-1$, that also coincide with the charged objects since two linked lines of charges a and b give rise to a braiding phase $e^{2\pi i \frac{ab}{N}}$. This example has many generalizations, but the important thing to notice is that this kind of anomaly by linking can arise whenever we have two higher-form symmetries of degree p and q, whose topological defects have the right dimensionality to link, namely $p + q = d - 1$.

As for continuous symmetries, also anomalies of discrete symmetries can be canceled by inflow with a $(d+1)-$dimensional invertible TQFT, which it is coupled with A. Hence, the partition function of this anomaly theory is $Z_{\text{inflow}} = e^{i \int_{X_{d+1}} \mathcal{L}_{\text{inflow}}[A]}$, with $\mathcal{L}_{\text{inflow}}[A]$ a $(d+1)-$form that depends locally on A. An alternative description of the gauge field A, is as a map

$$A : X \to B^{p+1}\mathbb{A} \tag{1.4.8}$$

with $B^{p+1}\mathbb{A} = K(\mathbb{A}, p+1)$ the *classifying space*, or, in this case, the *Eilenberg-MacLane space* [67]. We can generate $\mathcal{L}_{\text{inflow}}[A]$ taking a class $\omega \in H^{d+1}(B^p\mathbb{A}, \mathbb{R}/\mathbb{Z})$ and setting

$$\mathcal{L}_{\text{inflow}}[A] = 2\pi i A^*(\omega) \ . \tag{1.4.9}$$

Indeed it turns out that the group $H^{d+1}(B^{p+1}\mathbb{A}, \mathbb{R}/\mathbb{Z})$ classifies (bosonic) anomalies for a $p-$form symmetry $\mathbb{A}$ in $d-$dimensions.

The specific form of the anomaly theory, written in terms of the background, can often distinguish between the types of anomalies. For instance, the Pachner move type of anomaly for $\mathbb{Z}_N$ 0-form symmetries exists for d even, is classified by $H^{d+1}(B\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$ and has inflow action

$$\frac{2\pi i k}{N} \int_{X_{d+1}} A \cup \beta(A)^{d/2} \ , \qquad k = 0, ..., N-1 \ , \tag{1.4.10}$$

with $\beta : H^1(X, \mathbb{Z}_N) \to H^2(X, \mathbb{Z}_N)$ the Bockstein map associated with the sequence $1 \to \mathbb{Z}_N \to \mathbb{Z}_{N^2} \to \mathbb{Z}_N \to 1$. On the other hand, the braiding type of anomaly can exist as a mixed anomaly between a $p-$form symmetry $\mathbb{A}$ and a $q-$form symmetry $\mathbb{B}$, if one group can be understood as the set of charges for the other, namely $\mathbb{B} = \text{Hom}(\mathbb{A}, \mathbb{R}/\mathbb{Z}) = \mathbb{A}^\vee$, and $p + q = d - 1$. The two backgrounds are respectively $A \in Z^{p+1}(X, \mathbb{A})$, $B \in Z^{d-p}(X, \mathbb{A}^\vee)$, and the anomaly theory is

$$2\pi i \int_{X_{d+1}} A \cup B \tag{1.4.11}$$

with $\cup$ the cup product associated with the canonical pairing $\mathbb{A} \times \mathbb{A}^\vee \to \mathbb{R}/\mathbb{Z}$. If $p = q = \frac{d-1}{2}$ the two symmetries can also coincide, but for the anomaly to be non-trivial we need p odd. This is precisely the case of $U(1)_N$ CS theory.

1.4.2 Gauging: twisted sectors and dual symmetry

When the anomaly is non-trivial, the symmetry cannot be gauged. If it vanishes, the gauge invariant content of the background field is obtained by modding out coboundaries; hence it is described by the singular cohomology group $H^{p+1}(X_d, \mathbb{A})$. Now we assume to have a theory $\mathcal{T}$ with non-anomalous $p-$form symmetry $\mathbb{A}$, and we want to describe the theory $\mathcal{T}/\mathbb{A}$ after gauging the symmetry. The partition function is, up to a normalization constant,

$$Z_{\mathcal{T}/\mathbb{A}} = \sum_{A \in H^{p+1}(X_d,\mathbb{A})} Z_{\mathcal{T}}[A] \ . \tag{1.4.12}$$

The topological operators $U_a(\Sigma_{d-p-1})$ that generates $\mathbb{A}$ in the theory $\mathcal{T}$, become trivial in $\mathcal{T}/\mathbb{A}$. Indeed, if we try to insert a topological operator, it can be reabsorbed into the network of defects we use for the gauging, and since we are summing over all possible networks this operation has no effect. Equivalently, if we turn on a background A' the summand becomes $Z_{\mathcal{T}}[A + A']$, and we can simply shift $A \mapsto A - A'$ in the sum.

However, we can turn on a different background field, that is $(d-p-1)-$form gauge field valued in $\mathbb{A}^\vee = \mathrm{Hom}(\mathbb{A}, \mathbb{R}/\mathbb{Z})$, the Pontryagin dual: given $B \in Z^{d-p-1}(X_d, \mathbb{A}^\vee)$ we can modify the path integral (1.4.12) as

$$Z_{\mathcal{T}/\mathbb{A}}[B] = \sum_{A \in H^{p+1}(X_d,\mathbb{A})} \exp\left(2\pi i \int_{X_d} B \cup A\right) Z_{\mathcal{T}}[A] \ . \tag{1.4.13}$$

Here $\cup$ is the cup product associated with the natural pairing $\mathbb{A} \times \mathbb{A}^\vee \to \mathbb{R}/\mathbb{Z}$. This suggests that $\mathcal{T}/\mathbb{A}$ has a *dual symmetry* of degree $d - p - 2$ and based on the group $\mathbb{A}^\vee$ [11][10]. Notice that B must be a cocycle to not spoil gauge invariance under $A \mapsto A + \delta\lambda$. It is already clear from (1.4.13) that $\mathbb{A}^\vee$ is anomaly free ($B \mapsto B + \delta\xi$ leves $Z_{\mathcal{T}/\mathbb{A}}[B]$ invariant since $\delta A = 0$) and gauging it in $\mathcal{T}/\mathbb{A}$ brings back to $\mathcal{T}$ (eventually up to an outer automorphism of $\mathbb{A}$):

$$\left(\mathcal{T}/\mathbb{A}\right)/\mathbb{A}^\vee \cong \mathcal{T} \ . \tag{1.4.14}$$

Indeed (1.4.13) is a sort of discrete Fourier transform, that it is involutive

$$Z_{(\mathcal{T}/\mathbb{A})/\mathbb{A}^\vee}[A'] = \sum_{B \in H^{d-p-1}(X_d,\mathbb{A}^\vee)} \sum_{A \in H^{p+1}(X_d,\mathbb{A})} \exp\left(2\pi i \int_{X_d} \left(A' \cup B + B \cup A\right)\right) Z_{\mathcal{T}}[A] = Z_{\mathcal{T}}\left[\phi \cdot A\right] \ . \tag{1.4.15}$$

Here $\phi : \mathbb{A} \to \mathbb{A}$ is 1 if d is odd, or d and $p + 1$ are both even, while it is $\phi(a) = -a$ if d is even and $p + 1$ is odd.

We can describe the dual symmetry concretely. The upshot is that the topological operators $\widetilde{U}_\alpha(\Sigma_{p+1})$ are the holonomies of the dynamical gauge field

$$\widetilde{U}_\alpha(\Sigma_{p+1}) = \exp\left(2\pi i \alpha \cdot \int_{\Sigma_{p+1}} A\right) \quad , \quad \alpha : \mathbb{A} \to \mathbb{R}/\mathbb{Z} \tag{1.4.16}$$

while the charged operators are the *twisted* sectors of $\mathbb{A}$.

[10] In two dimension this was well known [68].

Twisted sectors. Let us expand on the second statement. In theory $\mathcal{T}$ we have topological operators $U_a(\Sigma_{d-p-1})$, and we can try to open the support on a boundary $\partial\Sigma_{d-p-1} = \gamma_{d-p-2}$. This operation is all but trivial: the operator placed on an open manifold is, in general, non-gauge invariant unless we specify something on the boundary, eventually placing there a non-gauge invariant operator $\tau_a(\gamma_{d-p-2})$ that compensates for the anomalous variation. This is called a *twisted sector operator* (or twist defect) of theory $\mathcal{T}$: it is a non-genuine $(d-p-2)-$dimensional operator living at the boundary of a topological operator. As such, the twist defects are labeled by $a \in \mathbb{A}$.

One important observation is that, if the symmetry $\mathbb{A}$ acts faithfully, namely there are operators charged by linking under it, the twist defects cannot be topological: if they were so, we could trivialize the linking by cutting the topological operator on the twisted sector and contracting it.

Similarly, if the symmetry is faithfully acting but the charged operators are also topological, then the twisted sector must be empty. Indeed, it must be impossible to cut the topological operator (even on a non-topological twisted sector), or otherwise we could trivialize the linking by moving the charged object. This is related with the fact that a symmetry whose charged objects are topological has an 't Hooft anomaly.

To give a simple concrete example, consider 4d Maxwell theory, and we focus on the subgroup $\mathbb{Z}_N \subset U(1)$ of the magnetic 1-form symmetry. Their topological operators are

$$U_a(\Sigma_2) = \exp\left(2\pi i \frac{a}{N}\int_{\Sigma_2}\frac{F}{2\pi}\right) , \qquad a = 0, ..., N-1 \ . \tag{1.4.17}$$

Making Σ_2 open, $\partial\Sigma_2 = \gamma_1$, is inconsistent unless we add a wrongly quantized Wilson line on the boundary

$$W_{\frac{a}{N}} = e^{\frac{ia}{N}\int_{\gamma_1} A} \ . \tag{1.4.18}$$

Therefore this wrongly quantized line lives in the twisted sector of the magnetic 1-form symmetry.

A general explicit description of the twisted sectors can be given in the cochain description of the background field $A \in C^{p+1}(X_d, \mathbb{A})$. A sensible background field is closed $\delta A = 0$, and this is equivalent to satisfying the cocycle condition at all junctions. If, however, we want to include an open defect, we need to relax $\delta A = 0$ at the location of the boundary. Specifically we impose

$$\delta A = z \ , \qquad z \in Z^{p+2}(X_d, \mathbb{A}) \ . \tag{1.4.19}$$

The cocycle z specifies the location of the twist defect: it is the Poincare' dual of γ_{d-p-2}, where the defect is located. This means that if we allow $U_a(\Sigma_{d-p-1})$ to end on $\tau_a(\gamma_{d-p-2})$, the cocycle z assigns $a \in \mathbb{A}$ to the $(p+2)-$simplex Δ_{p+2} orthogonal to γ_{d-p-2}. In particular, the boundary $\Gamma_{p+1} = \partial\Delta_{p+2}$ links with γ_{d-p+2}, and we have

$$a = \int_{\Delta_{p+2}} z = \int_{\Gamma_{p+1}} A = \mathrm{hol}(\Gamma_{p+1}) \ . \tag{1.4.20}$$

This has a clear interpretation: the holonomy of A around a cycle that links with the support of the twist defect is nothing but the symmetry transformation going around this cycle, that is the label of the twist defect.

Notice that a twisted sector can contain several operators. For instance if the theory has genuine $(d-p-2)-$dimensional operators, we can bring them at the boundary of $U_a(\Sigma_{d-p-1})$, thus producing various different twist defects that specify the same holonomy of A around them. One case to notice is when $p = \frac{d-2}{2}$, since $d-p-2 = p$ and so the operators charged under $\mathbb{A}$ have the same dimensionality of the twist defects, and we can construct new operators by bringing them at the boundary of $U_a(\Sigma_{d-p-1})$. The resulting twist defects will also get a charge by linking under $\mathbb{A}$.

When we gauge $\mathbb{A}$ we only keep $\mathbb{A}$-invariant operators: inserting a non-invariant operator in a correlation function gives a vanishing answer. Moreover, the $\mathbb{A}$ symmetry defect becomes trivial, and as a consequence the $\mathbb{A}$-singlet operators living in the twisted sectors become genuine. These are $\tau_a(\gamma_{d-p-2})$, explaining why the dual symmetry is of degree $d-p-2$, and the underlying group is $\mathbb{A}^\vee$: $\mathbb{A} \cong (\mathbb{A}^\vee)^\vee$ is now the set of charges. In fact, from the description we just gave of the twisted sectors, it is clear why $\tau_a(\gamma_{d-p-2})$ is charged under the new topological operators $\widetilde{U}_\alpha(\Sigma_{p+1})$ defined in (1.4.16): they evaluate the holonomy around Σ_{p+1} in the representation $\alpha \in \mathbb{A}^\vee$, but by construction the holonomy around a twist defect is a, hence we get a braiding phase $e^{2\pi i \alpha(a)}$.

An equivalent way of deriving this fact is by noting that inserting $\widetilde{U}_\alpha(\Sigma_{p+1})$ with Σ_{p+1} a contractible surface that links with γ_{d-p-2}, is equivalent to activating a specific configuration of $B = \delta\xi$ in $Z_{\mathcal{T}/\mathbb{A}}[B]$, with $\int_{\gamma_{d-p-2}} \xi = \alpha$. This does not modify the partition function, but it is a non-trivial operation if we insert a twist defect $\tau_a(\gamma_{d-p-2})$: A fails to be closed there, and

$$2\pi i \int_{X_d} B \cup A = (-1)^{d-p} 2\pi i \int_{X_d} \xi \cup z = (-1)^{d-p} 2\pi i \int_{\gamma_{d-p-2}} \xi(a) = 2\pi i \alpha(a) \ . \tag{1.4.21}$$

Discrete torsion. When we gauge the $p-$form symmetry $\mathbb{A}$ we can, in general, twist the sum (1.4.13) by adding *discrete torsion*. We take a cohomology class $\nu \in H^d(B^p\mathbb{A}, \mathbb{R}/\mathbb{Z})$ and construct the theory $\mathcal{T}/\mathbb{A}_\nu$ whose partition function is

$$Z_{\mathcal{T}/\mathbb{A}_\nu}[B] = \sum_{A \in H^{p+1}(X_d, \mathbb{A})} \exp\left(2\pi i \int_{X_d} B \cup A + A^*(\nu)\right) Z_{\mathcal{T}}[A] \ . \tag{1.4.22}$$

This can be understood as a two-steps process:

1. First, we stack a $d-$dimensional SPT phase on $\mathcal{T}$. The SPT phase has $p-$form symmetry $\mathbb{A}$ and partition function $Z_{\text{SPT}}[A] = e^{2\pi i \int_{X_d} A^*(\nu)}$. This operation is the same as modifying $\mathcal{T}$ by a local counterterm.

2. Second, we gauge $\mathbb{A}$ is the combined system $\mathcal{T}$+SPT.

The first operation makes the twist defects $\tau_a(\gamma_{d-p-2})$ no longer transparent to the action of the topological operators $U_b(\Sigma_{d-p-1})$. In fact, inserting a twist defect modifies the cocycle condition to $\delta A = z$, and the term $2\pi i \int A^*(\nu)$ will generally fail to be gauge invariant under $A \mapsto A + \delta\lambda$.

It is complicated (and not very illuminating) to try to determine in full generality the interplay between the twist defects and the symmetry $\mathbb{A}$, since the net effects are qualitatively different depending on d, p and the discrete torsion ν. Moreover, depending on the actual effect, the sharp consequence of the gauging is different. We will see various examples later on. Here we just point out one possibility. If $p = \frac{d-2}{2}$, notice that $(d-p-2)+(d-p-1) = d-1$, so the symmetry defect $U_b(\Sigma_{d-p-1})$ can link with $\tau_a(\gamma_{d-p-2})$. Moreover, for dimensional reasons, in this case $A^*(\nu)$ can be a quadratic function of A. For example, for $d=2, p=0$, an element $\nu \in H^2(B\mathbb{A}, \mathbb{R}/\mathbb{Z})$ defines an antisymmetric product $\chi_\nu : \mathbb{A} \times \mathbb{A} \to \mathbb{R}/\mathbb{Z}$ [69] and

$$2\pi i \int_{X_2} A^*(\nu) = 2\pi i \int_{X_2} \chi_\nu\left(A, \cup A\right) \ . \tag{1.4.23}$$

Similarly for $d=4, p=1$, $\nu \in H^4(B^2\mathbb{A}, \mathbb{R}/\mathbb{Z})$ defines a symmetric product $\chi_\nu : \mathbb{A} \times \mathbb{A} \to \mathbb{R}/\mathbb{Z}$ that allows to write a similar expression. In both cases, if we perform a gauge transformation $A \mapsto A + \delta\lambda$ that corresponds to shrinking the defect $U_b(\Sigma_{d-p-1})$ around $\tau_a(\gamma_{d-p-2})$ (namely $\int_{\gamma_{d-p-2}} \lambda = b$), we see that the discrete torsion makes the twist defect charged under $\mathbb{A}$, giving a braiding phase $\exp\left(2\pi i \chi_\nu(a,b)\right)$. Because of this, when we gauge $\mathbb{A}$ the twist defects are not gauge invariant and do not become genuine $(d-p-2)-$dimensional operator of $\mathcal{T}/\mathbb{A}_\nu$. Rather, we need to combine them with other charged objects under $\mathbb{A}$, which also become non-gauge invariant, to create gauge invariant defects.

The result is that the theory $\mathcal{T}/\mathbb{A}_\nu$ still has dual symmetry $\mathbb{A}^\vee$, but the charged objects are a combination of the twist defects and the operators charged under $\mathbb{A}$ in $\mathcal{T}$. The story for $p \neq \frac{d-2}{2}$ and different forms of the discrete torsion can be qualitatively different.

1.4.3 Examples

2d Ising CFT. The 2d Ising CFT is the simplest minimal model (see [70]). It has $c = \frac{1}{2}$ and three primaries:

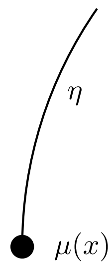
$$1 \; : \; (h, \overline{h}) = (0,0) \; , \qquad \sigma(x) \; : \; (h, \overline{h}) = \left(\frac{1}{16}, \frac{1}{16}\right) \; , \qquad \epsilon(x) \; : \; (h, \overline{h}) = \left(\frac{1}{2}, \frac{1}{2}\right) \; . \tag{1.4.24}$$

The CFT can be realized as the phase transition of the 2d ϕ^4 theory

$$\mathcal{L} = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4 \; . \tag{1.4.25}$$

This has a $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$. For large $m^2 > 0$ the theory is gapped and $\mathbb{Z}_2$ is preserved, while for large $m^2 < 0$ the theory is still gapped, but $\mathbb{Z}_2$ is spontaneously broken and there are two vacua. Hence, there is a phase transition on the m^2 axis (conventionally we can take it at $m^2 = 0$), which turns out to be of second order, and is described by the Ising CFT. $\sigma(x)$ is the low energy limit of $\phi(x)$, while $\epsilon(x) \leftrightarrow \phi^2(x)$. Hence, the $\mathbb{Z}_2$ 0-form symmetry acts on the CFT flipping the sign of $\sigma(x)$, while leaving $\epsilon(x)$ invariant.

Let us denote by η the topological line that generates the $\mathbb{Z}_2$ symmetry. It can end on a point operator $\mu(x)$, the *disorder operator*



Modular invariance of the torus partition function with the η line inserted, relates the trace of the identity operator on the twisted Hilbert space, with the trace of η on the untwisted Hilbert space. This can be used to show that $\mu(x)$ has the same conformal weights $(h, \overline{h}) = \left(\frac{1}{16}, \frac{1}{16}\right)$ as $\sigma(x)$ (see, e.g., [71] for a detailed discussion). Moreover, $\mu(x)$ is neutral under the $\mathbb{Z}_2$ symmetry[11].

Gauging $\mathbb{Z}_2$ amounts to sum over all networks of η, or equivalently a gauge field $A \in H^1(X_2, \mathbb{Z}_2)$. η becomes transparent, while we get a new topological line

$$\widetilde{\eta} = e^{i\pi \int A} \tag{1.4.26}$$

that generates the dual $\mathbb{Z}_2 = \mathbb{Z}_2^\vee$ 0-form symmetry. $\mu(x)$ becomes a genuine local operator and is charged under the dual symmetry. On the other hand $\sigma(x)$ is no longer a gauge invariant operator, since it transforms under a non-trial representation of the gauge symmetry. We can make a gauge

[11] There are other two operators $\psi_L(x), \psi_R(x)$ in the twisted sector, with weights $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$, and they are charged under $\mathbb{Z}_2$.

invariant operator by attaching $\sigma(x)$ at the end of a Wilson line constructed with the $\mathbb{Z}_2$ gauge field: this is nothing but $\widetilde{\eta}$. Therefore, in Ising/$\mathbb{Z}_2$, $\sigma(x)$ lives in the twisted sector of the dual $\mathbb{Z}_2$ symmetry.

Notice that the situation is perfectly symmetric between $\sigma(x)$ and $\mu(x)$ in the two theories, Ising and Ising/$\mathbb{Z}_2$. Gauging $\mathbb{Z}_2$ has just reshuffled the information between the twisted and untwisted sectors, but nothing physical really changed.

Moreover, while most of the things we said are valid in any 2d theory with a $\mathbb{Z}_2$ symmetry, the Ising CFT is completely determined by its primaries and their conformal dimensions. Therefore, the CFT Ising/$\mathbb{Z}_2$ is physically indistinguishable from Ising: the two theories are dual. This is the famous Kramers-Wannier duality [72].

Finally, we notice that we cannot twist this story with discrete torsion, since $H^2(B\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = 0$.

Four-dimensional gauge theories. We consider 4d gauge theories whose gauge group has Lie algebra $\mathfrak{su}(N)$. The discussion here applies to both Yang-Mills theories or gauge theories with only adjoint matter. As emphasized in [42], to completely specify the theory, one needs to choose a global form of the gauge group, and to specify what the set of genuine line operators is inside the lattice of all possible Wilson-'t Hooft lines [73]. All of these choices are obtained one from the other by gauging the 1-form symmetry, possibly with discrete torsion [30].

Let us start from the SU(N) gauge theory. Here we have Wilson lines in all representations that are charged under the electric 1-form symmetry $\mathbb{Z}_N$. To understand the result of the gauging, we use again the description in patches $\mathcal{U}_i$ of the 1-form symmetry, which acts by redefining the transition functions $g_{ij} \mapsto g_{ij}t_{ij}$, $t_{ij} \in \mathcal{Z}(\mathrm{SU}(N)) = \mathbb{Z}_N$, and $t_{ij}t_{jk}t_{ki} = 1$. Turn on a background field $B \in H^2(X_4, \mathbb{Z}_N)$ for the 1-form symmetry. Since $\delta B = 0$, locally $B = \delta t$, with t a locally defined 1-cochain, valued in $\mathbb{Z}_N$. This modifies the transition functions as $g_{ij} \mapsto \widetilde{g}_{ij} = g_{ij}t_{ij}$. However, the new transition functions do not satisfy the SU(N) cocycle condition, but

$$\widetilde{g}_{ij}\ \widetilde{g}_{jk}\ \widetilde{g}_{ki} = w_{ijk} \in \mathbb{Z}_N\ , \qquad w_{ijk} = t_{ij}\ t_{jk}\ t_{ki} = B_{ijk}\ . \tag{1.4.27}$$

This is not a good SU(N) bundle: it is a "singular" bundle, that is a $\mathrm{PSU}(N) = \mathrm{SU}(N)/\mathbb{Z}_N$ bundle whose characteristic class $[w] \in H^2(X_d, \mathbb{Z}_N)$ is fixed to be $w = B$.

The bottom line is that the partition function $Z_{\mathrm{SU}(N)}[B]$ of the SU(N) theory coupled with a background $B \in H^2(X_4, \mathbb{Z}_N)$ for the 1-form symmetry is obtained by performing the path integral over all PSU(N) connections whose characteristic class is $w = B$. If follows immediately that by gauging the 1-form symmetry we get the PSU(N) gauge theory:

$$\sum_{B \in H^2(X_4, \mathbb{Z}_N)} \int_{w=B} D[A_{\mathrm{PSU}(N)}]\, e^{-S} = \int D[A_{\mathrm{PSU}(N)}]\, e^{-S} \tag{1.4.28}$$

The dual symmetry is generated by the surfaces operators

$$\widetilde{U}_\alpha(\Sigma_2) = e^{i\frac{\alpha}{N}\int_{\Sigma_2} B} \tag{1.4.29}$$

and because $B = w$, these are nothing but the topological operators (1.3.28) for the magnetic 1-form symmetry $\pi(\mathrm{PSU}(N))^\vee = \mathbb{Z}_N^\vee \cong \mathbb{Z}_N$: the dual of the electric symmetry is the magnetic one.

Regarding the charged objects, we already know that line operators charged under the magnetic symmetry are the 't Hooft lines. In the SU(N) theory they are indeed non-genuine, and only exist at the boundary of the electric 1-form symmetry generators. In other words, they are twist defects that become genuine after the electric symmetry is gauged. Conversely, the Wilson lines of the SU(N) theory become non gauge invariant in the PSU(N) version. Since we know that gauging the magnetic symmetry of PSU(N) should bring us back to the SU(N) theory, the Wilson lines must be non-genuine and live in the twisted sector of the magnetic symmetry. Specifically, a Wilson line in a representation $\mathfrak{R}$ on $N-$ality $\alpha \in \mathbb{Z}_N$ lives at the boundary of $\widetilde{U}_\alpha$.

From this presentation is appears manifest the parallel with the Ising story: we just replaced 2d with 4d, 0-form with 1-form, and $\mathbb{Z}_2$ with $\mathbb{Z}_N$. Clearly, however, a generic SU(N) gauge theory has many more element and operators that Wilson and 't Hooft lines, hence differently from the Ising case we cannot claim that the SU(N) and the PSU(N) theories are equivalent. It turns out, however, that is some special theories the matching of the 1-form symmetries is one of the ingredients that strongly suggest the existence of a duality between the two theories. This is the case in $\mathcal{N}=4$ SYM theory at $\tau = i$, and the duality is the Montonen-Olive duality, or S-duality [74].

Obviously, there are other global forms of the gauge group. If N has a divisor k, then there is a subgroup $\mathbb{Z}_k \subset \mathbb{Z}_N$ of the center and we can have the group SU(N)/$\mathbb{Z}_k$. It should be clear from the above discussion that this can be realized by gauging the subgroup $\mathbb{Z}_k \subset \mathbb{Z}_N$ of the electric 1-form symmetry of SU(N).

These are not all *global variants* of SU(N) gauge theories: there can be theories with the same gauge group, but a different choice of genuine lines. They are obtained from the SU(N) theory by gauging the 1-form symmetry with discrete torsion. Indeed $H^4(B^2\mathbb{Z}_N, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_{\gcd(N,2)N}$, and we can modify the SU(N) theory coupled with a background $B \in H^2(X_4, \mathbb{Z}_N)$ by stacking an SPT phase[12]

$$\frac{2\pi i r}{2N} \int_{X_4} B \cup B \ , \qquad r = 0, ..., N-1 \ . \tag{1.4.30}$$

As we explained below (1.4.23) this modifies the charges of the twisted sectors, namely the 't Hooft lines here, under the electric 1-form symmetry. In particular, a 't Hooft line H_a of charge $a \in \mathbb{Z}_N$ under the magnetic 1-form symmetry of PSU(N), in the theory SU(N) + $\frac{2\pi i r}{2N} \int_{X_4} B \cup B$ becomes charged under the electric 1-form symmetry, with charge $(ra) \operatorname{mod}(N)$. If this is non-trivial, after gauging H_a will not become a genuine line. This does not mean that the dual symmetry is modified or acts non-faithfully: we can combine H_a with a Wilson line of charge $-ra \operatorname{mod}(N)$ under the electric symmetry, to create a dyonic line. This dyonic line lives in the same twisted sector as H_a is the SU(N) variant, but being uncharged under the electric symmetry becomes genuine after gauging $\mathbb{Z}_N$. The resulting theory is denoted by PSU$(N)_r$. It is a gauge theory with gauge group PSU(N), but where the lines with fundamental charge under the magnetic 1-form symmetry is HW_{-r}, with H the fundamental 't Hooft line and W_{-r} the Wilson line of $N-$ality $(-r) \operatorname{mod}(N)$.

For instance, if N is a prime number, all 't Hooft lines are charged under the electric symmetry, and hence in PSU$(N)_r$ all the lines are dyonic if $r \neq 0$. In total, there are $N+1$ global variants, the SU(N) theory and N PSU$(N)_r$ theories with $r = 0, ..., N-1$.

An other interpretation of r, or more precisely of $\frac{2\pi r}{N}$, is as a *discrete theta-angle*. To understand this, let us consider how the story considered so far is modified by adding a theta-term in the SU(N) theory

$$S_\theta = \frac{i\theta}{8\pi^2} \int_{X_4} \mathrm{Tr}(F \wedge F) = i\theta \int_{X_4} c_2(A) \ . \tag{1.4.31}$$

Here $c_2(A) \in H^4(X_4, \mathbb{Z})$ is the second Chern-class of the SU(N) bundle, and its integral is an integer. Therefore, θ is 2π periodic.

We then gauge $\mathbb{Z}_N$ (without discrete torsion) to obtain the PSU(N) theory. The crucial step (1.4.28) is still valid, but is less innocent than for $\theta = 0$. In fact, in a PSU(N) bundle, the instanton number $\frac{1}{8\pi^2} \int \mathrm{Tr}(F \wedge F)$ is not necessarily an integer, but can be an integral multiple of $1/N$. PSU(N)

[12]Here we assume X_4 to be spin, so that $\int_{X_4} B \cup B$ is even. Hence $r \sim r + N$ also for N even.

bundles with integral instanton numbers are precisely those that can be lifted to $\mathrm{SU}(N)$ bundles, while in general

$$\frac{1}{8\pi^2}\int_{X_4}\mathrm{Tr}(F\wedge F)=\int_{X_4}c_2(A)+\frac{2\pi}{2N}\int_{X_4}w\cup w\ . \tag{1.4.32}$$

The immediate consequence is that θ is no-longer 2π-periodic, but $\theta\sim\theta+2\pi N$. Moreover, since w is identified with $B\in H^2(X_4,\mathbb{Z}_N)$, shifting $\theta\mapsto\theta+2\pi r$ in the $\mathrm{PSU}(N)$ theory is equivalent to modifying the action by

$$\Delta S=\frac{2\pi ir}{2N}\int_{X_4}B\cup B\ . \tag{1.4.33}$$

This is precisely the discrete torsion (SPT) (1.4.30) we added to construct the $\mathrm{PSU}(N)_r$ theory. Therefore, different gauge theories with $\mathrm{PSU}(N)$ gauge group are obtained one from the other shifting θ by an integer multiple of 2π.

This fact also provides a nice physical interpretation for the modification of the genuine line operators in the various versions: the 't Hooft lines get a charge under the electric 1-form symmetry if $r\neq 0$ because of the Witten effect [75]. A magnetic monopole of magnetic charge q_m acquires an electric charge in the presence of a theta-angle

$$q_e=\frac{q_m\theta}{2\pi}\ . \tag{1.4.34}$$

The integer part of $\frac{\theta}{2\pi}$ precisely reproduce the charge of the 't Hooft line under the electric 1-form symmetry.

4d $\mathcal{N}=1$ SU(N) SYM. 4d $\mathcal{N}=1$ SYM theory with gauge group $\mathrm{SU}(N)$ provides an interesting example of a mixed 't Hooft anomaly involving the 1-form symmetry. The theory has a chiral $\mathbb{Z}_{2N}$ 0-form symmetry acting on the gaugino

$$\mathbb{Z}_{2N}:\qquad \lambda\to e^{\frac{2\pi im}{2N}}\lambda\ ,\qquad\qquad m=0,...,2N-1 \tag{1.4.35}$$

and we have already seen that it has a pure anomaly. However, there is also a 1-form symmetry $\mathbb{Z}_N$, and the two have a mixed anomaly.

To see this we use that the the theory has a classical symmetry $U(1)_R$, $\lambda\to e^{i\alpha}\lambda$ which does not leave invariant the path integral measure, changing it by the exponential of

$$\delta_\alpha S=\frac{i\alpha N}{4\pi^2}Tr(F\wedge F)\ . \tag{1.4.36}$$

This is equivalent to a shift of the theta-angle

$$\lambda\to e^{i\alpha}\lambda\iff\theta\to\theta+2N\alpha\ . \tag{1.4.37}$$

The subgroups $\mathbb{Z}_{2N}\subset U(1)_R$ remains at the quantum level because for $\alpha=\frac{2\pi r}{2N}$, θ shifts by $2\pi r$. However, this is not a trivial operation as soon as we turn on a background field $B\in H^2(X_4,\mathbb{Z}_N)$ for the 1-form symmetry. As we have seen, the action shifts by (1.4.33). Unlike what happened in the $\mathrm{PSU}(N)$ theory, we do not go to a different global variant, but we multiply the path integral by a B-dependent phase. To be more precise, consider coupling the theory both to B and a background field $\mathcal{A}\in H^1(X_4,\mathbb{Z}_{2N})$ for the $\mathbb{Z}_{2N}$ 0-form R-symmetry. A gauge transformation $\mathcal{A}\to\mathcal{A}+\delta\eta$ with $\eta=r\in\mathbb{Z}_{2N}$ constant is equivalent to $\theta\to\theta+2\pi r$ and produces a phase in the path integral

$$\exp\left(\frac{2\pi ir}{2N}\int_{X_4}B\cup B\right)\ . \tag{1.4.38}$$

This cannot be cancelled by modifying the action with a local counterterm of $\mathcal{A}$ and B. It represents a mixed anomaly between $\mathbb{Z}_{2N}$ and $\mathbb{Z}_N$ with inflow action

$$S_{\text{inflow}} = \frac{2\pi i}{2N} \int_{X_5} \mathcal{A} \cup B \cup B \ . \tag{1.4.39}$$

A very similar anomaly exists in pure SU(N) Yang-Mills (YM) theory at $\theta = \pi$, involving time reversal and the 1-form center symmetry, and provides a very strong evidence that time reversal is spontaneously broken [76].

1.5 Symmetry fractionalization

The coexistence of various symmetries, generally of different degrees, can make the computation of their anomalies ambiguous, as pointed out in [15, 53, 54] which we briefly review. The phenomenon discussed here will be relevant for Chapter 6 and Section 9.3.3. The slogan is that:

- Since anomalies are a lack of invariance of the path-integral under background gauge transformations, they can be uniquely determined only once we specify completely how the theory couples to background fields.

If the theory has various symmetries, the coupling to the background can mix them by activating certain *fractionalization classes.*

Consider, for instance, a 4d theory with a finite Abelian 0-form symmetry G, and a finite Abelian 1-form symmetry $\mathbb{A}$, with mixed anomaly[13]

$$S_{\text{mixed}} = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}(B) \tag{1.5.2}$$

with $A \in H^1(X, G)$, $B \in H^2(X, \mathbb{A})$ and $\mu \in H^1\left(BG, H^4(B^2\mathbb{A}, \mathbb{R}/\mathbb{Z})\right)$ the class specifying the anomaly. The mixed anomaly of 4d $\mathcal{N} = 1$ SYM is an example of this for $G = \mathbb{Z}_{2N}$ and $\mathbb{A} = \mathbb{Z}_N$.

The 0-form symmetry can also have a pure, cubic anomaly

$$S_{\text{pure}} = 2\pi i k \int_{X_5} A \cup \beta(A)^2 \tag{1.5.3}$$

specified by a class $k \in H^5(BG, \mathbb{R}/\mathbb{Z})$. However, its value is ambiguous. It can be shifted by modifying how the theory couples to the background fields, formally replacing

$$B \mapsto B' = B + A^*(\eta) \tag{1.5.4}$$

where $\eta \in H^2(BG, \mathbb{A})$ is called *fractionalization class.* This modifies S_{pure} by

$$\Delta S_{\text{pure}} = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}(A^*(\eta)) = 2\pi i \Delta k(\mu, \eta) \int_{X_5} A \cup \beta(A)^2 \ . \tag{1.5.5}$$

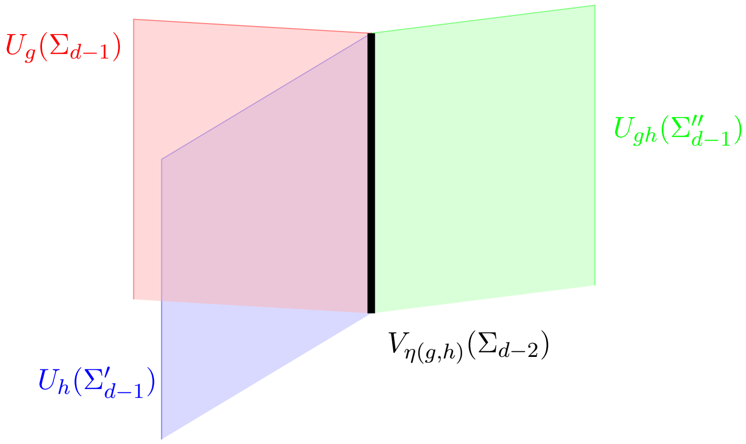
$\Delta k(\mu, \eta) \in H^5(BG, \mathbb{R}/\mathbb{Z})$ is the shift of the anomaly.

[13] See Appendix C for discussion of this topological action. The bottom line is that $\mathfrak{P}(B) = B^*\mathfrak{P} \in H^4(X, \Gamma(\mathbb{A}))$, with $\mathfrak{P} \in H^4(B^2\mathbb{A}, \Gamma(\mathbb{A}))$ the Pontryagin square, and $\Gamma(\mathbb{A}) \cong H^4(B^2\mathbb{A}, \mathbb{R}/\mathbb{Z})^\vee$ the universal quadratic group. The anomaly is specified by a class $\mu \in H^1\left(BG, H^4(B^2\mathbb{A}, \mathbb{R}/\mathbb{Z})\right)$ and written as

$$2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}(B) \ , \tag{1.5.1}$$

with the cup product associated with the canonical pairing $\Gamma(\mathbb{A})^\vee \times \Gamma(\mathbb{A}) \to \mathbb{R}/\mathbb{Z}$.

The shift (1.5.4) has the following interpretation. If we started from a reference coupling of G and $\mathbb{A}$ to the backgrounds, the shift tells us that the *new* background for $\mathbb{A}$ is $B + A^*(\eta)$. In particular, a background A for the 0-form symmetry also activates a background for the 1-form symmetry. At the level of topological defects, a network of G defects must also introduce $\mathbb{A}$ defects. Since $\eta \in H^2(BG, \mathbb{A})$, $A^*(\eta)$ is quadratic in A, which means that the (codimension two) junctions of codimension one defects $U_g(\Sigma_{d-1})$ of the 0-form symmetry are *dressed* by a defect $V_a(\Sigma_{d-2})$ of the 1-form symmetry, with $a = \eta(g, h)$:



The fact that $\eta(g, h)$ must be a 2-cocycle is a consequence of requiring associativity of the fusion of G-defects. This is the same as asking the absence of a non-split 2-group structure [15, 53].[14].

Since there are lines charged under the defects $V_a(\Sigma_{d-2})$ of the 1-form symmetry $\mathbb{A}$, a non-trivial fractionalization class $\eta \in H^2(BG, \mathbb{A})$ implies that these lines transform non-trivially under the 0-form symmetry G. Since this effect arises only by putting two G-defects together, η specifies a *projective representation* of the lines under the 0-form symmetry.

This observation has a beautiful physical interpretation that explains the name *symmetry fractionalization*. The theory we are discussing can arise as the IR of some microscopic model where the 1-form symmetry is absent because of the presence of massive degrees of freedom that screen the lines at higher energy. The IR lines are thought of as the world-lines of these massive particles. The fact that they transform projectively under G, means that the massive particles in the UV also transform projectively. Hence the faithfully acting 0-form symmetry group in the microscopic theory is a central extension

$$1 \longrightarrow \mathbb{A} \longrightarrow \widetilde{G} \longrightarrow \mathbb{A} \longrightarrow 1 \tag{1.5.6}$$

specified by $\eta \in H^2(BG, \mathbb{A})$.

From the UV point of view, we just started from a larger 0-form symmetry group $\widetilde{G}$, where the only degrees of freedom charged under $\mathbb{A} \subset G$ are also charged under the center of a gauge group $\mathcal{G}$. More precisely, the total (global times gauge) UV group is

$$\left(\widetilde{G} \times \mathcal{G}\right) / \mathbb{A}\ . \tag{1.5.7}$$

The matter charged under $\mathbb{A}$ is massive, and integrating it out gives rise to an emergent 1-form symmetry $\mathbb{A}$ at low energy. The non-trivial extension determines a non-trivial fractionalization class. If the emergent 1-form symmetry has a mixed anomaly with the quotient $G = \widetilde{G}/\mathbb{A}$, then this also determines an emergent anomaly for G (see Chapter 7 for a class of very similar phenomena).

[14]This is measured by the Postnikov class $\beta \in H^3(BG, \mathbb{A})$, that is in fact sometimes called *obstruction to symmetry fractionalization* [77].

Let us remark that not only a mixed anomaly, but also a pure anomaly for the 1-form symmetry can do the job. For instance in 3d the 1-form symmetry can have a pure anomaly that is quadratic in the background field (e.g. if we have a gauge group with a Chern-Simons level), and a non-trivial fractionalization class can induce a pure anomaly for the 0-form symmetry[15].

Although this nice interpretation in terms of symmetry extension in the UV is really tight to the mixture of 0- and 1-form symmetries, the idea that changing the coupling to the background of the higher-form symmetry can modify the 0-form symmetry anomaly is general. If we have a $p-$form symmetry $\mathbb{A}$ and a 0-form symmetry G, the generic junction of $(p+1)$ 0-form symmetry defects $U_{g_1}, ..., U_{g_{p+1}}$ to create a further one $U_{g_1 \cdots g_{p+1}}$, is of codimension $p+1$, and can be dressed by a defect $V_{\eta(g_1,\dots,g_{p+1})}$ of the $p-$form symmetry. Here $\eta \in H^{p+1}(BG, \mathbb{A})$. This amounts to modify the coupling with background fields by shifting

$$B_{p+1} \mapsto B'_{p+1} = B_{p+1} + A^*(\eta) \ , \quad B_{p+1} \in H^{p+1}(X_d, \mathbb{A}) \ , \quad A \in H^1(X_d, G) \ . \tag{1.5.8}$$

A mixed $G - \mathbb{A}$ anomaly, or a pure anomaly for $\mathbb{A}$, will modify the pure anomaly for G. We will continue to refer to this phenomenon as (higher) symmetry fractionalization, although the name is more appropriate to the case $p = 1$[16]. In Section 9.3.3 we will also see a generalization to the case of continuous symmetries.

[15] This requires G to be the product of at least two cyclic groups, since $H^4(B\mathbb{Z}_N, \mathbb{R}/\mathbb{Z}) = 0$.

[16] In Chapter 6, to offer a unified viewpoint, we will use the same name also in the case $p = 0$, where what happens is just coupling the background to a diagonal symmetry.

Chapter 2

A TQFT primer

In this chapter we introduce the basic tools of Topological Quantum Field Theories (TQFTs) that will be of great importance in the following chapters. Physically, TQFTs are a special type of QFTs with vanishing stress-energy tensor. As such, they provide the low-energy effective description of gapped systems. All observables are independent on the space-time metric; hence these theories can be formulated on smooth manifolds, without introducing a Riemannian structure. Intuitively, this means that the theory does not have propagating degrees of freedom. This is the basic fact that allows to define in a mathematically rigorous way the path integral of the theory. We start by reviewing these definitions somewhat abstractly, and we gradually move to more concrete approaches and techniques that will be used in the rest of the thesis.

2.1 TQFT axioms

The idea to define a $d-$dimensional TQFT [78], is to extrapolate the properties that a path integral based on a generally covariant action should have, and promote them to axioms (see [79] for a detailed review on the material of this subsection). We view the path integral as a function of the space-time manifold $\mathcal{M}_d$. This that can be closed, hence the path integral must give back a number $Z(\mathcal{M}_d) \in \mathbb{C}$, the partition function, or can have a boundary $X_{d-1} = \partial\mathcal{M}_d$. In the second case, to get a number, we need to fix boundary conditions for the field $\Phi|_{X_{d-1}} = \varphi$. Hence we have a functional $Z(\mathcal{M}_d)[\varphi]$ of the boundary condtion φ, that we view as an element of a vector space $\mathcal{H}_{X_{d-1}}$ associated with the boundary X_{d-1}. This is a model of a space-like slice, to which the quantum theory associates a states-space. Moreover, if $\mathcal{M}_d$ has two boundaries, the path integral computes the *evolution* from the state in one boundary to that in the other. This is a linear map between the two vector spaces.

The *space of manifolds*, on which the path integral is seen as a function, is formalized in terms of *oriented bordism category* $\text{Bord}_d^{\text{SO}}$. Objects are closed-oriented $(d-1)-$dimensional manifolds X_{d-1}, while morphisms $Y_d : X_{d-1} \to X'_{d-1}$ are oriented bordisms. This is a $d-$dimensional oriented manifold Y_d with boundary $\partial Y_d = X_{d-1} \sqcup \overline{X}'_{d-1}$ (the bar denotes the orientation reversal), together with a choice of incoming and outgoing components, X_{d-1} and $\overline{X}'_{d-1}$, respectively. This choice is a pair of embeddings $\iota_{\text{in}} : X_{d-1} \to \partial Y_d$, $\iota_{\text{out}} : X'_{d-1} \to \partial Y_d$, where the first preserves the orientation, and the second reverses it. Notice that the same manifold Y_d can be viewed as a bordism in various ways, depending on the choices of the in- and out-components. The composition of two bordisms $Y_d : X_{d-1} \to X'_{d-1}$, $Y'_d : X'_{d-1} \to X''_{d-1}$ is obtained as follows. By definition $\partial Y_d = X_{d-1} \sqcup \overline{X}'_{d-1}$, $\partial Y'_d = X'_{d-1} \sqcup \overline{X}''_{d-1}$, so that we can construct a manifold $Y'_d \# Y_d$ by gluing along X'_{d-1}, which appears with opposite orientations in the two factors. The result of the gluing is seen as a bordism $Y'_d \circ Y_d$:

$X_{d-1} \to X''_{d-1}$. The category $\mathrm{Bord}_d^{\mathrm{SO}}$ also has a *monoidal structure*, namely there is a sense of compositing two objects, simply taking the disjoint union of closed $(d-1)-$dimensional manifolds.

Atiyah [78] defines a $d-$dimensional TQFT as a symmetric monoidal functor

$$Z : \mathrm{Bord}_d^{\mathrm{SO}} \to \mathrm{Vec}_{\mathbb{C}} \ . \tag{2.1.1}$$

The target is the monoidal category of topological complex vector spaces, where the monoidal structure is given by the tensor product. The functor assigns a vector space $\mathcal{H}_{X_{d-1}} = Z(X_{d-1})$ to any closed manifold, with the property that

$$Z(X_{d-1} \sqcup X'_{d-1}) = Z(X_{d-1}) \otimes Z(X'_{d-1}) \ . \tag{2.1.2}$$

Given a bordism $Y_d : X_{d-1} \to X'_{d-1}$, the functor Z associates a linear map

$$Z(Y_d) : \mathcal{H}_{X_{d-1}} \to \mathcal{H}_{X'_{d-1}} \ . \tag{2.1.3}$$

Functoriality means that, if $Y_d : X_{d-1} \to X'_{d-1}$ and $Y_d : X'_{d-1} \to X''_{d-1}$ are glued to create $Y'_d \circ Y_d : X_{d-1} \to X''_{d-1}$, the linear map $Z(Y'_d \circ Y_d) : \mathcal{H}_{X_{d-1}} \to \mathcal{H}_{X''_{d-1}}$ is the composition

$$Z(Y'_d \circ Y_d) = Z(Y'_d) \circ Z(Y_d) \ . \tag{2.1.4}$$

This is the basic property that we want to ask to the path integral, namely *locality*: if we perform the path integral on a manifold, we must be able to cut it in smaller pieces, compute the path integral on each of them and then reconstruct the full answer by putting the pieces together.

Let us list some immediate consequences of the definition.

- The empty $d-1$ dimensional manifold is the identity for the disjoint union, hence $\mathcal{H}_{\emptyset} = \mathbb{C}$.
- A closed manifold Y_d can be viewed as a bordism $\emptyset \to \emptyset$, hence the functor produces a linear map $\mathbb{C} \to \mathbb{C}$ that is uniquely identified with a number

  $$Z(Y_d) \in \mathbb{C} \ . \tag{2.1.5}$$

  This is the partition function on the closed manifold.
- Given a manifold Y_d with boundary $\partial Y_d = \overline{X}_{d-1}$, we can view it as bordism in two ways. Either $Y_d^{(\mathrm{out})} : \emptyset \to X_{d-1}$ or $Y_d^{(\mathrm{in})} : \overline{X}_{d-1} \to \emptyset$. In the first case, the functor produces a linear map $\mathbb{C} \to \mathcal{H}_{X_{d-1}}$ that selects a vector

  $$V(Y_d) = |Y_d\rangle \in \mathcal{H}_{X_{d-1}} \tag{2.1.6}$$

  as the image of $1 \in \mathbb{C}$. In the second case, we get $\mathcal{H}_{\overline{X}_{d-1}} \to \mathbb{C}$, namely a linear functional

  $$\alpha(Y_d) = \langle Y_d| \in \mathcal{H}^{\vee}_{\overline{X}_{d-1}} \tag{2.1.7}$$

- The orientation reversal $\overline{Y}_d$ has boundary $\partial \overline{Y}_d = X_{d-1}$, hence we have a bordism $\overline{Y}_d^{(\mathrm{out})} : \emptyset \to \overline{X}_{d-1}$ and a bordism $\overline{Y}_d^{(\mathrm{in})} : X_{d-1} \to \emptyset$. These define respectively $|\overline{Y}_d\rangle \in \mathcal{H}_{\overline{X}_{d-1}}$ and $\langle \overline{Y}_d| \in \mathcal{H}^{\vee}_{X_{d-1}}$. By composing $Y_d^{\mathrm{out}} : \emptyset \to X_{d-1}$ above with $\overline{Y}_d^{\mathrm{in}} : X_{d-1} \to \emptyset$ we get a bordism $\mathcal{M}_d : \emptyset \to \emptyset$ associated with the closed manifold constructing by gluing Y_d with its orientation reversal along the common boundary. The functor gives

  $$\mathcal{Z}(\mathcal{M}_d) = \langle \overline{Y}_d | Y_d \rangle = \big\langle V(Y_d) \, , \ V(Y_d) \big\rangle \in \mathbb{C} \ , \tag{2.1.8}$$

that is the *norm* of the vector $V(Y_d)$. More generally, given two manifolds Y_d, Y'_d with the same boundary $\overline{X}_{d-1}$ we can glue them constructing a closed manifold $Y_d \# \overline{Y}'_d$ and its partition function is

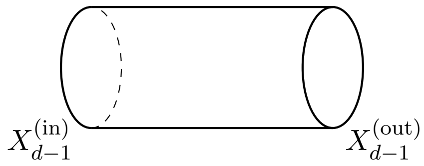
$$Z(Y_d \# \overline{Y}'_d) = \langle \overline{Y}'_d | Y_d \rangle = \big\langle V(Y'_d) \, , \, V(Y_d) \big\rangle \in \mathbb{C} \, , \tag{2.1.9}$$

which is the scalar product between states $V(Y_d), V(Y'_d)$.

Notice that we have *not* endowed the Hilbert spaces $\mathcal{H}_{X_{d-1}}$ with an extra inner structure: this is automatically given by the functor Z and the composition of bordisms. However, if we want a unitary TQFT we need to further require the scalar product to be Hermitian and positive definite.

Cylinder constructions Given a closed X_{d-1} we can construct the cylinder $X_{d-1} \times [0, 1]$. It can be viewed as a bordism in three distinct ways.

1. As $I : X_{d-1} \to X_{d-1}$. We can represent it graphically as a *straight* cylinder:

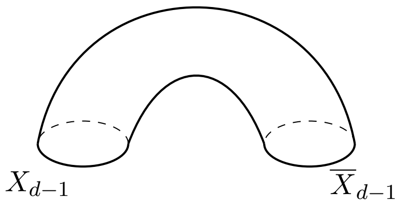
$I =$ 

   This satisfies $I^2 = I$, that is, $Z(I) : \mathcal{H}_{X_{d-1}} \to \mathcal{H}_{X_{d-1}}$ is a projector. Up to restricting every vector space to the image of $Z(I)$, we can assume[1]

$$Z(I) = \text{id}_{\mathcal{H}_{X_{d-1}}} \, . \tag{2.1.10}$$

   Physically, this is the statement that the time-evolution is trivial, hence the Hamiltonian on any space-like slice is zero, and so is the stress-tensor. In particular, the theory does not have any propagating degree of freedom.

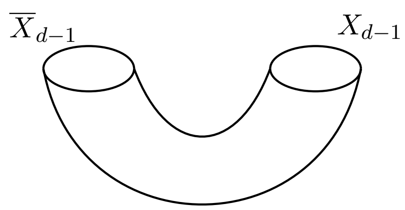
2. As a bordism $X_{d-1} \sqcup \overline{X}_{d-1} \to \emptyset$, that we represented as the *downwards horseshoe*



   The image under the functor $\eta(X_{d-1}) : \mathcal{H}_{X_{d-1}} \otimes \mathcal{H}_{\overline{X}_{d-1}} \to \mathbb{C}$ is a bilinear pairing between $\mathcal{H}_{X_{d-1}}$ and $\mathcal{H}_{\overline{X}_{d-1}}$. Choosing two basis $\{e_a\}$, $\{\overline{e}_a\}$ this can be represented by

$$\eta_{ab} = \eta(e_a, \overline{e}_b) \in \mathbb{C} \, . \tag{2.1.11}$$

3. As a bordism $\emptyset \to \overline{X}_{d-1} \sqcup X_{d-1}$, that we represent graphically as the *upwards horseshoe*



[1] Without this restriction the theory is non-unitary

The image under the functor $\gamma(X_{d-1}) : \mathbb{C} \to \mathcal{H}_{\overline{X}_{d-1}} \otimes \mathcal{H}_{X_{d-1}}$ gives a distinguished vector, as the image of 1. In a basis

$$\gamma(X_{d-1})(1) = \sum_{a,b} \gamma^{ab}\, \overline{e}_a \otimes e_b \ . \tag{2.1.12}$$

These three maps constructed out of the cylinder have a relation among them: gluing the left out-going boundary of the upwards horseshoe with the right in-coming boundary of the downwards horseshoe we get the cylinder $X_{d-1} \to X_{d-1}$. Similarly gluing the right out-going boundary of the upwards horseshoe with the left in-coming boundary of the downwards horseshoe gives the oppositely oriented cylinder $\overline{X}_{d-1} \to \overline{X}_{d-1}$. These translate into a conditions for η_{ab} and γ^{ab}:

$$\sum_{ab} \eta(v, \overline{e}_a) e_b \gamma^{ab} = v \ , \qquad \sum_{ab} \eta(e_b, \overline{v}) \overline{e}_a \gamma^{ab} = \overline{v} \ , \qquad \forall v \in \mathcal{H}_{X_{d-1}} \ , \ \ \forall \overline{v} \in \mathcal{H}_{\overline{X}_{d-1}} \ . \tag{2.1.13}$$

Setting $v = e_c, \overline{v} = \overline{e}_c$ we get

$$\sum_{a} \eta_{ca} \gamma^{ab} = \delta_c^b \ , \qquad \sum_{b} \gamma^{ab} \eta_{bc} = \delta_b^a \tag{2.1.14}$$

namely γ and η are one the inverse of the other. In particular $\eta(X_{d-1})$ gives a non-degenerate pairing that produces an isomorphism

$$\mathcal{H}_{\overline{X}_{d-1}} \cong \mathcal{H}^\vee_{X_{d-1}} \ . \tag{2.1.15}$$

Gluing rules Practically, to define a TQFT in this language, one wants to define a certain set of data, namely the result under the functor of simple bordisms that can be glued together to construct more complicated ones. The objects just introduced (η and γ) are important to *reverse the orientations.* In particular η can convert an out-going boundary X_{d-1} into in in-coming boundary $\overline{X}_{d-1}$, while γ does the opposite.

In practice, we choose a basis for any Hilbert space $\mathcal{H}_{X_{d-1}}$, and for any $d-$dimensional manifold Y_d with connected boundary components incoming and outgoing, respectively $X^i_{d-1,\text{in}}$, $i = 1, 2, ...$ and $X^j_{d-1,\text{out}}$, $j = 1, 2, ...$ we assign a tensor $Z(Y_d)_{\{a_i\},\{b_j\}}$. This specifies the linear map $\bigotimes_i \mathcal{H}_{\text{in},i} \to \bigotimes_j \mathcal{H}_{\text{out},j}$

$$Z(Y_d) \left(e_{a_1} \otimes e_{a_2} \otimes \cdots \right) = \sum_{b_j} Z(Y_d)_{\{a_i\},\{b_j\}} \left(e_{b_1} \otimes e_{b_2} \otimes \cdots \right) \ . \tag{2.1.16}$$

Clearly, a disconnected bordism gives a tensor that is just the product of the two. Then we can start attaching various pieces and generate other tensors. The common boundaries along which we glue must have opposite orientations. However, while the gluing is simply the composition in the case in which one boundary is incoming and the other is outgoing, when are both incoming (or both outgoing) we use $\eta_{a,b}$. More concretely, let Y_d be a (possibly disconnected) bordism $\sqcup_i X^i_{d-1,\text{in}} \to \sqcup_j X^j_{d-1,\text{out}}$. If $X^1_{d-1,\text{in}} = X^1_{d-1,\text{out}}$, then we can generate $\widetilde{Y}_d$ by gluing the two, and the associated tensor is

$$Z\left(\widetilde{Y}_d\right)_{\{a_2,\ldots,\},\{b_2,\ldots\}} = \sum_{a_1} Z\left(Y_d\right)_{\{a_1,a_2,\ldots,\},\{a_1 b_2,\ldots\}} \ . \tag{2.1.17}$$

If instead $X^1_{d-1,\text{in}} = \overline{X}^2_{d-1,\text{in}}$, the tensor associated with the manifold obtained by gluing along these boundary components is

$$Z\left(\widetilde{Y}_d\right)_{\{a_3,\ldots\},\{b_1\}} = \sum_{a_1,a_2} Z(Y_d)_{\{a_1,a_2,a_3,\ldots\},\{b_1,\ldots\}} \eta\left(X^1_{d-1,\text{in}}\right)_{a_1,a_2} \ . \tag{2.1.18}$$

An immediate consequence of these considerations concerns the partition function on $X_{d-1} \times S^1$. In fact, this manifold can be obtained from the straight cylinder by gluing the outgoing boundary with the incoming one. Equivalently, it is obtained gluing the upwards and downwards horseshoes, and the result is the trace of $\mathrm{id}_{\mathcal{H}_{X_{d-1}}}$, that is

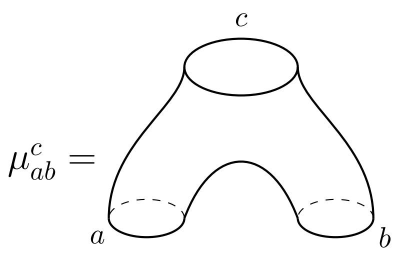
$$Z(X_{d-1} \times S^1) = \dim \mathcal{H}_{X_{d-1}} \tag{2.1.19}$$

All of these pieces of data cannot be assigned randomly. Indeed, if a manifold can be obtained in two different ways by gluing smaller pieces, the results must coincide. Ideally, one would like to define the tensors associated with the few bordisms that constitute the building blocks of all possible manifolds, and write down a finite set of consistency conditions that ensure the independence on the decomposition. In practice, this is incredibly complicated except in very low dimensions (2 and 3), because we do not know what these building blocks are.

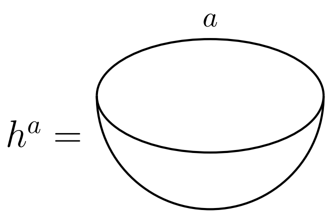
Two-dimensional TQFTs In 2d the situation is particularly easy for two reasons.

- Any 1-dimensional closed manifold is the disjoint union of circles, hence the only Hilbert space we need to define is $\mathcal{H}_{S^1}$.
- Any 2-dimensional manifold has a decomposition in pair of pants.

Because of the second point, the only other bordism that one needs on top of the horseshoes, is the pair of pants $\mu : \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1}$



required to be symmetric. This endows $\mathcal{H}_{S^1}$ with an algebra structure. Sometimes it is also necessary, in order to construct closed 2-manifolds, to *fill the holes* attaching a disk, hence we also need to specify



that defines a distinguished state

$$|HH\rangle = \sum_a h^a e_a \in \mathcal{H}_{S^1} \tag{2.1.20}$$

called the *Hartle-Hawking state*. It behaves as the unit for the algebra defined by μ. Hence μ^c_{ab} and h^a are required to satisfy

$$\sum_b \mu^c_{ab} h^b = \delta^c_a \ . \tag{2.1.21}$$

The only other consistency condition is the independence on the pair of pants decomposition, which is reduced to the so-called *Froboenius condition*. This is nothing but the associativity of the product μ:

$$\sum_c \mu^c_{a,b} \mu^e_{c,d} = \sum_c \mu^e_{a,c} \mu^c_{b,d} \ . \tag{2.1.22}$$

These facts completely characterize 2d TQFTs, and imply the famous result that these theories are classified by commutative Froboenius algebras [80].

Example: 2d $\mathbb{Z}_N$ gauge theories Let us consider a simple non-trivial example in 2d. We take $\mathcal{H}_{S^1} \cong \mathbb{C}^N$ with a basis $|a\rangle$, $a = 0, ..., N$. It is easy to verify that the data

$$h^a = \delta^{a,0} \ , \qquad \eta_{ab} = \delta_{a,-b} \ , \qquad \mu_{ab}^c = \delta_{a+b,c} \tag{2.1.23}$$

satisfy (2.1.21) and (2.1.22), hence they define a good TQFT. The sphere S^2 is obtained by gluing two disks, hence we have $Z(S^2) = 1$, while we already know that $Z(T^2) = \dim \mathcal{H}_{S^1} = N$. A torus with a boundary, instead, is constructed by gluing one in-coming boundary of a pair of pants with the out-going boundary. Denoting by a the label of the remaining boundary the result is

$$Z(T^2 \setminus P_a) = \sum_b \mu_{ab}^b = N\delta_{a,0} \ . \tag{2.1.24}$$

This piece allows to attach a handle to other 2d-manifolds. For instance a genus-two surface can be constructed by gluing two of these pieces along the common boundary, the result being $Z(\Sigma_2) = N^2$. Proceeding in this way we can construct any Riemann surface and in general

$$Z(\Sigma_g) = N^g = \sqrt{|H^1(\Sigma_g, \mathbb{Z}_N)|} \ . \tag{2.1.25}$$

2.2 Discrete gauge theories and BF theories

While the approach of the previous subsection is mathematically rigorous and clean, it is not very manageable especially when we want to discuss operators and defects. Moreover, in physics, TQFTs have many connections with non-topological theories, either because a dynamical theory can flow to a TQFT in the IR, or because we may want to couple a dynamical theory with a TQFT. To this aim, we introduce a more heuristic approach based on topological actions and path integrals. However, it is important to keep in mind the axiomatic formulation, which is sometimes useful to fix some subtleties in the path integral formulation.

2.2.1 Discrete gauge theories

Cohomological formulation We start with path integrals in terms of discrete gauge fields $A \in H^p(X_d, \mathbb{Z}_N)$. First, consider a vanishing action, hence the path integral is just a finite sum

$$Z = C(X_d) \sum_{A \in H^p(X_d, \mathbb{Z}_N)} \ , \tag{2.2.1}$$

where $C(X_d)$ is normalization constant, generically depending on X_d, that we will fix later on by comparison with the axiomatic formalism. This theory is called the pure $p-$form $\mathbb{Z}_N$ gauge theory.

One class of operators that we can consider are *electric* defects

$$U_\chi(\gamma_p) = \exp\left(2\pi i \frac{\chi}{N} \int_{\gamma_p} A\right) \ , \qquad \chi \in \mathbb{Z}_N^\vee \cong \mathbb{Z}_N \ , \tag{2.2.2}$$

with γ_p any $p-$cycle, possibly also homologically trivial. Since A is closed, the operators are topological. There are also *magnetic* (or 't Hooft) defects $V_a(\gamma'_{d-p-1})$ supported on $d-p-1$ dimensional cycles γ'_{d-p-1}, and labeled by $a \in \mathbb{Z}_N$. This is defined declaring that inserting it into a correlator modifies the path integral, summing over discrete gauge fields on $X_d \setminus \gamma'_{d-p-1}$ such that

$$\int_{S^p} A = a \tag{2.2.3}$$

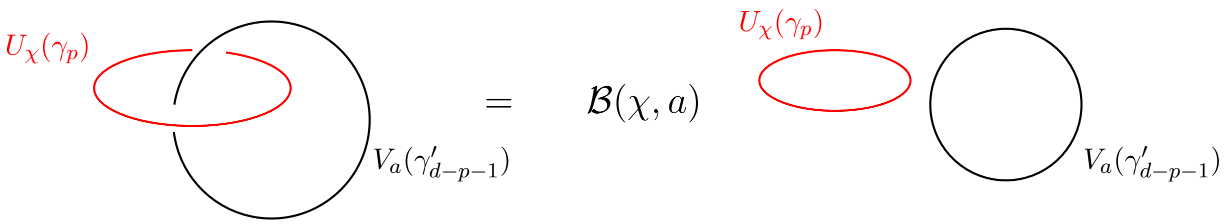
	Group	Topological operators	Charged objects
$p-$ form	$\mathbb{A}$	$V_{a\in\mathbb{A}}(\gamma'_{d-p-1})$	$U_{\chi\in\mathbb{A}^\vee}(\gamma_p)$
$(d-p-1)-$ form	$\mathbb{A}^\vee$	$U_{\chi\in\mathbb{A}^\vee}(\gamma_p)$	$V_{a\in\mathbb{A}}(\gamma'_{d-p-1})$

Table 2.1: Symmetries of the pure $p-$form $\mathbb{A}$ gauge theory in $d-$dimensions.

where S^p is a sphere linking with γ'_{d-p-1}. Hence, by construction

$$\left\langle U_\chi(\gamma_p)\, V_a(\gamma'_{d-p-1})\right\rangle = \exp\left(\frac{2\pi i \chi a}{N}\mathrm{Lk}(\gamma_p,\gamma'_{d-p-1})\right) \tag{2.2.4}$$

where $\mathrm{Lk}(\gamma_p,\gamma'_{d-p-1})$ is the linking number. More generally we can establish an operator equation, namely an identity valid in any correlation function:



$\mathcal{B}(\chi,a)$ is called *braiding phase* and is given by

$$\mathcal{B}(\chi,a) = \exp\left(\frac{2\pi i \chi a}{N}\right) . \tag{2.2.5}$$

Moreover, both classes of U_χ and V_a can be fused, and they follow the group low of $\mathbb{Z}_N^\vee$ and $\mathbb{Z}_N$ respectively.

All of this can be summarized saying that the theory has two (generically) higher-form symmetries: a $p-$forms symmetry $\mathbb{Z}_N$ generated by $V_a(\gamma'_{d-p-1})$ with charged objects $U_\chi(\gamma_p)$, and a $(d-p-1)-$form symmetry $\mathbb{Z}_N^\vee$ generated by $U_\chi(\gamma_p)$ with charged objects $V_a(\gamma'_{d-p-1})$. Notice that this discussion stays untouched if we replace $\mathbb{Z}_N$ with any other finite Abelian group $\mathbb{A}$: the $p-$form symmetry is $\mathbb{A}$, while the $(d-p-1)-$form symmetry is given by its Pontryagin dual $\mathbb{A}^\vee$. This is summarized in table 2.1. The braiding phase is given by the canonical pairing $\mathbb{A}\times\mathbb{A}^\vee \to \mathbb{R}/\mathbb{Z}$:

$$\mathcal{B}(\chi,a) = \exp\left(2\pi i \chi(a)\right) \in U(1) . \tag{2.2.6}$$

Co-chain formulation From this discussion notice that there is a symmetry for the replacement

$$\mathbb{A} \longleftrightarrow \mathbb{A}^\vee \quad , \qquad p \longleftrightarrow p^\vee = d-p-1 \tag{2.2.7}$$

A pure $p-$form $\mathbb{A}$ gauge theory is equivalent to a pure $p^\vee-$form $\mathbb{A}^\vee$ gauge theory. The only difference is that the defects defined *electrically* in one formulation become *magnetic* in the other and vice-versa. This is called electro-magnetic duality of the discrete gauge theory.

However, the way we presented the theory is not manifestly symmetric. A manifestly self-dual formulation is obtained by performing the path integral over general (non-necessarily closed) co-chains $A \in C^p(X_d,\mathbb{A})$, integrating in another co-chain $B\in C^{d-p-1}(X_d,\mathbb{A}^\vee)$, and using the action

$$S = 2\pi i \int_{X_d} A \cup \delta B . \tag{2.2.8}$$

Here $\cup$ is the cup product associated with the canonical pairing $\mathbb{A} \times \mathbb{A}^\vee \to \mathbb{R}/\mathbb{Z}$. Integrating out B we get a delta function imposing $\delta A = 0$, hence we recover the original formulation of the $p-$form $\mathbb{A}$ gauge theory. If we integrate out A instead, the path integral over B is reduced to co-cycles and we get the formulation as a $(d-p-1)-$from $\mathbb{A}^\vee$ gauge theory. In this way of formulating the path integral, we need to mod-out by the redundancies

$$A \mapsto A + \delta\lambda\ , \quad \lambda \in C^{p-1}(X_d, \mathbb{A})\ , \quad B \mapsto B + \delta\eta\ , \quad \eta \in C^{d-p-2}(X_d, \mathbb{A}^\vee)\ . \tag{2.2.9}$$

These are gauge transformations, and modding them out has the very important role of reducing the final path integral from a sum over co-cycles to a sum over cohomology. The formulation (2.2.8) is manifestly symmetric in the two fields, and both classes of defects have an *electric* presentation

$$U_\chi(\gamma_p) = \exp\left(2\pi i \chi \int_{\gamma_p} A\right)\ , \qquad V_a(\gamma'_{d-p-1}) = \exp\left(2\pi i a \int_{\gamma_{d-p-1}} B\right)\ . \tag{2.2.10}$$

Electro-magnetic symmetry A special feature arises if d is odd and

$$p = \frac{d-1}{2}\ . \tag{2.2.11}$$

In this case $p^\vee = p$, and electro-magnetic duality gives rise to a 0-form symmetry of the theory that exchange A and B or, more intrinsically, exchanges U_χ and V_a. Notice that this requires that we specify a map of the labels, namely an isomorphism

$$\phi : \mathbb{A} \xrightarrow{\ \sim\ } \mathbb{A}^\vee\ . \tag{2.2.12}$$

A finite Abelian group is always isomorphic to its Pontryagin dual, but it is non canonically so, hence defining this symmetry requires a choice. Importantly, ϕ cannot be any isomorphism: it must preserve the braiding phase (2.2.6)

$$\phi(a)\left(\phi^{-1}(\chi)\right) = \chi(a)\ . \tag{2.2.13}$$

This condition is made more explicit by introducing the non-degenerate bi-character $\gamma : \mathbb{A} \times \mathbb{A} \to \mathbb{R}/\mathbb{Z}$ as $\gamma(a,b) = \phi(a)b$. Hence the condition is that γ must be symmetric[2].

Naively one may think this symmetry is $\mathbb{Z}_2$. This is not always true. To see this, notice that for $p = p^\vee$ the most general defects are dyons

$$D_{\chi,a}(\gamma_p) = U_\chi(\gamma_p) V_a(\gamma_p) = \exp\left(2\pi i \int_{\gamma_p} (\chi A + aB)\right)\ , \tag{2.2.14}$$

and we have a $p-$forms symmetry group $\mathbb{A} \times \mathbb{A}^\vee$. The braiding $\mathcal{B}\big((\chi_1, a_1), (\chi_2, a_2)\big)$ is a bilinear form on this group, and involve both $\chi_1(a_2)$ and $\chi_2(a_1)$. However there is possibly a relative sign between these two terms, that arise because to compare the two one needs to use the action (2.2.8) integrated by parts, namely $B \cup \delta A$, and this picks a minus sign if p is even. Therefore

$$\mathcal{B}\Big((\chi_1, a_1), (\chi_2, a_2)\Big) = \exp\Big(2\pi i \left(\chi_1(a_2) + (-1)^{p+1} \chi_2(a_1)\right)\Big) \tag{2.2.15}$$

The electro-magnetic symmetry is an automorphism of the $p-$forms symmetry group, $\Phi : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A} \times \mathbb{A}^\vee$, that must preserve the braiding. Since the two terms in (2.2.15) will be exchanged by the action of the symmetry, for p even we need to twist the naive action by a minus sign:

$$\Phi(a, \chi) = \begin{cases} \left(\phi^{-1}(\chi), \phi(a)\right) & p \text{ odd} \\ \left(-\phi^{-1}(\chi), \phi(a)\right) & p \text{ even} \end{cases} \tag{2.2.16}$$

[2]This is the same as requiring that $\phi^\vee = \phi$, where for any homomorphism of Abelian groups $f : \mathbb{A} \to \mathbb{B}$ we define $f^\vee : \mathbb{B}^\vee \to \mathbb{A}^\vee$ as $f^\vee(\beta)a = \beta(f(a))$.

For this reason $\Phi^2 = -1$ if p is even. The -1 automorphism is typically called charge conjugation and denoted by C. Hence charge conjugation extends non-trivially electromagnetic duality for p even, while it is a decoupled factor for p odd. Thus the electro-magnetic symmetry is

$$G_{\text{em}} = \begin{cases} \mathbb{Z}_2 & p \text{ odd} \\ \\ \mathbb{Z}_4 & p \text{ even} \end{cases} \tag{2.2.17}$$

Dijkgraaf-Witten theories: two dimensions The discrete gauge theories have certain *twisted* modifications, known as Dijkgraaf-Witten (DW) theories [81]. Let us first focus on the case $p = 1$, and consider a group-cohomology class $\omega \in H^d(B\mathbb{A}, \mathbb{R}/\mathbb{Z})$. The discrete gauge field $A \in C^1(X_d, \mathbb{A})$ can be equivalently represented as map $A : X_d \to B\mathbb{A}$, that we can use to pull-back ω and produce $A^*\omega \in H^d(X_d, \mathbb{R}/\mathbb{Z})$. In the cohomological formulation, where $A \in H^1(X_d, \mathbb{A})$ and we do not introduce B, the DW theory consists in modifying the path integral into

$$Z = C(X_d) \sum_{A \in H^1(X_d, \mathbb{A})} e^{2\pi i \int_{X_d} A^*(\omega)} \tag{2.2.18}$$

In the cochain formulation, on the other hand, we modify the action as

$$S = 2\pi i \int_{X_d} \left(A \cup \delta B + A^*(\omega) \right) \ . \tag{2.2.19}$$

To make it gauge invariant we need to modify the gauge transformations, in a way that $A \mapsto A + \delta\lambda$ also acts on B. The precise modification must be studied case-by-case. Dijkgraaf-Witten theories can also be generalized to $p > 1$, but they are not classified by group cohomology. Indeed $A \in C^p(X_d, \mathbb{A})$ can be realized as a map $A : X_d \to B^{p+1}\mathbb{A}$, and this can be used to pull-back a class $\omega \in H^d(B^{p+1}\mathbb{A}, \mathbb{R}/\mathbb{Z})$.

Here we discuss in some detail the 2d case for illustration, leaving some higher-dimensional cases for the coming sections. A key fact (that will be of utmost importance in Chapter 6) is that $H^2(B\mathbb{A}, \mathbb{R}/\mathbb{Z})$ is canonically isomorphism to the group $\text{Alt}(\mathbb{A})$ of alternating bicharacters on $\mathbb{A}$, namely

$$\chi(a + a', b) = \chi(a, b) + \chi(a', b) \ , \qquad \chi(a, a) = 0 \ . \tag{2.2.20}$$

In particular χ is antisymmetric. The isomorphism is [69][3]

$$H^2(B\mathbb{A}, \mathbb{R}/\mathbb{Z}) \ni \omega \mapsto \chi_\omega(a, b) = \omega(a, b) - \omega(b, a) \ . \tag{2.2.21}$$

This is clearly a well defined homomorphism (it is independent on the representative of ω), and χ_ω is alternating. The proof that it is an isomorphism is more complicated and can be found in [69]. The alternating bicharacter defines an antisymmetric pairing $\mathbb{A} \times \mathbb{A} \to \mathbb{R}/\mathbb{Z}$. This allows us to construct a cup product $\cup_{\chi_\omega}$, that is non-vanishing on cochains of odd degree. The DW action is

$$S = 2\pi i \int_{X_2} A \cup \delta B + A \cup_{\chi_\omega} A \ . \tag{2.2.22}$$

For instance if $\mathbb{A}$ is the direct product of K $\mathbb{Z}_N$ factors, an alternating bicharacter is given by an antisymmetric $K \times K$ matrix χ_{ij} with $\mathbb{Z}_N$ entries, a 1-cochain is given by $A = (A_1, ..., A_K)$ with A_i a $\mathbb{Z}_N$ gauge field (and similarly for the 0-cochain $B = (B_1, ..., B_K)$) and the DW twist is [82]

$$\frac{2\pi i}{N} \int_{X_2} \sum_{i<j} \chi_{ij} A_i \cup A_j \ . \tag{2.2.23}$$

[3] Here we are using the additive notation for all Abelian groups. In other places it will be more convenient to use the multiplicative notation, and this should be clear from the context.

Here $\cup$ is the standard cup product associated with the ring structure of $\mathbb{Z}_N$.

For $\mathbb{A} = \mathbb{Z}_N$ there is no alternating bicharacter, and the simplest example of non-trivial DW twist arises for, $\mathbb{A} = \mathbb{Z}_N \times \mathbb{Z}_N$ (see [30] for more details on this example). An alternating bicharacter is determined by a single number $p \in \mathbb{Z}_N$ the corresponding DW theory is

$$S = \frac{2\pi i}{N} \int_{X_2} (A_1 \cup \delta B_1 + A_2 \cup \delta B_2 + pA_1 \cup A_2) \ . \tag{2.2.24}$$

The gauge transformations are[4]

$$A_1 \mapsto A_1 + \delta\lambda_1 \ , \qquad A_2 \mapsto A_2 + \delta\lambda_2 \ , \qquad B_1 \mapsto B_1 - p\lambda_2 \ , \qquad B_2 \mapsto B_2 + p\lambda_1 \ . \tag{2.2.25}$$

This modifies the set of gauge invariant operators. In fact $V_m^{(i)}(x) = e^{\frac{2\pi i m}{N} B_i(x)}$ is no longer gauge invariant, but can be made so by attaching a line defect to it:

$$\widetilde{V}_m^{(1)}(x; L) = \exp\left(\frac{2\pi i m}{N} B_1(x) + \frac{2\pi i p m}{N} \int_L A_2\right) , \quad \widetilde{V}_m^{(2)}(x; L) = \exp\left(\frac{2\pi i m}{N} B_2(x) - \frac{2\pi i p m}{N} \int_L A_1\right) . \tag{2.2.26}$$

Here L is an open line ending on x. The topological operators supported on open manifolds are essentially trivial[5]: there is no non-trivial configuration of defects we can make with them, and we can always deform them topologically to the trivial defect.

If p and N are coprime, all local operators are non-genuine, and all line operators $e^{\frac{2\pi i n}{N}\int A}$ can be opened on some point operator. Hence the theory is essentially trivial. If $\gcd(N, p) \neq 1$, some local operators become genuine and some of the lines cannot be cut-opened. Defining $k = \frac{N}{\gcd(N,p)}$, we notice that $\widetilde{V}_{lk}^{(i)}$ loose their dependence on L, hence

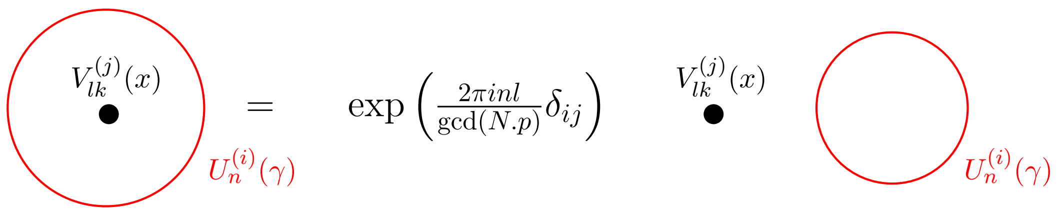
$$\widetilde{V}_{lk}^{(i)}(x; L) = V_{lk}^{(i)}(x) \ , \qquad l = 0, ..., \gcd(N, p) - 1 \in \mathbb{Z}_{\gcd(N,p)} \subset \mathbb{Z}_N \tag{2.2.27}$$

are genuine topological operators.

In addition, the lines that can be opened and become trivial are $U_n^{(i)}(\gamma)$ with $n = pa \bmod(N)$ for some a, or equivalently $n = \gcd(N, p)a' \bmod(N)$ for some a'. These form the subgroup $\mathbb{Z}_k \subset \mathbb{Z}_N$, hence the unbreakable lines are

$$U_n^{(i)}(\gamma) \ , \qquad\qquad n = 0, ..., \gcd(N, p) - 1 \in \mathbb{Z}_N/\mathbb{Z}_k \cong \mathbb{Z}_{\gcd(N,p)} \ . \tag{2.2.28}$$

These lines detect the genuine local operators through non-trivial braiding:



At the level of genuine and unbreakable operators, this theory is essentially equivalent to the untwisted pure $\mathbb{Z}_{\gcd(N,p)} \times \mathbb{Z}_{\gcd(N,p)}$ gauge theory in 2d. However, the theory is different in the fact that the trivialized subgroup $\mathbb{Z}_k \times \mathbb{Z}_k$ has a non-trivial SPT phase, as can be detected by studying the theory on manifolds with boundary [30]. In particular, if $\gcd(N, p) = 1$, the DW theory is an SPT for $\mathbb{Z}_N \times \mathbb{Z}_N$.

[4] B_i are 0-forms, hence they do not have their own gauge transformations.

[5] We will see shortly that non-triviality of these objects arise on space-times with boundary. Intuitively, that is because the end-points can be on the boundary.

2.2.2 Continuous fields formulation

We can introduce another formulation of discrete gauge theories, in terms of standard $U(1)$ gauge fields [30, 83–85]. This approach is mathematically less rigorous, but is often more manageable, and allows one to incorporate TQFTs into dynamical theories. For definiteness, we consider $\mathbb{A} = \mathbb{Z}_N$, but everything can be extended to any finite Abelian group. We consider the *BF theory*

$$S = \frac{iN}{2\pi} \int_{X_d} A_p \wedge dB_{d-p-1} \ . \tag{2.2.29}$$

A_p and B_{d-p-1} are $U(1)$ gauge fields, respectively a $p-$form and a $(d-p-1)-$from. The gauge transformations that we have to mod out in the path integral are

$$A_p \mapsto A_p + d\lambda_{p-1} \ , \qquad B_{d-p-1} \mapsto B_{d-p-1} + d\xi_{d-p-2} \ , \tag{2.2.30}$$

with $d\lambda_{p-1}$ and $d\xi_{d-p-2}$ closed forms with periods multiple of 2π. When they are exact (vanishing periods), this is a small gauge transformation, otherwise it is a large gauge transformation.

This way of writing the theory does not really make sense, since it involves integration of the gauge potential. The problem is very similar to that of correctly defining Wilson lines, and the solutions are also analogous (see [86] for a discussion)

- If we assume the existence of a $(d+1)-$dimensional manifold Y_{d+1} such that $\partial Y_{d+1} = X_{d+1}$, then we can define
$$S := 2\pi i N \int_{Y_{d+1}} \frac{dA_p}{2\pi} \wedge \frac{dB_{d-p-1}}{2\pi} \tag{2.2.31}$$
from which we immediately read the quantization $N \in \mathbb{Z}$ as a requirement for the independence on the extension Y_{d+1}, due to the integrality of the periods of $\frac{dA_p}{2\pi}, \frac{dB_{d-p-1}}{2\pi}$. However, this definition requires the existence of the extension Y_{d+1}, which is not guaranteed for $d \geq 4$, where the bordism group is nontrivial.

- One can define the action patchwise, choosing a local trivialization of all bundles and local representatives of the connection. This approach is quite cumbersome, since it requires precisely defining higher-form bundles, formalized in terms of *gerbes* [87].

- Finally, it is possible to correctly define the action using *differential cohomology* [88] (see [89] for a review).

The gauge invariant operators are

$$U_n(\gamma_p) = \exp\left(in \int_{\gamma_p} A_p\right) \ , \qquad V_m(\gamma'_{d-p-1}) = \exp\left(im \int_{\gamma'_{d-p-1}} B_{d-p-1}\right) \tag{2.2.32}$$

with $n, m \in \mathbb{Z}$ to ensure gauge invariance under large gauge transformations. The equations of motion

$$dA_p = 0 \ , \qquad dB_{d-p-1} = 0 \tag{2.2.33}$$

impose that A_p and B_{d-p-1} are flat connections. Hence, the operators U_n, V_m are topological. Moreover, by taking into account that A_p and B_{d-p-1} are not globally defined differential forms, but $U(1)$ connection, we also have

$$U_N = V_N = 1 \tag{2.2.34}$$

hence $n \sim n+N, m \sim m+N$, and therefore $n, m \in \mathbb{Z}_N$. To show this fact we can decompose one of the fields, say B_{d-p-1}, into a globally defined part $\widetilde{B}_{d-p-1}$ and a representative β of the bundle, namely

$$\frac{d\beta}{2\pi} \in H^{d-p}(X_d, \mathbb{Z}) \ . \tag{2.2.35}$$

The action is rewritten as

$$\frac{iN}{2\pi}\int_{X_d}\left(A_p \wedge d\beta - (-1)^p \widetilde{B}_{d-p-1} \wedge dA_p\right) = iN \int_{\mathrm{PD}^{-1}\left(\frac{d\beta}{2\pi}\right)} A_p - (-1)^p \frac{iN}{2\pi}\int_{X_d} \widetilde{B}_{d-p-1} \wedge dA_p \tag{2.2.36}$$

where $\mathrm{PD}^{-1}\left(\frac{d\beta}{2\pi}\right) \in H_p(X_d, \mathbb{Z})$ is the Poincare' dual cycle of $\frac{d\beta}{2\pi}$, and we can decompose it into a basis

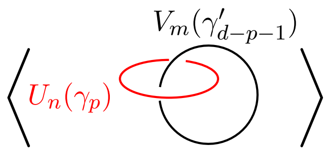
$$\mathrm{PD}^{-1}\left(\frac{d\beta}{2\pi}\right) = \sum_{j=1}^{b_p(X_d)} a_j \Sigma_j \ , \qquad \Sigma_j \in H_p(\Sigma_p, \mathbb{Z}) \ , \quad a_j \in \mathbb{Z} \ . \tag{2.2.37}$$

The path integral over B_{d-p-1} decomposes into a path integral over $\widetilde{B}_{d-p-1}$, and sums over the integers $a_j \in \mathbb{Z}$. The first gives a delta function that imposes $dA_p = 0$. Hence, the path integral over A_p is reduced to an integral over flat $U(1)$ connections. Moreover the sums over a_j give

$$\prod_j \sum_{a_j \in \mathbb{Z}} \exp\left(iN a_j \int_{\Sigma_j} A_p\right) = \prod_j \delta\left(e^{iN\int_{\Sigma_j} A_p} - 1\right) \tag{2.2.38}$$

and this implies $U_N = 1$. The same argument reversing the role of A_p and B_{d-p-1} proves that $V_N = 1$.

It is not hard to compute the correlation function of linking defects



(2.2.39)

Here γ_p, γ_{d-p-1} are both homologically trivial inside X_d, but each one is non-trivial in the space-time where the other is removed. Inserting the defects is equivalent to modifying the action into

$$\int_{X_d}\left(\frac{iN}{2\pi}A_p \wedge dB_{d-p-1} - inA_p \wedge \mathrm{PD}(\gamma_p) - imB_{d-p-1} \wedge \mathrm{PD}(\gamma'_{d-p-1})\right) \ . \tag{2.2.40}$$

Integrating out A_p, we produce a delta function that imposes

$$\frac{N}{2\pi} dB_{d-p-1} = n\mathrm{PD}(\gamma_p) \ . \tag{2.2.41}$$

This is solved by

$$B_{d-p-1} = \frac{2\pi n}{N}\mathrm{PD}(D_{p+1}) + B'_{d-p-1} \tag{2.2.42}$$

with D_{p+1} a disk with boundary γ_p, and B'_{d-p-1} a flat connection. The path integral over it reproduces the partition function, cancelled by the denominator in normalized correlation functions. The result is given by the remaining piece of the action, evaluated in the non-homogeneous term of the solution (2.2.42)

$$\frac{2\pi i n m}{N}\int_{X_d} \mathrm{PD}(D_{p+1}) \wedge \mathrm{PD}(\gamma'_{d-p-1}) \ . \tag{2.2.43}$$

The integral counts the number of intersections of D_{p+1} and γ'_{d-p-1}, namely $\mathrm{Lk}(\gamma_p, \gamma'_{d-p-1}) = 1$:

$$\left\langle U_n(\gamma_p)\, V_m(\gamma'_{d-p-1})\right\rangle = e^{\frac{2\pi i n m}{N}} \ . \tag{2.2.44}$$

This is the same as (2.2.4). Given the matching of gauge-invariant operators and their correlators, the BF theory (2.2.29) is simply another representation of the pure $\mathbb{Z}_N$ gauge theory[6] [84, 85].

We can also deduce a heuristic rule for going from discrete to continuous gauge fields. The discrete gauge field formulation is[7]

$$S = \frac{2\pi i}{N}\int_{X_d} \mathcal{A}_p \cup \delta\mathcal{B}_{d-p-1} \ , \quad \mathcal{A}_p \in C^p(X_d,\mathbb{Z}_N) \ , \quad \mathcal{B}_{d-p-1} \in C^{d-p-1}(X_d,\mathbb{Z}_N) \ . \tag{2.2.46}$$

A discrete field $\mathcal{A}_p$ has integers periods defined mod (N), while the continuous field A_p has periods that are integer multiples of $\frac{2\pi}{N}$. Also, by matching the operators (2.2.2) with (2.2.32), we are led to identify $\mathcal{A}_p$ with $\frac{N}{2\pi}A_p$. The $\cup$ is identified with the $\wedge$. To match δ with the ordinary differential, one notices that $\frac{\delta\mathcal{B}_{d-p-1}}{N}$ is the Bockstein homomorphism, that is the discrete analogue of the first Chern-class $\frac{dB_{d-p-1}}{2\pi}$. To summarize we got the dictionary

$$\mathcal{A}_p \longleftrightarrow \frac{N}{2\pi}A_p \ , \qquad \cup \longleftrightarrow \wedge \ , \qquad \frac{\delta\mathcal{B}_{d-p-1}}{N} \longleftrightarrow \frac{dB_{d-p-1}}{2\pi} \ . \tag{2.2.47}$$

As a check notice that this replacement maps (2.2.46) into (2.2.29).

Mixed anomaly The $p-$form $\mathbb{A}$ gauge theory has a $p-$form symmetry $\mathbb{A}$ and a $(d-p-1)-$form symmetry $\mathbb{A}^\vee$, and, in virtue of the braiding, the two symmetries have a mixed anomaly: inserting the defects of the two symmetries is equivalent to coupling them to background fields, and un-linking two defects is a background gauge transformation that modifies the partition function by a phase.

Using the BF theory formulation (2.2.29), allows us to describe the inflow action concretely. Here we focus on $\mathbb{A} = \mathbb{Z}_N$. We describe background fields with flat $U(1)$ gauge fields $\alpha_{p+1}, \beta_{d-p}$ with periods multiple of $\frac{2\pi}{N}$. Coupling the $p-$form symmetry with α_{p+1} amounts to modifying the action into

$$S[\alpha_{p+1}] = \frac{iN}{2\pi}\int_{X_d} \left(A_p \wedge dB_{d-p-1} + (-1)^p \alpha_{p+1} \wedge B_{d-p-1}\right) \ . \tag{2.2.48}$$

Indeed, a way to understand the $p-$form symmetry in the continuous fields formulation is as a shift $A_p \mapsto A_p + \lambda_p$, with $d\lambda_p = 0$ and $\int_{\gamma_p}\lambda_p \in \frac{2\pi}{N}\mathbb{Z}$. The coupling above makes the action invariant, even if it is $d\lambda p \neq 0$, provided that we transform $\alpha_{p+1} \mapsto \alpha_{p+1} + d\lambda_p$.

Similarly, coupling the $(d-p-1)-$symmetry with β_{d-p} is achieved by

$$S[\beta_{d-p}] = \frac{iN}{2\pi}\int_{X_d} \left(A_p \wedge dB_{d-p-1} - A_p \wedge \beta_{d-p}\right) \ , \tag{2.2.49}$$

with $\beta_{d-p} \mapsto \beta_{d-p} + d\xi_{d-p-1}$ while shifting $B_{d-p-1} \mapsto B_{d-p-1} + \xi_{d-p-1}$.

[6] There is a caveat in this statement. In the BF theory, it makes sense to consider non-genuine $p-$dimensional and $(d-p-1)-$dimensional operators,

$$\widetilde{U}_\alpha(D_{p+1}) = \exp\left(i\alpha\int_{D_{p+1}}\frac{dA_p}{2\pi}\right) \ , \qquad \widetilde{V}_\beta(D'_{d-p}) = \exp\left(i\beta\int_{D'_{d-p}}\frac{dB_{d-p-1}}{2\pi}\right) \tag{2.2.45}$$

with D_{p+1} and D'_{d-p} disks bounded by γ_p and γ'_{d-p-1}. Here $\alpha, \beta \in \mathbb{R}$, and these operators become genuine and coincide with U_n, V_m for integer values. For $\alpha, \beta \in (0,1)$, they have no analogue in the discrete $\mathbb{Z}_N$ gauge theory. If the boundary γ_p of D_{p+1} links with the support γ'_{d-p-1} of V_m, it creates a braiding phase $\exp\frac{2\pi i\alpha m}{N}$ that breaks the identification $m \sim m+N$. Thus, with these additional operators, $U_N = V_N = 1$ is not true. Indeed, the argument around (2.2.36) is invalid if such operators are in the path integral. However, it is consistent to consider the BF theory truncation excluding these continuous operators. This truncation coincides with the $\mathbb{Z}_N$ gauge theory.

[7] We are changing notation with respect to the previous discussion to distinguish discrete from continuous.

The ability of making the action gauge invariant by coupling the two symmetries singularly to backgrounds, reflects the absence of pure anomalies. The mixed anomaly, on the other hand, is detected by coupling *both* symmetries to backgrounds:

$$S[\alpha_{p+1}, \beta_{d-p}] = \frac{iN}{2\pi} \int_{X_d} \Big(A_p \wedge dB_{d-p-1} + (-1)^p \alpha_{p+1} \wedge B_{d-p-1} - A_p \wedge \beta_{d-p} \Big) \, . \tag{2.2.50}$$

Notice that there is no topological local counterterm that can be added. This is not gauge invariant:

$$S[\alpha_{p+1}+d\lambda_p, \beta_{d-p}+d\xi_{d-p-1}] = S[\alpha_{p+1}, \beta_{d-p}]+\frac{iN}{2\pi} \int_{X_d} \Big((-1)^p \alpha_{p+1}\wedge\xi_{d-p-1}+(-1)^p d\lambda_p\wedge\xi_{d-p-1}-\lambda_p\wedge\beta_{d-p} \Big) \, . \tag{2.2.51}$$

The anomalous variation can be cancelled by inflow with the SPT

$$S_{\text{inflow}} = \frac{iN}{2\pi} \int_{X_{d+1}} \alpha_{p+1} \wedge \beta_{d-p} \, . \tag{2.2.52}$$

Anomalies that are quadratic in the background fields are tight in having symmetry generators charged among themselves by braiding.

Example: canonical quantization 2d $\mathbb{Z}_N$ gauge theories. The continuous fields formulation is useful to canonically quantize the TQFT. We consider the example of $d = 2$ and $\mathbb{A} = \mathbb{Z}_N$, which allows us to make a connection with the axiomatic formulation of Section 2.1 and the example discussed at the end of that Section (see Appendix E for a discussion of the non-compact case, relevant for Chapters 8 and 9).

We discuss the theory

$$S = \frac{iN}{2\pi} \int_{X_2} \phi \, dA \tag{2.2.53}$$

where $\phi \sim \phi + 2\pi$ is a compact scalar and A is a $U(1)$ gauge field. We want to construct the Hilbert space on the circle $\mathcal{H}_{S^1}$ by canonical quantization. We place the theory on $S^1 \times \mathbb{R}$ with time $t \in \mathbb{R}$, and decompose $A = A_0^t dt + \widetilde{A}$, $d = \partial_t dt + \widetilde{d}$ (all the tildas denote the spatial S^1 part). Hence

$$dA = -\partial_t \widetilde{A} \wedge dr + \widetilde{d} A_0^t \wedge dt \qquad S = -\frac{iN}{2\pi} \int_{S^1\times\mathbb{R}} \Big(\phi \partial_t \widetilde{A} dt + \widetilde{d}\phi A_0^t dt \Big) \, . \tag{2.2.54}$$

We can choose the temporal gauge $A_0^t = 0$, that imposes the Gauss law $\widetilde{d}\phi = 0$. This means that ϕ only depends on time, and after introducing the holonomy

$$\chi = \int_{S^1} \widetilde{A} \tag{2.2.55}$$

(that also depends on time) the action becomes a quantum mechanical system

$$S = -\frac{iN}{2\pi} \int dt \, \phi(t) \, \partial_t \chi(t) \, . \tag{2.2.56}$$

Notice that $\phi \sim \phi + 2\pi$, $\chi \sim \chi + 2\pi$. This system can be quantized canonically, identifying coordinate and canonical momentum as

$$q = \chi \, , \qquad p = -\frac{N}{2\pi} \phi \tag{2.2.57}$$

with canonical commutation relation:

$$[\phi, \chi] = \frac{2\pi i}{N} \qquad \Longrightarrow \qquad e^{i\chi} \, e^{i\phi} = e^{\frac{2\pi i}{N}} \, e^{i\phi} \, e^{i\chi} \, . \tag{2.2.58}$$

The operators $e^{iN\phi}, e^{iN\chi}$ commute with all other operators (because ϕ, χ are not good operators), hence they are proportional to the identity in any irreducible representation. Without loss of generality, we take them to be the identity. This means that the eigenvalues of $e^{i\phi}, e^{i\chi}$ are N'th roots of unity. To construct a representation we start from an eigenstate of $e^{i\chi}$:

$$e^{i\chi}|k\rangle = e^{\frac{2\pi i k}{N}}|k\rangle \quad , \qquad k = 0, ..., N-1 \ . \tag{2.2.59}$$

Acting on $|k\rangle$ with $e^{i\phi}$ we get an eigenstate of $e^{i\chi}$ with eigenvalue $e^{\frac{2\pi i(k+1)}{N}}$. Since $e^{i\phi}$ is unitary we can take

$$e^{i\phi}|k\rangle = |k+1\rangle \ . \tag{2.2.60}$$

We conclude that the Hilbert space $\mathcal{H}_{S^1}$ is N dimensional and generated by states $|k\rangle$, $k = 0, ..., N-1$.

The same Hilbert space $\mathcal{H}_{S^1}$ can also be obtained by radial quantization, placing the theory on $X_2 = \mathbb{R}^2$ with time going radially. The states on S^1 are obtained, by state/operator correspondence, placing local operators $e^{ik\phi(x)}$ at the origin. Identifying $|0\rangle$ with the state prepared by the empty disk, namely the Hartle-Hawking state, we recognize that

$$\text{state/operator map:} \qquad\qquad |k\rangle \longleftrightarrow e^{ik\phi(x)} \ . \tag{2.2.61}$$

This observation allows to make connections with both the Euclidean treatment in terms of defects and their braiding, as well as with the axiomatic formulation. Regarding the first, the operator $e^{in\chi}$ is nothing but the line operator $U_n(\gamma) = e^{in\int_\gamma A}$ placed on the space-like slice. In radial quantization we can start contracting it toward the origin and making it disappear. However, at some point, it must pass the operator $e^{ik\phi(x)}$ that creates the state, thus producing a phase $\exp\left(\frac{2\pi i k n}{N}\right)$ that is precisely the eigenvalue of $e^{i\chi}$.

The connection with the axiomatic formulation is that we are now able to derive the data (2.1.23). The Hartle-Hawking state is obtained with the path integral on the disk with no insertion, hence it is $|0\rangle$, in agreement with $h^k = \delta^{k,0}$. Since the states $|k\rangle$ are created with the unitary operators $e^{ik\phi}$, taking the orientation reversal amounts to consider the state created by $\left(e^{ik\phi}\right)^\dagger = e^{-ik\phi}$, hence $\eta_{k,k'}$ that determines the isomorphism $\mathcal{H}^\vee_{S^1} \cong \mathcal{H}_{\overline{S}^1}$ is $\eta_{k,k'} = \delta_{k,-k'}$. Regarding $\mu^{k''}_{k,k'}$, this defines an algebra structure on $\mathcal{H}_{S^1}$, but by state/operator correspondence this must coincide with the OPE of local operators $e^{ik\phi}e^{ik'\phi} = e^{i(k+k')\phi}$, and hence $\mu^{k''}_{k,k'} = \delta_{k+k',k''}$.

We can also use the axiomatic formulation to fix the normalization of the path integral in (2.2.1) for $d = 2$. In fact, we already computed, using the axiomatic formulation, that $Z(\Sigma_g) = \sqrt{|H^1(\Sigma_g, \mathbb{Z}_N)|}$. Hence we have

$$C(\Sigma_g) = |H^1(\Sigma_g, \mathbb{Z}_N)|^{-1/2} \ . \tag{2.2.62}$$

Fixing the normalization constant for $d > 2$ and for general degree p of the form is slightly more complicated, but can always be done by comparison with the axiomatic formulation that guarantees the consistency under cutting and gluing [90].

2.2.3 3d Dijkgraaf-Witten theory

We consider 3d 1-form Dijkgraaf-Witten theory with twist $\omega \in H^3(B\mathbb{A}, \mathbb{R}/\mathbb{Z})$ [81]. We do not attempt to write the topological action for general $\mathbb{A}$, since it would involve several different terms depending on ω (see e.g. [82]). Some of them are cubic in the gauge fields, and are morally similar to the DW twist we discussed in 2d. Instead, for simplicity we just study $\mathbb{A} = \mathbb{Z}_N$ (for which these cubic terms are absent) and we can write the most general action, that involves a genuinely new term.

We have $H^3(B\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$, and the DW theory with twist $k \in \mathbb{Z}_N$ is

$$S = \frac{2\pi i}{N}\int_{X_3}\left(A \cup \delta B + \frac{k}{N} A \cup \delta A\right) \tag{2.2.63}$$

with A, B 1-form $\mathbb{Z}_N$ gauge fields. To arrive at this form of the action starting from $A^*(\omega_k)$, $\omega_k \in H^3(B\mathbb{Z}_N, \mathbb{R}/\mathbb{Z})$, there are several steps with an interpretation that is interesting to explain. The explicit expression of the cocycle $\omega_k : \mathbb{Z}_N^3 \to \mathbb{R}/\mathbb{Z}$ is (see e.g. [91])

$$\omega_k(a,b,c) = \left(kc\frac{(a+b)-(a+b)\,\mathrm{mod}(N)}{N^2}\right) \mathrm{mod}\,(1) = \begin{cases} 0 & \text{if } a+b < N \\ \frac{kc}{N}\,\mathrm{mod}\,(1) & \text{if } a+b \geq N \end{cases} \tag{2.2.64}$$

This can be canonically identified with the cocycle $\epsilon_k \in H^2(B\mathbb{Z}_N, \mathbb{Z}_N^\vee) \cong \mathbb{Z}_N$ whose explicit expression is (here we think $\mathbb{Z}_N^\vee$ as $\mathrm{Hom}(\mathbb{Z}_N, \mathbb{R}/\mathbb{Z})$):

$$\epsilon_k(a,b)\cdot c = \left(kc\frac{(a+b)-(a+b)\,\mathrm{mod}(N)}{N^2}\right) \mathrm{mod}\,(1) = \omega_k(a,b,c)\ . \tag{2.2.65}$$

ϵ_k determines an Abelian extension $\Gamma_{N,k}$:

$$1 \to \mathbb{Z}_N^\vee \to \Gamma_{N,k} \to \mathbb{Z}_N \to 1\ . \tag{2.2.66}$$

In turns, this determines a long exact sequence of singular cohomology groups, with connecting map the Bockstein $\beta_k : H^n(X, \mathbb{Z}_N) \to H^{n+1}(X, \mathbb{Z}_N^\vee)$. In particular, from the gauge field $A \in C^1(X_3, \mathbb{Z}_N)$ we can construct $\beta_k(A) \in C^2(X_3, \mathbb{Z}_N^\vee)$, hence the term

$$S_{\mathrm{DW}} = 2\pi i \int_{X_3} A \cup \beta_k(A) \tag{2.2.67}$$

with $\cup$ associated with the canonical pairing $\mathbb{Z}_N \times \mathbb{Z}_N^\vee \to \mathbb{R}/\mathbb{Z}$. To get the expression (2.2.63) from (2.2.67) we use an isomorphism $\mathbb{Z}_N^\vee \cong \mathbb{Z}_N$ to convert the cup product and divide by N, and we notice that $\beta_k = k\beta$, with $\beta(A) = \frac{\delta A}{N}$ the Bockstein associated with the sequence $1 \to \mathbb{Z}_N \to \mathbb{Z}_{N^2} \to \mathbb{Z}_N \to 1$.

For $k = 1$, we have $\Gamma_{N,k=1} \cong \mathbb{Z}_{N^2}$. Indeed, using the expression of ϵ_k we see that N times the generator of the quotient $\mathbb{Z}_N$, gives the generator of the subgroup $\mathbb{Z}_N^\vee$. For general $k = 0, ..., N-1$, N times the generator of $\mathbb{Z}_N$ gives k times the generator of $\mathbb{Z}_N^\vee$, hence

$$\Gamma_{N,k} \cong \left\langle x, y \mid x^N = y^k\ , y^N = 1\ , xy = yx \right\rangle \cong \mathbb{Z}_{N^2/\mathrm{gcd}(N,k)} \times \mathbb{Z}_{\mathrm{gcd}(N,k)}\ . \tag{2.2.68}$$

We want to show that all this mathematics has a nice physical interpretation in the TQFT (2.2.63). Unlike the 2d case, the presence of δ in the DW twist makes its effect very different. It does not modify the gauge transformations, but it modifies the braiding. It is easier to discuss this in the continuum formulation. Using the dictionary (2.2.47) we get the continuous field theory

$$S = \frac{iN}{2\pi}\int_{X_3} A \wedge dB + \frac{ik}{2\pi}\int_{X_3} A \wedge dA\ . \tag{2.2.69}$$

It turns out that the second term can be further divided by 2, at the prize of introducing an extra structure on X_3, namely a *spin-structure*, on which the theory will depend [81]

$$S = \frac{iN}{2\pi}\int_{X_3} A \wedge dB + \frac{ik}{4\pi}\int_{X_3} A \wedge dA\ . \tag{2.2.70}$$

In this notation $k \in \mathbb{Z}_{2N}$, but needs to be even in order for the theory to be strictly topological. TQFTs that can be only formulated on spin-manifolds are called spin-TQFTs [92]. We will assume k even for simplicity in the following.

The most general (topological) line operator is a dyon labeled by $a, b \in \mathbb{Z}$

$$U_{a,b}(\gamma) = \exp\left(ia\int_\gamma A + ib\int_\gamma B\right)\ . \tag{2.2.71}$$

With the same computation that led to (2.2.44) one obtains

$$\left\langle U_{a,b}(\gamma)\, U_{a',b'}(\gamma')\right\rangle = \exp\left[\frac{2\pi i}{N}\left(ab' + a'b - \frac{k}{N}bb'\right)\right] . \tag{2.2.72}$$

Because of the last term, this is no longer invariant under separate shifts of a and b by N. Instead the correct identifications are

$$(a,b) \sim (a+N, b) \ , \qquad (a,b) \sim (a+k, b+N) \ . \tag{2.2.73}$$

Therefore the 1-form symmetry Γ of the theory is not $\mathbb{Z}_N \times \mathbb{Z}_N$, but it is the quotient of $\mathbb{Z}\times\mathbb{Z}$ by the identification above. Γ has a subgroup isomorphic to $\mathbb{Z}_N$, generated by $U_{1,0}$. The quotient $\Gamma/\mathbb{Z}_N \cong \mathbb{Z}_N$ can be taken to be generated by $U_{0,-1}$, and has the property that

$$U_{0,-1}^N = U_{0,-N} = U_{k,0} = U_{1,0}^k \ . \tag{2.2.74}$$

Therefore we conclude that the 1-form symmetry is precisely the group extension discussed above

$$\Gamma = \Gamma_{N,k} \cong \mathbb{Z}_{N^2/\gcd(N,k)} \times \mathbb{Z}_{\gcd(N,k)} \ . \tag{2.2.75}$$

2.2.4 Kapustin-Seiberg TQFT in four-dimensions

In 4d a 2-form $\mathbb{A}$ gauge theory is equivalent to a 1-form $\mathbb{A}^\vee$ gauge theory, since in the cochain formulation this is

$$S = 2\pi i \int_{X_4} B \cup \delta A \ , \qquad B \in C^2(X_4, \mathbb{A}) \ , \quad A \in C^1(X_4, \mathbb{A}^\vee) \ . \tag{2.2.76}$$

Viewed as a 2-form $\mathbb{A}$ gauge theory, it has an interesting modification discussed in [14, 93] on the lattice, and in [13, 30] in the continuum. It consists of adding a twist, that is the integral of $B^*\omega \in H^4(X_4, \mathbb{R}/\mathbb{Z})$, with $\omega \in H^4(B^2\mathbb{A}, \mathbb{R}/\mathbb{Z})$. This group is isomorphic to the group $\mathcal{Q}(\mathbb{A})$ of $\mathbb{R}/\mathbb{Z}-$valued quadratic functions on $\mathbb{A}$. A quadratic function $q : \mathbb{A} \to \mathbb{R}/\mathbb{Z}$ satisfies

$$\chi(a,b) = q(a+b) - q(a) - q(b) \quad \text{bilinear in } a, b \ , \qquad q(-a) = q(a) \ . \tag{2.2.77}$$

Intuitively the twist is given by a quadratic function of B, but to understand precisely the form of the term we need to introduce the *universal quadratic group* $\Gamma(\mathbb{A})$ and the *Pontryagin square* (see [15, 93] for more details).

$\Gamma(\mathbb{A})$ is a group endowed with a quadratic function $Q : \Gamma(\mathbb{A}) \to \mathbb{A}$, such that for any Abelian group V and a quadratic function $q : \mathbb{A} \to V$ there is a linear map $\widetilde{q} : \Gamma(\mathbb{A}) \to V$ such that

$$q(a) = \widetilde{q}\left(Q(a)\right) \ . \tag{2.2.78}$$

This means that the group $\mathcal{Q}(\mathbb{A}, V)$ of V-valued quadratic functions on $\mathbb{A}$ is isomorphic to $\mathrm{Hom}(\Gamma(\mathbb{A}), V)$. The universal quadratic group enters in the game because one can show that [67]

$$H_4(B^2\mathbb{A}, \mathbb{Z}) = \Gamma(\mathbb{A}) \ . \tag{2.2.79}$$

The universal coefficients theorem then implies that

$$H^4(B^2\mathbb{A}, V) \cong \mathrm{Hom}(\Gamma(\mathbb{A}), V) \cong \mathcal{Q}(\mathbb{A}, V) \ . \tag{2.2.80}$$

There are two interesting choices of V. First, taking $V = \mathbb{R}/\mathbb{Z}$, we identify $H^4(B^2\mathbb{A}, \mathbb{R}/\mathbb{Z})$ with $\mathbb{R}/\mathbb{Z}-$valued quadratic functions, as we stated above. Second, taking $V = \Gamma(\mathbb{A})$, we can consider the identity in $\mathrm{Hom}(\Gamma(\mathbb{A}),\Gamma(\mathbb{A}))$, and it will correspond to a distinguished element $\mathfrak{P}\in H^4(B^2\mathbb{A},\Gamma(\mathbb{A}))$, called the Pontryagin square. In turn, this can be identified with a quadratic function $\mathbb{A}\to\Gamma(\mathbb{A})$.

Now, given $B\in C^2(X_4,\mathbb{A})$ we can consider

$$\mathfrak{P}(B) := B^*\mathfrak{P}\in H^4(X_4,\Gamma(\mathbb{A}))\ . \tag{2.2.81}$$

Then for any discrete torsion class in $H^4(B^2\mathbb{A},\mathbb{R}/\mathbb{Z})$, represent it as a quadratic function $q:\mathbb{A}\to\mathbb{R}/\mathbb{Z}$, and let $\widetilde{q}:\Gamma(\mathbb{A})\to\mathbb{R}/\mathbb{Z}$ be the associated linear map, we construct the action

$$S = 2\pi i\int_{X_4}\Big(B\cup\delta A+\widetilde{q}\,(\,\mathfrak{P}(B)\,)\Big) \tag{2.2.82}$$

Following Seiberg and Kapustin [30], for $\mathbb{A}=\mathbb{Z}_N$ we can describe this TQFT with continuous $U(1)$ gauge fields A,B, respectively a $1-$form and a $2-$form. The most general quadratic function on $\mathbb{Z}_N$ is $q(a)=\frac{pa^2}{2N}\,\mathrm{mod}(1)$, where $p\in\mathbb{Z}_{2N}$ and must be even if N is odd, hence taking values in $\mathbb{Z}_N\subset\mathbb{Z}_{2N}$. This agrees with the fact that $\Gamma(\mathbb{Z}_N)\cong\mathbb{Z}_N$ for N odd, while $\Gamma(\mathbb{Z}_N)=\mathbb{Z}_{2N}$ for N even. The twist should be a quadratic term in B, hence the only possibility is to write

$$S=\frac{iN}{2\pi}\int_{X_4}B\wedge dA+\frac{iNp}{4\pi}\int_{X_4}B\wedge B\ . \tag{2.2.83}$$

p is identified with the number determining the quadratic function. To argue this, we have to show that it must be an integer defined modulo $2N$, and must be even if N is odd. Using the dictionary (2.2.47) the action is

$$S=\frac{2\pi i}{N}\int_{X_4}B\cup\delta A+\frac{2\pi ip}{2N}\int_{X_4}B\cup B\ . \tag{2.2.84}$$

Hence p must be integer, and the second term is trivial for $p=2N$, so that $p\sim p+2N$. Moreover, to make the second term invariant (up to $2\pi i$) under shifts of B by a cochain proportional to N we need $Np\in 2\mathbb{Z}$.

To analyze the theory, let us use the continuous fields formulation for definiteness. The quadratic term in B modifies the gauge transformation into

$$B\mapsto B+d\lambda\ ,\qquad A\mapsto A+d\phi-p\lambda. \tag{2.2.85}$$

ϕ is a compact scalar, while λ a 1-form $U(1)$ gauge field, namely the periods of $\frac{d\lambda}{2\pi}$ are integers. Under (2.2.85) the action changes by

$$\delta S=2\pi iN\int_{X_4}\frac{d\lambda}{2\pi}\wedge\frac{dA}{2\pi}-2\pi i\frac{Np}{2}\int_{X_4}\frac{d\lambda}{2\pi}\wedge\frac{d\lambda}{2\pi}\ . \tag{2.2.86}$$

The first term is multiple of $2\pi i$, and the same is true for second provided $Np\in 2\mathbb{Z}$. In this formulation we also see that the second condition can be relaxed on a spin 4-manifold, since the integral of $\frac{d\lambda}{2\pi}\wedge\frac{d\lambda}{2\pi}$ is even. The theory for N and p both odd however, will depend on the choice of a spin-structure. This is similar to the 3d DW theory we discussed in the last Section, and as in that case we will assume p even if N is odd for simplicity.

Let us look at the gauge invariant operators. While

$$U_b(\gamma_2)=\exp\left(ib\int_{\gamma_2}B\right) \tag{2.2.87}$$

are gauge invariant provided $b\in\mathbb{Z}$, the lines of A are generically not gauge invariant under $B\mapsto B+d\lambda$, $A\mapsto A-p\lambda$, and require an open surface of B attached:

$$L_a(\gamma_1;D_2)=\exp\left(ia\int_{\gamma_1}A+ipa\int_{D_2}B\right)\ . \tag{2.2.88}$$

Here $\partial D_2 = \gamma_2$. The braiding between a (generically non-genuine) line and a surface is given by

$$\left\langle L_a(\gamma_1; D_2)\, U_b(\gamma_2) \right\rangle = \exp\left(\frac{2\pi i a b}{N} \text{Lk}(\gamma_1, \gamma_2)\right) , \tag{2.2.89}$$

from which we read the usual identification $a \sim a + N$, $b \sim b + N$.

However, the non-genuine lines are essentially trivial operators, as well as the surfaces that can be cut-opened. The situation is very similar to that of the 2d $\mathbb{Z}_N \times \mathbb{Z}_N$ DW theories discussed above. L_a becomes genuine when the dependence on the surface disappears, namely if pa is proportional to N. Introducing $k = \frac{N}{\gcd(N,p)}$, this is the case in which a is proportional to k. Hence there are $\gcd(N,p)$ genuine lines $L_a(\gamma_1)$, $a = nk$, $n = 0, ..., \gcd(N,p) - 1$, generating the subgroup $\mathbb{Z}_{\gcd(N,p)} \subset \mathbb{Z}_N$, that is the non-trivial 2-form symmetry of the theory. Moreover, surfaces U_b can be cut open on a line if b is proportional to $\gcd(N,p)$, hence the subgroup $\mathbb{Z}_k \subset \mathbb{Z}_N$ forms a trivial 1-form symmetry. The non-trivial one is described by the quotient $\mathbb{Z}_N/\mathbb{Z}_K \cong \mathbb{Z}_{\gcd(N,p)}$. These surfaces are also the charges of the non-trivial 2-form symmetry.

As in 2d DW theory, at the level of genuine and unbreakable operators, the theory seems equivalent to the untwisted $\mathbb{Z}_{\gcd(N,p)}$ 2-form gauge theory. However, the $\mathbb{Z}_k$ subgroup has a non-trivial SPT phase. In particular, the Kapustin-Seiberg theory is by itself a 1-form $\mathbb{Z}_N$ SPT if $\gcd(N,p) = 1$ [30].

2.3 Abelian TQFTs in three-dimensions

Three space-time dimensions are very special for TQFTs, because of the existence of Chern-Simons theory [47]. The Abelian theory with gauge group $U(1)$ has Lagrangian description

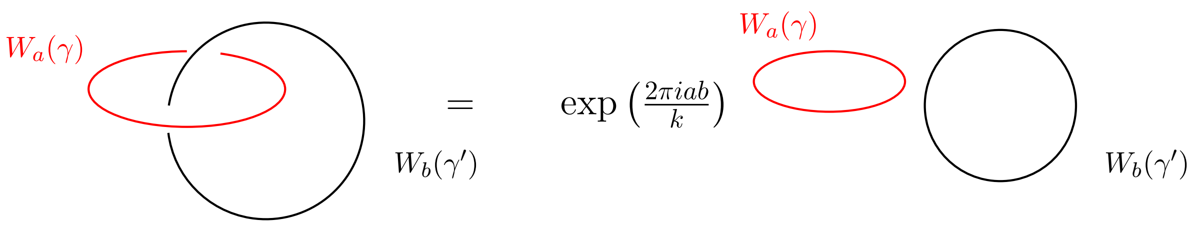
$$S = \frac{ik}{4\pi} \int_{X_3} A \wedge dA \tag{2.3.1}$$

with A a $U(1)$ gauge field and $k \in \mathbb{Z}$. If k is odd, the theory requires the choice of a spin structure. As for the BF theory, the correct definition of the action requires extending $X_3 = \partial Y_4$ and writing the integral of $\frac{F}{2\pi} \wedge \frac{F}{2\pi}$ on Y_4. The independence on the choice of Y_4 implies that k is an integer on spin manifolds, and an even integer on non-spin manifolds.

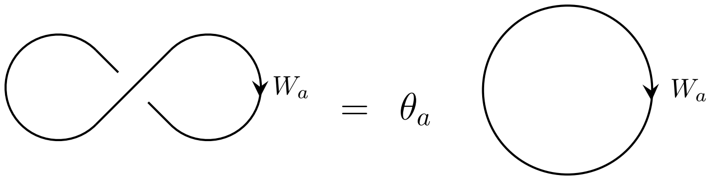
Everything is encoded in the topological lines

$$W_a(\gamma) = \exp\left(in \int_\gamma A\right) \tag{2.3.2}$$

that have braiding



It seems this identifies $a \sim a + k$ and reduces the set of lines to $\mathbb{Z}_k$. This is not quite true because of a subtle effect called *framing anomaly* [47]: the lines of Chern-Simons theory are *framed*, namely they require a choice of a section of the normal bundle to $\gamma \subset X_3$. Intuitively, this is a specification of a normal direction to any point of the curve that can be interpreted by saying that the lines are effectively *ribbons*. The effect is detected by a non-trivial phase θ_a arising in



(2.3.3)

θ_a is called the *topological spin* of the line, and measures a sort of self-braiding due to the ribbon nature of the line. Notice that the braiding $\mathcal{B}(a,b) = \exp\left(\frac{2\pi i ab}{k}\right)$ defines a bilinear form on the set of lines. By comparison between the braiding and the definition of θ_a, we see that θ_a is a quadratic refinement of $\mathcal{B}(a,b)$ (in comparing with (2.2.77) notice that this is written in the multiplicative notation):

$$\mathcal{B}(a,b) = \frac{\theta_{a+b}}{\theta_a \theta_b} \ . \tag{2.3.4}$$

Up to an arbitrary (and unphysical) gauge choice, we have

$$\theta_a = \exp\left(\frac{2\pi i a^2}{2k}\right) \ . \tag{2.3.5}$$

If $k \in 2\mathbb{Z}$ this formula is well defined over $\mathbb{Z}_k$ (namely $\theta_{a+k} = \theta_a$), but if k is odd, $\theta_k = -1$. In this case W_k has trivial braiding with all other lines but has nontrivial (-1) spin: it is called a *transparent fermion*. The set of lines for $k \in 2\mathbb{Z}+1$ should be enlarged to $\mathbb{Z}_{2k}$, where the second k lines are obtained by fusing the first k lines with the transparent fermion.

We will focus on $k \in 2\mathbb{Z}$ for simplicity. In a sense, the $U(1)_k$ Chern-Simons theory is the *square root* of the 3d $\mathbb{Z}_k$ gauge theory. The latter has a $\mathbb{Z}_k \times \mathbb{Z}_k$ 1-form symmetry with a mixed anomaly, while CS only has $\mathbb{Z}_k$ with a self-anomaly: the generators are charged among themselves. The inflow action, written in terms of a background $B \in H^2(X, \mathbb{Z}_k)$, is

$$S_{\text{inflow}} = \frac{2\pi i}{2k} \int_{X_4} B \cup B \ . \tag{2.3.6}$$

This can be derived in the very same way we arrived at the mixed anomaly for the $\mathbb{Z}_N$ gauge theory (2.2.52)

The intuition that $U(1)_k$ is a square root of the $\mathbb{Z}_k$ gauge theory is actually precise: $U(1)_k \times U(1)_{-k}$ is exactly equivalent to the $\mathbb{Z}_k$ gauge theory, as can be shown by taking the sum and difference of the two Chern-Simons gauge fields. More generally, it turns out that any 3d Abelian TQFT can be written as a Chern-Simons theory with gauge group $U(1)^r$:

$$S_K = \sum_{i,j=1}^{r} \frac{iK_{ij}}{4\pi} \int_{X_3} A_i \wedge dA_j \ . \tag{2.3.7}$$

Here $A_{i=1,\ldots,r}$ are $U(1)$ gauge fields, and K is an $r \times r$ symmetric non-degenerate matrix with integer entries. The theory is bosonic if $K_{ii} \in 2\mathbb{Z}$, while it requires a spin structure otherwise. We focus on the bosonic case (see [92] for the general case). The line operators are labelled by a vector of integers $\underline{a} = (a_1, \ldots, a_r)^\mathsf{T} \in \mathbb{Z}^r$:

$$W_{\underline{a}}(\gamma) = \exp\left(i \sum_{i=1}^{r} a_i \int_\gamma A_i\right) \tag{2.3.8}$$

and have braiding

$$\mathcal{B}(\underline{a}, \underline{b}) = \exp\left(2\pi i \underline{a}^\mathsf{T} K^{-1} \underline{b}\right) \ . \tag{2.3.9}$$

The lines $W_{\underline{b}}$ with $\underline{b} = K\underline{n}$, $\underline{n} \in \mathbb{Z}^r$, have trivial braiding with all other lines, and are therefore trivial.

The set of non-trivial lines is given by the quotient of lattices

$$D(K) = \frac{\mathbb{Z}^r}{K\mathbb{Z}^r} \ . \tag{2.3.10}$$

This is a finite Abelian group called *discriminant group*. This is the 1-form symmetry group of the theory, and it is easy to see that

$$|D(K)| = \det(K) \ . \tag{2.3.11}$$

The braiding $\mathcal{B}(\underline{a}, \underline{b})$ defines a symmetric bilinear form on the discriminant group. A quadratic refinement of it gives the topological spin $\theta_{\underline{a}}$ of the lines

$$\mathcal{B}(\underline{a}, \underline{b}) = \frac{\theta_{\underline{a}+\underline{b}}}{\theta_{\underline{a}}\theta_{\underline{b}}} \ . \tag{2.3.12}$$

We conclude that the spins completely determine the theory.

2.4 Topological boundary conditions

An important application of TQFTs in this thesis will be the Symmetry Topological Field Theory (SymTFT), which will be introduced in Chapter 4, and used in all the following chapters. This consists in coupling a dynamical QFT to a TQFT in one dimension higher and then studying its *topological boundary conditions*. For this reason, in this section we want to briefly explain the basic methods to deal with these boundary conditions. Many other aspects will be explained in detail in the coming chapters when necessary. Here we just want to illustrate the very important principle that

P: Topological boundary conditions are specified by maximal sets of mutually transparent objects.

We will refer to these maximal sets as *Lagrangian algebras*. This correspondence, strictly speaking, is *not* one to one for $d > 3$: from any topological boundary condition we can always produce others by stacking a $(d-1)-$dimensional TQFT on the boundary[8]. However, what is always true is that a Lagrangian algebra specifies a topological boundary condition. Roughly speaking, defects in the Lagrangian algebra are allowed to terminate on the boundary.

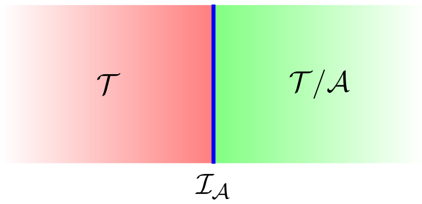
In general, a condensable algebra $\mathcal{A}$ of a TQFT is a set of topological defects that constitute an anomaly free symmetry, hence can be gauged. The condensable algebra is called *Lagrangian*, and denoted by $\mathcal{L}$, if it is maximal, in the sense that we cannot add other defects to it while preserving the anomaly free condition. This means that any other defect is *charged* under at least one element of the Lagrangian algebra. Hence by gauging (or condensing) $\mathcal{L}$ we get rid off all defects not in $\mathcal{L}$. On the other hand, the defects inside $\mathcal{L}$ becomes trivial, and the resulting theory is completely empty. One may wonder about the dual symmetry arising from the gauging. However this would act on the twisted sectors, and those are always empty in TQFT. We conclude that an equivalent characterization of Lagrangian algebras is the following

C: Lagrangian algebras of a TQFT are anomaly free symmetries whose gauging trivialize the theory.

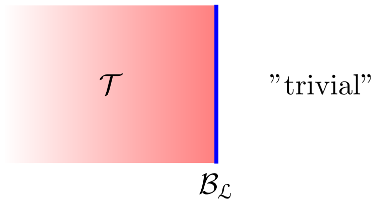
This explains how to get a topological boundary condition form $\mathcal{L}$. In general, given any condensable (anomaly free) algebra $\mathcal{A}$ of a TQFT $\mathcal{T}$, and given a codimension one submanifold $X_{d-1} \subset X_d$ dividing

[8]For $d \leq 3$, a $(d-1)-$dimensional TQFT, in order to be non-trivial, has topological local operators, and they would make the boundary non *simple*.

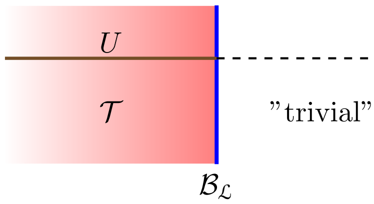
X_d into two regions, we can construct a topological interface $\mathcal{I}_\mathcal{A}$ between $\mathcal{T}$ and $\mathcal{T}/\mathcal{A}$, simply gauging $\mathcal{A}$ in one of the two halves



If $\mathcal{A} = \mathcal{L}$ is Lagrangian, then the theory on the right is trivial, and the interface becomes a boundary condition $\mathcal{B}_\mathcal{L}$:



This also explains the previous heuristic definition that, on the boundary condition $\mathcal{B}_\mathcal{L}$ defined by $\mathcal{L}$, we allow all the topological defects inside $\mathcal{L}$ to terminate. Indeed, if $U \in \mathcal{L}$, it will be transparent on the right hand, side, and appears to be *absorbed by the boundary*:



Example: discrete gauge theories. To illustrate this abstract discussion, let us discuss a concrete example. We consider the $\mathbb{Z}_N$ $p-$form gauge theory in $d-$dimensions, that we can present as a BF theory (2.2.29). In this Lagrangian presentation, boundary conditions can be analyzed by requiring a good variational principle in presence of a boundary $\partial X_d = X_{d-1} \neq \emptyset$. The variation of the action produces the equations of motion $dA_p = dB_{d-p-1} = 0$ in the bulk, provided we cancel the boundary term

$$\delta S\big|_\partial = -\frac{i(-1)^p N}{2\pi} \int_{\partial X_d} A_p \wedge \delta B_{d-p-1} \ . \tag{2.4.1}$$

A topological boundary condition should set this to zero imposing some condition on the fields that does not require introducing some extra structure on the boundary other than the smooth structure[9]. This is achieved by setting *Dirichlet* boundary conditions for B_{d-p-1}:

$$\delta B_{d-p-1}\big|_\partial = 0 \ . \tag{2.4.2}$$

A note on the terminology, which could be confusing otherwise. It really makes sense to call this boundary condition "Dirichlet" if we think about this theory as a $(d-p-1)-$form gauge theory in terms of B_{d-p-1}. Thinking about it as a $p-$form gauge theory for A_p, the same boundary condition is typically called "Neumann". It is clear that there is no intrinsic meaning in TQFT, since the terminology really depends on the presentation.

[9]For instance if the boundary condition requires to use a conformal structure of the boundary it is called a conformal boundary condition.

Coming back to the boundary condition, this can be interpreted in terms of the defects

$$U_n(\gamma_p) = \exp\left(in \int_{\gamma_p} A_p \right) , \qquad V_m(\gamma'_{d-p-1}) = \exp\left(im \int_{\gamma'_{d-p-1}} B_{d-p-1} \right) . \tag{2.4.3}$$

Since B_{d-p-1} is a gauge field, fixing its value on the boundary requires to freeze its gauge transformations there: $B_{d-p-1} \mapsto B_{d-p-1} + d\xi_{d-p-2}$ requires $\xi_{d-p-2}\big|_\partial = 0$. Therefore, it makes perfect sense to consider the defect V_m supported on an open manifold, provided that its boundary $\partial\gamma'_{d-p-1}$ is inside ∂X_{d-1}. Hence, the topological boundary condition can be described by allowing all the defects V_m to topologically terminate on the boundary. This corresponds to the Lagrangian algebra

$$\mathcal{L} = \left\{ V_m \mid m \in \mathbb{Z}_N \right\} . \tag{2.4.4}$$

In fact, this set of defects is non-anomalous, and is maximal since any other defect, that is one of the U_n, is charged under the V_m.

There is another obvious boundary condition, that we can call Neumann for B_{d-p-1} or Dirichlet for A_p, namely

$$\delta A_p\big|_\partial = 0 . \tag{2.4.5}$$

To get a good variational principle out of this, we first need to add a boundary term proportional to $A_p \wedge B_{d-p-1}$, that is equivalent to present the action in the *integrated by parts* form:

$$S = -\frac{i(-1)^p N}{2\pi} \int_{X_d} dA_p \wedge B_{d-p-1} . \tag{2.4.6}$$

This other boundary condition allows all the defects U_n to terminate on the boundary, and corresponds to the Lagrangian algebra

$$\mathcal{L} = \left\{ U_n \mid n \in \mathbb{Z}_N \right\} \tag{2.4.7}$$

that is also anomaly free and maximal.

One of the powers of characterizing topological boundary conditions in terms of Lagrangian algebras is that there are certain boundary conditions that are less evident in terms of explicit conditions on the fields (at least in terms of continuous fields). For example, suppose $N = pq$. Recall that the braiding between U_n and V_m defects is $\mathcal{B}(n,m) = \exp\left(\frac{2\pi i n m}{N}\right)$. Then the defects U_{px}, V_{qy}, with $x = 0, ..., q-1$, and $y = 0, ..., p-1$ are all mutually transparent, hence

$$\mathcal{L} = \left\{ U_{px} \,, V_{qy} \mid x \in \mathbb{Z}_q \,, y \in \mathbb{Z}_p \right\} \cong \mathbb{Z}_q \times \mathbb{Z}_p \tag{2.4.8}$$

is an anomaly free symmetry. It is also maximal, since any other defect braids non-trivially with at least one defect inside $\mathcal{L}$. Hence $\mathcal{L}$ is a Lagrangian algebra and can be used to define a topological boundary condition.

One general fact is that the dimension of a Lagrangian algebra is always the square root of the total dimension of the symmetry defects of the TQFT. This also motivates the name *Lagrangian*, resonating with the same notion in symplectic geometry.

Which TQFTs have topological boundaries? Not all TQFTs admit topological boundary conditions, hence Lagrangian algebras. The prototypical example is $U(1)_k$ Chern-Simons theory discussed in Section 2.3. The reason is very simple: $U(1)_k$ has a $\mathbb{Z}_k$ 1-form symmetry whose lines have braiding

$$\mathcal{B}(a,b) = \exp\left(\frac{2\pi i a b}{k} \right) . \tag{2.4.9}$$

The whole 1-form symmetry is anomalous, and a $\mathbb{Z}_p \subset \mathbb{Z}_k$ subgroup (here $k = pq$) is generated by lines W_{qx}, $x \in \mathbb{Z}_p$, that have braiding

$$\mathcal{B}(qx, qy) = \exp\left(\frac{2\pi i q x y}{p}\right) . \tag{2.4.10}$$

Hence $\mathbb{Z}_p$ is anomaly free only if p divides q, namely $q = lp$ and $k = lp^2$. Moreover, if $l > 1$ (that is, $p \neq q$), the lines W_{px} that form the subgroup $\mathbb{Z}_q$ are transparent to those of the subgroup $\mathbb{Z}_p$, but the two do not coincide; hence $\mathbb{Z}_p$ is not maximal. It might be that l is itself divisible by a perfect square and we can continue this way, but the process will stop unless k is not a perfect square. We conclude that

- $U(1)_k$ admits topological boundaries if and only if $k = p^2$ is a perfect square.

It is interesting to ask: when does a TQFT admit topological boundary conditions? Using the description in terms of Lagrangian algebras, the answer is very simple and elegant: topological boundary conditions exist if and only if the theory can be presented as a discrete gauge theory for some discrete (generalized) symmetry $\mathcal{C}$. In fact, gauging the Lagrangian algebra corresponding to the topological boundary produces a trivial theory $\mathcal{L}$. As we have already argued, the theory is trivial because the dual symmetry $\mathcal{C}$ does not act on anything, the twisted sector being empty. But precisely for this reason, we can reconstruct the original TQFT by gauging back $\mathcal{C}$, and presenting the theory as a pure $\mathcal{C}$ gauge theory.

The argument presented in this form might appear to be restricted to TQFTs whose set of topological defects forms a group-like symmetry. But it is not: the same argument makes sense also for TQFTs with *non-invertible* symmetries, replacing the gauging with a generalized gauging. This is the basic principle behind the fact that all $d-$dimensional TQFTs that admits topological boundaries have a so-called *state-sum* construction from the datum of a $(d-2)$ fusion category [94], which generalize the famous Turaev-Viro model [38].

Returning to $U(1)_k$ Chern-Simons, we have seen that if $k = p^2$ it admits topological boundary conditions. It is natural to ask if it is indeed a gauge theory. It is easy to check that the answer is affirmative: it is nothing but Dijkgraaf-Witten theory $\mathbb{Z}_p$ with twist 1:

$$\frac{ip}{2\pi}\int_{X_3} B \wedge dC + \frac{i}{4\pi}\int_{X_3} C \wedge dC . \tag{2.4.11}$$

Indeed as we discussed in Section 2.2.3 the presence of the twist combines the two $\mathbb{Z}_p$ symmetries into $\mathbb{Z}_{p^2}$, and the braiding can be also checked to be the correct one.

Chapter 3

Introduction to non-invertible symmetries

In this chapter, after a qualitative explanation motivating the appearance of category theory in the study of symmetries, we will review four interesting constructions of non-invertible higher categorical symmetries in QFT. A main ingredient is given by *condensation defects* introduced in [26], that we review in Section 3.2. These are an almost *trivial* type on non-invertible symmetries which do not impose dynamical constraints, but appear universally in all other types of categorical symmetries. We will then review duality defects [24] (that will be the heroes of Chapters 5 6), KOZ defects [25], and the non invertible chiral symmetry in 4d [28, 29].

3.1 Symmetries and category theory

One crucial realisation of the last years is that all topological operators in QFT have many properties that are natural generalisations of those of symmetries. They are robust, being preserved along RG flows and continuous variation of the parameters of the theory. Inserting them into the path integral can be viewed as a generalised notion of *coupling with backgrounds*, and as such they give rise to notions of 't Hooft anomalies that impose dynamical constraints. In addition, if these anomalies vanish, the topological defects can be gauged. For this reason, it has become common law to call *symmetries* all possible topological defects of a QFT *symmetries*, even if they do not fuse according to any group. They are typically called *non-invertible* or *categorical* symmetries. The reason for the second name is that it has been recognised [8, 20, 24–27, 91, 95] that the most general mathematical structure governing topological defects is that of *higher fusion categories* [96, 97].

A higher category is an algebraic structure with objects, morphsims between objects, 2-morphisms between morphisms, 3-morphsims between 2-morphisms, and so on. An n-category has n levels of morphisms. All the morphisms can be composed. If there is also a concept of composing (or fusing) the objects, the higher-category is called *monoidal*[1]. The word fusion is mathematically appropriate only under certain finitness conditions. In general we do not want to assume them, since QFTs also have continuous symmetries, but we will use the abuse of notation of calling fusion higher categories any monoidal higher category.

The idea is that d-dimensional QFTs have codimension one topological defects (0-form symmetries) that can be fused without encountering UV divergences, and they are objects of a $(d-1)$-category.

[1]The reason is that a monoidal n-category can be thought of as a $(n+1)$ category with a unique (monoid) object, and the objects of the n-category are realized as morphisms of it.

Topological defects of codimension two (1-form symmetries) can generically be interfaces between two codimension one defects, and are identified as (1-)morphisms. If they live in the bulk, they are thought of as endomorhisms of the identity object. Then topological defects of codimension three (2-form symmetries) separate two codimension two objects, and are 2-morphsims, and so on. In general, codimension $p+1$ objects are $(p+1)$ morphisms, and they can exist until $p = d-1$ (topological local operators).

This structure can be very complicated, and a complete understanding of its consequences for QFT is still lacking. A great simplification takes place in two dimensions, where the formalism of fusion (1-)categories is very well developed (see, e.g. [98] for a comprehensive review). Two-dimensional QFTs are in fact the place where non-invertible symmetries were first discovered and analyzed in great detail [20, 21, 91, 99], following the very well-known existence of topological defect lines in 2d CFTs [18, 100–102]. We will not furnish a detailed treatment of fusion category symmetries in 2d, that is a well established subject, and for which the reader can consult the references above. In the following, instead, we want to emphasize the more general viewpoint that emerged in the last years, without relying on the power tight to two dimensions.

We will not present a general mathematically abstract theory of higher categorical symmetries, but instead we will use a physics based approach of constructing these symmetries and analyzing their consequences in QFT. We hope that this bottom-up analysis can furnish insights on the more general and unified structures.

Truthfully speaking, a hint towards a unification driven by the physical approach has already emerged: it is the Symmetry Topological Field Theory (SymTFT) that we will introduce in the next chapter. This is the idea that for any categorical symmetry $\mathcal{C}$ in d-dimensions, we can associate a $(d+1)$-dimensional TQFT $\mathcal{Z}(\mathcal{C})$ that encodes all the properties of $\mathcal{C}$. This approach essentially moved the problem of studying complicated algebraic structures, the fusion higher categories, into a problem of TQFTs that is often more accessible. This will be the subject of the Chapters 5 6 7 8 9. In the remaining sections of this chapter, we will review some of the most important constructions of categorical symmetries in higher dimensions.

3.2 Condensation defects

Condensation defects are one of the main ingredients in *higher categorical symmetries*. In a sense they are the most *trivial* type of non-invertible symmetries in higher dimensions, but they universally appear in all other types of non-invertible symmetries, making them an indispensable tool. Moreover, they play a very important role in TQFTs, where 0-form symmetry defects have an explicit realisation in terms of condensation defects. This makes them very valuable in the SymTFT (Chapter 4) and will be extensively used in Chapter 5.

3.2.1 Generalities

A d-dimensional QFT with a p-form symmetry $\mathbb{A}$, with $p > 0$, automatically has a class of topological defects supported on n-dimensional manifold, for any $n > d-p-1$, called *condensation defects* [26, 50, 103]. The idea is very simple. Given $\Sigma_n \subset X_d$, the defects $U_a(\gamma_{d-p-1}) = U(\widehat{\gamma}_{d-p-1})$ generating the p-form symmetry ($\gamma_{d-p-1} \subset X_d$ is the geometric cycle, $\widehat{\gamma}_{d-p-1} \in Z_{d-p-1}(X_d, \mathbb{A})$ takes into account also $a \in \mathbb{A}$) can be placed on Σ_n, where they are of co-dimension $n-d+p+1$, and they formally generate a q-form symmetry there, with

$$q = n - d + p \, . \tag{3.2.1}$$

If this symmetry in non-anomalous, we can construct a n-dimensional defect $C(\Sigma_n)$ by formally gauging this symmetry [26]:

$$C(\Sigma_n) = N(\Sigma_n) \sum_{\widehat{\gamma}_{d-p-1} \in H_{d-p-1}(\Sigma_n, \mathbb{A})} U(\widehat{\gamma}_{d-p-1}) = N(\Sigma_n) \sum_{B \in H^{q+1}(\Sigma_n, \mathbb{A})} \mathrm{PD}^{-1}(B) \ . \tag{3.2.2}$$

Here $N(\Sigma_n)$ is a normalization constant, that is the same appearing in pure n-dimensional q-form $\mathbb{A}$ gauge theory. $\mathrm{PD}^{-1}(B)$ is the (inverse) Poincare' dual of the gauge field B, namely the network of topological defects corresponding to that specific background. In other words, inserting $C(\Sigma_n)$ into a correlator produces a sum of correlators, each with a specific network of $U_a(\gamma_{d-p-1})$ defects inserted along Σ_n. This has straightforward generalizations, gauging subgroups $\mathbb{B} \subset \mathbb{A}$, and also adding discrete torsion $\nu \in H^n(B^{q1}bB, \mathbb{R}/\mathbb{Z})$ if allowed.

The procedure of gauging the symmetry in some sub-manifold has been called *higher-gauging* in [26]. It must be emphasized that it is a completely different procedure from ordinary gauging of a discrete symmetry. The latter produces a new theory, while higher-gauging does not change the theory but defines a defect in that theory.

$C(\Sigma_n)$ (or its generalizations) is automatically topological, because so is $U_a(\gamma_{d-p-1})$, hence we regard it as a generalized symmetry. However, it should be noted that these defects are a sort of *bonus* whenever we have higher-form symmetries: they are not really new symmetries, and they are not expected to imply new consequences for theory, on top of those implied by $\mathbb{A}$. From a mathematical perspective [97, 103], we should take them into account to nicely *complete* the *categorical symmetry* of the theory. This is called *Karoubi* or *condensation completion.* Physically, there are two natural question we need to ask:

1. How do these defects *interact* with the other operators in the theory?
2. Where are these defects likely to show up in a natural way?

Since $C(\Sigma_n)$ is made by defects of the p-form symmetry $\mathbb{A}$, it is intuitively clear that the only operators $C(\Sigma_n)$ can interact with are those that are charged under $\mathbb{A}$. Therefore, even though $C(\Sigma_n)$ generates a $(d-n-1)$-form symmetry, defects of dimension $< p$ are transparent for $C(\Sigma_n)$. In particular, since $d-n-1 < p$, $C(\Sigma_n)$ cannot act by linking. This is because the defect is *porous*.

Regarding the second question, there are at least two situations in which condensation defects appear. One is in TQFTs where, for $n = d-1$, they turn out to generate all possible unitary 0-form symmetries that act on the lower-dimensional topological defects as automorphisms. The other is more surprising: they universally appear in the fusion rules of all non-invertible symmetries in $d > 2$.

It turns out that the condensation defects themselves are often non-invertible, with fusion rules of projectors: $C(\Sigma_n) \times C(\Sigma_n) \sim C(\Sigma_n)$. More precisely, the condensation defect is non-invertible whenever the higher-gauging we perform in n-dimensions would also be allowed in the bulk d-dimensional space. For instance, the $\mathbb{Z}_N$ 1-form symmetry of a 4d SU(N) gauge theory is non-anomalous and we can gauge it in the bulk. Therefore, if we construct a 3d condensation defect $C(\Sigma_3)$ by higher-gauging, this is non-invertible. On the other hand, higher-gauging the $\mathbb{Z}_N$ 1-form symmetry of the 3d $U(1)_N$ Chern-Simons theory on a two-dimensional surface would produce an invertible defect.

Moreover, even higher-gauging a non-anomalous symmetry can sometimes produce an invertible defect, if the higher-gauge utilizes a discrete torsion that is not allowed in the bulk. We will discuss examples of this in 5d TQFTs in Chapter 5. A 5d $\mathbb{Z}_N$ gauge theory has 2-form symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$ with an anomaly, but each of the two $\mathbb{Z}_N$ factors is anomaly free. However, while there is no 5-dimensional SPT phase for a $\mathbb{Z}_N$ 2-form symmetry, higher-gauging the surfaces on a 4-manifold is equivalent to gauging a 1-form symmetry, for which we can add discrete torsion $r \in H^4(B^2\mathbb{Z}_N, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{\gcd(N,2)\cdot N}$. We will see that if this is non-trivial, the condensation defect will be invertible.

We will now discuss in some more detail condensation defects in TQFTs, focusing on three-dimensions, that is the first non-trivial cases. This mostly riviewes [26]. Many higher-dimensional generalizations will be discussed in great detail in Chapter 5.

3.2.2 Condensation defects in TQFTs

In TQFTs condensation defects can be constructed and analyzed very explicitly. Consider the 3d $\mathbb{Z}_N$ gauge theory

$$S = \frac{iN}{2\pi} \int_{X_3} B \wedge dA \ . \tag{3.2.3}$$

This has 1-form symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$, generated by dyons

$$W_{a,b}(\gamma) = \exp\left(i \int_\gamma (aA + bB) \right) \ , \qquad a, b \in \mathbb{Z}_N \tag{3.2.4}$$

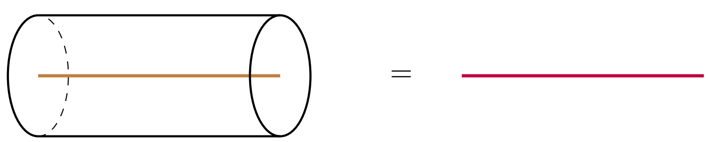
with braiding and topological spin given by

$$\mathcal{B}\big((a,b);(a',b')\big) = \exp\left(\frac{2\pi i}{N}(ab' + a'b) \right) \ , \qquad \theta_{(a,b)} = \exp\left(\frac{2\pi i ab}{N} \right) \tag{3.2.5}$$

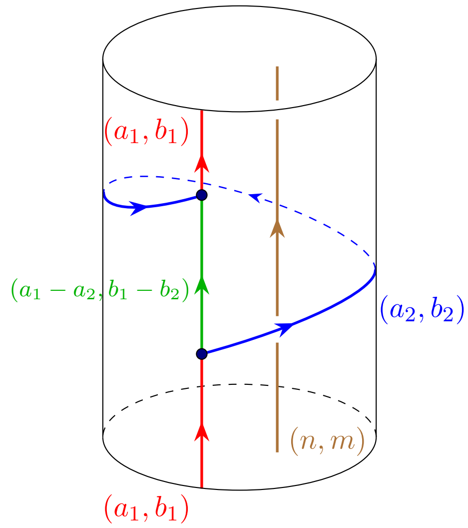
Hence the full 1-form symmetry is anomalous, while the two $\mathbb{Z}_N$ subgroups generated by $W_{(a,0)}$ and $W_{(0,b)}$ are non-anomalous. However the anomaly is only caused by the braiding, hence when the topological lines are thought as generating a 0-form symmetry on a two-dimensional surface, even the full $\mathbb{Z}_N \times \mathbb{Z}_N$ can be higher-gauged. Hence we can construct a condensation defect

$$C(\Sigma_2) = \frac{1}{\Big| H_1\left(\Sigma_2, \mathbb{Z}_N \times \mathbb{Z}_N\right) \Big|^{1/2}} \sum_{\gamma \in H_1(\Sigma_2, \mathbb{Z})} \sum_{a,b \in \mathbb{Z}_N} W_{(a,b)}(\gamma) \ . \tag{3.2.6}$$

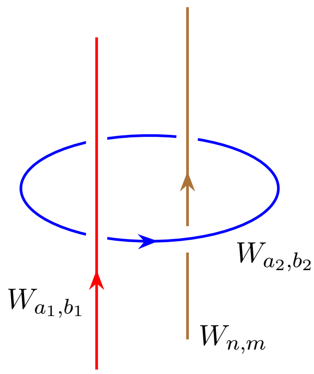
Although it is a codimension-one operator, hence generating a 0-form symmetry, it does not act on local operators, as there are none. Even in presence of local operators, it would not act on them, since it is made out of lines that cannot have a non-trivial interactions with points in 3d. $C(\Sigma_2)$ has an action on lines, not by standard lining, but by surrounding them:



When the cylinder is shrinked around a line $W_{n,m}(L)$ we remain with a generically different line, that could also be non-simple, but a sum of lines. To determine this action we use the definition of $C(\Sigma_2)$ as sum of lines on the cycles of the cylinder. We assume N odd for simplicity here. The generic element of the sum takes the form



This must be summed over $a_1, b_1, a_2, b_2 \in \mathbb{Z}_N$. This network can be resolved in four steps. First we pass the blue line through the upper piece of the red line, costing a factor of half-braiding $\exp\left(\frac{2\pi i}{2N}(a_1 b_2 + a_2 b_1)\right)$. The factor of 2 at the denominator is understood as multiplying the numerator by 2^{-1}, that exists because N is odd. Second, junctions among the red, green and blue lines can be resolved without any additional cost into



Third, we pass the blue line through the brown one, getting a braiding phase $\exp\left(\frac{2\pi i}{N}(a_2 m + b_2 n)\right)$, and the blue line can be made to disappear. Finally, we fuse the red and brown lines into W_{a_1+n, b_1+m}.

We conclude that the action of $C(\Sigma_2)$ on the line $W_{n,m}(L)$ is given by

$$C(\Sigma_2) \cdot W_{n,m}(L) = \frac{1}{N^2} \sum_{a_1, b_1, a_2, b_2 \in \mathbb{Z}_N} \exp\left(\frac{2\pi i}{N}(2^{-1} a_1 b_2 + 2^{-1} a_2 b_1 + a_2 m + b_2 n)\right) W_{a_1+n, b_1+m} \quad (3.2.7)$$

The sum over a_2, b_2 cancel the N^2 factor and gives two Kronecker delta imposing

$$a_1 = -2n \;, \qquad b_1 = -2m \;. \quad (3.2.8)$$

This is solved by the sums over a_1, b_1, and the result is

$$C(\Sigma_2) \cdot W_{n,m} = W_{-n,-m} \;. \quad (3.2.9)$$

We conclude that $C(\Sigma_2)$ is the topological operator implementing charge conjugation symmetry in the TQFT: $(n, m) \mapsto (-n, -m)$.

This example has a twisted generalization, including discrete torsion $r \in H^2(B\mathbb{Z}_N \times \mathbb{Z}_N, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_N$ in the higher-gauging, producing a different condensation defect $C_r(\Sigma_2)$. Recall that $H^2(B\mathbb{Z}_N \times \mathbb{Z}_N, \mathbb{R}/\mathbb{Z})$ is given by alternating bicharacters

$$\chi_r\left((a_1, b_1); (a_2, b_2)\right) = \frac{2\pi i r}{N}(a_1 b_2 - a_2 b_1) \tag{3.2.10}$$

hence this factor must be included at the exponent in the twisted higher-gauging. We assume $2r + 1, 2r - 1$ to be invertible numbers in $\mathbb{Z}_N$. The same computation above gives

$$C_r(\Sigma_2) \cdot W_{n,m} = W_{qn, q^{-1}m} \quad , \qquad q = (2r+1)^{-1}(2r-1) \ . \tag{3.2.11}$$

Therefore $C_r(\Sigma_r)$ implements the action of the automorphism $(n, m) \mapsto (qn, q^{-1}m)$, that is indeed a symmetry of the TQFT, since it preserves the spin.

It turns out that all automorphism 0-form symmetries of 3d TQFTs can be realized in a similar way with condensation defects [26]. As an other interesting example, that will be relevant in Chapter 6, we consider electro-magnetic duality of pure $\mathbb{A}$ gauge theories, introduced in Section 2.2.1. We recall that the theory has lines labeled by $(a, \alpha) \in \mathbb{A} \times \mathbb{A}^\vee$ with braiding $\mathcal{B}\left((a_1, \alpha_1), (a_2, \alpha_2)\right) = \exp\left(2\pi i(\alpha_2(a_1) + \alpha_1(a_2)\right)$, and electro-magnetic duality is a $\mathbb{Z}_2$ symmetry subordinated to a choice of a symmetric isomorphism $\phi : \mathbb{A} \to \mathbb{A}^\vee$ and given by

$$\Phi : \ (a, \alpha) \mapsto \big(\phi^{-1}(\alpha), \phi(a)\big) \ . \tag{3.2.12}$$

To construct a topological operator that implements this 0-form symmetry, we first look at the subset of Φ-fixed lines

$$F = \{(a, \alpha) \mid \Phi \cdot (a, \alpha) = (a, \alpha)\} = \{(a, \phi(a)) \mid a \in \mathbb{A}\} \tag{3.2.13}$$

and its orthogonal with respect to the braiding

$$F^\perp = \big\{(a, \alpha) \mid \mathcal{B}\big((a, \alpha); (a', \alpha')\big) = 1 \ , \forall (a', \alpha') \in F\big\} = \big\{\big(a, -\phi(a)\big) \mid a \in \mathbb{A}\big\} \ . \tag{3.2.14}$$

We claim that the action of Φ is implemented by a condensation defect $C_\phi(\Sigma_2)$ resulting from higher-gauging $F^\perp$ without torsion. We can follow the same steps as above. First we notice that, using the symmetry of ϕ, the half-braiding between $(a_1, -\phi(a_1))$ and $(a_2, -\phi(a_2))$ is given by

$$\left[\mathcal{B}\Big(\big(a_1, -\phi\left(a_1\right)\big), \big(a_2, -\phi\left(a_2\right)\big)\Big)\right]^{1/2} = \exp\left(-2\pi i \phi\left(a_1\right) a_2\right) \ . \tag{3.2.15}$$

Therefore the action of $C\phi(\Sigma_2)$ on $W_{a,\alpha}$ is

$$C_\phi(\Sigma_2) \cdot W_{a,\alpha} = \frac{1}{|\mathbb{A}|} \sum_{a_1, a_2 \in \mathbb{A}} \exp\left(2\pi i \Big(-\phi\left(a_1\right) a_2 - \phi\left(a_2\right) a + \alpha(a_2)\Big)\right) W_{a+a_1, \alpha - \phi(a_1)} \ . \tag{3.2.16}$$

The sum over a_2 cancels the factor $|\mathbb{A}|$ and imposes that $a_1 = -a + \phi^{-1}(\alpha)$. Hence the result is

$$C_\phi(\Sigma_2) \cdot W_{a,\alpha} = W_{\phi^{-1}(\alpha), \phi(a)} \ . \tag{3.2.17}$$

Non-invertible condensation defects. We conclude this section with one example of a non-invertible condensation defect in the 3d $\mathbb{Z}_N$ gauge theory. We notice that the subgroup $\mathbb{Z}_N\left\{(a, 0)\right\} \subset \mathbb{Z}_N \times \mathbb{Z}_N$ of the 1-form symmetry is non-anomalous, hence we expect that its higher-gauging on Σ_2 produces a non-invertible defect. We can verify this by computing its action on lines, using the procedure followed so far. Since we are condensing a non-anomalous symmetry, the half-braiding factor is trivial and we get

$$C_{\mathbb{Z}_N}(\Sigma_2)\cdot W_{n,m}=\frac{1}{N}\sum_{a_1,a_2}\exp\left(\frac{2\pi i}{N}a_2 m\right)W_{n+a_1,m}=\delta_{m,0}\sum_a W_{a,0}\ . \tag{3.2.18}$$

This right hand side is zero unless the line $W_{n,m}$ also belong the subgroup condensed to produce $C_{\mathbb{Z}_N}(\Sigma_2)$, and in this case the result is not a single line, but the sum of all lines in that subgroup. This is clearly not an invertible operation: $C_{\mathbb{Z}_N}(\Sigma_2)$ is a projector.

Applying $C_{\mathbb{Z}_N}(\Sigma_2)$ on both sides of (3.2.18) we deduce the fusion rule of the condensation defect with itself

$$C_{\mathbb{Z}_N}(\Sigma_2)\times C_{\mathbb{Z}_N}(\Sigma_2)=N C_{\mathbb{Z}_N}(\Sigma_2)\ . \tag{3.2.19}$$

The fusion coefficient N appearing here can be given an interesting interpretation by computing the fusion rule on general 2-manifolds Σ_2 other then the cylinder. This is done in [26], and we present several methods in a slightly more general higher dimensional context in Chapter 5. The general result is

$$C_{\mathbb{Z}_N}(\Sigma_2)\times C_{\mathbb{Z}_N}(\Sigma_2)=Z_{\mathbb{Z}_N}(\Sigma_2)\, C_{\mathbb{Z}_N}(\Sigma_2) \tag{3.2.20}$$

where $Z_{\mathbb{Z}_N}(\Sigma_2)=\Big|H^1(\Sigma_2,\mathbb{Z}_N)\Big|^{1/2}$ is the partition function of the 2d $\mathbb{Z}_N$ gauge theory on Σ_2. As we will see, this turns out to be a generic feature of non-invertible categorical symmetries in $d>2$: the fusion coefficients are not numbers, but partition functions of TQFTs, that is manifold-depended numbers [26, 104].

3.3 Duality defects

Duality defects are a type of non-invertible (categorical) defects that exist in even-dimensional QFTs with a *self-duality.* The latter is a quantum equivalence between the "*same*" theory in different points of its parameter space, that however involves a non-trivial map of operators. The word "*same*" is in quotation because in general this action can involve changing the global structure. Certain points of the parameter space might be fixed under this action, and the self-duality becomes a symmetry of the theory. However, this is strictly true only if the action does not involve a change of the global structure. It turns out that even when such a change is involved, there is a topological defect, but it is non-invertible [24]. These defects will be the main characters of Chapters 5 and 6, where many of their properties will be studied in great detail and the dynamical consequences will be analyzed. For this reason, in this section we will only review the basic construction and some examples, leaving many other aspects for those chapters.

3.3.1 Two-dimensional compact boson

T-duality. To illustrate the general discussion above, consider the 2d compact boson (see [99])

$$S_R[\Phi]=\frac{R^2}{4\pi}\int_{X_2} d\Phi\wedge * d\Phi \tag{3.3.1}$$

that has a parameter $R\in\mathbb{R}^+$. The theory has a self-duality, T-duality, that can be expressed as an equivalence of path integrals:

$$\int D[\Phi]e^{-S_R[\Phi]}=\int D[\widetilde{\Phi}]e^{-S_{\widetilde{R}}[\widetilde{\Phi}]}\ , \qquad \widetilde{R}=\frac{1}{R}\ . \tag{3.3.2}$$

This is obtained by a change of variable in the path integral, under which we identify

$$R^2 * d\Phi = -i d\widetilde{\Phi} \ . \tag{3.3.3}$$

It follows from the discussion in Section 1.2 around (1.2.9), that T-duality gives the following map of operators:

$$T(U_\alpha) = \widetilde{V}_\alpha \ , \qquad T(V_\beta) = \widetilde{U}_\beta \ , \qquad T(\mathcal{O}_n) = \widetilde{H}_n \ , \qquad T(H_w) = \widetilde{\mathcal{O}}_w \ . \tag{3.3.4}$$

Notice, in particular, that the theory $S_R[\Phi]$ has 0-form symmetries $U(1)_M \times U(1)_W$ while $S_{\widetilde{R}}[\widetilde{\Phi}]$ has $\widetilde{U(1)}_M \times \widetilde{U(1)}_W$, and T-duality specifies two isomorphisms

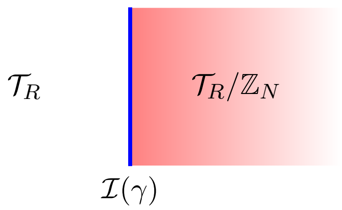
$$\phi_M : U(1)_M \to \widetilde{U(1)}_W \ , \qquad \phi_W : U(1)_W \to \widetilde{U(1)}_M \ . \tag{3.3.5}$$

For generic R, T-duality is an equivalence of two theories. However, at the special value

$$R = 1 \tag{3.3.6}$$

we have $R = \widetilde{R}$ (self-dual radius): the two theories are the same, and we have $U_\alpha = \widetilde{U}_\alpha, V_\beta = \widetilde{V}_\beta$ and so on. Therefore, the compact boson at the self-dual radius has an extra $\mathbb{Z}_2$ 0-from symmetry given by T-duality.

Half-space gauging. Consider now the compact boson $\mathcal{T}_R$ with a generic value of R, and place it on a manifold X_2 divided into two parts X_2^L, X_2^R by a curve $\gamma \subset X_2$. Choose $N \in \mathbb{N}$, and gauge $\mathbb{Z}_N \subset U(1)_M$ *only* in the right half X_2^R



This means that we introduce a $\mathbb{Z}_N$ gauge field $A \in H^1(X_2^R, \mathbb{Z}_N)$ coupled with the theory, and we make it dynamical. We also need to choose a boundary condition for A on $\gamma = \partial X_2^R$. We choose Dirichlet boundary conditions

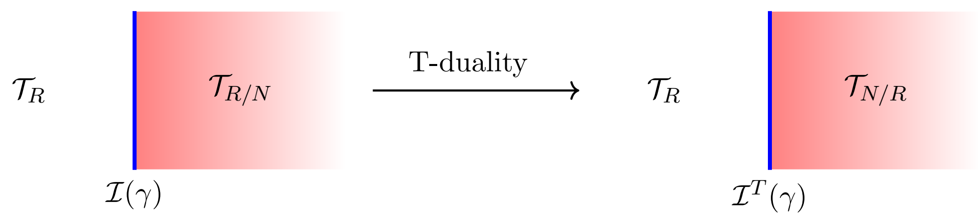
$$\delta A\big|_\gamma = 0 \ . \tag{3.3.7}$$

More intrinsically, this means that the lines $e^{ia\int A}$, $a \in \mathbb{Z}_N^\vee$ of the quantum symmetry can terminate topologically on γ. This operation defines a topological *interface* $\mathcal{I}(\gamma)$ between $\mathcal{T}_R$ and its orbifold $\mathcal{T}_R/\mathbb{Z}_N$.

The orbifold, however, is merely an other compact boson at radius $R' = R/N$:

$$\mathcal{T}_R/\mathbb{Z}_N = \mathcal{T}_{R/N} \ . \tag{3.3.8}$$

This just follows from the fact that the vertex operators $\mathcal{O}_n(x)$ that are invariant under $\mathbb{Z}_N$ are those with $n = mN$, $m \in \mathbb{Z}$, hence the field with 2π periodicity is $\Phi' = N\Phi$. By applying T-duality on the right side, the interface $\mathcal{I}(\gamma)$ gives rise to an *equivalent* interface

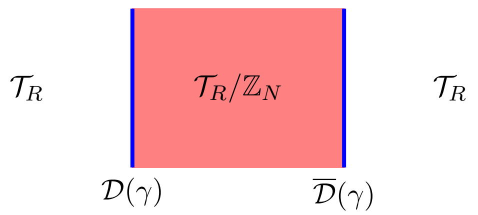


Now, if the square radius R^2 is integer[2], we are free to choose

$$N = R^2 \tag{3.3.9}$$

and $\mathcal{I}^T(\gamma)$ is not only an interface, but a *topological defect* in the theory $\mathcal{T}_R$. This is called a *duality defect* and we denote it by $\mathcal{D}(\gamma)$.

Notice that $D(\gamma)$ coincides with its orientation reversal $\overline{\mathcal{D}}(\gamma)$. This follows from T-duality being an order-two operation, so we can reverse left and right in the argument above. The duality defect is non-invertible. To establish this, we compute the fusion $\mathcal{D}(\gamma) \times \overline{\mathcal{D}}(\gamma)$ (equivalently $\mathcal{D}(\gamma) \times \mathcal{D}(\gamma)$), simply composing the gauging operations:



Using T-duality, the theory in the middle is also $\mathcal{T}_R$. The slab has the topology of $\gamma \times [0, 1]$, and since we are gauging $\mathbb{Z}_N$ there, squeezing the slab leaves the sum of all topological lines $U_{\alpha=\frac{2\pi a}{N}}(\gamma)$ generating $\mathbb{Z}_N$:

$$\mathcal{D}(\gamma) \times \mathcal{D}(\gamma) = \sum_{a=0}^{N-1} U_{\frac{2\pi a}{N}}(\gamma) \ . \tag{3.3.10}$$

We can also consider fusing $\mathcal{D}(\gamma)$ with $U_{\frac{2\pi a}{N}(\gamma)}$. Since these lines can end topologically on $\mathcal{D}(\gamma)$ (because of the Dirichlet boundary condition), they are absorbed by the duality defect:

$$\mathcal{D}(\gamma) \times U_{\frac{2\pi a}{N}}(\gamma) = \mathcal{D}(\gamma) \ . \tag{3.3.11}$$

Equations (3.3.10) and (3.3.11) are the fusion algebra of the so-called *Tambara-Yamagami fusion category* associated with the Abelian group $\mathbb{Z}_N$ [106].

Action on operators. To determine the action of $\mathcal{D}(\gamma)$ on the local operators $\mathcal{O}_n(x), H_w(x)$, we first analyze what happens to them in the orbifold $\mathcal{T}_{R'} = \mathcal{T}_R/\mathbb{Z}_N$. If n is not a multiple of N, $\mathcal{O}_n(x)$ goes into a twisted sector of the orbifold theory. If $n = mN$, instead, $\mathcal{O}_n(x)$ becomes the operator $\mathcal{O}'_m(x) = e^{im\Phi'(x)}$ of the orbifold. More generally, denoting with $\mathcal{O}', H'$ respectively the vertex and the vortex operator of the orbifold theory $\mathcal{T}_{R'} = \mathcal{T}_R/\mathbb{Z}_N$, the gauging interface $\mathcal{I}(\gamma)$ implements the map

$$\mathcal{I}(\gamma) : \begin{cases} \mathcal{O}_n(x) \mapsto \mathcal{O}'_{n/N}(x) \\ H_w(x) \mapsto H'_{Nw}(x) \ . \end{cases} \tag{3.3.12}$$

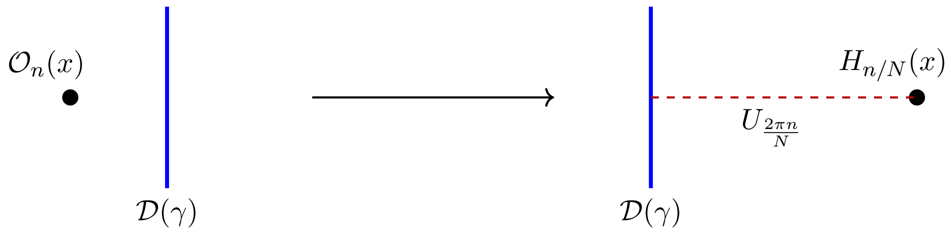
$\mathcal{O}'_{n/N}$ is a (generically) non-genuine operator living at the end-point of the line $V'_{\frac{2\pi n}{N}}$ of the winding symmetry of $\mathcal{T}_{R'}$.

Combining this with the action of T-duality in the orbifold theory, we get the action of the duality defect:

$$\mathcal{D}(\gamma) : \begin{cases} \mathcal{O}_n(x) \mapsto H_{n/N}(x) \\ H_w(x) \mapsto \mathcal{O}_{Nw}(x) \ . \end{cases} \tag{3.3.13}$$

[2]This can be easily generalized to any $R^2 = p/q \in \mathbb{Q}$, by gauging a non-anomalous subgroup $\mathbb{Z}_p \times \mathbb{Z}_q \subset U(1)_M \times U(1)_W$. Recently [105] discussed a further generalization to any radius, including irrational values.

In particular, the operators charged under $\mathbb{Z}_N \subset U(1)_M$ are mapped into twisted sector of the same symmetry:



This is the hallmark of non-invertible symmetries: they map untwisted to twisted sectors (and vice versa). Focusing only on $\mathbb{Z}_N \subset U(1)_M$, notice that while the charged operators are labeled by the Pontryagin dual $\mathbb{Z}_N^\vee$, the twisted sectors are labeled by elements of the symmetry group $\mathbb{Z}_N$. The two are isomorphic, but the isomorphism is non-canonical and requires a choice. In this case, this boils down to the isomorphism $\phi_M : U(1)_M \to \widetilde{U(1)}_W$ of (3.3.5).

3.3.2 Kramers-Wannier duality of Ising CFT

A similar story exists for the Ising CFT, replacing T-duality with Kramer-Wannier (KW) duality [72]. In the lattice model, KW-duality relates the high-temperature regime with the low-termperature one. At criticality, that is in the Ising CFT, it is a self-duality between Ising and Ising/$\mathbb{Z}_2$, that involves the map (we follow the notations introduced in 1.4.3)

$$\sigma(x) \mapsto \mu'(x) \ , \qquad \mu(x) \mapsto \sigma'(x) \ , \qquad \epsilon(x) \mapsto \epsilon(x)' \ . \tag{3.3.14}$$

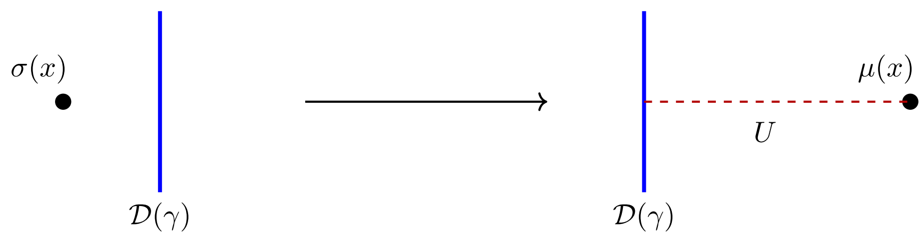
The prime denotes the same operators, but in Ising/$\mathbb{Z}_2$. In particular $\mu'(x)$ is a genuine local operator, while $\sigma'(x)$ lives in the twisted sector of the dual symmetry $\mathbb{Z}_2^\vee \cong \mathbb{Z}_2$.

We can apply the half-gauging construction introduced in the compact boson. We first construct a topological interface $\mathcal{I}(\gamma)$ that separates Ising from Ising/$\mathbb{Z}_2$, and then use KW duality to relate the latter to Ising, making $\mathcal{I}(\gamma)$ into a topological defect $\mathcal{D}(\gamma)$. Denoting by $U(\gamma)$ the generator of the $\mathbb{Z}_2$ symmetry, the same argument used in the compact boson implies the fusion rule

$$\mathcal{D}(\gamma) \times \mathcal{D}(\gamma) = 1 + U(\gamma) \ . \tag{3.3.15}$$

This is the fusion category Tambara-Yamagami for $\mathbb{Z}_2$, denoted by TY($\mathbb{Z}_2$), and sometimes called *Ising fusion category* [20, 21].

We can follow the same argument as before to determine the action on operators. If we start from $\sigma(x)$ in Ising, and we pass the topological gauging interface through it, it becomes the same operator in Ising/$\mathbb{Z}_2$, namely $\sigma'(x)$. The latter is attached to the line $V(\gamma)$ of the dual symmetry. Applying the KW map, this is then related with $\mu(x)$ attached to $U(\gamma)$:



3.3.3 Duality defects in four-dimensions

A completely analogous story exists in four-dimensions, as first pointed out in [24, 25, 107]. As we have seen, non invertible duality defects arise if a theory $\mathcal{T}$ has some symmetry G, and the gauged theory $\mathcal{T}/G$ has a non-trivial duality map with $\mathcal{T}$. The non-trivial map must involve identifying G with the dual symmetry. In 4d the dual of a 0-form symmetry is a 2-form symmetry, hence self-duality under gauging 0-form symmetries are not possible. It is possible, however, to have self-duality under gauging a 1-form symmetry, since its dual is again a 1-form symmetry.

A notable example is 4d $\mathcal{N}=4$ SYM. It has a conformal manifold parametrized by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \in \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \ . \tag{3.3.16}$$

The famous Montonen-Olive duality (or S-duality) states that the $\mathrm{SU}(N)$ theory at τ is equivalent to the $\mathrm{PSU}(N)$ at $-1/\tau$. The precise map is complicated, involving non-trivial actions on both local and extended operators. What will be important for us is not the action on local operators[3], but the action on lines. S-duality maps genuine (non-genuine) Wilson ('t Hooft) lines of the $\mathrm{SU}(N)$ theory, into genuine (non-genuine) 't Hooft (Wilson) lines of the $\mathrm{PSU}(N)$ theory. Notice that Wilson lines of $\mathrm{SU}(N)$ can be labeled with $\mathcal{Z}(\mathrm{SU}(N))^\vee$, while 't Hooft lines of $\mathrm{PSU}(N)$ are labeled by $\pi_1(\mathrm{PSU}(N))^{\vee\vee} \cong \pi_1(\mathrm{PSU}(N)) \cong \mathcal{Z}(\mathrm{SU}(N))$, where these isomorphisms are canonucal. Hence the duality must involve a map

$$\phi : \mathcal{Z}(\mathrm{SU}(N)) \to \pi_1(\mathrm{PSU}(N))^\vee \ . \tag{3.3.17}$$

The underlying map is provided by the celbrated *Langlands duality*, that relates a Lie group G with its Langlands dual ${}^L G$ (see [108] and references therein).

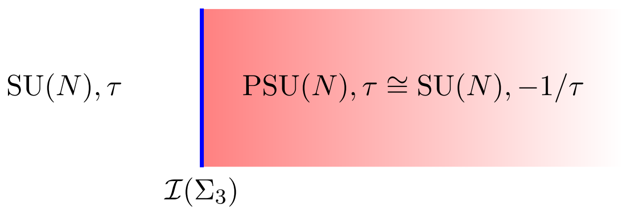
This duality is morally similar to T-duality and Kramers-Wannier duality in 2d, with a notable difference of being of order-four (as opposite to two). In fact, applying S twice, while mapping τ back to itself, it maps a Wilson line in rapresentation $\mathfrak{R}$ with that in the conjugate representation $\overline{\mathfrak{R}}$. Hence S duality squares to charge-conjugation

$$S^2 = C \ . \tag{3.3.18}$$

Finally, combining S with the shift of the theta angle $T : \tau \mapsto \tau + 1$ form the duality group

$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle S, T \;\middle|\; S^2 = C \ , \quad (ST)^3 = 1 \right\rangle \ . \tag{3.3.19}$$

We can apply the half-gauging construction again. Now the interface is three-dimensional $\mathcal{I}(\Sigma_3)$, and saparates the $\mathrm{SU}(N)$ theory from the $\mathrm{PSU}(N) = \mathrm{SU}(N)$ theory, both at τ



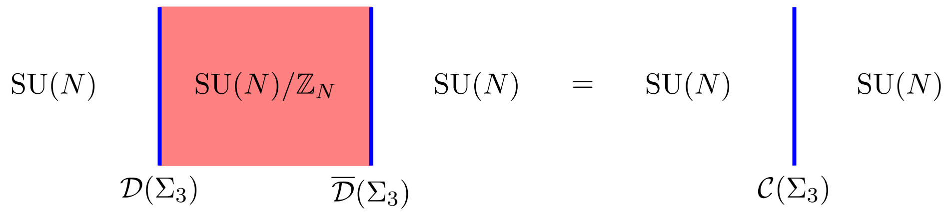
In the right side we used S-duality to get again the $\mathrm{SU}(N)$ theory, but at $-1/\tau$. At the special value

$$\tau = i \tag{3.3.20}$$

the two sides are the same theory, hence the interface become a topological defect $\mathcal{D}(\Sigma_3)$ of the $\mathrm{SU}(N)$ theory.

[3] Notice that the $\mathrm{SU}(N)$ and $\mathrm{PSU}(N)$ theories have the same local operators.

The fusion $\mathcal{D}(\Sigma_3) \times \overline{\mathcal{D}}(\Sigma_3)$ can be computed with the same technique as in the compact boson. The main difference, however, is that the topological defects of the 1-form symmetry, $U_a(\Sigma_2)$, are two-dimensional, while $\mathcal{D}(\Sigma_3)$ is a three-dimensional defect. Gauging the 1-form symmetry in the slab, amounts to inserting $U_a(\Sigma_2)$ on all possible cycles of it, and summing over a. When we squeeze the slab, this reduces to summing over insertion of these defects on the cycles of Σ_3:



The result is nothing but the higher-gauging of the 1-form symmetry on Σ_3, namely a condensation defect [24]:

$$\mathcal{D}(\Sigma_3) \times \overline{\mathcal{D}}(\Sigma_3) = \mathcal{C}(\Sigma_3) \ . \tag{3.3.21}$$

Let us make some comments

- If Σ_3 does not contain nontrivial 2-cycles, $\overline{\mathcal{D}}(\Sigma_3)$ is the inverse of $\mathcal{D}(\Sigma_3)$ and the symmetry is effectively invertible.
- For more complicated Σ_3, we see the emergence of interesting structures and $\mathcal{D}$ has no inverse. The algebraic structure is a categorical extension of the group $\mathbb{Z}_4$: it behaves as $\mathbb{Z}_4$ on certain manifolds, but it is modified by condensation defects.

Moreover, because the Dirichlet boundary conditions for the $\mathbb{Z}_N$ gauge field on the gauing interface, the generators of the 1-form symmetry $U_a(\Sigma_2)$ are completely absorbed by $\mathcal{D}(\Sigma_3)$ if we push Σ_2 on top of Σ_3:

$$\mathcal{D}(\Sigma_3) \times U_a(\Sigma_2) = \mathcal{D}(\Sigma_3) \ . \tag{3.3.22}$$

Finally, following the by now standard argument, we can determine the action of $\mathcal{D}(\Sigma_3)$ on operators. First of all let us notice that the gauging of the 1-form symmetry does not affect the local operators, hence the map on them is exactly the one given by S-duality, and in particular it is invertible. The non-invertiblity arises in the action on line operators: a Wilson line passing the gauging interface becomes non-genuine and attached to surface defects for the magnetic symmetry, and then S-duality maps it to a 't Hooft line attached to $U_a(\Sigma_2)$.

3.3.4 General duality defects

Having discussed some examples, we can illustrate the general story. Consider a d-dimensional QFT $\mathcal{T}$ with a non-anomalous finite p-form symmetry $\mathbb{A}$. We assume d to be even, and

$$p = \frac{d}{2} - 1 \ . \tag{3.3.23}$$

Since $d - p - 2 = p$, the gauged theory $\mathrm{T}/\mathbb{A}$ has a dual p-form symmetry $\mathbb{A}^\vee$. We denote by $U_a(\Sigma_{d-p-1})$ and $W_\alpha(\gamma_p)$ respectively the topological operators and the charged object for $\mathbb{A}$ in $\mathcal{T}$, and by $U'_\alpha(\Sigma_{d-p-1})$ and $H'_a(\gamma_p)$ the corresponding object for $\mathbb{A}^\vee$ in $\mathcal{T}'$=$\mathcal{T}/\mathbb{A}$. We also denote with $H_a(\gamma_p)$ the non-genuine operator of $\mathcal{T}$ in the twisted sector of $a \in \mathbb{A}$, and with $W'_\alpha(\gamma_p)$ the twisted sector of $\alpha \in \mathbb{A}^\vee$ in $\mathcal{T}'$. Notice that, upon gauging, $H_a \leftrightarrow H'_a$ and $W'_\alpha \leftrightarrow W_\alpha$.

We further assume that there is a duality

$$\mathcal{T}/\mathbb{A} \cong \mathcal{T} \ . \tag{3.3.24}$$

The precise map of this duality depends on the specific example, and can involve an action of operators of various dimensionality, different from p, that are not affected by the discrete gauging. What is universal is that the duality must relate the p-form symmetry $\mathbb{A}$ of $\mathcal{T}$ with the dual p-form symmetry $\mathbb{A}^\vee$ of T$/\mathbb{A}$. This makes sense because any finite Abelian group is ismorphic to its Pontryagin dual. However the isomorphism is non-canonical, and the duality must involve the specification of

$$\phi : \mathbb{A} \xrightarrow{\ \sim\ } \mathbb{A}^\vee \tag{3.3.25}$$

that determines how the topological operators transform under duality. The inverse map $\phi^{-1} : \mathbb{A}^\vee \to \mathbb{A}$, on the other hand, determines how the charged p-dimensional operators are mapped by the duality:

$$U_a \longmapsto U'_{\phi(a)} \ , \qquad W_\alpha \longmapsto H_{\phi^{-1}(\alpha)} \ , \qquad H_a \mapsto W'_{\phi(a)} \ . \tag{3.3.26}$$

The isomorphism ϕ cannot be generic, since the action of the symmetry on the charged operators must be preserved:

$$\phi(a)\left(\phi^{-1}(\alpha)\right) = \alpha(a) \ . \tag{3.3.27}$$

This means that the bicharacter $\gamma : \mathbb{A} \times \mathbb{A} \to \mathbb{R}/\mathbb{Z}$, $\gamma(a,b) = \phi(a) \cdot b$ must be symmetric.

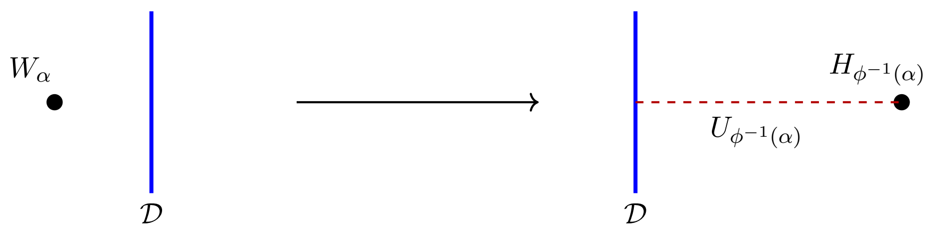
The observant reader may have recognized the resemblance between this discourse and the discussion on electro-magnetic duality in finite-group gauge theories near (2.2.12). This is not coincidental, as both are linked by the SymTFT framework, which we will clarify in Chapter 6.

Under the assumptions spelled out so far, we can proceed with the half-gauging construction as in the various examples. We first construct a topological gauging interface $\mathcal{I}(\Sigma_{d-1})$ separating $\mathcal{T}$ from $\mathcal{T}' = \mathcal{T}/\mathbb{A}$, imposing Dirichlet boundary conditions for the $\mathbb{A}$ gauge field on the right side. We then use the duality to map the right side $\mathcal{T}'$ back to $\mathcal{T}$, hence the gauging interface becomes a topological defect $\mathcal{D}(\Sigma_{d-1})$, with fusion rule

$$\mathcal{D}(\Sigma_{d-1}) \times \overline{\mathcal{D}}(\Sigma_{d-1}) = \mathcal{C}_{\mathbb{A}}(\Sigma_{d-1}) \tag{3.3.28}$$

where $\mathcal{C}_{\mathbb{A}}(\Sigma_{d-1})$ is the condensation defect of the full symmetry $\mathbb{A}$. Only in the special case $p = 0$, $d = 2$, this condensation defect formally reduces to the direct sum of $U_a(\Sigma_1)$, $a \in \mathbb{A}$, while for $p > 1$ it is a generalization of it.

The action on operators not affected by the symmetry $\mathbb{A}$ is non-universal and determined by the specific duality. The action on the p-dimensional operators $W_a(\gamma_p)$ is instead universal. In the gauged theory $W_\alpha(\gamma_p)$ becomes $W'_\alpha(\gamma_p)$, living at the end of the topological operator $U'_\alpha(\Sigma_{d-p-1})$, and the duality maps this into $H_{\phi^{-1}(\alpha)}$ living at the end of $U_{\phi^{-1}(\alpha)}$



The mathematical structure behind duality defects in a d-dimensional QFT is that of fusion $(d-1)$-category. The objects are provided by the duality defects. Since they absorb all symmetry defects of the p-form symmetry, there are no non-trivial endomorphisms on the duality defect. There are, however, non-trivial $(d/2-1)$-morphisms of the identity object (the bulk), provided by the higher-form symmetry. If these are the only elements of the higher-category we would be in trouble, since the fusion (3.3.28) does not close. We need to *complete* the category by adding all possible condensation defects, of all possible dimensionalities [103, 109], an operation called *Karoubi completion.*

3.4 Non-invertible symmetries from gauging

Many types of non-invertible symmetries can be obtained from a theory with only invertible symmetries, and gauging some non-anomalous discrete symmetry. One example is provided by theories with an invertible higher-form symmetry $\mathbb{A}$, together with a 0-form symmetry G that acts on it as an outer automorphism. Gauging G produces a non-invertible higher-form symmetry out of $\mathbb{A}$ [8, 23, 27]. In this section we discuss an other instance, where we start form a theory with a 0-form and a 1-form symmetry, without any action of one on the other, but with a mixed anomaly of a specific type. Gauging the 1-form symmetry makes the 0-form symmetry non-invertible [25].

3.4.1 Vacua in 4d $\mathcal{N}=1$ SYM

Consider the 4d SU(N) gauge theory with one adjoint fermion ψ, namely $\mathcal{N}=1$ SYM theory. There is a discrete $\mathbb{Z}_{2N}$ R-symmetry $\psi \mapsto e^{\frac{2\pi i}{2N}}\psi$, that is spontaneously broken down to $\mathbb{Z}_2 \subset \mathbb{Z}_{2N}$ by the fermion condensate

$$\langle\psi\psi\rangle = e^{\frac{2\pi i p}{N}}\Lambda^3 \quad , \qquad p = 0, ..., N-1 \ . \tag{3.4.1}$$

There are N vacua labeled by $p \in \mathbb{Z}_N = \mathbb{Z}_{2N}/\mathbb{Z}_2$. The vacua are permuted by the action of the broken symmetry, hence they are equivalent, and pairs of them are separated by domain walls, supporting a Chern-Simons theory [110]. This whole structure can be argued purely from symmetry considerations, deriving from the $\mathbb{Z}_{2N}$ R-symmetry and its anomalies [111].

As we have seen in Chapter 1 around (1.4.38), the R-symmetry has a mixed anomaly with the 1-form center symmetry $\mathbb{Z}_N$ with inflow action

$$\frac{2\pi i}{2N}\int_{X_5} \mathcal{A}\cup B\cup B \ . \tag{3.4.2}$$

Here $\mathcal{A} \in H^1(X,\mathbb{Z}_{2N})$, $B \in H^2(X,\mathbb{Z}_{2N})$ are backgrounds for the 0-form and the 1-form symmetry respectively. This means in the PSU(N) theory, obtained by gauging the 1-form symmetry, the R-symmetry is broken by a discrete analog of the ABJ anomaly.

How can we determine the vacuum structure of the PSU(N) theory, without having the R-symmetry at our disposal? We can proceed as follows. Gauging a discrete symmetry is a topological manipulation that commutes with the RG flow, so the IR of PSU(N) is obtained from the IR of SU(N) by gauging the 1-form symmetry. In each of the N vacua the 1-form symmetry is preserved, so that the deep IR theory is a trivial theory with 1-form symmetry $\mathbb{Z}_N$, namely an SPT. The absolute value of this SPT is meaningless (it can be changed by a counterterm of B), but the relative phase between two vacua is physical. Because of the 't Hooft anomaly (3.4.2), the relative SPT phase between two vacua related by the element $p \in \mathbb{Z}_N$ of the broken R-symmetry is [11]

$$S_{SPT} = \frac{2\pi i p}{2N}\int_{X_4} B\cup B \ . \tag{3.4.3}$$

After we make B dynamical each vacuum becomes a (possibly non-trivial) TQFT, namely a pure $\mathbb{Z}_N$ gauge theory with twist

$$\frac{2\pi i}{N}\int_{X_4} C \cup \delta B + \frac{2\pi i p}{2N}\int_{X_4} B \cup B. \tag{3.4.4}$$

Here $C \in C^1(X_4, \mathbb{Z}_N), B \in C^2(X_4, \mathbb{Z}_N)$. This is the Kapustin-Seiberg TQFT that we reviewed in Section 2.2.4. If N and p are coprime, this is a trivial theory, but in general it is not. This means that the various vacua are *not* all equivalent. For instance for SO(3) we have 2 vacua

- One is trivial ($N = 2$, $p = 1$)

- The other supports a non-trivial TQFT

$$\frac{2\pi i}{2}\int_{X_4} C \cup dB \tag{3.4.5}$$

  which has 2 lines and 2 surface operators

$$e^{ia\int_\gamma C} \quad , \quad e^{in\int_\Sigma B} \tag{3.4.6}$$

  $a, b = 0, 1$ with non-trivial mutual braiding. The lines are the low energy limit of the 't Hooft lines which get perimeter law, namely they condense breaking spontaneously the magnetic 1-form symmetry, and become topological at large distance.

This story was essentially known ten years ago [42], but there were some conceptual open questions, whose complete answer required a point missed until very recently.

1. The vacuum structure seems not to be determined by a symmetry. After all, all the N vacua are inequivalent, so they cannot be permuted by a symmetry group. On the other hand, by a quite involved analysis it is still the R-symmetry, but of another theory, which explains the vacuum structure. Conceptually, it seems natural that this structure is determined by an actual symmetry of the PSU(N) theory, but which symmetry?

2. In the SU(N) theory the only interesting physical part of the low energy theory is about the domain walls. Their infrared can be understood as the topological symmetry operators of the R-symmetry. What can we say about the domain walls in PSU(N)? Few years ago [111] analyzed this problem and claimed the the question is not well-posed: since some vacuum is not trivially gapped the physics is not purely three-dimensional and it makes no sense to talk about the domain wall itself.

While essentially correct, the answer of [111] to the second question has been very recently made more complete by Kaidi, Ohmori, and Zheng (KOZ) in [25], who also answered the first question. The R-symmetry can be rescued in the PSU(N) theory, in the sense that there are still three-dimensional topological defects implementing the R-symmetry action on local operators, but these defects are non-invertible at the level of their action on lines. Therefore, it also makes sense to talk about the domain walls, but they are non-invertible walls. We now review the KOZ construction a bit more in general, and then we will come back to the vacuum structure of PSU(N) $\mathcal{N} = 1$ SYM.

3.4.2 The KOZ construction

Consider a 4d theory $\mathcal{T}$ on a spin manifold X_4 with a 0-form symmetry $\mathbb{Z}_M$ and a 1-form symmetry $\mathbb{Z}_N$, with mixed anomaly (denote $k = \gcd(N, M)$)[4]

$$S_{\text{inflow}} = \frac{2\pi i}{k} \int_{X_5} A \cup \frac{B \cup B}{2} \tag{3.4.7}$$

Gauging the 1-form symmetry we get a theory $\mathcal{T}'=\mathcal{T}/\mathbb{Z}_N$ where the 0-form symmetry appears to be broken. In terms of topological defects this is understood as follows. Let $U_a(\Sigma_3)$, $a \in \mathbb{Z}_M$ be the defects of the 0-form symmetry in $\mathcal{T}$, and denote by $A \in H^1(X, \mathbb{Z}_M)$ a background field, that it the Poincare' dual of $U_a(\Sigma_3)$: given a 3-cocycle ω_3 we have

$$\int_{X_4} A \cup \omega_3 = a \int_{\Sigma_3} \omega_3 \ . \tag{3.4.8}$$

In the inflow picture we realize $\Sigma_3 = \partial\widetilde{\Sigma}_4$, with $\widetilde{\Sigma}_4 \subset X_5$ extending in the bulk, where we also pick an extension of A, whose Poincare' dual (in X_5) is a defect $\widetilde{U}_a(\widetilde{\Sigma}_4)$.

Activating a background $B \in H^2(X, \mathbb{Z}_N)$ for the 1-form symmetry, the system with $U_a(\Sigma_a)$ inserted becomes gauge invariant with the addition of the inflow action that reduces to

$$S_{\text{defect-inflow}}\Big(U_a(\Sigma_3); B\Big) = \frac{2\pi i a}{2k} \int_{\widetilde{\Sigma}_4} B \cup B \ . \tag{3.4.9}$$

This can be understood as an anomaly inflow for the defect. Under $B \mapsto B + \delta\nu$ it changes by

$$\delta S_{\text{defect-inflow}}\Big(U_a(\Sigma_3); B\Big) = \frac{2\pi i a}{k} \int_{\widetilde{\Sigma}_4} \delta\nu \cup B = \frac{2\pi i a}{k} \int_{\Sigma_3} \nu \cup B \tag{3.4.10}$$

This means that, *without* the inflow action, correlators of $\mathcal{T}$ with $U_a(\Sigma_3)$ inserted are not gauge invariant under $B \to B + \delta\nu$ but change by a phase:

$$\Big\langle U_a(\Sigma_3) \cdots \Big\rangle_{\mathcal{T}} \longmapsto \exp\left(-i\frac{2\pi i a}{k} \int_{\Sigma_3} \nu \cup B\right) \Big\langle U_a(\Sigma_3) \cdots \Big\rangle_{\mathcal{T}} \tag{3.4.11}$$

One may try to modify the definition of $U_a(\Sigma_3)$ by a B-dependent local counterterm to get rid of this phase, but there is no way to do that. Thus $U_a(\Sigma_3)$ is not well defined and we cannot sum over background fields B with this defect inserted.

The authors of [25] introduced a sophisticated procedure to obtain a topological co-dimension one defect in the theory $\mathcal{T}'=\mathcal{T}/\mathbb{Z}_N$. Preliminarly, we notice that the defect is non-anomalous under the subgroup $\mathbb{Z}_{N/k} \subset \mathbb{Z}_N$ of the 1-form symmetry. Hence this can be gauged without subtleties, and we can replace B with a gauge field B' for the quotient $\mathbb{Z}_k$.

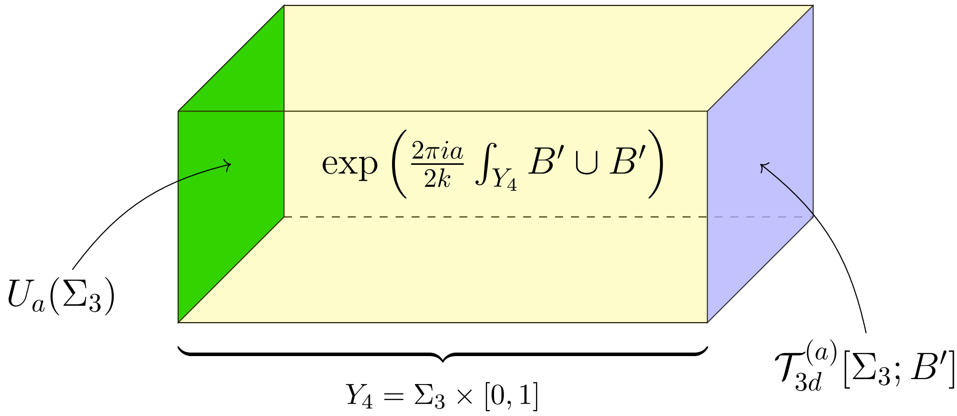
The anomalous phase cannot be eliminated by a local counterterm, essentially because it is an anomaly. But then it can be cancelled by inflow with a 4d term on an open region $Y_4 \subset X_4$, $\partial Y_4 = \Sigma_3$:

$$\exp\left(\frac{2\pi i a}{2k} \int_{\Sigma_4} B' \cup B'\right) \ . \tag{3.4.12}$$

This is formally the same as (3.4.9), but differently from $\widetilde{\Sigma}_4$, Y_4 does not extend to a 5d bulk. Hence, this defines a non-genuine defect of $\mathcal{T}$. The idea is to further cancel this inflow action by stacking, on

[4] Here the cup product in $B \cup B$ is the standard one associated with the ring structure of $\mathbb{Z}_N$. On the other hand the cup product between A and $\frac{B \cup B}{2}$ is associated with the pairing $\mathbb{Z}_M \times \mathbb{Z}_N \to \mathbb{Z}_{\gcd(N,M)}$ given by $(a, b) \mapsto (ab) \bmod(\gcd(N, M))$.

Σ_3, some 3d TQFT $\mathcal{T}_{3d}^{(a)}$ with a 1-form symmetry that has an anomaly canceled by the same inflow action (3.4.12). Pictorially this can be represented as follows



(3.4.13)

Squeezing the slab $Y_4 = \Sigma_3 \times [0,1]$ defines a genuine 3d topological defect

$$\mathcal{N}_a(\Sigma_3) = U_a(\Sigma_3)\mathcal{Z}_{\mathcal{T}_{3d}^{(a)}}[\Sigma_3; B'] \ . \tag{3.4.14}$$

Under $B' \mapsto B' + \delta\nu$ the partition function $\mathcal{Z}_{\mathcal{T}_{3d}^{(a)}}[\Sigma_3; B']$ changes by a phase equal and opposite to that of $U_a(\Sigma_3)$, so that $\mathcal{N}_a(\Sigma_3)$ is gauge invariant.

It seems that there is some level of arbitrariness in choosing the theory $\mathcal{T}_{3d}^{(a)}$, provided that it has the correct anomaly. One crucial result proved in [111], however, states that any 3d TQFT with 1-form symmetry $\mathbb{Z}_n$ and anomaly $p \in \mathbb{Z}_{\gcd(n,2)n}$, if $\gcd(n,p) = 1$, is the stacking of a *minimal theory*, denoted by $\mathcal{A}^{n,p}$, with a decoupled theory. This minimal theory is always some Abelian Chern-Simons theory, up to some gauiging. For $p = \pm 1$ it is simply $\mathcal{A}^{n,p=\pm 1} = U(1)_{\pm n}$. Applied to the present situation, this result implies that the fundamental defect ($a = 1$) has a minimal representative given by

$$\mathcal{N}_1(\Sigma_3) = U_1(\Sigma_3)\mathcal{Z}_{U(1)_{-k}}[\Sigma_3; B'] \ . \tag{3.4.15}$$

Any other choice of $\mathcal{T}_{3d}$ would just stack the defect with a decoupled 3d TQFT, that behaves as the identity operator in the theory.

Representing B' with a continuous $U(1)$ gauge field that is flat, and has periods given by k'th roots of unity, we can give a convenient path integral presentation of $\mathcal{N}_1(\Sigma_1)$

$$\mathcal{N}_1(\Sigma_3) = U_1(\Sigma_3)\int D[z]\exp\left(\frac{ik}{4\pi}\int_{\Sigma_3} z\wedge dz + \frac{ik}{2\pi}\int_{\Sigma_3} z\wedge B'\right) \ . \tag{3.4.16}$$

Regarding the other defects $a \neq 1$, the theory to be stacked is $\mathcal{A}^{k,-a}$ if $\gcd(a,k) = 1$. Otherwise we notice that the subgroup $\mathbb{Z}_{\gcd(a,k)} \subset \mathbb{Z}_k$ is not anomalous for the defect $U_a(\Sigma_3)$, and we can gauge it reducing the anomaly to $\frac{2\pi i a'}{2k'}\int_{Y_4} B''\cup B''$, with $a' = a/\gcd(a,k)$, $k' = k/\gcd(a,k)$, and $B'' \in H^2(X, \mathbb{Z}_{k'})$ a $\mathbb{Z}_{k'}$ gauge field. Now $\gcd(a',k') = 1$, and the gauge-invariant topological defect is

$$\mathcal{N}_a(\Sigma_3) = U_a(\Sigma_3)\,\mathcal{Z}_{\mathcal{A}^{k',a'}}[\Sigma_3; B''] \ . \tag{3.4.17}$$

When we make B'' dynamical, finally getting the gauged theory $\mathcal{T}'{=}\mathcal{T}/\mathbb{Z}_N$, all the defects $\mathcal{N}_a(\Sigma_3)$ survive and implement a 0-form symmetry of $\mathcal{T}'$.

Fusion rules. This symmetry, however, is non-invertible. Indeed, while the naked part $U_a(\Sigma_3)$ fuses following the $\mathbb{Z}_M$ group law, the TQFT dressings fuse in a non-invertible fashion. For the sake of explicitness, consider the fusion of $\mathcal{N}_1(\Sigma_3)$ with its orientation reversal. This involves computing

$$\mathcal{Z}_{U(1)_{-k}}[\Sigma_3; B'] \times \mathcal{Z}_{U(1)_k}[\Sigma_3; B'] = \int D[z_1, z_2] \exp\left(\frac{ik}{4\pi}\int_{\Sigma_3}\Big((z_1 \wedge dz_1 - z_2 \wedge dz_2) + 2(z_1 - z_2) \wedge B'\Big)\right) \tag{3.4.18}$$

Changing variable in the path integral, eliminating z_1 in favour of $z = z_1 - z_2$, the action becomes

$$\frac{k}{2\pi}\int_{\Sigma_3} z_2 \wedge dz + \frac{k}{4\pi} z \wedge dz + \frac{k}{2\pi}\int_{\Sigma_3} z \wedge B' \ . \tag{3.4.19}$$

This is nothing but the 3d $\mathbb{Z}_k$ DW theory with twist $p = k \in \mathbb{Z}_{2k}$ (2.2.70), coupled with B'.

The theory $\mathcal{T}' = \mathcal{T}/\mathbb{Z}_N$ has a dual 1-form symmetry $\mathbb{Z}_N^\vee \cong \mathbb{Z}_N$ generated by the surfaces

$$V_\alpha(\gamma_2) = e^{i\alpha\int_{\gamma_2} B} \ . \tag{3.4.20}$$

The result of the fusion is nothing condensation defect obtained by higher-gauging the subgroup $\mathbb{Z}_k \subset \mathbb{Z}_N^\vee$ of the dual 1-form symmetry, with discrete torsion k. Therefore

$$\mathcal{N}_1(\Sigma_3) \times \overline{\mathcal{N}}_1(\Sigma_3) = C_k(\Sigma_3) \ , \tag{3.4.21}$$

and $\mathcal{N}_1(\Sigma_3)$ doe not have an inverse.

Fusing the 1-form symmetry with the non-invertible defect. We can also consider the effect of pushing a 1-form symmetry defect $V_\alpha(\gamma_2)$ onto the non-invertible symmetry operator. Consider first the case of M multiple of N, so that $k = N$ and $B' = B$. Then bringing $V_\alpha(\gamma_2)$ on $\mathcal{N}_1(\Sigma_3)$, using the path integral presentation (3.4.16) we get

$$\mathcal{N}_1(\Sigma_3) \times V_\alpha(\gamma_2) = U_1(\Sigma_3)\int D[z] \exp\left(\frac{ik}{4\pi}\int_{\Sigma_3} z \wedge dz + i\int_{\Sigma_3}\left(\frac{k}{2\pi} z + \alpha \mathrm{PD}(\gamma_2)\right) \wedge B\right) \tag{3.4.22}$$

with $\mathrm{PD}(\gamma_2)$ the Poincare' dual cycle of γ_2 inside Σ_3. Hence shifting $z \mapsto z - \frac{2\pi\alpha}{k}\mathrm{PD}(\gamma_2)$, and using that $\mathrm{PD}(\gamma_2)$ is closed, we get

$$\mathcal{N}_1(\Sigma_3) \times V_\alpha(\gamma_2) = \mathcal{N}_1(\Sigma_3) \ , \tag{3.4.23}$$

namely the 1-form symmetry defect is absorbed.

For $k \neq N$ we decompose $B = \frac{k}{N}B' + \widetilde{B}$, where $B', \widetilde{B}$ have periods that are, respectively, k'th and N/k'th roots of unity. The 1-form symmetry defects can be written as

$$V_\alpha(\gamma_2) = e^{i\alpha\int_{\gamma_2} B} = e^{i\alpha'\int_{\gamma_2} B' + i\widetilde{\alpha}\int_{\gamma_2}\widetilde{B}} \ , \qquad \alpha' \in \mathbb{Z}_k \subset \mathbb{Z}_N^\vee \ , \quad \widetilde{\alpha} \in \mathbb{Z}_{N/k} = \mathbb{Z}_N^\vee/\mathbb{Z}_k \ . \tag{3.4.24}$$

Since only B' appears in the expression (3.4.16), only the 1-form symmetry defects of the subgroup $\mathbb{Z}_k \subset \mathbb{Z}_N^\vee$ are absorbed, leaving those of the quotient.

Categorical structure. As in the case of duality defects, $\mathcal{N}_a(\Sigma_3)$ are *effectively invertible* on Σ_3 without non-trivial 2-cycles. The non-invertible defects constitute a categorical extension of the group $\mathbb{Z}_M$. More precisely, the algebraic structure is that of a *fusion 3-category* [97]. Here, however, it is slightly more intricate than for duality defects, because not all 1-form symmetry defects are absorbed by the non-invertible symmetry:

- Objects are given by the $\mathcal{N}_a(\Sigma_3)$, $a \in \mathbb{Z}_M$, but also by all possible condensation defects of the 1-form symmetry surfaces.

- The 1-morphisms are given by the co-dimension surfaces of the 1-form symmetry $V_\alpha(\gamma_2)$, and they form a 2-category. More precisely, the category of surface defects (1-morphisms) depends on the object we are looking at. On the identity object, we really have all the $V_\alpha(\gamma_2)$. But, as we have seen, some of them are trivialized as we push them on $\mathcal{N}_a(\Sigma_3)$.

- There are no 2- and 3-morphisms.

3.4.3 Back to PSU(N) SYM

Now we can go back to 4d $\mathcal{N} = 1$ PSU(N) SYM. Here we have $M = 2N$, so $k = N$. The $\mathbb{Z}_{2N}$ R-symmetry is not absent, but is non-invertible. Since the non-invertibility becomes manifest only if Σ_3 has non-trivial 2-cycles, the action on local operators is unchanged with respect to the SU(N) theory. In particular, the fermion condensate $\langle\psi\psi\rangle \neq 0$ still breaks the symmetry spontaneously, predicting the existence of N isolated vacua.

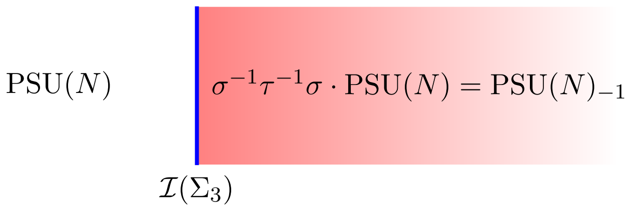
However, the vacua are mapped among them by a non-invertible symmetry defect. If $\Sigma_3 \subset X_4$ divides space-time into two regions and we insert $\mathcal{N}_1(\Sigma_3)$, the theory will have different vacua on either side with a phase difference $e^{\frac{2\pi i}{N}}$ of the fermion condensate. The IR phases on the two sides are different. This was previously understood by gauging the IR of the SU(N) theory, but can now be seen as a result of the non-invertible symmetry.

To derive this, we need to discuss the action of the non-invertible defect on the line operators. This is more clearly explained by employing a different construction of $\mathcal{N}_1(\Sigma_3)$, using a procedure analogous to the higher-gauging trick of Section 3.3. In any 4d gauge theory with 1-form symmetry $\mathbb{Z}_N$ we can consider the topological manipulations

σ : Gauging the 1-form symmetry.

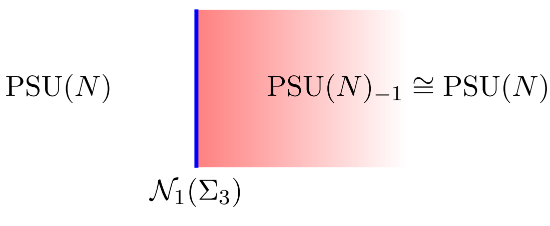
τ : Stacking an SPT for the 1-form symmetry $\frac{2\pi i}{2N}\int_{X_4} B \cup B$.

We perform the topological manipulation $\sigma^{-1}\tau^{-1}\sigma$ in half-space on the PSU(N) theory



Indeed σ maps PSU(N) to SU(N), τ^{-1} stacks the SPT with coefficient -1 for the electric 1-form symmetry, and gauging the 1-form symmetry back produces the global variant PSU$(N)_{-1}$.

However, because of the mixed 't Hooft anomaly between the $\mathbb{Z}_{2N}$ R-symmetry and the 1-form symmetry, the stacking done by τ^{-1} (this is equivalent to $\theta \mapsto \theta - 2\pi$) can be undone by an R-symmetry transformation. This means that PSU$(N)_{-1}$ has a *duality* with PSU(N). The map of the duality on operators is exactly the action of the R-symmetry generator. Using this duality $\mathcal{I}(\Sigma_3)$ is converted into a defect of PSU(N), that is $\mathcal{N}_1(\Sigma_3)$:



Consider now a 't Hooft line $H_a(\gamma_1)$ of the PSU(N) theory, and let us pass $\mathcal{N}_1(\Sigma_3)$ through it. We can adapt the same argument as for the duality defects. The gauging interface will send $H_a(\gamma)$ to the same line, but in the PSU$(N)_{-1}$ theory. Recall that in the PSU$(N)_{-1}$ theory the genuine lines are dyons. Hence $H_a(\gamma_1)$ is non-genuine, and attached to a surface. The duality, by Witten effect [75], maps this line to a dyon of PSU(N), that is also non-genuine.

Coming back to the non equivalence of the vacua, suppose that the vacuum on the left side is that with magnetic confinement, namely the 't Hooft line condenses and the IR is the TQFT

$$\frac{iN}{2\pi}\int_{X_4} A\wedge dB\ . \tag{3.4.25}$$

The lines of $H_a(\gamma_1)=e^{ia\int_{\gamma_1}A}$ are the IR limit of the 't Hooft lines, while the surfaces $V_\alpha(\gamma_2)=e^{i\alpha\int_{\gamma_2}B}$ generates the magnetic 1-form symmetry. The (non-genuine) Wilson line, instead, has area law, as well as all dyons.

As we pass $V_\alpha(\gamma_2)$ through the wall $\mathcal{N}_1(\Sigma_3)$ it gets absorbed and disappears. This means that on the right vacuum the surfaces are trivialized, and as a consequence the lines $H_a(\gamma_1)$ should also become trivial. We can say more: as we derived above, $H_a(\gamma_1)$ is non-genuine and attached to a surface. We recognize that the TQFT on the right side is the Kapustin-Seiberg TQFT with twist $p=1$.

$$\frac{iN}{2\pi}\int_{X_4} A\wedge dB+\frac{iN}{4\pi}\int_{X_4} B\wedge B\ . \tag{3.4.26}$$

In particular, this vacuum is trivially gapped, but supports a non-trivial SPT phase (recall that the Kapustin-Seiberg TQFT is an SPT if $\gcd(N,p)=1$). This is also in agreement with the fact that in the PSU$(N)_{-1}$ the genuine line is a dyon, that has area law. This phase is often called *oblique confinement.*

Proceeding in this way it is not hard to show that two vacua separated by the non-invertible defect $\mathcal{N}_a(\Sigma_3)$ are both Kapustin-Seiberg TQFTs, but with twists that differ by

$$p_R-p_L=a\ . \tag{3.4.27}$$

In particular, by acting with $\mathcal{N}_1(\Sigma_3)$ on the magnetically confined vacuum, we get a vacuum supoorting the TQFT

$$\frac{iN}{2\pi}\int_{X_4} A\wedge dB+\frac{iNa}{4\pi}\int_{X_4} B\wedge B\ , \tag{3.4.28}$$

that has $\gcd(N,a)$ non-trivial lines and surfaces.

3.5 Non-invertible chiral symmetry

The KOZ construction rescued a 0-form symmetry that naively is broken by a discrete version of the ABJ anomaly. It is then natural to ask if a similar construction can rescue an axial $U(1)$ symmetry broken by the ordinary ABJ anomaly [112, 113]. If the gauge field responsible for the breaking is Abelian, it was shown in [28, 29] that the answer is affirmative. The prize to pay here is twofold: in addition to making the rescued symmetry non-invertible (as in the discrete case), only the dense set of defects labeled by $\mathbb{Q}/\mathbb{Z}\subset U(1)$ survives at the quantum level[5]. We now review the construction of this *non-invertible chiral symmetry*, which will be the subject of Sections 8.4.2 and 9.4.2 where we will study, respectively, its SymTFT and its spontaneous breaking.

[5]There are some alternative proposals [114, 115] with the goal of preserving the full $U(1)$, while [116] recently made a proposal for modifying the original construction of [28, 29] using the *non-compact TQFTs* of [5, 117] to rescue axial rotations at irrational angles.

Consider a 4d free massless Dirac fermion Ψ. It has has two $U(1)$ symmetries:

$$\begin{aligned} U(1)_{\mathrm{V}}: &\qquad \Psi \to e^{i\alpha}\Psi \ , &\qquad \alpha \in [0,2\pi) \\ U(1)_{\mathrm{A}}: &\qquad \Psi \to e^{i\alpha\gamma_5}\Psi \ , &\qquad \alpha \in [0,2\pi) \ . \end{aligned} \tag{3.5.1}$$

Decomposing the Dirac fermion in terms of two Weyls $\psi_{\mathrm{L}}, \psi_{\mathrm{R}}$, $U(1)_{\mathrm{V}}$ rotates the two with opposite charge, while $U(1)_{\mathrm{A}}$ with the same charge. The $\mathbb{Z}_2$ subgroup is in common and is fermion parity $(-1)^F$. The Dirac mass term is charged under $U(1)_{\mathrm{A}}$ (with charge 2), hence preventing its generation is any symmetry preserving deformation of the theory. The standard triangle diagram computation shows that, if $U(1)_{\mathrm{V}}$ is coupled to a background field A_{V}, the current J_{A} of axial symmetry has an anomalous conservation equation

$$d * J_{\mathrm{A}} = \frac{1}{4\pi^2} F_{\mathrm{V}} \wedge F_{\mathrm{V}} \ . \tag{3.5.2}$$

In the modern language, $U(1)_{\mathrm{A}} \times U(1)_{\mathrm{A}}$ has a mixed 't Hooft anomaly with inflow action

$$S_{\text{inflow}} = \frac{i}{4\pi^2} \int_{X_5} A_{\mathrm{A}} \wedge F_{\mathrm{V}} \wedge F_{\mathrm{V}} \ , \qquad \partial X_5 = X_4 \ . \tag{3.5.3}$$

The axial symmetry also has a pure anomaly $\frac{in}{24\pi^2}\int_{X_5} A_{\mathrm{A}} \wedge F_{\mathrm{A}} \wedge F_{\mathrm{A}}$, with $n = 2$.

The vector symmetry is anomaly free and can be gauged, resulting in massless QED. Because of (3.5.2) the theory does not have the $U(1)_{\mathrm{A}}$ symmetry, that is broken by an ABJ anomaly.

In massless QED the fermion mass is no longer protected and one may suspect that the dynamics generates it. Of course everyone knows this is not going to happen (after all this is a weakly coupled theory), but why is it so? Is there a symmetry principle that we can invoke?

One standard argument is that there are no Abelian instantons in flat space. This is not completely satisfactory:

- What about more general topologies?
- The exact vanishing of some quantity is supposed to be explained by symmetry principles, while saying "there are no Abelian instantons" has the flavor of *dynamical* reason.

Needless to say, there is a symmetry principle, but it is a non-invertible one. In the following we omit the subscripts in the gauge fields, and A will denote the dynamical $U(1)_{\mathrm{V}}$ gauge field. The basic idea is a very old one: the anomalous conservation equation (3.5.2) leads to a true conservation equation for

$$\widetilde{J}_{\mathrm{A}} = J_{\mathrm{A}} + \frac{1}{4\pi^2} * \left(A \wedge dA\right) \ , \tag{3.5.4}$$

which, however, is not gauge invariant. From a defect point of view the topological operator is

$$\widehat{U}_\alpha(\Sigma_3) = \exp\left(i\alpha \int_{\Sigma_3} *J_{\mathrm{A}}\right) \exp\left(-i\frac{\alpha}{4\pi^2}\int_{\Sigma_3} A \wedge dA\right) \ . \tag{3.5.5}$$

The second term is a CS at level α/π, and is consistent only for $\alpha = 0, \pi$, that is fermion parity.

The inconsistency for other values of α is a sort of anomaly. The Chern-Simons term is not gauge invariant under large gauge transformations. This can be cured by replacing it with a bulk term $\frac{-i\alpha}{4\pi^2}\int_{Y_4} F \wedge F$, thus making the 3d defect a non-genuine one. We are in a similar situation to that in 4d $\mathcal{N} = 1$ SYM. We want to imitate the same logic, by stacking a 3d TQFT that cancels the anomaly. However, the anomaly here is continuous, and cannot be canceled by a TQFT. Then the idea is to consider any finite subgroup $\mathbb{Z}_n \subset U(1)$, for which the anomaly becomes discrete, and cancel it with a TQFT. Since n is arbitrary, we should be able to obtain infinitely many topological defects.

More precisely, for $\alpha = \pi/n$ the bulk is $-i\frac{1}{4\pi n}\int_{Y_4} F \wedge F$, and setting

$$B = \frac{F}{n} \tag{3.5.6}$$

this is nothing but the anomaly inflow for $U(1)_n$ Chern-Simons theory. Hence we obtain a good gauge invariant topological operator by replacing the bad piece of with $\mathcal{Z}_{U(1)_n}[\Sigma_3; F/n]$, the partition function of $U(1)_n$, coupled with $B = F/n$.

An equivalent way to derive this, is by notice that for $\alpha = \pi/n$ the bad term is a Chern-Simons at fractional level $k = 1/n$. In condensed matter theory, in particular in the context of the fractional quantum hall effect (FQHE) (see [118] for a review) there is a very well-known trick to make sense of it. One considers $U(1)_n$ in terms of a dynamical 3d gauge field a, coupled with A through

$$\frac{n}{4\pi}\int_{\Sigma_3} a \wedge da + \frac{1}{2\pi}\int_{\Sigma_3} a \wedge dA \ . \tag{3.5.7}$$

If we integrate out a blindly, setting $a = -1/n\, A$, plugging it back we obtain a fractional CS of A. This operation does not really make sense, and the full path integral over a with action (3.5.7) is the correct way of treating this fractional CS. We arrived at the same answer as before, that we should consider the topological operators

$$\mathcal{N}_{\alpha=\pi/n}(\Sigma_3) = \exp\left(i\frac{\pi}{n}\int_{\Sigma_3} *J_{\text{A}}\right)\mathcal{Z}_{U(1)_n}[\Sigma_3; F/n] \tag{3.5.8}$$

The extension for general fractional angles $\alpha = \pi p/n \in \pi\mathbb{Q}/\mathbb{Z}$ is straightforward (here $\gcd(p, n) = 1$). The TQFT we need to stack is the minimal theory $\mathcal{A}^{n,p}$ of [111], whose $\mathbb{Z}_n$ 1-form symmetry has anomaly $p \in \mathbb{Z}_n$:

$$\mathcal{N}_{\alpha=\pi p/n}(\Sigma_3) = \exp\left(i\frac{\pi p}{n}\int_{\Sigma_3} *J_{\text{A}}\right)\mathcal{Z}_{\mathcal{A}^{n,p}}[\Sigma_3; F/n] \ . \tag{3.5.9}$$

Notice that, a crucial ingredient to construct the topological operator is that F is closed, that is the existence of a conserved current $*\frac{F}{2\pi}$ for the magnetic $U(1)$ 1-form symmetry. The same defect has also an alternative construction using the half-space gauging the subgroup $\mathbb{Z}_n \subset U(1)$ of the magnetic 1-form symmetry (see [28, 29] for details).

With the same manipulations as around (3.4.18) we find the fusion rule

$$\mathcal{N}_{\alpha=\pi p/n}(\Sigma_3) \times \overline{\mathcal{N}}_{\alpha=\pi p/n}(\Sigma_3) = C_{\mathbb{Z}_n}(\Sigma_3) \tag{3.5.10}$$

with $C_{\mathbb{Z}_n}(\Sigma_3)$ a condensation defect of the $\mathbb{Z}_n$ subgroup of the magnetic 1-form symmetry. This is obtained by higher-gauging the surface defects $e^{ia/n\int_{\gamma_2} F}$, $a = 0, ..., n-1$.

Chapter 4

The Symmetry Topological Field Theory

In this chapter we review the idea that topological manipulations of a d-dimensional QFT can be realized in terms of a (d+1)-dimensional TQFT, called its Symmetry Topological Field Theory (Symmetry TFT or SymTFT) [11, 32–36]. More precisely, given a symmetry $\mathcal{C}$ in d-dimensions, the SymTFT $\mathcal{Z}(\mathcal{C})$ is associated with the symmetry, and encodes all its data in a way that is often more explicit then how it appear in a direct d-dimensional analysis. For instance it effectively encodes anomalies and global structures, as well as the representation theory for the symmetry. We mostly focus on invertible symmetries in this Chapter, as several categorical generalization will be discussed in great detail in the coming chapters.

4.1 Topological manipulations and their invariant

Consider a finite p-form symmetry $\mathbb{A}$ in d-dimensions, eventually with anomaly $\omega \in H^{d+1}(B^{p+1}\mathbb{A}, \mathbb{R}/\mathbb{Z})$[1]. We denote with $U_a(\Sigma_{d-p-1})$ the topological defects, and with $W_\alpha(\gamma_p)$ the charged objects. Let $\mathbb{B} \subset \mathbb{A}$ be a subgroup such that the anomaly ω pulls back to a trivial cocycle of $H^{d+1}(B^{p+1}\mathbb{B}, \mathbb{R}/\mathbb{Z})$. Then $\mathbb{B}$ is anomaly free and we can gauge it, eventually with discrete torsion $\nu \in H^d(B^{p+1}\mathbb{B}, \mathbb{R}/\mathbb{Z})$. This topological manipulation produces a different theory with the same local physics but different global properties. It has a p-form symmetry $\mathbb{A}/\mathbb{B}$ and a $(d-p-2)$-form symmetry Rep($\mathbb{B}$) (this is $\mathbb{B}^\vee$ if $\mathbb{B}$ is Abelian), possibly combined in some way: a group extension if $d-p-2=2$, or more generally an higher-group structure. The dual symmetry has topological defects $\widetilde{U}_\beta(\Sigma_{p+1}) = e^{i\beta \int_{\Sigma_{p+1}} B_{p+1}}$, with $B_{p+1} \in H^{p+1}(X, \mathbb{B})$ the $\mathbb{B}$ gauge field, and, for $\nu = 0$, acts on the twisted sectors $\widetilde{W}_b(\gamma_{d-p-2})$, that lived at the boundary of $U_b(\Sigma_{d-p-1})$ in the original theory. For $\nu \neq 0$ the charged objects might be combination of twisted and untwisted sectors of the original theory, as we discussed in 1.4. On the other hand, the operators $W_\beta(\gamma_p)$ that were charged under $\mathbb{B}$ in the original theory, now live at the boundary of $\widetilde{U}_\beta(\Sigma_{p+1})$, hence are in twisted sectors.

One can repeat various such topological manipulations, gauging non-anomalous subgroups with all possible discrete torsions. The resulting theories are all possible *global variants* of the original theory. They all have the same local dynamics but different global properties. Essentially, they differ for the choice of which operators are genuine and which live in twisted sectors, as well as their charges under the symmetries. It has been pointed out in [34] that the set of global variants can be endowed with the

[1] Actually anomalies are classified by some appropriate cobordism group. Here we restrict to bosonic anomalies, classified by the cohomology of the classifying space.

structure of a groupoid, dubbed *orbifold groupoid*[2]: objects are the global variants, and topological manipulations are morphisms among them, while the automorphisms of the symmetry of each global variant give endomorphisms.

Notice that the operators charged under $\mathbb{B}$ and $\mathbb{B}^\vee$ are mutually non-local, and the corresponding topological operators $U_b(\Sigma_{d-p-1}), \widetilde{U}_\beta(\Sigma_{p+1})$ do not exist simultaneously in any global structure. The basic idea of the Symmetry Topological Field Theory (SymTFT), introduced in various forms in [11, 32–36], is to make these two sets of operators to exist simultaneously in a (d+1)-dimensional TQFT, that be denote by $\mathcal{Z}(\mathbb{A}^{(p)})$. The QFT of interest lives at the boundary of $\mathcal{Z}(\mathbb{A}^{(p)})$, and the choice of which of the two sets of operator exist in the QFT is determined by the boundary conditions.

To understand what TQFT is the the SymTFT, let us start from the non-anomalous case $\omega = 0$. $\mathcal{Z}(\mathbb{A}^{(p)}$ must have both types of topological operators $U_a(\Sigma_{d-p-1})$ and $\widetilde{U}_\alpha(\Sigma_{p+1})$, and they should interact non-trivially. Indeed we observe that their dimensionalies add up to $(d-p-1)+(p+1) = d = (d+1)-1$, hence if we lift them to one dimension higher, they have the correct dimensionality to link. There is a natural TQFT that produces these defects with non-trivial linking, namely a $(p+1)$-form $\mathbb{A}$ gauge theory:

$$S = 2\pi i \int_{X_{d+1}} A_{p+1} \cup \delta C_{d-p-1} \ . \tag{4.1.1}$$

Here $A \in H^{p+1}(X_{d+1}, \mathbb{A})$ and $C \in H^{d-p-1}(X_{d+1}, \mathbb{A}^\vee)$, and cup product is associated with the natural pairing $\mathbb{A} \times \mathbb{A}^\vee \to \mathbb{R}/\mathbb{Z}$. The topological operators are identified as

$$U_a(\Sigma_{d-p-1}) = \exp\left(ia \int_{\Sigma_{d-p-1}} C_{d-p-1} \right) , \qquad \widetilde{U}_\alpha(\Sigma_{p+1}) = \exp\left(i\alpha \int_{\Sigma_{p+1}} A_{p+1} \right) . \tag{4.1.2}$$

The two have non-trivial braiding phase $e^{2\pi i \alpha(a)}$.

The generalization to the anomalous case is straightforward. From $\omega \in H^{d+1}(B^{p+1}\mathbb{A}, \mathbb{R}/\mathbb{Z})$, interpreting A_{p+1} as a map $X_{d+1} \to B^{p+1}\mathbb{A}$, we can construct the class $A^*(\omega) \in H^{d+1}(X_d, \mathbb{R}/\mathbb{Z})$. Thus we consider the TQFT

$$S = 2\pi i \int_{X_{d+1}} \left(A_{p+1} \cup \delta C_{d-p-1} + A^*(\omega) \right) \ . \tag{4.1.3}$$

It is natural to arrive at the following general proposal: the SymTFT for invertible symmetries is a non-trivial TQFT obtained by gauging the anomaly inflow theory. Notice that the inflow theory defines a trivial (invertible) TQFT with the given symmetry, and the partition function coupled with a background is the exponential of the inflow action. The SymTFT is obtained by making the background dynamical. Notice that even for non-anomalous symmetries this results in a non-trivial theory.

We can be even more general, and introduce the idea that, for any finite symmetry structure $\mathcal{C}$ acting on d-dimensional QFTs, we can construct a $(d+1)$-dimensional TQFT $\mathcal{Z}(\mathcal{C})$ by gauging gauging the symmetry in one dimension higher. More precisely, we first *extend* the symmetry to a trivially acting symmetry in $(d+1)$ dimensions. This amount to view the defects as boundary of defects extending in the extra dimension, and they generate a trivially acting symmetry, since we do not introduce charged operators in the extra dimension. We then gauge this symmetry in $(d+1)$-dimensions, obtaining a non-trivial TQFT. Notice that there is no obstruction in performing this operation: even if the symmetry is anomalous it cannot be gauged in d-dimensions, but its anomaly, from the (d+1)-dimensional viewpoint, is a feature rather than an obstruction. For this reason, as

[2]A groupoid is a categorical generalization of a group. A group can be viewed as a category with one objects, and the endomorphisms are all invertible and identified with the elements of the group. In a groupoid there can be several diffent objects, but all morphisms are invertible.

we will show in the next section for invertible symmetries and in more generality in Chapter 6, the SymTFT $\mathcal{Z}(\mathcal{C})$ also encodes the anomalies of $\mathcal{C}$.

This type of construction is well known in mathematics under the name of *state-sum construction*, and was pioneered by Turaev and Viro in the case of a fusion category in two-dimensions to construct a 3-dimensional TQFT [37, 38]. In general, state-sums produce a (d+1)-dimensional TQFT from the datum of a fusion (d-1)-category. The full set of defects of $\mathcal{Z}(\mathcal{C})$ form a fusion d-category, called the *Drinfeld center* of $\mathcal{C}$, and by abuse of notation we denote it also with $\mathcal{Z}(\mathcal{C})$.[3]

In tbe way we arrived at the construction of $\mathcal{Z}(\mathcal{C})$ is implicit that if $\mathcal{C}'$ is a symmetry structure obtained from $\mathcal{C}$ with a topological manipulation, then

$$\mathcal{Z}(\mathcal{C}') \cong \mathcal{Z}(\mathcal{C}) \ . \tag{4.1.4}$$

In other words the SymTFT is an invariant on the class of global variants, called *Morita equivalent class* in mathematics. For instance it also implies the equivalent of the pure $(p+1)$-form $\mathbb{A}$ gauge theory and the $(d-p-1)$-form $\mathbb{A}^\vee$ gauge theories in (d+1)-dimensions, as we discussed in 2.2.1. This equivalence is just manifest in the co-chain formulation and there is nothing surprising about it. However, the equivalence of SymTFTs for Morita equivalent symmetries sometimes implies highly non-trivial identification of TQFTs with a priori very different presentations.

It is also sometimes a very useful fact, since it might be much easier to obtain the SymTFT for a given symmetry $\mathcal{C}$ by actually writing down the one for a different global variant. The typical example is that of non-invertible symmetries that arise from gauging of invertible symmetries, as in the KOZ construction [25] reviewed in Section 3.4.2. Even though we do not know how to perform the state-sum construction starting from the datum of the fusion 3-category of interest, we can just make dynamical the inflow action (3.4.12) and obtain

$$\mathcal{Z}(\text{KOZ}) = \frac{2\pi i}{N}\int_{X_5} C_3 \cup \delta A_1 + \frac{2\pi i}{M}\int_{X_5} T_2 \cup \delta B_2 + \frac{2\pi i}{k}\int_{X_5} A_1 \cup \frac{B_2 \cup B_2}{2} \ . \tag{4.1.5}$$

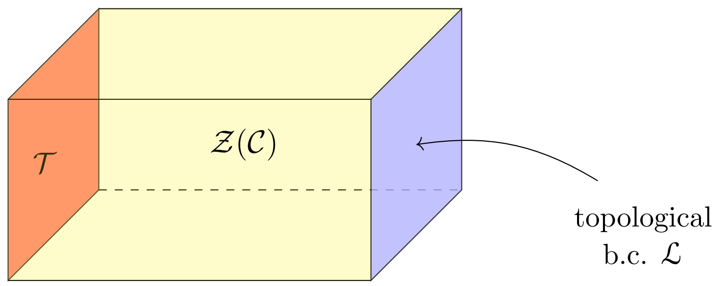
4.2 The slab construction

In the last section we simply describe a construction of a (d+1)-dimension TQFT from a symmetry of a d-dimension QFT, but have not been precise in the relation between the two theories. Now we introduce the slab construction, that allows to extract all the properties of a given symmetry $\mathcal{C}$ of a given QFT$_d$ $\mathcal{T}$ from its SymTFT $\mathcal{Z}(\mathcal{C})$.

First, we notice that, since $\mathcal{T}$ has symmetry $\mathcal{C}$, can live at the boundary of $\mathcal{Z}(\mathcal{C})$. In a sense, $\mathcal{T}$ specifies a (generically) non-topological boundary condition for the SymTFT $\mathcal{Z}(\mathcal{C})$. The setup consists in coupling the dynamical $\mathcal{C}$ gauge field of $\mathcal{Z}(\mathcal{C})$ with the $\mathcal{C}$-symmetry of $\mathcal{T}$, and this gauge field stays dynamical at the boundary. Notice that if we replace $\mathcal{T}$ with $\mathcal{T}'$ obtained by some topological manipulation, the result of the coupling with the bulk TQFT is the same.

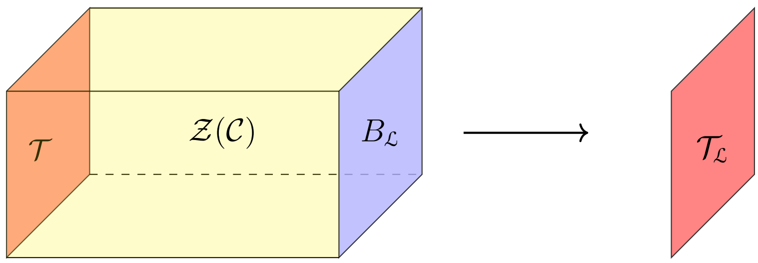
[3] The abuse of notation is actually using $\mathcal{Z}(\mathcal{C})$ for the (d+1)-dimensional TQFT, while using it for the Drinfeld center is well-established in the mathematical literature.

The idea is to place the SymTFT $\mathcal{Z}(\mathcal{C})$ on a slab with two boundaries



The left one is the physical theory $\mathcal{T}$, coupled in the way described above. On the other boundary $B_{\mathcal{L}}$, we impose topological boundary conditions, determined by a Lagrangian algebra $\mathcal{L}$ of $\mathcal{Z}(\mathcal{C})$ (see Section 2.4). The topological boundary condition can be thought of as the interface between $\mathcal{Z}(\mathcal{C})$ and the trivial theory obtained by gauging $\mathcal{L}$. In particular, defects inside $\mathcal{L}$ can end topologically on $B_{\mathcal{L}}$. Equivalently, $V \in \mathcal{L}$ is trivialized if it is pushed on the topological boundary condition. On the other hand, defects $U \notin \mathcal{L}$ give rise to non-trivial topological defects when pushed to the right boundary, and are identified with the defects obtained by fusing those inside $\mathcal{L}$.

The slab geometry is an interval times the boundary, and since the bulk theory topological the interval can be compactified producing a d-dimensional theory



The resulting d-dimensional theory $\mathcal{T}_{\mathcal{L}}$ is determined by the topological boundary condition $\mathcal{L}$, and it is a global variant of the original theory T. In particular there exists a *canonical* Lagrangian algebra $\mathcal{L}_{\mathcal{C}}$ that gives rise to the original theory

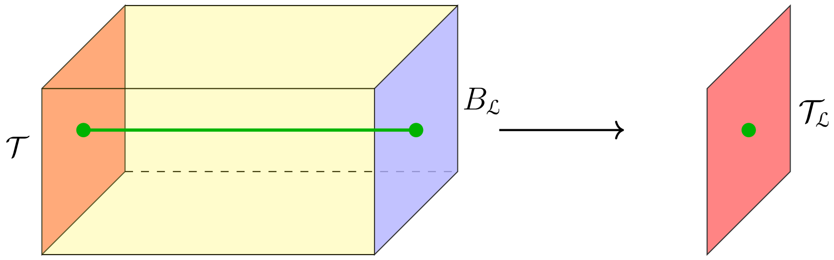
$$\mathcal{T}_{\mathcal{L}_{\mathcal{C}}} = \mathcal{T} \ , \tag{4.2.1}$$

that has symmetry $\mathcal{C}$. This canonical boundary condition is called *Dirichlet*. The intuition for the name is that, if $\mathcal{C}$ is an invertible symmetry, the SymTFT $\mathcal{Z}(\mathcal{C})$ is a pure $\mathcal{C}$ gauge theory in (d+1)-dimensions written in terms of a dynamical $\mathcal{C}$ gauge field A, and $\mathcal{L}_{\mathcal{C}}$ corresponds to $\delta A\big|_{\partial} = 0$. More generally the anomaly free symmetry generated by $\mathcal{L}_{\mathcal{C}}$ is the dual symmetry arising from the gauging of $\mathcal{C}$ in (d+1)-dimensions.

By playing with the topological boundary conditions, choosing the various Lagrangian algebras $\mathcal{L}$, we obtain all possible global variants $\mathcal{T}_{\mathcal{L}}$ of T. An other way to think about this is that specifying a topological boundary condition of $\mathcal{Z}(\mathcal{C})$ determines a d-d-dimensional TQFT of edge modes, and the slab construction provides a coupling between $\mathcal{T}$ and this TQFT.

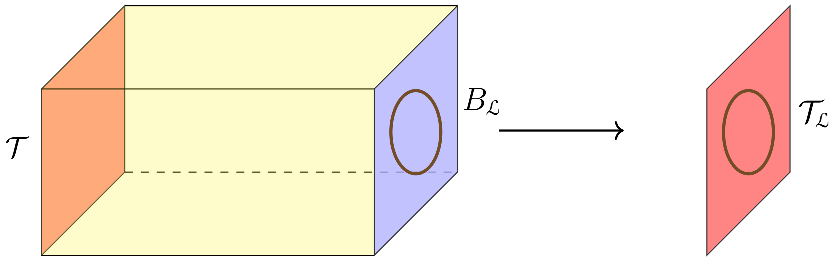
One reason way this construction is useful is that it completely disentangle the dynamical theory from the topological aspects. The piece of the slab made by $\mathcal{Z}(\mathcal{C})$ and the topological boundary, is totally independent on $\mathcal{T}$, and one can analyze it by its own. As emphasized by Freed, Moore and Teleman [36], that called $\mathcal{Z}(\mathcal{C})$ together with $\mathcal{L}$ a *quiche*, this can be taken as the abstract notion of *symmetry in QFT*. Attaching the left boundary $\mathcal{T}$, hence closing the quiche, dictates how the abstract symmetry acts on a concrete QFT.

Even though it might be sometimes complicated to understand what is the concrete topological manipulation corresponding to $\mathcal{L}$, we can analyze the quiche on its own and understand how it determines the topological defects and the charged objects. Both types of operators arise from topological defects of the bulk, but in different ways. The physical boundary condition of $\mathcal{Z}(\mathcal{C})$, that is the dynamical theory $\mathcal{T}$, leaves all the bulk fields dynamical there, hence all defects can end. The endpoint is generically non-topological, but might in some case be also topological, or can be even the trivial operator. Regarding the topological boundary, any defect $V(\Sigma_{p+1})$ belonging to $\mathcal{L}$ can terminate on $B_{\mathcal{L}}$, hence we can have configuration as



The result of the interval compactification is a (n-1)-dimensional operator $W(\gamma_p)$. This inherits the dynamical of the end-point of $V(\Sigma_{p+1})$ on T: it can be non-topological, topological, or trivial if that end-point has these properties. One can consider more refined adjectives, for instance if $U(\Sigma_n)$ terminate on $\mathcal{T}$ on a conformal defect, then $W(\gamma_p)$ will be conformal.

Consider instead a bulk defect $U(\Sigma_n)$ that does not belong to $\mathcal{L}$. By maximality of $\mathcal{L}$ this means that some defects in $\mathcal{L}$ are charged under $U(\Sigma_n)$. This defect cannot end on $B_{\mathcal{L}}$. Thus if we push it on the topological boundary it stays a non-trivial topological defect

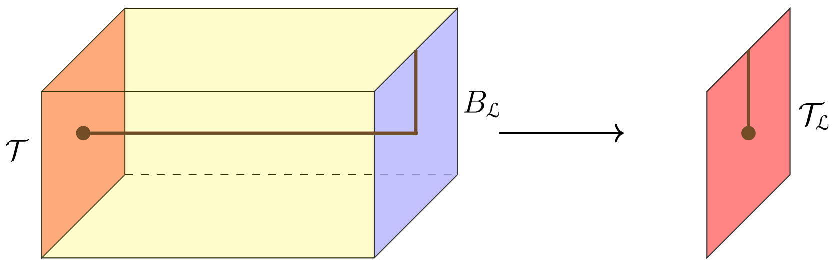


After compactification this produces a topological defect $U(\Sigma_n)$ of $\mathcal{T}_{\mathcal{L}}$ that generates a $(d-n-1)$-form symmetry. The charged objects for this symmetry are among the operators $W(\gamma_p)$ discussed above, since the non-trivial braiding in the bulk induces a non-trivial action on the boundary. Moreover, non-trivial topological defects, if fused with a defect inside $\mathcal{L}$, give rise to the same boundary defect, since elements of $\mathcal{L}$ are trivialized on $B_{\mathcal{L}}$.

We conclude that the topological defects of the global variant $\mathcal{T}_{\mathcal{L}}$ are given by the quotient

$$\mathcal{C}(\mathcal{L}) = \mathcal{Z}(\mathcal{C})/\mathcal{L} \ , \tag{4.2.2}$$

while the Lagrangian algebra $\mathcal{L}$ labels the charges (representation) under this symmetry [119]. Twisted sector operators also arise quite naturally in this framework. While defects $U(\Sigma_n) \notin \mathcal{L}$ cannot end on $B_{\mathcal{L}}$, they can end on $\mathcal{T}$, allowing to construct the configuaration



The result is an operator $\widetilde{W}(\gamma_{n-1})$ in the twisted sector of $U(\Sigma_n)$.

Everything we said so far is valid for any possible finite symmetry structure. It can be invertible or non-invertible, of any form degree, and can have a complicated categorical structure. Of course it might be computationally complicated to determine all possible Lagrangian algebras of $\mathcal{Z}(\mathcal{C})$ and analyze the property of all defects arising in a given choice of $\mathcal{L}$. However, we reduced the problem of studying a symmetry structure of a QFT $\mathcal{T}$, to a purely TQFT problem: the dynamics of the boundary theory does not play any role in the analysis. The idea of the SymTFT realizes concretely the philosophy of using tools from TQFTs in non-topological theories. We will see in the coming chapters how to employ several techniques to extract interesting data of the symmetry from the SymTFT, and use these data to derive dynamical constraints for the QFT of interest.

Example: invertible symmetries without anomalies Let us demonstrate this general discussion in a simple example of a p-form finite Abelian symmetry $\mathbb{A}$ in d dimensions. The SymTFT is (4.1.1), with topological defects (4.1.2). The Dirichlet boundary condition fixes the value of A_{p+1} on $B_{\mathcal{L}_{\mathbb{A}}}$, hence allowing all the $\widetilde{U}_\alpha(\Sigma_{p+1})$ to terminate topologically. The corresponding Lagrangian algebra is

$$\mathcal{L}_{\mathbb{A}} = \left\{ \widetilde{U}_\alpha \mid \alpha \in \mathbb{A}^\vee \right\} \cong \mathbb{A}^\vee \tag{4.2.3}$$

that labels the charges of a p-form $\mathbb{A}$ symmetry. The topological defects are $U_a(\Sigma_{d-p-1})$ pushed at the boundary. The canonical braiding of $U_a(\Sigma_{d-p-1})$ and $\widetilde{U}_\alpha(\Sigma_{p+1})$ in the bulk implies that the operators $W_\alpha(\gamma_p)$ arising as boundaries of $\widetilde{U}_\alpha(\Sigma_{p+1})$ are charged with respect to $U_a(\Sigma_{d-p-1})$ with charge $\alpha \in \mathbb{A}^\vee$. There is another topological boundary condition, the *Neumann* boundary condition, that fixes C_{d-p-1}, hence allowing $U_a(\Sigma_{d-p-1})$ to end topologically. The Lagrangian algebra is

$$\mathcal{L} = \{ U_a \mid a \in \mathbb{A} \} \in \mathbb{A} \tag{4.2.4}$$

that labels the charges under a $(d-p-2)$-form symmetry $\mathbb{A}^\vee$ generated by $\widetilde{U}_\alpha(\Sigma_{p+1})$. This is the dual symmetry obtained by gauging the p-form symmetry $\mathbb{A}$ above. On top of Dirichlet and Neumann, there are also boundary conditions corresponding to gauging a subgroup $\mathbb{B} \subset \mathbb{A}$. These are obtained by allowing the defect inside

$$\mathcal{L} = \left\{ U_b \ , \ \widetilde{U}_\beta \mid b \in \mathbb{B} \ , \ \beta \in N(\mathbb{B}) \right\} \tag{4.2.5}$$

to end. Here we introduced the notation $N(\mathbb{B}) = \{\beta \in \mathbb{A}^\vee \mid \beta(b) = 0 \ , \ \forall b \in \mathbb{B}\}$. It is easy to show that there are canonical isomorphisms $N(\mathbb{B}) \cong (\mathbb{A}/\mathbb{B})^\vee$ and $\mathbb{A}^\vee/N(\mathbb{B}) \cong \mathbb{B}^\vee$[4]. Therefore the symmetry of this global variant is

$$\frac{\mathbb{A} \times \mathbb{A}^\vee}{\mathcal{L}} = \mathbb{A}/\mathbb{B} \times \mathbb{B}^\vee \ . \tag{4.2.6}$$

In certain cases there are other topological boundary conditions, corresponding to additional topological manipulations, namely gauging with discrete torsion. For concretness let us specialize the discussion to $\mathbb{A} = \mathbb{Z}_N$ and consider the two cases $d = 2, p = 0$ and $d = 4, p = 1$. In both cases the two sets of defects $U_a, \widetilde{U}_\alpha$ have the same dimensionalities (1 and 2 respectively), hence we can consider dyons

$$D_{(a,\alpha)}(\Sigma_{p+1}) = U_a(\Sigma_{p+1}) \times \widetilde{U}_\alpha(\Sigma_{p+1}) \ . \tag{4.2.7}$$

[4] The first one is obvious, and by dualizing it we get a short exact sequence $1 \to \mathbb{B} \to \mathbb{A} \to N(\mathbb{B})^\vee \to 1$, whose dual sequence is the stament that $\mathbb{A}^\vee/N(\mathbb{B}) \cong \mathbb{B}^\vee$.

As we discussed around (2.2.15), the braiding is symmetric for $d=2$ and antisymmetric for $d=4$:

$$\mathcal{B}\left((a,\alpha),(b,\beta)\right) = \begin{cases} exp\left(\frac{2\pi i}{N}(a\beta+b\alpha)\right) & d=2 \\ exp\left(\frac{2\pi i}{N}(a\beta-b\alpha)\right) & d=4 \end{cases} \tag{4.2.8}$$

For $d=2$ this braiding cannot be trivialized by maximal sets other than those discussed above. However, for $d=4$, thanks to the minus sign we can consider, for any $r=0,...,N-1$

$$\mathcal{L}_r = \left\{D_{(a,ra)} \mid a \in \mathbb{Z}_N\right\} \ . \tag{4.2.9}$$

This is a Lagrangian algebra, and the corresponding boundary condition gives rise to a global variant in which the genuine line operators charged under the 1-form symmetry are dyons. We recognize that this is obtained by gauging the 1-form symmetry $\mathbb{Z}_N$ with discrete torsion r.

4.3 SymTFT and anomalies

The SymTFT captures all information of the symmetry, in all possible global forms. A fundamental datum of a given symmetry is its 't Hooft anomaly that, in general, can be defined as an obstruction to gauging the symmetry. There is a natural way in which $\mathcal{Z}(\mathcal{C})$ captures the anomaly for $\mathcal{C}$: the absense of the Neumann boundary condition, whose defining Lagrangian algebra has trivial intersaction with the one defining the Dirichlet boundary condition [3, 43–45]. Indeed, as we have seen in the example above, the Neumann boundary condition corresponds to gauging the full symmetry, hence an anomaly is an obstruction to the existence of such boundary condition. In Chapter 6, following [3] we will discuss a more refined notion, that focus on any single defect $\mathcal{D} \in \mathcal{C}$. This is defined to be anomalous if there is no Lagrangian algebra of $\mathcal{Z}(\mathcal{C})$ that includes it. In a sense the defect is anomalous if it cannot be trivialized by a topological manipulation. These two notitions are slightly different for general categorical symmetries, and this will be the subject of Chapter 6. Here we just want to demonstrate in a simple invertible example the principle that anomalies are detected by the absence of the Neumann boundary condition.

Consider a $\mathbb{Z}_N$ 0-form symmetry in 3d. Its anomaly is classified by $H^3(B\mathbb{Z}_N, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_N$, and the inflow action is given by

$$S_{\text{inflow}} = \frac{2\pi i k}{N} \int_{X_3} A \cup \beta(A) \ , \tag{4.3.1}$$

with $A \in H^1(X, \mathbb{Z}_N)$ and $\beta(A) = \frac{\delta A}{N}$ the Bockstein map. To construct the SymTFT we make A dynamical, and for convenience we pass to the co-chain formulation

$$\mathcal{Z}\left(\mathbb{Z}_N^{(k)}\right) = \frac{2\pi i}{N} \int_{X_3} \left(A \cup \delta B + \frac{k}{N} A \cup \delta A\right) \tag{4.3.2}$$

This is the 3d DW theory that we discussed in 2.2.3. As we have seen there, it has topological lines

$$U_{a,b}(\gamma) = \exp\left(\frac{2\pi i a}{N}\int_\gamma A + \frac{2\pi i b}{N}\int_\gamma B\right) \tag{4.3.3}$$

with braiding

$$\mathcal{B}\left((a_1,b_1),(a_2,b_2)\right) = \exp\left(\frac{2\pi i}{N}\left(a_1 b_2 + a_2 b_1 - \frac{k}{N} b_1 b_2\right)\right) \ . \tag{4.3.4}$$

Since the Dirichlet boundary condition allows all the lines $U_{a,0}$ to end, the would be Neumann boundary condition is associated with the orthogonal algebra

$$\mathcal{L}_N = \{U_{0,b} \mid b \in \mathbb{Z}_N\} \ . \tag{4.3.5}$$

However, from the braiding (4.3.4) we see that for $k \neq 0$ these defects are charged among themselves, hence they do not form a consistent condensable algebra. We conclude that for $k \neq 0$ the Neumann boundary condition is inconsistent, signaling the 't Hooft anomaly.

Chapter 5

Non-invertible symmetries and holography

In this chapter, we study how non-invertible self-duality defects arise in theories with a holographic dual, focusing on the paradigmatic example of $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM (see [8] for the extension to class-$\mathcal{S}$ theories), that has non-invertible duality and triality defects at $\tau = i$ and $\tau = e^{2\pi i/3}$, respectively. The main idea is that at these points in the gravitational moduli space, the gauged $SL(2,\mathbb{Z})$ duality symmetry of type IIB string theory (that is generically Higgsed down to $\mathbb{Z}_2$) is Higgsed to either $G = \mathbb{Z}_4$ or $G = \mathbb{Z}_6$, giving rise to a discrete emergent G gauge field. After we reduce on the internal manifold, the low-energy physics turns out to be dominated by an 5d BF theory theory, further gauged by G, that we analyze and which gives rise to the self-duality defects in the boundary theory. Using the five-dimensional bulk theory, we compute the fusion rules of those defects in detail.

5.1 Setup and general idea

Our interest here is in understanding how categorical symmetries appear in holography. The common lore is that a global symmetry of the boundary theory appears as a gauge symmetry, accompanied by a gauge field, in the bulk. This is confirmed and well understood in the case of invertible symmetries — both ordinary 0-form, continuous and discrete, as well as higher form. What happens for a non-invertible symmetry? What plays the role of a "non-invertible" gauge field?

We investigate this question in the specific case of *self-duality defects*.[1] introduced in [24] that we reviewed in Section 3.3. The idea is simple. The conformal manifold $\mathfrak{M}$ of the boundary theory is dual to a moduli space of bulk solutions in the gravitational description, while the choice of a global structure on the boundary corresponds to a certain boundary condition in gravity. The duality group Γ is a discrete gauge symmetry of string theory, which however is completely Higgsed at generic points x at which Γ acts faithfully on $\mathfrak{M}$. At a special point x which is stabilized by $G \subset \Gamma$, the duality symmetry Γ is only Higgsed to G, and in the low-energy description appears an emergent G gauge field that acts on the supergravity fields. In particular, it also acts on a low-energy topological sector of string theory whose topological (or conformal) boundary conditions encode the possible global structures of the boundary theory. It is this structure that plays the role of a "non-invertible gauge field", at least in this class of examples. The derivation and explanation of how the supergravity theory with extra gauge field gives rise to the non-invertible fusion rules is the subject of this chapter.

[1] The holographic description of non-invertible defects of the KOZ and orbifold type has recently been investigated in [120–122].

We focus on the four-dimensional $\mathcal{N}=4$ SYM theory with gauge algebra $\mathfrak{su}(N)$, that is holographically dual to type IIB string theory on asymptotically $\mathrm{AdS}_5 \times S^5$ spaces [123]. The boundary theory, however, is characterized by a specific gauge *group* with the given algebra, and thus this piece information must be encoded in the bulk theory. As explained by Witten [86], kinematical properties of the boundary theory, such as the global form of the gauge group, are captured by the long-distance behavior of the gravitational theory (or equivalently, by the behavior close to the boundary), which is encoded in the terms in the Lagrangian with the lowest number of derivatives, namely in the topological terms. One can more conveniently work with the effective 5d theory in AdS_5 obtained by reducing on the internal manifold. The 10d type IIB supergravity action contains the topological term

$$S_{\mathrm{IIB}} \supset \int_{X_{10}} B_2\, dC_2\, F_5 \,, \tag{5.1.1}$$

where B_2 is the NS-NS 2-form potential, C_2 the R-R 2-form potential, and F_5 is the field strength of the R-R 4-form potential. In the following we will denote $B_2 = b, C_2 = c$. In compactification on $\mathcal{M}_5 \times S^5$ with N units of 5-form flux on S^5, one obtains at low energies the 5d Chern-Simons action (5.1.4) [86, 124].

The continuous 2-form gauge fields b, c are dual to a $U(1) \times U(1)$ global 1-form symmetry of the boundary theory, whose two factors act on 't Hooft and Wilson line operators, respectively. This symmetry does not have to act faithfully on the boundary theory: it only acts faithfully on the full set of boundary theories with all possible global structures. This follows from the necessity of choosing boundary conditions. If we choose topological boundary conditions, the action (5.1.4) restricts b, c to be $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge fields [85] and accordingly restricts the 1-form symmetry. Boundary conditions $\rho(\mathcal{L})$ further set to zero a linear combination of b, c along a Lagrangian subgroup $\mathcal{L} \subset \mathbb{Z}_N \times \mathbb{Z}_N$, only leaving a 1-form symmetry of order N. Thus, the choice of boundary conditions specifies the global structure of the SYM theory [86, 125] and the spectrum of extended (here line) operators [42]. For instance, if we set $b = 0$ at the boundary, the boundary theory is $\mathrm{SU}(N)$. Fundamental strings (that couple to b) can end on the boundary producing Wilson line operators in generic representations [126, 127], their $\mathbb{Z}_N$ charge being measured by the topological operators $e^{i\int c}$, while 't Hooft lines only exist with vanishing $\mathbb{Z}_N$ charge. On the contrary, if we set $c = 0$ we obtain the $\mathrm{PSU}(N)_0$ theory. D1-branes (that couple to c) can end on the boundary producing 't Hooft line operators with generic $\mathbb{Z}_N$ charge, the latter being measured by $e^{i\int b}$, while Wilson lines only exist in representations with trivial N-ality. One can also choose conformal boundary conditions $b = *\,c$: they give rise to an extra singleton sector [124, 128] and describe the theory $U(N)$, for which the 1-form symmetry is indeed $U(1) \times U(1)$.

Type IIB string theory also enjoys an $SL(2,\mathbb{Z})$ symmetry. As in any theory of quantum gravity, this must be a *gauge* symmetry. It acts on the axiodilaton field $\tau = C_0 + ie^{-\phi}$ by standard fractional linear transformations

$$\tau \to \frac{\underline{a}\,\tau + \underline{b}}{\underline{c}\,\tau + \underline{d}}\,, \qquad \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{5.1.2}$$

(only $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ acts on τ) and on b, c as on a doublet $\mathcal{B} = (b,c)^{\mathsf{T}}$ in the fundamental representation,

$$\begin{pmatrix} b \\ c \end{pmatrix} \to \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \,. \tag{5.1.3}$$

At generic points τ in the moduli space, $SL(2,\mathbb{Z})$ is spontaneously broken to its $\mathbb{Z}_2$ center, and thus the corresponding gauge field does not appear in the low-energy supergravity description.[2] However,

[2]The $\mathbb{Z}_2$ center of $SL(2,\mathbb{Z})$, that maps $(b,c) \mapsto (-b,-c)$ but does not act on τ, is however always preserved and then the corresponding $\mathbb{Z}_2$ gauge field should be included.

special values of τ are left invariant by a larger subgroup $G \subset SL(2,\mathbb{Z})$ which therefore remains unbroken. The corresponding gauge field should then be included in the supergravity description, where it appears as an emergent gauge field for the low-energy observer. Specifically, $G = \mathbb{Z}_4$ at $\tau = i$ and $G = \mathbb{Z}_6$ at $\tau = e^{2\pi i/3}$. After compactification on S^5, we obtain a discrete G gauge field a in five dimensions, coupled to a G subgroup of the $SL(2,\mathbb{Z}_N)$ symmetry of

$$S_{\text{CS}} = \frac{N}{2\pi}\int b\, dc \equiv \frac{N}{4\pi}\int \mathcal{B}^{\mathsf{T}}\epsilon\, d\mathcal{B}\ . \tag{5.1.4}$$

Here $\mathcal{B} = \left(\begin{smallmatrix} b \\ c \end{smallmatrix}\right)$, while $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This is an interesting subsector of the full theory on its own. Our aim is to show that a is the gauge field corresponding to the non-invertible symmetries of the boundary theory.[3]

5.2 The 5d Chern-Simons theory and its symmetries

Consider the five-dimensional Chern-Simons action [86] (see also [11, 89, 124, 128, 129])[4]

$$S[Q] = \frac{1}{4\pi}\int Q(\mathcal{B},\, d\mathcal{B})\ , \tag{5.2.1}$$

where $\mathcal{B}$ is a vector of $2n$ 2-form gauge fields, Q is an integer-valued $2n \times 2n$ non-degenerate antisymmetric matrix (or symplectic form), and we used the notation $Q(x,y) = x^{\mathsf{T}}Qy$. We study this theory on spin manifolds. The theory has topological surface operators

$$U_m = e^{im^{\mathsf{T}}\int \mathcal{B}}\ , \tag{5.2.2}$$

where m is an integer-valued vector in $\mathbb{Z}^{2n}$, and the integral is over a 2-dimensional surface. These operators generate an (anomalous) 2-form symmetry. The operators U_m have nontrivial linking (see Figure 5.1 left) given by the antisymmetric braiding matrix

$$B_{mm'} = e^{2\pi i\, Q^{-1}(m,m')}\ . \tag{5.2.3}$$

Any operator for which $m = Q\,k$ with $k \in \mathbb{Z}^{2n}$ is completely transparent and thus trivial. Those operators generate a lattice Λ_Q, and the 2-form symmetry defect operators are labelled by the elements of the discriminant group

$$\mathcal{D}_Q = \mathbb{Z}^{2n}/\Lambda_Q\ . \tag{5.2.4}$$

This is the 2-form symmetry of the theory. Notice that $|\mathcal{D}_Q| = |\det Q|$.

The case relevant to type IIB string theory compactified on S^5 is $n = 1$ and $Q = N\epsilon$ with $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We denote by b and c the two components of $\mathcal{B}$. The action reads:[5]

$$S = \frac{N}{4\pi}\int \mathcal{B}^{\mathsf{T}}\epsilon\, d\mathcal{B} = \frac{N}{4\pi}\int \langle \mathcal{B},\, d\mathcal{B}\rangle = \frac{N}{4\pi}\int \big(b\, dc - c\, db\big)\ . \tag{5.2.5}$$

[3]The non-invertibility of duality and triality defects is *only* up to condensates. It is perhaps then not surprising that the corresponding bulk gauge field is a standard discrete connection for G, though coupled to a nontrivial topological sector S_{CS}.

[4]This action, as written, is not well defined [86]. When the spacetime manifold M_5 is the boundary of a six-manifold Z, one can define $S[Q] = \frac{1}{4\pi}\int_Z Q(d\mathcal{B}, d\mathcal{B})$. However, the bordism group in five dimensions is non-trivial and thus this cannot be done in general. One could instead use the formalism of Cheeger-Simons differential characters [88].

[5]We work with an antisymmetric 5d Lagrangian, which is manifestly invariant under $SL(2,\mathbb{Z})$ symmetry. One should however keep in mind that, as written, the action is not well defined (see footnote 4), and thus conclusions drawn from it should be taken with care. It turns out [86] that for N odd, the theory is $SL(2,\mathbb{Z})$ invariant only on spin manifolds, while on non-spin manifolds it is invariant under the subgroup $\Gamma(2)$ generated by S and T^2.

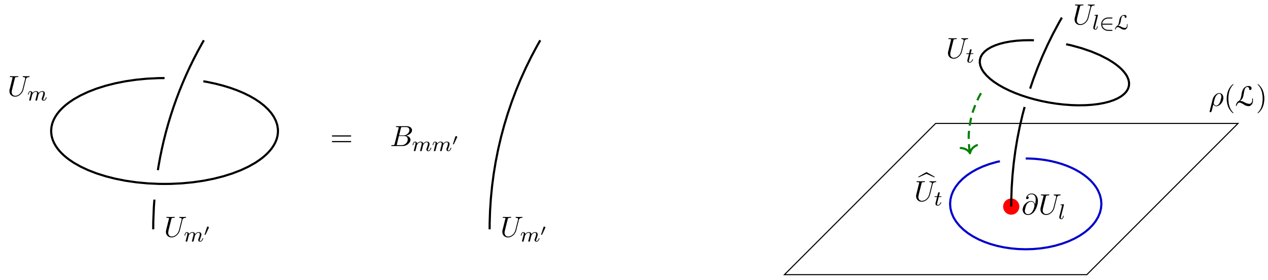


Figure 5.1: Left: Antisymmetric braiding $B_{mm'}$ between 2-dimensional defects U_m in 5d Chern-Simons theory. Right: Induced braiding between 2-dimensional defects $\widehat{U}_t$ and line defects $\partial U_{l\in\mathcal{L}}$ on gapped boundaries $\rho(\mathcal{L})$.

We introduced the antisymmetric Dirac pairing $\langle x, y\rangle = x_{\rm e} y_{\rm m} - x_{\rm m} y_{\rm e}$, where $x = (x_{\rm e}, x_{\rm m})$ is the expression of a vector in components. When describing the surface operators U_m, it might be convenient to package the information about m and the geometric 2-cycle wrapped by U_m into $\gamma \in H_2(M_5, \mathbb{Z}_N \times \mathbb{Z}_N)$, or its Poincaré-dual cocycle $\mathrm{PD}(\gamma) \in H^3(M_5, \mathbb{Z}_N \times \mathbb{Z}_N)$. In this case $U(\gamma)$ is described by the insertion of

$$U(\gamma) = \exp\left(i \int \mathcal{B}^\mathsf{T}\, \mathrm{PD}(\gamma)\right) \tag{5.2.6}$$

in the path integral.

In the general case, gapped boundary conditions $\rho(\mathcal{L})$ are in bijection with Lagrangian subgroups $\mathcal{L}$ of $\mathcal{D}_Q$. A subgroup is called Lagrangian if all its elements are mutually transparent, *i.e.*, if $B_{ll'} = 1$ for all $l, l' \in \mathcal{L}$, and if any element outside $\mathcal{L}$ braids non-trivially with at least one $l \in \mathcal{L}$ (*i.e.*, $\mathcal{L}$ is maximal). Defining a gapped boundary $\rho(\mathcal{L})$ is equivalent to gauging the Lagrangian subgroup $\mathcal{L}$ of the 2-form symmetry [130–132].[6]

Only dyons U_l with $l \in \mathcal{L}$ may terminate on the gapped boundary, defining in this way topological line operators ∂U_l there. Besides, dyons in $\mathcal{L}$ are absorbed by the gapped boundary if they are moved to lie within it, in other words the dyons $U_{l\in\mathcal{L}}$ are completely transparent (they do not contribute to correlation functions) when placed on the gapped boundary. The boundary has non-trivial topological surface operators corresponding to $m \notin \mathcal{L}$, obtained by moving $U_{m\notin\mathcal{L}}$ to lie within the boundary, however, because of the property just mentioned, those operators $\widehat{U}_t$ are labeled by conjugacy classes $t \in \mathcal{D}_Q/\mathcal{L} \equiv \mathcal{S}$. Thus the operators $\widehat{U}_t$ generate a 1-form symmetry $\mathcal{S}$ there. The charges under that symmetry are carried by the lines $\partial U_{l\in\mathcal{L}}$, as follows from the 5d braiding (see Figure 5.1 right):

$$\widehat{U}_t(\partial U_l) = e^{2\pi i\, Q^{-1}(t,l)}\, \partial U_l \;, \tag{5.2.7}$$

where, with some abuse of notation, we indicated by t any representative of its class in $\mathcal{D}_Q$.

Some properties become clear in the Lagragian description (5.2.1): a gapped boundary on X is defined by Dirichelet boundary conditions

$$l^\mathsf{T}\mathcal{B}\Big|_X = 0 \quad \text{(up to gauge transformations)} \qquad \text{for all } l \in \mathcal{L} \;. \tag{5.2.8}$$

Introducing a rectangular matrix L whose columns are the generators of $\mathcal{L}$ in $\mathbb{Z}^{2n}$, so that $L^\mathsf{T} Q L = 0$, the boundary condition is $L^\mathsf{T}\mathcal{B}\big|_X = 0$ (up to gauge transformations). This can be imposed by a

[6]More precisely, gauging the discrete symmetry $\mathcal{L}$ is equivalent to inserting a network of symmetry defects for $\mathcal{L}$ in the spacetime manifold. This is also equivalent to removing a tubular neighborhood of the network from the spacetime manifold, and placing the topological boundary condition $\rho(\mathcal{L})$ there. Thus, $\rho(\mathcal{L})$ is a topological interface between the ungauged theory and the trivial theory obtained by gauging $\mathcal{L}$ (such a theory is trivial because $\mathcal{L}$ is Lagrangian).

boundary TQFT:

$$S_{\substack{\text{gapped}\\ \text{boundary}}}[L] = \frac{1}{2\pi}\int Q\Big(\eta,\, L^{\mathsf{T}}(\mathcal{B}-d\xi)\Big) \;+\; \text{counterterms}\;, \tag{5.2.9}$$

where η is a 2-form gauge field in $\mathbb{R}^{2n}/\langle\mathcal{L}\rangle$, ξ is a 1-form gauge field in $\langle\mathcal{L}\rangle$, and $\langle\mathcal{L}\rangle$ is the real span of $\mathcal{L}$. The counterterms only involve $\mathcal{B}$, and are fixed by overall gauge invariance. To give an example, consider the type IIB case $Q = \left(\begin{smallmatrix} 0 & N \\ -N & 0\end{smallmatrix}\right)$ and take the *electric boundary* $\rho(\mathcal{L})$ where $\mathcal{L}$ is generated by $l = (1,0)$, corresponding to the boundary condition $b\big|_X = 0$ (up to gauge transformations). The boundary action is

$$S_{\substack{\text{electric}\\ \text{boundary}}} = \frac{N}{2\pi}\int\left[\eta\,(b-d\xi) - \frac{1}{2}\,bc\right]. \tag{5.2.10}$$

If we introduce a coordinate r transverse to the boundary, place the boundary at $r=0$ and the bulk in the region $r<0$, the full bulk plus boundary system has action

$$S_{\substack{\text{bulk plus}\\ \text{boundary}}} = \frac{N}{4\pi}\int_{r<0}\big(b\,dc - c\,db\big) + \frac{N}{2\pi}\int_{r=0}\left[\eta\,(b-d\xi) - \frac{1}{2}\,bc\right]. \tag{5.2.11}$$

The equations of motion fix the following conditions on the boundary:

$$b = d\xi\;, \qquad c = \eta\;, \qquad \eta \in H^2(M_5,\mathbb{Z}_N)\;. \tag{5.2.12}$$

Thus, b is set to be pure gauge, while η is the pull-back of c to the boundary and c remains unconstrained ($c \in H^2(M_5,\mathbb{Z}_N)$ is already imposed by the bulk EOMs). The system is invariant under the following gauge transformations:

$$b \,\to\, b + d\alpha_{\mathrm{e}}\;, \qquad c \,\to\, c + d\alpha_{\mathrm{m}}\;, \qquad \eta \,\to\, \eta + d\alpha_{\mathrm{m}}\;, \qquad \xi \,\to\, \xi + \alpha_{\mathrm{e}}\;. \tag{5.2.13}$$

Interpreting instead $\mathcal{L}$ as a subgroup of $\mathcal{D}_Q$ that is gauged, the dyons $U_{l\in\mathcal{L}}$ become trivial in the bulk because they are pure gauge and can be absorbed by the network of defects. On the contrary, the operators with $m \notin \mathcal{L}$ are projected out in the bulk (using the fact that $\mathcal{L}$ is Lagrangian) and can only exist on the boundary.

In the holographic setup, the 2-form symmetry $\mathcal{L}$ that we gauge in the bulk dictates what is the spectrum of physical lines in the holographic boundary [86, 133]. Thus, the surfaces U_l with $l \in \mathcal{L}$ become trivial in the bulk, but if they are attached to the holographic boundary, their endlines $\partial U_l \equiv W_l$ are the physical line operators of the boundary theory (notice that these are no longer topological, due to the holographic boundary conditions).[7] The 1-form symmetry of the boundary theory under which the lines W_l are charged is generated by the surface operators $\widehat{U}_t$, that can only live on the boundary.

Coming back to type IIB string theory, where $Q = N\epsilon$, the simplest case to discuss is when N is prime. We label the bulk surfaces U_m by $m = (m_{\mathrm{e}}, m_{\mathrm{m}})$, where m_{e} and m_{m} are the electric and magnetic charges, respectively. The topological sector has $N+1$ gapped boundary conditions:[8]

- An electric gapped boundary $\rho(e)$, for which $\mathcal{L}$ is generated by $l = (1,0)$. As a gauging, this is obtained by condensing the electric surfaces $(m_{\mathrm{e}}, 0) \in \mathcal{L}$ (while in terms of a gapped boundary, this is implemented by setting $b = 0$ there). It corresponds to the global variant $\mathrm{SU}(N)$ of the boundary theory. The Wilson lines $W_{l\in\mathcal{L}}$ are endpoints of bulk surfaces U_l. For instance, the

[7] In the picture in which the bulk with gauged $\mathcal{L}$ is substituted by a slab of bulk between the holographic boundary and a gapped boundary $\rho(\mathcal{L})$, the operators U_l can be stretched between a copy of W_l in the holographic boundary and a copy of W_l in the gapped boundary.

[8] See [134] for a recent in-depth study of gapped boundary conditions in the 5d Chern-Simons theory.

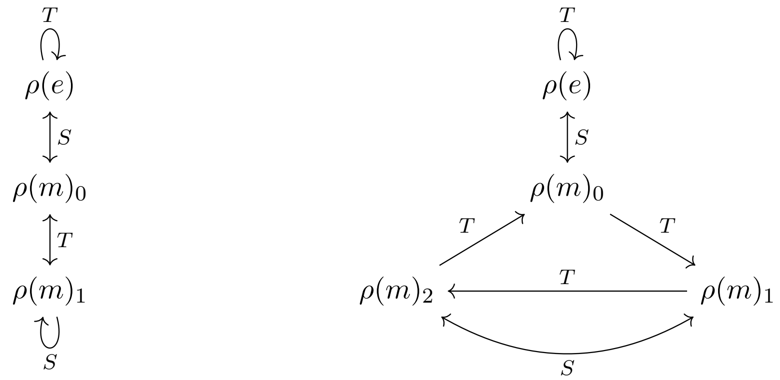


Figure 5.2: Action of $PSL(2,\mathbb{Z}_2) \cong S_3$ (left) and $PSL(2,\mathbb{Z}_3) \cong A_4$ (right) on Lagrangian subgroups (gapped boundaries).

Wilson line in the fundamental representation is the endpoint of a fundamental string [126, 127], which couples as $e^{i\int b}$ to the NS-NS B-field b. The boundary 1-form symmetry is generated by the surfaces $\widehat{U}_t$, and we can take for $\mathcal{S} \cong \mathbb{Z}_N$ the representatives $t = (0, t_{\mathrm{m}})$.

- N magnetic gapped boundaries $\rho(m)_r$ with $r = 0, \ldots, N-1$, for which $\mathcal{L}$ is generated by $l = (r, 1)$. They are obtained by condensing the dyonic surfaces $(r m_{\mathrm{m}}, m_{\mathrm{m}}) \in \mathcal{L}$ (or by setting $rb + c = 0$ on a gapped boundary). They correspond to the global variants $\mathrm{PSU}(N)_r$ of the boundary theory [42]. The 't Hooft or dyonic lines are endpoint of bulk surfaces $U_{l \in \mathcal{L}}$, for instance for $r = 0$ the basic 't Hooft line is the endpoint of a D1-brane, which couples as $e^{i\int c}$ to the R-R field c. The boundary 1-form symmetry is generated by surfaces $\widehat{U}_t$, represented for instance by $t = (t_{\mathrm{e}}, 0)$.

If N is not prime there is a larger number $\sigma_1(N) = \sum_{k|N} k$ of Lagrangian subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$, corresponding to global variants of the boundary theory of the form $\big(\mathrm{SU}(N)/\mathbb{Z}_k\big)_r$.

5.2.1 Global 0-form symmetries

The theories (5.2.1) can have 0-form symmetries as well. On spin manifolds, a (unitary) 0-form symmetry ω is an automorphism of the discriminant group $\mathcal{D}_Q$ that preserves the quadratic form:

$$\omega^{\mathsf{T}} Q^{-1} \omega = Q^{-1} \mod 1 \; . \tag{5.2.14}$$

Since ω is invertible, it maps Lagrangian subgroups to Lagrangian subgroups. We say that a gapped boundary $\rho(\mathcal{L})$ is ω-invariant if the corresponding Lagrangian subgroup is:

$$\omega\,\mathcal{L} = \mathcal{L} \; . \tag{5.2.15}$$

In the type IIB example, the 0-form symmetry group Γ is $SL(2,\mathbb{Z}_N)$, whose generators act on electric and magnetic charges as follows:

$$S : (e,m) \mapsto (-m, e) \; , \qquad T : (e,m) \mapsto (e+m, m) \; , \qquad C : (e,m) \mapsto (-e,-m) \; . \tag{5.2.16}$$

They satisfy $S^2 = C$, $T^N = \mathbb{1}$, and $(ST)^3 = C$. If M is the matrix acting on charges, then M^{T} gives the action on the gauge fields $\mathcal{B}$, as it follows from (5.2.2). This means that in our conventions

$$S : (b,c) \mapsto (c, -b) \; , \qquad T : (b,c) \mapsto (b, c+b) \; , \qquad C : (b,c) \mapsto (-b,-c) \; . \tag{5.2.17}$$

All subgroups of $\mathcal{D}_Q$ are invariant under C. For N prime we also have:

$$\rho(m)_r \xrightarrow{T} \rho(m)_{r+1} , \qquad \rho(e) \xleftrightarrow{S} \rho(m)_0 , \qquad \rho(m)_r \xleftrightarrow{S} \rho(m)_{r_S} \quad \text{for } r \neq 0 \tag{5.2.18}$$

where $r_S = -r^{-1}$ in $\mathbb{Z}_N$, while $\rho(e)$ is invariant under T (see Figure 5.2 for two examples). Lagrangian subgroups form two-terms orbits under S, except for $\rho(m)_r$ with $r^2 = -1 \bmod N$ which are invariant. Similarly, they form three-terms orbits under ST, except for $\rho(m)_r$ with $r(r+1) = -1 \bmod N$ which are invariant. Gapped boundaries corresponding to ω-invariant subgroups $\mathcal{L}$ allow for a 0-form symmetry action of the subgroup $G \subset \Gamma$ which stabilizes them.[9] This is clear from the Lagrangian description of the gapped boundaries, $S[L]$ in (5.2.9). The action of the 0-form symmetry does not leave the coupling to η invariant, but it can be reabsorbed in a redefinition of the generators L of $\mathcal{L}$.

5.2.2 Symmetry defects from higher gauging

In unitary TQFTs without local operators, all 0-form symmetries are expected to be generated by codimension-1 condensation defects, that we reviewed in Section 3.2. This statement can be proven in the context of three-dimensional modular tensor categories (MTCs) [26, 135], while it seems plausible for higher dimensional TQFTs [26]. In this section we construct the $SL(2,\mathbb{Z}_N)$ symmetry generators of the 5d CS theory (5.2.5), in terms of condensations of the 2-form symmetry on 4d submanifolds. The $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry generated by the topological surface operators U_m in the 5d bulk becomes a 1-form symmetry on a 4d submanifold Σ on which we perform the condensation.

We assume that the fusion algebra of surface operators is strictly associative, and since surfaces cannot braid in 4d, we can condense any subgroup $\mathcal{A} \subset \mathbb{Z}_N \times \mathbb{Z}_N$ of the 2-form symmetry. While condensing on a (spin) 4-manifold Σ, we have the possibility to add discrete torsion in the form of Dijkgraaf-Witten terms [81]. When we gauge the full group $\mathbb{Z}_N \times \mathbb{Z}_N$, the torsion is classified by $\mathbb{Z}_N^3$ and we label it by $x, y, z \in \mathbb{Z}_N$. In terms of the background $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ that we decompose into $\varphi_{\mathrm{e}}, \varphi_{\mathrm{m}} \in H^2(\Sigma, \mathbb{Z}_N)$, the phase of discrete torsion is given by

$$\Theta_{x,y,z} = \exp\left[\frac{2\pi i}{2N}\int_\Sigma \Big(y\,\mathfrak{P}(\varphi_{\mathrm{e}}) + z\,\mathfrak{P}(\varphi_{\mathrm{m}}) + 2x\,\varphi_{\mathrm{e}} \cup \varphi_{\mathrm{m}} \Big)\right] . \tag{5.2.19}$$

Here $\mathfrak{P} : H^2(\Sigma, \mathbb{Z}_N) \to H^4(\Sigma, \mathbb{Z}_{N \gcd(N,2)})$ is the Pontryagin square operation [136]. For N even, $\mathfrak{P}(\varphi)$ takes values in $\mathbb{Z}_{2N}$ and on spin manifolds it is an even class, therefore $y, z \in \mathbb{Z}_N$. For N odd, $\mathfrak{P}(\varphi)$ takes values in $\mathbb{Z}_N$ and we interpret the exponent as $\frac{2\pi i}{N}\, 2^{-1} y \int \mathfrak{P}(\varphi_{\mathrm{e}})$ where $2^{-1} = \frac{N+1}{2} \bmod N$, and similarly for $z\,\mathfrak{P}(\varphi_{\mathrm{m}})$, therefore $y, z \in \mathbb{Z}_N$ once again. On the other hand, when we gauge a $\mathbb{Z}_N$ subgroup, the torsion is classified by $\mathbb{Z}_N$ and then only a combination of x, y, z appears. For simplicity, we will only consider the case that N is a prime number, because then $\mathbb{Z}_N$ does not contain non-trivial proper subgroups, and all its non-zero elements are invertible.

We want to compute the action of the 0-form condensation defects V on the 2-form defects U_l. To that purpose, we place U_l along $\mathbb{R}^2$ and wrap V around them, namely we place V on $\mathbb{R}^2 \times S^2$ with S^2 surrounding U_l. It turns out that it is more clear to perform condensation on compact submanifolds, therefore we substitute $\mathbb{R}^2$ with T^2. Eventually, we place U_l on T^2 and V on $\Sigma \equiv T^2 \times S^2$ around U_l (as in Figure 5.3 center).

To condense on Σ, we decompose the 2-form symmetry background $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ into a pair of backgrounds $\{\phi^{T^2}, \phi^{S^2}\}$ on the two factors of Σ, and we denote by $n = (n_{\mathrm{e}}, n_{\mathrm{m}})$ the holonomy of ϕ^{S^2} on S^2 (representing defects on T^2) and by $m = (m_{\mathrm{e}}, m_{\mathrm{m}})$ the holonomy of ϕ^{T^2} on T^2 (representing

[9] This is true if G is a normal subgroup of Γ. This will always be so in the cases of interest to us.

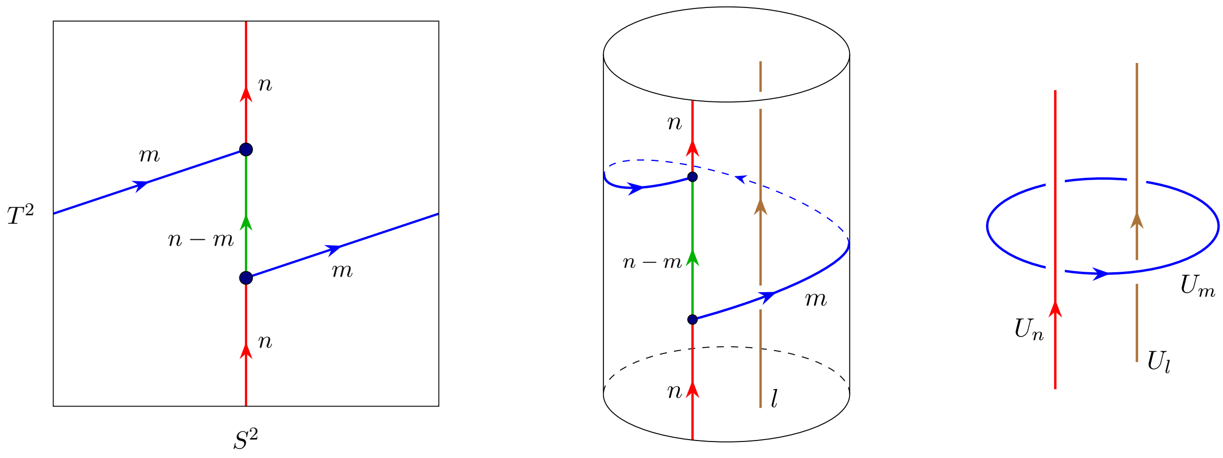


Figure 5.3: The condensation defect on $\Sigma = T^2 \times S^2$ with its network of 2d defects (left) surrounds a topological defect U_l placed on $T^2 \times \{0\}$ (center), where $0 \in B^3$ is the center of the 3-ball whose boundary is S^2. Up to a phase (5.2.22), the network can be resolved into a collection of closed surfaces with no junctions (right).

defects on S^2). Given a class $(x, y, z) \in \mathbb{Z}_N^3$ representing the choice of discrete torsion (5.2.19), its contribution to the path integral is

$$\Theta_{x,y,z}(n,m) = \exp\left[\frac{2\pi i}{N}\Big(x\left(n_{\mathrm{e}}m_{\mathrm{m}} + n_{\mathrm{m}}m_{\mathrm{e}}\right) + y\, n_{\mathrm{e}}m_{\mathrm{e}} + z\, n_{\mathrm{m}}m_{\mathrm{m}}\Big)\right] = \exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}}\mathcal{T}n\right] \tag{5.2.20}$$

where we introduced the symmetric matrix of discrete torsions

$$\mathcal{T} = \begin{pmatrix} y & x \\ x & z \end{pmatrix} , \tag{5.2.21}$$

whose entries are in $\mathbb{Z}_N$. We can label the condensation defects of the 5d Chern-Simons theory as $V[\mathcal{A}, \mathcal{T}]$, where $\mathcal{A}$ is the condensed subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ and $\mathcal{T}$ is the matrix of discrete torsions. When $\mathcal{A} = \mathbb{Z}_N \times \mathbb{Z}_N$ we omit it, while when $\mathcal{A}$ is one-dimensional we denote it by one of its generators (p, q).

To compute the action of the condensation defects on the surface operators we proceed similarly as in Section 3.2. The condensation on Σ involves a network of 2-dimensional defects, as in Figure 5.3 left. Instead of working with a network (that requires to understand the trivalent junctions), we can resolve it into a pair of 2-dimensional defects: one U_n along T^2 on an outer copy of Σ, and one U_m along S^2 on an inner copy of Σ (Fig. 5.3 right). This operation involves a phase, and is equivalent to a normal ordering prescription. More generally, for N odd we can write

$$U(\gamma_1 + \gamma_2) = \exp\left[-\frac{2\pi i}{N}\, 2^{-1} \langle \gamma_1, \gamma_2 \rangle\right] U(\gamma_1)\, U(\gamma_2) \,. \tag{5.2.22}$$

(The case of N even is discussed below.) Here $\gamma_i \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ represent two defects on Σ, while $\langle\, , \rangle$ is the product of the (symmetric) cup product on Σ and the (antisymmetric) Dirac pairing in $\mathbb{Z}_N \times \mathbb{Z}_N$. On the right-hand side, $U(\gamma_1)$ is outer while $U(\gamma_2)$ is inner. This is essentially a square root of the braiding matrix

$$B_{mn} = e^{\frac{2\pi i}{N}\langle m,n\rangle} \tag{5.2.23}$$

as in Figure 5.1 left. We obtain:

$$
\begin{aligned}
V[\mathcal{A},\mathcal{T}]\, U_l &= \frac{1}{\big|H^2(\Sigma,\mathcal{A})\big|^{1/2}} \sum_{n,m\in\mathcal{A}} \Theta_{x,y,z}(n,m)\, e^{-\frac{2\pi i}{N} 2^{-1}\langle n,m\rangle}\, B_{ml}\, U_{l+n} \\
&= \frac{1}{|\mathcal{A}|} \sum_{n,m\in\mathcal{A}} \exp\bigg[\frac{2\pi i}{N}\bigg(m_\mathrm{e}\Big(y\, n_\mathrm{e} + x\, n_\mathrm{m} + \tfrac{1}{2} n_\mathrm{m} + l_\mathrm{m}\Big) \\
&\qquad\qquad + m_\mathrm{m}\Big(x\, n_\mathrm{e} + z\, n_\mathrm{m} - \tfrac{1}{2} n_\mathrm{e} - l_\mathrm{e}\Big)\bigg)\bigg]\, U_{l+n}\ .
\end{aligned}
\tag{5.2.24}
$$

The sum over m produces a delta function for n. When this has exactly one solution, the sum over n selects a defect U_{Ml} where $M \in SL(2,\mathbb{Z}_N)$ is the group element corresponding to the condensation defect $V[\mathcal{A},\mathcal{T}] \equiv V_M$. The cases in which there are multiple or no solutions, even though they are not relevant to our purposes, will be discussed at the end. We summarize all cases in Table 5.2 at the end.

If $\mathcal{A} \cong \mathbb{Z}_N$ is generated by (p,q), we can write

$$
m_\mathrm{e} = \mu\, p\ , \qquad m_\mathrm{m} = \mu\, q\ , \qquad n_\mathrm{e} = \nu\, p\ , \qquad n_\mathrm{m} = \nu\, q\ . \tag{5.2.25}
$$

Notice that the phase (5.2.22) trivializes. The sum over μ produces a delta function that fixes $p\, l_\mathrm{m} - q\, l_\mathrm{e} + \xi\,\nu = 0$ and selects one value for ν (as long as $\xi \neq 0$), where $\xi = 2pqx + yp^2 + zq^2$. This reproduces the action of[10]

$$
M = T^k_\mathcal{H} \equiv \mathcal{H} T^k \mathcal{H}^{-1} \tag{5.2.26}
$$

for $k = -\xi^{-1}$ and

$$
\mathcal{H} = \begin{pmatrix} p & * \\ q & * \end{pmatrix} \in SL(2,\mathbb{Z}_N)\ . \tag{5.2.27}
$$

The three parameters x, y, z enter only in the combination ξ, as expected since the discrete torsion is classified by $\mathbb{Z}_N$. Since T^k leaves invariant the vector $v = (1,0)$, then $T^k_\mathcal{H}$ leaves invariant the vector $\mathcal{H}v = (p,q)$, and we obtain the defect implementing $T^k_\mathcal{H}$ by condensing the algebra generated by (p,q) (with a non-vanishing torsion determined by k). For instance, T^k is obtained by condensing the electric surfaces $(n_\mathrm{e}, 0)$, while its electromagnetic dual ST^kS^{-1} is realized by condensing the magnetic surfaces $(0, n_\mathrm{m})$. An element of $SL(2,\mathbb{Z}_N)$ (with N prime) can be written as $\mathcal{H}T^k\mathcal{H}^{-1}$ if and only if its trace is 2 mod N. There are N^2 such elements, including the identity.[11] Indeed condensation produces $N-1$ defects (as we change the torsion ξ) for each of the $N+1$ $\mathbb{Z}_N$ subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$, besides the identity (which is formally obtained by condensing the trivial subgroup $(0,0)$). We will comment on the case with vanishing torsion below.

The elements of $SL(2,\mathbb{Z}_N)$ (N prime) with trace different from 2 are obtained by condensing the full $\mathbb{Z}_N \times \mathbb{Z}_N$. The sum over m produces a delta function that fixes[12] $\big(\mathcal{T} + \frac{\epsilon}{2}\big)n + \epsilon l = 0$ and selects one value of n (as long as $\big(\mathcal{T} + \frac{\epsilon}{2}\big)$ is invertible). This reproduces the action of

$$
M = \Big(\mathcal{T} + \frac{\epsilon}{2}\Big)^{-1} \Big(\mathcal{T} - \frac{\epsilon}{2}\Big)\ . \tag{5.2.28}
$$

---

[10] One has $T^k_\mathcal{H} = \begin{pmatrix} 1-kpq & kp^2 \\ -kq^2 & 1+kpq \end{pmatrix}$, $\nu = kpl_\mathrm{m} - kql_\mathrm{e}$ and so $T^k_\mathcal{H} \begin{pmatrix} l_\mathrm{e} \\ l_\mathrm{m} \end{pmatrix} = \begin{pmatrix} l_\mathrm{e} + \nu p \\ l_\mathrm{m} + \nu q \end{pmatrix}$.

[11] All matrices $M \in SL(2,\mathbb{Z}_N)$ with $\mathrm{Tr}\, M = 2$ can be written as $M = \big(\begin{smallmatrix} 1-\alpha & \beta \\ \gamma & 1+\alpha \end{smallmatrix}\big)$ with $\alpha^2 = \beta\gamma$ mod N. This equation, for N prime, has $N^2 - 1$ solutions with at least one of α, β, γ not zero. One can also easily show that, for N prime, any such matrix M can be written as in footnote 10. The total number of elements in $SL(2,\mathbb{Z}_N)$ (N prime) is instead $N^3 - N$.

[12] When working in $\mathbb{Z}_N$ with N prime, by fractions we always mean the inverse element mod N.

$M \in SL(2,\mathbb{Z}_N)$	$M\cdot(l_1,l_2)$	$\mathcal{A}$	(x,y,z)
$C = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$	$(-l_1,\, -l_2)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$(0,0,0)$
$CT^k = \left(\begin{smallmatrix} -1 & -k \\ 0 & -1 \end{smallmatrix}\right)$	$(-l_1 - kl_2,\, -l_2)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$(0,0,\frac{1}{4}k)$
$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$(-l_2,\, l_1)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$\big(0,\frac{1}{2},\frac{1}{2}\big)$
$ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$	$(-l_2,\, l_1 + l_2)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$\big(\frac{1}{2},1,1\big)$
$(ST)^2 = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$(-l_1 - l_2,\, l_1)$	$\mathbb{Z}_N \times \mathbb{Z}_N$	$\big(\frac{1}{6},\frac{1}{3},\frac{1}{3}\big)$
$T^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$	$(l_1 + kl_2,\, l_2)$	$\big\langle(1,0)\big\rangle$	$\xi = -k^{-1}$

Table 5.1: Examples of $SL(2,\mathbb{Z}_N)$ condensation defects, obtained by condensing $\mathcal{A} \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ with torsion (x,y,z), for $N > 3$ prime (for $N = 2,3$ some of those formulas are different).

Note that $\det\big(\mathcal{T} \pm \frac{\epsilon}{2}\big) = (2 - \operatorname{Tr} M)^{-1}$, therefore all elements $M \in SL(2,\mathbb{Z}_N)$ with $\operatorname{Tr} M \neq 2$ can be obtained this way. The relation can be inverted:

$$\mathcal{T} = \frac{\epsilon}{2}\,\big(1 + M\big)\,\big(1 - M\big)^{-1} \,. \tag{5.2.29}$$

Notice that the two factors on the right-hand side commute. Moreover, in $SL(2,\mathbb{Z}_N)$ we have $\det(1 \pm M) = 2 \pm \operatorname{Tr} M$. The following relation is also useful:

$$\mathcal{T} + \frac{\epsilon}{2} = \epsilon\,\big(1 - M\big)^{-1} \,. \tag{5.2.30}$$

Explicitly, the discrete torsion that produces the symmetry defect for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}_N)$ with $\operatorname{Tr} M \neq 2$ is $x = \frac{d-a}{2(2-a-d)}$, $y = \frac{c}{2-a-d}$, $z = -\frac{b}{2-a-d}$. Finally, assuming that $\big(\mathcal{T} + \frac{\epsilon}{2}\big)$ is invertible, $\det \mathcal{T} = \operatorname{Tr}(1+M)\big[4\operatorname{Tr}(1-M)\big]^{-1}$ therefore $\mathcal{T}$ is invertible if and only if $\operatorname{Tr} M \neq -2 \bmod N$. The case $\mathcal{T} = 0$ corresponds to $M = -\mathbb{1} \equiv C$ which is charge conjugation. The case that $\mathcal{T}$ has rank 1 corresponds (for N prime) to $M = C\mathcal{H}T^k\mathcal{H}^{-1}$ where

$$\mathcal{T} = \frac{k}{4}\begin{pmatrix} q^2 & -pq \\ -pq & p^2 \end{pmatrix} = \frac{k}{4}\,(\epsilon v)\cdot(\epsilon v)^{\mathsf{T}} \,, \qquad v = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \text{and} \qquad \mathcal{H} = \begin{pmatrix} p & * \\ q & * \end{pmatrix} \,. \tag{5.2.31}$$

In Table 5.1 we summarize a few examples.

Small values of N. Some of the previous formulas are ill-defined for small N. For $N = 2$, and more generally for N even, we cannot use the normal ordering prescription in (5.2.22) because 2^{-1} is ill-defined. However, notice that the phase that enters in the definition (5.2.24) of the operator $V[\mathcal{A},\mathcal{T}]$ is the product of the torsion and the normal ordering phases:

$$\exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}} \begin{pmatrix} y & \tilde{x} \\ \tilde{x} - 1 & z \end{pmatrix} n\right] \equiv \exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}} \widetilde{\mathcal{T}} n\right] \,, \tag{5.2.32}$$

where $\tilde{x} = x + \frac{1}{2}$ and $\widetilde{\mathcal{T}} = \mathcal{T} + \frac{\epsilon}{2}$. The quantities $\tilde{x} \in \mathbb{Z}_N$ and $\widetilde{\mathcal{T}}$ are well defined, even for N even, and we can use them to classify the torsion. The group $SL(2,\mathbb{Z}_2) \cong PSL(2,\mathbb{Z}_2) \cong S_3$ (note that $C \cong \mathbb{1}$) has 6 elements, 4 of which have trace equal to 2 mod 2:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \,, \qquad STS^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \,, \qquad S = (TS)T(TS)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \tag{5.2.33}$$

besides the identity. The corresponding defect operators are obtained by condensing the $\mathbb{Z}_2$ subgroups generated by $(1,0)$, $(0,1)$ and $(1,1)$, respectively, with non-vanishing torsion $\xi = pq + yp^2 + zq^2 = 1$. The remaining elements,

$$ST = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad (ST)^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} , \tag{5.2.34}$$

have trace equal to 1 and are described by gauging the full $\mathbb{Z}_2 \times \mathbb{Z}_2$. The relation between torsion and symmetry action M is as in (5.2.28) and (5.2.30), as long as one parametrizes the torsion using $\widetilde{\mathcal{T}}$, therefore $\widetilde{\mathcal{T}} = \epsilon\,(1-M)^{-1}$. One finds that ST is obtained from torsion $(\tilde{x}, y, z) = (1,1,1)$, while $(ST)^2$ is obtained from $(\tilde{x}, y, z) = (0,1,1)$. These two values of the torsion are the only possible ones providing a matrix $\widetilde{\mathcal{T}}$ invertible in $\mathbb{Z}_2$.

For $N = 3$, the element $(ST)^2$ in Table 5.1 has trace equal to 2 mod 3. Indeed we can write $(ST)^2 = \mathcal{H}T^2\mathcal{H}^{-1}$ with $\mathcal{H} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and thus the corresponding defect operator is obtained by condensing the $\mathbb{Z}_3$ subgroup generated by $(1,1)$ with torsion $\xi = 1$.

Fusion. The fusion rules of (invertible) condensation defects correctly satisfy the product of $SL(2,\mathbb{Z}_N)$. The method we describe below is general, however for brevity we only exhibit the product of defects obtained by condensing the full group $\mathbb{Z}_N \times \mathbb{Z}_N$. The defect operators $V[\mathcal{T}]$ on Σ defined in (5.2.24) can be rewritten as

$$V[\mathcal{T}] = \frac{1}{N^2} \sum_{n,m\,\in\,\mathbb{Z}_N\times\mathbb{Z}_N} \exp\left[\frac{2\pi i}{N}\, m^{\mathsf{T}}\left(\mathcal{T} + \frac{\epsilon}{2}\right) n\right] U_n[T^2]\, U_m[S^2] \tag{5.2.35}$$

where we indicated whether the two-dimensional defects U are placed on T^2 or S^2, and rightmost operators are inner. Using the braiding matrix (5.2.23), we obtain

$$\begin{aligned} &V[\mathcal{T}_2]\, V[\mathcal{T}_1] = \\ &= \frac{1}{N^4} \sum_{\substack{n,m \\ n',m'}} \exp\left[\frac{2\pi i}{N}\left(m^{\mathsf{T}}\left(\mathcal{T}_2 + \frac{\epsilon}{2}\right)n + m'^{\mathsf{T}}\left(\mathcal{T}_1 + \frac{\epsilon}{2}\right)n' + m^{\mathsf{T}}\epsilon n'\right)\right] U_{n+n'}[T^2]\, U_{m+m'}[S^2] \,. \end{aligned} \tag{5.2.36}$$

Setting $n = l - n'$, $m = k - m'$ and performing the sum over m' produces a delta function on $(\mathcal{T}_1 + \mathcal{T}_2)n' = \left(\mathcal{T}_2 + \frac{\epsilon}{2}\right)l$. When $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible, one eliminates n' obtaining

$$V[\mathcal{T}_2] \times V[\mathcal{T}_1] = V[\mathcal{T}_{21}] \tag{5.2.37}$$

with

$$\mathcal{T}_{21} = \mathcal{T}_2 - \left(\mathcal{T}_2 - \frac{\epsilon}{2}\right)(\mathcal{T}_1 + \mathcal{T}_2)^{-1}\left(\mathcal{T}_2 + \frac{\epsilon}{2}\right) . \tag{5.2.38}$$

The relation (5.2.38) can be rewritten as

$$\mathcal{T}_{21} + \frac{\epsilon}{2} = \left(\mathcal{T}_1 + \frac{\epsilon}{2}\right)(\mathcal{T}_1 + \mathcal{T}_2)^{-1}\left(\mathcal{T}_2 + \frac{\epsilon}{2}\right) . \tag{5.2.39}$$

Together with (5.2.30), with a little bit of algebra, it implies $M_{21} = M_2 M_1$ as expected.

When $\mathcal{T}_1 + \mathcal{T}_2 = 0$, the sum over m' and n' sets $l = k = 0$. We conclude that

$$V[\mathcal{T}] \times V[-\mathcal{T}] = \mathbb{1} \,, \tag{5.2.40}$$

in agreement with the fact that $M(-\mathcal{T}) = M(\mathcal{T})^{-1}$. The case in which $\mathcal{T}_1 + \mathcal{T}_2$ has rank 1 can be treated in a similar way. In particular, from (5.2.30) it follows that

$$(1 - M_2)\,\epsilon^{-1}(\mathcal{T}_1 + \mathcal{T}_2)(1 - M_1)\,\epsilon^{-1} = (1 - M_2 M_1)\,\epsilon^{-1} \,. \tag{5.2.41}$$

Taking the determinant on both sides and using that $\det(1-M) = \text{Tr}(1-M)$ for M in $SL(2,\mathbb{Z})$, we conclude that $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible if and only if $M_2 M_1$ has trace different from 2 mod N, whilst $(\mathcal{T}_1 + \mathcal{T}_2)$ has rank 1 if and only if $M_2 M_1$ has trace equal to 2 mod N but is not the identity, and thus the corresponding symmetry operator is described by the condensation of a subgroup $\mathcal{A} \cong \mathbb{Z}_N$.

Degenerate torsion and non-invertible surfaces. Besides the $SL(2,\mathbb{Z}_N)$ symmetry defects, higher gauging can produce projectors when the symmetry we condense on a submanifold could also be condensed in the bulk [26]. This is the case when the condensed group is non-anomalous and the discrete torsion would be allowed in 5d. One example is the condensation of $\mathcal{A} = \big\langle (p,q) \big\rangle \cong \mathbb{Z}_N$ with vanishing torsion. From the analysis that follows (5.2.24) we see that the delta function introduced by the sum over μ either has no solution, or has $|\mathcal{A}| = N$ solutions, and the operator $V[\mathcal{A},0]$ acts on the surfaces U_l as

$$V[\mathcal{A},0]\, U_l = \begin{cases} 0 & \text{if } l \notin \mathcal{A} \;, \\ \sum_{n\in\mathcal{A}} U_n & \text{if } l \in \mathcal{A} \;. \end{cases} \tag{5.2.42}$$

This is consistent with the following non-invertible composition law [26]:

$$V[\mathcal{A},0] \times V[\mathcal{A},0] = \big|H^2(\Sigma,\mathcal{A})\big|^{1/2}\, V[\mathcal{A},0] \;, \tag{5.2.43}$$

where the coefficient on the right-hand side is the partition function of a TQFT.

Besides, we obtain a non-invertible surface when we condense the full $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry (which is anomalous in the bulk) with a torsion matrix $\mathcal{T}$ such that $\big(\mathcal{T} + \frac{\epsilon}{2}\big)$ is not invertible. Notice that, since $\mathcal{T}$ is symmetric and ϵ antisymmetric, if $\big(\mathcal{T} + \frac{\epsilon}{2}\big)$ is non-invertible then it has rank 1.[13] In that case, there exist two integer vectors $v_{1,2} \in \mathbb{Z}_N \times \mathbb{Z}_N$ such that

$$\mathcal{T} + \frac{\epsilon}{2} = (\epsilon v_1) \cdot (\epsilon v_2)^{\mathsf{T}} \qquad \text{and} \qquad v_1^{\mathsf{T}} \epsilon\, v_2 = 1 \;. \tag{5.2.44}$$

The second condition comes from the antisymmetric part of the matrix. The sum over m in (5.2.24) gives a delta function on the solutions to $\big(\mathcal{T} + \frac{\epsilon}{2}\big) n = -\epsilon l$, that takes the form

$$\big(v_2^{\mathsf{T}} \epsilon\, n\big)\, v_1 = l \;. \tag{5.2.45}$$

Let $\mathcal{A}_{1,2} \cong \mathbb{Z}_N$ be the two subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ generated by $v_{1,2}$, respectively. If $l \notin \mathcal{A}_1$ then (5.2.45) has no solution for n. On the contrary, if $l \in \mathcal{A}_1$ then the solutions are $n = -l + \nu v_2$ with any $\nu \in \mathbb{Z}_N$. We obtain:

$$V[\mathcal{T}]\, U_l = \begin{cases} 0 & \text{if } l \notin \mathcal{A}_1 \;, \\ \sum_{n\in\mathcal{A}_2} U_n & \text{if } l \in \mathcal{A}_1 \;. \end{cases} \tag{5.2.46}$$

In fact, given two different $\mathbb{Z}_N$ subgroups $\mathcal{A}_1 \neq \mathcal{A}_2$ (then, for N prime, $\mathcal{A}_1 \cap \mathcal{A}_2 = (0,0)$ necessarily), one easily checks the composition law

$$V[\mathcal{A}_2,0]\, V[\mathcal{A}_1,0] = V[\mathcal{T}] \tag{5.2.47}$$

where, on the right-hand side, $\mathcal{T}$ is given by (5.2.44).

[13] This is also true for $N=2$, because the matrix $\widetilde{\mathcal{T}} = \big(\begin{smallmatrix} y & \tilde{x} \\ \tilde{x}-1 & z \end{smallmatrix}\big)$ cannot be zero.

<table>
<tr><th>Condensed subgroup $\mathcal{A}$</th><th>Torsion</th><th>Elements of $SL(2,\mathbb{Z}_N)$</th><th>Specifications</th><th>Projector</th></tr>
<tr><td rowspan="4">$\mathbb{Z}_N \times \mathbb{Z}_N$</td><td rowspan="3">$\left(\mathcal{T}+\frac{\epsilon}{2}\right)$ invertible</td><td rowspan="3">$\operatorname{Tr} M \neq 2$</td><td>$\mathcal{T}$ invertible $\leftrightarrow$ $\operatorname{Tr} M \neq -2$</td><td rowspan="3"></td></tr>
<tr><td>$\mathcal{T}$ rank 1 $\leftrightarrow$ $M = C\mathcal{H}T^k\mathcal{H}^{-1}$</td></tr>
<tr><td>$\mathcal{T}=0$ $\leftrightarrow$ $M = -\mathbb{1} \equiv C$</td></tr>
<tr><td>$\left(\mathcal{T}+\frac{\epsilon}{2}\right)$ not invertible
($\Rightarrow$ rank 1)</td><td></td><td></td><td>✓</td></tr>
<tr><td rowspan="2">$\mathbb{Z}_N$</td><td>$\xi \neq 0$</td><td>$\operatorname{Tr} M = 2$
$M \neq \mathbb{1}$</td><td>$M = \mathcal{H}T^k\mathcal{H}^{-1}$</td><td></td></tr>
<tr><td>$\xi = 0$</td><td></td><td></td><td>✓</td></tr>
<tr><td>$\{0\}$</td><td></td><td>$M = \mathbb{1}$</td><td></td><td></td></tr>
</table>

Table 5.2: Summary of condensation defects, obtained by condensing $\mathcal{A} \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ with or without torsion. Some of them implement the 0-form symmetry $SL(2,\mathbb{Z}_N)$, while other ones are projectors. We assume that N is an odd prime.

5.2.3 Continuum description of symmetry defects

In view of describing the twisted sectors of the $SL(2,\mathbb{Z}_N)$ symmetry, it is useful to reformulate the previous discussion in terms of continuum Lagrangians. We take N odd. When the condensed group is $\mathcal{A} = \mathbb{Z}_N \times \mathbb{Z}_N$, the defect $V[\mathcal{T}]$ is described by a 4d TQFT with two dynamical 2-forms $\Phi = (\varphi_\mathrm{e}, \varphi_\mathrm{m})$ and four 1-forms $\Psi = (\psi_\mathrm{e}, \psi_\mathrm{m})$, $\Gamma = (\gamma_\mathrm{e}, \gamma_\mathrm{m})$ with action [30]:

$$S[\mathcal{T}] = \frac{N}{2\pi} \int_\Sigma \left[\mathcal{B}^\intercal \left(\Phi + d\Gamma\right) + \Phi^\intercal d\Psi + \frac{1}{2}\, \Phi^\intercal \mathcal{T} \Phi \right] . \tag{5.2.48}$$

The torsion is parametrized by the symmetric matrix $\mathcal{T}$ with entries in $\mathbb{Z}_N$ and such that $\mathcal{T} + \frac{\epsilon}{2}$ is invertible.[14] On the other hand, when $\mathcal{A} \cong \mathbb{Z}_N$ is generated by (p,q) we only keep one 2-form φ and two 1-forms ψ, γ with action:

$$S\big[\langle (p,q) \rangle,\, \xi\big] = \frac{N}{2\pi} \int_\Sigma \left[\big(pb + qc\big)\, \big(\varphi + d\gamma\big) + \varphi\, d\psi + \frac{\xi}{2}\, \varphi\varphi \right] . \tag{5.2.49}$$

The torsion is parametrized by a non-vanishing $\xi \in \mathbb{Z}_N$.

Integrating over Ψ and Γ in (5.2.48) forces Φ and the pull-back of $\mathcal{B}$ to be in $H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$. Then Φ can be identified with the Poincaré dual to a 2-cycle $\sigma \in H_2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$. Since Φ couples to $\mathcal{B}$, $\Phi = \mathrm{PD}(\sigma)$ represents a two-dimensional defect $U[\sigma]$ wrapped on σ, and the theory (5.2.48) reproduces higher gauging of the $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry on Σ with torsion $\mathcal{T}$. A similar discussion applies to (5.2.49). The action (5.2.48) is invariant under the following gauge transformations:

$$\mathcal{B} \to \mathcal{B} + d\alpha \,, \qquad \Phi \to \Phi + d\lambda \,, \qquad \Psi \to \Psi - \mathcal{T}\lambda - \alpha + d\mu \,, \qquad \Gamma \to \Gamma - \lambda + d\nu \,. \tag{5.2.50}$$

Considering $\mathcal{B}$ as a background field, the theory (5.2.48) is of a different type depending on whether $\mathcal{T}$ is an invertible matrix over $\mathbb{Z}_N$ or not.

If $\mathcal{T}$ is invertible in $\mathbb{Z}_N$, then (5.2.48) is an invertible TQFT. Indeed, adapting the discussion in [30] to our case, all closed surfaces $\exp\!\big(i m^\intercal \oint \Phi\big)$ are gauge invariant and implement a $\mathbb{Z}_N \times \mathbb{Z}_N$

[14] After integrating over Ψ, the periods of Φ are multiples of $\frac{2\pi}{N}$. Thus on spin manifolds Σ, shifts of the entries of $\mathcal{T}$ by N leave e^{iS} invariant [111].

1-form symmetry, however, because of the equation of motion $\mathcal{T}\Phi = -d\Psi$, when m is in the image of the map $\mathcal{T} : \mathbb{Z}_N^2 \to \mathbb{Z}_N^2$, the surface acts trivially. Therefore only $(\mathbb{Z}_N \times \mathbb{Z}_N)/\operatorname{\mathbb{I}m}\mathcal{T}$ acts faithfully, and if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$ then there is no faithful action at all. On the other hand, the line integrals of Ψ might not be gauge invariant by themselves and need to be the boundary of an open surface $\mathcal{D}$: $\exp\big(ik^{\mathsf{T}}\oint_\ell \Psi + ik^{\mathsf{T}}\mathcal{T}\int_{\mathcal{D}}\Phi\big)$ with $\ell = \partial\mathcal{D}$. They become pure line operators when the surface is transparent, *i.e.*, when $\mathcal{T}k = 0 \bmod N$. Hence the 2-form symmetry of the theory is $\ker\mathcal{T} \subset \mathbb{Z}_N \times \mathbb{Z}_N$, which is trivial if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$. Summarizing, if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$ then the theory (5.2.48) has no topological operators, and is thus an invertible TQFT. This implies that we could integrate out the fields Φ and Ψ. Their equations of motion say that $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ and $\mathcal{T}^{-1}(\mathcal{B}+d\Psi)+\Phi = \check{\Phi}$, where $\check{\Phi} \in H^2(\Sigma, \mathbb{Z}_1 \times \mathbb{Z}_1)$ is a gauge field with integer periods, while $\mathcal{T}^{-1}$ is the inverse of $\mathcal{T}$ in $\mathbb{Z}_N$. Substituting into the action, one obtains

$$S_{\text{invertible}}[\mathcal{T}] = \frac{N}{2\pi}\int_\Sigma \left[\mathcal{B}^{\mathsf{T}} d\widetilde{\Gamma} - \frac{1}{2}\,\mathcal{B}^{\mathsf{T}}\mathcal{T}^{-1}\mathcal{B}\right] , \tag{5.2.51}$$

up to total derivatives and multiples of 2π, where $\widetilde{\Gamma} = \Gamma - \mathcal{T}^{-1}\Psi$ transforms as $\widetilde{\Gamma} \to \widetilde{\Gamma} + \mathcal{T}^{-1}\alpha$ under gauge transformations.

If, on the contrary, $\mathcal{T}$ is a non-invertible matrix, then the 4d theory is a non-trivial TQFT with surface and line operators labeled by $(\mathbb{Z}_N \times \mathbb{Z}_N)/\operatorname{\mathbb{I}m}\mathcal{T}$ and $\ker\mathcal{T}$, respectively. Recall that this case corresponds to $SL(2,\mathbb{Z}_N)$ matrices M with $\operatorname{Tr} M = -2 \bmod N$, which are of the form $M = C\mathcal{H}T^k\mathcal{H}^{-1}$. In the special case $\mathcal{T} = 0$ (that corresponds to $M = C$) the 4d theory (5.2.48) is a pure $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge theory, whose 1-form symmetry is coupled to the background field $\mathcal{B}$.

We can verify that $S[\mathcal{T}]$ in (5.2.48) implements the correct transformation of 2d defects U_l. We introduce a coordinate r transverse to the 4d defect, such that $\Sigma = \{r = 0\}$, and consider the bulk-plus-defect action

$$S_{\substack{\text{bulk plus}\\\text{defect}}} = \frac{N}{4\pi}\int \mathcal{B}^{\mathsf{T}}\epsilon\, d\mathcal{B} + \frac{N}{2\pi}\int_{r=0}\left[\mathcal{B}^{\mathsf{T}}(\Phi + d\Gamma) + \Phi^{\mathsf{T}} d\Psi + \frac{1}{2}\,\Phi^{\mathsf{T}}\mathcal{T}\Phi\right] . \tag{5.2.52}$$

Integrating out the gauge field Φ we obtain an effective description of the interface, which induces a discontinuity

$$\mathcal{B}_L = M^{\mathsf{T}}\mathcal{B}_R \tag{5.2.53}$$

in the gauge field $\mathcal{B}$ (L, R stand for left/right at $r < 0$ and $r > 0$, respectively). Here M is transposed because the $SL(2,\mathbb{Z})$ action on fields is dual to the one on charges, that we previously denoted by M. Indeed, imagine placing a 2d defect operator U_l in the region $r < 0$ (see Figure 5.4) which, compared with our previous setup in Figure 5.3 center, would be the interior region. The expectation value of the operator is $\exp\big(il^{\mathsf{T}}\int\mathcal{B}_L\big) = \exp\big(il^{\mathsf{T}}M^{\mathsf{T}}\int\mathcal{B}_R\big)$. Thus, for an external observer, the compound system of the 4d defect on Σ wrapping the 2d operator U_l appears as a 2d operator U_{Ml}. Let us determine M from (5.2.52). After choosing a gauge (λ, α) in which Γ and Ψ are zero, the equations of motion for $\mathcal{B}$ and Φ read

$$0 = \big(\mathcal{B} + \mathcal{T}\Phi\big)\,\delta(r)dr \tag{5.2.54}$$

$$\epsilon\, d\mathcal{B} = -\Phi\,\delta(r)dr\ . \tag{5.2.55}$$

The gauge field Φ acts as a source for $\mathcal{B}$. Working in a gauge in which $\mathcal{B}_{ri} = 0$, we have $\partial_r\mathcal{B}(r) = \epsilon\,\Phi\,\delta(r)$. This differential equation can be solved: $\mathcal{B}(r) = \mathcal{B}_L + \epsilon\,\Phi\,\theta(r)$, where $\mathcal{B}_L$ is the value of $\mathcal{B}$ for $r < 0$. Multiplying by $\delta(r)$, integrating in a neighbourhood of $r = 0$ and using $\delta(r) = \partial_r\theta(r)$, we

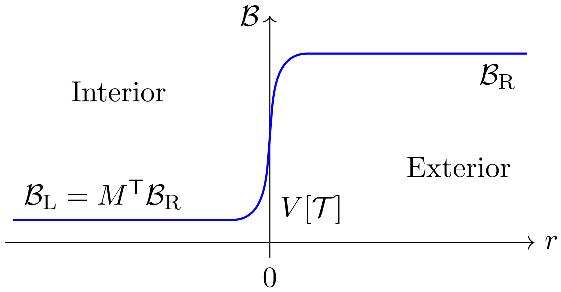


Figure 5.4: The 4d symmetry defect $V[\mathcal{T}]$ induces a discontinuity in the gauge field $\mathcal{B}$ across its surface. Compared with the setup of Figure 5.3 center, the region $r<0$ is the interior of the cylinder while $r>0$ is the exterior.

obtain $\mathcal{B}(0) = \mathcal{B}_L + \frac{\epsilon}{2}\,\Phi = -\mathcal{T}\Phi$. The second equality follows from (5.2.54). Finally, evaluating at $r>0$ we find $\mathcal{B}_R = \mathcal{B}_L + \epsilon\,\Phi$ which implies

$$\mathcal{B}_R = \left[1 - \epsilon\Big(\mathcal{T} + \frac{\epsilon}{2}\Big)^{-1}\right]\mathcal{B}_L \;. \tag{5.2.56}$$

This discontinuity, when written in terms of M using (5.2.30), is exactly (5.2.53). If $\mathcal{T}$ is invertible, one can repeat the computation using (5.2.51) obtaining the same result.

When $\mathcal{A} \cong \mathbb{Z}_N$ one should use the defect Lagrangian (5.2.49) with only one gauge field φ. For instance, when the defect action is coupled to b (*i.e.*, $(p,q) = (1,0)$) and with torsion $\xi \neq 0$, the equation of motion from c simply sets $db = 0$ implying $b_L = b_R$, while the equation of motion from b, after substituting for the solution $\varphi = -\xi^{-1}b(0)$, gives

$$c_L = c_R - \xi^{-1} b_R \;. \tag{5.2.57}$$

This corresponds to the action of T^k with $k = -\xi^{-1}$.

Fusion of defects. We can derive the fusion of defects — that we already analyzed around (5.2.37) in terms of the discrete formalism — using continuum Lagrangians. We place two defects, with action as in (5.2.48), along two codimension-1 surfaces $\Sigma_{1,2}$ at positions $r_{1,2}$ with $r_1 < r_2$. They act as sources for the bulk gauge fields $\mathcal{B}$:

$$\epsilon\, d\mathcal{B} = -\big(\Phi_1 + d\Gamma_1\big)\,\delta(r-r_1)dr - \big(\Phi_2 + d\Gamma_2\big)\,\delta(r-r_2)dr \;. \tag{5.2.58}$$

Since $d\Phi_{1,2} = 0$ from the equations of motion, we can solve the equation as

$$\mathcal{B} = \mathcal{B}_0 + \epsilon\,\big(\Phi_1 + d\Gamma_1\big)\,\theta(r-r_1) + \epsilon\,\big(\Phi_2 + d\Gamma_2\big)\,\theta(r-r_2) \;. \tag{5.2.59}$$

Here $\mathcal{B}_0$ is a background value for $\mathcal{B}$, before adding the effect of the defects. It turns out that a crucial role in computing the fusion is played by the slab of bulk theory in between the two defects, which produces a phase factor. There are two contributions. One comes from substituting (5.2.59) in the bulk action:

$$\begin{aligned}
&\frac{N}{4\pi}\int_{r_2} (\Phi_1 + d\Gamma_1)^{\mathsf{T}} \epsilon\,(\Phi_2 + d\Gamma_2)\,\theta(r_2 - r_1) + \frac{N}{4\pi}\int_{r_1} (\Phi_2 + d\Gamma_2)^{\mathsf{T}} \epsilon\,(\Phi_1 + d\Gamma_1)\,\theta(r_1 - r_2) \\
&= \frac{N}{4\pi}\int_{r_2} \big(\Phi_1 + d\Gamma_1\big)^{\mathsf{T}} \epsilon\,\big(\Phi_2 + d\Gamma_2\big) \;.
\end{aligned} \tag{5.2.60}$$

Another one comes from substituting (5.2.59) in the two defect actions. The defect at $r = r_2$ produces $-\frac{N}{2\pi}\int_{r_2} (\Phi_1 + d\Gamma_1)^{\mathsf{T}} \epsilon\,(\Phi_2 + d\Gamma_2)$, while the one at $r = r_1$ does not give any contribution. In those substitutions we did not include the background $\mathcal{B}_0$, that we will couple to the final effective action. Collecting the contributions, we obtain the following action for the product of defects:

$$S[\mathcal{T}_{21}] = S[\mathcal{T}_1] + S[\mathcal{T}_2] - \frac{N}{4\pi} \int \big(\Phi_1 + d\Gamma_1\big)^{\mathsf{T}} \epsilon \, \big(\Phi_2 + d\Gamma_2\big) \;. \tag{5.2.61}$$

We can interpret the effect of the last term in the path integral as a phase due to the braiding between 2-dimensional defects U_m. To write out the effective action, we identify $r_1 = r_2 = 0$ and simply write $\mathcal{B}_0 \to \mathcal{B}$ for the background field. We also change variables to $\Phi = \Phi_1 + \Phi_2$, $\widetilde{\Psi} = \Psi_1 - \Psi_2$ and $\Gamma = \Gamma_1 + \Gamma_2$. We obtain:

$$\begin{aligned} S[\mathcal{T}_{21}] &= \frac{N}{2\pi} \int_\Sigma \Big[\mathcal{B}^{\mathsf{T}} (\Phi + d\Gamma) + \Phi^{\mathsf{T}} \Big(d\Psi_2 + \frac{\epsilon}{2}\, d\Gamma_1 \Big) + \frac{1}{2}\, \Phi^{\mathsf{T}} \mathcal{T}_2 \Phi \Big] + S_{\text{int}}(\Phi, \Phi_1) \\ S_{\text{int}} &= \frac{N}{2\pi} \int_\Sigma \Big[\Phi_1^{\mathsf{T}} \Big(d\widetilde{\Psi} - \Big(\mathcal{T}_2 + \frac{\epsilon}{2} \Big) \Phi - \frac{\epsilon}{2}\, d\Gamma \Big) + \frac{1}{2}\, \Phi_1^{\mathsf{T}} (\mathcal{T}_1 + \mathcal{T}_2) \Phi_1 - \frac{1}{2}\, d\Gamma_1^{\mathsf{T}} \epsilon \, d\Gamma \Big] \;. \end{aligned} \tag{5.2.62}$$

The field Φ_1, which is forced to be a cochain in $H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ by the equations of motion, does not directly couple to the bulk. The last term is a total derivative that vanishes on closed manifolds.

When $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible in $\mathbb{Z}_N$, then Φ_1 appears quadratically and can be integrated out, obtaining:

$$\begin{aligned} S[\mathcal{T}_{21}] = \frac{N}{2\pi} \int_\Sigma \Big[& \mathcal{B}^{\mathsf{T}} (\Phi + d\Gamma) + \Phi^{\mathsf{T}} d\Psi + \frac{1}{2}\, \Phi^{\mathsf{T}} \mathcal{T}_{21} \Phi \\ & - \frac{1}{2}\, d\Big(\Big(\widetilde{\Psi} - \frac{\epsilon}{2}\, \Gamma \Big)^{\mathsf{T}} (\mathcal{T}_1 + \mathcal{T}_2)^{-1} d\Big(\widetilde{\Psi} - \frac{\epsilon}{2}\, \Gamma \Big) \Big) \Big] \;, \end{aligned} \tag{5.2.63}$$

where $\mathcal{T}_{21}$ is the matrix (5.2.38), we defined $\Psi = \Psi_2 + \frac{\epsilon}{2}\Gamma_1 + \big(\mathcal{T}_2 - \frac{\epsilon}{2}\big)\big(\mathcal{T}_1 + \mathcal{T}_2\big)^{-1}\big(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\big)$, and $(\mathcal{T}_1 + \mathcal{T}_2)^{-1}$ is the inverse in $\mathbb{Z}_N$. The last term is a total derivative and can be ignored on closed manifolds. We reproduce the action of a single defect with discrete torsion $\mathcal{T}_{21}$, which corresponds to $M_{21} = M_2 M_1$.

When $\mathcal{T}_2 = -\mathcal{T}_1 \equiv \mathcal{T}$, then Φ_1 is a Lagrange multiplier imposing $\Phi = \big(\mathcal{T} + \frac{\epsilon}{2}\big)^{-1} d\big(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\big)$ and the defect Lagrangian, up to total derivatives, simply becomes

$$S = \frac{N}{2\pi} \int_\Sigma \mathcal{B}^{\mathsf{T}} d\widehat{\Gamma} \;, \tag{5.2.64}$$

where $\widehat{\Gamma} = \Gamma + \big(\mathcal{T} + \frac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\big)$. On closed manifolds, this reproduces the result $V[\mathcal{T}] \times V[-\mathcal{T}] = \mathbb{1}$. Indeed the action (5.2.64) simply imposes that the pullback of $\mathcal{B}$ be in $H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ without any discontinuity between the L and R regions.

The other cases can be dealt with in a similar way. When $\mathcal{T}_1 + \mathcal{T}_2$ has rank one, the component of Φ_1 living in the kernel of $\mathcal{T}_1 + \mathcal{T}_2$ acts as a Lagrange multiplier, setting to zero one component of Φ, while the component in the cokernel produces the torsion term for the remaining component of Φ. Fusions involving defects from the condensation of $\mathcal{A} \cong \mathbb{Z}_N$ can be studied similarly.

5.3 Twisted sectors and non-invertible defects

Whenever a theory has a discrete 0-form symmetry Γ, one can consider its twisted sectors. In particular, there exist codimension-2 operators that live at the boundary of the codimension-1 defect operators implementing Γ. We call them the codimension-2 operators in the twisted sector. Gauging a (non-anomalous) subgroup $G \subset \Gamma$, the corresponding defects become transparent and the codimension-2 operators at their boundary get promoted to genuine operators of the gauged theory.[15] For instance, in 2d CFTs the twisted sectors are described by local operators at the end of defect (or twist) lines, and their inclusion in the gauged theory is required by modular invariance. In 3d TQFT the twisted sectors are described by line operators at the end of defect surfaces, and the modular tensor category (MTC) of lines gets promoted to a G-crossed MTC [77], also in order to assure modularity.

The situation in higher dimensions is less well understood. In this section we study the twisted sectors of the 5d Chern-Simons theory, exploiting the Lagrangian description of codimension-1 symmetry defects that implement $SL(2,\mathbb{Z}_N)$. In particular, we describe the 3d twist defects $D[\mathcal{T}]$ and $D[\mathcal{A},\xi]$ (or more compactly D_M) at the boundary of 4d symmetry defects $V[\mathcal{T}]$ and $V[\mathcal{A},\xi]$ (or V_M), respectively.

5.3.1 Lagrangian description of $D[\mathcal{T}]$

We can obtain a Lagrangian description of the 3d twisted-sector operators — that we dub $D[\mathcal{T}]$ — at the boundary of 4d $SL(2,\mathbb{Z}_N)$ symmetry defect operators $V[\mathcal{T}]$ from the Lagrangian description (5.2.48) of the latter.[16] As we will see in a moment, it is convenient to perform an integration by parts of the couplings $\mathcal{B}^\mathsf{T} d\Gamma$ and $\Phi^\mathsf{T} d\Psi$ and use the following equivalent Lagrangian for the 4d defect operators $V[\mathcal{T}]$:

$$S[\mathcal{T}] = \frac{N}{2\pi}\int_\Sigma \left[\mathcal{B}^\mathsf{T}\Phi + \Gamma^\mathsf{T} d\mathcal{B} + \Psi^\mathsf{T} d\Phi + \frac{1}{2}\,\Phi^\mathsf{T}\mathcal{T}\Phi\right] . \tag{5.3.1}$$

In the presence of a boundary $Y = \partial\Sigma$, this action is not invariant under the gauge transformations (5.2.50), rather, it shifts by a boundary term (up to integer multiples of 2π):

$$S \,\to\, S + \frac{N}{2\pi}\int_Y \left[\mathcal{B}^\mathsf{T}(\lambda - d\nu) + \Phi^\mathsf{T}(\alpha + \mathcal{T}\lambda - d\mu) + \alpha^\mathsf{T} d\lambda + \frac{1}{2}\,\lambda^\mathsf{T}\mathcal{T} d\lambda\right] . \tag{5.3.2}$$

This can by canceled by the following boundary action:

$$S_\text{twist}[\mathcal{T}] = \frac{N}{2\pi}\int_Y \left[\mathcal{B}^\mathsf{T}\Gamma + \Phi^\mathsf{T}\Psi + \Gamma^\mathsf{T} d\Psi - \frac{1}{2}\,\Gamma^\mathsf{T}\mathcal{T} d\Gamma\right] . \tag{5.3.3}$$

The reason why we wrote the 4d action as in (5.3.1) is that the 4d fields Γ and Ψ only appear as Lagrange multipliers with no derivatives, and thus their path-integrals at different spacetime points are independent. On the contrary, they appear dynamically (with derivatives) in the 3d action (5.3.3) and therefore their restrictions to Y can be treated as independent 3d fields, or edge modes. From the 3d point of view, the fields $\mathcal{B}$ and Φ appear as background fields (that can be integrated afterwards in 5 and 4 dimensions, respectively).[17] The coupled 4d-3d system is gauge invariant. We call the 3d defect defined by $S_\text{twist}[\mathcal{T}]$ a twist defect $D[\mathcal{T}]$ associated to the $SL(2,\mathbb{Z}_N)$ element $M(\mathcal{T})$ (5.2.28).

The actions (5.3.1) and (5.3.3) are invariant under all elements $M' \in SL(2,\mathbb{Z}_N)$ that commute with M, if we supplement the transformation $\mathcal{B} \to M'^\mathsf{T}\mathcal{B}$ with[18]

$$\Phi \,\to\, M'^{-1}\Phi\,, \qquad\qquad \Gamma \,\to\, M'^{-1}\Gamma\,, \qquad\qquad \Psi \,\to\, M'^\mathsf{T}\Psi\,. \tag{5.3.4}$$

Such an invariance is expected since, in general, acting with a 0-form symmetry h on a twisted sector D_g gives an element of $D_{hgh^{-1}}$. This will be important when gauging a subgroup of $SL(2,\mathbb{Z}_N)$.

[15] When G is Abelian, these are the codimension-2 operators charged under the $(d-2)$-form symmetry $\widehat{G}$ dual to G ($\widehat{G}$ is the Pontryagin dual) and implemented by the Wilson lines of G.

[16] A similar discussion would apply to the defects $D[\mathcal{A},\xi]$ at the boundary of $V[\mathcal{A},\xi]$, derived from (5.2.49).

[17] Using the equivalent action (5.2.48) one obtains the boundary action $S'_\text{twist} = \frac{N}{2\pi}\int_Y \left[\Gamma^\mathsf{T} d\Psi - \frac{1}{2}\Gamma^\mathsf{T}\mathcal{T} d\Gamma\right]$ in which the couplings to $\mathcal{B}$ and Φ are not manifest.

[18] Invariance of the last term follows from the fact that M' commutes with M if and only if $M'^\mathsf{T}\mathcal{T}M' = \mathcal{T}$.

Let us analyze the content of the three-dimensional theory $D[\mathcal{T}]$. For simplicity, we only consider the cases in which $\mathcal{T}$ is invertible in $\mathbb{Z}_N$, or $\mathcal{T}=0$. We start with the former. Setting $\mathcal{B}=\Phi=0$, (5.3.3) is the action of an Abelian Chern-Simons theory with four gauge fields, whose level matrix K and its inverse are

$$K = N\begin{pmatrix} -\mathcal{T} & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \,, \qquad K^{-1} = N^{-1}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & \mathcal{T} \end{pmatrix} \,. \tag{5.3.5}$$

There are $|\det K| = N^4$ line operators, given by $e^{i\int(n^\mathsf{T}\Gamma + m^\mathsf{T}\Psi)}$ with $n,m\in\mathbb{Z}_N\times\mathbb{Z}_N$. Not all of them, however, are genuine 3d line operators in the coupled 4d-3d system (keeping the 5d bulk as a background), rather some of them live at the end of a bulk surface $e^{in^\mathsf{T}\int\Phi}$. This follows from the gauge transformations (5.2.50). A basis of genuine line operators is given by

$$W_n = \exp\Big[in^\mathsf{T}\int\Big(\mathcal{T}\Gamma - \Psi\Big)\Big] \,. \tag{5.3.6}$$

We have chosen the parametrization such that W_n has charge $n=(n_\mathrm{e},n_\mathrm{m})$ under the $\mathbb{Z}_N\times\mathbb{Z}_N$ 1-form symmetry that couples to $\mathcal{B}$.[19] These lines have spin

$$\theta[W_n] = \exp\Big(-\frac{\pi i}{N}\,n^\mathsf{T}\mathcal{T}n\Big) \,, \tag{5.3.7}$$

and give a $\mathbb{Z}_N\times\mathbb{Z}_N$ generalization of the $\mathcal{A}^{N,p}$ minimal TQFTs introduced in [111] (see Appendix F there). Indeed, these lines have braiding $B_{ab} = \frac{\theta_{a+b}}{\theta_a\,\theta_b} = \exp\big[-\frac{2\pi i}{N}\,a^\mathsf{T}\mathcal{T}b\big]$ and, taken in isolation, give rise to a consistent MTC with unitary S-matrix $S_{ab} = \frac{1}{N}B_{ab}$. We will use the notation $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ to denote the theory of these lines:

$$\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) \;\subset\; D[\mathcal{T}] \,. \tag{5.3.8}$$

There is some redundancy in the nomenclature of the theories $\mathcal{A}^{N,-\mathcal{T}}$: for all matrices $\mathcal{Q}$ invertible in $\mathbb{Z}_N$, the theory $\mathcal{A}^{N,-\mathcal{Q}^\mathsf{T}\mathcal{T}\mathcal{Q}}$ (where the product of matrices is in $\mathbb{Z}_N$) is equivalent to $\mathcal{A}^{N,-\mathcal{T}}$ up to a relabelling of the lines $n\to\mathcal{Q}n$. They are distinguished, however, by how they couple to $\mathcal{B}$. We will refer to the theory (5.3.7) in which W_n has charge n as $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$. Notice that this theory is not coupled to the 4d field Φ.

The remaining lines are not genuine in the coupled 4d-3d system, and are generated by

$$L_m = \exp\Big[-im^\mathsf{T}\Big(\int_{\partial X}\Psi + \mathcal{T}\int_X\Phi\Big)\Big] \tag{5.3.9}$$

in addition to W_n, where X is a two-dimensional open surface ending on $D[\mathcal{T}]$. The twisted sector, as an isolated 3d theory, is formed by *both* genuine and non-genuine line operators. We chose the generators L_m such that in 3d (*i.e.*, switching the background Φ off) they have trivial braiding with W_n. Indeed, the twisted sector can be decomposed as

$$D[\mathcal{T}] = \mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) \times \mathcal{A}^{N,\mathcal{T}}(\mathcal{B}+\mathcal{T}\Phi) \,, \tag{5.3.10}$$

where the two factors are the MTCs of W_n and L_m, respectively.[20] However, as we will see in Section 5.4, once a subgroup of the $SL(2,\mathbb{Z}_N)$ 0-form symmetry is gauged in the bulk, some of the 4d operators become transparent and only the subcategory $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ of genuine operators survives.

[19]Indeed, under the transformation $\mathcal{B}\to\mathcal{B}+d\alpha$, $\Psi\to\Psi-\alpha$, the operator gets a phase $W_n\to e^{in^\mathsf{T}\int\alpha}W_n$.

[20]One could also consider the non-genuine operators $\ell_m = \exp\big[im^\mathsf{T}\big(\int_{\partial X}\Gamma + \int_X\Phi\big)\big]$ which do not couple to $\mathcal{B}$, however they have vanishing spin and do not form a MTC by themselves.

The $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry of $\mathcal{A}^{N,-\mathcal{T}}$ is anomalous, since the lines W_n that generate it have non-trivial braiding. Turning on the background field $\mathcal{B}$ coupled to the 1-form symmetry, the anomaly is canceled [111] by the following four-dimensional inflow action:[21]

$$I_{\mathcal{T}}(\mathcal{B}) = \frac{N}{2\pi} \int_\Sigma \left[\mathcal{B}^\mathsf{T} d\widetilde{\Gamma} - \frac{1}{2}\, \mathcal{B}^\mathsf{T} \mathcal{T}^{-1} \mathcal{B} \right] , \tag{5.3.11}$$

where the dynamical field $\widetilde{\Gamma}$ imposes $\mathcal{B} \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$ on shell, and $\mathcal{T}^{-1}$ is the inverse of $\mathcal{T}$ in $\mathbb{Z}_N$. This implies that $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ is not invariant under the gauge transformation $\mathcal{B} \to \mathcal{B} + d\alpha$, $\widetilde{\Gamma} \to \widetilde{\Gamma} + \mathcal{T}^{-1}\alpha$ but rather its path integral picks up a phase:

$$\exp\left[-\frac{iN}{2\pi} \int_Y \left(\alpha^\mathsf{T} d\widetilde{\Gamma} + \frac{1}{2}\, \alpha^\mathsf{T} \mathcal{T}^{-1} d\alpha \right) \right] . \tag{5.3.12}$$

Indeed one can check that the anomaly inflow action (5.3.11) for $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) \times \mathcal{A}^{N,\mathcal{T}}(\mathcal{B} + \mathcal{T}\Phi)$, if supplemented by the condition that $\Phi \in H^2(\Sigma, \mathbb{Z}_N \times \mathbb{Z}_N)$, coincides with the 4d action (5.2.48) for the defect $V[\mathcal{T}]$. Alternatively, one can start with the action (5.3.1) for $V[\mathcal{T}]$ and integrate out Φ. This is possible because, as stressed after (5.2.50), the theory is trivial as long as $\mathcal{T}$ is invertible in $\mathbb{Z}_N$. We already did this computation in (5.2.51): one is left with the invertible TQFT (5.3.11) in the 4d bulk and $\mathcal{A}^{N,-\mathcal{T}}$ on the 3d boundary. Either way, the coupled 4d-3d system is anomaly free.

The case of $\mathcal{T} = 0$, which describes the charge conjugation operator V_C, needs a separate discussion. Contrary to the previous case, there is no consistent MTC that describes the lines W_n decoupled from Φ. Those lines have trivial spin and braiding among themselves. This phenomenon was already observed in [111] and is a consequence of the non-invertibility of the 4d 2-form gauge theory for Φ. The action for the twisted sector $D[\mathcal{T} = 0] \equiv D_C$ is

$$S_{\text{twist}}[\mathcal{T} = 0] = \frac{N}{2\pi} \int_Y \left[\mathcal{B}^\mathsf{T} \Gamma + \Phi^\mathsf{T} \Psi + \Gamma^\mathsf{T} d\Psi \right] . \tag{5.3.13}$$

This is a 3d $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge theory (described by the 3d fields Γ, Ψ) coupled to the backgrounds fields $\mathcal{B}$ and Φ for the two copies of the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry, and we denote it by $(\mathbb{Z}_N \times \mathbb{Z}_N)_0(\mathcal{B}, \Phi)$.

Degeneracies. We ask what is the degeneracy of the twisted sectors, *i.e.*, how many boundaries an $SL(2, \mathbb{Z}_N)$ symmetry defect V can have. In three-dimensional TQFTs with a 0-form symmetry Γ, the number of simple lines in a twisted sector labeled by $g \in \Gamma$ is equal to the number of g-invariant simple lines in the untwisted sector [77]. In our case, the 5d CS theory has no genuine codimension-2 operators (besides the trivial one), therefore we expect every twisted sector to be unique. One could argue that we should also consider the operators obtained by fusing $D[\mathcal{T}]$ with codimension-2 condensation defects obtained from the bulk 2-form symmetry.

We can show that for defects $V[\mathcal{T}]$ obtained by condensing the full $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry, the boundary $D[\mathcal{T}]$ is left invariant by every such fusion, up to stacking with a decoupled TQFT. Indeed, fusing $D[\mathcal{T}]$ with a 2d symmetry defect $U(\gamma)$ with $\gamma \in H_2(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$ is equivalent to adding the following coupling to the action (5.3.3) of $D[\mathcal{T}]$:

$$\delta S_{\text{twist}}[\mathcal{T}] = \int_Y \mathcal{B}^\mathsf{T} \Gamma_\gamma , \qquad\qquad \Gamma_\gamma = \text{PD}(\gamma) \tag{5.3.14}$$

where $\text{PD}(\gamma) \in H^1(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$ is the Poincaré dual to γ on Y. Given a continuum description of the class Γ_γ, for instance through a delta 1-form, the extra coupling can be reabsorbed by the field

[21] In the conventions of [111], the 1-form symmetry is generated by the lines $\widetilde{W}_n \equiv W_{-\mathcal{T}^{-1}n}$ which have charge $-\mathcal{T}^{-1}n$ and spin $\exp\left(-\frac{\pi i}{N}\, n^\mathsf{T} \mathcal{T}^{-1} n\right)$. This theory, that [111] would call $\mathcal{A}^{N,-\mathcal{T}^{-1}}$, has an anomaly that is canceled by (5.3.11).

redefinition $\Gamma \to \Gamma - \frac{2\pi}{N}\Gamma_\gamma$, $\Phi \to \Phi + \frac{2\pi}{N} d\Gamma_\gamma$, $\mathcal{B} \to \mathcal{B} - \frac{2\pi}{N}\mathcal{T} d\Gamma_\gamma$, which however produces a phase

$$\exp\biggl(- \frac{2\pi i}{N} \int \frac{1}{2}\, \Gamma_\gamma^\mathsf{T} \mathcal{T} d\Gamma_\gamma \biggr) = \exp\biggl(- \frac{2\pi i}{N} \int \frac{1}{2}\, \Gamma_\gamma^\mathsf{T}\, N\mathcal{T}\, \beta(\Gamma_\gamma) \biggr) \equiv Q_{N\mathcal{T}}(\Gamma_\gamma) \;. \tag{5.3.15}$$

Notice that, in the continuum description on the left-hand side, $d\Gamma_\gamma$ is a class with values in N times $\mathbb{Z} \times \mathbb{Z}$ rather than identically zero. On the right-hand side we wrote the phase in a more precise way in terms of $\Gamma_\gamma \in H^1(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$ and the Bockstein map associated to the short exact sequence $0 \to \mathbb{Z}_N \xrightarrow{N} \mathbb{Z}_{N^2} \xrightarrow{\text{mod } N} \mathbb{Z}_N \to 0$ so that $\beta(\Gamma_\gamma) \in H^2(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$. The integrals in (5.3.15) are well defined on generic manifolds if $N\mathcal{T}$ is an even matrix, and on spin manifolds if $N\mathcal{T}$ is a more general integer matrix. Hence

$$U(\gamma) \times D[\mathcal{T}] = e^{i Q_{N\mathcal{T}}(\Gamma_\gamma)}\, D[\mathcal{T}] \;. \tag{5.3.16}$$

A similar effect has already been appreciated in dealing with N-ality defects in [25, 104].

Now, a 3d condensation defect for the $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry can be thought of as a 3d $\mathbb{Z}_N \times \mathbb{Z}_N$ Dijkgraaf-Witten (DW) theory, possibly with torsion $\mathcal{P}$, coupled to the dynamical field $\mathcal{B}$. The coupling is precisely (5.3.14) with Γ_γ substituted by the dynamical gauge field of the DW theory. We denote the 3d condensation defect as $\mathcal{C}_\mathcal{P}^{\mathbb{Z}_N \times \mathbb{Z}_N}$, and omit the subscript when there is no torsion. Stacking the condensation defect on $D[\mathcal{T}]$ replaces the coupling to $\mathcal{B}$ with the torsion term $Q_{N\mathcal{T}}(\Gamma_\gamma)$: this produces a shift $\delta\mathcal{P} = -N\mathcal{T}$ of the torsion of the DW theory. It turns out (see below) that if N is odd and the theory is spin, then shifts of the torsion components by multiples of N give equivalent theories, and so in our case the shift is immaterial. We conclude that

$$\mathcal{C}_\mathcal{P}^{\mathbb{Z}_N \times \mathbb{Z}_N} \times D[\mathcal{T}] = (\mathbb{Z}_N \times \mathbb{Z}_N)_\mathcal{P}\; D[\mathcal{T}] \;. \tag{5.3.17}$$

The factor on the right-hand side is a decoupled Dijkgraaf-Witten TQFT. A similar argument applies to any other 3d condensate in which only a subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ is condensed (possibly with torsion): they can all be absorbed by $D[\mathcal{T}]$. We conclude that there is no degeneracy in these twisted sectors.

When, on the other hand, the defect $V[\mathcal{A}, \xi]$ is obtained by condensing a subgroup $\mathcal{A}$ of $\mathbb{Z}_N \times \mathbb{Z}_N$, then only condensates of surfaces in $\mathcal{A}$ can similarly be absorbed by D, while more general surfaces in $\mathbb{Z}_N \times \mathbb{Z}_N$ cannot and give rise to a genuine degeneracy of the twisted sector. Since surfaces in $\mathcal{A}$ are absorbed, the degeneracy is given by all condensates (with torsion) of the quotient group $(\mathbb{Z}_N \times \mathbb{Z}_N)/\mathcal{A}$ (or its subgroups).

The last case is the 4d indentity interface $V_\mathbb{1}$, on which we do not gauge any symmetry. Its sector, which is the untwisted sector, consists of all possible 3d condensates in $\mathbb{Z}_N \times \mathbb{Z}_N$.

Dijkgraaf-Witten theories

The 3d $\mathbb{Z}_N^k$ Dijkgraaf-Witten theories can be described by the following Abelian Chern-Simons action:

$$S_\text{DW}[\mathcal{T}] = \int_Y \biggl[\frac{N}{2\pi}\, x^\mathsf{T} dy + \frac{1}{4\pi}\, y^\mathsf{T} \mathcal{P} dy \biggr] \tag{5.3.18}$$

where x, y are k-dimensional vectors of Abelian gauge fields and $\mathcal{P}$ is a $k \times k$ symmetric integer matrix. The theory is bosonic if $\mathcal{P}$ is even (*i.e.*, if its diagonal entries are even), otherwise it is spin. The level matrix is $K = \bigl(\begin{smallmatrix} 0 & N\mathbb{1} \\ N\mathbb{1} & \mathcal{P} \end{smallmatrix} \bigr)$. The theory has N^{2k} lines labelled by $n \in \mathbb{Z}_N^{2k}$ with spin

$$\theta[n] = \exp\bigl(\pi i\, n^\mathsf{T} K^{-1} n \bigr) \qquad \text{where} \qquad K^{-1} = \frac{1}{N^2} \begin{pmatrix} -\mathcal{P} & N\,\mathbb{1} \\ N\,\mathbb{1} & 0 \end{pmatrix} \;. \tag{5.3.19}$$

In all cases, a shift of $\mathcal{P}$ by N times an even integer matrix gives an equivalent theory, *i.e.*, the diagonal entries of $\mathcal{P}$ are defined modulo $2N$ while the off-diagonal entries modulo N. This follows from the field redefinition $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \to \left(\begin{smallmatrix} \mathbb{1} & \mathcal{Q} \\ 0 & \mathbb{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ where $\mathcal{Q}$ is an integer matrix, or equivalently, from the relabelling $n \to \left(\begin{smallmatrix} \mathbb{1} & 0 \\ \mathcal{Q} & \mathbb{1} \end{smallmatrix}\right)n$ of the lines. If N is odd, in addition, theories in which the entries of $\mathcal{P}$ differ by multiples of N are equivalent as spin theories.[22] This follows from the fact that the relabelling $n \to \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 2^{-1}\mathcal{Q} & \mathbb{1} \end{smallmatrix}\right)n$ (where 2^{-1} is the inverse in $\mathbb{Z}_N$) preserves the spin modulo a sign, which can be cancelled by fusing with the transparent fermion.

The coupling of the electric 1-form symmetry to a $\mathbb{Z}_N^k$ background field $\mathcal{B}$ is described by

$$S_{\mathrm{DW}}[\mathcal{T}](\mathcal{B}) = \int_Y \left[\frac{N}{2\pi}\left(\mathcal{B}^\mathsf{T} y + x^\mathsf{T} dy\right) + \frac{1}{4\pi}\, y^\mathsf{T}\mathcal{P} dy\right] , \tag{5.3.20}$$

invariant under $\mathcal{B} \to \mathcal{B} + d\alpha$, $x \to x - \alpha$. The lines labelled by $n = (n_\mathrm{e}, n_\mathrm{m})$ have charge $-n_\mathrm{e}$ under the electric 1-form symmetry. This statement persists under shifts of the components of $\mathcal{P}$ by multiples of N.

5.3.2 Fusion of twist defects

We now study the fusion of two twist defects $D[\mathcal{T}_1]$ and $D[\mathcal{T}_2]$. As expected, the fusion is compatible with the group product rule $M_{21} = M_2 M_1$ of 4d $SL(2,\mathbb{Z}_N)$ defect operators $V[\mathcal{T}]$, *i.e.*, of twisted sectors, however we would like to understand which condensates and decoupled TQFTs can be generated.

As already discussed in Section 5.2.3 for the fusion of 4d defects, the 5d bulk provides a crucial contribution to the fusion of 3d twist defects as well. The bulk contribution was computed in (5.2.61), thus the total action for the system of two 4d defects with boundary located on the same 3d (spin) manifold Y is

$$S = S[\mathcal{T}_1] + S[\mathcal{T}_2] - \frac{N}{4\pi}\int (\Phi_1 + d\Gamma_1)^\mathsf{T} \epsilon\, (\Phi_2 + d\Gamma_2) + S_\mathrm{twist}[\mathcal{T}_1] + S_\mathrm{twist}[\mathcal{T}_2] , \tag{5.3.21}$$

where, this times, we use the 4d action (5.3.1) for the symmetry defects.

The computation in the 4d bulk is similar to the one we did in Section 5.2.3. One introduces $\Phi = \Phi_1 + \Phi_2$, $\Gamma = \Gamma_1 + \Gamma_2$, $\widetilde{\Psi} = \Psi_1 - \Psi_2$, and eliminates Φ_2, Γ_2, Ψ_1. If $(\mathcal{T}_1 + \mathcal{T}_2)$ is an invertible matrix in $\mathbb{Z}_N$, the field Φ_1 can be integrated out leaving the bulk theory

$$S_\mathrm{bulk} = \frac{N}{2\pi}\int_\Sigma \left[\mathcal{B}^\mathsf{T}\Phi + \Gamma^\mathsf{T} d\mathcal{B} + \Psi^\mathsf{T} d\Phi + \frac{1}{2}\Phi^\mathsf{T}\mathcal{T}_{21}\Phi\right] , \tag{5.3.22}$$

where $\mathcal{T}_{21}$ is given in (5.2.38) and $\Psi = \Psi_2 + \frac{\epsilon}{2}\Gamma_1 + \left(\mathcal{T}_2 - \frac{\epsilon}{2}\right)(\mathcal{T}_1 + \mathcal{T}_2)^{-1}\left(\widetilde{\Psi} - \frac{\epsilon}{2}\Gamma\right)$. This is the theory $S[\mathcal{T}_{21}]$. There are leftover boundary terms, that together with $S_\mathrm{twist}[\mathcal{T}_1] + S_\mathrm{twist}[\mathcal{T}_2]$ give

$$\begin{aligned} S_\mathrm{boundary} = \frac{N}{2\pi}\int_Y \Big[&\mathcal{B}^\mathsf{T}\Gamma + \Phi^\mathsf{T}\Psi + \Gamma^\mathsf{T} d\Psi_2 - \frac{1}{2}\Gamma^\mathsf{T}\mathcal{T}_2 d\Gamma + \Gamma_1^\mathsf{T} d\Big(\widetilde{\Psi} + \left(\mathcal{T}_2 - \tfrac{\epsilon}{2}\right)\Gamma\Big) \\ &- \frac{1}{2}\Gamma_1^\mathsf{T}(\mathcal{T}_1 + \mathcal{T}_2) d\Gamma_1 - \frac{1}{2}\left(\widetilde{\Psi} - \tfrac{\epsilon}{2}\Gamma\right)^\mathsf{T}(\mathcal{T}_1 + \mathcal{T}_2)^{-1} d\left(\widetilde{\Psi} - \tfrac{\epsilon}{2}\Gamma\right)\Big] . \end{aligned} \tag{5.3.23}$$

The gauge transformations of the new fields are

$$\mathcal{B} \to \mathcal{B} + d\alpha , \quad \Phi \to \Phi + d\lambda , \quad \Psi \to \Psi - \mathcal{T}_{21}\lambda - \alpha + d\mu , \quad \Gamma \to \Gamma - \lambda + d\nu \tag{5.3.24}$$

where $\lambda = \lambda_1 + \lambda_2$. The theory (5.3.23) is not trivial and we cannot integrate other fields out. We perform a more rigorous analysis of it below, but for now, in order to understand the physics, let us perform an approximate computation. We introduce a new 1-form field

$$H = \Psi_1 - \mathcal{T}_1\Gamma_1 - \Psi_2 + \mathcal{T}_2\Gamma_2 = \widetilde{\Psi} + \mathcal{T}_2\Gamma - (\mathcal{T}_1 + \mathcal{T}_2)\,\Gamma_1 . \tag{5.3.25}$$

[22] This is not true, in general, if N is even. A counterexample for $k = 1$ is the family of four $\mathbb{Z}_2$ theories.

This combination is special because it is invariant under the gauge transformations (5.2.50) parametrized by λ_1, λ_2, α. We eliminate Γ_1 in favor of H: this is not a legit operation since $(\mathcal{T}_1+\mathcal{T}_2)$ is not a unimodular integer matrix, but let us proceed anyway and treat $(\mathcal{T}_1+\mathcal{T}_2)^{-1}$ as the inverse in $\mathbb{Q}$. Up to total derivatives, we obtain

$$S_{\text{boundary}} \sim \frac{N}{2\pi}\int_Y \left[\mathcal{B}^\mathsf{T}\Gamma + \Phi^\mathsf{T}\Psi + \Gamma^\mathsf{T} d\Psi - \frac{1}{2}\Gamma^\mathsf{T}\mathcal{T}_{21} d\Gamma\right] - \frac{N}{4\pi}\int_Y H^\mathsf{T}(\mathcal{T}_1+\mathcal{T}_2)^{-1} dH \,. \tag{5.3.26}$$

The first term is the expected action $S_{\text{twist}}[\mathcal{T}_{21}]$ of the twisted sector $D[\mathcal{T}_{21}]$. The second term is a decoupled TQFT, described by a Chern-Simons action with fractional level-matrix. Perturbatively, it behaves as the theory $\mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2}$ (while it is not well defined at the non-perturbative level).

If $(\mathcal{T}_1+\mathcal{T}_2)$ is not invertible in $\mathbb{Z}_N$ then the procedure has to be slightly changed. Let us discuss the case $\mathcal{T}_2 = -\mathcal{T}_1 \equiv \mathcal{T}$, corresponding to the fusion of a defect with its "inverse". This case is interesting because the fusion of two defects in inverse twisted sectors must produce an operator in the untwisted sector, which however contains all three-dimensional condensation defects. Starting with (5.3.21) and performing the field redefinitions to Φ, Γ, $\widetilde{\Psi}$, in the 4d bulk one finds Φ_1 to be a Lagrange multiplier imposing $\Phi = d\big(\mathcal{T}+\frac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi}-\frac{\epsilon}{2}\Gamma\big)$. It is convenient to define

$$\begin{aligned} \widehat{\Gamma} &= \Gamma + \big(\mathcal{T}+\tfrac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi}-\tfrac{\epsilon}{2}\Gamma\big) \,, \\ \widehat{\Psi} &= \Psi_2 + \mathcal{T}\Gamma_1 + \mathcal{T}\big(\mathcal{T}+\tfrac{\epsilon}{2}\big)^{-1}\big(\widetilde{\Psi}-\tfrac{\epsilon}{2}\Gamma\big) \,. \end{aligned} \tag{5.3.27}$$

Then the bulk action simply reduces to the completely trivial theory

$$S_{\text{bulk}} = \frac{N}{2\pi}\int_\Sigma \widehat{\Gamma}^\mathsf{T} d\mathcal{B} \tag{5.3.28}$$

that describes the identity operator $V_{\mathbb{1}}$. The boundary terms instead give

$$S_{\text{boundary}} = \frac{N}{2\pi}\int_Y \left[\mathcal{B}^\mathsf{T}\widehat{\Gamma} + \widehat{\Gamma}^\mathsf{T} d\widehat{\Psi} - \frac{1}{2}\widehat{\Gamma}^\mathsf{T}\mathcal{T} d\widehat{\Gamma}\right] . \tag{5.3.29}$$

The fields $\widehat{\Gamma}$, $\widehat{\Psi}$ are invariant under the gauge transformations λ_1, λ_2, indeed this 3d theory does not need to be attached to any 4d theory. On the other hand, $\widehat{\Psi} \to \widehat{\Psi} - \alpha$ under gauge transformations of $\mathcal{B}$ (while $\widehat{\Gamma}$ is invariant). The action (5.3.29) describes a 3d $\mathbb{Z}_N \times \mathbb{Z}_N$ Dijkgraaf-Witten theory with torsion equal to $-N\mathcal{T}$, in which a $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry is coupled to $\mathcal{B}$ — as in (5.3.20). Alternatively, this can be though of as a 3d condensation defect for the $\mathbb{Z}_N \times \mathbb{Z}_N$ global 2-form symmetry of the 5d bulk theory: $\widehat{\Psi}$ forces $\widehat{\Gamma} \in H^1(Y, \mathbb{Z}_N \times \mathbb{Z}_N)$, then $e^{i\frac{N}{2\pi}\int \mathcal{B}^\mathsf{T}\widehat{\Gamma}}$ is a two-dimensional operator of the 5d theory placed on the Poincaré dual to $\widehat{\Gamma}$ within Y, and the last term in (5.3.29) produces a phase weighing the sum over surfaces. We dubbed such a 3d condensation defect $\mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N}_{-N\mathcal{T}} \equiv \mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N}$, since we are considering N odd. Therefore, the fusion of a twist defect with its "inverse" is given by

$$D[\mathcal{T}] \times D[-\mathcal{T}] = D[\mathcal{T}] \times \overline{D}[\mathcal{T}] = \mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N} \,. \tag{5.3.30}$$

A more rigorous analysis of twisted sectors. The analysis of the fusion of twist defects we performed in (5.3.26) using the Lagrangian formulation, while suggesting the correct result, was imprecise. We can obtain a more rigorous and precise derivation by studying the algebra of topological operators.

As discussed in Section 5.3.1, if $\mathcal{T}$ is invertible in $\mathbb{Z}_N$ then the twist operator $D[\mathcal{T}]$ hosts a MTC of local line operators (which are not coupled to the 4d defect) forming the minimal TQFT $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$. When we fuse two twist operators $D[\mathcal{T}_1]$ and $D[\mathcal{T}_2]$, the set of local line operators is not simply the stacking of the two TQFTs because of the bulk contribution. Taken separately, the two minimal

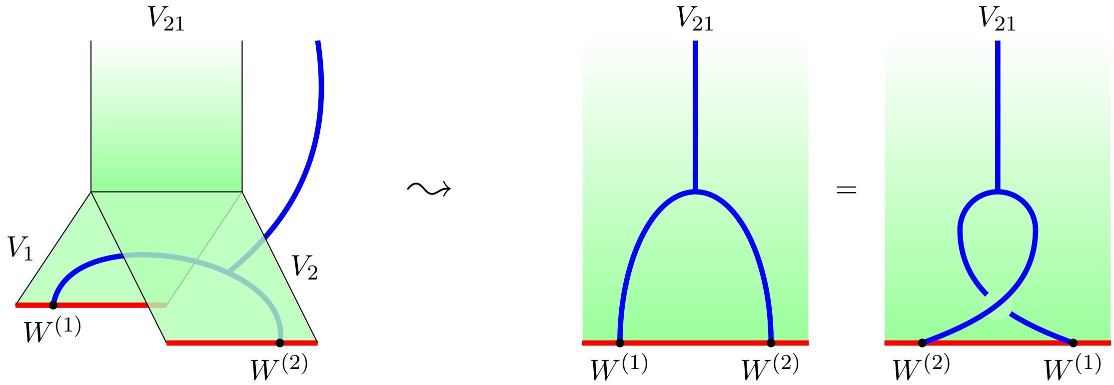


Figure 5.5: Braiding between lines $W^{(1)}$ and $W^{(2)}$ from bulk ordering. We represented the lines $W^{(i)}$ by black points, the 3d twist sectors $D[\mathcal{T}_i]$ by red lines, the surfaces $U(\gamma)$ by blue lines, and the 4d condensation defects V_i by green surfaces. Left: bulk definition of fusion. Right: two different ordering procedures, related by the half-braiding phase of the bulk 5d theory. In canonical quantization, time runs horizontally.

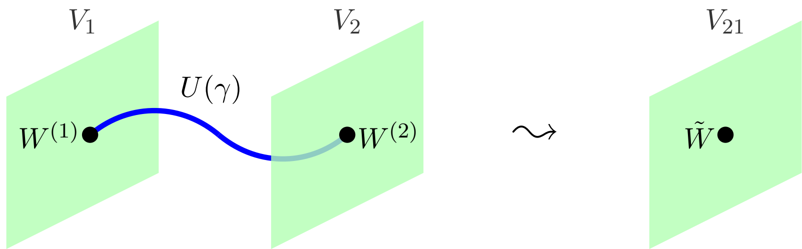


Figure 5.6: The lines $\tilde{W}$ of V_{21} that are decoupled from $\mathcal{B}$ can be seen as products of endlines that are attached to a surface $U(\gamma)$ stretched between the two defects V_1 and V_2. Once the defects are fused, the lines $\tilde{W}$ become local in V_{21}.

TQFTs have lines $W^{(1)}_{n_1}$ and $W^{(2)}_{n_2}$, respectively. The 5d dynamical bulk field $\mathcal{B}$, however, generates a non-trivial braiding between the two sets of lines:

$$B_{W^{(1)},W^{(2)}} = \exp\left(\frac{2\pi i}{N}\, n_1^\intercal \frac{\epsilon}{2} n_2\right) \tag{5.3.31}$$

where we are taking N odd. This follows from the boundary term $-\frac{N}{2\pi}\int_Y d\Gamma_1^\intercal \frac{\epsilon}{2} d\Gamma_2$ in (5.3.21) and the expression (5.3.6) for the local lines. It can also be understood as follows. In canonical quantization, the braiding matrix appears as a non-trivial commutator

$$W^{(1)}\, W^{(2)} = B_{W^{(1)},W^{(2)}}\, W^{(2)}\, W^{(1)}\,, \tag{5.3.32}$$

where the operators are time ordered. If $W^{(i)}$ were local lines in the full theory, this would be trivial because the lines would live on separate defects. However, in the full theory $\mathcal{B}$ is dynamical and thus both $W^{(1)}$ and $W^{(2)}$, which are coupled to $\mathcal{B}$, must be the endlines of suitable bulk surfaces $U(\gamma) = e^{i\int_\gamma \mathcal{B}}$. Likewise, also the product $W^{(1)}W^{(2)}$ must be attached to a bulk surface with the correct charge (see Figure 5.5). Commuting the order in which the endlines are fused has the effect of half-braiding the attached bulk surfaces, which is captured by the normal ordering phase $\exp\left(\frac{2\pi i}{N}\, 2^{-1}\langle n_1, n_2\rangle\right)$ we already introduced in (5.2.22). This is precisely the braiding (5.3.31).

We indicate the product of the two sectors $\mathcal{A}^{N,-\mathcal{T}_i}$ deformed by the extra braiding (5.3.31) as $\mathcal{A}^{N,-\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}$, in order to distinguish it from the standard decoupled tensor product. We label the lines of this theory by $\mathcal{N} = (n_1, n_2)$. The spin of the lines of $W^{(1)}$ and $W^{(2)}$ is undeformed, while the spin of product of lines can be computed using $\theta_{a+b} = \theta_a \theta_b B_{ab}$. We obtain

$$\theta[W_{\mathcal{N}}] = \exp\Big(\pi i\, \mathcal{N}^{\mathsf{T}} \mathcal{K}_{21} \mathcal{N}\Big) \qquad\qquad \mathcal{K}_{21} = \frac{1}{N} \begin{pmatrix} -\mathcal{T}_1 & \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & -\mathcal{T}_2 \end{pmatrix} . \tag{5.3.33}$$

The line $W_{\mathcal{N}}$ has charge $n_1 + n_2$ under the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry coupled to $\mathcal{B}$. We can identify a subset of lines that are decoupled from $\mathcal{B}$ and, under certain conditions, form a consistent, independent, and local 3d MTC. These are the lines with $\mathcal{N} = (l, -l)$: they exist without an attached bulk surface, and can be thought of as sitting at opposite ends of a $\mathcal{B}$ surface before fusion, see Figure 5.6. The spin of these lines is $\exp\big(-\frac{\pi i}{N}\, l^{\mathsf{T}} (\mathcal{T}_1 + \mathcal{T}_2) l\big)$ and thus, as long as $(\mathcal{T}_1 + \mathcal{T}_2)$ is invertible in $\mathbb{Z}_N$, they form the consistent MTC $\mathcal{A}^{N, -\mathcal{T}_1 - \mathcal{T}_2}$. The remaining lines are coupled to $\mathcal{B}$. We can identify a subset that has trivial braiding with the lines of $\mathcal{A}^{N, -\mathcal{T}_1 - \mathcal{T}_2}$. They are given by $\mathcal{N}_\eta = (\xi, -\xi + \eta)$ with $\xi = (\mathcal{T}_1 + \mathcal{T}_2)^{-1} \big(\mathcal{T}_2 + \frac{\epsilon}{2}\big) \eta$ and their spin is $\exp\big(-\frac{\pi i}{N}\, \eta^{\mathsf{T}} \mathcal{T}_{21} \eta\big)$ where the matrix $\mathcal{T}_{21}$ is the one in (5.2.38). Since the line $\mathcal{N}_\eta$ has charge η under the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry, they form the MTC $\mathcal{A}^{N, -\mathcal{T}_{21}}(\mathcal{B})$. Hence we arrive to the result

$$\mathcal{A}^{N, -\mathcal{T}_2}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{A}^{N, -\mathcal{T}_1}(\mathcal{B}) = \mathcal{A}^{N, -\mathcal{T}_1 - \mathcal{T}_2} \times \mathcal{A}^{N, -\mathcal{T}_{21}}(\mathcal{B}) . \tag{5.3.34}$$

The product on the right-hand side is the standard tensor product. The result is in accord with the factorization theorem of [111]. We have thus shown that:

$$D[\mathcal{T}_2] \times D[\mathcal{T}_1] = \mathcal{A}^{N, -\mathcal{T}_1 - \mathcal{T}_2}\, D[\mathcal{T}_{21}] , \tag{5.3.35}$$

as long as as both $(\mathcal{T}_1 + \mathcal{T}_2)$ and $\mathcal{T}_{21}$ are invertible in $\mathbb{Z}_N$, as suggested by (5.3.26).

The result could be confronted with the known composition of minimal TQFTs $\mathcal{A}^{N,p}$ [104], namely $\mathcal{A}^{N,p} \times \mathcal{A}^{N,q} = \mathcal{A}^{N, p+q} \times \mathcal{A}^{N, (p^{-1} + q^{-1})^{-1}}$ valid when $\gcd(p + q, N) = 1$. While we found an equivalent expression for the decoupled lines on the right-hand side, the lines coupled to $\mathcal{B}$ fuse differently because of the bulk dynamics.

Let us mention two cases in which the decomposition (5.3.34) fails. One case is when $\mathcal{T}_1 + \mathcal{T}_2 = 0$, namely when we consider the fusion $D[\mathcal{T}] \times D[-\mathcal{T}]$ in the untwisted sector. Set $\mathcal{T}_2 = -\mathcal{T}_1 = \mathcal{T}$. The lines decoupled from $\mathcal{B}$ have vanishing spin and form a Lagrangian subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$, signaling that $\mathcal{A}^{N, -\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N, -\mathcal{T}_1}$ must be a Dijkgraaf-Witten theory. Indeed, exploiting (5.3.33), we can exhibit the set of lines $E_n = \big(\begin{smallmatrix} \mathbb{1} \\ -\mathbb{1} \end{smallmatrix}\big) \big(\mathcal{T} - \frac{\epsilon}{2}\big)^{-1} n$ decoupled from $\mathcal{B}$ and with vanishing spin, a set of lines $M_m = (m, 0)$ with charge m under $\mathcal{B}$ and with spin $\exp\big(\frac{\pi i}{N} m^{\mathsf{T}} \mathcal{T} m\big)$, and show that the two sets have canonical braiding $\exp\big(\frac{2\pi i}{N} n^{\mathsf{T}} m\big)$. This is precisely the content of the theory (5.3.29). We thus reproduce the result (5.3.30).

Another special case is when $\mathcal{T}_{21} = 0$, namely when we consider two defects $V[\mathcal{T}_2]$, $V[\mathcal{T}_1]$ that fuse into the charge-conjugation defect $V_C \equiv V[\mathcal{T} = 0]$. The two torsion matrices must be related by $\mathcal{T}_1 = -\frac{\epsilon}{2} \mathcal{T}_2^{-1} \frac{\epsilon}{2}$. When this happens, the product $\mathcal{A}^{N, -\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N, -\mathcal{T}_1}$ is not a MTC because it contains a subcategory of transparent lines — the lines in the C twisted sector which couple only to $\mathcal{B}$ and not to Φ. In this case, we can study the fusion of the full twisted sectors, including the lines coupled to Φ. The final result is:

$$D[\mathcal{T}_1] \times D[\mathcal{T}_2] = \big(\mathbb{Z}_N \times \mathbb{Z}_N\big)_0 (\Phi, \Phi_1)\; D[0] . \tag{5.3.36}$$

We can compare this result with our standard computation by using the factorization:[23]

$$\big(\mathbb{Z}_N \times \mathbb{Z}_N\big)_0 (\Phi, \Phi_1) = \mathcal{A}^{N, -\mathcal{T}_1 - \mathcal{T}_2} \times \mathcal{A}^{N, \mathcal{T}_1 + \mathcal{T}_2} (\Phi, \Phi_1) . \tag{5.3.37}$$

[23]Note that $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 - \frac{\epsilon}{2} \mathcal{T}_1^{-1} \frac{\epsilon}{2} = \big(\mathcal{T}_1 + \frac{\epsilon}{2}\big) \mathcal{T}_1^{-1} \big(\mathcal{T}_1 - \frac{\epsilon}{2}\big)$ which is invertible under our assumptions.

Thus the result coincides with the remaining cases if we discard the term coupling to Φ_1 (which is integrated out in the 4d bulk computation).

Summarizing, we have obtained the following bulk fusion rules:

$$\begin{aligned} U(\gamma) \times D[\mathcal{T}] &= e^{iQ_N\mathcal{T}(\Gamma_\gamma)}\, D[\mathcal{T}] \;, \\ D[\mathcal{T}_2] \times D[\mathcal{T}_1] &= \mathcal{A}^{N,\,-\mathcal{T}_1-\mathcal{T}_2}\, D[\mathcal{T}_{21}] \;, \\ D[\mathcal{T}] \times \overline{D}[\mathcal{T}] &= \mathcal{C}^{\mathbb{Z}_N\times\mathbb{Z}_N} \;. \end{aligned} \tag{5.3.38}$$

We now extend our analysis to the physically relevant case of fusion on a gapped boundary.

5.3.3 Fusion on gapped boundaries

In Section 5.2 we discussed gapped boundaries $\rho(\mathcal{L})$ of the bulk 5d theory. These are defined by choosing a Lagrangian subgroup $\mathcal{L} \subset \mathcal{D}_Q \equiv \mathbb{Z}_N \times \mathbb{Z}_N$. The boundary condition sets to 1 the surface operators U_n with $n \in \mathcal{L}$, which are then screened on the boundary:

$$U_n\big|_X = 1 \qquad \text{if } n \in \mathcal{L} \;. \tag{5.3.39}$$

In terms of fields, one imposes Dirichelet boundary conditions $l^\mathsf{T}\mathcal{B}\big|_X = 0$ (up to gauge transformations) for all $l \in \mathcal{L}$. For N odd prime, the $N+1$ Lagrangian subgroups of $\mathcal{D}_Q$ are all isomorphic to $\mathbb{Z}_N$ and are generated by a single vector l. Thus the gapped boundaries are implemented by

$$l^\mathsf{T}\mathcal{B}\big|_X \equiv b_l = 0 \qquad \text{(up to gauge transformations)} \;. \tag{5.3.40}$$

In Section 5.2 we introduced the 1-form symmetry group $\mathcal{S} = \mathcal{D}_Q/\mathcal{L}$ of the gapped boundary. Here we also introduce the lattice $\mathcal{L}_\perp$ dual to $\mathcal{L}$ with respect to the Dirac pairing, and the vector $l_\perp = \epsilon\, l$ that generates $\mathcal{L}_\perp$. It satisfies $l_\perp^\mathsf{T} l = 0 \bmod N$.[24] Using this vector we can solve the boundary conditions by setting:

$$\mathcal{B}\big|_X = \tilde{b}_\perp\, l_\perp \;. \tag{5.3.41}$$

Notice that this is a condition on the field and not on the charges.

In this section we want to understand the fate of various types of defects once they are placed on the gapped boundary, or when they terminate on it. We already discussed the case of the 2d surfaces U_p with $p \in \mathcal{L}$: they can terminate on the gapped boundary, and become trivial if they are placed on top of it. On the other hand, if we fuse a 4d defect V_M (implementing the action of $M \in SL(2,\mathbb{Z}_N)$ on the gauge field $\mathcal{B}$) with a gapped boundary $\rho(\mathcal{L})$ we obtain a new gapped boundary $\rho(M\mathcal{L})$.

Let us now discuss the properties of the twist defects $D[\mathcal{T}]$ on a gapped boundary. Focusing on the case that $V[\mathcal{T}]$ comes from the condensation of $\mathbb{Z}_N \times \mathbb{Z}_N$ and that $\mathcal{T}$ is invertible in $\mathbb{Z}_N$, in Section 5.3.1 we discussed the 3d sector $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ of lines W_n on $D[\mathcal{T}]$ that are decoupled from $V[\mathcal{T}]$ but that cancel its anomaly (5.3.11). Those lines are charged under $\mathcal{B}$, and thus are the endlines of surfaces U_n in the fully dynamical theory. The subsector of lines $W_{n=sl}$ ($s \in \mathbb{Z}_N$) with charge proportional to l are attached to 2d surfaces of b_l, and form a consistent MTC $\mathcal{A}^{N,-t_l}$ for a 1-form symmetry $\mathcal{L} \cong \mathbb{Z}_N$, where $t_l \in \mathbb{Z}_N$ is

$$t_l = l^\mathsf{T}\mathcal{T} l \;, \tag{5.3.42}$$

[24] The lattice $\mathcal{L}$ could be self-dual, in which case $l_\perp = s\,l$ for some $s \in \mathbb{Z}$. In particular, for N prime, the self-dual lattices are in one-to-one correspondence with the roots $s^2 = -1$ and are generated by $l = (1, s)$.

provided that $t_l \neq 0$, namely, that the boundary $\rho(\mathcal{L})$ is *not* invariant under $V[\mathcal{T}]$.[25] (The case that $\mathcal{L}$ is invariant under $V[\mathcal{T}]$ will be dealt with in Section 5.3.3.) On the gapped boundary we set $b_l = 0$ (up to gauge transformations), therefore this sector becomes a decoupled TQFT. This allows us to define a minimal boundary twist defect $D_{\mathcal{L}}[\mathcal{T}]$, obtained by discarding the decoupled TQFT $\mathcal{A}^{N,-t_l}$.[26] The lines that braid trivially with $\mathcal{A}^{N,-t_l}$ can be generated by $n = \mathcal{T}^{-1} l_\perp$ and form a MTC $\mathcal{A}^{N,-t_\perp}$ with $t_\perp \in \mathbb{Z}_N$ defined as

$$t_\perp = l_\perp^\intercal \mathcal{T}^{-1} l_\perp \ . \tag{5.3.43}$$

These lines are coupled to the gauge field $b_\perp \equiv l_\perp^\intercal \mathcal{T}^{-1}\mathcal{B}$.[27] (We omit the dependence of $t_\perp$ and $b_\perp$ on $\mathcal{T}$ in order not to clutter.) We have then proved the factorization:

$$\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) = \mathcal{A}^{N,-t_l}(b_l) \times \mathcal{A}^{N,-t_\perp}(b_\perp) \ . \tag{5.3.44}$$

When we move the twist defect $D[\mathcal{T}]$ on top of a gap boundary, the first factor on the r.h.s. decouples yielding $D[\mathcal{T}]\big|_{\text{boundary}} = \mathcal{A}^{N,-t_l} \times D_{\mathcal{L}}[\mathcal{T}]$. We obtain:

$$D_{\mathcal{L}}[\mathcal{T}] = \mathcal{A}^{N,-t_\perp}(b_\perp) \qquad\qquad \text{for } M\mathcal{L} \neq \mathcal{L} \ . \tag{5.3.45}$$

Notice that $M\mathcal{L} = \mathcal{L}$ if and only if $\mathcal{T}\mathcal{L} = \mathcal{L}_\perp$ (see footnote 25). As we will see, this definition of $D_{\mathcal{L}}[\mathcal{T}]$ is consistent under fusion. Notice also that the twist defect $D_{\mathcal{L}}[\mathcal{T}]$, as opposed to $D[\mathcal{T}]$, is stuck on the gapped boundary.

As a check of (5.3.44), one can take the anomaly inflow action (5.3.11) and impose the boundary condition $b_l = 0$. This can be done by parametrizing a gauge field in the quotient group as $\mathcal{B} = t_\perp^{-1} b_\perp l_\perp$, which yields:

$$I(b_\perp) = \frac{N}{2\pi} \int_{4\text{d}} \left[b_\perp d\widetilde{\gamma} - \frac{1}{2}\, b_\perp t_\perp^{-1} b_\perp \right] \tag{5.3.46}$$

as expected (here $\widetilde{\gamma} = t_\perp^{-1} l_\perp^\intercal \widetilde{\Gamma}$). Thus, the theory $D_{\mathcal{L}}[\mathcal{T}]$ is the minimal one required to cancel the anomaly on the gapped boundary.

In order to compute the fusion $D_{\mathcal{L}}[\mathcal{T}_2] \times D_{\mathcal{L}}[\mathcal{T}_1]$ on a gapped boundary, we need to understand how to impose the boundary condition on the product theory $\mathcal{A}^{N,-\mathcal{T}_2} \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}$. Following our previous reasoning, the lines $W^{(1)}_{s_1 l}$ and $W^{(2)}_{s_2 l}$ with charges in $\mathcal{L}$ are the endlines of surfaces of b_l but decouple from $\mathcal{B}$ on the boundary. Since they are all in the same Lagrangian subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$, the two groups maintain trivial mutual braiding even after the deformation by $\mathcal{B}$. We have thus identified a subset of lines that couple to b_l and form the MTC $\mathcal{A}^{N,-t_{2,l}}(b_l) \times \mathcal{A}^{N,-t_{1,l}}(b_l)$, where $t_{j,l} = l^\intercal \mathcal{T}_j l$. The lines $W_{\mathcal{N}}$ that braid trivially with that subset, as we will see, form a MTC $\mathcal{A}^{N,-\mathcal{R}_{21}}$ coupled to $\mathcal{B}$ for some matrix $\mathcal{R}_{21}$:

$$\mathcal{A}^{N,-\mathcal{T}_2}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}(\mathcal{B}) = \mathcal{A}^{N,-t_{2,l}}(b_l) \times \mathcal{A}^{N,-t_{1,l}}(b_l) \times \mathcal{A}^{N,-\mathcal{R}_{21}}(\mathcal{B}) \ . \tag{5.3.47}$$

On the gapped boundary, the first two factors on the right-hand side decouple and moreover are precisely the two factors that are discarded in the definition of $D_{\mathcal{L}}[\mathcal{T}_1]$ and $D_{\mathcal{L}}[\mathcal{T}_2]$. After imposing $b_l\big|_X = 0$, the third factor only couples to a projection of $\mathcal{B}$. As we will see, such a projection is the very

[25] If $\mathcal{L}$ is invariant under M, then $Ml = sl$ for some $s \in \mathbb{Z}_N$. Note that $s \neq 0$, and since we are considering here defects V_M such that $\operatorname{Tr} M \neq 2 \bmod N$, then $s \neq 1$. From (5.2.29) one finds $l^\intercal \mathcal{T} l = \frac{1+s}{1-s}\, l^\intercal \frac{\epsilon}{2} l = 0$. On the contrary, if $l^\intercal \mathcal{T} l = 0$ then $\mathcal{T} l = r\, \frac{\epsilon}{2} l$ for some $r \in \mathbb{Z}_N$ and here $r \neq -1$. From (5.2.28) one finds $Ml = \frac{r-1}{r+1}\, l$. This shows that $\mathcal{L}$ is invariant under M if and only if $t_l = 0$. Besides, when $\mathcal{T}$ is invertible and thus $\operatorname{Tr} M \neq -2 \bmod N$, then a similar argument also shows an if and only if $t_\perp = 0$.

[26] The operation of discarding $\mathcal{A}^{N,-t_l}$ can be implemented as $D_{\mathcal{L}} = \left[D\big|_{\text{boundary}} \times \mathcal{A}^{N,t_l} \right] / \mathbb{Z}_N$ [111].

[27] The splitting of $\mathcal{B}$ into b_l and $b_\perp$ is well defined as long as the boundary is not invariant under $V[\mathcal{T}]$. Otherwise, $\mathcal{T} l \propto l_\perp$ and so $b_\perp \propto b_l$ which vanishes on the gapped boundary.

one predicted by fusion, namely to $b_\perp = l_\perp^\intercal \mathcal{T}_{21}^{-1}\mathcal{B}$. Besides, we expect the MTC $\mathcal{A}^{N,-\mathcal{R}_{21}}(b_\perp)$ to be the product of a MTC $\mathcal{N}_{21}$ that does not couple to $b_\perp$, and the MTC $\mathcal{A}^{N,-t_{21}^\perp}(b_\perp)$ (where $t_{21}^\perp = l_\perp^\intercal \mathcal{T}_{21}^{-1} l_\perp$) that lives on the twisted sector $D_\mathcal{L}[\mathcal{T}_{21}]$. We will verify this expectation, and show that

$$\mathcal{A}^{N,-\mathcal{R}_{21}}(b_\perp) = \mathcal{N}_{21} \times \mathcal{A}^{N,-t_{21}^\perp}(b_\perp) \;. \tag{5.3.48}$$

These relations imply the fusion rules

$$D_\mathcal{L}[\mathcal{T}_2] \times D_\mathcal{L}[\mathcal{T}_1] = \mathcal{N}_{21}\; D_\mathcal{L}[\mathcal{T}_{21}] \;, \tag{5.3.49}$$

where the decoupled TQFT $\mathcal{N}_{21}$ plays the role of a fusion coefficient.

Let us compute $\mathcal{N}_{21}$. The N^2 lines $W_\mathcal{N}$ of $\mathcal{A}^{N,-\mathcal{R}_{21}}$, that braid trivially with the first two factors on the r.h.s. of (5.3.47), have charges $\mathcal{N} = (\xi_1, \xi_2)$ determined by solving the equations

$$\begin{aligned} \mathcal{T}_1\, \xi_1 - \frac{\epsilon}{2}\, \xi_2 &= a_1\, l_\perp \\ \mathcal{T}_2\, \xi_2 + \frac{\epsilon}{2}\, \xi_1 &= a_2\, l_\perp \end{aligned} \tag{5.3.50}$$

for some coefficients $a_{1,2} \in \mathbb{Z}_N$ that depend on the line. In fact, one can use a_1, a_2 to parametrize the solutions. We first consider the simple case $\mathcal{T}_1 = \mathcal{T}_2$, then the generic case, and finally the exceptional case $\mathcal{T}_1 = -\mathcal{T}_2$.

Case $\mathcal{T}_1 = \mathcal{T}_2 \equiv \mathcal{T}$. This case computes the square of a defect $D_\mathcal{L}[\mathcal{T}]$. Noticing from (5.2.38) that $\mathcal{T}_{21} = \frac{1}{2}\big(\mathcal{T} + \frac{\epsilon}{2}\mathcal{T}^{-1}\frac{\epsilon}{2}\big)$, we find:

$$\xi_1 = \frac{1}{2}\, \mathcal{T}_{21}^{-1}\Big(a_1 + a_2\, \frac{\epsilon}{2}\, \mathcal{T}^{-1}\Big) l_\perp \;, \qquad \xi_2 = \frac{1}{2}\, \mathcal{T}_{21}^{-1}\Big(a_2 - a_1\, \frac{\epsilon}{2}\, \mathcal{T}^{-1}\Big) l_\perp \;. \tag{5.3.51}$$

The charge of a line under $\mathcal{B}$ is $\xi_1 + \xi_2$. One can check that the lines with $a_1 = a_2$ have charge proportional to $\mathcal{T}_{21}^{-1} l_\perp$, and so they couple to $b_\perp$. With some algebra[28] and (5.3.33), one can check that those lines braid trivially with the lines with $a_1 = -a_2$. This suggests to label the lines in terms of $a, c \in \mathbb{Z}_N$ and set $a_1 = a - c$, $a_2 = a + c$. The spin of a line labelled by (a,c) is found to be

$$\theta\big[W_{(a,c)}\big] = \exp\bigg(-\frac{\pi i}{N}\, t_{21}^\perp\, \big(a^2 + c^2\big)\bigg) \;, \tag{5.3.52}$$

where $t_{21}^\perp = l_\perp^\intercal \mathcal{T}_{21}^{-1} l_\perp$. As long as $t_{21}^\perp \neq 0$, such lines form the theory $\mathcal{A}^{N,-\mathcal{R}_{21}}$ with

$$\mathcal{R}_{21} = \begin{pmatrix} t_{21}^\perp & 0 \\ 0 & t_{21}^\perp \end{pmatrix} \;. \tag{5.3.53}$$

The subset of lines $(a,0)$ form the MTC $\mathcal{A}^{N,-t_{21}^\perp}(b_\perp)$, as expected. The lines $(0,c)$ have charges under $\mathcal{B}$ proportional to $\mathcal{T}_{21}^{-1}\epsilon\mathcal{T}^{-1} l_\perp$, which has vanishing contraction with $l_\perp^\intercal$ and thus is proportional to l. On the gapped boundary $b_l = 0$ and hence these lines form a decoupled MTC

$$\mathcal{N}_{21} = \mathcal{A}^{N,-t_{21}^\perp} \;. \tag{5.3.54}$$

We have obtained the fusion rule

$$D_\mathcal{L}[\mathcal{T}] \times D_\mathcal{L}[\mathcal{T}] = \mathcal{A}^{N,-t_{21}^\perp}\; D_\mathcal{L}[\mathcal{T}_{21}] \;. \tag{5.3.55}$$

Notice that this fusion rule is the same (with the same $\mathcal{N}_{21}$) on all gapped boundaries $\rho(\mathcal{L})$ belonging to the same orbit under $V[\mathcal{T}]$. This follows from footnote 18.

[28] One should use that $\mathcal{T}_{21}\epsilon\mathcal{T} = \mathcal{T}\epsilon\mathcal{T}_{21}$. It also implies that such a matrix is antisymmetric.

Generic case. In order to treat the general case it is convenient to parametrize the lines $(\xi_1,\xi_2) = (v, \eta - v)$ in terms of two vectors v, η, so that the charge of a line under $\mathcal{B}$ is η, and redefine the numbers $a_1 = p+q$, $a_2 = q$. The equations (5.3.50) become

$$\begin{aligned} \big(\mathcal{T}_1 + \mathcal{T}_2\big) v - \big(\mathcal{T}_2 + \tfrac{\epsilon}{2}\big)\eta &= p\, l_\perp \\ \mathcal{T}_2\, \eta - \big(\mathcal{T}_2 - \tfrac{\epsilon}{2}\big) v &= q\, l_\perp \;. \end{aligned} \tag{5.3.56}$$

Defining $\Gamma = \big(\mathcal{T}_2 - \frac{\epsilon}{2}\big)\big(\mathcal{T}_1 + \mathcal{T}_2\big)^{-1}$, the solutions are

$$v = q\, \Gamma^\mathsf{T} \mathcal{T}_{21}^{-1} l_\perp + p \Big[\big(\mathcal{T}_1 + \mathcal{T}_2\big)^{-1} + \Gamma^\mathsf{T} \mathcal{T}_{21}^{-1} \Gamma \Big] l_\perp \;, \qquad \eta = q\, \mathcal{T}_{21}^{-1} l_\perp + p\, \mathcal{T}_{21}^{-1} \Gamma l_\perp \tag{5.3.57}$$

and can be labelled by $q, p \in \mathbb{Z}_N$. Substituting in (5.3.33), the spins of the lines are

$$\theta\big[W_{(q,p)}\big] = \exp\bigg(- \frac{\pi i}{N}\, (q,p) \mathcal{R}_{21} \binom{q}{p} \bigg) \qquad \text{with} \qquad \mathcal{R}_{21} = \begin{pmatrix} t_{21}^\perp & c_\text{o} \\ c_\text{o} & c_\text{d} \end{pmatrix} \;, \tag{5.3.58}$$

where

$$c_\text{o} = l_\perp^\mathsf{T}\, \mathcal{T}_{21}^{-1} \Gamma\, l_\perp \;, \qquad c_\text{d} = l_\perp^\mathsf{T} \Big[\big(\mathcal{T}_1 + \mathcal{T}_2\big)^{-1} + \Gamma^\mathsf{T} \mathcal{T}_{21}^{-1} \Gamma \Big]\, l_\perp = l_\perp^\mathsf{T} \, \big(\mathcal{T}_1 + \tfrac{\epsilon}{2} \mathcal{T}_2^{-1} \tfrac{\epsilon}{2} \big)^{-1}\, l_\perp \;. \tag{5.3.59}$$

The subset of lines $(q,0)$ have charges η proportional to $\mathcal{T}_{21}^{-1} l_\perp$ and thus couple to $b_\perp$. Their spins show that they form the MTC $\mathcal{A}^{N,-t_{21}^\perp}(b_\perp)$. On the other hand, the subset of lines (q,p) with $q = -(t_{21}^\perp)^{-1} c_\text{o}\, p$ braid trivially with the former subset and constitute the theory $\mathcal{N}_{21}$. Their charges η are such that $l_\perp^\mathsf{T} \eta = 0$, therefore they are decoupled from $\mathcal{B}$ on the gapped boundary. Their spins show that

$$\mathcal{N}_{21} = \mathcal{A}^{N,-n_{21}} \qquad \text{with} \qquad n_{21} = c_\text{d} - (t_{21}^\perp)^{-1} c_\text{o}^2 = (t_{21}^\perp)^{-1} \det \mathcal{R}_{21} \;. \tag{5.3.60}$$

One should recall that, in the absence of a coupling to $\mathcal{B}$, the theories $\mathcal{A}^{N,-p}$ and $\mathcal{A}^{N,-pr^2}$ are equivalent for any invertible $r \in \mathbb{Z}_N$, and thus for N odd prime the only physical information in $n_{21} \neq 0$ is whether it is a quadratic residue or not. This is detected by the Legendre symbol $n_{21}^{(N-1)/2} \bmod N \in \{1,-1\}$.[29]

Case $\mathcal{T}_2 = -\mathcal{T}_1 \equiv \mathcal{T}$. This is the case leading to condensation. The equations for lines in $\mathcal{A}^{N,-\mathcal{R}_{21}}$ are just $\mathcal{T}\xi_1 + \frac{\epsilon}{2}\xi_2 = -a_1 l_\perp$ and $\mathcal{T}\xi_2 + \frac{\epsilon}{2}\xi_1 = a_2 l_\perp$. The general solution is

$$\xi_1 = \bigg[a \Big(\mathcal{T} + \frac{\epsilon}{2} \Big)^{-1} - c \Big(\mathcal{T} - \frac{\epsilon}{2} \Big)^{-1} \bigg]\, l_\perp \;, \qquad \xi_2 = \bigg[a \Big(\mathcal{T} + \frac{\epsilon}{2} \Big)^{-1} + c \Big(\mathcal{T} - \frac{\epsilon}{2} \Big)^{-1} \bigg]\, l_\perp \tag{5.3.61}$$

where we redefined $a_1 = c - a$ and $a_2 = c + a$. For these lines:

$$\theta\big[W_{(a,c)}\big] = \exp\bigg(- \frac{2\pi i}{N}\, ac\; 2 l_\perp^\mathsf{T} \Big(\mathcal{T} + \frac{\epsilon}{2} \Big)^{-1} l_\perp \bigg) \;. \tag{5.3.62}$$

Lines with either a or $c=0$ have vanishing spin, which indicates that we are dealing with a DW type theory. The lines with $a=0$ (electric) do not couple to $\mathcal{B}$ since they have $\xi_1 + \xi_2 = 0$. Redefining $a \to \big[2 l_\perp^\mathsf{T} \big(\mathcal{T} + \frac{\epsilon}{2}\big)^{-1} l_\perp \big]^{-1}$ gives the canonical braiding $B_{ac} = e^{\frac{2\pi i}{N} ac}$. Thus

$$\mathcal{A}^{N,-\mathcal{R}_{21}}(b_\perp) = (\mathbb{Z}_N)_0(b_\perp) = \mathcal{C}^{\mathbb{Z}_N} \;. \tag{5.3.63}$$

We conclude that:

$$D_\mathcal{L}[\mathcal{T}] \times \overline{D}_\mathcal{L}[\mathcal{T}] = \mathcal{C}^{\mathbb{Z}_N} \;. \tag{5.3.64}$$

The condensate $\mathcal{C}^{\mathbb{Z}_N}$ is for the 1-form symmetry $\mathcal{S} = (\mathbb{Z}_N \times \mathbb{Z}_N)/\mathcal{L} \cong \mathbb{Z}_N$ that exists on the gapped boundary.

[29] In the higher-rank case the situation is similar. For N odd prime, one can always bring a symmetric matrix $\mathcal{T}$ with values in $\mathbb{Z}_N$ to a diagonal form $U^\mathsf{T}\mathcal{T}U = \text{diag}(t_1,\ldots,t_r)$ using an invertible matrix U (see, *e.g.*, [137]). The TQFT is then characterized by the number of $+1$ and -1 Legendre symbols of the t_i's.

Examples. We can now apply our formalism to the known cases of duality and triality defects. We consider a generic boundary $\rho(\mathcal{L})$, but assume that it is not invariant under any symmetry defect appearing below (apart from C, which leaves every boundary invariant). For the application to self-duality defects, we must compute the fusion $D_{\mathcal{L}}[S] \times D_{\mathcal{L}}[S]$. This is a special case, since the r.h.s. involves charge conjugation. The complete fusion gives a coefficient which is a product of DW theories, these all admit a universal boundary condition which allows us to set them to one. This corresponds to the Dirichlet boundary of the DW theory. After this we find:

$$\begin{aligned} D_{\mathcal{L}}[S] \times D_{\mathcal{L}}[S] &= (\mathbb{Z}_N)(\tilde{b}_\perp, \phi_\perp)\ D_{\mathcal{L}}^{\text{triv}}[0] \\ D_{\mathcal{L}}[S] \times \overline{D}_{\mathcal{L}}[S] &= \mathcal{C}^{\mathbb{Z}_N}\ . \end{aligned} \tag{5.3.65}$$

For triality defects we compute:[30]

$$\begin{aligned} D_{\mathcal{L}}[ST] \times D_{\mathcal{L}}[ST] &= \mathcal{A}^{N,-p_{ST}}\ D_{\mathcal{L}}\left[(ST)^2\right] \\ D_{\mathcal{L}}[CST] \times D_{\mathcal{L}}[CST] &= \mathcal{A}^{N,-p_{ST}}\ D_{\mathcal{L}}\left[(ST)^2\right] \\ D_{\mathcal{L}}\left[(ST)^2\right] \times D_{\mathcal{L}}\left[(ST)^2\right] &= \mathcal{A}^{N,\,p_{ST}}\ D_{\mathcal{L}}[CST] \end{aligned} \tag{5.3.66}$$

where

$$p_{ST} = \begin{cases} 1 & \text{if } \mathcal{L} = \mathcal{L}(e)\ , \\ r^2 + r + 1 & \text{if } \mathcal{L} = \mathcal{L}(m)_r\ . \end{cases} \tag{5.3.67}$$

These fusions agree with those computed in [104] on the electric boundary.[31] Notice that $p_{ST} \neq 0 \bmod N$ as long as the boundary Lagrangian subgroup $\mathcal{L}$ is not invariant under ST.

Other defects. Another interesting case is when the 4d defects $V_M = V[\mathcal{A}, \xi]$ are obtained by condensing a $\mathbb{Z}_N$ subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$, corresponding to the elements $M \in SL(2, \mathbb{Z}_N)$ that are conjugate to T^k for some k. For simplicity let us consider the case $M = T^k$ with $k = 1, \ldots, N-1$. The twisted sectors $D_{T^k, \mathcal{L}}$ are described by the minimal theories $\mathcal{A}^{N,k^{-1}}(b)$ for $\mathbb{Z}_N$ coupled to the bulk field b. For the lines in these theories there is no extra contribution to the braiding when we stack the theories, and thus they fuse in the standard way:

$$D_{T^k, \mathcal{L}} \times D_{T^{k'}, \mathcal{L}} = \mathcal{A}^{N,\, k^{-1}+k'^{-1}}\ D_{T^{k+k'}, \mathcal{L}} \tag{5.3.68}$$

as long as $k + k' \neq 0 \bmod N$. This formula is in agreement with the fusion law of N-ality defects found in [25, 104]. Notice that these twist sectors are not unique since they can be fused to 3d condensates for the magnetic symmetry. However, on the magnetic boundaries $\mathcal{L}(m)$ (on which the twisted sector D_{T^k} hosts a minimal theory) we can take the condensates to be generated by the magnetic symmetry $l(m) \in \mathcal{L}(m)$.[32]. On the magnetic boundary these condensates however become all decoupled DW theories since $l(m)^{\mathsf{T}} \mathcal{B}\big|_X = 0$.

Twist defects and boundary-changing operators

Consider starting with a twist defect D_M (attached to a 4d symmetry defect V_M) in the bulk and moving it on top of a gapped boundary $\rho(\mathcal{L})$. We are here interested in the case that $\rho(\mathcal{L})$ is not

[30]To get to the result we use the property $\mathcal{A}^{N,pr^2} = \mathcal{A}^{N,p}$ for $\gcd(r, N) = 1$.

[31]One uses that $\mathcal{A}^{N,1} = U(1)_N$ [111]. Notice that the conventions of [104] defined in their eqns. (6.7)–(6.9) differ from ours, and their defects are the orientation reversal of ours, leading to a sign change in the level.

[32]We can always arrive at this choice since any two magnetic lattices differ by electric ones, which can be absorbed by D_{T^k}

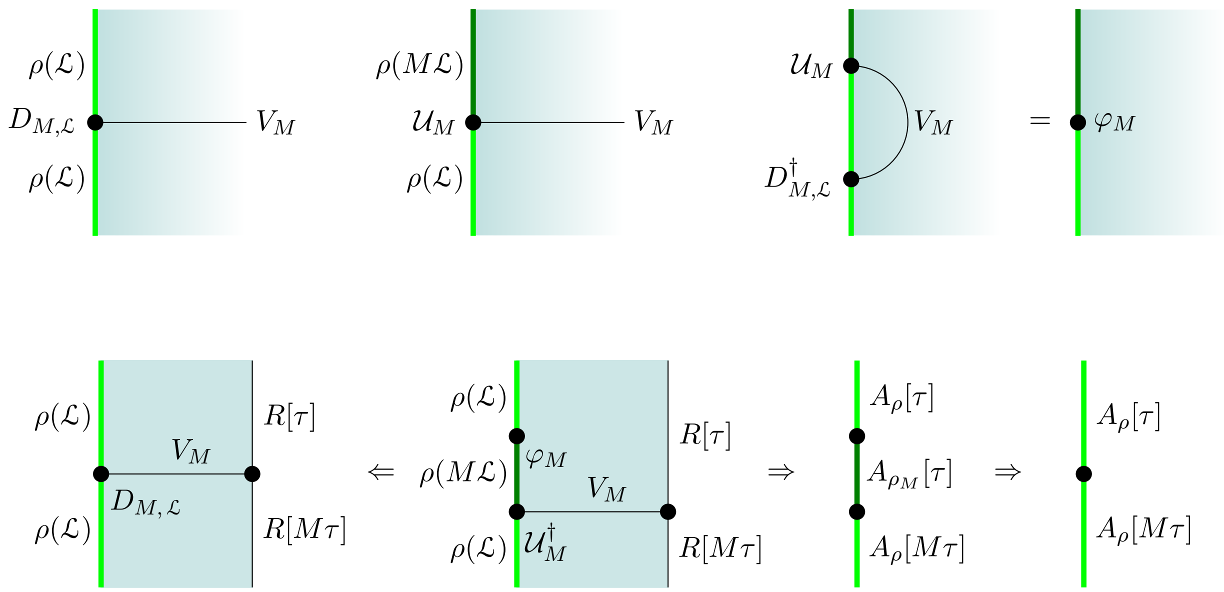


Figure 5.7: The way in which the SymTFT implements the construction of [24, 52]. Above: definition of various morphisms. Below: construction of the duality interface $D_{M,\mathcal{L}}$.

invariant under M. As discussed before (5.3.45), the defect D_M on the boundary decomposes into $D_{M,\mathcal{L}}$ and a decoupled TQFT. We conclude that $D_{M,\mathcal{L}}$ is an interface between two copies of $\rho(\mathcal{L})$, or using categorical terms, it defines a morphism $D_{M,\mathcal{L}} : M \times \rho(\mathcal{L}) \to \rho(\mathcal{L})$. This is depicted in Figure 5.7 left. On the other hand, if we bring the symmetry defect V_M on top of the boundary we obtain an action $V_M \times \rho(\mathcal{L}) = \rho(M\mathcal{L})$. Thus, we can construct an interface between $\rho(\mathcal{L})$ and $\rho(M\mathcal{L})$ by fusing the boundary with V_M only on a half-space and then letting V_M escape in the bulk, as in Figure 5.7 center. This defines a morphism $\mathcal{U}_M : M \times \rho(\mathcal{L}) \to \rho(M\mathcal{L})$. Since both interfaces sit at the end of a symmetry defect V_M, it is possible to define a local boundary-changing operator as the morphism $\varphi_M = \mathcal{U}_M \circ D^\dagger_{M,\mathcal{L}} : \rho(\mathcal{L}) \to \rho(M\mathcal{L})$, as in Figure 5.7 right.

Recall that, in the ungauged theory, one can expect to define only duality *interfaces*. The interface is a composite object given by a discrete gauging operation composed with an invertible duality transformation. On the TQFT side this is described by acting with $\mathcal{U}^\dagger_M$ to map the boundary to $\rho(M\mathcal{L})$ and then using φ_M to go back to $\rho(\mathcal{L})$. After compactifying the slab of SymTFT this gives an interface : $A_\rho[\tau] \to A_{\rho_M}[\tau] \to A_\rho[M\tau]$ between absolute theories. Shrinking the middle part of the drawing gives the duality interface. On the other hand the fusion $\varphi_M \times \mathcal{U}^\dagger_M = D_{M,\mathcal{L}}$ holds, since $D_{M,\mathcal{L}}$ is unique as a twist defect on the gapped boundary. We can thus identify the defect $D_{M,\mathcal{L}}$ on the bottom-left of Figure 5.7 with the duality interface in the absolute theory $A_\rho[\tau]$.

Boundaries with a stabilizer

Let us also discuss the properties of a twist defect $D_\mathcal{L}[\mathcal{T}]$ on a gapped boundary $\rho(\mathcal{L})$ that is invariant under the corresponding symmetry defect $V[\mathcal{T}]$. This means that M is in the stabilizer H of $\mathcal{L}$ in $SL(2,\mathbb{Z}_N)$. We can gather information on the degrees of freedom living on $D_\mathcal{L}[\mathcal{T}]$ by computing the anomaly inflow. The invertible TQFT living on $V[\mathcal{T}]$ is (5.3.11). On the gapped boundary we parametrize[33] $\mathcal{B} = \tilde{b}_\perp l_\perp$ and obtain

$$I_\mathcal{T}\big|_{\rho(\mathcal{L})} = \frac{N}{2\pi}\int\left[\tilde{\gamma}_\perp\, d\tilde{b}_\perp - \frac{1}{2}\, t_\perp\, \tilde{b}_\perp \tilde{b}_\perp\right] \tag{5.3.69}$$

[33]Here the normalization is different than before (5.3.46), because $t_\perp = 0$ when $\mathcal{L}$ is invariant under M.

(where $\tilde{\gamma}_\perp = l_\perp^\intercal \tilde{\Gamma}$). Since now $t_\perp = 0$ (see footnote 25), the anomaly is trivialized.

What happens to the lines in the twisted sector can be understood using the minimal theory description. We start with the twist defect $D[\mathcal{T}]$ in the bulk and push it onto an invariant boundary $\mathcal{L}$. Normally we would now separate the degrees of freedom which decouple on the boundary, which form a $\mathcal{A}^{N,-t_l}(b_l)$ factor. This is generated by lines $L_s \equiv W_{n=s\,l}$. If the boundary is invariant then $t_l = 0$ and this procedure is ill defined as the L_s all have vanishing spin. They thus form a Lagrangian subalgebra. This means that $\mathcal{A}^{N,-\mathcal{T}}$ should rather be thought of as a DW theory coupled to $\tilde{b}_\perp$. Since the lines with trivial spin are also uncharged under $\tilde{b}_\perp$ this can be thought of as a condensate:

$$\mathcal{A}^{N,\mathcal{T}}(\mathcal{B})\big|_X = \mathcal{C}^{\mathbb{Z}_N} \, . \tag{5.3.70}$$

To be more precise we can choose a generator u of $\mathcal{S}$. Since by definition $u^\intercal l_\perp \neq 0$ lines $\tilde{L}_r \equiv W_{n=ru}$ are charged under $\tilde{b}_\perp$. These lines have spin:

$$\theta\left[\tilde{L}_r\right] = \exp\left(-\frac{\pi i}{N}\, r^2\, u^\intercal \mathcal{T} u\right) \tag{5.3.71}$$

and braid with the electric lines L_s:

$$B_{r,s} = \exp\left(-\frac{2\pi i}{N}\, rs\, u^\intercal \mathcal{T} l\right) \neq 1\, , \tag{5.3.72}$$

since on the invariant boundary $\mathcal{T}\mathcal{L} = \mathcal{L}_\perp$. Properly redefining L_s we can make this braiding into canonical one. As we have already commented there is no canonical choice for u, since we are free to shift it by vectors in $\mathcal{L}$. The shift $u \to u + l$ does not affect the braiding with L_r but it does affect the spin of $\tilde{L}_s$:

$$\theta\left[\tilde{L}_r\right] \to \theta\left[\tilde{L}_r\right] \exp\left(-\frac{2\pi i}{N}\, r^2\, u^\intercal \mathcal{T} l\right) \tag{5.3.73}$$

For N odd and on spin manifolds we can use this to set $\theta[L_r]$ to one.

Since the defect $V[\mathcal{T}]$ has trivial anomaly on $\rho(\mathcal{L})$, it can end there without adding new degrees of freedom. Therefore the twist defect $D_\mathcal{L}[\mathcal{T}]$ is trivial (invertible) on an invariant boundary:

$$D[\mathcal{T}]\big|_X = \mathcal{C}^{\mathbb{Z}_N}\, D_\mathcal{L}^{\text{triv}}[\mathcal{T}]\, , \tag{5.3.74}$$

where the superscript is useful to remember this fact.

The same phenomenon appears if we consider a fusion $D_\mathcal{L}[\mathcal{T}_1] \times D_\mathcal{L}[\mathcal{T}_2]$ in which $V[\mathcal{T}_{21}]$ leaves the boundary $\rho(\mathcal{L})$ invariant, but neither $V[\mathcal{T}_1]$ nor $V[\mathcal{T}_2]$ do. We proceed as in the usual case by separating out the lines coupling to b_l from both terms in the fusion. This is a well defined procedure since t_1^l, $t_2^l \neq 0$ (due to $\mathcal{L}$ not being invariant under neither $\mathcal{T}_1$ nor $\mathcal{T}_2$). Based on the previous remarks we expect $\mathcal{A}^{N,-\mathcal{R}_{2,1}}$ to also be a condensate. It is clear that the theory contains a Lagrangian algebra generated by $W_{(q,0)}$ in (5.3.58). In the generic discussion these lines were coupled to $b_\perp$, however if the boundary is invariant they are not.[34] These form the set of "electric" lines. The magnetic lines $W_{(0,p)}$ instead couple to $\tilde{b}_\perp$, but have nontrivial spin:

$$\theta\left[W_{(0,p)}\right] = \exp\left(-\frac{\pi i}{N} c_\text{d}\, p^2\right)\, , \tag{5.3.75}$$

As before, we can redefine the magnetic lines by summing a multiple of the electric ones to set this to zero. Notice that the discussion here is also consistent with the example of $\mathcal{T}_2 = -\mathcal{T}_1$ discussed before, when the final result is a condensate and the identity defect leaves all boundaries invariant.

[34] The charge under the gauge symmetry for $\tilde{b}_\perp$ is $q l_\perp^\intercal \mathcal{T}_{2,1}^{-1} l_\perp$, which vanishes when the boundary is invariant.

We are now in a position to write down the full result of the boundary fusion for $D_{\mathcal{L}}[\mathcal{T}]$:

$$\begin{aligned} D_{\mathcal{L}}[\mathcal{T}_2] \times D_{\mathcal{L}}[\mathcal{T}_1] &= \mathcal{N}_{21}\, D_{\mathcal{L}}[\mathcal{T}_{21}]\,, && \text{if } V[\mathcal{T}_{2,1}]|\rho(\mathcal{L})\rangle \neq |\rho(\mathcal{L})\rangle\,, \\ D_{\mathcal{L}}[\mathcal{T}_2] \times D_{\mathcal{L}}[\mathcal{T}_1] &= \mathcal{C}^{\mathbb{Z}_N}\, D_{\mathcal{L}}^{\text{triv}}[\mathcal{T}]\,, && \text{if } V[\mathcal{T}_{2,1}]|\rho(\mathcal{L})\rangle = |\rho(\mathcal{L})\rangle\,, \\ D_{\mathcal{L}}[\mathcal{T}] \times \overline{D_{\mathcal{L}}[\mathcal{T}]} &= \mathcal{C}^{\mathbb{Z}_N}\,. \end{aligned} \tag{5.3.76}$$

This will have a more natural interpretation in the gauged theory. In that case we will see that anomaly cancellation forces the Gukov Witten operator GW$[\mathcal{T}]$ to exist only as a bound state with the twist defect $D[\mathcal{T}]$ for $V[\mathcal{T}]$. When the boundary $\mathcal{L}$ is $\mathcal{T}$-invariant there is no anomaly to cancel and GW$[\mathcal{T}]$ can exist as a genuine defect on the gapped boundary. The fusion rule above tell us that, when two bound operator fuse onto an invariant one, such fusion is always accompanied by the appearence of a condensation defect. This is consistent with the fact that defects $D_{\mathcal{L}}[\mathcal{T}]$ absorb surface defects $e^{i\int b_\perp}$, which survive on the gapped boundary. In the absence of the condensation defect the r.h.s. cannot absorb such lines and fusion would be inconsistent.

5.4 The gauged theory

Finally, we discuss the effect of gauging a discrete subgroup $G \subset SL(2, \mathbb{Z}_N)$ in the bulk TQFT. In the application to $\mathcal{N} = 4$ SYM, the only relevant groups (including the action of charge conjugation) are $\mathbb{Z}_4$ and $\mathbb{Z}_6$ generated by S and ST, respectively. Notice that they are both Abelian. The construction we present below applies to a generic Abelian G, while the non-Abelian case requires modifications that might be important in discussing theories of class S (we comment on that in the conclusions).

We will first describe abstractly the spectrum of operators in the gauged theory. We follow the rules for discrete gauging described for 3d MTCs in [77] and recently extended to higher dimensions in [27]. Particular care will be needed in describing the Gukov-Witten operators of G gauge theory, as they get dressed by the corresponding twist defects $D[\mathcal{T}]$. We will present the construction of these operators, that we dub $\mathfrak{D}[\mathcal{T}]$. Finally, we will study gapped boundaries $|\rho^*\rangle$ in the gauged theory in terms of orbits of boundaries $|\rho\rangle$ in the ungauged theory. This allows for a simple derivation of the fusion rules. We will also comment on the differences arising when the boundary has a nontrivial stabilizer.

In the following we will restrict to the study of twisted sectors $D[\mathcal{T}]$ for which $M(\mathcal{T})$ is an element of G. Together with the assumption that G is Abelian, this ensures that different twisted sectors do not mix among each other and that the genuine codimension-2 operators $\mathfrak{D}[\mathcal{T}]$ of the gauged theory are still labelled by group elements.[35]

5.4.1 Spectrum of bulk operators

The spectrum of topological operators in the gauged theory can be obtained, at least at a formal level, by applying standard rules for gauging a discrete 0-form symmetry to the ungauged theory. These are nicely summarized in [27]. Let us start with the surface defects U_n that implement the 2-form symmetry. These operators are in general not gauge invariant, as G acts on them nontrivially. We can build gauge-invariant combinations by considering orbits under G:

$$U^*_{[n]} = \frac{1}{\left|\text{Stab}(n)\right|} \sum_{g \in G} U(g\, n)\,, \tag{5.4.1}$$

[35] In the general case they are labelled by conjugacy classes under the adjoint action of G.

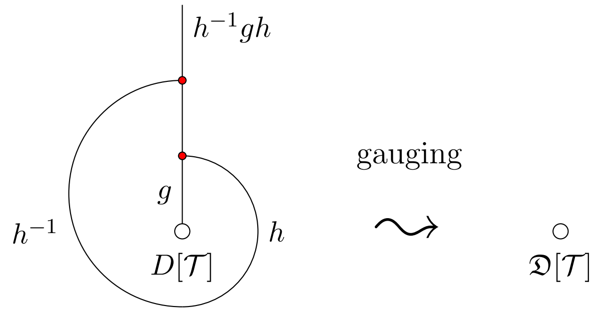


Figure 5.8: In the gauged theory, a twist defect $D[\mathcal{T}]$ (with $M(\mathcal{T}) \equiv g \in G$) is dressed by codimension-1 surfaces of G labelled by h.

where Stab(n) is the stabilizer group for n as an element of $\mathbb{Z}_N \times \mathbb{Z}_N$. When n admits a nontrivial stabilizer, the surface $U^*_{[n]}$ supports nontrivial line defects labelled by representations of Stab(n). In the cases considered here, that is N prime and $G = \mathbb{Z}_4$ or $\mathbb{Z}_6$, the only surface with a nontrivial stabilizer is the identity, while all others ones do not host any line.

As an example, in the case of the $\mathbb{Z}_4$ subgroup of $SL(2, \mathbb{Z}_N)$ generated by S, a dyon (e, m) is mapped to an orbit

$$[e, m] = (e, m) + (m, -e) + (-e, -m) + (-m, e) \ . \tag{5.4.2}$$

These objects are non-invertible and their fusion is

$$[e, m] \times [e', m'] = [e + e', m + m'] + [e + m', m - e'] + [e - e', m - m'] + [e - m', m + e'] \ . \tag{5.4.3}$$

More interesting is the situation for codimension-2 operators. We have already discussed that in the ungauged theory, genuine 3d operators are necessarily condensation defects. After the discrete gauging the situation is different. The twist defects $D[\mathcal{T}]$ for surfaces $V[\mathcal{T}]$ generating G become “liberated” — in the sense that they become genuine 3d operators — since the surfaces $V[\mathcal{T}]$ are transparent in the gauged theory. One could think of the liberated defects as arising from the“lassoed” configuration shown in Figure 5.8 after summing over G. Since G is Abelian, each twist sector is left fixed by the action of the lassos and it gives rise to a single genuine operator $\mathfrak{D}[\mathcal{T}]$.[36] The action of a lasso $V_{M'}$ reduces to a 0-form symmetry action on $D[\mathcal{T}]$, which maps $W_n \mapsto W_{M'n}$. This is indeed a symmetry of the theory, since

$$M'^{\mathsf{T}}\, \mathcal{T}\, M' = \mathcal{T} \tag{5.4.4}$$

and thus it preserves the braiding. Summing over such action means that the 0-form symmetry on the defect is gauged, so we would like to conclude that $\mathfrak{D}[\mathcal{T}]$ is $D[\mathcal{T}]/G$.

This description is slightly imprecise, because $D[\mathcal{T}]$ lives at the boundary of $V[\mathcal{T}]$. Indeed, the gauging process can be thought of as coupling the original system to a discrete G gauge theory. Its gauge field $a \in H^1(M_5, G)$ couples minimally to the 0-form symmetry defects $V_{M\in G}$ of the original theory (more details in Section 5.4.2). In this setup, inserting a twist defect $D[\mathcal{T}]$ is only consistent at locations where a is not closed: it must instead satisfy $\delta a = g$ schematically. Another way of saying this is that a exhibits a nontrivial holonomy g around the 3-cycle Y on which $D[\mathcal{T}]$ lies. This is the description of Gukov-Witten defect operators in G gauge theory, that we indicate as GW_g. We infer that a more precise definition of the new operators is:

$$\mathfrak{D}[\mathcal{T}] = \mathrm{GW}_{M(\mathcal{T})} \times D[\mathcal{T}]/G \ . \tag{5.4.5}$$

[36]When instead G is non-Abelian, twist defects also combine into orbits and the situation is more subtle.

The appearance of this "bound state" has a simple explanation: In the original theory, the defect $D[\mathcal{T}]$ was not gauge invariant due to anomaly inflow from $V[\mathcal{T}]$. The GW operator is not gauge invariant either, as it carries the anomaly of $V[\mathcal{T}]$. Their combination is a well defined operator in the gauged theory. This is a close cousin of the mechanism described in [25]. We also learn that $\mathfrak{D}[\mathcal{T}]$ is charged under the dual $\widehat{G}$ 3-form symmetry.

The exception is the twist defect $D[\mathcal{T}=0] \equiv D_C$ for charge conjugation. In this case there is no anomaly inflow and therefore the GW operator for C defines a genuine, group-like object in the gauged theory. This suggests that we should interpret the contributions from D_C arising upon fusion as decoupled condensates after gauging.

The following table summarizes the properties of some objects in the gauged theory:

Original object	Gauged object	Emergent lines	Grouplike?
$(0,0)$	$[0,0]$	$\mathrm{Rep}(G)$	YES
(e,m)	$[e,m] = \oplus_{g\in G}\, g(e,m)$	none	NO
$D[\mathcal{T}]$	$\mathfrak{D}[\mathcal{T}] = \mathrm{GW}_{M(\mathcal{T})} \times D[\mathcal{T}]/G$	$\mathrm{Rep}(G)$	NO
D_C	GW_C	none	YES

5.4.2 Hybrid formulation of the gauged theory

In order to give a Lagrangian description of the gauging of the subgroup $G \subset SL(2,\mathbb{Z}_N)$ in the 5d Chern-Simons theory, we employ a sort of hybrid formulation in which the Chern-Simons theory is described by continuum gauge fields, while the gauge field for G is described using singular cochains (see, *e.g.*, [93, 138] or the appendix in [15]).

First of all, on the spacetime manifold M_5 one chooses a simplicial triangulation. This is made of vertices or 0-simplices p_i with an arbitrary ordering for the index i, edges or 1-simplices p_{ij} (with $i<j$) connecting the vertices p_i and p_j, 2-simplices p_{ijk} (with $i<j<k$) bounded by p_{ij}, p_{jk} and p_{ik}, and so on. All simplices are contractible, and M_5 is the union of all 5-simplices. A gauge field a for the discrete gauge group G is a 1-cochain $a \in C^1(M_5,G)$ that assigns an element $a_{ij} \in G$ to each 1-simplex p_{ij} (with $i<j$), with the constraint that $da = \mathbb{1}$. We use multiplicative notation and define the differential as $(da)_{ijk} = a_{jk}a_{ik}^{-1}a_{ij}$ (with $i<j<k$). We will only consider the case that G is Abelian. Gauge transformations then map $a_{ij} \mapsto (d\lambda)_{ij}a_{ij}$ where $d\lambda_{ij} = \lambda_j\lambda_i^{-1}$ and $\lambda \in C^0(M_5,G)$ in a 0-cochain. The gauging of G is described by a sum over $a \in H^1(M_5,G)$ in cohomology.

Then we construct a covering of M_5 by closed patches that is dual to the triangulation, as follows. Each patch U_i is a 5d contractible manifold with boundary that contains the 0-simplex p_i. Then each non-empty intersection $U_{i_1\ldots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$ (with $i_1 < \ldots < i_k$ and $k = 2,\ldots,6$) is a $(6-k)$-dimensional contractible manifold with boundary that intersects the $(k-1)$-simplex $p_{i_1\ldots i_k}$ at one point. We give a graphical representation of this covering in Figure 5.9.

On every patch U_i we define gauge fields $\mathcal{B}_i$ with values in an Abelian group $\mathcal{A}$ (either continuous or discrete), and along the intersections U_{ij} we glue them using a group homomorphism $\theta: G \to \mathrm{Aut}(\mathcal{A})$ and the gauge field a:[37]

$$\mathcal{B}_i = \theta(a_{ij})\,\mathcal{B}_j \qquad \text{across } U_{ij} \,. \tag{5.4.6}$$

The gauge field $\mathcal{B}$ is thus a piecewise-smooth field with $\mathcal{B}\big|_{U_i} = \mathcal{B}_i$. Closeness of a guarantees that each $\mathcal{B}_i$ can be smooth and have a well-defined limit at triple intersections U_{ijk}. In particular, we can always find a gauge in which $a_{ij} = a_{jk} = a_{ik} = 1$ around a given triple intersection U_{ijk}, and in that gauge $\mathcal{B}$ can be smooth at the intersection.

[37]Besides, one could also have gauge transformations of $\mathcal{B}_i$, but we keep them implicit here.

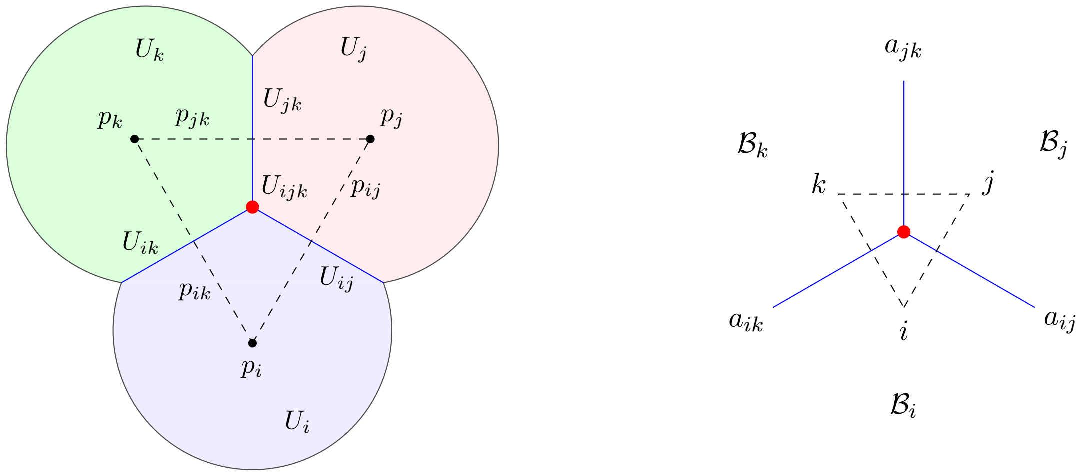


Figure 5.9: Left: representation of the simplicial triangulation and the covering by closed sets U_i, U_j, U_k near a triple intersection. Right: assignment of fields $\mathcal{B}_i$ and a_{ij}.

The construction is quite general. In our case $\mathcal{B}_i$ are continuous 2-form gauge fields valued in $\mathcal{A} = U(1)^2$, while $SL(2,\mathbb{Z})$ has the natural action on $\mathcal{A}$ and $G \subset SL(2,\mathbb{Z})$ is an Abelian subgroup. We should now understand how to construct the action. Integrating $S = \frac{N}{4\pi}\int \langle \mathcal{B},\, d\mathcal{B}\rangle$ with the discontinuous gluing conditions (5.4.6) leads to singularities, in particular the derivative $d\mathcal{B}$ has delta-function singularities along the surfaces U_{ij}. To remedy, we introduce a covariant derivative d_a that removes those singularities:

$$d_a\mathcal{B} = d\mathcal{B} - \sum_{U_{ij}} \delta^{(1)}(U_{ij})\big(\mathcal{B}_j - \mathcal{B}_i\big) = d\mathcal{B} - \sum_{U_{ij}} \delta^{(1)}(U_{ij})\, \sigma(a_{ij})\, \mathcal{B}_j \equiv \Big(d - \delta_a^{(1)}\sigma(a)\Big)\mathcal{B}\;, \tag{5.4.7}$$

where $\delta^{(1)}(U_{ij})$ is a delta-1-form, $\sigma(a) \equiv \mathbb{1} - \theta(a)$, and in the last expression we used a more compact notation. In this way, $d\mathcal{B}$ is a piecewise-smooth field such that $d\mathcal{B}\big|_{\mathring{U}_i} = d\mathcal{B}_i$ with discontinuities across U_{ij} but no delta-function singularities. We can then construct the action

$$S = \frac{N}{4\pi}\int \langle \mathcal{B}, d_a\mathcal{B}\rangle = \sum_{U_i} \frac{N}{4\pi}\int_{U_i} \langle \mathcal{B}_i, d\mathcal{B}_i\rangle \;. \tag{5.4.8}$$

The covariant derivative d_a can be integrated by parts, and the action is invariant under gauge transformations of a.

In order to discuss 1-form gauge transformations, we need to compute the square d_a^2 of the covariant derivative. It turns out that, to do that, we ought to be more careful and write $d_a\mathcal{B} = d\mathcal{B} - \sum_{U_{ij}} \delta^{(1)}(U_{ij})\big(\mathcal{B}_j^{(ij)} - \mathcal{B}_i^{(ij)}\big)$ where the label (ij) reminds us that we are taking the limit of $\mathcal{B}_i$ or $\mathcal{B}_j$ towards U_{ij}. Then $d_a d\mathcal{B} = -\sum_{U_{ij}} \delta^{(1)}(U_{ij})\big(d\mathcal{B}_j^{(ij)} - d\mathcal{B}_i^{(ij)}\big)$, and finally

$$\begin{aligned} d_a^2\, \mathcal{B} &= -\sum_{U_{ijk}} \delta^{(2)}(U_{ijk})\Big[\big(\mathcal{B}_j^{(ij)} - \mathcal{B}_i^{(ij)}\big) + \big(\mathcal{B}_k^{(jk)} - \mathcal{B}_j^{(jk)}\big) - \big(\mathcal{B}_k^{(ik)} - \mathcal{B}_i^{(ik)}\big)\Big] \\ &\equiv -\sum_{U_{ijk}} \delta^{(2)}(U_{ijk})\, \sigma(da_{ijk})\, \mathcal{B}\;. \end{aligned} \tag{5.4.9}$$

In the first equality we used that $d\big(\delta^{(1)}(U_{ij})\big) = \delta^{(2)}(\partial U_{ij})$ and that the boundary of a double intersection is a collection of triple intersections (with suitable signs due to orientations). In the second line we introduced a compact notation. Indeed, if a is closed ($da = \mathbb{1}$) then each $\mathcal{B}_i$ can be smooth and taking the limit towards U_{ijk} in each patch, the first line of (5.4.9) equals zero. If, instead, a is

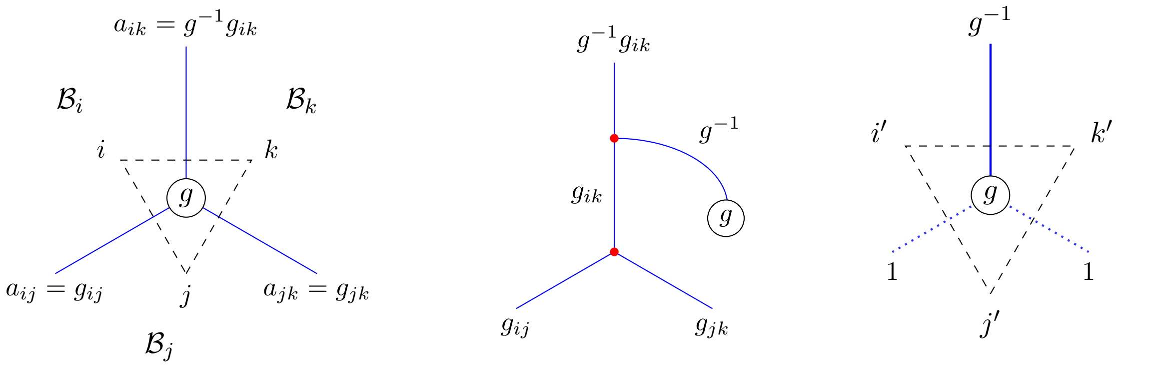


Figure 5.10: Left: A triple intersection U_{ijk} hosts a GW operator for $g \in G$. We parametrized the gauge field a in terms of $g_{ij}g_{jk} = g_{ik}$ and $da_{ijk} = g$. Center: A refinement of the triangulation such that the GW operator is pulled away from the junctions. Right: A zoom on the gauge field configuration around the isolated GW operator.

not closed, then the G bundle can have non-trivial holonomies around the triple intersections and the $\mathcal{B}_i$'s cannot be smooth there. Given $(da)_{ijk} = g$, consider a gauge in which $a_{ij} = a_{jk} = 1$, $a_{ik} = g^{-1}$ (see Figure 5.10 right). Then the contribution to (5.4.9) from U_{ijk} becomes $-\delta^{(2)}(U_{ijk})\,\sigma(da_{ijk})\,\mathcal{B}_i^{(ik)}$. We thus write the compact formula

$$d_a^2 = -\delta_a^{(2)}\,\sigma(da)\ . \tag{5.4.10}$$

In the presence of a background for a, 1-form gauge transformations of $\mathcal{B}$ become

$$\mathcal{B} \to \mathcal{B} + d_a\alpha\ , \tag{5.4.11}$$

and the action (5.4.8) remains gauge invariant as long as a is flat.

The theory in which G is gauged involves a sum over choices of a on double intersections U_{ij} that satisfy the closeness condition $da = \mathbb{1}$. A single symmetry defect $U(\gamma)$ in the ungauged theory is mapped to a sum over its G-orbit in the gauged theory. These are precisely the $[e,m]$ defect operators we introduced before.

On the other hand, we can introduce Gukow-Witten operators in the gauged theory [62]. These are codimension-2 disorder operators defined by a nontrivial holonomy $g \in G$ for a around a 3d submanifold γ'. In the hybrid formulation, such a GW operator displaced along a collection of triple intersections U_{ijk} is defined by a sum in the path integral over cochains a such that

$$da_{ijk} = g \qquad \text{whenever } U_{ijk} \subset \gamma'\ , \tag{5.4.12}$$

as in Figure 5.10. More generally, a collection of GW operators is described by an exact cochain $h \in C^2(M_5, G)$, and it prescribes to sum over cochains a with $da = h$ in the gauged theory. As mentioned above, requiring the $\mathcal{B}_i$'s to be smooth in their own patches in a neighborhood of a triple intersection, forces them to be invariant under g there.[38] This is a boundary condition naturally implemented on the GW operators, consistent with the fact that g-twisted sectors absorb the surfaces of $\mathcal{B}$ not stabilized by g.

Indeed, we can identify a double intersection U_{ij} with gauge field a_{ij} as an alternative description for the 4d symmetry defect V_M with $M = \theta(a_{ij})^\mathsf{T}$. This is already apparent if we compare the relation

[38] If $\mathcal{B}_i$ is smooth at U_{ijk}, then it has a well-define limit there. The limits in the three patches U_i, U_j, U_k are related by $\mathcal{B}_i = \theta(a_{ij})\,\mathcal{B}_j = \theta(a_{ik})\,\mathcal{B}_k$ and $\mathcal{B}_j = \theta(a_{jk})\,\mathcal{B}_k$. Recalling that G is Abelian, this implies $\theta(da_{ijk})\,\mathcal{B}_i = \mathcal{B}_i$ and similarly for $\mathcal{B}_j$ and $\mathcal{B}_k$.

$\mathcal{B}_i = \theta(a_{ij})\,\mathcal{B}_j$ between the fields on the two sides of the intersection and (5.2.53), but it also follows from the action. Let us rewrite (5.4.8) as

$$S = \frac{N}{4\pi}\int \langle \mathcal{B}, d\mathcal{B}\rangle - \frac{N}{4\pi}\sum_{U_{ij}}\int_{U_{ij}} \langle \mathcal{B}, \mathcal{B}_j - \mathcal{B}_i\rangle \ . \tag{5.4.13}$$

The first term imposes that $\mathcal{B}$ is a $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge field. When $\mathcal{T}(M)$ is invertible, we can identify the second term with the reduced defect action (5.2.51). Recall from the discussion in Section 5.2.3 that the relation between M and the torsion matrix $\mathcal{T}$ follows from determining the field on the defect $\mathcal{B}(0) = \frac{1}{2}(\mathcal{B}_{\rm L} + \mathcal{B}_{\rm R}) = -\mathcal{T}\Phi$, where $\epsilon\,\Phi = \mathcal{B}_{\rm R} - \mathcal{B}_{\rm L}$, in terms of the left/right fields $\mathcal{B}_{\rm L/R}$. Substituting $\mathcal{B}_{\rm R} - \mathcal{B}_{\rm L} = -\epsilon\,\mathcal{T}^{-1}\mathcal{B}(0)$ into (5.4.13), the second term becomes

$$-\frac{N}{4\pi}\sum_{U_{ij}}\int_{U_{ij}} \mathcal{B}^{\mathsf{T}}\mathcal{T}^{-1}\mathcal{B} \tag{5.4.14}$$

that reproduces (5.2.51).

To compute how a GW operator transforms under gauge transformations (5.4.11) we simply evaluate the variation of the action (5.4.8) on a non-closed gauge configuration as in (5.4.12):

$$\delta_\alpha S = -\sum_{U_{ijk}} \frac{N}{4\pi}\int_{U_{ijk}} \big\langle 2\mathcal{B} + d_a\alpha,\, \sigma(da_{ijk})\,\alpha\big\rangle \ . \tag{5.4.15}$$

In the gauge of Figure 5.10 right, as above, $\sigma(da_{ijk})\,\alpha = \alpha_i^{(ik)} - \alpha_k^{(ik)} = \epsilon\,\mathcal{T}^{-1}\alpha(0)$ in terms of the gauge transformation parameter on the defect. Substituting in (5.4.15) and using that the boundary conditions fix $\mathcal{B} = 0$ on the GW operator, we obtain

$$\delta_\alpha S = \sum_{U_{ijk}} \frac{N}{4\pi}\int_{U_{ijk}} \alpha^{\mathsf{T}}\mathcal{T}^{-1}d\alpha \ . \tag{5.4.16}$$

As in the description of Section 5.3 in terms of symmetry defects V_M, also in the hybrid formulation we find that pure GW operators are not gauge invariant in this theory. We can construct gauge-invariant operators by dressing the GW operators with the twisted sectors $D[\mathcal{T}]$, whose variation (5.3.12) is opposite to (5.4.16).

In the case of the symmetry defect V_C, the field on the defect is simply $\mathcal{B}(0) = 0$ and thus its gauge transformation parameter $\alpha(0)$ vanishes as well. This means that the gauge variation (5.4.15) vanishes and the GW operator for C is a well-defined gauge-invariant (invertible) topological operator in the gauged theory.

5.4.3 Gapped boundaries and non-invertible fusion rules

We consider now gapped boundaries in the gauged theory. We can use to our advantage the study and classification we already did in the ungauged theory. In order to construct a gauged boundary $|\rho^*\rangle$, we proceed in two steps. First we take a boundary $|\rho\rangle$ in the ungauged theory and make it invariant under the G action:

$$|\rho\rangle \quad\to\quad \frac{1}{\big|\mathrm{Stab}(\rho)\big|}\sum_{g\in G}|\rho_g\rangle \ . \tag{5.4.17}$$

As long as G is Abelian, we can associate a stabilizer $H \subset G$ in a consistent way also to the gauged boundary $|\rho^*\rangle$, since $\mathrm{Stab}(\rho_g) = \mathrm{Stab}(\rho)$. This does not specify a boundary condition completely, since it does not prescribe boundary conditions for neither the $\mathrm{Rep}(G)$ dual symmetry lines, nor the codimension-2 defects $\mathfrak{D}[\mathcal{T}]$. They form a canonically-conjugated pair of variables, since they braid nontrivially. Therefore, the second step is to choose boundary conditions for them. We choose to impose Dirichlet boundary conditions on a:[39]

$$|\mathrm{Dir}\rangle : \quad a = 0 \, . \tag{5.4.18}$$

Then the operators $\mathfrak{D}[\mathcal{T}]$ still exist on the gapped boundary as confined excitations.

These are not the only meaningful boundary conditions one could consider. Indeed it would be interesting to understand the effect of Dirichlet boundary conditions on the $\mathfrak{D}[\mathcal{T}]$'s, or of mixed ones. That they might be useful to describe theories in which either charge conjugation C (this has been studied, *e.g.*, in [27, 139, 140]) or the full categorical symmetry, are gauged. We hope to come back to these questions in the future.[40]

With Dirichlet boundary conditions on a, we define:

$$|\rho^*\rangle = \frac{1}{\big|\mathrm{Stab}(\rho)\big|} \left(\sum_{g \in G} |\rho_g\rangle \right) \times |\mathrm{Dir}\rangle \, . \tag{5.4.20}$$

The Dirichlet boundary condition on a greatly simplifies the discussion. The operators $\mathfrak{D}[\mathcal{T}]$, which away from the boundary host a $\mathrm{Rep}(G)$ worth of lines constructed with the gauge field a, on the gapped boundary reduce to a direct product $\mathrm{GW}_{M(\mathcal{T})} \times D[\mathcal{T}]$. The Gukov-Witten operators still exist on $|\rho^*\rangle$ and have group-like fusion. We will now show that the fusion of the twist operators $D_{\mathcal{L}}[\mathcal{T}]$ is the same on each gapped boundary $|\rho_g\rangle$ in the $|\rho^*\rangle$ orbit. This allows to use the results already derived for the boundary fusion.

We need to show that the various minimal theories we constructed in Section 5.3.3 in order to study the fusion of twist defects, are isomorphic for boundaries in the same G-orbit. Let $M_g \in G \subset SL(2, \mathbb{Z}_N)$ be a generator of G, and $M(\mathcal{T})$ be the element of G associated to the twist defect $D[\mathcal{T}]$ we want to study. Let $\big|\rho(\mathcal{L})\big\rangle$ be a gapped boundary defined by the Lagrangian subgroup $\mathcal{L}$ with generator l. The Lagrangian subgroup of $|\rho_g\rangle$ is $M_g\mathcal{L}$, and since $\mathcal{L}^{\mathsf{T}}\mathcal{L}_\perp = 0$, we have

$$\mathcal{L}_\perp^g = M_g^{-1\mathsf{T}}\, \mathcal{L}_\perp \, . \tag{5.4.21}$$

The generators l and $l_\perp$ transform in a similar way. Since G is Abelian and $\epsilon M_g = M_g^{-1\mathsf{T}}\epsilon$, then $\mathcal{T} = M_g^{\mathsf{T}} \mathcal{T} M_g$ and so both t_l and $t_\perp$ are invariant along the orbit. Besides, $\Gamma M_g = M_g \Gamma$ and thus the theory $\mathcal{R}_{21}$ is invariant as well. Since all relevant building blocks are isomorphic on boundaries that sit inside the same G-orbit, we conclude that fusion only depends on the orbit $|\rho^*\rangle$.

A new ingredient appears when fusion produces a defect $\mathcal{D}[\mathcal{T}_{21}]$ such that $M(\mathcal{T}_{21})$ stabilizes $|\rho\rangle$. As we discussed, in these cases the minimal theory is replaced by a condensate. After gauging G, we are left with the GW operator $\mathrm{GW}_{M(\mathcal{T}_{21})}$.

Using all of the above, we finally obtain the categorical fusion rules in the boundary theory specified

[39] This is the same choice made in the holographic setup of [133].

[40] In the same spirit, we could consider boundaries twisted by the dual $\check{G}$ symmetry. This amounts to choosing a representation α of G and define, for a boundary with a trivial stabilizer,

$$|\rho^*_\alpha\rangle = \textstyle\sum_{g \in G} \chi_\alpha(g)\, |\rho_g\rangle \times |\mathrm{Dir}\rangle \, . \tag{5.4.19}$$

These boundaries have vanishing overlap with the relative theory if we assume absolute theories in the same orbit to have the same partition function. When a stabilizer is present we can only twist by characters of $G/\mathrm{Stab}(\rho)$, while boundaries split into copies labelled by representations of $\mathrm{Stab}(\rho)$. We do not know how to interpret these splitted boundaries from the point of view of the 4d QFT, thus we only consider the ones labelled by the trivial representation.

by the gapped boundary $|\rho^*\rangle$:

$$\begin{aligned} \mathfrak{D}_{\rho^*}[\mathcal{T}_2] \times \mathfrak{D}_{\rho^*}[\mathcal{T}_1] &= \mathcal{N}_{21}\, \mathfrak{D}_{\rho^*}[\mathcal{T}_{21}] && M_{21} \notin \mathrm{Stab}(\rho^*)\ , \\ \mathfrak{D}_{\rho^*}[\mathcal{T}_2] \times \mathfrak{D}_{\rho^*}[\mathcal{T}_1] &= \mathcal{C}^{\mathbb{Z}_N}\ \mathrm{GW}_{M(\mathcal{T}_{21})} && M_{21} \in \mathrm{Stab}(\rho^*)\ , \\ \mathrm{GW}_{M(\mathcal{T}_2)} \times \mathrm{GW}_{M(\mathcal{T}_1)} &= \mathrm{GW}_{M(\mathcal{T}_{21})} && M_1, M_2 \in \mathrm{Stab}(\rho^*)\ . \end{aligned} \tag{5.4.22}$$

In the second line, the condensate is for the 1-form symmetry $\mathcal{S}$ on the gapped boundary, and the DW description couples to $\tilde{b}_\perp$. For defects in which only one $\mathbb{Z}_N$ factor is gauged, on the other hand, the fusions are as follows:[41]

$$\begin{aligned} \mathfrak{D}_{T^k,\,\rho^*} \times \mathfrak{D}_{T^{k'},\,\rho^*} &= \mathcal{A}^{N,\,k^{-1}+k'^{-1}}\ \mathfrak{D}_{T^{k+k'},\,\rho^*} && T \notin \mathrm{Stab}(\rho^*)\ , \\ \mathrm{GW}_{T^k} \times \mathrm{GW}_{T^k} &= \mathrm{GW}_{T^{k+k'}} && T \in \mathrm{Stab}(\rho^*)\ . \end{aligned} \tag{5.4.23}$$

The same can be said for conjugacy classes $g = \mathcal{H}^{-1} T \mathcal{H}$.

[41] These can be thought of as the case of T^k modulo conjugation.

Chapter 6

Anomalies of non-invertible self-duality symmetries

In this chapter we study anomalies of non-invertible duality symmetries in both 2d and 4d, employing the tool of the Symmetry Topological Field Theory (SymTFT). In the 2d case, the results are already known from [21] using the technology of fusion categories, but we find it useful to rephrase them in a way easily generalizable to higher dimensions. In both 2d and 4d we find two obstructions to gauging duality defects. The first obstruction requires the existence of a duality-invariant Lagrangian algebra in a Dijkgraaf-Witten theory in one dimension more, that is the SymTFT for the invertible part of the symmetry. In particular, intrinsically non-invertible duality symmetries [46] are necessarily anomalous. The second obstruction requires the vanishing of a pure anomaly for the invertible duality symmetry. However, this depends on further data. In 2d this can be analyzed explicitly using techniques from modular tensor categories, but it is harder to determine in 4d. To solve this problem, we propose and verify that it is equivalent to a choice of symmetry fractionalization for the invertible duality symmetry. We also comment on various possible applications of our results to self-dual theories.

6.1 Anomalies of duality defects from the SymTFT

A natural question, whenever a new type of global symmetry is discovered, is to define and characterize its 't Hooft anomalies. This, for instance, imposes universal dynamical constraints on symmetry-preserving RG flows. While for invertible (higher) symmetries a complete classification of 't Hooft anomalies is given by the appropriate cobordism group [60, 61, 141, 142], for non-invertible symmetries the correct general framework remains unclear. A standard approach is to define anomalies as obstructions to the gauging of a symmetry $\mathcal{C}$. Gauging (or condensation) in higher fusion categories is however a subtle procedure, as it requires the specification of a certain type of consistent algebra objects $\mathcal{A} \in \mathcal{C}$. While the mathematical theory governing such objects has been developed for 1-categories [50, 143] and recently for 2-categories [144], a complete characterization of the required consistency conditions is to this day still missing.

Building on the observation in Section 4.3 we aim to characterize the 't Hooft anomaly in terms of an obstruction to the existence of certain (Neumann) boundary conditions of the SymTFT. This allows us to translate a very hard problem of higher-category theory into a more concrete TQFT problem. One may think that is still complicated because we need to determine the SymTFT $\mathcal{Z}(\mathcal{C})$, and for categorical symmetries $\mathcal{C}$ this can be a intricate object to work with.

There is a very interesting class of categorical symmetries for which we can concretely attach these problems with success. These are the self-duality symmetries that we reviewed in Section 3.3. We consider $d = 2n$ dimensional QFTs $\mathcal{T}$ that are mapped back to itself after gauging a discrete $(n-1)$-form symmetry $\mathbb{A}$

$$\mathcal{T}/\mathbb{A} \cong \mathcal{T} \,, \tag{6.1.1}$$

As we showed in Section 3.3, this implies that the invertible symmetry $\mathbb{A}$ fits inside a $(d-1)$-category that includes duality defects $\mathcal{N}_g$, labeled by $g \in G$, the duality group. The corresponding symmetry category $\mathcal{C}$ is described as a category graded by the group G:

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \,, \qquad \mathcal{C}_0 = n\mathrm{Vec}_{\mathbb{A}} \,, \qquad \mathcal{C}_g = \{\mathcal{N}_g\} \,. \tag{6.1.2}$$

Here $n\mathrm{Vec}_{\mathbb{A}}$ is the category describing an anomaly-free $(n-1)$-form symmetry $\mathbb{A}$, and in the last equality we meant that the connected component[1] $\pi_0(\mathcal{C}_g)$ has a single simple object $\mathcal{N}_g$ for $g \neq 0$. The fusion rules of the $\mathcal{N}_g$'s respect the G-grading up to condensates $C_{\mathbb{A}}$ of the symmetry $\mathbb{A}$ [26]. In particular

$$\mathcal{N}_g \times \overline{\mathcal{N}}_g = C_{\mathbb{A}} \,. \tag{6.1.3}$$

We will consider the cases of $G = \mathbb{Z}_2$ in $d = 2$ and $G = \mathbb{Z}_4$, $\mathbb{Z}_3$ in $d = 4$.

We will define an 't Hooft anomaly for a non-invertible duality defect as the obstruction to constructing a condensable algebra $\mathcal{A}$ containing all the $\mathcal{N}_g$'s. It is believed (although only proven in 2d) that compatibility with a trivially-gapped phase is equivalent to the existence of such an algebra which furthermore contains all objects of the category $\mathcal{C}$ with sufficiently large multiplicities. [2] In two dimensions for Tambara-Yamagami (TY) categories [106], this viewpoint has been examined in [21] by exploiting the concept of *fiber functor*. In this case, condensable algebras are of the form $\mathcal{A} = \mathbb{B} \oplus n_\nu \mathcal{N}$ with $\mathbb{B} \subset \mathbb{A}$ a subgroup and n_ν an integer. The symmetry admits a trivially-gapped realization only if $\mathbb{B} = \mathbb{A}$, and in this case $n_\nu = \sqrt{|\mathbb{A}|}$. If instead $\mathbb{B} \subsetneq \mathbb{A}$, the symmetry only admits a duality-invariant TQFT. We will regard $\mathcal{N}$ to be anomaly-free in both cases.

The reason why the analysis of anomalies is accessible for duality defects, is that the SymTFT is particularly simple [2, 46], being closely related to the one of $n\mathrm{Vec}(\mathbb{A})$ (the category describing an anomaly-free $(n-1)$-form symmetry $\mathbb{A}$). The SymTFT for $n\mathrm{Vec}(\mathbb{A})$ is a generalized untwisted Dijkgraaf-Witten theory,

$$\mathcal{Z}\big(n\mathrm{Vec}(\mathbb{A})\big) = \mathrm{DW}(\mathbb{A}) \,, \tag{6.1.5}$$

and $n\mathrm{Vec}(\mathbb{A})$ is associated to the canonical (or electric) Dirichlet boundary condition in $\mathrm{DW}(\mathbb{A})$. Concretely, for $d = 2n$ dimensional boundaries, $\mathrm{DW}(\mathbb{A})$ is a $(d+1)$-dimensional pure n-form gauge theory for $\mathbb{A}$ with action

$$S = 2\pi i \int_{X_{d+1}} A \cup \delta B \,, \qquad A \in C^n(X_{d+1}, \mathbb{A}) \,, \qquad B \in C^n(X_{d+1}, \mathbb{A}^\vee) \,. \tag{6.1.6}$$

[1] Given a (higher) category $\mathcal{C}$, $\pi_0(\mathcal{C})$ denotes the set of simple objects of $\mathcal{C}$ modded out by the equivalence relation $x \sim y$ if $\mathrm{Hom}(x, y)$ is nontrivial [145]. Physically, the modding procedure corresponds to the condensation of symmetries localized on the defects.

[2] In 2d the compatibility with a trivially gapped phase is equivalent to the existence of a condensable algebra of the form

$$\mathcal{A} = \bigoplus_x d_x L_x \tag{6.1.4}$$

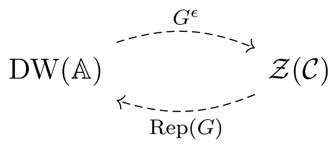
with L_x all the simple objects of the fusion category, and d_x their quantum dimensions. In higher dimensions is not clear what should replace the condition that the multiplicities equate the quantum dimensions. An interesting proposal could be that the coefficient of $\mathcal{N}$ is a TQFT whose partition function on any given manifold is equal to the expectation value of $\mathcal{N}$ on the same manifold.

This theory has an n-form symmetry $\mathbb{A} \times \mathbb{A}^\vee$ generated by the Wilson surface operators of B and A, respectively. The canonical Dirichlet boundary condition simply sets to zero the pull-back of A to the boundary. The duality symmetry G is a subgroup of the full 0-form symmetry of the Dijkgraaf-Witten theory,[3] and it acts by exchanging electric and magnetic operators according to an isomorphism

$$\phi : \mathbb{A} \to \mathbb{A}^\vee \, . \tag{6.1.7}$$

We will focus on the cases that G is isomorphic to $\mathbb{Z}_2$ for $n = 1$, and to either $\mathbb{Z}_3$ or $\mathbb{Z}_4$ for $n = 2$, corresponding to duality or triality symmetries. Therefore in the following we assume that G is Abelian (although a similar discussion could be made for non-Abelian G). Generically G also acts on boundary conditions through its action on the associated Lagrangian algebras $\mathcal{L}$.

Gauging the symmetry G , possibly with discrete torsion ϵ, gives the sought-after SymTFT $\mathcal{Z}(\mathcal{C})$:

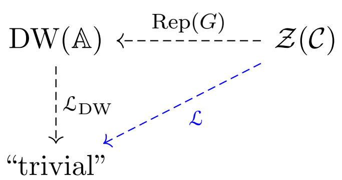


(6.1.8)

In these diagrams dashed lines represent gaugings in the bulk. The upper arrow indicates gauging with discrete torsion, while the lower one the "inverse" operation of gauging[4] the dual symmetry Rep(G) (for G Abelian, Rep$(G) \cong G^\vee$). We will argue that the choice of ϵ acts as a kind of pure G anomaly for the duality defects. As we discussed at length in Chapter 5, from the bulk perspective, the duality defects $\mathcal{N}_g$ are related to the liberated twist defects Σ_g of the 0-form symmetry G in DW($\mathbb{A}$) which are the objects carrying charge under the quantum symmetry Rep(G).

Roughly speaking, the possibility of going back and forth between the SymTFT for duality and the simple DW($\mathbb{A}$) as in (6.1.8) allows us to study topological boundary conditions (Lagrangian algebra $\mathcal{L}$) of $\mathcal{Z}(\mathcal{C})$ in terms of those of DW($\mathbb{A}$). Hence most of our analysis in this chapter will be about the DW theory, with the specification of the duality symmetry.

Let us now be more precise in how to translate information from DW($\mathbb{A}$) into $\mathcal{Z}(\mathcal{C})$. The gapped boundary conditions $\mathcal{L}_{\text{DW}}$ for DW($\mathbb{A}$) are classified by the maximal (Lagrangian) sublattices $\mathcal{L}_{\text{DW}}$ of mutually local charges. Condensing such objects leads in the bulk to a trivial theory (*i.e.*, an invertible TQFT, or SPT phase) whose unique state is the gapped boundary condition [130, 132]. From (6.1.8), we can always induce a gapped boundary condition $\mathcal{L}$ for $\mathcal{Z}(\mathcal{C})$ from a gapped boundary $\mathcal{L}_{\text{DW}}$ of DW($\mathbb{A}$) by first condensing Rep(G) and then $\mathcal{L}_{\text{DW}}$:



(6.1.9)

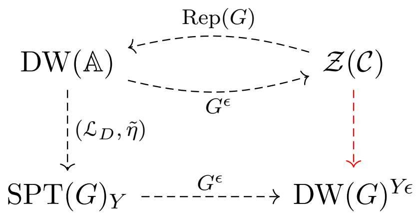
Here "trivial" is the trivial $(d{+}1)$-dimensional theory (SPT) obtained after condensing $\mathcal{L}_{\text{DW}}$ in DW($\mathbb{A}$). When $\mathcal{L}_{\text{DW}}$ is the canonical Dirichelet boundary condition for DW($\mathbb{A}$), this two-step gauging defines a canonical Dirichlet boundary condition $\mathcal{L}_{\mathcal{C}}$ for the SymTFT $\mathcal{Z}(\mathcal{C})$. Since the liberated twist defects Σ_g in $\mathcal{Z}(\mathcal{C})$ are charged under the Rep(G) symmetry, they are confined to the boundary $\mathcal{L}_{\mathcal{C}}$, which thus describes a system with a non-invertible self-duality symmetry.

[3]Depending on $\mathbb{A}$, in general there are other 0-form symmetries in the theory that map $\mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A} \times \mathbb{A}^\vee$ (*e.g.*, charge conjugation that acts separately on $\mathbb{A}$ and $\mathbb{A}^\vee$). Any subgroup G of the full symmetry could be considered. Here, for simplicity, we focus on symmetries G that descend to "dualities" (and are thus controlled by the map (6.1.7)) and are generically present for any $\mathbb{A}$. However, the discussion that follows is quite general.

[4]Precisely, the gauging of Rep(G) should be accompanied by stacking with the inverse SPT ϵ^{-1}, that we leave implicit.

Gauging the non-invertible symmetry $\mathcal{N}_g$ on the boundary, on the contrary, must correspond to a gapped boundary on which the twist defects Σ_g are trivialized. Thus, in order to detect the absence of a self-duality anomaly, we must construct a different set of boundary conditions $\mathcal{N}_{\mathcal{C}}$ for $\mathcal{Z}(\mathcal{C})$ whose symmetry is trivially charged under $\mathrm{Rep}(G)$. We will refer to these as Neumann boundary conditions, since the G gauge field remains dynamical on the boundary.

The crucial insight comes from considering — when it exists — a G-invariant Lagrangian algebra $\mathcal{L}_D$ in $\mathrm{DW}(\mathbb{A})$. This ensures that gauging $\mathcal{L}_D$ leads, in the bulk, to an SPT phase for G, rather than to a completely trivial theory as in (6.1.9). The SPT phase is completely specified by an element Y living in the appropriate cobordism group.[5] It turns out that the datum Y cannot be fixed by the choice of $\mathcal{L}_D$ alone, but it requires a further piece of data, which we dub $\tilde{\eta}$, describing how the symmetry G acts on the algebra morphisms of $\mathcal{L}_D$. This is called an *equivariantization* of $\mathcal{L}_D$ [98, 146]. We denote the equivariantized algebra by a pair $(\mathcal{L}_D, \tilde{\eta})$. The SPT phase Y also contains a nonempty G-twisted sector with a unique simple object M_g for each $g \in G$. In the 3d setting, these can be formally described as local modules twisted by a G-action, see Appendix A. Since $\mathcal{L}_D$ is G-invariant, the operation of gauging G commutes with the condensation of $(\mathcal{L}_D, \tilde{\eta})$, and composing the two operations we end up with a bulk Dijkgraaf-Witten theory for G with twist $Y\epsilon$:

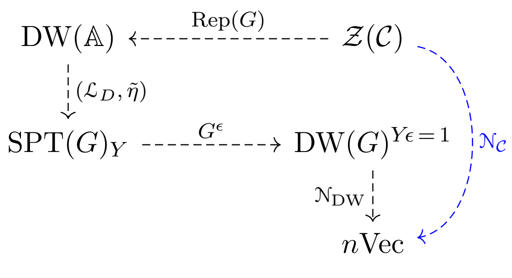


(6.1.10)

We stress that while $\mathrm{DW}(\mathbb{A})$ is an n-form gauge theory, $\mathrm{DW}(G)^{Y\epsilon}$ is always a standard (1-form) gauge theory. Its magnetic operators are the former twist defects M_g. In 3d their spin is determined by the total twist $Y\epsilon$.[6]

The theory $\mathrm{DW}(G)^{Y\epsilon}$ always admits a canonical (electric) Dirichlet boundary condition, corresponding to gauging $\mathrm{Rep}(G)$, that gives rise to an invertible symmetry G on the boundary (such occurrences have been dubbed *non-intrinsic* in [46]). This also coincides with one of the algebras $\mathcal{L}$ we have previously introduced in (6.1.9), in the special case that $\mathcal{L}_{\mathrm{DW}} = \mathcal{L}_D$ is G-invariant.

However, if $Y\epsilon = 1$, then also the magnetic defects M_g are condensable. This allows us to define another boundary condition, the Neumann boundary condition $\mathcal{N}_{\mathcal{C}}$ we were looking for:



(6.1.11)

Thus, the existence of a duality-invariant Lagrangian algebra $(\mathcal{L}_D, \tilde{\eta})$ in $\mathrm{DW}(\mathbb{A})$ and the triviality of $Y\epsilon$ are sufficient conditions for the self-duality symmetry $\mathcal{C}$ to be anomaly-free.

Let us argue that they are also necessary.[7] Suppose that there exists an algebra $\mathcal{N}_{\mathcal{C}}$ of $\mathcal{Z}(\mathcal{C})$ containing the liberated twist defects Σ_g, *i.e.*, such that $\mathrm{Hom}(\mathcal{N}_{\mathcal{C}}, \Sigma_g) \neq \emptyset$. It follows that $\mathcal{N}_{\mathcal{C}}$ has a

[5] For $d = 2$ this is just a bosonic SPT $\in H^3(G, U(1))$. For $d = 4$ instead we will work on spin manifolds and the correct group to consider is either $\mathrm{Tor}(\Omega_5^{\mathrm{spin}_G}(\mathrm{pt}))$ or $\mathrm{Tor}(\Omega_5^{\mathrm{spin}}(BG))$ depending on whether $(-1)^F$ sits inside the duality group or not, respectively. The same observations apply to the discrete torsion ϵ.

[6] In the 5d case, instead, the twist determines a triple linking between magnetic defects [43].

[7] See [147] for another argument, in the case $\mathbb{A} = \mathbb{Z}_n$, in favour of the necessity of a duality-invariant algebra.

natural grading in terms of the charge of its elements under $\mathrm{Rep}(G)$:

$$\mathcal{N}_{\mathcal{C}} = \bigoplus\nolimits_{g \in G} \mathcal{N}^g \, . \tag{6.1.12}$$

Algebra objects come equipped with a product (or morphism) $\times_{\mathcal{N}_{\mathcal{C}}}$ (see Appendix A, where it is called m) mapping $\mathcal{N}^g \times_{\mathcal{N}_{\mathcal{C}}} \mathcal{N}^h \to \mathcal{N}^{gh}$ and respecting the grading. The consistency conditions for $\mathcal{N}_{\mathcal{C}}$ to be a (gaugeable) algebra object are also graded over G, and in particular imply that $\mathcal{N}^0$ must itself be an algebra, although not a maximal (*i.e.*, Lagrangian) one. They also imply that $\mathcal{N}^{g\neq 0}$ are (local) modules over $\mathcal{N}^0$, *i.e.*, they survive the condensation of $\mathcal{N}^0$. Physically this corresponds to the fact that one can gauge $\mathcal{N}_{\mathcal{C}}$ sequentially. One first condenses just $\mathcal{N}^0$. This preserves the $\mathrm{Rep}(G)$ symmetry, since $\mathcal{N}^0$ has trivial grading, and besides the $\mathcal{N}^g$'s become invertible G defects. Since the $\mathcal{N}^g$'s were part of the algebra $\mathcal{N}_{\mathcal{C}}$, the symmetry $\mathrm{Rep}(G)$ after the first condensation must be anomaly free.

Assuming that $\mathcal{N}^g$ with $g \in G$ and $\mathrm{Rep}(G)$ are the only defects surviving the condensation of $\mathcal{N}^0$, it follows that the resulting theory is the Dijkgraaf-Witten theory for G with trivial twist, $\mathrm{DW}(G)$. In other words, we can identify the vertical red arrow in (6.1.10) on the right with the condensation of $\mathcal{N}^0$, as in (6.1.13). The assumption can be rigorously proven in 3d,[8] whilst we do not know how to do that in 5d, which is why our argument remains heuristic. Now, further gauging the $\mathrm{Rep}(G)$ symmetry (and stacking with a discrete torsion ϵ^{-1}) leads us to an SPT phase $Y = \epsilon^{-1}$ for G. Chasing the diagram below shows that we can define a gauging of $\mathrm{DW}(\mathbb{A})$ by sequentially gauging G^ϵ - $\mathcal{N}^0$ - $\mathrm{Rep}(G)$:

$$\begin{array}{ccc} \mathrm{DW}(\mathbb{A}) & \xrightarrow{\;G^\epsilon\;} & \mathcal{Z}(\mathcal{C}) \\ \downarrow{\scriptstyle (\mathcal{L}_D, \tilde{\eta})} & & \downarrow{\scriptstyle \mathcal{N}^0} \\ \mathrm{SPT}(G)_{\epsilon^{-1}} & \xleftarrow{\;\mathrm{Rep}(G)\;} & \mathrm{DW}(G) \end{array} \tag{6.1.13}$$

Since it produces an invertible TQFT with an action of G, the so-defined gauging must correspond to (*i.e.*, it must induce) a duality-invariant Lagrangian algebra $(\mathcal{L}_D, \tilde{\eta})$ in $\mathrm{DW}(\mathbb{A})$. Intuitively, one can think of $\mathcal{N}^0$ as the image of $\mathcal{L}_D$ under the gauging of G. We have thus argued that there is necessarily a duality-invariant Lagrangian algebra in $\mathrm{DW}(\mathbb{A})$.

As we stressed, our reasoning is mathematically rigorous only in the case of 3d TFTs, where the concepts above can be explicitly implemented. We, however, expect the same ideas to apply also to higher categorical settings, once a complete definition of the higher SymTFT is given. We arrive at the following proposal for the (lack of) anomalies of duality defects:

First obstruction. There must exist, in $\mathrm{DW}(\mathbb{A})$, a G-invariant boundary condition $(\mathcal{L}_D, \tilde{\eta})$. This allows to make the symmetry G invertible. In the language of [46], the self-duality symmetry is non-intrinsic.

Second obstruction. The invertible self-duality symmetry is anomaly-free. This is equivalent to the vanishing of the total Dijkgraaf-Witten twist, which depends on the equivariantization data $\tilde{\eta}$. In practice, the invertible self-duality symmetry suffers from a mixed anomaly with the symmetry $\mathcal{S}$ on the non-intrinsic boundary which can be computed explicitly. We conjecture (and show in examples) that the equivariantization data encode how the 0-form symmetry fractionalizes with the $(n-1)$-form

[8] The assumption and its consequence can be proven for 3d MTCs. The fact that $\mathcal{N}^g$ are invertible as bimodules and the fact that $\mathcal{N}_{\mathcal{C}}$ is Lagrangian imply that $\dim(\mathcal{N}^g) = \dim(\mathcal{N}^0) = |\mathbb{A}|$. After condensing $\mathcal{N}^0$, the resulting MTC has dimension $\mathcal{D} = |G|$, which is saturated by the $|G|$ invertible lines $\mathcal{N}^g$ times the $|G|$ elements of $\mathrm{Rep}(G)$. The fact that the $\mathcal{N}^g$'s is charged under $\mathrm{Rep}(G)$ gives the canonical braiding of the DW theory.

symmetry $\mathcal{S}$. Following [53], this can be used to shift the value of the anomaly ϵ, *i.e.*, to change the SPT phase Y in the bulk.

In the rest of this chapter we will unpack these two abstract obstructions, developing a concrete obstruction theory for gauging duality defects in two and four dimensions.

6.2 Anomalies of duality symmetries in 1+1 dimensions

This section is devoted to the discussion of anomalies in two-dimensional QFTs whose symmetries are described by Tambara-Yamagami (TY) categories [106], that we denote by $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$. The results are already known both in the physics and mathematics literature [21, 148] but here we present their derivation from the point of view of the SymTFT which can be generalized to higher dimensions.

The question of which TY categories are anomalous is not a purely academic one, as it can imply strong constraints on duality-preserving RG flows [20]. For instance, the tricritical Ising model has an *anomalous* non-invertible symmetry as well as a duality-preserving relevant deformation. As a direct consequence of the anomaly, the resulting RG flows cannot end in a trivially gapped theory. Depending on the sign of the deformation, the theory either flows to the gapless Ising model or to a gapped theory with three degenerate vacua.

Our SymTFT analysis gives a simple characterization of the known obstruction theory in two steps, as already pointed out in Section 6.1. The first obstruction is equivalent to the absence of a duality-invariant Lagrangian algebra $\mathcal{L}_D$, which otherwise gives rise to a global variant with invertible symmetry $\mathcal{S} \rtimes_\rho \mathbb{Z}_2$. The second obstruction is instead the standard 't Hooft anomaly for $\mathbb{Z}_2$ subgroups of $\mathcal{S} \rtimes_\rho \mathbb{Z}_2$ in that global variant. When there exists such an anomaly-free $\mathbb{Z}_2$ subgroup, it can be gauged. The combined sequential gauging of $\mathcal{L}_D$ and of $\mathbb{Z}_2$ corresponds, in the original theory, to a gauging that involves the non-invertible symmetry defect.

6.2.1 Algebras in TY categories and anomalies

We recall that, given an Abelian group $\mathbb{A}$, the Tambara-Yamagami symmetry is a $\mathbb{Z}_2$-graded fusion category

$$\mathcal{C} = \mathcal{C}_{\underline{0}} \oplus \mathcal{C}_{\underline{1}} \,, \tag{6.2.1}$$

where $\mathcal{C}_{\underline{0}} = \mathrm{Vec}_{\mathbb{A}}$ (the category of $\mathbb{A}$-graded vector spaces) describes an Abelian 0-form symmetry $\mathbb{A}$ with trivial anomaly, while $\mathcal{C}_{\underline{1}}$ has a single simple object $\mathcal{N}$, the duality defect. The fusion rules consistent with the grading are uniquely fixed and read:

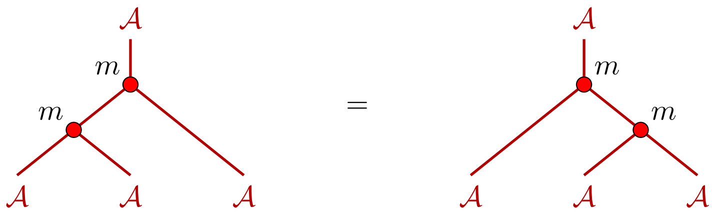
$$a \times b = (a+b) \,, \qquad a \times \mathcal{N} = \mathcal{N} \times a = \mathcal{N} \,, \qquad \mathcal{N} \times \mathcal{N} = \bigoplus\nolimits_{a\in\mathbb{A}} a \,. \tag{6.2.2}$$

Here $a, b \in \mathbb{A}$ are the simple objects in $\mathcal{C}_{\underline{0}}$, and $+$ in the first equation is the binary group operation in $\mathbb{A}$. The category is uniquely determined by a triplet $(\mathbb{A}, \gamma, \epsilon)$ where $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$ is a non-degenerate symmetric bicharacter, whilst $[\epsilon] \in H^3\big(\mathbb{Z}_2, U(1)\big) \cong \mathbb{Z}_2$ is the Frobenius-Schur indicator of the self-dual defect $\mathcal{N}$. This data enters in the associator, or F-symbol, of the TY category:

$$\Big[F^{a,\mathcal{N},b}_{\mathcal{N}}\Big]_{\mathcal{N},\mathcal{N}} = \Big[F^{\mathcal{N},a,\mathcal{N}}_{b}\Big]_{\mathcal{N},\mathcal{N}} = \gamma(a,b) \,, \qquad \Big[F^{\mathcal{N},\mathcal{N},\mathcal{N}}_{\mathcal{N}}\Big]_{a,b} = \frac{\epsilon}{\sqrt{|\mathbb{A}|}}\, \gamma(a,b)^{-1} \,, \tag{6.2.3}$$

where $\epsilon = \pm 1$, while all other non-vanishing associators are equal to 1. In Section 3.3, constructing the duality defect via half-gauging, we interpreted the bicharacter γ as specifying the isomorphism between the symmetries $\mathbb{A}$ and $\mathbb{A}^\vee$ on the two sides of the defect.

We now review the classification of *bosonic* gaugeable symmetries $\mathcal{A}$ in $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$, which are described by symmetric Frobenius algebras in $\mathcal{C}$ [50]. These are defined by an object $\mathcal{A} \in \mathcal{C}$ together with a choice of three-valent junction $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is strictly associative:



(6.2.4)

This diagram encodes the cancellation of 't Hooft anomalies for the symmetry $\mathcal{A}$. We give a brief review of the relevant concepts in Appendix A. The classification problem has been solved in the mathematics literature in [148] and given a physical interpretation from the viewpoint of TQFTs in [21].

Such algebras for the TY category can be divided into two types depending on whether $\mathcal{A}$ also contains the element $\mathcal{N}$ or not. If not, the gaugeable algebras correspond to gauging a subgroup $\mathbb{B}$ of $\mathbb{A}$ with discrete torsion $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$. From the latter one constructs its commutator[9]

$$\chi_\nu(b_1, b_2) = \frac{\nu(b_1, b_2)}{\nu(b_2, b_1)} \; . \tag{6.2.5}$$

This defines a map $[\nu] \to \chi_\nu$ from $H^2\big(\mathbb{B}, U(1)\big)$ to the group of alternating bicharacters which turns out to be an isomorphism [69]. The Frobenius algebra corresponding to the pair $\big(\mathbb{B}, [\nu]\big)$ is:

$$\mathcal{A} \equiv \mathcal{A}_{\mathbb{B},\,\nu} = \bigoplus\nolimits_{b \in \mathbb{B}} b \; , \qquad\qquad m_{b,\,b'}^{b+b'} = \nu(b, b') \; . \tag{6.2.6}$$

On the other hand, when including also $\mathcal{N}$ the most general algebra reads:[10]

$$\mathcal{A} \equiv \mathcal{A}_{\underline{0}} \oplus \mathcal{A}_{\underline{1}} = \mathcal{A}_{\mathbb{B},\,\nu} \oplus n_\nu \mathcal{N} \; , \qquad\qquad n_\nu \geq 1 \; . \tag{6.2.7}$$

The choices of $\mathbb{B}$, ν and n_ν for which such an object can be consistently defined on orientable 2-manifolds are severely restricted by the following two obstructions [21, 69, 148].

First obstruction. We introduce the subgroup $N(\mathbb{B}) \subset \mathbb{A}^\vee$ of characters annihilating $\mathbb{B}$:

$$N(\mathbb{B}) = \Big\{ \beta \in \mathbb{A}^\vee \;\Big|\; \beta(b) = 1 \; , \; \forall b \in \mathbb{B} \Big\} \; . \tag{6.2.8}$$

This group is canonically isomorphic to $(\mathbb{A}/\mathbb{B})^\vee$, while the quotient $\mathbb{A}^\vee / N(\mathbb{B})$ is canonically isomorphic to $\mathbb{B}^\vee$. We also define the *radical* $\mathrm{Rad}(\nu) \subset \mathbb{B}$ of the class $[\nu]$:

$$\mathrm{Rad}(\nu) = \Big\{ b \in \mathbb{B} \;\Big|\; \chi_\nu(b, b') = 1 \; , \; \forall b' \in \mathbb{B} \Big\} \; . \tag{6.2.9}$$

The alternating bicharacter χ_ν is non-degenerate on $\mathbb{B}/\mathrm{Rad}(\nu)$.

For the first obstruction to vanish these subgroups must be related as

$$\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B}) \; . \tag{6.2.10}$$

[9] Notice that χ_ν is well defined (it is independent from the choice of representative ν), alternating (meaning that $\chi_\nu(a, a) = 1$) and antisymmetric (meaning that $\chi_\nu(a, b) = \chi_\nu(b, a)^{-1}$). One can prove that $\chi_\nu : \mathbb{B} \times \mathbb{B} \to U(1)$ is bilinear (in the multiplicative sense), see for instance [149]. Then alternating implies antisymmetric, while the opposite is false and in fact dropping the alternating property one can describe fermionic Lagrangian algebras, see also after (6.2.40).

[10] Notice that, if we restrict to spin manifolds, there are more candidate algebras because it is possible to gauge discrete symmetries with a nontrivial Arf twist (*i.e.*, a discrete torsion constructed out of the spin connection, see *e.g.* [34, 150]). This difference will become apparent in the SymTFT.

Besides, there must exist an involutive automorphism

$$\sigma : \mathbb{B}/\mathrm{Rad}(\nu) \to \mathbb{B}/\mathrm{Rad}(\nu) \qquad \text{with} \qquad \sigma^2 = 1 \tag{6.2.11}$$

such that the symmetric and alternating bicharacters, when restricted to $\mathbb{B}$ and then projected to $\mathbb{B}/\mathrm{Rad}(\nu)$, satisfy

$$\gamma\big(\sigma(a), b\big) = \chi_\nu(a,b) \qquad \text{for} \quad a, b \in \mathbb{B}/\mathrm{Rad}(\nu) \; . \tag{6.2.12}$$

Note that the projections from $\mathbb{B}$ to $\mathbb{B}/\mathrm{Rad}(\nu)$ are well defined. One can prove, using the equation above, that $\nu(a,b)$ and $\nu\big(\sigma(b), \sigma(a)\big)$, when projected to $\mathbb{B}/\mathrm{Rad}(\nu)$, define equivalent cohomology classes in $H^2\big(\mathbb{B}/\mathrm{Rad}(\nu), U(1)\big)$.[11] Thus there exists a 1-cochain $\tilde{\eta} \in H^1\big(\mathbb{B}/\mathrm{Rad}(\nu), U(1)\big)$ such that[12]

$$\frac{\nu(a,b)}{\nu\big(\sigma(b), \sigma(a)\big)} = d\tilde{\eta}(a,b) \;, \qquad \tilde{\eta}(a)\, \tilde{\eta}\big(\sigma(a)\big) = 1 \; . \tag{6.2.14}$$

From (6.2.8)–(6.2.10) it easily follows that

$$|\mathbb{A}|\, n_\nu^2 = |\mathbb{B}|^2 \;, \tag{6.2.15}$$

where[13] $n_\nu^2 = \big|\mathbb{B}/\mathrm{Rad}(\nu)\big|$. The positive integer n_ν appearing here turns out to be the same as the one in (6.2.7). Notice in particular that a necessary condition to satisfy the first obstruction is that $|\mathbb{A}|$ is a perfect square. Since, as it follows from (6.2.2), the quantum dimension of $\mathcal{N}$ is $|\mathbb{A}|^{\frac{1}{2}}$, this reproduces the known fact that gauging is not possible in presence of non-integer quantum dimensions [20].

The rough idea that leads to these formulas is the following. Decomposing the defining equation of a Frobenius algebra into its graded components, one finds that $\mathcal{A}_{\underline{1}}$ must be an invertible $\mathcal{A}_{\underline{0}}$-bimodule: $\mathcal{A}_{\underline{1}} \times_{\mathcal{A}_{\underline{0}}} \mathcal{A}_{\underline{1}} = \mathbb{1}_{\mathcal{A}_{\underline{0}}}$, where $\times_{\mathcal{A}_{\underline{0}}}$ is the tensor product in the category of $\mathcal{A}_{\underline{0}}$-bimodules [50, 98, 152]. Physically this means that we can gauge $\mathcal{A}_{\underline{0}}$, and then $\mathcal{A}_{\underline{1}}$ will become an invertible $\mathbb{Z}_2$ global symmetry of the gauged theory. Eqns. (6.2.10)–(6.2.12) are necessary in order to endow $\mathcal{A}_{\underline{1}}$ with a bimodule structure, and in particular (6.2.14) ensures that the bimodule is invertible. This furthermore implies that $\dim(\mathcal{A}_{\underline{1}}) = \dim(\mathcal{A}_{\underline{0}})$ which reproduces (6.2.15) in terms of the integer n_ν in (6.2.7).

Second obstruction. Having discussed the structure of $\mathcal{A}_{\underline{1}}$, we can simply gauge $\mathcal{A}$ sequentially by first gauging $\mathcal{A}_{\underline{0}}$ and then $\mathcal{A}_{\underline{1}}$. After the first step, $\mathcal{A}_{\underline{0}}$ becomes the identity defect while $\mathcal{A}_{\underline{1}}$ becomes an invertible $\mathbb{Z}_2$ symmetry: $\mathcal{A}_{\underline{1}}^2 = 1$. In order to be able to gauge the full algebra, it must happen

[11] To prove it, one checks that the 2-cochains $\nu(a,b)$ and $\nu\big(\sigma(b),\sigma(a)\big)$ produce the same bicharacter χ_ν in (6.2.5) and so, by isomorphism, must be different representatives of the same cohomology class.

[12] It is always possible to choose $\tilde{\eta}$ such that it satisfies both relations. Consider the case $\mathbb{B} = \mathbb{A}$ and define the two subgroups $\mathbb{A}^\sigma = \{a \in \mathbb{A} \,|\, a = \sigma(a)\}$ as well as $\mathbb{A}_\sigma = \{a + \sigma(a) \,|\, a \in \mathbb{A}\}$, clearly $\mathbb{A}_\sigma \subset \mathbb{A}^\sigma \subset \mathbb{A}$. One checks (see [69]) that γ can be consistently reduced to a bicharacter $\bar{\gamma}$ on the quotient $\mathbb{A}^\sigma/\mathbb{A}_\sigma$: $\gamma\big(a + b + \sigma(b), a' + b' + \sigma(b')\big) = \gamma(a, a')$ for any $a, a' \in \mathbb{A}^\sigma$. Let μ^{-1} be a quadratic refinement of $\bar{\gamma}$ (see Sec. 6.2.1). Now, take the first equation and restrict it to $\mathbb{A}^\sigma$:

$$a, b \in \mathbb{A}^\sigma : \quad d\tilde{\eta}(a,b) = \frac{\nu(a,b)}{\nu(b,a)} = \chi_\nu(a,b) = \gamma(a,b) = \bar{\gamma}\big(\pi(a), \pi(b)\big) = d\mu\big(\pi(a), \pi(b)\big) \tag{6.2.13}$$

using (6.2.12), and π is the projection $\mathbb{A}^\sigma \xrightarrow{\pi} \mathbb{A}^\sigma/\mathbb{A}_\sigma$. It follows that $\tilde{\eta}\big|_{\mathbb{A}^\sigma} = \xi \cdot \pi^*\mu$ for some $\xi \in H^1\big(\mathbb{A}^\sigma, U(1)\big) \cong (\mathbb{A}^\sigma)^\vee$. Since the map $\mathbb{A}^\vee \to (\mathbb{A}^\sigma)^\vee$ given by restriction is surjective, it is always possible to find another solution $\tilde{\eta}' = \zeta \cdot \tilde{\eta}$ where $\zeta \in \mathbb{A}^\vee$ is a character such that $\zeta\big|_{\mathbb{A}^\sigma} = \xi^{-1}$ and thus $\tilde{\eta}'\big|_{\mathbb{A}^\sigma} = \pi^*\mu$. This implies $\tilde{\eta}'\big|_{\mathbb{A}_\sigma} = \tilde{\eta}'\big(a + \sigma(a)\big) = 1$ for all $a \in \mathbb{A}$. Using $d\tilde{\eta}\big(a, \sigma(a)\big) = 1$ from the first equation in (6.2.14), the second equation follows. The general case for $\mathbb{B} \subset \mathbb{A}$ is a straightforward generalization.

[13] Since χ_ν is a non-degenerate alternating bicharacter on $\mathbb{B}/\mathrm{Rad}(\nu)$ with values in $U(1)$, there exists an isotropic subgroup $\mathbb{G}$ such that $\mathbb{B}/\mathrm{Rad}(\nu) = \mathbb{G} \times \mathbb{G}^\vee$ and in particular $\big|\mathbb{B}/\mathrm{Rad}(\nu)\big| = |\mathbb{G}|^2 = n_\nu^2$ is a perfect square, where $n_\nu = |\mathbb{G}|$. See, *e.g.*, Lemma 5.2 in [151].

that $\mathcal{A}_{\underline{1}}$ has a trivial self-anomaly ϵ_{tot}. This comes in two parts: a "bare" contribution from the original Frobenius-Schur indicator ϵ of the duality defect, and a further contribution Y coming from the bimodule morphism $\mathcal{A}_{\underline{1}} \times \mathcal{A}_{\underline{1}} \to \mathbb{1}$. The latter turns out to be given by the Arf invariant of $\tilde{\eta}$ restricted to the elements of $\mathbb{B}/\text{Rad}(\nu)$ invariant under the involution σ:

$$Y = \text{sign}\Bigg(\sum_{\substack{b \in \mathbb{B}/\text{Rad}(\nu) \\ \sigma(b)=b}} \tilde{\eta}(b) \Bigg) = \text{Arf}(\tilde{\eta}) \; . \tag{6.2.16}$$

We stress that here we are using multiplicative notation for $\tilde{\eta}$, so that Y is the sign of a sum of phases (alternatively, in (6.2.95) we indicate the correct normalization). The second obstruction then vanishes if and only if

$$\epsilon_{\text{tot}} = \epsilon\, Y = 1 \; . \tag{6.2.17}$$

Later on, around eqn. (6.2.122), we will find an alternative formula for the spectrum of values that Y can take as we explore the possible consistent choices of $\tilde{\eta}$ — the so-called fractionalization classes.

A note on quadratic refinements

At various points in this chapter we use the existence and properties of quadratic refinements. A function $q : \mathbb{A} \to U(1)$ (with $\mathbb{A}$ a finite Abelian group) is called a quadratic function if $q(a) = q(-a)$ and (using multiplicative notation)

$$\zeta(a,b) \,\equiv\, \frac{q(a+b)}{q(a)\, q(b)} \tag{6.2.18}$$

is a symmetric bicharacter. One easily derives that $q(0) = 1$, $q(ta) = q(a)^{t^2}$ for any $t \in \mathbb{Z}$, and

$$\zeta(a,a) = q(a)^2 \; . \tag{6.2.19}$$

Any quadratic function q, by definition, comes equipped with an associated symmetric bicharacter ζ as in (6.2.18). However also the converse is true: any symmetric bicharacter ζ arises from a (not necessarily unique) quadratic function q, which is called a quadratic refinement of ζ. The set of quadratic refinements forms a torsor over $\text{Hom}(\mathbb{A}, \mathbb{Z}_2)$, indeed one easily proves that the ratio of two quadratic refinements is a $\mathbb{Z}_2$-valued character on $\mathbb{A}$.[14]

A closely related statement is that any symmetric 2-cocycle $\nu \in Z^2\big(\mathbb{A}, U(1)\big)$ is exact. This follows from the isomorphism between altenating bicharacters χ_ν (6.2.5) and classes $[\nu] \in H^2\big(\mathbb{A}, U(1)\big)$. In the special case that the symmetric 2-cocycle $\nu(a,b)$ is bilinear, exactness is equivalent to the existence of a quadratic refinement.

6.2.2 SymTFT description

It is possible to reformulate the properties of Tambara-Yamagami categories $\text{TY}(\mathbb{A})_{\gamma,\epsilon}$ in terms of their 3d SymTFT, *i.e.*, using the language of modular tensor categories (MTCs).[15] Let us review this fact, that will be useful in order to discuss anomalies in the following sections. In particular let us describe how the data $(\mathbb{A}, \gamma, \epsilon)$ appears from the bulk viewpoint.

[14] The set of quadratic functions $q : \mathbb{A} \to U(1)$ is an extension of the group of symmetric bicharacters $\zeta : \mathbb{A} \times \mathbb{A} \to U(1)$ by $\text{Hom}(\mathbb{A}, \mathbb{Z}_2)$. For each bicharacter, a quadratic function is easily constructed. For $\mathbb{A} = \mathbb{Z}_n$ the bicharacters are $\zeta_r(a,b) = \exp\big(\frac{2\pi i r}{n} ab\big)$ with $r \in \mathbb{Z}_n$. Given one of them, a quadratic refinement is $q_r(a) = \exp\big(\frac{\pi i r(n+1)}{n} a^2\big)$. If n is odd then $r \in \mathbb{Z}_n$ and the quadratic function is unique. If n is even then $r \in \mathbb{Z}_{2n}$ and the quadratic functions for r and $r+n$ produce the same bicharacter $\mathbb{Z}_r$. The case that $\mathbb{A}$ is a product of cyclic factors is similar.

[15] Given a fusion category $\mathcal{C}$ as the symmetry of some 2d theory, the corresponding 3d SymTFT is given via the Turaev-Viro construction [37] by the TQFT whose MTC is the *Drinfeld center* of $\mathcal{C}$ denoted $\mathcal{Z}(\mathcal{C})$.

One starts from a pure 3d gauge theory for $\mathbb{A}$ (*i.e.*, a Dijkgraaf-Witten theory for $\mathbb{A}$ with no torsion), which is the SymTFT describing the invertible symmetry $\mathrm{Vec}_{\mathbb{A}}$.[16] The spectrum of lines of the $\mathbb{A}$ gauge theory is $\mathbb{A} \times \mathbb{A}^\vee$, the lines being labelled by pairs $(a, \alpha) \in \mathbb{A} \times \mathbb{A}^\vee$. The braiding is canonically determined by the pairing between $\mathbb{A}$ and $\mathbb{A}^\vee$:

$$\mathcal{B}_{(a_1,\alpha_1),(a_2,\alpha_2)} = \alpha_1(a_2)\,\alpha_2(a_1) \,. \tag{6.2.20}$$

It follows that the topological spins are

$$\theta_{(a,\alpha)} = \alpha(a) \,. \tag{6.2.21}$$

Crucially, as shown in 2.2.1, the theory enjoys *electric-magnetic* (EM) duality specified by an isomorphism $\phi : \mathbb{A} \to \mathbb{A}^\vee$ as

$$\begin{aligned} \Phi : \quad \mathbb{A} \times \mathbb{A}^\vee \quad &\to \quad \mathbb{A} \times \mathbb{A}^\vee \\ (a, \alpha) \quad &\mapsto \quad \big(\phi^{-1}(\alpha), \phi(a)\big) \,. \end{aligned} \tag{6.2.22}$$

However not all choices of isomorphism are consistent EM dualities since Φ needs to preserve the braiding. This condition is equivalent to the bicharacter $\gamma(a, b) = \phi(a)\, b$ associated with ϕ being symmetric. Note that Φ squares to 1, so that the duality group is $G \cong \mathbb{Z}_2$.

If the boundary theory is self-dual under gauging, we can construct the full SymTFT that includes the duality defect by gauging the duality symmetry G [46]. The gauging operation comes with a choice of discrete torsion $\epsilon \in H^3\big(G, U(1)\big) \cong \mathbb{Z}_2$ which translates to the Frobenius-Schur indicator of the duality defect $\mathcal{N}$ on the boundary. To summarize, the data $(\mathbb{A}, \gamma, \epsilon)$ of the boundary Tambara-Yamagami category appears from the bulk viewpoint as the choice of a duality symmetry $G \cong \mathbb{Z}_2$ of the $\mathbb{A}$ Dijkgraaf-Witten theory and of a discrete torsion for the gauging.

To properly discuss the gauged theory, we first describe the data of the 3d Dijkgraaf-Witten theory enriched by the 0-form symmetry (a G-crossed category in the language of [77]). This includes data describing the topological twist defects for the $G \cong \mathbb{Z}_2$ symmetry. The full tensor category is graded:

$$\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\mathbb{Z}_2} = \mathcal{Z}(\mathrm{Vec}_{\mathbb{A}}) \oplus \mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_\Phi \,, \tag{6.2.23}$$

where $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_\Phi$ describes the twisted sector for the $\mathbb{Z}_2$ symmetry. The number of simple components of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_\Phi$ is the same as the number of Φ-invariant anyons [77]. The latter are all of the form $\big(a, \phi(a)\big)$ with $a \in \mathbb{A}$. Thus there are $|\mathbb{A}|$ simple objects in $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_\Phi$ which we denote as σ_a, $a \in \mathbb{A}$ (not to be confused with the involution σ). The fusion and braiding data for the $\mathbb{Z}_2$ extension has been computed in [153], although we use here a slightly different notation similar to the one employed in [46]. We find

$$\begin{aligned} (a, \alpha) \times (b, \beta) &= (a + b,\, \alpha + \beta) \,, \qquad\qquad (a, \alpha) \times \sigma_b = \sigma_{b+a+\phi^{-1}(\alpha)} \\ \sigma_a \times \sigma_b &= \bigoplus_{c \in \mathbb{A}} \big(a + b + c,\, \phi(-c)\big) \,. \end{aligned} \tag{6.2.24}$$

These fusion rules are derived by realizing the G symmetry defects as 2d condensates [26] of the anti-diagonal lines $\big(a, \phi(-a)\big)$ (see *e.g.* [46] for the case $\mathbb{A} = \mathbb{Z}_n$).[17] Since the quantum dimension of (a, α) is 1, we also have

$$\dim(\sigma_a) = \sqrt{|\mathbb{A}|} \,. \tag{6.2.25}$$

[16]Mathematically this corresponds to the fact that the Drinfeld center of $\mathrm{Vec}_{\mathbb{A}}$ is $\mathbb{A} \times \mathbb{A}^\vee$.

[17]Indeed, the anti-diagonal lines are absorbed by the σ_b's, and $\sigma_b \times \sigma_{-b}$ is a 1d condensate of anti-diagonal lines.

Object	Definition	Dim	# of Objects	Spin θ
$L_{(a,\,x)}$	$\eta^x \times \big(a,\, \phi(a)\big)$	1	$2\,\lvert\mathbb{A}\rvert$	$\gamma(a,a)$
$X_{(a,\,b)}$	$\big(a,\, \phi(b)\big) \oplus \big(b,\, \phi(a)\big)$	2	$\lvert\mathbb{A}\rvert\,\big(\lvert\mathbb{A}\rvert-1\big)/2$	$\gamma(a,b)$
$\Sigma_{(a,\,x)}$	$\eta^x \times \sigma_a$	$\sqrt{\lvert\mathbb{A}\rvert}$	$2\lvert\mathbb{A}\rvert$	$(-1)^x \sqrt{\frac{\epsilon}{\lvert\mathbb{A}\rvert^{1/2}} \sum_{b\,\in\,\mathbb{A}} f_a(b)^{-1}}$

Table 6.1: Objects (lines) of the 3d SymTFT $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$.

The non-vanishing R-matrices, in a gauge, are given by [153]:

$$\begin{aligned} &R^{(a_1+a_2,\,\alpha_1+\alpha_2)}_{(a_1,\alpha_1),\,(a_2,\alpha_2)} = \alpha_2(a_1)\;, && R^{\sigma_{a_1+a_2+\phi^{-1}(\alpha_1)}}_{(a_1,\alpha_1),\,\sigma_{a_2}} = f_{a_2}(a_1)\;, \\ &R^{\sigma_{a_1+a_2+\phi^{-1}(\alpha_2)}}_{\sigma_{a_1},\,(a_2,\alpha_2)} = 1\;, && R^{(a_3,\alpha_3)}_{\sigma_{a_1},\,\sigma_{a_2}} = f_{a_1}(-a_3)^{-1}\;. \end{aligned} \tag{6.2.26}$$

In the last entry, (a_3, α_3) must be a fusion channel of $\sigma_{a_1} \times \sigma_{a_2}$. Besides, $f_a : \mathbb{A} \to U(1)$ is a collection (for $a \in \mathbb{A}$) of functions given by $f_a = \phi(a) \cdot f_0$, or more explicitly $f_a(b) = \gamma(a, b)\, f_0(b)$, required to satisfy

$$f_a(b)\; f_a(b') = \gamma(b, b')\; f_a(b + b')\;. \tag{6.2.27}$$

Notice that the equations for different values of a are all equivalent. In these equations, distinct choices for f_0 differ by an $\mathbb{A}$-character and only reshuffle the f_a's, therefore the set of f_a's forms a torsor over $\mathbb{A}^\vee$. However f_0 should be chosen such that $f_0(b) = f_0(-b)$, in other words f_0 is a quadratic refinement of γ^{-1}, which alwasys exists (see Section 6.2.1). Possible different choices are related by $\mathrm{Hom}(\mathbb{A}, \mathbb{Z}_2)$ and correspond to different gauge choices. The (gauge dependent) spins of the twisted sector lines are [153]:

$$\theta(\sigma_a) = \sqrt{\frac{1}{\lvert\mathbb{A}\rvert^{1/2}} \sum_{b\,\in\,\mathbb{A}} f_a(b)^{-1}}\;, \tag{6.2.28}$$

where the choice of sign for the square root is gauge.

We can now discuss the gauging of the symmetry $G \cong \mathbb{Z}_2$ with a twist $\epsilon \in H^3\big(G, U(1)\big) \cong \mathbb{Z}_2$. The gauged theory $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\mathbb{Z}_2}/\mathbb{Z}_2$ is isomorphic to $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$ and is graded by the quantum $\mathbb{Z}_2$ 1-form symmetry whose charged objects are the liberated twisted sectors σ_a. There are three types of objects, whose properties are summarized in Table 7.7. In the first line, $L_{(a,\,x)}$ arise from the Φ-invariant elements $\big(a, \phi(a)\big)$ in the ungauged theory. The label $x \in \{\underline{0}, \underline{1}\}$ specifies the dressing by the $\mathbb{Z}_2$ line $\eta \equiv L_{(0,\underline{1})}$ generating the dual 1-form symmetry $\mathrm{Rep}(\mathbb{Z}_2)$. The lines $X_{(a,\,b)}$ with $a \neq b$ arise from long orbits of generic invertible objects and absorb the $\mathbb{Z}_2$ line η. Finally, $\Sigma_{(a,\,x)}$ are the liberated twisted sectors, which are the charged objects under the dual $\mathrm{Rep}(\mathbb{Z}_2)$ symmetry and thus span the non-trivially graded component. The total dimension of the category is

$$\dim\Big(\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)\Big) = \left(\sum_{\ell \text{ simple}} \dim(\ell)^2\right)^{1/2} = 2\,\lvert\mathbb{A}\rvert\;. \tag{6.2.29}$$

The topological manipulations of the theory with TY symmetry correspond to Lagrangian algebras of this SymTFT. By definition of Drinfeld center, there should exist a Lagrangian algebra corresponding to the global variant with full TY symmetry. As an object, this is given by

$$\mathcal{L}_{\mathrm{TY}} = \mathbb{1} \oplus \eta \oplus \bigoplus_{b \neq 0} X_{(0,\,b)}\;, \tag{6.2.30}$$

and indeed:

$$\dim\big(\mathcal{L}_{\rm TY}\big) = 2\,|\mathbb{A}| = \dim\Big(\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)\Big)\ . \tag{6.2.31}$$

This is the algebra induced by the electric Lagrangian subgroup $\mathcal{L}_{\rm e} = \bigoplus_{\alpha\in\mathbb{A}^\vee}(0,\alpha)$ in the pure $\mathbb{A}$ gauge theory, following our discussion in Section 6.1. While $\mathcal{L}_{\rm e}$ is clearly not duality invariant, it can be uplifted to an algebra in $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$ by adding to it its images under Φ.[18] The resulting object is well defined in $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$, it has vanishing spin (see Table 7.7) and has dimension $2\,|\mathbb{A}|$ so it is Lagrangian. This provides an explicit realization of the sequential gauging procedure outlined in (6.1.9). The symmetry on the corresponding boundary can be computed using the sequential gauging prescription. In the trivially-graded sector $\mathcal{C}_0$ the simple objects are the elements of the quotient $(\mathbb{A}\times\mathbb{A}^\vee)/\mathbb{A}^\vee \simeq \mathbb{A}$. They generate the 0-form symmetry and we label them simply by a. On the other hand, in the $\mathcal{C}_\Phi$ sector all of the twist defects fall into a single orbit, without fixed points under fusion with $\mathcal{L}_{\rm e}$ as can be checked from (6.2.24). Let us denote this object by $\mathcal{N}$. The bulk fusion rules imply (6.2.2), giving back the $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ symmetry.

6.2.3 First obstruction and Lagrangian algebras

Our first goal is to describe how the first obstruction appears from the SymTFT perspective. We have already mentioned in Section 6.1 that the first obstruction precludes the existence of a discrete gauging $(\mathbb{B},\nu)$ which renders the duality symmetry $\mathcal{N}$ invertible. Since, from the SymTFT perspective, discrete gauging operations correspond to different choices of gapped boundary condition $\mathcal{L}$, it is natural to rephrase the first obstruction in the language of Lagrangian algebras of the DW theory. A similar logic has been followed recently in [44], where the obstructions to gauge the entire symmetry category (*i.e.* the case $\mathbb{B}=\mathbb{A}$ in our notation) when $|\mathbb{A}|$ is odd have been found counting the number of bulk lines with trivial spin. However such method is hard to generalize to higher dimensions (which is the main aim in this chapter) since it relies on the notion of topological spin which has no known analog in higher categories. In this and the next two sections, instead, we provide a complete bulk classification of the obstruction theory for $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ and besides we develop methods that allow us to extend the results to higher-dimensional cases.

The crucial point which makes this problem accessible is that the SymTFT for the TY category is a $G\cong\mathbb{Z}_2$ gauging of the Dijkgraaf-Witten theory DW($\mathbb{A}$) [46]. By gauging G back and forth, we can rephrase the problem in terms of gauging Lagrangian algebras of a bulk theory that only consists of invertible symmetries. As already argued in Section 6.1, a sufficient condition for $\mathcal{N}$ to be anomalous is the absence of G-invariant Lagrangian algebras in DW($\mathbb{A}$), namely, of Lagrangian algebras $\mathcal{L}_D$ satisfying

$$\Phi(\mathcal{L}_D) = \mathcal{L}_D\ . \tag{6.2.32}$$

A duality-invariant Lagrangian algebra of DW($\mathbb{A}$) also gives rise to a duality-invariant boundary condition, where the duality symmetry becomes invertible. Hence we realize that, in the terminology of [46], *intrinsic non-invertible symmetries are anomalous.*

Notice that the obstruction we are discussing here is a priori distinct from the first obstruction we discussed in Section 6.2.1. However the main result of this section is to show that the two obstructions are equivalent. In order to do so, we make the first obstruction more explicit by classifying all Lagrangian algebras of DW($\mathbb{A}$) and providing explicit equivalent conditions for their duality invariance in terms of the data $(\mathbb{A},\gamma)$.

[18]If a line $\big(a,\,\phi(a)\big)$ (like the identity in this case) is duality invariant, we must add $L_{(a,\underline{0})}\oplus L_{(a,\underline{1})}$ to the algebra.

The 3d theory DW($\mathbb{A}$) can be thought of as the SymTFT of any theory with a non-anomalous 0-form symmetry $\mathbb{A}$, and as such the correspondence between topological manipulations and bulk Lagrangian algebras is particularly explicit, but yet non-trivial. The (bosonic) topological manipulations of the boundary are determined by two pieces of data [34]:

- The choice of a subgroup $\mathbb{B} \subset \mathbb{A}$ to be gauged.
- The choice of a class $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$ which plays the role of the discrete torsion.

The resulting symmetry after gauging is an extension of $\mathbb{A}/\mathbb{B}$ by the quantum symmetry $\mathbb{B}^\vee$ [154] (see Appendix B.1 for details).

On the other hand, global variants of the boundary theory correspond to different interfaces between DW($\mathbb{A}$) and the trivial 3d theory (*i.e.*, to gapped boundaries), and thus are specified by gauging a subgroup $\mathcal{L} \subset \mathbb{A} \times \mathbb{A}^\vee$, Lagrangian with respect to the braiding. Correspondingly, the lines of $\mathcal{L}$ can end on the boundary, and the topological lines of the boundary theory generating the 0-form symmetry are labelled by the quotient group

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L} \; . \tag{6.2.33}$$

Thus we expect a correspondence between pairs $\big(\mathbb{B}, [\nu]\big)$ and Lagrangian algebras $\mathcal{L}$ such that (6.2.33) coincides with the symmetry after gauging $\mathbb{B}$ with discrete torsion $[\nu]$. Notice that the braiding induces a canonical isomorphism[19]

$$\mathcal{L} \cong \mathcal{S}^\vee \; . \tag{6.2.34}$$

The simplest case is when $H^2\big(\mathbb{A}, U(1)\big) = 0$ (*e.g.*, if $\mathbb{A} = \mathbb{Z}_n$) so that the topological manipulations are simply labelled by the gauged subgroup $\mathbb{B} \subset \mathbb{A}$.[20] Then we consider

$$\mathcal{L}_\mathbb{B} \;\equiv\; \mathbb{B} \times N(\mathbb{B}) \;\subset\; \mathbb{A} \times \mathbb{A}^\vee \tag{6.2.35}$$

which has cardinality $|\mathbb{A}|$ and is made of lines of vanishing spin (in particular it trivializes the braiding, see (6.2.8)), hence it is Lagrangian. Moreover

$$\mathcal{S}_\mathbb{B} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L}_\mathbb{B} = (\mathbb{A}/\mathbb{B}) \times \mathbb{B}^\vee \tag{6.2.36}$$

is precisely the symmetry after gauging $\mathbb{B}$.

In the general case we define the linear map $\psi_\nu : \mathbb{B} \to \mathbb{B}^\vee$ associated to χ_ν:

$$\psi_\nu(b_1)\, b_2 = \chi_\nu(b_1, b_2) \; . \tag{6.2.37}$$

Given the pair $\big(\mathbb{B}, [\nu]\big)$ we construct the subgroup $\mathcal{L}_{\mathbb{B},[\nu]} \subset \mathbb{A} \times \mathbb{A}^\vee$ as follows. Since $\mathbb{B}^\vee = \mathbb{A}^\vee/N(\mathbb{B})$, any element of $\mathbb{A}^\vee$ can be presented as a pair $(\beta, \eta) \in N(\mathbb{B}) \times \mathbb{B}^\vee$ (even though the sum is different from the one in $\mathbb{A}^\vee$) and we denote this element simply as $\beta\eta \in \mathbb{A}^\vee$. The association is not canonical, however different choices agree on η (which is the projection from $\mathbb{A}^\vee$ to $\mathbb{B}^\vee$) while may differ on β. We denote by $\tilde{c} \in H^2\big(\mathbb{B}^\vee, N(\mathbb{B})\big)$ the cocycle which makes $\mathbb{A}^\vee$ an extension of $\mathbb{B}^\vee$ by $N(\mathbb{B})$. Then we construct

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big(b,\, \beta\psi_\nu(b)\big) \in \mathbb{A} \times \mathbb{A}^\vee \;\Big|\; b \in \mathbb{B}, \quad \beta \in N(\mathbb{B}) \Big\} \; . \tag{6.2.38}$$

[19] This can be seen as follows. The braiding is a bilinear non-degenerate pairing on $\mathbb{A} \times \mathbb{A}^\vee$ and thus induces an isomorphism $\mathbb{A} \times \mathbb{A}^\vee \to (\mathbb{A} \times \mathbb{A}^\vee)^\vee$. Saying that $\mathcal{L}$ is Lagrangian is equivalent to saying that its image under this isomorphism is the subgroup of linear functions on $\mathbb{A} \times \mathbb{A}^\vee$ which vanish over $\mathcal{L}$. The latter is canonically isomorphic to the Pontryagin dual of $(\mathbb{A} \times \mathbb{A}^\vee)/\mathcal{L} = \mathcal{S}$.

[20] Indeed if $H^2\big(\mathbb{A}, U(1)\big) = 0$ then $H^2\big(\mathbb{B}, U(1)\big) = 0$ for every subgroup $\mathbb{B}$ of $\mathbb{A}$.

This contains $N(\mathbb{B})$ as a subgroup (for $b=0$), while its quotient by $N(\mathbb{B})$ is isomorphic to $\mathbb{B}$, hence $\mathcal{L}_{\mathbb{B},[\nu]}$ is a group extension

$$1 \;\to\; N(\mathbb{B}) \;\to\; \mathcal{L}_{\mathbb{B},[\nu]} \;\to\; \mathbb{B} \;\to\; 1 \tag{6.2.39}$$

whose corresponding cocycle is $\psi_\nu^*(\tilde{c}) \in H^2\big(\mathbb{B}, N(\mathbb{B})\big)$ (see Appendix B.1 for details). Moreover $\mathcal{L}_{\mathbb{B},[\nu]}$ has cardinality $|\mathbb{A}|$, and since χ_ν is alternating the spin of the lines is trivial:

$$\theta_{(b,\,\beta)} = \chi_\nu(b,b) = 1 \;. \tag{6.2.40}$$

Here (b,β) is a shorthand for $\big(b,\, \beta\psi_\nu(b)\big)$, and β does not contribute because it belongs to $N(\mathbb{B})$. One could weaken the alternating condition and just ask χ_ν to be antisymmetric. In that case the spins would be ± 1 and one would allow for fermionic Lagrangian algebras, which correspond to fermionizations of the boundary symmetry. We will not discuss such cases here, but note that they are a natural candidate to explain why certain duality symmetries — such as $\mathrm{TY}(\mathbb{Z}_2)_{\gamma,1}$ — can be gauged on spin manifolds.

We have thus shown that $\mathcal{L}_{\mathbb{B},[\nu]}$ is a Lagrangian algebra with respect to the braiding. In Appendix B.1 we prove that any Lagrangian algebra of $\mathbb{A}\times\mathbb{A}^\vee$ arises in this way. This classification of boundary conditions of the Dijkgraaf-Witten theory coincides with previously known results from category theory [155]. The boundary condition corresponding to $\mathcal{L}_{\mathbb{B},[\nu]}$ is obtained from the original one by gauging $\mathbb{B}$ with discrete torsion $[\nu]$. Indeed the symmetry on that boundary is

$$\mathcal{S} = \big(\mathbb{A}\times\mathbb{A}^\vee\big)/\mathcal{L}_{\mathbb{B},[\nu]} \;\cong\; \big(\mathcal{L}_{\mathbb{B},[\nu]}\big)^\vee \;, \tag{6.2.41}$$

which is the group extension dual to (6.2.39), namely

$$1 \;\to\; \mathbb{B}^\vee \;\to\; \mathcal{S} \;\to\; \mathbb{A}/\mathbb{B} \;\to\; 1 \;. \tag{6.2.42}$$

The cocycle is $\psi_\nu \circ c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$, where $c \in H^2(\mathbb{A}/\mathbb{B},\mathbb{B})$ determines $\mathbb{A}$ as an extension of $\mathbb{A}/\mathbb{B}$ by $\mathbb{B}$. One can show that this is indeed the symmetry after gauging $\mathbb{B}$ with discrete torsion $[\nu]$ (see Appendix B.1 for the proof).

We should now determine whether $\mathrm{DW}(\mathbb{A})$ admits duality-invariant Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$. In the simplest case that $[\nu]=0$ and hence $\mathcal{L}_\mathbb{B} = \mathbb{B}\times N(\mathbb{B})$, duality invariance is simply equivalent to $\phi(\mathbb{B}) = N(\mathbb{B})$. Since $\big|N(\mathbb{B})\big| = |\mathbb{A}|/|\mathbb{B}|$ this requires $|\mathbb{B}|^2 = |\mathbb{A}|$ and in particular the cardinality of $\mathbb{A}$ must be a perfect square ($n_\nu = 1$ in (6.2.7) in this case). However this is in general not sufficient: $\phi(b) \in \mathbb{A}^\vee$ must vanish on $\mathbb{B}$, so that $\mathbb{B}$ must be a Lagrangian subgroup of $\mathbb{A}$ with respect to the symmetric bicharacter γ associated with ϕ.

In the cases with discrete torsion, we observe that

$$\Phi\big(\mathcal{L}_{\mathbb{B},[\nu]}\big) = \Big\{ \Big(\phi^{-1}\big(\beta\psi_\nu(b)\big),\, \phi(b) \Big) \in \mathbb{A}\times\mathbb{A}^\vee \;\Big|\; b\in\mathbb{B}, \quad \beta \in N(\mathbb{B}) \Big\} \tag{6.2.43}$$

is equal to $\mathcal{L}_{\mathbb{B},[\nu]}$ if and only if for all $b\in\mathbb{B}$ and $\beta\in N(\mathbb{B})$ there exist $b'\in\mathbb{B}$ and $\beta'\in N(\mathbb{B})$ such that

$$b' = \phi^{-1}\big(\beta\psi_\nu(b)\big) \;, \qquad\qquad b = \phi^{-1}\big(\beta'\psi_\nu(b')\big) \;. \tag{6.2.44}$$

Before stating the general condition under which these equations can be solved, consider the simpler case $\mathbb{B}=\mathbb{A}$ for which $N(\mathbb{B})=0$. Define the group homomorphism $\sigma = \phi^{-1}\circ\psi_\nu : \mathbb{A}\to\mathbb{A}$ in terms of which the two conditions become $b'=\sigma(b)$, $b=\sigma(b')$. They have a solution if and only if $\sigma^2=1$. In particular both σ and ψ_ν must be automorphisms.

When $\mathbb{B}\subsetneq\mathbb{A}$ is a proper subgroup, there are further conditions for duality invariance. The proof is technical and we report it in Appendix B.2.1. Let us remind that the *radical* of the class $[\nu]$ is

$$\mathrm{Rad}(\nu) = \mathrm{Ker}(\psi_\nu) \;\subset\; \mathbb{B} \;. \tag{6.2.45}$$

Besides, the projection of χ_ν to $\mathbb{B}/\operatorname{Rad}(\nu)$ being non-degenerate gives an isomorphism

$$\psi_\nu : \mathbb{B}/\operatorname{Rad}(\nu) \to \big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee \ . \tag{6.2.46}$$

Then duality invariance of $\mathcal{L}_{\mathbb{B},[\nu]}$ is equivalent to the following conditions:

1. $\phi\big(\operatorname{Rad}(\nu)\big) = N(\mathbb{B})$. In particular $N(\mathbb{B}) \subset \phi(\mathbb{B})$, and $|\mathbb{B}| = n_\nu\,|\mathbb{A}|^{1/2} \geq |\mathbb{A}|^{1/2}$. In other words, $\mathbb{B}$ cannot be smaller than Lagrangian and $|\mathbb{A}|$ must be a perfect square, hence reproducing the known obstruction induced by non-integer quantum dimensions [20].

2. Assuming condition 1., also ϕ projects to an isomorphism $\phi : \mathbb{B}/\operatorname{Rad}(\nu) \to \big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee$. Then we can define an automorphism

$$\sigma \;\equiv\; \phi^{-1} \circ \psi_\nu : \mathbb{B}/\mathrm{Rad}(\nu) \to \mathbb{B}/\mathrm{Rad}(\nu) \tag{6.2.47}$$

which, by construction, satisfies $\gamma\big(\sigma(a), b\big) = \chi_\nu(a,b)$. The second condition is that

$$\sigma^2 = 1 \ . \tag{6.2.48}$$

Notice that the conditions we obtained for $\mathcal{L}_{\mathbb{B},[\nu]}$ to be duality invariant are equivalent to the first obstruction we reviewed in Section 6.2.1. We thus arrive to the punchline of this section: the first obstruction is equivalent to the absence of duality-invariant Lagrangian algebras in $\mathrm{DW}(\mathbb{A})$, or in other words, to the non-invertible duality symmetry being intrinsic.

A straightforward consequence of the conditions above concerns the action of the duality symmetry G on the symmetry $\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L}_{\mathbb{B},[\nu]}$ of the invariant boundary. To this purpose, it is convenient to present $\mathcal{S}$ as a group extension (6.2.42) and further view $\mathbb{B}^\vee$ as an extension of $\mathrm{Rad}(\nu)^\vee$ by $\big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee$, hence presenting the elements of $\mathcal{S}$ as triplets $(\beta, \eta, \tilde{a})$ with $\beta \in \big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee$, $\eta \in \mathrm{Rad}(\nu)^\vee$ and $\tilde{a} \in \mathbb{A}/\mathbb{B}$. Using that $\mathcal{S} = \mathcal{L}^\vee_{\mathbb{B},[\nu]}$ we find that the duality exchanges $\mathrm{Rad}(\nu)^\vee$ with $\mathbb{A}/\mathbb{B}$, while it acts on $\big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee$ as the automorphism $\sigma^\vee$:

$$\Phi :\; (\beta, \eta, \tilde{a}) \;\to\; \Big(\sigma^\vee(\beta),\; \phi(\tilde{a}),\; \phi^{-1}(\eta)\Big) \ . \tag{6.2.49}$$

When the data $\big(\mathbb{B}, [\nu]\big)$ defines a duality-invariant Lagrangian subgroup, using the definition of σ in (6.2.47) and $\sigma^2 = 1$ we can relate the symmetric and the antisymmetric bicharacters as

$$\chi_\nu(b_1, b_2) = \gamma\big(\sigma(b_1), b_2\big) \ , \qquad\qquad \gamma(b_1, b_2) = \chi_\nu\big(\sigma(b_1), b_2\big) \ . \tag{6.2.50}$$

This in turn implies a condition for the class $[\nu]$:

$$\nu(b_1, b_2)\, \nu\big(\sigma(b_1), \sigma(b_2)\big) = d\tilde{\zeta}(b_1, b_2) \qquad \text{or equivalently} \qquad \frac{\nu(b_1, b_2)}{\nu\big(\sigma(b_2), \sigma(b_1)\big)} = d\tilde{\eta}(b_1, b_2) \ . \tag{6.2.51}$$

This is because the l.h.s. of both equations is a symmetric 2-cocycle (see Section 6.2.1 or footnote 11). Those relations coincide with the known relation (6.2.14) (also appearing in the equivariantization of the algebras in TY categories, see Section 6.2.4).

We can neatly express the condition (6.2.51) by noticing that the action $\rho : G \to \mathrm{Aut}(\mathbb{A})$ of any group G on a generic Abelian group $\mathbb{A}$ induces an action on $H^2\big(\mathbb{A}, U(1)\big)$ given by

$$(\rho_g\, \xi)(a_1, a_2) = \xi\big(\rho_g^{-1}(a_1),\; \rho_g^{-1}(a_2)\big) \tag{6.2.52}$$

for each $g \in G$ and $\xi \in H^2\big(\mathbb{A}, U(1)\big)$. Then (6.2.51) can be expressed as

$$\sigma\,[\nu] = \rho_{\underline{1}}\,[\nu] = [\nu^{-1}] \ , \tag{6.2.53}$$

where $\underline{1}$ is the generator of $G \cong \mathbb{Z}_2$. This reformulation will be convenient later on.

Examples

To make concrete the discussion above, we show a few examples. For convenience, here we use additive notation for the phases by thinking of them as elements of $\mathbb{R}/\mathbb{Z}$ instead of $U(1)$.

1. The simplest example is $\mathbb{A} = \mathbb{Z}_n$ for which there is no discrete torsion, and the subgroups are in correspondence with the divisors of n. Let $n = pq$, and $\mathbb{B} = \{qx \mid x = 0, \ldots, p-1\} \cong \mathbb{Z}_p$ so that $N(\mathbb{B}) = \{py \mid y = 0, \ldots, q-1\} \cong \mathbb{Z}_q$.

When we gauge $\mathbb{B}$ on the boundary, the global symmetry is the direct product of the dual symmetry $\mathbb{Z}_p$ and the quotient $\mathbb{Z}_q$. From the bulk perspective, the prescription is that this boundary condition is obtained by allowing the lines of the form (qx, py) to terminate on the boundary hence becoming transparent there. On the other hand, the 0-form symmetry is generated by the remaining lines stacked at the boundary, which indeed form the group $\mathbb{Z}_p \times \mathbb{Z}_q$.

For what concerns duality invariance, we need $\mathbb{B} \cong N(\mathbb{B})$ and hence $p = q$: this implies that $n = p^2$ must be a perfect square. Any symmetric bicharacter takes the form $\gamma(a, b) = rab/n \pmod 1$ for some $r \in \mathbb{Z}_n$ (r must be coprime with n for the bicharacter to be non-degenerate), and we notice indeed that $\mathbb{Z}_p \subset \mathbb{Z}_{p^2}$ is Lagrangian in all cases:

$$\gamma(px, py) = 0 \; . \tag{6.2.54}$$

The integer coefficient introduced in (6.2.7) here is $n_\nu = |\mathbb{B}|/|\mathbb{A}|^{1/2} = 1$.

2. A less trivial example is $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with n a prime number. There are $n+3$ subgroups: the trivial one, the $n+1$ subgroups isomorphic to $\mathbb{Z}_n$ generated by $(1,0)$ and $(s,1)$ with $s = 0, \ldots, n-1$, and the full $\mathbb{A}$. Only the last one admits non-trivial discrete torsion $[\nu] \in H^2\big(\mathbb{Z}_n \times \mathbb{Z}_n, U(1)\big) \cong \mathbb{Z}_n$ which could be represented as

$$\nu\big((x_1, x_2),\, (y_1, y_2)\big) = \frac{r}{n}\, x_1 y_2 \qquad \text{or equivalently as} \qquad \nu\big((x_1, x_2),\, (y_1, y_2)\big) = -\frac{r}{n}\, x_2 y_1 \; . \tag{6.2.55}$$

The corresponding alternating bicharacter is given by the matrix

$$\chi_\nu = \frac{1}{n}\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} , \qquad \text{with} \qquad r \in \mathbb{Z}_n \; . \tag{6.2.56}$$

In total there are $2n+2$ boundary theories. One can explicitly see that these are in one-to-one correspondence with the Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$ in $\mathbb{A} \times \mathbb{A}^\vee$.

Let us show that the induced global symmetry at the boundary is the one obtained by gauging $\mathbb{B}$ with discrete torsion ν. The cases $\mathbb{B} = \{0\}$ or $\mathbb{B} \cong \mathbb{Z}_n$ are similar to the one discussed above and the corresponding Lagrangian algebras are, respectively:

$$\begin{aligned} \mathcal{L}_{\mathbb{B},[0]} &= \Big\{ \big((0,0); (a_1, a_2)\big) \;\Big|\; a_1, a_2 \in \mathbb{Z}_n \Big\} && \text{for } \mathbb{B} = \{0\} \; , \\ \mathcal{L}_{\mathbb{B},[0]} &= \Big\{ \big((a_1, 0); (0, a_2)\big) \;\Big|\; a_1, a_2 \in \mathbb{Z}_n \Big\} && \text{for } \mathbb{B} \cong \mathbb{Z}_n \text{ generated by } (1,0) \; , \\ \mathcal{L}_{\mathbb{B},[0]} &= \Big\{ \big((sa_1, a_1); (a_2, -sa_2)\big) \;\Big|\; a_1, a_2 \in \mathbb{Z}_n \Big\} && \text{for } \mathbb{B} \cong \mathbb{Z}_n \text{ generated by } (s,1) \; . \end{aligned} \tag{6.2.57}$$

When $\mathbb{B} = \mathbb{Z}_n \times \mathbb{Z}_n$ the resulting boundary theory has symmetry $\mathbb{B}^\vee \cong \mathbb{Z}_n \times \mathbb{Z}_n$ even for non-trivial discrete torsion. According to our prescription, and using that $N(\mathbb{B})$ is trivial and the map ψ_ν has the same matrix form of χ_ν defined in (6.2.56), the corresponding Lagrangian subgroup of $\mathbb{A} \times \mathbb{A}^\vee$ is

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big((a_1, a_2); (-ra_2, ra_1)\big) \;\Big|\; a_1, a_2 \in \mathbb{Z}_n \Big\} \; , \tag{6.2.58}$$

and indeed $(\mathbb{A}\times\mathbb{A}^\vee)/\mathcal{L} = \mathbb{Z}_n\times\mathbb{Z}_n$. To see the effect of the discrete torsion we use a Lagrangian description of the DW theory:

$$S = \frac{2\pi i}{n}\int_{X_3}\big(A_1\cup dB_1 + A_2\cup dB_2\big)\ . \tag{6.2.59}$$

The generic line with charges $\big((a_1,a_2);(b_1,b_2)\big)$ is

$$\exp\bigg[\frac{2\pi i}{n}\int_\gamma \big(a_1A_1+a_2A_2+b_1B_1+b_2B_2\big)\bigg]\ . \tag{6.2.60}$$

Hence, as a boundary condition, $\mathcal{L}$ in (6.2.58) corresponds to $A_1+rB_2 = A_2 - rB_1 = 0$. Changing variables according to $A_1\to A_1+rB_2$, $A_2\to A_2-rB_1$ we obtain the same bulk Lagrangian as in (6.2.59) but with an extra boundary term

$$\delta S_\text{bdry} = \frac{2\pi i r}{n}\int_{\partial X_3} B_1\cup B_2\ , \tag{6.2.61}$$

which is precisely the discrete torsion for the gauging on the boundary.

Let us discuss which of those algebras are duality invariant, and in particular which symmetric bicharacters admit duality-invariant algebras. There are two natural classes of non-degenerate symmetric bicharacters, diagonal and off-diagonal:

$$\gamma^{(\text{D})}_{t_1,t_2} = \frac1n\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix} \qquad\text{and}\qquad \gamma^{(\text{O})}_t = \frac1n\begin{pmatrix} 0 & t\\ t & 0\end{pmatrix}\ . \tag{6.2.62}$$

Here non-degeneracy requires t_1, t_2, t to be invertible elements of $\mathbb{Z}_n$.[21] Note that $\mathbb{B}=\{0\}$ cannot lead to duality-invariant algebras because it is smaller than Lagrangian.

Consider the case of $\gamma^{(\text{O})}_t$. First we look at Lagrangian algebras associated with subgroups $\mathbb{B}\cong\mathbb{Z}_n$ which, according to our general analysis, need to be Lagrangian with respect to $\gamma^{(\text{O})}_t$ because $[\nu]=0$. The two subgroups $\mathbb{B}=\big\langle(1,0)\big\rangle$, $\big\langle(0,1)\big\rangle$ are always Lagrangian, while $\mathbb{B}=\big\langle(s,1)\big\rangle$ is Lagrangian only if

$$2st = 0 \mod n \tag{6.2.63}$$

which can never be satisfied if n is odd. Then we look at the cases with $\mathbb{B}=\mathbb{A}$. In order to satisfy $\phi\big(\text{Rad}(\nu)\big) = N(\mathbb{B}) = \{0\}$ in (6.2.10) we need a discrete torsion (6.2.56) with $r\neq 0$. From (6.2.47) we find

$$\sigma = \begin{pmatrix} t^{-1}r & 0\\ 0 & -t^{-1}r\end{pmatrix}\ . \tag{6.2.64}$$

The duality-invariant condition $\sigma^2=1$ reads $(t^{-1}r)^2 = 1 \bmod n$, which can always be satisfied by the values $r=\pm t$.

Consider now the case of $\gamma^{(\text{D})}_{t_1,t_2}$. The subgroups $\mathbb{B}\cong\mathbb{Z}_n$ are Lagrangian with respect to $\gamma^{(\text{D})}_{t_1,t_2}$ only when $\mathbb{B}=\big\langle(s,1)\big\rangle$ with $t_1\,s^2+t_2 = 0 \bmod n$. For $\mathbb{B}=\mathbb{A}$, instead, we need a non-vanishing discrete torsion (6.2.56), and since

$$\sigma = \begin{pmatrix} 0 & -t_1^{-1}r\\ t_2^{-1}r & 0\end{pmatrix}\ , \tag{6.2.65}$$

the duality-invariance condition reads $r^2 = -t_1t_2 \bmod n$. This equation and the previous one for s do not always have solutions. For instance, if $t_1=t_2=1$, then r (or s) must be a square root of -1 which exists for $n=2,5,13,\ldots$ but not for $n=3,7,11,\ldots$

[21] For n prime, they are just non-vanishing. However these two bicharacters will be used also for n non prime, hence t_1, t_2, t must be coprime with n.

In summary, while $\mathrm{TY}(\mathbb{Z}_n \times \mathbb{Z}_n)_{\gamma,\epsilon}$ with off-diagonal bicharacter γ always trivializes the first obstruction, when the bicharacter is diagonal the category is necessarily anomalous for certain values of n for which the first obstruction forbids the gauging. We also notice that in all of these examples, when there is a duality-invariant Lagrangian algebra associated with $\mathbb{B} \cong \mathbb{Z}_n$ we have $n_\nu = 1$, while for $\mathbb{B} \cong \mathbb{A}$ we have $n_\nu = n$.

3. We conclude with a more complicated example which is representative of the general case $\mathbb{B} \subsetneq \mathbb{A}$ but $[\nu] \neq 0$, hence $\mathbb{B}$ is non-Lagrangian. Take $\mathbb{A} = \mathbb{Z}_4 \times \mathbb{Z}_4$ which, besides the subgroups we already considered, also has the subgroup

$$\mathbb{B} = \big\{(x, 2y) \bigm| x \in \mathbb{Z}_4\,,\ y \in \mathbb{Z}_2\big\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \tag{6.2.66}$$

(as well as the similar one with the two factors swapped) hence realizing

$$N(\mathbb{B}) = \big\{(0, 2\tilde{y}) \bigm| \tilde{y} \in \mathbb{Z}_2\big\} \cong \mathbb{Z}_2 \subset \mathbb{A}^\vee\ . \tag{6.2.67}$$

The most general alternating bicharacter on $\mathbb{B}$ is

$$\chi_\nu = \frac{1}{4}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \qquad \text{with} \qquad 2(a+b) = 0 \mod 4\ , \tag{6.2.68}$$

hence $a, b \in \mathbb{Z}_4$ must be either both even or both odd. If a, b are both even then $\mathrm{Rad}(\nu) = \mathbb{B}$ and duality invariance cannot be satisfied. If a, b are odd, instead,

$$\mathrm{Rad}(\nu) = \big\{(2z, 0) \bigm| z \in \mathbb{Z}_2\big\} \cong \mathbb{Z}_2 \subset \mathbb{Z}_4 \times \mathbb{Z}_2\ . \tag{6.2.69}$$

The condition $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B})$ cannot be satisfied with the diagonal bicharacter $\gamma^{(\mathrm{D})}_{t_1,t_2}$, while with the off-diagonal one $\gamma^{(\mathrm{O})}_t$ the condition is met (for both the invertible elements $t = 1, 3$). The second condition for duality invariance involves

$$\sigma = \phi^{-1} \circ \psi_\nu = \begin{pmatrix} tb & 0 \\ 0 & ta \end{pmatrix}. \tag{6.2.70}$$

The condition $\sigma^2 = 1$ is equivalent to $(tb)^2 = (ta)^2 = 1$ which is automatically satisfied. In this case we get $n_\nu = 2$.

6.2.4 Second obstruction and equivariantization

In the previous section we rephrased the first obstruction to the gauging of a 2d duality symmetry in terms of the existence of a duality-invariant Lagrangian algebra $\mathcal{L}_D$ in the 3d TQFT $\mathrm{DW}(\mathbb{A})$. Gauging $\mathcal{L}_D$ leads to a bulk SPT phase $Y \in H^3\big(G, U(1)\big)$ for the duality symmetry $G \cong \mathbb{Z}_2$, which determines the DW twist of the corresponding G gauge theory as explained in (6.1.10). The total twist $\epsilon_{\mathrm{tot}} = \epsilon\, Y$ in turn determines whether a Neumann boundary condition is allowed (and $\mathcal{N}$ is anomaly-free).

In order to understand the origin of Y we must describe in detail how to make the gauging of $\mathcal{L}_D$ consistent with the presence of a 0-form symmetry. Naively this should amount to the requirement that $\mathcal{L}_D$ be G-invariant as an object: $\Phi(\mathcal{L}_D) = \mathcal{L}_D$ as stressed in (6.2.32). This is however not sufficient, as the algebra $\mathcal{L}_D$ comes with a specific choice of morphism $m : \mathcal{L}_D \times \mathcal{L}_D \to \mathcal{L}_D$ that is associative and commutative (see Appendix A for the definitions) and a set of projections $\pi_x \in \mathrm{Hom}(\mathcal{L}_D, x)$. An *equivariantization* of $\mathcal{L}_D$ is the definition of a consistent action of the 0-form symmetry on the projections that leaves m invariant (for more details we refer the reader to Appendix A and [146] for a thorough treatment). To define this structure the proper context is that of G-crossed MTCs [77]. In this framework a symmetry defect U_g acts on the junction spaces $V^z_{x,y}$, where $x, y, z \in \mathbb{A}\times\mathbb{A}^\vee$ label simple objects[22] , by a unitary automorphism $[\mathcal{U}_g]^z_{x,y} : V^z_{x,y} \to V^{g(z)}_{g(x),\,g(y)}$ as

$$U_g\big(v^z_{x,y}\big) = [\mathcal{U}_g]^z_{x,y}\cdot v^{g(z)}_{g(x),\,g(y)} \tag{6.2.71}$$

where v is a chosen basis vector of $V^z_{x,y}$ (see Figure 6.1). The phases $[\mathcal{U}_g]^z_{x,y}$ have to satisfy several compatibility conditions with the data of the underlying category, in particular consistency with the braiding requires

$$[\mathcal{U}_g]^z_{x,y}\, R^z_{x,y} = R^{g(z)}_{g(x),\,g(y)}\, [\mathcal{U}_g]^z_{y,x}\,. \tag{6.2.72}$$

Using the R-matrices (6.2.26) and the $G\cong\mathbb{Z}_2$ action on elements of DW($\mathbb{A}$) one easily sees that this equation admits a simple solution

$$[\mathcal{U}_g]^{(a+b,\alpha+\beta)}_{(a,\alpha),\,(b,\beta)} = \alpha(b)\,, \tag{6.2.73}$$

for $g=\underline{1}$ the generator of $\mathbb{Z}_2$.

Now let us come to the equivariantization. For the algebras discussed in Section 6.2.3, a consistent[23] choice of m is

$$m^{x+x'}_{x,x'} = \nu(b',\, b) \qquad \text{where} \qquad x=\big(b,\,\beta\psi_\nu(b)\big)\in\mathcal{L}_D\,. \tag{6.2.74}$$

In the following we will use $x, y, z, \ldots$ to denote elements of $\mathcal{L}_D$ in order to lighten the notation. Working in components we expand

$$m = \bigoplus_{x,y} m^z_{x,y} \qquad \text{and} \qquad m^z_{x,y}\in V^z_{x,y} \tag{6.2.75}$$

where $z = x+y$. The defects U_g act on the projectors $\pi_x : \mathcal{L}_D \to x$ by an automorphism $\tilde\eta_g(x) : \pi_x \to \pi_{g(x)}$ as follows (see Figure 6.1)

$$U_g(\pi_x) = \tilde\eta_g(x)\cdot\pi_{g(x)} \tag{6.2.76}$$

Using these transformations, m is invariant if [24]

$$m^{g(z)}_{g(x),\,g(y)} = m^z_{x,y}\,[\mathcal{U}_g]^z_{x,y}\,\frac{\tilde\eta_g(z)}{\tilde\eta_g(x)\,\tilde\eta_g(y)}\,. \tag{6.2.77}$$

The equivariantization datum $\tilde\eta$ can be neatly interpreted in cohomology. First acting with gauge transformations $\pi_x\to\mu(x)\,\pi_x$ on the vector spaces associated to π_x and $\pi_{g(x)}$ we can identify

$$\tilde\eta_g(x)\,\sim\,\tilde\eta_g(x)\,\frac{\mu\big(g(x)\big)}{\mu(x)}\,. \tag{6.2.78}$$

Second, consistency with the group composition law demands that

$$\tilde\eta_g(x)\,\tilde\eta_h\big(g(x)\big) = \tilde\eta_{gh}(x)\,. \tag{6.2.79}$$

Interpreting $\tilde\eta_g$ as a cochain in $C^1\big(\mathcal{L}_D, U(1)\big)$, so that $\tilde\eta\in C^1\big(G,\,C^1\big(\mathcal{L}_D,U(1)\big)\big)$, we can rewrite (6.2.79) and (6.2.78) in terms of a differential. Using now additive notation, for the sake of clarity and for later convenience, they look, respectively, as

$$d_\rho\tilde\eta = 0\,, \qquad\qquad \tilde\eta\,\sim\,\tilde\eta + d_\rho\mu\,, \tag{6.2.80}$$

[22]Throughout this section we leave implicit that all simple objects are invertible and hence all junction spaces are one-dimensional.

[23]Commutativity of the algebra requires $m^z_{x,x'} = m^z_{x',x}R^z_{x,x'}$, which, in our case, becomes $m^z_{x,x'}/m^z_{x',x} = \chi_\nu(b',b)$.

[24]Here we use that all objects in the algebra $\mathcal{L}_D$ are invertible and appear with multiplicity one in the DW($\mathbb{A}$) theory.

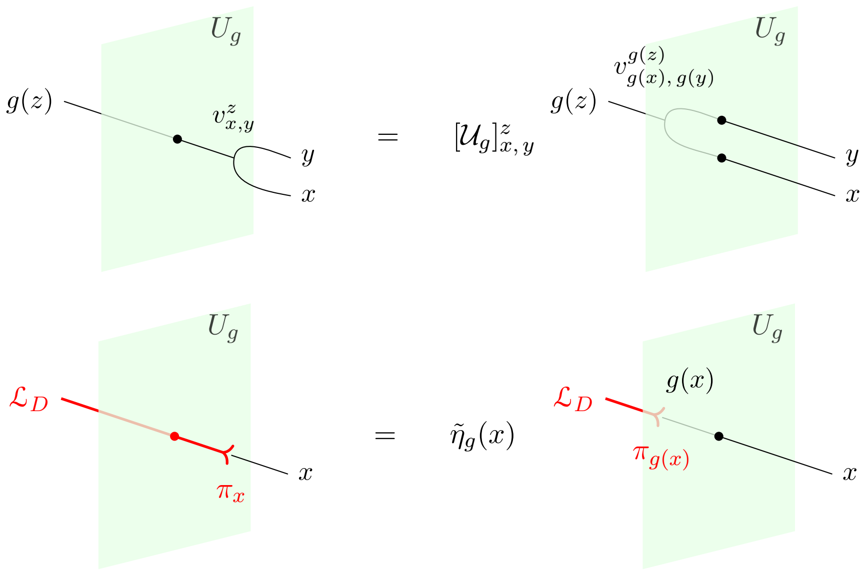


Figure 6.1: Graphical representation of the action of a symmetry defect U_g on the junction spaces $V^z_{x,\,y}$ (above) and on the projectors π_x (below).

for any $\mu \in C^0\big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\big) \cong C^1\big(\mathcal{L}_D, U(1)\big)$. Here d_ρ is a twisted differential, while ρ is the G-action on anyons. We obtain that $\tilde{\eta}$ is naturally an object in twisted group cohomology (see *e.g.* [15] and Appendix C for a review):

$$\tilde{\eta} \in H^1_\rho\Big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\Big) \,. \tag{6.2.81}$$

Restricting the solution ((6.2.73)) to elements of $\mathcal{L}_D$ we find

$$[\mathcal{U}_g]^{x+x'}_{x,\,x'} = \chi_\nu\big(b, b'\big) = \chi_\nu\big(\sigma(b), \sigma(b')\big)^{-1} \tag{6.2.82}$$

where in the second step we used the relations between the symmetric and antisymmetric bicharacters (6.2.50). Since $m^{g(x+x')}_{g(x),\,g(x')} = \nu\big(\sigma(b'), \sigma(b)\big)$ from (6.2.74), then (6.2.77) becomes

$$\frac{\nu(b, b')}{\nu\big(\sigma(b'), \sigma(b)\big)} = d\tilde{\eta}_g \tag{6.2.83}$$

which we recognize as the first equation in (6.2.14) with a caveat. The set of solutions for $\tilde{\eta}_g$, with $g = \underline{1}$, form a torsor over $\mathcal{L}_D^\vee$ while the solutions of (6.2.14) are related by elements of $\big(\mathbb{B}/\text{Rad}(\nu)\big)^\vee$, therefore, strictly speaking, the solutions sets of the two equations differ. However we will see below that the two sets of equations give rise to the same second obstruction.[25]

For later convenience we also notice that the set of solutions to (6.2.83) for $\tilde{\eta}$ forms a torsor over

$$H^1_\rho\big(G,\, \mathcal{L}_D^\vee\big) = H^1_\rho(G, \mathcal{S}) \,, \tag{6.2.84}$$

whose elements we denote by η. This will be useful for the upcoming reinterpretation of the second obstruction in terms of symmetry fractionalization in Section 6.2.5. All in all we found that an equivariantization of a duality-invariant Lagrangian algebra $\mathcal{L}_D$ is specified by the choice of an element $\tilde{\eta} \in H^1_\rho\big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\big)$ satisfying (6.2.83), and that any two choices differ by an element η of $H^1_\rho\big(G,\, \mathcal{L}_D^\vee\big)$.

[25]Physically the extra solutions correpond to symmetry fractionalization patterns between $\mathbb{Z}_2$ and $\mathcal{S}$ for which there is no mixed 't Hooft anomaly.

Given an equivariantization $\tilde{\eta}$ of $\mathcal{L}_D$, we ask what is the SPT phase $Y \in H^3\big(G, U(1)\big)$ for G that we obtain after gauging $(\mathcal{L}_D, \tilde{\eta})$. Indeed, the theory after gauging has a single genuine line $\mathbb{1}$ (and thus is an invertible TQFT) but also a single non-genuine topological twist line M_g for each $g \in G$. The spins θ_{M_g} of such objects are gauge dependent by a G-character [77]. In the presence of a discrete torsion Y, the θ_{M_g}'s do not form a G-character: their deviation from being a character is physical and is induced by the SPT phase Y. In the present case that $G \cong \mathbb{Z}_2$,[26] θ_M does not square to 1 but instead

$$\theta_M = \sqrt{Y(\underline{1}, \underline{1}, \underline{1})}\ , \tag{6.2.85}$$

the sign of the square root being pure gauge. We can thus detect the $\mathbb{Z}_2$ SPT phase through the gauge-invariant quantity $\theta_M^2 = Y$. We now show how to reproduce (6.2.16). A key fact is that, given a choice of equivariantization $\tilde{\eta}$ for $\mathcal{L}_D$, there is a unique non-genuine twist line M after gauging $(\mathcal{L}_D, \tilde{\eta})$. It is then possible to show, using the defining equation (A.3.10) for a twisted local module M that

$$f_a(b)^{-1} = \tilde{\eta}(b) \qquad \text{for} \qquad b \in \mathbb{B}/\mathrm{Rad}(\nu) \qquad \text{and} \qquad \sigma(b) = b\ , \tag{6.2.86}$$

where f_a is the function introduced in (6.2.27). The equation holds for all the values of a for which $\mathrm{Hom}(\sigma_a,\, M) \neq 0$.[27] We will use the notation $M^{(a)}$ to account for the different choices one has for the equivariantization $\tilde{\eta}$: upon gauging, each choice leads to a theory with a unique non-genuine operator, however different choices lead to different SPT phases Y and the label a (whose possible values depend on $\mathcal{L}_D$ in a complicated way) keeps track of the equivariantization chosen.

Since θ_M^2 must be well defined, the spins squared of the components of M must coincide. Since, as an object, $M^{(a)}$ can be described as the orbit of the twist defect σ_a under fusion with the lines of $\mathcal{L}_D$, using the fusions in (6.2.24) we get:[28]

$$M^{(a)} = \bigoplus_u \sigma_{a+u} \qquad \text{where} \qquad u = b + \phi^{-1}\big(\beta\psi_\nu(b)\big) \qquad \text{for all} \qquad \big(b,\, \beta\psi_\nu(b)\big) \in \mathcal{L}_D\ . \tag{6.2.87}$$

Consistency with the existence of a unique local module requires that $\theta^2_{\sigma_a} = \theta^2_{\sigma_{a+u}}$, *i.e.*

$$\theta^2_{M^{(a)}} = \frac{1}{\sqrt{|\mathbb{A}|}} \sum_{c\,\in\,\mathbb{A}} f_a(c)^{-1} \overset{!}{=} \frac{1}{\sqrt{|\mathbb{A}|}} \sum_{c\,\in\,\mathbb{A}} f_{a+u}(c)^{-1} = \frac{1}{\sqrt{|\mathbb{A}|}} \sum_{c\,\in\,\mathbb{A}} f_a(c)^{-1}\, \gamma(u,c)^{-1}\ , \tag{6.2.88}$$

from which we can extract some consequences. For our purposes it will be enough to consider $u = b + \phi^{-1}(\psi_\nu(b))$ with $b \in \mathbb{B}$, we then impose

$$\begin{aligned} \theta^2_{M^{(a)}} &= \frac{1}{|\mathbb{B}|} \sum_{b\in\mathbb{B}} \theta^2_{M^{(a)}} = \frac{1}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{b\in\mathbb{B}\\ c\in\mathbb{A}}} f_a(c)^{-1}\, \gamma(b + \phi^{-1}(\psi_\nu(b)), c)^{-1} \\ &= \frac{1}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{b\in\mathbb{B}\\ c\in\mathbb{A}}} f_a(c)^{-1}\, \gamma(b,c)^{-1}\, \gamma(\phi^{-1}(\psi_\nu(b)), c)^{-1}\ . \end{aligned} \tag{6.2.89}$$

Any $b \in \mathbb{B}$ can be split as

$$b = \iota(\phi^{-1}(\beta)) + s(x) \tag{6.2.90}$$

[26] In order not to clutter we will suppress the label g in what follows, as there is only one nontrivial G defect anyway.

[27] To show that the result holds, consider (A.3.10) and set $g(x_i) = x_i$. The matrix r_L can then be eliminated on the two sides. Decomposing the module M_g in its components and using the formulas (6.2.26) for the R matrix gives the desired result.

[28] Besides identifying twist defects related by fusion with the lines of $\mathcal{L}_D$, one also has to impose locality conditions, that depend on $\tilde{\eta}$ (see Appendix A). Together these constraints single out a unique twist defect for each choice of equivariantization.

with $\beta \in N(\mathbb{B})$ and $x \in \mathbb{B}/\text{Rad}(\nu)$. Here ι is the inclusion of $\text{Rad}(\nu)$ in $\mathbb{B}$ and $s : \mathbb{B}/\text{Rad}(\nu) \to \mathbb{B}$ is a section. Using linearity of γ and that $\psi_\nu(\phi^{-1}(\beta)) = 0$ we see that the only β-dependent factor in the summand is $\beta(c)$, so that the sum over β constraints $c \in \mathbb{B}$. We then have

$$\theta^2_{M^{(a)}} = \frac{|\text{Rad}(\nu)|}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{b' \in \mathbb{B} \\ x \in \mathbb{B}/\text{Rad}(\nu)}} f_a(b')^{-1}\, \gamma(s(x), b')^{-1}\, \gamma(\sigma(s(x)), b')^{-1} \,. \tag{6.2.91}$$

We now split also b' as (6.2.90) obtaining

$$\theta^2_{M^{(a)}} = \frac{|\text{Rad}(\nu)|}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{\beta' \in N(\mathbb{B}) \\ x, x' \in \mathbb{B}/\text{Rad}(\nu)}} f_a(\phi^{-1}(\beta'))^{-1} f_a(s(x'))^{-1}\, \gamma(s(x), s(x'))^{-1}\, \gamma(\sigma(s(x)), s(x'))^{-1} \tag{6.2.92}$$

where we noticed that $f_a(\phi^{-1}(\beta) + b) = f_a(\phi^{-1}(\beta)) f_a(b)$ for any $\beta \in N(\mathbb{B})$ and $b \in \mathbb{B}$. Because of this f_a restricted on $\text{Rad}(\nu)$ is a character, hence the sum over β' yields $\theta^2_{M^{(a)}} = 0$ unless $f_a(\phi^{-1}(\beta')) = 1$ for any $\beta' \in N(\mathbb{B})$, *i.e.* f_a must restrict to the trivial character on $\text{Rad}(\nu)$ to avoid an inconsistent answer. Plugging this in we get

$$\theta^2_{M^{(a)}} = \frac{|\text{Rad}(\nu)|^2}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{\beta' \in N(\mathbb{B}) \\ x, x' \in \mathbb{B}/\text{Rad}(\nu)}} f_a(s(x'))^{-1}\, \gamma(s(x) + \sigma(s(x)), s(x'))^{-1} \tag{6.2.93}$$

notice that, due to the property of f_a mentioned above, this expression is independent of the sections chosen hence we shall drop them in the following. Using (6.2.50) we rewrite

$$\gamma(\sigma(x), x') = \gamma(x, \sigma(x'))^{-1} \tag{6.2.94}$$

so that summing over x constraints x' to be fixed by σ. All in all the spin of the twist defect is

$$\theta^2_{M^{(a)}} = \frac{|\text{Rad}(\nu)|}{\sqrt{|\mathbb{A}|}} \sum_{\substack{b \in \mathbb{B}/\text{Rad}(\nu) \\ \sigma(b) = b}} f_a(b)^{-1} \tag{6.2.95}$$

hence, due to (6.2.86), confirming that

$$\theta^2_{M^{(a)}} = \text{Arf}(\tilde{\eta}) = Y \,. \tag{6.2.96}$$

Notice that this computation automatically provides with the proper normalization to ensure that $\text{Arf}(\tilde{\eta}) = \pm 1$.

Example. Consider $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with off-diagonal bicharacter $\gamma_1^{(\text{O})}$. The invariant algebra is

$$\mathcal{L}_D = \left\{ \big((a_1, a_2); (-a_2, a_1)\big) \;\big|\; a_1, a_2 \in \mathbb{Z}_n \right\} . \tag{6.2.97}$$

Our choice for the functions f_a in (6.2.27) is

$$f_{(a_1,a_2)}(b_1, b_2) = \exp\!\left(-\frac{2\pi i}{n}\, b_1 b_2\right) \gamma(a, b) \,. \tag{6.2.98}$$

From this it is simple to show that

$$\theta^2_{\sigma_a} = \exp\!\left(-\frac{2\pi i}{n}\, a_1 a_2\right) . \tag{6.2.99}$$

A module $M^{(a)}$ is given, as an object, by

$$M^{(a)} = \begin{cases} \bigoplus_{b\,\in\,\mathbb{Z}_n} \sigma_{b,\,a_2} & \text{for } n \text{ odd}\,, \\ \bigoplus_{b\,\in\,\mathbb{Z}_n} \sigma_{a_1+2b,\,a_2} & \text{for } n \text{ even}\,. \end{cases} \tag{6.2.100}$$

Imposing the spin $\theta^2_{\sigma_a}$ to be constant on the orbit $M^{(a)}$ strongly constrains the possible local module candidates. One finds that for n odd there is only one consistent choice of module M, namely $M^{(0,0)}$ while for n even there are four, corresponding to $(a_1, a_2) = \left(s_1,\, \frac{n}{2}s_2\right)$ and $s_{1,2} \in \{0,1\}$. Their spins squared are:

$M^{(a)}$	$M^{(0,\,0)}$	$M^{(1,\,0)}$	$M^{(0,\,n/2)}$	$M^{(1,\,n/2)}$
θ^2_M	1	1	1	-1

(6.2.101)

It is possible to check that all four satisfy the locality condition (A.3.10) for the four inequivalent choices of $\tilde{\eta}$, parametrized by $H^1_\rho(\mathbb{Z}_2,\, \mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_2 \times \mathbb{Z}_2$. We will see in the next section how the same result can be obtained in terms of symmetry fractionalization.

6.2.5 Second obstruction and symmetry fractionalization

The discussion in the previous section gave us a description of the second obstruction from a purely bulk perspective. However, it requires precise knowledge of the full categorical data of the 3d MTC, hence it is hard to generalize it to higher-dimensional cases. Moreover, it leaves one conceptual problem to address: what is the physical interpretation of the different choices of equivariantization from the point of view of the boundary? We make here a proposal that solves both issues: different choices of equivariantization in the bulk lead to different ways to couple the symmetry to backgrounds fields. This is the *symmetry fractionalization* phenomenon that we introduced in Section 1.5 following [15, 53, 54].

Even though we do not know how to turn on background fields for the non-invertible symmetry directly, we can use the vanishing of the first obstruction to reduce the problem to the discussion of inequivalent couplings to standard $\mathbb{Z}_2$ background fields on the invertible boundary. There we also have the 0-form symmetry $\mathcal{S} = \mathcal{Z}(\mathbb{A})/\mathcal{L}_D$, which crucially has a mixed anomaly with G. It is known [15, 53, 54] that in such cases the cubic G anomaly might not have an intrinsic value: it can be changed by choosing different symmetry fractionalization classes. Analyzing this phenomenon will lead to the required condition for the vanishing of the second obstruction.

Let us start by determining the mixed anomaly between G and $\mathcal{S}$. The duality action Φ, which leaves $\mathcal{L}_D$ invariant, descends to an action on the quotient $\mathcal{S} = (\mathbb{A} \times \mathbb{A}^\vee)/\mathcal{L}_D$, which we already described in detail in Section 6.2.3. For simplicity we consider here the case $\mathbb{B} = \mathbb{A}$, so that $\mathcal{S} = \mathbb{A}^\vee$. The general case is qualitatively analogous and we report it in Appendix B.2.2. We use the duality isomorphism ϕ to write the background for $\mathbb{A}^\vee$ as $\phi(B)$ with $B \in H^1(X, \mathbb{A})$. The partition function of the invertible boundary theory coupled to a background B can be easily expressed in terms of the reference electric boundary:

$$Z_{\text{inv}}\left[\phi(B)\right] = \sum_{b\,\in\,H^1(X,\mathbb{A})} \exp\left[2\pi i \int_X b^*\nu + 2\pi i \int_X b \cup \phi(B)\right] Z_{\text{e}}[b]\;. \tag{6.2.102}$$

Here $b^*\nu \in H^2\big(X, U(1)\big)$ is the pull-back of $\nu \in H^2\big(\mathbb{A}, U(1)\big)$, understood in additive notation. The duality maps Z_{e} to the partition function of the magnetic theory Z_{m}, which in turn can be expressed as a discrete gauging of the electric theory:

$$\Phi \cdot Z_{\text{e}}[b] = Z_{\text{m}}\left[\phi(b)\right] = \sum_{a\,\in\,H^1(X,\mathbb{A})} \exp\left[2\pi i \int_X \phi(a) \cup b\right] Z_{\text{e}}[a]\;. \tag{6.2.103}$$

The action of Φ on the invertible boundary can be derived combining (6.2.102) with (6.2.103), using that Φ only acts on the partition functions Z, and it reads

$$\Phi \cdot Z_{\text{inv}}\big[\phi(B)\big] = \exp\left[2\pi i \int_X B^* \nu\right] Z_{\text{inv}}\big[\phi(\sigma B)\big] \ . \tag{6.2.104}$$

The overall phase stems from a mixed 't Hooft anomaly between G and $\mathcal{S}$. Crucially, from (6.2.104) we find that $G \cong \mathbb{Z}_2$ acts non trivially on $\mathcal{S}$ through an automorphism $\rho : G \to \text{Aut}(\mathcal{S})$ such that

$$\rho_{\underline{1}}(B) = \sigma B \ , \tag{6.2.105}$$

so that the total symmetry of the invertible boundary is a semidirect product $\mathcal{S} \rtimes_\rho G$. Thanks to

$$\exp\left[2\pi i \int_X B^* \big(\nu \circ \sigma\big)\right] = \exp\left[-2\pi i \int_X B^* \nu\right] \ , \tag{6.2.106}$$

which is the integrated additive version of (6.2.51), the aforementioned anomaly is consistent with the $\mathbb{Z}_2$ symmetry. Let us write the inflow action for the anomalous phase, introducing a background filed $A \in H^1(X, \mathbb{Z}_2)$ for G. The general construction is detailed in Appendix C.1. The bottom line of that discussion is that such anomalies are classified by $\mu \in H^1_\rho\big(\mathbb{Z}_2, H^2\big(\mathbb{A}, U(1)\big)\big)$ in terms of which the 3d inflow action is

$$S_\mu = 2\pi i \int_{X_3} \mu(A) \cup B \cup B \ . \tag{6.2.107}$$

In components this is defined as

$$\Big(\mu(A) \cup B \cup B\Big)_{ijkl} = \mu(A_{ij})\big(\rho_{A_{ij}} B_{jk} \, , \, \rho_{A_{ik}} B_{kl}\big) \ . \tag{6.2.108}$$

A gauge variation $A \to A + d\lambda$ produces a boundary term

$$S_\mu \to S_\mu + 2\pi i \int_{\partial X_3} \mu(\lambda) \cup B \cup B \ . \tag{6.2.109}$$

The class μ can be thought of as a function $\mu : \mathbb{Z}_2 \to H^2\big(\mathbb{A}, U(1)\big)$ satisfying the twisted cocycle condition (using addivite notation)

$$\rho_g \, \mu(h) + \mu(g) = \mu(g+h) \ , \tag{6.2.110}$$

and subject to the the identification

$$\mu(g) \cong \mu(g) + \rho_g \, \xi - \xi \qquad \text{for any} \qquad \xi \in H^2\big(\mathbb{A}, U(1)\big) \ . \tag{6.2.111}$$

Because of the relation (6.2.51), which in additive notation reads $\sigma \cdot \nu = -\nu$, we can consistently choose

$$\mu(\underline{0}) = 0 \ , \qquad \mu(\underline{1}) = \nu \ . \tag{6.2.112}$$

Notice that this makes sense because Φ^2 leaves Z_{inv} invariant. With this choice, taking a background such that the pull-back of A to the boundary ∂X_3 is $\underline{0}$ and performing a gauge transformation $A \to A + d\lambda$ with $\lambda\big|_{\partial X_3} = \underline{1}$, one reproduces the anomalous phase (6.2.104). This construction also provides a convenient way to determine whether the anomalous phase (6.2.104) corresponds to a true anomaly or can be cancelled by a local counterterm. Indeed the latter situation occurs if and only if μ is cohomologically trivial, namely

$$\nu = \sigma \cdot \xi - \xi \tag{6.2.113}$$

for some $\xi \in H^2(\mathbb{A}, U(1))$. In this case the anomalous phase is eliminated by modifying the action coupled to $B \in H^1(X_2, \mathbb{A})$ by the addition of the local counterterm

$$S_{\text{c.t.}} = 2\pi i \int_{X_2} B^* \xi \ . \tag{6.2.114}$$

If there exists no ξ satisfying (6.2.113) then the anomalous phase cannot be cancelled and there is an anomaly. To show the power of this method, let us discuss the example of $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with diagonal symmetric bicharacter $\gamma_{1,1}^{(\text{D})}$ and

$$\nu\big((x_1, x_2),\, (y_1, y_2)\big) = \frac{r}{n}\, x_1 y_2 \qquad \text{with} \qquad r^2 = -1 \mod n\ . \tag{6.2.115}$$

Then σ acts on $\mathbb{A}$ as $\sigma(x_1, x_2) = (rx_2, -rx_1)$, and since the most general $\xi \in H^2(\mathbb{A}, U(1))$ is represented as $\xi\big((x_1, x_2), (y_1, y_2)\big) = \frac{s}{n} x_1 y_2$ or equivalently as $\xi\big((x_1, x_2), (y_1, y_2)\big) = -\frac{s}{n} x_2 y_1$ then

$$\big(\sigma \cdot \xi - \xi\big)\big((x_1, x_2), (y_1, y_2)\big) = -\frac{2s}{n}\, x_1 y_2\ . \tag{6.2.116}$$

For n odd, we can always choose $s = -2^{-1} r$ hence the anomalous phase can be cancelled by a local counterterm and it is not an anomaly. On the other hand, for n even, r is necessarily odd and thus no choice of s can cancel the anomalous phase: in this case this is a genuine anomaly.

As already argued, the question of what is the value of the pure $G \cong \mathbb{Z}_2$ anomaly on the invertible boundary is not well-posed until we specify how G couples to a background field $A \in H^1(X_2, G)$. In the boundary global variant where the full symmetry category is invertible, the presence of another 0-form symmetry $\mathcal{S}$ allows one to make discrete choices for that coupling labelled by a class

$$\eta \in H^1_\rho(G, \mathcal{S})\ , \tag{6.2.117}$$

which satisfies the twisted cocycle condition $\rho_g\, \eta(h) + \eta(g) = \eta(g+h)$ and is subject to the identification $\eta(g) \cong \eta(g) + \rho_g\, c - c$ for any $c \in \mathcal{S}$, similarly to (6.2.110)–(6.2.111). So η specifies a (twisted) homomorphism from G to $\mathcal{S}$ which allows one to modify the minimal coupling prescription for A, declaring that the latter effectively couples to the diagonal subgroup of G and the image $\eta(G) \subset \mathcal{S}$. The anomaly cannot be unambiguously determined until we specify η because different choices correspond to different $\mathbb{Z}_2$ subgroups of the full 0-form symmetry group and, due to the mixed anomaly (6.2.107), they can have different anomalies.

This is nothing but *symmetry fractionalization* [15, 53, 54] (Section 1.5)[29] (which will be relevant for the 4d/5d case), but we will use the same terminology to emphasize a unified description. The crucial point is that in general there is no canonical choice and we can only talk about differences of anomalies induced by a certain class η. This is easy to implement at the level of background fields. When $A \in H^1(X_2, \mathbb{Z}_2)$ is activated, the symmetry fractionalization class changes the background $B \in H^1(X_2, \mathcal{S})$ to

$$B' = B + A^* \eta = B + \eta(A)\ . \tag{6.2.118}$$

By plugging this expression into the mixed anomaly (6.2.107) we change the pure $\mathbb{Z}_2$ anomaly, classified by $H^3(\mathbb{Z}_2, U(1))$, by an extra piece

$$S_{\text{pure}} = 2\pi i \int_{X_3} \mu(A) \cup \eta(A) \cup \eta(A) \ \equiv\ 2\pi i \int_{X_3} A^* y \tag{6.2.119}$$

[29] See, *e.g.*, [77, 156–159] and references therein for discussions of symmetry fractionalization in the condensed matter literature.

that can be written in terms of a class $y \in H^3\big(\mathbb{Z}_2, U(1)\big)$. An explicit expression for $y(g_1, g_2, g_3)$ can be derived by recasting $\mu(A) \cup \eta(A) \cup \eta(A)$ as

$$\Big(\mu(A) \cup \eta(A) \cup \eta(A)\Big)_{ijkl} = -\mu(-A_{ij})\Big(\eta(A_{jk})\,,\ \rho_{A_{jk}}\eta(A_{kl})\Big)\ . \tag{6.2.120}$$

This is useful because A appears with only three different pairs of indices, and we conclude that

$$y(g_1, g_2, g_3) = -\mu(-g_1)\Big(\eta(g_2)\,,\ \rho_{g_2}\eta(g_3)\Big)\ . \tag{6.2.121}$$

The possible non-triviality of this 3-cocycle is determined by its value at $g_1 = g_2 = g_3 = \underline{1}$, and we will denote simply by μ and η their values at $g = \underline{1}$. Since $\mu = \nu$ and $\rho\,\eta = -\eta$ we obtain

$$y \;\equiv\; y(\underline{1}, \underline{1}, \underline{1}) = \nu(\eta, \eta)\ . \tag{6.2.122}$$

Going back to multiplicative notation, we obtain that

$$Y = \nu(\eta,\, \eta)\ , \tag{6.2.123}$$

we will see in examples that coincides with the SPT phase obtained by the equivariantization procedure in Section 6.2.4.

Examples

Let us apply the general discussion to the previously discussed examples.

1. The first example is $\mathbb{A} = \mathbb{Z}_n$ where a duality-invariant lattice is present only for $n = p^2$. The choice of discrete torsion ν is trivial, so there is no way to shift the "bare" Frobenius-Schur indicator ϵ and the second obstruction vanishes if and only if $\epsilon = 1$.

2. Next we consider $\mathrm{TY}(\mathbb{Z}_n \times \mathbb{Z}_n)$. Choosing the diagonal bicharacter $\gamma^{(\mathrm{D})}_{1,1}$ in (6.2.62), the duality-invariant boundaries are obtained by gauging the full $\mathbb{A}$ with discrete torsion ν such that (see (6.2.56))

$$\chi_\nu = \frac{1}{n}\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} \qquad\qquad \text{with} \qquad r^2 = -1 \mod n\ . \tag{6.2.124}$$

Thus the action $\rho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{A}^\vee)$ is $\rho_{\underline{1}}(a_1, a_2) = (ra_2, -ra_1)$. We look for all possible symmetry fractionalization classes $\eta \in H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n)$, which are determined by $\eta \equiv \eta(\underline{1}) = (x_1, x_2)$ constrained by $x_2 = rx_1$. Taking into account the identification

$$(x_1, rx_1) \sim (x_1, rx_1) + (rc_2 - c_1, -rc_1 - c_2) \tag{6.2.125}$$

and setting $c_2 = 0$, $c_1 = -1$ we realize that $x_1 \sim x_1 + 1$ and hence all cocycles are exact:

$$H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n) = 0\ . \tag{6.2.126}$$

Thus the phenomenon of symmetry fractionalization is absent in this case and there is only a single equivariantization for $\mathcal{L}_D$. The second obstruction again vanishes if and only if $\epsilon = 1$.

Choosing instead the off-diagonal bicharacter $\gamma^{(\mathrm{O})}_1$ is more interesting. As already discussed, the duality-invariant boundaries are associated with the alternating bicharacters

$$\chi_\nu = \frac{1}{n}\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} \qquad\qquad \text{with} \qquad r^2 = 1 \mod n\ . \tag{6.2.127}$$

Then $\rho_{\underline{1}}(a_1, a_2) = (-ra_1, ra_2)$ and the most general cocycle $\eta \in H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n)$ has $\eta = (x_1, x_2)$ with

$$(r-1)\, x_1 = 0 \mod n \, , \qquad (r+1)\, x_2 = 0 \mod n \, , \tag{6.2.128}$$

and is subject to the identifications $x_1 \sim x_1 - (r+1)c_1$, $x_2 \sim x_2 - (r-1)c_2$. Without loss of generality we can take $r = 1$, so that $2x_2 = 0$ and $x_1 \sim x_1 + 2$. Hence for n odd there is no symmetry fractionalization while for n even:

$$H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad (n \text{ even}) \tag{6.2.129}$$

generated by $\eta_{s_1,\, s_2} = \left(s_1, \frac{n}{2} s_2\right)$ with $s_{1,2} \in \{0, 1\}$. A representative for ν is

$$\nu(a, b) = \exp\left(\frac{2\pi i}{n}\, a_1 b_2\right) \tag{6.2.130}$$

and therefore

$$Y = \nu(\eta, \eta) = \exp\big(\pi i\, s_1 s_2\big) = \mathrm{Arf}(\tilde{\eta}) \, . \tag{6.2.131}$$

Thus the second obstruction vanishes if and only if

$$\epsilon = 1 \quad \text{and} \quad s_1 s_2 = 0 \, , \qquad \text{or} \qquad \epsilon = -1 \quad \text{and} \quad s_1 s_2 = 1 \, , \tag{6.2.132}$$

in agreement with the discussion in [21] for the case $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ and with our computations using the equivariantization of $\mathcal{L}_D$ around (6.2.101).

6.3 Anomalies of duality symmetries in 3+1 dimensions

We now extend the classification of anomalies for non-invertible duality defects to the four-dimensional case. As in 2d, we find that there are two obstructions to gauging a non-invertible duality symmetry. The first obstruction again hinges upon the absence of a duality-invariant bulk Lagrangian algebra $\mathcal{L}_D$. This maps to the fact that the 4d theory $\mathcal{T}$ coupled to the SymTFT must admit a duality-invariant global variant. The second obstruction is the presence of a cubic anomaly:

$$\epsilon_{\text{tot}} \in \Omega_5^{\text{spin}}(BG) \, , \tag{6.3.1}$$

which can be contaminated by a mixed anomaly involving the 0-form symmetry G and a 1-form symmetry $\mathcal{S}$ through a symmetry fractionalization mechanism similar to the 2d case, now encoded in a class $\eta \in H^2_\rho(G, \mathcal{S})$.

Well-known examples of 4d theories with self-duality symmetries are the free Maxwell theory, super-Yang-Mills theories with $\mathcal{N} = 4$ supersymmetry and whose gauge algebra is invariant under Langlands duality (*i.e.*, ADEFG as well as $B_2 \cong C_2$) [2, 24, 52, 104] and various theories of class $\mathcal{S}$ [9, 160]. Understanding the anomalies in these symmetries has immediate interesting consequences. For example, it has been recently observed [161] that the $\mathcal{N} = 1^*$ massive deformation of $\mathcal{N} = 4$ SYM preserves a self-duality symmetry. The well-known results about vacuum degeneracy in $\mathcal{N} = 1^*$ can then be reinterpreted as anomaly matching conditions. A second natural application is to constrain which $\mathcal{N} = 3$ theories can be described through a discrete gauging of $\mathcal{N} = 4$, which we comment upon in the conclusions.

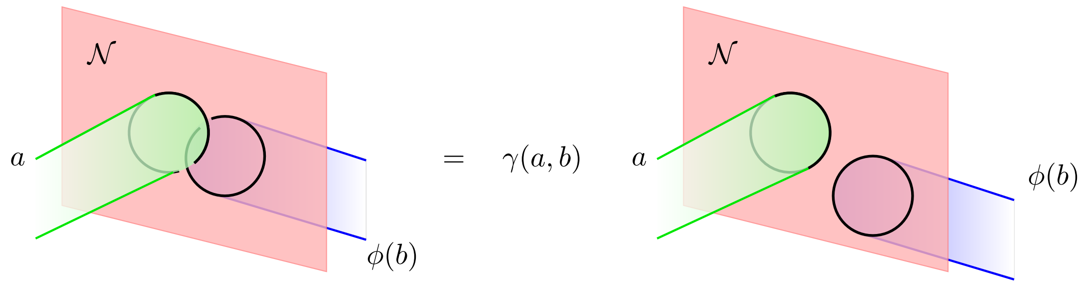


Figure 6.2: Braiding of lines W_a^L and $W_{\phi(b)}^R$ on the duality defect $\mathcal{N}$. Unlinking the line configuration gives rise to the symmetric bicharacter $\gamma(a,b)$.

6.3.1 Duality defects

Much of our analysis in Section 6.2 can be generalized to self-duality defects in four-dimensional theories that are self-dual under the gauging of a 1-form symmetry $\mathbb{A}$, possibly with discrete torsion [24, 52, 104]. Again, the self-duality must be supplied with a choice of isomorphism $\phi : \mathbb{A} \to \mathbb{A}^\vee$. While a complete description of the underlying fusion 3-category $\mathcal{C}$ is still out of reach, some of the relevant data can be spelled out explicitly.[30] As stated in the introduction, this is a graded category with the grading being implemented by the duality group G, that for now we take to be cyclic. The fusion rules take the form

$$a \times \mathcal{N}_g = \mathcal{N}_g \times a = \mathcal{N}_g \,, \qquad \mathcal{N}_g(\Sigma) \times \overline{\mathcal{N}}_g(\Sigma) = \sum_{a \in H_2(\Sigma, \mathbb{A})} a = C_{\mathbb{A}}(\Sigma) \,, \tag{6.3.2}$$

where $\mathcal{N}_{g\neq 0}$ are the duality interfaces, Σ the 3-manifold where they live, a a 1-form symmetry surface, and $C_{\mathbb{A}}(\Sigma)$ the condensate of $\mathbb{A}$ on Σ. We studied the fusion of $\mathcal{N}_g \times \mathcal{N}_h$ in Chapter 5 (see also [9]), and is group-like at the level of connected components, *i.e.*, forgetting the appearance of condensates (see footnote 1). Assuming that G is cyclic, let $\mathcal{N}$ be a generator of it.

A first piece of categorical data can be obtained by noticing that the 1-form symmetry surfaces can end topologically on $\mathcal{N}$ thus defining topological line operators W_a^L and W_β^R, where L/R encode the side (Left or Right) on which the 1-form symmetry surfaces a, β end.[31] These line defects must compose according to the $\mathbb{A}$ group law, modulo undetectable decoupled objects:[32]

$$W_a^L \times W_b^L = W_{a+b}^L \tag{6.3.3}$$

and similarly for W_β^R. Following the same logic as in the Tambara-Yamagami case, we consider the braiding between endlines of 1-form symmetry surfaces a and $\beta = \phi(b)$ ending on the two sides of the duality defect $\mathcal{N}$. The endline of W_β^R is an 't Hooft line T_β^L from the point of view of the left side, and hence it braids canonically with W_a^L. We conclude that the braiding between W_a^L and $W_{\phi(b)}^R$ is given by a symmetric bicharacter γ:

$$\mathcal{B}_{W_a^L, W_{\phi(b)}^R} = \phi(b)\, a = \gamma(a,b) \,, \tag{6.3.4}$$

where the symmetry of γ follows from the fact that we should get the same result if we worked in the magnetic frame instead. The configuration is depicted in Figure 6.2.

[30] The SymTFT analysis offers a complementary viewpoint on the data constituting the duality category on the boundary, which might be easier to handle. We explain how the data we describe here is matched between bulk and boundary in Section 6.3.2.

[31] One could think of those as 2-morphisms $W_a^R : a \times \mathbb{1}_{\mathcal{N}} \to \mathbb{1}_{\mathcal{N}}$ and $W_\beta^L : \mathbb{1}_{\mathcal{N}} \times \beta \to \mathbb{1}_{\mathcal{N}}$, where $\mathbb{1}_{\mathcal{N}}$ is the identity endomorphism of $\mathcal{N}$.

[32] The importance of modding out such decoupled TQFTs has been recently emphasized in [162] in a related context.

The lines W_a^L and W_β^R form a $3d$ TQFT $\mathcal{A}$, but such a description is clearly non-minimal: lines of the form $\mathcal{K}_a = W_a^L \times W_{\phi(-a)}^R$ are decoupled from the bulk 1-form symmetry and constitute an undetectable sector $\mathcal{A}_0$. Quotienting this out gives the minimal description $\mathcal{A}_{\text{min}}$ of the category of lines living on the defect.[33] This produces, in general, a set of lines L_a forming a minimal TQFT $\mathcal{A}^{\mathbb{A},q}$ with 1-form symmetry $\mathbb{A}$ [111], where q a quadratic refinement of the symmetric bilinear form γ. This resonates with previous results obtained from the SymTFT perspective in Chapter 5.

Finally, as in the 2d case, we can associate to $\mathcal{N}$ a pure G anomaly ϵ. This is a higher analogue of the Frobenius-Schur indicator. Indeed, ϵ can be understood as a standard G 't Hooft anomaly on four-manifolds with trivial $H_2(X,\mathbb{A})$. On these manifolds, $\{\mathcal{N}_g\}$ behave as a standard invertible symmetry according to the fusion rules (6.3.2). At the level of the SymTFT, the presence of a nontrivial ϵ gives a DW twist for the theory $\mathcal{Z}(\mathcal{C})$. All in all, we find that the known data defining a self-duality category in 4d, or at least a subset of it, is given by a pure anomaly ϵ for the self-duality group and a symmetric bicharacter $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$.

In the ensuing analysis we will make two simplifying assumptions. First, we will consider duality defects on spin manifolds, $w_2(TX) = 0$. The classification of discrete gauging operations (global variants of a gauge theory) is different on non-spin manifolds, as the set of discrete theta angles is larger.[34] Physically this amounts to the possibility of assigning a well-defined spin to lines as this cannot be screened by heavy neutral fermions [163]. This restriction has physical consequences on the obstruction theory outlined above: some duality defects can be anomaly free on spin manifolds, but anomalous in the presence of a nontrivial w_2.[35]. As a prototypical example, consider the $\mathfrak{su}(2)$ $\mathcal{N}=4$ SYM theory. This admits an S-invariant global variant $\mathrm{SO}(3)_-$ on spin manifolds. On non-spin (but orientable) manifolds this variant splits into $\mathrm{SO}(3)_-^{\mathrm{b}}$ and $\mathrm{SO}(3)_-^{\mathrm{f}}$, where b/f (bosonic/fermionic) refer to the spin of the generator of the lattice of genuine lines. According to [163] (Appendix C) the two objects are interchanged by S. Thus, although the duality symmetry in SU(2) $\mathcal{N}=4$ SYM might be non-anomalous on spin manifolds, it is anomalous on generic orientable manifolds.

Our second assumption is to consider duality defects for which G does not contain fermion parity. This for example excludes the vanilla S-duality of the $\mathcal{N}=4$ SYM theory, for which $S^4 = (-1)^F$, but includes the situation where S is twisted by a discrete R-symmetry [161]. At the practical level, this implies that the relevant cobordism classification for cubic G anomalies is given by $\Omega_5^{\text{spin}}(BG)$ as opposed to $\Omega_5^{\text{spin}_G}(\text{pt})$. Both groups have been computed, *e.g.*, in [164, 165].

6.3.2 SymTFT and Lagrangian algebras

The SymTFT for 4d duality defects, as discussed in Chapter 5, can be described in close analogy with the 2d case. We start from a 5d Dijkgraaf-Witten theory for a 1-form symmetry $\mathbb{A}$ with trivial twist. This has topological surface operators labelled by pairs $(a,\alpha) \in \mathbb{A} \times \mathbb{A}^\vee$ with antisymmetric canonical braiding

$$\mathcal{B}_{(a_1,\alpha_1),(a_2,\alpha_2)} = \alpha_1(a_2)\, \alpha_2(a_1)^{-1} \quad \in U(1)\,. \tag{6.3.5}$$

As in three dimensions, the 5d pure 2-form gauge theory for $\mathbb{A}$ enjoys *electric-magnetic* duality, corresponding to a choice of isomorphism ϕ. As explained in 2.2.1, since in 5d the braiding is antisymmetric,

[33] Formally one stacks $\mathcal{A}$ with the orientation reversal of $\mathcal{A}_0$ and gauges the diagonal symmetry $\mathbb{A}$: $\mathcal{A}_{\text{min}} = (\mathcal{A} \times \overline{\mathcal{A}}_0)/\mathbb{A}$.

[34] As an illuminating example, consider $\mathbb{A} = \mathbb{Z}_n$ with n even. On generic manifolds, discrete torsion terms are classified by $H^4\big(B^2\mathbb{Z}_n, U(1)\big) = \mathbb{Z}_{2n}$, while on spin manifolds the order-two element of this group vanishes due to the Wu formula $B \cup B = B \cup \big(w_2 + w_1^2\big)$ mod 2, where $w_j(TX)$ are the Stiefel-Whitney classes of X. This discussion generalizes to arbitrary $\mathbb{A}$ in a straightforward manner.

[35] Loosely speaking, this is some kind of mixed anomaly with gravity, due to the dependence on $w_2(TX)$.

the duality symmetry is

$$\begin{aligned} S: \quad \mathbb{A}\times\mathbb{A}^\vee \quad &\to \qquad \mathbb{A}\times\mathbb{A}^\vee \\ (a,\alpha) \quad &\mapsto \quad \big(-\phi^{-1}(\alpha),\, \phi(a)\big) \end{aligned} \tag{6.3.6}$$

that is an order-four automorphism $S^2(a,\alpha) = (-a,-\alpha)$, namely $S^2 = C$. The 5d DW($\mathbb{A}$) theory enjoys a larger set of 0-form symmetries, for any group $\mathbb{A}$. Indeed we can define another generator

$$T: (a,\,\alpha) \mapsto \big(a + \phi^{-1}(\alpha),\, \alpha\big)\ , \tag{6.3.7}$$

and in this way construct an order-three automorphism of $\mathbb{A}\times\mathbb{A}^\vee$:

$$CST: (a,\,\alpha) \mapsto \big(\phi^{-1}(\alpha),\, -\alpha - \phi(a)\big) \qquad \text{such that} \qquad (CST)^3 = \mathbb{1}\ . \tag{6.3.8}$$

The SymTFT for the duality or triality defects is then defined by gauging the group G generated by S or CST, respectively. This gauging admits a choice of discrete torsion, which on spin manifolds is classified by

$$\epsilon \in \Omega_5^{\text{spin}}(BG)\ , \tag{6.3.9}$$

and can be thought of as the higher analogue of the Frobenius-Schur indicator we introduced before.

Notice that if we gauge the group $G = \mathbb{Z}_4$ generated by S, the generator maps $(b,0) \overset{S}{\mapsto} \big(0,\phi(b)\big)$ and thus the isomorphism ϕ appearing in (6.3.6) is precisely the one extracted from the boundary theory using (6.3.4). The same is true if we gauge $G = \mathbb{Z}_6$ generated by ST since $(b,0) \overset{ST}{\mapsto} \big(0,\phi(b)\big)$. On the other hand, if we gauge $G = \mathbb{Z}_3$ generated by CST, the isomorphisms in (6.3.6) and (6.3.4) differ by C.

The same argument for the first obstruction corresponding to the absence of G-invariant Lagrangian algebras in the DW($\mathbb{A}$) theory carries over to the 5d case. We are thus led to study the properties of gapped boundaries of the pure 2-form gauge theory for $\mathbb{A}$. These are labelled by two discrete choices, as in 2d:

- a subgroup $\mathbb{B}\subset\mathbb{A}$ to be gauged;
- a class $[\nu] \in H^4\big(B^2\mathbb{B}, U(1)\big)$ specifying the discrete torsion.

Recall that in 2d the discrete-torsion classes are classified by alternating bicharacters. The analog here is the identification of $H^4\big(B^2\mathbb{B}, U(1)\big)$ with the dual of the universal quadratic group $\Gamma(\mathbb{B})$ (see [15, 93] for details):

$$H^4\big(B^2\mathbb{B}, U(1)\big) \;\cong\; \Gamma(\mathbb{B})^\vee\ . \tag{6.3.10}$$

This means that any discrete torsion class $[\nu]$ is represented by a quadratic function $q_\nu : \mathbb{B}\to U(1)$. The group $\Gamma(\mathbb{B})$ is equipped with a quadratic function $\mathcal{Q}: \mathbb{B}\to\Gamma(\mathbb{B})$ and is such that for any Abelian group V, any quadratic function $q : \mathbb{B}\to V$ factorizes as $q = \tilde{q}\circ\mathcal{Q}$ with $\tilde{q}:\Gamma(\mathbb{B})\to V$ a group homomorphism. Therefore, a quadratic function $q_\nu : \mathbb{B}\to U(1)$ is represented by a group homomorphism $\tilde{q}_\nu : \Gamma(\mathbb{B})\to U(1)$. The topological term implementing the discrete torsion is

$$S_{\text{torsion}} = \int_{X_4} B^*\nu = \int_{X_4} \tilde{q}_\nu\big(\mathfrak{P}(B)\big)\ . \tag{6.3.11}$$

Here $\mathfrak{P}\in H^4\big(B^2\mathbb{B},\Gamma(\mathbb{B})\big)$ is the special element whose representative homomorphism[36] is the identity map, $\tilde{q}_{\mathfrak{P}} : \Gamma(\mathbb{B}) \overset{\text{id}}{\to} \Gamma(\mathbb{B})$, called the universal Pontryagin square class. Then one constructs its pull back $\mathfrak{P}(B) \equiv B^*\mathfrak{P} \in H^4\big(X_4,\Gamma(\mathbb{B})\big)$ which is called the Pontryagin square of B, whilst $\tilde{q}_\nu\in\Gamma(\mathbb{B})^\vee$ is the homomorphism associated with the quadratic function q_ν.

[36]Indeed (6.3.10) generalizes to $H^4\big(B^2\mathbb{B},\mathbb{C}\big)\cong \text{Hom}\big(\Gamma(\mathbb{B}),\mathbb{C}\big)$ for any Abelian group $\mathbb{C}$.

As already explained in Section 6.2.1, each quadratic function q_ν has an associated symmetric bicharacter $\chi_\nu : \mathbb{B} \times \mathbb{B} \to U(1)$. Crucially, if X_4 is a four-dimensional spin manifold, then two discrete torsions ν, ν' leading to two quadratic functions q_ν, $q_{\nu'}$ which are different quadratic refinements of the same bicharacter, lead to the same topological term [111, 163]: $\int_{X_4} B^*\nu = \int_{X_4} B^*\nu'$. Thus, by working on spin manifolds, we can safely label topological manipulations of the boundary theory in terms of a choice of subgroup $\mathbb{B} \subset \mathbb{A}$ and of a symmetric bicharacter χ_ν. Then most of the results will be closely analogous to the 2d/3d case, just replacing antisymmetric with symmetric bicharacters.

As explained, on spin manifolds we can label the Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$ in terms of the data $(\mathbb{B}, \chi_\nu)$. The corresponding gapped boundary has a 1-form symmetry

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L}_{\mathbb{B},[\nu]} \ . \tag{6.3.12}$$

One can easily adapt the 3d discussion in order to explicitly write the form of the general Lagrangian algebra. The symmetric bicharacter $\chi_\nu : \mathbb{B} \times \mathbb{B} \to U(1)$ induces a group homomorphism $\psi_\nu : \mathbb{B} \to \mathbb{B}^\vee$ as in the 3d case. Given a pair $(\mathbb{B}, \chi_\nu)$ we construct the Lagrangian algebra $\mathcal{L}_{\mathbb{B},[\nu]} \subset \mathbb{A} \times \mathbb{A}^\vee$ as

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big(b,\, \beta\psi_\nu(b)\big) \in \mathbb{A} \times \mathbb{A}^\vee \ \Big|\ b \in \mathbb{B}\,, \quad \beta \in N(\mathbb{B}) \Big\} \ . \tag{6.3.13}$$

This has cardinality $|\mathbb{A}|$ and is Lagrangian since $\mathcal{B}_{(b_1,\beta_1),(b_2,\beta_2)} = \chi_\nu(b_2, b_1)\chi_\nu(b_1, b_2)^{-1} = 1$, where (b, β) is a shorthand for $\big(b, \beta\psi_\nu(b)\big)$ and we used the symmetry of χ_ν. As in the 3d case (see Appendix B.1) one can show that all Lagrangian algebras of the 5d DW($\mathbb{A}$) theory are of this form.

6.3.3 First obstruction

After fixing a choice of electric-magnetic duality, we ask what are the conditions for a duality-invariant Lagrangian algebra $\mathcal{L}_D = \Phi(\mathcal{L}_D)$ to exist. We will study two cases: $\Phi = S$ (duality) and $\Phi = CST$ (triality). Other cyclic 0-form symmetry groups, when they exist, can be treated similarly. As we previously showed, all Lagrangian algebras are of the form (6.3.13). To verify whether a lattice is Φ-invariant, as in 3d, we impose that the pairing between $\mathcal{L}$ and $\Phi(\mathcal{L})$ be trivial. The analysis is analogous to the 3d case. For both choices of Φ, we find the necessary condition

$$\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B}) \ , \tag{6.3.14}$$

where $\mathrm{Rad}(\nu)$ is the kernel of ψ_ν. As in the 3d case, this implies that $|\mathbb{B}|^2 = k|\mathbb{A}|$ for some positive integer $k = \big|\mathbb{B}/\,\mathrm{Rad}(\nu)\big| \in \mathbb{N}$, and again $\mathbb{B}$ cannot be smaller than Lagrangian. Notice however that since χ_ν is now symmetric rather than antisymmetric, we cannot conclude that $|\mathbb{A}|$ (and in particular k) is a perfect square. Indeed we will see explicit counterexamples, hence showing that in higher categories the obstruction from non-integer quantum dimensions of [20] does not hold.

The remaining conditions depend on Φ and are listed below.

Duality. The automorphism $\sigma = \phi^{-1}\psi_\nu$ of $\mathbb{B}/\mathrm{Rad}(\nu)$ must satisfy

$$\sigma^2 = -1 \ . \tag{6.3.15}$$

In particular σ allows us to relate the two symmetric bicharacters as

$$\gamma\big(\sigma(a),\, b\big) = \chi_\nu(a, b) \ . \tag{6.3.16}$$

From the two equations above it follows that σ is an order-two automorphism of the group of symmetric bilinear forms on $\mathbb{B}/\mathrm{Rad}(\nu)$:

$$\chi_\nu\big(\sigma(a), \sigma(b)\big)\ \chi_\nu(a, b) = 1 \ . \tag{6.3.17}$$

Triality. The automorphism $\tau = \phi^{-1}\psi_\nu$ must satisfy

$$1 + \tau + \tau^2 = 0 \,. \tag{6.3.18}$$

It is simple to show that the above implies that τ is an order-three operation: $\tau^3 = 1$. Also in this case, the restriction to $\mathbb{B}/\mathrm{Rad}(\nu)$ of

$$\gamma\big(\tau(a),\, b\big) = \chi_\nu(a,b) \tag{6.3.19}$$

holds. Using the two above equations it follows that τ is an order-three automorphism of the group of symmetric bilinear forms on $\mathbb{B}/\mathrm{Rad}(\nu)$:

$$\chi_\nu\big(\tau^2(a), \tau^2(b)\big)\; \chi_\nu\big(\tau(a), \tau(b)\big)\; \chi_\nu(a,b) = 1 \,. \tag{6.3.20}$$

Examples

1. Let us study the case of $\mathbb{A} = \mathbb{Z}_n$ with the standard symmetric bicharacter $\gamma(a_1,a_2) = \exp\big(\frac{2\pi i}{n}\, a_1 a_2\big)$. Consider a factorization $n = pq$ and a subgroup

$$\mathbb{B} = \big\{ b\, q \;\big|\; b = 0, \ldots, p-1 \big\} \cong \mathbb{Z}_p \tag{6.3.21}$$

so that $N(\mathbb{B}) \cong \mathbb{Z}_q$. Since duality invariance requires $\mathbb{B}$ to contain $\phi^{-1}\big(N(\mathbb{B})\big)$ as a subgroup, q must divide p and we set $p = \ell q$. A choice of ψ_ν is associated with another symmetric bicharacter χ_ν defined on $\mathbb{B}$:

$$\chi_\nu(b_1,b_2) = \exp\bigg(\frac{2\pi i r}{p}\, b_1 b_2 \bigg) \,, \tag{6.3.22}$$

where $r \in \{0, \ldots, p-1\}$. Notice that $\mathrm{Rad}(\nu) \cong \mathbb{Z}_{\gcd(r,p)}$ hence imposing $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B})$ forces $\gcd(r,p) = q$, namely $r = sq$ with $\gcd(s,\ell) = 1$. Furthermore, since the restriction of γ to $\mathbb{B}$ is $\gamma(qb_1, qb_2) = \exp\big(\frac{2\pi i}{\ell}\, b_1 b_2\big)$, over $\mathbb{B}/\mathrm{Rad}(\nu) \cong \mathbb{Z}_p/\mathbb{Z}_q \cong \mathbb{Z}_\ell$ we have

$$\sigma(b) = \phi^{-1}\psi_\nu(b) = s\, b \mod \ell \,. \tag{6.3.23}$$

Thus we find that:

1. On spin manifolds, there is a duality-invariant $\mathcal{L}_D$ for $\mathbb{A} = \mathbb{Z}_n$ if and only if there exists an ℓ such that $n = \ell q^2$ and -1 is a quadratic residue mod ℓ, *i.e.*, there exists also an s such that

   $$s^2 = -1 \mod \ell \,. \tag{6.3.24}$$

   This equation has solutions for $\ell = 1, 2, 5, 10, 13, 17, 25, 26, \ldots$

2. On spin manifolds, there is a triality-invariant $\mathcal{L}_D$ for $\mathbb{A} = \mathbb{Z}_n$ if and only if there exist ℓ, s such that $n = \ell q^2$ and

   $$s^2 + s + 1 = 0 \mod \ell \,. \tag{6.3.25}$$

   This equation has solutions for $\ell = 1, 3, 7, 13, 19, 21, 31, 37, \ldots$

These results coincide with the recent classification [166] of 4d topological $\mathbb{Z}_n$ gauge theories that are duality or triality invariant on spin manifolds.[37]

[37]Let us also notice a few facts. In the case of duality invariance, the possible values of ℓ are those that can be written as $\ell = x^2 + y^2$ for coprime x, y. The condition can never be satisfied by ℓ multiple of 4; indeed, if ℓ is even then s must be odd, but then $s^2 = 1$ mod 4. In the case of triality invariance, the possible values of ℓ are those that can be written as $\ell = x^2 + xy + y^2$ for coprime x, y. The condition can never be satisfied by ℓ multiple of 9.

2. Another interesting case to consider is $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ which is the 1-form symmetry group of a Spin$(4k)$ gauge theory. On $\mathbb{Z}_2 \times \mathbb{Z}_2$ there are four symmetric non-degenerate quadratic forms:

$$\gamma^{(D)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \gamma^{(O)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^{+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^{-} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} . \tag{6.3.26}$$

In this case -1 acts as the identity on $\mathbb{A}$ and duality is an involution. Thus given any choice of γ, the first obstruction is cancelled by choosing $\mathbb{B} = \mathbb{A}$, $\sigma = 1$ and $\chi_\nu = \gamma$. The case of triality is slightly more involved. Let us consider $\mathbb{B} = \mathbb{A}$. It is simple to show that the only two $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphisms τ satisfying $\tau^2 + \tau + 1 = 0$ are $\tau^{\pm} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which are inverses to each other. If $\gamma = \gamma^{(D)}$ we can solve the triality obstruction by taking $\chi_\nu = \gamma^{\pm}$ and $\tau = \tau^{\pm}$, similarly if $\gamma = \gamma^{\pm}$ we can take $\chi_\nu = \gamma^{(D)}$ and $\tau = \tau^{\mp}$. On the other hand, if $\gamma = \gamma^{(O)}$ then $\gamma\big(\tau^{\pm}(a), b\big)$ is not symmetric and the obstruction is present for $\mathbb{B} = \mathbb{A}$. Let us then consider $\gamma = \gamma^{(O)}$ and $\mathbb{B} = \mathbb{Z}_2$. Since $N(\mathbb{B})$ is also $\mathbb{Z}_2$, we must have that $\mathbb{B} = \phi\big(N(\mathbb{B})\big)$. It is simple to verify that taking $\mathbb{B}$ to be the diagonal $\mathbb{Z}_2$ this is indeed satisfied. We conclude that the first obstruction for $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ vanishes for both duality and triality.

This example, combined with the previous one, allows us to discuss the first obstruction for $\mathcal{N} = 4$ Spin$(2m)$ SYM (and its global variants). Recall that the 1-form symmetry group is

$$\mathbb{A} = \begin{cases} \mathbb{Z}_4 & \text{if } m = 2k+1 \, , \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } m = 2k \, . \end{cases} \tag{6.3.27}$$

We thus find that the first obstruction vanishes in all cases.

6.3.4 Second obstruction and symmetry fractionalization

While in absence of duality- (or triality-) invariant Lagrangian algebras the non-invertible self-duality symmetry is anomalous, when such an invariant algebra does exist the anomalies are determined by those on the invariant boundary, where the symmetry is invertible. The philosophy is the same as in the 2d/3d case: due to a mixed anomaly between the 1-form symmetry $\mathcal{S} = \mathbb{A} \times \mathbb{A}^\vee / \mathcal{L}_D$ and the invertible duality symmetry G we can shift the value of the pure G anomaly by changing the symmetry fractionalization class $\eta \in H^2_\rho(G, \mathcal{S})$. We now determine the mixed anomaly in the simpler case $\mathbb{B} = \mathbb{A}$, the generalization to proper subgroups being straightforward but technically tedious.

Duality. In the case of $\Phi = S$ and so $G = \mathbb{Z}_4$ the invariant partition function is given by:[38]

$$Z_{\text{inv}}\big[\phi(B)\big] = \sum_{b \in H^2(X,\mathbb{A})} \exp\left(2\pi i \int_{X_4} b^* \nu + 2\pi i \int_{X_4} \phi(B) \cup b\right) Z_{\text{e}}[b] \, , \tag{6.3.28}$$

where Z_{e} is the partition function corresponding to the reference electric boundary condition, while ν is defined through a bicharacter χ_ν such that

$$\gamma\big(\sigma(a), b\big) = \chi_\nu(a,b) \qquad \text{and} \qquad \sigma^2 = -1 \, . \tag{6.3.29}$$

The action of S-duality on Z_{inv} is easily determined using the action of S-duality on the electric theory:

$$S \cdot Z_{\text{e}}[B] = \sum_{a \in H^2(X,\mathbb{A})} \exp\left(2\pi i \int_{X_4} \phi(B) \cup a\right) Z_{\text{e}}[a] \, . \tag{6.3.30}$$

[38]For simplicity we omit the normalization factors due to gauging.

We find

$$S \cdot Z_{\text{inv}}\big[\phi(B)\big] = G_\nu \exp\!\left(2\pi i \int_{X_4} B^*\nu\right) Z_{\text{inv}}\big[\phi(\sigma B)\big]\ , \tag{6.3.31}$$

where $G_\nu \equiv \sum_{b\in H^2(X,\mathbb{B})} \exp\!\big(2\pi i \int_X b^*\nu\big)$. Here, assuming that X_4 is spin, we used the simplifying relation

$$\exp\!\left(2\pi i \int_{X_4} B^*\big(\nu\circ\sigma\big)\right) = \exp\!\left(-2\pi i \int_{X_4} B^*\nu\right)\ . \tag{6.3.32}$$

Assuming that X_4 is simply connected (and thus $H^2(X_4,\mathbb{Z})$ has no torsion classes) and spin, one can show that the Gauss sum G_ν is equal to 1 [166]. In a similar way we can verify that

$$S^2 \cdot Z_{\text{inv}}\big[\phi(B)\big] = Z_{\text{inv}}\big[-\phi(B)\big] = C\cdot Z_{\text{inv}}\big[\phi(B)\big]\ . \tag{6.3.33}$$

Eqn. (6.3.31) implies that the $\mathbb{Z}_4$ symmetry generated by S acts on the 1-form symmetry of the theory through σ, *i.e.*, the symmetry is a split 2-group with nontrivial action $\rho : G \to \text{Aut}(\mathbb{A})$ [15, 17, 93] given by $\rho_{\underline{1}}(a) = \sigma\, a$. Furthermore, the overall phase $\exp\!\big(2\pi i \int_X B^*\nu\big)$ should be thought of as encoding a mixed anomaly

$$\mu \in H^1_\rho\Big(\mathbb{Z}_4,\, H^4\big(B^2\mathbb{A},\, U(1)\big)\Big) \qquad \text{where} \qquad \mu(\underline{1}) = \nu \tag{6.3.34}$$

and $\underline{1}$ is the generator of $G=\mathbb{Z}_4$, much as in the 2d case.

Triality. For $\Phi = CST$ and so $G=\mathbb{Z}_3$ we have the same expression (6.3.28) for $Z_{\text{inv}}\big[\phi(B)\big]$, but with the class ν now satisfying (6.3.20) in terms of a τ such that $\tau^2+\tau+1=0$. T-duality acts on the electric boundary as

$$T\cdot Z_e[B] \equiv \exp\!\left(-2\pi i \int_{X_4} B^*\gamma\right) Z_e[B]\,. \tag{6.3.35}$$

Then

$$(CST)\cdot Z_{\text{e}}[B] = \exp\!\left(2\pi i \int_{X_4} B^*\gamma\right) \sum_{a\in H^2(X,\mathbb{A})} \exp\!\left(2\pi i \int_{X_4} \phi(B)\cup a\right) Z_{\text{e}}[a] \tag{6.3.36}$$

with $B^*\gamma$ any class stemming from a quadratic refinement of γ, *i.e.* the Pontryagin square induced by γ, and we find

$$(CST)\cdot Z_{\text{inv}}\big[\phi(B)\big] = G_{\gamma+\nu}\exp\!\left(2\pi i \int_{X_4} B^*\nu\right) Z_{\text{inv}}\big[\phi(\tau B)\big] \tag{6.3.37}$$

Here we used that, on spin manifolds, $\exp\!\big[2\pi i \int B^*\big(\nu + \nu\circ\tau + \nu\circ\tau^2\big)\big] = 1$. It also holds that

$$(CST)^3\cdot Z_{\text{inv}}\big[\phi(B)\big] = Z_{\text{inv}}\big[\phi(B)\big]\ . \tag{6.3.38}$$

As before, the result is interpreted by saying that the split 2-group is twisted by the $\mathbb{Z}_3$ symmetry and the overall phase comes from a mixed anomaly

$$\mu \in H^1_\rho\Big(\mathbb{Z}_3,\, H^4\big(B^2\mathbb{A},\, U(1)\big)\Big) \qquad \text{where} \qquad \mu(\underline{1}) = \nu\ . \tag{6.3.39}$$

We thus conclude that, similarly to the 3d case, the 5d mixed anomaly is determined by a class

$$\mu \in H^1_\rho\Big(G,\, H^4\big(B^2\mathbb{A}, U(1)\big)\Big) \cong H^1_\rho(G, \Gamma(\mathbb{A})^\vee) \tag{6.3.40}$$

namely a function from G to the group of quadratic functions over $\mathbb{A}$ satisfying

$$\rho_g\,\mu(h) + \mu(g) = \mu(g+h) \tag{6.3.41}$$

and subject to the the identification

$$\mu(g) \cong \mu(g) + \rho_g \xi - \xi \qquad \text{for any} \qquad \xi \in H^4\big(B^2\mathbb{A}, U(1)\big) . \tag{6.3.42}$$

The full detailed derivation of the anomaly inflow is given in Appendix C.2 and we find

$$S_\mu = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}_\rho(B) . \tag{6.3.43}$$

To reproduce the anomalous phase arising in the boundary theory we have to compare this phase with the boundary term arising in S_μ from $A + d\lambda$ when we set the pull-back of A to the boundary to zero, as well as the boundary value of λ equal to the element of the group G for which we compute the variation. This determines all the values of $\mu(g)$ for $g \in G$. We can check that the consistency (6.3.41) of these values is satisfied. In the case of duality $G = \mathbb{Z}_4$, since ν satisfies $\nu\big(\sigma(a), \sigma(b)\big) = -\nu(a, b)$, we deduce that

$$\mu(\underline{1}) = \mu(\underline{3}) = \nu , \qquad \mu(\underline{0}) = \mu(\underline{2}) = 0 . \tag{6.3.44}$$

It is obvious that (6.3.41) is satisfied.

For triality $G = \mathbb{Z}_3$ the crucial relation is

$$\gamma(a, b) + \gamma\big(\tau(a), \tau(b)\big) + \gamma\big(\tau^2(a), \tau^2(b)\big) = 0 . \tag{6.3.45}$$

By looking at the anomalous phases that we got this implies that

$$\mu(\underline{0}) = 0 , \qquad \mu(\underline{1})(a, b) = \gamma\big(\tau(a), \tau(b)\big) , \qquad \mu(\underline{2})(a, b) = \gamma\big(\tau(a), \tau(b)\big) + \gamma(a, b) . \tag{6.3.46}$$

Among the consistency relations (6.3.41), the only non-trivial (and independent) ones to check are: $\tau\mu(\underline{1}) + \mu(\underline{1}) = \mu(\underline{2})$, $\tau\mu(\underline{2}) + \mu(\underline{1}) = 0$ and $\tau^2\mu(\underline{2}) + \mu(\underline{2}) = \mu(\underline{1})$, which are indeed satisfied thanks to (6.3.45).

Given such a mixed anomaly, we are now able to discuss the pure G anomaly. The philosophy is the same as in the 2d/3d case: combining the choice of symmetry fractionalization with the mixed anomaly we can induce an extra contribution to the pure anomaly for the invertible duality symmetry. The details are however slightly different.

In 4d symmetry fractionalization is classified by $\eta \in H^2_\rho(G, \mathbb{A})$, which, as opposed to the 2d case where it corresponds to the choice of a G subgroup of the full symmetry, here it corresponds to the choice of a 1-form symmetry defect $\eta(g, h) \in \mathbb{A}$ inserted along the junction of the intersection of g, h and gh defects. This amounts to redefine the coupling of the 0-form symmetry to a background, prescribing that B is shifted to

$$B' = B + A^*\eta \in H^2_\rho(X, \mathbb{A}) . \tag{6.3.47}$$

By plugging this expression into the mixed anomaly (6.3.43) we shift the pure G anomaly by an extra piece

$$S_{\text{pure}} = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}_\rho(A^*\eta) \equiv 2\pi i \int_{X_5} A^* y \tag{6.3.48}$$

that can be written in terms of a class $y \in H^5\big(G, U(1)\big)$. In order to work out an explicit expression for this class we rely on a working assumption. We note that the Pontryagin square operation, when the homology group $H_2(X_5, \mathbb{Z})$ is torsion-free, can be written as a cup product[39] [93]:

$$\mathfrak{P}_\rho(A^*\eta) = A^*\eta \cup A^*\eta . \tag{6.3.49}$$

[39] The expression (6.3.49) should be interpreted as follows. One writes $\mathbb{A} = \oplus_i \mathbb{Z}_{n_i}$ and lift $A^*\eta$ to $\oplus_i\mathbb{Z}$, which is always possible for finite Abelian groups. In $\oplus_i\mathbb{Z}$ we can take the product among the various components of the lift, then (6.3.49) is obtained restricting the result to $\Gamma(\oplus_i\mathbb{Z}_{n_i}) = \bigoplus_i \Gamma(\mathbb{Z}_{n_i}) \oplus \bigoplus_{i<j} \mathbb{Z}_{n_i} \otimes \mathbb{Z}_{n_j}$. If X_5 has torsion 1-cycles the Pontryagin square is not a cup product and in order to write it in components we need Steenrod's cup products (see *e.g.* [15]).

On the other hand the pure G anomaly is non-trivial when the homology group $H_1(X_5,\mathbb{Z})$ contains torsion [165]. Therefore, in order to do the computation, we pick a bulk spin manifold X_5^* with torsion 1-cycles but with torsion-free 2-cycles so as to write (6.3.48) as

$$S_{\text{pure}} = 2\pi i \int_{X_5^*} \mu(A) \cup A^*\eta \cup A^*\eta \ . \tag{6.3.50}$$

Then it is easy to conclude that

$$y\big(g_1,g_2,g_3,g_4,g_5\big) = \langle -\mu(-g_1)\,, \eta(g_2,g_3)\,\rho_{g_2+g_3}\,\eta(g_4,g_5)\rangle \ . \tag{6.3.51}$$

where the product in the second entry should be interpreted as in footnote 39. When the second entry is the image of a quadratic function $\gamma : \mathbb{A} \to \Gamma(\mathbb{A})$ the above expression can be rewritten in a simpler form using the universal property defining $\Gamma(\mathbb{A})$ (see the discussion around (6.3.10)). In particular if we can find a representative for η that is invariant under the ρ action, setting $g_2 = g_4$ and $g_3 = g_5$, we have

$$y\big(g_1,g_2,g_3,g_2,g_3\big) = \langle -\mu(-g_1)\,, \eta(g_2,g_3)\,\eta(g_2,g_3)\rangle = -\mu(-g_1)(\eta(g_2,g_3)) \ . \tag{6.3.52}$$

Examples

We now discuss how this general story applies to examples where $\mathbb{A} = \mathbb{Z}_n$ and G is either $\mathbb{Z}_4$ or $\mathbb{Z}_3$, namely duality and triality respectively. This has some consequence for the anomaly structure of $\mathcal{N}=4$ SYM theories with gauge group SU(n) at $\tau = i, e^{\frac{2\pi i}{3}}$ respectively.

Several technical details on the computations of the twisted cohomology groups are based on the following known result (see *e.g.*[138]). If $G \cong \mathbb{Z}_k$, denoting $f = \rho_{\underline{1}} \in \text{Aut}(\mathbb{A})$ (note that $f^k = 1$), then

$$H^n_\rho(G,\mathbb{A}) \cong \begin{cases} \dfrac{\text{Ker}(1-f)}{\text{Im}(1+f+f^2+\ldots+f^{k-1})} & \text{if } n \text{ is even} \\[2ex] \dfrac{\text{Ker}(1+f+f^2+\ldots+f^{k-1})}{\text{Im}(1-f)} & \text{if } n \text{ is odd} \end{cases} \tag{6.3.53}$$

The symmetry fractionalization classes are classified by $H^2_\rho(G,\mathbb{A})$, and we notice that in both the duality and triality examples we have

$$1+f+f^2+\ldots+f^{k-1} = 0 \tag{6.3.54}$$

by virtue of the relations $\sigma^2 = -1$, $\tau^2+\tau+\tau = 1$. Hence for us

$$H^2_\rho(G,\mathbb{A}) = \text{Ker}(1-f) = \big\{a \in \mathbb{A} \mid \rho_{\underline{1}}(a) = a\big\} = \text{Fix}_{\rho_{\underline{1}}}(\mathbb{A}) \ . \tag{6.3.55}$$

This also gives a hint for the form of the explicit representatives of the non-trivial twisted cocycles as

$$\eta_x(\underline{1},\underline{1}) = x \ , \quad x \in \text{Fix}_{\rho_{\underline{1}}}(\mathbb{A}) \tag{6.3.56}$$

Duality. For the case of duality $G \cong \mathbb{Z}_4$ we have

$$\rho_{\underline{1}}(a) = ta \ , \quad t^2 = -1 \bmod(n) \ . \tag{6.3.57}$$

Using (6.3.55) we get

$$H^2_\rho(\mathbb{Z}_4,\mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \tag{6.3.58}$$

and in the even case the cocycles can be represented

$$\eta_s(\underline{1},\underline{1}) = \eta_s(\underline{3},\underline{3}) = \eta_s(\underline{1},\underline{3}) = \eta_s(\underline{3},\underline{1}) = \frac{n}{2}s \ , \qquad s = 0,1 \tag{6.3.59}$$

with all the other values vanishing. By setting $n = 2m$, the pure anomaly is determined by the value of the 5-cocycle $Y \in H^5(\mathbb{Z}_4, U(1))$ in $g_1 = ... = g_5 = \underline{1}$ and we get

$$Y = q_\nu(\eta_s(\underline{1},\underline{1})) = e^{2\pi i t s^2 \frac{m}{4}} \, . \tag{6.3.60}$$

We conclude that for n odd the pure duality anomaly on the invertible boundary is the bare one, given by $\epsilon \in H^5(\mathbb{Z}_4, U(1)) \cong \mathbb{Z}_4$, while for n even the cancellation depends on the possible values of Y. Recall that the first obstruction never vanishes when m is even. Therefore the possible values of Y are

$$Y = \exp\left(\frac{\pi i}{2} t(2k+1)\right) \qquad \text{for } n = 2(2k+1) \, . \tag{6.3.61}$$

In the $\mathcal{N} = 4$ theory with gauge group SU(n) at $\tau = i$ the non-invertible duality symmetry is anomalous whenever it is intrinsically non-invertible, on spin manifolds we have given the relevant condition for $\mathbb{A} = \mathbb{Z}_n$ around equation (6.3.24). If the defect is non-intrinsically non-invertible the anomaly automatically vanishes provided we combine the duality with an appropriate R-symmetry rotation in order to have a $\mathbb{Z}_4$ operation (see *e.g.* [167, 168]). Indeed following [169] and using that $\Omega_5^{spin}(B\mathbb{Z}_4) \cong \mathbb{Z}_4$ one gets [161]

$$\epsilon = 60(n-1) - 24(n^2-1) \bmod(4) = 0 \, , \tag{6.3.62}$$

therefore one should choose the trivial fractionalization class to cancel the second obstruction. One could also consider other definitions of S-duality which do not involve the R-symmetry, in such cases the relevant bordism group $\Omega^{\text{spin}-\mathbb{Z}_8}(pt) = \mathbb{Z}_{32} \oplus \mathbb{Z}_2$ is larger and our techniques would need to be refined in order to appropriately account for the cubic anomaly.

A similar conclusion applies to Maxwell theory, for which $S^4 = 1$ and the anomaly 56 mod$(4) = 0$ also identically vanishes.

Triality. In the triality case $G \cong \mathbb{Z}_3$,

$$\rho_{\underline{1}}(a) = ta \ , \qquad t^2 + t + 1 = 0 \tag{6.3.63}$$

for which we get

$$H^2_\rho(\mathbb{Z}_3, \mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_3 & \text{if } n = 0 \bmod(3) \\ 0 & \text{otherwise} \end{cases} \tag{6.3.64}$$

and the (non)trivial cocycles are

$$\eta_s(\underline{1},\underline{1}) = \eta_s(\underline{2},\underline{2}) = \eta_s(\underline{1},\underline{2}) = \eta_s(\underline{2},\underline{1}) = \frac{n}{3}s \ , \qquad s = 0,1,2 \tag{6.3.65}$$

with all the other values vanishing. Setting $n = 3m$, the class $Y \in H^5(\mathbb{Z}_3, U(1)) \cong \mathbb{Z}_3$ is determined by[40]

$$Y = \left[\mu(\underline{2})\Big(\eta_s(\underline{1},\underline{1}),\, \rho_{\underline{2}}\, \eta_s(\underline{1},\underline{1})\Big)\right]^{-1} = q_\gamma\left(\eta_s(\underline{1},\underline{1})\right) = \begin{cases} \exp\left(2\pi i \dfrac{k}{3}\right) & \text{if } m = 2k \\ \\ \exp\left(2\pi i \dfrac{4k+2}{3}\right) & \text{if } m = 2k+1 \end{cases} . \tag{6.3.66}$$

[40]One can easily check that, when $n = 3m$ is also even, so $m = 0$ mod (2), the choice of quadratic refinement for q_γ is immaterial.

Again we can apply these results to the case of triality symmetry appearing in $\mathcal{N}=4$ SYM at $\tau=e^{2i\pi/3}$. The triality defect is non-intrinsic when there exist $t\in\mathbb{Z}_n$ such that $1+t+t^2=0 \bmod(n)$. When this is the case we can ask about the second obstruction. To apply our methods we are not forced to combine the naive CST operation with an R-symmetry rotation to eliminate fermion parity, since $(CST)^3=\mathbb{1}$. Then, by the same token as the duality case and knowning that $\Omega_5^{spin}(B\mathbb{Z}_3)\cong\mathbb{Z}_9$, we have

$$\epsilon = 60(n-1) \bmod(9) = -3(n-1) \bmod(9)\,. \tag{6.3.67}$$

Notice that Y is valued in the $\mathbb{Z}_3$ subgroup of the $\mathbb{Z}_9$ anomaly group, then to compare Y to the ϵ above we need to multiply by 3. When $n=1 \bmod(3)$ then $\epsilon=0$ and there is no choice of fractionalization, therefore the second obstruction vanishes. For $n=2 \bmod(3)$ we find $\epsilon=6$ and the triality defect is always anomalous. Finally when $n=0 \bmod 3$ we have $\epsilon=3$ and a simple computation shows that the second obstruction can be cancelled only when $n=3m$ with $m=1 \bmod (3)$.

In Maxwell theory instead the anomaly is $56 \bmod(9)=2$ and cannot be cancelled by any choice of symmetry fractionalization. We conclude that the triality symmetry in Maxwell theory is always anomalous due to the second obstruction.

6.4 A check from dimensional reduction

As a check of our results, we show that the obstruction theory of Section 6.3 is consistent with the one for Tambara-Yamagami categories upon dimensional reduction on an orientable 2-manifold W. We treat explicitly the case that W is a torus T^2, but the generalization to any Riemann surface $\Sigma_{\mathfrak{g}}$ is straightforward. Physically this should be expected, indeed the simplest example of a 4d theory enjoying self-duality is Maxwell theory, which upon compactification on T^2 reduces to the theory of two compact bosons.[41] In this example the complexified gauge coupling τ is mapped to the position of the 2d CFT on the Narain moduli space. Such a theory is well known to enjoy Tambara-Yamagami-type symmetries if the point on the conformal manifold is chosen appropriately [99].

Compactifying the 5d Dijkgraaf-Witten theory for $\mathbb{A}$ on the torus is a simple exercise. The resulting 3d TQFT has a 1-form symmetry $\widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}}^\vee$ where

$$\widetilde{\mathbb{A}} = \mathbb{A}\times\mathbb{A}\,, \tag{6.4.1}$$

together with a 0-form and a 2-form symmetry, both being $\mathbb{A}\times\mathbb{A}^\vee$, which we neglect in the following discussion. Given a choice ϕ for the isomorphism that enters into the 5d duality symmetry, the defect Φ also implements a $\mathbb{Z}_4$ symmetry in 3d:

$$\Phi(a_1,\,a_2;\,\alpha_1,\,\alpha_2) = \left(-\phi^{-1}(\alpha_2),\,\phi^{-1}(\alpha_1);\,-\phi(a_2),\,\phi(a_1)\right)\,, \tag{6.4.2}$$

where $(a_1,a_2)\in\widetilde{\mathbb{A}}$ and $(\alpha_1,\alpha_2)\in\widetilde{\mathbb{A}}^\vee$. To get a $\mathbb{Z}_2$ symmetry we compose this transformation with the internal S-duality of the torus, which also squares to charge conjugation and sends $(a_1,\,a_2;\,\alpha_1,\,\alpha_2)\to(a_2,\,-a_1;\,\alpha_2,\,-\alpha_1)$. The resulting $\mathbb{Z}_2$ symmetry, which we dub $\widetilde{\Phi}$ acts as:

$$\widetilde{\Phi}(a_1,\,a_2;\,\alpha_1,\,\alpha_2) = (\phi^{-1}(\alpha_1),\,\phi^{-1}(\alpha_2);\,\phi(a_1),\,\phi(a_2))\,, \tag{6.4.3}$$

or, using the $\widetilde{\mathbb{A}}$

$$\begin{array}{rcl}\widetilde{\Phi}:\widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}}^\vee & \longrightarrow & \widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}}^\vee \\ (\widetilde{a},\widetilde{\alpha}) & \longrightarrow & (\widetilde{\phi}^{-1}(\widetilde{\alpha}),\widetilde{\phi}(\widetilde{a}))\end{array} \tag{6.4.4}$$

with $\widetilde{\phi}:\mathbb{A}\times\mathbb{A}\to\mathbb{A}^\vee\times\mathbb{A}^\vee$ given by $\widetilde{\phi}(a_1,a_2)=(\phi(a_1),\phi(a_2))$.

[41] Plus a decoupled 2d Maxwell sector that we ignore. Such a sector has a 1-form and a (-1)-form symmetry (associated to a 2π shift of the theta angle), associated to the 0-form and 2-form symmetries of the SymTFT.

First and second obstruction upon dimensional reduction

We now discuss how the first obstruction in 5d is mapped to the first obstruction in 3d language after compactification. Clearly not all Lagrangian algebras $\mathcal{L}$ in the 3d description can descend from a 5d description, so we must first characterize them. Recall that, in 5d, algebras where described by a choice of subgroup $\mathbb{B}$ of $\mathbb{A}$ together with a discrete torsion $[\nu] \in H^4\big(B^2\mathbb{B},\, U(1)\big)$. Upon reduction on T^2 this should map to a specific class $[\tilde{\nu}] \in H^2\big(B\widetilde{\mathbb{B}},\, U(1)\big)$, where $\widetilde{\mathbb{B}} = \mathbb{B} \times \mathbb{B}$. Expanding the 5d background $B = B_1\theta_1 + B_2\theta_2$ with θ_i a basis of $H^1(T^2, \mathbb{Z})$ (we neglect the 0-form and 2-form symmetries), we find:

$$\int_{T^2} B^*\nu = B_1 \cup_\nu B_2 - B_2 \cup_\nu B_1 \ , \tag{6.4.5}$$

where $\cup_\nu$ is the cup product induced by the symmetric bilinear form χ_ν. The bicharacter corresponding to $\tilde{\nu}$ is then, in matrix and additive notation,

$$\chi_{\tilde{\nu}} = \begin{pmatrix} 0 & \chi_\nu \\ -\chi_\nu & 0 \end{pmatrix} . \tag{6.4.6}$$

A 3d Lagrangian algebra $\widetilde{\mathcal{L}}$ induced from 5d then is of the form

$$\widetilde{\mathcal{L}} = \Big\{ \big(\tilde{b},\, \tilde{\beta}\psi_{\tilde{\nu}}(\tilde{b})\big) \ \Big|\ \tilde{b} \in \widetilde{\mathbb{B}}\,, \quad \tilde{\beta} \in N(\widetilde{\mathbb{B}}) \Big\} \ , \tag{6.4.7}$$

where $\psi_{\tilde{\nu}} : \widetilde{\mathbb{B}} \to \widetilde{\mathbb{B}}^\vee$ is the homomorphism associated with the antisymmetric bicharacter (6.4.6). Since $\mathrm{Rad}(\tilde{\nu}) = \mathrm{Rad}(\nu) \times \mathrm{Rad}(\nu)$ the 5d condition $\phi(N(\mathbb{B})) = \mathrm{Rad}(\nu)$ implies $\phi(N(\widetilde{\mathbb{B}})) = \mathrm{Rad}(\widetilde{\nu})$ in 3d. On the other hand, the map $\tilde{\sigma} = \tilde{\phi}^{-1}\psi_{\tilde{\nu}}$ is given by:

$$\tilde{\sigma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} , \tag{6.4.8}$$

which is an involution $\tilde{\sigma}^2 = 1$. We have thus shown that solutions to the first obstruction in 5d always descend to solutions to the first obstruction in 3d.

Let us now discuss the second obstruction. We notice that the 5d discrete torsion ϵ, when reduced on T^2, trivializes. This is because the torus (as well as any Riemann surface) does not have torsion 1-cycles. Thus it is not possible to detect the 5d second obstruction in 3d after compactification on a Riemann surface. Indeed, from the point of view of symmetry fractionalization, we have $G \cong \mathbb{Z}_n$ and for any Abelian group $\mathbb{A}$ we get

$$H^1_\rho(\mathbb{Z}_n,\, \mathbb{A}) = \mathrm{Ker}(1+f)/\mathrm{Im}(1-f) \ , \tag{6.4.9}$$

with $f = \rho_{\underline{1}}$. Applying this to the case $\mathbb{A} = \widetilde{\mathbb{B}}/N(\widetilde{\mathbb{B}})$ and $f = \tilde{\sigma}$ it is simple to prove that the twisted cohomology group is trivial for any choice of $\mathbb{B}$.[42] Thus there are no fractionalization classes and therefore the second obstruction always trivializes.

6.5 Comments on applications

Let us conclude by mentioning some immediate applications of our results, as well as some interesting open problems.

[42] Using that $\sigma^2 = -1$ we find that $\mathrm{Ker}(1+\tilde{\sigma})$ is spanned by elements $(b_1, b_2) \in \widetilde{\mathbb{B}}/N(\widetilde{\mathbb{B}})$ such that $b_2 = \sigma(b_1)$. An element of $\mathrm{Im}(1-\tilde{\sigma})$ instead is of the form $(b_1, b_2) = \big(x - \sigma(y),\, \sigma(x) + y\big)$. A simple manipulation shows that this is equivalent to $b_2 = \sigma(b_1)$.

4d $\mathcal{N} = 3$ theories. It has been appreciated in the past that a class of 4d $\mathcal{N} = 3$ theories may be obtained from a discrete gauging of the $\mathcal{N} = 4$ duality symmetry for special values of τ [167, 168]. More precisely, given a $\mathbb{Z}_k$ subgroup of $SL(2, \mathbb{Z})$ and a fixed coupling τ_k, where $k = 2, 3, 4, 6,$[43] we can combine this transformation with a $\mathbb{Z}_k$ R-symmetry rotation in the Cartan of SU(4) so that the combined action preserves $\mathcal{N} = 3$ supersymmetry. As the gauge coupling $\tau = \tau_k$ must be fixed to its self-dual value, these theories have no exactly marginal deformation and are inherently strongly coupled. The case of $k = 2$ is special, as the symmetry is charge conjugation, hence it preserves the full $\mathcal{N} = 4$ supersymmetry, and is invertible. We will thus concentrate on the cases $k = 4$ (corresponding to the S transformation) and $k = 3$ (corresponding to the CST transformation) and gauge group SU(n). As the duality symmetry is non-invertible, it must be gauged together with (a subgroup of) the $\mathbb{Z}_n$ 1-form symmetry and our results imply that this is only consistent if the first obstruction vanishes. Thus there is a severe constraint on the possible $\mathcal{N} = 3$ theories which can be obtained in this way. For example our results show that there is no such theory for $n = 3$ and $k = 4$. We must also check the vanshing of the second obstruction. The joint duality/R-symmetry anomaly is given by [161]:

$$60(n-1) - 24(n^2-1) \begin{cases} \bmod 4\,, & \text{if } k = 4 \\ \bmod 9\,, & \text{if } k = 3 \end{cases} . \tag{6.5.1}$$

For the duality case the cubic anomaly is identically trivial, thus the vanishing of the first obstruction is a sufficient condition for the gauging to be consistent. For triality instead it is given by 6 mod 9 when $n = 3m + 2$ and is zero otherwise. It has been checked in [161] that this anomaly identically trivializes when the first obstruction vanishes. Therefore also in the triality case the gauging is consistent if the first obstruction vanishes. This also implies that, when $n = 3m$, we must choose the trivial fractionalization class $\eta \in H^2_\rho(\mathbb{Z}_3, \mathbb{Z}_{3m})$.

In some special cases the S-fold construction of [170] gives rise to discrete gaugings of $\mathcal{N} = 4$ SYM [171]. These are engineered by 2 D3-branes probing a $k = 3, 4, 6$ S-fold and lead to a discrete gauging of SU(3), SO(5) and G_2 $\mathcal{N} = 4$ SYM respectively. Our analysis can be applied to the first two cases which, following the discussed examples, indeed are free of anomalies for triality and duality respectively. It would certainly be interesting to understand whether our methods can give some insight also on $\mathcal{N} = 3$ theories which cannot be obtained by a discrete gauging procedure from $\mathcal{N} = 4$ and, in particular, if they enlarge the list of generalized symmetries of S-folds described recently in [172, 173].

A mixed anomaly. We have mentioned in Section 6.2.3 that the space of duality-invariant Lagrangian algebras is larger on spin manifolds. Similarly one can argue, for example following [166], that the first obstruction in the 4d case has less solutions if the spacetime X is not spin. This should be rephrased as the presence of a mixed 't Hooft anomaly between the non-invertible symmetry $\mathcal{N}$ and gravity, sourced by a nontrivial second Stiefel-Whitney class $w_2(X)$. A well known example is the symmetry $\mathrm{TY}(\mathbb{Z}_2)_{1,1}$ of the Ising CFT. As a bosonic symmetry this is anomalous as the first obstruction cannot be cancelled. However, if we consider it on spin manifolds X only, the obstruction is absent since the bulk algebra $\mathcal{L}_D = \{(0,0),\,(1,1)\}$ is manifestly duality invariant. Such an algebra can only be condensed on spin manifolds as $\theta_{(1,1)} = -1$. On the field theory side it is well known [20, 99, 174] that fermionizing the Ising CFT into a Majorana fermion the duality symmetry $\mathcal{N}$ becomes

[43] To be precise, since the duality group is Mp(2, $\mathbb{Z}$) the discrete groups are actually $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_{12}$ as charge conjugation squares to fermion number $C^2 = (-)^F$. The combined duality - R symmetry transformation however lies is $\mathbb{Z}_k$ with k as in the main text.

the invertible $(-1)^{F_L}$ which is anomaly free. A similar example in 4d, as already stated before, if the $\mathcal{N} = 4$ SU(2) SYM theory, whose duality symmetry is anomaly-free on spin manifolds (after combining it with an R-symmetry rotation) it is anomalous by the first obstruction when X is non-spin. It would be nice to make this idea more precise.

Duality-invariant RG flows. In both 2d and 4d, duality-symmetric theories allow for a plethora of interesting RG flows which preserve the non-invertible symmetry. In the former case they have been studied in [99], while in the latter an initial study has appeared recently [161]. As in the 2d case, the anomalies for the duality symmetry can lead to strong constraints on the possible low energy phases. A simple example is the $\mathcal{N} = 1^*$ [175–179] deformation of $\mathcal{N} = 4$ SYM at $\tau = i$, which, in the presence of the first obstruction, necessarily leads in the IR either to spontaneous symmetry breaking of the non-invertible symmetry, or to a self-dual Coulomb phase [161]. A related problem deserving further study in the light of our results is the deformation of the SU(2), SU(3), SU(4) $\mathcal{N} = 4$ theory by the Konishi operator. This must lead in the IR either to an $\mathcal{N} = 0$ CFT or to chiral symmetry breaking in order to match the cubic SU(4) anomaly. Consistency of these scenarios with the intricate pattern of non-invertible symmetries and their anomalies might allow to put stringent constraints on the possible IR phases. This problem is currently under investigation.

Intrinsic versus anomalous. In this chapter we have seen that, in the context of duality symmetries, the concept of "instrinsic" [46] implies that the duality symmetry is anomalous. Such concept is not unique to duality symmetries, and can be rephrased as the statement that the symmetry category $\mathcal{C}$ is not Morita equivalent to any category of the form $n\mathrm{Vec}_G$ for some (higher) group G. It would be interesting to understand how far the relationship between 't Hooft anomalies and intrinsic defects extends.

Chapter 7

Intrinsically gapless topological phases in various dimensions

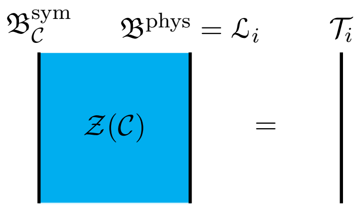
In this chapter we use the SymTFT approach approach to study gapless phases in (1+1)d and (3+1)d (we also make comments on (2+1)d). We classify the gapless symmetry-protected (gSPT) phases in these setups, with particular focus on intrinsically gSPTs (igSPTs). These are symmetry protected critical points which cannot be deformed to a trivially gapped phase without spontaneously breaking the symmetry. Although these are by now well-known in (1+1)d, we demonstrate their existence in (3+1)d gauge theories, in the presence of both 1-form symmetries. Here, they have a clear physical interpretation in terms of an obstruction to confinement, even though the full 1-form symmetry does not suffer from 't Hooft anomalies. These igSPT phases provide a new way to realize 1-form symmetries in CFTs, that has no analog for gapped phases. The SymTFT approach allows for a direct generalization from invertible symmetries to non-invertible duality symmetries, for which we study gSPT and igSPT phases as well.

7.1 Generalities on (intrinsically) gapless topological phases

Once symmetry considerations are taken into account, phases of matter in any spacetime dimensions have a rich and intricate structure. The inclusion of generalized and categorical symmetries refines the classification of these phases, and allows to predict new phase transitions. According to the Landau paradigm (and its *categorical* generalization [180]), a phase is characterized by the symmetries and their realization on the ground state.

If we focus on finite (categorical) symmetries $\mathcal{C}$ and *gapped phases*, the most general phase consists in spontaneously breaking $\mathcal{C}$ to some subsymmetry $\mathcal{C}_p \subset \mathcal{C}$, which can be possibly realized in a non-trivial way, namely $\mathcal{C}_p$ can have a symmetry protected topological (SPT) phase. Suppose, for instance, that $\mathcal{C}$ is a p-form group symmetry G. Then the gapped phases are classified by the preserved subgroup $H \subset G$ and the SPT phase $\omega \in H^d(B^{p+1}H, U(1))$. Notice that these data are also those determining a *topological manipulation* with the symmetry G.

The correspondence between gapped phases and topological manipulation can be explained by using the SymTFT. A gapped phase is realized, in the SymTFT, by taking the physical boundary to be topological [119, 180–182]

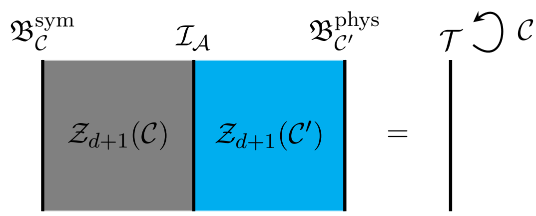


(7.1.1)

The symmetry boundary $\mathfrak{B}^{\text{sym}}_{\mathcal{C}}$ is specified by the canonical Lagrangian algebra $\mathcal{L}_{\mathcal{C}}$ that determines the symmetry $\mathcal{C}$ on the boundary. The physical boundary, being also topological, is determined by choosing any Lagrangian algebra $\mathcal{L}_i$, and the result is a gapped phase $\mathcal{T}_i$. In particular, SPT phases are characterized by requiring the intersection to be trivial. Thus Lagrangian algebras determines both topological manipulations and gapped phases, explaining the correspondence between the two.

The SymTFT characterization of the phase is that defects that can terminate on both gapped boundaries, namely $l \in \mathcal{L}_{\mathcal{C}} \cap \mathcal{L}_i$, give rise to topological operators charged under the spontaneously broken symmetry. The preserved subsymmetry, instead, is determined by the defects of the bulk that are not trivialized by the symmetry boundary (hence they are not part of $\mathcal{L}_{\mathcal{C}}$), but are trivialized by the gapped physical boundary (hence they are part of $\mathcal{L}_i$). Different $\mathcal{L}_i$ with the same intersection with $\mathcal{L}_{\mathcal{C}}$ determine different SPT phases for the preserved part of the symmetry.

Following this idea, we can also study gapless phases. More precisely, gapless phases can have topological features, and these can be classified with the lens of the SymTFT. The idea [183–186] is to use a non-topological (gapless) physical boundary, but not for the SymTFT $\mathcal{Z}(\mathcal{C})$ of the full symmetry. Instead, the gapless physical boundary is a boundary for a *reduced* theory $\mathcal{Z}(\mathcal{C}')$, obtained by gauging a condensable algebra $\mathcal{A}$ of $\mathcal{Z}(\mathcal{C})$



(7.1.2)

Indeed, condensable algebras, if non-Lagrangian, define a topological interface $\mathcal{I}_{\mathcal{A}}$ (as opposite to a boundary). Only $\mathcal{C}'$ (that, roughly speaking, is a quotient of $\mathcal{C}$ by the preserved symmetry $\mathcal{C}_p$) acts faithfully on the gapless sector. The intersection $\mathcal{L}_{\mathcal{C}} \cap \mathcal{A}$ gives rise to topological order parameter, hence again a part of the symmetry that is spontaneously broken[1]. The rest of the symmetry, however, acts on a non-trivial CFT arising from the physical boundary, and hence its order parameters can be non-topological. In other words, we are describing a phase sitting at low energy of some microscopic theory with symmetry $\mathcal{C}$, such that along the RG flow some order parameters become massive, are integrated out, and the low energy theory has a smaller symmetry. Remarkably, this reduction from $\mathcal{C}$ to $\mathcal{C}'$ can happen in topologically non-trivial ways: the trivialized subsymmetry can have an SPT. These kind of phases have been dubbed *gapless topological phases* [187]. If there are no topological order parameter, namely if $\mathcal{L}_{\mathcal{C}} \cap \mathcal{A} = 1$, there is no part of the symmetry that is spontaneously broken, and the phase is called a *gapless Symmetry Protected Topological* (gSPT) phase.

Even more remarkably, there are certain gapless phases whose topological features have no analogue in gapped phases, hence are called *intrinsically gapless topological phases* [188]. To deform these phases

[1] This gives rise to multiple local vacua if the symmetry is 0-form, or a topological order if the symmetry is of higher-form.

into gapped ones, it is necessary to break spontaneously some additional part of the symmetry. If no part of the symmetry was orginally spontaneously broken the phase is called *intrinsically gapless Symmetry Protected Topological* (igSPT) phase. These phases have been intensely studied in (1+1)d [183, 184, 186, 188–190]. The main subject of this subject of this chapter is to extend these studies to higher-dimensional theories, and in the presence of 1-form symmetries.

One important remark is in order. While 0-form symmetries can be preserved (non-spontaneously broken) in gapless theory even though there are non-trivial local order parameters (with vanishing VEV), the same is not true for 1-form symmetries. If a CFT has a 1-form symmetry, either it acts trivially or it is spontaneously broken: there are conformal line defects charged under the symmetry that takes VEV.[2] However it is very important to notice that there are two conceptually distinct ways to break spontaneously a 1-form symmetry:

- Topological order: there are charged line operators with perimeter law, and become topological line operators in the IR. As a consequence there is a (possibly emergent) $(d-2)$-form symmetry in the low energy theory.
- Conformal 1-form symmetry breaking: the IR is a gapless theory with conformal line defects whose VEV is a constant. These lines are not topological, and there is no $(d-2)$-form symmetry.

When we talk about gSPT phases for a 1-form symmetry, strictly speaking the quotient of the symmetry that acts non-trivially on the gapless sector is spontaneously broken, but in the conformal way. The igSPT phases for 1-form symmetries are obstruction for a CFT with conformal 1-form symmetry breaking to be deformed into a gapped symmetry preserving phase.

Before moving on with the various examples and classification of gapless phases, let us make an other important general remark. If the symmetry we start with has a 't Hooft anomaly, there is no sensible concept of igSPT. Indeed it is already implied by the anomaly that the symmetry cannot be realized in a gapped phase without spontaneously breaking the symmetry. On the other hand, starting with a non-anomalous symmetry $\mathcal{C}$ of the microscopic theory, an igSPT phase with a quotient $\mathcal{C}'$ acting non-trivially on the gapless degree of freedom boils down in having an *emergent anomaly* for $\mathcal{C}'$. For group symmetry this connects with the Wang-Wen-Witten result [191], that for certain dimensions and for certain degrees of the form, a group symmetry G with an anomaly $\omega \in H^{d+1}(B^{p+1}G, U(1))$ can be centrally extended $1 \to H \to \widetilde{G} \to G \to 1$ in such a way that the anomaly pulls back to a trivial anomaly $\widetilde{\omega} = 0$ for $\widetilde{G}$. This means that starting with a microscopic theory with non-anomalous symmetry $\widetilde{G}$, along the RG flow the subgroup $H \subset \widetilde{G}$ can be trivialized in a way that leaves a non-trivial anomaly for the quotient $G = \widetilde{G}/H$. An igSPT phase realizes this instance.

7.2 gSPT phases for Abelian 0-Form Symmetries in (1+1)d

For general finite groups G the possible gSPT and igSPT phases were classified in [184] using the results of [192] on condensable algebras in the SymTFT. Nevertheless, in view of later generalizations to TY categories and to higher dimensions, we find it useful to provide a direct classification in the Abelian group case. Consider in this section an Abelian 0-form symmetry $\mathbb{A}^{(0)}$ in (1+1)d.

7.2.1 SymTFT Characterization of Gapped Phases

The SymTFT is the (2+1)d Dijkgraaf-Witten theory for $\mathbb{A}^{(0)}$. In general, a Lagrangian algebra of $\mathcal{Z}(\mathcal{C})$ of a fusion category $\mathcal{C}$ has to satisfy various consistency conditions (see, e.g. the Appendices of

[2] A non-trivial line of a CFT cannot have area law.

[186, 193]). In the present simplified setting, we need a maximal algebra of lines that are mutually local. In particular, these have spin 1.

The Drinfeld center for the 0-form symmetry $\mathbb{A}$, $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$, is isomorphic to $\mathbb{A} \times \mathbb{A}^\vee$. Anyons are lines labeled by (a, α) that are charged under the electric and magnetic symmetries in the bulk. The braiding $\mathcal{B}$ and topological spins θ of these anyons are

$$\mathcal{B}\left[(a,\alpha)\,,\,(b,\beta)\right] = \beta(a)\,\alpha(b)\ \ , \qquad\qquad \theta_{(a,\alpha)} = \alpha(a)\,. \tag{7.2.1}$$

The canonical Lagrangian algebra corresponding to the symmetry boundary where the $\text{Vec}_{\mathbb{A}}$ symmetry is realized is

$$\mathcal{L}_{\text{sym}} = \left\{ (0,\alpha) \;\middle|\; \alpha \in \mathbb{A}^\vee \right\}. \tag{7.2.2}$$

The quotient

$$\mathcal{C} = \mathbb{A} \times \mathbb{A}^\vee / \mathcal{L}_{\text{sym}} \cong \mathbb{A} \tag{7.2.3}$$

represents the symmetry group, while $\mathcal{L}_{\text{sym}} \cong \mathbb{A}^\vee$ describes all possible irreducible representations (characters) of $\mathbb{A}$. Different choices of Lagrangian algebras for $\mathcal{L}_{sym}$ lead to categorical symmetry which are gauge-related (Morita equivalent) to $\text{Vec}_{\mathbb{A}}$.

Given a subgroup $\mathbb{B} \subset \mathbb{A}$, denote by $N(\mathbb{B}) \subset \mathbb{A}^\vee$ the subgroup of characters annihilating $\mathbb{B}$

$$N(\mathbb{B}) = \left\{ \beta \in \mathbb{A}^\vee \;\middle|\; \beta(b) = 1\ ,\forall b \in \mathbb{B} \right\} \cong (\mathbb{A}/\mathbb{B})^\vee\,. \tag{7.2.4}$$

As we shown in Chapter 6 (see Appendix B.1 for the details) that the Lagrangian algebras of $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ are classified by a choice of subgroup $\mathbb{B} \subset \mathbb{A}$ and a cocycle $\omega \in H^2(\mathbb{B}, U(1))$:

$$\mathcal{L}_{\mathbb{B},\omega} = \left\{ (b, \beta\psi(b)) \;\middle|\; b \in \mathbb{B}\ , \beta \in N(\mathbb{B}) \right\}. \tag{7.2.5}$$

Here $\psi : \mathbb{B} \to \mathbb{B}^\vee$ is a group homomorphism determined by $\omega \in H^2(\mathbb{B}, U(1))$ as follows. Using the well-known isomorphism [69] between the group of alternating bicharacters and $H^2(\mathbb{B}, U(1))$, we construct the alternating bicharacter $\chi : \mathbb{B} \times \mathbb{B} \to U(1)$:

$$\chi(b, b') = \frac{\omega(b, b')}{\omega(b', b)}\ . \tag{7.2.6}$$

Then ψ is given by

$$\psi(b)b' = \chi(b, b')\ . \tag{7.2.7}$$

Characterization of SPT phases. A (gapped) SPT phase is realized in the SymTFT setup, by choosing the physical boundary to be gapped, i.e. given by a Lagrangian algebra $\mathcal{L}_{\mathbb{B},\omega}$ such that there is no genuine charged operator after the interval compactification. This means that no anyon is allowed to end on both boundaries

$$\mathcal{L}_{\text{sym}} \cap \mathcal{L}_{\mathbb{B},\omega} = 1\,. \tag{7.2.8}$$

From the realization (7.2.5) we see that $\mathcal{L}_{\text{sym}} \cap \mathcal{L}_{\mathbb{B},\omega} = N(\mathbb{B})$. Therefore, SPT phases are obtained by choosing $\mathbb{B} = \mathbb{A}$ and we recover the usual group-cohomology classification of SPTs in terms of $\omega \in H^2(\mathbb{A}, U(1))$ [59].

Similar considerations apply even if $\mathcal{L}_{sym}$ is not of the form (7.2.2). For any symmetry associated with $\mathcal{L}_{\text{sym}} = \mathcal{L}_{\mathbb{B},\omega}$ we can repeat the above analysis finding that an SPT for that symmetry is again a Lagrangian $\mathcal{L}_{\mathbb{B}',\omega'}$ satisfying

$$\mathcal{L}_{\mathbb{B},\omega} \cap \mathcal{L}_{\mathbb{B}',\omega'} = 1\,. \tag{7.2.9}$$

7.2.2 Classification of gSPT and igSPT Phases

While Lagrangian algebras determine gapped phases, non-maximal condensable algebras define an interface with a reduced topological order [184, 185, 194, 195]. Picking a (generically non-gapped) physical boundary of the reduced topological order, one constructs a generic, not necessarily gapped phase, and hence we refer to these as gapless phases. Some of them are incompatible with a gapped realization and hence are called intrinsically gapless phases. Therefore, gapless phases are classified by condensable algebras $\mathcal{A}$. Intuitively, the anyons in common between $\mathcal{A}$ and $\mathcal{L}_{\text{sym}}$ give rise to topological local operators, describing discrete vacua, while the reduced topological order describes a part of the symmetry which only acts non-trivially on the gapless sector.

Condensable Algebras of $\mathcal{Z}(\mathbf{Vec}_{\mathbb{A}})$

Condensable algebras $\mathcal{A}$ of $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ can be parametrized by subgroups made of bosonic lines, but they are not necessarily maximal. We can characterize them similarly to the Lagrangian ones. Consider the projection $\pi_{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A}$ on the first factor, and define $\mathbb{B} := \pi_{\mathbb{A}}(\mathcal{A}) \subset \mathbb{A}$ so that there is a short exact sequence

$$1 \to \ker(\pi_{\mathbb{A}}|_{\mathcal{A}}) \to \mathbb{A} \to \mathbb{B} \to 1\,. \tag{7.2.10}$$

For a generic condensable algebra, $\ker(\pi_{\mathbb{A}}|_{\mathcal{A}})$ is a subgroup of $N(\mathbb{B})$

$$\mathbb{D} = \ker(\pi_{\mathbb{A}}|_{\mathcal{A}}) \subset N(\mathbb{B})\,. \tag{7.2.11}$$

It is convenient to represent $\mathbb{A}^\vee$ as a group extension

$$1 \to \mathbb{D} \to \mathbb{A}^\vee \to \mathbb{A}^\vee/\mathbb{D} \to 1\;, \tag{7.2.12}$$

so that any character is written as a pair $\alpha = \delta\xi$, $\delta \in \mathbb{D}, \xi \in \mathbb{A}^\vee/\mathbb{D}$. Notice that $\mathbb{A}^\vee/\mathbb{D}$ is a group extension of $\mathbb{B}^\vee$ by $N(\mathbb{B})/\mathbb{D}$. The algebra $\mathcal{A}$ can then be represented by

$$\mathcal{A} = \Big\{ \big(b, \delta\psi(b)\big) \;\big|\; b \in \mathbb{B}\;, \delta \in \mathbb{D} \Big\}\,, \tag{7.2.13}$$

where $\psi : \mathbb{B} \to \mathbb{A}^\vee/\mathbb{D}$ is a group homomorphism. The trivial spin condition translates into

$$\psi(b)b = 1 \quad , \qquad \forall b \in \mathbb{B}\,. \tag{7.2.14}$$

We conclude that condensable algebras are labelled by triples $(\mathbb{B}, \mathbb{D}, \psi)$ where $\mathbb{B} \subset \mathbb{A}$, $\mathbb{D} \subset N(\mathbb{B})$ and $\psi : \mathbb{B} \to \mathbb{A}^\vee/\mathbb{D}$ is a group-homorphism such that $\psi(b)b = 1$[3]. We denote condensable algebras by

$$\mathcal{A}_{\mathbb{B},\mathbb{D},\psi} : \qquad \mathbb{B} \subset \mathbb{A}\,, \quad \mathbb{D} \subset N(\mathbb{B}) \subset \mathbb{A}^\vee\,, \quad \psi : \mathbb{B} \to \mathbb{A}^\vee/\mathbb{D}\,. \tag{7.2.16}$$

gSPT Phases

Gapless phases are obtained by condensable but non-maximal algebras, which define interfaces to a reduced topological order – see (7.1.2). The set of charges realized on the ground states is given by anyons of $\mathcal{A}$ which can also end on the symmetry boundary. A gapless SPT (gSPT) phase is a gapless

[3] Notice that it makes sense to evaluate an element (here $\psi(b)$) of $\mathbb{A}^\vee/\mathbb{D}$ on elements of $\mathbb{B}$ because of the short exact sequence

$$1 \to N(\mathbb{B})/\mathbb{D} \to \mathbb{A}^\vee/\mathbb{D} \to \mathbb{B} \to 1\,. \tag{7.2.15}$$

phase in which the only charge realized on the vacuum is the trivial one. Hence, the condensable algebra for a gSPT has to satisfy

$$\mathcal{A}_{\mathrm{gSPT}} \cap \mathcal{L}_{\mathrm{sym}} = 1\,. \tag{7.2.17}$$

By the classification above of condensable algebras, it follows that $\mathcal{A}_{\mathbb{B},\mathbb{D},\psi} \cap \mathcal{L}_{\mathrm{sym}} = \mathbb{D}$, and the condition (7.2.17) is nothing but $\mathbb{D} = 1$. Thus whenever $\mathbb{B} \subsetneq \mathbb{A}$ is a proper subgroup $\mathcal{A}_{\mathbb{B},1,\psi}$ describes a gSPT. In summary gSPT phases are classified by algebras $\mathcal{A}_{\mathbb{B},\psi} \equiv \mathcal{A}_{\mathbb{B},1,\psi}$, and parametrized by the following data:

1. A choice of a proper subgroup $\mathbb{B} \subsetneq \mathbb{A}$.
2. A choice of a group homomorphism $\psi : \mathbb{B} \to \mathbb{A}^\vee$ such that $\psi(b)b = 1$.

igSPT Phases

A gSPT phase is called *intrinsic* (or igSPT for short) if it cannot be deformed to an ordinary (gapped) SPT phase. This means that the condensable algebra $\mathcal{A}_{\mathbb{B},\psi}$ is not a subalgebra of a Lagrangian algebra corresponding to an SPT. This would be a Lagrangian algebra containing $\mathcal{L}_{\mathbb{B},1,\psi}$ of the form

$$\left\{(a, \widehat{\psi}(a)) \mid a \in \mathbb{A}\,, \right\}\,, \tag{7.2.18}$$

where $\widehat{\psi} : \mathbb{A} \to \mathbb{A}^\vee$ is a homomorphism such that $\widehat{\psi}(a)a = 1$,$\forall a \in \mathbb{A}$, and corresponds to a class in $H^2(\mathbb{A}, U(1))$. We conclude that a gSPT classified by $(\mathbb{B}, \psi)$ is an igSPT if and only if $\psi : \mathbb{B} \to \mathbb{A}^\vee$ cannot be extended to a homomorphism $\widehat{\psi} : \mathbb{A} \to \mathbb{A}^\vee$ while preserving the property $\widehat{\psi}(a)a = 1$.

7.2.3 Examples

We now provide several explicit examples of igSPTs in $(1+1)$d. Since we will be dealing with cyclic groups, we will use *additive* notation for ease of reading.

Minimal (i)gSPT Example: $\mathbb{A} = \mathbb{Z}_4$

The case of $\mathbb{Z}_4$ is very well known [99, 196]. Let us nevertheless consider it in the context of our general classification. $\mathbb{Z}_4$ has one non-trivial proper subgroup $\mathbb{B} = \mathbb{Z}_2 = \{0, 2\} \subset \mathbb{Z}_4$. There are two possible homomorphisms $\psi : \mathbb{Z}_2 \to \mathbb{Z}_4$, classified by the choice of possible order-two elements of $\mathbb{Z}_4$ to assign $\psi(2)$: the trivial homomorphism and $\psi(2) = 2$. The first case gives a (nonintrinsic) gSPT. The other possibility also defines a gSPT since $\psi(2)2 = 4 \bmod(4) = 0$. The last case is also an igSPT. In fact, the extension of $\psi : \mathbb{Z}_2 \to \mathbb{Z}_4^\vee$ to $\widehat{\psi} : \mathbb{Z}_4 \to \mathbb{Z}_4$ must satisfy $\widehat{\psi}(2) = 2\widehat{\psi}(1)$, so $\widehat{\psi}(1)$ is 1 or 3, and in both cases $\widehat{\psi}(1)1 \neq 0 \bmod(4)$. The igSPT algebra is

$$\mathcal{A}_{\mathbb{Z}_2,1,\psi} = \{(2x, 2x) \mid x = 0, 1\ \}\ . \tag{7.2.19}$$

Cyclic Groups $\mathbb{A} = \mathbb{Z}_\mathbf{n}$

Subgroups $\mathbb{B} \subset \mathbb{Z}_n$ are labelled by divisors p of n. Let $n = pq$, then the subgroup is

$$\mathbb{B}_p = \{qx \mid x = 0, ..., p-1\} \cong \mathbb{Z}_p\,. \tag{7.2.20}$$

Homomorphsims $\psi : \mathbb{B}_p \to \mathbb{Z}_n^\vee$ are determined by picking an order p element $y \in \mathbb{Z}_n^\vee \cong \mathbb{Z}_n$[4] and declaring that $\psi(q) = y$. Clearly the elements of order p of $\mathbb{Z}_n^\vee$ forms the subgroup

$$N(\mathbb{B}_q) = \{qm \mid m = 0, ..., p-1\} \cong (\mathbb{Z}_n/\mathbb{B}_q)^\vee \cong \mathbb{Z}_p\,, \tag{7.2.21}$$

[4]We pick the (non-canonical) isomorphism $\mathbb{Z}_n \to \mathbb{Z}_n^\vee$ that assign to $a \in \mathbb{Z}_n$ the character $\chi_a(b) = \exp\left(\frac{2\pi i a b}{n}\right)$.

hence we have p possible homomorphisms $\psi_m : \mathbb{B}_p \to \mathbb{Z}_n^\vee$ labelled by $m \in \mathbb{Z}_p$, and defined by

$$\psi_m(q) = qm\,. \tag{7.2.22}$$

We see that

$$\psi_m(qx)\,qx = q^2x^2m \text{ mod}(n) \tag{7.2.23}$$

so that this is alternating if and only if p divides qm. This implies that m must be proportional to $p/\text{gcd}(p,q)$, namely it takes value in the subgroup $\mathbb{Z}_{\text{gcd}(p,q)} \subset \mathbb{Z}_p$. This subgroup classifies the gSPT phases for $\mathbb{Z}_n$.

Let us consider the igSPT phases: there is only a trivial SPT phase for these symmetries, and these algebras for the igSPT are not contained within this. Let us show this in a way which is helpful in other cases[5]. We should ask when ψ_m with $m = pr/\text{gcd}(p,q)$ can be extended to $\widehat{\psi}_m : \mathbb{Z}_n \to \mathbb{Z}_n^\vee$ in such a way that $\widehat{\psi}_m(a)a = 0 \,\text{mod}(n)$ for all $a \in \mathbb{Z}_n$. Notice that $q\widehat{\psi}_m(1) = qm \text{ mod}(n)$, so that

$$\widehat{\psi}_m(1) = m + kp\ ,\ \ k = 0, ..., q-1\,. \tag{7.2.24}$$

Thus the condition $\widehat{\psi}_m(1)1 = 0\,\text{mod}(n)$ becomes

$$m + kp = 0 \text{ mod}(n)\,. \tag{7.2.25}$$

Clearly this can never be satisfied by a non-trivial m. Thus we conclude that any nontrivial $\mathbb{Z}_n$ gSPT phase is also an igSPT. The igSPT algebras are

$$\mathcal{A}_{\mathbb{Z}_p,1,\psi_m} = \{(qx, mqx)|\ l = 0, ..., p-1\}\ , \qquad m = pk/\text{gcd}(p,q)\ , \quad k \in \mathbb{Z}_{\text{gcd}(p,q)}\ . \tag{7.2.26}$$

To summarize, the gSPT phases for $\mathbb{Z}_n$ are classified by divisors p of n and an element in $\mathbb{Z}_{\text{gcd}(p,q)}$. Hence we have

$$G(n) = \sum_{p|n} \text{gcd}\,(p,q) \tag{7.2.27}$$

many gSPT phases.

igSPTs for $\mathbb{A} = \mathbb{Z}_\mathbf{n} \times \mathbb{Z}_\mathbf{n}$

We do not attempt to classify all gapless phases for $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$, rather we focus on igSPTs existing for $n = pq$, $\text{gcd}(p,q) \neq 1$, which are representative of the general scenario. This is instructive since $\mathbb{Z}_n \times \mathbb{Z}_n$ also admits gapped SPTs, so not all non-trivial gSPTs are automatically intrinsic.

We look at the subgroup $\mathbb{B} = \{(qx, qy)\} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. The most general homomorphism $\mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_n \times \mathbb{Z}_n$ is determined by four numbers mod(p)

$$\psi_{s_1,s_2,r_1,r_2}(qx, qy) = (s_1qx + r_1qy, s_2qx + r_2qy) \tag{7.2.28}$$

and this is alternating if and only if $s_1, r_2, r_1 + s_2$ are proportional to $p/\text{gcd}(p,q)$. However, it admits an alternating extension $\widehat{\psi} : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \times \mathbb{Z}_n$ only if $s_1 = r_2 = r_1 + s_2 = 0\,\text{mod}(p)$, the value of $r \equiv r_1 = -s_2 = 0, ..., n-1$ (this is a lift of $r_1 = -s_2$ to $\mathbb{Z}_n$) being the value of the ordinary gapped SPT classified by $H^2(\mathbb{Z}_n \times \mathbb{Z}_n, U(1)) = \mathbb{Z}_n$.

Hence for $\text{gcd}(p,q) \neq 1$ we can choose $s_1, r_2, r_1 + s_2$ to be non-trivial, producing igSPT phases. We have $p\left(\text{gcd}(p,q)^3 - 1\right)$ such phases. Those with s_1, r_2 non-trivial but $r_1 + s_2 = 0$ are the igSPTs

[5] Either for $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ or, as will be mostly important for us, in (3+1)d.

for the two $\mathbb{Z}_n$ factors that we already discussed, while $r_1 + s_2 = kp/\gcd(p,q)$, $k \neq 0$ are new. For example, for $n = 9$, an igSPT is given by $p = q = 3$, $s_1 = r_2 = 0$, $r_1 = s_2 = 1$. The algebra is[6]

$$\mathcal{A}_{\mathbb{Z}_3\times\mathbb{Z}_3,1,\psi_{0,1,1,0}} = \{(3x, 3y; 3y, 3x)\ |a, b = 0, 1, 2\}\ . \tag{7.2.29}$$

More generally, letting $\ell = p/\gcd(p,q)$, we denote these algebras as $\mathcal{A}_{\mathbb{Z}_p\times\mathbb{Z}_p,1,\psi}$ with

$$\psi(qx, qy) = \Big(k_1\ell qx + (r + k_2\ell)\, qy\, ,\, -rqx + k_3\ell qy\Big) \tag{7.2.30}$$

determined by $k_1, k_2, k_3 \in \mathbb{Z}_{\gcd(p,q)}$ and $r \in \mathbb{Z}_p$. These define a class of igSPTs for $\mathbb{Z}_n \times \mathbb{Z}_n$.

7.2.4 Physical Construction of igSPTs

In this section we describe how to use a KT transformation to generate examples of igSPTs, following [190, 197]. We then describe the effect of embedding their construction into a UV complete QFT.

KT Transformation to igSPT Phases

We will now illustrate these (i)gSPTs with concrete (1+1)d Field theories. This is a slight generalization of the results in [197], which treated the case $\mathbb{A} = \mathbb{Z}_4$. The procedure works in four steps:

1. Consider a theory $\mathfrak{T}_0$ with zero-form symmetry $\mathbb{Z}_n^{(0)}$ (here $n = pq$), gauge $\mathbb{Z}_p \subset \mathbb{Z}_n^{(0)}$ to construct $\mathfrak{T}_0/\mathbb{Z}_p$.

2. Stack a trivial theory with $\mathbb{Z}'_q$ symmetry.

3. Gauge $\mathbb{Z}_p^\vee \times \mathbb{Z}'_q$, with a discrete torsion class $m \in H^2(\mathbb{Z}_p \times \mathbb{Z}_q, U(1)) = \mathbb{Z}_{\gcd(p,q)}$. This produces a theory with $\mathbb{Z}_n \times \mathbb{Z}_q$ symmetry.

4. Identify the gauge field for $\mathbb{Z}_q$ with that for the $\mathbb{Z}_q$ quotient in $\mathbb{Z}_n$.

This will prove instrumental in discussing examples with 1-form symmetry in Section 7.3. Let us now give some further details.

As we have seen, an igSPT for $\mathbb{Z}_n$ is given by a subgroup $\mathbb{Z}_p \subset \mathbb{Z}_n$ such that

$$\gcd(p,q) \neq 1\ , \qquad n = pq\ , \tag{7.2.31}$$

together with the choice of $m \in \mathbb{Z}_{\gcd(p,q)}$ (the igSPT is non-trivial if $m \neq 1$). This second choice can be understood as an element

$$m \in H^2(\mathbb{Z}_p \times \mathbb{Z}_q, U(1)) = \mathbb{Z}_{\gcd(p,q)}\ , \tag{7.2.32}$$

namely an SPT phase for $\mathbb{Z}_p \times \mathbb{Z}_q$. The authors of [196] gave a construction of a continuum QFT that realizes the igSPT for $\mathbb{Z}_4$. We can generalize this construction to produce a continuum QFT realizing of all the igSPT phases that we have classified.

We start from any 2d CFT $\mathfrak{T}_0$ with $\mathbb{Z}_n^{\text{in}}$ symmetry. We gauge the subgroup $\mathbb{Z}_p \subset \mathbb{Z}_n^{\text{in}}$, producing a theory $\mathfrak{T}_0/\mathbb{Z}_p$. Since $\gcd(p,q) \neq 1$ the sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_n^{\text{in}} \to \mathbb{Z}_q \to 1 \tag{7.2.33}$$

[6]Our notation is $(e_1, e_2; m_1, m_2)$, where $e_{1,2}, m_{1,2}$ label the electric and magnetic lines of the two $\mathbb{Z}_n$ groups, respectively.

Group	$\mathbb{Z}_p$	$\mathbb{Z}_p^\vee$	$\mathbb{Z}_p^{\vee\vee}$	$\mathbb{Z}_q$	$\mathbb{Z}'_q$	$\mathbb{Z}'^\vee_q$
Background Field	A_p	$\widehat{A}_p$	$\widetilde{A}_p$	B_q	B'_q	$\widehat{B'}_q$

Table 7.1: Groups and background fields.

does not split, hence the resulting theory has symmetry $\mathbb{Z}_p^\vee \times \mathbb{Z}_q$ with mixed anomaly [154]

$$\frac{2\pi i}{p}\int_{X_3} \widehat{A}_p \cup \beta(B_q)\,, \tag{7.2.34}$$

where $\widehat{A}_p$ is the background field for $\mathbb{Z}_p^\vee \cong \mathbb{Z}_p$, B_q a background for $\mathbb{Z}_q = \mathbb{Z}_n^{\text{in}}/\mathbb{Z}_p$, and $\beta : H^1(X,\mathbb{Z}_q) \to H^2(X,\mathbb{Z}_p)$ is the Bockstein associated with the sequence (7.2.33).[7]

Clearly, if we now make $\widehat{A}_p$ dynamical, we recover the theory $\mathfrak{T}_0$. Instead, we perform a slightly different operation. First we declare that the system has a further trivially acting $\mathbb{Z}_q$ symmetry that we denote by $\mathbb{Z}'_q$ to distinguish it from the quotient $\mathbb{Z}_q = \mathbb{Z}_n^{\text{in}}/\mathbb{Z}_p$. This can be thought of as stacking a decoupled trivially gapped system with a $\mathbb{Z}'_q$ symmetry. Denote the background field for this by B'_q. The notation for various groups and background fields is summarized in table 7.1.

We then gauge $\mathbb{Z}_p^\vee \times \mathbb{Z}'_q$, but crucially adding a discrete torsion

$$\exp\left(\frac{2\pi i m}{\gcd(p,q)}\int_{X_2} \widehat{A}_p \cup B'_q\right)\,, \qquad m \in H^2(\mathbb{Z}_p \times \mathbb{Z}'_q, U(1)) = \mathbb{Z}_{\gcd(p,q)}\,. \tag{7.2.35}$$

Denoting by $\widetilde{A}_p$ and $\widehat{B}'_q$ the backgrounds for the two dual symmetries $(\mathbb{Z}_p^\vee)^\vee \cong \mathbb{Z}_p$, $(\mathbb{Z}'_q)^\vee$ that arise from this gauging, the resulting partition function is

$$\begin{aligned} Z_{\mathfrak{T}}[\widetilde{A}_p, \widehat{B}'_q, B_q] = \sum_{\widehat{A}_p, B'_q} \exp\bigg(&\frac{2\pi i m}{\gcd(p,q)}\int_{X_2} \widehat{A}_p \cup B'_q + \frac{2\pi i}{p}\int_{X_2} \widetilde{A}_p \cup \widehat{A}_p + \\ &+ \frac{2\pi i}{q}\int_{X_2} \widehat{B}'_q \cup B'_q\bigg) Z_{\mathfrak{T}_0/\mathbb{Z}_p}[\widehat{A}_p, B_q]\,. \end{aligned} \tag{7.2.36}$$

To obtain the correct cocycle conditions of the new backgrounds, we need to impose gauge invariance under both $\widehat{A}_p \to \widehat{A}_p + \delta\lambda$ and $B'_q \to B'_q + \delta\eta$. This imposes

$$\delta\widehat{B'}_q = 0\,, \qquad \delta\widetilde{A}_p = \beta(B_q)\,. \tag{7.2.37}$$

The dual symmetry $\mathbb{Z}_p^{\vee\vee} = \mathbb{Z}_p$ now extends $\mathbb{Z}_q$ producing again a new $\mathbb{Z}_n$ symmetry. This is to be distinguished from the original $\mathbb{Z}_n^{\text{in}}$, as it is not faithfully acting: as we will see shortly the $\mathbb{Z}_p$ subgroup does not act at all. To show this fact let us manipulate the partition function. Rewriting $Z_{\mathfrak{T}_0/\mathbb{Z}_p}$ in terms of $Z_{\mathfrak{T}_0}$, we can perform the sum over $\widehat{A}_p$, that imposes a delta function, that can be solved by the sum over A_p, and we remain with

$$Z_{\mathfrak{T}}[\widetilde{A}_p, \widehat{B'}_q, B_q] = \sum_{B'_q} \exp\left(\frac{2\pi i}{q}\int_{X_2} \widehat{B'}_q \cup B'_q\right) Z_{\mathfrak{T}_0}\left[q\left(\widetilde{A}_p + m\frac{p}{\gcd(p,q)}B'_q\right) + B_q\right]\,. \tag{7.2.38}$$

To gain some intuition about the result, consider the particular case $q = p$, $m = 1$ (for $p = 2$ this is the case discussed in [198]). We omit the subscripts. In the sum over B' we can shift $B' \mapsto$

[7] In our case it is given explicitly by $\beta(B_q) = \frac{\delta B_q}{p}$ where B_q in the right hand side is an arbitrary lift to $\mathbb{Z}_n$ of the $\mathbb{Z}_q$ gauge field.

Group	$\mathbb{Z}_n$	$\widehat{\mathbb{Z}}_p$	$\mathbb{Z}_p$	$\mathbb{Z}'_p$
Embedding	$\mathbb{Z}_n^m$	$\widehat{\mathbb{Z}}_p^w$	$\mathbb{Z}_n^m/\mathbb{Z}_p^m$	$\mathbb{Z}_p^{w\prime}$
Twist field	$W = e^{i\widetilde{X}/p^2}$	$\widehat{V} = e^{iX}$	$W = e^{i\widetilde{X}/p^2}$	$V' = e^{iX'/p}$

	$\widehat{\mathbb{Z}}_p$	$\mathbb{Z}_p$	$\mathbb{Z}'_p$
$\widehat{V}$	1	$e^{-2\pi i/p^2}$	$e^{-2\pi i/p}$
V'	$e^{2\pi i/p}$	1	1
W	$e^{2\pi i/p^2}$	1	1

Table 7.2: Embedding of symmetries, relevant twist fields and their charge after stacking the SPT for the free boson example.

$B' - \widetilde{A}$ to eliminate it from $Z_{\mathfrak{T}_0}$. Hence, this shifted sum over B' reproduces the partition function of $Z_{\mathfrak{T}_0/\mathbb{Z}_p}[\widehat{B'}, A]$, but there is an addition phase factor:

$$Z_{\mathfrak{T}}[\widetilde{A}, \widehat{B'}, B] = \exp\left(-\frac{2\pi i}{p}\int_{X_2} \widehat{B'} \cup \widetilde{A}\right) Z_{\mathfrak{T}_0/\mathbb{Z}_p}[\widehat{B'}, B]\,. \tag{7.2.39}$$

Naively, this seems like stacking a CFT $\mathfrak{T}_0/\mathbb{Z}_p$ and an invertible phase, but it's more nuanced. The theory has a symmetry $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ with backgrounds $p\widetilde{A} + B$ and $\widehat{B}$, respectively. Focusing on the gapless sector $\mathfrak{T}_0/\mathbb{Z}_p$, the $\mathbb{Z}_p \subset \mathbb{Z}_{p^2}$ does not act, leaving a faithful $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry (with the first factor being $\mathbb{Z}_{p^2}/\mathbb{Z}_p$) and a mixed anomaly: a gauge transformation $\widehat{B'} \to \widehat{B'} + \delta\lambda$ multiplies $Z_{\mathfrak{T}_0/\mathbb{Z}_p}$ by a phase. Including the symmetry $\mathbb{Z}_p$ with background $\widetilde{A}$ acting trivially, the anomaly is cancelled by the Green-Schwarz mechanism:

$$\exp\left(-\frac{2\pi i}{p}\int_{X_2} \widehat{B'} \cup \widetilde{A}\right) \to \exp\left(-\frac{2\pi i}{p}\int_{X_2} \delta\lambda \cup \widetilde{A}\right) = \exp\left(\frac{2\pi i}{p}\int_{X_2} \lambda \cup \beta(B)\right).$$

Physically, for the CFT $\mathfrak{T}_0/\mathbb{Z}_p$, the anomaly implies that the system cannot be gapped while preserving $\mathbb{Z}_p \times \mathbb{Z}_p$. However, since the anomaly is canceled by a symmetry acting on an invertible phase, by realizing the latter as the IR of a trivially gapped theory, the anomaly is absent in the full theory. From an IR viewpoint, this can be seen as an example of the general story of [199]. Thus, in the full theory, there is no obstruction to gapping it. At low energy, without considering the additional symmetry, we cannot gap the system unless the additional degrees of freedom become massless, closing the symmetry gap, thus encountering a phase transition and allowing further deformations to gap the theory.

Embedding into a UV theory

This construction neatly describes the IR phase. To embed it into a full-fledged UV theory, we simply consider the product of theory $\mathfrak{T}_0$ — or an arbitrary QFT flowing to it— with a theory $\mathfrak{T}'$ flowing to a trivially gapped phase and carrying the $\mathbb{Z}'_q$ symmetry action. Let us focus on the case $n = p^2$ and give a concrete example. Consider $\mathfrak{T}_0$ to be a free compact scalar X of radius R and $\mathfrak{T}'$ to be a compact scalar X' of large radius R' deformed by $\lambda\cos(X')$.[8] We take the radius R' large enough so that this term is relevant and the embedding of symmetries is as in table 7.2.

Stacking the $\mathbb{Z}_p^\vee \times \mathbb{Z}_p'$ SPT $\exp\left(\frac{2\pi i}{p}\int \widehat{A}_p \cup A_p'\right)$ gives charge to twist fields according to table 7.2.[9] The charges can be used to describe the local field content of the theory after performing the gauging procedure. Surprisingly, we learn that neither X nor X' are well defined field anymore, but rather we should consider:

$$Y = X + X'/p\,, \quad Z = X' - \widetilde{X}\,. \tag{7.2.40}$$

Written in terms of these fields the cosine potential becomes:

$$\lambda\cos\left(Z + \widetilde{Y}\right)\,, \tag{7.2.41}$$

which pins the momentum modes of one field to the winding modes of the other. The faithfully acting (unbroken) symmetry on the IR scalar is just the $\mathbb{Z}_p$ diagonal between momentum and winding, which is anomalous.

7.3 (i)gSPTs for 1-form Symmetries in (3+1)d

The formalism we developed in (1+1)d can be adapted to discuss phases – gapped and gapless – with 1-form symmetries in (3+1)d. We carry out a SymTFT analysis, showing the existence – and providing a classification – of igSPT phases for 1-form symmetries. This refines the standard classification of phases of gauge theories. The SymTFT approach also aids in the construction of concrete physical examples, and we provide an interpretation of these phases as topological obstructions to confinement.

7.3.1 SymTFT and Gapped Phases for 1-form Symmetries

The SymTFT for a 1-form symmetry $\mathbb{A}^{(1)}$ in (3+1)d is a five-dimensional TQFT whose topological defects are surfaces that form a group $\mathcal{Z}(\mathbb{A}^{(1)}) = \mathbb{A} \times \mathbb{A}^\vee$ governing their fusion. The braiding is anti-symmetric

$$\mathcal{B}\left((a,\alpha),(b,\beta)\right) = \alpha(b)\,\beta(a)^{-1}\,. \tag{7.3.1}$$

The canonical Lagrangian algebra that leads to the 1-form symmetry $\mathbb{A}$ is

$$\mathcal{L}_{\text{sym}} = \left\{(0,\alpha)\;\Big|\;\alpha\,\in\,\mathbb{A}^\vee\right\}. \tag{7.3.2}$$

Everything is very similar to the (1+1)d case, with the only (but crucial) difference that the braiding is anti-symmetric.

For simplicity, we assume all manifolds are spin. This technical assumption serves to identify different global variants with the same 1-form symmetry and the same choice of charges, but where the line operators have different statistics [163]. The group of discrete torsions

$$H^4(B^2\mathbb{A}, U(1)) \cong \{q : \mathbb{A} \to U(1)\;,\quad \text{quadratic form}\} \tag{7.3.3}$$

is a central extension of the group of symmetric bilinear forms Symm($\mathbb{A}$). The projection map is the polarization

$$\chi_q(a,b) = q(a+b) - q(a) - q(b)\,. \tag{7.3.4}$$

[8]We normalize the fields so that the periodicity is always 2π to simplify the notation.

[9]A simple derivation of this fact, following e.g. [9], is to realize a twist defect through an open background $\delta A = u$, with u the charge of the twisted sector. Performing a gauge transformation and using the SPT action with this background gives non-trivial charges of twist defects.

The fiber is isomorphic to $\mathrm{Hom}(\mathbb{A}, \mathbb{Z}_2)$, and its elements are called characteristic elements. From a quadratic form q and a gauge field $B \in H^2(X_4, \mathbb{A})$ one produces the discrete torsion

$$\int_{X_4} q(B)\,. \tag{7.3.5}$$

The key fact is that different quadratic forms with same polarization give the same integral on any *spin* 4-manifold. Hence we can mod-out $\mathrm{Hom}(\mathbb{A}, \mathbb{Z}_2)$ and work with symmetric bicharacters $\chi \in \mathrm{Symm}(\mathbb{A})$.

As in (1+1)d, Lagrangian algebras of $\mathcal{Z}(\mathbb{A}^{(1)})$ are classified by a subgroup $\mathbb{B} \subset \mathbb{A}$ and a symmetric homomorphism $\psi : \mathbb{B} \to \mathbb{B}^\vee$ as

$$\mathcal{L}_{\mathbb{B},\psi} = \{(b, \beta\psi(b)) \mid b \in \mathbb{B}\ ,\ \beta \in N(\mathbb{B})\}\ . \tag{7.3.6}$$

The symmetric condition means that $\chi : \mathbb{B} \times \mathbb{B} \to U(1)$, $\chi(b, b') = \psi(b)b'$ is a symmetric bicharacter, and this ensures that the elements of $\mathcal{L}_{\mathbb{B},\psi}$ do not braid among themselves. It also identifies ψ with a discrete torsion element in $H^4(B^2\mathbb{A}, U(1))/\mathrm{Hom}(\mathbb{A}, \mathbb{Z}_2) \cong \mathrm{Symm}(\mathbb{A})$.

In this section, we discuss gapped phases with 1-form symmetry; hence let us make some general comment on them. In the UV there are line operators labeled by their charges valued in $\mathbb{A}^\vee$. Since the phase is gapped, at low energy for each line there are two possibilities: either it has area low and flows to the trivial line (confined lines), or it has perimeter law and flows to a non-trivial topological line (deconfined lines). According to 't Hooft a phase is described by a Lagrangian lattice of dyons [42]. More explicitly, given the algebra $\mathcal{L}_{\mathbb{B},\psi}$, perimeter law is assigned to the lines:

$$W^\beta\,, \quad T^b W^{\psi(b)}\,. \tag{7.3.7}$$

where T, W are Wilson and 't Hooft lines for the universal cover of the gauge group and we only indicate their 1-form symmetry charge through our notation. The set of charges of deconfined lines forms a subgroup $\mathbb{D} \subset \mathbb{A}^\vee$, while $\mathbb{A}^\vee/\mathbb{D}$ is the quotient that labels the confined lines. Its Pontryagin dual $\mathbb{B} = (\mathbb{A}^\vee/\mathbb{D})^\vee$ is the preserved subgroup of the 1-form symmetry, while the quotient $\mathbb{A}/\mathbb{B}$ is spontaneously broken.

We can make this discussion more systematic using the SymTFT approach. Gapped phases with 1-form symmetry $\mathbb{A}^{(1)}$ are classified by Lagrangian algebras $\mathcal{L}_{\mathbb{B},\psi}$. The physical interpretation is the following. Fixing the symmetry to be $\mathbb{A}^{(1)}$ means that the symmetry boundary is determined by $\mathcal{L}_{\mathrm{sym}} = \{(0, \alpha) \mid \alpha \in \mathbb{A}^\vee\}$, hence the symmetry operators are the surfaces $(a, 0) \in \mathcal{Z}(\mathbb{A}^{(1)})$ pushed at the boundary. Looking for gapped phases means that the physical boundary is also topological and determined by a Lagrangian algebra. The surfaces that can end on *both* boundaries give rise to non-trivial topological line operators, namely the deconfined lines. They form the group

$$\mathcal{L}_{\mathrm{sym}} \cap \mathcal{L}_{\mathbb{B},\psi} \cong \mathbb{D} = N(\mathbb{B})\ . \tag{7.3.8}$$

Hence deconfined lines are completely transparent under the subgroup $\mathbb{B} \subset \mathbb{A}$ of the 1-form symmetry, while are detected by the quotient $\mathbb{A}/\mathbb{B}$. The group of non-trivial lines $N(\mathbb{B})$ is the Pontryagin dual of $\mathbb{A}/\mathbb{B}$, representing its set of charges. Moreover $\mathbb{A}^\vee/N(\mathbb{B}) \cong \mathbb{B}^\vee$ is the set of confined lines and is the dual of the trivialized subgroup of the 1-form symmetry.

The presence of $\mathbb{B}$ may also be detected by looking at the twisted sectors. These arise because some surfaces $(b, \psi(b)) \in \mathcal{L}_{\mathbb{B},\psi}$ cannot end on $\mathcal{L}_{\mathrm{sym}}$ and produce non-genuine lines in the twisted sector of $b \in \mathbb{B}$. Importantly, these non-genuine lines are also in general charged under the subgroup $\mathbb{B}$ of the 1-form symmetry $\mathbb{A}^{(1)}$: passing a surface labelled by $b' \in \mathbb{B}$ through a non-genuine line $(b, \psi(b))$ we pick a phase

$$\psi(b)b' = \chi(b, b')\ . \tag{7.3.9}$$

We conclude that the (3+1)d gapped phase is the spontaneous breaking of $\mathbb{A}$ down to $\mathbb{B}$ whose SPT phase is ψ. In particular, the phases with $\mathbb{B} = \mathbb{A}$ are SPT phases for the whole 1-form symmetry, and are determined by ψ. Thus, we recover the usual classification by $H^4(B^2\mathbb{A}, U(1))$ [14] (more precisely by Symm($\mathbb{A}$) on spin manifolds).

As an example, consider $\mathbb{A} = \mathbb{Z}_n$. Subgroups are given by divisors $p|n$:

$$\mathbb{B}_p = \{qx \mid x = 0, ..., p-1\} \cong \mathbb{Z}_p \quad , \quad n = pq\,. \tag{7.3.10}$$

Identifying $\mathbb{Z}_n^\vee \cong \mathbb{Z}_n$ we have

$$N(\mathbb{B}_p) = \{py \mid y = 0, ..., q-1\} \cong \mathbb{Z}_q\ , \tag{7.3.11}$$

while

$$\mathbb{B}_p^\vee \cong \mathbb{A}/N(\mathbb{B}_p) = \{x \sim x + p \mid x = 0, ..., p-1\}\ . \tag{7.3.12}$$

A homomorphism $\psi : \mathbb{B}_q \to \mathbb{B}_q^\vee$ is the multiplication by a number $r = 0, ..., p-1$, and is automatically symmetric. Therefore

$$\mathcal{L}_{p,r} = \{(qx, rx + py) \mid x = 0, ..., p-1\ ,\quad y = 0, ..., q-1\}\ . \tag{7.3.13}$$

SPT phases are obtained by setting $p = n$ and are classified by $r \in \mathbb{Z}_n$, i.e.

$$\text{SPT}: \qquad \mathcal{L}_r = \{(x, rx) \mid x = 0, ..., n-1\}\ , \qquad r = 0, \cdots, n-1\,. \tag{7.3.14}$$

7.3.2 (i)gSPT Phases protected by 1-form Symmetries

Now we look at gapless phases with 1-form symmetry $\mathbb{A}^{(1)}$. Each line operator, labeled by $\alpha \in \mathbb{A}^\vee$ can either flow to a trivial line, or to a non-trivial line, and the latter form a subgroup $N(\mathbb{B}) \subset \mathbb{A}^\vee$ (hence $\mathbb{B} \subset \mathbb{A}$ is trivial in the IR). However, differently from gapped phases, among the non-trivial lines, some can be topological, while others are not. The first set forms a subgroup $\mathbb{D} \subset N(\mathbb{B})$, and the presence of a gapless sector is characterized by the non-triviality of the quotient $N(\mathbb{B})/\mathbb{D}$ that labels the charges of non-topological lines of the gapless sector.

All of this can be formalized considering non-maximal condensable algebras of the SymTFT. The classification is as follows: the condensable algebras[10] are

- A subgroup $\mathbb{B} \subset \mathbb{A}$.
- A subgroup $\mathbb{D} \subset N(\mathbb{B})$
- A group homomorphism $\psi : \mathbb{B} \to \mathbb{A}^\vee/\mathbb{D}$ with the property that

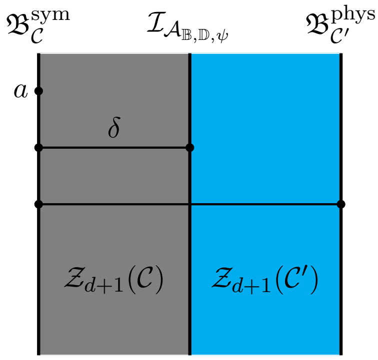
$$\psi(b)b' = \psi(b')b\ , \quad \forall b, b' \in \mathbb{B}\,. \tag{7.3.15}$$

The corresponding algebra is

$$\mathcal{A}_{\mathbb{B},\mathbb{D},\psi} = \Big\{(b, \delta\,\psi(b)) \;\Big|\; b \in \mathbb{B}\,, \delta \in \mathbb{D}\Big\}\ . \tag{7.3.16}$$

[10]We note that this reference to algebra is not quite accurate as these defects form a sub-higher-category, but we refrain from this in this context unnecessary embellishment in terminology.

In general, non-maximal condensable algebras describe gapless phases. The interpretation of (7.3.16) is clear from the "Club-Sandwich" picture:



(7.3.17)

Condensing $\mathcal{A}_{\mathbb{B},\mathbb{D},\psi}$ in half space we construct an interface between $\mathcal{Z}(\mathcal{C})$ and a reduced topological order $\mathcal{Z}(\mathcal{C}')$, with $\mathcal{C} = \mathbb{A}^{(1)}$. After interval compactification, we can distinguish three types of object, represented in the figure above from bottom to top:

1. Dynamical degrees of freedom charged under $\mathcal{C}'$, described by surface operators extending throughout the bulk.
2. Topological lines $(0, \delta)$ ending both on the $\mathcal{I}_{\mathcal{A}_{\mathbb{B},\mathbb{D},\psi}}$ interface and the symmetry boundary $\mathfrak{B}^{\text{sym}}$. These describe a (deconfined) SSB phase dressing the gapless degrees of freedom.
3. Topological surfaces labelled by $a \in \mathbb{A}$ confined on $\mathfrak{B}^{\text{sym}}$ and describing the full symmetry.

Up to this point everything is the same as in the gapped case. The difference is that now the quotient $\mathbb{A}/\mathbb{B}$ does not act faithfully on topological lines:

$$\mathcal{L}_{\text{sym}} \cap \mathcal{A}_{\mathbb{B},\mathbb{D},\psi} \cong \mathbb{D}\,, \tag{7.3.18}$$

which is in general smaller than $(\mathbb{A}/\mathbb{B})^{\vee} = N(\mathbb{B})$. Hence the subgroup $(N(\mathbb{B})/\mathbb{D})^{\vee} \subset \mathbb{A}/\mathbb{B}$ acts trivially on the topological lines, and can only act on gapless degrees of freedom coming from the physical boundary.

gSPTs and igSPTs. Gapless SPT phases are those in which there is no non-trivial topological line, and hence

$$\mathbb{D} = 1\ . \tag{7.3.19}$$

$\mathbb{B}$ is trivial at low energy, while $\mathbb{A}/\mathbb{B}$ only acts on a gapless sector. Therefore, gSPT phases for a 1-form symmetry in (3+1)d are classified by pairs $(\mathbb{B}, \psi)$, with $\mathbb{B} \subset \mathbb{A}$ and $\psi : \mathbb{B} \to \mathbb{A}^{\vee}$ a homomorphism satisfying the property (7.3.15).

This phase is *intrinsically gapless* (igSPT) if and only if there is no Lagrangian algebra of the from $\mathcal{L}_{\mathbb{A},\widehat{\psi}}$ (this ensures that $\mathcal{L}_{\mathbb{A},\widehat{\psi}} \cap \mathcal{L}_{\text{sym}} = 1$) such that $\mathcal{A}_{\mathbb{B},1,\psi} \subset \mathcal{L}_{\mathbb{A},\widehat{\psi}}$. This means that ψ must not admit an extension to $\widehat{\psi} : \mathbb{A} \to \mathbb{A}^{\vee}$ preserving the property (7.3.15).

7.3.3 Examples

Cyclic groups $\mathbb{A} = \mathbb{Z}_n$

Any homomorphism between two cyclic groups satisfies $\psi(b)b' = \psi(b')b$. Hence, gSPT phases are classified by a divisor $p|n$ that determines

$$\mathbb{B}_p = \{qx \mid x = 0, ..., p-1\}\ , \quad n = pq \tag{7.3.20}$$

and an order p element of $\mathbb{Z}_n^\vee \cong \mathbb{Z}_n$. The latter is of the form qm, $m = 0, ..., p-1$ and determines

$$\psi(qx) = qmx\,. \tag{7.3.21}$$

Any homomorphism $\psi : \mathbb{Z}_q \to \mathbb{Z}_n^\vee$ has a symmetric extension to $\mathbb{Z}_n$, so there are no igSPT phases.

Minimal igSPT $\mathbb{A} = \mathbb{Z}_4 \times \mathbb{Z}_2$

The smallest 1-form symmetry group that admits intrinsically gapless SPT phases is $\mathbb{Z}_4 \times \mathbb{Z}_2$. The subgroup $\mathbb{B} \subset \mathbb{A}$ that is part of the classification data $(\mathbb{B}, \psi)$, here is $\mathbb{B} = \mathbb{Z}_2 = \langle (2,0) \rangle \subset \mathbb{A}$ (for all other subgroups there are no igSPTs). The most general homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_4 \times \mathbb{Z}_2$ is

$$\psi_{s_1,s_2}(2x, 0) = (2s_1 x, s_2 x) \tag{7.3.22}$$

with $s_i = 0, 1$. Since $\psi_{s_1,s_2}(2x,0) \cdot (2x', 0) = 1$, this is automatically symmetric. Moreover, if $s_2 = 1$ it does not admit any extension (neither non-symmetric) $\widehat{\psi} : \mathbb{A} \to \mathbb{A}^\vee$, hence it represents an igSPT phase. We have two of them with $s_1 = 0, 1$. We will come back to this example shortly, providing a concrete model that realizes these phases in the vanilla case of $s_1 = 0$ (turning on s_1 corresponds to stacking a standard gapped SPT), whose condensable algebra is

$$\mathcal{A}_{\mathbb{Z}_2,1,\psi} = \{(2x, 0; 0, x) \mid x = 0, 1\ \}\ . \tag{7.3.23}$$

The fact that this defines an igSPT can be understood from the reduced topological order, too. Using standard methods one discovers that the reduced theory is the (4+1)d Dijkgraaf-Witten theory for $\mathbb{Z}_4$, whose by electric and magnetic surfaces are generated by:

$$E = (1, 0; 0, 0)\,, \qquad M = (0, 1; 1, 0)\,. \tag{7.3.24}$$

In order to determine the symmetry $\mathcal{C}'$ we ask which of these surfaces can terminate topologically on $\mathfrak{B}^{\text{sym}}$. The allowed surfaces are:

$$E^2 = (0, 0; 0, 1), \quad M^2 = (0, 0; 2, 0)\,. \tag{7.3.25}$$

This boundary condition defines a polarization whose symmetry is $\mathcal{C}' = \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ with a mixed anomaly [154]:

$$I = \pi i \int B_1\, \beta(B_2)\,, \quad \beta(B_2) = \frac{1}{2} dB_2\,. \tag{7.3.26}$$

This proves that the IR symmetry is anomalous, i.e. we are describing an igSPT.

Examples: $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$

A wider class of examples of gSPT and igSPT phases arises for $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$, provided that $n = pq$ can be written as the product of two non coprime integers. Let us present all the details in the $n = 4$ case, and then sketch the generalization to other values of n.

The group $\mathbb{Z}_4 \times \mathbb{Z}_4$ has nine non-trivial proper subgroups:

$$(\mathbb{Z}_4)_L\,, (\mathbb{Z}_4)_R\,, (\mathbb{Z}_4)_D\,, (\mathbb{Z}_2)_L\,, (\mathbb{Z}_2)_R\,, (\mathbb{Z}_2)_D\,, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2\,. \tag{7.3.27}$$

For all subgroups isomorphic to a cyclic group, it can be checked that any symmetric homomorphism $\psi : \mathbb{B} \to \mathbb{Z}_4 \times \mathbb{Z}_4$ has a symmetric extension, therefore there are no igSPTs associated with these subgroups.

Let us look at $\mathbb{B} = \mathbb{Z}_2 \times \mathbb{Z}_2$. There are 16 homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_4 \times \mathbb{Z}_4$ that we label with four parameters $s_1, s_2, r_1, r_2 = 0, 1$:

$$\psi_{s_1,s_2,r_1,r_2}(2x, 2y) = (2s_1 a + 2r_1 y, 2s_2 x + 2r_2 y) \; . \tag{7.3.28}$$

Notice that $\psi_{s_1,s_2,r_1,r_2}(2x, 2y)(2x', 2y') = 1$, so these are all symmetric and define gSPTs. Each homomorphism ψ_{s_1,s_2,r_1,r_2} has 16 extensions, parametrized by other four labels $\sigma_1, \sigma_2, \rho_1, \rho_2 = 0, 1$:

$$\widehat{\psi}^{\sigma_1,\sigma_2,\rho_1,\rho_2}_{s_1,s_2,r_1,r_2}(x, y) = \Big((s_1 + 2\sigma_1)x + (r_1 + 2\rho_1)y, (s_2 + 2\sigma_2)x + (r_2 + 2\rho_2)y\Big) \, , \tag{7.3.29}$$

that is symmetric if and only if $r_1 + 2\rho_1 = s_2 + 2\sigma_2$. We conclude that if $r_1 \neq s_2$ there is no symmetric extension of $\widehat{\psi}_{s_1,s_2,r_1,r_2}$, hence this represents an igSPT phase. We have eight igSPTs of this kind.

The last subgroup to consider is $\mathbb{B} = \mathbb{Z}_4 \times \mathbb{Z}_2$, for which there are no igSPTs. Indeed the most general homomorphism is

$$\widehat{\psi}_{s_1,s_2,r_1,r_2}(x, 2y) = (s_1 x + 2r_1 y, s_2 x + 2r_2 y) \; , \tag{7.3.30}$$

where $s_1, s_2 = 0, 1, 2, 3$ while $r_1, r_2 = 0, 1$, and is symmetric if and only if $s_2 \bmod(2) = r_1$. There are four extensions

$$\widehat{\psi}^{\rho_1,\rho_2}_{s_1,s_2,r_1,r_2}(x, y) = \Big(s_1 x + (r_1 + 2\rho_1)y, s_2 x + (r_2 + 2\rho_2)y\Big) \; , \tag{7.3.31}$$

for which the symmetric condition is $s_2 = r_1 + 2\rho_1$, that has solution precisely if $s_2 \bmod(2) = r_1$.

The igSPTs of $\mathbb{A} = \mathbb{Z}_4 \times \mathbb{Z}_4$ have a natural generalization for $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$. Consider $n = pq$, and we look at the subgroup $\mathbb{B} = \mathbb{Z}_p \times \mathbb{Z}_p$. Symmetric homomorphisms $\mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_n \times \mathbb{Z}_n$ are

$$\psi_{s1,s_2,r_1,r_2}(qx, qy) = \Big(qs_1 x + qr_1 y, qs_2 x + qr_2 y\Big) \tag{7.3.32}$$

with $s_1, s_2, r_1, r_2 \in \mathbb{Z}_p$ and

$$r_1 = s_2 \bmod \left(\frac{p}{\gcd(p, q)}\right) \, . \tag{7.3.33}$$

The difference $r_1 - s_2$ can then take $\gcd(p, q)$ values. Therefore there are $p^3\gcd(p, q)$ gSPTs. To check which of them are igSPTs we look for the extensions $\widehat{\psi} : \mathbb{A} \to \mathbb{A}^\vee$, that are parametrized by $\sigma_i, \rho_i \in \mathbb{Z}_q$

$$\widehat{\psi}^{\sigma_1,\sigma_2,\rho_1,\rho_2}_{s_1,s_2,r_1,r_2}(x, y) = \Big((s_1 + p\sigma_1)x + (r_1 + p\rho_1)y, (s_2 + p\sigma_2)x + (r_2 + p\rho_2)y\Big) \, , \tag{7.3.34}$$

and for this to be symmetric, i.e. for an extension to exist, the condition is

$$r_1 + p\rho_1 = s_2 + p\sigma_2 \mod(n) \, . \tag{7.3.35}$$

Thus if $r_1 = s_2 \bmod p$ there is a symmetric extension. Otherwise, there is no one, and in the latter case we get an igSPT. We conclude that, among the $p^3\gcd(p, q)$ gSPTs, p^3 are non-intrinsic, while the remaining $p^3(\gcd(p, q) - 1)$ are igSPTs.

7.3.4 Physical Realization of igSPT Phases

We can take as input any of the igSPT phases we found and produce a physical IR theory that realizes it similarly to the (1+1)d construction of [196]. To illustrate the idea, we consider the realization of the minimal example of $\mathbb{A} = \mathbb{Z}_4 \times \mathbb{Z}_2$, which is associated with $\mathbb{B} = \mathbb{Z}_2$ and $\psi(2a, 0) = (0, a)$. We take a CFT $\mathfrak{T}_0$ with 1-form symmetry $\mathbb{Z}_4^{\text{in}}$ and construct $\mathfrak{T}_0/\mathbb{Z}_2$ by gauging the $\mathbb{Z}_2 \subset \mathbb{Z}_4^{\text{in}}$ subgroup of

Group	$\mathbb{Z}_2^e$	$\mathbb{Z}_2^m$	$\mathbb{Z}_2^{e'}$	$\mathbb{Z}_2^{m\vee}$	$\mathbb{Z}_2^{e'\vee}$	$\mathbb{Z}_4$
Background Field	B_e	B_m	B'_e	$\widehat{B}_m$	$\widehat{B'}_e$	$2\widehat{B}_m + B_e$

Table 7.3: 1-Form symmetry groups and background fields.

the 1-form symmetry. Since $\mathbb{Z}_4^{\text{in}}$ is a non-trivial extension, in the resulting theory the dual $\mathbb{Z}_2^m = \mathbb{Z}_2^\vee$ 1-forms symmetry and the quotient $\mathbb{Z}_2^e = \mathbb{Z}_4^{\text{in}}/\mathbb{Z}_2$ have a mixed 't Hooft anomaly

$$\frac{2\pi i}{2}\int_{X_5} B_e \cup \beta(B_m)\ , \tag{7.3.36}$$

with B_m the background for the dual symmetry $\mathbb{Z}_2^m$, and B_e the background for the quotient $\mathbb{Z}_2^e$. We then stack a completely trivial theory with 1-form symmetry $\mathbb{Z}_2^{e'}$, and use ψ to construct an SPT involving B_m and the background B'_e for the decoupled $\mathbb{Z}_2^{e'}$:

$$\frac{2\pi i}{2}\int_{X_4} B_m \cup B'_e\ . \tag{7.3.37}$$

Finally, we gauge $\mathbb{Z}_2^m \times \mathbb{Z}_2^{e'}$ with this SPT. Following similar steps as in Section 7.2.4 we find that the resulting theory has symmetry $\mathbb{Z}_4 \times \mathbb{Z}_2$. The $\mathbb{Z}_4$ part arises as a non-trivial extension:

$$1 \to \mathbb{Z}_2^{m\vee} \to \mathbb{Z}_4 \to \mathbb{Z}_2^e\,. \to 1 \tag{7.3.38}$$

Denoting by $\widehat{B}_m$ the background field for $\mathbb{Z}_2^{m\vee}$, there is a modified cocycle condition

$$\delta\widehat{B}_m = \beta(B_e)\ . \tag{7.3.39}$$

The partition function of the resulting theory, that we denote by $\mathfrak{T}$, is

$$Z_{\mathfrak{T}}\left[\widehat{B}_m, B_e, \widehat{B'}_e\right] = \exp\left(-\frac{2\pi i}{2}\int_{X_4} \widehat{B}_m \cup \widehat{B'}_e\right) Z_{\mathfrak{T}_0/\mathbb{Z}_2}\left[\widehat{B'}_e, B_e\right]\ . \tag{7.3.40}$$

Here $\widehat{B'}_e$ is the background for the dual symmetry $\mathbb{Z}_2^{e'\vee}$ (see table 7.3 for a summary of the symmetries and background fields involved). As we will see shortly in a concrete model, this piece of the symmetry acts trivially on the dynamical CFT, that is $\mathfrak{T}_0/\mathbb{Z}_2$. The same is clearly true also for the $\mathbb{Z}_2 \subset \mathbb{Z}_4$ subgroup, as its background $\widehat{B}_m$ only appears in the multiplying phase. This is the same result as in (1+1)d: the the anomaly of the dynamical part $\mathfrak{T}_0/\mathbb{Z}_2$ is cancelled by the Green-Schwarz mechanism due to the modified cocycle condition of $\widehat{B}_m$ and the presence of the multiplying phase. Hence the CFT has an emergent anomaly that forbids from trivially gapping it by IR deformations. However, remembering the presence of the symmetry $\mathbb{Z}_2^{m\vee}$ only acting on gapped degrees of freedom, the anomaly is cancelled and it is possible to drive the system to a trivially gapped phase by a UV deformation. This would, however, require to make gapless some of the degrees of freedom on which $\mathbb{Z}_2^{m\vee}$ is acting, hence closing the symmetry gap and encountering a phase transition.

7.3.5 An SU(4) × SU(2) Gauge Theory Realization of igSPTs

This somewhat formal discussion can be made concrete in a model model where all the above ingredients are embedded into an asymptotically free gauge theory. For instance, the CFT $\mathfrak{T}_0$ with 1-form symmetry $\mathbb{Z}_4^{\text{in}}$ can be realized as the fixed point of a SU(4) gauge theory with enough massless adjoint

	$\mathbb{Z}_2^e$	$\mathbb{Z}_2^m$	$\mathbb{Z}_2^{e'}$	Twisted Sector
T	1	i	1	$\mathbb{Z}_2^e$
W	i	1	-1	$\mathbb{Z}_2^m$
T'	1	-1	1	$\mathbb{Z}_2^{e'}$

Table 7.4: Charges of twisted lines under the various symmetries.

fermions ψ to land in the conformal window. For the trivial theory with $\mathbb{Z}_2$ symmetry we could simply take pure SU(2) YM theory, but without affecting the IR we can replace it with SU(2) gauge theories with massive adjoint fermions χ.[11] This fact will be relevant at the end of the analysis.

Let us analyze the various line operators, and re-derive the above result in a more physical way, interpreting it in terms of confinement/deconfinement. We denote by W^a and T^b, $a, b = 0, ..., 3$ the Wilson and (non-genuine) 't Hooft lines of the SU(4) sector, while by W', T' the analogous lines in the SU(2) sector. Since the first sector is designed to flow in the conformal window, all the W^a lines have perimeter law, while T^b have area law and disappear from the CFT. Vice versa, W' and T' have, respectively, area and perimeter law. In the original SU(4) $\times$ SU(2) global variants, all the Wilson lines are genuine, while the 't Hooft lines are non-genuine, i.e. live in twisted sectors. In the IR this flows to a CFT with $\mathbb{Z}_4$ 1-form symmetry times a trivially gapped (confined) phase with $\mathbb{Z}_2$ zero-form symmetry. The phase is not protected by symmetry, as turning on a mass for the adjoints ψ leads to a trivially gapped (confined) theory.

We then pass to the SU(4)/$\mathbb{Z}_2$ global variant. Now the 1-form symmetry is

$$\mathbb{Z}_2^e \times \mathbb{Z}_2^m \times \mathbb{Z}_2^{e'} \,, \tag{7.3.41}$$

and the genuine lines are W^2, T^2 and W', which are charged respectively under the three $\mathbb{Z}_2$ factors. We then gauge $\mathbb{Z}_2^m \times \mathbb{Z}_2^{e'}$ adding the discrete torsion term (7.3.37), whose effect is to change the lines that become genuine. In fact, before promoting B_m, B'_e to dynamical fields, the lines $W(W^3), T, T'$ are in twisted sectors, and the presence of the counterterm (7.3.37) changes their charges as in table 7.4. The factors of -1 stem from the stacking with the SPT phase, while the fractionalized charge i describes the mixed 't Hooft anomaly

$$\pi i \int B_e \cup \beta(B_m)\,. \tag{7.3.42}$$

After gauging $\mathbb{Z}_2^m \times \mathbb{Z}_2^{e'}$, on top of keeping only the invariant lines (only W^2 in this case), we have to add the twisted sector lines that are *not* charged under $\mathbb{Z}_2^m \times \mathbb{Z}_2^{e'}$. Hence, the lattice of genuine lines is generated by

$$WW' \qquad \text{and} \qquad T'T^2\,, \tag{7.3.43}$$

as opposed to W and T', and the symmetry is $\mathbb{Z}_4 \times \mathbb{Z}_2^{e'\vee}$. The lines W and W' both go into the twisted sector of $\mathbb{Z}_2^{e'\vee}$ (whilst their product WW' is genuine), T goes into the twisted sector of $\mathbb{Z}_4$, while T' into the twisted sector of the subgroup $\mathbb{Z}_2^{m\vee}$ (indeed $T'T^2$ is genuine).

[11]The reason for this choice is that, with the correct number of SU(2) adjoints, in the UV we may play with their mass, eventually reaching the massless point so that the SU(2) sector also reaches a conformal point.

Line	WW'	W^2	$T'T^2$	W	W'	T'	T
Area/Perimeter	A	P	A	P	A	P	A
Genuine/Twisted	genuine	genuine	genuine	$\mathbb{Z}_2^{e'\vee}$	$\mathbb{Z}_2^{e'\vee}$	$\mathbb{Z}_2^{m\vee}$	$\mathbb{Z}_4$

Table 7.5: Confinement pattern of both genuine and non-genuine lines in the final theory $\mathfrak{T}$.

Following our dynamical assumptions, before the discrete manipulation, W and T' are deconfined, while W' and T are confined. Thus in the low-energy theory the only genuine line present is W^2, as both WW' and $T'T^2$ have area law. This indicates that only a $\mathbb{Z}_2$ quotient of the 1-form symmetry acts in the IR theory, which does not contain topological lines. Hence, this $\mathbb{Z}_2$ quotient acts on the gapless sector. In the twisted sectors, instead, T and W' have area law, while T' and W have perimeter law. Thus, both $\mathbb{Z}_4^e$ and $\mathbb{Z}_2^{m'}$ can be detected at low energy by looking at the twisted sectors. The reader can consult table 7.5 for a clear synthesis.

Deformation by Fermion Masses

We can now characterize the peculiar feature of this topological phase in this physical setup. Suppose that we try to deform the IR in order to reach a fully confined phase. Since SSB for 1-form symmetry is detected by reducing on an S^1 we can think of a finite-temperature setup where a monopole potential, as well as fermion masses, can be turned on if allowed in the IR. Let us first focus on fermion masses. To completely confine the theory, we should confine the W^2 line. This can be done, for example, by turning on an equal mass for all SU(4) adjoints[12]. This would imply that T has perimeter law. Notice, however, that T is no longer present in the low energy description. Furthermore, if we give T a perimeter law, then $T'T^2$ is also deconfined, and we would break $\mathbb{Z}_2^{e'\vee}$ spontaneously.

At first glance, this may seem akin to a (3+1)d CFT with an anomalous 1-form symmetry, which prevents the theory from being trivially gapped. However, the unique aspect of this phase is the absence of an anomaly for the full UV symmetry $\mathcal{C}$. This indicates that the theory might become fully confined if we can render some gapped degrees of freedom massless initially. For example, by tuning to zero the UV mass for the SU(2) adjoints we can drive this sector to conformality in the IR.

Consequently, the WW' line has perimeter law, and the full $\mathbb{Z}_4$ acts non-trivially on the gapless degrees of freedom. This represents a gapless SPT phase, but it is the non-intrinsic one. In fact, it can be deformed into the trivial phase by uniformly increasing the masses for the low-energy adjoints of both SU(4) and SU(2).

However, this deformation is not possible in the igSPT phase since, there, the SU(2) adjoints have already been integrated out, and their mass operator is not part of the CFT. Furthermore, to make this deformation possible, certain degrees of freedom need to become massless, leading to a phase transition, supporting the assertion that the igSPT we are discussing is a distinct phase.

Deformation by Monopole Potentials

Finally, let us comment on robustness of this phase against another type of deformation, which is important to distinguish this phase from more familiar ones. The fact that giving a mass to the SU(4) adjoint fermions drives the CFT into a $\mathbb{Z}_2$ SSB phase should not be surprising, and we would have

[12]We assume that the fermion mass operator is still relevant in the IR CFT.

found a similar result starting from PSU(4) gauge theory instead. However, in the latter, the SSB phase is not protected by any mechanism.

In fact, the situation is different once we study the deformations obtained by reducing on S^1 and turning on monopole potentials. Here we use the abuse of terminology by which any local operator obtained wrapping a line operator on S^1 is called a monopole, independently of whether it is charged under an electric or a magnetic symmetry.

In the PSU(4) theory on $S^1 \times \mathbb{R}^3$ we can turn on a monopole potential

$$V(\mathcal{W}\mathcal{W}^\dagger)\,, \tag{7.3.44}$$

where $\mathcal{W}$ is the Polyakov loop. By carefully choosing the potential, this condenses $\mathcal{W}$ and leads to the PSU(4) confined phase. Notice that we can turn on this potential in the IR since W has a perimeter law.

In the igSPT case, instead, we would like to turn on a potential for $\mathcal{M}\mathcal{M}^\dagger$:

$$V(\mathcal{M}\mathcal{M}^\dagger)\,, \tag{7.3.45}$$

where $\mathcal{M}$ is the reduction of the 't Hooft line on S^1. However, table 7.5 shows that T has area law, thus $\mathcal{M}$ is absent in the IR and this deformation is not available in the low-energy theory. We can only turn on a potential for $\mathcal{W}^2(\mathcal{W}^2)^\dagger$, which would lead us too to a SSB phase for $\mathbb{Z}_4^e$. Thus, the igSPT phase cannot be trivially gapped by IR deformations.

7.4 Gapless Phases with Duality Symmetries

We now turn to exploring (intrinsically) gapless SPT phases for selected non-invertible symmetries. In this section we will focus on duality-type symmetries in both $(1+1)$d and $(3+1)$d, giving a SymTFT construction and discussing the constraints on the IR physics from the perspective of the gapless sector.[13]

7.4.1 Phases with Duality Symmetries: (1+1)d

We want to generalize the analysis of the gapless phases in (1+1)d to a larger class of fusion categories $\mathcal{C}$ whose Drinfeld center $\mathcal{Z}(\mathcal{C})$ is obtained from some Abelian topological order $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ by gauging a finite invertible 0-form symmetry G [14]. Examples are $\text{Vec}_{D_{2n}}$, whose Drinfeld center is obtained by gauging charge conjugation in $\mathcal{Z}(\text{Vec}_{\mathbb{Z}_n})$, and Tambara-Yamagami categories $\mathsf{TY}(\mathbb{A},\,\gamma,\,\epsilon)$ [21, 91, 99, 202], whose center is obtained from $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ gauging electro-magnetic duality [3, 46, 186, 203]. Later we will extend our approach to $(3+1)$d. Although the center of TY-categories is well-known [203], our approach gives a construction of the center discussed in Chapter 6, which extends more easily to higher dimensions (see also [46]). Our goal will be to determine the condensable algebras, including SPT, gSPT, and most interestingly igSPTs, through the gauging.

Structure of the Center

We consider a faithful 0-form symmetry group G acting on $\mathbb{A} \times \mathbb{A}^\vee$ by exchanging anyons, while preserving fusion and braiding. Practically, there exists a group homomorphism $\Phi : G \to \text{Aut}(\mathbb{A} \times \mathbb{A}^\vee)$

[13]The methods and classification developed here generalize in a straightforward manner to G-ality defects. See [99, 200, 201] for recent studies.

[14]Without loss of generality we can assume G to act faithfully on $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$.

such that

$$\theta_{\Phi_g(a,\alpha)} = \theta_{(a,\alpha)}\,, \qquad \forall a \,\in\, \mathbb{A}\,, \alpha \,\in\, \mathbb{A}^\vee\,. \tag{7.4.1}$$

Faithfulness means that Φ defines an embedding $G \subset \mathrm{Aut}_0(\mathbb{A}\times\mathbb{A}^\vee)$ in the subgroup $\mathrm{Aut}_0(\mathbb{A}\times\mathbb{A}^\vee) \subset \mathrm{Aut}(\mathbb{A}\times\mathbb{A}^\vee)$ that preserves the pairing $(a,\alpha)\mapsto \alpha(a) \,\in\, U(1)$.

Structure of $\mathbf{Aut_0}(\mathbb{A}\times\mathbb{A}^\vee)$. $\mathrm{Aut}_0(\mathbb{A}\times\mathbb{A}^\vee)$ is generated by two subgroups: one is isomorphic to $\mathrm{Aut}(\mathbb{A})$, the other to $\mathbb{Z}_2$. The embedding $\mathrm{Aut}(\mathbb{A}) \subset \mathrm{Aut}_0(\mathbb{A}\times\mathbb{A}^\vee)$ is canonical. Given an automorphism $\rho : \mathbb{A} \to \mathbb{A}$, we have $\rho^{-1\,\vee} : \mathbb{A}^\vee \to \mathbb{A}^\vee$ given by $\rho^{-1\,\vee}(\alpha)(a) = \alpha(\rho^{-1}(a))$. Hence $P_\rho \,\in\, \mathrm{Aut}_0(\mathbb{A}\times\mathbb{A}^\vee)$ is given by

$$P_\rho(a,\alpha) = \Big(\rho(a), \rho^{-1\,\vee}(\alpha)\Big)\,. \tag{7.4.2}$$

The $\mathbb{Z}_2$ subgroup, instead, is generated by electro-magnetic duality S and its identification is non-canonical. Given a *choice* of isomorphism $\phi : \mathbb{A} \to \mathbb{A}^\vee$ such that $\phi^\vee = \phi$ – namely $\phi(a)b = \gamma(a,b)$ is a symmetric bicharacter – then S is defined as[15]

$$S(a,\alpha) = \left(\phi^{-1}(\alpha), \phi(a)\right)\,, \tag{7.4.3}$$

and preserves the spins because of $\phi(a)b = \phi(b)a$. $\mathbb{Z}_2 = \langle S\rangle$ and $\mathrm{Aut}(\mathbb{A})$ do not commute, but

$$SP_\rho S = P_{\rho^\phi}\,, \qquad \text{where} \qquad \rho^\phi = \phi^{-1}\rho^{-1\,\vee}\phi \,\in\, \mathrm{Aut}(\mathbb{A})\,. \tag{7.4.4}$$

With this $\mathbb{Z}_2$-action on $\mathrm{Aut}(\mathbb{A})$ we have $\mathrm{Aut}_0(\mathbb{A}\times\mathbb{A}^\vee) \cong \mathrm{Aut}(\mathbb{A}) \rtimes \mathbb{Z}_2$.

Anyons of the gauged center We consider the fusion categories $\mathcal{C}$ such that $\mathcal{Z}(\mathcal{C}) = \mathcal{Z}(\mathrm{Vec}_\mathbb{A})/G$. To fix our notation and language, let us briefly review the anyon content of $\mathcal{Z}(\mathcal{C})$, in a way that can be generalized to higher dimensions [2, 3, 45, 46]. As this is a well-known procedure, we refer the reader to [77] for details. In the following, we only consider the case of G Abelian and, in particular, $G = \mathbb{Z}_2$. Anyons in $\mathcal{Z}(\mathcal{C})$ fall into three classes:

- G-invariant combinations (orbits) of anyons of $\mathcal{Z}(\mathrm{Vec}_\mathbb{A})$. For $G = \mathbb{Z}_2$ they can be long orbits $X_{a,b}$ or invariant lines $L_{a,\pm}$.
- Lines η_r, $r \,\in\, \mathsf{Rep}(G) \cong G^\vee$ of the dual symmetry.
- Twist defects $\Sigma_{(\alpha,\,x)}$, coming from the G-twisted sectors σ_a prior to the gauging. For $G = \mathbb{Z}_2$, we have $x = \pm$. These are the charged objects under the $\mathsf{Rep}(G)$ symmetry.

The suffix $\pm$ indicates that the $\mathsf{Rep}(G)$ symmetry line can fuse with the defect giving rise to a new topological object. We summarize their structure, in the case of Tambara-Yamagami categories, in table 7.7.

Let us exemplify this discussion in the example of Vec_{D_8}. D_8 can be realized as a semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$, where $G = \mathbb{Z}_2$ acts as $a \mapsto -a$ on $\mathbb{Z}_4$. Hence $\mathcal{Z}(\mathrm{Vec}_{D_8})$ can be obtained following the procedure described in Section 7.4.1, starting from $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_4})$ and gauging the $G = \mathbb{Z}_2$ symmetry acting as charge conjugation

$$C : (a,b) \mapsto (-a,-b)\,, \qquad (a,b) \,\in\, \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_4}) = \mathbb{Z}_4 \times \mathbb{Z}_4\,. \tag{7.4.5}$$

The invariant lines, that coincide with their own orthogonal (with respect to the braiding), are

[15] It is easy to check that two different choices ϕ_1, ϕ_2 lead to S_2, S_1 such that $S_2 = P_\rho S_1$, with $\rho = \phi_2^{-1}\phi_1 \,\in\, \mathrm{Aut}(\mathbb{A})$.

Outer Auto	$([g], \rho)$	Anyon label	Dim	T
$L^+_{0,0}$	$(1, 1)$	1	1	1
$L^-_{0,0}$	$(1, 1_a)$	e_{RG}	1	1
$L^-_{0,2}$	$(1, 1_x)$	e_R	1	1
$L^+_{0,2}$	$(1, 1_{ax})$	e_G	1	1
$L_{0,1}$	$(1, E)$	m_B	2	1
$L^-_{2,0}$	$(a^2, 1)$	e_{RGB}	1	1
$L^+_{2,0}$	$(a^2, 1_a)$	e_B	1	1
$L^+_{2,2}$	$(a^2, 1_x)$	e_{GB}	1	1
$L^-_{2,2}$	$(a^2, 1_{ax})$	e_{RB}	1	1
$L_{2,1}$	(a^2, E)	f_B	2	-1
$L_{1,0}$	$(a, 1)$	m_{RG}	2	1
$L_{1,1}$	(a, i)	s_{RGB}	2	i
$L_{1,2}$	$(a, -1)$	f_{RG}	2	-1
$L_{1,3}$	$(a, -i)$	$\bar{s}_{RGB}$	2	$-i$
$\Sigma^+_{1,1}$	$(x, +, +)$	m_{GB}	2	1
$\Sigma^+_{1.0}$	$(x, +, -)$	m_G	2	1
$\Sigma^-_{1,0}$	$(x, -, -)$	f_G	2	-1
$\Sigma^-_{1,1}$	$(x, -, +)$	f_{GB}	2	-1
$\Sigma^+_{0,1}$	$(ax, +, +)$	m_{RB}	2	1
$\Sigma^+_{0,0}$	$(ax, +, -)$	m_R	2	1
$\Sigma^-_{0,0}$	$(ax, -, -)$	f_R	2	-1
$\Sigma^-_{0,1}$	$(ax, -, +)$	f_{RB}	2	-1

Table 7.6: The table lists all anyons of $\mathcal{Z}(\mathrm{Vec}_{D_8})$ using three distinct notations, see [186]. The first column uses our notation, which is natural in the context of gauging. The second column employs the standard notation for $\mathcal{Z}(\mathrm{Vec}_G)$ for any finite group G, expressed through conjugacy classes $[g]$ and stabilizer representations ρ. The third column shows the corresponding labels in terms of three copies of the toric code, as referenced in [204]. The final two columns display the quantum dimension and spin.

$$F = F^\perp = \{(2x, 2y) \mid x, y = 0, 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \,. \tag{7.4.6}$$

The twist defects in the G-crossed extension of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_4})$ can be labelled as

$$\sigma_{(a,b)} \ , \qquad (a, b) \in \{(0, 0), (1, 0), (0, 1), (1, 1)\} = \frac{\mathbb{Z}_4 \times \mathbb{Z}_4}{F^\perp} \,. \tag{7.4.7}$$

Notice that all twist defects are charge conjugation invariant.

Now we gauge $G = \mathbb{Z}_2$. We will denote by $\pm$ the trivial and non-trivial representations of $\mathbb{Z}_2$. From the invariant bulk lines we obtain Abelian anyons:

$$L^\pm_{0,0} \ , \quad L^\pm_{2,0} \ , \quad L^\pm_{0,2} \ , \quad L^\pm_{2,2} \,. \tag{7.4.8}$$

From the non-invariant bulk anyons we get the dimension-two anyons:

$$L_{1,0} \ , \quad L_{0,1} \ , \quad L_{1,1} \ , \quad L_{2,1} \ , \quad L_{1,2} \ , \quad L_{1,3} \,. \tag{7.4.9}$$

Finally since the twist defects are all charge conjugation invariant we get

$$\Sigma^\pm_{0,0} \ , \quad \Sigma^\pm_{1,0} \ , \quad \Sigma^\pm_{0,1} \ , \quad \Sigma^\pm_{1,1} \,. \tag{7.4.10}$$

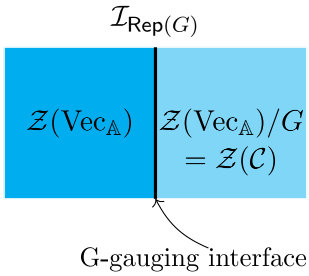
Object	Definition	Dim	# of Objects	Spin θ
$L_{(a,x)}$	$\eta^x \times \big(a,\, \phi(a)\big)$	1	$2\,\lvert\mathbb{A}\rvert$	$\gamma(a,a)$
$X_{(a,b)}$	$\big(a,\, \phi(b)\big) \oplus \big(b,\, \phi(a)\big)$	2	$\lvert\mathbb{A}\rvert\,\big(\lvert\mathbb{A}\rvert-1\big)/2$	$\gamma(a,b)$
$\Sigma_{(a,x)}$	$\eta^x \times \sigma_a$	$\sqrt{\lvert\mathbb{A}\rvert}$	$2\lvert\mathbb{A}\rvert$	$(-1)^x \sqrt{\dfrac{\epsilon}{\lvert\mathbb{A}\rvert^{1/2}} \sum_{b\in\mathbb{A}} f_a(b)^{-1}}$

Table 7.7: Objects (lines) of the 3d SymTFT for the Tambara-Yamagami symmetry.

In table 7.6 we present the translation between these labels and other labels for $\mathcal{Z}(\text{Vec}_{D_8})$ used in the literature [186].

Condensable Algebras in $\mathcal{Z}(\mathcal{C})$

In the present context, useful information about $\mathcal{Z}(\mathcal{C})$ can be inferred from the fact that $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ and $\mathcal{Z}(\mathcal{C})$ are connected to each other by gauging G or $\mathsf{Rep}(G)^{(1)} = G^\vee$. To start with, we consider the open club sandwiches, defined by condensable algebras. These define a topological interface[16]

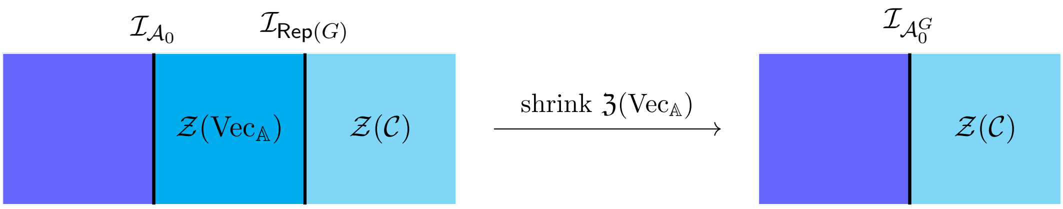


(7.4.11)

The topological interface is a G-gauging interface, defined by gauging G in the right-half of the space with Dirichlet boundary conditions for the G gauge field. Alternatively, it is associated with the condensation of the algebra

$$\mathcal{A}_G = \bigoplus_{r\,\in\,\mathsf{Rep}(G)} \eta_r \tag{7.4.12}$$

of $\mathcal{Z}(\mathcal{C})$. From the point of view of $\mathcal{Z}(\mathcal{C})$, $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ can be understood as a reduced topological order. We can construct a lift from *any* condensable algebra $\mathcal{A}_0$ of $\mathcal{Z}(\text{Vec}_{\mathbb{A}})$ to a condensable algebra $\mathcal{A}_0^G$ of $\mathcal{Z}(\mathcal{C})$. Any condensable algebra $\mathcal{A}_0$ defines an interface $\mathcal{I}_{\mathcal{A}_0}$ (and viceversa), and $\mathcal{A}_0^G$ is the algebra corresponding to the stacking of the two interfaces



(7.4.13)

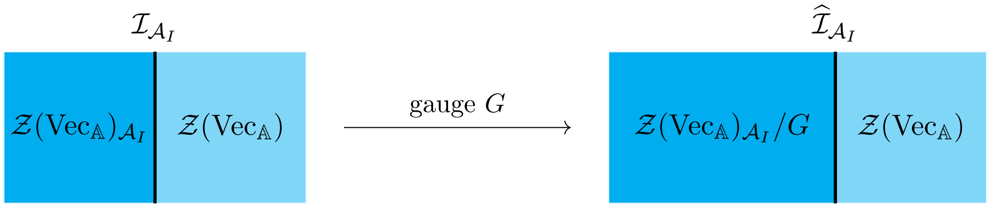
We call these algebras *induced algebras* from $\mathcal{A}_0$. These do not exhaust all the possible condensable algebras in $\mathcal{Z}(\mathcal{C})$ as they will all condense the $\mathsf{Rep}(G)$ lines. As appreciated in Chapter 6, the missing

[16]The study of such interfaces has a long history, dating back to [50, 130, 131], see also [185, 205] for recent studies in $(1+1)$d.

algebras are lifts of G-invariant algebras $\mathcal{A}_I$ in $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$:[17]

$$\Phi(\mathcal{A}_I) = \mathcal{A}_I \,. \tag{7.4.14}$$

Having found an invariant algebra, we must also specify a way in which the symmetry G acts on the algebra structure of $\mathcal{A}_I$. This defines an equivariantization of $\mathcal{A}_I^\eta$ of $\mathcal{A}_I$, and can be characterized precisely (see Section 6.2.4). We will often omit this datum unless relevant for the phase described. An intuitive justification of this construction follows from considering the reduced topological order $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})/\mathcal{A}_I$ and gauging the G symmetry:



(7.4.15)

This setup is transparent to the $\mathsf{Rep}(G)$ lines and thus defines a new type of interface, denoted by $\widehat{\mathcal{I}}_{\mathcal{A}_I}$. This describes the most general reduced topological order in which the G symmetry is still present. It is possible to find physical examples in which the G symmetry does not act faithfully on gapless degrees of freedom without being broken. We will not discuss these examples, but let us briefly give a flavor of how they are achieved.

Consider a lift of an invariant algebra $\mathcal{A}_I$. Since it is duality-invariant, we can often enlarge it by including twisted sector operators $\Sigma_{a,x}$. This eliminates the G symmetry in the IR description by gauging it. Alternatively, if invariant anyons L_a are present in $\mathcal{A}_I$, and we call $\mathbb{B}_{inv}$ the subgroup that they form, we can decorate $\widehat{\mathcal{A}}_I$ with dual symmetry lines η_r using a group homomorphism $\xi : \mathbb{B}_{inv} \to G$. This eliminates the G symmetry from the IR description as it is nonlocal with respect to a nontrivial decoration.

gSPT and igSPT phases for $\mathcal{C}$

The structure we have just introduced allows for an efficient description of (i)gSPT phases for $\mathcal{C}$. First of all, notice that the canonical Lagrangian algebra $\mathcal{L}_{\mathrm{symm}}$ that fixes the symmetry to be $\mathcal{C}$ is nothing but $\mathcal{L}_0^G$, induced by

$$\mathcal{L}_0 = \left\{(0,\alpha)|\ \alpha \in \mathbb{A}^\vee\right\} \,, \tag{7.4.16}$$

which in turn is the canonical Lagrangian algebra of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$ that gives the symmetry $\mathrm{Vec}_{\mathbb{A}}$. As $\mathcal{L}_{\mathrm{symm}}$ contains all the $\mathsf{Rep}(G)$ lines, a necessary condition for a condensable algebra $\mathcal{A}$ to correspond to a gSPT phase is:

$$\mathcal{A} \cap \mathsf{Rep}(G) = \{1\} \,. \tag{7.4.17}$$

Thus $\mathcal{A}$ must be a lift of an invariant algebra $\mathcal{A}_I$. Clearly $\mathcal{A}_I$ is not just *any* invariant algebra in $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$, but must be a gSPT algebra in order to avoid an SSB phase of $\mathbb{A}$.

However, what we are really interested in is determining which of these gapless SPT phases are intrinsic. We then have to ask if a gSPT algebra $\widehat{\mathcal{A}}_I$ is a subalgebra of a Lagrangian algebra $\mathcal{L}$ corresponding to a (gapped) SPT. We will specialize to the case in which the duality symmetry acts faithfully on gapless degrees of freedom. The general case is also treatable through our formalism, but the complete classification becomes cumbersome. To continue the discussion, we use the result

[17] Strictly speaking, preserving a subgroup of G is sufficient, but our examples will all be for $G = \mathbb{Z}_2$.

of Chapter 6 that (gapped) SPTs of $\mathcal{Z}(\mathcal{C})$ can only arise from G-invariant Lagrangian algebras $\mathcal{L}_{\mathsf{I}}$ of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$. The corresponding SPT algebra $\widetilde{\mathcal{L}}_{\mathsf{I}}$ of $\mathcal{Z}(\mathcal{C})$ is obtained as follows. First, we construct $\widehat{\mathcal{L}}_{\mathsf{I}}$ that is condensable but non-maximal in $\mathcal{Z}(\mathcal{C})$. The reduced topological order is a Dijkgraaf-Witten theory $\mathcal{Z}(\mathrm{Vec}_G^{\omega})$. If the cocycle $\omega \in H^3(G, U(1))$ is trivial[18] one can further condense the magnetic lines of $\mathcal{Z}(\mathrm{Vec}_G)$. This sequence of condensations defines a gapped boundary of $\mathcal{Z}(\mathcal{C})$, and $\widetilde{\mathcal{L}}_{\mathsf{I}}$ is the corresponding Lagrangian algebra.

Using these facts it is easy to see that if $\mathcal{A}_{\mathsf{I}}$ is a G-invariant igSPT algebra of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$, then $\widehat{\mathcal{A}}_{\mathsf{I}}$ is also intrinsic.[19]Interestingly, the vice-versa is not necessarily true. We may have a G-invariant gSPT algebra $\mathcal{A}_{\mathsf{I}}$ of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$ that is non-intrinsic –namely $\mathcal{A}_{\mathsf{I}} \subset \mathcal{L}_0$ for some SPT Lagrangian algebra $\mathcal{L}_0$– but the latter is not G-invariant and hence it does not give rise to a gapped SPT of $\mathcal{Z}(\mathcal{C})$. Finally, we can have a Lagrangian algebra $\mathcal{L}_I^{\eta}$ with non-trivial equivariantization datum η, whose reduced topological order describes a phase with $\mathrm{Vect}_{\mathbb{Z}_2}^{\omega}$ symmetry with nontrivial anomaly ω.

To summarize, we can distinguish three types of igSPT phases for theories with $\mathcal{C}$ symmetry:

- **Type I:** igSPT phases of $\mathrm{Vec}_{\mathbb{A}}$ whose condensable algebra is G-invariant.
- **Type II:** gSPT phases of $\mathrm{Vec}_{\mathbb{A}}$ which are *not* intrisic and whose condensable algebra is G-invariant, but such that none of the SPT Lagrangian algebras containing it is G-invariant.
- **Type III:** SPT phases for the $\mathbb{A}$ symmetry which are described by a duality invariant Lagrangian algebra $\mathcal{L}_I$, but with nontrivial choice of equivariantization $\mathcal{L}_I^{\eta}$, which makes them igSPT phases when taking into account duality.

The physical interpretation of Type I and Type II igSPT phases is quite different. Consider, for instance, the case of duality defects, in which $\mathcal{C}$ extends $\mathrm{Vec}_{\mathbb{A}}$ by adding a non-invertible defect $\mathcal{N}$. A Type I igSPT is an intrinsically gapless phase if we forget the duality, that simply remains so when we remember it, but the presence of $\mathcal{N}$ does not play any fundamental role: the prize to pay for gapping the theory is spontaneously breaking $\mathbb{A}$. On the other hand, a Type II igSPT is such that it is non-intrinsic if we forget duality, hence it can be deformed to a gapped phase if we discard the non-invertible symmetry, but it is intrinsic precisely because of its presence. Hence it is a topological phase protected by the non-invertible symmetry, and if we want to gap the theory, we need to spontaneously break it. Type III phases are similar to Type II phases, in which the full $\mathbb{A}$ symmetry has been realized in a trivially gapped fashion. From the perspective of the gapless degrees of freedom, the duality symmetry is realized in very different manners:[20]

- In **Type I** igSPTs the duality defect is realized as an *invertible* and *anomaly-free* symmetry.
- In **Type II** igSPTs, it is instead realized as a *non-invertible*, but *anomalous* symmetry.
- In **Type III** igSPTs the duality symmetry in the IR is *invertible*, but anomalous.

At this point, it is useful to discuss some concrete example. It turns out that the natural igSPT phases in (1+1)d are of Type I or III, while we will see examples of Type II igSPTs in (3+1)d.

[18] ω depends not only on the symmetry and the algebra $\mathcal{L}_{\mathsf{I}}$, but also on its equivariantization $\mathcal{L}_I^{\eta}$.

[19] If it was not, then $\widehat{\mathcal{A}}_{\mathsf{I}} \subset \widetilde{\mathcal{L}}_{\mathsf{I}}$ for some SPT algebra of $\mathcal{Z}(\mathcal{C})$, but then $\mathcal{A}_{\mathsf{I}} \subset \mathcal{L}_{\mathsf{I}}$.

[20] For readers familiar with the obstruction theory describing anomalies of Tambara-Yamagami categories [3, 21, 148], Type I igSPTs realize anomalous 1-form symmetries in the IR, Type II igSPTs realize anomalous Tamabara-Yamagami type categories with a nontrivial first obstruction, and Type III igSPTs realize anomalous Tambara-Yamagami categories with trivial first obstruction and nontrivial second obstuction.

Type I igSPT phases for $\mathbf{Vec}_{D_8}$. The Drinfeld center of Vec_{D_8} is obtained from $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_4})$ by gauging charge conjugation $C : (a, b) \mapsto (-a, -b)$.

The group-symmetry Vec_{D_8} admits two igSPT phases of Type I. Indeed the well known $\mathrm{Vec}_{\mathbb{Z}_4}$ igSPT with algebra

$$\mathcal{A}_{\mathbb{Z}_2,1,\psi} = \{(2x, 2x) \; | x = 0, 1\} \tag{7.4.18}$$

is C-invariant. Moreover since the line $(2, 2)$ is by itself C-invariant, this algebra gives rise to two decorated igSPT algebras in $\mathcal{Z}(\mathrm{Vec}_{D_8})$ in

$$\widehat{\mathcal{A}}^{\pm}_{\mathbb{Z}_2,1,\psi} = L_{0,0} + L^{\pm}_{2,2} \; . \tag{7.4.19}$$

Finally, since the only SPT Lagrangian algebra of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_4})$, namely $\mathcal{L}_{\mathrm{SPT}} = \{(a, 0)\}$, is C-invariant, it gives rise to an SPT for Vec_{D_8} and there cannot be Type II igSPT phases.

This result matches the finding of [186] by direct analysis of the modular data of the Drinfeld center and the structure of its Hasse diagram. We can also use the present formalism to generalize this example to find igSPT phases for all $\mathrm{Vec}_{D_{2n}}$.

Type I igSPT phases in TY categories. Tambara-Yamagami categories $\mathsf{TY}(\mathbb{A}, \gamma, \epsilon)$ are classified by a finite Abelian group $\mathbb{A}$, a symmetric bicharacter $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$ and a Froboenius-Schur indicator $\epsilon = \pm 1$ [202]. As we have seen in Chapter 6, these data appear naturally in the Drinfeld center, which can be obtained from $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})$ by gauging the electro-magnetic duality (7.4.3) determined by a symmetric isomorphism $\phi : \mathbb{A} \to \mathbb{A}^{\vee}$ ($\gamma(a, b) = \phi(a)b$ is the associated bicharacter) with discrete torsion $\epsilon \in H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$.

Consider the case $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with off-diagonal bicharacter $\phi_O(x, y) = (y, x)$ and trivial FS indicator, which always admits gapped SPT phases (see Chapter 6). As we have seen, if $n = pq$, $\gcd(p, q) \neq 1$ there is a class of igSPT for $\mathrm{Vec}_{\mathbb{A}}$, $\mathcal{A}_{\mathbb{Z}_p \times \mathbb{Z}_p,1,\psi}$ with ψ given by (7.2.30). These algebras produce igSPT phases for $TY(\mathbb{Z}_n \times \mathbb{Z}_n, \gamma_O, +)$ if they are duality invariant, and a necessary condition for this to happen is that the image of ψ must be $\mathbb{Z}_p \times \mathbb{Z}_p$. Setting $r = 0$ ($r \neq 0$ corresponds to stacking a gapped SPT) this requires $p = q$: otherwise $t = p/\gcd(p, q)$ is never invertible over $\mathbb{Z}_p$, and the image will be $t\mathbb{Z}_p \times t\mathbb{Z}_p \subset \mathbb{Z}_p \times \mathbb{Z}_p$, which is a proper subset.

The simplest example is $n = 9$, $p = q = 3$, and we choose $k_1 = k_3 = r = 0$, $k_2 = 1$. This gives the algebra

$$\mathcal{A}_{\mathbb{Z}_3 \times \mathbb{Z}_3,1,\psi} = \{(3x, 3y; 3y, 3x) \mid x, y = 0, 1, 2\,\} \tag{7.4.20}$$

that is duality invariant for the off-diagonal bicharacter and produces an igSPT phase of of Type I with algebra

$$\widehat{\mathcal{A}}_{\mathbb{Z}_3 \times \mathbb{Z}_3,1,\psi} = \bigoplus_{x,y=0}^{2} (3x, 3y; 3y, 3x)\,. \tag{7.4.21}$$

Notice that, even though all 9 anyons appearing are individually duality invariant, no decoration of this algebra is possible. Indeed, all non-trivial anyons are of order three, hence dressing one of them with the non-trivial generator of the quantum symmetry $\mathsf{Rep}(\mathbb{Z}_2)$ would imply the presence of the dressed identity line, that is not consistent with an SPT.[21]

[21] An example where decoration is possible is $n = 16$, $p = q = 4$, and again $k_1 = k_3 = r = 0$, $k_2 = 1$:

$$\mathcal{A}_{\mathbb{Z}_4 \times \mathbb{Z}_4,1,\psi} = \{(4x, 4y; 4y, 4x) \mid x, y = 0, ..., 3\} \; . \tag{7.4.22}$$

This is essentially the same as before, but since as a group $\mathcal{A}_{\mathbb{Z}_4 \times \mathbb{Z}_4,1,\psi} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, we can freely assign a representation to each of the two generators of $\mathbb{Z}_4 \times \mathbb{Z}_4$, and this algebra produces $2^2 = 4$ igSPT phases of Type I.

We also comment on the symmetry realization on the gapless sector. After condensing $\mathcal{A}_{\mathbb{Z}_3\times\mathbb{Z}_3,1,\psi}$, the resulting theory describes $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_9})$, with electric and magnetic lines generated by $E=(1,0;0,-1)$ and $M=(0,1;2,0)$, respectively. A simple computation shows that our UV choice for duality symmetry descends to charge conjugation C. Furthermore, the gapped boundary condition $\mathfrak{B}^{\mathrm{sym}}$ is mapped to the Lagrangian algebra $\mathcal{L}_{\mathbb{Z}_3,1}$, which describes a theory with $\mathbb{Z}_3\times\mathbb{Z}_3$ symmetry - with a mixed anomaly - stacked to an invertible $\mathbb{Z}_2$ duality symmetry (see Chapter 6).

Type III igSPTs in TY categories. An example of Type III igSPT can also be found in the TY category. Consider $\mathrm{Rep}(D_8)=\mathrm{TY}(\mathbb{Z}_2\times\mathbb{Z}_2,\gamma_O,+)$. This was the first non-invertible igSPT discovered [186]. We will show that the present formalism reproduces it correctly. This symmetry allows for a duality-invariant SPT by using the Lagrangian algebra:

$$\mathcal{L}=\{(x,y;x,y)\,,\quad x,y=0,1\}\,. \tag{7.4.23}$$

The duality symmetry can act on this algebra in multiple manners, namely the different *equivariantizations* discussed in 6. In practice they are in one-to-one correspondence with one cochains η which form a torsor over:

$$H^1_\sigma(\mathbb{Z}_2,\,\mathcal{L}^\vee)\,, \tag{7.4.24}$$

satisfying

$$d_\sigma\eta(b,b')=\frac{\chi[\psi](b,b')}{\chi[\psi](\sigma(b'),\sigma(b))}\,. \tag{7.4.25}$$

Importantly, they induce a 't Hooft anomaly $\omega\in H^3(\mathbb{Z}_2,U(1))$ for the (invertible) duality symmetry after gauging, given by:

$$\omega=\mathrm{Arf}(\eta)\,. \tag{7.4.26}$$

In our example $\sigma=1$ and the r.h.s. of (7.4.25) vanishes. Since $H^1(\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Z}_2)=\mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Z}_2)=\mathbb{Z}_2\times\mathbb{Z}_2$ there are four inequivalent choices. We have shown in Chapter 6 (see also [21]) that the η map sending the generator of $\mathbb{Z}_2$ to $(1,1)\in\mathbb{Z}_2\times\mathbb{Z}_2$ has negative Arf invariant. This describes an igSPT between $\mathrm{Rep}(D_8)$ and $\mathbb{Z}_2^\omega$, as already shown in[186].

7.4.2 Phases with Duality Symmetries: (3+1)d

The formalism developed in the last subsection can be extended to discuss duality defects in (3+1)d, that arise when a theory is self-dual under gauging a 1-form symmetry $\mathbb{A}^{(1)}$ [24, 25]. In Chapter 6 we showed how to obtain the Drinfeld center by gauging electro-magnetic duality of $\mathcal{Z}(\mathbb{A}^{(1)})$, and how to use it to characterized (gapped) SPT phases. We then use the ideas of the last subsection to characterize intrinsically gapless SPT phases, and discuss examples.[22]

SPTs for Duality Defects

Self-duality Symmetry. As for 0-form symmetries in (1+1)d, also the SymTFT $\mathcal{Z}(\mathbb{A}^{(1)})$ for 1-form symmetries in (3+1)d has a universal electro-magnetic symmetry determined by a symmetry isomorphism $\phi:\mathbb{A}\to\mathbb{A}^\vee$. The only difference is that, since the braiding in five dimensions (8.2.18) is antisymmetric, the duality symmetry is

$$G=\langle S\rangle:(a,\alpha)\mapsto\big(-\phi^{-1}(\alpha),\phi(a)\big)\,, \tag{7.4.27}$$

[22]It should be noted that, in higher dimensions, a Lagrangian algebra does not uniquely specify a gapped phase, as decoration by boundary topological order is possible, see e.g. [206–209]. Our results remain valid, however including these further data will lead to a larger set of igSPTs.

n	2	5	10	13	17	25	26	...
r	1	$2,3$	$3,7$	$5,8$	$4,13$	$7,18$	$5,21$	...

Table 7.8: First few values of n for which the equation $r^2 = -1 \text{ mod}(n)$ has solutions.

and has order 4. By gauging it, eventually with discrete torsion $\epsilon \in H^5(\mathbb{Z}_4, U(1)) \cong \mathbb{Z}_4$, we get the SymTFT for theories self-dual under gauging $\mathbb{A}^{(1)}$, which include a non-invertible symmetry defect $\mathcal{N}$ [2, 3, 45, 46].

Let us set $\epsilon = 0$ for simplicity. As shown in Chapter 6 (see also [45]) the SPT phases for the self-duality symmetries are given by G-invariant SPT phases for the 1-form symmetry $\mathbb{A}^{(1)}$. These are Lagrangian algebras $\mathcal{L}_{\mathbb{A},\psi}$ such that $S(\mathcal{L}_{\mathbb{A},\psi}) = \mathcal{L}_{\mathbb{A},\psi}$. This last condition can be translated into the requirements that [3] $\psi : \mathbb{A} \to \mathbb{A}^\vee$ is an isomorphism, and $\sigma := \phi^{-1} \circ \psi$ is a square root of the inversion:

$$\sigma^2 = -1\,. \tag{7.4.28}$$

For $\mathbb{A} = \mathbb{Z}_n$ this is equivalent to the existence a quadratic residue of -1. In table 7.8 we report the first few values of n for which this exists.

Since it will be important for us, let us also consider $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$. There are two natural choices of isomorphism ϕ in the $\mathbb{Z}_n \times \mathbb{Z}_n$ case

- The diagonal $\phi_D(x, y) = (x, y)$.
- The off-diagonal $\phi_O(x, y) = (y, x)$.

In the first case a duality invariant SPT is given by a symmetric matrix

$$\psi = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix} \tag{7.4.29}$$

such that $\psi^2 = -1$. This means that

$$\alpha^2 + \beta^2 = \delta^2 + \beta^2 = -1\,, \quad (\alpha + \delta)\beta = 0\,. \tag{7.4.30}$$

Taking $\beta = 0$, this again reduces to the existence of a quadratic residue of -1. We can also take $\alpha = -\delta$, and the condition becomes[23]

$$\alpha^2 + \beta^2 = -1 \text{ mod}(n)\,. \tag{7.4.31}$$

This choice of algebra prescribes perimeter law for the dyonic lines

$$\begin{aligned} &W_1^{-\alpha} T_1\,, \quad W_2^{-\delta} T_2\,, && \beta = 0 \\ &W_1^{-\alpha} W_2^{-\beta} T_1\,, \quad W_2^{\alpha} W_1^{-\beta} T_2\,, && \beta^2 = -1 \text{ mod}(n)\,. \end{aligned} \tag{7.4.32}$$

On the other hand for ϕ_O the existence of duality invariant SPTs is that

$$(\phi_O^{-1}\psi)^2 = \begin{pmatrix} \beta & \delta \\ \alpha & \beta \end{pmatrix}^2 = \begin{pmatrix} \beta^2 + \alpha\delta & 2\delta\beta \\ 2\alpha\beta & \beta^2 + \alpha\delta \end{pmatrix} = -1\,. \tag{7.4.33}$$

[23]This is a weaker condition because even if -1 is not a square, it might be a sum of two squares. For example, this is what happens for $n = 3$. However, for $n = 4$ even this weaker condition cannot be satisfied. In general, if $n = 0 \,\text{mod}(4)$ there is no solution, whereas for all other values of n there is at least one.

This can always be solved by taking δ to be coprime with n and

$$\alpha = -\delta^{-1} \,, \quad \beta = 0 \,. \tag{7.4.34}$$

Moreover, if there is a quadratic residue of -1, it is also solved by $\alpha = \delta = 0$ and :

$$\beta^2 = -1 \bmod(n) \,. \tag{7.4.35}$$

Thus, with the off-diagonal isomorphism, the duality defect for a $\mathbb{Z}_n \times \mathbb{Z}_n$ 1-form symmetry has an SPT for any n.

The above SPTs are realized by assigning perimeter law to the dyonic lattice generated by:

$$\begin{array}{lll} W_1^{-\alpha} T_1 \,, & W_2^{\alpha^{-1}} T_2 \,, & \beta = 0 \\ W_1^{-\beta} T_2 \,, & W_2^{-\beta} T_1 \,, & \beta^2 = -1 \bmod(n) \,. \end{array} \tag{7.4.36}$$

gSPT and igSPT for Duality Defects

We now look at gapless SPT phases protected by duality symmetry. The general story of Sections 7.4.1 and 7.4.1 applies also here. The only difference concerns the decoration: the topological defects of $\mathcal{Z}(\mathbb{A}^{(1)})$ related with the 1-form symmetry are surfaces, hence they have a different dimensionality of the lines generating the quantum symmetry $\mathsf{Rep}(G)$. Invariant lines, strictly speaking, do not come in copies labeled by representations but can be stacked with condensates of the quantum symmetry [8, 210, 211], and condensable algebras can sometimes be decorated with these condensates. We will not analyze this here, and we leave the analysis of the physical relevance of these decoration for the igSPT phases for future studies.

Starting from the condensable algebras for the 1-form symmetry $\mathbb{A}^{(1)}$ studied in Section 7.3, we again have a classification into Type I, II and III igSPT phases. Interestingly, there are natural examples of all these types in (3+1)d. In particular, Type II igSPTs are gapless phases which would not be intrinsic in absence of the non-invertible symmetry, but the presence of the latter forbids to gap the theory, while Type III igSPTs are characterized by the presence of an anomalous, invertible duality symmetry in the infrared.

Type I igSPTs for $\mathbb{Z}_\mathbf{n} \times \mathbb{Z}_\mathbf{n}$ and $\phi_\mathbf{D}$. We look at theories with 1-form symmetry $\mathbb{Z}_n \times \mathbb{Z}_n$ and duality defects associated with the diagonal bicharacter $\phi_D(x, y) = (x, y)$. Clearly, interesting examples of igSPT phases are those that do not have a UV anomaly, so the theory would be compatible with a gapped SPT. Thus we look at integers $n \neq 0 \bmod(4)$. As we have seen in Section 7.3, the 1-form symmetry $\mathbb{Z}_n \times \mathbb{Z}_n$ admits igSPT phases only if n can be written as the product of two non-coprime integers. The smallest such integer, which is $\neq 0 \bmod(4)$ is $n = 9$, for which $p = q = 3$. The most general igSPT for the 1-form symmetry is determined by the condensable algebra

$$\mathcal{A}_{\mathbb{Z}_3 \times \mathbb{Z}_3, 1, \psi} = \left\{ \Big((3x, 3y); (\psi(3x, 3y)) \Big) \mid x, y = 0, 1, 2 \right\} \tag{7.4.37}$$

with ψ represented by the matrix

$$\psi = \begin{pmatrix} s_1 & r_1 \\ s_2 & r_2 \end{pmatrix} \tag{7.4.38}$$

with $r_1 \neq s_2 \bmod(3)$ for the phase to be intrinsic. The condition of duality invariance is $\psi^2 = -1$, that is

$$s_1^2 + r_1 s_2 = r_2^2 + r_1 s_2 = -1 \;\; , \qquad r_1(s_1 + r_2) = s_2(s_1 + r_2) = 0 \;. \tag{7.4.39}$$

Setting $s_1 = r_2 = 0$ (since they represent stacking with an ordinary SPT), this can be solved by

$$s_2 = -1 \quad , \qquad r_1 = 1 \qquad \text{(or vice versa)} \; . \tag{7.4.40}$$

Since $r_1 \neq s_2$, this leads to an algebra $\widehat{\mathcal{A}}_{\mathbb{Z}_3\times\mathbb{Z}_3,1,\psi}$ representing an igSPT phase for the full categorical structure including the duality defect.

A Type II igSPT for $\mathbb{Z}_4 \times \mathbb{Z}_4$ and ϕ_O Consider $\mathbb{A} = \mathbb{Z}_4 \times \mathbb{Z}_4$, and duality associated with the off-diagonal bicharacter $\phi_O(x,y) = (y,x)$. This symmetry is non-anomalous and compatible with gapped SPT phases[24]. It is easy to check that there are no Type I igSPTs in this case. There are, however, igSPT phases of Type II. Consider the subgroup $(\mathbb{Z}_2)_L \subset \mathbb{Z}_4 \times \mathbb{Z}_4$ with the homomorphism:

$$\psi(2a,0) = (0,2a) \,. \tag{7.4.42}$$

The algebra $\mathcal{A}_0$ defined by this homomorphism

$$\mathcal{A}_0 = \{(0,0;0,0),(2,0;0,2)\} \tag{7.4.43}$$

is duality invariant for the off-diagonal isomorphism. We have seen that for the 1-form symmetry this is a gSPT, but not igSPT. Indeed there are symmetric extensions, and the most general one is

$$\widehat{\psi}_{\sigma_1,\sigma_2,k}(x,y) = (2\sigma_1 x + (1+2\sigma_2)y, (1+2\sigma_2)x + ky) \;, \tag{7.4.44}$$

where $\sigma_1, \sigma_2 = 0,1$, while $k = 0,1,2,3$. Since $\mathcal{A}_0$ is a duality invariant gSPT for the 1-form symmetry, it produces a gSPT $\widehat{\mathcal{A}}$ for the full categorical duality symmetry. To determine whether $\widehat{\mathcal{A}}$ is an igSPT (of Type II) or not we need to check if some of the extensions $\widehat{\psi}_{\sigma_1,\sigma_2,k}$ define duality invariant SPTs – if they do not, these are igSPTs. We have

$$\left(\phi_O^{-1}\widehat{\psi}_{\sigma_1,\sigma_2,k}\right)^2 = \begin{pmatrix} 1+2k\sigma_1 & 2k \\ 0 & 1+2k\sigma_1 \end{pmatrix} \,. \tag{7.4.45}$$

For this to be -1, we would need k to be even. But then $1+2k\sigma_1 = 1 \bmod(4)$. We conclude that there is no duality-invariant symmetric extension of ψ. Hence $\widehat{\mathcal{A}}$ is an igSPT protected by the categorical duality symmetry.

A complementary perspective: line order parameters. Let us study the $\mathbb{Z}_4 \times \mathbb{Z}_4$ example from the perspective of line operators. The algebra $\mathcal{A}_0 = \{(0,0;0,0),(2,0;0,2)\}$ describes a gapless phase where the dyon

$$W_2^2 T_1^2 \,, \tag{7.4.46}$$

has perimeter law. The gapped dressing is duality-invariant with ϕ_O as the condensed line is. Now we ask whether we can condense the required dyons to reach a duality-invariant trivially gapped phase. Following our previous classification, for $n = 4$ these are described by $\alpha = 1,3\,;\beta = 0$ and correspond to the condensation of

$$W_1 T_1 \,,\; W_2^3 T_2 \,, \quad \text{or} \quad W_1^3 T_1 \,,\; W_2 T_2 \,, \tag{7.4.47}$$

[24] There are duality invariant Lagrangian algebras for the 1-form symmetry, for instance:

$$\mathcal{L} = \{(x,-y;x,-y), \quad x,y = 0,...,3\} \,. \tag{7.4.41}$$

respectively. We notice, however, that all four of these lines are non-local w.r.t. $W_2^2 T_1^2$ and thus have area law. We conclude that the necessary monopole potential is inaccessible in the deep IR and the phase is protected by the duality symmetry.

Furthermore, we can also show that, breaking duality, it is possible to gap out the theory by an IR deformation. Consider the dyons $T_2 W_1$ and $T_1^{-1} W_2$, which are mutually local with respect to the condensed one and have perimeter law in the IR. We can turn on a monopole potential for them and let them condense. This corresponds to the completion

$$\mathcal{A} = \{(a, b; b, -a)\} \,, \tag{7.4.48}$$

for the algebra $\mathcal{A}_0$.

We can also be more specific about the realization $\mathcal{C}'$ of the symmetry in the gapless degrees of freedom and describe the anomaly. Condensing $\mathcal{A}_0$ gives rise to a $\mathbb{Z}_4 \times \mathbb{Z}_2$ DW theory, with lines generated by:

$$E = (0, 1; 1, 0)\,, \quad M = (0, 0; 0, 1)\,, \tag{7.4.49}$$

$$E' = (1, 0; 0, 1)\,, \quad M' = (0, 0; 2, 0)\,. \tag{7.4.50}$$

The UV duality symmetry acts by:

$$\begin{aligned} S(E) &= -E + M'\,, \quad S(M) = M - E'\,, \\ S(E') &= -E' + 2M\,, \quad S(M') = M' + 2E\,. \end{aligned} \tag{7.4.51}$$

We can ask whether this duality symmetry is anomalous. The most general SPT for the 1-form symmetry is parametrized by algebras $\mathcal{L}_{l,s,s'}$ with generators:

$$E + lM + sM'\,, \quad E' + 2sM + s'M'\,. \tag{7.4.52}$$

We then ask whether any of these is duality-invariant. It is straightforward, although tedious, to show that none of these Lagrangian algebras are duality invariant. We thus conclude that the duality symmetry that acts on the gapless degrees of freedom is *anomalous*.

A Type III igSPT for $\mathbb{Z}_2$. Type III igSPTs are also quite easy to derive. Consider the simplest case $\mathbb{A} = \mathbb{Z}_2$, where an SPT for the duality symmetry is described by the dyonic algebra $\mathcal{L}_D$ generated by $(1, 1)$. On the gapped boundary corresponding to $\mathcal{L}_D$ the symmetry is $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ with a mixed anomaly [25]:

$$I = \pi i \int A \cup \frac{1}{2}\mathfrak{P}(B)\,. \tag{7.4.53}$$

An equivariantization of $\mathcal{L}_D$ contains a choice of symmetry fractionalization class, $\eta \in H^2_\sigma(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. The non-trivial choice $\eta(A) = \beta(A) = \frac{1}{2} dA$ shifts the mixed anomaly and gives a term

$$I' = \pi i \int A \cup \beta(A)^2\,, \tag{7.4.54}$$

describing an anomalous (invertible) duality symmetry. Thus, this algebra realizes a Type III igSPT from the non-invertible duality symmetry to $\mathbb{Z}_2^\omega$.

7.5 igSPTs for 2-Groups in (2+1)d

We conclude by briefly discussing an example of a igSPTs in (2+1)d for 2-group symmetries. The idea is that in a 2-group the 1-form symmetry part is a subgroup, while the 0-form is a quotient. If along an RG flow the charged lines confine, the 1-form symmetry is trivialized and we may end up in a CFT with an anomalous 0-form symmetry (even though the full 2-group is not anomalous).

Consider a discrete 2-group Γ, with 0-form symmetry $\mathbb{A}$ and a 1-form symmetry $\mathbb{B}$:

$$1 \longrightarrow \mathbb{B} \longrightarrow \Gamma \longrightarrow \mathbb{A} \longrightarrow 1\,. \tag{7.5.1}$$

We consider a trivial $\mathbb{A}$ action on $\mathbb{B}$ *but* a non-trivial Postinikov class $\beta \in H^3(\mathbb{A}, \mathbb{B})$ [15]. The SymTFT for this system admits a Lagrangian description[25]

$$S = 2\pi i \int_{4d} C \cup dB + 2\pi i \int_{4d} T \cup dA + 2\pi i \int_{4d} C \cup \beta(A)\,, \tag{7.5.2}$$

with $C \in C^1(X, \mathbb{B}^\vee)$ and $T \in C^2(X, \mathbb{A}^\vee)$, while $B \in C^2(X, \mathbb{B})$ and $A \in C^1(X, \mathbb{A})$. Integrating out C imposes the 2-group constraint:

$$dB = -\beta(A)\,. \tag{7.5.3}$$

The 2-group is realized by imposing Dirichlet boundary conditions for B and A, which become the background fields B and A for the 2-group. Now suppose that, as groups, $\mathbb{B}$ is a subgroup of $\mathbb{A}^\vee$, and let $\iota : \mathbb{B} \hookrightarrow \mathbb{A}^\vee$ be the corresponding embedding. This induces a (surjective) map $\iota^\vee : \mathbb{A} \to \mathbb{B}^\vee$ defined by $\iota^\vee(a)(b) = \iota(b) \cdot a$.

This allows us to define a new interface $\mathcal{I}_\iota$, by: $C - \iota^\vee(A) = 0$, or, in terms of lines, by condensing the algebra generated by

$$\exp\left(2\pi i \int_\gamma (C - \iota^\vee(A))\right)\,. \tag{7.5.4}$$

Since these lines braid non-trivially with the surfaces $e^{i\int_\Sigma B}$, the reduced topological order is

$$S_{\mathcal{I}_\iota} = 2\pi i \int_{4d} T \cup dA + 2\pi i \int_{4d} \iota^\vee(A) \cup \beta(A)\,. \tag{7.5.5}$$

This is a Dijkgraaf-Witten theory for $\mathbb{A}$, with twist $\omega \in H^4(\mathbb{A}, U(1))$ such that

$$A^*(\omega) = \iota^\vee(A) \cup \beta(A)\,. \tag{7.5.6}$$

Physically we are describing a situation where the lines charged under the 1-form symmetry part of the 2-group are completely confined, and there is a gapless theory capturing the low energy dynamics below the scale of confinement. The 1-form symmetry is trivialized in the IR, while the 0-form symmetry $\mathbb{A}$ acts on the gapless sector. If $\omega \in H^4(\mathbb{A}, U(1)$ is non-trivial, the 0-form symmetry has an emergent anomaly and the gapless phase is an igSPT.

An simplest example is $\mathbb{B} = \mathbb{Z}_n$ and $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$, with $\iota : \mathbb{Z}_n \to \mathbb{Z}_n \times \mathbb{Z}_n$ the embedding in the right factor.[26] Decomposing $A = (A_1, A_2)$ as a pair of two $\mathbb{Z}_n$ gauge fields, we have $\iota^\vee(A) = A_2$. If the Postnikov class is such that

$$\beta(A) = A_1 \cup \text{Bock}(A_1)\,, \tag{7.5.8}$$

then the emergent anomaly for the IR symmetry $\mathbb{Z}_n \times \mathbb{Z}_n$ is non-trivial

$$I = 2\pi i \int A_2 \cup A_1 \cup \text{Bock}(A_1)\,. \tag{7.5.9}$$

[25]Here $\beta(a) \in C^3(X, \mathbb{B})$ denotes the pull-back $a^*(\beta)$, with a realized as a map $X \to B\mathbb{A}$.

[26]Anomalies for $\mathbb{Z}_n \times \mathbb{Z}_n$ in 3d are classified by $H^4(\mathbb{Z}_n \times \mathbb{Z}_n, U(1)) \cong \mathbb{Z}_n \times \mathbb{Z}_n$. The two generators are

$$2\pi i \int_{4d} A_1 \cup A_2 \cup \text{Bock}(A_2)\,, \qquad 2\pi i \int_{4d} A_2 \cup A_1 \cup \text{Bock}(A_1)\,. \tag{7.5.7}$$

Chapter 8

Anomaly and gauging of U(1) symmetries

In this chapter we extend the SymTFT framework to include continuous symmetries. We mostly focus on theories with a $U(1)$ symmetry in arbitrary dimension, and make some comments on the non-Abelian cases. We provide a Lagrangian description of the SymTFT, written in terms of gauge fields for group $\mathbb{R}$ — as opposed to $U(1)$. These theories are TQFTs with a continuum of operators, hence they constitute new mathematical objects with few properties that deviate from the standard well-known cases (see Appendix E for a first attempt to a mathematical definition of these TQFTs). The SymTFT describes the structure of the symmetry, its anomalies, and the possible topological manipulations, *i.e.*, to gaugings of discrete symmetry subgroups, or alternatively of the whole $U(1)$ but in a flat way [1]. We also propose an operation that produces the SymTFT for the theory obtained by dynamically gauging the $U(1)$ symmetry. We provide a few selected examples of our construction. For instance, we give the SymTFT description of the chiral anomaly in two and four dimensions, as well as the SymTFT for a 4d Abelian gauge theory with 2-group symmetry. We also provide a 3d example, illustrating interesting phenomena even in the absence of anomalies (that do not exist in odd dimensions). A particularly interesting outcome of our construction is the SymTFT for the non-invertible $\mathbb{Q}/\mathbb{Z}$ symmetry that arises from a $U(1)$ chiral symmetry with ABJ (Adler-Bell-Jackiw) anomaly in 4d Abelian gauge theories [28, 29].

8.1 SymTFT for $U(1)$ symmetries

The SymTFT contains all the categorical data of the global symmetry of the boundary QFT$_d$. For the simple Abelian TQFTs considered in this chapter, both the topological symmetry defects of the boundary QFT that (in the language of [11]) generate the symmetry, and the charges that the operators can carry, are described by the bulk operators [2]. A choice of boundary condition corresponds to a maximal set of mutually-transparent bulk operators (which we call a Lagrangian algebra $\mathcal{L}$) that can terminate on the boundary. In other words, the boundary condition sets those operators to be trivial on the boundary. The endpoints (or more generally end-surfaces) of those operators correspond to the charged operators in the boundary theory, therefore $\mathcal{L}$ is also the set of charges that the operators of the boundary theory can have. On the contrary, we can produce topological operators of QFT$_d$ by

[1] The flat gauging of a $U(1)$ p-form symmetry consists in summing over all flat bundles. It is a topological manipulation that does not introduce new degrees of freedoms, and yields a dual $(d{-}p{-}2)$-form symmetry $\mathbb{Z}$.

[2] More generally, the SymTFT contains complete information also on generalized charges, namely representations of p-form symmetries on q-dimensional objects, where $q \geq p$ [119].

laying the bulk operators on the topological boundary. Therefore the symmetry defects that generate the symmetry of the boundary theory are the operators of the SymTFT modulo $\mathcal{L}$.

We propose that the SymTFT for continuous $U(1)$ symmetries, either 0-form or higher p-form, is a BF theory of gauge fields for gauge group $\mathbb{R}$, as opposed to $U(1)$. To explain this point, it is useful to first review the ordinary BF theory description of $\mathbb{Z}_N$ gauge theory.

***U*(1) gauge fields.** The $\mathbb{Z}_N$ gauge theory in $d+1$ dimensions can be formulated as a BF theory of standard $U(1)$ gauge fields using the action [30, 84, 85]

$$\mathcal{Z} = \frac{iN}{2\pi} \int_{X_{d+1}} B_{d-p} \wedge dA_p \,. \tag{8.1.1}$$

The gauge fields are not globally-defined forms: they are patched using $U(1)$ gauge transformations $\delta A_p = d\lambda_{p-1}$ and $\delta B_{d-p} = d\lambda_{d-p-1}$. This leads to the Dirac quantization condition

$$\frac{1}{2\pi} \int_{\gamma_{p+1}} dA_p \in \mathbb{Z} \,, \qquad \frac{1}{2\pi} \int_{\gamma_{d-p+1}} dB_{d-p} \in \mathbb{Z} \,, \tag{8.1.2}$$

where γ_j are closed j-cycles. We can define extended operators

$$U_\alpha(\gamma_{d-p}) = e^{i\alpha \int_{\gamma_{d-p}} B_{d-p}} \,, \qquad W_\beta(\gamma_p) = e^{i\beta \int_{\gamma_p} A_p} \,, \tag{8.1.3}$$

and invariance under large gauge transformations requires $\alpha, \beta \in \mathbb{Z}$.

In order to obtain the EOMs, it is convenient to decompose each gauge field into a representative of nontrivial $U(1)$ bundles, over which one has to sum, and a globally-defined form describing fluctuations within a given bundle. Variations with respect to the fluctuations give $dA_p = 0$ and $dB_{d-p} = 0$, which guarantee that the operators (8.1.3) are topological. The sum over bundles produces delta functions imposing

$$\frac{N}{2\pi} \int_{\gamma_p} A_p \in \mathbb{Z} \,, \qquad \frac{N}{2\pi} \int_{\gamma_{d-p}} B_{d-p} \in \mathbb{Z} \,. \tag{8.1.4}$$

This implies that the operators (8.1.3) with $\alpha \to \alpha + N$ or $\beta \to \beta + N$ are equivalent. Hence the theory has the following nonequivalent topological operators [3]:

$$U_n(\gamma_{d-p}) \,, \qquad W_m(\gamma_p) \,, \qquad \text{with } n, m \in \mathbb{Z}_N \,. \tag{8.1.5}$$

The braiding between these operators can be computed by inserting one operator in (8.1.1) as a source, and then evaluating the VEV of the other one with the EOMs. The result for the braiding is

$$\mathcal{B}(\alpha, \beta) = \exp\left(\tfrac{2\pi i}{N}\, \alpha\beta\right) \,. \tag{8.1.6}$$

This is the phase picked up by correlation functions when an operator U_α is moved across an operator W_β. The data (8.1.5) and (8.1.6) characterizes the $\mathbb{Z}_N$ gauge theory.

There are various topological boundary conditions.

- A natural boundary condition is that the operators W_m (with $m \in \mathbb{Z}_N$) can terminate on the boundary. The defects U_n with $n \in \mathbb{Z}_N$ can lie on the boundary and play the role of symmetry defects for a

$$\mathbb{Z}_N \ (p-1)\text{-form symmetry} \,. \tag{a}$$

  We can express the condition as a Dirichlet boundary condition for A_p which then plays the role of the background field for the $(p-1)$-form symmetry. Anomalies are obtained by adding topological terms to $\mathcal{Z}$ written in terms of A_p. These terms affect the EOMs and in general change the list of topological operators, the braiding, the possible boundary conditions, *etc.*...

[3] The theory might contain other topological operators, for instance condensates [26] or theta-defects [31, 211].

- Another boundary condition is to let the defects U_n (with $n \in \mathbb{Z}_N$) terminate on the boundary. This gives

$$\mathbb{Z}_N \ (d{-}p{-}1)\text{-form symmetry} \tag{b}$$

on the boundary. Indeed, the two symmetries are related by the flat gauging of $\mathbb{Z}_N$ in the boundary theory.

Other boundary conditions might exist, for instance when N has divisors, or when $p = d - p$.

$\mathbb{R}$ **gauge fields.** In [133] it was found that the theory of a 2d free compact scalar — that has $U(1)^2$ global symmetry — admits a dual holographic description in terms of a certain 3d Chern-Simons theory of two gauge fields for group $\mathbb{R}$, as opposed to $U(1)$. This suggests to study the Abelian BF theory with action

$$\mathcal{Z} = \frac{1}{2\pi} \int_{X_{d+1}} b_{d-p} \wedge da_p \,, \tag{8.1.7}$$

where a_p and b_{d-p} are gauge fields for the group $\mathbb{R}$. This means that they are globally-defined forms, subject to small but not large gauge transformations. $\mathbb{R}$-bundles are necessarily trivial, namely the Dirac quantization conditions collapse to

$$\int_{\gamma_{p+1}} da_p = \int_{\gamma_{d-p+1}} db_{d-p} = 0 \,, \tag{8.1.8}$$

which is Stokes' theorem. We can always rescale a_p or b_{d-p} by a real constant, so that the overall coefficient in $\mathcal{Z}$ is unphysical and we have fixed it to 1.

The EOMs simply set $da_p = 0$ and $db_{d-p} = 0$, so that the operators U_α and W_β are topological. Since there are no large gauge transformations, those operators are gauge invariant with no restrictions on α and β. Since there are no nontrivial bundles to sum over, there are no restrictions on the holonomies. We conclude that the theory has the following topological operators:

$$U_\alpha(\gamma_{d-p}) \,, \qquad W_\beta(\gamma_p) \,, \qquad \text{with } \alpha, \beta \in \mathbb{R} \,. \tag{8.1.9}$$

The braiding is as in (8.1.6) but with $N = 1$. Various topological boundary conditions are possible.

- Let all defects $W_\beta(\gamma_p)$ terminate on the boundary. They represent charged operators along $\partial\gamma_p$ with generic charges $\beta \in \mathbb{R}$. All defects $U_\alpha(\gamma_{d-p})$ can lie on the boundary and play the role of the symmetry defects for an

$$\mathbb{R} \ (p\,{-}1)\text{-form symmetry} \,. \tag{c}$$

Such a symmetry indeed allows for generic charges. An example of a theory with this symmetry is a free $\mathbb{R}$ $(p-1)$-form gauge field (for $p = 1$ this is a free noncompact scalar).

- Similarly, let all defects $U_\alpha(\gamma_{d-p})$ terminate on the boundary. This describes an

$$\mathbb{R} \ (d{-}p{-}1)\text{-form symmetry} \tag{d}$$

on the boundary. It is obtained from (c) by gauging the whole $\mathbb{R}$ on the boundary (the Pontryagin dual to $\mathbb{R}$ is $\mathbb{R}$). This gauging, in order to be topological, must be a *flat gauging.* This means that the field strength of the boundary gauge field is identically zero and we only sum over flat connections.

- Let the defects U_n and W_m with $n, m \in \mathbb{Z}$ terminate on the boundary. From (8.1.6), this choice constitutes a maximal set of mutually-transparent operators and is thus a Lagrangian algebra. The coset classes of operators that can lie on the boundary are given by U_α and W_β with $\alpha, \beta \in \mathbb{R}/\mathbb{Z} =$

$U(1)$. Thus there are two factors, such that charged operators have integer charges while defects are valued in $U(1)$. This describes a

$$U(1)^{(p-1)} \times U(1)^{(d-p-1)} \text{ symmetry}\,. \tag{e}$$

The two factors have a mixed anomaly. This is obtained from (c) by gauging a $\mathbb{Z}$ subgroup of $\mathbb{R}$ ($U(1)$ is the Pontryagin dual to $\mathbb{Z}$) and the mixed anomaly follows from the fact that the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ does not split [154]. An example of a theory with this symmetry is a free $U(1)$ $(p-1)$-form gauge field (for $p = 1$ this is a free compact scalar, for $p = 2$ a free photon), dual to a free $U(1)$ $(d-p-1)$-form gauge field. We refer to these theories as generalized Maxwell theories.

In the last case, one could more generally consider the defects U_{nR} and $W_{mR^{-1}}$ for any real constant R. This would amount to rescale the radii of the two $U(1)$ factors. The SymTFT does not determine the actual value of the radius, but can compare the radii arising in two different topological boundaries. For instance, for $p = 1$ we get a compact boson whose radius is R times larger than the one in case (e). Similarly, for $p = 2$ the choice of R corresponds to a rescaling of the electric charge in Maxwell's theory. The special case of a rescaling by $R = N \in \mathbb{Z}$ also corresponds to gauging a $\mathbb{Z}_N$ subgroup of $U(1)^{(d-p-1)}$. The cases (c) and (d) with symmetry $\mathbb{R}$ correspond to the decompactification limits of the original field or of its dual, respectively.

$U(1)/\mathbb{R}$ gauge fields. Lastly, consider the case

$$\mathcal{Z} = \frac{i}{2\pi} \int_{X_{d+1}} b_{d-p} \wedge dA_p \tag{8.1.10}$$

where A_p and b_{d-p} are $U(1)$ and $\mathbb{R}$ gauge fields, respectively. As before, the overall coefficient of $\mathcal{Z}$ can always be set to 1. This time Dirac's quantization conditions read $\frac{1}{2\pi}\int dA_p \in \mathbb{Z}$ and $\int db_{d-p} = 0$. By the same arguments as before, one concludes that the theory has the following nonequivalent topological operators:

$$U_\alpha(\gamma_{d-p}) \text{ with } \alpha \in \mathbb{R}/\mathbb{Z} = U(1)\,, \qquad W_m(\gamma_p) \text{ with } m \in \mathbb{Z}\,. \tag{8.1.11}$$

Some interesting boundary conditions are the following.

- Let the defects W_m (with $m \in \mathbb{Z}$) terminate on the boundary. Then the defects U_α with $\alpha \in U(1)$ can lie on the boundary and represent a

$$U(1) \ (p-1)\text{-form symmetry}\,. \tag{f}$$

- Let all defects U_α with $\alpha \in U(1)$ terminate on the boundary. Then the defects W_m with $m \in \mathbb{Z}$ can lie on the boundary and represent a

$$\mathbb{Z} \ (d-p-1)\text{-form symmetry}\,. \tag{g}$$

  This is obtained from (f) by flat gauging the whole $U(1)$. For $p = d-1$, an example of a theory with 0-form symmetry $\mathbb{Z}$ is a scalar field ϕ with periodic potential, as in band theory.

- Let the defects W_{bm} with $m \in \mathbb{Z}$ and $b > 1$ an integer constant, as well as $U_{n/b}$ with $n \in \mathbb{Z}_b$, terminate on the boundary. Then the nonequivalent defects that can lie on the boundary are U_α with $\alpha \in \mathbb{R}/\big(\frac{1}{b}\mathbb{Z}\big) \cong U(1)$ and W_k with $k \in \mathbb{Z}_b$. They represent a

$$U(1)^{(p-1)} \times \mathbb{Z}_b^{(d-p-1)} \text{ symmetry}\,. \tag{h}$$

  This is obtained from (f) by gauging a $\mathbb{Z}_b$ subgroup of $U(1)$. Indeed the charged operators have integer charges that are multiples of b. There is a mixed anomaly between $U(1)$ and $\mathbb{Z}_b$ that follows from the short exact sequence $0 \to \mathbb{Z}_b \to U(1) \to U(1) \to 0$.

As a check, consider the case (e) that we derived from the SymTFT (8.1.7) using only $\mathbb{R}$ gauge fields. It should also arise from the SymTFT of two $U(1)$ symmetries with a mixed anomaly:

$$\mathcal{Z} = \frac{i}{2\pi} \int \Big[b_{d-p} \wedge dB_p + a_p \wedge dA_{d-p} - B_p \wedge dA_{d-p} \Big] . \tag{8.1.12}$$

Indeed the field A_{d-p} can be integrated out producing a delta function that enforces $B_p = a_p$ (restricting B_p to be an $\mathbb{R}$ gauge field), and integrating out B_p one reproduces the action (8.1.7). Alternatively, and more precisely, one can list the topological operators and compute their correlations functions. One realizes that $e^{in\int B_p}$ has identical correlation functions to those of $e^{i\alpha\int a_p}$ for $\alpha = n$, hence the two operators are identified. Similarly for $e^{im\int A_{d-p}}$ and $e^{i\beta\int b_{d-p}}$ for $\beta = m$. The alternative presentation (8.1.12) of the SymTFT for a generalized Maxwell field will be important in Section 8.3.

Note: in the following we will keep denoting $U(1)$ and $\mathbb{R}$ gauge fields with upper and lower case letters respectively, while the subscript denote the diemsion of the form.

8.2 Examples

Let us present a few interesting examples. Other examples, which require us to understand how to dynamically gauge a $U(1)$ symmetry from the point of view of the SymTFT, are described in Section 8.4.

8.2.1 Chiral anomaly in 2d

The SymTFT for a two-dimensional theory with $U(1)$ 0-form symmetry and an 't Hooft anomaly is obtained from the action (8.1.10) with $d = 2$ and $p = 1$ by adding a term that describes the chiral anomaly:

$$\mathcal{Z} = \int_{X_3} \left[\frac{i}{2\pi} \, b_1 \wedge dA_1 + \frac{ik}{4\pi} \, A_1 \wedge dA_1 \right] , \tag{8.2.1}$$

where k is constrained to be integer (when k is odd the theory requires a spin structure [81]).

The theory has topological line operators given by

$$U_{(n,\alpha)}(\gamma_1) = \exp\left(i \int_{\gamma_1} (nA_1 + \alpha\, b_1) \right) \tag{8.2.2}$$

with $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. The braiding between them is

$$\mathcal{B}\big[(n_1, \alpha_1), (n_2, \alpha_2) \big] = \exp\Big[2\pi i \big(n_1\alpha_2 + n_2\alpha_1 - k\alpha_1\alpha_2 \big) \Big] \tag{8.2.3}$$

and the exponentiated spin is given by a quadratic refinement thereof:

$$\theta_{(n,\alpha)} = \exp\Big[2\pi i \, \alpha \big(n - \tfrac{k}{2}\, \alpha \big) \Big] . \tag{8.2.4}$$

The line $(0, 1)$ has spin $\theta = (-1)^k$ and is a transparent fermion for k odd. Both spin and braiding are invariant under the following identifications:

$$\begin{cases} (n, \alpha) \sim (n + k,\, \alpha + 1) & \text{for } k \text{ even,} \\ (n, \alpha) \sim (n + 2k,\, \alpha + 2) & \text{for } k \text{ odd.} \end{cases} \tag{8.2.5}$$

Let us discuss boundary conditions. First, we can let all lines $(n, 0)$ terminate on the boundary. For k even, this is a maximal set $\mathcal{L}$ of mutually-transparent lines and they are all bosonic. For k odd, we should also let the lines $(n, 1)$ terminate on the boundary in order to have a maximal set $\mathcal{L}$, and the extra lines have spin -1. Thus, the Lagrangian algebra $\mathcal{L}$ is bosonic for k even, and spin for k odd. The nonequivalent topological line operators that can lie on the boundary are labeled by $\alpha \in \mathbb{R}/\mathbb{Z} \cong U(1)$, so this describes a $U(1)$ 0-form symmetry.

Let us use the lines $(0,\alpha)$ as representatives of the symmetry defect operators of the boundary theory. For k even, the bosonic lines $(n,0)$ that end on the 2d boundary represent boundary local operators, and their charge measured by the braiding (8.2.3) is n. For k odd, the lines $(n,0)$ and $(n+k,1)$ that end on the boundary represent local operators with charge n and spin ± 1, respectively. Thus the local operators can have arbitrary and independent integer charges and statistics.

It is possible to specialize to theories with a spin-charge relation, in which even-charge operators are bosonic and odd-charge operators are fermionic (*e.g.*, a free complex Weyl fermion). In this case the SymTFT (8.2.1) is written in terms of a spin$_c$ connection A_1 [212, 213] whose Dirac quantization condition on 2-cycles is modified according to the second Stiefel-Whitney class of the manifold: $\frac{1}{2\pi}dA_1 = \frac{1}{2}w_2 \bmod 1$ (one also needs to add a gravitational term to the action). Since A_1 is not an ordinary connection but $2A_1$ is, gauge invariance restricts the operators $U_{(m,\alpha)}$ to have even m. This in turn implies the spin-charge relation for the boundary local operators: the endpoints of $(n,0)$ are bosonic for n even, and the endpoints of $(n+k,1)$ are fermionic for n odd.

When $k \neq 0$, the operators $U_{(0,\alpha)}$ are no longer mutually transparent and therefore there is no boundary condition corresponding to (g). This is a manifestation of the chiral anomaly: the $U(1)^{(0)}$ symmetry of the boundary theory cannot be gauged. Indeed the anomaly of a continuous group is uniquely determined by the anomaly of all its discrete subgroups [214].

However, it might still be possible to gauge a discrete subgroup of $U(1)$. Consider the bosonic case of k even. Given an integer d that divides $k/2$, consider the set $\mathcal{L}$ of line operators $\big(\frac{k}{2d}m + d\,\ell, \frac{m}{d}\big)$ with $\ell, m \in \mathbb{Z}$ (modulo identifications). Such operators have spin $\theta = 1$ and are thus bosons. From (8.2.3), a line that has trivial braiding with all elements of $\mathcal{L}$ must be in $\mathcal{L}$, thus $\mathcal{L}$ is maximal and is a Lagrangian algebra. This shows that when $k/2$ is divisible by d, the $\mathbb{Z}_d$ subgroup of $U(1)$ is anomaly free and can be gauged in the boundary theory. The nonequivalent topological lines that can lie on the boundary are $U_{(n,\alpha)}$ with $n \in \mathbb{Z}_d$ and $\alpha \in \mathbb{R}/\big(\frac{1}{d}\mathbb{Z}\big)$. They represent a symmetry $U(1)^{(0)} \times {\mathbb{Z}_d}^{(0)}$ with an 't Hooft anomaly for $U(1)$ and a mixed anomaly between $U(1)$ and $\mathbb{Z}_d$.

As a check, we can simply restrict to $\mathbb{Z}_d$ the anomaly-inflow action $\mathcal{Z}_{\text{inflow}} = \frac{k}{4\pi}\int A \wedge dA$, where A is seen as an extension from 2d to 3d of the background gauge field for the $U(1)$ symmetry [215]. This is achieved by replacing

$$A \mapsto \frac{2\pi}{d}\,\mathcal{A}\,, \qquad \frac{dA}{2\pi} \mapsto \beta(\mathcal{A})\,, \tag{8.2.6}$$

where $\mathcal{A} \in H^1(X;\mathbb{Z}_d)$ is (an extension of) the background field for $\mathbb{Z}_d$, while $\beta : H^1(X;\mathbb{Z}_d) \to H^2(X;\mathbb{Z})$ is the Bockstein homomorphism associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_d \to 0$. The inflow action reduces to $2\pi\frac{k}{2d}\int \mathcal{A} \cup \beta(\mathcal{A})$, where the integral is an integer modulo d, which is indeed an integer multiple of 2π whenever d divides $k/2$ and thus $\mathbb{Z}_d$ is anomaly free. The converse, to determine which $\mathbb{Z}_d$ subgroups are actually anomalous, is a delicate issue [216, 217]: one should determine whether the reduced anomaly-inflow action is trivial when evaluated on the generator(s) of the relevant bordism group $\Omega_3^{\text{SO}}(B\mathbb{Z}_d)$.

In the fermionic case of k odd one can perform a similar analysis. Given d that divides k (in particular d is odd), the set $\mathcal{L}$ of line operators $\big(\frac{k}{d}\frac{d+1}{2}m + d\,\ell, \frac{m}{d}\big)$ labelled by $\ell, m \in \mathbb{Z}$ (here $\frac{d+1}{2} = 2^{-1} \bmod d$) is a maximal set of mutually-transparent lines with spins $\theta = (-1)^m = \pm 1$, suggesting that the $\mathbb{Z}_d$ subgroup can be gauged. This should be compared with the evaluation of the reduced inflow action on the generator(s) of the bordism group $\Omega_3^{\text{spin}}(B\mathbb{Z}_d)$.

We ask whether there can be other more exotic topological boundary conditions. To be concrete, take $k = 1$ and consider the following set:

$$\mathcal{C} = \left\{ \left(n, n + \ell\sqrt{2}\right) \;\middle|\; n, \ell \in \mathbb{Z} \right\}. \tag{8.2.7}$$

The set is closed under sum. Moreover, mapping the second component to the interval $[0, 2)$ (by shifting also the first component), we obtain a dense irrational subset. The operators in $\mathcal{C}$ have $\theta = (-1)^n$ and are all mutually transparent, however the set is not maximal (hence it does not describe a topological boundary condition) and in fact it cannot be made into a maximal set. A line mutually transparent with $\mathcal{C}$ is of the form $\left(m, m + \frac{h}{\sqrt{2}}\right)$ for some $m, h \in \mathbb{Z}$. For h odd, these lines are not contained in $\mathcal{C}$, however they cannot be included in $\mathcal{C}$ because they have nontrivial spin $\theta = i\,(-1)^{m+1}$. We have thus found an example of an Abelian algebra that cannot be completed into a Lagrangian algebra. Interestingly, this is a peculiarity of TQFTs with a continuum of lines, as can be intuitively understood from the necessity of taking the square root. Indeed a finite semi-simple modular tensor category admitting Lagrangian algebras is a Drinfeld center, and the condensation of an algebra of a Drinfeld center yields another Drinfeld center [218]. As we showed, this result is no longer true in the presence of a continuum of lines.

8.2.2 $U(1)$ symmetry in 3d

In odd dimensions there are no anomalies for $U(1)$ symmetries. Nevertheless there are useful pieces of information encoded in the SymTFT. Here we consider the 3d case, hence the basic 4d SymTFT describing a $U(1)$ symmetry is (8.1.10) with $d = 3$, $p = 1$. There are two more terms that can be added: a theta-term $\frac{\theta}{8\pi^2}\int_{X_4} dA_1 \wedge dA_1$ and a phi-term $\frac{\phi}{8\pi^2}\int_{X_4} b_2 \wedge b_2$. In the absence of the latter, the theta-term is unphysical since it can be reabsorbed by a shift of b_2. Here we study the effect of the phi-term, while we leave the analysis of the most general action with both terms for the future.

The 4d TQFT we consider is

$$\mathcal{Z} = \frac{i}{2\pi}\int_{X_4} b_2 \wedge dA_1 + \frac{i\phi}{8\pi^2}\int_{X_4} b_2 \wedge b_2\,. \tag{8.2.8}$$

This can be thought of as the non-compact version of the theory studied in [30]. The gauge transformations are

$$\delta b_2 = d\lambda_1\,, \qquad \delta A_1 = d\rho_0 - \tfrac{\phi}{2\pi}\,\lambda_1\,. \tag{8.2.9}$$

The gauge-invariant operators are the surfaces

$$U_\alpha(\gamma_2) = e^{i\alpha\int_{\gamma_2} b_2} \tag{8.2.10}$$

and the (generically) non-genuine lines

$$W_n(\gamma_1, D_2) = \exp\!\left(in\int_{\gamma_1} A_1 + i\,\frac{n\phi}{2\pi}\int_{D_2} b_2\right), \tag{8.2.11}$$

both topological. Here $n \in \mathbb{Z}$ whilst D_2 is a disk with $\partial D_2 = \gamma_1$. A non-trivial correlator on the sphere is the braiding of $U_\alpha(\gamma_2)$ and $W_n(\gamma_1, D_2)$ in a configuration in which γ_1 and γ_2 link, hence γ_2 intersects D_2 at a point:

$$\big\langle U_\alpha(\gamma_2)\, W_n(\gamma_1, D_2)\big\rangle = e^{2\pi i\alpha n}\,. \tag{8.2.12}$$

We read off that $\alpha \sim \alpha + 1$. This implies that if $\frac{n\phi}{2\pi}$ is an integer in (8.2.11), there is no dependence of W_n on the disk D_2: choosing a different disk would give an operator which differs by $U_\alpha(\gamma_2)$ for some integer α and where γ_2 is the union (with opposite orientations) of the two disks, hence that difference is trivial. In particular we read off that ϕ is a periodic parameter:

$$\phi \sim \phi + 2\pi \,. \tag{8.2.13}$$

An interesting observation is that, while for irrational values of $\frac{\phi}{2\pi}$ the theory has no genuine lines, hence it is essentially trivial in the bulk, when $\phi = 2\pi p/q$ with $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$, the lines W_{mq} are genuine.

Let us study topological boundary conditions. First, we can let all surfaces U_α terminate on the boundary. This corresponds to a Dirichlet boundary condition for b_2. Since the gauge transformations of b_2 are forced to vanish at the boundary, there we can construct the genuine line operators $\mathcal{W}_n = e^{in\int A_1}$ (equivalently, since b_2 is a background field at the boundary, the dependence of W_n on it can be removed by a counterterm). This boundary condition thus describes a $\mathbb{Z}$ 1-form symmetry, whose topological operators are the lines $\mathcal{W}_n$ and whose charged operators are the endlines of U_α on the boundary. The phi-term represents an anomaly for this 1-form symmetry, and if $\frac{\phi}{2\pi}$ is irrational there are no non-anomalous subgroups of $\mathbb{Z}$. Indeed in this case the bulk does not have other genuine topological operators besides U_α, hence there are no other topological boundary conditions. In particular, there is no boundary with a $U(1)$ symmetry.

On the contrary, consider the case $\frac{\phi}{2\pi} = p/q \in \mathbb{Q}$. The lines W_{qm} with $m \in \mathbb{Z}$ are genuine, as we noticed, and we can let them terminate on the boundary. In order to have a maximal set of mutually-transparent objects, we should let the surfaces U_α with $\alpha = l/q$ and $l \in \mathbb{Z}$ terminate on the boundary as well. This boundary condition describes the symmetry ${\mathbb{Z}_q}^{(1)} \times U(1)^{(0)}$ with a mixed anomaly and a pure anomaly for ${\mathbb{Z}_q}^{(1)}$, and is obtained from the previous one by gauging the subgroup $q\mathbb{Z} \subset \mathbb{Z}$ of the 1-form symmetry, which is non-anomalous in this case. The symmetry $U(1)^{(0)}$ is generated by U_α with identification $\alpha \sim \alpha + 1/q$ (the reduced range is due to the boundary condition). The local operators $\mathcal{M}_m$ charged under $U(1)^{(0)}$ are the endpoints of the genuine lines W_{qm}.

To get some intuition on the nature of the QFTs described by this boundary condition, let us present a Lagrangian example. On a manifold with boundary, the variation of the action (8.2.8) under a gauge transformation (8.2.9) generates a boundary term:

$$\delta \mathcal{Z} = \frac{i}{2\pi} \int_{\partial X_4} \left[\lambda_1 \wedge dA_1 - \frac{\phi}{4\pi} \, \lambda_1 \wedge d\lambda_1 \right] \tag{8.2.14}$$

where the boundary value of A_1 is fixed. This can be cancelled by edge modes. Since the local operators $\mathcal{M}_m$ charged under $U(1)^{(0)}$ are the endpoints of W_{qm}, the corresponding background field is qA_1 at the boundary. Since the line operators charged under ${\mathbb{Z}_q}^{(1)}$ are the endlines of $U_{l/q}$, the background field is b_2/q. At the boundary we can place the Chern-Simons theory $U(1)_{pq}$:

$$S_\partial = -i \int_{\partial X_4} \left[\frac{pq}{4\pi} \, \mathcal{B} \wedge d\mathcal{B} + \frac{1}{2\pi} (qA_1) \wedge d\mathcal{B} \right] . \tag{8.2.15}$$

This theory has 1-form symmetry $\mathbb{Z}_{pq} = \mathbb{Z}_p \times \mathbb{Z}_q$, but we only consider the $\mathbb{Z}_q$ subgroup which has anomaly p [111]. The generator of $\mathbb{Z}_q$ acts as $\delta\mathcal{B} = \frac{1}{q}\lambda_1$ so that the variation of S_∂ cancels (8.2.14). The Chern-Simons theory also has a magnetic $U(1)$ 0-form symmetry with current $*d\mathcal{B}$, which is coupled to the background field (qA_1). The local operators $\mathcal{M}_m$ are monopoles. In this example the current is trivial at separated points, but it still has an interesting contact term [219] (to make the current nontrivial at separated points, one could consider Maxwell-Chern-Simons theory instead). Because of the 4d EOM, we can identify $dA_1 = -p\big(\frac{1}{q} b_2\big)$ with the background for the 1-form symmetry, hence the theory is coupled to the latter as well, in a subtle way. Another possibility is to use the topological theory $\mathcal{A}^{q,p}\big[\frac{1}{q}dA_1\big]$ [111] as boundary theory. Because of the identifications among those theories, this in particular shows that $p \sim p+q$ in accord with (8.2.13).

8.2.3 Chiral anomaly in 4d

For a four-dimensional theory with a $U(1)$ 0-form symmetry and an 't Hooft anomaly, the SymTFT is

$$\mathcal{Z} = \int_{X_5} \left[\frac{i}{2\pi}\, b_3 \wedge dA_1 + \frac{ik}{24\pi^2}\, A_1 \wedge dA_1 \wedge dA_1 \right] \tag{8.2.16}$$

where A_1 is a $U(1)$ gauge field while b_3 is an $\mathbb{R}$ gauge field. The parameter k is an integer in fermionic theories, while in general bosonic theories it should be a multiple of 6.

The theory has topological line and surface operators

$$W_n(\gamma_1) = e^{in\int_{\gamma_1} A_1} \quad , \qquad U_{(\beta,m)}(\gamma_3) \;= \exp\left[i \int_{\gamma_3} \Big(\beta\, b_3 + \frac{m}{4\pi}\, A_1 \wedge dA_1 \Big) \right] \tag{8.2.17}$$

with $n, m \in \mathbb{Z}$. The quantization of m corresponds to spin-Chern-Simons theories. The perturbative EOMs $dA_1 = db_3 = 0$ guarantee that these operators are topological. The observables of the theory include the linking of W_n with $U_{(\beta,m)}$, and the triple-linking between three operators $U_{(\beta_i,m_i)}$ on surfaces ${\gamma_3}^{(i)}$. The latter probes the linking between the intersection $\widetilde{\gamma}_1 = {\gamma_3}^{(1)} \cap {\gamma_3}^{(2)}$ and ${\gamma_3}^{(3)}$ (one can show that this is symmetric in the three surfaces). The braiding can be defined as the following expectation value on the sphere:

$$\big\langle W_n(\gamma_1)\, U_{(\beta,m)}(\gamma_3) \big\rangle = \exp\!\big(2\pi i\, n\beta\, \mathrm{Lk}(\gamma_1, \gamma_3) \big) \tag{8.2.18}$$

where Lk is the geometric linking number. Alternatively, it is the phase picked up by generic correlation functions when W_n is moved across $U_{(\beta,m)}$. The triple-linking number on the sphere is the following expectation value:

$$\big\langle U_{(\beta_1,m_1)}\big(\gamma_3^{(1)}\big)\, U_{(\beta_2,m_2)}\big(\gamma_3^{(2)}\big)\, U_{(\beta_3,m_3)}\big(\gamma_3^{(3)}\big) \big\rangle = \exp\!\big[2\pi i \big(m_1\beta_2\beta_3 + m_2\beta_1\beta_3 + m_3\beta_1\beta_2 - k\beta_1\beta_2\beta_3 \big) \text{Ł} \big] \tag{8.2.19}$$

where $\text{Ł} = \mathrm{Lk}\big(\widetilde{\gamma}_1, {\gamma_3}^{(3)}\big)$. Note that, differently from the 2d case, there is no operator $U_{(\beta,m)}$ with trivial triple-linking with all other pairs of operators. Hence there are no identifications of labels in this case.

Using (8.2.18) and (8.2.19) we can look for topological boundary conditions corresponding to the condensation of a "Lagrangian algebra". By this we mean a set of line and surface operators which are: (i) closed under fusion, (ii) mutually transparent with respect to both braiding and triple linking, and (iii) maximal in the sense that any operator transparent with the set belongs itself to the set. These conditions guarantee that the Lagrangian algebra can be condensed, and the result is the trivial TQFT. We find the following possibilities.

First, the lines W_n and the surfaces $U_{(\beta=j,m)}$ with $n, j, m \in \mathbb{Z}$ are mutually transparent and maximal. Condensing all of them we obtain the boundary condition for a theory with $U(1)$ 0-form symmetry. Its nonequivalent topological symmetry operators are $U_{(\beta,0)}(\gamma_3)$ where $\beta \in \mathbb{R}/\mathbb{Z}$ with $\beta \sim \beta + 1$ because of the condensation.

Second, if $k = 0$, another Lagrangian algebra is given by all the operators $U_{(\beta,0)}$. This correspond to a 2-form symmetry $\mathbb{Z}$ whose topological symmetry operators are the lines W_n with $n \in \mathbb{Z}$, as in the general case (g) [4]. Such a boundary condition is obtained from the previous one by flat gauging the

[4] The boundary theory also inherits the topological surfaces $U_{(0,m)}(\gamma_3)$ which are theta-defects [31, 211] constructed out of the lines $W_n(\gamma_1)$.

$U(1)$ symmetry on the boundary. If $k \neq 0$, however, this algebra does not exist, consistently with the statement that the SymTFT describes an anomalous symmetry.

Lastly, given an integer d such that $3d$ divides k, one can construct a Lagrangian algebra made of

$$W_{d\rho} \quad , \qquad U_{(\beta,m)} \text{ with } (\beta, m) = \left(\frac{\nu}{d}, \frac{k}{3d}\nu + d^2\mu\right) \tag{8.2.20}$$

and labelled by $\rho, \nu, \mu \in \mathbb{Z}$. The endpoints of the lines are the charged objects. Since the charges are multiples of d, we conclude that this boundary corresponds to gauging the non-anomalous $\mathbb{Z}_d$ subgroup of $U(1)^{(0)}$. The symmetry on this boundary is $U(1)^{(0)} \times \mathbb{Z}_d^{(2)}$ where the first factor is the $\mathbb{Z}_d$ quotient of the original symmetry, while the second factor is the dual 2-form symmetry generated by the lines W_p with $p \in \mathbb{Z}_d$.

8.3 Non-topological manipulations

The SymTFT for $U(1)$ symmetries that we discussed is the straightforward generalization of the discrete case. Its topological boundaries correspond to *topological* manipulations that use the $U(1)$ symmetry. These include gauging discrete subgroups, possibly with discrete torsion, as well as performing the flat gauging the whole $U(1)$. However, differently from the discrete ones, continuous $U(1)$ symmetries also allow for non-topological, *dynamical* manipulations — such as coupling to a dynamical photon — that introduce new degrees of freedom. For instance, a 4d free complex scalar field has a $U(1)^{(0)}$ symmetry and its SymTFT is (8.1.10) with $p = 1$, $d = 4$. By gauging dynamically the $U(1)$ we obtain scalar QED which has a $U(1)^{(1)}$ symmetry, hence its SymTFT is again (8.1.10) but with $p = 2$, $d = 4$. We see that dynamical manipulations are not described by different topological boundary conditions of the same SymTFT, but rather by a map between two different SymTFTs. As we will argue, this is a controlled operation.

8.3.1 Gauging dynamically a $U(1)$

For concreteness we are going to focus on 0-form symmetries, but the generalization to higher forms is straightforward. The initial SymTFT is (8.1.10) with $p = 1$. The idea is to add to it the SymTFT of a d-dimensional photon, and to couple the two TFTs in the bulk in a way that reproduces the coupling of the current to the gauge field on the boundary. It is convenient to use the alternative formulation (8.1.12) for the SymTFT of the photon, which we report here for convenience:

$$\mathcal{Z} = \frac{i}{2\pi}\int \Big[g_{d-2} \wedge dG_2 + f_2 \wedge dF_{d-2} - G_2 \wedge dF_{d-2}\Big]. \tag{8.3.1}$$

The coupling to the fields appearing in the "matter" part $\frac{1}{2\pi}\int b_{d-1} \wedge dA_1$ must be such that, on the boundary, the Wilson lines of the Maxwell field are endable on the local operators charged under $U(1)^{(0)}$. The Wilson lines are the endlines of the surfaces of G_2, while the above-mentioned local operators are the endpoints of the lines of A_1. Hence, the coupling must allow the surfaces of G_2 to end on the lines of A_1. We are led to the total action:

$$\mathcal{Z} = \frac{i}{2\pi}\int_{X_{d+1}} \Big[b_{d-1} \wedge dA_1 + b_{d-1} \wedge G_2 + g_{d-2} \wedge dG_2 + f_2 \wedge dF_{d-2} - G_2 \wedge dF_{d-2}\Big]. \tag{8.3.2}$$

Because of the added coupling $b_{d-1} \wedge G_2$, the standard gauge transformation of G_2 must also act on A_1:

$$\delta G_2 = d\lambda_1 \,, \qquad \delta A_1 = -\lambda_1 \,. \tag{8.3.3}$$

Hence the lines of A_1 are no longer gauge invariant, but we can construct the non-genuine line operators

$$L_n(\gamma_1, D_2) = \exp\left(in\int_{\gamma_1} A_1 \,+\, in\int_{D_2} G_2\right) \tag{8.3.4}$$

where $\partial D_2 = \gamma_1$. This implies that all the surfaces of G_2 can end, and on the boundary all Wilson lines can be cut open, achieving the coupling of the photon with matter.

Similarly, the gauge transformations of b_{d-1} must act on g_{d-2}:

$$\delta b_{d-1} = d\eta_{d-2}\,, \qquad \delta g_{d-2} = (-1)^d\,\eta_{d-2}\,. \tag{8.3.5}$$

As a consequence the surfaces of g_{d-2} are not gauge invariant, and must be attached to a disk where b_{d-1} is integrated:

$$U_\alpha(\gamma_{d-2}, D_{d-1}) = \exp\left(i\alpha\int_{\gamma_{d-2}} g_{d-2} \,-\, (-1)^d i\alpha\int_{D_{d-1}} b_{d-1}\right). \tag{8.3.6}$$

This also has a clear physical interpretation. The operators $\exp\big(i\alpha\int b_{d-1}\big)$ that used to generate on the boundary the global $U(1)$ 0-form symmetry we are gauging, can now be opened and trivialized. Gauging a symmetry trivializes the topological operators that generate it.

By inspection we notice that the only genuine operators that cannot be opened are the surfaces of f_2 and F_{d-2}. All the other ones become trivial because of the coupling $b_{d-1}\wedge G_2$. Therefore the theory is equivalent to

$$\mathcal{Z} = \frac{i}{2\pi}\int_{X_{d+1}} f_2\wedge dF_{d-2}\,. \tag{8.3.7}$$

This is nothing but the action (8.1.10) with $p = 2$, and it describes the dual magnetic symmetry appearing after the dynamical gauging.

We also notice that going from $\frac{i}{2\pi}\int b_{d-1}\wedge dA_1$ to (8.3.7) can be understood as the replacement:

$$dA_1 \,\mapsto\, f_2\,, \qquad b_{d-1} \,\mapsto\, dF_{d-2}\,. \tag{8.3.8}$$

The first substitution means that A_1, which was previously flat, can now describe a curved background with field strength f_2, which is free — hence dynamical — at the boundary. The second substitution means that the previous (Hodge dual) current b_{d-1} of the $U(1)^{(0)}$ symmetry is trivialized in terms of a form of lower degree.

The prescription (8.3.8) is the basic ingredient to construct maps between different SymTFTs, implementing the dynamical manipulations. In Section 8.4 we will apply it to more complicated examples in order to construct the SymTFTs of various symmetry structures.

8.3.2 The Anomaly Polynomial TFT

It is important to emphasize that, while for discrete symmetries the SymTFT is unique and its topological boundaries describe all possible manipulations, here we need a step further. Indeed we find distinct SymTFTs, related by the map (8.3.8), that describe the dynamical manipulations, while each SymTFT admits various topological boundaries that describe the topological manipulations. We observe, however, that one can construct a (unique) $(d+2)$-dimensional TQFT whose $(d+1)$-dimensional topological boundaries correspond to the distinct SymTFTs. We dub this the *Anomaly Polynomial* TFT. It is written entirely in terms of $\mathbb{R}$ gauge fields, which can be identified with the field strengths of the gauged symmetries in the various theories related by dynamical manipulations.

We illustrate the idea in the simplest example of a 2d theory with a single $U(1)^{(0)}$ symmetry. Denoting by $k \in \mathbb{Z}$ the anomaly, the Anomaly Polynomial TFT is a 4d TQFT with action:

$$\mathcal{P} = \frac{i}{2\pi} \int_{X_4} \left[g_1 \wedge df_2 + \frac{k}{2} \, f_2 \wedge f_2 \right] \tag{8.3.9}$$

and gauge transformations $\delta g_1 = d\rho_0 - k\lambda_1$, $\delta f_2 = d\lambda_1$. On a manifold with boundary the gauge variation produces a boundary term. It is not uniquely determined because we have the freedom to add a boundary term proportional to $g_1 \wedge f_2$, but independently of this choice the boundary gauge variation cannot be cancelled unless we couple the 4d theory to a 3d theory of edge modes.

Consider first the case $k = 0$. The gauge variation of (8.3.9) produces the boundary term

$$\delta\mathcal{P} = -\frac{i}{2\pi} \int_{\partial X_4} d\rho_0 \wedge f_2 \,. \tag{8.3.10}$$

In order to cancel it, we place on the boundary the following 3d TQFT of edge modes with a 1-form symmetry coupled to f_2, whose boundary value we regard as a background field:

$$\mathcal{Z} = \frac{i}{2\pi} \int_{\partial X_4} \Big[b_1 \wedge dA_1 - b_1 \wedge f_2 \Big] \,. \tag{8.3.11}$$

The 1-form symmetry acts on the lines $e^{in\int A_1}$ and shifts $\delta A_1 = \lambda_1$. The gauge variation (8.3.10) is canceled by imposing the gluing condition $g_1|_\partial = -b_1$, which is the boundary EOM from the variation of the total action with respect to f_2, and is compatible with the gauge transformation $\delta b_1 = -d\rho_0$. Turning off the background f_2, we recognize (8.3.11) as the SymTFT for a 2d theory with $U(1)$ 0-form symmetry.

There is another boundary theory we can use. It is more cleanly presented if we first add to (8.3.9) the boundary term $g_1 \wedge f_2$, which is equivalent to recasting the Anomaly Polynomial TFT as $\mathcal{P}' = \frac{i}{2\pi} \int_{X_4} dg_1 \wedge f_2$. The gauge variation produces the boundary term

$$\delta\mathcal{P}' = \frac{i}{2\pi} \int_{\partial X_4} g_1 \wedge d\lambda_1 \,. \tag{8.3.12}$$

Now we regard the boundary value of g_1 as a background field for the 0-form symmetry of a boundary 3d TQFT:

$$\mathcal{Z}' = \frac{i}{2\pi} \int_{\partial X_4} \Big[h_2 \wedge d\Theta_0 - h_2 \wedge g_1 \Big] \,. \tag{8.3.13}$$

The 0-form symmetry shifts $\delta\Theta_0 = \rho_0$, and the variation (8.3.12) is canceled by imposing the gluing condition $f_2|_\partial = h_2$, which follows for the variation of the total action with respect to g_1 and is compatible with the gauge transformation $\delta h_2 = d\lambda_1$. We recognize (8.3.13) as the SymTFT for a $U(1)^{(-1)}$ symmetry [220], related in 2d by dynamical gauging to the $U(1)^{(0)}$ symmetry. It corresponds to a shift of the theta angle $\frac{\theta}{2\pi} \int_{X_2} \mathcal{F}$ for the dynamical $U(1)$ gauge field. We observe indeed that turning off background fields, (8.3.13) is obtained from (8.3.11) by using the map (8.3.8) that implements dynamical gauging in the SymTFT.

In the case with anomaly $k \neq 0$, a gauge variation produces the boundary term

$$\delta\mathcal{P} = \frac{i}{2\pi} \int_{\partial X_4} \left[-\big(d\rho_0 - k\lambda_1 \big) \wedge f_2 + \frac{k}{2} \, \lambda_1 \wedge d\lambda_1 \right] . \tag{8.3.14}$$

It is canceled by the following modification of (8.3.11):

$$\mathcal{Z} = \frac{i}{2\pi} \int_\partial \left[b_1 \wedge dA_1 + \frac{k}{2} A_1 \wedge dA_1 - \big(b_1 + kA_1 \big) \wedge f_2 \right] . \tag{8.3.15}$$

The current in parenthesis that multiplies the background field f_2 is $\partial\mathcal{Z}/\partial(dA_1)$. One uses the transformations $\delta A_1 = \lambda_1$, $\delta b_1 = -d\rho_0$ as well as the gluing condition $g_1|_\partial = -b_1 - kA_1$ that follows from varying the total action with respect to f_2. With the background field f_2 off, we recognize (8.3.15) as the SymTFT (8.2.1) for a $U(1)^{(0)}$ symmetry with chiral anomaly k in 2d. It turns out that the other boundary theory (8.3.13) for $k = 0$ cannot be modified in any way to cancel the gauge variation once $k \neq 0$, hence becoming an inconsistent boundary condition. This is a signal of the 't Hooft anomaly, from the Anomaly Polynomial TFT viewpoint.

There is an analogous story for $U(1)$ symmetries in 4d, for which the Anomaly Polynomial TFT takes the form:

$$\mathcal{P} = \frac{i}{2\pi} \int_{X_6} \left[g_3 \wedge df_2 - \frac{k}{12\pi}\, f_2 \wedge f_2 \wedge f_2 \right], \tag{8.3.16}$$

with transformations $\delta g_3 = d\rho_2 + \frac{k}{2\pi}\lambda_1 \wedge f_2 + \frac{k}{4\pi}\lambda_1 \wedge d\lambda_1$ and $\delta f_2 = d\lambda_1$. For $k = 0$ there are two possible boundary theories, which are the SymTFTs for a 0-form and 1-form $U(1)$ symmetries, respectively, related by dynamical gauging. For $k \neq 0$, instead, only the first one is consistent and it takes the form:

$$\mathcal{Z} = \frac{i}{2\pi} \int_{\partial X_6} \left[b_3 \wedge dA_1 + \frac{k}{12\pi}\, A_1 \wedge dA_1 \wedge dA_1 - \left(b_3 + \frac{k}{4\pi}\, A_1 \wedge dA_1 - \frac{k}{4\pi} A_1 \wedge f_2 \right) \wedge f_2 \right] \tag{8.3.17}$$

where the terms in parenthesis describe the coupling of the 1-form symmetry to the background f_2 and correspond to $\partial\mathcal{Z}/\partial(dA_1)$. One finds the gluing condition $g_3|_\partial = -b_3 - \frac{k}{4\pi} A_1 \wedge dA_1 + \frac{k}{2\pi} A_1 \wedge f_2$ from the variation of the total action with respect to f_2, compatible with the transformations $\delta A_1 = \lambda_1$, $\delta b_3 = -d\big(\rho_2 - \frac{k}{4\pi}\lambda_1 \wedge A_1\big)$.

8.4 More examples

We present here two other examples that can be derived by dynamically gauging a $U(1)$ 0-form symmetry.

8.4.1 Abelian 2-group symmetry in 4d

A 0-form and a 1-form symmetry can combine into one algebraic structure known as a 2-group [11, 14–16]. Four-dimensional theories with a continuous Abelian 2-group symmetry were discussed in [16]. Consider a 4d Abelian gauge theory coupled to chiral fermions. The photon has a magnetic 1-form symmetry $U(1)^{(1)}$ whose current is $J_2 = *\,(\mathcal{F}/2\pi)$ written in terms of the dynamical field strength $\mathcal{F}$. This current is topologically conserved: $d * J_2 = d\mathcal{F}/2\pi = 0$. The chiral fermions transform under a 0-form symmetry $U(1)^{(0)}$ with current J_1. Suppose that there is a gauge-flavor-flavor triangle anomaly, so that $d * J_1 = \frac{k}{4\pi^2}\,(dA_1) \wedge \mathcal{F}$ where A_1 is the background gauge field coupled to J_1. It is easy to arrange the charges such that there are no other anomalies, for instance:

$$\begin{array}{rcccc} \text{Weyl fermions}: & \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \text{gauge charges } q_i: & 1 & -1 & 1 & -1 \\ \text{flavor charges } f_i: & 2 & 1 & -2 & -1 \end{array} \tag{8.4.1}$$

One checks that $\sum q_i^3 = \sum q_i^2 f_i = \sum f_i^3 = 0$, while $\sum q_i f_i^2 \equiv 2k = 6$ in this example [5]. The theory has therefore a modified conservation equation:

$$d\,{*J_1} - \frac{k}{2\pi}\,(dA_1) \wedge {*J_2} = 0\,, \qquad d\,{*J_2} = 0\,. \tag{8.4.2}$$

[5]The minimal mixed $U(1)$ Chern-Simons terms that are well defined are $\frac{1}{4\pi^2} A\, dB\, dB$ or $\frac{1}{8\pi^2}(A\, dB\, dB + B\, dA\, dA)$.

This type of symmetry is called a 2-group symmetry. If we couple this symmetry to $U(1)$ background gauge fields A_1 and C_2 through the Lagrangian terms

$$\mathcal{L}_4 \supset (*J_1) \wedge A_1 + (*J_2) \wedge C_2 \,, \tag{8.4.3}$$

the modified conservation equations imply modified gauge transformations for the backgroud fields:

$$\delta A_1 = d\lambda_0 \,, \qquad \delta h_2 = d\xi_1 \,, \qquad \delta b_3 = d\gamma_2 - \frac{k}{2\pi} d\xi_1 \wedge A_1 \,, \qquad \delta C_2 = d\eta_1 + \frac{k}{2\pi} d\lambda_0 \wedge A_1 \,. \tag{8.4.4}$$

Here we also included the conjugated $\mathbb{R}$ gauge fields b_3 and h_2 necessary to write a 5d Lagrangian. A gauge-invariant 5d SymTFT action we can write is:

$$\mathcal{Z} = \frac{i}{2\pi} \int \left[b_3 \wedge dA_1 + h_2 \wedge dC_2 + \frac{k}{2\pi} h_2 \wedge A_1 \wedge dA_1 \right]. \tag{8.4.5}$$

This is the SymTFT for a 2-group symmetry.

Indeed we can derive this theory from the SymTFT of the free-fermion theory by the procedure of gauging the $U(1)$. The SymTFT of the free-fermion theory is

$$\mathcal{Z} = \frac{i}{2\pi} \int \left[b_3 \wedge dA_1 + f_3 \wedge dG_1 + \frac{k}{2\pi} G_1 \wedge dA_1 \wedge dA_1 \right] \tag{8.4.6}$$

where A_1 and G_1 are the $U(1)$ background fields for the flavor symmetry and the to-be gauge symmetry, respectively. Gauging the symmetry is implemented by the replacement $dG_1 \mapsto h_2$ and $f_3 \mapsto dC_2$, as in (8.3.8), which indeed produces the action in (8.4.5).

8.4.2 $\mathbb{Q}/\mathbb{Z}$ non-invertible symmetry in 4d

Using the building blocks introduced so far, we can derive the SymTFT describing the non-invertible chiral symmetry of QED-like theories [28, 29], that we reviewd in 3.5. We start from the 5d SymTFT for two 0-form symmetries $U(1)_{\text{A}} \times U(1)_{\text{V}}$ with a mixed AVV anomaly and a pure AAA anomaly:

$$\mathcal{Z} = \frac{i}{2\pi} \int_{X_5} \left[b_3 \wedge dA_1 + c_3 \wedge dV_1 + \frac{l}{4\pi} A_1 \wedge dV_1 \wedge dV_1 + \frac{k}{12\pi} A_1 \wedge dA_1 \wedge dA_1 \right]. \tag{8.4.7}$$

We assumed that the VAA triangle anomaly vanishes, therefore l must be even. For $l = k = 2$ this can be thought of as, for instance, the SymTFT for a Dirac fermion in four dimensions.

The symmetry $U(1)_{\text{V}}$ has no pure anomaly, hence it can be gauged dynamically by coupling it to a photon. This is implemented in the SymTFT as explained in Section 8.3, and the net result is to replace $dV_1 \mapsto f_2$ and $c_3 \mapsto dG_2$. We obtain:

$$\mathcal{Z} = \frac{1}{2\pi} \int_{X_5} \left[b_3 \wedge dA_1 + f_2 \wedge dG_2 + \frac{l}{4\pi} A_1 \wedge f_2 \wedge f_2 + \frac{k}{12\pi} A_1 \wedge dA_1 \wedge dA_1 \right]. \tag{8.4.8}$$

We propose that this is the SymTFT for the non-invertible $\mathbb{Q}/\mathbb{Z}$ chiral symmetry in 4d.

Let us study this theory more carefully. For simplicity we set $l = 2$ (the generalization to other even values of l being straightforward). While the gauge transformations of b_3 and G_2 are standard, the presence of the non-derivative term $A_1 \wedge f_2 \wedge f_2$ forces the gauge transformations of f_2 and A_1 to also act on b_3 and G_2:

$$\delta A_1 = d\rho_0 \,, \quad \delta f_2 = d\lambda_1 \,, \,, \quad \delta b_3 = -\tfrac{2}{4\pi} \lambda_1 \wedge d\lambda_1 - \tfrac{2}{2\pi} \lambda_1 \wedge f_2 \,, \quad \delta G_2 = -\tfrac{2}{2\pi} \rho_0 \left(f_2 + d\lambda_1 \right) - \tfrac{2}{2\pi} \lambda_1 \wedge A_1 \,. \tag{8.4.9}$$

Keeping this into account, we analyse the operator content of the theory. First we have

$$V_\alpha(\gamma_2) = e^{i\alpha\int_{\gamma_2} f_2}\,, \qquad W_n(\gamma_1) = e^{in\int_{\gamma_1} A_1} \tag{8.4.10}$$

which are both topological and gauge invariant because of the EOMs $dA_1 = 0$ and $df_2 = 0$. On the other hand, the integrals of b_3 and G_2 do not lead to gauge-invariant operators because of (8.4.9).

We can try to construct non-genuine operators:

$$\widetilde{U}_\alpha(\gamma_3, D_4) = \exp\left(i\alpha\int_{\gamma_3} b_3 + i\,\frac{2\alpha}{4\pi}\int_{D_4} f_2\wedge f_2\right), \quad \widetilde{\mathcal{T}}_n(\gamma_2, D_3) \;= \exp\left(in\int_{\gamma_2} G_2 + i\,\frac{2n}{2\pi}\int_{D_3} A_1\wedge f_2\right). \tag{8.4.11}$$

They depend on the open regions D_4 and D_3, whose boundaries are γ_3 and γ_2, respectively. They are gauge invariant and topological because of the EOMs. On a boundary condition which is Dirichlet for A_1 and G_2, that correspond to the QED-like theory, the boundary values of A_1 and G_2 play the role of backgrounds fields for the would-be axial $U(1)_{\rm A}$ and the magnetic $U(1)^{(1)}$ symmetry, respectively. The endlines of $\widetilde{\mathcal{T}}_n$ would seem to be 't Hooft lines, charged under the operators V_α, while the $\widetilde{U}_\alpha$ lying on the boundary would seem to generate the axial symmetry, whose charged operators are the endpoints of W_n. However this conclusion is not correct. In the definition (8.4.11), γ_3 and γ_2 are boundaries and hence homologically trivial. Since the operators are topological, they are essentially trivial and cannot be used to define boundary conditions. Moreover, differently from other cases considered above, the non-genuine operators $\widetilde{U}_\alpha$ do not become genuine even on the boundary, since f_2 is not set to zero there. One can however do better.

Consider $\widetilde{\mathcal{T}}_n$ first. The bulk term $\frac{2n}{2\pi}A_1\wedge f_2$ can be thought of as the inflow action for a 2d pure $\mathbb{Z}_{2n}$ gauge theory, where A_1 and f_2 are viewed as the background fields for the $\mathbb{Z}_{2n}$ 0-form and 1-form symmetry, respectively. This implies that if we take D_3 to be a tube $\gamma_2\times[0,1]$, we can place $\exp\bigl(in\int_{\gamma_2} G_2\bigr)$ on one end of the tube, and the 2d $\mathbb{Z}_{2n}$ gauge theory coupled to A_1 and f_2 on the other end. Then we shrink the tube, so as to define a genuine 2d operator:

$$\mathcal{T}_n(\gamma_2) = \widetilde{\mathcal{T}}_n(\gamma_2)\;\mathbb{Z}_{2n}[\gamma_2; A_1, f_2]\,. \tag{8.4.12}$$

We can perform a similar operation with $\widetilde{U}_\alpha$. Since α is a continuous parameter, we should be careful. Indeed a 3d TQFT on which $\frac{2\alpha}{4\pi}\int_{D_4} f_2\wedge f_2$ can topologically terminate only exists if $2\alpha = p/q \in \mathbb{Q}$ (we take $\gcd(p,q)=1$) [6]. This is the minimal TQFT with 1-form symmetry $\mathbb{Z}_q$ and anomaly p introduced in [111] and denoted by $\mathcal{A}^{q,p}$. By coupling its 1-form symmetry to f_2 we are able to define a genuine 3-dimensional topological operator

$$U_{p/2q}(\gamma_3) = \widetilde{U}_{\alpha=p/2q}(\gamma_3)\;\mathcal{A}^{q,p}[\gamma_3; f_2]\,. \tag{8.4.13}$$

The operators at irrational α, on the other hand, remain non-genuine.

The procedure we illustrated is the bulk analogue of [28, 29], with the novelty that not only the generator of the chiral symmetry, but also the 2-dimensional operators $\mathcal{T}_n$ whose endlines are the 't Hooft lines, require dressing with a TQFT in order to be well defined. This fact is harder to deduce directly from the boundary because the 't Hooft lines are not topological. One of the advantages of the SymTFT description is that even the topological aspects of non-topological charged objects can be derived from properties of the topological operators in the bulk. Our result is an example of this phenomenon.

[6] For $\alpha = 1/2$ we can simply use the $U(1)_1$ trivial spin-Chern-Simons theory. Indeed this is the only value that corresponds to an invertible symmetry, namely $(-1)^F$.

The necessity of dressing even the 't Hooft lines is not surprising, since the non-invertibility of the $\mathbb{Q}/\mathbb{Z}$ symmetry is precisely encoded in the action on 't Hooft lines. When a line crosses the symmetry defect it emerges with a 2-dimensional topological operator attached [28, 221]. It would be interesting to derive this action from the bulk, studying in detail the braiding and crossing of the various operators we introduced.

Finally let us mention that $U_{p/2q}(\gamma_3)$ can be further dressed with an other genuine 3-dimensional operator corresponding to a Chern-Simons term for A_1, as we did in Section 8.2.3:

$$U_{(p/2q,m)}(\gamma_3) = U_{p/2q}(\gamma_3) \times \exp\left(i\frac{m}{4\pi}\int_{\gamma_3} A_1 \wedge dA_1 \right) . \tag{8.4.14}$$

Studying the triple linking of these operators allows one to characterize the anomaly of the non-invertible chiral symmetry, for which the term $A_1 \wedge dA_1 \wedge dA_1$ in (8.4.8) is responsible.

8.5 Non-Abelian symmetries

Let us make some comments on the extension to non-Abelian symmetries (this has been further discussed in [222]). A natural candidate for the SymTFT is the non-Abelian BF theory of [83]:

$$\mathcal{Z} = \frac{i}{2\pi}\int_{X_{d+1}} \mathrm{Tr}_{\mathfrak{g}}\big(b_{d-1} \wedge F_2\big) . \tag{8.5.1}$$

Here the fundamental fields are a standard gauge field A for the Lie group G, and a collection b_{d-1} of as many $\mathbb{R}$ $(d-1)$-form gauge fields as $\dim(\mathfrak{g})$ (where $\mathfrak{g}$ is the Lie algebra of G) that transform in the adjoint representation of G. In other words, b_{d-1} is a section of the adjoint bundle. Then $F_2 = dA - iA \wedge A$ is the non-Abelian field strength of A, and $\mathrm{Tr}_{\mathfrak{g}}$ is the Killing form on $\mathfrak{g}$ [7]. Thus the theory has two sets of gauge transformations:

$$A \mapsto \Lambda\, A\, \Lambda^{-1} + i\, d\Lambda\, \Lambda^{-1} , \qquad\qquad b_{d-1} \mapsto \Lambda\, b_{d-1}\, \Lambda^{-1} \tag{8.5.2}$$

as well as

$$b_{d-1} \mapsto b_{d-1} + D_A \lambda_{d-2} . \tag{8.5.3}$$

Here λ_{d-2} is a globally-defined section of the adjoint bundle defined by A, while $D_A = d + i[A, \cdot\,]_\pm$ is the covariant derivative that acts on p-forms valued in the Lie algebra as

$$D_A \eta_p = d\eta_p + i\big(A \wedge \eta_p - (-1)^p\, \eta_p \wedge A\big) . \tag{8.5.4}$$

The EOMs are $F_2 = 0$ and $Db_{d-1} = 0$, therefore the theory has topological Wilson line operators

$$W_{\mathfrak{R}}(\gamma_1) = \mathrm{Tr}_{\mathfrak{R}}\ \mathrm{Pexp}\big(i \textstyle\int_{\gamma_1} A\big) \tag{8.5.5}$$

where $\mathfrak{R}$ are representations of G. We expect that there exists a boundary condition in which these lines can end on the boundary, and describe local operators transforming in representations $\mathfrak{R}$ of the symmetry G. But what about the symmetry generators of G? On the boundary they must be of co-dimension one and labeled by G, thus they cannot simply be $(d-1)$-dimensional operators of the bulk pushed at the boundary: they would be co-dimension two operators, hence they cannot have a non-Abelian fusion rule.[8] In the bulk, there exist co-dimension two operators, the Gukov-Witten

[7] In fact, as spelled out in [83], one can use a b_{d-1} that takes values in the dual to $\mathfrak{g}$, so that a trace is not used and $\mathfrak{g}$ does not need to be semisimple.

[8] This issue arises whenever the topological defects of a symmetry are, in some sense, charged among themselves. A similar story happens for Chern-Simons theory, see [208].

$U_{[g]}(\Sigma_{d-1})$ operators. These are defined by specifying the holonomy of A on a circle that links with Σ_{d-1}. However, since only the conjugacy class of the holonomy is gauge invariant, these operators are labeled not by G, but by its set of conjugacy classes. One possibility, explored in [222], is that the Gukov-Witten operators pushed at the boundary are not simple defects: they become sums over all possible operators labelled by group element in the given conjugacy class. Hence the boundary theory has more $(d-1)$-dimensional operators then the bulk. A related, but slightly different approach consists in regarding the symmetry generators of the boundary as arising from *non-genuine* operators of the bulk, labeled by $g \in G$. In other words, one can define Gukov-Witten operators labeled by group elements, and make them gauge invariant by attaching a d-dimensional topological surface to it.

Chapter 9

Holographic duals of symmetry broken phases

In this chapter we explore a novel interpretation of SymTFTs as theories of gravity, proposing a holographic duality where the bulk SymTFT (with the gauging of a suitable Lagrangian algebra) is dual to the universal effective field theory (EFT) that describes spontaneous symmetry breaking on the boundary. We test this conjecture in various dimensions and with many examples involving different continuous symmetry structures, including non-Abelian and non-invertible symmetries, as well as higher groups. For instance, we find that many Abelian SymTFTs are dual to free theories of Goldstone bosons or generalized Maxwell fields, while non-Abelian SymTFTs relate to non-linear sigma models with target spaces defined by the symmetry groups. We also extend our analysis to include the non-invertible $\mathbb{Q}/\mathbb{Z}$ axial symmetry, finding it to be dual to axion-Maxwell theory, and a non-Abelian 2-group structure in four dimensions, deriving a new parity-violating interaction that has implications for the low-energy dynamics of $U(N)$ QCD.

9.1 Topological field theories as holographic duals

A profound insight by E. Witten is that Topological Quantum Field Theories (TQFTs), due to their general covariance, can be seen as theories of quantum gravity [223]. Unlike in more conventional examples, general covariance is not achieved by integrating over metrics but rather by not introducing them at all. Consequently, these theories lack any semiclassical description involving weakly interacting gravitons. In traditional gravitational theories, one selects a background metric and expands around it, thereby breaking general covariance spontaneously. Therefore, TQFTs can be viewed as theories of quantum gravity with unbroken general covariance — where gravitons are, in a certain sense, confined.

This old story requires some important refinements. A full quantum-gravity theory should not depend on the background topology. TQFTs, on the other hand, are sensitive to spacetime topology through their global symmetries. One way to achieve such an independence is to sum over all topologies, which can be done in low dimensions [224–228]. Alternatively, one can use TQFTs that do not even depend on topology [133], hence that are free of global symmetries and then trivial (or invertible) [58, 141]. These can be obtained by gauging a maximal non-anomalous set of topological defects, namely a *Lagrangian algebra*, in a nontrivial TQFT.

TQFTs with Lagrangian algebras also admit topological boundary conditions and can be regarded SymTFTs $\mathcal{Z}(\mathcal{C})$ for some symmetry structure $\mathcal{C}$. In this chapter we focus of continuous symmetries, for which the SymTFTs have been discussed in chapter 8[1]. After the inclusion of continuoys symmetries in the game, the picture is that for *any* internal symmetry structure $\mathcal{C}$ in d-dimensions one can canonically associate a $(d+1)$-dimensional TQFT, its SymTFT $\mathcal{Z}(\mathcal{C})$.

Our aim here is to give a different interpretation to these TQFTs $\mathcal{Z}(\mathcal{C})$, not as SymTFTs but as theories of gravity. More precisely, we want to establish holographic dualities in which the bulk theory is a SymTFT. The main proposal of this chapter is the following:

- Thought of as a theory of gravity, the SymTFT $\mathcal{Z}(\mathcal{C})$ for a symmetry $\mathcal{C}$ is the holographic dual to the universal effective field theory (EFT) that describes the spontaneous breaking of $\mathcal{C}$.

It is a general principle of quantum field theory that any theory with a certain continuous global symmetry that is spontaneously broken, in the far infrared (IR) flows to the same universal theory of Goldstone bosons [230, 231]. This is roughly speaking always a sigma model, although the target space can be infinite dimensional (*e.g.*, it is the classifying space B^pG in the case of higher-form symmetries).[2] As for the SymTFT, this EFT is also canonically determined by the symmetry $\mathcal{C}$ without any further information. For this reason, it is natural to expect that, even though they appear to be completely different objects — a $(d+1)$-dimensional TQFT and a d-dimensional EFT — the two can be somehow related as they both have the same input datum. We will prove by means of many examples that this correspondence is holography.

A crucial part of the story is the proper choice of boundary conditions. These will be non-topological and of the Dirichlet type for some combination of the bulk fields. Since bulk fields are gauge fields A, these boundary conditions break some gauge invariance, making it a global symmetry of the boundary theory. This agrees with the general principle in holography that boundary global symmetries correspond to bulk gauge fields. The non-triviality of the system really comes from the boundary conditions that, being non-topological, generate dynamics on the boundary. The boundary theory can be thought of as a theory of edge modes. Our setup has several similarities with, and may be understood as a generalization of, the Chern–Simons/WZW correspondence [47, 48] and its reinterpretation as a full-flagged holographic duality by means of bulk anyon condensation [133].

To explain the basic setup, consider the simple example of the SymTFT for a $U(1)$ p-form symmetry in d-dimensions studied in chapter 8

$$S = \frac{i}{2\pi}\int_{X_{d+1}} b_{d-p-1}\wedge dA_{p+1} \tag{9.1.1}$$

In SymTFT, (9.1.1) is placed on a slab with two boundaries, one of which is topological and determines the symmetry after the slab is squeezed. This topological boundary is characterized by a maximal set of mutually transparent objects, which we generically refer to as a Lagrangian algebra $\mathcal{L}$. In this example a natural Lagrangian algebra consists of all $V_n(\gamma_{p+1})$, while the $U_\beta(\gamma_{d-p-1})$ become the generators of the $U(1)$ p-form symmetry of the boundary theory.

In this chapter, instead, we consider a different setting in which (9.1.1) is placed on a manifold X_{d+1} with a unique connected boundary $\mathcal{M}_d = \partial X_{d+1}$, which we endow with a Riemannian structure. On $\mathcal{M}_d$ we fix non-topological boundary conditions[3]

$$A_{p+1} + iC\star b_{d-p-1} = \mathcal{A}_{p+1}\,. \tag{9.1.2}$$

[1] See also [117, 222] and [229] for a different proposal involving non-topological theories.

[2] It is not clear to us how to make this precise for non-invertible symmetries, for instance for the $\mathbb{Q}/\mathbb{Z}$ chiral symmetry discovered in [28, 29].

[3] Such boundary conditions in BF theory, and the edge modes they lead to, have recently been studied in [232].

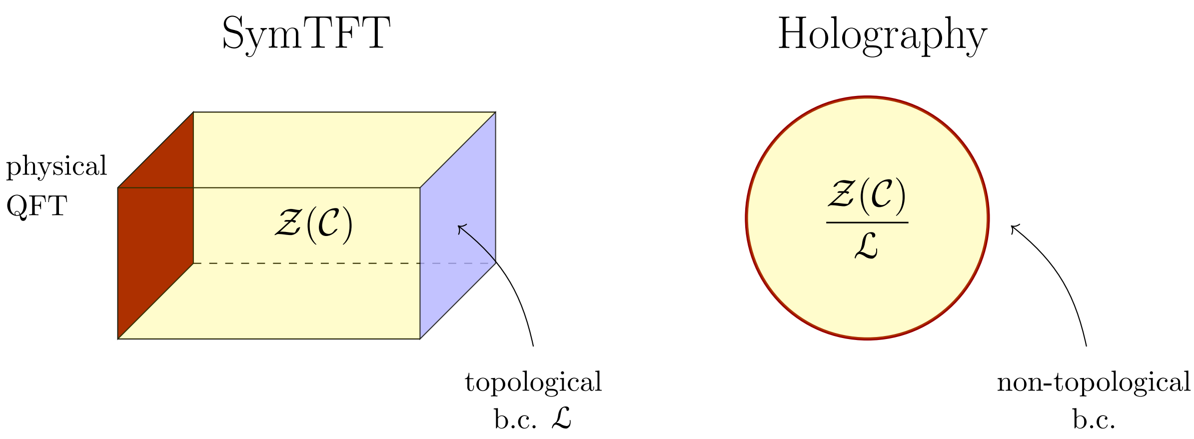


Figure 9.1: Left: the SymTFT setup. The TQFT is placed on a slab, whose right boundary is topological and determined by a Lagrangian algebra $\mathcal{L}$. Right: the holographic setup considered here. There is only one boundary with non-topological boundary conditions, while the Lagrangian algebra $\mathcal{L}$ is gauged to make the bulk invertible.

Here $\star$ is the Hodge star operator of the boundary, $\mathcal{A}_{p+1}$ is a fixed $(p+1)$-form on the boundary, and C is a generically dimensionful constant with mass dimension $[C] = 2p + 2 - d$.[4] Moreover, the Lagrangian algebra $\mathcal{L}$ that was used to define the topological boundary in the SymTFT setup must now be gauged in the bulk X_{d+1}, and the final bulk theory $\mathcal{Z}(\mathcal{C})/\mathcal{L}$ is an invertible TQFT. See Fig. 9.1 for a comparison of the two setups.

In this second setup we want to establish a precise holographic duality with a certain local QFT$_d$ living on the boundary, which we need to determine. More precisely, the equality we need to show is the standard one [123, 233, 234]:

$$Z_{\text{TQFT}_{d+1}}\big[\varphi\big|_\partial = \mathcal{A}\big] \;=\; Z_{\text{QFT}_d}[\mathcal{M}_d, \mathcal{A}]\,. \tag{9.1.3}$$

Here TQFT$_{d+1}$ is the result of gauging $\mathcal{L}$ in $\mathcal{Z}(\mathcal{C})$, φ denotes generically some bulk fields (for instance $\varphi = A_{p+1} + iC \star b_{d-p-1}$ in the example (9.1.1)), while $\mathcal{A}$ is introduced as a boundary value from the bulk viewpoint and plays the role of a background field for the boundary QFT. Although SymTFT superficially resembles holography, the two are fundamentally different. SymTFT only captures symmetries and disregards dynamics, allowing any QFT with the specified symmetry. In contrast, in holography the dual QFT$_d$ is uniquely determined by the bulk theory and its boundary conditions, encoding both symmetries and dynamics as in (9.1.3).[5]

We will determine the dual QFT$_d$ explicitly in the many examples considered below, providing strong evidence for the conjecture that the dual theory to $\mathcal{Z}(\mathcal{C})/\mathcal{L}$ is always the symmetry-breaking EFT for $\mathcal{C}$. Some of these checks are quite subtle and highly nontrivial. For instance, the Goldstone theory for a $U(1)$ symmetry with a cubic 't Hooft anomaly in 4d is still a compact boson with no additional terms as in the non-anomalous case,[6] but the background field for the symmetry is coupled non-minimally to the theory. We discuss this in Section 9.3.3 (in particular (9.3.20) is the additional coupling) to which we refer for more details. The SymTFT for a 4d anomalous $U(1)$ is

$$S = \frac{i}{2\pi}\int_{X_5} b_3 \wedge dA_1 + \frac{ik}{24\pi^2}\int_{X_5} A_1 \wedge dA_1 \wedge dA_1\,. \tag{9.1.4}$$

[4]The introduction of such a scale is necessary since the components of A_{p+1} have dimension $p+1$ while those of b_{d-p-1} have dimension $d-p-1$. In this way the forms A_{p+1} and b_{d-p-1} are dimensionless, the action in (9.1.1) is dimensionless, but $\star b_{d-p-1}$ has dimension $d-2p-2$.

[5]See [235] for a general description of symmetry operators in holography.

[6]This is different from the non-Abelian case, in which an anomaly implies a WZW term in the sigma model.

Forgetting about the boundary value $\mathcal{A}_1$ appearing in the boundary condition (9.1.2), the additional cubic term does not affect the dual boundary QFT_4. However we will show in Section 9.3.2 that keeping track of $\mathcal{A}_1$ we reproduce exactly the non-minimal coupling expected for an anomalous $U(1)$.

Before moving to the various examples, let us clarify a conceptual point. The assertion that certain dynamical QFTs have a TQFT as holographic dual might be perplexing at first. The origin of the confusion is that, even though TQFTs are good theories of gravity, the non-appearance of a metric tensor $g_{\mu\nu}$ is puzzling for holography: the metric should be dual to the stress-energy tensor $T_{\mu\nu}$ of the boundary QFT. While this observation is in general correct, in a few special cases it might have a loophole: the stress tensor might not be an independent operator. For instance, this is the case in the CS/WZW correspondence [47, 48]. In 2d WZW models the stress tensor of the CFT, using the Sugawara construction, is made out of the currents which are dual to the gauge fields of the 3d Chern–Simons bulk theory. Something very similar happens in our examples. Indeed, the EFTs for symmetry breaking are very special QFTs in which everything, including the stress-energy tensor, is determined by the currents and their correlation functions. This is at the core of the *universality* of those EFTs. For instance, in the theory of a $U(1)$ Goldstone boson with action

$$S = \frac{R^2}{4\pi} \int_{\mathcal{M}_d} d\Phi \wedge \star\, d\Phi\,, \tag{9.1.5}$$

the $U(1)$ current is $J_\mu = \frac{iR^2}{2\pi} \partial_\mu \Phi$ and the stress tensor is a composite operator of J_μ:

$$T_{\mu\nu} = \frac{R^2}{4\pi} \left(\partial_\mu \Phi\, \partial_\nu \Phi - \frac{1}{2}\, \delta_{\mu\nu} \, (\partial\Phi)^2 \right) = \frac{\pi}{R^2} \left(\frac{1}{2}\, \delta_{\mu\nu}\, J^2 - J_\mu J_\nu \right). \tag{9.1.6}$$

Through the boundary conditions, the bulk SymTFT provides background fields for the global symmetries of the boundary theory, which are sources for the boundary currents. Hence the TQFT can compute correlation functions of the currents, and by universality correlation functions of all operators, including those of the stress tensor, even without an explicit source $g_{\mu\nu}$. This is a general statement: in the EFTs for spontaneous breaking the currents completely determine all operators and the holographic duals do not need a graviton field.

It is expected, however, that embedding our models into RG flows and taking into account non-universal features would require to reintroduce dynamical gravity into the game. Indeed, a related observation is that the boundary theories we obtain are either free or non-renormalizable. The reason why a TQFT, which is expected to be UV complete and finite, can be dual to a non-renormalizable theory is the choice of non-topological boundary conditions, which introduce an energy scale in the theory. This scale sets a limit below which both the bulk and boundary theories are well defined. Above this threshold, the boundary theory requires the inclusion of more and more operators to tame UV divergencies. This issue has to carry over to the bulk TQFT as well — albeit in a way unclear to us — making the TQFT description incomplete. The expectation is that, to make sense of the bulk theory above the scale of the boundary condition, one has to allow for dynamical gravity in the bulk in a way that is similar to the embedding of an EFT for spontaneous breaking into a UV complete theory. It would be interesting to understand this point better.

9.2 $U(1)$ Goldstone bosons

The simplest cases to test our conjecture are those of $U(1)$ symmetries of generic degree. We warm up with the textbook example of a spontaneously broken $U(1)$ 0-form symmetry in generic dimension and then move on to the case of higher-form symmetries, whose Goldstone bosons are (free) $U(1)$ higher-form gauge fields [11].

9.2.1 0-form symmetries

Consider the following TQFT in $d+1$ dimensions

$$S = \frac{i}{2\pi} \int_{X_{d+1}} b_{d-1} \wedge dA_1 \,, \tag{9.2.1}$$

where A_1 is a $U(1)$ gauge field while b_{d-1} is an $\mathbb{R}$ $(d-1)$-form gauge field. We endow the boundary $\mathcal{M}_d = \partial X_{d+1}$ with a Riemmanian metric and impose the boundary condition

$$\star A_1 = -\frac{i}{R^2}\, b_{d-1} + \star \mathcal{A}_1 \,. \tag{9.2.2}$$

Here R is a parameter of mass dimension $(d-2)/2$, while $\mathcal{A}_1$ is a fixed background 1-form on $\mathcal{M}_d$. Notice that only in $d=2$ this boundary condition is conformally invariant. In order to get a consistent variational principle with this boundary condition we must add a boundary term S_∂ to (9.2.1). Indeed, the variation of the action produces a boundary piece

$$\delta S\big|_{\mathcal{M}_d} = (-1)^{d-1}\, \frac{i}{2\pi} \int_{\mathcal{M}_d} b_{d-1} \wedge \delta A_1 = \frac{1}{2\pi R^2} \int_{\mathcal{M}_d} b_{d-1} \wedge \star\, \delta b_{d-1} \,, \tag{9.2.3}$$

which requires a boundary term

$$S_\partial = -\frac{1}{4\pi R^2} \int_{\mathcal{M}_d} b_{d-1} \wedge \star\, b_{d-1} \,. \tag{9.2.4}$$

Since the boundary condition (9.2.2) breaks gauge invariance on the boundary, we have to be careful in specifying the group of transformations we quotient by in the bulk: we choose to allow only gauge transformations that are trivial on the boundary. This implies that the bulk gauge symmetries become global on the boundary. For any global symmetry we should be able to turn on a background. In our setup this operation has a very natural realization: instead of freezing gauge transformations on the boundary, we allow them but transform the boundary data so as to render the boundary condition invariant. For instance, we can make (9.2.2) gauge invariant under gauge transformations of A_1 by demanding that $A_1 \mapsto A_1 + d\lambda_0$ is accompanied by a transformation of the fixed background $\mathcal{A}_1$:

$$\mathcal{A}_1 \mapsto \mathcal{A}_1 + d\lambda_0 \,. \tag{9.2.5}$$

With this choice, $\mathcal{A}_1$ is interpreted as a background gauge field for the global $U(1)$ symmetry on the boundary. Notice that with our choice of boundary term the whole system is gauge invariant.

We can also restore the gauge transformations $b_{d-1} \mapsto b_{d-1} + d\nu_{d-2}$ by transforming

$$\mathcal{A}_1 \;\mapsto\; \mathcal{A}_1 - (-1)^d \frac{i}{R^2} \star d\nu_{d-2} \,, \tag{9.2.6}$$

which however are not proper background gauge transformations. A clearer and equivalent possibility is to parametrize the boundary condition as

$$\star A_1 = -\frac{i}{R^2} \left(b_{d-1} - \mathcal{B}_{d-1} \right) , \tag{9.2.7}$$

where $\mathcal{B}_{d-1}$ is another fixed background on the boundary that transforms as $\mathcal{B}_{d-1} \mapsto \mathcal{B}_{d-1} + d\nu_{d-2}$. It can be understood as a background field for the global $(d-2)$-form symmetry on the boundary. Yet another possibility is to restore both gauge transformations, for instance through the parametrization

$$\star \left(A_1 - \mathcal{A}_1 \right) = -\frac{i}{R^2} \left(b_{d-1} - \mathcal{B}_{d-1} \right) . \tag{9.2.8}$$

We can use it to discover information about the boundary theory. Indeed, with the choice of boundary term in (9.2.4), the system is not gauge invariant, rather under a gauge transformation we find

$$\delta(S+S_\partial) = (-1)^{d-1}\frac{i}{2\pi}\int_{\mathcal{M}_d} d\nu_{d-2}\wedge\mathcal{A}_1 - \frac{1}{4\pi R^2}\int_{\mathcal{M}_d}\big(2\,d\nu_{d-2}\wedge\star\mathcal{B}_{d-1} + d\nu_{d-2}\wedge\star d\nu_{d-2}\big)\,. \tag{9.2.9}$$

The second piece can be cancelled by modifying the boundary term with the addition of

$$\frac{1}{4\pi R^2}\int_{\mathcal{M}_d}\mathcal{B}_{d-1}\wedge\star\mathcal{B}_{d-1}\,, \tag{9.2.10}$$

that can be understood as a local counterterm. However the first piece in (9.2.9) cannot be removed while preserving background gauge invariance for the $U(1)$ 0-form symmetry. This is a sign that the two symmetries have a mixed 't Hooft anomaly. Indeed, as we are going to see, the theory we are describing is the holographic dual to a d-dimensional compact boson. In what follows we will turn on only the background for the $U(1)$ 0-form symmetry, *i.e.*, we will use the boundary condition (9.2.2).

In order to rewrite the path integral of this TQFT as that of the compact boson we proceed in analogy with [48, 49, 236] (see also [84]). We assume that X_{d+1} contains an S^1 factor parametrized by $t\sim t+\beta$, interpreted as Euclidean time, hence $X_{d+1} = X_d\times S^1$ and $\partial X_{d+1}\equiv\mathcal{M}_d = \mathcal{M}_{d-1}\times S^1$. For simplicity, we also choose the metric of ∂X_{d+1} to be diagonal in $\mathcal{M}_{d-1}$ and S^1 so that

$$\star\, dt = (-1)^{d-1}\,\mathrm{Vol}_{\mathcal{M}_{d-1}}\ \in\ \Omega^{d-1}(\mathcal{M}_{d-1}) \tag{9.2.11}$$

with $\mathrm{Vol}_{\mathcal{M}_{d-1}}$ the volume form of $\mathcal{M}_{d-1}$. We decompose the bulk fields as

$$A_1 = A_0^t\,dt + \widetilde{A}_1\,, \qquad\qquad b_{d-1} = b_{d-2}^t\wedge dt + \widetilde{b}_{d-1}\,, \tag{9.2.12}$$

where forms with a tilde live on the *spatial* manifold X_d. The time components A_0^t and b_{d-2}^t appear linearly and can be treated as Lagrange multipliers. Integrating them out enforces

$$\tilde{d}\widetilde{A}_1 = 0\,, \qquad\qquad \tilde{d}\,\widetilde{b}_{d-1} = 0\,. \tag{9.2.13}$$

We now make a choice for X_d and take it to be a d-dimensional ball so that $\mathcal{M}_d = S^{d-1}\times S^1$. Then (9.2.13) are solved by introducing a compact scalar Φ_0 and a $(d-2)$-form $\mathbb{R}$ gauge field ω_{d-2} as

$$\widetilde{A}_1 = \tilde{d}\Phi_0\,, \qquad\qquad \widetilde{b}_{d-1} = \tilde{d}\omega_{d-2}\,. \tag{9.2.14}$$

Rewriting both the bulk action and the boundary term using Φ_0 and ω_{d-2}, the system reduces to the boundary action

$$\begin{aligned} S = \frac{i}{2\pi}\int_{\mathcal{M}_d}\Big[&(-1)^d\,\tilde{d}\omega_{d-2}\wedge\big(\partial_t\Phi_0 - \mathcal{A}_0^t\big)dt \\ &- \frac{i}{2}\Big(R^2\big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\big)\wedge\star\big(\tilde{d}\Phi_0-\widetilde{\mathcal{A}}_1\big) + \frac{1}{R^2}\,\tilde{d}\omega_{d-2}\wedge\star\,\tilde{d}\omega_{d-2}\Big)\Big]\,. \end{aligned} \tag{9.2.15}$$

This action is not covariant, and time derivatives appear linearly. For $d=2$, the action contains two scalars and is a manifestly self-dual formulation of the compact boson known in the condensed matter literature as the Luttinger liquid Lagrangian (see, *e.g.*, [237] for a recent discussion). It has the advantage of making both $U(1)$ symmetries explicit, at the expense of hiding Lorentz invariance. The action (9.2.15) is a d-dimensional generalization of it and it makes both the 0-form and the $(d-2)$-form $U(1)$ symmetries manifest.

Path integrals with an action linear in time derivatives are interpreted as phase-space path integrals. One can typically obtain a configuration-space path integral by integrating out the momenta that appear quadratically. Indeed, here $\tilde{d}\omega_{d-2}$ is the conjugate momentum to Φ_0 and we can recast the theory in a Lorentz-invariant form by integrating out ω_{d-2}. An important observation is that the action has zero modes that need to be eliminated. One way to see this is via the equations of motion for ω_{d-2}. These are

$$\tilde{d}\left[\left(\partial_t\Phi_0 - \mathcal{A}_0^t\right)dt \,+\, (-1)^d\frac{i}{R^2}\star\tilde{d}\omega_{d-2}\right] = 0 \tag{9.2.16}$$

with solution

$$\tilde{d}\omega_{d-2} = iR^2\left(\partial_t\Phi_0 - \mathcal{A}_0^t\right)\star dt - iR^2\star\tilde{d}\gamma_0\,. \tag{9.2.17}$$

Notice that, since $\left(\partial_t\Phi_0 - \mathcal{A}_0^t\right)\star dt$ is a $(d-1)$-form supported only on space, we have $\tilde{d}\star\tilde{d}\gamma_0 = 0$. The scalar γ_0 is integrated over but its path integral is naively divergent because γ_0 has vanishing action, *i.e.*, it is a zero-mode. Therefore in order to get a consistent theory we have to gauge fix $\gamma_0 = 0$. Plugging $\tilde{d}\omega_{d-2}$ in (9.2.15) we get the final action

$$S = \frac{R^2}{4\pi}\int_{\mathcal{M}_d}\left(d\Phi_0 - \mathcal{A}_1\right)\wedge\star\left(d\Phi_0 - \mathcal{A}_1\right), \tag{9.2.18}$$

corresponding to a d-dimensional compact boson with radius R. Had we integrated out Φ_0 from (9.2.15), we would have found the dual formulation in terms of the $(d-2)$-form ω_{d-2}. The background field $\mathcal{A}_1$ corresponds to the $U(1)$ shift symmetry of the boson and the anomalous shift we discussed above corresponds to the mixed 't Hooft anomaly with the winding symmetry.

One might be puzzled by the fact that we have one bulk gauge symmetry $U(1)$, but we still obtain two global symmetries on the boundary, which might seem to clash with the usual holographic expectations. However, for the compact boson this is not really a contradiction: all correlation functions of one current can be obtained from those of the other. Indeed, the backgrounds of the two symmetries are obtained one from the other using the $\star$ operator (modulo counterterms, which correspond to contact terms in correlators); thus, functional derivatives of the partition function with respect to a single background already contain the information of all correlators of both currents (see [129] for a related discussion).

Before going on, let us mention an alternative, quicker way to arrive at the final result that does not pass through the Luttinger-liquid-like formulation (9.2.15). It requires X_{d+1} to be a ball, and hence $\mathcal{M}_d = S^d$. After determining the boundary conditions (9.2.2) and the boundary term (9.2.4), we just integrate the entire b_{d-1} out, imposing $dA_1 = 0$. Since the bulk is now topologically trivial, this is solved by $A_1 = d\Phi_0$. Using the boundary condition to express the boundary term (9.2.4) in terms of A_1, and plugging back $A_1 = d\Phi_0$, we immediately get (9.2.18).

9.2.2 Higher-form symmetries

The higher-form case is very similar and we only flash the 1-form symmetry example, just to highlight one small subtlety. The TQFT we start with has action

$$S = \frac{i}{2\pi}\int_{X_{d+1}} f_{d-2}\wedge dG_2\,, \tag{9.2.19}$$

with f_{d-2} and G_2 being an $\mathbb{R}$ and $U(1)$ gauge field, respectively. On X_{d+1} with boundary $\mathcal{M}_d$, that we endow with a Riemannian metric (if $d = 4$ a conformal structure is enough) we set the boundary condition (see also [232]):

$$\star G_2 = (-1)^{d+1}\frac{ie^2}{\pi}f_{d-2} + \star\mathcal{G}_2\,, \tag{9.2.20}$$

where $[e^2] = 4 - d$. We must also add a boundary term

$$S_\partial = -\frac{e^2}{4\pi^2} \int_{\partial X_{d+1}} f_{d-2} \wedge \star f_{d-2} \,. \tag{9.2.21}$$

When solving the constraints imposed by the integral over time components as

$$\widetilde{f}_{d-2} = \tilde{d}\omega_{d-3} \,, \qquad\qquad \widetilde{G}_2 = \tilde{d}A_1 \,, \tag{9.2.22}$$

we introduce (time-dependent) forms ω_{d-3} and A_1 only on the spatial manifold X_d, namely without time components. The boundary action one obtains is

$$\begin{aligned} S = \frac{i}{2\pi} \int_{\mathcal{M}_d} \Big[&(-1)^d \, \tilde{d}\omega_{d-3} \wedge \big(\partial_t A_1 + \mathcal{G}_1^t\big) \wedge dt \\ &- \frac{i}{2} \bigg(\frac{e^2}{\pi} \, \tilde{d}\omega_{d-3} \wedge \star \, \tilde{d}\omega_{d-3} + \frac{\pi}{e^2} \big(\tilde{d}A_1 - \widetilde{\mathcal{G}}_2\big) \wedge \star \big(\tilde{d}A_1 - \widetilde{\mathcal{G}}_2\big) \bigg) \Big] \,. \end{aligned} \tag{9.2.23}$$

This is a higher-form generalization of (9.2.15) and integrating out ω_{d-3} we obtain

$$S = \frac{1}{4e^2} \int_{\mathcal{M}_d} \big(dA_1 - \mathcal{B}_2\big) \wedge \star \big(dA_1 - \mathcal{B}_2\big) \,, \tag{9.2.24}$$

where $\mathcal{B}_2 = -\mathcal{G}_1^t \wedge dt + \widetilde{\mathcal{G}}_2$ is a 2-form background field. This is a Maxwell action in d dimensions coupled to a background field $\mathcal{B}_2$ for its electric 1-form symmetry.

The subtlety we want to point out is that A_1 does not have the time component, hence this is a gauge-fixed Maxwell action.[7] There is a gauge choice that arises naturally in this reduction procedure, that is, the temporal gauge. The same story goes through for any higher-form gauge field: the boundary action is always a generalized Maxwell theory in the temporal gauge (see [84] for a discussion on this point). It is important to keep this small subtlety in mind when looking at more complicated TQFTs that produce further interactions involving the photon. For instance, in Section 9.4 we will obtain Chern–Simons terms on the boundary, and we will have to keep in mind that they always arise in the temporal gauge.

9.2.3 Lagrangian algebras and topological sectors

There is one very important caveat in the discussion of the previous two sections. Let us focus on the 0-form symmetry case for definiteness. We have shown that with the boundary condition we chose, the path integral of the TQFT can be rewritten as a path integral with the action of a compact boson (9.2.18). However, the domain is not the one of the physical theory. The reason is that when we solve (9.2.13) introducing Φ_0 and ω_{d-2} as in (9.2.14), these fields cannot wind around the time circle S^1. Hence what we established in Section 9.2.1 is that the TQFT partition function is equal to the zero-winding sector of a compact boson.[8]

However, it turns out that we can produce the path integral in *any* fixed winding sector, simply by inserting a Wilson line $e^{in\int_{S^1} A_1}$ along the time circle in the bulk. The line pierces the spatial manifold X_d at a point P, creating a nontrivial $(d-1)$-cycle $\Sigma_{d-1} \subset X_d$ and introducing a monodromy for $\widetilde{b}_{d-1}$ around it:

$$\int_{\Sigma_{d-1}} \widetilde{b}_{d-1} = 2\pi n \,. \tag{9.2.25}$$

[7] This subtlety does not arise in the quicker procedure described at the end of the last section.

[8] For $d = 2$ the boundary spatial manifold is S^1, and since Φ_0 is compact the path integral includes a sum over all windings around that spatial circle, but not around the time circle.

To get the TQFT partition function with this insertion, consider a generator $\frac{\eta_{d-1}}{2\pi}$ of $H^{d-1}(X_d \setminus P;\mathbb{Z})$, namely $\int_{\Sigma_{d-1}} \eta_{d-1} = 2\pi$. The second equation in (9.2.13) is now solved by

$$\widetilde{b}_{d-1} = n\,\eta_{d-1} + \tilde{d}\omega_{d-2}\,. \tag{9.2.26}$$

With the same steps as before we obtain a path integral on boundary fields Φ_0 and ω_{d-2}, again over configurations of Φ_0 with zero winding around the time circle, but with a modified action with respect to (9.2.15):

$$\begin{aligned} S_n = \frac{i}{2\pi}\int_{\mathcal{M}_d}\bigg[&(-1)^d\,\tilde{d}\omega_{d-2}\wedge\big(\partial_t\Phi_0 - \mathcal{A}_0^t\big)dt \\ &-\frac{i}{2}\bigg(R^2\,\big(\tilde{d}\Phi_0-\widetilde{\mathcal{A}}_1\big)\wedge\star\big(\tilde{d}\Phi_0-\widetilde{\mathcal{A}}_1\big)+\frac{1}{R^2}\,\tilde{d}\omega_{d-2}\wedge\star\,\tilde{d}\omega_{d-2}\bigg)\bigg] \\ -(-1)^d\,\frac{in}{2\pi}\int_{\mathcal{M}_d}&\mathcal{A}_0^t\,\widehat{\eta}_{d-1}\wedge dt + \frac{n^2}{4\pi R^2}\int_{\mathcal{M}_d}\widehat{\eta}_{d-1}\wedge\star\,\widehat{\eta}_{d-1}\,. \end{aligned} \tag{9.2.27}$$

Here $\widehat{\eta}_{d-1}$ is the pull-back of η_{d-1} on $\mathcal{M}_d$. It is a top form on $\partial X_d \equiv \mathcal{M}_{d-1}$ and one can make a choice for the representative η_{d-1} in (9.2.26) such that $\widehat{\eta}_{d-1} = \frac{2\pi}{v}\,\mathrm{Vol}_{\mathcal{M}_{d-1}}$ with $v = \int_{\mathcal{M}_{d-1}}\mathrm{Vol}_{\mathcal{M}_{d-1}}$ the volume of the boundary spatial slice. In particular $\star\,\widehat{\eta}_{d-1} = \frac{2\pi}{v}\,dt$. Plugging this back into (9.2.27) we obtain

$$S_n = S_0 - in\theta + \frac{\pi\beta n^2}{vR^2} \qquad \text{where} \qquad \theta = (-1)^d\int_{S^1}\mathcal{A}_0^t\,dt\,. \tag{9.2.28}$$

Here S_0 is the action (9.2.15) written in terms of the periodic scalar in the Luttinger liquid form, which could be rewritten in the Lorentz covariant form (9.2.18) that makes manifest its nature as a boson of radius R. Notice that $\theta \sim \theta + 2\pi$ has the interpretation of a chemical potential for the $U(1)$ 0-form symmetry. The partition function with the line inserted is then

$$Z_n = Z_{\text{pert}}\,\exp\bigg(in\theta - \frac{\pi\beta}{vR^2}\,n^2\bigg) \tag{9.2.29}$$

where Z_{pert} is the perturbative contribution due to a periodic boson.

We want to show our claim that, after we condense a Lagrangian algebra in the bulk, the partition function includes the sum over all topological sectors of the compact scalar, hence reproducing the physical partition function. The simplest Lagrangian algebra contains all the lines $W_n = e^{in\int A_1}$ and no surfaces $V_\alpha = e^{i\alpha\int b_{d-1}}$. Due to our choice of geometry, gauging this algebra is the same as summing over all lines inserted along the time circle, hence summing over all n in (9.2.29). The bulk interpretation of this sum is that we are computing the partition function of the SPT phase obtained by gauging the algebra, which we are taking as our theory of gravity. Hence using Poisson's summation formula we find[9]

$$Z_{\text{gravity}} = \sum_{n\in\mathbb{Z}} Z_n = Z_{\text{pert}}\sum_{w\in\mathbb{Z}}\exp\bigg[-\frac{\pi v R^2}{\beta}\Big(w+\frac{\theta}{2\pi}\Big)^2\bigg]\,. \tag{9.2.30}$$

The right hand side is precisely the partition function of a compact boson of radius R (with chemical potential θ).

More generally, the bulk TQFT has other Lagrangian algebras consisting of the lines W_{km} and the surfaces $V_{m'/k}$ for an integer number $k \in \mathbb{Z}$. Condensing one of them produces a different SPT phase in the bulk, hence a different theory of gravity. In the SymTFT story this corresponds to gauging the $\mathbb{Z}_k$ subgroups of the $U(1)$ symmetry at the boundary. Because of the chosen geometry, there are no

[9] Here we are neglecting an extra factor $\sqrt{\beta/vR^2}$, since normalizations of the path integrals do not play a role here. A similar factor is neglected in (9.2.31).

$(d-1)$-cycles in the bulk and hence condensing this algebra simply means summing over all Wilson lines of charge multiple of k. The result is

$$Z'_{\text{gravity}} = \sum_{m\in\mathbb{Z}} Z_{km} = Z_{\text{pert}} \sum_{w\in\mathbb{Z}} \exp\left[-\frac{\pi v}{\beta}\left(\frac{R}{k}\right)^2 \left(w + \frac{k\theta}{2\pi}\right)^2\right] \tag{9.2.31}$$

and the right-hand side can be interpreted as the partition function of a compact boson of radius $R' = R/k$. This is an orbifold of the previous boundary theory, which could be thought of as a different global form of the same theory.

We want to comment on a slightly different way to obtain a holographic dual to compact bosons, which also fits our proposal. We could have started with the TQFT of two $\mathbb{R}$ gauge fields described by the action

$$S = \frac{i}{2\pi}\int_{X_{d+1}} b_{d-1}\wedge da_1\,. \tag{9.2.32}$$

In this TQFT the charges of the Wilson lines $W_\alpha = e^{i\alpha\int a_1}$ are not quantized, and since there is no sum over fluxes,[10] there is no identification among the charges of $V_\beta = e^{i\beta\int b_{d-1}}$. The spectrum of bulk operators is then larger, labelled by $\mathbb{R}\times\mathbb{R}$, and the corresponding braiding is the phase $e^{2\pi i\alpha\beta}$. Lagrangian algebras are classified by the choice of a real number $Q\in\mathbb{R}_+$ and are given by [133]

$$\mathcal{L}_Q = \left\{W_{Qn}, V_{Q^{-1}m}\;\middle|\; n,m\in\mathbb{Z}\right\}. \tag{9.2.33}$$

We showed in chapter 8 that this TQFT is the SymTFT for two $U(1)$ symmetries, namely a 0-form and a $(d-2)$-form, with a mixed anomaly. While this is a different symmetry structure from just a single $U(1)$, the second higher-form symmetry arises universally in the IR whenever the 0-form symmetry is spontaneously broken. Hence the two symmetry structures share the same EFT that describes the broken phase and, according to our proposal, they should both be the holographic dual to a compact boson. Indeed there is no much difference between the two theories: the non-topological boundary conditions can be chosen to be the same, and the computations of Section 9.2.1 give the same result.

The considerations explained in this section can be repeated for any higher-form symmetry. However, in order to detect the various global structures of a boundary p-form Maxwell theory, one needs to properly choose the geometry. Indeed the fluxes are supported on $(p+1)$-dimensional cycles, and thus a natural choice is to take $X_{d+1} = B_{d-p}\times T^{p+1}$ with B_{d-p} a ball. One of the S^1 factors of the torus plays the role of a time circle, and $X_d = B_{d-p}\times T^p$. The bulk TQFT has action

$$S = \frac{i}{2\pi}\int_{X_{d+1}} b_{d-p-1}\wedge dA_{p+1} \tag{9.2.34}$$

where b_{d-p-1} is an $\mathbb{R}$ gauge field whilst A_{p+1} is a $U(1)$ gauge field. One can obtain an SPT phase by gauging the Lagrangian algebra given by $W_n = e^{in\int A_{p+1}}$, and this is realized by inserting these defects along the T^{p+1} factor in the bulk. This sum indeed reproduces the sum over fluxes of the p-form Maxwell theory on the boundary. The choice of other Lagrangian algebras modifies the value of the electric charge and corresponds to discrete gaugings of the 1-form symmetry.

9.3 Abelian anomalies and higher groups

We can enrich the analysis of $U(1)$ symmetries by including anomalies (Sections 9.3.1 and 9.3.2) or a 2-group structure (Section 9.3.4). We show here that, when doing it, the dual boundary theory

[10] An $\mathbb{R}$ gauge field admits a gauge in which the connection is globally defined, therefore the field strength is an exact form and its integrals on compact submanifolds vanish.

gets coupled to background fields in a non-minimal way. In Sections 9.3.3 and 9.3.5 we provide a field-theoretic interpretation of our results in terms of symmetry fractionalization.

9.3.1 Chiral anomaly in 2d

The SymTFT for an anomalous $U(1)$ symmetry in 2d has action:

$$S = \frac{i}{2\pi} \int_{X_3} b_1 \wedge dA_1 + \frac{ik}{4\pi} \int_{X_3} A_1 \wedge dA_1 \,. \tag{9.3.1}$$

The additional bulk Chern–Simons term significantly affects the consistent boundary conditions. To establish a proper variational principle with a non-topological boundary condition, it is essential to include the boundary term

$$S_\partial = -\frac{1}{4\pi R^2} \int_{\partial X_3} \left(b_1 + \frac{k}{2} A_1 \right) \wedge \star \left(b_1 + \frac{k}{2} A_1 \right) \tag{9.3.2}$$

together with the following Dirichlet boundary condition:[11]

$$\star \, \delta A_1 = -\frac{i}{R^2} \, \delta \left(b_1 + \frac{k}{2} A_1 \right) . \tag{9.3.3}$$

In order to properly turn on a background for the boundary $U(1)$ symmetry we have to render the boundary condition invariant under gauge transformations of A_1. This is most naturally done by introducing a 1-form $\mathcal{A}_1$ as

$$\star \, (A_1 - \mathcal{A}_1) = -\frac{i}{R^2} \left(b_1 + \frac{k}{2} \, (A_1 - \mathcal{A}_1) \right) . \tag{9.3.4}$$

This boundary condition is invariant under $\delta \mathcal{A}_1 = \delta A_1 = d\lambda_0$, allowing us to interpret $\mathcal{A}_1$ as a background field for the $U(1)$ symmetry on the boundary. Notice that our choice does not modify (9.3.3) and is thus just a particularly convenient parametrization.

Before deriving the dual boundary theory, we can already establish that it has an 't Hooft anomaly. Indeed, under a gauge transformation $\delta A_1 = \delta \mathcal{A}_1 = d\lambda_0$ the total action $S + S_\partial$ transforms as

$$\delta(S + S_\partial) = -\frac{ik}{4\pi} \int_{\mathcal{M}_2} d\lambda_0 \wedge \mathcal{A}_1 - \frac{k^2}{16\pi R^2} \int_{\mathcal{M}_2} \Big(2 \, d\lambda_0 \wedge \star \mathcal{A}_1 + d\lambda_0 \wedge \star \, d\lambda_0 \Big) \tag{9.3.5}$$

where $\mathcal{M}_2 = \partial X_3$. The second term can be cancelled by adding the following counterterm to the boundary action:

$$S_{\text{c.t.}} = \frac{k^2}{16\pi R^2} \int_{\mathcal{M}_2} \mathcal{A}_1 \wedge \star \, \mathcal{A}_1 \,. \tag{9.3.6}$$

However the remaining total gauge variation

$$\delta \big(S + S_\partial + S_{\text{c.t.}} \big) = -\frac{ik}{4\pi} \int_{\mathcal{M}_2} d\lambda_0 \wedge \mathcal{A}_1 \tag{9.3.7}$$

cannot be cancelled by any local boundary counterterm: it is precisely the anomalous variation corresponding to a perturbative $U(1)$ anomaly.

To derive the boundary theory we follow the steps outlined in Section 9.2. The constraints imposed by the path integral over time components again allow us to write $\widetilde{A}_1 = \tilde{d}\Phi_0$ and $\widetilde{b}_1 = \tilde{d}\omega_0$. The

[11] One can check, by writing all possible boundary terms and imposing consistency of the variational principle, that these boundary data are the only possible choice.

boundary action expressed in terms of these variables, after introducing $\mathcal{F} = \mathcal{A}_1 - \frac{ik}{2R^2} \star \mathcal{A}_1$ for convenience, reads:

$$S = \frac{i}{2\pi} \int_{\mathcal{M}_2} \Big[\Big(\tilde{d}\omega_0 + \tfrac{k}{2}\, \tilde{d}\Phi_0 \Big) \big(\partial_t \Phi_0 - \mathcal{F}_0^t \big) \wedge dt \tag{9.3.8}$$
$$- \frac{i}{2} \Big(R^2 \Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{F}}_1 \Big) \wedge \star \Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{F}}_1 \Big) + \frac{1}{R^2} \Big(\tilde{d}\omega_0 + \tfrac{k}{2}\, \tilde{d}\Phi_0 \Big) \wedge \star \Big(\tilde{d}\omega_0 + \tfrac{k}{2}\, \tilde{d}\Phi_0 \Big) \Big) \Big] + S_{\text{c.t}}\,.$$

This is the same action as in (9.2.15) for $d = 2$ but with $\omega_0 \mapsto \omega_0 + \frac{k}{2}\, \Phi_0$. Integrating ω_0 out we find

$$S = \frac{R^2}{4\pi} \int_{\mathcal{M}_2} \big(d\Phi_0 - \mathcal{A}_1 \big) \wedge \star \big(d\Phi_0 - \mathcal{A}_1 \big) + \frac{ik}{4\pi} \int_{\mathcal{M}_2} \Phi_0 \, d\mathcal{A}_1\,. \tag{9.3.9}$$

This action describes a compact boson of radius R, but with an unusual coupling to a background for the momentum symmetry. Such a coupling reproduces the anomalous shift (9.3.7) that is indeed cancelled by the inflow action

$$S_{\text{inflow}} = -\frac{ik}{4\pi} \int_{3\text{d}} \mathcal{A}_1 \wedge d\mathcal{A}_1\,. \tag{9.3.10}$$

Notice that the extra coupling $\Phi_0\, d\mathcal{A}_1$ in (9.3.9) has a form similar to the coupling with the winding symmetry. In a sense, we are prescribing that a background $\mathcal{A}_1$ for the momentum symmetry also activates a background $\mathcal{B}_1 = k\mathcal{A}_1$ for the winding symmetry. In other words, $\mathcal{A}_1$ is not coupled with the momentum symmetry but rather with a diagonal combination of momentum and winding.[12] Since the two symmetries have a mixed anomaly, this diagonal $U(1)$ inherits a pure anomaly.

9.3.2 Chiral anomaly in 4d

The treatment of anomalies in higher dimensions presents a further conceptual difference. As a representative case, we consider $d = 4$ and the TQFT with action

$$S = \frac{i}{2\pi} \int_{X_5} b_3 \wedge dA_1 + \frac{ik}{24\pi^2} \int_{X_5} A_1 \wedge dA_1 \wedge dA_1\,. \tag{9.3.11}$$

To get a good variational principle we need to impose

$$\star\, \delta A_1 = -\frac{i}{R^2}\, \delta \Big(b_3 + \frac{k}{6\pi} A_1 \wedge dA_1 \Big) \tag{9.3.12}$$

and add a boundary term

$$S_\partial = -\frac{1}{4\pi R^2} \int_{\partial X_5} \Big(b_3 + \frac{k}{6\pi} A_1 \wedge dA_1 \Big) \wedge \star \Big(b_3 + \frac{k}{6\pi} A_1 \wedge dA_1 \Big)\,. \tag{9.3.13}$$

These choices however do not allow us to turn on a background by simply changing the parametrization of the boundary condition, as we did in 2d. Indeed, if we try to restore the gauge transformations of A_1, the boundary condition shifts by terms that depend on the field A_1 itself and cannot be cancelled by adding counterterms in the background only. Turning on a background in $d > 2$ requires us to change the boundary data in a nontrivial way. In Appendix D we explain an iterative procedure that, starting from the data above, produces a consistent variational principle together with a gauge-invariant boundary condition. The result for $d = 4$ is

$$\star\, (A_1 - \mathcal{A}_1) = -\frac{i}{R^2} \Big(b_3 + \frac{k}{6\pi} (A_1 - \mathcal{A}_1) \wedge dA_1 + \frac{k}{12\pi} (A_1 - \mathcal{A}_1) \wedge d\mathcal{A}_1 \Big) \tag{9.3.14}$$

[12] More precisely, it is the diagonal combination between momentum and a $\mathbb{Z}_k$ extension of the winding symmetry.

with boundary term

$$S_\partial = -\frac{1}{4\pi R^2}\int_{\partial X_5}\left(b_3+\frac{k}{6\pi}\big(A_1-\mathcal{A}_1\big)\wedge dA_1+\frac{k}{12\pi}A_1\wedge d\mathcal{A}_1\right)^2+\frac{ik}{24\pi^2}\int_{\partial X_5}\mathcal{A}_1\wedge A_1\wedge dA_1\,. \quad (9.3.15)$$

When setting $\mathcal{A}_1 = 0$ we recover the previous boundary data, but in general there are new terms that mix background and dynamical fields. As in 2d, one can show that the system has an anomaly performing a gauge transformation $\delta A_1 = \delta\mathcal{A}_1 = d\lambda_0$: up to a counterterm the gauge variation is

$$\delta\big(S+S_\partial+S_{\text{c.t.}}\big)=\frac{ik}{24\pi^2}\int_{\partial X_5}\lambda_0\, d\mathcal{A}_1\wedge d\mathcal{A}_1\,. \quad (9.3.16)$$

The procedure to determine the dual boundary theory is completely analogous to the examples we have already presented. Integrating the time components out, we introduce $\widetilde{A}_1=\tilde{d}\Phi_0$ and $\widetilde{b}_3=\tilde{d}\omega_2$. To simplify our expressions, we denote $\mathcal{F}_1=\mathcal{A}_1-\frac{ik}{12\pi R^2}\star(\mathcal{A}_1\wedge d\mathcal{A}_1)$. Then the boundary action, in its non-covariant presentation, is

$$S=\frac{i}{2\pi}\int_{\mathcal{M}_4}\Big[\Big(\tilde{d}\omega_2+\tfrac{k}{12\pi}\tilde{d}\Phi_0\wedge\tilde{d}\widetilde{\mathcal{A}}_1\Big)\big(\partial_t\Phi_0-\mathcal{F}_0^t\big)dt-\frac{i}{2}\Big(R^2\Big(\tilde{d}\Phi_0-\widetilde{\mathcal{F}}_1\Big)\wedge\star\Big(\tilde{d}\Phi_0-\widetilde{\mathcal{F}}_1\Big)$$
$$+\frac{1}{R^2}\Big(\tilde{d}\omega_2+\tfrac{k}{12\pi}\tilde{d}\Phi_0\wedge\tilde{d}\widetilde{\mathcal{A}}_1\Big)\wedge\star\Big(\tilde{d}\omega_2+\tfrac{k}{12\pi}\tilde{d}\Phi_0\wedge\tilde{d}\widetilde{\mathcal{A}}_1\Big)\Big)\Big]+S_{\text{c.t.}} \quad (9.3.17)$$

where $\mathcal{M}_4=\partial X_5$. As before we can integrate out ω_2 and the final action reads

$$S=\frac{R^2}{4\pi}\int_{\mathcal{M}_4}\big(d\Phi_0-\mathcal{A}_1\big)\wedge\star\big(d\Phi_0-\mathcal{A}_1\big)+\frac{ik}{24\pi^2}\int_{\mathcal{M}_4}\Phi_0\, d\mathcal{A}_1\wedge d\mathcal{A}_1\,. \quad (9.3.18)$$

This represents a compact scalar with a non-standard coupling to a background associated with the shift symmetry, akin to the situation in 2d. The additional interaction accounts for the anomalous variation described by (9.3.16). Nevertheless, unlike in the 2d scenario, we cannot view this altered interaction as a combination of the shift and winding symmetries since the two have different degree.

9.3.3 Anomaly matching in the broken phase

Let us provide a purely field-theoretic interpretation of the result in the previous section. For any Lie-group symmetry G, the Goldstone theory describing the symmetry breaking phase is a non-linear sigma model with target space G. In even spacetime dimensions d, the symmetry G can suffer from perturbative anomalies and the question is how these are matched in the sigma model.

For non-Abelian G it is well known that the anomaly is reproduced by a WZW term [238]. This is an additional interaction with important dynamical consequences. Perturbative anomalies are classified by $H^{d+2}(BG;\mathbb{Z})$, which determines a $(d+1)$-dimensional Chern–Simons action that cancels the anomaly by inflow. On the other hand, WZW terms in d dimensions are classified by $H^{d+1}(G;\mathbb{Z})$. Anomaly matching is mathematically represented by a map

$$\tau : H^{d+2}(BG;\mathbb{Z})\to H^{d+1}(G;\mathbb{Z}) \quad (9.3.19)$$

called *transgression* [239]. For $d = 2$ this map also underlines the map of levels in the CS/WZW correspondence [81]. For the simple Lie group $G = \mathrm{SU}(n)$, the transgression map τ is injective [81], meaning that any perturbative anomaly is matched by a WZW term.[13] However this is not the general case, and if τ has a nontrivial kernel, the corresponding anomalies require some new ingredient to be matched in the sigma model.

[13]The transgression map is expected to be injective for all simple Lie groups.

Here we focus on the extreme case $G = U(1)$ for which $H^{d+1}\big(U(1);\mathbb{Z}\big) = 0$, namely there is no WZW term at all, and any anomaly must be matched in a different way. From our holographic analysis we know the answer to this question: the dynamics of the sigma model is unchanged with respect to the non-anomalous case, but the symmetry is coupled non-minimally to the background $\mathcal{A}_1$ through the extra topological term

$$\frac{ik}{(2\pi)^{d/2}\left(\frac{d}{2}+1\right)!}\int_{\mathcal{M}_d}\Phi_0\left(d\mathcal{A}_1\right)^{d/2}. \tag{9.3.20}$$

This term reproduces the anomaly, but at this level it seems a bit ad hoc. We want to clarify why it arises from a UV viewpoint and how we understand it in the IR. This is important to understand why there is a difference in how anomaly matching works in the Abelian and non-Abelian cases.

We can show in a simple model that when the background field is turned on in the UV, the additional coupling (9.3.20) is generated along the RG flow by integrating out massive fields. Consider a 4d theory with a massless Dirac fermion ψ and a complex scalar ϕ, coupled via a Yukawa interaction:

$$\mathcal{L}\supset\phi\,\overline{\psi}\psi\,. \tag{9.3.21}$$

The theory has an axial symmetry $U(1)_A$ under which both Weyl components of ψ have charge 1, while ϕ has charge -2. $U(1)_A$ has a cubic anomaly with $k=2$. Choosing a potential $V(\phi)$ that induces condensation of ϕ, the axial symmetry gets spontaneously broken to $\mathbb{Z}_2 = (-1)^F$. By decomposing $\phi = \rho\, e^{i\Theta}$ into its radial and angular parts, the VEV $\langle\rho\rangle = v$ gives mass to both ρ and ψ. The angular part Θ remains massless and is the only degree of freedom at low energy: it is the Goldstone boson. The faithful symmetry in the IR is $U(1) = U(1)_A/\mathbb{Z}_2$ that shifts Θ. In order to reproduce the anomaly, the coupling to a background $\mathcal{A}$ must include the term

$$\frac{i}{24\pi^2}\,\Theta\,(d\mathcal{A})^2\,. \tag{9.3.22}$$

Indeed this term arises when integrating out the fermion. To see this notice that, for fixed ϕ and $\mathcal{A}$, if ϕ is real and positive then the fermion path integral can be regularized in a way such that the measure is positive [240–242]. Clearly this is not true on a generic configuration, but we can make it true by performing an axial rotation of parameter $e^{i\alpha}$, with $\alpha = -\frac{1}{2}\Theta$. A textbook computation [243, 244] shows that the path integral measure of the fermion changes by a phase

$$D[\psi]\;\mapsto\;D[\psi]\,\exp\!\left(\frac{ik}{24\pi^2}\int\alpha\,(d\mathcal{A})^2\right). \tag{9.3.23}$$

Setting $\alpha = -\frac{1}{2}\Theta$ this precisely reproduces the coupling (9.3.22). Now the Yukawa coupling becomes $\rho\,\overline{\psi}\psi$, that for fixed ρ is essentially a positive mass term for the fermion, hence integrating out the fermion becomes a safe operation that does not introduce extra phases.

Returning to the general case, we want to interpret the extra coupling (9.3.20) as specifying a (higher) symmetry fractionalization class for the $U(1)$ symmetry (see Section 1.5). This reinterpretation will be crucial to understand the analogous story for higher groups in the following sections. A 0-form symmetry G can fractionalize in the presence of a discrete 1-form symmetry Γ. This means that when two topological defects $g,h\in G$ fuse to produce $gh\in G$, their codimension-two junction gets covered by a topological defect $\omega(g,h)\in\Gamma$ of the 1-form symmetry [15, 53, 54], where $\omega\in H^2(BG;\Gamma)$. Equivalently, a background $\mathcal{A}_1$ for G turns on a background $\mathcal{B}_2 = \mathcal{A}_1^*\,\omega$ for the 1-form symmetry. In this formula, we think of $\mathcal{A}_1$ as a map $\mathcal{M}_d\to BG$ and of $\mathcal{B}_2$ as an element of $H^2(\mathcal{M}_d,\Gamma)$ so that we can use $\mathcal{A}_1$ to pull back ω. With this interpretation it becomes clear that, if G and Γ have a mixed anomaly, a non-trivial fractionalization class modifies the pure anomaly for G, possibly making it nontrivial even when it vanished originally [53, 54]. This has a natural generalization to the case that Γ is a discrete p-form symmetry: when $p+1$ topological defects $g_1, \ldots, g_{p+1} \in G$ fuse in generic position, they create a codimension-$(p+1)$ junction that can be dressed by a defect $\omega(g_1, \ldots, g_{p+1})$ of the p-form symmetry Γ, where ω is a class in $H^{p+1}(BG; \Gamma)$. Equivalently, a background $\mathcal{A}_1$ turns on a background $\mathcal{B}_{p+1} = \mathcal{A}_1^* \, \omega$ for Γ.

The compact boson theory that describes the breaking of a $U(1)$ 0-form symmetry also possesses a $U(1)$ $(d-2)$-form winding symmetry, and the two have a mixed anomaly. For this reason, a pure anomaly for the 0-form symmetry can be induced by fractionalizing it with the $(d-2)$-form symmetry. One minor modification with respect to what we described above is necessary because the p-form symmetry (here $p = d-2$) is continuous. Its most natural description is not in terms of a background potential $\mathcal{B}_{p+1}$, which is not a cohomology class in general, but in terms of its field strength $\frac{1}{2\pi} \, d\mathcal{B}_{p+1} \in H^{p+2}(\mathcal{M}_d; \mathbb{Z})$. As a consequence the fractionalization class, instead of being an element of $H^{p+1}\big(BU(1); U(1)\big)$, is more naturally an element of $H^{p+2}(BU(1); \mathbb{Z}) \cong \mathbb{Z}$. This is the datum that determines a $(p+1)$-dimensional Chern–Simons level, or equivalently the corresponding Chern class in $(p+2)$ dimensions. Hence, in analogy with the discrete case, we prescribe that a background $\mathcal{A}_1$ for the 0-form symmetry activates a background $\mathcal{B}_{d-1}$ for the $(d-2)$-form symmetry whose field strength is

$$\frac{1}{2\pi} \, d\mathcal{B}_{d-1} = \frac{k}{(2\pi)^{d/2} \left(\frac{d}{2}+1\right)!} \left(d\mathcal{A}_1\right)^{d/2} . \tag{9.3.24}$$

Recalling that the $(d-2)$-form symmetry is coupled to its background field through the action term $\frac{i}{2\pi} \int_{\mathcal{M}_d} \Phi_0 \, d\mathcal{B}_{d-1}$, this reproduces the coupling (9.3.20) in agreement with our holographic result.

9.3.4 Abelian 2-groups

We consider a 2-group symmetry in four dimensions formed by a $U(1)$ 0-form symmetry and a $U(1)$ 1-form symmetry. This can be obtained by starting from a theory with two $U(1)$ 0-form symmetries with a cubic mixed anomaly and gauging the $U(1)$ that appears linearly in the anomaly polynomial [16, 154]. The 1-form symmetry participating in the 2-group structure is the magnetic symmetry of the photon. As derived in chapter 8, the SymTFT for such a 2-group symmetry has action:

$$S = \frac{i}{2\pi} \int_{X_5} \left(b_3 \wedge dA_1 + h_2 \wedge dC_2 + \frac{k}{2\pi} \, h_2 \wedge A_1 \wedge dA_1 \right) . \tag{9.3.25}$$

This is invariant under the gauge transformation (8.4.4)[14].

We place this TQFT on a manifold with boundary, $X_5 = B_4 \times S^1$ for simplicity, and we interpret it as a theory of gravity, holographically dual to some 4d quantum field theory on the boundary. The last term in (9.3.25) contains a derivative, therefore it affects the boundary contribution to the variational principle, similarly to the case of chiral anomalies. To fix the boundary terms S_∂ and the boundary conditions on the fields, we use the same logic as in that case. We find the boundary conditions

$$\star \left(A_1 - \mathcal{A}_1\right) = -\frac{i}{R^2} \left[b_3 + \frac{k}{2\pi} \, h_2 \wedge \left(A_1 - \mathcal{A}_1\right) \right] , \quad \star h_2 = \frac{ie^2}{\pi} \left(C_2 - \mathcal{C}_2 - \frac{k}{2\pi} \, \mathcal{A}_1 \wedge A_1 \right) \tag{9.3.26}$$

[14] There is some freedom in the choice of transformations that leave (9.3.25) invariant. In particular, the transformation $\delta A_1 = d\lambda_0$ could be accompanied by an action on both b_3 and C_2 as $\delta b_3 = -\epsilon \frac{k}{2\pi} d\lambda_0 \wedge h_2$ and $\delta C_2 = (1-\epsilon) \frac{k}{2\pi} d\lambda_0 \wedge A_1$ for any choice of ϵ. Here we chose $\epsilon = 0$ which matches the transformations in the boundary theory.

and a corresponding boundary term

$$S_\partial = -\frac{i}{2\pi}\int_{\partial X_5} h_2 \wedge \left(C_2 - \tfrac{k}{2\pi}\,\mathcal{A}_1 \wedge A_1\right) - \frac{e^2}{4\pi^2}\int_{\partial X_5}\left(C_2 - \tfrac{k}{2\pi}\,\mathcal{A}_1 \wedge A_1\right) \wedge \star \left(C_2 - \tfrac{k}{2\pi}\,\mathcal{A}_1 \wedge A_1\right)$$
$$- \frac{1}{4\pi R^2}\int_{\partial X_5}\left[b_3 + \frac{k}{2\pi}\,h_2 \wedge \left(A_1 - \mathcal{A}_1\right)\right] \wedge \star \left[b_3 + \frac{k}{2\pi}\,h_2 \wedge \left(A_1 - \mathcal{A}_1\right)\right]. \tag{9.3.27}$$

Here $\mathcal{A}_1$, $\mathcal{C}_2$ are fixed gauge fields on the boundary that transform as a proper 2-group background:

$$\delta\mathcal{A}_1 = d\lambda_0\,, \qquad \delta\mathcal{C}_2 = d\eta_1 + \frac{k}{2\pi}\,d\lambda_0 \wedge \mathcal{A}_1\,. \tag{9.3.28}$$

This makes the boundary conditions gauge invariant, provided we add a counterterm $\frac{e^2}{4\pi^2}\int_{\partial X_5}\mathcal{C}_2\wedge\star\,\mathcal{C}_2$.

With the usual procedure, we obtain that the dual boundary theory has action:

$$\begin{aligned} S = \frac{R^2}{4\pi}\int_{\partial X_5}\left(d\Phi_0 - \mathcal{A}_1\right)\wedge\star\left(d\Phi_0 - \mathcal{A}_1\right) + \frac{1}{4e^2}\int_{\partial X_5} da_1 \wedge \star\, da_1 \\ + \frac{i}{2\pi}\int_{\partial X_5}\mathcal{C}_2 \wedge da_1 + \frac{ik}{4\pi^2}\int_{\partial X_5}\Phi_0\; da_1 \wedge d\mathcal{A}_1\,. \end{aligned} \tag{9.3.29}$$

Naively one may think that a_1 is an $\mathbb{R}$ gauge field, because it comes from the trivialization of h_2. However, we have to take into account the condensation of the appropriate Lagrangian algebra in the bulk, necessary to trivialize the TQFT and making it independent of the topology. Specifically, here the relevant Lagrangian algebra is

$$\mathcal{L} = \left\{ e^{in\oint A_1},\, e^{im\oint C_2} \;\middle|\; n, m \in \mathbb{Z}\right\}. \tag{9.3.30}$$

Following the same logic as in Section 9.2.3, this introduces a sum over the fluxes of da_1 that effectively makes a_1 into a $U(1)$ gauge field.

Turning off the background $\mathcal{A}_1$ we obtain a free compact scalar and a free photon (coupled to a background field $\mathcal{C}_2$ for its magnetic symmetry), enjoying a $U(1)$ 0-form symmetry with conserved current $J_1 = \frac{iR^2}{2\pi}\,d\Phi_0$, and a $U(1)$ 1-form symmetry with conserved current $J_2 = \frac{1}{2\pi}\star da_1$, respectively. However, as soon as we turn on a background $\mathcal{A}_1$ for the 0-form symmetry, the 2-group structure manifests itself through the nonstandard coupling between the photon and the scalar, which modifies the currents and the background gauge transformations [16]. This is very similar to what happened in the case of the chiral anomaly, and we will provide a similar interpretation in terms of symmetry fractionalization in the next section.

Let us show that the theory in (9.3.29) reproduces the 2-group symmetry [16]. First, notice that the gauge transformation

$$\delta\Phi_0 = \lambda_0\,, \qquad \delta\mathcal{A}_1 = d\lambda_0\,, \qquad \delta\mathcal{C}_2 = d\eta_1 + \frac{k}{2\pi}\,d\lambda_0 \wedge \mathcal{A}_1 \tag{9.3.31}$$

leaves the action invariant. This is indeed the background gauge transformation for a 2-group. Second, in the presence of a background the currents get modified to:[15]

$$J_1 = \frac{iR^2}{2\pi}\left(d\Phi_0 - \mathcal{A}_1\right) + \frac{k}{4\pi}\star\left(d\Phi_0 \wedge da_1\right), \qquad J_2 = \frac{1}{2\pi}\star da_1\,, \tag{9.3.32}$$

and these satisfy modified conservation equations

$$d\star J_1 + \frac{k}{2\pi}\,d\mathcal{A}_1 \wedge \star J_2 = 0\,, \qquad d \star J_2 = 0\,, \tag{9.3.33}$$

that are the correct conservation equations for a 2-group symmetry.

[15]For a $U(1)$ p-form symmetry we use the convention that the current J_{p+1} is defined by $\star J_{p+1} = -i\frac{\delta S}{\delta\mathcal{A}_{p+1}}$ where $\mathcal{A}_{p+1}$ is the background field.

9.3.5 Abelian 2-groups in the broken phase

The unusual coupling to the background $\mathcal{A}_1$ in (9.3.29), responsible for the 2-group structure of the symmetry, is quite similar to the coupling (9.3.20) responsible for a chiral anomaly, that we interpreted in terms of symmetry fractionalization. Indeed we can give a similar interpretation here too. While it is intuitively clear why symmetry fractionalization can induce a pure anomaly, and this fact has been studied extensively [53, 54], the necessity of symmetry fractionalization to match higher-group structures has not been much appreciated. There is indeed one important difference, namely the nature of the symmetry used to fractionalize the $U(1)$ 0-form symmetry in question: it is a *composite symmetry* [245].

In general, if we have two $U(1)$ symmetries of degrees p and q with currents J_{p+1} and J_{q+1} respectively, if $p+q \geq d-1$ we can construct a third $U(1)$ symmetry simply because the current

$$J_{p+q-d+2} = \star\big((\star\, J_{p+1}) \wedge (\star\, J_{q+1})\big) \tag{9.3.34}$$

is automatically conserved. This symmetry is of degree $p+q-d+1$. In general, it is not a particularly interesting symmetry because its consequences are already implied by the constituent symmetries. However, it plays a role in our discussion. The IR theory of a 4d compact boson has an emergent 2-form symmetry: the winding symmetry of the scalar with current $J_3 = -\frac{1}{2\pi} \star d\Phi_0$. This is the symmetry we used to fractionalize the 0-form symmetry in the case of the chiral anomaly. In this case, since we also have the magnetic 1-form symmetry of the photon with current $J_2 = \frac{1}{2\pi} \star da_1$, we can construct

$$\widehat{J}_1 = \star\big((\star\, J_3) \wedge (\star\, J_2)\big) = \frac{1}{4\pi^2} \star \big(d\Phi_0 \wedge da_1\big) \tag{9.3.35}$$

that generates a 0-form symmetry. Using this symmetry to fractionalize the shift symmetry of the compact boson, as described in Section 9.3.3, we obtain precisely the non-canonical coupling in (9.3.29).

9.4 Boundary Chern–Simons-like terms

In this section we study bulk models obtained by adding terms without derivatives. These do not affect the boundary terms in the variational principle and hence do not modify the boundary conditions. Thus the dual theory couples minimally to the background fields, but it contains extra interactions, typically Chern–Simons-like terms. Our main motivation here is to verify our conjecture in a case with a non-invertible symmetry, the $\mathbb{Q}/\mathbb{Z}$ chiral symmetry in four dimensions [28, 29] reviewed in Section 3.5,[16] and to provide a framework to study aspects of its spontaneous breaking. We also consider in Section 9.4.1 a bulk 4d TQFT introduced in Section 8.2.2, which was argued to be related to 3d gauge theories with Chern–Simons interactions. We use our formalism to establish a precise holographic duality confirming this expectation.

9.4.1 Holographic dual to Maxwell–Chern–Simons theory

We consider the 4d TQFT studied in Section 8.2.2, that has action

$$S = \frac{i}{2\pi} \int_{X_4} \left(A_1 \wedge db_2 + \frac{\phi}{4\pi}\, b_2 \wedge b_2 \right). \tag{9.4.1}$$

Recall that the gauge-invariant operators include surfaces $U_\alpha(\gamma_2) = e^{i\alpha \int_{\gamma_2} b_2}$ and the generically non-genuine lines $W_n(\gamma_1, D_2) = e^{in\int_{\gamma_1} A_1 + \frac{in\phi}{2\pi}\int_{D_2} b_2}$ that need an attached two-disk D_2 bounded by γ_1. The label $\alpha \sim \alpha + 1$ is circle valued, while $n \in \mathbb{Z}$.

[16]See [116] for a recent proposal to recover the full $U(1)$ chiral symmetry.

We will be mostly interested in the case

$$\phi = \frac{2\pi}{k} \qquad \text{with} \qquad k \in \mathbb{Z}\,. \tag{9.4.2}$$

In this case the lines W_{mk} become genuine, and an interesting Lagrangian algebra[17] is obtained by taking all the genuine lines together with the surfaces $U_{l/k}$ with $l \in \mathbb{Z}_k$. Used in SymTFT, this Lagrangian algebra describes the symmetry $U(1)^{[0]} \times \mathbb{Z}_k^{[1]}$: the first factor is a 0-form symmetry, the second factor is an anomalous 1-form symmetry (with coefficient 1), and there is a mixed anomaly between the two.

We place the theory on a manifold with boundary, where we impose the boundary condition

$$\star\big(A_1 - \mathcal{A}_1\big) = -\frac{i\pi}{k^2e^2}\, b_2\,. \tag{9.4.3}$$

In order to have a good variational principle we must add the boundary term

$$S_\partial = -\frac{k^2e^2}{4\pi^2}\int_{\partial X_4} A_1 \wedge \star A_1 = \frac{1}{4k^2e^2}\int_{\partial X_4}\bigg(b_2 + \frac{ik^2e^2}{\pi}\star\mathcal{A}_1\bigg)\wedge\star\bigg(b_2 + \frac{ik^2e^2}{\pi}\star\mathcal{A}_1\bigg)\,. \tag{9.4.4}$$

The gauge transformation $\delta A_1 = d\rho_0$ is restored by $\delta\mathcal{A}_1 = d\rho_0$ that makes (9.4.3) invariant. The full system is gauge invariant, provided that we also add a counterterm $S_{\text{c.t.}} = \frac{k^2e^2}{4\pi^2}\int_{\partial X_4}\mathcal{A}_1\wedge\star\mathcal{A}_1$.

We take the bulk to be the product of a three-dimensional ball B_3 and the time circle S^1, so that $\partial X_4 \equiv \mathcal{M}_3 = S^2 \times S^1$. Integrating out the time components A_0^t, b_1^t we get delta functions imposing

$$\tilde{d}\,\widetilde{b}_2 = 0\,, \qquad \tilde{d}\widetilde{A}_1 + \tfrac{1}{k}\,\widetilde{b}_2 = 0\,, \tag{9.4.5}$$

that are solved introducing Φ_0 and $\hat{a}_1$ through

$$\widetilde{b}_2 = \tilde{d}\,\hat{a}_1\,, \qquad \widetilde{A}_1 = \tilde{d}\Phi_0 - \tfrac{1}{k}\,\hat{a}_1\,. \tag{9.4.6}$$

With this, the bulk path integral reduces to a boundary path integral with action

$$\begin{aligned} S + S_\partial + S_{\text{c.t.}} = \frac{i}{2\pi}\int_{\mathcal{M}_3}\bigg[&\partial_t\hat{a}_1\wedge\Big(\tilde{d}\Phi_0 - \tfrac{1}{k}\,\hat{a}_1\Big)\wedge dt + \frac{1}{2k}\,\hat{a}_1\wedge d\hat{a}_1 + \tilde{d}\,\hat{a}_1\wedge\mathcal{A}_0^t\,dt \\ &- \frac{i}{2}\bigg(\frac{\pi}{k^2e^2}\,\tilde{d}\,\hat{a}_1\wedge\tilde{d}\,\hat{a}_1 + \frac{k^2e^2}{\pi}\Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1 - \tfrac{1}{k}\,\hat{a}_1\Big)\wedge\star\Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1 - \tfrac{1}{k}\,\hat{a}_1\Big)\bigg)\bigg]\,. \end{aligned} \tag{9.4.7}$$

Attempting to integrate out $\hat{a}_1$ to derive a covariant action for the scalar field, as we did in Section 9.2.1, results in a non-local action.[18] However, there is no problem in integrating out Φ_0 from (9.4.7) and we obtain a local and covariant boundary theory with action

$$S = \frac{1}{4k^2e^2}\int_{\mathcal{M}_3} d\hat{a}_1\wedge\star d\hat{a}_1 + \frac{i}{4\pi k}\int_{\mathcal{M}_3}\hat{a}_1\wedge d\hat{a}_1 + \frac{i}{2\pi}\int_{\mathcal{M}_3} d\hat{a}_1\wedge\mathcal{A}_1\,. \tag{9.4.8}$$

This might seem like a $U(1)$ gauge theory with an improperly quantized Chern–Simons level. However we must be careful in identifying the correct $U(1)$ gauge field, by considering the condensation of the Lagrangian algebra that trivializes the bulk. This includes all genuine lines as well as k surfaces:

$$\mathcal{L} = \Big\{W_{km} = e^{ikm\int A_1}\,,\; U_{l/k} = e^{\frac{il}{k}\int b_2}\;\Big|\; m\in\mathbb{Z}\,,\;\; l\in\mathbb{Z}_k\Big\}\,. \tag{9.4.9}$$

[17]A more natural Lagrangian algebra consists of all surfaces U_α. Used in SymTFT it describes an exotic $\mathbb{Z}$ 1-form symmetry with anomaly parametrized by ϕ, while holographically we expect it to describe its breaking.

[18]A similar (even though less transparent) problem would have arisen if we tried to obtain the boundary theory using the second method described at the end of Section 9.2.1, *i.e.*, by integrating out directly the whole b_2: the latter does not appear linearly in the bulk action.

On the geometry that we are considering, condensing $\mathcal{L}$ amounts to inserting the lines W_{km} along the time circle and summing over m, while the surfaces have no effect. The insertion of W_{km} modifies the path integral so as to impose that $\int_{S^2} b_2 = 2\pi k m$ for any two-sphere in B_3 that surrounds the Wilson line. This in particular includes the boundary spatial manifold. From the boundary theory viewpoint, this is a topological sector of the path integral with flux

$$\int_{S^2} \frac{d\hat{a}_1}{2\pi} = km\,. \tag{9.4.10}$$

Hence the canonically normalized $U(1)$ gauge field is $a_1 = \hat{a}_1/k$, in terms of which the boundary theory has action

$$S = \frac{1}{4e^2}\int_{\mathcal{M}_3} da_1 \wedge \star\, da_1 + \frac{ik}{4\pi}\int_{\mathcal{M}_3} a_1 \wedge da_1 + \frac{ik}{2\pi}\int_{\mathcal{M}_3} da_1 \wedge \mathcal{A}_1\,. \tag{9.4.11}$$

This is Maxwell–Chern–Simons theory at level k, coupled to a background field for the topological $U(1)$ symmetry acting on monopoles. More precisely, the background field for this symmetry is $\mathcal{A}'_1 = k\mathcal{A}_1$, while $\mathcal{A}_1$ is the background for a larger non-faithful $U(1)$ symmetry obtained by extending the topological symmetry with a trivially-acting $\mathbb{Z}_k$.[19]

It should be noted that this example has a slightly different flavor than all other ones discussed in this chapter. The UV symmetry is $U(1)^{[0]} \times \mathbb{Z}_k^{[1]}$, however only $\mathbb{Z}_k$ is spontaneously broken, indeed there are no Goldstone bosons in the IR since the photon is massive due to the Chern–Simons term. We consider this example has a warm up for the next one.

9.4.2 Spontaneously broken non-invertible $\mathbb{Q}/\mathbb{Z}$ chiral symmetry

Consider now the non-invertible chiral symmetry in 4d reviewed in Section 8.2.1, for which we established the SymTFT in chapter 8:

$$S = \frac{i}{2\pi}\int_{X_5}\left(b_3 \wedge dA_1 + f_2 \wedge dG_2 + \frac{k}{4\pi}\, A_1 \wedge f_2 \wedge f_2\right). \tag{9.4.12}$$

The gauge transformations are

$$\begin{aligned} \delta A_1 &= d\rho_0\,, & \delta b_3 &= d\xi_2 - \frac{k}{4\pi}\,\lambda_1 \wedge d\lambda_1 - \frac{k}{2\pi}\,\lambda_1 \wedge f_2\,, \\ \delta f_2 &= d\lambda_1\,, & \delta G_2 &= d\eta_1 - \frac{k}{2\pi}\,\rho_0\left(f_2 + d\lambda_1\right) - \frac{k}{2\pi}\,\lambda_1 \wedge A_1\,. \end{aligned} \tag{9.4.13}$$

As shown in chapter 8, the gauge-invariant genuine topological defects are:

$$\begin{aligned} W_n(\gamma_1) &= e^{in\int_{\gamma_1} A_1}\,, & U_{\frac{p}{kq}}(\gamma_3) &= e^{i\frac{p}{kq}\int_{\gamma_3} b_3}\,\mathcal{A}^{q,p}(\gamma_3; f_2)\,, \\ V_\alpha(\gamma_2) &= e^{i\alpha\int_{\gamma_2} f_2}\,, & \mathcal{T}_m(\gamma_2) &= e^{im\int_{\gamma_2} G_2}\,\mathbb{Z}_{km}(\gamma_2; A_1, f_2)\,. \end{aligned} \tag{9.4.14}$$

Here $n, m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}/\mathbb{Z}$, while $p/q \in \mathbb{Q}$ with $\gcd(p,q) = 1$ and $p \sim p + kq$ so that the label $p/kq \in \mathbb{Q}/\mathbb{Z}$. Then $\mathbb{Z}_{km}(\gamma_2; A_1, f_2)$ denotes a pure 2d $\mathbb{Z}_{km}$ gauge theory on γ_2, whose 0-form and 1-form symmetries are coupled, respectively, to A_1 and f_2. Similarly, $\mathcal{A}^{q,p}(\gamma_3; f_2)$ is the minimal Abelian TQFT with $\mathbb{Z}_q$ 1-form symmetry and anomaly labeled by p introduced in [111], whose 1-form symmetry is coupled to f_2. Stacking these TQFTs is necessary in order to make the operators gauge invariant and topological. The theories $\mathcal{A}^{q,p}$ are nontrivial for any $q \neq 1$, so that only a $\mathbb{Z}_k$ subgroup

[19] The reason why we got this coupling is that the TQFT we started with describes this larger symmetry, implemented by the operators $e^{i\alpha\int b_2}$, but the subgroup $\mathbb{Z}_k$ was condensed in the bulk, and acts trivially in the boundary theory. As discussed in Section 9.2.1, we did not explicitly introduce a background for the $\mathbb{Z}_k$ 1-form symmetry.

of the operators $U_{\frac{p}{kq}}$ (those with $q=1$) are invertible, while all other ones obey non-invertible fusion rules. Similarly, $\mathcal{T}_m$ are non-invertible. In the SymTFT approach it is natural to choose topological boundary conditions associated with the Lagrangian algebra

$$\mathcal{L} = \left\{ W_n \,,\, \mathcal{T}_m \,\middle|\, n,m \in \mathbb{Z} \right\} . \tag{9.4.15}$$

The remaining operators $U_{\frac{p}{kq}}(\gamma_3)$ and $V_\alpha(\gamma_2)$ implement the non-invertible symmetry and the magnetic 1-form symmetry, respectively.

Continuing with the approach we have followed so far, we want to consider a theory of gravity based on (9.4.12) with the condensation of $\mathcal{L}$ in the bulk. We place this theory on a manifold X_5 with a boundary and impose the non-topological boundary conditions

$$\star A_1 = -\frac{i}{R^2}\, b_3 + \star \mathcal{A}_1 \,, \qquad\qquad \star G_2 = -\frac{i\pi}{e^2}\, f_2 + \star \mathcal{G}_2 \,. \tag{9.4.16}$$

We need to add a boundary term:

$$S_\partial = -\frac{1}{4\pi R^2}\int_{\partial X_5} b_3 \wedge \star b_3 - \frac{1}{4e^2}\int_{\partial X_5} f_2 \wedge \star f_2 \,. \tag{9.4.17}$$

As before we would like to assign gauge transformation rules to the boundary fields $\mathcal{A}_1$, $\mathcal{G}_2$ in order to restore some of the gauge transformations on the boundary, corresponding to the symmetries that become global there. However, while we can restore $\delta G_2 = d\eta_1$ by transforming $\delta\mathcal{G}_2 = d\eta_1$, the gauge transformation $\delta A_1 = d\rho_0$ cannot be restored. Indeed, while the first eqn. in (9.4.16) could be made gauge invariant by prescribing that $\delta\mathcal{A}_1 = d\rho_0$, the second one would not be invariant because G_2 transforms as $\delta G_2 = -\frac{k}{2\pi}\rho_0 f_2$. This term cannot be reabsorbed by modifying the gauge transformations of $\mathcal{G}_2$, since f_2 is a dynamical field. Thus the only way to make the boundary conditions gauge invariant is to freeze the boundary value of ρ_0, as those of λ_1 and ξ_2.

To get the boundary theory, as before, we integrate out the time components imposing

$$\tilde{d}\widetilde{A}_1 = 0\,, \qquad \tilde{d}\widetilde{f}_2 = 0\,, \qquad \tilde{d}\,\widetilde{b}_3 + \frac{k}{4\pi}\,\widetilde{f}_2 \wedge \widetilde{f}_2 = 0\,, \qquad \tilde{d}\,\widetilde{G}_2 + \frac{k}{2\pi}\,\widetilde{A}_1 \wedge \widetilde{f}_2 = 0\,, \tag{9.4.18}$$

which are solved by

$$\widetilde{A}_1 = \tilde{d}\Phi_0\,, \qquad \widetilde{f}_2 = \tilde{d}a_1\,, \qquad \widetilde{b}_3 = \tilde{d}\omega_2 - \frac{k}{4\pi}\,a_1 \wedge \tilde{d}a_1\,, \qquad \widetilde{G}_2 = \tilde{d}C_1 - \frac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1\,. \tag{9.4.19}$$

The total action reduces to a boundary theory with action:

$$\begin{aligned} S = \frac{i}{2\pi}\int_{\mathcal{M}_4}\Bigg[& \Big(\tilde{d}\omega_2 - \tfrac{k}{4\pi}\,a_1\wedge\tilde{d}a_1\Big)\wedge\big(\partial_t\Phi_0 - \mathcal{A}_0^t\big)dt - \Big(\tilde{d}C_1 - \tfrac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1\Big)\wedge\partial_t a_1\wedge dt \\ & - \frac{i}{2}\bigg(R^2\Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\Big)\wedge\star\Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\Big) + \frac{1}{R^2}\Big(\tilde{d}\omega_2 - \tfrac{k}{4\pi}\,a_1\wedge\tilde{d}a_1\Big)\wedge\star\Big(\tilde{d}\omega_2 - \tfrac{k}{4\pi}\,a_1\wedge\tilde{d}a_1\Big)\bigg) \\ & - \frac{i}{2}\bigg(\frac{\pi}{e^2}\,\tilde{d}a_1\wedge\star\tilde{d}a_1 + \frac{e^2}{\pi}\Big(\tilde{d}C_1 - \tfrac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1 - \widetilde{\mathcal{G}}_2\Big)\wedge\star\Big(\tilde{d}C_1 - \tfrac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1 - \widetilde{\mathcal{G}}_2\Big)\bigg) \\ & + \tilde{d}a_1\wedge\mathcal{G}_1^t\wedge dt + \frac{ik}{4\pi}\,\Phi_0\,da_1\wedge da_1\Bigg] \end{aligned} \tag{9.4.20}$$

where $\mathcal{M}_4 = \partial X_5$. We can then integrate out both ω_2 and C_2 obtaining

$$S = \int_{\mathcal{M}_4}\left[\frac{R^2}{4\pi}\left(d\Phi_0 - \mathcal{A}_1\right)\wedge\star\left(d\Phi_0 - \mathcal{A}_1\right) + \frac{1}{4e^2}\,da_1\wedge\star da_1 + \frac{ik}{8\pi^2}\,\Phi_0\,da_1\wedge da_1 + \frac{i}{2\pi}\,da_1\wedge\mathcal{G}_2\right]. \tag{9.4.21}$$

As in the cases of the Abelian 2-group and of Maxwell–Chern–Simons theory, gauging the Lagrangian algebra introduces fluxes for a_1 turning it into a standard $U(1)$ gauge field. The theory in (9.4.21) describes a compact boson Φ_0 and a photon a_1 interacting via an axion coupling. This is called axion-Maxwell theory, and the full structure of its symmetries (including some emergent ones) has been studied in great detail in [221]. From the discovery of the non-invertible chiral symmetry, it has been suspected that axion-Maxwell theory universally describes its symmetry breaking [29, 221, 246].[20] Our result confirms that. Notably, this is the first interacting boundary theory we found among the examples considered so far.

Some comments on the coupling to the background fields are in order. As we already noticed after (9.4.17), there is no sensible gauge transformation rules that we could assign to $\mathcal{A}_1$ and $\mathcal{G}_2$ to make the boundary condition invariant under $\delta A_1 = d\rho_0$, hence we needed to freeze it. In the action (9.4.21), $\mathcal{A}_1$ should not be thought of as the background field for the 0-form non-invertible symmetry, but rather just as an external source that couples with the operator $J_1^{(A)}$. This is enough for holography, but it might seem a bit unsatisfactory from a symmetry viewpoint. However, this is really the hallmark of the non-invertible nature of the symmetry: ordinary background gauge fields seem not to exist, and they are effectively replaced by boundary values of dynamical fields in one dimension higher [2]. The underlying reason is that non-invertible symmetries map untwisted sectors to twisted sectors, hence the gauge transformations of a background gauge field necessarily involve an interplay among backgrounds that do not exist simultaneously in the theory, but only in the SymTFT (or in holography) where all global variants are on the same footing. This is the reason why SymTFT is the main tool for discussing anomalies [3, 43–45].

9.5 Non-Abelian Goldstone bosons

A very interesting class of examples are those of spontaneously broken non-Abelian symmetries. In these cases the boundary EFTs that we derive are interacting and generically non-renormalizable. In the 2d/3d case we will be able to recover and somewhat generalize the CS/WZW correspondence outside of the conformal point, while in higher dimensions we will obtain the pion Lagrangian on the boundary. We start with the non-Abelian generalization of the theories considered in Section 9.2 and then add an anomaly term, which corresponds to WZW terms in various dimensions. Finally we show how our setup is able to produce an EFT for spontaneously broken non-Abelian 2-group symmetries.

9.5.1 Holographic dual to the pion Lagrangian

Let G be a connected and compact Lie group (with Lie algebra $\mathfrak{g}$). As discussed in Section 8.5, the SymTFT for a non-Abelian 0-form symmetry G in d dimensions is the TQFT with action:[21]

$$S = \frac{i}{2\pi} \int_{X_{d+1}} \mathrm{Tr}\big(b_{d-1} \wedge F_2\big)\,, \tag{9.5.1}$$

where $F_2 = dA_1 + iA_1 \wedge A_1$ is the field strength of a G connection A_1 while b_{d-1} is a $\mathfrak{g}$-valued $(d-1)$-form.

We use the following non-topological boundary condition and boundary term on $\mathcal{M}_d = \partial X_{d+1}$:

$$\star\big(A_1 - \mathcal{A}_1\big) = -\frac{i}{f_\pi^2}\, b_{d-1}\,, \qquad\qquad S_\partial = -\frac{1}{4\pi f_\pi^2} \int_{\mathcal{M}_d} \mathrm{Tr}\big(b_{d-1} \wedge \star\, b_{d-1}\big)\,. \tag{9.5.2}$$

[20] For instance, the 4d $\mathbb{CP}^1$ non-linear sigma model enjoys a $\mathbb{Q}/\mathbb{Z}$ non-invertible symmetry [247] and it was argued in [246] that its breaking leads to axion-Maxwell theory.

[21] For $d = 3$ this theory was first considered by Horowitz [83]. Curiously, the motivation was to view it as an exactly solvable theory of gravity.

We can recover the gauge transformations on the boundary by assigning the transformation rule $\mathcal{A}_1 \mapsto \Lambda\mathcal{A}_1\Lambda^{-1} + id\Lambda\Lambda^{-1}$ so that $\mathcal{A}_1$ is interpreted as a background field for a global symmetry G.[22] We can proceed with the usual steps to derive the dual boundary theory. Taking the spacetime to be $X_{d+1} = B_d \times S^1$, the path integral over time components imposes

$$\widetilde{F}_2 = 0\,, \qquad\qquad D_{\widetilde{A}_1}\widetilde{b}_{d-1} = 0\,. \tag{9.5.3}$$

The first equation can be solved in terms of a G-valued scalar field U as

$$\widetilde{A}_1 = i\,\tilde{d}U\,U^{-1}\,. \tag{9.5.4}$$

To solve the second one, since the covariant derivative with respect to a flat connection squares to zero (*i.e.*, it becomes a differential), we set

$$\widetilde{b}_{d-1} = \widetilde{D}\omega_{d-2} \tag{9.5.5}$$

where ω_{d-2} is a $\mathfrak{g}$-valued $(d-2)$-form, and $\widetilde{D}$ denotes the covariant derivative with respect to $i\,\tilde{d}U\,U^{-1}$. By plugging these back, the theory reduces to a boundary action:

$$\begin{aligned} S = (-1)^d\frac{i}{2\pi}\int_{\mathcal{M}_d}\mathrm{Tr}\Big[\widetilde{D}\omega_{d-2}\wedge\Big(i\,\partial_t U\,U^{-1} - \mathcal{A}_0^t\Big)dt\Big] \\ + \frac{1}{4\pi}\int_{\mathcal{M}_d}\mathrm{Tr}\Big[\frac{1}{f_\pi^2}\,\widetilde{D}\omega_{d-2}\wedge\star\,\widetilde{D}\omega_{d-2} + f_\pi^2\Big(i\,\tilde{d}U\,U^{-1} - \widetilde{\mathcal{A}}_1\Big)\wedge\star\Big(i\,\tilde{d}U\,U^{-1} - \widetilde{\mathcal{A}}_1\Big)\Big]\,. \end{aligned} \tag{9.5.6}$$

One important difference with respect to the Abelian case is that U and ω_{d-2} do not appear symmetrically. While U appears in a complicated way, the action is still quadratic in ω_{d-2} that can thus be integrated out using its equation of motion

$$\widetilde{D}\Big(\partial_t U\,U^{-1} + i\mathcal{A}_0^t\Big)\wedge dt + \frac{(-1)^{d-1}}{f_\pi^2}\,\widetilde{D}\star\widetilde{D}\omega_{d-2} = 0\,. \tag{9.5.7}$$

Eliminating a zero-mode as in the Abelian case, we can use this equation to determine $\widetilde{D}\omega_{d-2}$, and we find the manifestly covariant form of the boundary theory:

$$S = \frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_d}\mathrm{Tr}\Big[\Big(i\,dU\,U^{-1} - \mathcal{A}_1\Big)\wedge\star\Big(i\,dU\,U^{-1} - \mathcal{A}_1\Big)\Big]\,. \tag{9.5.8}$$

This describes a sigma model with target G, coupled to a background field $\mathcal{A}_1$ for the symmetry G that acts as $U \mapsto gU$ with $g \in G$. The sigma model is a non-renormalizable theory that provides the leading universal term in an expansion in number of derivatives (in 4d this is chiral perturbation theory), describing the EFT of any theory with spontaneously broken symmetry G [230, 231].

9.5.2 Non-Abelian chiral anomaly

For any even d we can add a Chern–Simons term to the bulk theory (9.5.1):[23]

$$S_{\mathrm{CS}} = \frac{i\kappa_d}{2\pi}\int_{X_{d+1}}\mathrm{Tr}\big(\mathrm{CS}_{d+1}(A_1)\big)\,, \qquad \kappa_d = \frac{k}{(2\pi)^{\frac{d}{2}-1}\big(\frac{d}{2}+1\big)!}\,, \qquad k\in\mathbb{Z}\,, \tag{9.5.9}$$

[22] Differently from the Abelian case, here we cannot turn on another background to rescue the other gauge symmetry as well. The reason is that the gauge transformation (8.5.3) of b_{d-1} cannot be reabsorbed in the boundary condition by replacing b_{d-1} with $b_{d-1} - \mathcal{B}_{d-1}$ and assigning a transformation rule to $\mathcal{B}_{d-1}$. Indeed, this transformation would necessarily involve the dynamical field A_1, instead of the background $\mathcal{A}_1$.

[23] Here we assume G to be simple and simply connected.

that describes the presence of a perturbative anomaly for G. In this case, differently from the Abelian one, anomaly matching requires a WZW term in the spontaneously broken phase [238]. We want to show that this fact is implied by our conjecture. We also consider the case of $d = 2$ where, strictly speaking, our conjecture does not apply because there is no spontaneous breaking of a continuous symmetry in two dimensions.

Two dimensions

In the case of $d = 2$, we use the boundary condition

$$\star\left(A_1 - \mathcal{A}_1\right) = -\frac{i}{f_\pi^2}\left(b_1 + \frac{k}{2}\left(A_1 - \mathcal{A}_1\right)\right) \tag{9.5.10}$$

that is gauge invariant under $A_1 \mapsto \Lambda A_1 \Lambda^{-1} + id\Lambda\,\Lambda^{-1}$, $\mathcal{A}_1 \mapsto \Lambda \mathcal{A}_1 \Lambda^{-1} + id\Lambda\,\Lambda^{-1}$, and add the boundary term

$$S_\partial = -\frac{1}{4\pi f_\pi^2}\int_{\partial X_3} \mathrm{Tr}\Big[\left(b_1 + \tfrac{k}{2}\,A_1\right) \wedge \star\left(b_1 + \tfrac{k}{2}\,A_1\right)\Big] \tag{9.5.11}$$

to make the variational principle well defined.

As a preliminary consistency check, we compute the gauge variation. The total gauge-transformed action differs by

$$\Delta\big(S + S_\partial + S_{\text{c.t.}}\big) = \frac{ik}{4\pi}\int_{\partial X_3} \mathrm{Tr}\big(\mathcal{A}_1 \wedge i\Lambda^{-1}d\Lambda\big) + \frac{k}{24\pi}\int_{X_3} \mathrm{Tr}\big((i\Lambda^{-1}d\Lambda)^3\big) \tag{9.5.12}$$

from the original one.[24] Upon expanding $\Lambda = \mathbb{1} + \lambda_0$ and retaining only the linear order in λ_0, this reduces to the usual form of the consistent anomaly:

$$\delta\big(S + S_\partial + S_{\text{c.t.}}\big) = \frac{ik}{4\pi}\int_{\partial X_3} \mathrm{Tr}\big(\mathcal{A}_1 \wedge i d\lambda_0\big)\,. \tag{9.5.13}$$

One can proceed in determining the dual boundary theory similarly to the non-anomalous case. Since the boundary condition is essentially the same (simply written in a different parametrization), the only difference is the bulk Chern–Simons term which gives rise to a WZW term in the boundary theory:

$$\begin{aligned} S = \frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_2} \mathrm{Tr}\Big[\left(i\,dU\,U^{-1} - \mathcal{A}_1\right) \wedge \star\left(i\,dU\,U^{-1} - \mathcal{A}_1\right)\Big] + \frac{k}{12\pi}\int_{X_3} \mathrm{Tr}\Big[\left(iU^{-1}dU\right)^3\Big] \\ - \frac{ik}{4\pi}\int_{\mathcal{M}_2} \mathrm{Tr}\Big[\mathcal{A}_1 \wedge i\,dU\,U^{-1}\Big]\,. \end{aligned} \tag{9.5.14}$$

We notice that there is also a non-standard coupling to the background field, that in our approach arises because of the boundary conditions, similarly to the Abelian case. Differently from that case, however, in a purely field theoretic analysis this is not interpreted as a coupling to a diagonal symmetry (since a winding symmetry is absent here), but rather it arises from the standard trial-and-error procedure to couple the G symmetry to a background in the presence of the WZW term, similarly to the 4d analysis in [238].

For generic values of f_π^2 the theory is not conformally invariant at the quantum level. However choosing $f_\pi^2 = \frac{k}{2}$ the theory has a conserved holomorphic current which generates a Kac–Moody symmetry algebra, and it displays conformal invariance [248]. In this case we recover a form of the CS/WZW correspondence, which is more general on one side, being valid even outside of the conformal point, but less general on the other side, since in the conformal case it automatically produces the full physical WZW model instead of its chiral halves.

[24] Here $S_{\text{c.t.}} = \dfrac{k^2}{8\pi f_\pi^2}\displaystyle\int_{\partial X_3} \mathrm{Tr}\big(\mathcal{A}_1 \wedge \star\mathcal{A}_1\big)$ is a counterterm we add to simplify the final result.

Four dimensions

In the case of $d=4$, the 5d Chern–Simons term is

$$\operatorname{Tr}\big(\mathrm{CS}_5(A_1)\big) = \operatorname{Tr}\bigg(A_1 \wedge (dA_1)^2 + \frac{3i}{2}\, A_1^3 \wedge dA_1 - \frac{3}{5} A_1^5\bigg) \,. \tag{9.5.15}$$

As one might suspect already from the Abelian case, in order to obtain a gauge-invariant boundary condition with a consistent variational principle we need to introduce extra terms in the boundary condition that mix background and dynamical fields. We use the same iterative procedure discussed in Appendix D for the Abelian anomaly, even though the computations are clearly more tedious here. We find the following solution. The boundary condition is

$$\star\big(A_1 - \mathcal{A}_1\big) - \frac{i\kappa_4}{f_\pi^2}\left(\frac{1}{2}\left(\mathcal{A}_1\, \mathcal{F}_2 + \mathcal{F}_2\, \mathcal{A}_1\right) - \frac{i}{2}\, \mathcal{A}_1^3\right) = -\frac{i}{f_\pi^2}\, \Omega_3 \tag{9.5.16}$$

where $\mathcal{F}_2$ is the field strength of $\mathcal{A}_1$ while

$$\Omega_3 = b_3 + \kappa_4\bigg(F_2\big(A_1 - \mathcal{A}_1\big) + \big(A_1 - \mathcal{A}_1\big)F_2 - \frac{i}{2}\Big(\big(A_1 - \mathcal{A}_1\big)^3 + \mathcal{A}_1\Big)^3 + \frac{1}{2}\big(\mathcal{A}_1\mathcal{F}_2 + \mathcal{F}_2\mathcal{A}_1\big)\bigg) \tag{9.5.17}$$

and the boundary term is

$$\begin{aligned} S_\partial &= -\frac{1}{4\pi f_\pi^2}\int_{\partial X_5} \operatorname{Tr}\big(\Omega_3 \wedge \star\, \Omega_3\big) \;+\; S_{\text{top}} \;+\; S_{\text{c.t.}}\,, \\ S_{\text{top}} &= \frac{i\kappa_4}{2\pi}\int_{\partial X_5} \operatorname{Tr}\bigg[\frac{1}{2}\, F_2\, \mathcal{A}_1\, A_1 + \frac{1}{2}\, \mathcal{A}_1\, F_2\, A_1 - \frac{i}{4}\, A_1\, \mathcal{A}_1\, A_1\, \mathcal{A}_1 + \frac{i}{2}\, A_1^3\, \mathcal{A}_1\bigg]. \end{aligned} \tag{9.5.18}$$

The counterterm $S_{\text{c.t.}}$ is used to simplify the final expression, and it is convenient to choose it as

$$S_{\text{c.t.}} = \frac{\kappa_4^2}{4\pi f_\pi^2}\int_{\partial X_5} \operatorname{Tr}\Big[\phi(\mathcal{A}_1) \wedge \star\, \phi(\mathcal{A}_1)\Big]\,, \qquad \phi(\mathcal{A}_1) = \frac{1}{2}\Big(\mathcal{A}_1 \wedge d\mathcal{A}_1 + d\mathcal{A}_1 \wedge \mathcal{A}_1 + i\mathcal{A}_1^3\Big)\,. \tag{9.5.19}$$

The boundary condition is gauge invariant under the transformation $A_1 \mapsto \Lambda A_1 \Lambda^{-1} + i d\Lambda\, \Lambda^{-1}$, $\mathcal{A}_1 \mapsto \Lambda \mathcal{A}_1 \Lambda^{-1} + i d\Lambda\, \Lambda^{-1}$ and one can compute the total gauge variation

$$\begin{aligned} \Delta\big(S + S_\partial\big) = &-\frac{i\kappa_4}{2\pi}\int_{\partial X_5} \operatorname{Tr}\Big[\big(i\Lambda^{-1} d\Lambda\big) \wedge \phi(\mathcal{A}_1) + \frac{i}{4}\big(\mathcal{A}_1 \wedge i\Lambda^{-1} d\Lambda\big)^2 - \frac{i}{2}\big(i\Lambda^{-1} d\Lambda\big)^3 \wedge \mathcal{A}_1\Big] \\ &- \frac{i\kappa_4}{20\pi}\int_{X_5} \operatorname{Tr}\Big[\big(i\Lambda^{-1} d\Lambda\big)^5\Big]\,. \end{aligned} \tag{9.5.20}$$

Expanding $U = \mathbb{1} + \lambda_0$ to linear order, we recover the usual form of the consistent anomaly in four dimensions:

$$\delta\big(S + S_\partial\big) = -\frac{ik}{48\pi^2}\int_{\partial X_5} \operatorname{Tr}\Big[i d\lambda_0 \wedge \Big(\mathcal{A}_1 \wedge d\mathcal{A}_1 + d\mathcal{A}_1 \wedge \mathcal{A}_1 + i\mathcal{A}_1^3\Big)\Big]\,. \tag{9.5.21}$$

We can then proceed, as before, with the reduction of the action on the boundary. We find

$$\begin{aligned} S = &\frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_4} \operatorname{Tr}\Big[\Big(idU\, U^{-1} - \mathcal{A}_1\Big) \wedge \star\Big(idU\, U^{-1} - \mathcal{A}_1\Big)\Big] - \frac{ik}{240\pi^2}\int_{X_5} \operatorname{Tr}\Big[\big(iU^{-1} dU\big)^5\Big] \\ &+ \frac{ik}{48\pi^2}\int_{\mathcal{M}_4} \operatorname{Tr}\Big[idU\, U^{-1} \wedge \Big(\mathcal{A}_1 \wedge \mathcal{F}_2 + \mathcal{F}_2 \wedge \mathcal{A}_1 - \mathcal{A}_1^3\Big)\Big] \\ &+ \frac{k}{48\pi^2}\int_{\mathcal{M}_4} \operatorname{Tr}\Big[\frac{1}{2}\, idU\, U^{-1} \wedge \mathcal{A}_1 \wedge idU\, U^{-1} \wedge \mathcal{A}_1 - \big(idU\, U^{-1}\big)^3 \wedge \mathcal{A}_1\Big]\,. \end{aligned} \tag{9.5.22}$$

Turning off the background gauge field $\mathcal{A}_1$ we recognize a non-linear sigma model with target space G with a properly normalized WZW term, that describes the dynamics of Goldstone bosons. The coupling to the background $\mathcal{A}_1$ is completely fixed by the requirement of a gauge-invariant boundary condition, and correctly captures the anomaly of the non-linearly realized G symmetry.

9.5.3 Non-Abelian 2-group symmetries

In 4d one can have 2-group symmetries whose 0-form part is a non-Abelian group G, while the 1-form part is $U(1)$. These symmetry structures arise, *e.g.*, if one starts from a theory with a 0-form symmetry group $U(1) \times G$ with an 't Hooft anomaly that is linear in $U(1)$ and quadratic in G:

$$S_{\text{inflow}} = \frac{ik}{8\pi^2} \int_{X_5} dV_1 \wedge \text{Tr}\Big(A_1 \wedge dA_1 + \frac{2i}{3} A_1^3 \Big) , \tag{9.5.23}$$

and then gauges the $U(1)$ symmetry [16]. The 1-form symmetry involved in the 2-group is the magnetic symmetry of the gauged $U(1)$.

The SymTFT for this non-Abelian 2-group symmetry can be derived using the dynamical gauging procedure described in Section 8.3. Indeed one starts from the SymTFT for the $U(1) \times G$ 0-form symmetry:

$$S' = \frac{i}{2\pi} \int_{X_5} \Big[g_3 \wedge dV_1 + \text{Tr}\big(b_3 \wedge F_2\big) + \frac{k}{4\pi} dV_1 \wedge \text{Tr}\Big(A_1 \wedge dA_1 + \frac{2i}{3} A_1^3 \Big) \Big] \tag{9.5.24}$$

where g_3 and V_1 are an $\mathbb{R}$ and a $U(1)$ gauge field, respectively, b_3 is $\mathfrak{g}$-valued and A_1 is a G connection (F_2 is its field strength). Then one applies the map introduced in Section 8.3 that implements the dynamical gauging of $U(1)$ on the boundary from the viewpoint of the SymTFT. Recall that the net effect is the replacement $dV_1 \mapsto h_2$, $g_3 \mapsto dC_2$, thus the resulting SymTFT has action

$$S = \frac{i}{2\pi} \int_{X_5} \Big[h_2 \wedge dC_2 + \text{Tr}\big(b_3 \wedge F_2\big) + \frac{k}{4\pi} h_2 \wedge \text{Tr}\Big(A_1 \wedge dA_1 + \frac{2i}{3} A_1^3 \Big) \Big] . \tag{9.5.25}$$

The gauge transformations are:[25]

$$\begin{aligned} & h_2 \mapsto h_2 + d\xi_1 , && A_1 \mapsto \Lambda A_1 \Lambda^{-1} + i d\Lambda\, \Lambda^{-1} , \\ & b_3 \mapsto b_3 - \frac{k}{4\pi} \xi_1 \wedge F_2 , && C_2 \mapsto C_2 + d\eta_1 - \frac{k}{4\pi} \text{Tr}\big(A_1 \wedge i\Lambda^{-1} d\Lambda\big) + \frac{ik}{6\pi} \text{Tr}\,\Theta_2 , \end{aligned} \tag{9.5.27}$$

where Θ_2 is a locally defined real 2-form with the property that $\text{Tr}\big((i\Lambda^{-1}d\Lambda)^3\big) = d\,\text{Tr}\,\Theta_2$.

Again, we can use an iterative procedure to determine a set of gauge-invariant boundary conditions together with a boundary term that provide a good variation principle. The boundary conditions are

$$\star\,(A_1 - \mathcal{A}_1) = -\frac{i}{R^2}\Big(b_3 + \frac{k}{4\pi}(A_1 - \mathcal{A}_1) \Big) , \quad \star\, h_2 = \frac{ie^2}{\pi} \Big(C_2 - \mathcal{C}_2 - \frac{k}{4\pi} \text{Tr}\big(\mathcal{A}_1 \wedge A_1\big) \Big) \tag{9.5.28}$$

while the boundary term is

$$\begin{aligned} S_\partial = & -\frac{i}{2\pi} \int_{\partial X_5} h_2 \wedge \Big(C_2 - \frac{k}{4\pi} \text{Tr}\big(\mathcal{A}_1 \wedge A_1\big) \Big) \\ & - \frac{e^2}{4\pi^2} \int_{\partial X_5} \Big(C_2 - \frac{k}{4\pi} \text{Tr}\big(\mathcal{A}_1 \wedge A_1\big) \Big) \wedge \star \Big(C_2 - \frac{k}{4\pi} \text{Tr}\big(\mathcal{A}_1 \wedge A_1\big) \Big) \\ & - \frac{1}{4\pi R^2} \int_{\partial X_5} \text{Tr}\Big[\Big(b_3 + \frac{k}{4\pi}(A_1 - \mathcal{A}_1) \Big) \wedge \star \Big(b_3 + \frac{k}{4\pi}(A_1 - \mathcal{A}_1) \Big) \Big] . \end{aligned} \tag{9.5.29}$$

The boundary condition becomes gauge invariant by assigning the following transformations to the backgrounds $\mathcal{A}_1$ and $\mathcal{C}_2$:

$$\mathcal{A}_1 \mapsto \Lambda \mathcal{A}_1 \Lambda^{-1} + i d\Lambda\, \Lambda^{-1} , \qquad \mathcal{C}_2 \mapsto \mathcal{C}_2 + d\eta_1 - \frac{ik}{4\pi} \text{Tr}\big(\mathcal{A}_1 \wedge \Lambda^{-1} d\Lambda\big) + \frac{ik}{12\pi} \text{Tr}\,\Theta_2 . \tag{9.5.30}$$

[25] Recall that the variation of the three-dimensional Chern–Simons term is:

$$\text{Tr}\big(\text{CS}_3(A_1)\big) \;\mapsto\; \text{Tr}\big(\text{CS}_3(A_1)\big) + d\,\text{Tr}\big(A_1 \wedge i\Lambda^{-1} d\Lambda\big) - \frac{i}{3} \text{Tr}\Big(\big(i\Lambda^{-1} d\Lambda\big)^3 \Big) . \tag{9.5.26}$$

These reproduce the background gauge transformation of [16] for a non-Abelian 2-group symmetry upon expanding $U = \mathbb{1} + \lambda_0$ at first order:

$$\delta \mathcal{A}_1 = i D_{\mathcal{A}_1} \lambda_0 \,, \qquad \delta \mathcal{C}_2 = d\eta_1 - \frac{ik}{4\pi} \operatorname{Tr}\big(\mathcal{A}_1 \wedge d\lambda_0\big) \,. \tag{9.5.31}$$

It is also easy to see that the whole bulk-boundary system is gauge invariant under transformations of A_1 and C_2 provided we add a counterterm $S_{\text{c.t.}} = \frac{e^2}{4\pi^2} \int_{\partial X_5} \mathcal{C}_2 \wedge \star \mathcal{C}_2$.

We can apply our usual machinery to get the dual boundary theory. We obtain a G-valued scalar field U from A_1, and a Maxwell field a_1 from h_2, with the following boundary action:

$$\begin{aligned} S = {} & \frac{f_\pi^2}{4\pi} \int_{\mathcal{M}_4} \operatorname{Tr}\Big[\Big(idU\,U^{-1} - \mathcal{A}_1\Big) \wedge \star \Big(idU\,U^{-1} - \mathcal{A}_1\Big)\Big] + \frac{1}{4e^2} \int_{\mathcal{M}_4} da_1 \wedge \star da_1 \\ & + \frac{k}{24\pi^2} \int_{\mathcal{M}_4} a_1 \wedge \operatorname{Tr}\Big[\big(iU^{-1}dU\big)^3\Big] \\ & + \frac{i}{2\pi} \int_{\mathcal{M}_4} da_1 \wedge \operatorname{Tr}\Big[\mathcal{A}_1 \wedge iU^{-1}dU\Big] + \frac{i}{2\pi} \int_{\mathcal{M}_4} \mathcal{C}_2 \wedge da_1 \,. \end{aligned} \tag{9.5.32}$$

In the first line we recognize a non-linear sigma model with target space G and a Maxwell theory. The last line describes the coupling to the background field $\mathcal{C}_2$ for the magnetic $U(1)$ 1-form symmetry, as well as a nonstandard coupling to the background $\mathcal{A}_1$ for the symmetry G, similar to the one arising in the Abelian case in Section 9.3.4. The most interesting new thing here is the term in the second line that describes a coupling between the photon and the pions. This is a linear coupling of the photon to the current of a topological symmetry that exists in any sigma model with target G. According to our conjecture, this model is the universal EFT that describes the IR of any theory with a spontaneously broken non-Abelian 2-group symmetry. To the best of our knowledge, this universal EFT was not derived elsewhere.

Some comments on the extra Wess–Zumino-like coupling are in order. First, in any RG flow that breaks the 2-group spontaneously, this coupling must be generated as a consequence of the 2-group matching. In a sense, it is similar to the presence of the WZW term in the EFT of a spontaneously broken anomalous non-Abelian symmetry. Quite like that term, it breaks a symmetry of the EFT that would be there if $k = 0$. Indeed, for $k = 0$ the theory is separately invariant under four $\mathbb{Z}_2$ symmetries: parity $P_0 \colon x_i \mapsto -x_i$ for $i = 1, 2, 3$; photon charge conjugation $C_1 \colon a_1 \mapsto -a_1$; non-Abelian charge conjugation[26] $C_2 \colon U \mapsto U^\mathsf{T}$; pion number mod-2 $(-1)^{N_\pi} \colon U \mapsto U^{-1}$. All these four symmetries are violated by the photon-pion coupling, but the product of any two of them is preserved. Therefore the discrete symmetry for $k \neq 0$ is $(\mathbb{Z}_2)^3$ generated by

$$P = P_0 \, (-1)^{N_\pi} \,, \qquad C = C_1 \, C_2 \,, \qquad \widetilde{C} = C_1 \, (-1)^{N_\pi} \,. \tag{9.5.33}$$

The photon-pion coupling allows, for instance, a process involving three pions and one photon, which would have been forbidden otherwise. We summarize the various symmetry actions and charges in Table 9.1.

Second, the 2-group symmetry we started with could suffer from a perturbative cubic chiral anomaly for G as well. This would be described by the addition of a 5d Chern–Simons term (9.5.15) to the bulk action in (9.5.25), and would result in an extra WZW term $S_{\text{WZW}} = -\frac{ik}{240\pi^2} \int_{X_5} \operatorname{Tr}\big[(iU^{-1}dU)^5\big]$ in the 4d boundary action (9.5.32).[27] This term would further break the discrete symmetry of the EFT to $(\mathbb{Z}_2)^2$ generated by P and C, as it is clear from Table 9.1.

[26]The reason for this name will be clear in the upcoming discussion of $U(N)$ QCD.

[27]We did not work out the detailed form of the coupling to the background fields.

	Definition	x_i	$a_1\,\mathrm{Tr}\Big[\big(iU^{-1}dU\big)^3\Big]$	$\mathrm{Tr}\Big[\big(iU^{-1}dU\big)^5\Big]$
P_0	$x_i \mapsto -x_i$	-1	-1	-1
C_1	$a_1 \mapsto -a_1$	1	-1	1
C_2	$U \mapsto U^{\mathsf{T}}$	1	-1	1
$(-1)^{N_\pi}$	$U \mapsto U^{-1}$	1	-1	-1

Table 9.1: The four $\mathbb{Z}_2$ symmetries, and the corresponding phases acquired by the coordinates, the photon-pion coupling term, and the standard WZW term, respectively. Notice that while $\mathrm{Tr}\big[(iU^{-1}dU)^5\big]$ is invariant under $U \mapsto U^{\mathsf{T}}$, the term $\mathrm{Tr}\big[(iU^{-1}dU)^3\big]$ changes sign.

An application: $U(N)$ QCD. Let us present a concrete application of the effective action (9.5.32). Consider a 4d gauge theory with $U(N)$ gauge group and N_f flavors of massless Dirac fermions, so that there is a chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. It can be obtained by gauging the baryon number symmetry $U(1)_B$ in ordinary $\mathrm{SU}(N)$ QCD, hence it contains an Abelian gauge field A_μ on top of the non-Abelian gauge fields. Being weakly coupled at low energy, A_μ is not expected to drastically modify the strong coupling dynamics of the non-Abelian sector. Hence for N_f small enough, the quark bilinear takes VEV and spontaneously breaks the chiral symmetry:[28]

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \;\to\; \mathrm{SU}(N_f)_V \tag{9.5.34}$$

producing at low energy massless pions that interact as a non-linear sigma model with target space $\mathrm{SU}(N_f)$. The pions are neutral under the non-Abelian gauge symmetry $\mathrm{SU}(N)$, whose gluons are confined. However the Abelian gauge field A_μ remains even in the deep IR and there is no reason why it should be decoupled from the non-linear sigma model. Indeed, while the pion fields themselves are neutral under $U(1)$, being bound states of quarks it is a priori unclear whether there is a low-energy remnant of the quark-photon interaction.

We can answer this question using our result, and showing that the photon is not decoupled. Indeed there is a $U(1)$ magnetic 1-form symmetry from the Abelian gauge field (that is its Goldstone boson), which forms a non-trivial 2-group with $\mathrm{SU}(N_f)_L$ (and also with $\mathrm{SU}(N_f)_R$, but we can just focus on one of the two). To see this, we notice that there is a triangle anomaly $U(1)$ - $\mathrm{SU}(N_f)^2_L$ whose anomaly polynomial is

$$\mathcal{P}_{U(1)\,\text{-}\,\mathrm{SU}(N_f)^2_L} = \frac{N}{8\pi^2}\, dA \wedge \mathrm{Tr}\big(\mathcal{F} \wedge \mathcal{F}\big)\,, \tag{9.5.35}$$

where $\mathcal{F} = d\mathcal{G} + i\,\mathcal{G} \wedge \mathcal{G}$ is the field strength of the background field $\mathcal{G}$ for $\mathrm{SU}(N_f)_L$. The coefficient N comes because all left-moving fermions have charge 1 under $U(1)$ and are in the fundamental representation of the non-Abelian gauge symmetry $\mathrm{SU}(N)$. By comparison with (9.5.23) we read off that the $U(1)$ 1-form symmetry and $\mathrm{SU}(N_f)_L$ form a 2-group with $k = N$. Because of chiral symmetry breaking and spontaneous breaking of the 1-form symmetry, the 2-groups is fully broken and, from our result above, the low-energy EFT describing pions and photon is (9.5.32), plus the standard WZW

[28] Notice that the usual argument [12] based on 't Hooft anomaly matching in $\mathrm{SU}(N)$ QCD is also valid here, hence we do not really need to make the assumption that the photon does not affect chiral symmetry breaking.

term (also with coefficient N) for the pions due to the cubic $\mathrm{SU}(N_f)_L$ anomaly:[29]

$$\begin{aligned} S_{\mathrm{IR}} = \frac{f_\pi^2}{4\pi} \int_{\mathcal{M}_4} \mathrm{Tr}\Big[\big(idU\,U^{-1}\big) \wedge \star \big(idU\,U^{-1}\big)\Big] + \frac{1}{4e^2} \int_{\mathcal{M}_4} dA \wedge \star dA \\ + \frac{N}{24\pi^2} \int_{\mathcal{M}_4} A \wedge \mathrm{Tr}\Big[\big(iU^{-1}dU\big)^3\Big] - \frac{iN}{240\pi^2} \int_{X_5} \mathrm{Tr}\Big[\big(iU^{-1}dU\big)^5\Big] \,. \end{aligned} \tag{9.5.36}$$

Thus, while the pions themselves are uncharged under the $U(1)$ gauge group, the photon A is coupled with an effective current

$$J_B = -\frac{N}{24\pi^2} \star \mathrm{Tr}\Big[\big(iU^{-1}dU\big)^3\Big]. \tag{9.5.37}$$

This current is conserved, and in the absence of the pion-photon interaction it generate a global $U(1)$ symmetry of the sigma model: the topological symmetry due to the non-trivial homotopy group $\pi_3\big(\mathrm{SU}(N_f)\big) = \mathbb{Z}$. The integral of $\star J_B$ gives indeed the winding number:

$$w(\mathcal{M}_3) = -\frac{i}{24\pi^2} \int_{\mathcal{M}_3} \mathrm{Tr}\Big[\big(iU^{-1}dU\big)^3\Big] \,\in \mathbb{Z}\,. \tag{9.5.38}$$

In the $U(N)$ theory, configurations with nontrivial winding have a $U(1)$ gauge charge. These configurations are Skyrmions: solitonic objects which, in the $\mathrm{SU}(N)$ theory, are identified with the baryons [238, 250]. This is confirmed by our finding: the $U(N)$ theory is obtained from ordinary $\mathrm{SU}(N)$ QCD by gauging the baryon number symmetry, hence the baryons are no longer gauge invariant, but rather are coupled with A.

We can make this more precise as follows. In the absence of the photon-pion coupling, the operators charged under the topological $U(1)$ symmetry are local operators $\mathcal{B}_q(x)$ defined as disorder operators which impose that

$$w(S^3) = q \in \mathbb{Z} \tag{9.5.39}$$

on a 3-sphere S^3 that links with x. Similarly to the monopole operator in Chern–Simons theory, $\mathcal{B}_q(x)$ gets a gauge charge Nq due to the coupling with the photon.

Also, in the absence of the 2-group structure, the low-energy effective theory would have an emergent electric $U(1)$ 1-form symmetry shifting $A \to A + \lambda$ (with the periods of λ in the interval $[0, 2\pi]$) and acting on the Wilson lines $W_n(\gamma) = e^{in\int_\gamma A}$. Because of the photon-pion coupling, however, only a $\mathbb{Z}_N \subset U(1)$ subgroup of this 1-form symmetry emerges. Indeed using the quantization (9.5.38), shifting $A \to A + \lambda$ leaves the exponentiated action invariant only if the periods of λ are multiples of $\frac{2\pi}{N}$. An equivalent way to see this is that the Wilson line $W_{n=N}$ can terminate on the Baryon operator $\mathcal{B}_1(x)$. Notice that the microscopic theory does not have this $\mathbb{Z}_N$ 1-form symmetry, because the quarks have unit charge under the gauged $U(1)_B$. The emergence of $\mathbb{Z}_N$ has a clear interpretation: the quarks are confined and the only dynamical particles charged under $U(1)_B$ at low energy are baryons, with charges multiple of N.

As a final comment, notice that among the three $\mathbb{Z}_2$ symmetries P, C, $\widetilde{C}$ defined in (9.5.33) that are preserved by the photon-pion coupling, only P and C are preserved also by the standard WZW term, while $\widetilde{C}$ is explicitly broken (see Table 9.1). This has to do with the fact that in $U(N)$ QCD, $C_2 : U \mapsto U^{\mathsf{T}}$ is the low-energy remnant of the non-Abelian charge conjugation that, in the UV, also acts on the $\mathrm{SU}(N)$ gauge bosons, confined in the IR. In the $U(N)$ theory this charge conjugation is

[29] 2-group structures in sigma models arising in the IR of QCD-like theories have been recently considered also in [249]. The IR there, however, is purely scalar, and the 2-group is not fully spontaneously broken (the 1-form symmetry is preserved). The interaction responsible for the 2-group is not a photon-pion coupling, but rather a coupling between pions parametrizing two different target spaces. Indeed the UV model studied in [249] can be obtained from $U(N)$ QCD by adding scalars charged under $U(1)_B$ that Higgs the Abelian gauge field.

not independent from the Abelian charge conjugation C_1 acting on the photon, since the fermions are in the fundamental representation of both. Hence, only the product $C = C_1 C_2$ is a symmetry of the theory.

Appendix A

Gauging in fusion categories

In this appendix we briefly review well known material about gauging in fusion categories and modular tensor categories (possibly extended by a 0-form symmetry). A complete review of the underlying formalism can be found for instance in [50] and [77], respectively.

A.1 Gauging and algebras

Gauging a generalized symmetry in two dimensions corresponds to the definition of a special symmetric Frobenius algebra $\mathcal{A} \subset \mathcal{C}$. This is described by a triplet:

$$\mathcal{A} \equiv (\mathcal{A},\, m\,, \eta)\ , \qquad m \in \mathrm{Hom}(\mathcal{A} \times \mathcal{A}, \mathcal{A})\ , \qquad \eta \in \mathrm{Hom}(\mathbb{1}, \mathcal{A})\ , \tag{A.1.1}$$

where $\mathcal{A} = \bigoplus_{x_i} Z_i(\mathcal{A})\, x_i$ is an object in $\mathcal{C}$, and we define $Z_i(\mathcal{A}) = \dim\big(\mathrm{Hom}(\mathcal{A}, x_i)\big)$. We use x_i to denote the simple objects in $\mathcal{C}$. The maps π_i are projectors $\pi_i : \mathcal{A} \to x_i$ onto the simple components of $\mathcal{A}$ and can be used to recast the commuting diagrams below as tensor-valued expressions. The algebra morphism m trivializes the associator: $m \circ (m \times \mathrm{id}_\mathcal{A}) = m \circ (\mathrm{id}_\mathcal{A} \times m)$. Furthermore $m \circ \eta = \mathrm{id}_\mathcal{A}$. We will henceforth suppress η for simplicity. The algebra also has a dual structure

$$(\Delta, \bar{\eta})\ , \qquad \Delta \in \mathrm{Hom}(\mathcal{A}, \mathcal{A} \times \mathcal{A})\ , \qquad \bar{\eta} \in \mathrm{Hom}(\mathcal{A}, \mathbb{1}) \tag{A.1.2}$$

satisfying $m \circ \Delta = \bar{\eta} \circ \Delta = \mathrm{id}_\mathcal{A}$. Furthermore Δ and m satisfy the so-called Frobenius condition, namely that the following diagram commutes, ensuring that crossing moves from any direction can be performed safely:

$$\begin{array}{ccccc} \mathcal{A} \times \mathcal{A} & & \xrightarrow{\Delta \times \mathrm{id}_\mathcal{A}} & & \mathcal{A} \times \mathcal{A} \times \mathcal{A} \\ \big\downarrow{\scriptstyle \mathrm{id}_\mathcal{A} \times \Delta} & \searrow^{m} & \mathcal{A} & \searrow^{\Delta} & \big\downarrow{\scriptstyle \mathrm{id}_\mathcal{A} \times m} \\ \mathcal{A} \times \mathcal{A} \times \mathcal{A} & & \xrightarrow{m \times \mathrm{id}_\mathcal{A}} & & \mathcal{A} \times \mathcal{A} \end{array} \tag{A.1.3}$$

In three dimensions an algebra must satisfy an additional condition which ensures that it is compatible with the braided structure:

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{b} & \mathcal{A} \times \mathcal{A} \\ & \searrow^{m} & \big\downarrow{\scriptstyle m} \\ & & \mathcal{A} \end{array} \tag{A.1.4}$$

Such an algebra is called *commutative*. The gauging of a symmetry $\mathcal{A}$ is implemented by inserting a network of $\mathcal{A}$ defects with morphisms m and Δ at three-valent junctions, along a graph that is dual to a triangulation of the spacetime manifold.

To understand the symmetry of the theory after gauging we must introduce the concept of modules. First, let us introduce the category of (left) $\mathcal{A}$-modules $\mathrm{Mod}_\mathcal{A}$. Its elements are doublets (M, r_L) with

M an object an $r_L \in \text{Hom}(\mathcal{A} \times M, M)$ a morphism allowing the algebra object to end on M. The morphism r_L must satisfy a natural compatibility condition:

$$\begin{array}{ccc} \mathcal{A}\times\mathcal{A}\times M & \xrightarrow{r_L} & \mathcal{A}\times M \\ \big\downarrow m & & \big\downarrow r_L \\ \mathcal{A}\times M & \xrightarrow{\quad r_L \quad} & M \end{array} \tag{A.1.5}$$

This equation allows us to interpret r_L as a sort of representation of the algebra $\mathcal{A}$ on M. Physically the category $\text{Mod}_{\mathcal{A}}$ describes an $\mathcal{A}$-invariant boundary condition. In two dimensions the category describing the symmetry after gauging $\mathcal{A}$ is the bimodule category $\text{Bimod}_{\mathcal{A}-\mathcal{A}}$ of $\mathcal{A}$-bimodules. A bimodule (B, r_L, r_R) is both a left and a right module for $\mathcal{A}$, such that the left and right actions commute:

$$\begin{array}{ccc} \mathcal{A}\times B\times\mathcal{A} & \xrightarrow{r_L} & B\times\mathcal{A} \\ \big\downarrow r_R & & \big\downarrow r_R \\ \mathcal{A}\times B & \xrightarrow{\quad r_L \quad} & B \end{array} \tag{A.1.6}$$

In three dimensions, instead, the category describing the symmetry after gauging $\mathcal{A}$ is that of local modules $\text{Mod}_{\mathcal{A}}^{\text{loc}}$ of the commutative algebra $\mathcal{A}$. These are modules which are compatible with braiding with $\mathcal{A}$. In particular, given a left-module morphism r_L, we define the right morphism r_R as

$$r_R = r_L \circ b \tag{A.1.7}$$

with the consistency condition:

$$\begin{array}{ccc} \mathcal{A}\times M & \xrightarrow{b\circ b} & \mathcal{A}\times M \\ & \searrow^{r_L} & \big\downarrow r_L \\ & & M \end{array} \tag{A.1.8}$$

This implements the intuition that the objects remaining after gauging $\mathcal{A}$ must braid trivially with $\mathcal{A}$. It is known that the dimension of the category of local modules is

$$\dim\big(\text{Mod}_{\mathcal{A}}^{\text{loc}}\big) = \frac{\dim(\mathcal{C})}{\dim(\mathcal{A})} \,, \qquad\qquad \dim(\mathcal{A}) \equiv \sum_{x_i \text{ simple}} Z_i(\mathcal{A}) \dim(x_i) \,. \tag{A.1.9}$$

Since the dimension of a fusion category must be ≥ 1, there is a notion of maximality in gauging commutative algebras, which implies that

$$\dim(\mathcal{A}) \leq \dim(\mathcal{C}) \,. \tag{A.1.10}$$

When the inequality is saturated the algebra $\mathcal{A}$ is called *Lagrangian* and is denoted by the letter $\mathcal{L}$.

There exist standard techniques to construct the category of modules, which employ the fact that the formal tensor product $\text{Ind}_{\mathcal{A}}(x_i) = \mathcal{A} \times x_i$ gives a (reducible) left $\mathcal{A}$-module. Such modules are called "induced" and the construction of $\text{Mod}_{\mathcal{A}}$ boils down to the decomposition of induced modules. The interested reader can consult [50, 152] for a review of these techniques.

A.2 Theories with a 0-form symmetry

Let us also recall some facts about 3d theories enriched with a 0-form symmetry G. These are the so-called G-crossed extensions and we refer to [77] for a complete review. A G-crossed extension is described by a graded tensor category

$$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g \tag{A.2.1}$$

with $\mathcal{C}_g$ the g-twisted sector of the 0-form symmetry. This can be thought of as a 2-category $\Sigma\mathcal{C}$ with a single connected component, $\left|\pi_0(\Sigma\mathcal{C})\right| = 1$, in which the twist defects provide a basis for the homomorphisms $\sigma_g \in \text{Hom}(U_g, \mathbb{1})$. We use x_i to denote simple objects in the untwisted sector, $\sigma_{g,i}$ to denote simple twist defects, and X_g to denote generic twist defects. The fusion product on twist defects is graded:

$$\mathcal{C}_g \times \mathcal{C}_h \subset \mathcal{C}_{gh} \; . \tag{A.2.2}$$

The 0-form symmetry naturally acts on the defects in $\mathcal{C}$ via an automorphism U of the fusion algebra: we write $U_g[X_h] = g(X)_{ghg^{-1}} \in \mathcal{C}_{ghg^{-1}}$. In the following we will restrict to Abelian 0-form symmetries G. The symmetry then acts on the junction spaces $V^{(gh,k)}_{(g,i),(h,j)}$ by (unitary) isomorphisms

$$\mathcal{U}_g : V^{(g_1 g_2,k)}_{(g_1,i),(g_2,j)} \to V^{(g_1 g_2,g(k))}_{(g_1,g(i)),(g_2,g(j))} \; , \tag{A.2.3}$$

while the G composition law is encoded in a morphism

$$\lambda_{x_i}(g,h) : \; g\big(h(x_i)\big) \to gh(x_i) \; . \tag{A.2.4}$$

The category comes with graded associator α and braiding isomorphism $b : X_g \times Y_h \to g(Y)_h \times X_g$, satisfying G-crossed versions of the pentagon and hexagon equations. The number of simple objects $\sigma_{g,i}$ in the g-twisted sector is equal to the number of g-invariant local lines $x_i \in \mathcal{C}_0$, such that $g(x_i) = x_i$. This follows from modularity of the Hilbert space on T^2 with G backgrounds. The dimension of each graded category $\mathcal{C}_g$ is the same, thus:

$$\dim(\mathcal{C}_g) = \dim(\mathcal{C}_0) \; , \qquad\qquad \dim(\mathcal{C}) = |G|\dim(\mathcal{C}_0) \; . \tag{A.2.5}$$

A.3 Gauging and equivariantization

There are two natural operations that can be introduced in this setting. The first one is gauging the 0-form symmetry G (or a subgroup thereof). This leads to a larger modular tensor category $\mathcal{C}/G$ which has dimension:

$$\dim(\mathcal{C}/G) = |G|\dim(\mathcal{C}) \; . \tag{A.3.1}$$

The category $\mathcal{C}/G$ has an anomaly-free 1-form symmetry $\text{Rep}(G) = G^\vee$ that assigns charges $\in G$ to the liberated g-twisted sectors. The category after gauging is thus still graded by this charge:

$$\mathcal{C}/G = \bigoplus\nolimits_{g\in G} \mathcal{D}_g \; . \tag{A.3.2}$$

The way in which simple objects of $\mathcal{C}/G$ are constructed is familiar from the theory of orbifolds. A simple object $\sigma_{g,i}$ before gauging is equivariantized into an orbit $\Sigma_{g,i}$ after gauging:

$$\Sigma_{g,i} = \bigoplus_{h\in G/\text{Stab}(\sigma_{g,i})} h(\sigma_{g,i}) \; , \tag{A.3.3}$$

where $\text{Stab}(X) = \{g \in G : g(X) = X\}$ is the stabilizer group of X. The object $\Sigma_{g,i}$ can furthermore be dressed by symmetry lines carrying a representation π of $\text{Stab}(\sigma_{g,i})$. We thus get the lines $\Sigma^\pi_{g,i}$, whose number is $\left|\text{Stab}(\sigma_{g,i})\right|$.

The second operation is gauging an algebra $\mathcal{A} \subset \mathcal{C}_0$. Let $H \subset G$ be the stabilizer of $\mathcal{A}$, namely $H = \{g \in G : g(\mathcal{A}) = \mathcal{A}\}$. We say that $\mathcal{A}$ preserves a subgroup H of the 0-form symmetry. In order to fully specify an H-invariant algebra, we must also associate a consistent H-action to the data (m, η). This constitutes an *equivariantization* of $\mathcal{A}$ and it is generally not unique nor it is guaranteed

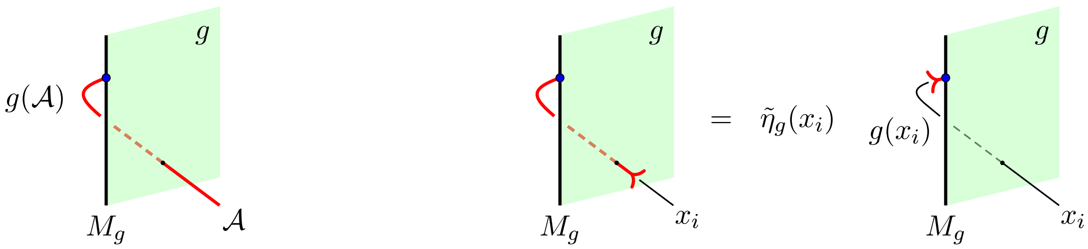


Figure A.1: Action of G on $\mathcal{A}$, both abstractly (left) and in components (right).

to exist. The required conditions are simple to summarize. First, we require the stabilizer H to leave the algebra morphism fixed:

$$m^{g(z)}_{g(x),\,g(y)} = m^z_{x,y}\ [\mathcal{U}_g]^z_{x,y}\ \frac{\tilde{\eta}_g(z)}{\tilde{\eta}_g(x)\,\tilde{\eta}_g(y)} \qquad \text{for all } g \in H. \tag{A.3.4}$$

In order to write this equation in components, one needs to define the projectors $\pi_x : \mathcal{A} \to x$ and the maps $\tilde{\eta}_g(x)$ that represent the action of H on the projectors:

$$\pi_{x_i} \to \tilde{\eta}_g(x_i)\,\pi_{g(x_i)}\ . \tag{A.3.5}$$

Furthermore, the maps $\tilde{\eta}_g(x)$ must compose nicely under the H action:

$$\tilde{\eta}_g(x_i)\ \tilde{\eta}_h\big(g(x_i)\big) = \tilde{\eta}_{gh}(x_i)\ \lambda_{x_i}(g,h)\ , \tag{A.3.6}$$

where the morphisms $\lambda_x(g,h)$ are the ones we defined in (A.2.4).

A solution to the equations (A.3.4)–(A.3.6) is not guaranteed to exist, and its existence is tied to the splitting of a certain short exact sequence [146]. Even if a solution exists, it must be modded out by the appropriate gauge transformations. Suppose that $\mathrm{Hom}(\mathcal{A}, x_i)$ is at most one-dimensional, then $\tilde{\eta}_g$ is a 1-cochain and we can redefine

$$\pi_{x_i} \to \mu(x_i)\,\pi_{x_i}\ , \qquad \tilde{\eta}_g(x_i) \to \tilde{\eta}_g(x_i)\,\frac{\mu(g(x_i))}{\mu(x_i)}\ . \tag{A.3.7}$$

Once this is settled, gauging $\mathcal{A}$ preserves the subgroup H of the 0-form symmetry. The resulting category is: $\mathcal{C}/\mathcal{A} = \bigoplus_{h\in H} \mathcal{C}_h/\mathcal{A}$, and each entry has dimension $\dim(\mathcal{C}_h/\mathcal{A}) = \dim(\mathcal{C}_0)/\dim(\mathcal{A})$.

Lastly, let us describe the objects of the twisted category $\mathcal{C}/\mathcal{A}$. Since $\mathcal{A}$ has trivial grading, it is possible to define twisted module categories $\mathrm{Mod}^g_{\mathcal{A}}$ in terms of doublets (M_g, r_L) where $M_g \in \mathcal{C}_g$ and r_L is a left map $r_L : \mathcal{A} \times M_g \to M_g$. The interesting part of the construction involves making these modules local. In particular, the braiding map $b : M_g \times \mathcal{A} \to \mathcal{A} \times M_g$ induces a nontrivial action of g on the module morphism r_L that in components maps

$$r_L(x_i)\ \to\ \tilde{\eta}_g(x_i)\, r_L\big(g(x_i)\big)\ , \tag{A.3.8}$$

as in the pictures of Figure A.1. The local bimodule condition is encoded in the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A} \times M_g & \xrightarrow{\ b\ } & M_g \times \mathcal{A} & \xrightarrow{\ b\ } & g(\mathcal{A}) \times M_g \\ & \searrow^{r_L} & & & \downarrow{\scriptstyle g(r_L)} \\ & & & & M_g \end{array} \tag{A.3.9}$$

or, in components,

$$r_L(x_i) = \tilde{\eta}_g(x_i)\ R_{x_i,\,M_g} \cdot R_{M_g,\,x_i} \cdot r_L\big(g(x_i)\big)\ . \tag{A.3.10}$$

Thus the specification of $\tilde{\eta}$ influences the structure of the H-twisted sectors after gauging $\mathcal{A}$.

Appendix B

Some technical results about Lagrangian algebra in DW theories

B.1 Lagrangian algebras for DW theories

Here we prove that any Lagrangian algebra of DW($\mathbb{A}$) is of the form

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big(b, \beta\psi_\nu(b)\big) \;\Big|\; b \in \mathbb{B}\,,\ \beta \in N(\mathbb{B}) \Big\} \tag{B.1.1}$$

for some subgroup $\mathbb{B} \subset \mathbb{A}$ and a class $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$, and that the associated boundary condition corresponds to a theory obtained from the electric boundary by gauging $\mathbb{B}$ with discrete torsion $[\nu]$.

We denote by $\pi_{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A}$ and $\pi_{\mathbb{A}^\vee} : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A}^\vee$ the projections on the two factors. Let $\mathcal{L} \subset \mathbb{A} \times \mathbb{A}^\vee$ be Lagrangian. We define a subgroup of $\mathbb{A}$

$$\mathbb{B} = \pi_{\mathbb{A}}(\mathcal{L}) \ \subset \mathbb{A}\,. \tag{B.1.2}$$

Notice that $(\mathbb{B}, 0)$ is not necessarily a subgroup of $\mathcal{L}$. On the other hand, any element of the form $(0, \beta)$ with $\beta \in N(\mathbb{B})$ has trivial braiding with any element of $\mathcal{L}$, and since $\mathcal{L}$ is maximal, it follows that $\big(0, N(\mathbb{B})\big)$ is a subgroup of $\mathcal{L}$ and thus $N(\mathbb{B}) \subset \pi_{\mathbb{A}^\vee}(\mathcal{L})$. Using the short exact sequence

$$1 \longrightarrow N(\mathbb{B}) \longrightarrow \mathbb{A}^\vee \longrightarrow \mathbb{B}^\vee \longrightarrow 1 \tag{B.1.3}$$

we realize any element of $\mathbb{A}^\vee$, and in particular of $\pi_{\mathbb{A}^\vee}(\mathcal{L})$, as a pair $\beta\omega$ with $\beta \in N(\mathbb{B})$ and $\omega \in \mathbb{B}^\vee$. All elements of $\mathcal{L}$ are then of the form $(b, \beta\omega)$ with $b \in \mathbb{B}$, $\beta \in N(\mathbb{B})$ and $\omega \in \mathbb{B}^\vee$, but since $|\mathcal{L}| = |\mathbb{A}| = |\mathbb{B}|\big|N(\mathbb{B})\big|$ there must exist a homomorphism $\psi : \mathbb{B} \to \mathbb{B}^\vee$ such that

$$\omega = \psi(b)\,. \tag{B.1.4}$$

The fact that $\mathcal{L}$ is Lagrangian and so all its elements have vanishing spin implies a constraint on ψ. Defining a bicharacter $\chi : \mathbb{B} \times \mathbb{B} \to U(1)$ as $\chi(b_1, b_2) = \psi(b_1)\, b_2$, and then imposing that $\big(b, \beta\psi(b)\big)$ has trivial spin, we obtain

$$1 = \theta_{(b,\,\beta\psi(b))} = \chi(b, b)\,. \tag{B.1.5}$$

Thus χ is alternating and it defines a class $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$, hence $\mathcal{L} = \mathcal{L}_{\mathbb{B},[\nu]}$.

Now we aim to prove that the boundary defined by $\mathcal{L}_{\mathbb{B},[\nu]}$, where the symmetry is

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L}_{\mathbb{B},[\nu]} \ \cong\ \mathcal{L}^\vee_{\mathbb{B},[\nu]}\,, \tag{B.1.6}$$

is obtained from the electric boundary by gauging $\mathbb{B}$ with discrete torsion $[\nu]$. First we notice that $\mathcal{L}_{\mathbb{B},[\nu]}$ is an extension of $\mathbb{B}$ by $N(\mathbb{B})$ determined as follows. Let $\tilde{c} \in H^2\big(\mathbb{B}^\vee, N(\mathbb{B})\big)$ be the class associated with the short exact sequence (B.1.3). This class is determined by Pontryagin duality from $1 \to \mathbb{B} \to \mathbb{A} \to \mathbb{A}/\mathbb{B} \to 1$, which is associated with a class $c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B})$.[1] The class $\tilde{c}$ enters in the composition rule of elements of $\mathbb{A}^\vee$ when they are represented as pairs $\beta\eta$, $\beta \in N(\mathbb{B})$, $\eta \in \mathbb{B}^\vee$:

$$\beta_1\eta_1 + \beta_2\eta_2 = \big(\beta_1 + \beta_2 - \tilde{c}(\eta_1, \eta_2)\big)\,(\eta_1 + \eta_2)\;. \tag{B.1.8}$$

The inverse is $-\beta\eta = \big(-\beta + \tilde{c}(\eta, -\eta)\big)(-\eta)$. The elements of $\mathcal{L}_{\mathbb{B},[\nu]}$ can be realized as pairs of $b \in \mathbb{B}$ and $\beta \in N(\mathbb{B})$ with $[b, \beta] \equiv \big(b,\, \beta\psi_\nu(b)\big)$, and their composition law is

$$[b_1, \beta_1] + [b_2, \beta_2] = \Big[b_1 + b_2\,,\; \beta_1 + \beta_2 - \tilde{c}\big(\psi_\nu(b_1), \psi_\nu(b_2)\big)\Big]\;. \tag{B.1.9}$$

We conclude that $\mathcal{L}_{\mathbb{B},[\nu]}$ is an extension

$$1 \longrightarrow N(\mathbb{B}) \longrightarrow \mathcal{L}_{\mathbb{B},[\nu]} \longrightarrow \mathbb{B} \longrightarrow 1 \tag{B.1.10}$$

determined by the class $\psi_\nu^*(\tilde{c}) \in H^2\big(\mathbb{B}, N(\mathbb{B})\big)$. Taking the Pontryagin dual of (B.1.10) we get

$$1 \longrightarrow \mathbb{B}^\vee \xrightarrow{\;\iota\;} \mathcal{S} \xrightarrow{\;\pi\;} \mathbb{A}/\mathbb{B} \longrightarrow 1\;, \tag{B.1.11}$$

whose associated class is $\widehat{c} \equiv \psi_\nu \circ c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$.

To show that this is the correct symmetry structure of the boundary theory obtained by gauging $\mathbb{B}$ with discrete torsion $[\nu]$, we consider its partition function coupled to a background

$$B = \iota(B_1) + s(B_2) \quad \in H^1(X_2, \mathcal{S})\;, \tag{B.1.12}$$

where $s : \mathbb{A}/\mathbb{B} \to \mathcal{S}$ is a section of π and B_1, B_2 are gauge fields valued in $\mathbb{B}^\vee$ and $\mathbb{A}/\mathbb{B}$, respectively. Closure $dB = 0$ implies that $dB_2 = 0$, whilst the differential of B_1 is equal to the pull-back through B_2 of the extension class $\widehat{c} \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$, namely $(dB_1)_{ijk} = \widehat{c}\,(B_{2ij}, B_{2jk}) \equiv (B_2^*\,\widehat{c}\,)_{ijk}$. On the other hand, the dynamical gauge field B' valued in $\mathbb{B}$ must satisfy $dB' = B_2^*\,c$ in the presence of a background B_2, and the partition function is thus

$$Z_{\mathbb{B},[\nu]} = \sum_{B' \text{ s.t. } dB' = B_2^* c} \exp\left[\int_{X_2} \Big(B'^*\nu + B_1 \cup B'\Big)\right]\, Z_{\mathrm{e}}[B', B_2]\;. \tag{B.1.13}$$

The exponent is not gauge invariant under $B' \to B' + d\rho$ unless B_1 satisfies

$$\psi_\nu(dB') - dB_1 = 0\;. \tag{B.1.14}$$

This determines the modified cocycle condition for B_1 as

$$dB_1 = B_2^*(\psi_\nu \circ c)\;, \tag{B.1.15}$$

hence proving that $\mathcal{S}$ is the correct symmetry after gauging $\mathbb{B}$ with discrete torsion $[\nu]$.

---

[1] Given an Abelian extension $1 \to \mathbb{A} \xrightarrow{i} \mathbb{B} \xrightarrow{\pi} \mathbb{C} \to 1$ with section $s : \mathbb{C} \to \mathbb{B}$, the class $[\epsilon] \in H^2(\mathbb{C}, \mathbb{A})$ has representative $i\big(\epsilon(c_1, c_2)\big) = s(c_1 + c_2) - s(c_1) - s(c_2)$ which is symmetric. For each $\alpha \in \mathbb{A}^\vee$, $\alpha\epsilon : \mathbb{C} \times \mathbb{C} \to U(1)$ is a symmetric 2-cocycle and is thus exact (see Sec. 6.2.1), therefore there exists $\beta : \mathbb{C} \times \mathbb{A}^\vee \to U(1)$ such that (in additive notation):

$$\alpha\epsilon(c_1, c_2) = \beta(c_1 + c_2, \alpha) - \beta(c_1, \alpha) - \beta(c_2, \alpha) \qquad \forall\;\; c_1, c_2 \in \mathbb{C},\;\; \alpha \in \mathbb{A}^\vee\;. \tag{B.1.7}$$

Construct $\Omega(c, \alpha_1, \alpha_2) = \beta(c, \alpha_1 + \alpha_2) - \beta(c, \alpha_1) - \beta(c, \alpha_2) \in U(1)$. One checks that this is linear in the first entry in $\mathbb{C}$, and thus it defines a map $\epsilon^\vee : \mathbb{A}^\vee \times \mathbb{A}^\vee \to \mathbb{C}^\vee$. This is the class of the Abelian extension $1 \to \mathbb{C}^\vee \to \mathbb{B}^\vee \to \mathbb{A}^\vee \to 1$, that reproduces the sum in (B.1.8) if we use the pairing $(\gamma, \alpha)(a, c) = \gamma(c) + \alpha(a) + \beta(c, \alpha)$.

B.2 General duality-invariant Lagrangian algebras

Now we report technical details regarding Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$ for general $\mathbb{B}$. In particular, we give the conditions for their duality invariance and compute the mixed 't Hooft anomaly with the invertible duality symmetry in those cases, extending the discussion for $\mathbb{B}=\mathbb{A}$ given in the main text. For concreteness we look at the 2d/3d case, but the 4d/5d one is analogous.

B.2.1 Proof of duality invariance

In Section 6.2.3 after (6.2.46) we claimed that a Lagrangian algebra $\mathcal{L}_{\mathbb{B},[\nu]}$ as in (6.2.38) is duality invariant, namely the isomorphism Φ in (6.2.22) acts as $\Phi\big(\mathcal{L}_{\mathbb{B},[\nu]}\big)=\mathcal{L}_{\mathbb{B},[\nu]}$, if and only if

1. $\phi\big(\mathrm{Rad}(\nu)\big)=N(\mathbb{B})$;
2. the isomorphism $\sigma=\phi^{-1}\circ\psi_\nu$ acting on $\mathbb{B}/\mathrm{Rad}(\nu)$ satisfies $\sigma^2=\mathbb{1}$.

Let us prove the claim. To prove it, we first notice that since $\mathcal{L}_{\mathbb{B},[\nu]}$ and $\Phi\big(\mathcal{L}_{\mathbb{B},[\nu]}\big)$ are both Lagrangian, they are equal if and only if all their lines are mutually transparent. In other words, if and only if

$$\phi(b')\,b\,\cdot\,\beta\big[\phi^{-1}\big(\beta'\psi_\nu(b')\big)\big]\,\cdot\,\psi_\nu(b)\big[\phi^{-1}\big(\beta'\psi_\nu(b')\big)\big]=1 \tag{B.2.1}$$

for all $b,b'\in\mathbb{B}$ and $\beta,\beta'\in N(\mathbb{B})$.

First we prove that the two conditions above are necessary. Recall that $\mathrm{Rad}(\nu)=\mathrm{Ker}(\psi_\nu)$, and notice that $\phi(b)\,b'=\phi(b')\,b$ while $\psi_\nu(b)\,b'=\big[\psi_\nu(b')\,b\big]^{-1}$. Specializing (B.2.1) to $\beta=1$ (in multiplicative notation) and $b\in\ker(\psi_\nu)$ we get $\phi(b)\in N(\mathbb{B})$ and thus $\phi\big(\mathrm{Ker}(\psi_\nu)\big)\subset N(\mathbb{B})$. Specializing (B.2.1) to $\beta=1$ and $b'=1$ we get

$$1=\psi_\nu(b)\big(\phi^{-1}(\beta')\big)=\big[\psi_\nu\big(\phi^{-1}(\beta')\big)\,b\big]^{-1}\;, \tag{B.2.2}$$

thus $\phi^{-1}\big(N(\mathbb{B})\big)\subset\mathrm{Ker}(\psi_\nu)$. We conclude that $\phi\big(\mathrm{Ker}(\psi_\nu)\big)=N(\mathbb{B})$ which is condition 1. Specializing (B.2.1) to $\beta=\beta'=1$ we get $\gamma(b',b)=\chi_\nu\big(\phi^{-1}\circ\psi_\nu(b'),b\big)$ for all $b,b'\in\mathbb{B}$. Assuming condition 1., both sides project consistently to $\mathbb{B}/\mathrm{Rad}(\nu)$, and thus $\phi(b')=\psi_\nu\big(\sigma(b')\big)\in\mathbb{B}/\mathrm{Rad}(\nu)$ for all $b'\in\mathbb{B}/\mathrm{Rad}(\nu)$. We conclude that $\sigma^2=\mathbb{1}$, which is condition 2.

Conversely, we prove that the two conditions are also sufficient. From condition 1. it follows that $\phi^{-1}(\beta')\in\mathrm{Ker}(\psi_\nu)\subset\mathbb{B}$, therefore $\beta\big(\phi^{-1}(\beta')\big)=\psi_\nu\big(\phi^{-1}(\beta')\big)=1$. Similarly $\beta\big(\phi^{-1}\circ\psi_\nu(b')\big)=1$. Eqn. (B.2.1) then reduces to

$$\phi(b')\,b\,\cdot\,\psi_\nu(b)\big(\phi^{-1}\circ\psi_\nu(b')\big)=1\;, \tag{B.2.3}$$

that can be rewritten as $\gamma(b',b)=\chi_\nu\big(\sigma(b'),b\big)=\gamma\big(\sigma^2(b'),b\big)$ using the definition of σ. Both sides project consistently to $\mathbb{B}/\mathrm{Rad}(\nu)$, and the equation is satisfied using condition 2. This completes the proof.

It will be useful to discuss a few consequence of the theorem. Each of the commuting diagrams below expresses the fact that ϕ is a group isomorphism between the respective Abelian groups.

- Since $\phi\big(\mathrm{Rad}(\nu)\big)=N(\mathbb{B})$, then the short exact sequence $1\to\mathrm{Rad}(\nu)\to\mathbb{A}\to\mathbb{A}/\mathrm{Rad}(\nu)\to 1$ is the image under ϕ^{-1} of $1\to N(\mathbb{B})\to\mathbb{A}^\vee\to\mathbb{B}^\vee\to 1$. In other words there is a commutative diagram:

$$\begin{array}{llllllllll} S_1: & 1 & \longrightarrow & N(\mathbb{B}) & \longrightarrow & \mathbb{A}^\vee & \longrightarrow & \mathbb{B}^\vee & \longrightarrow & 1 \\ & & & \uparrow\phi & & \uparrow\phi & & \uparrow\phi & & \\ S_2: & 1 & \longrightarrow & \mathrm{Rad}(\nu) & \longrightarrow & \mathbb{A} & \longrightarrow & \mathbb{A}/\mathrm{Rad}(\nu) & \longrightarrow & 1 \end{array} \tag{B.2.4}$$

- Taking the Pontryagin dual of the diagram (B.2.4) and using the symmetry of ϕ, namely that $\phi^\vee = \phi$, we obtain an other commutative diagram:

$$\begin{array}{lccccccccc} S_3 = S_1^\vee : & 1 & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{A} & \longrightarrow & \mathbb{A}/\mathbb{B} & \longrightarrow & 1 \\ & & & \downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \\ S_4 = S_2^\vee : & 1 & \longrightarrow & N\big(\mathrm{Rad}(\nu)\big) & \longrightarrow & \mathbb{A}^\vee & \longrightarrow & \mathrm{Rad}(\nu)^\vee & \longrightarrow & 1 \end{array} \tag{B.2.5}$$

- It is simple to prove that there is a canonical isomorphism $N\big(\mathrm{Rad}(\nu)\big)/N(\mathbb{B}) \cong \big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee$. Then using that $\phi(\mathbb{B}) = N\big(\mathrm{Rad}(\nu)\big)$ and $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B})$, we find a commutative diagram:

$$\begin{array}{lccccccccc} S_5 : & 1 & \longrightarrow & \mathrm{Rad}(\nu) & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{B}/\mathrm{Rad}(\nu) & \longrightarrow & 1 \\ & & & \downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \\ S_6 : & 1 & \longrightarrow & N(\mathbb{B}) & \longrightarrow & N\big(\mathrm{Rad}(\nu)\big) & \longrightarrow & \big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee & \longrightarrow & 1 \end{array} \tag{B.2.6}$$

as well as its Pontryagin dual:

$$\begin{array}{lccccccccc} S_7 = S_5^\vee : & 1 & \longrightarrow & \big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee & \longrightarrow & \mathbb{B}^\vee & \longrightarrow & \mathrm{Rad}(\nu)^\vee & \longrightarrow & 1 \\ & & & \uparrow\phi & & \uparrow\phi & & \uparrow\phi & & \\ S_8 = S_6^\vee : & 1 & \longrightarrow & \mathbb{B}/\mathrm{Rad}(\nu) & \longrightarrow & \mathbb{A}/\mathrm{Rad}(\nu) & \longrightarrow & \mathbb{A}/\mathbb{B} & \longrightarrow & 1 \end{array} \tag{B.2.7}$$

B.2.2 Mixed anomaly in the general case

The discussion in this appendix is technical and it involves some notation. We will use several short exact sequences which we denote uniformly as

$$S_m : \quad 1 \longrightarrow \mathbb{B}_m \xrightarrow{\iota_m} \mathbb{A}_m \xrightarrow{\pi_m} \mathbb{A}_m/\mathbb{B}_m \longrightarrow 1 \;, \tag{B.2.8}$$

where ι_m, π_m, s_m denote respectively the inclusion, the projection, and a section of π_m. Each sequence S_m induces an extension class $c_m \in H^2(\mathbb{A}_m/\mathbb{B}_m, \mathbb{B}_m)$. The sequences that will be used are the $S_1, \ldots, S_8$ introduced in Appendix B.2.1 above.

Moreover, we will systematically decompose gauge fields valued in $\mathbb{A}_m$ in terms of gauge fields values in the subgroup and the quotient according to

$$a_m = \iota_m(b_m) + s_m(b'_m) \;. \tag{B.2.9}$$

As discussed after (B.1.12), closure $da_m = 0$ of the gauge field implies that

$$db'_m = 0 \;, \qquad\qquad db_m = b'^{*}_m(c_m) \;. \tag{B.2.10}$$

All fields have a subscript labelling the corresponding short exact sequence, and both the field valued in the subgroup $\mathbb{B}_m$ and in the quotient are denoted by the same letter, but the one in the quotient is always primed. An important remark is in order. The relations (B.2.10) mean that in the presence of a non-trivial extension, the background b_m for the subgroup is the sum of an ordinary cohomology class and a particular co-chain solving the constraint (B.2.10), which depends on the background for the quotient. Hence all path integrals are intended to be done in order: one first integrates the cohomology part of the background b_m for the subgroup, and then the background b'_m for the quotient.

Let $\mathcal{L}_D$ be a duality-invariant algebra associated with $(\mathbb{B}, [\nu])$. We want to compute the mixed anomaly between $\mathcal{S} = \mathcal{Z}(\mathbb{A})/\mathcal{L}_D$ and the duality $G \cong \mathbb{Z}_2$ on the invertible boundary. This is obtained from the electric boundary by gauging $\mathbb{B}$ with discrete torsion $[\nu]$. A gauge field $A \in H^1(X, \mathbb{A})$ can be decomposed according to the sequence S_3 in (B.2.5) as

$$A = \iota_3(b_3) + s_3(b'_3) \;. \tag{B.2.11}$$

After gauging $\mathbb{B}$ with torsion, the dual symmetry $\mathcal{S}$ is an extension of $\mathbb{A}/\mathbb{B}$ by $\mathbb{B}^\vee$ with extension class $\widehat{c} = \psi_\nu \circ c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$ (see Appendix B.1), and a background field for $\mathcal{S}$ is described by a pair B, b_3' valued in $\mathbb{B}^\vee$ and $\mathbb{A}/\mathbb{B}$, respectively, with $dB = b_3'^*(\widehat{c})$. The partition function on the invertible boundary is

$$Z_{\text{inv}}[B, b_3'] = \sum_{b_3} \exp\left[2\pi i \int \Big(b_3^*(\nu) + B \cup b_3\Big)\right] Z_{\text{e}}[b_3, b_3'] \,. \tag{B.2.12}$$

By acting with the duality on the electric boundary we get the magnetic one, corresponding to the gauging of $\mathbb{A}$ with trivial torsion:

$$\Phi \cdot Z_{\text{e}}[b_3, b_3'] = \sum_{a \in H^1(X,\mathbb{A})} \exp\left[2\pi i \int \phi(a) \cup \Big(\iota_3(b_3) + s_3(b_3')\Big)\right] Z_{\text{e}}[a] \,. \tag{B.2.13}$$

We decompose the $\mathbb{A}$-valued field a according to the sequence S_3: $a = \iota_3(a_3) + s_3(a_3')$. Because of the commutative diagram (B.2.5), $\phi(a) \in H^1(X, \mathbb{A}^\vee)$ has a decomposition using S_4:

$$\phi(a) = \iota_4(x_4) + s_4(x_4') \qquad \text{with} \qquad x_4 = \phi(a_3) \,, \qquad x_4' = \phi(a_3') \,. \tag{B.2.14}$$

Furthermore, it is useful to decompose b_3 using S_5: $b_3 = \iota_5(y_5) + s_5(y_5')$. Hence, using that ν vanishes on $\text{Rad}(\nu)$, we have

$$\begin{aligned} \Phi \cdot Z_{\text{inv}}[B, b_3'] = \sum_{y_5,\, y_5',\, x_4,\, x_4'} \exp\Bigg[2\pi i \int \Bigg(& y_5'^*\nu + B \cup \Big(\iota_5(y_5) + s_5(y_5')\Big) + \\ & + \phi(a) \cup \Big(\iota_3\iota_5(y_5) + \iota_3 s_5(y_5') + s_3(b_3')\Big)\Bigg)\Bigg] Z_{\text{e}}[a_3, a_3'] \,. \end{aligned} \tag{B.2.15}$$

We can perform the sum over y_5 and y_5', in this order. Since y_5 appears linearly, the sum over it gives a delta function imposing

$$\pi_7\Big(B + \pi_1\big(\phi(a)\big)\Big) = 0 \qquad \Leftrightarrow \qquad B + \pi_1\big(\phi(a)\big) \in \iota_7\Big(\big(\mathbb{B}/\,\text{Rad}(\nu)\big)^\vee\Big) \,. \tag{B.2.16}$$

We notice that $\pi_7\pi_1 = \pi_4$, and since $\phi(a) = \iota_4(x_4) + s_4(x_4')$, it follows that (B.2.16) can be rewritten as $\pi_7(B) + x_4' = 0$. This delta function will be resolved by the sum over x_4', which however must be performed only after the sum over x_4. We can then integrate out y_5'. Since it appears quadratically, the sum over it can be performed by solving its equation of motion

$$\iota_7\big(\psi_\nu(y_5')\big) + B + \pi_1\big(\phi(a)\big) = 0 \,. \tag{B.2.17}$$

This equation makes sense in virtue of (B.2.16). This equation can be inverted in virtue of (B.2.16) and using that $\psi_\nu : \mathbb{B}/\text{Rad}(\nu) \to \big(\mathbb{B}/\text{Rad}(\nu)\big)^\vee$ is invertible. Plugging the result back we get

$$\Phi \cdot Z_{\text{inv}}[B, b_3'] = \sum_{a_3,\, a_3'} \exp\left[2\pi i \int \bigg(\phi(a) \cup s_3(b_3') - \Big(\psi_\nu^{-1}\big(B + \pi_1\, \phi(a)\big)\Big)^{\!*}\nu\bigg)\right] \delta\big(\pi_7\, B + x_4'\big)\, Z_{\text{e}}[a] \,. \tag{B.2.18}$$

We decompose B using S_7 as

$$B = \iota_7(B_7) + s_7(B_7') \,, \tag{B.2.19}$$

and the duality maps

$$\Phi(B_7,\, B_7',\, b_3') = \big(\sigma^\vee(B_7),\, \phi(b_3'),\, \phi^{-1}(B_7')\big) \,. \tag{B.2.20}$$

The sum over a_3' resolves the delta function, while the one over a_3 reconstructs Z_{inv} up to a multiplicative factor which gives the anomaly:

$$\Phi \cdot Z_{\text{inv}}[B_7, B_7', b_3'] = \exp\left[-\int \big(\phi^{-1}(\sigma^\vee B_7)\big)^* \nu\right] Z_{\text{inv}}\big[\sigma^\vee(B_7),\, \phi(b_3'),\, \phi^{-1}(B_7')\big] \,. \tag{B.2.21}$$

Appendix C

Twisted cohomology and anomalies

Here we provide details on the topological actions that we use in 3d and 5d to cancel the mixed anomaly between the self-duality symmetry and the 0-form (in 2d) or the 1-form (in 4d) symmetry, when we go to the invariant boundary. In the 2d case this is an anomaly for a semi-direct product, while in 4d it is an anomaly for a split 2-group. In both cases we do not discuss the full anomaly, but only the piece linear in the gauge field $A \in H^d(X, G)$ for the self-duality symmetry.

C.1 Anomaly for a semi-direct product in 2d

We consider a semi-direct product $\mathbb{A} \rtimes_\rho G$ ($\mathbb{A}$ and G being both Abelian) with homomorphism $\rho : G \to \mathrm{Aut}(\mathbb{A})$. This is associated with a short exact sequence

$$1 \longrightarrow \mathbb{A} \xrightarrow{\ \iota\ } \mathbb{A} \rtimes_\rho G \xrightarrow{\ \pi\ } G \longrightarrow 1 \tag{C.1.1}$$

which splits, namely it admits a section $s : G \to \mathbb{A} \rtimes_\rho G$ which is a group homomorphism. Any element can be written uniquely as $\iota(a)\, s(g)$, $a \in \mathbb{A}$, $g \in G$, with product rule

$$\iota(a_1)\, s(g_1) \cdot \iota(a_2)\, s(g_2) = \iota\big(a_1 + \rho_{g_1}(a_2)\big)\, s(g_1 + g_2) \;. \tag{C.1.2}$$

In particular

$$s(g)\, \iota(a)\, s(g^{-1}) = \iota\big(\rho_g(a)\big) \;. \tag{C.1.3}$$

Semi-direct products are generically non-Abelian, and accordingly we only consider standard 1-form gauge fields. These are classes $\mathcal{A} \in H^1(X, \mathbb{A} \rtimes_\rho G)$, namely

$$(d\mathcal{A})_{ijk} = \mathcal{A}_{jk} \mathcal{A}_{ik}^{-1} \mathcal{A}_{ij} = 1 \;, \qquad\qquad \mathcal{A}_{ij} \sim \Lambda_i^{-1} \mathcal{A}_{ij} \Lambda_j \;, \tag{C.1.4}$$

where the order of multiplication matters. Since $\mathcal{A}_{ij} \in \mathbb{A} \rtimes_\rho G$, we can write

$$\mathcal{A}_{ij} = \iota(B_{ij})\, s(A_{ij}) \tag{C.1.5}$$

where $B \in C^1(X, \mathbb{A})$ and $A \in C^1(X, G)$. Using the commutation relation (C.1.3), the cocycle condition $(d\mathcal{A})_{ijk} = 1$ is equivalent to

$$\big(d_{\rho(A)} B\big)_{ijk} = \rho_{A_{ij}} B_{jk} - B_{ik} + B_{ij} = 0 \;, \qquad\qquad (dA)_{ijk} = A_{jk} - A_{ik} + A_{ij} = 0 \;. \tag{C.1.6}$$

The identification $\mathcal{A}_{ij} \sim \Lambda_i^{-1} \mathcal{A}_{ij} \Lambda_j$, upon decomposing $\Lambda_i = \iota(\theta_i)\, s(\lambda_i)$, becomes

$$B_{ij} \sim \rho_{\lambda_i}^{-1}\big(B_{ij} + \rho_{A_{ij}} \theta_j - \theta_i\big) = \rho_{\lambda_i}^{-1}\big(B + d_{\rho(A)}\theta\big)_{ij} \;, \qquad A_{ij} \sim A_{ij} + \lambda_j - \lambda_i = (A + d\lambda)_{ij} \;. \tag{C.1.7}$$

Hence A defines a class in the cohomology group $H^1(X,G)$, while B a class in the *twisted* cohomology group $H^1_\rho(X,\mathbb{A})$ — also called cohomology with local coefficients.

We are interested in the anomaly for $\mathbb{A}\rtimes_\rho G$ whose 3d inflow action is quadratic in B and "linear" in A. The word linear is in quotes since B is a twisted class, and thus A will appear not only linearly, but also in the twisting. This anomaly is identified by a characteristic class of $\mathbb{A}\rtimes_\rho G$ bundles, which lives in $H^1_\rho\big(G,\, H^2(\mathbb{A},U(1))\big)$ [14]. Such a class can be thought of as a function μ on G with values in the group of alternating bicharacters over $\mathbb{A}$, satisfying (in additive notation):

$$\rho_g\,\mu(h)+\mu(g)=\mu(g+h)\;. \tag{C.1.8}$$

The G-action on bicharacters is given in (6.2.52). Besides, the function μ is subject to the identification

$$\mu(\,\cdot\,)\;\sim\;\mu(\,\cdot\,)+\rho_{(\cdot)}\xi-\xi \qquad\qquad \text{for any }\;\xi\in H^2\big(\mathbb{A},U(1)\big)\;. \tag{C.1.9}$$

Notice that $\mu(0)=0$, so that $\mu(-g)=-\rho_g^{-1}\mu(g)$.

Given $A\in H^1(X,G)$, we construct $\mu(A)\in C^1\big(X,\,H^2(\mathbb{A},U(1))\big)$ (both notations $\mu(A)$ and $A^*\mu$ could be used). This is a cochain $\mu(A_{ij}):\mathbb{A}\times\mathbb{A}\to U(1)$ satisfying the twisted cocycle condition:

$$\big(d_{\rho(A)}\mu(A)\big)_{ijk}\;\equiv\;\rho_{A_{ij}}\mu(A_{jk})-\mu(A_{ik})+\mu(A_{ij})=0\;. \tag{C.1.10}$$

Moreover, under a gauge transformation $A\to A+d\lambda$, it changes by

$$\begin{aligned}\mu(A_{ij})\;\to\;\mu\big(A_{ij}+\lambda_j-\lambda_i\big)&=\rho_{\lambda_i}^{-1}\,\mu\big(A_{ij}+\lambda_j\big)+\mu(-\lambda_i)\\ &=\rho_{\lambda_i}^{-1}\Big(\rho_{A_{ij}}\,\mu(\lambda_j)+\mu(A_{ij})-\mu(\lambda_i)\Big)=\rho_{\lambda_i}^{-1}\Big(\mu(A_{ij})+\big(d_{\rho(A)}\,\mu(\lambda)\big)_{ij}\Big)\;,\end{aligned} \tag{C.1.11}$$

hence $\mu(A)\in H^1_\rho\big(X,\,H^2(\mathbb{A},U(1))\big)$.

Given $B\in H^1_\rho(X,\mathbb{A})$, we can form the cup product $\mu(A)\cup B\cup B\in H^3\big(X,U(1)\big)$ as:

$$\big(\,\mu(A)\cup B\cup B\,\big)_{ijkl}=\mu(A_{ij})\Big(\rho_{A_{ij}}B_{jk}\,,\,\rho_{A_{ik}}B_{kl}\Big)\;, \tag{C.1.12}$$

see App. A of [15]. Under a gauge variation $A\to A+d\lambda$, $B\to\rho(\lambda)^{-1}B$ as in (C.1.7) we find:

$$\begin{aligned}\big(\,\mu(A)\cup B\cup B\,\big)_{ijkl}\;\to\;&\rho_{\lambda_i}^{-1}\Big(\mu(A_{ij})+\big(d_{\rho(A)}\,\mu(\lambda)\big)_{ij}\Big)\Big(\rho_{\lambda_i}^{-1}\rho_{A_{ij}}B_{jk}\,,\,\rho_{\lambda_i}^{-1}\rho_{A_{ik}}B_{kl}\Big)\\ &=\big(\,\mu(A)\cup B\cup B\,\big)_{ijkl}+\big(d_{\rho(A)}\,\mu(\lambda)\big)_{ij}\Big(\rho_{A_{ij}}B_{jk}\,,\,\rho_{A_{ik}}B_{kl}\Big)\;.\end{aligned} \tag{C.1.13}$$

This means that we get a linear variation

$$\delta\Big(\mu(A)\cup B\cup B\Big)=\big(d_{\rho(A)}\,\mu(\lambda)\big)\cup B\cup B=d\Big(\mu(\lambda)\cup B\cup B\Big)\;. \tag{C.1.14}$$

We write the inflow action as

$$S_\mu=2\pi i\int_{X_3}\mu(A)\cup B\cup B\;. \tag{C.1.15}$$

When X_3 is closed this is gauge invariant, however if $\partial X_3=X_2$ we get a boundary term:

$$S_\mu\;\to\;S_\mu+2\pi i\int_{X_2}\mu(\lambda)\cup B\cup B\;. \tag{C.1.16}$$

C.2 Anomaly for a split 2-group in 4d

In 4d we have an analog story, where $\mathbb{A}$ is now a 1-form symmetry. The full symmetry structure is a split 2-group, which is a higher categorical version of a semi-direct product. The definitions can be found in [17] and a more physical discussion is in [15, 93]. Here we simply use two facts which from our viewpoint can be motivated as being the straightforward generalization of the discussion on semi-direct products.

First, a background field for a split 2-group is made of an ordinary cohomology class $A \in H^1(X,G)$ and a twisted cohomology class $B \in H^2_\rho(X,\mathbb{A})$. The latter means that

$$\big(d_{\rho(A)}B\big)_{ijkl} = \rho_{A_{ij}} B_{jkl} - B_{ikl} + B_{ijl} - B_{ijk} = 0 \;, \tag{C.2.1}$$

and there is an identification (or gauge transformation)

$$B_{ijk} \sim \rho^{-1}_{\lambda_i}\big(B_{ijk} + \rho_{A_{ij}}\theta_{jk} - \theta_{ik} + \theta_{ij}\big) \;, \qquad\qquad A_{ij} \sim A_{ij} + \lambda_j - \lambda_i \;, \tag{C.2.2}$$

which are the obvious generalizations of (C.1.7).

Second, the piece of the anomaly for a split 2-group which is "linear" in A and quadratic in B is labelled by a characteristic class of 2-group gauge bundles:

$$\mu \in H^1_\rho\Big(G,\, H^4\big(B^2\mathbb{A}, U(1)\big)\Big) \;. \tag{C.2.3}$$

One can show [93] that $H^4\big(B^2\mathbb{A},U(1)\big)$ is isomorphic to $\Gamma(\mathbb{A})^\vee$, the Pontryagin dual of the universal quadratic group of $\mathbb{A}$, which can be identified with the group of quadratic functions $q : \mathbb{A} \to U(1)$ (see [15, 93] for precise definitions and details, as well as the discussion around (6.3.11)). The G-action on them is naturally given by

$$(\rho_g q)(a) = q\big(\rho_g^{-1} a\big) \;. \tag{C.2.4}$$

The construction of the 5d anomaly inflow is very similar to the semi-direct product case, thus we skip many details. Given $A \in H^1(X,G)$, we construct $\mu(A)$ which satisfies (C.1.10) and (C.1.11), thus defining a class in $H^1_\rho\big(X,\Gamma(\mathbb{A})^\vee\big)$. Recall that $H^4\big(B^2\mathbb{A},\Gamma(\mathbb{A})\big) \cong \mathrm{Hom}\big(\Gamma(\mathbb{A}),\Gamma(\mathbb{A})\big)$ has a distinguished element $\mathfrak{P}$ (the identity map) called the universal Pontryagin class, such that

$$B \in H^2_\rho(X,\mathbb{A}) \qquad \rightsquigarrow \qquad \mathfrak{P}_\rho(B) \equiv B^*\mathfrak{P} \in H^4_\rho\big(X,\Gamma(\mathbb{A})\big) \;. \tag{C.2.5}$$

The action of G on $\Gamma(\mathbb{A})$ is induced by the one on $\Gamma(\mathbb{A})^\vee$ in such a way to make the natural pairing $\langle\;,\;\rangle : \Gamma(\mathbb{A}) \times \Gamma(\mathbb{A})^\vee \to U(1)$ invariant. Under $A \to A + d\lambda$ the latter transforms as

$$\mathfrak{P}_\rho(B)_{i_0,\dots,i_4} \;\to\; \rho^{-1}_{\lambda_{i_0}}\, \mathfrak{P}_\rho(B)_{i_0,\dots,i_4} \;. \tag{C.2.6}$$

Using the pairing between $\Gamma(\mathbb{A})$ and $\Gamma(\mathbb{A})^\vee$ we construct $\mu(A) \cup \mathfrak{P}_\rho(B) \in H^5\big(X,U(1)\big)$ as:

$$\big(\mu(A) \cup \mathfrak{P}_\rho(B)\big)_{i_0,\dots,i_5} = \Big\langle \mu(A)_{i_0 i_1},\, \rho_{A_{i_0 i_1}} \mathfrak{P}_\rho(B)_{i_1,\dots,i_4} \Big\rangle \;. \tag{C.2.7}$$

Under $A \to A + d\lambda$ we have

$$\mu(A) \cup \mathfrak{P}_\rho(B) \;\to\; \mu(A) \cup \mathfrak{P}_\rho(B) + d\Big(\mu(\lambda) \cup \mathfrak{P}_\rho(B)\Big) \;. \tag{C.2.8}$$

We conclude that the 5d inflow action is

$$S_\mu = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}_\rho(B) \;, \tag{C.2.9}$$

and its gauge variation on a manifold X_5 with boundary $X_4 = \partial X_5$ is

$$S_\mu \;\to\; S_\mu + 2\pi i \int_{X_4} \mu(\lambda) \cup \mathfrak{P}_\rho(B) \;. \tag{C.2.10}$$

Appendix D

Anomalous boundary conditions

In this appendix we present an iterative procedure to consistently turn on a background for boundary theories with a $U(1)$ anomalous symmetry in generic even dimension. For the sake of concreteness we present this procedure in the simplest case of a $U(1)$ symmetry with anomaly, but the same idea can be used for higher groups and in the non-Abelian cases discussed in the main text. In general, the method presented here is necessary to determine consistent boundary conditions whenever the simple BF theory is modified by some non-Gaussian term containing derivatives.

Consider the TQFT with action

$$S = \frac{i}{2\pi}\int_{X_{d+1}} \left(b_{d-1}\wedge dA_1 + \kappa_d\, A_1\wedge (dA_1)^{\frac{d}{2}}\right), \qquad \kappa_d = \frac{k}{(2\pi)^{\frac{d}{2}-1}\left(\frac{d}{2}+1\right)!}\,, \tag{D.0.1}$$

and $k\in\mathbb{Z}$. In the presence of a boundary, the variation of the action produces a term

$$-\frac{i}{2\pi}\int_{\partial X_{d+1}}\left(b_{d-1}+\frac{d}{2}\,\kappa_d\, A_1\wedge (dA_1)^{\frac{d}{2}-1}\right)\delta A_1\,. \tag{D.0.2}$$

This can be cancelled by imposing the boundary condition

$$\star A_1 = -\frac{i}{R^2}\underbrace{\left(b_{d-1}+\frac{d}{2}\,\kappa_d\, A_1\wedge (dA_1)^{\frac{d}{2}-1}\right)}_{\mathcal{T}_0} + \star\mathcal{A}_1 \tag{D.0.3}$$

and adding the boundary term

$$S_\partial^{(0)} = -\frac{1}{4\pi R^2}\int_{\partial X_{d+1}}\left(b_{d-1}+\frac{d}{2}\,\kappa_d\, A_1\wedge (dA_1)^{\frac{d}{2}-1}\right)\wedge\star\left(b_{d-1}+\frac{d}{2}\,\kappa_d\, A_1\wedge (dA_1)^{\frac{d}{2}-1}\right). \tag{D.0.4}$$

However, there is no gauge transformation of $\mathcal{A}_1$ that makes the boundary condition gauge invariant. The only way to have a gauge-invariant boundary condition is to add terms that mix $\mathcal{A}_1$ with the dynamical fields. The simplest such modification is to replace $\mathcal{T}_0$ in (D.0.3) with

$$\mathcal{T}_0' = \mathcal{T}_0 - \frac{d}{2}\,\kappa_d\,\mathcal{A}_1\wedge (dA_1)^{\frac{d}{2}-1}\,. \tag{D.0.5}$$

Consequently we must modify the boundary term into

$$-\frac{1}{4\pi R^2}\int_{\partial X_{d+1}}\mathcal{T}_0'\wedge\star\mathcal{T}_0'\,. \tag{D.0.6}$$

However, since the boundary condition now imposes $\delta\mathcal{T}_0' = iR^2\star\delta A_1$, we get an extra unwanted term in the variational principle:

$$-\frac{i}{2\pi}\int_{\partial X_{d+1}}\frac{d}{2}\,\kappa_d\,\mathcal{A}_1\wedge (dA_1)^{\frac{d}{2}-1}\wedge\delta A_1\,. \tag{D.0.7}$$

This can be cancelled by adding a topological term proportional to $\mathcal{A}_1 \wedge A_1 \wedge (dA_1)^{\frac{d}{2}-1}$ to the boundary term. Indeed

$$\int_{\partial X_{d+1}} \delta\Big(\mathcal{A}_1\, A_1\, (dA_1)^{\frac{d}{2}-1}\Big) = \int_{\partial X_{d+1}} \Big(\tfrac{d}{2}\, \mathcal{A}_1\, (dA_1)^{\frac{d}{2}-1}\, \delta A_1 - \big(\tfrac{d}{2}-1\big)\, d\mathcal{A}_1\, A_1\, (dA_1)^{\frac{d}{2}-2}\, \delta A_1\Big)\,. \tag{D.0.8}$$

However, this also produces an extra term that must be cancelled. This is easily achieved by modifying both the boundary condition and the boundary term by the addition of this extra term to $\mathcal{T}_0'$. This produces

$$\mathcal{T}_1 = \mathcal{T}_0' + \kappa_d \big(\tfrac{d}{2}-1\big)\, d\mathcal{A}_1\, A_1\, (dA_1)^{\frac{d}{2}-2}\,. \tag{D.0.9}$$

At the same time we modify the boundary term that, including the new topological term, becomes

$$S_\partial^{(1)} = -\frac{1}{4\pi R^2}\int_{\partial X_{d+1}} \mathcal{T}_1 \wedge \star\, \mathcal{T}_1 + \frac{i}{2\pi}\int_{\partial X_{d+1}} \kappa_d\, \mathcal{A}_1 \wedge A_1 \wedge (dA_1)^{\frac{d}{2}-1}\,. \tag{D.0.10}$$

These new boundary condition and boundary term give a consistent variational principle. However, the boundary condition is again non gauge invariant because of the last term we added to $\mathcal{T}_1$, and we have to repeat the procedure above.

At each step, the non-gauge-invariant piece in the boundary condition becomes of one lower degree in A_1 (and one higher in $\mathcal{A}_1$). Hence, the procedure stops when we reach a term linear in A_1: we can make the boundary condition gauge invariant by adding a term purely in $\mathcal{A}_1$, which does not modify the variational principle. The procedure stops after $(d/2-1)$ steps, yielding the boundary condition

$$\star\big(A_1 - \mathcal{A}_1\big) = -\frac{i}{R^2}\Big(\Omega_{d-1} - \kappa_d\, \mathcal{A}_1\, (\mathcal{A}_1)^{\frac{d}{2}-1}\Big) \tag{D.0.11}$$

where

$$\Omega_{d-1} = b_{d-1} + \kappa_d \sum_{r=0}^{\frac{d}{2}-2} \big(\tfrac{d}{2}-r\big)\, (d\mathcal{A}_1)^r \big(A_1 - \mathcal{A}_1\big)\, (dA_1)^{\frac{d}{2}-1-r} + \kappa_d\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\, A_1\,. \tag{D.0.12}$$

The corresponding boundary term is

$$S_\partial = -\frac{1}{4\pi R^2}\int_{\partial X_{d+1}} \Omega_{d-1} \wedge \star\, \Omega_{d-1} + \frac{i\kappa_d}{2\pi}\sum_{r=0}^{\frac{d}{2}-2}\int_{\partial X_{d+1}} \mathcal{A}_1\, (d\mathcal{A}_1)^r\, A_1\, (dA_1)^{\frac{d}{2}-r-1}\,. \tag{D.0.13}$$

As a sanity check, we can verify that the boundary theory is anomalous under $U(1)$ gauge transformations. Under $\delta A_1 = \delta\mathcal{A}_1 = d\lambda_0$ the topological terms on the boundary produce

$$\begin{aligned} &\frac{i\kappa_d}{2\pi}\sum_{r=0}^{\frac{d}{2}-2}\int_{\partial X_{d+1}} \Big(d\lambda_0\, (d\mathcal{A}_1)^r\, A_1\, (dA_1)^{\frac{d}{2}-r-1} + \mathcal{A}_1\, (d\mathcal{A}_1)^r\, d\lambda_0\, (dA_1)^{\frac{d}{2}-r-1}\Big) \\ &\quad = \frac{i\kappa_d}{2\pi}\sum_{r=0}^{\frac{d}{2}-2}\int_{\partial X_{d+1}} \lambda_0\, \Big((d\mathcal{A}_1)^{r+1}(dA_1)^{\frac{d}{2}-r-1} - (d\mathcal{A}_1)^r (dA_1)^{\frac{d}{2}-r}\Big) \\ &\quad = \frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} \Big(\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\, (d\mathcal{A}_1) - \lambda_0\, (dA_1)^{\frac{d}{2}}\Big)\,. \end{aligned} \tag{D.0.14}$$

Then, using the boundary condition,

$$\begin{aligned} \delta S_\partial = &\frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\big(A_1 - \mathcal{A}_1\big) - \frac{\kappa_d^2}{2\pi R^2}\int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} \wedge \star\Big((d\mathcal{A}_1)^{\frac{d}{2}-1}\, A_1\Big) \\ &- \frac{\kappa_d^2}{4\pi R^2}\int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} \wedge \star\Big(d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\Big) + \frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} \Big(\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1}(d\mathcal{A}_1) - \lambda_0\, (dA_1)^{\frac{d}{2}}\Big)\,. \end{aligned} \tag{D.0.15}$$

The bulk contributes with a term

$$\delta S = -\frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} d\lambda_0\, A_1\, (dA_1)^{\frac{d}{2}-1} \tag{D.0.16}$$

which, together with the last term in (D.0.14), combines to a total derivative (on the boundary) and can be neglected. We remain with

$$\delta S_{\text{tot}} = -\frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\,\mathcal{A}_1 - \delta\left[\frac{\kappa_d^2}{4\pi R^2}\int_{\partial X_{d+1}} \left(\mathcal{A}_1\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\right)\wedge\star\left(\mathcal{A}_1\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\right)\right]. \tag{D.0.17}$$

We can isolate the anomalous variation adding a final counterterm

$$S_{\text{c.t.}} = \frac{\kappa_d^2}{4\pi R^2}\int_{\partial X_{d+1}} \left(\mathcal{A}_1\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\right)\wedge\star\left(\mathcal{A}_1\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\right). \tag{D.0.18}$$

Appendix E

Non-compact TQFTs

In this appendix we provide a mathematical definition and details on the TQFTs with infinitely many operators introduced in chapter 8 and used in 9 as holographic duals. The main issue is defining the theory with cutting and gluing while avoiding infinities from inserting a complete basis of states. We argue that this is possible if all manifolds have at least one non-empty boundary component. On the other hand, the partition functions on closed manifolds will be generically infinite.

E.1 Two-dimensional non-compact Dijkgraaf-Witten theory

We refer to section 2.1 for a review of the basics of axiomatic TQFTs in the standard case. Already the fact

$$Z\big(X_{d-1} \times S^1\big) = \sum\nolimits_a \delta_{a,a} = \dim\big(\mathcal{H}_{X_{d-1}}\big)\,. \tag{E.1.1}$$

suggests that in the non-compact case closed bordisms should not be included in the definition. We want to argue that, avoiding closed manifolds, there are classes of manifolds in which we can give a precise definition of the $U(1)/\mathbb{R}$ BF-like theories

$$S = \frac{i}{2\pi}\int_{\mathcal{M}_d} b_{d-p-1} \wedge dA_p\,. \tag{E.1.2}$$

The Hilbert space: canonical quantization. As an illustration, we consider the case of $d = 2$ with $p = 1$. Hence $b_0 = \phi$ is a non-compact scalar, and A is a $U(1)$ gauge field. The Hilbert space $\mathcal{H}_{S^1}$ can be constructed by canonical quantization. We set $\mathcal{M}_2 = S^1 \times \mathbb{R}$, with $\mathbb{R}$ parametrized by t, and split $A = \widetilde{A} + A_0^t\, dt$. Then

$$S = -\frac{i}{2\pi}\int_{S^1\times\mathbb{R}} \Big(A_0^t\, \tilde{d}\phi \wedge dt + \phi\, \partial_t \widetilde{A} \wedge dt\Big)\,. \tag{E.1.3}$$

We choose the temporal gauge $A_0^t = 0$, and we need to impose the Gauss law $\tilde{d}\phi = 0$, namely $\phi = \phi(t)$ is independent of the spatial coordinate. Introducing

$$q(t) = \int_{S^1} \widetilde{A}\,, \qquad\qquad p(t) = \frac{1}{2\pi}\phi(t)\,, \tag{E.1.4}$$

we see that $q(t) \sim q(t) + 2\pi$ is a periodic variable, and the action becomes

$$S = -i\int_{\mathbb{R}} p\, \partial_t q\, dt\,. \tag{E.1.5}$$

This is a free infinitely-massive particle on a circle of radius 2π. The quantization is straightforward. We have the commutation relations

$$[\hat{q}, \hat{p}] = i \qquad\qquad \Rightarrow \qquad\qquad e^{i\alpha\hat{p}} \cdot e^{in\hat{q}} = e^{i\alpha n}\, e^{in\hat{q}} \cdot e^{i\alpha\hat{p}}\,. \tag{E.1.6}$$

Here $n \in \mathbb{Z}$ because of the periodicity of $\hat{q}$, while α is a generic real number. However the operator $e^{2\pi i\hat{p}}$ commutes with the whole operator algebra, hence it is a number that we can set to 1. Therefore the operators acting on the Hilbert space are

$$\widehat{\mathcal{O}}_\alpha = e^{i\alpha\hat{p}} \quad \text{with} \quad \alpha \in [0, 2\pi)\,, \qquad\qquad \widehat{W}_n = e^{in\hat{q}} \quad \text{with} \quad n \in \mathbb{Z}\,, \tag{E.1.7}$$

with algebra

$$\widehat{\mathcal{O}}_\alpha\, \widehat{\mathcal{O}}_\beta = \widehat{\mathcal{O}}_{\alpha+\beta\ (\mathrm{mod}\ 2\pi)}\,, \qquad \widehat{W}_n\, \widehat{W}_m = \widehat{W}_{n+m}\,, \qquad \widehat{\mathcal{O}}_\alpha\, \widehat{W}_n = e^{i\alpha n}\, \widehat{W}_n\, \widehat{\mathcal{O}}_\alpha\,. \tag{E.1.8}$$

Starting from a simultaneous eigenstate of the $\widehat{W}_n$'s such that

$$\widehat{W}_n\, |\theta\rangle = e^{in\theta}\, |\theta\rangle\,, \tag{E.1.9}$$

using the algebra we find

$$\widehat{\mathcal{O}}_\alpha\, |\theta\rangle = |\theta - \alpha\rangle\,. \tag{E.1.10}$$

Hence we get a basis labelled by a compact continuous variable $\theta \in U(1)$. We can also use a non-compact but countable basis, starting with an eigenstate of $\widehat{\mathcal{O}}_\alpha$:

$$\widehat{\mathcal{O}}_\alpha\, |k\rangle = e^{i\alpha k}\, |k\rangle\,. \tag{E.1.11}$$

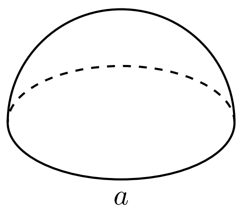
It must be $k \in \mathbb{Z}$ to respect the periodicity $\alpha \sim \alpha + 2\pi$. Then using the algebra we infer

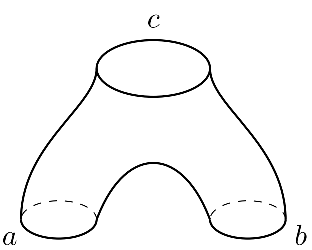
$$\widehat{W}_n\, |k\rangle = |k + n\rangle\,. \tag{E.1.12}$$

The relation between the two basis is

$$|k\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta\, e^{ik\theta}\, |\theta\rangle\,, \qquad\qquad |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{-ik\theta}\, |k\rangle\,. \tag{E.1.13}$$

TQFT data. Since the Hilbert space is infinite dimensional, the partition function on T^2 is infinite. Let us show that, on the other hand, we can consistently define a functor on the category of open oriented bordisms. In 2d the huge computational simplifications are that the only Hilbert space is $\mathcal{H}_{S^1}$, and that every 2d manifold has a pair of pants decomposition. Eventually, one also needs to *fill holes* by attaching a disk. Hence, on top of the horseshoe η_{ab}, the only other data one needs to assign are the disk and the pair of pants:

$h_a =$ 

$\mu^c_{ab} =$ 

The numbers h_a define a distinguished state $|HH\rangle = \sum_a h_a |a\rangle$, called the Hartle–Hawking state. These two data must satisfy the obvious condition that if we fill one of the two incoming holes of the pair of pants with the Hartle–Hawking state we get the cylinder:

$$\textstyle\sum_b \mu^c_{ab}\, h_b = \delta_{a,c}\,. \tag{E.1.14}$$

The only other consistency condition is the independence from the chosen pair of pants decomposition, that reduces to the Froboenius condition [251]:

$$\sum_c \mu_{a,b}^{c}\,\mu_{c,d}^{e} = \sum_c \mu_{a,c}^{e}\,\mu_{b,d}^{c}\,. \tag{E.1.15}$$

Let us use the continuous basis $|\theta\rangle$. The cylinder (identity) becomes a delta function $\delta(\theta_1 - \theta_2)$. Moreover, we define

$$h_\theta = \delta(\theta)\,, \qquad \eta_{\theta_1,\theta_2} = \delta(\theta_1+\theta_2)\,, \qquad \mu_{\theta_1,\theta_2}^{\theta_3} = \delta(\theta_1+\theta_2-\theta_3)\,. \tag{E.1.16}$$

Also, all sums are replaced by integrals on $[0,2\pi)$ in this basis. The condition (E.1.14) is obviously satisfied, while the Froboenius condition (E.1.15) reads

$$\int_0^{2\pi} d\theta\;\delta(\theta_1+\theta_2-\theta)\,\delta(\theta+\theta_3-\theta_4) = \int_0^{2\pi} d\theta\;\delta(\theta_1+\theta-\theta_4)\,\delta(\theta_2+\theta_3-\theta) \tag{E.1.17}$$

which is satisfied since both sides are equal to $\delta(\theta_1+\theta_2+\theta_3-\theta_4)$. The choice of these data is motivated by the fact that the continuous basis $|\theta\rangle$ is related, by the state/operator correspondence, with the local operators $\mathcal{O}_\alpha(x) = e^{i\frac{\alpha}{2\pi}\phi(x)}$, and the pair of pants must reproduce their OPE $\mathcal{O}_\alpha\,\mathcal{O}_\beta = \mathcal{O}_{\alpha+\beta}$. Then the Hartle–Hawking state is fixed by (E.1.14).

Cutting an gluing: path integrals on open surfaces. With these pieces of data, we can compute the value of the functor for arbitrary bordisms with a non-empty boundary. The simplest nontrivial such manifold is the torus with a puncture. This can be obtained from the pair of pants by gluing one of the two incoming boundaries with the outgoing one. Denoting by θ the label of the puncture, namely the non-glued circle, the result is[1]

$$Z\big(\Sigma_1 \setminus P_\theta\big) = \int_0^{2\pi} d\theta'\;\delta(\theta) = 2\pi\,\delta(\theta)\,. \tag{E.1.18}$$

This is a projector on the Hartle–Hawking state. Another simple example is the torus with two punctures that can be obtained from the previous result by gluing the remaining boundary to the outgoing boundary of another pair of pants. Hence, the result is

$$Z\Big(\Sigma_1 \setminus \{P_{\theta_1},P_{\theta_2}\}\Big) = \int_0^{2\pi} d\theta'\;\delta(\theta_1+\theta_2-\theta')\,2\pi\delta(\theta') = 2\pi\,\delta(\theta_1+\theta_2)\,. \tag{E.1.19}$$

We can now put these two examples together, gluing the boundary of a torus with one puncture to one of the two boundaries of the torus with two punctures, resulting in a genus-two surface with a puncture:

$$Z\big(\Sigma_2 \setminus P_\theta\big) = \int_0^{2\pi} d\theta'\;2\pi\delta(\theta+\theta')\,2\pi\delta(\theta') = (2\pi)^2\,\delta(\theta)\,. \tag{E.1.20}$$

Proceeding in this way it is not hard to prove the general result. The value of the functor an a genus g surface with n incoming boundaries labelled by $\theta_1,\ldots,\theta_n$ and m outgoing boundaries labelled by $\theta'_1,\ldots,\theta'_m$ is given by

$$Z\Big(\Sigma_g \setminus \{P_{\theta_1},\ldots,P_{\theta_n},P_{\theta'_1},\ldots,P_{\theta'_m}\}\Big) = (2\pi)^g\;\delta\big(\theta_1+\ldots+\theta_n-\theta'_1-\ldots-\theta'_m\big)\,. \tag{E.1.21}$$

The important observation is that the partition function on compact Riemann surfaces is infinite. Indeed, a compact Riemann surface of genus g is obtained by closing the hole of a one-punctured Riemann surface $\Sigma_g \setminus P_\theta$ by means of gluing the Hartle–Hawking state. The result is clearly infinite:

$$Z(\Sigma_g) = \int_0^{2\pi} d\theta\;(2\pi)^g\,\delta(\theta)\,\delta(\theta) = (2\pi)^g\,\delta(0)\,. \tag{E.1.22}$$

[1] We denote a genus g Riemann surface as Σ_g.

We conclude that the TQFT is well defined on the category of open oriented bordisms.

Let us remark that, given the Hilbert space we constructed, there is another set of data that can be formulated, which is essentially the same as the one we discussed but in the discrete basis $|k\rangle$:

$$h'_k = \delta_{k,0}\,, \qquad \eta'_{k_1,k_2} = \delta_{k_1,-k_2}\,, \qquad (\mu')^{k_3}_{k_1,k_2} = \delta_{k_1+k_2,k_3}\,. \tag{E.1.23}$$

With these data one gets infinite answers even on open manifolds, as soon as they have a non-trivial topology. It must be noticed that, indeed, these are not merely the data (E.1.16) written in a different basis: translating (E.1.16) in the discrete basis using (E.1.13) we get

$$h_k = \frac{1}{\sqrt{2\pi}}\,, \qquad \eta_{k_1,k_2} = \delta_{k_1,k_2}\,, \qquad \mu^{k_3}_{k_1,k_2} = \sqrt{2\pi}\,\delta_{k_1,k_2}\,\delta_{k_1,k_3}\,. \tag{E.1.24}$$

We conclude that (E.1.16) and (E.1.23) really define two different TQFTs.

How did we choose one instead of the other? As we already pointed out, in 2d TQFT the choice is really dictated by the fact that the pair of pants is related with the OPE of local operators. The data (E.1.23) would then be relevant for the TQFT with Lagrangian formulation

$$S' = \frac{i}{2\pi}\int_{\mathcal{M}_2} \Phi\, da_1\,, \tag{E.1.25}$$

where $\Phi \sim \Phi + 2\pi$ is a compact scalar, while a_1 an $\mathbb{R}$ gauge field. Canonical quantization produces the same Hilbert space as the theory with non-compact scalar and $U(1)$ gauge field; however, here the local operators $\mathcal{O}_n(x) = e^{in\Phi(x)}$ are labeled by an integer, and hence are related with the discrete basis by the state/operator correspondence. For this reason, in contrast to the previous case, the quantization of this theory produces the data (E.1.23) in which the pair of pants gives the Abelian fusion algebra in the discrete basis.

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