

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Exact Methods for Spectral Problems via Conformal Field Theory, Supersymmetric Gauge Theory, and Topological Strings

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Abstract

In this thesis, we develop and apply methods inspired by two-dimensional conformal field theory, supersymmetric gauge theory, and topological string theory to the study of spectral problems. In the first part, we investigate periodic spectral problems governed by the Lamé and Heun equations, including confluent limits of the latter. These equations arise both as classical limits of BPZ equations satisfied by degenerate Virasoro conformal blocks and as quantized Seiberg–Witten curves of four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theories. The two descriptions are related by the AGT correspondence, under which degenerate classical conformal blocks are mapped to surface-defect partition functions in the Nekrasov–Shatashvili (NS) limit. We solve the connection problem for the Lamé equation and develop a systematic resummation procedure, based on suitable limits of blow-up equations, for bulk and defect NS functions. The resummation resolves their singular instanton expansions at resonant values of the Floquet exponent and reveals the branch structure of the accessory parameters and Floquet solutions, whose branch points determine the edges of spectral bands and gaps. For the Lamé equation, the resonant limit of the resummed accessory parameter agrees with the values obtained independently from isomonodromic and orbifold-defect constructions. The latter also provides the corresponding (anti-)periodic solutions. For the Heun equation, we extend the resummation procedure to include the Floquet solutions, obtaining at resonance both the accessory parameters and the corresponding (anti-)periodic solutions. We further construct their logarithmic companions and analyze the resonant monodromy, distinguishing the mass loci where spectral gaps close from the smaller loci where the resonant monodromy becomes semisimple.

In the second part, we study finite-difference equations obtained by quantizing the mirror curves of the $Y^{N,0}$ geometries, which engineer five-dimensional $\mathcal{N} = 1$ $SU(N)$ gauge theories. In the framework of the open topological string/spectral theory (TS/ST) correspondence, we construct exact generalized eigenfunctions that are expected to be entire in the open and closed moduli and to become square-integrable when the spectral moduli belong to the appropriate exact quantization locus. Their four-dimensional limit gives explicit analytic solutions to the eigenvalue problem associated with $H_N = 2\Lambda^N \cosh(\mathbf{p}) + V_N(\mathbf{x})$, where $V_N(x)$ is an arbitrary monic polynomial of degree N . These operators are solvable finite-difference deformations of the Schrödinger Hamiltonians $\mathbf{p}^2 + V_N(\mathbf{x})$, and our solutions describe both bound states and resonances. We also identify special loci at which the eigenfunctions exhibit enhanced decay. For confining potentials, this enhancement leads to spectral degeneracies, while for potentials unbounded from below, it produces square-integrable states with real energies. We finally discuss the related operators with a $\sinh(\mathbf{p})$ kinetic term, an inverted potential $-V_N(\mathbf{x})$, or both.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. The results presented in this thesis are based on the following works:

- G. Bonelli, P. Gavrylenko, T. Pedroni, and A. Tanzini, *Blowing-up the edge: connection formulae and stability chart of the Lamé equation*, [arXiv:2507.04860](#), submitted to *Letters in Mathematical Physics* (under review).
- M. François, A. Grassi, and T. Pedroni, *Eigenfunctions of deformed Schrödinger equations*, [arXiv:2511.10636](#), submitted to *SIGMA* (under review).
- S. Jeong and T. Pedroni, *Split Heun functions via blown-up surface defects*, [arXiv:2607.24920](#), submitted to *Letters in Mathematical Physics* (under review).

For all three works, I contributed to all the results presented, either through their derivation or through independent calculations to verify them. The writing and revision of each manuscript were a collaborative effort involving all its authors.

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Introduction

Spectral problems are ubiquitous in mathematics and the physical sciences. They play a fundamental role in quantum mechanics and quantum field theory, in general relativity and gravitational physics, as well as in string theory, condensed-matter physics, and many other areas. Despite the diversity of these applications, the underlying questions often take a common form: one seeks to determine the spectrum of an operator, construct the corresponding eigenfunctions, and understand how these spectral data depend on the physical parameters of the problem.

The development of exact methods for the study of spectral problems is therefore of central importance. Exact results can reveal structures that may be difficult to identify through perturbative or purely numerical analyses and can uncover connections between apparently unrelated physical systems. Symmetry, in its many manifestations, often provides the organizing principle underlying such exact results.

The main goal of this thesis is to develop and apply new methods for the study of spectral problems by drawing on ideas and techniques from supersymmetric gauge theory, conformal field theory and string theory. A recurring theme of this work is that spectral quantities can be encoded in a broad family of special functions arising naturally in these subjects. These include Nekrasov partition functions [1–4], superconformal indices [5–8], conformal blocks associated with Virasoro and affine current algebras [9–13], and refined topological string partition functions [14–17].

Although these objects do not constitute a single formally defined class, it is useful to regard them as modern special functions: they possess distinguished series or integral representations, satisfy differential, difference, and functional equations, and exhibit rich transformation laws and analytic properties. This point of view is rooted in a long representation-theoretic tradition in the theory of special functions [18] and is closely connected with the web of correspondences relating supersymmetric gauge theory, conformal field theory, topological string theory, and quantum integrable systems [19–24].

The fact that these functions arise in several areas of mathematics and physics, and admit different interpretations in each, often makes it possible to transfer information from one setting to another. For example, physical principles may constrain the analytic properties of conformal blocks, which can in turn determine the connection formulae governing the analytic continuation of solutions to differential equations [25]. Similarly, the requirement of background independence in topological string theory can lead to the construction of entire solutions to difference equations, which become eigenfunctions of the associated operators once the appropriate spectral conditions are imposed [17, 26–28]. This transfer of information between different descriptions underlies much of the work presented in this thesis.

The thesis develops along two main directions. The first concerns periodic spectral problems associated with two classical second-order differential equations of Fuchsian type:

the Lamé equation on the torus,

$$\left(\partial_z^2 - \mu(\mu - 1)\wp(z) + E\right)\psi(z) = 0,$$

and the Heun equation on the sphere,

$$\left(\partial_z^2 + \frac{\delta_0}{z^2} + \frac{\delta_q}{(z - q)^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_\infty - \delta_0 - \delta_q - \delta_1}{z(z - 1)} + \frac{u}{z(z - 1)(z - q)}\right)\psi(z) = 0.$$

Their spectral data and solutions can be encoded, respectively, in bulk and surface-defect Nekrasov partition functions of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $SU(2)$ [29–32]. Through the AGT correspondence [22, 33, 34], these partition functions are in turn related to conformal blocks of the Virasoro algebra. This connects the analytic and spectral theory of these differential equations to conformal field theory and supersymmetric gauge theory.

The $SU(2)$ Nekrasov partition functions depend on the Coulomb-branch parameter a , the masses of the hypermultiplets, and two equivariant parameters ε_1 and ε_2 . Their instanton parts are naturally represented as power-series expansions in an instanton-counting parameter q . The quantities relevant to the analysis of the differential equations above emerge in the Nekrasov–Shatashvili (NS) limit,

$$\varepsilon_2 \longrightarrow 0, \quad \varepsilon_1 = \hbar,$$

where the parameter ε_1 plays the role of Planck’s constant. The differential equations above are written in units where $\hbar = 1$ for simplicity.

The analytic dependence of the NS functions on a plays an important role in our analysis. In the associated spectral problems, the dimensionless ratio $2a/\hbar$ is identified with the Floquet exponent characterizing quasi-periodic solutions [29, 35, 36]. A significant difficulty arises at the nonzero resonant loci

$$\frac{2a}{\hbar} \in \mathbb{Z} \setminus \{0\},$$

where the Floquet multipliers coalesce and take the values ± 1 , corresponding respectively to periodic or anti-periodic solutions. At these loci, the coefficients of the NS functions develop poles whose orders increase with the instanton number. As a result, the naive instanton expansion ceases to be well defined precisely at the values relevant to the periodic spectral problem.

A substantial part of this thesis is devoted to overcoming this obstacle and extending the gauge-theory and conformal-field-theory descriptions to the resonant loci. This requires a reorganization of the instanton expansions that makes the analytic structure of the underlying functions manifest [37, 38]. Our approach is based on the derivation of a new family of functional identities involving only NS functions from the blow-up equations satisfied by the full Nekrasov partition functions [3].

To illustrate the basic idea, let us consider the pure $SU(2)$ gauge theory, whose associated differential equation is the Mathieu equation:

$$\left(\partial_x^2 + \lambda + 8\pi^2\sqrt{q}\cos 2\pi x\right)\psi(x, a; q) = 0.$$

The latter can be obtained as a suitable confluent limit of the Heun equation, as well as through a degeneration of the Lamé equation. Despite its relative simplicity, this example already exhibits the essential features of the more general problem. For the pure

SU(2) gauge theory, a vanishing blow-up equation for the bulk Nekrasov partition function $\mathcal{Z}(a; \varepsilon_1, \varepsilon_2; \mathbf{q})$ takes the form

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathcal{Z}(a + n\varepsilon_1; \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + n\varepsilon_2; \varepsilon_2, \varepsilon_1 - \varepsilon_2; \mathbf{q}) = 0.$$

This identity provides a strong functional constraint on the partition function. In the NS limit, the logarithm of $\mathcal{Z}$ admits the asymptotic expansion

$$\log \mathcal{Z}(a; \hbar, \varepsilon_2; \mathbf{q}) \simeq \frac{1}{\varepsilon_2} \mathcal{W}_0(a, \hbar; \mathbf{q}) + \mathcal{W}_1(a, \hbar; \mathbf{q}) + \mathcal{O}(\varepsilon_2),$$

where $\mathcal{W}_0$ is the effective twisted superpotential and $\mathcal{W}_1$ is its first correction in the ε_2 expansion. Substituting this expansion into the two factors of the blow-up equation and taking the diagonal limit $\varepsilon_2 \rightarrow \varepsilon_1 = \hbar$, one obtains a functional relation that closes on $\mathcal{W}_0$ and $\mathcal{W}_1$:

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp(2\mathcal{W}_1(a + n\hbar, \hbar; \mathbf{q}) - n\partial_a \mathcal{W}_0(a + n\hbar, \hbar; \mathbf{q}) - \partial_{\hbar} \mathcal{W}_0(a + n\hbar, \hbar; \mathbf{q})) = 0.$$

The NS function $\mathcal{W}_0$ obtained from the pure SU(2) Nekrasov partition function $\mathcal{Z}$ determines the spectrum of the Mathieu operator through the quantum Matone relation [39, 40]

$$\lambda = -4\pi^2 \mathbf{q} \partial_{\mathbf{q}} \mathcal{W}_0(a, 1; \mathbf{q}).$$

The resulting NS blow-up equation can then be used to determine a partial resummation of the instanton expansions of $\mathcal{W}_0$ and $\mathcal{W}_1$, making their analytic dependence on the Coulomb-branch parameter a , and in particular their branch structure near the resonant loci, manifest. Through the AGT correspondence, $\mathcal{W}_0$ is related to the classical limit of an irregular Virasoro conformal block [41, 42], so that the same resummation admits a direct interpretation in conformal field theory.

The same basic idea is developed for the NS functions that determine the accessory parameters of the Heun equation and its confluent limits, as well as the spectral parameter of the Lamé equation. It is then extended to the defect NS functions that encode Floquet solutions of the Heun equation and its confluences. At the resonant loci, this construction provides access to periodic and anti-periodic solutions and to their logarithmic companions.

Alongside this resummation analysis, we solve the connection problem for the Lamé equation by deriving connection formulae from the fusion and braiding transformations of degenerate Virasoro conformal blocks. These results, together with the resummed NS functions, are used to investigate the periodic spectral problems associated with the two fundamental cycles of the torus, including their periodic and anti-periodic eigenvalues, eigenfunctions and band-gap structure.

We also obtain these spectral quantities through two complementary constructions. The periodic and anti-periodic eigenvalues can be recovered from an isomonodromic deformation problem governed by an elliptic form of the Painlevé VI equation [43]. Alternatively, orbifold surface-defect partition functions provide a direct construction of both these eigenvalues and the corresponding eigenfunctions [36]. These approaches rely on mechanisms distinct from the resummation of the bulk NS functions and provide independent checks of the resulting spectral data.

The second direction of the thesis concerns finite-difference equations and operators obtained by quantizing the mirror curves of the toric Calabi–Yau threefolds associated

with the $Y^{N,0}$ geometries [44]. These equations take the form

$$\Lambda^N [\varphi(x + i\hbar) + \varphi(x - i\hbar)] + \sum_{\ell=0}^N \kappa_\ell e^{(\frac{N}{2}-\ell)x} \varphi(x) = 0, \quad \kappa_0 = \kappa_N = 1.$$

The parameters Λ and $\kappa = \{\kappa_\ell\}_{\ell=1}^{N-1}$ are moduli of the mirror curve and become parameters of the corresponding quantum operator.

Within the framework of the topological string/spectral theory (TS/ST) correspondence [15–17], we construct non-perturbative solutions of this difference equation that are conjecturally entire functions of the variable x and of the closed-string moduli κ . When the latter are placed on the appropriate exact quantization locus, these entire solutions are expected to become square-integrable and therefore define physical eigenfunctions of the corresponding operator. Our main interest, however, lies in a particular scaling limit of this construction, known as the four-dimensional limit, and in the resulting spectral quantities. In this limit, the relevant Hamiltonian takes the form

$$H_N = 2\Lambda^N \cosh(\mathbf{p}) + V_N(\mathbf{x}), \quad [\mathbf{x}, \mathbf{p}] = i\hbar,$$

where $V_N(x)$ is an arbitrary monic polynomial potential of degree N . The resulting operator can thus be regarded as a finite-difference deformation of the standard quantum-mechanical operator

$$\hat{H}_N = \mathbf{p}^2 + V_N(\mathbf{x}).$$

In the four-dimensional limit, the TS/ST correspondence conjecturally provides exact entire solutions to this deformed Schrödinger problem, which become square-integrable when a single quantization condition is imposed on the energy. Furthermore, the corresponding operators exhibit phenomena with no direct counterpart in ordinary one-dimensional quantum mechanics [44, 45]. For even N , ground-state degeneracies can occur at special points in parameter space, including configurations corresponding to symmetric double-well potentials. For odd N , there exist special loci at which square-integrable states with real energies occur even though the corresponding polynomial potential is unbounded from below. Quantum integrability plays an important role in understanding these special loci, as we will discuss in greater detail in the main body of this thesis.

Structure of the thesis

Chapter I reviews the necessary background on Fuchsian differential equations, two-dimensional conformal field theory, supersymmetric gauge theory and topological string theory. Section 1 focuses on local solutions of Fuchsian differential equations, their monodromy properties and the associated connection problem. Section 2 introduces conformal blocks, BPZ equations and their classical limits. Section 3 reviews Nekrasov partition functions, the blow-up equations, the AGT and Bethe/gauge correspondence. Finally, Section 4 discusses refined topological-string partition functions, which in the geometries relevant to this thesis can be identified with five-dimensional Nekrasov partition functions, together with local mirror symmetry and the main ideas underlying the TS/ST correspondence. The original contributions of the thesis begin in Chapter II. We solve the connection problem for the Lamé equation using two-dimensional conformal field theory and develop the resummation procedure for the accessory parameters of the Lamé equation and the Heun equation, and confluent limits of the latter. We also investigate the periodic spectral problem associated with the Lamé equation. In Chapter III, we extend the resummation procedure to Floquet solutions of the Heun equation and its confluent

limits. We then analyse the resonant A-cycle monodromy, with particular emphasis on its semisimple locus in the parameter space of the equation. Chapter [IV](#) is devoted to the difference operators arising from the quantization of mirror curves of the $Y^{N,0}$ geometries. We construct their exact solutions within the framework of the TS/ST correspondence and study their four-dimensional limit, which leads to finite-difference deformations of quantum-mechanical anharmonic oscillators. Finally, we conclude by summarizing the main results and discussing possible directions for future research.

Chapter I

Background

Section 1

Fuchsian Differential Equations

In this first background section, we review the theory of second-order Fuchsian ordinary differential equations, focusing on their local solutions, analytic continuation and monodromy, and concluding with a discussion of connection problems. We discuss equations on the sphere and the torus separately, reviewing the most relevant examples: the hypergeometric, Heun and Lamé equations.

General references. For background and more detailed explanations of the material reviewed in this section, see [46–50].

1.1 Second-order Fuchsian equations on the sphere

Let z denote the standard affine coordinate on $\mathbb{C} \subset \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. We consider equations of the form

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0, \quad (1.1)$$

where p and q are meromorphic functions on $\mathbb{CP}^1$, therefore rational functions of z . A point $z_0 \in \mathbb{C}$ is called an *ordinary* point of (1.1) if both p and q are holomorphic at z_0 . Otherwise, it is a *singular* point. A singular point z_0 is called *regular* if

$$(z - z_0)p(z) \quad \text{and} \quad (z - z_0)^2q(z) \quad (1.2)$$

extend holomorphically to $z = z_0$. Equivalently, p has at most a simple pole and q has at most a double pole at z_0 . The point at infinity is studied by introducing the local coordinate $w = 1/z$ and setting

$$\tilde{y}(w) = y\left(\frac{1}{w}\right).$$

In terms of w , equation (1.1) becomes

$$\begin{aligned} 0 &= \tilde{y}''(w) + \left[\frac{2}{w} - \frac{1}{w^2}p\left(\frac{1}{w}\right) \right] \tilde{y}'(w) + \frac{1}{w^4}q\left(\frac{1}{w}\right) \tilde{y}(w) \\ &\equiv \tilde{y}''(w) + \tilde{p}(w)\tilde{y}'(w) + \tilde{q}(w)\tilde{y}(w). \end{aligned} \quad (1.3)$$

The point $z = \infty$ is ordinary or singular according to whether $w = 0$ is ordinary or singular for (1.3). In particular, infinity is at most a regular singular point precisely when

$$p(z) = \mathcal{O}(z^{-1}), \quad q(z) = \mathcal{O}(z^{-2}), \quad z \rightarrow \infty. \quad (1.4)$$

Equation (1.1) is called *Fuchsian* if all its singular points on $\mathbb{CP}^1$ are regular singular points. Let $z_1, \dots, z_m \in \mathbb{C}$ be the finite singular points of (1.1). Since p and q are rational functions, the Fuchsian condition implies that they can be written as

$$p(z) = \sum_{i=1}^m \frac{r_i}{z - z_i}, \quad q(z) = \sum_{i=1}^m \left(\frac{s_i}{(z - z_i)^2} + \frac{t_i}{z - z_i} \right), \quad \sum_{i=1}^m t_i = 0, \quad (1.5)$$

with $r_i, s_i, t_i \in \mathbb{C}$, $i = 1, \dots, m$. Indeed, polynomial parts are excluded by the conditions (1.4), which also imply $\sum_i t_i = 0$.

The corresponding coefficients at infinity are then determined by the parameters associated with the finite singular points. Indeed, passing to the local coordinate w , the coefficients of (1.3) have the expansions

$$\begin{aligned} \tilde{p}(w) &= \frac{r_\infty}{w} + \mathcal{O}(1), & r_\infty &= 2 - \sum_{i=1}^m r_i, \\ \tilde{q}(w) &= \frac{s_\infty}{w^2} + \frac{t_\infty}{w} + \mathcal{O}(1), & s_\infty &= \sum_{i=1}^m (s_i + z_i t_i), \quad t_\infty = \sum_{i=1}^m (2z_i s_i + z_i^2 t_i). \end{aligned} \quad (1.6)$$

It is often convenient to remove from (1.1) the first-derivative term. In order to do so, locally we may set

$$y(z) = \exp \left(-\frac{1}{2} \int^z p(\zeta) d\zeta \right) \psi(z). \quad (1.7)$$

Equation (1.1) is then transformed into the normal form

$$\psi''(z) + T(z)\psi(z) = 0, \quad (1.8)$$

where

$$T(z) = q(z) - \frac{1}{2}p'(z) - \frac{1}{4}p(z)^2, \quad (1.9)$$

is a rational function of the variable z . For an equation written in normal form, a finite singular point is regular when $T(z)$ has at most a double pole there, while the point at infinity is regular singular when

$$T(z) = \mathcal{O}(z^{-2}), \quad z \rightarrow \infty. \quad (1.10)$$

Let again $z_1, \dots, z_m \in \mathbb{C}$ be the finite singular points of (1.1). It follows from (1.5) that $T(z)$ has at most double poles at the points z_i and satisfies $T(z) = \mathcal{O}(z^{-2})$ at infinity. It can therefore be written as

$$T(z) = \sum_{i=1}^m \left(\frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right), \quad \sum_{i=1}^m c_i = 0. \quad (1.11)$$

Meanwhile, expanding $T(z)$ near infinity gives

$$T(z) = \frac{1}{z} \sum_{i=1}^m c_i + \frac{1}{z^2} \sum_{i=1}^m (\delta_i + z_i c_i) + \mathcal{O}(z^{-3}) = \frac{1}{z^2} \sum_{i=1}^m (\delta_i + z_i c_i) + \mathcal{O}(z^{-3}), \quad (1.12)$$

so that the coefficient controlling the leading behaviour at infinity is

$$\delta_\infty = \sum_{i=1}^m (\delta_i + z_i c_i). \quad (1.13)$$

Let now $z_0 \in \mathbb{C}$ be an ordinary point. By the existence and uniqueness theorem for holomorphic ordinary differential equations, for every choice of initial data

$$y(z_0) = a, \quad y'(z_0) = b, \quad a, b \in \mathbb{C}, \quad (1.14)$$

there exists a unique solution of (1.1) holomorphic in a neighbourhood of z_0 . In particular, the solutions y_1 and y_2 determined by

$$\begin{aligned} y_1(z_0) &= 1, & y_1'(z_0) &= 0, \\ y_2(z_0) &= 0, & y_2'(z_0) &= 1 \end{aligned} \quad (1.15)$$

form a basis of the two-dimensional space of local solutions. The linear independence of two solutions is measured by their Wronskian,

$$W(y_1, y_2)(z) = y_1(z)y_2'(z) - y_1'(z)y_2(z). \quad (1.16)$$

Using (1.1), one finds

$$W'(z) = -p(z)W(z), \quad (1.17)$$

and hence, on a simply connected neighbourhood of z_0 ,

$$W(y_1, y_2)(z) = W(y_1, y_2)(z_0) \exp\left(-\int_{z_0}^z p(\zeta) d\zeta\right). \quad (1.18)$$

It follows that two solutions which are linearly independent at one point remain linearly independent throughout their common domain of definition. For an equation in normal form, $p(z) = 0$, and the Wronskian is therefore constant.

Let z_i be a finite regular singular point. Then, the coefficients of (1.1) admit Laurent expansions of the form

$$p(z) = \frac{r_i}{z - z_i} + \mathcal{O}(1), \quad q(z) = \frac{s_i}{(z - z_i)^2} + \mathcal{O}\left((z - z_i)^{-1}\right). \quad (1.19)$$

Substituting the leading ansatz $y(z) \sim (z - z_i)^\rho$ into (1.1) gives the indicial equation

$$\rho(\rho - 1) + r_i\rho + s_i = 0, \quad (1.20)$$

whose roots ρ_i^+ and ρ_i^- are called *local exponents* of the equation at z_i . For an equation in normal form, if

$$T(z) = \frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} + \mathcal{O}(1), \quad z \rightarrow z_i, \quad (1.21)$$

the indicial equation reads instead

$$\rho(\rho - 1) + \delta_i = 0. \quad (1.22)$$

It is often convenient to parametrize the coefficient of the double pole as

$$\delta_i = \frac{1}{4} - \theta_i^2, \quad \text{or equivalently} \quad \delta_i = \alpha_i(1 - \alpha_i), \quad (1.23)$$

where the two parametrizations are related by $\alpha_i = \frac{1}{2} + \theta_i$, up to the transformations $\theta_i \mapsto -\theta_i$ and $\alpha_i \mapsto 1 - \alpha_i$. The corresponding local exponents are then

$$\rho_i^\pm = \frac{1}{2} \pm \theta_i, \quad \text{or equivalently} \quad \rho_i^+ = \alpha_i, \quad \rho_i^- = 1 - \alpha_i. \quad (1.24)$$

If the difference of the local exponents is not an integer, $\rho_i^+ - \rho_i^- \notin \mathbb{Z}$, the Frobenius theorem guarantees the existence of a basis of local solutions

$$y_i^\pm(z) = (z - z_i)^{\rho_i^\pm} \sum_{n=0}^{\infty} a_{i,n}^\pm (z - z_i)^n, \quad a_{i,0}^\pm \neq 0, \quad (1.25)$$

called *Frobenius solutions*. After fixing branches of $(z - z_i)^{\rho_i^\pm}$, the normalizations of the Frobenius solutions are uniquely fixed by the conditions $a_{i,0}^\pm = 1$.

Suppose instead that the exponents are *resonant*, namely that their difference is an integer. After exchanging the labels if necessary, we write

$$\rho_i^+ - \rho_i^- = N, \quad N \in \mathbb{Z}_{\geq 0}. \quad (1.26)$$

Then, the solution associated with the exponent ρ_i^+ can be chosen in Frobenius form, while a second linearly independent solution generally contains a logarithmic term:

$$\begin{aligned} y_i^+(z) &= (z - z_i)^{\rho_i^+} \sum_{n=0}^{\infty} a_{i,n}^+ (z - z_i)^n, \\ y_i^-(z) &= C_i y_i^+(z) \log(z - z_i) + (z - z_i)^{\rho_i^-} \sum_{n=0}^{\infty} b_{i,n} (z - z_i)^n. \end{aligned} \quad (1.27)$$

When $N > 0$ the constant C_i , which in general depends on the parameters of the equation, may vanish. In this case both solutions can be chosen in Frobenius form, without logarithmic terms. When $N = 0$, the logarithmic term is necessary in order to obtain a second linearly independent solution. In either case, the second solution is defined only up to the addition of a constant multiple of y_i^+ .

After choosing a branch of $\log(z - z_i)$, the Frobenius solutions are single-valued on a simply connected domain of a punctured neighbourhood of z_i . The radius of convergence of the power series appearing in these solutions is at least the distance from z_i to the nearest other singular point of the equation. The same construction applies at $z = \infty$ after rewriting the equation in the local coordinate w as in (1.3).

The preceding local analysis distinguishes the parameters which determine the local exponents from those which affect only the global properties of the equation. For the general equation (1.1), the indicial equations (1.20) give

$$r_i = 1 - \rho_i^+ - \rho_i^-, \quad s_i = \rho_i^+ \rho_i^-, \quad i \in \{1, \dots, m, \infty\}, \quad (1.28)$$

at each singular point. Combining this relation with the definition of r_∞ in (1.6) gives the Fuchs relation

$$\sum_{i=1}^m (\rho_i^+ + \rho_i^-) + \rho_\infty^+ + \rho_\infty^- = m - 1. \quad (1.29)$$

Thus, the local exponents at the different singular points cannot be chosen independently.

Once the positions of the singularities and a compatible set of local exponents have been fixed, the coefficients r_i and s_i are completely determined. The remaining coefficients t_i satisfy

$$\sum_{i=1}^m t_i = 0, \quad \sum_{i=1}^m (s_i + z_i t_i) = s_\infty, \quad (1.30)$$

and the $m - 2$ independent combinations of the t_i are the *accessory parameters*. They are not determined by the local exponents, but control the global monodromy of the equation.

For an equation in normal form, one has $p(z) = 0$ and $q(z) = T(z)$, so that the coefficients in the discussion above specialize to $r_i = 0$, $s_i = \delta_i$ and $t_i = c_i$ at the finite singular points, while $r_\infty = 2$ and $s_\infty = \delta_\infty$. Therefore, once the positions and local exponents are fixed, the coefficients c_i satisfy

$$\sum_{i=1}^m c_i = 0, \quad \sum_{i=1}^m (\delta_i + z_i c_i) = \delta_\infty, \quad (1.31)$$

and the $m - 2$ independent combinations of the c_i are the accessory parameters of the equation in normal form.

The local exponents also determine the eigenvalues of the local monodromy, although in the resonant case they do not by themselves determine its Jordan form. For a non-resonant Frobenius basis $y_i^\pm$ at z_i , analytic continuation once counterclockwise around the singularity acts as

$$y_i^\pm \mapsto e^{2\pi i \rho_i^\pm} y_i^\pm. \quad (1.32)$$

Thus, with respect to the basis vector $(y_i^+, y_i^-)^T$, the local monodromy matrix is

$$M_i = \begin{pmatrix} e^{2\pi i \rho_i^+} & 0 \\ 0 & e^{2\pi i \rho_i^-} \end{pmatrix}. \quad (1.33)$$

In the resonant case, since $\rho_i^+ - \rho_i^- \in \mathbb{Z}$, the two monodromy eigenvalues coincide. Denoting their common value by σ_i , the logarithmic solutions in (1.27) transform under the same analytic continuation as

$$y_i^+ \mapsto \sigma_i y_i^+, \quad (1.34)$$

$$y_i^- \mapsto \sigma_i y_i^- + 2\pi i C_i \sigma_i y_i^+. \quad (1.35)$$

The corresponding local monodromy matrix is therefore

$$M_i = \sigma_i \begin{pmatrix} 1 & 0 \\ 2\pi i C_i & 1 \end{pmatrix}. \quad (1.36)$$

When $C_i = 0$, the local monodromy is diagonal, while for $C_i \neq 0$ it has a non-trivial Jordan block. In both cases its eigenvalues are $e^{2\pi i \rho_i^\pm}$.

Let now $\Sigma = \{z_1, \dots, z_m, \infty\} \subset \mathbb{CP}^1$ denote the set of singular points of the equation. Every point of the punctured sphere $\mathbb{CP}^1 \setminus \Sigma$ is ordinary and local solutions can therefore be analytically continued along any path which avoids Σ . Fixing a base point $z_* \in \mathbb{CP}^1 \setminus \Sigma$, we choose an ordered basis of local solutions near z_* ,

$$\mathbf{y}^{(z_*)} = \begin{pmatrix} y_1^{(z_*)} \\ y_2^{(z_*)} \end{pmatrix}. \quad (1.37)$$

Analytic continuation along a loop γ based at z_* produces another basis of local solutions $\mathbf{y}_\gamma^{(z_*)}$, which is related to the original one by a constant invertible matrix:

$$\mathbf{y}_\gamma^{(z_*)} = M_\gamma \mathbf{y}^{(z_*)}, \quad M_\gamma \in \mathrm{GL}(2; \mathbb{C}), \quad (1.38)$$

where the matrix M_γ depends only on the homotopy class of γ . Analytic continuation therefore defines the *monodromy representation*

$$\varrho : \pi_1(\mathbb{CP}^1 \setminus \Sigma, z_*) \longrightarrow \mathrm{GL}(2; \mathbb{C}), \quad \varrho([\gamma]) = M_\gamma, \quad (1.39)$$

its image being the *monodromy group* of the equation. A change of the basis $\mathbf{y}^{(z_*)}$ conjugates all the monodromy matrices simultaneously, so the monodromy representation is naturally defined only up to conjugation. Similarly, changing the base point and choosing a path connecting the two base points identifies the corresponding fundamental groups and produces an equivalent monodromy representation, again up to conjugation.

For an equation in normal form, the Wronskian of any two solutions is constant on every branch and analytic continuation along a loop γ multiplies it by $\det M_\gamma$. On the other hand, analytic continuation must preserve the value of the constant Wronskian. Since the Wronskian of a basis is nonzero, it follows that $\det M_\gamma = 1$ and therefore the monodromy representation of an equation in normal form takes values in $\mathrm{SL}(2; \mathbb{C})$. This conclusion applies in particular to the local Frobenius bases of an equation in normal form, both in the non-resonant and logarithmic cases.

To compare the local monodromies at different singular points, all the corresponding matrices must be expressed in the same basis near z_* . For each singular point z_i , we can choose a path from z_* to a punctured neighbourhood of z_i and use it to transport the local Frobenius basis to the base point. The monodromy of a small positively oriented loop around z_i , conjugated by this path, is then represented in the common base-point basis. Its conjugacy class is determined by the local matrices (1.33) and (1.36).

If $\Sigma = \{z_1, \dots, z_m, \infty\}$, the fundamental group of the punctured sphere is generated by based loops $\gamma_1, \dots, \gamma_m, \gamma_\infty$ around the singular points, subject to the relation

$$\gamma_1 \gamma_2 \cdots \gamma_m \gamma_\infty = 1, \quad (1.40)$$

for a compatible choice of orientations and ordering. The corresponding monodromy matrices, all expressed in the basis $\mathbf{y}^{(z_*)}$, therefore satisfy

$$M_1 M_2 \cdots M_m M_\infty = I. \quad (1.41)$$

At each finite regular singular point $z_i \in \Sigma$, let now

$$\mathbf{y}^{(z_i)} = \begin{pmatrix} y_i^+ \\ y_i^- \end{pmatrix}, \quad (1.42)$$

be a local basis of Frobenius solutions. At infinity, the corresponding basis is defined using the local coordinate $w = 1/z$. The domains of convergence of these local solutions do not, in general, cover the entire punctured sphere. Nevertheless, every ordinary point can be connected to a punctured neighbourhood of z_i by a path in $\mathbb{CP}^1 \setminus \Sigma$. Analytic continuation along such paths therefore extends a local Frobenius basis to a basis of solutions near any ordinary point.

Let γ be a path in $\mathbb{CP}^1 \setminus \Sigma$ connecting a neighbourhood of z_i to a neighbourhood of another singular point z_j . After analytic continuation along γ , the basis $\mathbf{y}^{(z_i)}$ can be expressed as a linear combination of the basis $\mathbf{y}^{(z_j)}$,

$$\mathbf{y}_\gamma^{(z_i)} = C_{ij}^\gamma \mathbf{y}^{(z_j)}, \quad C_{ij}^\gamma \in \mathrm{GL}(2; \mathbb{C}). \quad (1.43)$$

The matrix C_{ij}^γ is called the *connection matrix* between the two local bases and its entries are the corresponding *connection coefficients*. It is independent of z because, near the endpoint of γ , the analytically continued basis $\mathbf{y}_\gamma^{(z_i)}$ and the local basis $\mathbf{y}^{(z_j)}$ are bases of the same two-dimensional solution space.

The connection matrix depends, however, on the choices of branches and normalizations of the local solutions, as well as on the homotopy class of the path γ . Once these choices

have been fixed, the *connection problem* consists in determining C_{ij}^γ explicitly as a function of the parameters of the differential equation.

Frobenius bases associated with the singularities of the equation are not the only useful choices of solutions. One may choose a pants decomposition of the punctured sphere $\mathbb{CP}^1 \setminus \Sigma$ by cutting it along simple closed curves into three-punctured spheres. If the monodromy around one of these cutting curves is diagonalizable, one may choose a basis of solutions which diagonalizes it. On an annular neighbourhood of the cutting curve, these solutions can be regarded as Floquet solutions [32, 51, 52].

Connection problems can then be formulated between Frobenius bases near the punctures and bases adapted to the cutting curves. Curves dual to the cuts provide complementary monodromy data related to the gluing of the three-punctured components. This construction will be described explicitly for the four-punctured sphere in the discussion of the Heun equation.

Hypergeometric differential equation. Up to a Möbius transformation of the variable z and a gauge transformation of y , every second-order Fuchsian equation on the sphere with three regular singular points can be reduced to the hypergeometric equation

$$z(1-z)y'' + (c - (a+b+1)z)y' - aby = 0. \quad (1.44)$$

Its singular points are 0, 1 and ∞ , the corresponding pairs of local exponents being $(0, 1-c)$, $(0, c-a-b)$ and (a, b) , respectively. Then, for non-resonant values of the parameters a , b and c , Frobenius bases may be chosen as

$$y_1^{(0)}(z) = {}_2F_1(a, b, c; z), \quad (1.45a)$$

$$y_2^{(0)}(z) = z^{1-c} {}_2F_1(a-c+1, b-c+1, 2-c; z), \quad (1.45b)$$

$$y_1^{(1)}(z) = {}_2F_1(a, b, a+b-c+1; 1-z), \quad (1.45c)$$

$$y_2^{(1)}(z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1; 1-z), \quad (1.45d)$$

$$y_1^{(\infty)}(z) = (-z)^{-a} {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{z}\right), \quad (1.45e)$$

$$y_2^{(\infty)}(z) = (-z)^{-b} {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{z}\right). \quad (1.45f)$$

Here, ${}_2F_1(a, b, c; z)$ is the standard hypergeometric series

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (x)_n = \prod_{k=0}^{n-1} (x+k), \quad (x)_0 = 1. \quad (1.46)$$

The three pairs are local bases adapted to the three singular points. Their Frobenius series converge up to the nearest singularity and the corresponding local branches, together with analytic continuation, describe the solutions on the punctured sphere $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

The connection problem for the hypergeometric equation can be solved explicitly in terms of Gamma functions. Assuming again non-resonant values of the parameters a , b and c , we consider local basis vectors defined as

$$\mathbf{y}^{(0)}(z) = \begin{pmatrix} y_1^{(0)}(z) \\ y_2^{(0)}(z) \end{pmatrix}, \quad \mathbf{y}^{(1)}(z) = \begin{pmatrix} y_1^{(1)}(z) \\ y_2^{(1)}(z) \end{pmatrix}. \quad (1.47)$$

After choosing the branches determined by the real logarithms of z and $1-z$ for $z \in (0, 1)$, then extending by analytic continuation, one finds

$$\mathbf{y}^{(0)}(z) = C_{01} \mathbf{y}^{(1)}(z), \quad (1.48)$$

where

$$C_{01} = \begin{pmatrix} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} & \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \end{pmatrix}. \quad (1.49)$$

Since the three-punctured sphere has no accessory parameters, these connection coefficients are completely determined by the local exponents. Analogous connection matrices involving the basis at infinity can be found in [50].

Heun differential equation. Similarly to the previous example, up to a Möbius transformation of z and a gauge transformation of y , the general second-order Fuchsian equation on the sphere with four regular singular points is the Heun equation. We consider the normal form

$$y''(z) + T(z)y(z) = 0, \quad (1.50)$$

where

$$T(z) = \frac{\frac{1}{4} - \theta_0^2}{z^2} + \frac{\frac{1}{4} - \theta_q^2}{(z-q)^2} + \frac{\frac{1}{4} - \theta_1^2}{(z-1)^2} + \frac{\theta_0^2 + \theta_q^2 + \theta_1^2 - \theta_\infty^2 - \frac{1}{2}}{z(z-1)} + \frac{u}{z(z-1)(z-q)}. \quad (1.51)$$

The singular points are 0, q , 1 and ∞ . The parameters θ_0 , θ_q , θ_1 and θ_∞ determine the local exponents, while u is the accessory parameter. Assuming that the differences of exponents are non-integers, local Frobenius bases may be chosen in the form

$$y_i^\pm(z) = (z - z_i)^{\frac{1}{2} \pm \theta_i} (1 + \mathcal{O}(z - z_i)), \quad z_i \in \{0, q, 1\}, \quad (1.52)$$

and, near infinity,

$$y_\infty^\pm(z) = z^{\frac{1}{2} \pm \theta_\infty} (1 + \mathcal{O}(z^{-1})). \quad (1.53)$$

Thus, the local monodromy eigenvalues around z_i are given by $e^{2\pi i(\frac{1}{2} \pm \theta_i)}$, with possible non-trivial Jordan blocks in the resonant case.

Let now $0 < |q| < 1$. The annulus

$$\mathcal{A}_q = \{z \in \mathbb{C} \mid |q| < |z| < 1\} \quad (1.54)$$

contains no singular points and carries a distinguished homotopy class of closed curves encircling 0 and q [32, 52]. We call any representative of this class the A-cycle. Up to the choice of paths and base point, its monodromy matrix M_A is the product of the local monodromies around 0 and q , $M_A = M_0 M_q$.

If the A-cycle monodromy matrix M_A is diagonalizable, one can choose a basis of solutions $y_A^\pm$ consisting of its eigenvectors, where $\mu_A^\pm$ are the corresponding eigenvalues. The $y_A^\pm$ are *Floquet solutions* associated with the A-cycle and $\mu_A^\pm$ their *Floquet multipliers*. Choosing *Floquet exponents* $\alpha_\pm$ such that

$$\mu_A^\pm = e^{2\pi i \alpha_\pm}, \quad (1.55)$$

the Floquet solutions can be written as

$$y_A^\pm(z) = z^{\alpha_\pm} \phi_\pm(z), \quad \phi_\pm(z) = \sum_{n \in \mathbb{Z}} a_n^\pm z^n, \quad (1.56)$$

where $\phi_\pm$ are single-valued holomorphic functions on $\mathcal{A}_q$ and their Laurent series converge in the annulus. The Floquet exponents $\alpha_\pm$ are defined only modulo integers.

Since (1.50) is in normal form, its monodromy matrices belong to $\mathrm{SL}(2; \mathbb{C})$, and therefore $\mu_A^+ \mu_A^- = 1$. It is then convenient to introduce a monodromy parameter σ_A , defined modulo signs and integer shifts, such that

$$\mu_A^\pm = e^{\pm 2\pi i \sigma_A}, \quad \mathrm{tr} M_A = 2 \cos(2\pi \sigma_A). \quad (1.57)$$

The annulus $\mathcal{A}_q$ is naturally associated with a pants decomposition of the four-punctured sphere. Cutting along the A-cycle separates it into two three-punctured spheres, one containing the punctures 0 and q , the other containing 1 and ∞ . The conjugacy class of M_A specifies the monodromy along the separating curve, while the monodromy along a dual B-cycle depends on the connection matrix relating solution bases adapted to the two three-punctured spheres [32]. In contrast with the hypergeometric equation, these global monodromy data are not fixed by the local exponents: they depend on the accessory parameter u .

The assumption $0 < |q| < 1$ is not essential. For any other configuration of the punctures, one may choose an appropriate separating curve, possibly after a Möbius transformation, and repeat the same construction. The different choices of separating curve correspond to the different pants decompositions of the four-punctured sphere.

Let us define

$$\mathbf{y}^{(z_i)} = \begin{pmatrix} y_i^+ \\ y_i^- \end{pmatrix}, \quad \mathbf{y}^{(A)} = \begin{pmatrix} y_A^+ \\ y_A^- \end{pmatrix}. \quad (1.58)$$

The connection problem for the Heun equation consists in relating the Frobenius bases $\mathbf{y}^{(z_i)}$ near 0, q , 1, ∞ to one another, or to the Floquet basis $\mathbf{y}^{(A)}$ in the annulus. For example, after analytic continuation along a chosen path γ ,

$$\mathbf{y}_\gamma^{(z_i)} = C_{iA}^\gamma \mathbf{y}^{(A)}. \quad (1.59)$$

Unlike in the hypergeometric case, the entries of a Heun connection matrix are not generally given by elementary Gamma-function expressions [25, 32].

1.2 Second-order Fuchsian equations on the torus

Let $E_\tau = \mathbb{C}/\Lambda_\tau$ be a complex torus, where

$$\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}, \quad \mathrm{Im}(\tau) > 0. \quad (1.60)$$

We consider equations of the form

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0, \quad (1.61)$$

where z is a coordinate on the universal cover $\mathbb{C}$ of the torus, while p and q are meromorphic elliptic functions:

$$p(z + \lambda) = p(z), \quad q(z + \lambda) = q(z), \quad \lambda \in \Lambda_\tau. \quad (1.62)$$

The periodicity of the coefficients ensures that (1.61) descends to a differential equation on E_τ . However, its solutions on the universal cover are not, in general, elliptic functions. Their transformation under lattice translations will instead be described by the monodromy along the two fundamental cycles of the torus, as we will discuss in greater detail below.

Let $\zeta_0 \in E_\tau$, and let $z_0 \in \mathbb{C}$ be a representative on the universal cover. The point ζ_0 is ordinary if p and q are holomorphic at $z = z_0$, and it is a singular point otherwise. A singular point is then regular if $p(z)$ and $q(z)$ have at most a simple and a double pole

at $z = z_0$, respectively. Since the coefficients of the equation are elliptic, this condition is independent of the chosen representative of ζ_0 . The equation is called Fuchsian if all its singular points on E_τ are regular singular points.

Let $\zeta_1, \dots, \zeta_m \in E_\tau$ be the singular points of (1.61), with $z_1, \dots, z_m \in \mathbb{C}$ representatives on the universal cover. The Fuchsian condition requires the local expansions of $p(z)$ and $q(z)$ to take the form

$$p(z) = \frac{r_i}{z - z_i} + \mathcal{O}(1), \quad q(z) = \frac{s_i}{(z - z_i)^2} + \frac{t_i}{z - z_i} + \mathcal{O}(1), \quad z \rightarrow z_i. \quad (1.63)$$

To write these coefficients globally, we use the Weierstrass functions $\wp, \zeta$ associated with the lattice Λ_τ ; see Appendix A.1 for their definitions. Their local behaviour at the origin is

$$\zeta(z) = \frac{1}{z} + \mathcal{O}(z^3), \quad \wp(z) = \frac{1}{z^2} + \mathcal{O}(z^2). \quad (1.64)$$

Recall that $\wp$ is elliptic, while ζ is quasi-periodic:

$$\zeta(z + \lambda) = \zeta(z) + 2\eta_\lambda, \quad \lambda \in \Lambda_\tau, \quad (1.65)$$

where, for $\lambda = n + m\tau$ with $n, m \in \mathbb{Z}$,

$$\eta_\lambda = n\eta_1 + m\eta_\tau. \quad (1.66)$$

The most general elliptic coefficients with the prescribed pole structure are then

$$p(z) = p_0 + \sum_{i=1}^m r_i \zeta(z - z_i), \quad p_0 \in \mathbb{C}, \quad \sum_{i=1}^m r_i = 0, \quad (1.67a)$$

$$q(z) = q_0 + \sum_{i=1}^m (s_i \wp(z - z_i) + t_i \zeta(z - z_i)), \quad q_0 \in \mathbb{C}, \quad \sum_{i=1}^m t_i = 0. \quad (1.67b)$$

The constraints on r_i and t_i ensure that the quasi-periodic contributions of the ζ -functions cancel.

As on the Riemann sphere, the first-derivative term can be removed locally by rescaling the dependent variable from $y(z)$ to $\psi(z)$ as in (1.7). This transforms (1.61) into the normal form

$$\psi''(z) + T(z)\psi(z) = 0, \quad (1.68)$$

where $T(z)$, as a function of $q(z)$ and $p(z)$, reads as in (1.9). Since p and q are elliptic, T is also an elliptic function and therefore defines a global meromorphic potential on E_τ .

For an equation in normal form, the Fuchsian condition requires the elliptic potential $T(z)$ to have at most double poles at the singular points z_i . Its most general form is therefore

$$T(z) = \sum_{i=1}^m (\delta_i \wp(z - z_i) + c_i \zeta(z - z_i)) + E, \quad \sum_{i=1}^m c_i = 0. \quad (1.69)$$

The local theory on the torus is identical to that on the sphere. In particular, the expansions (1.63) lead to the indicial equation (1.20), with coefficients r_i and s_i while, for an equation in normal form, the analogous expansion of $T(z)$ leads to (1.22), with δ_i as the coefficient of the double pole. The resulting Frobenius local solutions, including the possible logarithmic terms in the resonant case, are therefore exactly as described in the previous subsection.

On the universal cover, the only difference is that the singularities are repeated over the lattice. Thus, the Frobenius series centered at z_i converge at least in the largest disc containing no other point of the lifted singular set

$$\tilde{\Sigma} = \{z_j + \lambda \mid j = 1, \dots, m, \lambda \in \Lambda_\tau\}. \quad (1.70)$$

As in the sphere case, the local exponents $\rho_i^\pm$ determine the parameters r_i and s_i in (1.67) through

$$r_i = 1 - \rho_i^+ - \rho_i^-, \quad s_i = \rho_i^+ \rho_i^-, \quad (1.71)$$

and the ellipticity condition $\sum_i r_i = 0$ gives the Fuchs relation

$$\sum_{i=1}^m (\rho_i^+ + \rho_i^-) = m. \quad (1.72)$$

Once the singular points and a compatible set of local exponents have been fixed, the constants p_0 and q_0 , together with $m - 1$ independent linear combinations of the coefficients t_i , are accessory parameters.

For an equation in normal form, the local exponents determine $\delta_i = \rho_i^+ \rho_i^-$, while the constant E and $m - 1$ independent linear combinations of the coefficients c_i are accessory parameters. The local monodromy around each regular singular point is described exactly as on the sphere. In particular, the local exponents determine its eigenvalues as in (1.33), while in the resonant case they do not by themselves determine its Jordan form, as illustrated by (1.36).

Let $\Sigma = \{\zeta_1, \dots, \zeta_m\}$, and consider the m -punctured torus $E_\tau \setminus \Sigma$. Repeating the construction given for the punctured sphere, analytic continuation of a basis chosen near a base point $\zeta_* \in E_\tau \setminus \Sigma$ defines a monodromy representation

$$\varrho : \pi_1(E_\tau \setminus \Sigma, \zeta_*) \longrightarrow \mathrm{GL}(2; \mathbb{C}). \quad (1.73)$$

As before, changing the basis conjugates all the monodromy matrices simultaneously. Changing the base point, after choosing a path connecting the two base points, likewise produces an equivalent representation up to conjugation. For an equation in normal form, the monodromy representation takes values in $\mathrm{SL}(2; \mathbb{C})$.

In contrast with the punctured sphere, the fundamental group of $E_\tau \setminus \Sigma$ contains two additional generators, which we denote by γ_A and γ_B , corresponding to the two fundamental cycles of the torus. Together with positively oriented loops γ_i around the punctures ζ_i , they satisfy

$$\gamma_A \gamma_B \gamma_A^{-1} \gamma_B^{-1} \gamma_1 \cdots \gamma_m = 1, \quad (1.74)$$

for a compatible choice of paths and ordering. Consequently, with all monodromy matrices expressed in the same base-point basis,

$$M_A M_B M_A^{-1} M_B^{-1} M_1 \cdots M_m = I. \quad (1.75)$$

After choosing a lift of the base point to $\mathbb{C}$, the cycles γ_A and γ_B lift to paths whose endpoints differ by the periods 1 and τ of the torus E_τ , respectively. In the presence of punctures, M_A and M_B need not commute: their commutator is related by (1.75) to the product of the local monodromies. In the absence of singularities, the relation reduces to $M_A M_B = M_B M_A$.

Connection problems on the torus are defined exactly as on the sphere. Given local Frobenius bases near two singular points ζ_i and ζ_j and a path γ connecting their neighbourhoods, analytic continuation defines the connection matrix C_{ij}^γ as in (1.43). Its dependence on the choices of local bases and on the homotopy class of γ is unchanged.

The new feature on the torus is that paths with the same endpoints may differ not only by loops around the singular points, but also by windings around the two fundamental cycles. On the universal cover, a lift of a continuation path from a neighbourhood of z_i may

end near $z_j + \lambda$, with $\lambda \in \Lambda_\tau$. After identifying $z_j + \lambda$ with z_j on the torus, the resulting connection matrix incorporates the corresponding period monodromy.

As on the punctured sphere, local Frobenius bases are not the only useful choices of solutions. The construction based on a pants decomposition extends naturally to the punctured torus [51]. For $m \geq 1$, a pants decomposition of an m -punctured torus involves m disjoint cutting curves. One of them may be chosen to be non-separating and to represent one of the fundamental cycles of the torus, while the remaining curves separate the punctures into three-punctured components. Dual curves take into account the corresponding gluing data. These cutting and dual curves depend on the chosen decomposition and are not additional independent homology cycles of the torus.

If the monodromy around a chosen cutting curve is diagonalizable, one may select a basis of solutions which diagonalizes it, exactly as in the sphere case. When the cutting curve represents one of the fundamental cycles of the torus, these solutions are naturally interpreted as Floquet solutions along that cycle. Connection problems can then be formulated between such a Floquet basis and a local Frobenius basis, or between bases adapted to different cycles.

In general, monodromy matrices associated with different cycles do not commute and hence cannot be diagonalized simultaneously. The choice of a Floquet basis is therefore tied to a particular cutting curve. The construction will be illustrated explicitly for the one-punctured torus in the discussion of the Lamé equation.

Lamé differential equation. The Lamé equation is the basic example of a second-order Fuchsian equation on a complex torus. In Weierstrass form, it reads

$$\left(\partial_z^2 - \mu(\mu - 1)\wp(z) + E\right)\psi(z) = 0, \quad (1.76)$$

where E is an accessory parameter, which plays the role of the spectral parameter. Since $\wp(z)$ has a double pole at the lattice points and no other poles, the equation has a single regular singular point on E_τ , located at $z = 0$ modulo Λ_τ .

Near the origin, one obtains the indicial equation

$$\rho(\rho - 1) - \mu(\mu - 1) = 0, \quad (1.77)$$

and hence the local exponents

$$\rho_+ = \mu, \quad \rho_- = 1 - \mu. \quad (1.78)$$

When $2\mu - 1 \notin \mathbb{Z}$, a normalized local Frobenius basis can be chosen in the form

$$\psi_+(z) = z^\mu \left(1 + \mathcal{O}(z^2)\right), \quad \psi_-(z) = z^{1-\mu} \left(1 + \mathcal{O}(z^2)\right), \quad (1.79)$$

the corresponding local monodromy eigenvalues being $e^{2\pi i\mu}$ and $e^{-2\pi i\mu}$. When $2\mu - 1 \in \mathbb{Z}$ logarithmic terms may instead occur. However, for $\mu = 0, 1$ the coefficient of $\wp(z)$ vanishes and $z = \lambda$, $\lambda \in \Lambda_\tau$, is an ordinary point, while for $\mu \in \mathbb{Z} \setminus \{0, 1\}$, $z = \lambda$ is an apparent singularity with trivial local monodromy.

Let M_A and M_B denote the monodromy matrices along the two fundamental cycles of the torus, and let M_0 be the local monodromy matrix around the puncture. The general relation (1.75) specializes to

$$M_A M_B M_A^{-1} M_B^{-1} M_0 = I. \quad (1.80)$$

If M_A is diagonalizable, one may choose Floquet solutions $\psi_A^\pm$ satisfying

$$\psi_A^\pm(z+1) = e^{\pm 2\pi i \sigma_A} \psi_A^\pm(z), \quad (1.81)$$

on the universal cover. Floquet solutions along the B-cycle are defined analogously. When $\mu \in \mathbb{Z}$, one has $M_0 = I$, and therefore M_A and M_B commute. If they are diagonalizable, they can be diagonalized simultaneously, giving solutions with prescribed Floquet multipliers along both periods.

The Lamé equation is closely related to the Heun equation through the double covering

$$x = \wp(z) : E_\tau \longrightarrow \mathbb{CP}^1. \quad (1.82)$$

The three zeros of $\wp'(z)$ are mapped to

$$e_1 = \wp\left(\frac{1}{2}\right), \quad e_2 = \wp\left(\frac{1+\tau}{2}\right), \quad e_3 = \wp\left(\frac{\tau}{2}\right). \quad (1.83)$$

while $z = 0$ is mapped to $x = \infty$. Using

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \quad (1.84)$$

and writing $\psi(z) = y(\wp(z))$, equation (1.76) becomes

$$y''(x) + \frac{1}{2} \left(\frac{1}{x - e_1} + \frac{1}{x - e_2} + \frac{1}{x - e_3} \right) y'(x) - \frac{\mu(\mu - 1)x - E}{4(x - e_1)(x - e_2)(x - e_3)} y(x) = 0, \quad (1.85)$$

which is a special Heun equation with regular singular points e_1, e_2, e_3 and ∞ .

The connection problem may therefore be described either on the torus or through the associated Heun equation. On the torus, we consider the basis vectors

$$\psi^{(0)} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi^{(A)} = \begin{pmatrix} \psi_A^+ \\ \psi_A^- \end{pmatrix}. \quad (1.86)$$

After analytic continuation along a suitable path γ , they are related by

$$\psi_\gamma^{(0)} = C_{0A}^\gamma \psi^{(A)}, \quad (1.87)$$

where the entries of C_{0A}^γ depend in general on the spectral parameter E , as discussed in Chapter II.

Section 2

Two-Dimensional Conformal Field Theory

In this second background section, we briefly review the aspects of two-dimensional conformal field theory that will be relevant in later chapters. Virasoro conformal blocks play a central role, together with the BPZ equations they satisfy in the presence of degenerate primary fields. We discuss their fusion and braiding properties, which provide analytic continuation formulas for their local expansions, as well as their classical limits, which establish a connection with the Fuchsian equations reviewed in the previous section. Once again, we treat the sphere and torus cases separately and show how the hypergeometric, Heun and Lamé equations arise in the context of two-dimensional conformal field theory.

General references. For background and more detailed explanations of the material reviewed in this section, see [53–60].

2.1 Conformal symmetry in two dimensions

In the following, we work in Euclidean signature, possibly after a Wick rotation from a Lorentzian theory. On a local coordinate patch, we consider the complex coordinates

$$z = x + iy, \quad \bar{z} = x - iy, \quad (2.1)$$

which are treated as independent complex variables. The physical Euclidean plane is recovered by restricting to the real slice $\bar{z} = z^*$. The distinctive feature of conformal symmetry in two dimensions is that local conformal transformations factorize as

$$z \mapsto f(z), \quad \bar{z} \mapsto \bar{f}(\bar{z}), \quad (2.2)$$

where f and $\bar{f}$ are invertible locally holomorphic functions. On the physical real slice they are related by complex conjugation, while in the “complexified” theory they can be treated independently. Since these functions are otherwise arbitrary, the local conformal symmetry algebra is infinite-dimensional and splits into commuting holomorphic and antiholomorphic sectors. In the following, we focus on the holomorphic sector and every statement has an analogous antiholomorphic counterpart.

Expanding an infinitesimal holomorphic conformal transformation in Laurent modes around a point gives the vector fields

$$\ell_n = -z^{n+1} \partial_z, \quad n \in \mathbb{Z}, \quad (2.3)$$

which generate the Witt algebra. Its unique non-trivial one-dimensional central extension is the Virasoro algebra, generated by L_n , $n \in \mathbb{Z}$, with commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}. \quad (2.4)$$

The parameter c is called the *central charge* and it is part of the defining data of the conformal theory. The representations most relevant to the following discussion are highest-weight representations. They are generated from a highest-weight, or *primary*, vector $|\Delta\rangle$ satisfying

$$L_0 |\Delta\rangle = \Delta |\Delta\rangle, \quad L_n |\Delta\rangle = 0, \quad n > 0. \quad (2.5)$$

The eigenvalue Δ is the *holomorphic conformal weight*. In the following we will often allow c and Δ to be complex, and we will use the Liouville parametrization

$$\Delta_P = P(Q - P), \quad c = 1 + 6Q^2, \quad Q = b + b^{-1}, \quad b \in \mathbb{C}^\times. \quad (2.6)$$

This parametrization is invariant under $b \mapsto b^{-1}$, while the conformal weight Δ_P is invariant under the reflection $P \mapsto Q - P$.

The Verma module associated with $|\Delta\rangle$ is obtained by acting freely on the highest-weight state with the negative Virasoro modes,

$$\mathcal{V}_{c,\Delta} = \text{span}_{\mathbb{C}} \{L_{-n_1} \cdots L_{-n_k} |\Delta\rangle \mid k \geq 0, \quad n_1 \geq \cdots \geq n_k \geq 1\}. \quad (2.7)$$

The vectors obtained in this way are called *descendants*. Their level is $N = \sum_{i=1}^k n_i$ and their L_0 eigenvalue is $\Delta + N$. At fixed level N , a basis is labelled by the partitions of N , or equivalently by Young diagrams with N boxes. The standard $\mathbb{C}$ -bilinear form on $\mathcal{V}_{c,\Delta}$, called the Shapovalov form, is fixed by

$$L_n^\dagger := L_{-n}, \quad \langle \Delta | \Delta \rangle := 1, \quad (2.8)$$

where $\dagger$ denotes the Virasoro anti-involution. For generic complex values of c and Δ , the Shapovalov form cannot be interpreted as a positive-definite Hermitian inner product.

A highest-weight representation is called *degenerate* when its Verma module contains a non-zero singular vector at some positive level N . This is a descendant which is itself primary:

$$|\chi\rangle = \sum_{\lambda \vdash N} a_\lambda L_{-\lambda} |\Delta\rangle, \quad L_{n>0} |\chi\rangle = 0, \quad L_0 |\chi\rangle = (\Delta + N) |\chi\rangle, \quad (2.9)$$

where $a_\lambda = a_{\lambda_1 \dots \lambda_\ell}$ and $L_{-\lambda} = L_{-\lambda_1} \cdots L_{-\lambda_\ell}$, for a partition $\lambda = \{\lambda_1, \dots, \lambda_\ell\}$ of N . Such a vector generates a proper submodule of $\mathcal{V}_{c,\Delta}$. With respect to the Shapovalov form, every vector in this submodule is orthogonal to every vector in $\mathcal{V}_{c,\Delta}$, including the other vectors in the same submodule. The irreducible highest-weight module is obtained by quotienting the Verma module by its maximal proper submodule $\mathcal{M}_{c,\Delta}$, generated by all its singular vectors and their descendants. Equivalently, all singular vectors and their descendants are identified with zero in the irreducible quotient.

The degenerate highest weights are classified by the Kac labels (r, s) , with

$$P_{r,s} = \frac{1-r}{2}b + \frac{1-s}{2}b^{-1}, \quad \Delta_{r,s} = \Delta_{P_{r,s}}, \quad r, s \in \mathbb{Z}_{>0}. \quad (2.10)$$

For generic b , the Verma module $\mathcal{V}_{c,\Delta_{r,s}}$ contains a singular vector at level rs . Up to overall normalization, the lowest-level examples are

$$|\chi_{1,1}\rangle = L_{-1} |\Delta_{1,1}\rangle, \quad (2.11a)$$

$$|\chi_{2,1}\rangle = (L_{-1}^2 + b^2 L_{-2}) |\Delta_{2,1}\rangle, \quad (2.11b)$$

$$|\chi_{1,2}\rangle = (L_{-1}^2 + b^{-2} L_{-2}) |\Delta_{1,2}\rangle. \quad (2.11c)$$

At special values of b , additional singular vectors may occur.

More generally, a two-dimensional conformal field theory may possess a chiral symmetry algebra larger than the Virasoro algebra. Important examples include affine Kac–Moody algebras and $\mathcal{W}_N$ algebras. In this thesis, we restrict the discussion to Virasoro symmetry, indicating when the constructions admit generalizations to larger chiral symmetry algebras.

2.2 2d CFT on the sphere

We now consider a conformal field theory on the sphere $\mathbb{CP}^1$, denoting by z the standard affine coordinate on $\mathbb{C} \subset \mathbb{CP}^1$, and by $\bar{z}$ its antiholomorphic counterpart. In a neighbourhood of infinity, a local coordinate is instead given by $w = 1/z$.

In radial quantization, the state-operator correspondence identifies states with local fields inserted at the origin. In particular, highest-weight states correspond to primary fields, while their Virasoro descendants correspond to descendant fields. More generally, every state in the state space is associated with a local field and vice versa.

For simplicity we focus on diagonal two-dimensional CFTs so that, in particular, $\Delta = \bar{\Delta}$ and $c = \bar{c}$. We therefore denote a highest-weight state $|\Delta, \bar{\Delta}\rangle = |\Delta, \Delta\rangle$ of $\text{Vir}_c \otimes \bar{\text{Vir}}_c$ simply by $|\Delta\rangle$. Similarly, we label a primary field $\phi_{\Delta, \bar{\Delta}}$, whose total scaling dimension is $\Delta + \bar{\Delta} = 2\Delta$, only by its holomorphic conformal weight or, equivalently, by the corresponding *Liouville momentum*:

$$\phi_{\Delta, \bar{\Delta}} \equiv \phi_{\Delta} \equiv \phi_P, \quad \Delta = P(Q - P), \quad P \sim Q - P. \quad (2.12)$$

Under a local conformal change of coordinates $w = f(z)$, $\bar{w} = \bar{f}(\bar{z})$, a primary field transforms according to

$$\phi'_{\Delta}(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\Delta} \phi_{\Delta}(z, \bar{z}). \quad (2.13)$$

The state associated with a primary field is obtained by inserting the field at the origin and letting it act on the conformally invariant vacuum:

$$|\Delta\rangle = \lim_{z, \bar{z} \rightarrow 0} \phi_{\Delta}(z, \bar{z}) |0\rangle = \phi_{\Delta}(0) |0\rangle. \quad (2.14)$$

The vacuum corresponds to the identity operator and is annihilated by the generators of global conformal transformations, as well as by all positive Virasoro modes:

$$L_n |0\rangle = \bar{L}_n |0\rangle = 0, \quad n \geq -1. \quad (2.15)$$

In radial quantization around the origin, the holomorphic and antiholomorphic components of the stress-energy tensor admit the Laurent expansions

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}, \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \frac{\bar{L}_n}{\bar{z}^{n+2}}. \quad (2.16)$$

Thus, the Virasoro generators L_n and $\bar{L}_n$ are the Laurent modes of the corresponding components of the stress-energy tensor.

The defining transformation property (2.13) of a primary field can be expressed through its commutators with these generators. For a primary field of holomorphic conformal weight Δ , one has

$$[L_n, \phi_\Delta(z, \bar{z})] = \left(z^{n+1} \partial_z + (n+1) \Delta z^n \right) \phi_\Delta(z, \bar{z}), \quad (2.17)$$

and similarly for the antiholomorphic case.

A compact way of encoding the local properties of fields is through the *operator product expansion* (OPE). The OPE is an expansion, valid inside correlation functions, which describes the behaviour of a product of local fields as their insertion points approach one another. For two primary fields, it takes the form

$$\begin{aligned} \phi_{P_i}(z_1, \bar{z}_1) \phi_{P_j}(z_2, \bar{z}_2) &= \sum_P C_{P_i P_j}^P z_{12}^{\Delta_P - \Delta_{P_i} - \Delta_{P_j}} \bar{z}_{12}^{\Delta_P - \Delta_{P_i} - \Delta_{P_j}} \\ &\times \sum_{N, \bar{N} \geq 0} z_{12}^N \bar{z}_{12}^{\bar{N}} \sum_{\substack{|\lambda|=N \\ |\bar{\lambda}|=\bar{N}}} \beta_{P_i P_j}^{P, \lambda} \bar{\beta}_{P_i P_j}^{P, \bar{\lambda}} L_{-\lambda} \bar{L}_{-\bar{\lambda}} \phi_P(z_2, \bar{z}_2), \end{aligned} \quad (2.18)$$

where $z_{12} = z_1 - z_2$, $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$, and $\lambda, \bar{\lambda}$ are partitions of $N, \bar{N}$ respectively. The sum over the momentum P is schematic: it runs over the primary operators of the theory under consideration and may be replaced by an integral in theories with a continuous spectrum. The coefficients $C_{P_i P_j}^P$ are the *OPE coefficients* of the theory. Once these coefficients are specified, the relative contributions of the Virasoro descendants are fixed by conformal symmetry in terms of the central charge and the conformal weights.

The conformal families that may appear in the OPE of two primary fields, together with their multiplicities, define the corresponding *fusion rule*:

$$[\phi_{P_i}] \times [\phi_{P_j}] = \sum_P N_{P_i P_j}^P [\phi_P]. \quad (2.19)$$

Here $[\phi_{P_i}]$ denotes the conformal family generated by the primary field ϕ_{P_i} , while $N_{P_i P_j}^P$ counts the number of independent ways in which the family $[\phi_P]$ can occur in the OPE.

The transformation law (2.17) of a primary field is then equivalently encoded in its OPE with the holomorphic stress-energy tensor:

$$T(w) \phi_\Delta(z, \bar{z}) \sim \frac{\Delta}{(w-z)^2} \phi_\Delta(z, \bar{z}) + \frac{1}{w-z} \partial_z \phi_\Delta(z, \bar{z}), \quad (2.20)$$

where the symbol $\sim$ indicates equality of the singular terms as $w \rightarrow z$. An analogous expression holds in the antiholomorphic sector. The holomorphic stress-energy tensor is not a Virasoro primary field when $c \neq 0$, but it is a quasi-primary, as follows from its self-OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \quad (2.21)$$

Equivalently, under a holomorphic change of coordinates $w = f(z)$, the stress-energy tensor transforms as

$$T'(w) = \left(\frac{dz}{dw} \right)^2 \left[T(z) - \frac{c}{12} \{w, z\} \right], \quad (2.22)$$

where

$$\{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2} \left(\frac{w''(z)}{w'(z)} \right)^2, \quad (2.23)$$

is the Schwarzian derivative. For global conformal transformations, the Schwarzian derivative vanishes and $T(z)$ transforms as a field of holomorphic weight two.

Under the state-operator correspondence, the Virasoro descendants of a primary state correspond to descendant fields. In particular, the state $L_{-n}|\Delta\rangle$ is associated with the field $(L_{-n}\phi_\Delta)(z, \bar{z})$ so that, for example, the holomorphic stress-energy tensor $T(z)$ is the L_{-2} -descendant of the identity operator. Using the mode expansion of the stress-energy tensor, one can write

$$(L_{-n}\phi_\Delta)(z, \bar{z}) = \frac{1}{2\pi i} \oint_z \frac{dw}{(w-z)^{n-1}} T(w) \phi_\Delta(z, \bar{z}), \quad n \geq 1, \quad (2.24)$$

where the contour is positively oriented around z and encloses no other operator insertions. An analogous construction using $\bar{T}(\bar{w})$ defines the antiholomorphic descendants. In particular, the OPE between $T(z)$ and a primary field implies

$$(L_{-1}\phi_\Delta)(z, \bar{z}) = \partial_z \phi_\Delta(z, \bar{z}). \quad (2.25)$$

The main observables of a two-dimensional conformal field theory are correlation functions of local fields, in particular of primary fields. For primary fields ϕ_i of holomorphic conformal weights Δ_i , we consider

$$\langle \phi_1(z_1, \bar{z}_1) \cdots \phi_N(z_N, \bar{z}_N) \rangle. \quad (2.26)$$

Then, using the contour representation of descendant fields (2.24), one finds the *Virasoro Ward identities*

$$\left\langle (L_{-n}\phi_\Delta)(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle = \mathcal{L}_{-n}(z) \left\langle \phi_\Delta(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle, \quad (2.27)$$

where

$$\mathcal{L}_{-n}(z) = \sum_{i=1}^N \left(\frac{(n-1)\Delta_i}{(z_i - z)^n} - \frac{1}{(z_i - z)^{n-1}} \partial_{z_i} \right). \quad (2.28)$$

This implies that correlation functions involving descendant fields can be obtained by applying suitable differential operators to correlation functions of primary fields.

Taking the primary field at z to be the identity and setting $n = 2$ reproduces the conformal Ward identity for an insertion of $T(z)$. Setting $n = 1$, together with (2.25), gives instead the following global conformal Ward identity

$$\left(\partial_z + \sum_{i=1}^N \partial_{z_i} \right) \left\langle \phi_\Delta(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle = 0. \quad (2.29)$$

The remaining holomorphic global conformal Ward identities on the sphere are obtained by integrating $T(w)$ against the globally defined holomorphic vector fields $w\partial_w$ and $w^2\partial_w$, and read

$$\left(z\partial_z + \Delta + \sum_{i=1}^N (z_i\partial_{z_i} + \Delta_i) \right) \left\langle \phi_\Delta(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle = 0 \quad (2.30a)$$

$$\left(z^2\partial_z + 2\Delta z + \sum_{i=1}^N (z_i^2\partial_{z_i} + 2\Delta_i z_i) \right) \left\langle \phi_\Delta(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle = 0 \quad (2.30b)$$

On the sphere, the above global conformal Ward identities allow three insertion points of a correlation function to be fixed by a $\text{PSL}(2; \mathbb{C})$ transformation, these points being conventionally chosen to be 0, 1 and ∞ , where we use the standard definition

$$\phi_\Delta(\infty) = \lim_{z, \bar{z} \rightarrow \infty} z^{2\Delta} \bar{z}^{2\Delta} \phi_\Delta(z, \bar{z}), \quad (2.31)$$

for a primary field inserted at infinity. Thus, for $N \geq 3$, an N -point function on the sphere can be reduced to a function of $N - 3$ independent complex variables, which may be chosen as cross-ratios

$$G_N(\Delta; \mathbf{x}, \bar{\mathbf{x}}) = \left\langle \phi_N(\infty) \phi_{N-1}(1) \prod_{i=2}^{N-2} \phi_i(x_i, \bar{x}_i) \phi_1(0) \right\rangle, \quad (2.32)$$

where the dependence of G_N on the central charge has been suppressed. Furthermore, the functional form of two- and three-point functions is completely fixed by global conformal invariance. Assuming a discrete spectrum and choosing a basis in which the two-point functions are orthonormal, one has

$$\langle \phi_i(z_i, \bar{z}_i) \phi_j(z_j, \bar{z}_j) \rangle = \frac{\delta_{P_i P_j}}{z_{ij}^{2\Delta_i} \bar{z}_{ij}^{2\Delta_i}}, \quad (2.33a)$$

$$\begin{aligned} \langle \phi_i(z_i, \bar{z}_i) \phi_j(z_j, \bar{z}_j) \phi_k(z_k, \bar{z}_k) \rangle &= \frac{C_{P_i P_j P_k}}{z_{ij}^{\Delta_i + \Delta_j - \Delta_k} z_{ik}^{\Delta_i + \Delta_k - \Delta_j} z_{jk}^{\Delta_j + \Delta_k - \Delta_i}} \\ &\times \frac{1}{\bar{z}_{ij}^{\Delta_i + \Delta_j - \Delta_k} \bar{z}_{ik}^{\Delta_i + \Delta_k - \Delta_j} \bar{z}_{jk}^{\Delta_j + \Delta_k - \Delta_i}}, \end{aligned} \quad (2.33b)$$

where $z_{ij} = z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$ and $C_{P_i P_j P_k}$ are related in a simple way to the OPE coefficients of the theory.

For $N > 3$, global conformal invariance does not completely determine the correlation functions, which retain a non-trivial dependence on the moduli of the punctured sphere. They can nevertheless be decomposed by repeatedly applying the OPE. A choice of the order in which the OPEs are performed defines an OPE channel, or equivalently a pants decomposition of the punctured sphere. For each intermediate primary representation, the contributions of the primary field and all its Virasoro descendants are collected into *Virasoro conformal blocks*, which are the subject of the next subsection.

2.2.1 Virasoro sphere conformal blocks

Virasoro sphere conformal blocks are universal functions of the moduli of the punctured sphere and of the representation-theoretic data, namely the central charge and the external and intermediate conformal weights. Their functional form is fixed by Virasoro symmetry, while the spectrum of representations appearing in the decomposition and the corresponding OPE coefficients are theory-dependent data. Consider, for example, the four-point function

$$G_4(\Delta; x, \bar{x}) = \langle \phi_4(\infty) \phi_3(1) \phi_2(x, \bar{x}) \phi_1(0) \rangle. \quad (2.34)$$

In the *s-channel*, obtained by first applying the OPE between ϕ_2 and ϕ_1 , its conformal-block decomposition takes the schematic form

$$G_4(\Delta; x, \bar{x}) = \sum_A C_{P_1 P_2}^A C_{A P_3 P_4} \mathcal{F}^{(s)}(A, \mathbf{P}; x) \bar{\mathcal{F}}^{(s)}(A, \mathbf{P}; \bar{x}), \quad (2.35)$$

where again the dependence on the central charge is understood. Assuming here and below the standard OPE normalization, the *s-channel* block has the local behaviour

$$\mathcal{F}^{(s)}(A, \mathbf{P}; x) = x^{\Delta_A - \Delta_{P_1} - \Delta_{P_2}} (1 + \mathcal{O}(x)), \quad x \rightarrow 0, \quad (2.36)$$

and is therefore a local expansion naturally adapted to the OPE limit $x \rightarrow 0$.

More generally, an N -point conformal block on the sphere is specified by a choice of OPE channel and by the $N - 3$ primary representations propagating in its internal channels, different OPE channels giving different decompositions of the same correlation function.

It is often useful to represent an OPE channel and its corresponding conformal block by a trivalent graph. The external edges correspond to the inserted primary fields, the internal edges are labelled by intermediate primary representations, and each trivalent vertex represents an elementary fusion of three representations [61]. For example, the s -channel four-point conformal block in (2.35) is represented by

$$\mathcal{F}^{(s)}(A, \mathbf{P}; x) = \begin{array}{c} P_3 \qquad P_2 \\ | \qquad | \\ \hline P_4 \qquad A \qquad P_1 \end{array} . \quad (2.37)$$

Once the external states and local coordinates at the punctures are fixed, conformal blocks can be viewed as locally holomorphic functions on the moduli space of punctured Riemann surfaces. The conformal blocks associated with a chosen OPE channel furnish a local basis of solutions to the Virasoro Ward identities. Geometrically, the choice of OPE channel is encoded by a pants decomposition [60–62]. The associated sewing parameters provide local coordinates on the corresponding region of moduli space, while the elements of the conformal-block basis are labeled by the representations propagating through the internal channels. In the corresponding OPE domain, each block admits a convergent power-series expansion in these parameters. Beyond this domain, the blocks are obtained by analytic continuation and, since they are generally multivalued, the result may depend on the continuation path.

More precisely, the analytic continuation of conformal blocks between the OPE domains associated with different pants decompositions is implemented by fusion and braiding transformations, which relate the corresponding local bases. Fusion changes the order of successive OPEs, while braiding exchanges two fields through analytic continuation of their insertion points. The corresponding kernels depend on the central charge, the external and intermediate conformal weights, the normalization of the blocks, and the continuation paths. Together with higher-genus modular transformations, they constitute the Moore–Seiberg data [61].

The four-point conformal blocks provide the prototypical example. Besides the s -channel decomposition given in (2.35), the same four-point function G_4 admits the following t -channel and u -channel decompositions

$$G_4(\Delta; x, \bar{x}) = \sum_A C_{P_2 P_3}^A C_{A P_1 P_4} \mathcal{F}^{(t)}(A, \mathbf{P}; x) \bar{\mathcal{F}}^{(t)}(A, \mathbf{P}; \bar{x}), \quad (2.38a)$$

$$G_4(\Delta; x, \bar{x}) = \sum_A C_{P_2 P_4}^A C_{A P_1 P_3} \mathcal{F}^{(u)}(A, \mathbf{P}; x) \bar{\mathcal{F}}^{(u)}(A, \mathbf{P}; \bar{x}). \quad (2.38b)$$

The corresponding conformal blocks are local expansions adapted to the OPE limits $x \rightarrow 1$ and $x \rightarrow \infty$, respectively, and with the standard OPE normalization they have the local behaviour

$$\mathcal{F}^{(t)}(A, \mathbf{P}; x) = (1 - x)^{\Delta_A - \Delta_{P_2} - \Delta_{P_3}} (1 + \mathcal{O}(1 - x)), \quad x \rightarrow 1, \quad (2.39a)$$

$$\mathcal{F}^{(u)}(A, \mathbf{P}; x) = x^{-\Delta_A - \Delta_{P_2} + \Delta_{P_4}} (1 + \mathcal{O}(x^{-1})), \quad x \rightarrow \infty. \quad (2.39b)$$

The local bases of conformal blocks associated with these three channels are related by fusion and braiding kernels, $\mathbb{F}$ and $\mathbb{B}$. In our conventions, these relations take the schematic form

$$\mathcal{F}^{(s)}(A, \mathbf{P}; x) = \sum_{A'} \mathbb{F}_{AA'} \begin{bmatrix} P_3 & P_2 \\ P_4 & P_1 \end{bmatrix} \mathcal{F}^{(t)}(A', \mathbf{P}; x), \quad (2.40a)$$

$$\mathcal{F}^{(s)}(A, \mathbf{P}; x) = \sum_{A'} \mathbb{B}_{AA'} \begin{bmatrix} P_3 & P_2 \\ P_4 & P_1 \end{bmatrix} \mathcal{F}^{(u)}(A', \mathbf{P}; x) \quad (2.40b)$$

The dependence of the kernels on the central charge, the chosen normalization and analytic-continuation paths has been left implicit. For generic Virasoro representations, these transformations are expressed by integral kernels over intermediate momenta. In rational CFTs or in degenerate sectors with finitely many allowed fusion channels, they reduce to finite-dimensional matrices [60].

The fusion and braiding transformations can also be conveniently represented diagrammatically as follows:

$$\begin{array}{c} P_3 \quad P_2 \\ | \quad | \\ P_4 \quad A \quad P_1 \end{array} = \sum_{A'} \mathbb{F}_{AA'} \begin{bmatrix} P_3 & P_2 \\ P_4 & P_1 \end{bmatrix} \begin{array}{c} P_3 \quad P_2 \\ \diagdown \quad \diagup \\ A' \\ | \\ P_4 \quad P_1 \end{array}, \quad (2.41a)$$

$$\begin{array}{c} P_3 \quad P_2 \\ | \quad | \\ P_4 \quad A \quad P_1 \end{array} = \sum_{A'} \mathbb{B}_{AA'} \begin{bmatrix} P_3 & P_2 \\ P_4 & P_1 \end{bmatrix} \begin{array}{c} P_2 \quad P_3 \\ | \quad | \\ P_4 \quad A' \quad P_1 \end{array}. \quad (2.41b)$$

2.2.2 BPZ equations

In an irreducible degenerate representation, the singular vector and all its descendants are identified with zero. Through the state-operator correspondence, the descendant field associated with the singular vector therefore decouples from correlation functions. Using the Virasoro Ward identities (2.27), this null-vector decoupling condition becomes a differential equation for correlation functions containing the corresponding degenerate primary, known as a BPZ equation [53].

More precisely, consider the correlation function

$$\left\langle \phi_{r,s}(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle, \quad (2.42)$$

where $\phi_{r,s} = \phi_{\Delta_{r,s}}$ is a primary field whose holomorphic conformal weight is equal to $\Delta_{r,s}$, as in (2.10). Then, the correlation function satisfies a holomorphic differential equation of order rs in z , together with an analogous antiholomorphic differential equation in $\bar{z}$. For example, the level-two singular-vector relations

$$(L_{-1}^2 \phi_{2,1})(z, \bar{z}) + b^2 (L_{-2} \phi_{2,1})(z, \bar{z}) = 0, \quad (2.43a)$$

$$(L_{-1}^2 \phi_{1,2})(z, \bar{z}) + b^{-2} (L_{-2} \phi_{1,2})(z, \bar{z}) = 0, \quad (2.43b)$$

imply, respectively,

$$\left(b^{-2} \partial_z^2 + \sum_{i=1}^N \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_{z_i} \right) \right) \left\langle \phi_{2,1}(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle = 0, \quad (2.44a)$$

$$\left(b^2 \partial_z^2 + \sum_{i=1}^N \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_{z_i} \right) \right) \left\langle \phi_{1,2}(z, \bar{z}) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle = 0, \quad (2.44b)$$

with analogous equations holding in the antiholomorphic coordinates. Since the BPZ equations are linear and (anti)holomorphic, they are also satisfied by the corresponding conformal blocks.

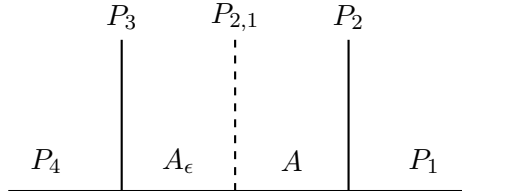
An OPE channel in which the degenerate field is fused with an external primary field at z_i produces a Frobenius-like basis of solutions near the corresponding singular point $z = z_i$. Its elements are naturally labelled by the primary representations allowed by the fusion rules of the two fields which, for the two level-two degenerate fields discussed above, are

$$[\phi_{2,1}] \times [\phi_P] = \sum_{\epsilon=\pm} \left[\phi_{P-\frac{\epsilon b}{2}} \right], \quad (2.45a)$$

$$[\phi_{1,2}] \times [\phi_P] = \sum_{\epsilon=\pm} \left[\phi_{P-\frac{\epsilon}{2b}} \right]. \quad (2.45b)$$

Consequently, for generic values of P and b and fixing all other representation-theoretical parameters, the corresponding second-order BPZ equations possess two linearly independent local solutions, labelled by $\epsilon = \pm$.

Not every OPE channel, however, is associated with a Frobenius-like expansion at a puncture. The degenerate field may instead be inserted between two intermediate representations along an internal channel of the pants decomposition, as in the following diagram



where $A_\epsilon = A - \epsilon b/2$, with $\epsilon = \pm$, according to (2.45). In a coordinate adapted to the cutting curve, the resulting blocks admit convergent Laurent expansions on an annulus containing no other insertions, after factoring out an overall power that determines their monodromy around the curve. These blocks form a basis adapted to the monodromy around the internal cutting curve, rather than to the local monodromy around an individual puncture.

In general, when one of the external fields is degenerate, the fusion rules allow only finitely many intermediate representations so, for a fixed OPE channel and fixing the remaining representation-theoretical data, the corresponding conformal blocks span a finite-dimensional space of solutions to the BPZ equation. The fusion and braiding kernels then reduce to connection matrices relating bases adapted either to individual punctures or to internal cutting curves.

2.2.3 Classical limit of sphere conformal blocks

Let us now review the behaviour of conformal blocks in the limit in which the central charge c is sent to infinity, which we refer to as the *classical limit* [9]. In the Liouville

parametrization (2.6), we will realize this limit by taking $b \rightarrow 0$. We mostly focus on the all-heavy regime, in which both the external and intermediate conformal weights scale linearly with c . Other scaling regimes can be analyzed by analogous semiclassical expansions.

Consider an N -point conformal block on the sphere $\mathcal{F}^{(\sigma)}$ in a fixed OPE channel σ . We scale the external momenta P_i , $i = 1, \dots, N$, and the intermediate momenta A_j , $j = 1, \dots, N-3$, as

$$P_i = \frac{p_i}{b}, \quad A_j = \frac{\alpha_j}{b}, \quad (2.46)$$

where the *classical momenta* p_i and α_j are kept fixed as $b \rightarrow 0$. The corresponding external conformal weights behave as

$$\Delta_{P_i} = P_i(Q - P_i) = \frac{\delta_i}{b^2} + p_i, \quad \delta_i = p_i(1 - p_i), \quad (2.47)$$

and analogously for the internal ones. The quantities δ_I are called *classical conformal weights*, where I may label either an external or an intermediate representation.

For fixed classical momenta and fixed sphere moduli $\mathbf{x} = (x_1, \dots, x_{N-3})$, the conformal block is expected to exponentiate in the limit $b \rightarrow 0$,

$$\mathcal{F}^{(\sigma)}(\mathbf{A}, \mathbf{P}; \mathbf{x}) \sim \exp \left[\frac{1}{b^2} f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; \mathbf{x}) \right], \quad (2.48)$$

with the function $f^{(\sigma)}$ being the corresponding *classical conformal block*. After choosing a branch of the logarithm in the OPE domain, it is defined by

$$f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; \mathbf{x}) = \lim_{b \rightarrow 0} b^2 \log \mathcal{F}^{(\sigma)}(\mathbf{A}, \mathbf{P}; \mathbf{x}). \quad (2.49)$$

More generally, the logarithm of the conformal block is expected to admit an asymptotic expansion of the form

$$\log \mathcal{F}^{(\sigma)} \sim \frac{1}{b^2} f_0^{(\sigma)} + f_1^{(\sigma)} + b^2 f_2^{(\sigma)} + b^4 f_3^{(\sigma)} + \dots = \sum_{k=0}^{\infty} b^{2k-2} f_k^{(\sigma)}, \quad (2.50)$$

where the leading coefficient $f_0^{(\sigma)} = f^{(\sigma)}$ is the classical conformal block, while the higher coefficients describe its subleading semiclassical corrections. The functions $f_k^{(\sigma)}$ are finite for generic fixed values of the classical momenta and the moduli within the chosen OPE domain. They may nevertheless develop singularities at special values of the momenta or at degeneration loci of the punctured surface.

A particularly important class of classical blocks is obtained by inserting a *light degenerate primary* field into an otherwise all-heavy conformal block. For a degenerate field $\phi_{r,1}$ of momentum $P_{r,1} = \frac{1-r}{2}b$, the classical conformal weight vanishes as $b \rightarrow 0$:

$$\lim_{b \rightarrow 0} b^2 \Delta_{r,1} = 0. \quad (2.51)$$

The field $\phi_{r,1}$ is therefore light and does not modify the leading contribution of order b^{-2} to the logarithm of the conformal block.

Let $\Psi_\epsilon^{(\sigma)}$ denote the conformal block obtained by inserting $\phi_{r,1}$ at the point $(z, \bar{z})$ into a correlator whose remaining insertions are all heavy. The label σ denotes the choice of channel, while the label ϵ denotes the choice of intermediate representation allowed by the

fusion rules of the degenerate field. In the classical limit, the dependence on z factors from the leading exponential behaviour:

$$\Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \mathbf{x}) \sim \exp\left[\frac{1}{b^2} f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; \mathbf{x})\right] \tilde{\psi}_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{x}), \quad b \rightarrow 0, \quad (2.52)$$

where $\tilde{\psi}_\epsilon^{(\sigma)}$ remains finite in this limit, and $f^{(\sigma)}$ is the classical limit of the all-heavy conformal block obtained by removing the degenerate insertion. We use the same label for the OPE channel, since a choice of channel for the degenerate conformal block uniquely determines the corresponding channel for the all-heavy conformal block. Equivalently,

$$\log \Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \mathbf{x}) = \frac{1}{b^2} f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; \mathbf{x}) + \log \tilde{\psi}_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{x}) + \mathcal{O}(b^2). \quad (2.53)$$

Thus, the leading divergent term is independent of the position z of the light degenerate field and all the dependence on z is contained, at leading finite order, in the function $\tilde{\psi}^{(\sigma)}$. More precisely, the classical degenerate block may be defined by the limit

$$\psi_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{x}) = \lim_{b \rightarrow 0} \frac{\Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \mathbf{x})}{\mathcal{F}_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; \mathbf{x})}, \quad (2.54)$$

which differs from $\tilde{\psi}_\epsilon^{(\sigma)}$ by an overall z -independent factor.

Since $\phi_{r,1}$ possesses a singular vector at level r , the corresponding conformal block satisfies a BPZ equation of order r . Substituting the semiclassical factorization of the degenerate block into this equation and taking $b \rightarrow 0$ produces an ordinary differential equation of order r for $\psi_\epsilon^{(\sigma)}$. Its coefficients are determined by the classical conformal block, in particular by derivatives of $f^{(\sigma)}$ with respect to the moduli. For $r = 2$, this procedure gives a second-order Fuchsian differential equation as we will discuss below.

Indeed, let us derive the classical BPZ equation for the level-two degenerate field $\phi_{2,1}$. The singular-vector relation (2.43a) implies that the conformal block $\Psi_\epsilon^{(\sigma)}$ satisfies the BPZ equation (2.44a), with $\epsilon = \pm$ according to (2.45). In the all-heavy classical limit, the dependence on the position z of the degenerate insertion factors, so that by dividing the BPZ equation by the appropriate non-degenerate conformal block, multiplying by b^2 , and keeping only the leading terms as $b \rightarrow 0$, one finds

$$\left(\partial_z^2 + T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\})\right) \psi_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}) = 0, \quad (2.55)$$

where

$$T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}) = \sum_{i=1}^N \left[\frac{\delta_i}{(z - z_i)^2} + \frac{c_i^{(\sigma)}}{z - z_i} \right], \quad c_i^{(\sigma)} = \partial_{z_i} f^{(\sigma)}. \quad (2.56)$$

The classical dimensions δ_i are associated with the external classical momenta p_i , while the coefficients $c_i^{(\sigma)}$ are the accessory parameters of the classical differential equation, through which the dependence on the internal classical momenta α_i enters.

For a fixed all-heavy channel, the choice of how the degenerate primary is fused determines a basis of solutions of the associated classical BPZ equation. Taking the OPE with a heavy insertion gives a Frobenius basis at that puncture, while inserting it along an internal channel gives a Floquet basis on the corresponding annulus. In the former case, the relevant external classical momentum fixes the local monodromy around the puncture through the Frobenius exponents

$$\rho_i^+ = p_i, \quad \rho_i^- = 1 - p_i, \quad (2.57)$$

while in the latter case, the relevant intermediate classical momentum fixes the monodromy around the corresponding cutting curve. Changing the underlying all-heavy channel generally changes the classical block and its accessory parameters. All the singular points of (2.55) are regular, making this classical BPZ equation a second-order Fuchsian equation.

Before fixing any of the insertion points, the classical limits of the three global conformal Ward identities (2.29) and (2.30) give

$$\sum_{i=1}^N c_i^{(\sigma)} = 0, \quad \sum_{i=1}^N (z_i c_i^{(\sigma)} + \delta_i) = 0, \quad \sum_{i=1}^N (z_i^2 c_i^{(\sigma)} + 2z_i \delta_i) = 0. \quad (2.58)$$

Consequently, only $N - 3$ accessory parameters are independent, in agreement with the number of complex moduli of the N -punctured sphere. Using global conformal invariance to place three heavy insertions at 0, 1, and ∞ , denoting the remaining positions by $\mathbf{x} = (x_1, \dots, x_{N-3})$ and relabeling the classical weights, the classical BPZ equation (2.55) becomes

$$\left(\partial_z^2 + T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{x}) \right) \psi_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{x}) = 0, \quad (2.59)$$

with

$$\begin{aligned} T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{x}) = & \frac{\delta_0}{z^2} + \frac{\delta_1}{(z-1)^2} + \sum_{a=1}^{N-3} \frac{\delta_{x_a}}{(z-x_a)^2} \\ & - \frac{\delta_0 + \delta_1 + \sum_{a=1}^{N-3} \delta_{x_a} - \delta_\infty}{z(z-1)} + \sum_{a=1}^{N-3} \frac{x_a(x_a-1)\partial_{x_a} f^{(\sigma)}}{z(z-1)(z-x_a)}. \end{aligned} \quad (2.60)$$

In this form, the independent accessory parameters u_a are parametrized as

$$u_a = x_a(x_a-1)\partial_{x_a} f^{(\sigma)}. \quad (2.61)$$

For example, for three heavy insertions, there are no moduli and no independent accessory parameters. The potential reduces to

$$T_{\text{cl}}(z) = \frac{\delta_0}{z^2} + \frac{\delta_1}{(z-1)^2} - \frac{\delta_0 + \delta_1 - \delta_\infty}{z(z-1)}, \quad (2.62)$$

and the classical BPZ equation (2.59) is equivalent to the hypergeometric equation. For four heavy insertions at 0, x , 1 and ∞ , one finds instead

$$T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, x) = \frac{\delta_0}{z^2} + \frac{\delta_x}{(z-x)^2} + \frac{\delta_1}{(z-1)^2} - \frac{\delta_0 + \delta_x + \delta_1 - \delta_\infty}{z(z-1)} + \frac{x(x-1)\partial_x f^{(\sigma)}}{z(z-1)(z-x)}, \quad (2.63)$$

and the classical BPZ equation (2.59) is the Heun equation in normal form, with accessory parameter given by

$$u^{(\sigma)} = x(x-1)\partial_x f^{(\sigma)}. \quad (2.64)$$

2.3 2d CFT on the torus

In this section we briefly review the formulation of conformal field theories on the complex torus [63]:

$$E_\tau = \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}, \quad \text{Im } \tau > 0, \quad (2.65)$$

where z denotes the coordinate on the universal cover and $z \sim z + m + n\tau$, with $m, n \in \mathbb{Z}$. All the local statements made on the sphere remain valid: the state-operator correspondence,

$$\mathfrak{q} = e^{2\pi i \tau}, \quad \bar{\mathfrak{q}} = e^{-2\pi i \bar{\tau}}, \quad (2.66)$$
$$\left\langle \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle_{\tau} := \text{Tr}_{\mathcal{H}} \left[\mathfrak{q}^{L_0 - \frac{c}{24}} \bar{\mathfrak{q}}^{\bar{L}_0 - \frac{\bar{c}}{24}} \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right], \quad (2.67)$$
$$Z(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}} \left[\mathfrak{q}^{L_0 - \frac{c}{24}} \bar{\mathfrak{q}}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right]. \quad (2.68)$$
$$\sum_{i=1}^N \partial_{z_i} \left\langle \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle_{\tau} = 0, \quad \sum_{i=1}^N \partial_{\bar{z}_i} \left\langle \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle_{\tau} = 0, \quad (2.69)$$
$$G_N^\tau(\mathbf{\Delta}; \mathbf{z}, \bar{\mathbf{z}}, \tau, \bar{\tau}) = \left\langle \phi_1(0) \prod_{i=2}^N \phi_i(z_i, \bar{z}_i) \right\rangle_\tau. \quad (2.70)$$
$$G_1^\tau(\Delta_P; \tau, \bar{\tau}) = \sum_A C_{APA} \mathcal{F}(A, P; \mathbf{q}) \bar{\mathcal{F}}(A, P; \bar{\mathbf{q}}), \quad (2.71)$$
$$\mathcal{F}(A, P; \mathfrak{q}) = \mathfrak{q}^{\Delta_A - \frac{c}{24}} (1 + \mathcal{O}(\mathfrak{q})) . \quad (2.72)$$
$$\mathcal{F}(A, P; \mathbf{q}) = \text{Diagram of a disk with a vertical line segment attached to its top boundary, labeled } P \text{ at the top and } A \text{ at the bottom.} \quad (2.73)$$

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As mentioned before, the local OPE with the stress-energy tensor is the same as on the sphere, but the conformal Ward identity for its insertion is different [65]. To write it, we use the Weierstrass functions associated with the lattice $\mathbb{Z} + \tau\mathbb{Z}$ and the quasi-period convention introduced in Section 1, see also Appendix A.1. It is also useful to introduce the shifted Weierstrass functions

$$\zeta_*(z) = \zeta(z) - 2\eta_1(\tau)z, \quad \wp_*(z) = -\partial_z \zeta_*(z) = \wp(z) + 2\eta_1(\tau), \quad (2.78)$$

which have singular behavior

$$\zeta_*(z) \simeq \frac{1}{z} + \mathcal{O}(z), \quad \wp_*(z) \simeq \frac{1}{z^2} + \mathcal{O}(1). \quad (2.79)$$

For an unnormalized torus correlator, or for each of its conformal blocks, the holomorphic Ward identity can be written as

$$\begin{aligned} & \left\langle T(w) \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle_\tau \\ &= \left[2\pi i \partial_\tau + \sum_{i=1}^N (\Delta_i \wp_*(w - z_i) + \zeta_*(w - z_i) \partial_{z_i}) \right] \left\langle \prod_{i=1}^N \phi_i(z_i, \bar{z}_i) \right\rangle_\tau, \end{aligned} \quad (2.80)$$

where the term $2\pi i \partial_\tau$, which has no analogue on the sphere, accounts for deformations of the complex structure.

Similarly, BPZ equations take a different form on the torus. Consider a torus conformal block $\Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \{z_i\}, \mathbf{q})$ with the insertion of a degenerate field $\phi_{2,1}(z, \bar{z})$ and N other primary fields. We use the same notation as in the previous subsection to denote the choice of channel σ and the choice of fusion branch associated with the degenerate insertion ϵ . Taking the regular part of the Ward identity (2.80) as $w \rightarrow z$, the singular-vector relation (2.43a) gives

$$\left[b^{-2} \partial_z^2 + 2\pi i \partial_\tau + 2\eta_1 \Delta_{2,1} + \sum_{i=1}^N (\Delta_i \wp_*(z - z_i) + \zeta_*(z - z_i) \partial_{z_i}) \right] \Psi_\epsilon^{(\sigma)} = 0. \quad (2.81)$$

The analogous antiholomorphic equation follows in the same way. Using translation invariance, we may fix one of the nondegenerate insertions at $z_0 = 0$. Relabeling the remaining ones by z_i , $i = 1, \dots, N-1$, equation (2.81) becomes

$$\begin{aligned} & \left[b^{-2} \partial_z^2 - \zeta_*(z) \partial_z + 2\pi i \partial_\tau + 2\eta_1 \Delta_{2,1} + \Delta_0 \wp_*(z) \right. \\ & \quad \left. + \sum_{i=1}^{N-1} \Delta_i \wp_*(z - z_i) + \sum_{i=1}^{N-1} [\zeta_*(z - z_i) - \zeta_*(z)] \partial_{z_i} \right] \Psi_\epsilon^{(\sigma)} = 0. \end{aligned} \quad (2.82)$$

As in the sphere case, in the classical limit the preceding BPZ equation reduces to a second-order Fuchsian ordinary differential equation on the torus, as we now review.

2.3.1 Classical limit of torus conformal blocks

The classical limit of torus conformal blocks is defined exactly as for sphere conformal blocks. We scale all external and intermediate momenta as $P_i = p_i/b$ and $A_j = \alpha_j/b$,

keeping both the classical momenta and the geometric moduli of the punctured torus fixed. A torus block with only heavy insertions exponentiates as

$$\mathcal{F}^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \mathbf{q}) \sim \exp \left[\frac{1}{b^2} f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{q}) \right], \quad b \rightarrow 0, \quad (2.83)$$

where the classical torus block $f^{(\sigma)}$ can be defined as in (2.49). Furthermore, the logarithm of the torus block $\mathcal{F}^{(\sigma)}$ admits the same type of asymptotic expansion in powers of b^2 as in (2.50).

The insertion of a light degenerate field $\phi_{r,1}$ again does not change the leading term of order b^{-2} , as in (2.52). In particular, the torus conformal block $\Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \{z_i\}, \mathbf{q})$ with the insertion of a degenerate field $\phi_{2,1}(z, \bar{z})$ and N other heavy primary fields factorizes as

$$\Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \{z_i\}, \mathbf{q}) \sim \exp \left[\frac{1}{b^2} f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{q}) \right] \tilde{\psi}_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}, \mathbf{q}), \quad (2.84)$$

where $\epsilon = \pm$ according to the fusion rule (2.45). The classical block $f^{(\sigma)}$ is the classical limit of the all-heavy conformal block obtained by removing the degenerate insertion, with its channel fixed by that of the degenerate conformal block $\Psi_\epsilon^{(\sigma)}$. Thus, we can define the classical limit of the degenerate conformal block by

$$\psi_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}, \mathbf{q}) = \lim_{b \rightarrow 0} \frac{\Psi_\epsilon^{(\sigma)}(\mathbf{A}, \mathbf{P}; z, \{z_i\}, \mathbf{q})}{\mathcal{F}^{(\sigma)}(\mathbf{A}, \mathbf{P}; \{z_i\}, \mathbf{q})}, \quad (2.85)$$

differing from $\tilde{\psi}_\epsilon^{(\sigma)}$ by an overall z -independent factor.

Thus, dividing equation (2.82) by the appropriate non-degenerate conformal block, multiplying by b^2 and taking $b \rightarrow 0$ gives the classical BPZ equation

$$\left(\partial_z^2 + T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}, \tau) \right) \psi_\epsilon^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}, \mathbf{q}) = 0, \quad (2.86)$$

where

$$\begin{aligned} T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}, \tau) &= \delta_0 \wp_*(z) + \sum_{i=1}^{N-1} \delta_i \wp_*(z - z_i) + 2\pi i \partial_\tau f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{q}) \\ &+ \sum_{i=1}^{N-1} [\zeta_*(z - z_i) - \zeta_*(z)] \partial_{z_i} f^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \mathbf{q}). \end{aligned} \quad (2.87)$$

As in the sphere case, the classical dimensions δ_j are associated with the external classical momenta p_j , while the dependence on the internal classical momenta α_j enters through the modular derivative $2\pi i \partial_\tau f^{(\sigma)}$ and the puncture-position accessory parameters $c_i^{(\sigma)} = \partial_{z_i} f^{(\sigma)}$. After rewriting the equation in elliptic normal form, the modular derivative combines with the constant terms generated by the shifted Weierstrass functions to give the energy parameter $E^{(\sigma)}$ below. After fixing $z_0 = 0$, the $N - 1$ independent accessory parameters associated with the relative puncture positions are

$$c_i^{(\sigma)} = \partial_{z_i} f^{(\sigma)}, \quad i = 1, \dots, N - 1. \quad (2.88)$$

The coefficient associated with the puncture at the origin is then defined by the classical limit of the global holomorphic Ward identity

$$c_0^{(\sigma)} := - \sum_{i=1}^{N-1} c_i^{(\sigma)} \quad \implies \quad \sum_{i=0}^{N-1} c_i^{(\sigma)} = 0. \quad (2.89)$$

Using the explicit form of the shifted Weierstrass functions in (2.78), the classical potential can be written in the elliptic normal form introduced in (1.69):

$$T_{\text{cl}}^{(\sigma)}(\boldsymbol{\alpha}, \mathbf{p}; z, \{z_i\}, \tau) = \sum_{i=0}^{N-1} \left[\delta_i \wp(z - z_i) + c_i^{(\sigma)} \zeta(z - z_i) \right] + E^{(\sigma)}, \quad (2.90)$$

with

$$E^{(\sigma)} = 2\pi i \partial_\tau f^{(\sigma)} + 2\eta_1 \left(\sum_{i=0}^{N-1} \delta_i + \sum_{i=1}^{N-1} z_i \partial_{z_i} f^{(\sigma)} \right), \quad (2.91)$$

where $z_0 = 0$, without loss of generality. The constraint (2.89) on the $c_i^{(\sigma)}$ makes the quasi-periodic ζ -contributions combine into an elliptic function. Then, the $N - 1$ independent coefficients $c_i^{(\sigma)}$ are conjugate to the relative puncture positions, while the additional accessory parameter $E^{(\sigma)}$ is associated with the complex-structure modulus. Their total number is therefore N , equal to the complex dimension of the moduli space of an N -punctured torus.

Near a heavy insertion, the local form of (2.86) is identical to the sphere case, since

$$\wp(z - z_i) = (z - z_i)^{-2} + \mathcal{O}((z - z_i)^2), \quad \zeta(z - z_i) = (z - z_i)^{-1} + \mathcal{O}((z - z_i)^3). \quad (2.92)$$

Then the local analysis is unchanged and, in particular, the local exponents are parametrized by the external classical momenta as in (2.57). Globally, however, the solutions also acquire monodromy around the two noncontractible cycles of the torus, with the choice of intermediate representations in the degenerate torus block determining the corresponding monodromy data.

Let us consider the example $N = 1$. In this case, the elliptic potential (2.90) reduces to

$$T_{\text{cl}}(\alpha, p; z, \tau) = \delta_p \wp(z) + E, \quad E = 2\pi i \partial_\tau f(\alpha, p; \mathbf{q}) + 2\eta_1 \delta_p, \quad (2.93)$$

corresponding to the Lamé equation (1.76). Thus, the derivative of the classical one-point torus block with respect to τ determines the Lamé accessory parameter [31].

Section 3

Supersymmetric Gauge Theory

In this third background section, we briefly review the relevant aspects of supersymmetric gauge theory, focusing on four-dimensional $\mathcal{N} = 2$ theories and their five-dimensional $\mathcal{N} = 1$ uplift. After a general introduction to the four-dimensional setting, we review the Seiberg–Witten solution and its relation to classical algebraic integrable systems. We then discuss Nekrasov partition functions and their blow-up equations, as well as the AGT and Bethe/gauge correspondences. We conclude by outlining the extension of these structures to five-dimensional gauge theories compactified on a circle.

General references. For background and more detailed explanations of the material reviewed in this section, see [66–70].

3.1 Four-dimensional $\mathcal{N} = 2$ gauge theories

3.1.1 Generalities

In this section we fix conventions and recall the basic ingredients of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, introducing, in particular, their matter content. In two-component spinor notation, the translation generators of the $\mathcal{N} = 2$ super-Poincaré algebra are denoted by $P_{\alpha\dot{\alpha}}$, while the Lorentz generators are $M_{\alpha\beta} = M_{\beta\alpha}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{M}_{\dot{\beta}\dot{\alpha}}$. The supercharges are $\mathcal{Q}_{\alpha}^I$ and $\bar{\mathcal{Q}}_{\dot{\alpha}I} = (\mathcal{Q}_{\alpha}^I)^{\dagger}$, where $I, J = 1, 2$ are $\text{SU}(2)_R$ indices.

The ordinary Poincaré commutation relations will be left implicit, as will the Lorentz and R -symmetry transformation laws of the generators, which are fixed by their indices and charges. The nonvanishing anticommutation relations involving the supercharges are

$$\{\mathcal{Q}_{\alpha}^I, \bar{\mathcal{Q}}_{\dot{\beta}J}\} = 2\delta_J^I P_{\alpha\dot{\beta}}, \quad (3.1a)$$

$$\{\mathcal{Q}_{\alpha}^I, \mathcal{Q}_{\beta}^J\} = 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{IJ}Z, \quad (3.1b)$$

$$\{\bar{\mathcal{Q}}_{\dot{\alpha}I}, \bar{\mathcal{Q}}_{\dot{\beta}J}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}\bar{Z}, \quad (3.1c)$$

where Z is the complex central charge allowed by the $\mathcal{N} = 2$ supersymmetry algebra. Its precise value depends on the vacuum and on the electric, magnetic and flavour charges of the state under consideration, as we will discuss below. The $\text{SU}(2)_R$ doublet structure of the supercharges and the possible central extension are characteristic features of extended supersymmetry. At the classical level, the R -symmetry group is

$$\text{SU}(2)_R \times \text{U}(1)_r, \quad (3.2)$$

possibly reduced quantum mechanically by anomalies or explicitly by deformations. When the theory is at a conformal fixed point, its $\mathcal{N} = 2$ super-Poincaré symmetry is enhanced

to $\mathcal{N} = 2$ superconformal symmetry. Such theories will play an important role in the discussion of class $\mathcal{S}$ theories and the AGT correspondence.

The field content of an $\mathcal{N} = 2$ gauge theory is organized into multiplets of the $\mathcal{N} = 2$ super-Poincaré algebra. The gauge field belongs to a *vector multiplet*, whose on-shell field content is

$$\mathcal{V} = \left(\phi, \lambda_\alpha^I, \bar{\lambda}_{\dot{\alpha}I}, A_{\alpha\dot{\alpha}} \right), \quad (3.3)$$

where ϕ is a complex scalar, λ_α^I and $\bar{\lambda}_{\dot{\alpha}I}$ are Weyl fermions, and $A_{\alpha\dot{\alpha}}$ is the gauge field. The scalar and the gauge field are singlets under $SU(2)_R$, while the fermions form a doublet. All components of $\mathcal{V}$ transform in the adjoint representation of the gauge group $\mathcal{G}_g$.

Matter fields are instead organized into *hypermultiplets*. A full hypermultiplet transforming in a representation W of the gauge group has the schematic on-shell field content

$$\mathcal{H}_W = \left(q^I, \psi_\alpha, \tilde{\psi}_\alpha \right), \quad q^I = \left(q, \tilde{q}^\dagger \right). \quad (3.4)$$

The two complex scalars q^I form a doublet of $SU(2)_R$, while the fermions ψ_α and $\tilde{\psi}_\alpha$ are $SU(2)_R$ singlets. More explicitly, q and ψ_α transform in W , while $\tilde{q}$ and $\tilde{\psi}_\alpha$ transform in the conjugate representation $\bar{W}$. Hypermultiplets may also transform under a global flavour symmetry group $\mathcal{G}_f$. Below we will mainly consider fundamental, adjoint and bifundamental hypermultiplets.

States in the Hilbert space of the theory that are annihilated by some supercharges belong to short, or protected, representations of the supersymmetry algebra and are called *BPS states*. They may arise either as elementary excitations of the fields or as solitonic objects, as in the case of BPS monopoles and dyons. The supersymmetry algebra implies for any state the BPS bound

$$M \geq \sqrt{2} |Z|, \quad (3.5)$$

where M is the mass of the state and Z is its central charge. States saturating this bound are precisely BPS states, and their masses are therefore fixed by

$$M_{\text{BPS}} = \sqrt{2} |Z|. \quad (3.6)$$

This relation is protected against quantum corrections except through the dependence of Z on the vacuum and on the charges of the state.

3.1.2 Lagrangian gauge theories

We now briefly review Lagrangian gauge theories, focusing for simplicity on the case $\mathcal{G}_g = SU(N)$. The bare Yang–Mills coupling g_{uv} and theta angle θ combine into the *holomorphic gauge coupling*

$$\tau_{\text{uv}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{uv}}^2}. \quad (3.7)$$

The vector-multiplet contribution to the classical scalar potential contains a term of the form

$$V_{\text{vec}}(\phi) \sim \frac{1}{g_{\text{uv}}^2} \text{Tr} \left([\phi, \phi^\dagger]^2 \right). \quad (3.8)$$

In the presence of hypermultiplets, the full scalar potential also contains additional terms involving their scalar components. In what follows we restrict our attention to the component of the moduli space of vacua on which the hypermultiplet scalars have vanishing vacuum expectation values.

Supersymmetric vacua have zero energy, so the potential must vanish and, in particular,

$$[\phi, \phi^\dagger] = 0. \quad (3.9)$$

Thus ϕ is a normal matrix and, modulo gauge transformations, its value in a classical vacuum can be chosen to lie in the Cartan subalgebra of $\mathrm{SU}(N)$:

$$\phi_0 = \mathrm{diag}(a_1, \dots, a_N), \quad \sum_{i=1}^N a_i = 0, \quad a_i \in \mathbb{C}. \quad (3.10)$$

The tracelessness condition leaves $r = N - 1$ independent complex parameters, where r is the rank of $\mathrm{SU}(N)$. Moreover, configurations related by the Weyl group of $\mathrm{SU}(N)$, which acts by permuting the eigenvalues a_i , are gauge equivalent. The classical *Coulomb branch* is therefore

$$\mathcal{B}_{\mathrm{cl}} = \mathbb{C}^r / W_{\mathrm{SU}(N)}, \quad (3.11)$$

and can equivalently be parametrized by the gauge-invariant complex coordinates

$$u_k^{\mathrm{cl}} = \frac{1}{k} \mathrm{Tr} \phi_0^k = \frac{1}{k} \sum_{i=1}^N a_i^k, \quad k = 2, \dots, N. \quad (3.12)$$

At a generic point of $\mathcal{B}_{\mathrm{cl}}$, the eigenvalues a_i are all distinct and the gauge symmetry is broken to the maximal torus

$$\mathrm{SU}(N) \longrightarrow \mathrm{U}(1)^r. \quad (3.13)$$

The off-diagonal components of the vector multiplet, namely the W -bosons and their superpartners, acquire masses proportional to $|a_i - a_j|$ for $i \neq j$. Thus the only massless vector multiplets at a generic point are the r abelian vector multiplets associated with the Cartan subalgebra.

At the quantum level, the microscopic nonabelian coupling renormalizes. In the holomorphic Wilsonian scheme, the one-loop running coupling at a scale μ is

$$\tau_{\mathrm{mic}}(\mu) = \tau_{\mathrm{uv}} - \frac{b}{2\pi i} \log \frac{\mu}{\Lambda_{\mathrm{uv}}}, \quad b > 0, \quad (3.14)$$

where Λ_{uv} is a UV cutoff scale and b is the one-loop beta-function coefficient. In the case of gauge group $\mathrm{SU}(N)$ with N_f fundamental hypermultiplets, $b = 2N - N_f$. For an asymptotically free theory, the running coupling can be traded for the RG-invariant *holomorphic dynamical scale*

$$\Lambda^b = \mu^b e^{2\pi i \tau_{\mathrm{mic}}(\mu)} = \Lambda_{\mathrm{uv}}^b e^{2\pi i \tau_{\mathrm{uv}}} \iff e^{2\pi i \tau_{\mathrm{mic}}(\mu)} = \left(\frac{\Lambda}{\mu} \right)^b. \quad (3.15)$$

In an $\mathcal{N} = 2$ theory, this Wilsonian running is perturbatively one-loop exact: there are no corrections beyond one loop. Non-perturbative corrections to holomorphic quantities in the low-energy effective action are nevertheless possible and arise from Yang–Mills *instantons* [71, 72].

An instanton of positive integer charge k is a $\frac{1}{2}$ -BPS solution of the self-duality equations of the Euclidean theory with classical action

$$S_k = -2\pi i k \tau_{\mathrm{uv}}, \quad (3.16)$$

and hence its contribution to the path integral is weighted by

$$e^{-S_k} = e^{2\pi i k \tau_{\mathrm{uv}}} = \mathbf{q}^k, \quad \mathbf{q} = e^{2\pi i \tau_{\mathrm{uv}}}. \quad (3.17)$$

Let us now consider the low-energy effective description on the Coulomb branch. Let $\mathcal{B}$ denote the quantum Coulomb branch and let r be its dimension. At a generic point of $\mathcal{B}$, away from its singular loci, the gauge group is broken to its maximal torus and the massive charged fields can be integrated out. The massless degrees of freedom are therefore r abelian $\mathcal{N} = 2$ vector multiplets.

At the two-derivative level, the Wilsonian effective action of these multiplets is determined locally by a holomorphic *prepotential* $\mathcal{F}(\mathbf{a}, \mathbf{m}; \tau_{\text{uv}})$ regarded as a function of the local special coordinates $\mathbf{a}$, the hypermultiplet masses $\mathbf{m}$ and the UV coupling [71, 72]. To write the effective action, one must first choose an electric-magnetic duality frame. The complex scalars of the vector multiplets then define *local special coordinates*

$$a^I, \quad I = 1, \dots, r, \quad (3.18)$$

with their magnetic duals being

$$a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I}. \quad (3.19)$$

The effective abelian gauge couplings are encoded in the symmetric matrix

$$\tau_{IJ} = \frac{\partial a_{D,I}}{\partial a^J} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}. \quad (3.20)$$

For $\text{SU}(2)$, this matrix reduces to a single low-energy abelian coupling $\tau_{\text{eff}}(a)$, which should not be confused with the microscopic nonabelian coupling $\tau_{\text{mic}}(\mu)$.

In a weakly coupled semiclassical region of the Coulomb branch, the special coordinates a^I approach the independent components of the adjoint scalar vacuum expectation value. Away from the semiclassical region, however, they are only locally defined. By contrast, gauge-invariant coordinates on the Coulomb branch are provided by the vacuum expectation values of the Coulomb-branch chiral operators,

$$u_k = \frac{1}{k} \langle \text{Tr } \phi^k \rangle, \quad k = 2, \dots, N. \quad (3.21)$$

In the semiclassical region, they reduce to the classical expressions

$$u_k = \frac{1}{k} \sum_{i=1}^N a_i^k + \dots, \quad (3.22)$$

where the dots denote quantum corrections to their relation with the special coordinates. Furthermore, in a weakly coupled region, the prepotential admits the semiclassical expansion

$$\mathcal{F} = \mathcal{F}_{\text{cl}} + \mathcal{F}_{1\text{-loop}} + \mathcal{F}_{\text{inst}}, \quad (3.23)$$

where the perturbative quantum corrections terminate at one loop, while $\mathcal{F}_{\text{inst}}$ contains the nonperturbative instanton corrections.

Suppressing fermions and higher-derivative terms, the bosonic part of the two-derivative effective Lagrangian takes the schematic form

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{4\pi} \left[\text{Im } \tau_{IJ} \partial_\mu a^I \partial^\mu \bar{a}^{\bar{J}} - \frac{1}{4} \text{Im } \tau_{IJ} F_{\mu\nu}^I F^{J\mu\nu} + \frac{1}{4} \text{Re } \tau_{IJ} F_{\mu\nu}^I \tilde{F}^{J\mu\nu} \right] + \dots. \quad (3.24)$$

Thus, the same matrix τ_{IJ} determines both the abelian gauge couplings and the metric on the Coulomb branch. The latter is locally

$$g_{I\bar{J}} = \text{Im } \tau_{IJ}, \quad (3.25)$$

and has the Kähler potential

$$K(\mathbf{a}, \bar{\mathbf{a}}) = \text{Im} \left(\bar{a}^I a_{D,I} \right). \quad (3.26)$$

The geometry defined by these data is called rigid special Kähler geometry [73]. The special coordinates also determine the central charge. For a state with electric, magnetic and flavour charges

$$\gamma = \left(n_{e,I}, n_m^I, s_f \right), \quad (3.27)$$

one has

$$Z_\gamma = \sum_{I=1}^r \left(n_{e,I} a^I + n_m^I a_{D,I} \right) + \frac{1}{\sqrt{2}} \sum_f s_f m_f. \quad (3.28)$$

Thus, a singularity of the abelian effective description occurs when a charged BPS state present in the spectrum becomes massless. At such a locus the corresponding light degrees of freedom must be restored to the effective action.

Globally, the special coordinates are naturally combined into the symplectic vector

$$\Pi = \begin{pmatrix} a_{D,I} \\ a^I \end{pmatrix}. \quad (3.29)$$

When transported around singular loci of $\mathcal{B}$, this vector undergoes electric-magnetic duality transformations in $\text{Sp}(2r, \mathbb{Z})$. The prepotential therefore exists only locally, while the special Kähler metric and the associated electromagnetic charge lattice are globally defined.

Determining the exact low-energy theory is thus equivalent to determining the special coordinates $(a^I, a_{D,I})$ and their monodromies over the Coulomb branch. The Seiberg–Witten solution provides precisely these data by realizing the special coordinates as periods of a meromorphic differential on a family of algebraic curves fibered over $\mathcal{B}$. These curves will also provide the link with classical algebraic integrable systems.

3.1.3 Seiberg–Witten solution & integrable systems

The Seiberg–Witten solution [71, 72] of an $\mathcal{N} = 2$ gauge theory consists of a family of Riemann surfaces $\Sigma_{\mathbf{u}}$, called *Seiberg–Witten curves*, parametrized by points $\mathbf{u} \in \mathcal{B}$ of the Coulomb branch, together with a meromorphic one-form λ , called the *Seiberg–Witten differential*. The special coordinates of the low-energy theory are recovered as periods of λ ,

$$a^I = \frac{1}{2\pi i} \oint_{A_I} \lambda, \quad a_{D,I} = \frac{1}{2\pi i} \oint_{B^I} \lambda, \quad (3.30)$$

and the effective abelian couplings are then determined by (3.20). Here $\{A_I, B^I\}$ is a symplectic basis of the relevant electromagnetic sublattice of $H_1(\Sigma_{\mathbf{u}}, \mathbb{Z})$.

The electromagnetic BPS charge lattice is naturally identified with this homology lattice, whose intersection form reproduces the Dirac pairing. A state with electric and magnetic charges $(n_{e,I}, n_m^I)$ is represented by the cycle

$$\gamma = n_{e,I} A_I + n_m^I B^I, \quad (3.31)$$

and its electromagnetic central charge is the corresponding period,

$$Z_\gamma^{\text{em}} = \frac{1}{2\pi i} \oint_\gamma \lambda. \quad (3.32)$$

In theories with matter, the full charge lattice including flavour charges is described by an appropriate extension of this homology lattice associated with the poles of the Seiberg–Witten differential, whose residues encode the mass parameters.

The basic example is pure $\mathcal{N} = 2$ Yang–Mills theory with gauge group $SU(2)$. Quantum mechanically, the Coulomb branch is a complex one-dimensional space parametrized by

$$u = \frac{1}{2} \langle \text{Tr } \phi^2 \rangle, \quad (3.33)$$

up to normalization conventions. In Gaiotto form [71, 74], the curve is written as

$$\Sigma_u : \quad \lambda^2 = \phi_2(z), \quad \phi_2(z) = \left(\frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \right) dz^2, \quad (3.34)$$

where z is a coordinate on $\mathbb{C}^\times$. Equivalently, this can be written as

$$x^2 = u + \Lambda^2 \left(z + \frac{1}{z} \right), \quad \lambda = x \frac{dz}{z}. \quad (3.35)$$

The special coordinates are then

$$a(u) = \frac{1}{2\pi i} \oint_A \lambda, \quad a_D(u) = \frac{1}{2\pi i} \oint_B \lambda, \quad (3.36)$$

and determine the effective coupling through

$$\tau(u) = \frac{da_D}{da}. \quad (3.37)$$

The curve degenerates at the two finite points $u = \pm 2\Lambda^2$, where a magnetic monopole and a dyon become massless.

A closely related example is the $\mathcal{N} = 2$ theory with gauge group $SU(2)$ and four fundamental hypermultiplets, whose massless limit is superconformal. In Gaiotto form [72, 74], the Seiberg–Witten curve is the double cover

$$\Sigma_u : \quad \lambda^2 = \phi_2(z), \quad (3.38)$$

where $z \in \mathbb{CP}^1 \setminus \{0, \mathfrak{q}, 1, \infty\}$. In the weakly coupled duality frame associated with the degeneration $\mathfrak{q} \rightarrow 0$, $\mathfrak{q}$ is identified with the exponentiated microscopic gauge coupling, $\mathfrak{q} = e^{2\pi i \tau_{\text{uv}}}$, up to conventions. The quadratic differential is then

$$\phi_2(z) = \left[\frac{\delta_0}{z^2} + \frac{\delta_{\mathfrak{q}}}{(z - \mathfrak{q})^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_\infty - \delta_0 - \delta_{\mathfrak{q}} - \delta_1}{z(z - 1)} + \frac{\mathfrak{q}(\mathfrak{q} - 1)\tilde{u}}{z(z - \mathfrak{q})(z - 1)} \right] dz^2, \quad (3.39)$$

where $\tilde{u}$ is a coordinate on the Coulomb branch whose relation to $u = \frac{1}{2} \langle \text{Tr } \phi^2 \rangle$ involves a coupling-dependent normalization and mass-dependent shifts. The coefficients δ_i are quadratic functions of the puncture masses, which are linear combinations of the four physical hypermultiplet masses. Here one recognizes the rational potential appearing in the normal form of the Heun equation, as given in (1.51) and (2.63).

The asymptotically free theories with $N_f < 4$ can be obtained by successively decoupling hypermultiplets from the $N_f = 4$ theory. The first decoupling limit is

$$m_4 \rightarrow \infty, \quad \mathfrak{q} \rightarrow 0, \quad \Lambda_3 \sim \mathfrak{q} m_4 \text{ fixed}, \quad (3.40)$$

where $\mathbf{q} = e^{2\pi i \tau_{uv}}$. Thus, the mass of one hypermultiplet is sent to infinity while the microscopic coupling approaches the weak-coupling cusp, leaving fixed the dynamical scale Λ_3 of the $N_f = 3$ theory.

The remaining flavours can be decoupled iteratively, with the corresponding scale-matching relations being

$$\Lambda_{N_f-1}^{5-N_f} \sim m_{N_f} \Lambda_{N_f}^{4-N_f}, \quad N_f = 3, 2, 1, \quad (3.41)$$

where m_{N_f} is the mass of the hypermultiplet being decoupled. These limits successively produce the theories with $N_f = 2, 1, 0$, with $N_f = 0$ denoting the pure SU(2) theory. The omitted proportionality constants depend on the normalization chosen for the masses and dynamical scales.

At the level of the quadratic differential, these decoupling limits are confluent limits in which regular singularities collide and produce *irregular singularities*. Their higher-order poles encode the dynamically generated scales. In particular, the pure SU(2) curve, with two irregular singularities, is recovered as the final decoupling limit of the four-flavour curve [66, 72, 74].

Another important example is the $\mathcal{N} = 2^*$ theory, namely $\mathcal{N} = 4$ super Yang–Mills deformed by a mass m for the adjoint hypermultiplet. For gauge group SU(2), the corresponding Seiberg–Witten curve is again written as a double cover [75]

$$\Sigma_u : \quad \lambda^2 = \phi_2(z), \quad (3.42)$$

with

$$\phi_2(z) = \left(u + m^2 \wp(z) \right) dz^2, \quad z \in \mathbb{C} / (\mathbb{Z} + \tau_{uv} \mathbb{Z}) =: E_{\tau_{uv}}, \quad (3.43)$$

up to shifts of u and normalization conventions. Here $\wp(z)$ is the Weierstrass function on the torus $E_{\tau_{uv}}$, the coefficient of its double pole at the puncture being fixed by the adjoint mass, and one recognizes the elliptic potential defining the Lamé equation, as given in (1.76) and (2.93). In the limit $m \rightarrow 0$ one recovers the $\mathcal{N} = 4$ theory, while the pure $\mathcal{N} = 2$ theory is instead obtained by decoupling the adjoint hypermultiplet,

$$m \rightarrow \infty, \quad \mathbf{q} = e^{2\pi i \tau_{uv}} \rightarrow 0, \quad \Lambda^4 \sim \mathbf{q} m^4 \text{ fixed.} \quad (3.44)$$

Geometrically, this limit degenerates the torus and produces the sphere with the two irregular singularities of the pure SU(2) curve.

Finally, for the pure SU(N) gauge theory, the Seiberg–Witten curve can be written as [76, 77]

$$\Sigma_h : \quad \Lambda^N \left(z + \frac{1}{z} \right) = x^N + \sum_{k=2}^N (-1)^k h_k x^{N-k}, \quad \lambda = x \frac{dz}{z}. \quad (3.45)$$

This is an N -sheeted cover of the z -sphere and has genus $N - 1$, matching the rank of the gauge group. The coefficients h_k form an alternative set of gauge-invariant Coulomb-branch coordinates. They are polynomially related, through the Newton identities, to

$$u_k = \frac{1}{k} \left\langle \text{Tr } \phi^k \right\rangle, \quad k = 2, \dots, N. \quad (3.46)$$

For example,

$$h_2 = -u_2, \quad h_3 = u_3, \quad h_4 = \frac{1}{2} u_2^2 - u_4, \quad h_5 = u_5 - u_2 u_3. \quad (3.47)$$

A quantization of this Seiberg–Witten curve will be studied in the final chapter of this thesis.

Soon after the work of Seiberg and Witten, it was realized that Seiberg–Witten curves admit a natural interpretation as spectral curves of classical algebraic integrable systems [75–77]. In this interpretation, the Coulomb branch $\mathcal{B}$ is the base of a Liouville torus fibration. Its generic fibers are the Jacobians of the Seiberg–Witten curves, or appropriate Prym subvarieties when the physical abelian variety is smaller than the full Jacobian. The Coulomb-branch coordinates u_i are identified with the mutually commuting Hamiltonians of the integrable system, while the Seiberg–Witten differential plays the role of the action differential and its periods give the corresponding action variables.

For the examples discussed above, the corresponding integrable systems are:

- SU(2) theory with $N_f = 4$: the four-punctured $\mathfrak{sl}_2$ Gaudin system and the inhomogeneous two-site XXX spin chain [78, 79];
- SU(N) $\mathcal{N} = 2^*$ theory: the N -particle elliptic Calogero–Moser system [75];
- pure SU(N) gauge theory: the N -particle periodic Toda chain [77].

For the last two systems, the center-of-mass degree of freedom is removed, in accordance with the rank $N - 1$ of SU(N).

This gauge-theory/integrable-system correspondence therefore provides a complementary interpretation of the Seiberg–Witten geometry. It becomes particularly systematic in the class $\mathcal{S}$ construction, where the relevant integrable systems arise as Hitchin systems. Moreover, as we will discuss below, the Nekrasov–Shatashvili limit of the $\Omega_{\varepsilon_1, \varepsilon_2}$ -background quantizes these classical systems and leads to the Bethe/gauge correspondence [29].

3.1.4 Supersymmetric localization & blow-up equations

Supersymmetric localization is a computational method allowing one to reduce certain path integrals to finite-dimensional integrals or discrete sums [70]. Consider a Euclidean supersymmetric theory on a background preserving a fermionic symmetry Q . In general, Q need not be nilpotent, but may satisfy

$$Q^2 = \delta_{\text{bos}}, \quad (3.48)$$

where δ_{bos} is a combination of bosonic symmetries, such as spacetime isometries, gauge transformations, flavour transformations and R -symmetry transformations.

Suppose that the action and path-integral measure are invariant under Q . One may then deform the action by a Q -exact term,

$$S \longrightarrow S_t = S + tQV, \quad (3.49)$$

where V is chosen to be invariant under Q^2 . The resulting partition function

$$Z(t) = \int \mathcal{D}\Phi e^{-S - tQV} \quad (3.50)$$

is t -independent and one can therefore evaluate the path integral in the limit $t \rightarrow \infty$. If the bosonic part of QV is positive semi-definite, the integral localizes onto the configurations satisfying

$$(QV)_{\text{bos}} = 0, \quad (3.51)$$

which are the BPS configurations fixed by the chosen supercharge, and the Gaussian approximation transverse to this localization locus becomes exact. If the localization locus decomposes into connected components $\mathcal{M}_{\text{loc}}^{(\alpha)}$, the original path integral reduces schematically to

$$Z = \sum_{\alpha} \int_{\mathcal{M}_{\text{loc}}^{(\alpha)}} d\mu_{\alpha} \exp \left(- S_{\text{cl}}|_{\mathcal{M}_{\text{loc}}^{(\alpha)}} \right) Z_{1\text{-loop}}^{(\alpha)}, \quad (3.52)$$

where $d\mu_{\alpha}$ is the measure on the collective coordinates, or equivalently on the zero modes tangent to $\mathcal{M}_{\text{loc}}^{(\alpha)}$. The factor $Z_{1\text{-loop}}^{(\alpha)}$ is the superdeterminant of the non-zero quadratic fluctuations transverse to the localization locus. Both the classical action and the one-loop determinant may depend on the point of $\mathcal{M}_{\text{loc}}^{(\alpha)}$. If the localization locus consists of isolated BPS configurations, the integrals reduce to a sum over these configurations. The same argument applies to expectation values of Q -closed observables $\mathcal{O}$, whose restrictions to the localization locus are inserted in the corresponding finite-dimensional integrals.

Nekrasov's construction [1, 2, 80] applies this principle to four-dimensional $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^4$, providing a microscopic computation of the low-energy effective prepotential $\mathcal{F}$. After a suitable topological twist, the non-perturbative localization locus decomposes into sectors described by moduli spaces of Yang–Mills instantons. Since both $\mathbb{R}^4$ and the instanton moduli spaces are non-compact, the construction is formulated equivariantly with respect to spacetime rotations and the Cartan tori of the gauge and flavour symmetries. The corresponding equivariant deformation of the supersymmetric theory defines the $\Omega_{\varepsilon_1, \varepsilon_2}$ -background, with ε_1 and ε_2 the equivariant parameters associated with spacetime rotations, and gives rise to the *Nekrasov partition function* $\mathcal{Z}$. The low-energy prepotential is recovered from it through

$$\mathcal{F} = \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \varepsilon_1 \varepsilon_2 \log \mathcal{Z}, \quad (3.53)$$

up to normalization conventions.

More precisely, focusing on a gauge theory with gauge group $U(N)$, the asymptotic value of the vector-multiplet scalar is set to have eigenvalues $\mathbf{a} = (a_1, \dots, a_N)$ which, on $\mathbb{R}^4$, are fixed boundary data rather than variables to be integrated over. The Nekrasov partition function on $(\mathbb{R}^4, \Omega_{\varepsilon_1, \varepsilon_2}) \simeq \mathbb{C}_{\varepsilon_1, \varepsilon_2}^2$ factorizes as

$$\mathcal{Z}(\mathbf{a}, \mathbf{m}; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \mathcal{Z}_{\text{cl}} \mathcal{Z}_{1\text{-loop}} \mathcal{Z}_{\text{inst}}, \quad (3.54)$$

where $\mathbf{m}$ denotes a set of mass parameters. The $SU(N)$ theory is then obtained by imposing $\sum_{\alpha=1}^N a_{\alpha} = 0$, up to possible decoupled $U(1)$ factors whose precise form depends on the matter content and normalization conventions.

The classical action contributes in two ways. Its evaluation on the perturbative Coulomb-branch vacuum selected by the asymptotic value $\mathbf{a}$ of the vector-multiplet scalar, in the presence of the $\Omega_{\varepsilon_1, \varepsilon_2}$ -background, gives

$$\mathcal{Z}_{\text{cl}} = \exp \left[- \frac{2\pi i \tau_{\text{uv}}}{2\varepsilon_1 \varepsilon_2} \sum_{\alpha=1}^N a_{\alpha}^2 \right] = \mathbf{q}^{-\frac{1}{2\varepsilon_1 \varepsilon_2} \sum_{\alpha=1}^N a_{\alpha}^2}, \quad \mathbf{q} = e^{2\pi i \tau_{\text{uv}}}, \quad (3.55)$$

while its topological part instead assigns the weight $\mathbf{q}^k$ to the sector of instanton number k , as will appear in the instanton expansion below.

The one-loop contribution $\mathcal{Z}_{1\text{-loop}}$ is the equivariantly regularized superdeterminant of quadratic fluctuations around the perturbative vacuum. It factorizes into a vector-multiplet contribution $z_{1\text{-loop}}^{\text{vec}}$ and one contribution for each hypermultiplet. These determinants can be regularized in terms of Barnes double gamma functions $\Gamma_2(x; \varepsilon_1, \varepsilon_2)$ [1, 2, 22], and

their explicit expressions for a vector multiplet and for (anti)fundamental and adjoint hypermultiplets of mass m read

$$z_{1\text{-loop}}^{\text{adj}}(\mathbf{a}, m) = \prod_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^N \exp[\gamma_{\varepsilon_1, \varepsilon_2}(a_\alpha - a_\beta - m)], \quad (3.56a)$$

$$z_{1\text{-loop}}^{\text{vec}}(\mathbf{a}) = \left(z_{1\text{-loop}}^{\text{adj}}(\mathbf{a}, 0) \right)^{-1}, \quad (3.56b)$$

$$z_{1\text{-loop}}^{\text{fund}}(\mathbf{a}, m) = \prod_{\alpha} \exp[\gamma_{\varepsilon_1, \varepsilon_2}(a_\alpha - m)], \quad (3.56c)$$

$$z_{1\text{-loop}}^{\text{antifund}}(\mathbf{a}, m) = z_{1\text{-loop}}^{\text{fund}}(-\mathbf{a}, \varepsilon_1 + \varepsilon_2 - m), \quad (3.56d)$$

where

$$\gamma_{\varepsilon_1, \varepsilon_2}(x) = -\log \Gamma_2(x + \varepsilon_1 + \varepsilon_2; \varepsilon_1, \varepsilon_2). \quad (3.57)$$

Here and below, contributions independent of the Coulomb parameters are omitted and the explicit dependence on ε_1 and ε_2 in the single-multiplet contributions (3.56) has been suppressed for notational simplicity.

Finally, $\mathcal{Z}_{\text{inst}}$ collects the k -dependent contributions of the non-trivial instanton sectors, all obeying the same asymptotic boundary condition specified by the Coulomb parameters $\mathbf{a}$. In the sector of instanton number k , the BPS configurations form the (resolved) moduli space $\mathcal{M}_{k,N}$ of framed k -instantons and the integration over their collective coordinates, together with the Gaussian integration over fluctuations transverse to the moduli space, defines an equivariant measure on $\mathcal{M}_{k,N}$.

More precisely, after extracting the perturbative determinant $\mathcal{Z}_{1\text{-loop}}$, the remaining vector-multiplet fluctuations determine the equivariant measure on $\mathcal{M}_{k,N}$ while hypermultiplet fluctuations define matter bundles over $\mathcal{M}_{k,N}$, whose characteristic classes are inserted in the integral. The classical topological term assigns the weight $\mathbf{q}^k$ to the sector of instanton number k , thus

$$\mathcal{Z}_{\text{inst}} = \sum_{k=0}^{\infty} \mathbf{q}^k Z_{\text{inst}}^{(k)}, \quad Z_{\text{inst}}^{(0)} = 1, \quad (3.58)$$

where

$$Z_{\text{inst}}^{(k)} = \int_{\mathcal{M}_{k,N}}^T \alpha_{\text{matter}}. \quad (3.59)$$

Here, $\int^T$ denotes T -equivariant integration, where the torus T is the product of the spacetime rotation torus $\text{U}(1)_{\varepsilon_1} \times \text{U}(1)_{\varepsilon_2}$ and the Cartan torus of the framing group $\text{U}(N)$. These act on $\mathcal{M}_{k,N}$ with equivariant parameters $(\varepsilon_1, \varepsilon_2)$ and $\mathbf{a}$, respectively. A flavour torus acting on the matter bundles may also be included, with the masses $\mathbf{m}$ as its equivariant parameters. The class α_{matter} denotes the corresponding equivariant characteristic class and is equal to 1 in the pure theory. In that case, $Z_{\text{inst}}^{(k)}$ is called the equivariant volume of $\mathcal{M}_{k,N}$.

For generic equivariant parameters, the induced torus action on the resolved instanton moduli space has isolated fixed points. The Atiyah–Bott–Berline–Vergne [81, 82] localization formula then gives

$$Z_{\text{inst}}^{(k)} = \sum_{p \in \mathcal{M}_{k,N}^T} \frac{\alpha_{\text{matter}}|_p}{e_T(T_p \mathcal{M}_{k,N})}, \quad (3.60)$$

where $e_T(T_p \mathcal{M}_{k,N})$ is the equivariant Euler class of the tangent space at the fixed point p . The computation of $\mathcal{Z}_{\text{inst}}$ therefore reduces to describing $\mathcal{M}_{k,N}$, identifying its torus-fixed

points and determining the torus characters of its tangent spaces and matter bundles. The ADHM construction provides precisely this information [1, 80, 83].

For gauge group $U(N)$, the resolved moduli space of framed k -instantons admits an explicit description in terms of two complex vector spaces

$$V \simeq \mathbb{C}^k, \quad W \simeq \mathbb{C}^N, \quad (3.61)$$

where k is the instanton number and N is the dimension of the framing space W . The ADHM data consist of

$$(B_1, B_2, I, J) \in \text{Hom}(V, V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W), \quad (3.62)$$

and obey the real and complex moment-map equations

$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \mathbf{1}_V, \quad (3.63a)$$

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0. \quad (3.63b)$$

The parameter $\zeta > 0$ is a Fayet–Iliopoulos (FI) parameter implementing the non-commutative resolution of the small-instanton singularities. The group $U(k)$ acts on the ADHM data as

$$(B_1, B_2, I, J) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}), \quad g \in U(k), \quad (3.64)$$

and the resolved instanton moduli space is the hyperkähler quotient

$$\mathcal{M}_{k,N} = \{(B_1, B_2, I, J) \mid \mu_{\mathbb{C}} = 0, \mu_{\mathbb{R}} = \zeta \mathbf{1}_V\} / U(k). \quad (3.65)$$

The maximal torus of the framing group $U(N)$ acts on W with equivariant parameters $\mathbf{a} = (a_1, \dots, a_N)$. Introducing the multiplicative variables $q_1 = e^{\varepsilon_1}$ and $q_2 = e^{\varepsilon_2}$, the equivariant torus

$$T = U(1)_{\varepsilon_1} \times U(1)_{\varepsilon_2} \times U(1)^N \quad (3.66)$$

acts on the ADHM data as

$$B_1 \mapsto q_1 B_1, \quad B_2 \mapsto q_2 B_2, \quad I \mapsto I g_{\mathbf{a}}^{-1}, \quad J \mapsto q_1 q_2 g_{\mathbf{a}} J, \quad (3.67)$$

where

$$g_{\mathbf{a}} = \text{diag}(e^{a_1}, \dots, e^{a_N}). \quad (3.68)$$

Since $\mathcal{M}_{k,N}$ is a quotient by $U(k)$, a point is fixed by T when the torus action on its ADHM data can be compensated by a transformation of V . For generic equivariant parameters, the fixed points are isolated and are labelled by N -tuples of Young diagrams

$$\mathbf{Y} = (Y_1, \dots, Y_N), \quad |\mathbf{Y}| = \sum_{\alpha=1}^N |Y_{\alpha}| = k. \quad (3.69)$$

To describe the contribution of each fixed point, recall that the *equivariant character* of a finite-dimensional T -representation with weights w_{ℓ} is

$$\text{ch}_T E = \sum_{\ell} e^{w_{\ell}}. \quad (3.70)$$

The character of the dual representation is denoted by $(\text{ch}_T E)^*$ and is obtained by reversing the sign of each weight. In the following, we use the same symbol for a representation and its equivariant character whenever no confusion can arise.

The character of the framing space is

$$W = \text{ch}_T W = \sum_{\alpha=1}^N e^{a_\alpha}, \quad (3.71)$$

while at the fixed point labelled by $\mathbf{Y}$ the character of V is

$$V_{\mathbf{Y}} = \text{ch}_T V_{\mathbf{Y}} = \sum_{\alpha=1}^N e^{a_\alpha} \sum_{(i,j) \in Y_\alpha} q_1^{i-1} q_2^{j-1}. \quad (3.72)$$

The ADHM deformation complex encodes the infinitesimal variations of the ADHM data, together with the moment-map constraint and the quotient by $U(k)$ [80]. Its character gives the character of the tangent space to the moduli space at the fixed point:

$$T_{\mathbf{Y}} \mathcal{M}_{k,N} = \text{ch}_T T_{\mathbf{Y}} \mathcal{M}_{k,N} = W V_{\mathbf{Y}}^* + q_1 q_2 V_{\mathbf{Y}} W^* - (1 - q_1)(1 - q_2) V_{\mathbf{Y}} V_{\mathbf{Y}}^*. \quad (3.73)$$

This character determines all the equivariant weights entering the vector-multiplet contribution. More precisely, for a virtual character

$$\chi = \sum_{\ell} n_{\ell} e^{w_{\ell}}, \quad (3.74)$$

define

$$\mathcal{E}[\chi] = \prod_{\ell} w_{\ell}^{n_{\ell}}. \quad (3.75)$$

This operation associates to a character its equivariant Euler class, with negative multiplicities producing denominator factors. The vector-multiplet contribution at the fixed point $\mathbf{Y}$ is therefore

$$z_{\text{inst}}^{\text{vec}}(\mathbf{a}, \mathbf{Y}) = \mathcal{E}(-T_{\mathbf{Y}} \mathcal{M}_{k,N}) = \frac{1}{\mathcal{E}(T_{\mathbf{Y}} \mathcal{M}_{k,N})}, \quad (3.76)$$

where the explicit dependence on ε_1 and ε_2 has been suppressed.

Matter multiplets are incorporated through the equivariant Euler classes of the corresponding matter bundles. For N_f fundamental hypermultiplets, introduce the flavour character

$$M = \sum_{f=1}^{N_f} e^{m_f}. \quad (3.77)$$

The corresponding matter character is

$$\chi_{\text{fund}} = q_1 q_2 V_{\mathbf{Y}} M^*, \quad (3.78)$$

and hence

$$\begin{aligned} Z_{\text{inst}}^{\text{fund}}(\mathbf{a}, \mathbf{Y}, \mathbf{m}) &= \prod_{f=1}^{N_f} z_{\text{inst}}^{\text{fund}}(\mathbf{a}, \mathbf{Y}, m_f) = \mathcal{E}[\chi_{\text{fund}}] \\ &= \prod_{f=1}^{N_f} \prod_{\alpha=1}^N \prod_{(i,j) \in Y_\alpha} (a_\alpha - m_f + i\varepsilon_1 + j\varepsilon_2). \end{aligned} \quad (3.79)$$

The factor $q_1 q_2$ shifts the matter weights by $\varepsilon_1 + \varepsilon_2$ and omitting this factor corresponds to a different convention for the hypermultiplet masses. For an adjoint hypermultiplet of mass m , the matter character is instead obtained by twisting the tangent-space character:

$$\chi_{\text{adj}}(m) = e^{-m} T_{\mathbf{Y}} \mathcal{M}_{k,N}, \quad z_{\text{inst}}^{\text{adj}}(\mathbf{a}, \mathbf{Y}, m) = \mathcal{E}[\chi_{\text{adj}}(m)]. \quad (3.80)$$

In particular, at $m = 0$ the adjoint contribution cancels the vector-multiplet contribution, as expected for the $\mathcal{N} = 4$ theory.

Combining the vector-multiplet and matter-multiplet factors, the complete fixed-point contribution is

$$Z_{\mathbf{Y}} = \mathcal{E}(-T[\mathbf{Y}]), \quad T[\mathbf{Y}] := T_{\mathbf{Y}}\mathcal{M}_{k,N} - \chi_{\text{matter}}. \quad (3.81)$$

At the fixed point labelled by $\mathbf{Y}$, the operation $\mathcal{E}$ maps the tangent-space character to the Euler class appearing in the localization denominator,

$$\mathcal{E}[T_{\mathbf{Y}}\mathcal{M}_{k,N}] = e_T(T_{\mathbf{Y}}\mathcal{M}_{k,N}), \quad (3.82)$$

while it maps the matter character to the restriction of the inserted matter class,

$$\mathcal{E}[\chi_{\text{matter}}] = \alpha_{\text{matter}}|_{\mathbf{Y}}. \quad (3.83)$$

Thus, (3.81) is precisely the fixed-point ratio prescribed by equivariant localization (3.60). It follows that

$$Z_{\text{inst}}^{(k)} = \sum_{|\mathbf{Y}|=k} Z_{\mathbf{Y}}, \quad (3.84)$$

and the instanton partition function becomes

$$\mathcal{Z}_{\text{inst}} = \sum_{\mathbf{Y}} \mathbf{q}^{|\mathbf{Y}|} Z_{\mathbf{Y}}. \quad (3.85)$$

More generally, this construction extends to quiver gauge theories [22, 84], with hypermultiplets in fundamental or adjoint representations of the individual gauge-group factors and in bifundamental representations connecting different factors. In particular, the one-loop and instanton contributions of a hypermultiplet in the $(N_1, \bar{N}_2)$ representation of $U(N_1) \times U(N_2)$ read, respectively,

$$z_{1\text{-loop}}^{\text{bif}}(\mathbf{a}, \mathbf{b}, m) = \prod_{\alpha=1}^{N_1} \prod_{\beta=1}^{N_2} \exp[\gamma_{\varepsilon_1, \varepsilon_2}(a_{\alpha} - b_{\beta} - m)], \quad (3.86a)$$

$$\begin{aligned} z_{\text{inst}}^{\text{bif}}(\mathbf{a}, \mathbf{Y}; \mathbf{b}, \mathbf{W}; m) &= \prod_{\alpha=1}^{N_1} \prod_{\beta=1}^{N_2} \left[\prod_{s \in Y_{\alpha}} \left(E_{Y_{\alpha}, W_{\beta}}(a_{\alpha} - b_{\beta}; s) - m \right) \right. \\ &\quad \left. \times \prod_{t \in W_{\beta}} \left(\varepsilon_1 + \varepsilon_2 - E_{W_{\beta}, Y_{\alpha}}(b_{\beta} - a_{\alpha}; t) - m \right) \right], \end{aligned} \quad (3.86b)$$

where, for two Young diagrams Y, W , we define

$$E_{Y,W}(x; s) = x - \varepsilon_1 L_W(s) + \varepsilon_2 (A_Y(s) + 1), \quad (3.87)$$

with $A_Y(s)$ and $L_W(s)$ the arm and leg lengths of a box s . All previous one-loop and instanton single-multiplet contributions can be obtained by specializing the bifundamental ones. In particular,

$$z_{\text{inst}}^{\text{adj}}(\mathbf{a}, \mathbf{Y}, m) = z_{\text{inst}}^{\text{bif}}(\mathbf{a}, \mathbf{Y}; \mathbf{a}, \mathbf{Y}; m), \quad (3.88a)$$

$$z_{\text{inst}}^{\text{vec}}(\mathbf{a}, \mathbf{Y}) = \left(z_{\text{inst}}^{\text{adj}}(\mathbf{a}, \mathbf{Y}, 0) \right)^{-1}. \quad (3.88b)$$

Similar localization constructions also apply to supersymmetric observables, including the chiral operators whose vacuum expectation values define the Coulomb-branch coordinates u_k in (3.21) [85], as well as BPS line and surface defects [33, 86].

Blow-up equations

Supersymmetric localization and the computation of Nekrasov partition functions can also be carried out on suitable toric four-manifolds [87, 88]. A particularly important example is the blow-up of $\mathbb{C}^2$ at the origin, $\hat{\mathbb{C}}^2$. We describe the resulting blow-up relations in the pure SU(2) theory, following Nakajima and Yoshioka [3, 69].

The blow-up geometry can be realized explicitly as

$$\hat{\mathbb{C}}^2 = \left\{ (z_1, z_2, [u : v]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid z_1 v = z_2 u \right\}, \quad (3.89)$$

and the blow-down map

$$\pi : \hat{\mathbb{C}}^2 \longrightarrow \mathbb{C}^2, \quad \pi(z_1, z_2, [u : v]) = (z_1, z_2), \quad (3.90)$$

is an isomorphism away from the origin. The preimage of the origin is the exceptional divisor

$$E = \pi^{-1}(0) \simeq \mathbb{CP}^1, \quad (3.91)$$

which parametrizes the complex directions through the blown-up origin. The standard torus action on $\mathbb{C}^2$,

$$(z_1, z_2) \longmapsto (q_1 z_1, q_2 z_2), \quad q_i = e^{\varepsilon_i}, \quad (3.92)$$

lifts to $\hat{\mathbb{C}}^2$ according to

$$(z_1, z_2, [u : v]) \longmapsto (q_1 z_1, q_2 z_2, [q_1 u : q_2 v]). \quad (3.93)$$

The exceptional divisor contains two torus-fixed points,

$$p_1 = (0, 0, [1 : 0]), \quad p_2 = (0, 0, [0 : 1]). \quad (3.94)$$

Their affine neighbourhoods are both isomorphic to $\mathbb{C}^2$, but the induced equivariant parameters are different. Around p_1 they are

$$(\varepsilon_1, \varepsilon_2 - \varepsilon_1), \quad (3.95)$$

while around p_2 they are

$$(\varepsilon_1 - \varepsilon_2, \varepsilon_2). \quad (3.96)$$

Equivariant localization on $\hat{\mathbb{C}}^2$ therefore expresses the instanton partition function in terms of two copies of the $\mathbb{C}^2$ partition function, one associated with each fixed point. In addition, since the exceptional divisor is a compact two-cycle, gauge configurations may carry a quantized magnetic flux through it. For the pure SU(2) theory, writing the Coulomb parameters as $\mathbf{a} = (a, -a)$, the flux can be parametrized by $\mathbf{n} = (n, -n)$. The effective Coulomb parameters in the two affine patches are then shifted to

$$a^{(1)} = a + n\varepsilon_1, \quad a^{(2)} = a + n\varepsilon_2. \quad (3.97)$$

Thus, localization on the blow-up gives a factorization of the schematic form

$$\widehat{\mathcal{Z}}_\delta(a; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \sum_{n \in \mathbb{Z} + \delta} \mathcal{Z}(a + n\varepsilon_1; \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + n\varepsilon_2; \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathbf{q}), \quad (3.98)$$

where δ labels the flux sector. This factorization computes the partition function on $\hat{\mathbb{C}}^2$. The Nakajima–Yoshioka *blow-up equations* follow by combining it with geometric relations between the instanton moduli spaces on $\hat{\mathbb{C}}^2$ and $\mathbb{C}^2$ induced by the blow-down map. For

suitable insertions and flux sectors, the partition function on the blow-up is either identified with the partition function on $\mathbb{C}^2$ or it vanishes. This produces, respectively, unity and vanishing blow-up equations.

For example, for the pure $SU(2)$ partition function, the unity relation takes the form

$$\mathcal{Z}(a; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \sum_{n \in \mathbb{Z}} \mathcal{Z}(a + n\varepsilon_1; \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + n\varepsilon_2; \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathbf{q}), \quad (3.99)$$

while the corresponding vanishing relation is

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathcal{Z}(a + n\varepsilon_1; \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + n\varepsilon_2; \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathbf{q}) = 0. \quad (3.100)$$

These functional identities relate the Nekrasov partition function at $(\varepsilon_1, \varepsilon_2)$ to products of partition functions with shifted Coulomb parameters and equivariant parameters. Since the terms with non-zero flux begin at higher instanton order, the relations can be used recursively to determine the instanton expansion.

Such blow-up relations admit several generalizations and, in particular, they can be derived for gauge theories in the presence of codimension-two BPS surface defects [52, 89].

3.1.5 Class $\mathcal{S}$ gauge theories, the AGT and Bethe/gauge correspondence

A large class of four-dimensional $\mathcal{N} = 2$ theories can be constructed by compactifying the six-dimensional $\mathcal{N} = (2, 0)$ theory on a punctured Riemann surface C with a partial topological twist [74]. The resulting four-dimensional theory is called a theory of class $\mathcal{S}$. The complex structure moduli of C are identified with exactly marginal gauge couplings of the four-dimensional theory, while the punctures encode flavour symmetries and mass parameters.

Let us first focus on the case of the A_1 -type six-dimensional $\mathcal{N} = (2, 0)$ theory, which gives theories $\mathcal{T}_2[C]$ with gauge group factors $SU(2)$. In this case a pants decomposition of C gives a weakly coupled duality frame. Each three-punctured sphere corresponds to a trifundamental half-hypermultiplet of $SU(2)^3$, while gluing two punctures by a long tube corresponds to gauging the diagonal $SU(2)$ flavour symmetry with an $\mathcal{N} = 2$ vector multiplet. The complex structure parameter of the tube is then identified with

$$\mathbf{q} = e^{2\pi i \tau_{uv}}, \quad (3.101)$$

where τ_{uv} is the corresponding gauge coupling. Different pants decompositions of the same surface give different weakly coupled descriptions of the same theory and are related by S-duality. For instance, the sphere with four regular punctures describes the $SU(2)$ theory with $N_f = 4$, while the torus with one regular puncture describes the $\mathcal{N} = 2^* SU(2)$ theory.

In this language, the Seiberg–Witten geometry is naturally encoded by a quadratic differential on C . For theories of type A_1 , the Seiberg–Witten curve takes the form

$$\Sigma_{\mathbf{u}} : \quad \lambda^2 = \phi_2(z), \quad (3.102)$$

where ϕ_2 is a meromorphic quadratic differential whose singular behaviour at the punctures is fixed by the mass parameters, while the remaining coefficients of ϕ_2 are Coulomb-branch parameters. For theories of type A_{N-1} this generalizes to an N -sheeted spectral curve

$$\Sigma_{\mathbf{u}} : \quad \lambda^N + \sum_{k=2}^N \phi_k \lambda^{N-k} = 0, \quad \lambda = x \, dz. \quad (3.103)$$

where ϕ_k are meromorphic k -differentials on C , whose regular punctures are classified by partitions of N .

The AGT correspondence relates this class $\mathcal{S}$ construction to two-dimensional conformal field theory [22]. For theories of type A_1 , the instanton part of the Nekrasov partition function is identified with a Liouville conformal block on the same punctured surface C , schematically

$$\mathcal{Z}_{\text{inst}}^{\mathcal{T}_2[C]}(\mathbf{a}, \mathbf{m}; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \mathcal{F}_C(\mathbf{A}, \mathbf{P}; \mathbf{q}), \quad (3.104)$$

up to simple convention-dependent prefactors. The sewing parameters of the conformal block are the gauge couplings $\mathbf{q}$, the internal Liouville momenta $\mathbf{A}$ are identified with the Coulomb moduli $\mathbf{a}$ and the external Liouville momenta $\mathbf{P}$ are identified with the mass parameters $\mathbf{m}$. The Liouville central charge is

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad b^2 = \frac{\varepsilon_2}{\varepsilon_1}. \quad (3.105)$$

In this dictionary, changing pants decomposition corresponds to changing OPE channel in the two-dimensional conformal block and crossing symmetry of Liouville theory is mapped to S-duality of the four-dimensional gauge theory.

The correspondence extends to theories $\mathcal{T}_N[C]$ of type A_{N-1} by replacing Liouville theory with A_{N-1} Toda conformal field theory, and Virasoro conformal blocks with $\mathcal{W}_N$ conformal blocks [90]. Other extensions of the correspondence involve, for example, affine chiral algebras [91], or identify BPS surface-defect partition functions with conformal blocks containing degenerate field insertions [33]. Furthermore, through the AGT dictionary, the gauge-theoretical blow-up equations of Nakajima and Yoshioka, and their generalizations, imply relations between conformal blocks of chiral algebras at different values of the central charge [92].

The AGT correspondence also has a non-chiral version. Supersymmetric localization on S^4 , and more generally on the ellipsoid S_b^4 , gives an integral over Coulomb parameters of the product of two Nekrasov partition functions associated with the two fixed points of the spacetime torus action [93, 94]. For the theory $\mathcal{T}_2[C]$ one finds, schematically,

$$\mathcal{Z}_{S_b^4}^{\mathcal{T}_2[C]} = \int d\mathbf{a} \, \mathcal{Z}_{\text{cl}} \mathcal{Z}_{1\text{-loop}} |\mathcal{Z}_{\text{inst}}|^2, \quad (3.106)$$

with $\varepsilon_2/\varepsilon_1 = b^2$. Under the correspondence, this reproduces the full Liouville correlator on C : the Nekrasov instanton factors give the holomorphic and antiholomorphic conformal blocks, while the classical and one-loop factors combine into the integration measure and three-point functions.

Finally, the Bethe/gauge correspondence relates four-dimensional $\mathcal{N} = 2$ gauge theories in the Nekrasov–Shatashvili (NS) limit of the $\Omega_{\varepsilon_1, \varepsilon_2}$ -background to quantizations of the classical algebraic integrable systems whose bases are the Coulomb branches of the corresponding gauge theories [21, 29, 51]. The NS limit is obtained by sending one equivariant parameter to zero, say ε_2 , while the other is kept fixed and identified with the quantization parameter, $\varepsilon_1 = \hbar$. In this limit,

$$\mathcal{W}_0(\mathbf{a}, \mathbf{m}, \hbar; \mathbf{q}) = \lim_{\varepsilon_2 \rightarrow 0} \varepsilon_2 \log \mathcal{Z}(\mathbf{a}, \mathbf{m}; \varepsilon_1 = \hbar, \varepsilon_2; \mathbf{q}) \quad (3.107)$$

is identified with the effective twisted superpotential of the two-dimensional $\mathcal{N} = (2, 2)$ theory governing the low-energy physics of the four-dimensional theory in the $\Omega_{\hbar, 0}$ -background. The Bethe/gauge correspondence further identifies this twisted superpotential with the

Yang–Yang functional of the associated quantum integrable system. Under this identification, supersymmetric vacua correspond to Bethe states, while operators in the twisted chiral ring map to the mutually commuting quantum Hamiltonians. The vacuum equations therefore take the schematic form

$$\frac{\partial \mathcal{W}_0}{\partial a_\alpha} = 2\pi i n_\alpha, \quad n_\alpha \in \mathbb{Z}. \quad (3.108)$$

This correspondence admits several refinements. For instance, common eigenfunctions of the mutually commuting Hamiltonians of the quantum integrable system can be engineered by taking the NS limit of appropriately normalized partition functions in the presence of codimension-two half-BPS surface defects. Partition functions associated with other types of surface defects can similarly be used to construct wavefunctions solving the corresponding quantum Seiberg–Witten curve [29, 32, 33, 95].

For gauge theories to which the AGT correspondence applies, the NS limit $\varepsilon_2 \rightarrow 0$ is mapped to a classical limit $c \rightarrow \infty$ of the corresponding conformal blocks. Consequently, the instanton contribution to the twisted superpotential is encoded in the leading classical exponent of the conformal block, up to convention-dependent normalization factors [31, 96]. Moreover, partition functions in the presence of suitable BPS surface defects are identified with conformal blocks containing degenerate vertex-operator insertions [33]. In the NS limit, the associated null-vector decoupling equations reduce to differential equations corresponding to quantum Seiberg–Witten curves [32, 96].

3.2 Five-dimensional $\mathcal{N} = 1$ gauge theories

Most of the preceding discussion admits a natural five-dimensional uplift. In this setting, one considers a five-dimensional $\mathcal{N} = 1$ gauge theory on $\mathbb{R}^4 \times S^1_\beta$, where β denotes the circumference of the compactification circle. The squared Yang–Mills coupling has dimensions of length, and the Yang–Mills Lagrangian is therefore non-renormalizable and should be regarded as a low-energy effective description. Nevertheless, for suitable choices of gauge group, matter content, and five-dimensional Chern–Simons coupling, these theories can arise as relevant deformations of interacting five-dimensional superconformal field theories, whose existence can often be understood through string-theory or M-theory constructions [97, 98].

At the level of propagating fields, the field content is organized into vector multiplets,

$$\mathcal{V} = (\phi, \lambda^A, A_M), \quad M = 1, \dots, 5, \quad A = 1, 2, \quad (3.109)$$

where ϕ is a real scalar, λ^A is an $\mathrm{SU}(2)_R$ doublet of symplectic-Majorana fermions and A_M is the gauge field, all in the adjoint representation of the gauge group, and hypermultiplets

$$\mathcal{H}_W = (q^A, \psi), \quad A = 1, 2, \quad (3.110)$$

where the complex scalars q^A form an $\mathrm{SU}(2)_R$ doublet, while ψ is a Dirac fermion, all transforming in the representation W of the gauge group.

On $\mathbb{R}^5$, vacuum expectation values of the real adjoint scalar ϕ parametrize a Coulomb branch of real dimension $\mathrm{rank}(\mathcal{G}_g)$. Upon compactification on S^1_β , however, the scalar ϕ combines with the holonomy of the gauge field around the circle to form complex Coulomb moduli. It is therefore natural to consider multiplicative Coulomb parameters of the schematic form

$$e^{\beta a_\alpha} = \exp \left(\beta \phi_\alpha + i \oint_{S^1_\beta} A_{5,\alpha} dx^5 \right). \quad (3.111)$$

The Seiberg–Witten geometry also admits a five-dimensional uplift. For a theory compactified on a circle, the Seiberg–Witten curve is naturally identified with the spectral curve of a relativistic deformation of the algebraic integrable system associated with its four-dimensional limit [98].

Let us now specialize to the pure $U(N)$ gauge theory with vanishing five-dimensional Chern–Simons level in order to illustrate the main new features of the five-dimensional uplift. A convenient dimensionless complexification of the gauge coupling on $\mathbb{R}^4 \times S^1_\beta$ is

$$\tau_{5d} = \frac{4\pi i\beta}{g_{5d}^2} + \frac{\theta}{2\pi}, \quad \mathfrak{q}_{5d} = e^{2\pi i\tau_{5d}}, \quad (3.112)$$

where θ is the theta angle of the effective four-dimensional theory. Introducing the multiplicative spectral coordinates $X = e^{\beta x}$ and $Y = e^y$, the Seiberg–Witten curve of the pure theory can be written as

$$\Sigma_\kappa : \quad \Lambda^N (Y + Y^{-1}) + \sum_{\ell=0}^N \kappa_\ell X^{\frac{N}{2}-\ell} = 0, \quad \kappa_0 = \kappa_N = 1. \quad (3.113)$$

The coefficients κ_0, κ_N can be fixed by rescaling X and multiplying the equation by an overall nonzero constant. The remaining parameters $\kappa = (\kappa_1, \dots, \kappa_{N-1})$ are gauge-invariant coordinates on the Coulomb branch, while Λ is related to the gauge coupling by

$$\mathfrak{q}_{5d} = (-1)^N \Lambda^{2N}. \quad (3.114)$$

This curve is the spectral curve of the relativistic periodic Toda chain after removal of the decoupled center-of-mass sector, with the compactification parameter β controlling the relativistic deformation. Taking the standard four-dimensional scaling of the spectral coordinate and Coulomb moduli, together with

$$\Lambda^{2N} \sim (-1)^N (\beta \Lambda_{4d})^{2N}, \quad (3.115)$$

the curve reduces, as $\beta \rightarrow 0$, to the Seiberg–Witten curve of the four-dimensional pure $SU(N)$ theory (3.45).

The $\Omega_{\varepsilon_1, \varepsilon_2}$ -background admits a particularly natural geometric realization in five dimensions. It can be described as the twisted product [2]

$$\mathbb{C}^2 \rtimes_\varepsilon S^1_\beta = \frac{\mathbb{C}^2 \times \mathbb{R}}{(z_1, z_2, t) \sim (q_1 z_1, q_2 z_2, t + \beta)}, \quad q_i = e^{\beta \varepsilon_i}. \quad (3.116)$$

After traversing the circle, the fields return to themselves up to rotations of the two complex planes of $\mathbb{C}^2$, accompanied by an $SU(2)_R$ transformation chosen so as to preserve supersymmetry. The five-dimensional Nekrasov partition function can be interpreted as an equivariant Witten index of the theory quantized on $\mathbb{C}^2$ [4, 99],

$$\mathcal{Z}^{5d}(\mathbf{a}; q_1, q_2; \mathfrak{q}_{5d}) = \text{Tr}_{\mathcal{H}_{\mathbb{C}^2}} \left[(-1)^F e^{-\beta H_Q} q_1^{J_1} q_2^{J_2} \mathfrak{q}_{5d}^K \right], \quad (3.117)$$

where $H_Q = \{Q, Q^\dagger\}$ is the supersymmetric Hamiltonian, J_1, J_2 denote the supersymmetry-preserving equivariant generators, combining spacetime rotations with the compensating R -symmetry action, and

$$K = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr} F \wedge F \quad (3.118)$$

is the instanton charge.

Configurations that appear as instantons in four dimensions become particle-like BPS solitons in five dimensions. In the Euclidean description relevant to the index, their worldlines wrap S^1_β and a configuration of instanton charge k is weighted by q_{5d}^k . The instanton contribution to the $U(N)$ partition function therefore admits an expansion

$$\mathcal{Z}_{\text{inst}}^{5d}(\mathbf{a}; q_1, q_2; q_{5d}) = \sum_{k=0}^{\infty} q_{5d}^k Z_{\text{inst}}^{5d, (k)}, \quad (3.119)$$

where $Z_{\text{inst}}^{5d, (k)}$ is the equivariant index on the moduli space $\mathcal{M}_{k,N}$ of $U(N)$ instantons of charge k .

As in the four-dimensional case, equivariant localization expresses this index as a sum over fixed points labelled by N -tuples $\mathbf{Y}$ of Young diagrams. The equivariant character of the tangent space retains the form (3.73), but is now interpreted as a character in equivariant K-theory, with the weights entering multiplicatively. Accordingly, the equivariant Euler class (3.74) is replaced by its K-theoretic analogue,

$$\mathcal{E}_\beta[\chi] = \prod_{\ell} \left(1 - e^{-\beta w_\ell}\right)^{n_\ell}, \quad \chi = \sum_{\ell} n_\ell e^{\beta w_\ell}. \quad (3.120)$$

Indeed, since $1 - e^{-\beta w_\ell} \sim \beta w_\ell$ as $\beta \rightarrow 0$, the cohomological Euler class is recovered in the four-dimensional limit, up to an overall power of β .

The resulting *K-theoretic instanton partition function* is therefore

$$\mathcal{Z}_{\text{inst}}^{5d}(\mathbf{a}; q_1, q_2; q_{5d}) = \sum_{\mathbf{Y}} q_{5d}^{|\mathbf{Y}|} z_{\text{inst}}^{5d, \text{vec}}(\mathbf{a}, \mathbf{Y}; q_1, q_2), \quad |\mathbf{Y}| = \sum_{\alpha=1}^N |Y^{(\alpha)}|, \quad (3.121)$$

where

$$\begin{aligned} z_{\text{inst}}^{5d, \text{vec}}(\mathbf{a}, \mathbf{Y}; q_1, q_2) &= \prod_{\alpha, \beta=1}^N \left(\prod_{s \in Y_\alpha} \left[1 - e^{-\beta(a_\alpha - a_\beta - \varepsilon_1 L_{Y_\beta}(s) + \varepsilon_2(A_{Y_\alpha}(s) + 1))} \right]^{-1} \right. \\ &\quad \times \left. \prod_{t \in Y_\gamma} \left[1 - e^{-\beta(a_\alpha - a_\gamma + \varepsilon_1(L_{Y_\alpha}(t) + 1) - \varepsilon_2 A_{Y_\gamma}(t))} \right]^{-1} \right). \end{aligned} \quad (3.122)$$

In the limit $\beta \rightarrow 0$, one has

$$q_i = 1 + \beta \varepsilon_i + \mathcal{O}(\beta^2), \quad e^{\beta a_\alpha} = 1 + \beta a_\alpha + \mathcal{O}(\beta^2), \quad (3.123)$$

and the four-dimensional instanton partition function is recovered after the appropriate matching of the instanton counting parameters.

This framework admits many further generalizations, including the addition of hypermultiplets, extensions to quiver gauge theories, the computation of expectation values of Wilson loops wrapping S^1_β , the inclusion of BPS surface defects, and the formulation of blow-up relations. Likewise, class $\mathcal{S}$ constructions and the AGT correspondence admit natural five-dimensional uplifts. In particular, the five-dimensional AGT correspondence identifies K-theoretic instanton partition functions with conformal blocks of q_1, q_2 -deformed chiral algebras, including the deformed Virasoro and $\mathcal{W}_N$ algebras [100, 101].

The Bethe/gauge correspondence also admits a five-dimensional, relativistic uplift. However, the naive extension of the four-dimensional NS prescription is generally incomplete. In many examples, the effective twisted superpotential extracted from the five-dimensional Nekrasov partition function contains unphysical poles for real values of the quantization

parameter $\hbar$, and the resulting NS quantization conditions fail to reproduce the exact eigenvalues of the associated relativistic quantum integrable system [102, 103]. To address this problem, two apparently different but equivalent non-perturbative prescriptions have been formulated [104]. Both amount to a *non-perturbative completion* of the effective twisted superpotential by additional contributions that are non-perturbative in $\hbar$. These contributions cancel the unphysical poles and lead to well-defined exact quantization conditions and the correct spectra. In particular, one of these prescriptions can be regarded as the gauge-theoretic counterpart of the TS/ST correspondence, which we review in the next section.

Section 4

Topological Strings

In this fourth and final background section, we briefly review those aspects of topological string theory that are relevant to the topological string/spectral theory (TS/ST) correspondence, discussed in the final subsection. The main objects of interest are refined and unrefined topological string partition functions and their associated free energies, in both the closed and open string sectors. We begin with a brief review of the topological A-model and B-model, followed by mirror symmetry, with particular emphasis on its local formulation, and conclude with the TS/ST correspondence. Since the Calabi–Yau backgrounds considered in this thesis engineer five-dimensional supersymmetric gauge theories, we also discuss the relation between refined topological string partition functions and the Nekrasov partition functions of the corresponding gauge theories, which often provide a more convenient means of computing the relevant free energies.

General references. For background and more detailed explanations of the material reviewed in this section, see [105–108].

4.1 Topological string theory

Topological string theory is a simpler version of string theory in which many quantities can be computed exactly, while still retaining deep information about the geometry of the target space. Like ordinary string theory, it is a quantum theory of maps

$$X : \Sigma_g \rightarrow \mathcal{M}, \quad (4.1)$$

from a Riemann surface Σ_g , possibly with boundary, to a target manifold $\mathcal{M}$. The case $\partial\Sigma_g = \emptyset$ corresponds to closed strings while, if $\partial\Sigma_g \neq \emptyset$, one obtains open topological strings. In the latter case the boundary of the worldsheet must satisfy appropriate boundary conditions, namely its image is constrained to lie on suitable topological branes $\mathcal{B}$ in $\mathcal{M}$,

$$X(\partial\Sigma_g) \subset \mathcal{B} \subset \mathcal{M}. \quad (4.2)$$

We will mostly focus on the closed case, but we will also discuss generalizations to open strings since they will play a role in the last chapter of this thesis.

In the applications relevant for us, $\mathcal{M}$ will be a Calabi–Yau (CY) threefold, namely a Kähler manifold of complex dimension three with trivial canonical bundle. By Yau’s theorem, a compact Kähler manifold with vanishing first Chern class admits a Ricci-flat Kähler metric. In particular, this applies when its canonical bundle is trivial. However, compactness will not be assumed in general: non-compact, or local, CY threefolds will be particularly important in what follows.

The worldsheet theory from which topological string theory is constructed is the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric non-linear sigma model with target $\mathcal{M}$, its bosonic fields being the maps $X : \Sigma_g \rightarrow \mathcal{M}$. Locally, one may describe these maps either by real coordinates X^μ on $\mathcal{M}$, or, since $\mathcal{M}$ is complex, by holomorphic and anti-holomorphic coordinates $X^i, \bar{X}^{\bar{i}}$. The fermionic fields are sections of bundles obtained by pulling back the holomorphic and anti-holomorphic tangent bundles of $\mathcal{M}$ to the worldsheet. Schematically, the bosonic part of the action has the form

$$S_{\text{bos}} = \frac{1}{4\pi\alpha'} \int_{\Sigma_g} (G_{\mu\nu}(X) dX^\mu \wedge *dX^\nu + iB_{\mu\nu}(X) dX^\mu \wedge dX^\nu), \quad (4.3)$$

where α' is the Regge slope, G is a Kähler metric on $\mathcal{M}$ and B is the Kalb–Ramond two-form. In complex coordinates the Kähler metric has only mixed components,

$$ds^2 = 2G_{i\bar{j}} dX^i d\bar{X}^{\bar{j}}, \quad \omega = iG_{i\bar{j}} dX^i \wedge d\bar{X}^{\bar{j}}. \quad (4.4)$$

where ω is the closed Kähler form, $d\omega = 0$. The $\mathcal{N} = (2, 2)$ supersymmetry implies the existence of two $U(1)$ R-symmetries, usually called the vector and axial R-symmetries, $U(1)_V$ and $U(1)_A$. A topological twist modifies the Lorentz spins of the fields by mixing the worldsheet rotation group with one of these R-symmetries [109].

There are two inequivalent topological twists of the $\mathcal{N} = (2, 2)$ sigma model. The A-twist uses the vector R-symmetry, while the B-twist uses the axial R-symmetry. After the twist, one of the supercharges becomes a scalar BRST operator Q , satisfying $Q^2 = 0$. Physical observables are then defined as cohomology classes of Q . Moreover, the stress-energy tensor is Q -exact, so correlation functions of Q -closed operators are independent of the worldsheet metric. In this sense the twisted theory is a two-dimensional topological quantum field theory. Topological string theory is obtained by coupling this topological field theory to topological gravity, which amounts to summing over worldsheet topologies and integrating over the moduli space of Riemann surfaces.

In the closed topological string theory on a CY threefold, the basic observable is the *topological string partition function*, or equivalently its logarithm, the *free energy*. Perturbatively, the latter is obtained by summing over connected closed worldsheets of fixed genus g , weighted by $g_s^{-\chi_g}$,

$$Z_{\text{closed}}(\mathbf{m}; g_s) = \exp(F_{\text{closed}}(\mathbf{m}; g_s)), \quad F_{\text{closed}}(\mathbf{m}; g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(\mathbf{m}), \quad (4.5)$$

where $\chi_g = 2 - 2g$ is the corresponding Euler characteristic. Here g_s is the *topological string coupling constant*, $\mathbf{m}$ denotes the closed-string moduli and $F_g(\mathbf{m})$ is the connected contribution from closed Riemann surfaces of genus g .

The open topological string theory is obtained by allowing the worldsheet to have boundaries, with boundary conditions specified by topological branes in the target space. For a fixed brane configuration, the connected open free energies are usually denoted by $F_{g,h}(\mathbf{m}, \mathbf{z})$, where g is the genus, h is the number of boundaries and $\mathbf{z}$ denotes the open-string moduli. They are naturally weighted by $g_s^{-\chi_{g,h}}$, where $\chi_{g,h} = 2 - 2g - h$ is the Euler characteristic of the bordered worldsheet, so that

$$Z_{\text{open}}(\mathbf{m}, \mathbf{z}; g_s) = \exp(F_{\text{open}}(\mathbf{m}, \mathbf{z}; g_s)), \quad F_{\text{open}}(\mathbf{m}, \mathbf{z}; g_s) = \sum_{g \geq 0} \sum_{h \geq 1} g_s^{2g-2+h} F_{g,h}(\mathbf{m}, \mathbf{z}). \quad (4.6)$$

The two twists lead to two different theories, called the *topological A-model* and *B-model*, which probe complementary aspects of the CY geometry. In the A-model the closed-string

moduli $\mathbf{m}$ are identified with the *complexified Kähler parameters* of $\mathcal{M}$. These are typically written as

$$t_i = \int_{C_i} (\omega + iB), \quad i = 1, \dots, b_2(\mathcal{M}), \quad (4.7)$$

where $\{C_i\}$ is a basis of two-cycles of $\mathcal{M}$, ω is the Kähler form and B is the Kalb–Ramond two-form. The A-model free energies depend on these parameters, but not on the complex structure of $\mathcal{M}$. The path integral localizes on holomorphic maps $\Sigma_g \rightarrow \mathcal{M}$ and the resulting free energies encode Gromov–Witten invariants.

In the B-model, instead, the closed-string moduli $\mathbf{m}$ are identified with the *complex structure moduli* κ of $\mathcal{M}$ and the B-model free energies are independent of the Kähler parameters $\mathbf{t}$. At genus zero, the theory is governed by the special geometry of the complex-structure moduli space: the periods of the holomorphic three-form Ω over a symplectic basis of three-cycles determine the genus-zero prepotential and hence the genus-zero free energy. Higher-genus free energies are then governed by the holomorphic anomaly equations.

In the open string sector this distinction is accompanied by the corresponding choice of branes. A-branes are supported on Lagrangian submanifolds and open A-model free energies count holomorphic maps from bordered Riemann surfaces with boundary on them. B-branes are supported on holomorphic submanifolds or more generally described by holomorphic data.

4.1.1 The A-model

Let us now focus on the A-model. As mentioned above, the closed-string moduli of the A-model are the complexified Kähler parameters $\mathbf{t} = \{t_i\}_{i=1}^{b_2}$ in (4.7). It is also convenient to introduce the exponentiated variables

$$Q_i = e^{-t_i}, \quad Q^{\mathbf{d}} = e^{-\mathbf{d} \cdot \mathbf{t}}, \quad (4.8)$$

where $\mathbf{d} = (d_1, \dots, d_{b_2})$ is a degree vector labelling a curve class in $H_2(\mathcal{M}, \mathbb{Z})$.

The A-model path integral localizes on holomorphic maps

$$X : \Sigma_g \rightarrow \mathcal{M}, \quad (4.9)$$

which are classified topologically by the image of the fundamental class of the worldsheet,

$$X_*[\Sigma_g] = \sum_i d_i [C_i] \in H_2(\mathcal{M}, \mathbb{Z}). \quad (4.10)$$

The contribution of maps from a genus g Riemann surface and degree $\mathbf{d}$ is encoded in the *Gromov–Witten invariant* $N_{g,\mathbf{d}} \in \mathbb{Q}$.

Near the large-radius point $Q_i = 0$, $i = 1, \dots, b_2(\mathcal{M})$, the genus g A-model free energy admits an expansion in the variables Q_i . Separating polynomial and constant-map contributions, for a smooth compact CY threefold $\mathcal{M}$, one can write schematically

$$F_0(\mathbf{t}) = \frac{1}{6} \sum_{i,j,k} a_{ijk} t_i t_j t_k + \sum_{\mathbf{d} \neq 0} N_{0,\mathbf{d}} Q^{\mathbf{d}}, \quad (4.11a)$$

$$F_1(\mathbf{t}) = \sum_i b_i t_i + \sum_{\mathbf{d} \neq 0} N_{1,\mathbf{d}} Q^{\mathbf{d}}, \quad (4.11b)$$

$$F_g(\mathbf{t}) = c_g \chi(\mathcal{M}) + \sum_{\mathbf{d} \neq 0} N_{g,\mathbf{d}} Q^{\mathbf{d}}, \quad g \geq 2. \quad (4.11c)$$

Here a_{ijk} are the triple intersection numbers, while the coefficients b_i are related to the second Chern class $c_2(T\mathcal{M})$. The term $c_g\chi(\mathcal{M})$ is the constant-map contribution, with

$$c_g = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}, \quad g \geq 2, \quad (4.12)$$

where B_n denotes the n -th Bernoulli number. For non-compact CY threefolds the polynomial and constant-map terms are more subtle, but the structure of the large-radius expansion in curve classes remains essentially the same.

The previous expansion can be reorganized in terms of integer invariants called *Gopakumar–Vafa invariants* $n_g^{\mathbf{d}}$ [110, 111]. Then, the non-polynomial part of the all-genus free energy can be written as

$$F_{\text{GV}}(\mathbf{t}; g_s) = \sum_{g \geq 0} \sum_{\mathbf{d} \neq 0} \sum_{w=1}^{\infty} \frac{1}{w} n_g^{\mathbf{d}} \left(2 \sin \frac{wg_s}{2} \right)^{2g-2} Q^{w\mathbf{d}}, \quad (4.13)$$

where the integer w is the multicovering number. In this representation, the rational Gromov–Witten invariants are reorganized in terms of the integers $n_g^{\mathbf{d}}$, which admit a physical interpretation in terms of BPS states. In M-theory compactified on the CY threefold, these states arise from M2-branes wrapping holomorphic curves in the class $\mathbf{d}$ and appear as particles in the remaining five dimensions. They preserve part of the supersymmetry and saturate the corresponding BPS bound. The invariants $n_g^{\mathbf{d}}$ encode a protected count of these states, with the index g organizing their spin content [110, 111].

In practice, computing the A-model free energies directly from the moduli spaces of stable maps is often prohibitively complicated. For toric CY threefolds, however, the topological vertex [112] provides an efficient combinatorial formalism for computing the full topological-string partition function. In this approach, the toric diagram of the CY threefold is decomposed into elementary trivalent vertices joined by edges and one assigns a Young diagram λ_e to each internal edge. Schematically, the closed topological-string partition function takes the form

$$Z_{\text{closed}}(\mathbf{Q}; q) = \sum_{\lambda_e} \prod_{\text{edges } e} Q_e^{|\lambda_e|} \prod_{\text{vertices } v} C_{\lambda_{v,1}\lambda_{v,2}\lambda_{v,3}}(q), \quad q = e^{ig_s}, \quad (4.14)$$

where $Q_e = e^{-t_e}$ is the exponentiated Kähler parameter associated with the internal edge e and $C_{\lambda\mu\nu}(q)$ is the topological vertex, an explicit combinatorial function of three Young diagrams. The precise gluing rules also involve framing factors and, depending on the orientations of the edges, transposed Young diagrams.

Furthermore, the topological vertex admits a refinement [14] in which the single topological-string parameter $q = e^{ig_s}$ is replaced by two equivariant fugacities q_1 and q_2 . Denoting the dimensionful gauge-theory equivariant parameters by $\varepsilon_i^{\text{GT}}$, we use

$$q_i = e^{\beta\varepsilon_i^{\text{GT}}} = e^{i\varepsilon_i}, \quad \beta\varepsilon_i^{\text{GT}} = i\varepsilon_i. \quad (4.15)$$

From the M-theory perspective, compactification on the CY threefold $\mathcal{M}$ engineers a five-dimensional $\mathcal{N} = 1$ theory whose BPS particles arise, in particular, from M2-branes wrapping holomorphic curves in $\mathcal{M}$. Such particles transform under the five-dimensional massive little group

$$\text{Spin}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R, \quad (4.16)$$

and the refinement keeps track separately of the corresponding spin quantum numbers j_L and j_R . The topological string partition function $Z_{\text{closed}}(\mathbf{Q}; q)$ is then recovered from the *refined* partition function $Z_{\text{closed}}^{\text{ref}}(\mathbf{Q}; q_1, q_2)$ in the unrefined limit,

$$Z_{\text{closed}}(\mathbf{Q}; q) = Z_{\text{closed}}^{\text{ref}}(\mathbf{Q}; e^{ig_s}, e^{-ig_s}). \quad (4.17)$$

When the non-compact CY threefold $\mathcal{M}$ geometrically engineers a five-dimensional $\mathcal{N} = 1$ gauge theory [19, 113], the refined topological-string partition function is identified with the K-theoretic Nekrasov partition function (3.117) of the gauge theory on the $\Omega_{\varepsilon_1, \varepsilon_2}$ -deformed spacetime $\mathbb{C}_{\varepsilon_1, \varepsilon_2}^2 \times S_\beta^1$ [14, 114],

$$Z_{\text{closed}}^{\text{ref}}(\mathbf{Q}; q_1, q_2) \simeq \mathcal{Z}^{5\text{d}}(\mathbf{a}, \mathbf{m}; q_1, q_2; \mathbf{q}_{5\text{d}}). \quad (4.18)$$

Under this identification, the Kähler parameters $\mathbf{t}$ are mapped to the Coulomb-branch parameters $\mathbf{a}$, mass parameters $\mathbf{m}$ and instanton counting parameter $\mathbf{q}_{5\text{d}}$ of the gauge theory. The equality can involve convention-dependent classical, constant-map and decoupled factors.

The A-model can also be defined in the presence of A-branes. We restrict here to the standard case of branes wrapping Lagrangian submanifolds $\mathcal{L} \subset \mathcal{M}$ and carrying flat connections. The relevant maps are then holomorphic maps from bordered Riemann surfaces,

$$X : (\Sigma_{g,h}, \partial\Sigma_{g,h}) \longrightarrow (\mathcal{M}, \mathcal{L}), \quad X(\partial\Sigma_{g,h}) \subset \mathcal{L}, \quad (4.19)$$

where h is the number of boundary components.

The refined topological vertex formalism extends naturally to this open sector. In the toric description, the presence of Lagrangian branes is implemented by assigning suitable representations and holonomy variables to the external legs on which the branes are placed. Gluing the resulting open refined vertices produces a refined open topological-string partition function $Z_{\text{open}}^{\text{ref}}$.

When $\mathcal{M}$ geometrically engineers a five-dimensional $\mathcal{N} = 1$ gauge theory, a Lagrangian A-brane lifts in M-theory to an M5-brane wrapping [115]

$$\mathcal{L} \times \mathbb{R}^2 \times S^1 \subset \mathcal{M} \times \mathbb{R}^4 \times S^1. \quad (4.20)$$

It therefore engineers a codimension-two BPS defect, supporting a three-dimensional $\mathcal{N} = 2$ theory coupled to the five-dimensional bulk theory. The refined open topological string partition function is accordingly identified with the K-theoretic Nekrasov partition function of the coupled 5d/3d bulk-defect system,

$$Z_{\text{open}}^{\text{ref}}(\mathbf{Q}, \mathbf{z}; q_1, q_2) \simeq \mathcal{Z}^{5\text{d}/3\text{d}}(\mathbf{a}, \mathbf{m}; q_1, q_2; \mathbf{x}, \mathbf{q}_{5\text{d}}). \quad (4.21)$$

This identification is understood after the appropriate map between geometric and gauge-theory parameters and up to framing, normalization and possible decoupled factors. Depending on conventions, the full open partition function gives the coupled bulk-defect partition function, while its ratio to the closed partition function gives the normalized expectation value of the defect.

4.1.2 The B-model

Let us now turn to the B-model. In contrast with the A-model, the B-model depends only on the complex structure of the CY threefold $\mathcal{M}$ and is independent of its Kähler parameters. In this case the path integral localizes on constant maps, so the theory is governed by the geometry of the complex-structure moduli space.

At genus zero the main object is the holomorphic three-form Ω , which varies with the complex structure of $\mathcal{M}$. Let

$$\{\alpha_I, \beta^I\}, \quad I = 0, \dots, h^{2,1}(\mathcal{M}), \quad (4.22)$$

be a symplectic basis¹ of three-cycles, satisfying

$$\alpha_I \cap \beta^J = \delta_I^J, \quad \alpha_I \cap \alpha_J = \beta^I \cap \beta^J = 0. \quad (4.23)$$

We denote the corresponding periods of Ω by

$$A^I = \int_{\alpha_I} \Omega, \quad B_I = \int_{\beta^I} \Omega. \quad (4.24)$$

Special geometry implies that there exists a homogeneous prepotential $\mathcal{F}_0(A)$ such that the dual periods are obtained as

$$B_I = \frac{\partial \mathcal{F}_0}{\partial A^I}. \quad (4.25)$$

Choosing a non-vanishing period, say A^0 , one introduces projective special coordinates

$$v^i = \frac{A^i}{A^0}, \quad i = 1, \dots, h^{2,1}(\mathcal{M}), \quad (4.26)$$

and, since $\mathcal{F}_0(A)$ is homogeneous of degree two, it can be written as

$$\mathcal{F}_0(A) = (A^0)^2 F_0(\mathbf{v}), \quad (4.27)$$

where $F_0(\mathbf{v})$ is the genus-zero B-model free energy in this frame. In practice, the periods of Ω can often be determined as solutions of a system of Picard–Fuchs differential equations.

At higher genus, the B-model free energies are instead governed by the BCOV holomorphic anomaly equations [116]. These equations recursively determine the antiholomorphic dependence of F_g in terms of lower-genus data and the special geometry of the complex-structure moduli space. Thus, the higher-genus amplitudes are not globally holomorphic functions of the complex-structure moduli. In practice, one solves the anomaly equations recursively and then takes an appropriate holomorphic limit. This procedure leaves a holomorphic ambiguity at each genus, which must be fixed by imposing boundary conditions at distinguished loci in moduli space, such as large-complex-structure, conifold or orbifold points.

For non-compact CY threefolds, this general structure continues to hold, although some of the relevant cycles may be non-compact and their periods must be understood in a relative or suitably regularized sense. Moreover, some complex-structure deformations can be non-normalizable and therefore behave as external parameters rather than dynamical moduli.

The B-model also admits an open-string sector, obtained by imposing boundary conditions associated with B-branes. In their simplest geometric description, B-branes wrap complex submanifolds $S \subset \mathcal{M}$ and carry holomorphic vector bundles.

Since the B-model localizes on constant maps, its open-string amplitudes do not count holomorphic curves with boundaries, as in the open A-model. Instead, they depend on the complex structure of $\mathcal{M}$ and on the holomorphic moduli of the branes, while remaining independent of the Kähler parameters. For space-filling B-branes, the corresponding open string field theory is holomorphic Chern–Simons theory [117].

4.1.3 Mirror symmetry

In essence, mirror symmetry states that the A-model on a CY threefold $\mathcal{M}$ is equivalent to the B-model on a mirror CY threefold $\widehat{\mathcal{M}}$, and conversely. In particular, the complex-structure moduli κ of $\widehat{\mathcal{M}}$ are mapped to the complexified Kähler moduli $\mathbf{t}$ of $\mathcal{M}$, this

¹For a compact CY threefold $\mathcal{M}$, $b_3(\mathcal{M}) = 2h^{2,1}(\mathcal{M}) + 2$.

relation being called the *mirror map*. For the closed string free energies at genus g , mirror symmetry implies schematically that

$$F_g^A(\mathbf{t}(\boldsymbol{\kappa})) = F_g^B(\boldsymbol{\kappa}), \quad (4.28)$$

where $\mathbf{t}(\boldsymbol{\kappa})$ is the mirror map. Mirror symmetry therefore relates the enumerative invariants of $\mathcal{M}$, encoded in the A-model free energies, to the complex-structure geometry of its mirror $\widehat{\mathcal{M}}$, encoded in the B-model free energies. The latter can be computed recursively using the holomorphic anomaly equations or, for local CY geometries admitting a mirror-curve description, by means of topological recursion [118].

For the purposes of the TS/ST correspondence, the most important case is that of non-compact toric CY threefolds. In this setting mirror symmetry takes a particularly explicit form, known as local mirror symmetry. As we discuss further in the next subsection, the mirror of such a threefold $\mathcal{M}$ can be written as

$$uv = H_{\mathcal{M}}(X, Y; \boldsymbol{\kappa}, \boldsymbol{\xi}), \quad u, v \in \mathbb{C}, \quad X, Y \in \mathbb{C}^\times, \quad X = e^x, \quad Y = e^y, \quad (4.29)$$

where $\boldsymbol{\kappa}$ denote the normalizable (or true) complex moduli, while $\boldsymbol{\xi}$ denote non-normalizable parameters, usually referred to as mass parameters. The essential information of the mirror threefold is encoded in the algebraic curve

$$\Sigma : \quad H_{\mathcal{M}}(e^x, e^y; \boldsymbol{\kappa}, \boldsymbol{\xi}) = 0, \quad (4.30)$$

called the *mirror curve*. The periods of the holomorphic three-form reduce to periods of the logarithmic differential

$$\lambda = y \, dx, \quad (4.31)$$

on Σ , with suitable choices of logarithmic branches and integration cycles. In an appropriate symplectic basis of one-cycles $\{\alpha_i, \beta^j\}$, one obtains schematically

$$t_i = \oint_{\alpha_i} \lambda, \quad \frac{\partial F_0}{\partial t_i} = \oint_{\beta^i} \lambda. \quad (4.32)$$

Thus, the A-periods determine the mirror map while the B-periods determine the genus-zero prepotential.

When $\mathcal{M}$ geometrically engineers a five-dimensional $\mathcal{N} = 1$ gauge theory, the same curve acquires a gauge-theoretic interpretation. After an appropriate identification of parameters and, possibly, a birational change of variables, Σ is the Seiberg–Witten curve of the five-dimensional theory compactified on a circle S^1_β and λ is its Seiberg–Witten differential [119]. The normalizable moduli $\boldsymbol{\kappa}$ provide the gauge-invariant coordinates on the Coulomb branch, while their A-periods give the special coordinates. The parameters $\boldsymbol{\xi}$ encode masses, gauge couplings and other non-dynamical data. In this way, the special geometry of the gauge theory is identified with the local special geometry of the mirror CY threefold.

The mirror curve is also the object that is quantized in the TS/ST correspondence. This quantization has a physical motivation in the open-string sector. Topological branes probing the mirror geometry have partition functions that behave as wavefunctions, whose semiclassical action is determined by the differential λ in (4.31). The mirror curve then appears as the classical limit of an operator equation satisfied by the brane wavefunction. In geometrically engineered gauge theories, this quantization is associated with the Ω -deformation: in the NS limit, the surviving deformation parameter plays the role of Planck’s constant [20, 29, 115].

Thus, one promotes x and y to non-commuting operators satisfying

$$[x, y] = i\hbar, \quad (4.33)$$

and, after specifying an ordering prescription, obtains the *quantum mirror curve*

$$H_{\mathcal{M}}(e^x, e^y; \kappa, \xi) \psi(x) = 0. \quad (4.34)$$

In the x -representation, $y = -i\hbar\partial_x$, so that exponentials of y act as finite translations and the quantum curve is generically a difference equation.

The quantization of the mirror curve also deforms its periods. Considering a formal solution of the form

$$\psi(x) = \exp\left[\frac{i}{\hbar} \int^x S(x'; \hbar) dx'\right], \quad S(x; \hbar) = y(x) + \mathcal{O}(\hbar), \quad (4.35)$$

one can define the quantum differential

$$\lambda_{\hbar} = S(x; \hbar) dx. \quad (4.36)$$

Then, its quantum A-periods determine the *quantum mirror map*:

$$t_i(\kappa, \xi; \hbar) = \oint_{\alpha_i} \lambda_{\hbar}, \quad \lim_{\hbar \rightarrow 0} t_i(\kappa, \xi; \hbar) = t_i(\kappa, \xi). \quad (4.37)$$

In the standard geometric engineering setting, only the Coulomb moduli receive non-trivial quantum corrections, while the mass parameters remain undeformed, up to conventional linear redefinitions. Quantum special geometry identifies the corresponding B-periods with derivatives of the Nekrasov–Shatashvili free energy,

$$\oint_{\beta^i} \lambda_{\hbar} = \hbar \frac{\partial F_{\text{NS}}}{\partial t_i}, \quad (4.38)$$

again schematically and up to normalization conventions. Here F_{NS} is obtained from the refined topological-string partition function in the limit $\varepsilon_1 = \hbar$, $\varepsilon_2 \rightarrow 0$ [115, 120, 121].

From the viewpoint of the engineered five-dimensional gauge theory, F_{NS} is identified, up to conventional normalizations, with the effective twisted superpotential $\mathcal{W}_0$, which in turn plays the role of the Yang–Yang function of the associated quantum integrable system. As discussed in subsections 3.1.5 and 3.2, its critical-point equations can therefore be interpreted as quantization conditions for the quantum mirror curve [29, 120].

As already mentioned in subsection 3.2, however, the quantization conditions obtained directly from the NS free energy capture only the perturbative $\hbar$ -sector and do not by themselves determine the exact spectrum at finite $\hbar$. Indeed, the BPS expansion of F_{NS} contains trigonometric denominators such as $\sin(w\hbar/2)$, with $w \in \mathbb{Z}_{>0}$, and thus develops a dense set of poles on the real $\hbar$ -axis. Furthermore, these quantization conditions miss contributions that are non-perturbative in $\hbar$. The problem of constructing a pole-free, non-perturbative completion that yields well-defined exact quantization conditions and reproduces the spectrum of the quantum mirror curve is one of the starting points of the closed TS/ST correspondence [15, 16].

The gauge-theory interpretation provides an alternative way of computing the quantum mirror map. The relevant observables are half-BPS Wilson loops wrapping S^1_{β} in the $\Omega_{\varepsilon_1, \varepsilon_2}$ -deformed background $\mathbb{R}^4_{\varepsilon_1, \varepsilon_2} \times S^1_{\beta}$, in the NS limit. For suitable representations R_i , their expectation values determine the gauge-invariant moduli of the quantum curve [95, 122, 123],

$$\kappa_i \propto \langle W_{R_i} \rangle_{\text{NS}}(\mathbf{t}, \xi; \hbar), \quad (4.39)$$

up to convention-dependent normalization. Solving these relations for $\mathbf{t}$ as functions of $\boldsymbol{\kappa}$ gives precisely the quantum mirror map.

The open-string sector also has a natural interpretation in this local mirror picture. In particular, the mirrors of suitable A-model branes are B-branes supported at points of the mirror curve, so that their open string moduli can be regarded as coordinates on Σ . This is the geometric reason why mirror curves play a central role not only in the closed TS/ST correspondence, but also in its open-string generalizations.

This relation becomes even more direct after quantization. After choosing x as the position variable, the open refined topological string partition function in the NS limit defines a formal wavefunction

$$\psi_{\text{WKB}}(x, \boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar) = Z_{\text{open}}^{\text{NS}}(x; \mathbf{t}(\boldsymbol{\kappa}, \boldsymbol{\xi}, \hbar), \boldsymbol{\xi}, \hbar), \quad (4.40)$$

whose large- x , large-radius expansion has coefficients with exact dependence on $\hbar$ and which, when formally expanded around $\hbar = 0$, satisfies the quantum mirror-curve equation order by order in $\hbar$,

$$\hat{H}_{\mathcal{M}} \psi_{\text{WKB}} = 0. \quad (4.41)$$

More precisely, its formal small- $\hbar$ expansion

$$\log \psi_{\text{WKB}} \sim \sum_{n \geq 0} \hbar^{n-1} s_n(x), \quad \hbar \rightarrow 0, \quad (4.42)$$

reproduces the all-orders WKB solution associated with the quantum curve [17, 26, 115].

Although this representation retains the full $\hbar$ -dependence of its coefficients, this is not enough to produce an acceptable wavefunction for the spectral problem. Indeed, the BPS expansion of $\log Z_{\text{open}}^{\text{NS}}$ contains denominators of the form $1 - e^{iw\hbar}$, $w \in \mathbb{Z}_{>0}$, and generically develops poles at a dense set of real values of $\hbar$, with $\hbar/(2\pi) \in \mathbb{Q}$. Moreover, as a function of the open modulus x , the resulting expression can have singularities, including poles, inside the strip of analyticity required by the difference operator. It also does not generally have the decay properties required for square integrability. Consequently, reproducing the all-orders WKB expansion does not by itself guarantee that the resulting function belongs to the domain of the quantum mirror-curve operator.

One of the starting points of the open TS/ST correspondence is precisely the problem of finding a non-perturbative completion of this formal wavefunction. The additional contributions must cancel the spurious poles in both $\hbar$ and x and restore the appropriate analyticity and asymptotic behaviour. The resulting functions are expected to solve the quantum mirror-curve equation and, when the appropriate closed-string moduli lie on the spectral locus, become normalizable eigenfunctions of the corresponding operator [17, 26, 28].

4.2 The TS/ST correspondence

The topological string/spectral theory correspondence, or TS/ST correspondence, proposes a non-perturbative formulation of topological string theory on local toric Calabi–Yau threefolds in terms of the spectral theory of quantum mirror curves and, conversely, conjecturally expresses exact spectral quantities of these curves in terms of topological-string data [15–17, 26–28].

Local mirror symmetry plays a prominent role in this correspondence. Let $\mathcal{M}$ be a toric CY threefold. Its toric data can be described by a set of distinct lattice vectors

$$\nu^{(a)} = (\nu_1^{(a)}, \nu_2^{(a)}) \in \mathbb{Z}^2, \quad a = 1, \dots, n. \quad (4.43)$$

Their convex hull is the Newton polygon, or toric diagram, of the geometry. The choice of these points is part of the definition of the toric CY: different polygons give different geometries, while different triangulations of the same polygon correspond to different crepant resolutions, or phases, related by flops.

The mirror curve is obtained from the Newton polynomial associated to this polygon,

$$H_{\mathcal{M}}(X, Y) = \sum_{a=1}^n c_a X^{\nu_1^{(a)}} Y^{\nu_2^{(a)}} = 0, \quad X, Y \in \mathbb{C}^\times, \quad X = e^x, \quad Y = e^y, \quad (4.44)$$

where we suppress the dependence on the complex-structure moduli for notational convenience. Multiplying $H_{\mathcal{M}}$ by an overall constant, and rescaling X and Y , removes three coefficients, leaving only $n - 3$ independent ones.

The coefficients of the Newton polynomial naturally split into true moduli κ and mass parameters ξ . Geometrically, the true moduli are associated with the internal points of the Newton polygon. Their number is equal to the genus g of the mirror curve and they become the spectral moduli in the TS/ST correspondence. The remaining independent parameters are mass parameters, which become coupling constants of the quantum mechanical operators studied in the TS/ST correspondence. Examples of genus-one mirror curves include

$$\text{local } \mathbb{P}^2 : \quad e^x + e^y + e^{-x-y} + \kappa = 0, \quad (4.45a)$$

$$\text{local } \mathbb{P}^1 \times \mathbb{P}^1 : \quad e^y + e^{-y} + e^x + \xi e^{-x} + \kappa = 0, \quad (4.45b)$$

$$\text{local } \mathbb{F}_1 : \quad e^x + e^y + e^{-x-y} + \xi e^{-x} + \kappa = 0. \quad (4.45c)$$

Here, as usual, “local S ” denotes the total space $\text{Tot}(K_S \rightarrow S)$ of the canonical bundle over the corresponding surface S .

The starting point of the TS/ST correspondence is the canonical quantization of the mirror curve. One promotes x and y to canonically conjugate operators

$$[x, y] = i\hbar, \quad (4.46)$$

and defines the quantum mirror curve by Weyl quantization,

$$H_{\mathcal{M}}(e^x, e^y) = 0 \quad \longrightarrow \quad \mathbf{H}_{\mathcal{M}}(e^x, e^y) \psi(x) = 0. \quad (4.47)$$

As mentioned in the previous subsection, in the x -representation the quantum mirror curve is naturally a finite-difference operator. For genus-one mirror curves, there is a single true modulus κ and the quantized curve can often be written in the form

$$\mathbf{H}_{\mathcal{M}}(e^x, e^y) = \mathbf{O}_{\mathcal{M}} + \kappa. \quad (4.48)$$

Setting $\kappa = -e^E$, the equation

$$(\mathbf{O}_{\mathcal{M}} - e^E) \psi = 0, \quad (4.49)$$

takes the form of an eigenvalue problem, with E playing the role of the energy.

The operator $\mathbf{O}_{\mathcal{M}}$, densely defined on $L^2(\mathbb{R})$, is typically unbounded. However, for appropriate real values of $\hbar$ and of the mass parameters ξ , its inverse

$$\rho_{\mathcal{M}} = \mathbf{O}_{\mathcal{M}}^{-1}, \quad (4.50)$$

is expected, and in many important examples can be proven, to be a positive trace class operator on $L^2(\mathbb{R})$ [124]. Thus, $\rho_{\mathcal{M}}$ has a purely discrete positive spectrum, which we write as

$$\rho_{\mathcal{M}} \psi_n = e^{-E_n} \psi_n, \quad E_n \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (4.51)$$

The spectrum can be neatly packaged into the Fredholm determinant of $\rho_{\mathcal{M}}$,

$$\Xi_{\mathcal{M}}(\kappa, \boldsymbol{\xi}, \hbar) = \det(1 + \kappa \rho_{\mathcal{M}}) = 1 + \sum_{N \geq 1} Z_{\mathcal{M}}(N, \boldsymbol{\xi}, \hbar) \kappa^N, \quad (4.52)$$

an entire function of κ , with zeros occurring at $\kappa = -e^{E_n}$. The coefficients $Z_{\mathcal{M}}(N, \boldsymbol{\xi}, \hbar)$ are known as fermionic spectral traces, while the vanishing of the spectral determinant, $\Xi_{\mathcal{M}}(\kappa, \boldsymbol{\xi}, \hbar) = 0$, provides the exact quantization condition that replaces the naïve five-dimensional NS quantization condition.

Another central ingredient in the TS/ST correspondence is the refined closed topological-string partition function $Z_{\text{closed}}^{\text{ref}}(\boldsymbol{Q}, \boldsymbol{\xi}; q_1, q_2)$, discussed in subsection 4.1.1. It depends on the two equivariant deformation parameters ε_1 and ε_2 through $q_1 = e^{i\varepsilon_1}$ and $q_2 = e^{i\varepsilon_2}$. In the unrefined limit $\varepsilon_1 = -\varepsilon_2 = g_s$, it reduces to the ordinary closed topological-string partition function. In particular, the non-polynomial part of its logarithm is the Gopakumar–Vafa free energy introduced above

$$F_{\text{GV}}(\boldsymbol{t}, \boldsymbol{\xi}; g_s) = \log Z_{\text{GV}}(\boldsymbol{t}, \boldsymbol{\xi}; e^{ig_s}), \quad (4.53)$$

where Z_{GV} is the non-polynomial part of the refined partition function $Z_{\text{closed}}^{\text{ref}}(\boldsymbol{Q}, \boldsymbol{\xi}; q_1, q_2)$ in the unrefined limit. Furthermore, in the Nekrasov–Shatashvili limit $\varepsilon_1 = \hbar$, $\varepsilon_2 \rightarrow 0$, the refined partition function encodes the quantum special geometry of the quantum mirror curve through the NS free energy,

$$F_{\text{NS}}(\boldsymbol{t}, \boldsymbol{\xi}; \hbar) = - \lim_{\varepsilon_2 \rightarrow 0} \varepsilon_2 \log Z_{\text{closed}}^{\text{ref}}(\boldsymbol{t}, \boldsymbol{\xi}; e^{i\hbar}, e^{i\varepsilon_2}), \quad (4.54)$$

as discussed around equations (4.37) and (4.38).

For a genus- g mirror curve, quantization gives a family of g related trace-class operators and a generalized spectral determinant. The TS/ST correspondence expresses this Fredholm determinant as a generalized theta function constructed from the *modified grand potential* $J_{\mathcal{M}}$ [15, 16],

$$\Xi_{\mathcal{M}}(\boldsymbol{\kappa}, \boldsymbol{\xi}; \hbar) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp[J_{\mathcal{M}}(\boldsymbol{\mu} + 2\pi i \boldsymbol{n}, \boldsymbol{\xi}; \hbar)], \quad \kappa_i = e^{\mu_i}. \quad (4.55)$$

Here the sum over integer shifts of the chemical potentials μ_i implements the $2\pi i$ periodicity implied by the relation $\kappa_i = e^{\mu_i}$. The modified grand potential has the schematic decomposition

$$J_{\mathcal{M}} = J_{\mathcal{M}}^{\text{WKB}} + J_{\mathcal{M}}^{\text{WS}}, \quad (4.56)$$

where the WKB part is constructed from the NS free energy F_{NS} , together with the appropriate perturbative contributions. The worldsheet part is constructed from the Gopakumar–Vafa free energy F_{GV} , evaluated at the dual string coupling

$$g_s = \frac{4\pi^2}{\hbar}, \quad (4.57)$$

and at the correspondingly rescaled Kähler parameters, including the appropriate B -field shift. This combination is essential: the worldsheet contribution provides corrections that are non-perturbative in $\hbar$ from the WKB viewpoint and, after the appropriate B -field shift, cancels the poles of the NS sector. The resulting Fredholm determinant is therefore well defined for real $\hbar$ and its zero locus yields the exact spectrum of the quantum mirror curve.

The correspondence therefore relates two a priori different objects. On one side one has the spectral data of the operator $\rho_{\mathcal{M}}$, such as its Fredholm determinant, spectral traces and

eigenvalues. On the other side one has the enumerative invariants of the toric CY threefold $\mathcal{M}$, organized by the ordinary and refined topological-string free energies. The Fredholm determinant can be regarded as a completion of the closed topological-string partition function that is non-perturbative in the string coupling g_s . Conversely, topological strings provide an exact description, valid at finite $\hbar$, of the spectral quantities associated with the quantum mirror curve, including effects that are non-perturbative in $\hbar$. In this sense, spectral theory and topological strings give mutually complementary non-perturbative descriptions of one another.

The open-string sector also has a natural place in this picture. As discussed in subsection 4.1.3, in the context of local mirror symmetry, B-branes are supported at points of the mirror curve, so that the open string modulus can be identified with a coordinate on the curve. In the NS limit, the refined open topological-string partition function provides a formal solution to the quantum mirror curve regarded as a difference equation. However, this solution is not generally a well-defined function for all real $\hbar$, nor does it satisfy the analyticity and asymptotic conditions required of proper eigenfunctions of the quantum mirror-curve operator $\mathcal{O}_{\mathcal{M}}$, or equivalently of its inverse $\rho_{\mathcal{M}}$. A non-perturbative open-string completion is moreover expected to be entire in the open string modulus and, off-shell, in the closed string moduli.

Among other objectives, the open TS/ST correspondence aims to provide a general prescription for constructing such eigenfunctions in terms of open topological string quantities. This prescription should provide well-defined eigenfunctions for arbitrary real $\hbar > 0$ and, through the relation $g_s = 4\pi^2/\hbar$, a non-perturbative definition of the open topological string partition function at finite g_s .

To our knowledge, prior to the work presented in the final chapter of this thesis, a complete construction of entire off-shell eigenfunctions at generic $\hbar$ had been obtained only for local $\mathbb{F}_0$ [28], building on [27]. Earlier works treated several geometries in the maximally supersymmetric case and provided checks at generic $\hbar$ [17, 26]. In the final chapter, we give a prescription for the exact solutions associated with an infinite family of quantum mirror curves geometrically engineering five-dimensional $\mathcal{N} = 1$ $SU(N)$ gauge theories. These solutions are conjectured to be entire in the open and closed string moduli and to become square-integrable eigenfunctions when evaluated on-shell.

Concluding remarks

Having introduced the main theoretical tools used in this thesis, we now turn to their application to concrete problems. In the next chapter, we mainly focus on the Lamé equation, using its relation to two-dimensional conformal field theory and supersymmetric gauge theory to study its connection problem, and the associated periodic spectral problems. Topological string theory and the TS/ST correspondence, discussed in this final background section, will become relevant in the final chapter, where we turn to finite-difference equations obtained by quantizing mirror curves.

Chapter II

Periodic Spectral Problems and the Lamé Equation

In this chapter, we begin developing and applying methods inspired by two-dimensional conformal field theory and supersymmetric gauge theory to the study of periodic spectral problems. The necessary background on Fuchsian differential equations was reviewed in Section 1, while the relevant material on conformal blocks, Nekrasov partition functions, the AGT correspondence, and the Nekrasov–Shatashvili (NS) limit was reviewed in Sections 2 and 3.

Our primary example is the Lamé equation and the results presented here are based on [125]. In Weierstrass form, the equation reads

$$\left(\partial_z^2 + E - \mu(\mu - 1)\wp(z)\right)\psi(z) = 0.$$

A central question in the study of periodic spectral problems is how the stability of their solutions depends on the parameters of the problem, as encoded in the corresponding stability chart and band-gap structure.

To address this question, we begin by studying the global analytic behavior of solutions to the Lamé equation. For a linear differential equation, this requires solving the connection problem, namely relating bases of local solutions adapted to different singular points or regions. As discussed in Section 2, the Lamé equation arises from the classical limit of the BPZ equation satisfied by a torus conformal block containing a degenerate insertion. This relation makes it possible to translate the crossing transformations of the underlying two-dimensional CFT into connection formulas for solutions of the Lamé equation. A similar strategy was applied to the Heun equation in [25]; see also [50, Section 31.2(iv)] and the related works [11, 32, 126, 127].

Through the AGT correspondence, the same conformal blocks are related to the Nekrasov partition function of the $\mathcal{N} = 2^* \text{SU}(2)$ gauge theory. Their classical limit is mapped to the NS limit of the gauge-theory partition function, providing an explicit combinatorial construction of the Lamé accessory parameter and its associated solutions. This relation has been investigated from several complementary perspectives; see, for example, [31, 96, 128–131].

We then move to investigate the characteristic band-gap spectrum of the Lamé equation. The boundaries of the allowed bands are determined by the eigenvalues associated with periodic and anti-periodic solutions. On the CFT side, conformal blocks possess a lattice of poles in the internal momentum. In the classical limit, the classical internal momentum is related to the Floquet exponent of a basis of Floquet solutions to the Lamé equation, and the singular values of the classical block correspond to the resonant Floquet exponents

that characterize the band edges. The ordinary classical or instanton expansion therefore becomes singular precisely in the region needed to study periodic and anti-periodic solutions.

The main idea is that in the NS limit, the lattice of poles coalesces along one direction and gives rise to a branch-cut structure. Capturing this analytic behavior requires an appropriate resummation of the classical conformal block or, equivalently, of the NS effective twisted superpotential. Earlier proposals for this resummation were obtained by examining the NS function at special values of the masses for which algebraic integrability permits high-order computations [37, 38, 132, 133]. In the Mathieu case, analogous information can be extracted from high-order eigenvalue expansions obtained, for example, through continued-fraction methods.

In this chapter, we develop a systematic derivation of the effective twisted superpotential from the Nakajima–Yoshioka blow-up equations [69] and their $\mathbb{C}^2/\mathbb{Z}_2$ counterparts [134, 135], the latter having first been derived for the $\mathcal{N} = 2^*$ theory in [43]. We show that the relevant NS limits of these equations close on the effective twisted superpotential $\mathcal{W}_0$ and its first ε_2 -correction $\mathcal{W}_1$. After introducing a suitable ansatz in terms of resummation functions, we find that the blow-up equations, supplemented by appropriate boundary conditions, conjecturally determine these functions completely. The resulting expressions reproduce the functions previously inferred in [38, 133] and directly exhibit the logarithmic and square-root branch structures of the resummed classical blocks.

The construction applies both to the $\mathcal{N} = 2^*$ theory associated with the Lamé equation and to $SU(2)$ theories with $N_f \leq 4$ fundamental hypermultiplets. In particular, the cases with $1 \leq N_f \leq 4$ will be examined in greater detail in the next chapter. Remarkably, a single blow-up equation is sufficient to determine both $\mathcal{W}_0$ and $\mathcal{W}_1$. The resulting resummed effective twisted superpotential remains well controlled at the resonant values of the Floquet exponent and therefore gives direct access to the periodic and anti-periodic eigenvalues at the band edges. It also determines an infinite family of periodic energy branches as functions of the Floquet exponent, thereby encoding the complete band structure; see Figure II.2.

As independent checks, we compute the periodic and anti-periodic eigenvalues of the Lamé equation through its isomonodromic deformation problem [43], while we obtain its (anti-)periodic eigenvalues and eigenfunctions through orbifold surface-defect partition functions, following [36]. The results agree with those obtained from the resummed NS free energy. These alternative methods provide direct access to particular eigenvalues and eigenfunctions, while the resummed classical block furnishes a single analytic object encoding the spectral information across the resonant points.

Structure of the chapter. In Section 5, we study the connection problem for degenerate conformal blocks. We first derive the braiding and fusion transformations of degenerate four-point blocks on the sphere and then use these data to obtain the connection formulas for degenerate two-point blocks on the torus. Taking the classical limit yields the corresponding connection formulas for solutions of the Lamé equation.

In Section 6, after fixing the relevant AGT conventions, we take the NS limit of the $\mathbb{C}^2$ blow-up equations. We derive the resulting constraints on $\mathcal{W}_0$ and $\mathcal{W}_1$, introduce the resummation ansatz, and determine the analytic structure of its profile functions. Section 7 develops the analogous construction using the $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations. In this formulation, the resummation functions satisfy ordinary differential equations that can be integrated explicitly, reproducing the results of the $\mathbb{C}^2$ analysis.

In Section 8, we use the resummed effective twisted superpotential to study the real periodic spectral problem of the Lamé equation. We first analyze the spectrum along the A-

cycle and its band-gap structure. We then compare the resulting periodic and anti-periodic eigenvalues with those obtained from isomonodromic deformations and construct the corresponding eigenfunctions using orbifold surface defects. Finally, we use the connection formulas derived in Section 5 to study the periodic spectral problem along the B-cycle.

Appendix A collects the conventions for the special functions and Nekrasov partition functions used throughout the chapter. Appendix B presents checks based on standard spectral methods, while Appendix C gives explicit examples of the equations obtained from the blow-up relations and their solutions.

5 Connecting solutions of the Lamé equation

In this section, we solve the connection problem associated with the Lamé equation following the approach of [25]. To this end, we begin by reviewing the connection problem for degenerate four-point conformal blocks on the sphere and deriving their braiding and fusion matrices. We then consider degenerate two-point blocks on the torus and compute their connection formulas using the data obtained in the spherical case. Finally, we take the classical limit, in which the BPZ equation reduces to the Lamé equation, and show that the resulting connection formulas for the classical conformal blocks coincide with those for the solutions of the Lamé equation. See Section 2 for the relevant background on two-dimensional CFT.

5.1 Degenerate 4-point blocks on the sphere

Consider the 4-point conformal block¹ $\mathcal{F}(a_4, a_3, -b/2, a_1; x)$ on the Riemann sphere, associated to the correlator

$$\langle V_{a_4}(\infty) V_{a_3}(1) \phi_{2,1}(x, \bar{x}) V_{a_1}(0) \rangle, \quad (5.1)$$

with V_{a_i} primary fields of holomorphic conformal dimension $\Delta_i = a_i(Q - a_i)$, $Q = b + b^{-1}$, and $\phi_{2,1}$ the degenerate field of Liouville momentum $a_{2,1} = -b/2$. Due to the fusion rules of the degenerate primary:

$$[\phi_{2,1}] \times [V_p] = \sum_{\epsilon=\pm} [V_{p_\epsilon}], \quad (5.2)$$

with $p_\epsilon = p - \epsilon b/2$, the space of such conformal blocks is two dimensional and we can denote a basis by $\mathcal{F}_\epsilon^{(\sigma)}(\{a_i\}; x)$, where σ specifies the choice of OPE channel, while ϵ specifies the choice of intermediate momentum. The block $\mathcal{F}_\epsilon^{(\sigma)}(\{a_i\}; x)$ satisfies the following hypergeometric BPZ equation:

$$\left(b^{-2} \frac{d^2}{dx^2} + \frac{2x-1}{x(1-x)} \frac{d}{dx} + \frac{\Delta_1}{x^2} + \frac{\Delta_3}{(1-x)^2} + \frac{\Delta_1 + \Delta_2 + \Delta_3 - \Delta_4}{x(1-x)} \right) \mathcal{F}(\{a_i\}; x) = 0, \quad (5.3)$$

which can be put in standard form by the ansatz $\mathcal{F}(\{a_i\}; x) = x^{ba_1} (1-x)^{ba_3} F(\{a_i\}; x)$. By considering power series solutions near the point $x = 0$, one finds a basis of s -channel conformal blocks:

$$\mathcal{F}_+^{(s)}(\{a_i\}; x) = x^{ba_1} (1-x)^{ba_3} {}_2F_1(\chi_1, \chi_2, \chi_3; x), \quad (5.4a)$$

$$\mathcal{F}_-^{(s)}(\{a_i\}; x) = x^{b(Q-a_1)} (1-x)^{ba_3} {}_2F_1(\chi_1 - \chi_3 + 1, \chi_2 - \chi_3 + 1, 2 - \chi_3; x), \quad (5.4b)$$

¹The Liouville momenta, denoted by uppercase Roman letters such as P_i and A in Section 2, are denoted here by lowercase Roman letters such as a_i , m and a . Unless otherwise stated, all parameters are assumed to be complex.

where

$$\chi_1 = b(a_1 + a_3 + a_4 - 3b/2) - 1, \quad (5.5a)$$

$$\chi_2 = b(a_1 + a_3 - a_4 - b/2), \quad (5.5b)$$

$$\chi_3 = b(2a_1 - b). \quad (5.5c)$$

Similarly, by looking for solutions near $x = 1$ one finds a basis of t -channel conformal blocks:

$$\mathcal{F}_+^{(t)}(\{a_i\}; x) = x^{ba_1}(1-x)^{ba_3} {}_2F_1(\chi_1, \chi_2, \chi_1 + \chi_2 + 1 - \chi_3; 1-x), \quad (5.6a)$$

$$\mathcal{F}_-^{(t)}(\{a_i\}; x) = x^{ba_1}(1-x)^{b(Q-a_3)} {}_2F_1(\chi_3 - \chi_1, \chi_3 - \chi_2, 1 + \chi_3 - \chi_1 - \chi_2; 1-x), \quad (5.6b)$$

and finally, near the point at infinity, one finds a basis of u -channel conformal blocks:

$$\mathcal{F}_+^{(u)}(\{a_i\}; x) = e^{-i\pi ba_3} x^{ba_1 - \chi_2} (1-x)^{ba_3} {}_2F_1(\chi_2, \chi_2 - \chi_3 + 1, \chi_2 - \chi_1 + 1; x^{-1}), \quad (5.7a)$$

$$\mathcal{F}_-^{(u)}(\{a_i\}; x) = e^{-i\pi ba_3} x^{ba_1 - \chi_1} (1-x)^{ba_3} {}_2F_1(\chi_1, \chi_1 - \chi_3 + 1, \chi_1 - \chi_2 + 1; x^{-1}), \quad (5.7b)$$

where the normalization has been chosen appropriately for the study of conformal blocks on the torus. Indeed, we want to identify solutions near the points $x = 0$ and $x = \infty$ by matching their local monodromies. With the previous choice we can see that, when $a_4 = a_1 = \alpha/b$ and we consider the classical limit $b \rightarrow 0$, the behavior of these local solutions around their respective expansion points matches:

$$\begin{aligned} \mathcal{F}_+^{(s)} &= x^\alpha (1 + \mathcal{O}(x)), & \mathcal{F}_-^{(s)} &= x^{1-\alpha} (1 + \mathcal{O}(x)), \\ \mathcal{F}_+^{(u)} &= x^\alpha (1 + \mathcal{O}(x^{-1})), & \mathcal{F}_-^{(u)} &= x^{1-\alpha} (1 + \mathcal{O}(x^{-1})). \end{aligned}$$

Due to their explicit expressions in terms of hypergeometric functions, we can obtain connection formulas between different local bases of solutions, see e.g. [136] and [50] (Section 15.10). Namely, we find the fusion matrix:

$$\mathcal{F}_\epsilon^{(s)}(\{a_i\}; x) = \sum_{\epsilon'=\pm} \mathbb{F}_{\epsilon\epsilon'} \begin{bmatrix} a_3 & -b/2 \\ a_4 & a_1 \end{bmatrix} \mathcal{F}_{\epsilon'}^{(t)}(\{a_i\}; x), \quad (5.9)$$

expressing the following change of local basis:

$$\begin{array}{c} a_3 \quad -b/2 \\ | \quad | \\ a_4 \quad a_{1\epsilon} \quad a_1 \end{array} = \sum_{\epsilon'=\pm} \mathbb{F}_{\epsilon\epsilon'} \begin{array}{c} a_3 \quad -b/2 \\ \diagdown \quad \diagup \\ a_4 \quad a_{3\epsilon'} \quad a_1 \end{array}, \quad (5.10)$$

and the braiding matrix:

$$\mathcal{F}_\epsilon^{(s)}(\{a_i\}; x) = \sum_{\epsilon'=\pm} \mathbb{B}_{\epsilon\epsilon'} \begin{bmatrix} a_3 & -b/2 \\ a_4 & a_1 \end{bmatrix} \mathcal{F}_{\epsilon'}^{(u)}(\{a_i\}; x), \quad (5.11)$$

expressing the change between the s -channel and u -channel bases:

$$\begin{array}{c} a_3 \quad -b/2 \\ | \quad | \\ a_4 \quad a_{1\epsilon} \quad a_1 \end{array} = \sum_{\epsilon'=\pm} \mathbb{B}_{\epsilon\epsilon'} \begin{array}{c} -b/2 \quad a_3 \\ | \quad | \\ a_4 \quad a_{4(-\epsilon')} \quad a_1 \end{array}. \quad (5.12)$$

These two connection matrices read respectively:

$$\mathbb{F} \begin{bmatrix} a_3 & -b/2 \\ a_4 & a_1 \end{bmatrix} = \begin{pmatrix} \frac{\Gamma(\chi_3)\Gamma(\chi_3-\chi_1-\chi_2)}{\Gamma(\chi_3-\chi_1)\Gamma(\chi_3-\chi_2)} & \frac{\Gamma(\chi_3)\Gamma(\chi_1+\chi_2-\chi_3)}{\Gamma(\chi_1)\Gamma(\chi_2)} \\ \frac{\Gamma(2-\chi_3)\Gamma(\chi_3-\chi_1-\chi_2)}{\Gamma(1-\chi_1)\Gamma(1-\chi_2)} & \frac{\Gamma(2-\chi_3)\Gamma(\chi_1+\chi_2-\chi_3)}{\Gamma(1+\chi_1-\chi_3)\Gamma(1+\chi_2-\chi_3)} \end{pmatrix}, \quad (5.13a)$$

$$\mathbb{B} \begin{bmatrix} a_3 & -b/2 \\ a_4 & a_1 \end{bmatrix} = \begin{pmatrix} \frac{e^{i\pi(ba_3-\chi_2)}\Gamma(\chi_3)\Gamma(\chi_1-\chi_2)}{\Gamma(\chi_3-\chi_2)\Gamma(\chi_1)} & \frac{e^{i\pi(ba_3-\chi_1)}\Gamma(\chi_3)\Gamma(\chi_2-\chi_1)}{\Gamma(\chi_3-\chi_1)\Gamma(\chi_2)} \\ \frac{e^{i\pi(ba_3-\chi_2+\chi_3-1)}\Gamma(2-\chi_3)\Gamma(\chi_1-\chi_2)}{\Gamma(1-\chi_2)\Gamma(\chi_1-\chi_3+1)} & \frac{e^{i\pi(ba_3-\chi_1+\chi_3-1)}\Gamma(2-\chi_3)\Gamma(\chi_2-\chi_1)}{\Gamma(1-\chi_1)\Gamma(\chi_2-\chi_3+1)} \end{pmatrix}. \quad (5.13b)$$

5.2 Degenerate 2-point blocks on the torus

Consider the 2-point conformal block on the torus $\Phi(z, a, m; \mathbf{q})$ [62, 63, 137, 138], associated with the torus correlator

$$\text{Tr}_{\mathcal{H}} \left(\mathbf{q}^{L_0 - \frac{c}{24}} \bar{\mathbf{q}}^{\bar{L}_0 - \frac{c}{24}} V_m(0) \phi_{2,1}(z, \bar{z}) \right). \quad (5.14)$$

Here z is a coordinate on the universal cover of the complex torus, $\mathbf{q} = e^{2\pi i \tau}$, $\bar{\mathbf{q}} = e^{-2\pi i \bar{\tau}}$ the modular parameters, $L_0, \bar{L}_0$ the diagonal Virasoro generators, the central charge is $c = 1 + 6Q^2$ and finally the trace is taken over the Hilbert space of the theory. The internal representation associated with the degenerate block $\Phi(z, a, m; \mathbf{q})$ has corresponding internal Liouville momentum a , and is generated by the primary operator V_a . Precisely as before, due to the fusion rules (5.2) of the degenerate primary $\phi_{2,1}$, the space of such conformal blocks is two dimensional and we can denote a basis by $\Phi_{\epsilon}^{(\sigma)}(z, a, m; \mathbf{q})$, where σ specifies the choice of OPE channel and $\epsilon = \pm$ specifies the choice of intermediate momentum. Making the change of variable² $y = e^{2\pi i z}$ and taking the OPE between the degenerate field and V_a , we find for the corresponding necklace channel basis of blocks:

$$\Phi_{\epsilon}^{(n)}(z, a, m; \mathbf{q}) = y^{\Delta_{a\epsilon} - \Delta_a} \hat{\Phi}_{\epsilon}^{(n)}(y, a, m; \mathbf{q}), \quad (5.15)$$

where $\hat{\Phi}_{\epsilon}^{(n)}$ is single-valued on the y -annulus and admits a Laurent expansion in this variable. Thus, the necklace-channel blocks $\Phi_{\epsilon}^{(n)}$ are quasi-periodic functions along the A-cycle of the torus:

$$\Phi_{\epsilon}^{(n)}(z + 1, a, m; \mathbf{q}) = e^{2\pi i(\Delta_{a\epsilon} - \Delta_a)} \Phi_{\epsilon}^{(n)}(z, a, m; \mathbf{q}). \quad (5.16)$$

Meanwhile, the B-cycle monodromy matrix is non-diagonal and can be computed via the Moore-Seiberg formalism [61, 137] by knowing the braiding matrix on the sphere (5.13b), see also Section 2. Indeed, performing the following sequence of transformations:

$$= \sum_{\epsilon' = \pm} \mathbb{B}_{\epsilon\epsilon'}^{-1} \quad , \quad (5.17)$$

we find:

$$\Phi_{\epsilon}^{(n)}(z + \tau, a, m; \mathbf{q}) = \sum_{\epsilon' = \pm} M_{B, \epsilon\epsilon'}^{(n)} \Phi_{\epsilon'}^{(n)}(z, a, m; \mathbf{q}), \quad (5.18)$$

²This change of variable maps the torus to an annulus with boundary circles identified, and introduces a factor of $y^{\Delta_{2,1}}$ which contributes to the monodromy of the block.

where the B-cycle monodromy matrix is:

$$M_{B, \epsilon \epsilon'}^{(n)} = \mathbb{B}_{\epsilon \epsilon'}^{-1} \begin{bmatrix} m & -b/2 \\ a_\epsilon & a_\epsilon \end{bmatrix} e^{-\epsilon \frac{b}{2} \partial_a}, \quad (5.19)$$

and the shift operator acts by $e^{k\partial_a} f(a) = f(a+k)$. Taking instead the OPE between V_m and $\phi_{2,1}$, and going to the annulus $y = e^{2\pi i z}$, one finds for the OPE channel basis:

$$\Phi_\epsilon^{(o)}(z, a, m; \mathbf{q}) = (1-y)^{\Delta_{m_\epsilon} - \Delta_m - \Delta_{2,1}} \mathcal{F}(a, m_\epsilon; \mathbf{q}) (1 + \mathcal{O}(1-y)). \quad (5.20)$$

Thus, the blocks are diagonal with respect to the analytic continuation around the $z=0$ point:

$$\Phi_\epsilon^{(o)}(e^{2\pi i} z, a, m; \mathbf{q}) = e^{2\pi i(\Delta_{m_\epsilon} - \Delta_m - \Delta_{2,1})} \Phi_\epsilon^{(o)}(z, a, m; \mathbf{q}). \quad (5.21)$$

We can again compute their A-cycle and B-cycle monodromy matrices by knowing the connection matrices on the sphere. Indeed, we can write the OPE channel blocks in terms of the necklace ones, of which we now know the monodromy properties:

$$= \sum_{\epsilon' = \pm} \mathbb{F}_{\epsilon \epsilon'}^{-1} \quad (5.22)$$

Thus, we find:

$$\Phi_\epsilon^{(o)}(z+1, a, m; \mathbf{q}) = \sum_{\epsilon'' = \pm} M_{A, \epsilon \epsilon''}^{(o)} \Phi_{\epsilon''}^{(o)}(z, a, m; \mathbf{q}), \quad (5.23a)$$

$$\Phi_\epsilon^{(o)}(z+\tau, a, m; \mathbf{q}) = \sum_{\epsilon''' = \pm} M_{B, \epsilon \epsilon'''}^{(o)} \Phi_{\epsilon'''}^{(o)}(z, a, m; \mathbf{q}), \quad (5.23b)$$

where

$$M_{A, \epsilon \epsilon''}^{(o)} = \sum_{\epsilon' = \pm} \mathbb{F}_{\epsilon \epsilon'}^{-1} \begin{bmatrix} m & -b/2 \\ a & a \end{bmatrix} e^{2\pi i(\Delta_{a_{\epsilon'}} - \Delta_a)} \mathbb{F}_{\epsilon' \epsilon''} \begin{bmatrix} m & -b/2 \\ a & a \end{bmatrix}, \quad (5.24a)$$

$$M_{B, \epsilon \epsilon'''}^{(o)} = \sum_{\epsilon', \epsilon'' = \pm} \mathbb{F}_{\epsilon \epsilon'}^{-1} \begin{bmatrix} m & -b/2 \\ a & a \end{bmatrix} M_{B, \epsilon' \epsilon''}^{(n)} \mathbb{F}_{\epsilon'' \epsilon'''} \begin{bmatrix} m & -b/2 \\ a & a \end{bmatrix}, \quad (5.24b)$$

and the shift operator $e^{k\partial_a}$ in $M_B^{(n)}$ acts on everything on its right. By going to the OPE channel basis (5.21), we can also compute the monodromy matrix for the analytic continuation of necklace channel blocks around the point $z=0$, which is given by:

$$M_{0, \epsilon \epsilon''}^{(n)} = \sum_{\epsilon' = \pm} \mathbb{F}_{\epsilon \epsilon'} \begin{bmatrix} m & -b/2 \\ a & a \end{bmatrix} e^{2\pi i(\Delta_{m_{\epsilon'}} - \Delta_m - \Delta_{2,1})} \mathbb{F}_{\epsilon' \epsilon''}^{-1} \begin{bmatrix} m & -b/2 \\ a & a \end{bmatrix}. \quad (5.25)$$

Similarly, by decomposing the computation in terms of the elementary fusion and braiding matrices, one can obtain the monodromy matrix for the analytic continuation of any n-point block, with the insertion of at least one degenerate $\phi_{2,1}$, around any point of insertion of other primaries and around the two non-trivial cycles of the torus.

5.3 The Lamé and the Mathieu equations

The torus conformal block $\Phi_\epsilon^{(\sigma)}(z, a, m; \mathbf{q})$ satisfies the BPZ equation:

$$\left(-b^{-2}\partial_z^2 + \zeta_*(z)\partial_z - \Delta_m \wp_*(z) - 2\Delta_{2,1}\eta_1 - 2\pi i\partial_\tau\right)\Phi(z, a, m; \mathbf{q}) = 0, \quad (5.26)$$

where $\zeta_*(z) = \zeta(z; 1, \tau) - 2\eta_1 z$ and $\wp_*(z) = \wp(z; 1, \tau) + 2\eta_1$ [31], $\wp$ and ζ being the Weierstrass elliptic functions, see Appendix A for the conventions adopted. In the classical limit $b \rightarrow 0$ [9, 131], with $a = \alpha/b$ and $m = \mu/b$, the leading exponential behavior of the degenerate two-point torus blocks is independent of the choice of channel:

$$\lim_{b \rightarrow 0} b^2 \log \Phi_\epsilon^{(\sigma)}\left(z, \frac{\alpha}{b}, \frac{\mu}{b}; \mathbf{q}\right) = f(\alpha, \mu; \mathbf{q}), \quad (5.27)$$

where

$$f(\alpha, \mu; \mathbf{q}) = \lim_{b \rightarrow 0} b^2 \log \mathcal{F}\left(\frac{\alpha}{b}, \frac{\mu}{b}; \mathbf{q}\right), \quad (5.28)$$

is the classical one-point torus conformal block. The dependence on z , together with possible channel-dependent and z -independent normalization factors, appears at order b^0 in the classical limit. We therefore define the normalized classical two-point blocks by³

$$\psi_\epsilon^{(\sigma)}(z, \alpha, \mu; \mathbf{q}) = \lim_{b \rightarrow 0} \frac{\Phi_\epsilon^{(\sigma)}(z, \alpha/b, \mu/b; \mathbf{q})}{\mathcal{F}(\alpha/b, \mu/b; \mathbf{q})}. \quad (5.29)$$

In this limit, equation (5.26) reduces to the Lamé equation in Weierstrass form:

$$\left(\partial_z^2 + E - \mu(\mu - 1)\wp(z)\right)\psi(z, \alpha, \mu; \mathbf{q}) = 0, \quad (5.30)$$

and the eigenvalue E is given in terms of the classical 1-point block as:

$$E = -2\mu(\mu - 1)\eta_1 - 4\pi^2 \mathbf{q} \partial_{\mathbf{q}} f(\alpha, \mu; \mathbf{q}). \quad (5.31)$$

To obtain connection formulas for the analytic continuation of solutions to the Lamé equation, we can simply take the classical limit of the monodromy matrices (5.19), (5.24) and (5.25) derived in the previous section. In this limit, the necklace-channel blocks (5.15) descend to a basis of Floquet solutions of the Lamé equation with Floquet exponents $\pm\nu$, $\nu = 2\alpha - 1$, as follows from their quasi-periodicity along the A-cycle:

$$\psi_\epsilon^{(n)}(z + 1, \alpha, \mu; \mathbf{q}) = e^{2\pi i \epsilon(\alpha - 1/2)} \psi_\epsilon^{(n)}(z, \alpha, \mu; \mathbf{q}). \quad (5.32)$$

When $\tau \in i\mathbb{R}_{>0}$ and $\mu \in \mathbb{R}$, restricting the equation to the horizontal line $z = x + \tau/2$, with $x \in \mathbb{R}$, gives a nonsingular real periodic spectral problem, which we will study in Section 8. In terms of ν , the $\mathbf{q}$ -expansion of the eigenvalue (5.31) takes the form

$$E/\pi^2 = \nu^2 - \frac{1}{3}\mu(\mu - 1) + \frac{8\mu(\mu - 1)(\nu^2 - 1 + \mu(\mu - 1))}{\nu^2 - 1}\mathbf{q} + \mathcal{O}(\mathbf{q}^2), \quad (5.33)$$

which can be easily verified by standard methods, see appendix B.1.

Let us now consider the B-cycle monodromy matrix (5.19). The shift operator $e^{-\epsilon \frac{b}{2} \partial_a}$ acts on the internal momentum as $a \rightarrow a - \epsilon \frac{b}{2}$, which in terms of the classical momentum

³Due to the AGT correspondence [22], the one-point torus block, the two-point torus in the necklace channel, and thus their classical limits, can be computed via simple combinatorial formulas, see equations (6.11), (A.31) and (A.37).

$\alpha = ba$ becomes $\alpha \rightarrow \alpha - \epsilon \frac{b^2}{2}$. Thus, using the definition of the normalized classical two-point block (5.29), we find

$$\begin{aligned} \lim_{b \rightarrow 0} \frac{\Phi_{\epsilon'}^{(n)}(z, a - \epsilon b/2, m; \mathbf{q})}{\mathcal{F}(a, m; \mathbf{q})} &= \lim_{b \rightarrow 0} \frac{\mathcal{F}(a - \epsilon b/2, m; \mathbf{q})}{\mathcal{F}(a, m; \mathbf{q})} \frac{\Phi_{\epsilon'}^{(n)}(z, a - \epsilon b/2, m; \mathbf{q})}{\mathcal{F}(a - \epsilon b/2, m; \mathbf{q})} \\ &= e^{-\frac{\epsilon}{2} \partial_\alpha f(\alpha, \mu; \mathbf{q})} \psi_{\epsilon'}^{(n)}(z, \alpha, \mu; \mathbf{q}). \end{aligned} \quad (5.34)$$

By taking the classical limit of the relation (5.18), normalizing by the one-point torus block $\mathcal{F}(a, m; \mathbf{q})$, we obtain

$$\psi_\epsilon^{(n)}(z + \tau, \alpha, \mu; \mathbf{q}) = \sum_{\epsilon' = \pm} M_{B, \epsilon \epsilon'}^{(n)} \psi_{\epsilon'}^{(n)}(z, \alpha, \mu; \mathbf{q}), \quad (5.35)$$

with the B-cycle monodromy matrix $M_B^{(n)}$ given by

$$M_B^{(n)} = \begin{pmatrix} e^{-\frac{1}{2} \frac{\partial f}{\partial \alpha}} \frac{\Gamma(2 - 2\alpha) \Gamma(1 - 2\alpha)}{\Gamma(2 - 2\alpha - \mu) \Gamma(1 - 2\alpha + \mu)} & e^{-\frac{1}{2} \frac{\partial f}{\partial \alpha}} (-i + \cot 2\pi\alpha) \sin \pi\mu \\ -e^{\frac{1}{2} \frac{\partial f}{\partial \alpha}} (i + \cot 2\pi\alpha) \sin \pi\mu & e^{\frac{1}{2} \frac{\partial f}{\partial \alpha}} \frac{\Gamma(2\alpha) \Gamma(2\alpha - 1)}{\Gamma(2\alpha - \mu) \Gamma(2\alpha + \mu - 1)} \end{pmatrix}. \quad (5.36)$$

Meanwhile, for generic values of μ , the OPE-channel blocks (5.20) descend to a basis of Frobenius solutions near $z = 0$. Since the leading OPE coefficient of $\Phi_\epsilon^{(o)}(z, a, m; \mathbf{q})$ is the one-point block $\mathcal{F}(a, m - \epsilon b/2; \mathbf{q})$, carefully taking the classical limit in the normalized solutions (5.29) gives

$$\psi_\epsilon^{(o)}(z, \alpha, \mu; \mathbf{q}) = e^{-\frac{\epsilon}{2} \partial_\mu f(\alpha, \mu; \mathbf{q})} (-2\pi i z)^{\frac{1}{2} + \epsilon(\mu - \frac{1}{2})} (1 + \mathcal{O}(z)), \quad (5.37)$$

where the $-2\pi i$ follows from $1 - e^{2\pi i z} = -2\pi i z + \mathcal{O}(z^2)$ in (5.20). The local monodromy of these solutions is then

$$\psi_\epsilon^{(o)}(e^{2\pi i} z, \alpha, \mu; \mathbf{q}) = e^{2\pi i \epsilon \mu} \psi_\epsilon^{(o)}(z, \alpha, \mu; \mathbf{q}). \quad (5.38)$$

All other connection matrices can be obtained in a similar way⁴. As a consistency check, one may compute the group commutator

$$M_A M_B M_A^{-1} M_B^{-1}, \quad (5.39)$$

which belongs to the same conjugacy class as M_0^{-1} , as required by the fundamental-group relation on the once-punctured torus. Since $M_0 \in \text{SL}(2; \mathbb{C})$, the matrices M_0^{-1} and M_0 themselves belong to the same conjugacy class.

Turning to the ordinary matrix commutator of the necklace-channel A-cycle and B-cycle monodromy matrices, for generic values of the Floquet exponent ν , one finds

$$[M_A, M_B] = 0 \quad \Longleftrightarrow \quad \mu \in \mathbb{Z}. \quad (5.40)$$

The vanishing of the commutator is independent of the choice of basis, therefore we find that integrality of μ is a necessary condition for the existence of elliptic solutions of the Lamé equation. It is not sufficient, since the accessory parameter E must satisfy additional conditions.

⁴The connection formulas for the Lamé equation can be obtained also via the probabilistic construction of conformal blocks [139].

For $\mu \in \mathbb{Z}$ and generic values $\nu \notin \mathbb{Z}$ of the Floquet exponent, the eigenvalues $e^{\pm\pi i\nu}$ of M_A are distinct. Since M_A and M_B commute, M_B preserves the one-dimensional eigenspaces of M_A and the two monodromies can therefore be simultaneously diagonalized, being semisimple. At discrete values of E , some of the resulting common eigenfunctions become periodic or anti-periodic and reduce to the classical Lamé polynomials [47].

By a further limit, it is known that the Lamé equation degenerates into the Mathieu equation. Using the representation (A.3) of the Weierstrass $\wp$ function in equation (5.30), setting $z = x + \tau/2$, and considering the limit $\mathfrak{q} \rightarrow 0$, $\mu \rightarrow \infty$ with $t = \mathfrak{q}\mu^4$ fixed, one finds:

$$\left(\partial_x^2 + \lambda + 8\pi^2\sqrt{t}\cos 2\pi x\right)\psi(x, \alpha; t) = 0, \quad (5.41)$$

where the eigenvalue λ is obtained from equation (5.31) as the limit:

$$\lambda = \left[E + 2\mu(\mu - 1)\eta_1\right]_{\mathfrak{q} \rightarrow 0, \mu \rightarrow \infty} = -4\pi^2 t \partial_t f(\alpha; t), \quad (5.42)$$

and similarly for the eigenfunctions. The function $f(\alpha; t)$ corresponds to the classical irregular block on the sphere [41]. In terms of the Floquet exponent $\nu = 2\alpha - 1$, the first few terms of the small t expansion of the eigenvalue read⁵:

$$\lambda/\pi^2 = \nu^2 + \frac{8t}{(\nu^2 - 1)} + \frac{8(7 + 5\nu^2)t^2}{(\nu^2 - 1)^3(\nu^2 - 4)} + \mathcal{O}(t^3), \quad (5.43)$$

which again can be verified via standard methods.

6 Resummation via $\mathbb{C}^2$ blow-up equations

In this section we investigate the analytic structure of classical conformal blocks as functions of the classical intermediate momentum α or better, the Floquet exponent $\nu = 2\alpha - 1$. Since in the following we will discuss the $\mathbb{C}^2$ and the $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations, following the conventions of [43, 140], we briefly review the dictionary of the AGT correspondence [22], leaving definitions to Appendix A.2. The Nekrasov partition function [1, 2] of the $SU(2)$ gauge theory with N_f fundamentals of masses $\boldsymbol{\mu} = \{\mu_i\}_{i=1}^{N_f}$ is the product of a classical, a 1-loop and an instanton part:

$$\mathcal{Z}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = \mathcal{Z}_{\text{cl}}^{[N_f]}(a; \varepsilon_1, \varepsilon_2; t) \mathcal{Z}_{1\text{-loop}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2) \mathcal{Z}_{\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t), \quad (6.1)$$

which depend on the vev of the adjoint scalar a , the two equivariant parameters $\varepsilon_1, \varepsilon_2$ and the coupling constant t . The AGT correspondence relates the $N_f = 4$ Nekrasov partition function to the 4-point conformal block on the sphere by⁶

$$\begin{aligned} \mathcal{F}(\tilde{a}; \alpha_0, m_0, m_1, \alpha_1; t) &= (1 - t)^{-2m_0(Q - m_1)} t^{-m_1(Q - m_1) - \alpha_1(Q - \alpha_1)} \\ &\cdot \mathcal{Z}_{\text{cl}}^{[4]}(a; \varepsilon_1, \varepsilon_2; t) \mathcal{Z}_{\text{inst}}^{[4]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t), \end{aligned} \quad (6.2)$$

where $\tilde{a}, Q$ are related to the gauge theory parameters by

$$\tilde{a} = \frac{a}{\sqrt{\varepsilon_1 \varepsilon_2}} + \frac{Q}{2}, \quad Q = b + b^{-1}, \quad b^2 = \frac{\varepsilon_2}{\varepsilon_1}, \quad (6.3)$$

⁵In order to compare with the expansion presented in [50], Section 28.15, one must identify $t = \frac{1}{16}q^2$.

⁶The ordering of the arguments of the four-point block follows the arrangement of the operator insertions in (5.1), while $\tilde{a}$ denotes the internal Liouville momentum.

while for the remaining ones the dictionary reads:

$$\begin{aligned}\alpha_0 &= \frac{1}{2} \left(Q + \frac{\mu_1 - \mu_2}{\sqrt{\varepsilon_1 \varepsilon_2}} \right), & m_0 &= \frac{1}{2} \left(Q + \frac{\mu_1 + \mu_2}{\sqrt{\varepsilon_1 \varepsilon_2}} \right), \\ \alpha_1 &= \frac{1}{2} \left(Q - \frac{\mu_3 - \mu_4}{\sqrt{\varepsilon_1 \varepsilon_2}} \right), & m_1 &= \frac{1}{2} \left(Q - \frac{\mu_3 + \mu_4}{\sqrt{\varepsilon_1 \varepsilon_2}} \right).\end{aligned}\tag{6.4}$$

The partition function can be equivalently written as:

$$\mathcal{Z}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = \exp \left(F^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) \right),\tag{6.5}$$

where the prepotential:

$$F^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = F_{\text{cl}}^{[N_f]}(a; \varepsilon_1, \varepsilon_2; t) + F_{1\text{-loop}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2) + F_{\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t),\tag{6.6}$$

admits an expansion in powers of ε_2 at fixed ε_1 . In particular, writing:

$$F^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) \simeq \sum_{n=0}^{\infty} \mathcal{W}_n^{[N_f]}(a, \boldsymbol{\mu}, \varepsilon_1; t) \varepsilon_2^{n-1},\tag{6.7}$$

and denoting the classical 4-point block on the sphere by $f^{[4]}(\nu, \hat{\boldsymbol{\mu}}; t)$, the latter can be expressed in terms of the NS prepotential [29] $\mathcal{W}_0^{[4]}(a, \boldsymbol{\mu}, \varepsilon_1; t)$ as

$$\varepsilon_1 f^{[4]}(\nu, \hat{\boldsymbol{\mu}}; t) = -\frac{\beta_1}{2\varepsilon_1} \log(1-t) - \frac{\beta_2}{2\varepsilon_1} \log t + \mathcal{W}_{0,\text{cl}}^{[4]} \left(\frac{\varepsilon_1 \nu}{2}, \varepsilon_1; t \right) + \mathcal{W}_{0,\text{inst}}^{[4]} \left(\frac{\varepsilon_1 \nu}{2}, \boldsymbol{\mu}, \varepsilon_1; t \right),\tag{6.8}$$

where

$$\beta_1 = (\varepsilon_1 + \mu_1 + \mu_2)(\varepsilon_1 + \mu_3 + \mu_4), \quad \beta_2 = \varepsilon_1^2 - \mu_3^2 - \mu_4^2,\tag{6.9a}$$

$$\hat{\mu}_i = \frac{\mu_i}{\varepsilon_1}, \quad \nu = \frac{2a}{\varepsilon_1}.\tag{6.9b}$$

The remaining equivariant parameter ε_1 is then identified with the reduced Planck constant $\varepsilon_1 = \hbar$ of the relevant quantum mechanical problem. Then, from the $N_f = 4$ instanton partition function we can obtain the $N_f < 4$ ones by decoupling the appropriate number of hypermultiplets. This is achieved by sending their masses to infinity and the coupling constant t to zero, while keeping their product fixed, in the 1-loop and instanton parts, for example:

$$\mathcal{Z}_{\text{inst}}^{[N_f < 4]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = \lim_{\mu_{N_f+1} \rightarrow \infty} \cdots \lim_{\mu_4 \rightarrow \infty} \mathcal{Z}_{\text{inst}}^{[4]} \left(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; \frac{t}{\prod_{i=N_f+1}^4 \mu_i} \right).\tag{6.10}$$

At the level of the classical 4-point conformal block $f^{[4]}(\nu, \hat{\boldsymbol{\mu}}; t)$, these decoupling limits result in the irregular classical blocks $f^{[N_f < 4]}(\nu, \hat{\boldsymbol{\mu}}; t \varepsilon_1^{N_f-4})$ [41]. Moving to the torus, the Nekrasov partition function $\mathcal{Z}(a, m; \varepsilon_1, \varepsilon_2; \mathfrak{q})$ of the $\mathcal{N} = 2^*$ SU(2) gauge theory depends instead on the mass of the adjoint hypermultiplet m and the coupling constant $\mathfrak{q} = e^{2\pi i \tau}$. Via the AGT correspondence, the 1-point torus conformal block is identified with the Nekrasov instanton partition function as:

$$\mathcal{F}(\tilde{a}, m/\sqrt{\varepsilon_1 \varepsilon_2}; \mathfrak{q}) = \mathfrak{q}^{-1/24} \mathcal{Z}_{\text{cl}}(a; \varepsilon_1, \varepsilon_2; \mathfrak{q}) \mathcal{Z}_{\text{inst}}(a, m; \varepsilon_1, \varepsilon_2; \mathfrak{q}),\tag{6.11}$$

where $\tilde{a}$ and Q are as in (6.3). Thus, the NS prepotential $\mathcal{W}_0(a, m, \varepsilon_1; \mathbf{q})$ is related to the classical conformal block $f(\nu, \hat{m}; \mathbf{q})$ by:

$$\varepsilon_1 f(\nu, \hat{m}; \mathbf{q}) = \mathcal{W}_{0, \text{cl}}\left(\frac{\varepsilon_1 \nu}{2}, \varepsilon_1; \mathbf{q}\right) + \mathcal{W}_{0, \text{inst}}\left(\frac{\varepsilon_1 \nu}{2}, m, \varepsilon_1; \mathbf{q}\right), \quad \hat{m} = \frac{m}{\varepsilon_1}. \quad (6.12)$$

In the limit $m \rightarrow \infty$, $\mathbf{q} \rightarrow 0$ with $t = m^4 \mathbf{q}$ fixed, the $\mathcal{N} = 2^*$ instanton partition function $\mathcal{Z}_{\text{inst}}(a, m; \varepsilon_1, \varepsilon_2; \mathbf{q})$ reduces to the pure gauge theory one, $\mathcal{Z}_{\text{inst}}^{[0]}(a; \varepsilon_1, \varepsilon_2; t)$, and similarly for the prepotential and the semi-classical block, after an appropriate normalization. Leveraging the identifications due to the AGT correspondence, in the following we will denote by $f_{\text{inst}}^{[N_f]}$ and f_{inst} the *instanton* part of the classical conformal blocks.

6.1 NS limit: $N_f \leq 4$ and $\mathcal{N} = 2^*$ theories

The blow-up equations are a set of extremely rigid constraints satisfied by Nekrasov partition functions [3, 52], and thus conformal blocks [92]. We consider the vanishing $\mathbb{C}^2$ blow-up equations, which, for the partition functions of $\text{SU}(2)$ gauge theories with $N_f \leq 4$ fundamental hypermultiplets, take the form:

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathcal{Z}^{[N_f]}(a + n\varepsilon_1, \tilde{\boldsymbol{\mu}}; \varepsilon_1, \varepsilon_2 - \varepsilon_1; t) \mathcal{Z}^{[N_f]}(a + n\varepsilon_2, \tilde{\boldsymbol{\mu}}; \varepsilon_2, \varepsilon_1 - \varepsilon_2; t) = 0, \quad (6.13)$$

where $\tilde{\mu}_{1,2} = \mu_{1,2} - (\varepsilon_1 + \varepsilon_2)/2$ and $\tilde{\mu}_{3,4} = -\mu_{3,4} + (\varepsilon_1 + \varepsilon_2)/2$, while for the $\mathcal{N} = 2^*$ $\text{SU}(2)$ gauge theory, we consider [43]:

$$\begin{aligned} & \theta_2(0|2\tau) \sum_{n \in \mathbb{Z}} \mathcal{Z}(a + n\varepsilon_1, m; \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + n\varepsilon_2, m; \varepsilon_2, \varepsilon_1 - \varepsilon_2; \mathbf{q}) \\ & - \theta_3(0|2\tau) \sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathcal{Z}(a + n\varepsilon_1, m; \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + n\varepsilon_2, m; \varepsilon_2, \varepsilon_1 - \varepsilon_2; \mathbf{q}) = 0. \end{aligned} \quad (6.14)$$

In particular, in the pure gauge theory limit $\mathbf{q} \rightarrow 0$, $m \rightarrow \infty$, with $t = \mathbf{q}m^4$ held fixed, equation (6.14) reduces to equation (6.13) for $N_f = 0$. To obtain relations involving the classical conformal blocks, we consider the $\varepsilon_1 - \varepsilon_2 \rightarrow 0$ limit of the previous blow-up equations, starting with (6.13). Expressing the partition function in terms of the prepotential, as in equation (6.5), we obtain:

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp\left(F^{[N_f]}(a + n\varepsilon_1, \tilde{\boldsymbol{\mu}}; \varepsilon_1, \varepsilon_2 - \varepsilon_1; t) + F^{[N_f]}(a + n\varepsilon_2, \tilde{\boldsymbol{\mu}}; \varepsilon_2, \varepsilon_1 - \varepsilon_2; t)\right) = 0. \quad (6.15)$$

Then, expanding the prepotential as:

$$F^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = \frac{1}{\varepsilon_2} \mathcal{W}_0^{[N_f]}(a, \boldsymbol{\mu}, \varepsilon_1; t) + \mathcal{W}_1^{[N_f]}(a, \boldsymbol{\mu}, \varepsilon_1; t) + \mathcal{O}(\varepsilon_2), \quad (6.16)$$

in the limit $\varepsilon_1 - \varepsilon_2 \rightarrow 0$, we obtain:

$$\begin{aligned} & F^{[N_f]}(a + n\varepsilon_1, \tilde{\boldsymbol{\mu}}; \varepsilon_1, \varepsilon_2 - \varepsilon_1; t) + F^{[N_f]}(a + n\varepsilon_2, \tilde{\boldsymbol{\mu}}; \varepsilon_2, \varepsilon_1 - \varepsilon_2; t) \\ & \simeq \frac{1}{\varepsilon_2 - \varepsilon_1} \mathcal{W}_0^{[N_f]}(a + n\varepsilon_1, \tilde{\boldsymbol{\mu}}, \varepsilon_1; t) + \mathcal{W}_1^{[N_f]}(a + n\varepsilon_1, \tilde{\boldsymbol{\mu}}, \varepsilon_1; t) + \varepsilon_1 \leftrightarrow \varepsilon_2 \\ & = 2\mathcal{W}_1^{[N_f]}(a + n\hbar, \tilde{\boldsymbol{\mu}}, \hbar; t) - n\partial_a \mathcal{W}_0^{[N_f]}(a + n\hbar, \tilde{\boldsymbol{\mu}}, \hbar; t) - \partial_{\hbar} \mathcal{W}_0^{[N_f]}(a + n\hbar, \tilde{\boldsymbol{\mu}}, \hbar; t), \end{aligned}$$

where we set $\varepsilon_1 = \varepsilon_2 = \hbar$. Thus, in this limit, equation (6.13) becomes:

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{2\mathcal{W}_1^{[N_f]}(a+n\hbar, \tilde{\mu}, \hbar; t) - n\partial_a \mathcal{W}_0^{[N_f]}(a+n\hbar, \tilde{\mu}, \hbar; t) - \partial_{\hbar} \mathcal{W}_0^{[N_f]}(a+n\hbar, \tilde{\mu}, \hbar; t)} = 0, \quad (6.17)$$

where the Coulomb branch parameter a and the masses μ_i are shifted after taking the $\hbar$ -derivative. In particular, after taking the limit, the shifted masses read $\tilde{\mu}_{1,2} = \mu_{1,2} - \hbar$ and $\tilde{\mu}_{3,4} = -\mu_{3,4} + \hbar$. Similarly, for the $\mathcal{N} = 2^*$ gauge theory, the NS limit of the blow-up equation (6.14) takes the form:

$$\begin{aligned} & \theta_2(0|2\tau) \sum_{n \in \mathbb{Z}} e^{2\mathcal{W}_1(a+n\hbar, m, \hbar; q) - n\partial_a \mathcal{W}_0(a+n\hbar, m, \hbar; q) - \partial_{\hbar} \mathcal{W}_0(a+n\hbar, m, \hbar; q)} \\ & - \theta_3(0|2\tau) \sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{2\mathcal{W}_1(a+n\hbar, m, \hbar; q) - n\partial_a \mathcal{W}_0(a+n\hbar, m, \hbar; q) - \partial_{\hbar} \mathcal{W}_0(a+n\hbar, m, \hbar; q)} = 0. \end{aligned} \quad (6.18)$$

6.2 Solving the blow-up equations: $N_f \leq 4$ theories

Let us begin by considering equation (6.17). To make it more explicit, we split the two prepotentials into classical, 1-loop and instanton parts, using the definitions (A.22a) and (A.22b). Then, dividing by a common factor of $t^{\frac{a^2}{\hbar^2} - \frac{3}{4}}$, we find:

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} t^{n^2} e^{2\mathcal{W}_{1, \text{1-loop}}^{[N_f]} - (\partial_{\hbar} + n\partial_a) \mathcal{W}_{0, \text{1-loop}}^{[N_f]} + 2\mathcal{W}_{1, \text{inst}} - (\partial_{\hbar} + n\partial_a) \mathcal{W}_{0, \text{inst}}^{[N_f]}} = 0, \quad (6.19)$$

where $\mathcal{W}_{j, \text{1-loop}}^{[N_f]} = \mathcal{W}_{j, \text{1-loop}}(a + n\hbar, \tilde{\mu}, \hbar)$ and $\mathcal{W}_{j, \text{inst}}^{[N_f]} = \mathcal{W}_{j, \text{inst}}(a + n\hbar, \tilde{\mu}, \hbar; t)$. Normalizing by the 1-loop term for $n = 0$, what remains are just simple rational functions of a , $\mu_{i=1}^{N_f}$ and $\hbar$, with poles for $a \in \frac{\hbar}{2}\mathbb{Z}$. Indeed, with this normalization and setting $\hbar = 1$, equation (6.19) reads:

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} t^{n^2} \mathcal{G}_{1\text{-loop}}^{[N_f]}(a, \mu, n) e^{2\mathcal{W}_{1, \text{inst}}^{[N_f]} - (\partial_{\hbar} + n\partial_a) \mathcal{W}_{0, \text{inst}}^{[N_f]}} = 0, \quad (6.20)$$

where $\mathcal{G}_{1\text{-loop}}^{[N_f]}(a, \mu, n)$ can be computed as the $\varepsilon_1 = \varepsilon_2 = 1$ limit of the l -factors $L^{(n, -n), 1}$ in Appendix B of [52]. Namely, defining the functions:

$$g_{1\text{-loop}}^{\text{hyp}}(a, \mu, n) = \prod_{j=0}^{n - \frac{1}{2}} (-a - j + \mu)^j (a + j + \mu)^j, \quad n \geq \frac{1}{2}, \quad (6.21a)$$

$$g_{1\text{-loop}}^{\text{vec}}(a, n) = \begin{cases} \prod_{j=1}^{2n} (-2a - j + 1)^j (2a + j)^{j-1}, & \text{if } n \geq \frac{1}{2}, \\ 1, & \text{if } n = 0, \end{cases} \quad (6.21b)$$

the 1-loop factor can be written as:

$$\mathcal{G}_{1\text{-loop}}^{[N_f]}(a, \mu, n) = \begin{cases} \frac{\prod_{i=1}^{N_f} g_{1\text{-loop}}^{\text{hyp}}(a, \tilde{\mu}_i, n)}{g_{1\text{-loop}}^{\text{vec}}(a, n)}, & \text{if } n \geq \frac{1}{2}, \\ \frac{\prod_{i=1}^{N_f} g_{1\text{-loop}}^{\text{hyp}}(-a, \tilde{\mu}_i, -n)}{g_{1\text{-loop}}^{\text{vec}}(-a, -n)}, & \text{if } n \leq -\frac{1}{2}. \end{cases} \quad (6.22)$$

where we recall that $\tilde{\mu}_{1,2} = \mu_{1,2} - 1$ and $\tilde{\mu}_{3,4} = -\mu_{3,4} + 1$ for $\hbar = 1$. Generalizing the considerations of [37, 38], to solve equation (6.20) we make the following ansatz:

$$\mathcal{W}_{0, \text{inst}}^{[N_f]}(a, \mu, \hbar; t) = R_0^{[N_f]}(a, \mu, \hbar; t)$$

$$+ \hbar \sum_{k,j=1}^{\infty} \left[\sum_{\pm} g_{k,j}^{[N_f]} \left(\left\{ \frac{\mu_i}{\hbar} \right\}, \frac{(t\hbar^{N_f-4})^{j/2}}{j \pm 2a/\hbar} \right) \right] (t\hbar^{N_f-4})^{k+j/2-1}, \quad (6.23a)$$

$$\begin{aligned} \mathcal{W}_{1,\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}, \hbar; t) &= R_1^{[N_f]}(\boldsymbol{\mu}, \hbar; t) \\ &+ \sum_{k,j=1}^{\infty} \left[\sum_{\pm} \tilde{g}_{k,j}^{[N_f]} \left(\left\{ \frac{\mu_i}{\hbar} \right\}, \frac{(t\hbar^{N_f-4})^{j/2}}{j \pm 2a/\hbar} \right) \right] (t\hbar^{N_f-4})^{k+j/2-1} \\ &+ \sum_{k,j=1}^{\infty} \left[\sum_{\pm} \tilde{f}_{k,j}^{[N_f]} \left(\left\{ \frac{\mu_i}{\hbar} \right\}, \frac{(t\hbar^{N_f-4})^{j/2}}{j \pm 2a/\hbar} \right) \right] (t\hbar^{N_f-4})^{k-1}, \end{aligned} \quad (6.23b)$$

where the *resummation functions* $g_{k,j}^{[N_f]}(\boldsymbol{\mu}, x)$, $\tilde{g}_{k,j}^{[N_f]}(\boldsymbol{\mu}, x)$ and $\tilde{f}_{k,j}^{[N_f]}(\boldsymbol{\mu}, x)$ must be analytic near $x = 0$ and vanish at $x = 0$. Moreover, in order for the ansatz to admit a power series expansion in integer powers of t , the functions must have definite parity in their last argument, namely, the $g_{k,j}^{[N_f]}$ and $\tilde{g}_{k,j}^{[N_f]}$ are assumed to be odd, while the functions $\tilde{f}_{k,j}$ are assumed to be even. The functions $R_i^{[N_f]}$ are also vanishing at $t = 0$ and take into account the regular part of the instanton expansion. They can be computed explicitly from $R_0^{[4]}$ and $R_1^{[4]}$ in equations (C.1) and (C.2) by taking appropriate decoupling limits. To obtain equations for the resummation functions we proceed in the following way. The blow-up equation (6.20) holds for any value of the parameter a , allowing us to set:

$$a = \frac{\hbar}{2} \left(m_1 + \frac{1}{x} t^{\frac{m_2}{2}} \right), \quad m_1 \in \mathbb{Z}, \quad m_2 \in \mathbb{N}, \quad (6.24)$$

with x a complex variable, and expand the resulting expression in t . Due to the symmetry of the equation for $a \rightarrow -a$ (accompanied by $n \rightarrow -n$), we may restrict to $m_1 \geq 0$. The choice in (6.24) allows us to focus on the functions with $j = m_2$, as can be seen by noting that, when $j = m_2$ and $2n = \mp j - m_1$, these functions and their derivatives enter the equations as:

$$g_{k,j}^{[N_f]} \left(\left\{ \frac{\mu_i}{\hbar} \right\}, \frac{(t\hbar^{N_f-4})^{j/2}}{j \pm (2n + m_1) \pm t^{m_2/2}/x} \right) = g_{k,m_2}^{[N_f]} \left(\left\{ \frac{\mu_i}{\hbar} \right\}, \pm x \hbar^{m_2(N_f-4)/2} \right).$$

Meanwhile, when $2n = \mp j - m_1$ and $j \neq m_2$ we have two possibilities: if $m_2 < j$ the functions will enter through their expansion around zero, while if $m_2 > j$ they will enter through their expansion around infinity. Finally, if $j \pm (2n + m_1) \neq 0$, the functions will enter simply through their expansion around zero. We now begin by deriving the equations satisfied by the resummation functions and show that they can be easily solved. Finally, we discuss the structure of the blow-up equation's expansion, taking into account the behavior of the resummation functions expanded around infinity.

6.2.1 Algebraic equations for the resummation functions

First of all, we consider the simple rational functions $\mathcal{G}_{1\text{-loop}}^{[N_f]}(a, \boldsymbol{\mu}, n)$ in equation (6.22). When both $a > 0$ and $n > 0$, its expansion in a around some fixed positive value will start from a constant term. Then, for any value of $n \in \mathbb{Z} + \frac{1}{2}$, $\mathcal{G}_{1\text{-loop}}(a, n)$ has always a simple pole for $a = 0$ and finally, for $n < 0$, the function has poles at $a = j/2$, $j \in \{1, \dots, 2|n|\}$, of

order $2j$ except for $j = 2|n|$, for which the order is $2|n| - 1$. Setting $2a = m_1 + t^{m_2/2}/x$, expanding in t and combining with the overall power of t^{n^2} , results in a predictable leading order behavior. In particular, for $2n = \mp m_2 - m_1$, $m_1 > 0$, we find:

$$\begin{aligned} t^{(m_1+m_2)^2/4} \mathcal{G}_{1\text{-loop}}^{[N_f]} \left(\frac{m_1 x + t^{m_2/2}}{2x}, \boldsymbol{\mu}, -\frac{m_2 + m_1}{2} \right) &\sim \mathcal{O}(t^{(m_1-m_2)^2/4}), \\ t^{(m_1-m_2)^2/4} \mathcal{G}_{1\text{-loop}}^{[N_f]} \left(\frac{m_1 x + t^{m_2/2}}{2x}, \boldsymbol{\mu}, \frac{m_2 - m_1}{2} \right) &\sim \mathcal{O}(t^{(m_1-m_2)^2/4}), \end{aligned}$$

while for values of n between $-(m_2 + m_1)/2$ and $(m_2 - m_1)/2$, the power of t at leading order can be negative. Let us now consider the instanton contributions. To understand the structure of the expansion and the form of the resulting equations, we can examine the n -th term of the blow-up equation (6.20), where n is such that $j \pm (2n + m_1) = 0$, for some $j \leq m_2$. These terms give the most interesting contributions, since for $j < m_2$ we will find the non-trivial expansion of the resummation functions around infinity and for $j = m_2$ we will find the functions valued at $\pm x$, while for $j > m_2$ the functions are expanded around zero. For example, we have:

$$\begin{aligned} \mathcal{W}_{1,\text{inst}}^{\mp, [N_f]} &= \mathcal{W}_{1,\text{inst}}^{[N_f]} \left(\frac{\mp j x + t^{m_2/2}}{2x}, \tilde{\boldsymbol{\mu}}, 1; t \right) \\ &= \sum_{k=1}^{\infty} t^{k-1} \left[\tilde{f}_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) \pm t^{j/2} \tilde{g}_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) \right] + h^{\mp, [N_f]}(x, \boldsymbol{\mu}, t), \end{aligned}$$

and similarly for $\mathcal{W}_{0,\text{inst}}^{[N_f]}$, where $y = x t^{(j-m_2)/2}$, while the functions $h^{\mp, [N_f]}$ take into account the remaining part of the expression, which will have an expansion starting from some positive power of t . The parity of the resummation functions has been used to simplify the previous formula. Thus, isolating only the relevant terms at the exponent in equation (6.20), we find the following expression:

$$\begin{aligned} 2\mathcal{W}_{1,\text{inst}}^{\mp, [N_f]} - (\partial_h + n\partial_a) \mathcal{W}_{0,\text{inst}}^{\mp, [N_f]} &= \sum_{k \geq 1} t^{k-1} \left[2\tilde{f}_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) \mp m_1 y^2 \partial_y g_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) \right. \\ &\quad \left. \pm t^{j/2} \left(2\tilde{g}_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) + \gamma_{k,j}^{[N_f]} g_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) + \delta_j^{[N_f]} y \partial_y g_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) \right) \right. \\ &\quad \left. + \sum_{i=1}^{N_f} \tilde{\mu}_i \partial_{\tilde{\mu}_i} g_{k,j}^{[N_f]}(\tilde{\boldsymbol{\mu}}, y) \right] + s^{\mp, [N_f]}(x, \boldsymbol{\mu}, t), \end{aligned} \quad (6.25)$$

where:

$$\gamma_{k,j}^{[N_f]} = (4 - N_f)(k + j/2) + N_f - 5, \quad (6.26a)$$

$$\delta_j^{[N_f]} = (4 - N_f)j/2 - 1, \quad (6.26b)$$

and the functions $s^{\mp, [N_f]}(x, \boldsymbol{\mu}, t)$ vanish at $t = 0$. By collecting terms of the same order in t from the expansion of the blow-up equation, these expressions, together with the 1-loop term, allow us to derive the form of the equations satisfied by the resummation functions and show that the t -expansion around the poles (6.24) is well-behaved. Namely, by setting $j = m_2$, assuming $m_1 > 0$ for simplicity⁷, omitting the $[N_f]$ label from the resummation functions, and performing the change of variables:

$$\tilde{f}_{1,m_2} = \frac{1}{2} \log u_{m_2}, \quad \partial_x g_{1,m_2} = \frac{1}{x^2} \log(x v_{m_2}), \quad (6.27)$$

⁷The equations for $m_1 = 0$ are automatically satisfied once a solution for the $m_1 > 0$ case is found.

we obtain at leading order:

$$u_{m_2}(v_{m_2}^{m_1} + c_{m_1, m_2} v_{m_2}^{-m_1}) + r_{m_1, m_2, 1} = 0 \quad (6.28)$$

where the constant c_{m_1, m_2} can be computed from the 1-loop factor (6.22) and reads:

$$c_{m_1, m_2}(\boldsymbol{\mu}) = (-1)^{m_1+1} \frac{\prod_{i=1}^{N_f} \tilde{c}_{m_1, m_2}(\mu_i + 1)}{m_2^{2m_1} [(m_2 - 1)!]^{4m_1}}, \quad (6.29)$$

with $\tilde{c}_{m_1, m_2}(\mu)$ defined in equation (C.6). We observe here, via the change of variables (6.27), the appearance of logarithmic branching, necessary to obtain a linear equation (6.28) in the function u_{m_2} . Meanwhile, at orders t^{k-1} and $t^{k+j/2-1}$, respectively, compared to the leading order equations, we find:

$$2(v_{m_2}^{m_1} + c_{m_1, m_2} v_{m_2}^{-m_1}) \tilde{f}_{k, m_2} + m_1 x^2 (v_{m_2}^{m_1} - c_{m_1, m_2} v_{m_2}^{-m_1}) \partial_x g_{k, m_2} + r_{m_1, m_2, k} = 0 \quad (6.30)$$

$$2\tilde{g}_{k, m_2} + \gamma_{k, m_2}^{[N_f]} g_{k, m_2} + \delta_{m_2}^{[N_f]} x \partial_x g_{k, m_2} + \sum_{i=1}^{N_f} \tilde{\mu}_i \partial_{\tilde{\mu}_i} g_{k, m_2} + \tilde{r}_{m_1, m_2, k} = 0 \quad (6.31)$$

The functions $r_{m_1, m_2, k}(x, \boldsymbol{\mu})$ and $\tilde{r}_{m_1, m_2, k}(x, \boldsymbol{\mu})$ account for the remaining contributions of the t -expansion at each fixed order and can be explicitly computed for any choice of m_1 and m_2 . In particular, they also include contributions from resummation functions that are assumed to have been determined from equations at lower orders in the expansion. For example, in equation (6.31), the terms involving the function $g_{k, j}$ could, in principle, be absorbed into $\tilde{r}_{m_1, m_2, k}$. However, we write them explicitly to highlight the simple relations that connect the functions at each step. The three families of equations (6.28), (6.30) and (6.31) determine the resummation functions up to a branching ambiguity, which can be fixed by imposing the appropriate behavior at zero and at infinity, see Appendix C.2 for an explicit example. Indeed, equation (6.28) determines u_{m_2} as a function of v_{m_2} , and by equating two such expressions for different values of m_1 , one finds:

$$\begin{aligned} & (v_{m_2}^{\tilde{m}_1} + c_{\tilde{m}_1, m_2} v_{m_2}^{-\tilde{m}_1}) r_{m_1, m_2, 1} - r_{\tilde{m}_1, m_2, 1} (v_{m_2}^{m_1} + c_{m_1, m_2} v_{m_2}^{-m_1}) = 0, \\ \implies & \left(v_{m_2} - \frac{c_{1, m_2}}{v_{m_2}} \right) = \begin{cases} \pm \sqrt{\frac{r_{4, m_2, 1}}{r_{2, m_2, 1}} - 2c_{1, m_2}}, & \text{for } m_2 \text{ odd,} \\ \pm \sqrt{\frac{r_{3, m_2, 1}}{r_{1, m_2, 1}} - c_{1, m_2}}, & \text{for } m_2 \text{ even,} \end{cases} \end{aligned} \quad (6.32)$$

where in the last step we set $m_1 = 2$, $\tilde{m}_1 = 4$ for odd m_2 , $m_1 = 1$, $\tilde{m}_1 = 3$ for even m_2 , respectively, and used the fact that $c_{m_1, m_2} = (-1)^{m_1+1} c_{1, m_2}^{m_1}$. After choosing a branch of the square root on the right-hand side, equation (6.32) becomes quadratic in v_{m_2} . Solving it makes the square-root dependence explicit, with the appropriate branches selected by the required behavior at $x = 0$. This provides the mechanism for the square-root branching observed in the explicit solutions. Then, two copies of equation (6.30), evaluated at different values of m_1 , determine the functions $\tilde{f}_{k, m_2}$ and g_{k, m_2} , with the integration constant of the latter fixed by requiring it to vanish at zero. Finally, equation (6.31) determines $\tilde{g}_{k, m_2}$ in terms of the previously determined resummation functions. Thus, we have shown that expanding the blow-up equation (6.20) for two distinct values of m_1 is sufficient to determine all the resummation functions. For example, the first few functions in the $N_f = 0$ case are given by:

$$\begin{aligned}
 g_{1,1}^{[0]}(x) &= -\frac{1 - \sqrt{1+4x^2} + \log\left(\sqrt{\frac{1}{4}+x^2} + \frac{1}{2}\right)}{x}, & g_{2,1}^{[0]}(x) &= \frac{(2-x^2)\sqrt{1+4x^2} - 3x^2 - 2}{12x^3}, \\
 g_{1,2}^{[0]}(x) &= \frac{\sqrt{1+x^2} - 1 - \log\left(\frac{1}{2}\sqrt{1+x^2} + \frac{1}{2}\right)}{x}, & g_{2,2}^{[0]}(x) &= -\frac{8(\sqrt{1+x^2} - 1)}{9x}, \\
 \tilde{g}_{1,1}^{[0]}(x) &= \frac{1 - \sqrt{1+4x^2}}{2x}, & \tilde{g}_{2,1}^{[0]}(x) &= \frac{1 + 3x^2 + \frac{5x^4 - 5x^2 - 1}{\sqrt{1+4x^2}}}{6x^3}, \\
 \tilde{g}_{1,2}^{[0]}(x) &= \frac{1 - \sqrt{1+x^2}}{4x}, & \tilde{g}_{2,2}^{[0]}(x) &= \frac{32 - \frac{2(13x^2+16)}{\sqrt{1+x^2}}}{27x}, \\
 \tilde{f}_{1,1}^{[0]}(x) &= -\frac{1}{4}\log(1+4x^2), & \tilde{f}_{2,1}^{[0]}(x) &= \frac{3x^2}{2(1+4x^2)}, \\
 \tilde{f}_{1,2}^{[0]}(x) &= -\frac{1}{4}\log(1+x^2), & \tilde{f}_{2,2}^{[0]}(x) &= \frac{4x^2}{9(1+x^2)}.
 \end{aligned}$$

while their general structure will be discussed in Section 6.4. However, two points may still appear unclear at this stage. First, the presence of logarithmic terms in the resummation functions could invalidate the expansion of the blow-up equation for $m_2 > 1$, as these functions will be expanded around infinity. We will show shortly that this is not the case. Second, it is not clear whether knowledge of the Taylor coefficients of the resummation functions with $j > m_2$ is required to determine the functions at the step $j = m_2$. While such terms do appear in the functions $r_{m_1, m_2, k}$ and $\tilde{r}_{m_1, m_2, k}$ and introduce some subtleties, discussed in detail through an example in Appendix C.2, these do not affect the overall procedure. In conclusion, we found evidence that a single blow-up equation is enough to determine all the resummation functions $g_{k,j}^{[N_f]}$, $\tilde{g}_{k,j}^{[N_f]}$ and $\tilde{f}_{k,j}^{[N_f]}$. Moreover, since $\mathcal{W}_{0, \text{inst}}^{[N_f]}$ and $\mathcal{W}_{1, \text{inst}}^{[N_f]}$, as in equations (6.23a) and (6.23b), coincide with the known instanton expansion when re-expanded around $t = 0$, they will automatically satisfy any other blow-up equation.

6.2.2 Structure of the expansion

Let us now examine more carefully the structure of the blow-up equation's expansion, taking into account the asymptotic behavior of the resummation functions at infinity. As already mentioned, there is a distinction between the cases $m_2 = 1$ and $m_2 > 1$, in that in the former no resummation function is expanded around infinity. Moreover, for $m_2 > 1$, certain summands in the blow-up equation (6.20) begin with negative powers of t ; specifically, this occurs for some values of n within the range $\{-\frac{m_2+m_1}{2}, \dots, \frac{m_2-m_1}{2}\}$. However, these terms combine in such a way that the first non-zero coefficient in the final expression appears at order $t^{(m_1-m_2)^2/4}$. To illustrate the difference between the two cases, let us first consider a single function. Take, for instance, a value of n such that $2n + j + m_1 = 0$ for some $j < m_2$. Then, for example:

$$\mathcal{W}_{1, \text{inst}}^{\mp, [N_f]} = \sum_{k=1}^{\infty} t^{k-1} \left[\tilde{f}_{k,j}^{[N_f]} \left(\tilde{\mu}, \frac{x}{t^{(m_2-j)/2}} \right) \pm t^{j/2} \tilde{g}_{k,j}^{[N_f]} \left(\tilde{\mu}, \frac{x}{t^{(m_2-j)/2}} \right) \right] + h^{\mp, [N_f]},$$

By explicitly solving for the resummation functions, as shown in the previous section, we find that the asymptotic behavior of $g_{1,j}^{[N_f]}$ and $\tilde{f}_{1,j}^{[N_f]}$ is given by:

$$g_{1,j}^{[N_f]}(\tilde{\mu}, y) = \text{const} - \frac{1}{2y} \log y^2 + \mathcal{O}(y^{-1}), \quad \tilde{f}_{1,j}^{[N_f]}(\tilde{\mu}, y) = -\frac{1}{4} \log y^2 + \text{const} + \mathcal{O}(y^{-2}).$$

In contrast, the asymptotic behavior of all other functions is purely power-like, starting from a constant term. Therefore, by absorbing the subleading terms into $h^{\mp, [N_f]}$, we obtain:

$$\mathcal{W}_{1, \text{inst}}^{\mp, [N_f]} = \frac{1}{4} \log \left(\frac{t^{m_2-j}}{x^2} \right) + \tilde{h}^{\mp, [N_f]}(x, \boldsymbol{\mu}, t).$$

Such logarithmic terms remain at the exponent in equation (6.20), effectively decreasing or increasing the overall power of t from which the expansion of a particular term in the blow-up equation begins. Indeed, considering the expression (6.25), we can isolate the terms containing logarithms, i.e. the $k = 1$ ones, and obtain:

$$2\mathcal{W}_{1, \text{inst}}^{\mp, [N_f]} - (\partial_h + n\partial_a) \mathcal{W}_{0, \text{inst}}^{\mp, [N_f]} = (\mp m_1 - 1/2) \log \frac{x^2}{t^{m_2-j}} + \tilde{s}^{\mp, [N_f]}(x, \boldsymbol{\mu}, t),$$

where $\tilde{s}^{\mp, [N_f]}$ is regular at $t = 0$. In particular, we notice that all the $t \log t$ terms cancel in the final expression. Thus, we have shown that the expansion of the blow-up equation (6.20) considered above indeed leads to a well-defined power series in the parameter t .

6.3 Solving the blow-up equations: $\mathcal{N} = 2^*$ theory

It is straightforward to repeat the previous analysis in the $\mathcal{N} = 2^*$ case. Denoting by μ the mass of the adjoint hypermultiplet, to avoid confusion with the integers m_1 and m_2 , and setting $\hbar = 1$, we can write the blow-up equation (6.18) as:

$$\begin{aligned} & \theta_2(0|2\tau) \sum_{n \in \mathbb{Z}} \mathbf{q}^{n^2} \mathcal{G}_{1\text{-loop}}(a, \mu, n) e^{2\mathcal{W}_{1, \text{inst}} - n\partial_a \mathcal{W}_{0, \text{inst}} - \partial_h \mathcal{W}_{0, \text{inst}}} \\ & - \theta_3(0|2\tau) \sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathbf{q}^{n^2} \mathcal{G}_{1\text{-loop}}(a, \mu, n) e^{2\mathcal{W}_{1, \text{inst}} - n\partial_a \mathcal{W}_{0, \text{inst}} - \partial_h \mathcal{W}_{0, \text{inst}}} = 0, \end{aligned} \quad (6.33)$$

where $\mathcal{W}_{i, \text{inst}} = \mathcal{W}_{i, \text{inst}}(a + n, \mu, 1; \mathbf{q})$. Explicitly, the 1-loop term reads:

$$\mathcal{G}_{1\text{-loop}}(a, \mu, n) = \begin{cases} \frac{g_{1\text{-loop}}^{\text{adj}}(a, \mu, n)}{g_{1\text{-loop}}^{\text{vec}}(a, n)}, & \text{for } n \geq 0, \\ \frac{g_{1\text{-loop}}^{\text{adj}}(-a, \mu, -n)}{g_{1\text{-loop}}^{\text{vec}}(-a, -n)}, & \text{for } n < 0, \end{cases} \quad (6.34)$$

where $g_{1\text{-loop}}^{\text{vec}}$ is defined in equation (6.21b), while:

$$g_{1\text{-loop}}^{\text{adj}}(a, \mu, n) = \begin{cases} \prod_{j=1}^{2n} (-2a - j - \mu + 1)^j (2a + j - \mu)^{j-1}, & \text{for } n > 0, \\ 1, & \text{for } n = 0. \end{cases} \quad (6.35)$$

To solve equation (6.33), we consider the ansatz:

$$\mathcal{W}_{0, \text{inst}}(a, \mu, \hbar; \mathbf{q}) = \hbar \sum_{j,k=1}^{\infty} \left[\sum_{\pm} g_{k,j} \left(\frac{\mu}{\hbar}, \frac{\mathbf{q}^{j/2}}{j \pm 2a/\hbar} \right) \right] \mathbf{q}^{k+j/2-1}, \quad (6.36a)$$

$$\begin{aligned} \mathcal{W}_{1, \text{inst}}(a, \mu, \hbar; \mathbf{q}) &= \log \varphi(\mathbf{q})^{-1} + \sum_{j,k=1}^{\infty} \left[\sum_{\pm} \tilde{g}_{k,j} \left(\frac{\mu}{\hbar}, \frac{\mathbf{q}^{j/2}}{j \pm 2a/\hbar} \right) \right] \mathbf{q}^{k+j/2-1} \\ &+ \sum_{j,k=1}^{\infty} \left[\sum_{\pm} \tilde{f}_{k,j} \left(\frac{\mu}{\hbar}, \frac{\mathbf{q}^{j/2}}{j \pm 2a/\hbar} \right) \right] \mathbf{q}^{k-1}, \end{aligned} \quad (6.36b)$$

and expand in $\mathfrak{q}$ the blow-up equation, setting:

$$a = \frac{\hbar}{2} \left(m_1 + \frac{1}{x} \mathfrak{q}^{\frac{m_2}{2}} \right), \quad m_1 \in \mathbb{Z}, \quad m_2 \in \mathbb{N}. \quad (6.37)$$

In this case, the regular parts of the resummation ansatz are determined by considering the $\mathcal{N} = 4$ limit of the partition function (A.30), i.e. the $\mu \rightarrow 0$ limit, which is known exactly. Then, we can compute the analogue of (6.25), which reads:

$$\begin{aligned} 2\mathcal{W}_{1,\text{inst}}^\mp - (\partial_{\hbar} + n\partial_a) \mathcal{W}_{0,\text{inst}}^\mp &= \sum_{k \geq 1} \mathfrak{q}^{k-1} \left[2\tilde{f}_{k,j}(\mu, y) \mp m_1 y^2 \partial_y g_{k,j}(\mu, y) \right. \\ &\quad \left. \pm \mathfrak{q}^{j/2} \left(2\tilde{g}_{k,j}(\mu, y) - g_{k,j}(\mu, y) - y \partial_y g_{k,j}(\mu, y) + \mu \partial_\mu g_{k,j}(\mu, y) \right) \right] + s^\mp(x, \mu, \mathfrak{q}) \end{aligned} \quad (6.38)$$

where $y = x\mathfrak{q}^{(j-m_2)/2}$. This expression allows us to write the explicit form of the equations, which read:

$$u_{m_2}(v_{m_2}^{m_1} + c_{m_1, m_2} v_{m_2}^{-m_1}) + r_{m_1, m_2, 1} = 0, \quad (6.39a)$$

$$2(v_{m_2}^{m_1} + c_{m_1, m_2} v_{m_2}^{-m_1}) \tilde{f}_{k, m_2} + m_1 x^2 (v_{m_2}^{m_1} - c_{m_1, m_2} v_{m_2}^{-m_1}) \partial_x g_{k, m_2} + r_{m_1, m_2, k} = 0, \quad (6.39b)$$

$$2\tilde{g}_{k, m_2} - g_{k, m_2} - x \partial_x g_{k, m_2} + \mu \partial_\mu g_{k, m_2} + \tilde{r}_{m_1, m_2, k} = 0, \quad (6.39c)$$

where each function depends on the variable x and on the adjoint mass μ only. In particular, the constants $c_{m_1, m_2}(\mu)$ can be computed from (6.34) to be:

$$c_{m_1, m_2}(\mu) = (-1)^{m_1+1} \frac{\prod_{j=0}^{m_2-1} (j + \mu)^{2m_1} (j - \mu + 1)^{2m_1}}{m_2^{2m_1} [(m_2 - 1)!]^{4m_1}}. \quad (6.40)$$

Since the equations (6.39) are essentially identical to those presented in the previous section, we do not repeat the discussion of their solution here. Instead, we note that the same logic applies and refer the reader to Appendix C.2 for an explicit example. The general structure of the solutions is likewise essentially unchanged, as can be seen, for instance, from the very first functions:

$$g_{1,1}(\mu, x) = \frac{\sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} - \log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} \right) - 1}{x},$$

$$\tilde{g}_{1,1}(\mu, x) = \frac{(-1 + \mu + \mu^2) \left(1 - \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} \right)}{2\mu(\mu - 1)x},$$

$$\tilde{f}_{1,1}(\mu, x) = -\frac{1}{4} \log \left(1 + 4\mu^2(\mu - 1)^2 x^2 \right).$$

One can also verify that, given the resulting asymptotic behavior of the resummation functions, the expansion of the blow-up equation (6.33) is well-defined. Finally, we have shown that also for the $\mathcal{N} = 2^*$ theory, a single blow-up equation completely determines the functions appearing in the resummation ansatz, which, if re-expanded, reproduces the usual Nekrasov expansions. Thus, any other blow-up equation will be satisfied automatically by the ansatz, as in the previous case.

6.4 General structure of the resummation functions

We conclude this section by outlining the general structure exhibited by the resummation functions⁸. In particular, our results are compatible with and extend those previously conjectured in the literature [37, 38]. Starting from the NS prepotentials, or classical blocks, we find:

$$g_{1,n}(\boldsymbol{\mu}, z) = -\frac{\log\left(\frac{1}{2} + \sqrt{\frac{1}{4} + w_n z^2 / \zeta_n^2}\right) + 1 - \sqrt{1 + 4w_n z^2 / \zeta_n^2}}{z}, \quad (6.41a)$$

$$g_{k \geq 2,n}(\boldsymbol{\mu}, z) = \frac{1}{z^{2k-1}} \left[\left(1 + \frac{4w_n z^2}{\zeta_n^2}\right)^{5/2-k} Q_{k,n}(\boldsymbol{\mu}, z^2) + P_{k,n}(\boldsymbol{\mu}, z^2) \right], \quad (6.41b)$$

where

$$w_n = \begin{cases} w_n^{[N_f]}, & \text{for } 0 \leq N_f \leq 4, \\ c_n^2, & \text{for } \mathcal{N} = 2^*, \end{cases} \quad (6.42)$$

and

$$w_n^{[N_f]} = \prod_{k=\frac{1-n}{2}}^{\frac{n-1}{2}} \prod_{i=1}^{N_f} (\mu_i + k), \quad c_n = \prod_{k=-n}^{n-1} (\mu + k), \quad \zeta_n = n!(n-1)!. \quad (6.43)$$

The functions $Q_{k,n}$ and $P_{k,n}$ are polynomials in z^2 of degree $2k-3$ and $k-1$, respectively, with coefficients that are rational functions of the masses. Meanwhile, for the first ε_2 corrections to the NS prepotentials, or equivalently, the first b^2 corrections to the classical blocks, we find:

$$\tilde{g}_{k,n}(\boldsymbol{\mu}, z) = \frac{1}{z^{2k-1}} \left[\left(1 + \frac{4w_n z^2}{\zeta_n^2}\right)^{\frac{3}{2}-k} U_{k,n}(\boldsymbol{\mu}, z^2) + V_{k,n}(\boldsymbol{\mu}, z^2) \right], \quad (6.44a)$$

$$\tilde{f}_{1,n}(\boldsymbol{\mu}, z) = -\frac{1}{4} \log\left(1 + 4w_n z^2 / \zeta_n^2\right), \quad \tilde{f}_{k \geq 2,n}(\boldsymbol{\mu}, z) = \frac{z^2 Z_{k,n}(\boldsymbol{\mu}, z^2)}{(1 + 4w_n z^2 / \zeta_n^2)^{k-1}}, \quad (6.44b)$$

where the functions $U_{k,n}$, $V_{k,n}$, and $Z_{k,n}$ are polynomials⁹ in z^2 of degree $2k-2$, $k-1$, and $k-2$, respectively, with coefficients that are again rational functions of the masses. In particular, for the $N_f \leq 4$ cases, we find that also the functions $\tilde{g}_{1,n}$ seem to have a simple universal structure, namely:

$$\tilde{g}_{1,n}(\boldsymbol{\mu}, z) = \frac{1 - \sqrt{1 + \frac{4w_n z^2}{\zeta_n^2}}}{2nz}, \quad (6.45)$$

while the same does not appear to be true in the $\mathcal{N} = 2^*$ case. As an example, the first few independent polynomials for $N_f = 1$ read:

$$\begin{aligned} Q_{2,1}(\mu_1, x) &= \frac{1}{\mu_1^2} \left(\frac{1}{48} (-1 + 8\mu_1^2) + \frac{1}{24} \mu_1 (1 - 2\mu_1^2) x \right), & Q_{2,2}(\mu_1, x) &= -\frac{8}{9} \mu_1 x, \\ U_{2,1}(\mu_1, x) &= -\frac{1}{6} - \frac{5}{6} \mu_1 x - \left(\frac{1}{4} - \frac{5}{6} \mu_1^2 \right) x^2, & U_{2,2}(\mu_1, x) &= -\frac{13}{54} \mu_1 (4\mu_1^2 - 1) x^2 - \frac{32}{27} \mu_1 x, \end{aligned}$$

⁸The emerging pattern has been tested for higher or lower values of k and n , depending on the computational complexity of the equations involved.

⁹The polynomials $P_{k,n}$ and $V_{k,n}$ are not independent of their partners $Q_{k,n}$ and $U_{k,n}$, since they are determined from them by requiring the functions $g_{k,n}$ and $\tilde{g}_{k,n}$ to be of $\mathcal{O}(z)$ as $z \rightarrow 0$.

$$Z_{2,1}(\mu_1, x) = -\frac{1}{4}(1 - 6\mu_1^2), \quad Z_{2,2}(\mu_1, x) = -\frac{1}{9}\mu_1(1 - 4\mu_1^2).$$

Moreover, for the $\mathcal{N} = 2^*$ case, we find that by parametrizing the resummation functions in the following way:

$$g_{1,n}^*(\mu, z) = -\frac{\log\left(\frac{1}{2} + \sqrt{\frac{1}{4} + c_n^2 z^2 / \zeta_n^2}\right) + 1 - \sqrt{1 + 4c_n^2 z^2 / \zeta_n^2}}{z}, \quad (6.47a)$$

$$g_{k \geq 2,n}^*(\mu, z) = \frac{1}{z(c_n z)^{2k-2}} \left[\left(1 + \frac{4c_n^2 z^2}{\zeta_n^2}\right)^{5/2-k} Q_{k,n}^*(\mu, c_n^2 z^2) + P_{k,n}^*(\mu, c_n^2 z^2) \right], \quad (6.47b)$$

$$\tilde{g}_{k,n}^*(\mu, z) = \frac{1}{(c_n z)^{2k-1}} \left[\left(1 + \frac{4c_n^2 z^2}{\zeta_n^2}\right)^{\frac{3}{2}-k} U_{k,n}^*(\mu, c_n^2 z^2) + V_{k,n}^*(\mu, c_n^2 z^2) \right], \quad (6.47c)$$

$$\tilde{f}_{1,n}^*(\mu, z) = -\frac{1}{4} \log(1 + 4c_n^2 z^2 / \zeta_n^2), \quad \tilde{f}_{k \geq 2,n}^*(\mu, z) = \frac{c_n^2 z^2 Z_{k,n}^*(\mu, c_n^2 z^2)}{(1 + 4c_n^2 z^2 / \zeta_n^2)^{k-1}}, \quad (6.47d)$$

the coefficients of the functions $Q_{k,n}^*$, $P_{k,n}^*$, $U_{k,n}^*$, $V_{k,n}^*$, and $Z_{k,n}^*$, which are polynomials of the same degree as their unstarred counterparts, are themselves polynomials in the variable μ , of degree $4(k-1)$, $4(k-1)$, $4(k-1) + 2n$, $4(k-1) + 2n$, and $4(k-1)$, respectively. The first few independent polynomials in this case read:

$$\begin{aligned} Q_{2,1}^*(\mu, x) &= -\frac{1}{6}(1 - 2\mu + \mu^2 + 2\mu^3 - \mu^4) + \frac{1}{12}(4 - 8\mu + 7\mu^2 + 2\mu^3 - \mu^4)x, \\ Q_{2,2}^*(\mu, x) &= -\frac{8}{9}\mu^2(\mu - 1)^2x, \quad Q_{2,3}^*(\mu, x) = -\frac{3}{16}\mu^2(\mu - 1)^2x, \\ U_{1,1}^*(\mu, x) &= \frac{1}{2}(1 - \mu - \mu^2), \quad U_{1,2}^*(\mu, x) = -\frac{1}{4}(4 - 2\mu - 13\mu^2 + 6\mu^3 + \mu^4), \\ U_{2,1}^*(\mu, x) &= \frac{1}{6}(1 - 3\mu + 3\mu^2 - \mu^3 - 3\mu^4 + 3\mu^5 - \mu^6) \\ &\quad + \frac{1}{12}(4 - 12\mu + 15\mu^2 - 10\mu^3 - 15\mu^4 + 24\mu^5 - 10\mu^6)x \\ &\quad + \frac{1}{6}(4 - 12\mu + 9\mu^2 + 23\mu^3 - 24\mu^4 - 9\mu^5 + 5\mu^6)x^2, \\ Z_{2,1}^*(\mu, x) &= -\frac{1}{2}(4 - 8\mu + 5\mu^2 + 6\mu^3 - 3\mu^4), \quad Z_{2,2}^*(\mu, x) = \frac{4}{9}\mu^2(\mu - 1)^2. \end{aligned}$$

7 Resummation via $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations

In this section, we carry out the same computational steps as in the previous section, now in the context of the first-order $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations [134, 135]. This analysis serves both as a consistency check of the previous results and as a means to better understand the interplay between the blow-up equations and the resummation ansatz, with a view toward possible generalizations. On the one hand, the $\mathbb{C}^2$ blow-up equations yield purely algebraic relations, but require more summands due to the sum being restricted to half-integer shifts. On the other hand, the $\mathbb{C}^2/\mathbb{Z}_2$ case leads to differential equations while benefiting from the inclusion of both integer and half-integer shifts. Moreover, the solvability of the ODEs arising from the $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations can be traced back [141] to the simple algebraic relations among the resummation functions, which themselves follow from the $\mathbb{C}^2$ blow-up equations. We continue to follow the conventions of [43, 140], to which we refer the reader for further details, including the discussion of higher-order blow-up equations.

7.1 NS limit: $N_f \leq 4$ and $\mathcal{N} = 2^*$ theories

The first order $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations for the partition functions of $SU(2)$ gauge theories with $N_f \leq 4$ fundamental hypermultiplets read:

$$\sum_{2n \in \mathbb{Z}} D_{2\varepsilon_1, 2\varepsilon_2}^1 \left(\mathcal{Z}^{[N_f]}(a + 2n\varepsilon_1, \boldsymbol{\mu}; 2\varepsilon_1, \varepsilon_2 - \varepsilon_1; t), \mathcal{Z}^{[N_f]}(a + 2n\varepsilon_2, \boldsymbol{\mu}; 2\varepsilon_2, \varepsilon_1 - \varepsilon_2; t) \right) = 0, \quad (7.1)$$

while for the $\mathcal{N} = 2^*$ $SU(2)$ gauge theory, it takes the form:

$$\begin{aligned} & \sum_{2n \in \mathbb{Z}} D_{2\varepsilon_1, 2\varepsilon_2}^1 (\mathcal{Z}(a + 2n\varepsilon_1, m; 2\varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}), \mathcal{Z}(a + 2n\varepsilon_2, m; \varepsilon_1 - \varepsilon_2, 2\varepsilon_2; \mathbf{q})) \\ &= (\varepsilon_1 + \varepsilon_2) \gamma_0(\mathbf{q}) \sum_{2n \in \mathbb{Z}} \mathcal{Z}(a + 2n\varepsilon_1, m; 2\varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{q}) \mathcal{Z}(a + 2n\varepsilon_2, m; \varepsilon_1 - \varepsilon_2, 2\varepsilon_2; \mathbf{q}). \end{aligned} \quad (7.2)$$

The k -th asymmetric logarithmic Hirota derivative $D_{\varepsilon_1, \varepsilon_2}^k$ is defined by:

$$f(\mathbf{q}e^{\lambda\varepsilon_1}) g(\mathbf{q}e^{\lambda\varepsilon_2}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} D_{\varepsilon_1, \varepsilon_2}^k (f, g), \quad (7.3)$$

while $\gamma_0(\mathbf{q}) = 2\mathbf{q}\partial_{\mathbf{q}} \log[\theta_3(0|\tau)/\varphi(\mathbf{q})]$. As expected, in the pure gauge theory limit $\mathbf{q} \rightarrow 0$, $m \rightarrow \infty$, with $t = \mathbf{q}m^4$ held fixed, equation (7.2) reduces to equation (7.1) for $N_f = 0$. Taking the $\varepsilon_1 - \varepsilon_2 \rightarrow 0$ limit of the previous equations and performing computations analogous to those in the preceding section, we obtain:

$$\sum_{2n \in \mathbb{Z}} e^{2\mathcal{W}_1^{[N_f]} - 2(n\partial_a + \partial_h)\mathcal{W}_0^{[N_f]}} \partial_{\ln t} \left[\mathcal{W}_1^{[N_f]} - \left(n\partial_a + \partial_h + \frac{1}{h} \right) \mathcal{W}_0^{[N_f]} \right] = 0, \quad (7.4)$$

where $2\varepsilon_1 = 2\varepsilon_2 = \hbar$ and the first argument of each function is shifted by $n\hbar$, after taking the $\hbar$ -derivative: $\mathcal{W}_j^{[N_f]} = \mathcal{W}_j^{[N_f]}(a + n\hbar, \boldsymbol{\mu}, \hbar; t)$, $j = 0, 1$. Similarly, for the $\mathcal{N} = 2^*$ gauge theory, the NS limit of the first order blow-up equation (7.2) reads:

$$\sum_{2n \in \mathbb{Z}} e^{2\mathcal{W}_1 - 2(n\partial_a + \partial_h)\mathcal{W}_0} \left[\partial_{\ln \mathbf{q}} \mathcal{W}_1 - \left(n\partial_a + \partial_h + \frac{1}{h} \right) \partial_{\ln \mathbf{q}} \mathcal{W}_0 - \frac{1}{2} \gamma_0(\mathbf{q}) \right] = 0, \quad (7.5)$$

where $\mathcal{W}_j = \mathcal{W}_j(a + n\hbar, m, \hbar; \mathbf{q})$.

7.2 Solving the blow-up equations: $N_f \leq 4$ theories

We begin by considering equation (7.4) and proceed analogously to the case of the $\mathbb{C}^2$ blow-up equations. Specifically, we explicitly write out the classical and 1-loop parts of the prepotentials $\mathcal{W}_i$, normalize by the 1-loop contribution at $n = 0$, divide by the common factor $t^{-2\frac{a^2}{h^2} + \frac{1}{2}}$ and perform the substitution $n \rightarrow n/2$. With this normalization and setting $\hbar = 1$, equation (7.4) becomes:

$$\sum_{n \in \mathbb{Z}} t^{n^2/2} \tilde{\mathcal{G}}_{1\text{-loop}}^{[N_f]}(a, \boldsymbol{\mu}, n) \tilde{\mathcal{G}}_{\text{exp}}^{[N_f]}(a, \boldsymbol{\mu}, n; t) \tilde{\mathcal{G}}_{\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}, n; t) = 0, \quad (7.6)$$

where

$$\tilde{\mathcal{G}}_{\text{exp}}^{[N_f]}(a, \boldsymbol{\mu}, n; t) = e^{2\mathcal{W}_{1, \text{inst}}^{[N_f]} - (2\partial_h + n\partial_a)\mathcal{W}_{0, \text{inst}}^{[N_f]}}, \quad (7.7a)$$

$$\tilde{\mathcal{G}}_{\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}, n; t) = n(a + \frac{n}{2}) + t\partial_t \left[\mathcal{W}_{1, \text{inst}}^{[N_f]} - \left(\partial_h + \frac{n}{2}\partial_a + 1 \right) \mathcal{W}_{0, \text{inst}}^{[N_f]} \right], \quad (7.7b)$$

and $\mathcal{W}_{j,\text{inst}}^{[N_f]} = \mathcal{W}_{j,\text{inst}}^{[N_f]}(a + n\hbar/2, \boldsymbol{\mu}, \hbar; t)$. The 1-loop term $\tilde{\mathcal{G}}_{1\text{-loop}}^{[N_f]}(a, \boldsymbol{\mu}, n)$ can be computed as the $\varepsilon_1 = \varepsilon_2 = 1/2$ limit of the l -factors $l_n^{[N_f]}$ in Eq. (2.15) of [140]. Namely, defining the functions:

$$\tilde{g}_{1\text{-loop}}^{\text{hyp}}(a, \mu, n) = \begin{cases} \prod_{j=\frac{n-1}{2}}^{\frac{n-1}{2}} (\mu - j - a)^{2j} (\mu + j + a)^{2j}, & \text{if } n = 2p, \\ \prod_{j=1}^{\frac{n-1}{2}} (\mu - j - a)^{2j} (\mu + j + a)^{2j}, & \text{if } n = 2p + 1, \\ 1, & \text{if } n = 0, 1, \end{cases} \quad (7.8a)$$

$$\tilde{g}_{1\text{-loop}}^{\text{vec}}(a, n) = \begin{cases} (-1)^n 2a(n+2a)^{2n-1} \prod_{j=1}^{n-1} (j+2a)^{4j}, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \end{cases} \quad (7.8b)$$

$$(7.8c)$$

for $p \in \mathbb{N}_{\geq 1}$, the 1-loop factor can be written as:

$$\tilde{\mathcal{G}}_{1\text{-loop}}^{[N_f]}(a, \boldsymbol{\mu}, n) = \begin{cases} \frac{\prod_{i=1}^{N_f} \tilde{g}_{1\text{-loop}}^{\text{hyp}}(a, \mu_i, n)}{\tilde{g}_{1\text{-loop}}^{\text{vec}}(a, n)}, & \text{if } n \geq 0, \\ \frac{\prod_{i=1}^{N_f} \tilde{g}_{1\text{-loop}}^{\text{hyp}}(-a, \mu_i, -n)}{\tilde{g}_{1\text{-loop}}^{\text{vec}}(-a, -n)}, & \text{if } n < 0. \end{cases} \quad (7.9)$$

Once again, we consider the ansatz (6.23a), (6.23b), and expand the blow-up equations in the parameter t , evaluated at the points (6.24). We restrict to $m_1 \geq 0$ without loss of generality and focus on the simpler $N_f = 0$ case. We begin by deriving the differential equations satisfied by the resummation functions, and then show that these equations can be solved in closed form due to the specific form of their coefficients. Finally, we analyze the structure of the blow-up equation's expansion, taking into account the asymptotic behavior of the resummation functions near infinity. On the way, we will comment on the generalization to $N_f > 0$, pointing to the relevant formulas collected in Appendix C.1. In particular, the form of the differential equations we will study remains exactly the same for every choice of $N_f \leq 4$, due to the fact that the dependence on the masses $\boldsymbol{\mu}_{i=1}^{N_f}$ of the resummation functions is completely fixed by the boundary conditions.

7.2.1 ODEs from blow-ups

First of all, let us consider the rational functions $\tilde{\mathcal{G}}_{1\text{-loop}}(a, n) = \tilde{\mathcal{G}}_{1\text{-loop}}^{[0]}(a, n)$, which we rewrite for convenience:

$$\tilde{\mathcal{G}}_{1\text{-loop}}(a, n) = \begin{cases} \frac{(-1)^n}{2a(n+2a)^{2n-1}} \frac{1}{\prod_{j=1}^{n-1} (j+2a)^{4j}}, & \text{if } n \geq 0, \\ \frac{(-1)^n}{2a(n+2a)^{-2n-1}} \frac{1}{\prod_{j=1}^{-n-1} (j-2a)^{4j}}, & \text{if } n < 0. \end{cases} \quad (7.10)$$

Similarly to the $\mathbb{C}^2$ case, setting $2a = m_1 + t^{m_2/2}/x$, expanding in t , and combining with the overall factor of $t^{n^2/2}$ results in a predictable leading-order behavior. This clearly applies as well to the $N_f > 0$ case (7.9). Let us consider, then, the instanton contributions (7.7a) and (7.7b). Isolating the relevant terms for the expansion of the equation, similarly to equation (6.25), we find:

$$\begin{aligned} \log \tilde{\mathcal{G}}_{\text{exp}}^{\mp} &= \log \tilde{\mathcal{G}}_{\text{exp}} \left(\frac{m_1 x + t^{m_2/2}}{2x}, \mp j - m_1; t \right) = 2 \sum_{k \geq 1} t^{k-1} \left[\tilde{f}_{k,j}(y) \mp m_1 y^2 g'_{k,j}(y) \right. \\ &\quad \left. \pm t^{j/2} \left(\tilde{g}_{k,j}(y) + (2j + 4k - 5) g_{k,j}(y) + (2j - 1) y g'_{k,j}(y) \right) \right] + \sigma_{\text{exp}}^{\mp}(x, t), \end{aligned} \quad (7.11)$$

$$\begin{aligned}
 \tilde{\mathcal{G}}_{\text{inst}}^{\mp} = & \sigma_{\text{inst}}^{\mp}(x, t) + \frac{j}{2}(j \pm m_1) + \sum_{k \geq 1} t^{k-1} \left\{ \left[(k-1) \tilde{f}_{k,j}(y) + \frac{j}{2} y \tilde{f}'_{k,j}(y) \right. \right. \\
 & \mp m_1(k+j-1) y^2 g'_{k,j}(y) \mp j \frac{m_1}{2} y^3 g''_{k,j} \left. \right] \pm t^{j/2} \left[(k+j/2-1) \tilde{g}_{k,j} + \frac{j}{2} y \tilde{g}'_{k,j} \right. \\
 & \left. \left. + 2(k+j/2-1)(2k+j-3) g_{k,j} + (1-k+j(3j+4k-6)) y g'_{k,j} + \frac{j}{2} (2j-1) y^2 g''_{k,j} \right] \right\}
 \end{aligned} \tag{7.12}$$

where $y = xt^{(j-m_2)/2}$, a prime denotes a derivative with respect to y , and the functions $\sigma_{\text{exp/inst}}^{\mp}(x, t)$ vanish at $t = 0$. By collecting terms of the same order in t from the expansion of the blow-up equation, these expressions allow us to derive the form of the differential equations and demonstrate that the t -expansion around the poles (6.24) is well-behaved. Namely, by setting $j = m_2$, assuming $m_1 > 0$ for simplicity¹⁰, and performing the change of variables¹¹:

$$\tilde{f}_{1,m_2} = \frac{1}{2} \log u_{m_2}, \quad g'_{1,m_2} = \frac{1}{2x^2} \log(x^2 \tilde{v}_{m_2}), \tag{7.13}$$

we obtain at leading order:

$$u'_{m_2} + a_{m_1,m_2,1} u_{m_2} + b_{m_1,m_2,1} = 0, \tag{7.14}$$

with coefficients given by:

$$\begin{aligned}
 a_{m_1,m_2,1} &= \partial_x \log \left[x^{2m_2} (\tilde{v}_{m_2}^{m_1} + d_{m_1,m_2} \tilde{v}_{m_2}^{-m_1}) \right], \\
 b_{m_1,m_2,1} &= \frac{4}{m_2 x^{2m_1+1}} \frac{\rho_{m_1,m_2,1}}{\tilde{v}_{m_2}^{m_1} + d_{m_1,m_2} \tilde{v}_{m_2}^{-m_1}}.
 \end{aligned}$$

The constants d_{m_1,m_2} can be computed from the 1-loop term (7.10) and read:

$$d_{m_1,m_2} = -m_2^{-4m_1} [(m_2 - 1)!]^{-8m_1}. \tag{7.15}$$

We note here that, with this method, the change of variables (7.13) is necessary to obtain a linear differential equation (7.14) for the function u_{m_2} , and it introduces a logarithmic branching. Meanwhile, at orders t^{k-1} and $t^{k+j/2-1}$, respectively, compared to the previous equations, we find:

$$\tilde{f}'_{k,m_2} + a_{m_1,m_2,k} \tilde{f}_{k,m_2} + b_{m_1,m_2,k} = 0, \tag{7.16}$$

$$\tilde{g}'_{k,m_2} + \tilde{a}_{m_1,m_2,k} \tilde{g}_{k,m_2} + \tilde{b}_{m_1,m_2,k} = 0, \tag{7.17}$$

¹⁰The equations for $m_1 = 0$ involve all three types of resummation functions simultaneously and are automatically satisfied once a solution for the $m_1 > 0$ case is found.

¹¹In particular, the functions $\tilde{v}_{m_2}$ are related to the functions v_{m_2} introduced in the previous section simply by $v_{m_2}^2 = \tilde{v}_{m_2}$, up to a branching ambiguity.

with coefficients given by:

$$\begin{aligned}
 a_{m_1, m_2, k} &= \partial_x \left\{ \log [u_{m_2} (\tilde{v}_{m_2}^{m_1} + d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1})] + 2 \left(m_2 + \frac{k-1}{m_2} \right) \log x \right\}, \\
 b_{m_1, m_2, k} &= \left\{ \partial_x \left[m_1 u_{m_2} (\tilde{v}_{m_2}^{m_1} - d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1}) x^{2 \left(m_2 + 1 + \frac{k-1}{m_2} \right)} g'_{k, m_2} \right] \right. \\
 &\quad \left. + 2 \frac{u_{m_2}}{m_2} x^{2 \left(m_2 - m_1 + \frac{k-1}{m_2} \right) - 1} \rho_{m_1, m_2, k} \right\} e^{-\int a_{m_1, m_2, k} dx}, \\
 \tilde{a}_{m_1, m_2, k} &= \partial_x \left\{ \log [u_{m_2} (\tilde{v}_{m_2}^{m_1} - d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1})] + 2 \left(m_2 + \frac{1}{2} + \frac{k-1}{m_2} \right) \log x \right\}, \\
 \tilde{b}_{m_1, m_2, k} &= \frac{2}{m_2 x^{2m_1+1}} \frac{\tilde{\rho}_{m_1, m_2, k}}{u_{m_2} (\tilde{v}_{m_2}^{m_1} - d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1})}.
 \end{aligned}$$

The functions $\rho_{m_1, m_2, k}(x)$ and $\tilde{\rho}_{m_1, m_2, k}(x)$, which introduce inhomogeneous terms in the differential equations, account for the remaining contributions of the t -expansion at each fixed order and can be explicitly computed for any choice of m_1 and m_2 , similarly to the functions $r_{m_1, m_2, k}$ and $\tilde{r}_{m_1, m_2, k}$ in the previous section. The second and third sets of equations have been put in normal form for $\tilde{f}_{k, m_2}$ and $\tilde{g}_{k, m_2}$ respectively. The specific form chosen for the coefficients anticipates the method of solution, which will become evident in the following discussion. The former set can also be written in normal form with respect to g'_{k, m_2} , obtaining:

$$g''_{k, m_2} + \alpha_{m_1, m_2, k} g'_{k, m_2} + \beta_{m_1, m_2, k} = 0, \quad (7.18)$$

where

$$\begin{aligned}
 \alpha_{m_1, m_2, k} &= \partial_x \left[\log [u_{m_2} (\tilde{v}_{m_2}^{m_1} - d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1})] + 2 \left(m_2 + 1 + \frac{k-1}{m_2} \right) \log x \right], \\
 \beta_{m_1, m_2, k} &= \frac{1}{m_1} e^{-\int \alpha_{m_1, m_2, k} dx} \left\{ \partial_x \left[x^{2 \left(m_2 + \frac{k-1}{m_2} \right)} u_{m_2} (\tilde{v}_{m_2}^{m_1} + d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1}) \tilde{f}_{k, m_2} \right] \right. \\
 &\quad \left. + 2 \frac{u_{m_2}}{m_2} x^{2 \left(m_2 - m_1 + \frac{k-1}{m_2} \right) - 1} \rho_{m_1, m_2, k} \right\}.
 \end{aligned}$$

As previously anticipated, the structure of these equations remains unchanged even for $N_f > 0$, as can be verified using formulas (C.4) and (C.5). The mass dependence is implicitly introduced through appropriate replacements, such as $d_{m_1, m_2} \rightarrow d_{m_1, m_2}(\boldsymbol{\mu})$, $\rho_{m_1, m_2, k}(x) \rightarrow \rho_{m_1, m_2, k}(x, \boldsymbol{\mu})$, $\tilde{\rho}_{m_1, m_2, k}(x) \rightarrow \tilde{\rho}_{m_1, m_2, k}(x, \boldsymbol{\mu})$ and so forth.

7.2.2 Solvability of the ODEs

We now examine in more detail the general structure of the equations derived in the previous subsection, showing that, upon providing appropriate boundary conditions, they admit a closed form solution in terms of elementary functions. In practice, they can be solved in the order in which they arise from the expansion. Let us begin by analyzing equations (7.14). These equations are straightforwardly solvable via the method of variation of parameters, and their general solution is given by:

$$u_{m_2} = \frac{\kappa_{m_1, m_2, 1} x^{-2m_2}}{\tilde{v}_{m_2}^{m_1} + d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1}} \quad (7.19)$$

where

$$\kappa_{m_1, m_2, 1}(x) = \kappa_{m_1, m_2, 1}^{(0)} - \frac{4}{m_2} \int x^{2(m_2 - m_1) - 1} \rho_{m_1, m_2, 1}(x) dx, \quad (7.20)$$

and $\kappa_{m_1, m_2, 1}^{(0)}$ is an integration constant. In the case $m_2 = 1$ and $m_1 = 1$, this integration constant can be fixed by requiring $u_1(0) = 1$ and $\tilde{v}_1(x) \simeq 1/x^2$, i.e. $\tilde{f}_1(0) = 0$ and $g'_1(0) = \text{const}$, while already for $m_2 = 1$ and $m_1 = 2$ this is not the case. However, in Appendix C.3, we show that this second integration constant is in fact fixed by the blow-up equation itself. To determine $\tilde{v}_{m_2}$ we can simply equate two u_{m_2} 's for different values of m_1 , obtaining:

$$\frac{\kappa_{m_1, m_2, 1}}{\tilde{v}_{m_2}^{m_1} + d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1}} = \frac{\kappa_{\tilde{m}_1, m_2, 1}}{\tilde{v}_{m_2}^{\tilde{m}_1} + d_{\tilde{m}_1, m_2} \tilde{v}_{m_2}^{-\tilde{m}_1}}.$$

Due to the form of d_{m_1, m_2} in (7.15), setting $m_1 = 1$ and $\tilde{m}_1 = 2$, the function $\tilde{v}_{m_2}$ solves a simple quadratic equation:

$$\tilde{v}_{m_2}^2 - \frac{\kappa_{2, m_2, 1}}{\kappa_{1, m_2, 1}} \tilde{v}_{m_2} + \frac{1}{m_2^4 [(m_2 - 1)!]^8} = 0. \quad (7.21)$$

We note that, once again, this is the origin of the square roots that appear in the resummation functions. The same holds also for $N_f > 0$, as can be verified from equation (C.7). Of the two solutions of the quadratic equation, we keep the only one compatible with $\tilde{v}_{m_2}(x) \simeq 1/x^2$, which is then determined uniquely once the remaining integration constant has been fixed. Moving on to equations (7.16), we observe that the coefficients $a_{m_1, m_2, k}$ are still total derivatives. As a result, the equations can be solved in closed form, with solutions given by:

$$\tilde{f}_{k, m_2} = \frac{\kappa_{m_1, m_2, k}}{\kappa_{m_1, m_2, 1}} x^{-2\frac{k-1}{m_2}}, \quad (7.22)$$

where:

$$\kappa_{m_1, m_2, k}(x) = \kappa_{m_1, m_2, k}^{(0)} - \int e^{\int a_{m_1, m_2, k} dx} b_{m_1, m_2, k} dx, \quad (7.23)$$

and $\kappa_{m_1, m_2, k}^{(0)}$ is an integration constant, which can be fixed by requiring the vanishing of resummation functions at zero. Once $\tilde{f}_{k, m_2}$ is expressed in terms of g'_{k, m_2} , we can substitute this expression into (7.18) for a different value of m_1 , thereby obtaining a linear differential equation for g'_k . This equation can again be solved by performing simple integrals, and the two integration constants for g_k are then fixed by requiring the function to vanish at zero. Finally, also equations (7.17) can be solved in closed form, with their solution reading:

$$\tilde{g}_{k, m_2} = \frac{\tilde{\kappa}_{m_1, m_2, k} \tilde{v}_{m_2}^{m_1} + d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1}}{\kappa_{m_1, m_2, 1} \tilde{v}_{m_2}^{m_1} - d_{m_1, m_2} \tilde{v}_{m_2}^{-m_1}} x^{-2\frac{k-1}{m_2} - 1}. \quad (7.24)$$

where:

$$\tilde{\kappa}_{m_1, m_2, k}(x) = \tilde{\kappa}_{m_1, m_2, k}^{(0)} - \frac{2}{m_2} \int x^{2\left(m_2 - m_1 + \frac{k-1}{m_2}\right)} \tilde{\rho}_{m_1, m_2, k}(x) dx, \quad (7.25)$$

and $\tilde{\kappa}_{m_1, m_2, k}^{(0)}$ is again an integration constant, which can be determined by requiring the functions $\tilde{g}_{k, m_2}$ to vanish at zero. Thus, we have seen that expanding the blow-up equation (7.6) at two distinct values of m_1 is also sufficient to determine all the resummation functions. One can then verify that, even in this case, the expansion of the blow-up equation is well-defined, similarly to the discussion in Section 6.2.2. Again, similarly to the $\mathbb{C}^2$ case, the Taylor coefficients of the resummation functions for $j > m_2$ introduce certain subtleties, which we discuss in Appendix C.3 through the example of the solution of the first few equations.

7.3 Solving the blow-up equations: $\mathcal{N} = 2^*$ theory

In the $\mathcal{N} = 2^*$ case, denoting again by μ the mass of the adjoint hypermultiplet, we can write the first-order blow-up equation (7.5) as:

$$\sum_{n \in \mathbb{Z}} \mathfrak{q}^{n^2/2} \tilde{\mathcal{G}}_{1\text{-loop}}(a, \mu, n) \tilde{\mathcal{G}}_{\text{exp}}(a, \mu, n; \mathfrak{q}) \tilde{\mathcal{G}}_{\text{inst}}(a, \mu, n; \mathfrak{q}) = 0, \quad (7.26)$$

where explicitly the three terms read:

$$\tilde{\mathcal{G}}_{1\text{-loop}}(a, \mu, n) = \begin{cases} \frac{(2a+\mu)(n-\mu+2a)^{2n-1}}{2a(n+2a)^{2n-1}} \frac{\prod_{j=1}^{n-1} (j+\mu+2a)^{2j+1} (j-\mu+2a)^{2j-1}}{\prod_{j=1}^{n-1} (j+2a)^{4j}}, & \text{if } n > 0, \\ \frac{(2a-\mu)(n+\mu+2a)^{-2n-1}}{2a(n+2a)^{-2n-1}} \frac{\prod_{j=1}^{-n-1} (j+\mu-2a)^{2j+1} (j-\mu-2a)^{2j-1}}{\prod_{j=1}^{-n-1} (j-2a)^{4j}}, & \text{if } n < 0, \\ 1, & \text{if } n = 0, \end{cases} \quad (7.27a)$$

$$\tilde{\mathcal{G}}_{\text{exp}}(a, \mu, n; \mathfrak{q}) = e^{2\mathcal{W}_{1,\text{inst}} - (2\partial_{\hbar} + n\partial_a)\mathcal{W}_{0,\text{inst}}}, \quad (7.27b)$$

$$\tilde{\mathcal{G}}_{\text{inst}}(a, \mu, n; \mathfrak{q}) = n(a + \frac{n}{2}) - \frac{1}{2}\gamma_0 + \mathfrak{q}\partial_{\mathfrak{q}} \left[\mathcal{W}_{1,\text{inst}} - \left(\partial_{\hbar} + \frac{n}{2}\partial_a + 1 \right) \mathcal{W}_{0,\text{inst}} \right], \quad (7.27c)$$

with $\mathcal{W}_{i,\text{inst}} = \mathcal{W}_{i,\text{inst}}(a + \frac{n}{2}, \mu, 1; \mathfrak{q})$. To solve equation (7.26), we consider the ansatz (6.36a), (6.36b) and expand in $\mathfrak{q}$ the blow-up equation (7.26), setting the Coulomb branch parameter a to be as in equation (6.37). Then, we can compute the analogues of (7.11) and (7.12), which read:

$$\text{log } \tilde{\mathcal{G}}_{\text{exp}}^{\mp} = 2 \sum_{k \geq 1} \mathfrak{q}^{k-1} \left[\tilde{f}_{k,j} \mp m_1 y^2 \partial_y g_{k,j} \mp \mathfrak{q}^{j/2} (g_{k,j} - \tilde{g}_{k,j} + y \partial_y g_{k,j} - \mu \partial_{\mu} g_{k,j}) \right] + \sigma_{\text{exp}}^{*\mp}, \quad (7.28)$$

$$\begin{aligned} \tilde{\mathcal{G}}_{\text{inst}}^{\mp} &= \frac{j}{2}(j \pm m_1) + \sum_{k \geq 1} \mathfrak{q}^{k-1} \left[(k-1) \tilde{f}_{k,j} + \frac{j}{2} y \partial_y \tilde{f}_{k,j} \mp m_1 (k+j-1) y^2 \partial_y g_{k,j} \right. \\ &\quad \mp \frac{j}{2} m_1 y^3 \partial_y^2 g_{k,j} \pm \mathfrak{q}^{j/2} \left((k+j/2-1) \tilde{g}_{k,j} + \frac{j}{2} y \partial_y \tilde{g}_{k,j} - 2(k+j/2-1) g_{k,j} \right. \\ &\quad \left. \left. - (k+2j-1) y \partial_y g_{k,j} - \frac{j}{2} y^2 \partial_y^2 g_{k,j} + \mu (k+j/2-1) \partial_{\mu} g_{k,j} + \frac{j}{2} \mu y \partial_{\mu} \partial_y g_{k,j} \right) \right] + \sigma_{\text{inst}}^{*\mp}, \end{aligned} \quad (7.29)$$

where we recall that $y = x\mathfrak{q}^{(j-m_2)/2}$. These two expressions allow us to write the explicit form of the equations, up to the usual inhomogeneous part. Indeed, the equations we obtain have precisely the same form as those presented in the previous section, with the appropriate replacements to incorporate the implicit μ -dependence. For this reason, we do not repeat the discussion of their solution. However, in Appendix C.4, we show that the solution of these differential equations closely follows the same steps as those for the analogous equations in the pure gauge theory case. One can, once again, compute explicitly the constant $d_{m_1, m_2}(\mu)$ from (7.27a), obtaining:

$$d_{m_1, m_2}(\mu) = - \left(\frac{\mu(m_2 - \mu)}{m_2[(m_2 - 1)!]^2} \prod_{j=1}^{m_2-1} (j - \mu)(j + \mu) \right)^{4m_1}, \quad (7.30)$$

and show that the resulting $\mathfrak{q}$ -expansion of the blow-up equation is well behaved.

8 The periodic Lamé spectral problem

In this section we study the real periodic spectral problem associated with the Lamé equation on the rectangular torus, $\tau \in i\mathbb{R}_{>0}$. For convenience, we rescale the torus coordinate by setting $x = \pi z$, so that the periods become π and $\pi\tau$ and the real section is parametrized by $x + \pi\tau/2$, with $x \in [0, \pi]$. Then, the Lamé equation reads:

$$\left(\partial_x^2 + \mathfrak{e} - \mu(\mu-1)\wp(x + \pi\tau/2; \pi, \pi\tau)\right) \psi(x, \nu, \mu; \mathfrak{q}) = 0, \quad (8.1)$$

where the rescaled eigenvalue $\mathfrak{e} = E/\pi^2$ is given by:

$$\mathfrak{e}(\nu, \mu; \mathfrak{q}) = -2\mu(\mu-1)\frac{\eta_1}{\pi^2} - 4\mathfrak{q}\partial_{\mathfrak{q}}f(\nu, \mu; \mathfrak{q}). \quad (8.2)$$

where f is the classical one-point torus block in (5.28) understood as a function of $\nu = 2\alpha - 1$. From the point of view of the periodic spectral problem, the resonant values $\nu \in \mathbb{Z}$ correspond to the edges between gaps and bands in the spectrum, thus to eigenvalues associated to periodic and anti-periodic solutions of the equation. Indeed, the partial resummation of the classical block allows one to obtain a finite answer for these eigenvalues, simply by taking first the limit $\nu \rightarrow n^\pm$, or equivalently $\nu \rightarrow -n^\mp$, $n \in \mathbb{N}$, and then expanding around $\mathfrak{q} = 0$. In particular, from this perspective, it is clear that the fractional powers of $\mathfrak{q}$ appearing in the expansion of the eigenvalues arise from the square roots present in the resummation functions. Meanwhile, the distinction between eigenvalues corresponding to even and odd solutions stems from the direction in which the limit $\nu \rightarrow n$ is taken. As an example, the eigenvalues associated to $\nu^2 = 1, 4$ are respectively¹²:

$$\mathfrak{e}_{1,\pm}(\mu; \mathfrak{q}) = 1 - \frac{1}{3}\mu(\mu-1) \pm 4\mu(\mu-1)\mathfrak{q}^{1/2} - 2\mu(\mu-1)(\mu^2 - \mu - 4)\mathfrak{q} + \mathcal{O}(\mathfrak{q}^{3/2}), \quad (8.3a)$$

$$\mathfrak{e}_{2,+}(\mu; \mathfrak{q}) = 4 - \frac{1}{3}\mu(\mu-1) + \frac{20}{3}\mu^2(\mu-1)^2\mathfrak{q} + \mathcal{O}(\mathfrak{q}^2), \quad (8.3b)$$

$$\mathfrak{e}_{2,-}(\mu; \mathfrak{q}) = 4 - \frac{1}{3}\mu(\mu-1) - \frac{4}{3}\mu(\mu-1)(\mu-4)(\mu+3)\mathfrak{q} + \mathcal{O}(\mathfrak{q}^2), \quad (8.3c)$$

where $\pm$ refers to the $\nu \rightarrow n^\pm$ limit. In the Mathieu limit $\mathfrak{q} = t/\mu^4$, $\mu \rightarrow \infty$, they reduce to the corresponding eigenvalues¹³:

$$\pi^{-2}\lambda_{1,\pm}(t) = 1 \pm 4t^{1/2} - 2t + \mathcal{O}(t^{3/2}), \quad (8.4a)$$

$$\pi^{-2}\lambda_{2,+}(t) = 4 + \frac{20t}{3} - \frac{763t^2}{54} + \mathcal{O}(t^3), \quad (8.4b)$$

$$\pi^{-2}\lambda_{2,-}(t) = 4 - \frac{4t}{3} + \frac{5t^2}{54} + \mathcal{O}(t^3). \quad (8.4c)$$

The resulting expansions can be checked against the ones obtained from standard methods, as shown in Appendix B.1, and allow one to construct the stability chart of the equation. It is interesting to observe that, by choosing an integer value of $\mu \geq 2$, all poles in ν vanish from the perturbative $\mathfrak{q}$ -expansion of the eigenvalue, except for the first $\mu - 1$. This occurs due to the finite-gap property of the Lamé potential: for these specific values of μ , only the first $\mu - 1$ gaps in the spectrum remain open, while all subsequent ones are

¹²As the explicit formulae (8.3) and (8.4) show, in some cases the coefficients of the fractional-power terms in the eigenvalue expansions associated with the same Floquet exponent ν differ only by a sign. However, this is not a general feature.

¹³As mentioned at the end of Section 5, in order to compare with the expansion presented in [50], Section 28.6, one must identify $t = \frac{1}{16}q^2$.

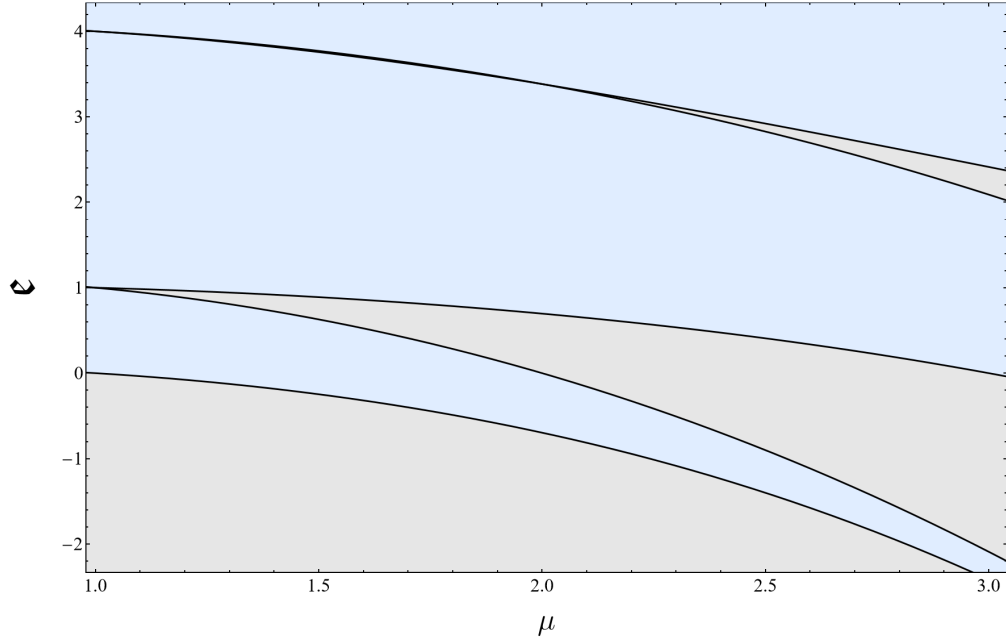


Figure II.1: A portion of the stability chart of the $\tau = i$ Lamé equation (8.1) is reconstructed by plotting the band-gap edges (8.3) (black lines) as functions of the coupling μ . Bands are shown in light blue, while gaps are shown in gray.

closed. This property is even more evident in the resummed classical block, where the functions $g_{k,j}^*$ as in (6.47) vanish when $\mu \in \{1 - j, \dots, j\}$, for any value of k . Moreover, the resummed classical block enables the construction of an infinite set of periodic functions of the Floquet exponent, corresponding to the bands in the spectrum. These functions are initially defined on the interval $\nu \in [0, 1]$ by

$$\mathfrak{e}^{(n)}(\nu, \mu; \mathfrak{q}) = \begin{cases} \mathfrak{e}(n + \nu, \mu; \mathfrak{q}), & \text{for } n \geq 0 \text{ even,} \\ \mathfrak{e}(n + 1 - \nu, \mu; \mathfrak{q}), & \text{for } n > 0 \text{ odd.} \end{cases} \quad (8.5)$$

They are then extended to $\nu \in [-1, 1]$ by imposing

$$\mathfrak{e}^{(n)}(-\nu, \mu; \mathfrak{q}) = \mathfrak{e}^{(n)}(\nu, \mu; \mathfrak{q}), \quad (8.6)$$

and finally to all real values of ν by requiring periodicity with period 2. At the resonant points $\nu = \pm 1$, this periodic continuation amounts to gluing the branch reached by crossing the cut at $\nu = 1$ from left to right to the branch defined immediately to the right of $\nu = -1$. Finally, we remark that studying the spectral problem (8.1) is equivalent to studying the Jacobi form of the Lamé equation:

$$\left(\partial_z^2 + E_J - \mu(\mu - 1)\kappa^2 \operatorname{sn}^2(z; \kappa^2) \right) \psi(z) = 0, \quad (8.7)$$

where κ is the elliptic modulus of the elliptic sine function, see Appendix A. The Jacobi-form energy E_J and the rescaled Weierstrass energy $\mathfrak{e}$ are related by

$$E_J(\nu, \mu; \kappa^2) = \left(\frac{\pi}{2K(\kappa^2)} \right)^2 \mathfrak{e}(\nu, \mu; \mathfrak{q}) - \mu(\mu - 1) \frac{e_3}{e_1 - e_3}, \quad (8.8)$$

where the rescaling of $\mathfrak{e}$ is due to the rescaling of the periods of the torus, which are $2K(\kappa^2)$ and $2iK(1 - \kappa^2)$ for equation (8.7). For instance, in the case of a square torus with $\tau = i$

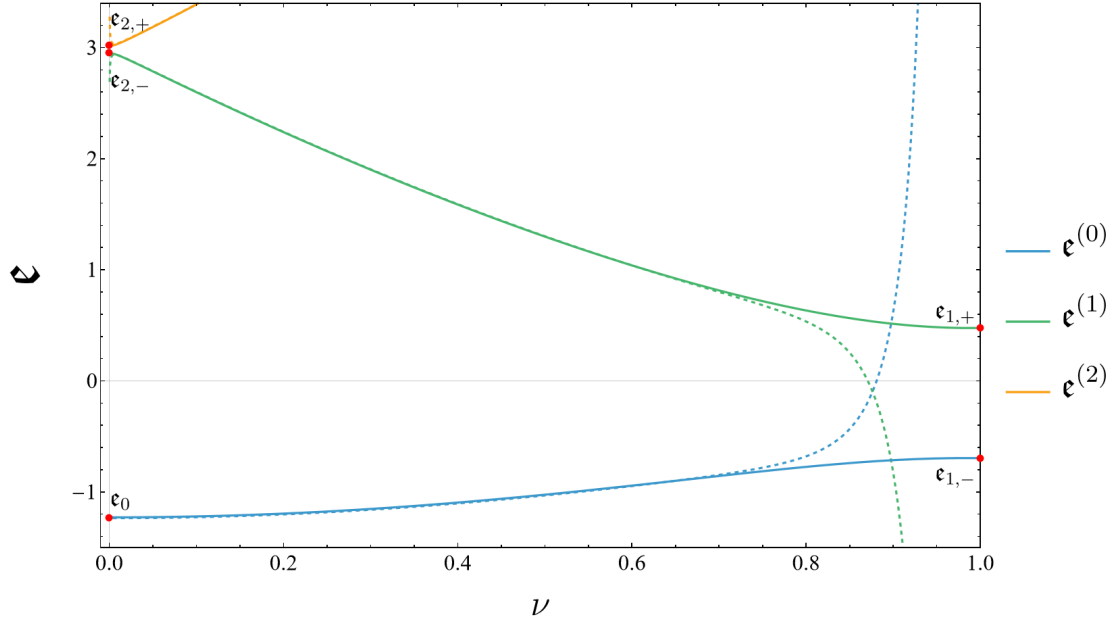


Figure II.2: The first three bands obtained from equation (8.5) using the standard NS prepotential (dashed lines) are compared with those obtained using the resummed prepotential (solid lines), for the coupling $\mu = 2.4$ and $\tau = i$. The two results show good agreement away from the band-gap edges, which are indicated by red dots.

and $\kappa^2 = 1/2$, we have the particularly simple relation:

$$E_J(\nu, \mu; \kappa^2) = \Gamma\left(\frac{3}{4}\right)^4 \frac{\mathfrak{e}(\nu, \mu; \mathfrak{q})}{\pi} + \frac{\mu(\mu - 1)}{2}. \quad (8.9)$$

8.1 (Anti-)Periodic eigenvalues from isomonodromic deformations

An alternative approach to the study of the periodic spectral problem (8.1) comes from the theory of isomonodromic deformations of linear ODEs. In this subsection, we closely follow the discussion in [43], to which the reader may refer for further details, and extend it to the periodic case. Consider the following Fuchsian linear system on the complex torus:

$$\partial_z Y(z, \tau) = A(z, \tau) Y(z, \tau), \quad (8.10)$$

with the matrix $A(z, \tau)$ given by

$$A(z, \tau) = \begin{pmatrix} p & mx(2Q, z) \\ mx(-2Q, z) & -p \end{pmatrix}, \quad (8.11)$$

where

$$x(u, z) = \frac{\theta_1(z - u|\tau)\theta_1'(0|\tau)}{\theta_1(z|\tau)\theta_1(u|\tau)},$$

is a Lamé function, while p and Q are complex auxiliary parameters. The matrix (8.11) has a simple pole at $z = 0$ and we can consider the associated isomonodromic deformation problem, thus the combined system of (8.10) and¹⁴

$$2\pi i \partial_\tau Y(z, \tau) = -MY(z, \tau), \quad M(z, \tau) = m \begin{pmatrix} \wp(2Q) & \partial_Q x(2Q, z) \\ \partial_Q x(-2Q, z) & \wp(2Q) \end{pmatrix}. \quad (8.12)$$

¹⁴The derivatives $\partial_Q x(\pm 2Q, z)$ are understood to mean $\partial_u x(u, z)$ evaluated at $u = \pm 2Q$.

Compatibility between the two linear systems then leads to:

$$p = 2\pi i \partial_\tau Q, \quad (8.13a)$$

$$(2\pi i)^2 \frac{d^2}{d\tau^2} Q = m^2 \wp'(2Q), \quad (8.13b)$$

the latter corresponding to the equation of motion for the classical non-autonomous 2-particle elliptic Calogero-Moser (eCM) system and a specific elliptic form of the Painlevé VI equation. Being a second order ODE, (8.13b) must be supplemented with two integration constants (σ, η) which are chosen such that:

$$Q|_{m=0} = \tau\sigma + \frac{\eta}{4\pi}. \quad (8.14)$$

These two parameters can be seen as coordinates on the space of monodromy data of the linear system (8.10). The non-autonomous eCM system is a classical integrable system, with Hamiltonian:

$$\mathcal{H} = p^2 - m^2 (\wp(2Q) + 2\eta_1(\tau)), \quad (8.15)$$

and corresponding tau-function $\mathcal{T}$ defined by:

$$\mathcal{H} = 2\pi i \partial_\tau \log \mathcal{T}. \quad (8.16)$$

This tau-function admits an explicit representation in terms of dual Nekrasov partition functions (8.18), which in turn are expressed in terms of self-dual Nekrasov partition functions¹⁵ [137, 142]. Namely:

$$\mathcal{T}(\sigma, m, \eta, \tau) = \eta(\tau) \frac{\mathcal{Z}_{1/2}^D(\sigma, m, \eta; \mathbf{q})}{\theta_2(2Q|2\tau)} = \eta(\tau) \frac{\mathcal{Z}_0^D(\sigma, m, \eta; \mathbf{q})}{\theta_3(2Q|2\tau)}, \quad (8.17)$$

where $Q = Q(\sigma, m, \eta, \tau)$ is a solution of (8.13b) and

$$\mathcal{Z}_\epsilon^D(\sigma, m, \eta; \mathbf{q}) = \sum_{n \in \mathbb{Z} + \epsilon} e^{in\eta} \frac{\prod_{\epsilon'=\pm} G(1 - m + 2\epsilon'(\sigma + n))}{\prod_{\epsilon'=\pm} G(1 + 2\epsilon'(\sigma + n))} \mathbf{q}^{(\sigma+n)^2 - \frac{1}{24}} \mathcal{Z}_{\text{inst}}(\sigma + n, m; \mathbf{q}). \quad (8.18)$$

The Lamé equation is obtained from (8.10) by turning the linear system into a second order equation for one of the components of $Y(z, \tau)$, provided the *singularity matching condition*:

$$Q(\sigma, m, \eta, \tau) = 0, \quad (8.19)$$

is satisfied, which can be shown to be equivalent to:

$$\theta_2(0|2\tau) \mathcal{Z}_0^D(\sigma, m, \eta; \mathbf{q}) - \theta_3(0|2\tau) \mathcal{Z}_{1/2}^D(\sigma, m, \eta; \mathbf{q}) = 0. \quad (8.20)$$

In particular, one finds:

$$\left(\partial_z^2 - m(m \pm 1) \wp(z) - E_\pm^B \right) \psi(z) = 0, \quad (8.21)$$

where $E_\pm^B$ denotes the accessory parameter in the convention of [43]. Since it enters the equation with a minus sign, it is related to our energy by $E = -E_\pm^B$, and hence to the rescaled eigenvalue by $\mathfrak{E} = -E_\pm^B/\pi^2$, with $\mu = m$ for the minus branch and $\mu = -m$ for

¹⁵In particular, $\mathcal{Z}_{\text{inst}}(\sigma, m; \mathbf{q}) = \mathcal{Z}_{\text{inst}}(\sigma, m; 1, -1; \mathbf{q})$, where the latter is defined in equation (A.31c), corresponds to a $c = 1$ conformal block via the identification (6.11).

the plus branch. The singularity matching condition expresses it in terms of the Painlevé tau-function $\mathcal{T}(\sigma, m, \eta, \tau)$ evaluated on a solution of (8.19):

$$E_{\pm}^B = \left[2\pi i \partial_{\tau} \left(\log \mathcal{Z}_0^D(\sigma, m, \eta; \mathbf{q}) + \log \frac{\eta(\tau)}{\theta_3(0|2\tau)} \right) \pm 2m \frac{\theta_3''(0|2\tau)}{\theta_3(0|2\tau)} + 2m(m \pm 1)\eta_1(\tau) \right] \Big|_{Q=0}. \quad (8.22)$$

The sign $\pm$ corresponds to the two possible branches of the solution to equation (8.13b) near $Q = 0$. In order to compute the eigenvalues associated to periodic and anti-periodic solutions to the spectral problem (8.1), we interpret the singularity matching condition (8.20) as a quantization condition for the system, which we solve in two non-generic limits¹⁶ of the monodromy data (σ, η) , corresponding to limits to logarithmic Painlevé tau-functions [143–145]. In particular, setting $\mathbf{t} = \mathbf{q}^{1/2} = e^{i\pi\tau}$, in order to obtain “periodic” eigenvalues we will solve (8.20) in the limit $\eta = \Omega\sigma$ and $\sigma \rightarrow 0$, as an equation for $\Omega = \Omega(m, \mathbf{t})$, while to obtain “anti-periodic” eigenvalues we will solve it in the limit $\eta = \Omega(\sigma - 1/2)$ and $\sigma \rightarrow 1/2$. Once a solution is found, this can be substituted in the corresponding limit of the eigenvalue (8.22) to obtain the desired result. In both periodic and anti-periodic cases, we find:

$$\Omega_{|\nu|}(m, \mathbf{t}) = \omega_{|\nu|}(m, \mathbf{t}) + 8iH(-1 - m) + 4i \log \mathbf{t}, \quad \nu \in \mathbb{Z}, \quad (8.23)$$

where $H(x)$ is the harmonic number function, and $\omega_{|\nu|}(m, \mathbf{t}) = \mathbf{t}^{-|\nu|}(\omega_{-|\nu|}(m) + \mathcal{O}(\mathbf{t}))$ depends on the absolute value of the specific integer Floquet exponent considered. In particular, we obtain:

$$\Omega_0(m, \mathbf{t}) = -4i \lim_{\nu \rightarrow 0} \frac{1}{\nu} \frac{\partial f}{\partial \nu} \left(\nu, \mu = m; \mathbf{t}^2 \right), \quad (8.24)$$

where the classical conformal block $f(\nu, \mu; \mathbf{q})$ in this formula contains also a 1-loop part, as in equation (8.43). Thus, we find a connection between $c = 1$ conformal blocks and classical ones, or equivalently between the self-dual limit of Nekrasov functions and the NS one. The corresponding eigenvalues are also observed to match, i.e. we find:

$$E_{-}^B(\nu, m; \mathbf{t}) \Big|_{\nu \rightarrow 0} = -\pi^2 \mathfrak{e} \left(0, m; \mathbf{t}^2 \right),$$

where $\mathfrak{e}(\nu, \mu; \mathbf{q})$ is as in equation (8.2). Meanwhile, for $|\nu| > 0$, we find a quadratic equation that determines the first coefficient $\omega_{-|\nu|}(m)$, which, in turn, selects the eigenvalue associated to odd or even solutions. Interestingly, we find that $\omega_{-|\nu|}(m)$ has the form:

$$\omega_{-|\nu|}^{\mp}(m) = \mp(-1)^{|\nu|} \frac{iN}{(m \mp |\nu|) \prod_{j=1-|\nu|}^{|\nu|-1} (m + j)}, \quad (8.25)$$

with N a positive integer. Once $\Omega_{|\nu|}(m, \mathbf{t})$ has been determined up to the desired order, the eigenvalues can be obtained as follows: the solution derived from $\omega_{-|\nu|}^{-}$ can be directly substituted into equation (8.22) for E_{-}^B , whereas the solution obtained from $\omega_{-|\nu|}^{+}$ must be substituted into the equation for E_{+}^B , with the additional step of sending $m \rightarrow -m$.

8.2 (Anti-)Periodic eigenfunctions from orbifold defects

Another approach to studying the periodic spectral problem emerges directly from the $\mathcal{N} = 2^* \text{SU}(2)$ gauge theory, in which equation (8.1) can be interpreted as a quantization

¹⁶From the gauge theory viewpoint, this should correspond to the alignment of the central charges of two BPS particles, defining a wall of marginal stability. It would be interesting to further investigate the problem from this perspective.

of the Seiberg–Witten curve [71, 72], capturing the dynamical content of the low-energy effective theory. In this framework, solutions to the Lamé equation correspond to the NS limit of expectation values of certain BPS surface defect operators [34, 146], which can be computed via localization in the Ω -background. At least two types of BPS surface defects can be considered; the first is associated with the insertion of a degenerate primary operator in the dual CFT picture [33, 91], precisely the one we examined at the beginning of this work. The second corresponds to an orbifold defect [36, 147], wherein the theory is defined on $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_2$ instead of on $\mathbb{C}^2 \simeq \mathbb{R}^4$. See also Chapter III, where the orbifold construction is reviewed for the $SU(2)$ gauge theory with four fundamental hypermultiplets.

To fully characterize this second type of defect, one must specify a coloring function $c : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, which determines the gauge group component of the embedding of $\mathbb{Z}_2$ into the symmetry group of the theory. In particular, we focus on regular orbifold defects, characterized by a surjective coloring function. In this case, there are only two possibilities: either $c = \text{id} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, or $c = (01) : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, the elementary transposition. Let $\mathbf{Y} = (Y_0, Y_1)$ denote a pair of Young diagrams. In the presence of such a defect, the elementary characters appearing in the equivariant localization formulas used to compute the instanton partition functions decompose as

$$N_\omega = \sum_{\alpha \in c^{-1}(\omega)} e^{a_\alpha}, \quad (8.26a)$$

$$K_\omega = \sum_{\alpha=0,1} \sum_{\substack{(r,s) \in Y_\alpha \\ c(\alpha)+s \equiv \omega+1 \pmod{2}}} e^{a_\alpha + \varepsilon_1(r-1) + \varepsilon_2(s-1)}, \quad (8.26b)$$

where $\omega \in \mathbb{Z}_2$. The full $\mathbb{Z}_2$ -invariant character is then given by:

$$T[\mathbf{Y}]_c = (1 - e^{-\mu}) \sum_{\omega \in \mathbb{Z}_2} [N_\omega K_\omega^* + q_1 q_2 N_\omega^* K_{\omega-1} - (1 - q_1) K_\omega K_\omega^* + q_2(1 - q_1) K_\omega K_{\omega+1}^*], \quad (8.27)$$

where $q_i = e^{\varepsilon_i}$ and the $*$ -operation reverses the sign of all the weights in the exponents. In particular, for the id case we have $N_0 = e^{a_0}$, $N_1 = e^{a_1}$ and

$$K_0 = \sum_{(r,s) \in Y_0, s \text{ odd}} e^{a_0 + \varepsilon_1(r-1) + \varepsilon_2(s-1)} + \sum_{(r,s) \in Y_1, s \text{ even}} e^{a_1 + \varepsilon_1(r-1) + \varepsilon_2(s-1)}, \quad (8.28)$$

$$K_1 = \sum_{(r,s) \in Y_0, s \text{ even}} e^{a_0 + \varepsilon_1(r-1) + \varepsilon_2(s-1)} + \sum_{(r,s) \in Y_1, s \text{ odd}} e^{a_1 + \varepsilon_1(r-1) + \varepsilon_2(s-1)}, \quad (8.29)$$

while, for the (01) case, we have $N_0 = e^{a_1}$, $N_1 = e^{a_0}$ and

$$K_0 = \sum_{(r,s) \in Y_0, s \text{ even}} e^{a_0 + \varepsilon_1(r-1) + \varepsilon_2(s-1)} + \sum_{(r,s) \in Y_1, s \text{ odd}} e^{a_1 + \varepsilon_1(r-1) + \varepsilon_2(s-1)}, \quad (8.30)$$

$$K_1 = \sum_{(r,s) \in Y_0, s \text{ odd}} e^{a_0 + \varepsilon_1(r-1) + \varepsilon_2(s-1)} + \sum_{(r,s) \in Y_1, s \text{ even}} e^{a_1 + \varepsilon_1(r-1) + \varepsilon_2(s-1)}. \quad (8.31)$$

Defining the fractional coupling constants $\mathbf{t}_0 = \mathbf{t}\xi$ and $\mathbf{t}_1 = \mathbf{t}/\xi$, such that $\mathbf{t}^2 = \mathbf{q} = e^{2\pi i \tau}$, the instanton part of the orbifold defect partition function is given by:

$$\Psi_c^{\text{inst}}(\xi, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}) = \sum_{\mathbf{Y}} \prod_{\omega \in \mathbb{Z}_2} \mathbf{t}_\omega^{k_\omega} \mathcal{E}(-T[\mathbf{Y}]_c), \quad (8.32)$$

where $k_\omega = \dim K_\omega$, $\nu = (a_0 - a_1)/\varepsilon_1$, $a_0 = -a_1 = \varepsilon_1 \nu/2$, and $\mathcal{E}$ acts by transforming sums of exponentials of weights into products of weights:

$$\mathcal{E} \left(\sum_{\alpha} e^{w_\alpha} - \sum_{\beta} e^{w_\beta} \right) = \frac{\prod_{\alpha} w_\alpha}{\prod_{\beta} w_\beta}.$$

Thus, setting $\xi = e^{2ix}$, we define the full defect partition functions as:

$$\Psi_{\text{id}}(x, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}) = e^{i\nu x} \theta_4(x/\pi|\tau)^{1-\mu} \Psi_{\text{id}}^{\text{inst}}(e^{2ix}, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}), \quad (8.33a)$$

$$\Psi_{(01)}(x, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}) = e^{-i\nu x} \theta_4(x/\pi|\tau)^{1-\mu} \Psi_{(01)}^{\text{inst}}(e^{2ix}, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}). \quad (8.33b)$$

Employing the qq-characters formalism [34], one can show that the NS limit of the two defect partition functions:

$$\psi_c(x, \nu, \mu; \mathbf{t}) = \lim_{\varepsilon_2 \rightarrow 0} \frac{\Psi_c(x, \nu, \mu; 1, \varepsilon_2; \mathbf{t})}{\sqrt{\mathcal{Z}_{\text{inst}}^{\text{U}(2)}(\nu/2, \mu; 1, \varepsilon_2; \mathbf{t}^2)}}, \quad (8.34)$$

where $\mathcal{Z}_{\text{inst}}^{\text{U}(2)}$ is as in equation (A.32), constitute a pair of linearly independent solutions to the Lamé equation (8.1). Indeed, defining the effective twisted superpotential $\mathcal{W}_0$ as

$$\mathcal{W}_0(\nu, \mu, \varepsilon_1; \mathbf{t}) = \lim_{\varepsilon_2 \rightarrow 0} \varepsilon_2 \log \Psi_c^{\text{inst}}(\xi, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}), \quad (8.35)$$

the analogue of formula (8.2) takes the form:

$$\mathfrak{e}(\nu, \mu; \mathbf{t}) = \nu^2 - 4\mathbf{t}\partial_{\mathbf{t}}\mathcal{W}_0(\nu, \mu, 1; \mathbf{t}) + \frac{1}{3}\mu(\mu-1)(1-2E_2(\tau)), \quad (8.36)$$

which yields the standard perturbative expansion of the eigenvalue. By instead considering the even and odd combinations:

$$\Psi_{\pm}(x, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}) = \Psi_{\text{id}}(x, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}) \pm \Psi_{(01)}(x, \nu, \mu; \varepsilon_1, \varepsilon_2; \mathbf{t}), \quad (8.37)$$

the effective twisted superpotential computed using these functions remains regular even for integer values of the Floquet exponent. Namely, if we define:

$$\mathcal{W}_0^{n,\pm}(\mu; \mathbf{t}) = \lim_{\varepsilon_2 \rightarrow 0} \varepsilon_2 \log \Psi_{\pm}(x, \nu = n, \mu; \varepsilon_1 = 1, \varepsilon_2; \mathbf{t}), \quad (8.38)$$

we can compute the periodic and anti-periodic eigenvalues simply through¹⁷:

$$\mathfrak{e}_{n,\pm}(\mu; \mathbf{t}) = n^2 - 4\mathbf{t}\partial_{\mathbf{t}}\mathcal{W}_0^{n,\pm}(\mu; \mathbf{t}) + \frac{1}{3}\mu(\mu-1)(1-2E_2(\tau)). \quad (8.39)$$

Finally, also periodic and anti-periodic eigenfunctions can be obtained from $\Psi_{\pm}$, by considering:

$$\psi_{n,\pm}(x, \mu; \mathbf{t}) = \lim_{\varepsilon_2 \rightarrow 0} e^{-\mathcal{W}_0^{n,\pm}(\mu; \mathbf{t})/\varepsilon_2} \Psi_{\pm}(x, \nu = n, \mu; \varepsilon_1 = 1, \varepsilon_2; \mathbf{t}). \quad (8.40)$$

8.3 B-cycle periodic spectral problem

Until now, we have studied the periodic spectral problem on the A-cycle (8.1), however this is not the only possibility. Indeed, we can also examine the periodic spectral problem on the B-cycle:

$$\left(\partial_x^2 - \mathfrak{e} + \mu(\mu-1)\wp(ix + \pi/2; \pi, \pi\tau)\right)\psi = 0, \quad x \in [0, \pi|\tau|], \quad (8.41)$$

¹⁷We note that the $\pm$ symbol here indicates the parity of the (anti-)periodic eigenfunction, in contrast to the lateral limits $\nu \rightarrow n^{\pm}$ used above. These two conventions do not necessarily agree.

where, once again, $\tau \in i\mathbb{R}_{>0}$, and the eigenvalue $\mathfrak{e}$ is given by equation (8.2). Due to the modular properties of the $\wp$ -function, this problem is actually equivalent to the previous one under the transformation $\tau \rightarrow -1/\tau$. Indeed, since:

$$\wp(z; \tau) = \frac{1}{\tau^2} \wp(z/\tau; -1/\tau),$$

by making the change of variable $\tilde{x} = ix/\tau$, the equation becomes:

$$\left(-\partial_{\tilde{x}}^2 - \mathfrak{e}\tau^2 + \mu(\mu - 1) \wp(\tilde{x} + \pi/2\tau; \pi, -\pi/\tau) \right) \psi = 0.$$

In particular, this implies that studying the B-cycle spectral problem (8.41) in the limit $\tau \rightarrow i\infty$ is equivalent to studying the A-cycle spectral problem (8.1) in the opposite limit $\tau \rightarrow 0$. To this end, we can leverage the knowledge of the monodromy matrices to gain insights into the spectrum in this regime. Recalling that $\nu = 2a = 2\alpha - 1$, the trace of the B-cycle monodromy matrix $M_B^{(n)}$ in equation (5.36) can be written as:

$$\text{Tr } M_B = \frac{\sin \pi(2a + \mu)}{\sin 2\pi a} e^{\eta/2} + \frac{\sin \pi(2a - \mu)}{\sin 2\pi a} e^{-\eta/2}. \quad (8.42)$$

Here, a term that coincides with the 1-loop contribution to the NS prepotential of the $\mathcal{N} = 2^*$ gauge theory has been absorbed into the definition of the conformal block $f(a, \mu; \mathfrak{q})$, so that the dual period η is given by

$$\eta(a, \mu; \mathfrak{q}) = \partial_a f(a, \mu; \mathfrak{q}) = -2a \log \mathfrak{q} + 2 \log \frac{\Gamma(1 + 2a)\Gamma(1 - 2a - \mu)}{\Gamma(1 - 2a)\Gamma(1 + 2a - \mu)} + \partial_a f_{\text{inst}}(a, \mu; \mathfrak{q}). \quad (8.43)$$

Since we are interested in the spectral problem for values of the coupling $\mu \geq 1$, it is more convenient to rewrite (8.42) as:

$$\text{Tr } M_B = -\frac{\sin \pi(2a - \mu)}{\sin 2\pi a} e^{\eta/2} - \frac{\sin \pi(2a + \mu)}{\sin 2\pi a} e^{-\eta/2}, \quad (8.44)$$

where the dual period now reads¹⁸

$$\eta(a, \mu; \mathfrak{q}) = \partial_a f(a, \mu; \mathfrak{q}) = -2a \log \mathfrak{q} + 2 \log \frac{\Gamma(1 + 2a)\Gamma(\mu - 2a)}{\Gamma(1 - 2a)\Gamma(\mu + 2a)} + \partial_a f_{\text{inst}}(a, \mu; \mathfrak{q}), \quad (8.45)$$

From the general theory of periodic differential equations, we can write:

$$\text{Tr } M_B = 2 \cos(\pi \nu_D), \quad (8.46)$$

where ν_D is the B-cycle Floquet exponent, such that when it is an integer, the corresponding non-logarithmic solutions are either periodic or anti-periodic along the B-cycle. Thus, we can solve this equation for a as a function of ν_D and the other variables, and then use the explicit expression for the eigenvalue (8.2) to determine the energy of the system as a function of these parameters.

¹⁸The dual period η , in either of the two forms (8.43) and (8.45), with f_{inst} given by the resummed instanton function obtained through (6.12) and (6.36a), satisfies $\lim_{a \rightarrow n/2} \exp \eta(a, \mu; \mathfrak{q}) = 1$, $n \in \mathbb{Z}$, to all orders accessible from the available resummation functions. See the end of subsection 10.2.2 for the analogous statement for SU(2) theories with fundamental hypermultiplets.

8.3.1 Scattering states

For generic values of μ , a first set of solutions can be obtained by considering the following double expansion of the Coulomb branch parameter:

$$a_n(\nu_D, \mu, \tau) = \frac{n}{2\tau} + \sum_{k \geq 0} c_k^{(n)}(\nu_D, \mu, \tau) \mathfrak{q}^k, \quad c_k^{(n)}(\nu_D, \mu, \tau) = \sum_{j \geq 2} c_{k,j}^{(n)}(\nu_D, \mu) \tau^{-j},$$

with $n \in \mathbb{Z} \setminus \{0\}$, which implies a corresponding double expansion for the eigenvalue. For physical values of the parameters, these solutions are purely imaginary and the very first terms read

$$a_n(\nu_D, \mu, \tau) = \frac{n}{2\tau} + \frac{\mathrm{i}n [\pi (\cos \pi \mu + \cos \pi(\nu_D + n)) \csc \pi \mu + 2H(\mu - 1)]}{2\pi \tau^2} + \mathcal{O}(\tau^{-3}), \quad (8.47)$$

while for the corresponding eigenvalues we find

$$\begin{aligned} \mathfrak{e}_n(\nu_D, \mu, \tau) = & -\frac{1}{3}\mu(\mu - 1) + \frac{n^2}{\tau^2} \\ & + \frac{2\mathrm{i}n^2 [\pi (\cos \pi(\nu_D + n) + \cos \pi \mu) \csc \pi \mu + 2H(\mu - 1)]}{\pi \tau^3} + \mathcal{O}(\tau^{-4}). \end{aligned} \quad (8.48)$$

In particular, one can notice that the dual Floquet exponent ν_D always appears in combination with the integer n as $\cos \pi(\nu_D + n)$. These expansions exhibit poles for integer values of μ , due to terms proportional to $\csc \pi \mu$, except if:

$$\cos \pi(\nu_D + n) + \cos \pi \mu = 0 \iff \cos \pi(\nu_D + n) = \cos \pi(\mu - 1),$$

which implies that the two monodromy exponents μ and ν_D are essentially identified, and in particular, ν_D is constrained to be an integer when μ is an integer. We will now show that, in the case where μ is an integer, this term is precisely what is missing from the expansion, without imposing any restriction on ν_D . Indeed, for $\mu = k \in \mathbb{Z}$, equation (8.46) simplifies to:

$$(-1)^{k-1} \cosh \frac{\eta}{2} = \cos \pi \nu_D, \quad (8.49)$$

and η also simplifies significantly; in particular, the Γ -functions disappear leaving behind only simple rational functions. In this case, we can solve directly for the B-cycle Floquet exponent $\nu_D = \nu_D(a, k, \tau)$ as:

$$\nu_D(a, k, \tau) = k - 1 + 2n \pm \frac{\mathrm{i}}{2\pi} \eta(a, k; \mathfrak{q}), \quad n \in \mathbb{Z}. \quad (8.50)$$

This relation can then be inverted to express a as $a = a(\nu_D, k, \tau)$, leading to an expansion analogous to (5.33) for the eigenvalue as a function of the B-cycle Floquet exponent. The first few terms of the resulting expression read:

$$\begin{aligned} \mathfrak{e}_n(\nu_D, k, \tau) = & -\frac{1}{3}k(k - 1) + \frac{\xi_n^2}{\tau^2} + \frac{4\mathrm{i}\xi_n^2 H(k - 1)}{\pi \tau^3} - \frac{12\xi_n^2 H(k - 1)^2}{\pi^2 \tau^4} + \mathcal{O}(\tau^{-5}) \\ & + 8k(k - 1) \left((1 + k - k^2) - k(k - 1) \xi_n^2 \left(\frac{1}{\tau^2} + \frac{\mathrm{i}(1 + 4H(k - 1))}{\pi \tau^3} \right) + \mathcal{O}(\tau^{-4}) \right) \mathfrak{q} + \mathcal{O}(\mathfrak{q}^2), \end{aligned} \quad (8.51)$$

where $\xi_n = \nu_D - (k - 1 + 2n)$. Interestingly, when also $\nu_D \in \mathbb{Z}$, equation (8.49) is equivalent to the usual NS quantization condition:

$$\partial_a f(a, \mu; \mathfrak{q}) = 2\pi \mathrm{i} \tilde{n}, \quad \tilde{n} \in \mathbb{Z}, \quad \tilde{n} \equiv k - 1 - \nu_D \pmod{2}. \quad (8.52)$$

The solutions are then characterized by the integer $\tilde{n}$ and the first few terms of their expansion reads:

$$a_{\tilde{n}}(k, \tau) = -\frac{\tilde{n}}{2\tau} - \frac{i\tilde{n}H(k-1)}{\pi\tau^2} + \frac{2\tilde{n}H(k-1)^2}{\pi^2\tau^3} + \mathcal{O}(\tau^{-4}) \\ + \left(\frac{2i\tilde{n}k^2(k-1)^2}{\pi\tau^2} - \frac{8\tilde{n}k^2(k-1)^2H(k-1)}{\pi^2\tau^3} + \mathcal{O}(\tau^{-4}) \right) \mathbf{q} + \mathcal{O}(\mathbf{q}^2), \quad (8.53)$$

while the corresponding eigenvalues are given by equation (8.51) where ξ_n is replaced by $\tilde{n}$. Finally, the quantization condition simplifies also in the case $\mu = k + 1/2$, $k \in \mathbb{Z}$, resulting in:

$$(-1)^k \cot(2\pi a) \sinh\left(\frac{\eta}{2}\right) = \cos(\pi\nu_D). \quad (8.54)$$

8.3.2 Quasi-bound states

A second set of qualitatively different solutions can instead be obtained by expanding the quantization condition (8.46) for a near the poles of the Γ -functions in (8.45). Assuming $\mu \geq 1$, we can consider $a \in \mathbb{R}$. In particular, the trace of the monodromy matrix (8.44) is invariant under $a \rightarrow -a$. Consequently, we can restrict to $a > 0$, with $a = 0$ being shared with the previous set of solutions. Then, to ensure that ν_D remains finite in the $\tau \rightarrow i\infty$ limit, the only possibility is:

$$\hat{a}_n(\nu_D, \mu; \mathbf{q}) = \frac{\mu - n}{2} + \sum_{i \geq 1} \sum_{j \geq 0} a_{i,j}(\nu_D, \mu; \log \mathbf{q}) \mathbf{q}^{i(\mu-n)/2+j}, \quad (8.55)$$

where $n \in \mathbb{Z}$, $1 \leq n < \mu$. This second set of solutions consists again of double expansions, this time in $\mathbf{q}^{\frac{\mu-n}{2}}$ and $\mathbf{q}$, which correspond to the bound states of the Pöschl–Teller potential in the $\tau \rightarrow i\infty$ limit, see Appendix B.2. The first terms in the corresponding expansion of the eigenvalue are

$$\hat{\mathbf{e}}_n(\nu_D, \mu; \mathbf{q}) = (\mu - n)^2 - \frac{1}{3}\mu(\mu - 1) - \frac{4 \cos \pi\nu_D}{\pi} \frac{\Gamma(1 + n - \mu)\Gamma(2\mu - n) \sin \pi\mu}{\Gamma(n)\Gamma(\mu - n)} \mathbf{q}^{\frac{\mu-n}{2}} \\ + \frac{8\mu(\mu - 1)(\mu(\mu - 1) - 1 + (\mu - n)^2)}{(\mu - n)^2 - 1} \mathbf{q} + \dots, \quad (8.56)$$

where, depending on the values of n and μ , some of the terms omitted from the expansion may be of lower order in $\mathbf{q}$ than the displayed term of order $\mathbf{q}^{\frac{\mu-n}{2}}$. Expanding for a near the remaining poles of the Γ -functions results in a divergent $\text{Tr } M_B$, except in the case where μ is an integer. In this situation, the poles at $a = n/2$, with $n \in \mathbb{N}$, coincide with those associated with quasi-bound states. Indeed, due to cancellations between the Γ -functions, equation (8.49) can be explicitly written as:

$$e^{2\pi a \text{Im}(\tau) + \frac{1}{2} \partial_a f_{\text{inst}}} \frac{(\mu - 1 - 2a) \cdots (1 - 2a)}{(\mu - 1 + 2a) \cdots (1 + 2a)} + (a \rightarrow -a) = 2(-1)^{\mu-1} \cos \pi\nu_D. \quad (8.57)$$

Assuming $\mu \geq 1$ and $a > 0$, the remaining poles of the Γ -functions are now located at $a = \frac{1}{2}, \dots, \frac{\mu-1}{2}$. This implies that analyzing these points requires the resummation of f_{inst} around these specific values of a . For instance, in the case $\mu = 2$, there is a unique solution near $a \sim \frac{1}{2}$. Using the methods introduced in the previous sections to compute $\partial_a f_{\text{inst}}$ around this singular point, we find:

$$\hat{\mathbf{e}}_1(\nu_D; \mathbf{q}) = \frac{1}{3} + 8 \cos \pi\nu_D \sqrt{\mathbf{q}} + 8(1 - 4(1 + \log \sqrt{\mathbf{q}}) \sin^2 \pi\nu_D) \mathbf{q} + \dots \quad (8.58)$$

Meanwhile, for $\mu = 3$, there are two solutions, $a \sim \frac{1}{2}$ and $a \sim 1$, for which the corresponding eigenvalues are given by:

$$\begin{aligned} \hat{\mathbf{e}}_1(\nu_D; \mathbf{q}) &= 2 + 48(3 + 2 \cos \pi \nu_D) \mathbf{q} \\ &\quad - 48(127 + 128 \cos \pi \nu_D - 8(1 - 3 \log \mathbf{q}) \sin^2 \pi \nu_D) \mathbf{q}^2 + \dots, \end{aligned} \quad (8.59)$$

$$\hat{\mathbf{e}}_2(\nu_D; \mathbf{q}) = -1 - 24 \cos \pi \nu_D \sqrt{\mathbf{q}} - 24(1 + (28 + 6 \log \mathbf{q}) \sin^2 \pi \nu_D) \mathbf{q} + \dots \quad (8.60)$$

Concluding remarks

In this chapter, we have seen how two-dimensional conformal field theory provides a solution to the connection problem for the Lamé equation. We have also studied the analytic dependence on the Floquet exponent of the accessory parameters of the Lamé equation, the Heun equation, and its confluent limits. For the Lamé equation, we have compared the resulting spectral data with those obtained through standard methods in Appendix B, and through complementary constructions based on isomonodromic deformations and orbifold surface defects.

Several questions remain open. These include a rigorous proof of the proposed Lamé connection formulas, as well as an all-order justification of the resummation construction. In particular, it remains to establish the existence and uniqueness of the solution to the NS blow-up equations within the proposed resummation ansatz and to prove the general structure of the resummation functions described in Section 6.4.

The resummation procedure also raises a natural question: can it be extended from the accessory parameters to the Floquet solutions themselves? As we have seen, these solutions can be constructed from orbifold surface-defect partition functions in the NS limit, and their instanton expansions exhibit a similar pattern of poles in the Floquet exponent. In the next chapter, we address this question for the Heun equation and its confluent limits. The starting point is a set of blow-up relations involving orbifold surface-defect partition functions, from which we derive NS equations governing the resummation of the corresponding Floquet solutions.

Chapter III

Periodic Spectral Problems and the Heun Equation

Building on the previous chapter, we further develop the conformal-field-theoretic and gauge-theoretic framework for studying periodic spectral problems. The focus now shifts to the Heun equation and to $SU(2)$ gauge theories coupled to fundamental hypermultiplets. The results presented in this chapter are based on [148]. Following the same general strategy, we derive a new class of NS blow-up equations involving surface-defect partition functions and use them to extend the resummation procedure from the bulk NS functions $\mathcal{W}_0$ and $\mathcal{W}_1$ to the solutions of the associated differential equations.

The main objects of interest are regular orbifold surface-defect partition functions. In the NS limit, they furnish a basis of Floquet solutions to the differential operator associated with the gauge theory [32, 147]. For the $SU(2)$ theory with four fundamental hypermultiplets, this is the Heun oper¹ on the four-punctured sphere. The theories obtained by decoupling some of the hypermultiplets similarly describe the confluent degenerations of the Heun equation. We denote these theories by $N_f = (n_0, n_1)$, where the pair records the flavor coloring associated with the orbifold surface defect. The bulk NS quantities for a theory with N_f fundamental hypermultiplets considered in the previous chapter correspond, in the notation of this chapter and up to simple redefinitions of the masses, to $n_0 + n_1 = N_f$.

At generic values of the Coulomb parameter a , the two defect partition functions provide two independent solutions with A-cycle Floquet multipliers $e^{\pm \frac{2\pi i a}{\hbar}}$. At the resonant loci

$$\frac{2a}{\hbar} = n \in \mathbb{Z},$$

these multipliers coalesce to $(-1)^n$. This is a global resonance of the A-cycle monodromy, rather than a collision of local exponents at one of the punctures. For generic values of the masses, the resonant monodromy is not semisimple: only one periodic or anti-periodic Floquet solution remains, while the second independent solution is its logarithmic companion.

As in the bulk problem studied in the previous chapter, the naive instanton expansion ceases to be well defined at nonzero resonance. The coefficients of the bulk NS functions

¹In the second-order case considered in this chapter, an oper can be represented by a differential operator $\widehat{\mathcal{D}} : K^{-1/2} \rightarrow K^{3/2}$, locally of the form $\widehat{\mathcal{D}} = \partial_z^2 + T(z)$, where K is the canonical bundle on $\mathbb{CP}^1$ and $T(z)$ is meromorphic [32, 149]. The Heun operator considered below takes this form after a scalar gauge transformation removing the first-derivative term and dividing by $\hbar^2$.

$\mathcal{W}_0$ and $\mathcal{W}_1$, as well as those of the defect NS functions $\hat{\mathcal{W}}_\beta$, develop poles at

$$2a = \pm j\hbar, \quad j \in \mathbb{Z}_{>0},$$

whose order grows with the instanton number. Consequently, neither the accessory parameter obtained from the quantum Matone relation [39, 40] nor the defect wavefunctions admit a term-by-term resonant limit. Constructing the resonant solutions therefore requires a simultaneous resummation of the bulk and defect NS functions.

The starting point of this chapter is the derivation of the blow-up equations that allow us to implement this resummation. We begin with the two blow-up relations satisfied by orbifold surface-defect partition functions, associated with the two possible flux sectors along the exceptional divisor [52, 150]. Taking appropriate limits of these relations and combining them, following the approach of Chapter II, yields a new class of equations involving exclusively the NS functions $\mathcal{W}_0$, $\mathcal{W}_1$ and $\hat{\mathcal{W}}_\beta$ defined in (9.17).

We first derive these equations for the $N_f = (2, 2)$ theory, see (9.25), and then obtain their $N_f = (n_0, n_1)$ counterparts by decoupling fundamental hypermultiplets, see (9.31).

Similarly to the previous chapter, near the j -th resonance, we reorganize the towers of Coulomb-branch poles in terms of the correlated variables

$$x_j^\pm = \frac{\mathfrak{t}^j}{j \pm 2a/\hbar}, \quad \mathfrak{q} = \mathfrak{t}^2.$$

Given the resummation functions for the bulk quantities $\mathcal{W}_0$ and $\mathcal{W}_1$, obtained through the procedure developed in Chapter II, the new blow-up equations recursively determine the corresponding defect resummation functions. In this way, the singular towers of poles are reorganized into controlled square-root and logarithmic branch structures. Although the resulting NS functions are not holomorphic at the resonant points, the resummed accessory parameter and suitably normalized defect wavefunctions possess finite lateral limits.

At a nonzero resonance $2a/\hbar = n$, the two lateral limits of the resummed accessory parameter define

$$u_\sigma^{(n)} := \lim_{2a/\hbar \rightarrow n^\sigma} u_{\text{res}}(a), \quad \sigma \in \{+, -\}.$$

The corresponding limits of the normalized defect partition functions yield periodic or anti-periodic solutions $\phi_\sigma^{(n)}$. For generic masses, the two accessory parameters are distinct,

$$u_+^{(n)} \neq u_-^{(n)},$$

so that the two solutions belong to different resonant Heun operators. On an appropriate real slice, these accessory parameters represent the two edges of the spectral gap associated with the n -th resonance.

For each lateral branch, the resonant A-cycle monodromy is generically non-semisimple. We construct the logarithmic companion of $\phi_\sigma^{(n)}$ in two different ways: by reduction of order and by taking a suitable collision limit of the two resummed Floquet solutions. In an appropriate basis, the resulting monodromy takes the form

$$M_A = (-1)^n \begin{pmatrix} 1 & 0 \\ 2\pi i K_\sigma^{(n)} & 1 \end{pmatrix}.$$

The coefficient $K_\sigma^{(n)}$ depends on the normalization of the periodic or anti-periodic solution, but its vanishing is invariant under regular changes of normalization. It therefore characterizes the semisimplicity of the corresponding resonant branch.

We further conjecture that, on each lateral branch, the resummed dual quantum period satisfies

$$\lim_{2a/\hbar \rightarrow n\sigma} \exp(a_D) = 1.$$

This in turn implies

$$\lim_{2a/\hbar \rightarrow n\sigma} \partial_a u_{\text{res}} = 0,$$

and, under the appropriate normalization and regularity assumptions, allows the logarithmic companion to be represented by the resonant limit of the a -derivative of a single resummed Floquet solution, see the end of Section 10.2.2.

A further result concerns the special loci in the mass-parameter space at which the resonant A-cycle monodromy becomes semisimple. These loci are controlled by the zeros of the polynomials

$$\mathfrak{p}_N(m) = \prod_{k=1-\frac{N}{2}}^{\frac{N}{2}} (m+k), \quad N = |n|,$$

in units where $\hbar = 1$. Cancellation of the singularities in the bulk instanton functions defines a locus $\mathcal{D}_{\text{bulk}}^{(N)}$ on which

$$u_+^{(n)} = u_-^{(n)},$$

and hence the N -th spectral gap closes. Regularity of both defect wavefunctions imposes stronger conditions and defines a smaller locus

$$\mathcal{D}_{\text{defect}}^{(N)} \subset \mathcal{D}_{\text{bulk}}^{(N)}.$$

For the $N_f = (2, 2)$ theory, this requires specializing at least one mass of each flavor color to a zero of $\mathfrak{p}_N$.

On $\mathcal{D}_{\text{defect}}^{(N)}$, the two appropriately normalized resonant defect wavefunctions remain linearly independent and solve the same Heun equation. The A-cycle monodromy is therefore semisimple and satisfies

$$M_A = (-1)^N I.$$

By contrast, on

$$\mathcal{D}_{\text{bulk}}^{(N)} \setminus \mathcal{D}_{\text{defect}}^{(N)},$$

the spectral gap closes, but the two defect wavefunctions reduce to the same periodic or anti-periodic solution after normalization, and the second independent solution remains logarithmic. Thus, gap closure alone does not imply semisimplicity of the resonant monodromy. Within the small- $\mathfrak{t}$ analysis developed here, the semisimple locus is identified precisely with $\mathcal{D}_{\text{defect}}^{(N)}$.

Finally, we conjecture closed-form expressions for the leading defect resummation functions at arbitrary resonance, see (10.31) and (10.32). These expressions are tested against the direct instanton expansion through order $\mathfrak{t}^{10}$ for several theories and resonances. We conclude the analysis with a detailed application to the $N_f = (1, 1)$ theory.

Structure of the chapter. In Section 9, we introduce the orbifold surface-defect partition functions and review how their NS limits give solutions of the Heun equation. We then present the two surface-defect blow-up relations, derive the new NS blow-up equations for the $N_f = (2, 2)$ theory, and obtain their $N_f = (n_0, n_1)$ counterparts through hypermultiplet decoupling limits.

In Section 10, we formulate the resummation ansatz for the bulk and defect NS functions and show how the new blow-up equations determine the defect resummation functions. We then construct the lateral periodic and anti-periodic solutions and their logarithmic companions, analyze the resonant A-cycle monodromy, and identify the special mass loci on which it becomes semisimple. The section concludes with the explicit example of the $N_f = (1, 1)$ theory.

Appendix A.3 collects the conventions and formulas for the relevant Nekrasov functions, while Appendix C.5 provides computational details for the $N_f = (1, 1)$ blow-up equation and its resummed solutions.

9 Blow-up equations for orbifold surface defects

In this section, we first review the orbifold construction of surface-defect partition functions in four-dimensional $\mathcal{N} = 2$ gauge theories, focusing in particular on the $SU(2)$ theory with $N_f = 4$, which we will denote by $N_f = (2, 2)$ for reasons that will become clear below. We then discuss the differential equations satisfied by the corresponding defect partition functions, together with the associated blow-up equations. In particular, we consider a special limit of the latter in which only Nekrasov–Shatashvili functions survive. Finally, we discuss the decoupling limits of the $N_f = (2, 2)$ gauge theory at the level of both the partition functions and the differential/blow-up equations.

9.1 Orbifold surface defects and solutions of the Heun equation

There are several ways to define BPS surface defects in $\mathcal{N} = 2$ gauge theories [86]. One approach is to prescribe singular boundary conditions for the fields along the surface, which can be modeled using an orbifold construction [147]. Consider an $\mathcal{N} = 2$ gauge theory on the orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_p)$, defined by the following $\mathbb{Z}_p$ action on $\mathbb{C}^2$:

$$\zeta : (z_1, z_2) \mapsto (z_1, \zeta z_2), \quad \zeta = \exp\left(\frac{2\pi i}{p}\right) \in \mathbb{Z}_p, \quad (9.1)$$

with the orbifold singularity located along the z_1 -plane. The orbifold can be mapped to $\mathbb{C}^2$ via the change of variables

$$(z_1, z_2) \mapsto (\tilde{z}_1 = z_1, \tilde{z}_2 = z_2^p), \quad (9.2)$$

in terms of which the field configurations are allowed to develop singularities along the surface $\tilde{z}_2 = 0$, thereby engineering a BPS surface defect.

To fully characterize the surface defect, one must specify a choice of coloring function

$$c : [N] \longrightarrow [p], \quad (9.3)$$

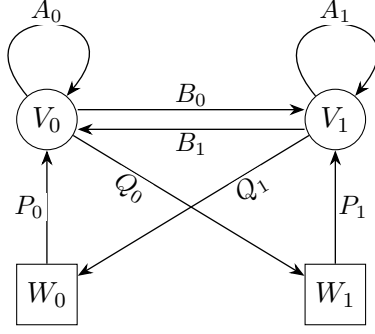
where $[N] := \{0, 1, \dots, N-1\}$ and $[p] := \{0, 1, \dots, p-1\}$. This specifies the lift of the $\mathbb{Z}_p$ -action to the gauge framing space and hence determines the Levi subgroup of the gauge symmetry preserved by the defect.

In the presence of fundamental hypermultiplets, one must also specify a lift of the orbifold action to their flavor space. This is encoded by a flavor-coloring function

$$c_f : [N_f] \longrightarrow [p]. \quad (9.4)$$

Equivalently, if $M \simeq \mathbb{C}^{N_f}$ denotes the flavor space, the coloring determines a $\mathbb{Z}_p$ -grading

$$M = \bigoplus_{\omega \in \mathbb{Z}_p} M_\omega, \quad n_\omega := \dim M_\omega = |c_f^{-1}(\omega)|. \quad (9.5)$$


 Figure III.1: $\mathbb{Z}_2$ chain-saw quiver

The masses assigned color ω are the equivariant weights of the flavor torus acting on M_ω . We will use the notation,

$$N_f = (n_0, n_1, \dots, n_{p-1}), \quad (9.6)$$

for the resulting colored matter content, so that the total number of fundamental hypermultiplets is $\sum_\omega n_\omega$. In the $\mathbb{Z}_2$ case relevant below, this becomes $N_f = (n_0, n_1)$, where n_0 and n_1 count the fundamental hypermultiplets of colors 0 and 1, respectively. For the parent theory with four fundamental hypermultiplets, we choose (m_3, m_4) to have color 0 and (m_1, m_2) to have color 1, and hence denote the theory by $N_f = (2, 2)$.

The Nekrasov instanton partition function of the $\mathcal{N} = 2$ gauge theory in the presence of such a defect is obtained by implementing the $\mathbb{Z}_p$ orbifold at the level of the ADHM construction. In this formulation, the ADHM data are decomposed according to the $\mathbb{Z}_p$ -weights, yielding a chain-saw quiver with p nodes [147].

For the purpose of studying the Heun oper and its degenerations, we restrict our attention to $SU(2)$ gauge theories with fundamental hypermultiplets. We also take the orbifold surface defects to be *regular*, which are characterized by a one-to-one coloring function (in particular, $p = 2$). In this case, there are only two possible choices, namely $c = \text{id}$ and $c = (01)$, the elementary transposition.

The chain-saw quiver is drawn in Figure III.1 and in the present $\mathbb{Z}_2$ case, all indices are understood modulo 2. Let V_ω and W_ω be vector spaces of dimensions

$$\dim V_\omega = k_\omega, \quad \dim W_\omega = 1, \quad \omega \in \mathbb{Z}_2.$$

A representation of the $\mathbb{Z}_2$ chain-saw quiver consists of linear maps between these vector spaces:

$$A_\omega \in \text{End}(V_\omega), \quad B_\omega \in \text{Hom}(V_\omega, V_{\omega+1}), \quad P_\omega \in \text{Hom}(W_\omega, V_\omega), \quad Q_\omega \in \text{Hom}(V_\omega, W_{\omega+1}).$$

The chain-saw moment map equations are

$$\mu_\omega := A_{\omega+1}B_\omega - B_\omega A_\omega + P_{\omega+1}Q_\omega = 0, \quad \omega \in \mathbb{Z}_2,$$

or, explicitly,

$$\begin{aligned} \mu_0 &= A_1 B_0 - B_0 A_0 + P_1 Q_0 = 0, \\ \mu_1 &= A_0 B_1 - B_1 A_1 + P_0 Q_1 = 0. \end{aligned}$$

Let $G_{\mathbf{k}} := \text{GL}(V_0) \times \text{GL}(V_1)$. We define the $\mathbb{Z}_2$ chain-saw quiver variety by the GIT quotient

$$\mathcal{M}_{\mathbf{k}}^{\text{cs}} := \mu^{-1}(0)^{\text{st}} / G_{\mathbf{k}},$$

where the superscript “st” indicates that we restrict to stable representations, meaning representations for which there is no proper pair of subspaces $S_\omega \subset V_\omega$ such that

$$A_\omega(S_\omega) \subset S_\omega, \quad B_\omega(S_\omega) \subset S_{\omega+1}, \quad P_\omega(W_\omega) \subset S_\omega,$$

for both $\omega = 0, 1$.

The torus acting on the chain-saw quiver variety is $T = \mathbb{C}_{\varepsilon_1}^\times \times \mathbb{C}_{\hat{\varepsilon}_2}^\times \times \mathbb{C}_{a_0}^\times \times \mathbb{C}_{a_1}^\times$. We write an element of T as

$$t = (t_1, t_2, u_0, u_1), \quad t_1 = e^{\varepsilon_1}, \quad t_2 = e^{\hat{\varepsilon}_2}, \quad u_\omega = e^{a_{c^{-1}(\omega)}},$$

where, for the $SU(2)$ theory, we set $a_0 = a$ and $a_1 = -a$. Here u_ω acts on the one-dimensional framing space $W_\omega \cong \mathbb{C}$. The induced action on the chain-saw data is

$$A_\omega \mapsto t_1 A_\omega, \quad B_\omega \mapsto t_2 B_\omega, \quad P_\omega \mapsto P_\omega u_\omega^{-1}, \quad Q_\omega \mapsto t_1 t_2 u_{\omega+1} Q_\omega, \quad \omega \in \mathbb{Z}_2,$$

where all indices are taken modulo 2. This action preserves the moment map equations, since they transform as $\mu_\omega \mapsto t_1 t_2 \mu_\omega$. Moreover, the T -action commutes with the action of $G_{\mathbf{k}}$, and hence descends to an action on $\mathcal{M}_{\mathbf{k}}^{\text{cs}}$.

The T -fixed points $(\mathcal{M}_{\mathbf{k}}^{\text{cs}})^T$ are isolated and are classified by colored partitions. Since $\dim W_0 = \dim W_1 = 1$, a fixed point is labeled by a pair of Young diagrams

$$\lambda = (\lambda^{(0)}, \lambda^{(1)}).$$

A box (r, s) in $\lambda^{(\eta)}$, where r and s denote respectively its row and column indices, has $\mathbb{Z}_2$ -weight $\eta + s - 1 \pmod{2}$. Thus, odd columns have weight η , while even columns have weight $\eta + 1 \pmod{2}$. The dimension vector $\mathbf{k} = (k_0, k_1)$ is recovered from the colored boxes by

$$k_0 = \sum_{\substack{s \geq 1 \\ s \text{ odd}}} \lambda_s^{(0),t} + \sum_{\substack{s \geq 1 \\ s \text{ even}}} \lambda_s^{(1),t}, \quad k_1 = \sum_{\substack{s \geq 1 \\ s \text{ even}}} \lambda_s^{(0),t} + \sum_{\substack{s \geq 1 \\ s \text{ odd}}} \lambda_s^{(1),t}.$$

Thus

$$(\mathcal{M}_{\mathbf{k}}^{\text{cs}})^T = \left\{ (\lambda^{(0)}, \lambda^{(1)}) \mid |\lambda|_0 = k_0, |\lambda|_1 = k_1 \right\},$$

where $|\lambda|_\omega$ denotes the number of boxes of color ω .

The equivariant Chern characters of the framing and tautological bundles, together with the characters of the flavor spaces, are then written as

$$N_\omega = e^{a_{c^{-1}(\omega)} - \omega \frac{\varepsilon_2}{2}}, \tag{9.7a}$$

$$K_\omega = \sum_{\eta=0,1} \sum_{\substack{(r,s) \in \lambda^{(\eta)} \\ \eta+s-1 \equiv \omega \pmod{2}}} e^{a_{c^{-1}(\eta)} - \eta \frac{\varepsilon_2}{2} + \varepsilon_1(r-1) + \frac{\varepsilon_2}{2}(s-1)}, \tag{9.7b}$$

$$M_0^{(n_0)} = \sum_{\alpha=1}^{n_0} e^{m_2 + \alpha}, \quad M_1^{(n_1)} = \sum_{\alpha=1}^{n_1} e^{m_\alpha - \frac{\varepsilon_2}{2}}, \tag{9.7c}$$

where $\omega \in \mathbb{Z}_2$, and we denote by $N_f = (n_0, n_1)$ the distribution of fundamental hypermultiplets,² $n_0, n_1 \leq 2$.

In the preceding quiver description, $\hat{\varepsilon}_2$ denotes the equivariant weight of the covering coordinate z_2 . In the characters above, we instead express the weights in terms of the rotation parameter ε_2 on the target space $\mathbb{C}^2$. Under the quotient map

$$\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_2) \longrightarrow \mathbb{C}^2, \quad (z_1, z_2) \longmapsto (\tilde{z}_1 = z_1, \tilde{z}_2 = z_2^2), \tag{9.8}$$

²We are considering here a fixed choice of flavor coloring function c_f and splitting the masses accordingly.

the target coordinate $\tilde{z}_2$ has equivariant weight ε_2 , thus $\hat{\varepsilon}_2 = \frac{\varepsilon_2}{2}$. This accounts for the replacement of the orbifold equivariant parameter by $\varepsilon_2/2$ in the characters above. Away from the ramification divisor $z_2 = 0$, a weight of $\mathbb{Z}_2$ -color 1 can be converted into one of color 0 by tensoring it with z_2^{-1} . In additive notation, this shifts its equivariant weight by

$$w^{(1)} \mapsto w^{(1)} - \frac{\varepsilon_2}{2}. \quad (9.9)$$

This explains the color-dependent shifts $a_{c^{-1}(1)} \mapsto a_{c^{-1}(1)} - \varepsilon_2/2$ and $m_\alpha \mapsto m_\alpha - \varepsilon_2/2$ if $c_f(\alpha) = 1$, while the color-0 weights remain unshifted. The trivialization by z_2^{-1} is singular along $z_2 = 0$; consequently, smooth $\mathbb{Z}_2$ -equivariant fields on the covering space descend to fields with the prescribed surface singularity along $\tilde{z}_2 = 0$.

The full equivariant character entering the localization formula is given by

$$\begin{aligned} T[\lambda]_c^{(n_0, n_1)} = \sum_{\omega \in \mathbb{Z}_2} & \left[N_\omega K_\omega^* + e^{\varepsilon_1 + \frac{\varepsilon_2}{2}} N_\omega^* K_{\omega-1} - (1 - e^{\varepsilon_1}) K_\omega K_\omega^* \right. \\ & \left. + e^{\frac{\varepsilon_2}{2}} (1 - e^{\varepsilon_1}) K_\omega K_{\omega+1}^* - M_\omega^{(n_\omega)} K_\omega^* \right]. \end{aligned} \quad (9.10)$$

The partition function of the orbifold theory is given by the generating function of the equivariant integrals of the Euler class of Dirac zero-mode bundle $\mathcal{F}$,

$$\begin{aligned} \Psi_{c, \text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; x, \mathbf{q}) &= \sum_{k_0, k_1=0}^{\infty} \prod_{\omega=0,1} \mathbf{q}_\omega^{k_\omega} \int_{\mathcal{M}_{\mathbf{k}}^{\text{cs}}} e_{\text{T}}(\mathcal{F}) \\ &= \sum_{\lambda} \prod_{\omega=0,1} \mathbf{q}_\omega^{k_\omega} \mathcal{E}(-T[\lambda]_c^{(n_0, n_1)}). \end{aligned} \quad (9.11)$$

We denote the *fractionalized* couplings by $(\mathbf{q}_\omega)_{\omega=0,1}$, which we choose to express in terms of the bulk gauge coupling $\mathbf{q}$ and the monodromy defect parameter x by

$$\mathbf{q}_0 = \frac{\mathbf{q}}{x}, \quad \mathbf{q}_1 = x. \quad (9.12)$$

Also, $k_\omega = \dim K_\omega$ and $\mathcal{E}$ denotes the operation that converts the equivariant Chern character into the equivariant Euler class,

$$\mathcal{E}\left(\sum_{\alpha} e^{w_\alpha} - \sum_{\beta} e^{w_\beta}\right) = \frac{\prod_{\alpha} w_\alpha}{\prod_{\beta} w_\beta}. \quad (9.13)$$

Finally, denoting by $\beta = 0, 1$ the two choices of coloring functions $c = \text{id}$ and $c = (01)$, respectively, the full orbifold surface defect partition function takes the form

$$\Psi_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; x, \mathbf{q}) = \Psi_{\beta, \text{cl}} \Psi_{\beta, 1\text{-loop}}^{(n_0, n_1)} \Psi_{\beta, \text{inst}}^{(n_0, n_1)}, \quad (9.14)$$

where the classical part of the partition function is

$$\Psi_{\beta, \text{cl}}(a, \varepsilon_1, \varepsilon_2; x, \mathbf{q}) = \mathbf{q}^{-\frac{a^2}{\varepsilon_1 \varepsilon_2}} x^{-\frac{a_\beta}{\varepsilon_1}}, \quad (9.15)$$

and the one-loop contribution is reported in Appendix A.3. Analogous expressions for the bulk Nekrasov partition function $\mathcal{Z}^{(n_0, n_1)}$ can also be found in Appendix A.3.

In the presence of a half-BPS surface defect, local observables in the defect and the bulk are subject to the chiral ring equation [151]. In the presence of the Ω -background,

this chiral ring equation uplifts to a differential equation satisfied by the partition function [32, 146]. In particular, in the $N_f = (2, 2)$ case, the orbifold partition functions (9.14) can be shown to obey

$$\left[\varepsilon_1^2 \partial_x^2 + \left(\frac{\varepsilon_1}{x} + \frac{m_1 + m_2}{1-x} + \frac{\mathfrak{q}}{x} \frac{m_3 + m_4}{\mathfrak{q} - x} \right) \varepsilon_1 \partial_x + \left(\frac{m_1 m_2}{x(x-1)} + \frac{\mathfrak{q}}{x} \frac{m_3 m_4}{x(\mathfrak{q} - x)} \right) + \frac{(\mathfrak{q} - 1) \varepsilon_1 \varepsilon_2}{x(x - \mathfrak{q})(x - 1)} \mathfrak{q} \partial_{\mathfrak{q}} \right] \Psi_{\beta}^{(2,2)} = 0, \quad \beta = 0, 1. \quad (9.16)$$

In the context of the AGT/BPS-CFT correspondence [22, 34, 91, 152], the regular orbifold surface-defect partition functions of the $SU(2)$ $N_f = (2, 2)$ theory are identified, up to elementary prefactors, with four-point conformal blocks of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$. Accordingly, equation (9.16) can also be viewed as a gauge-transformed KZ equation satisfied by the affine conformal block. For $\widehat{\mathfrak{sl}}_2$, the KZ-BPZ correspondence expresses this four-point KZ conformal block as an integral transform of a five-point Virasoro conformal block on the sphere with one insertion of the level-two degenerate primary field $\phi_{2,1}(x)$, which solves the corresponding BPZ equation [153–155]. In this particular case, the reduced KZ equation and the BPZ equation admit the same form, with monodromy parameters shuffled by the integral transform.

In the Nekrasov–Shatashvili (NS) limit, defined by $\varepsilon_2 \rightarrow 0$ and $\varepsilon_1 = \hbar$, the bulk and defect partition functions behave as

$$\log \mathcal{Z}^{(n_0, n_1)} \simeq \frac{1}{\varepsilon_2} \mathcal{W}_0^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1; \mathfrak{q}) + \mathcal{W}_1^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1; \mathfrak{q}) + \mathcal{O}(\varepsilon_2), \quad (9.17a)$$

$$\log \Psi_{\beta}^{(n_0, n_1)} \simeq \frac{1}{\varepsilon_2} \mathcal{W}_0^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1; \mathfrak{q}) + \hat{\mathcal{W}}_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1; x, \mathfrak{q}) + \mathcal{O}(\varepsilon_2), \quad (9.17b)$$

where $\mathcal{W}_0^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1; \mathfrak{q})$ coincides with the twisted superpotential of the effective two-dimensional $\mathcal{N} = (2, 2)$ theory supported on the z_2 -plane [29]. Upon normalizing the defect partition function by the bulk one, the limit $\varepsilon_2 \rightarrow 0$ leads to a well-defined wavefunction,

$$\begin{aligned} \psi_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \hbar; x, \mathfrak{q}) &:= \lim_{\varepsilon_2 \rightarrow 0} \frac{\Psi_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \hbar, \varepsilon_2; x, \mathfrak{q})}{\mathcal{Z}^{(n_0, n_1)}(a, \mathbf{m}, \hbar, \varepsilon_2; \mathfrak{q})} \\ &= \exp \left[\hat{\mathcal{W}}_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \hbar; x, \mathfrak{q}) - \mathcal{W}_1^{(n_0, n_1)}(a, \mathbf{m}, \hbar; \mathfrak{q}) \right]. \end{aligned} \quad (9.18)$$

In the $N_f = (2, 2)$ case, this wavefunction satisfies the NS limit of (9.16), which is precisely the Heun equation,

$$\left[\hbar^2 \partial_x^2 + \left(\frac{\hbar}{x} + \frac{m_1 + m_2}{1-x} + \frac{\mathfrak{q}}{x} \frac{m_3 + m_4}{\mathfrak{q} - x} \right) \hbar \partial_x + \left(\frac{m_1 m_2}{x(x-1)} + \frac{\mathfrak{q}}{x} \frac{m_3 m_4}{x(\mathfrak{q} - x)} \right) + \frac{(\mathfrak{q} - 1) u^{(2,2)}}{x(x - \mathfrak{q})(x - 1)} \right] \psi_{\beta}^{(2,2)} = 0, \quad \beta = 0, 1. \quad (9.19)$$

By the monodromies of its solutions, the Heun operator (9.19) defines a parabolic $\mathrm{PGL}(2; \mathbb{C})$ -local system on the four-punctured sphere $\widehat{\mathbb{C}} \setminus \{0, \mathfrak{q}, 1, \infty\}$. For generic values of the masses, the local monodromies around the four punctures are semisimple, with conjugacy classes fixed by the corresponding local exponents. For fixed $\mathfrak{q}$ and fixed local exponents, the space of such operators is an affine line, parametrized by the accessory parameter $u^{(2,2)}$. Through the relation

$$u^{(2,2)} = \hbar \mathfrak{q} \partial_{\mathfrak{q}} \mathcal{W}_0^{(2,2)} = -a^2 + \hbar \mathfrak{q} \partial_{\mathfrak{q}} \mathcal{W}_{0, \text{inst}}^{(2,2)}, \quad (9.20)$$

the Coulomb parameter a provides a local parametrization of this oper line. Under the Riemann–Hilbert map, this oper locus forms a holomorphic Lagrangian submanifold inside the moduli space of parabolic $\mathrm{PGL}(2; \mathbb{C})$ -local systems on the four-punctured sphere.

By (9.19), the monodromy surface defect partition functions provide solutions to the Heun oper. Note that they are convergent in the domain $|\mathfrak{q}_0|, |\mathfrak{q}_1| < 1$, which is equivalent to $|\mathfrak{q}| < |x| < 1$. The two choices of the coloring function $\beta \in \{0, 1\}$ label the two independent solutions in this domain at generic values of the Coulomb parameter a , diagonal under the monodromy along the A-cycle of the four-punctured sphere $\hat{\mathbb{C}} \setminus \{0, \mathfrak{q}, 1, \infty\}$.

In this Chapter, we depart from the generic values of the Coulomb parameter a and study Heun solutions near the special loci $a \in \frac{\hbar}{2}\mathbb{Z}$, where the naive series expansion (9.18) of the wavefunction ceases to be well-defined. The method is the resummation of the defect partition functions near the special loci, which becomes available through the blow-up equations.

9.2 Blow-up equations for surface defect partition functions

It was physically argued and tested in [52] that surface defect partition functions satisfy blow-up equations, extending the corresponding results known for bulk partition functions [3]. See also [89]. In this Chapter we will be interested in the blow-up equations obtained by considering the $\mathcal{N} = 2$ gauge theory on $\hat{\mathbb{C}}^2$, the blow-up of $\mathbb{C}^2$ at the origin. The blow-up equations are labeled by the first Chern class $c_1 = k[E]$ of the torsion-free sheaf on $\hat{\mathbb{C}}^2$, with $k \in \{0, 1\}$, where $E \hookrightarrow \hat{\mathbb{C}}^2$ denotes the exceptional divisor. Focusing once again on the $N_f = (2, 2)$ orbifold partition functions, the blow-up equations corresponding to $k = 0$ take the form

$$\Psi_\beta^{(2,2)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; x, \mathfrak{q}) = \sum_{n \in \mathbb{Z}} \Psi_\beta^{(2,2)}(a + n\varepsilon_1, \mathbf{m}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; x, \mathfrak{q}) \mathcal{Z}^{(2,2)}(a + n\varepsilon_2, \mathbf{m}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathfrak{q}), \quad (9.21)$$

while the $k = 1$ blow-up equations read

$$\begin{aligned} \Psi_\beta^{(2,2)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; x, \mathfrak{q}) &= -\mathfrak{q}^{-\frac{1}{4}x^{\frac{1}{2}}} \frac{1 - \mathfrak{q}}{1 - x} \\ &\sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_\beta^{(2,2)}\left(a + n\varepsilon_1, \mathbf{m} + \frac{\varepsilon_1}{2}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; x, \mathfrak{q}\right) \mathcal{Z}^{(2,2)}\left(a + n\varepsilon_2, \mathbf{m} + \frac{\varepsilon_2}{2}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathfrak{q}\right). \end{aligned} \quad (9.22)$$

These equations have been proved using CFT methods and the AGT correspondence in [150]. As discussed in Chapter II, it is possible to take an appropriate limit of the blow-up equations that gives constraints on the NS asymptotic functions in equation (9.17). In particular, in the limit $\varepsilon_1 = \varepsilon_2 = \hbar$, the $k = 0$ blow-up equations (9.21) reduce to

$$\Psi_\beta^{(2,2)}(a, \mathbf{m}, \hbar, \hbar; x, \mathfrak{q}) = \sum_{n \in \mathbb{Z}} \exp\left(\hat{\mathcal{W}}_\beta^{(2,2)} + \mathcal{W}_1^{(2,2)} - (\partial_\hbar + n\partial_a)\mathcal{W}_0^{(2,2)}\right), \quad (9.23)$$

where $\hat{\mathcal{W}}_\beta^{(2,2)} = \hat{\mathcal{W}}_\beta^{(2,2)}(a + n\hbar, \mathbf{m}, \hbar; x, \mathfrak{q})$, and similarly for $\mathcal{W}_0^{(2,2)}$ and $\mathcal{W}_1^{(2,2)}$. Meanwhile, the $k = 1$ blow-up equations (9.22) reduce, in this limit, to

$$\begin{aligned} \Psi_\beta^{(2,2)}(a, \mathbf{m}, \hbar, \hbar; x, \mathfrak{q}) &= -\mathfrak{q}^{-\frac{1}{4}x^{\frac{1}{2}}} \frac{1 - \mathfrak{q}}{1 - x} \\ &\times \sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp\left(\hat{\mathcal{W}}_\beta^{(2,2)} + \mathcal{W}_1^{(2,2)} - \left(\frac{1}{2}\partial_{\mathbf{m}} + \partial_\hbar + n\partial_a\right)\mathcal{W}_0^{(2,2)}\right), \end{aligned} \quad (9.24)$$

where $\partial_{\mathbf{m}} = \sum_i \partial_{m_i}$, $\hat{\mathcal{W}}_{\beta}^{(2,2)} = \hat{\mathcal{W}}_{\beta}^{(2,2)}(a + n\hbar, \mathbf{m} + \hbar/2, \hbar; x, \mathbf{q})$, and similarly for $\mathcal{W}_0^{(2,2)}$ and $\mathcal{W}_1^{(2,2)}$. Equating the two right-hand sides then yields a blow-up equation involving only the NS functions $\mathcal{W}_0^{(2,2)}$, $\mathcal{W}_1^{(2,2)}$ and $\hat{\mathcal{W}}_{\beta}^{(2,2)}$,

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} \exp\left(\hat{\mathcal{W}}_{\beta}^{(2,2)} + \mathcal{W}_1^{(2,2)} - (\partial_{\hbar} + n\partial_a)\mathcal{W}_0^{(2,2)}\right) \\ & + \mathbf{q}^{-\frac{1}{4}} x^{\frac{1}{2}} \frac{1 - \mathbf{q}}{1 - x} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp\left(\hat{\mathcal{W}}_{\beta}^{(2,2)} + \mathcal{W}_1^{(2,2)} - \left(\frac{1}{2}\partial_{\mathbf{m}} + \partial_{\hbar} + n\partial_a\right)\mathcal{W}_0^{(2,2)}\right) = 0. \end{aligned} \quad (9.25)$$

This equation will be the main focus of the present Chapter and will allow us to perform the resummation of the defect prepotentials $\hat{\mathcal{W}}_{\beta}^{(n_0, n_1)}$.

9.3 Decoupling limits

The differential and blow-up equations satisfied by the $N_f = (n_0, n_1)$ partition functions $\Psi_{\beta}^{(n_0, n_1)}$ can be obtained by considering appropriate decoupling limits of the corresponding equations for the $N_f = (2, 2)$ theory. Starting from equation (9.16), the decoupling of the masses m_{α} , $\alpha \in \{1, 2\}$, can be achieved by performing the substitutions $x \rightarrow x/m_{\alpha}$ and $\mathbf{q} \rightarrow \mathbf{q}/m_{\alpha}$, followed by the limit $m_{\alpha} \rightarrow \infty$. For instance, decoupling the mass m_2 leads to the following equation:

$$\left[\varepsilon_1^2 \partial_x^2 + \left(1 + \frac{\mathbf{q}}{x} \frac{m_3 + m_4}{\mathbf{q} - x}\right) \varepsilon_1 \partial_x + \left(\frac{\mathbf{q}}{x^2} \frac{m_3 m_4}{\mathbf{q} - x} - \frac{m_1}{x}\right) + \frac{\varepsilon_1 \varepsilon_2}{x(x - \mathbf{q})} \mathbf{q} \partial_{\mathbf{q}} \right] \Psi_{\beta}^{(2,1)} = 0. \quad (9.26)$$

Meanwhile, the decoupling of the masses $m_{2+\alpha}$, with $\alpha \in \{1, 2\}$, can be achieved by performing the substitution $\mathbf{q} \rightarrow \mathbf{q}/m_{2+\alpha}$, followed by the limit $m_{2+\alpha} \rightarrow \infty$. For instance, decoupling the mass m_4 leads to the equation

$$\left[\varepsilon_1^2 \partial_x^2 + \left(\frac{m_1 + m_2}{1 - x} - \frac{\mathbf{q}}{x^2}\right) \varepsilon_1 \partial_x + \left(\frac{m_1 m_2}{x(x - 1)} - \frac{\mathbf{q}}{x^3} m_3\right) + \frac{\varepsilon_1 \varepsilon_2}{x^2(1 - x)} \mathbf{q} \partial_{\mathbf{q}} \right] \Psi_{\beta}^{(1,2)} = 0. \quad (9.27)$$

In an analogous way, one can obtain all confluent degenerations of the Heun equation (9.19). The same limits must also be implemented in the quantum Matone relation (9.20), in order to recover the correct relation between the accessory parameter and the twisted effective superpotential of the theory obtained after hypermultiplet decoupling.

Turning next to the blow-up equations, and performing the same limits on (9.21) and (9.22), it is straightforward to see that the $k = 0$ blow-up equations always take the form

$$\begin{aligned} & \Psi_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; x, \mathbf{q}) \\ & = \sum_{n \in \mathbb{Z}} \Psi_{\beta}^{(n_0, n_1)}(a + n\varepsilon_1, \mathbf{m}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; x, \mathbf{q}) \mathcal{Z}^{(n_0, n_1)}(a + n\varepsilon_2, \mathbf{m}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathbf{q}), \end{aligned} \quad (9.28)$$

while in the $k = 1$ case the blow-up equations read

$$\begin{aligned} & \Psi_{\beta}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; x, \mathbf{q}) = c_{n_0, n_1}(x, \mathbf{q}) \times \\ & \sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_{\beta}^{(n_0, n_1)}\left(a + n\varepsilon_1, \mathbf{m} + \frac{\varepsilon_1}{2}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; x, \mathbf{q}\right) \mathcal{Z}^{(n_0, n_1)}\left(a + n\varepsilon_2, \mathbf{m} + \frac{\varepsilon_2}{2}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathbf{q}\right), \end{aligned} \quad (9.29)$$

where the prefactor c_{n_0, n_1} is defined as

$$c_{n_0, n_1}(x, \mathbf{q}) = -\mathbf{q}^{-\frac{1}{4}} x^{\frac{1}{2}} \cdot \begin{cases} \frac{1-\mathbf{q}}{1-x}, & \text{if } n_0 = n_1 = 2, \\ \frac{1}{1-x}, & \text{if } n_0 \leq 1 \text{ and } n_1 = 2, \\ 1, & \text{if } n_1 \leq 1. \end{cases} \quad (9.30)$$

Finally, in the general $N_f = (n_0, n_1)$ case, the NS blow-up equations (9.25) take the form

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} \exp\left(\hat{\mathcal{W}}_{\beta}^{(n_0, n_1)} + \mathcal{W}_1^{(n_0, n_1)} - (\partial_{\hbar} + n\partial_a)\mathcal{W}_0^{(n_0, n_1)}\right) \\ & - c_{n_0, n_1}(x, \mathbf{q}) \sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp\left(\hat{\mathcal{W}}_{\beta}^{(n_0, n_1)} + \mathcal{W}_1^{(n_0, n_1)} - \left(\frac{1}{2}\partial_{\mathbf{m}} + \partial_{\hbar} + n\partial_a\right)\mathcal{W}_0^{(n_0, n_1)}\right) = 0, \end{aligned} \quad (9.31)$$

where the dependence on the appropriately shifted arguments is always understood.

10 Resummation of surface defect partition functions

In this section we begin by briefly reviewing the properties of the NS instanton functions $\mathcal{W}_{0, \text{inst}}^{(n_0, n_1)}$, $\mathcal{W}_{1, \text{inst}}^{(n_0, n_1)}$ and $\hat{\mathcal{W}}_{\beta, \text{inst}}^{(n_0, n_1)}$, which motivate the resummation procedure and the form of the resummation ansatz that we will introduce [37, 38], see Chapter II for more details. This review is included also to avoid possible confusion with the notation used in the previous chapter. To this end, we perform the convenient change of variables $x = z\mathbf{t}$, $\mathbf{q} = \mathbf{t}^2$, in terms of which the Heun equation (9.19) takes the form

$$\begin{aligned} & \left[\hbar^2 \partial_z^2 + \left(\frac{\hbar}{z} - \frac{\mathbf{t} m_1 + m_2}{z} \frac{1}{\mathbf{t} - z^{-1}} + \frac{\mathbf{t} m_3 + m_4}{z} \frac{1}{\mathbf{t} - z} \right) \hbar \partial_z \right. \\ & \left. + \frac{\mathbf{t}}{z^2} \left(\frac{m_1 m_2}{\mathbf{t} - z^{-1}} + \frac{m_3 m_4}{\mathbf{t} - z} \right) + \frac{(1 - \mathbf{t}^2) u^{(2,2)}}{z^2 (\mathbf{t} - z)(\mathbf{t} - z^{-1})} \right] \psi_{\beta}^{(2,2)} = 0, \end{aligned} \quad (10.1)$$

and where the accessory parameter is given by

$$u^{(2,2)} = \frac{\hbar}{2} \mathbf{t} \partial_{\mathbf{t}} \mathcal{W}_0^{(2,2)}. \quad (10.2)$$

With a slight abuse of notation, we will denote

$$\mathcal{W}_{i, \text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \hbar; \mathbf{t}^2) \equiv \mathcal{W}_{i, \text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \hbar; \mathbf{t}), \quad i = 0, 1, \quad (10.3)$$

and, similarly,

$$\hat{\mathcal{W}}_{\beta, \text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \hbar; z\mathbf{t}, \mathbf{t}^2) \equiv \hat{\mathcal{W}}_{\beta, \text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \hbar; z, \mathbf{t}), \quad \beta = 0, 1. \quad (10.4)$$

The intended meaning will always be clear from the variables appearing in the arguments. We then proceed to show how the blow-up equations (9.31) can be used to determine recursively the resummation functions parametrizing the NS instanton function $\hat{\mathcal{W}}_{\beta, \text{inst}}^{(n_0, n_1)}$, using as input the resummation functions parametrizing $\mathcal{W}_{0, \text{inst}}^{(n_0, n_1)}$ and $\mathcal{W}_{1, \text{inst}}^{(n_0, n_1)}$. We then discuss the resulting pattern that emerges for the defect resummation functions, their analytic structure, and how these properties allow one to safely take the limit to half-integer values of $a/\hbar$, correctly reproducing the expected wavefunctions. Finally, we explain how the logarithmic companions of the resonant solutions can be obtained from the resummed wavefunctions, discuss the semisimple locus of the resonant A-cycle monodromy, and conclude with the explicit example of the $N_f = (1, 1)$ theory.

10.1 Resummation ansatz

In the following discussion we suppress the numbers of hypermultiplets n_0 and n_1 , and set $\hbar = 1$. The dependence on $\hbar$ can be reinstated by rescaling the remaining variables and the NS functions according to

$$\mathcal{W}_{0,\text{inst}}^{(n_0,n_1)}(a, \mathbf{m}, \hbar; \mathbf{t}) = \hbar \mathcal{W}_{0,\text{inst}}^{(n_0,n_1)}\left(a\hbar^{-1}, \mathbf{m}\hbar^{-1}, 1; \mathbf{t}\hbar^{\frac{n_0+n_1}{2}-2}\right), \quad (10.5a)$$

$$\mathcal{W}_{1,\text{inst}}^{(n_0,n_1)}(a, \mathbf{m}, \hbar; \mathbf{t}) = \mathcal{W}_{1,\text{inst}}^{(n_0,n_1)}\left(a\hbar^{-1}, \mathbf{m}\hbar^{-1}, 1; \mathbf{t}\hbar^{\frac{n_0+n_1}{2}-2}\right), \quad (10.5b)$$

$$\hat{\mathcal{W}}_{\beta,\text{inst}}^{(n_0,n_1)}(a, \mathbf{m}, \hbar; z, \mathbf{t}) = \hat{\mathcal{W}}_{\beta,\text{inst}}^{(n_0,n_1)}\left(a\hbar^{-1}, \mathbf{m}\hbar^{-1}, 1; z\hbar^{\frac{n_1-n_0}{2}}, \mathbf{t}\hbar^{\frac{n_0+n_1}{2}-2}\right). \quad (10.5c)$$

$\mathcal{W}_{0,\text{inst}}$. The instanton prepotential $\mathcal{W}_{0,\text{inst}}(a, \mathbf{m}, \hbar; \mathbf{t})$ is a symmetric function of the masses $\mathbf{m}$ and an even function of both a and $\mathbf{t}$. When expanded around $\mathbf{t} = 0$, it admits the power-series representation

$$\mathcal{W}_{0,\text{inst}}(a, \mathbf{m}, 1; \mathbf{t}) = R_0(a, \mathbf{m}, 1; \mathbf{t}) + \sum_{k=1}^{\infty} f_{0,k}(a, \mathbf{m}) \mathbf{t}^{2k}, \quad (10.6)$$

where the coefficients $f_{0,k}$ are rational functions of the remaining variables, with a predictable pole structure, namely with fixed locations and degrees of the poles at each order in the expansion, while R_0 denotes the part of the prepotential regular in a , see equation (A.58). In particular, upon decomposition into simple fractions, each function $f_{0,k}$ takes the form

$$f_{0,k}(a, \mathbf{m}) = \sum_{j=1}^k \sum_{p=1}^{\lfloor k/j \rfloor} \sum_{\pm} \frac{d_{j,p}^{(k)}(\mathbf{m})}{(j \pm 2a)^{2\lfloor k/j \rfloor - 2p + 1}}, \quad (10.7)$$

where the coefficients $d_{j,p}^{(k)}(\mathbf{m})$ are symmetric polynomials in the masses $\mathbf{m}$ with rational coefficients. It is also worth noting that $a = 0$ is a regular point of the instanton prepotential. This structure naturally leads to the resummation ansatz

$$\mathcal{W}_{0,\text{inst}}(a, \mathbf{m}, 1; \mathbf{t}) = R_0(a, \mathbf{m}, 1; \mathbf{t}) + \sum_{k,j=1}^{\infty} \sum_{\pm} \mathbf{t}^{2k+j-2} g_{k,j}\left(\mathbf{m}, \frac{\mathbf{t}^j}{j \pm 2a}\right), \quad (10.8)$$

where the functions³ $g_{k,j}(\mathbf{m}, x)$ are odd and analytic in a neighborhood of $x = 0$, so as to reproduce the correct pole structure of the standard instanton expansion. As described in subsection 6.4 of Chapter II, the general structure that seems to emerge for the resummation functions is the following:

$$g_{1,j}(\mathbf{m}, x) = -\frac{1}{x} \left(\log \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4w_j^{(n_0,n_1)} x^2 / \zeta_j^2} \right] + 1 - \sqrt{1 + 4w_j^{(n_0,n_1)} x^2 / \zeta_j^2} \right), \quad (10.9a)$$

$$g_{k \geq 2,j}(\mathbf{m}, x) = \frac{1}{x^{2k-1}} \left[\left(1 + 4w_j^{(n_0,n_1)} x^2 / \zeta_j^2 \right)^{\frac{5}{2}-k} Q_{k,j}(\mathbf{m}, x^2) + P_{k,j}(\mathbf{m}, x^2) \right], \quad (10.9b)$$

³In this section, the variable x denotes the last argument of the resummation functions under consideration and should not be confused with the variable appearing in the Heun equation (9.19), which is replaced throughout by $z\mathbf{t}$.

where $Q_{k,j}$ and $P_{k,j}$ are polynomials in x^2 of degrees $2k-3$ and $k-1$, respectively, while⁴

$$w_j^{(n_0, n_1)} = \prod_{k=1-\frac{j}{2}}^{\frac{j}{2}} \prod_{i=1}^{n_0} (m_{i+2} + k) \prod_{i=1}^{n_1} (m_i + k), \quad (10.10a)$$

$$\zeta_j = (-1)^{j-1} j! (j-1)!. \quad (10.10b)$$

Although $x = \infty$ is a branch point of the functions $g_{k,j}$, these functions remain finite as $x \rightarrow \pm\infty$. This makes it possible to define a resummed accessory parameter u_{res} through the relation (10.2), using the resummed expression (10.8) for the instanton contribution. Then, the resulting u_{res} remains well defined in the resonant limits $2a \rightarrow \pm n$, with $n \in \mathbb{N}$. Because of the branch structure introduced by the resummation functions, the value of the accessory parameter at resonance depends on the direction from which the limit is approached. More precisely, one obtains the two branches,

$$u_{\pm}^{(n)} = \lim_{2a \rightarrow n^{\pm}} u_{\text{res}} = \lim_{2a \rightarrow (-n)^{\mp}} u_{\text{res}}, \quad n \in \mathbb{N}. \quad (10.11)$$

The resummed accessory parameter can then be used to construct an infinite family of functions of a quasi-momentum κ :

$$E^{(n)}(\kappa, \mathbf{m}, \hbar; \mathbf{t}) = \begin{cases} -u_{\text{res}}\left(\frac{n}{2} + \kappa, \mathbf{m}, \hbar; \mathbf{t}\right), & n \geq 0 \text{ even}, \\ -u_{\text{res}}\left(\frac{n+1}{2} - \kappa, \mathbf{m}, \hbar; \mathbf{t}\right), & n > 0 \text{ odd}, \end{cases} \quad (10.12)$$

initially defined for $\kappa \in [0, 1/2]$. Each function $E^{(n)}$ is first extended to the interval $\kappa \in [-1/2, 1/2]$ by imposing the reflection symmetry

$$E^{(n)}(-\kappa, \mathbf{m}, \hbar; \mathbf{t}) = E^{(n)}(\kappa, \mathbf{m}, \hbar; \mathbf{t}). \quad (10.13)$$

The branches reached at the endpoints $\kappa = -\frac{1}{2}$ and $\kappa = \frac{1}{2}$ are then identified, and this gluing is repeated along the real κ -axis. In this way, each $E^{(n)}$ is extended to an even, 1-periodic function of the quasi-momentum κ . The resulting $E^{(n)}$ can then be interpreted as the dispersion relations describing the spectral bands of the periodic spectral problems⁵ associated with the Heun equation (10.1) and its confluent limits.

$\mathcal{W}_{1,\text{inst}}$. The prepotential $\mathcal{W}_{1,\text{inst}}(a, \mathbf{m}, \hbar; \mathbf{t})$ is also a symmetric function of the masses $\mathbf{m}$ and an even function of a and $\mathbf{t}$, admitting the power-series representation

$$\mathcal{W}_{1,\text{inst}}(a, \mathbf{m}, 1; \mathbf{t}) = R_1(a, \mathbf{m}, 1; \mathbf{t}) + \sum_{k=1}^{\infty} f_{1,k}(a, \mathbf{m}) \mathbf{t}^{2k}, \quad (10.14)$$

where the coefficients $f_{1,k}$ are again rational functions with a predictable pole structure, and R_1 denotes the part of $\mathcal{W}_{1,\text{inst}}$ that is regular in a , see equation (A.58). Upon decomposing the functions $f_{1,k}$ into simple fractions, we find

$$f_{1,k}(a, \mathbf{m}) = \sum_{j=1}^k \sum_{p=1}^{\lfloor k/j \rfloor} \sum_{\pm} \left(\frac{\alpha_{j,p}^{(k)}(\mathbf{m})}{(j \pm 2a)^{2\lfloor k/j \rfloor - 2p + 2}} + \frac{\beta_{j,p}^{(k)}(\mathbf{m})}{(j \pm 2a)^{2\lfloor k/j \rfloor - 2p + 1}} \right), \quad (10.15)$$

⁴The coefficients $w_j^{(n_0, n_1)}$ are related to the coefficients $w_j^{[N_f = n_0 + n_1]}$ introduced in Chapter II by a simple redefinition of the masses, while ζ_j differs from its counterpart in that chapter by an overall sign determined by the parity of $j \geq 1$.

⁵Under the change of variables $z = e^{2\pi i y}$, the Heun equation (10.1) becomes a differential equation with coefficients that are periodic in y with period one.

where $\alpha_{j,p}^{(k)}(\mathbf{m})$ and $\beta_{j,p}^{(k)}(\mathbf{m})$ are symmetric polynomials in the masses $\mathbf{m}$ with rational coefficients. Again, it is worth noting that $a = 0$ is a regular point of $\mathcal{W}_{1,\text{inst}}$. In contrast to $\mathcal{W}_{0,\text{inst}}$, the pole structure of each coefficient now involves both even and odd powers. This motivates the introduction of two distinct sets of resummation functions in the ansatz

$$\mathcal{W}_{1,\text{inst}}(a, \mathbf{m}, 1, \mathbf{t}) = R_1(a, \mathbf{m}, 1; \mathbf{t}) + \sum_{k,j=1}^{\infty} \sum_{\pm} \mathbf{t}^{2k-2} \left[\tilde{f}_{k,j} \left(\mathbf{m}, \frac{\mathbf{t}^j}{j \pm 2a} \right) + \mathbf{t}^j \tilde{g}_{k,j} \left(\mathbf{m}, \frac{\mathbf{t}^j}{j \pm 2a} \right) \right] \quad (10.16)$$

where the functions $\tilde{f}_{k,j}(\mathbf{m}, x)$ and $\tilde{g}_{k,j}(\mathbf{m}, x)$ are analytic near $x = 0$ and vanishing at $x = 0$, and are even and odd in x , respectively. As described in subsection 6.4 of Chapter II, their explicit form appears to follow the pattern

$$\tilde{g}_{k,j}(\mathbf{m}, x) = \frac{1}{x^{2k-1}} \left[\left(1 + \frac{4w_j^{(n_0, n_1)} x^2}{\zeta_j^2} \right)^{\frac{3}{2}-k} U_{k,j}(\mathbf{m}, x^2) + V_{k,j}(\mathbf{m}, x^2) \right], \quad (10.17)$$

for the $\tilde{g}$ -functions, while for the $\tilde{f}$ -functions we find

$$\tilde{f}_{1,j}(\mathbf{m}, x) = -\frac{1}{4} \log \left(1 + \frac{4w_j^{(n_0, n_1)} x^2}{\zeta_j^2} \right), \quad \tilde{f}_{k \geq 2,j}(\mathbf{m}, x) = \frac{x^2 Z_{k,j}(\mathbf{m}, x^2)}{\left(1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2 \right)^{k-1}}. \quad (10.18)$$

The functions $U_{k,j}$, $V_{k,j}$, and $Z_{k,j}$ are polynomials in x^2 of degrees $2k-2$, $k-1$, and $k-2$, respectively⁶. As one approaches a pole in a , the resummation functions $\tilde{g}_{k,j}$ above develop a branch-point non-analyticity while remaining finite. Similarly, the functions $\tilde{f}_{k,j}$ remain finite with the exception of $\tilde{f}_{1,j}$, whose singular part behaves as

$$\tilde{f}_{1,j} \left(\mathbf{m}, \frac{\mathbf{t}^j}{j \pm 2a} \right) = -\frac{1}{4} \log \frac{\mathbf{t}^{2j}}{(j \pm 2a)^2} + \mathcal{O}(1), \quad 2a \rightarrow \mp j. \quad (10.19)$$

This singular behavior can be canceled by adding to the NS prepotential the combination

$$\frac{1}{2} \log(\Gamma(1+2a)\Gamma(1-2a)) = -\frac{1}{2} \log(2a \pm j) + \mathcal{O}(1), \quad 2a \rightarrow \mp j, \quad (10.20)$$

which precisely compensates the singular contribution coming from $\tilde{f}_{1,j}$.

$\hat{\mathcal{W}}_{\beta,\text{inst}}$. The defect instanton prepotentials $\hat{\mathcal{W}}_{\beta,\text{inst}}(a, \mathbf{m}, \hbar; z, \mathbf{t})$ are symmetric functions of the two sets of masses $\{m_\alpha\}$ and $\{m_{2+\alpha}\}$, with $\alpha \in \{1, 2\}$, independently, namely they are invariant under permutations within each set, but not under permutations mixing the two sets. Since the two defect instanton functions are related by

$$\hat{\mathcal{W}}_{1,\text{inst}}(a, \mathbf{m}, \hbar, z, \mathbf{t}) = \hat{\mathcal{W}}_{0,\text{inst}}(-a, \mathbf{m}, \hbar, z, \mathbf{t}), \quad (10.21)$$

we can focus on the $\beta = 0$ case. The NS function admits the power-series representation

$$\hat{\mathcal{W}}_{0,\text{inst}}(a, \mathbf{m}, 1; z, \mathbf{t}) = \hat{R}_0(a, \mathbf{m}, 1; z, \mathbf{t}) + \sum_{k=1}^{\infty} \mathbf{f}_{0,k}(a, \mathbf{m}; z) \mathbf{t}^k, \quad (10.22)$$

⁶The coefficients of the polynomials $U_{k,j}$, $V_{k,j}$, and $Z_{k,j}$ differ from those in Chapter II because we use a different convention for the fundamental mass parameters.

where the coefficients $f_{0,k}$ are once again rational functions, while $\hat{R}_0$ denotes the part of the function that is regular in a , see equation (A.58). The functions $f_{0,k}$ admit the simple-fraction decomposition

$$f_{0,k}(a, \mathbf{m}; z) = \sum_{j=1}^k \sum_{p=1}^{\lfloor k/j \rfloor} \left(\frac{\hat{\alpha}_{j,p}^{(k)}(\mathbf{m}, z)}{(j+2a)^{\lfloor k/j \rfloor - p + 1}} + \frac{\hat{\beta}_{j,p}^{(k)}(\mathbf{m}, z)}{(j-2a)^{\lfloor k/j \rfloor - p + 1}} \right), \quad (10.23)$$

where the coefficients $\hat{\alpha}_{j,p}^{(k)}(\mathbf{m}, z)$ and $\hat{\beta}_{j,p}^{(k)}(\mathbf{m}, z)$ are symmetric functions of the two sets of masses $\{m_\alpha\}$ and $\{m_{2+\alpha}\}$, with $\alpha \in \{1, 2\}$, independently. As in the case of the NS functions discussed above, the defect NS prepotential $\hat{\mathcal{W}}_{0,\text{inst}}$ is also regular at $a = 0$. Given the pole structure discussed above, we can consider a resummation ansatz involving two families of functions:

$$\hat{\mathcal{W}}_{0,\text{inst}}(a, \mathbf{m}, 1; z, \mathbf{t}) = \hat{R}_0(a, \mathbf{m}, 1; z, \mathbf{t}) + \sum_{k,j=1}^{\infty} \sum_{\pm} \mathbf{t}^{k-1} h_{k,j}^{\pm} \left(\mathbf{m}, z, \frac{\mathbf{t}^j}{j \pm 2a} \right), \quad (10.24)$$

such that the functions $h_{k,j}^{\pm}(\mathbf{m}, z, x)$ are analytic near $x = 0$ and vanish at $x = 0$. Alternatively, the form of the blow-up equation (9.31) suggests incorporating $\mathcal{W}_{1,\text{inst}}$ into $\hat{\mathcal{W}}_{\beta,\text{inst}}$, which, from the perspective of the Heun equation (10.1), amounts to a change of wavefunction normalization and leads to the following resummation ansatz:

$$\begin{aligned} & \hat{\mathcal{W}}_{0,\text{inst}}(a, \mathbf{m}, 1; z, \mathbf{t}) + \mathcal{W}_{1,\text{inst}}(a, \mathbf{m}, 1; \mathbf{t}) \\ &= \hat{R}_0(a, \mathbf{m}, 1; z, \mathbf{t}) + R_1(a, \mathbf{m}, 1; \mathbf{t}) + \sum_{k,j=1}^{\infty} \sum_{\pm} \mathbf{t}^{k-1} \tilde{h}_{k,j}^{\pm} \left(\mathbf{m}, z, \frac{\mathbf{t}^j}{j \pm 2a} \right), \end{aligned} \quad (10.25)$$

where the functions $\tilde{h}_{k,j}^{\pm}(\mathbf{m}, z, x)$ are also analytic and vanish in a neighborhood of $x = 0$. The two families of resummation functions are then related by

$$\tilde{h}_{k,j}^{\pm}(\mathbf{m}, z, x) = h_{k,j}^{\pm}(\mathbf{m}, z, x) + \sum_{r \geq 1} \left(\delta_{k,2r-1} \tilde{f}_{r,j}(\mathbf{m}, x) + \delta_{k,2r+j-1} \tilde{g}_{r,j}(\mathbf{m}, x) \right). \quad (10.26)$$

10.2 Resummation procedure

The blow-up equations (9.31) allow one to determine the resummation functions $h_{k,j}^{\pm}$ in equation (10.24), provided that the resummation functions $g_{k,j}$, $\tilde{g}_{k,j}$ and $\tilde{f}_{k,j}$ for the NS prepotentials $\mathcal{W}_0$ and $\mathcal{W}_1$ are known. Meanwhile, to determine the resummation functions $\tilde{h}_{k,j}^{\pm}$ in equation (10.25), only the functions $g_{k,j}$ are necessary. Since the equations (9.31) for $\beta = 0$ and $\beta = 1$ are related by the transformation $a \rightarrow -a$, together with $n \rightarrow -n$, we restrict our analysis to the $\beta = 0$ case. By suppressing the “instanton” label, setting $\hbar = 1$, and applying the substitutions $\mathbf{q} = \mathbf{t}^2$ and $x = z \mathbf{t}$, the blow-up equation (9.31) can be rewritten as

$$\begin{aligned} 0 &= \sum_{n \in \mathbb{Z}} \mathbf{t}^{n(2n-1)} z^{-n} \ell_n^{(n_0, n_1)}(a, \mathbf{m}) e^{\hat{\mathcal{W}}_0 + \mathcal{W}_1 - (\partial_{\hbar} + n \partial_a) \mathcal{W}_0} \\ &\quad - c_{n_0, n_1}(z \mathbf{t}, \mathbf{t}^2) \sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathbf{t}^{n(2n-1)} z^{\frac{1}{2}-n} \ell_n^{(n_0, n_1)}(a, \mathbf{m}) e^{\hat{\mathcal{W}}_0 + \mathcal{W}_1 - (\frac{1}{2} \sum_i \partial_{m_i} + \partial_{\hbar} + n \partial_a) \mathcal{W}_0}, \end{aligned} \quad (10.27)$$

where the notation $\hat{\mathcal{W}}_0 = \hat{\mathcal{W}}_0(a + n, \mathbf{m}, 1; z, \mathbf{t})$ is understood, and similarly for $\mathcal{W}_0$ and $\mathcal{W}_1$. The ℓ -factors can be written as

$$\ell_n^{(n_0, n_1)}(a, \mathbf{m}) = g_n^v(a) \hat{g}_n^v(a) \prod_{i_0=1}^{n_0} g_n^f(a, m_{i_0+2}) \prod_{i_1=1}^{n_1} g_n^f(a, m_{i_1}) \hat{g}_n^f(a, m_{i_1}), \quad n \in \frac{1}{2} \mathbb{Z}, \quad (10.28)$$

where the various factors appearing in the product are given by

$$g_n^v(a) = \begin{cases} \prod_{j=1}^{2n} (-2a - j + 1)^{-j} (2a + j)^{1-j}, & n \geq 0, \\ g_{-n}^v(-a), & n < 0, \end{cases} \quad (10.29a)$$

$$g_n^f(a, m) = \begin{cases} \prod_{j_0=0}^{\lfloor n \rfloor} (-a - j_0 + m + 1)^{j_0} \prod_{j_1=1}^{\lfloor \frac{n+1}{2} \rfloor} (a + j_1 + m)^{j_1-1}, & n \geq 0, \\ g_{-n}^f(-a, m), & n < 0, \end{cases} \quad (10.29b)$$

$$\hat{g}_n^v(a) = \frac{\Gamma(2a + 2n)}{\Gamma(2a)}, \quad \hat{g}_n^f(a, m) = \frac{\Gamma(a - m)}{\Gamma(a - m + n - \xi_n)}. \quad (10.29c)$$

The constant ξ_n appearing in the last equation vanishes for $n \in \mathbb{Z}$, while $\xi_n = 1/2$ for $n \in \mathbb{Z} + 1/2$. To derive from (10.27) the equations determining the resummation functions $h_{k,j}^\pm$, we set

$$a = \frac{1}{2} \left(\sigma_1 + \frac{\mathfrak{t}^{\sigma_2}}{x} \right), \quad \sigma_i \in \mathbb{Z}, \quad \sigma_2 > 0, \quad (10.30)$$

where x is a complex variable, and expand (10.27) around $\mathfrak{t} = 0$. This yields a series expansion in $\mathfrak{t}$, which may start at a positive power. By setting the coefficients of this expansion to zero, starting from the first non-vanishing term, (10.27) provides algebraic equations, linear once we set $h_{1,j}^\pm = \log H_{1,j}^\pm$, constraining $h_{k,\sigma_2}^+(\mathbf{m}, z, x)$ and $h_{k,\sigma_2}^-(\mathbf{m}, z, x)$. Since the equations obtained for different values of σ_1 are independent, they can be straightforwardly solved for $h_{k,\sigma_2}^\pm$. However, determining the functions at level $j = \sigma_2 + 1$ requires knowledge of both the functions at level $j = \sigma_2$ and the functions $g_{k,j}$, $\tilde{g}_{k,j}$ and $\tilde{f}_{k,j}$ at level $j = \sigma_2 + 1$. Consequently, we proceed iteratively: we first compute the resummation functions for $\sigma_2 = 1$ up to a fixed value of k , then we proceed to $\sigma_2 = 2$, and so forth. A concrete example of this procedure is presented in Appendix C.5, where we write down and solve the first few equations arising from the expansion of (10.27) for the $N_f = (1, 1)$ case.

10.2.1 Leading-order resummation functions & split eigenfunctions

The previous resummation procedure is computationally demanding in the most general $N_f = (2, 2)$ case, especially when the mass parameters are kept generic, as we assume throughout this subsection. In this case, we obtained only the first few $j = 1$ and $j = 2$ resummation functions. In theories with fewer hypermultiplets, it is easier to reach higher orders, particularly in the $N_f = (0, 0)$ case. The $N_f = (1, 1)$ theory provides a useful intermediate case for understanding the structure of the general resummation functions at higher orders, since it retains much of the structure of the $N_f = (2, 2)$ case while being computationally less demanding. Guided by the explicit form of the first few leading resummation functions in these examples, we conjecture the following general structure for the leading resummation functions of the first family appearing in (10.24), for arbitrary j :

$$h_{1,j}^+(\mathbf{m}, z, x) = \log \left(\frac{\sqrt{1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2} + 2z^j v_{1,j}^{(n_1)} x / \zeta_j - 1}{2z^j v_{1,j}^{(n_1)} x / \zeta_j \left(1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2 \right)^{\frac{1}{4}}} \right), \quad (10.31a)$$

$$h_{1,j}^-(\mathbf{m}, z, x) = \log \left(\frac{\sqrt{1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2} + 2z^j v_{1,j}^{(n_1)} x / \zeta_j + 1}{2 \left(1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2 \right)^{\frac{1}{4}}} \right). \quad (10.31b)$$

Analogously, for the second family in (10.25), we find

$$\tilde{h}_{1,j}^+(\mathbf{m}, z, x) = \log \left(\frac{\sqrt{1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2} + 2z^j v_{1,j}^{(n_1)} x / \zeta_j - 1}{2z^j v_{1,j}^{(n_1)} x / \zeta_j \sqrt{1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2}} \right), \quad (10.32a)$$

$$\tilde{h}_{1,j}^-(\mathbf{m}, z, x) = \log \left(\frac{\sqrt{1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2} + 2z^j v_{1,j}^{(n_1)} x / \zeta_j + 1}{2\sqrt{1 + 4w_j^{(n_0, n_1)} x^2 / \zeta_j^2}} \right), \quad (10.32b)$$

where

$$v_{0,j}^{(n_0)} = \prod_{i=1}^{n_0} \mathfrak{p}_j(m_{i+2}), \quad v_{1,j}^{(n_1)} = \prod_{i=1}^{n_1} \mathfrak{p}_j(m_i), \quad (10.33a)$$

$$w_j^{(n_0, n_1)} = v_{0,j}^{(n_0)} v_{1,j}^{(n_1)}, \quad \zeta_j = (-1)^{j-1} j! (j-1)!, \quad (10.33b)$$

with the polynomials $\mathfrak{p}_j$ defined as follows:

$$\mathfrak{p}_j(m) = \prod_{k=1-\frac{j}{2}}^{\frac{j}{2}} (m+k). \quad (10.34)$$

These expressions have been tested against the instanton expansion of the combinations appearing in (10.24) and (10.25), and were found to reproduce the expected pole-contributions to high order. Moreover, their large- x asymptotics are consistent with their role in parametrizing solutions of the Heun equation (10.1). Indeed, approaching a pole of the Coulomb branch parameter a in the resummation ansatz (10.24) along the real axis, or equivalently taking the limit $x \rightarrow \pm\infty$, the functions in (10.31) behave as

$$h_{1,j}^+(\mathbf{m}, z, x) = -\frac{1}{2} \log |x| + \log \left(\frac{\sqrt{|\zeta_j|} \left(v_{1,j}^{(n_1)} \pm (-1)^{j-1} \sqrt{w_j^{(n_0, n_1)}} z^{-j} \right)}{\sqrt{2} v_{1,j}^{(n_1)} (w_j^{(n_0, n_1)})^{1/4}} \right) + \mathcal{O}(|x|^{-1}), \quad (10.35a)$$

$$h_{1,j}^-(\mathbf{m}, z, x) = \frac{1}{2} \log |x| + \log \left(\frac{\sqrt{w_j^{(n_0, n_1)}} \mp (-1)^j z^j v_{1,j}^{(n_1)}}{\sqrt{2|\zeta_j|} (w_j^{(n_0, n_1)})^{1/4}} \right) + \mathcal{O}(|x|^{-1}), \quad (10.35b)$$

while the functions in (10.32), appearing in the resummation ansatz (10.25), behave as

$$\tilde{h}_{1,j}^+(\mathbf{m}, z, x) = -\log |x| + \log \left(\frac{|\zeta_j| \left(v_{1,j}^{(n_1)} \pm (-1)^{j-1} z^{-j} \sqrt{w_j^{(n_0, n_1)}} \right)}{2 v_{1,j}^{(n_1)} \sqrt{w_j^{(n_0, n_1)}}} \right) + \mathcal{O}(|x|^{-1}), \quad (10.36a)$$

$$\tilde{h}_{1,j}^-(\mathbf{m}, z, x) = \log \left(\frac{1}{2} \mp \frac{(-1)^j z^j v_{1,j}^{(n_1)}}{2 \sqrt{w_j^{(n_0, n_1)}}} \right) + \mathcal{O}(|x|^{-1}). \quad (10.36b)$$

All other resummation functions $h_{k,j}^\pm$ and $\tilde{h}_{k,j}^\pm$, with $k > 1$, remain regular in the limits $x \rightarrow \pm\infty$, similarly to their bulk analogues. In view of the specific large- x behavior of the resummation functions above, we can define two linearly independent resummed Floquet solutions in slightly different but equivalent ways. These choices differ by z -independent

normalization factors which may themselves vanish or diverge at $a \rightarrow \pm j/2$. This is precisely what is needed to compensate the zeros or divergences produced by a given choice of resummation ansatz. For instance, one natural choice is

$$\tilde{\phi}_0(a, \mathbf{m}, \hbar; z, \mathbf{t}) = z^{-\frac{a}{\hbar}} \Gamma\left(1 + \frac{2a}{\hbar}\right) \exp\left(\hat{\mathcal{W}}_{0,\text{inst}}(a, \mathbf{m}, \hbar; z, \mathbf{t}) + \mathcal{W}_{1,\text{inst}}(a, \mathbf{m}, \hbar; \mathbf{t})\right), \quad (10.37a)$$

$$\tilde{\phi}_1(a, \mathbf{m}, \hbar; z, \mathbf{t}) = \tilde{\phi}_0(-a, \mathbf{m}, \hbar; z, \mathbf{t}), \quad (10.37b)$$

where $\hat{\mathcal{W}}_{0,\text{inst}} + \mathcal{W}_{1,\text{inst}}$ is given in (10.25), with $\hbar$ reintroduced by an appropriate rescaling of the other variables (10.5). These solutions are well defined for arbitrary values of the Coulomb branch parameter a , up to the choice of branches when taking the resonant limits $2a/\hbar \rightarrow n \in \mathbb{Z} \setminus \{0\}$. Indeed, in the $a \rightarrow j^\pm/2$ limit, one finds

$$\tilde{\phi}_\pm^{(j)}(\mathbf{m}, 1; z, \mathbf{t}) = \lim_{a \rightarrow j^\pm/2} \tilde{\phi}_0(a, \mathbf{m}, 1; z, \mathbf{t}) = \frac{j!}{2} \left(z^{-\frac{j}{2}} \pm \frac{(-1)^j v_{1,j}^{(n_1)}}{\sqrt{w_j^{(n_0, n_1)}}} z^{\frac{j}{2}} \right) + \mathcal{O}(\mathbf{t}), \quad (10.38)$$

with the same limit applied to the second solution defined above yielding the same resonant wavefunctions $\tilde{\phi}_\pm^{(j)}$, up to an overall normalization. The same conclusion holds upon taking instead the limit $a \rightarrow -j^\pm/2$, with the roles of $\tilde{\phi}_0$ and $\tilde{\phi}_1$ being exchanged. Thus, for any real value of a away from resonance, the two solutions in (10.37) form a basis of linearly independent Floquet solutions, describing eigenfunctions associated with points inside the spectral bands (10.12) of the periodic spectral problems related to the Heun equation (10.1) and its confluent limits. At a nonzero resonance, each resummed solution admits two lateral limits, giving the resonant (anti-)periodic solutions associated with the two corresponding band edges. At $a = 0$, by contrast, the two resummed solutions reduce to the same periodic wavefunction corresponding to the minimum of the first spectral band. Since the instanton expansions are regular at $a = 0$, this wavefunction can equivalently be obtained by setting $a = 0$ before performing the resummation. Generically, its second linearly independent companion is logarithmic and can be constructed by the methods discussed in the next subsection.

One may also consider the following z -independent renormalization of the solutions obtained by dividing the defect partition function by the bulk partition function:

$$\psi_0(a, \mathbf{m}, \hbar; z, \mathbf{t}) = \frac{z^{-\frac{a}{\hbar}}}{\Gamma\left(1 - \frac{2a}{\hbar}\right)} \exp\left(\hat{\mathcal{W}}_{0,\text{inst}}(a, \mathbf{m}, \hbar; z, \mathbf{t}) - \mathcal{W}_{1,\text{inst}}(a, \mathbf{m}, \hbar; \mathbf{t})\right), \quad (10.39a)$$

$$\psi_1(a, \mathbf{m}, \hbar; z, \mathbf{t}) = \psi_0(-a, \mathbf{m}, \hbar; z, \mathbf{t}), \quad (10.39b)$$

where the gamma function in the denominator has been included in order to cancel the singular large- x behavior of the leading resummation functions, which is given by

$$h_{1,j}^+(\mathbf{m}, z, x) - \tilde{f}_{1,j}(\mathbf{m}, x) = \log\left(1 \mp (-1)^j \frac{\sqrt{w_j^{(n_0, n_1)}}}{v_{1,j}^{(n_1)}} z^{-j}\right) + \mathcal{O}(|x|^{-1}), \quad (10.40a)$$

$$h_{1,j}^-(\mathbf{m}, z, x) - \tilde{f}_{1,j}(\mathbf{m}, x) = \log|x| + \log\left(\frac{\sqrt{w_j^{(n_0, n_1)}} \mp (-1)^j z^j v_{1,j}^{(n_1)}}{|\zeta_j|}\right) + \mathcal{O}(|x|^{-1}), \quad (10.40b)$$

as $x \rightarrow \pm\infty$.

Before moving to the next subsection, let us note that the factor $w_j^{(n_0, n_1)}$ controls the only square-root branch structure responsible for the splitting of the resummed accessory parameter and Floquet solutions in the resonant limit. For the accessory parameter, this follows from the structure of the bulk resummation functions in (10.9). The same square-root factor governs the resummation functions of $\mathcal{W}_1$ in (10.17) and (10.18), as well as the leading defect resummation functions in (10.31) and (10.32). Together with the linearity of the equations determining the subleading defect resummation functions, this shows that the latter cannot introduce an additional independent square-root branch responsible for the splitting. The vanishing of $w_j^{(n_0, n_1)}$ therefore removes this source of splitting, although it does not by itself guarantee that the resulting resonant limits are regular.

10.2.2 Logarithmic solutions

At the resonant loci $2a/\hbar = n \in \mathbb{Z}$, the two A-cycle Floquet multipliers coalesce:

$$e^{\pm 2\pi i \frac{a}{\hbar}} \rightarrow (-1)^n. \quad (10.41)$$

Thus, generically, the resonant monodromy is not semisimple and the second independent solution is logarithmic. In the following, we discuss two standard constructions which, when applied to our setting, allow us to obtain the logarithmic companions of the resonant (anti-)periodic solutions discussed in the previous subsection.

Reduction of order. We begin by recalling the reduction-of-order construction of a logarithmic companion solution. After setting $\hbar = 1$ in (10.1), the Heun equation takes the form

$$\psi'' + p(z)\psi' + q(z)\psi = 0, \quad (10.42)$$

with

$$p(z) = \frac{1}{z} - \frac{\mathbf{t}}{z} \frac{m_1 + m_2}{\mathbf{t} - z^{-1}} + \frac{\mathbf{t}}{z} \frac{m_3 + m_4}{\mathbf{t} - z}, \quad (10.43)$$

Thus, by Abel's identity, the Wronskian $W(\psi_1, \psi_2)$ of any two solutions ψ_1 and ψ_2 is proportional to

$$\mu(z) := \exp\left(-\int^z p(\xi) d\xi\right) \propto z^{-1}(1 - \mathbf{t}z)^{m_1+m_2}\left(1 - \frac{\mathbf{t}}{z}\right)^{m_3+m_4}, \quad (10.44)$$

where the proportionality constant is independent of z . Let $\psi^{(n)}$ be a nonzero resonant Floquet solution,

$$\psi^{(n)}(ze^{2\pi i}) = (-1)^n \psi^{(n)}(z). \quad (10.45)$$

Once the normalization of $\mu(z)$ has been fixed, we define its canonical companion by

$$\psi_{\log}^{(n)}(z) := \psi^{(n)}(z) \int^z \frac{\mu(w)}{\psi^{(n)}(w)^2} dw, \quad (10.46)$$

which, by construction, satisfies

$$W(\psi^{(n)}, \psi_{\log}^{(n)}) = \mu(z). \quad (10.47)$$

Therefore, for fixed $\psi^{(n)}$, the companion is uniquely determined by this Wronskian normalization up to

$$\psi_{\log}^{(n)} \mapsto \psi_{\log}^{(n)} + c(\mathbf{m}, \mathbf{t}) \psi^{(n)}, \quad (10.48)$$

where this ambiguity corresponds to changing the lower endpoint of the integral. In the annulus $|t| < |z| < |t|^{-1}$, the integrand admits a Laurent expansion in z and, upon integration, the coefficient of z^{-1} is the only term that produces a logarithm. Thus one can write

$$\psi_{\log}^{(n)}(z) = K^{(n)}(\mathbf{m}, t) \psi^{(n)}(z) \log z + \varphi^{(n)}(z), \quad (10.49)$$

where $\varphi^{(n)}$ has the same A-cycle monodromy as $\psi^{(n)}$,

$$\varphi^{(n)}(ze^{2\pi i}) = (-1)^n \varphi^{(n)}(z), \quad (10.50)$$

and

$$K^{(n)}(\mathbf{m}, t) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\mu(z)}{\psi^{(n)}(z)^2} dz, \quad (10.51)$$

where γ is a positively oriented closed contour contained in the annulus $|t| < |z| < |t|^{-1}$ and winding once around the origin $z = 0$. It follows that the canonical logarithmic solution transforms under the A-cycle monodromy as

$$\psi_{\log}^{(n)}(ze^{2\pi i}) = (-1)^n \psi_{\log}^{(n)}(z) + 2\pi i (-1)^n K^{(n)}(\mathbf{m}, t) \psi^{(n)}(z), \quad (10.52)$$

and, in the basis

$$(\psi^{(n)}, \psi_{\log}^{(n)})^T, \quad (10.53)$$

one obtains

$$M_A = (-1)^n \begin{pmatrix} 1 & 0 \\ 2\pi i K^{(n)} & 1 \end{pmatrix}. \quad (10.54)$$

Thus, the coefficient $K^{(n)}$ measures the nilpotent part of the resonant A-cycle monodromy. In particular,

$$K^{(n)} = 0 \iff M_A = (-1)^n I, \quad (10.55)$$

provided that, at the zeros of $K^{(n)}$, the accessory parameter and both solutions $\psi^{(n)}$ and $\psi_{\log}^{(n)}$ remain regular, and that their limiting values continue to form a linearly independent pair. In that case, the resonant monodromy is semisimple, or equivalently there exists a second independent resonant Floquet solution with the same multiplier.

However, as we discuss in more detail in the next subsection, $K^{(n)}$ is not an invariant scalar until a normalization of the resonant Floquet solution has been fixed. Indeed, keeping the normalization of $\mu(z)$ fixed, a parameter-dependent rescaling

$$\psi^{(n)} \mapsto f(\mathbf{m}, t) \psi^{(n)} \quad (10.56)$$

induces

$$K^{(n)} \mapsto f(\mathbf{m}, t)^{-2} K^{(n)}. \quad (10.57)$$

Thus $K^{(n)}$ should be regarded as the local representative, in a chosen local frame for the resonant Floquet line, of the nilpotent monodromy coefficient. In particular, its local expression depends on the frame, while its vanishing is invariant under invertible changes of frame.

Applying this construction to our setting, for any of the resonant representatives constructed above, such as $\psi^{(n)} = \tilde{\phi}_{\pm}^{(n)}$, we define⁷

$$\tilde{\phi}_{\pm, \log}^{(n)}(z) := \tilde{\phi}_{\pm}^{(n)}(z) \int^z \frac{\mu(w)}{\tilde{\phi}_{\pm}^{(n)}(w)^2} dw = K_{\pm}^{(n)}(\mathbf{m}, t) \tilde{\phi}_{\pm}^{(n)}(z) \log z + \tilde{\varphi}_{\pm}^{(n)}(z), \quad (10.58)$$

⁷We recall that the sign $\pm$ in the wavefunctions $\tilde{\phi}_{\pm}^{(n)}$ labels solutions of two distinct spectral problems, since for generic values of the mass parameters the corresponding accessory parameters $u_{\pm}^{(n)}$ are different.

where $K_{\pm}^{(n)}$ is obtained from (10.51) by replacing $\psi^{(n)}$ with $\tilde{\phi}_{\pm}^{(n)}$. The gauge-theory construction of the previous subsection gives the resonant solution, and hence its logarithmic companion together with the coefficient $K_{\pm}^{(n)}$, as an expansion in powers of $\mathfrak{t}$. However, it does not by itself specify a preferred global normalization of the resonant solution. For this reason, in practice one might have to impose a different normalization a posteriori, order by order in $\mathfrak{t}$, for example by fixing a chosen Laurent coefficient of $\tilde{\phi}_{\pm}^{(n)}$. Once a choice has been made, one can compute the corresponding representative of $K_{\pm}^{(n)}$ and the associated logarithmic solution as power series in $\mathfrak{t}$.

Different choices of normalization will lead to different local representatives. In particular, if a zero of the intrinsic nilpotent coefficient is not visible in a given representative, this may indicate that the chosen frame is singular at that locus: the rescaling used to define the normalized resonant solution may have introduced an artificial pole or zero. One should then pass to a different local normalization, regular on the locus under consideration, and recompute the corresponding representative of $K_{\pm}^{(n)}$.

Coalescence. In the generic non-semisimple case, an alternative construction of the logarithmic solution is obtained by coalescing two non-resonant independent Floquet solutions. If two limiting solutions ϕ_0 and ϕ_1 become proportional at resonance, one may choose a z -independent factor $A(a, \mathbf{m}; \mathfrak{t})$ such that

$$\lim_{2a \rightarrow n^{\pm}} (\phi_0(a; z) - A(a, \mathbf{m}; \mathfrak{t})\phi_1(a; z)) = 0. \quad (10.59)$$

Then the standard coalescence construction gives a second solution of the resonant equation:

$$\phi_{\pm, \log}^{(n)}(z) := \lim_{2a \rightarrow n^{\pm}} \frac{\phi_0(a; z) - A(a, \mathbf{m}; \mathfrak{t})\phi_1(a; z)}{2a - n}, \quad (10.60)$$

whenever the limit exists. Equivalently, if the numerator extends holomorphically to the resonant point, this may be written as

$$\phi_{\pm, \log}^{(n)}(z) := \lim_{2a \rightarrow n^{\pm}} \frac{1}{2} \partial_a (\phi_0(a; z) - A(a, \mathbf{m}; \mathfrak{t})\phi_1(a; z)). \quad (10.61)$$

This is the usual Frobenius/Floquet mechanism by which a logarithmic solution appears when two exponents collide. In the present problem, however, it cannot be applied directly before resummation, because the $\mathfrak{t}$ -expanded Floquet solutions develop poles of increasing order at the resonant values of a . Instead, after resummation, the numerator solves the Heun equation (10.1) for every non-resonant value of a , and its divided limit solves the limiting equation at resonance. The monodromy of $\phi_{\pm, \log}^{(n)}$ is fixed solely by differentiating the two Floquet multipliers. If

$$\phi_{\pm}^{(n)}(z) = \lim_{2a \rightarrow n^{\pm}} \phi_0(a; z) = \lim_{2a \rightarrow n^{\pm}} A(a, \mathbf{m}; \mathfrak{t})\phi_1(a; z), \quad (10.62)$$

then one finds, up to the overall sign convention in the definition of $\phi_{\pm, \log}^{(n)}$ and in the assignment of the two Floquet multipliers,

$$\phi_{\pm, \log}^{(n)}(ze^{2\pi i}) = (-1)^n \phi_{\pm, \log}^{(n)}(z) - 2\pi i (-1)^n \phi_{\pm}^{(n)}(z). \quad (10.63)$$

Thus, the logarithmic companion obtained in this collision basis has a mass-independent representative for the off-diagonal monodromy coefficient, due to the normalization of the logarithmic pair selected by the collision construction.

Single-derivative construction. In our setting, we can consider a slight variation of the previous construction, suggested by the analytic properties of the resummed accessory parameter $u_{\text{res}}^{(n_0, n_1)}$. Instead of differentiating a vanishing difference of two colliding solutions, one may try to differentiate a single Floquet solution. In general, this does not give another solution, since the Heun operator in (10.1) does not commute with ∂_a . However, the dependence on a enters the differential operator only through the accessory parameter $u_{\text{res}}^{(n_0, n_1)}$. Hence, after differentiating the equation with respect to a , the source term is proportional to $\partial_a u_{\text{res}}^{(n_0, n_1)}$ and if the resummed accessory parameter satisfies

$$\lim_{2a \rightarrow n} \partial_a u_{\text{res}}^{(n_0, n_1)} = 0, \quad (10.64)$$

then the inhomogeneous term vanishes in the resonant limit. Thus, if the prefactor $B(a, \mathbf{m}; \mathbf{t})$ is chosen so that both $B\phi_0$ and $B\partial_a\phi_0$ admit finite resonant limits, the function

$$\chi_{\pm, \log}^{(n)}(z) := \lim_{2a \rightarrow n^\pm} B(a, \mathbf{m}; \mathbf{t}) \partial_a \phi_0(a; z) \quad (10.65)$$

is again a solution of the resonant Heun equation.

Its logarithmic monodromy follows again from differentiating the Floquet multiplier. More precisely, denoting the resonant solution selected by the same prefactor by

$$\chi_{\pm}^{(n)}(z) := \lim_{2a \rightarrow n^\pm} B(a, \mathbf{m}; \mathbf{t}) \phi_0(a; z), \quad (10.66)$$

then, up to the overall sign convention for the Floquet multiplier of ϕ_0 , one obtains

$$\chi_{\pm, \log}^{(n)}(ze^{2\pi i}) = (-1)^n \chi_{\pm, \log}^{(n)}(z) - 2\pi i (-1)^n \chi_{\pm}^{(n)}(z). \quad (10.67)$$

As usual, $\chi_{\pm, \log}^{(n)}$ is defined only up to the addition of a multiple of the resonant solution, and this ambiguity depends on the chosen prefactor B . In this sense, the single-derivative construction is a stronger statement than the standard collision construction: it uses the special property (10.64) of the resummed accessory parameter. The condition (10.64) follows from a simple conjectural property of the resummed dual quantum period. Let

$$a_D^{(n_0, n_1)} = \partial_a \mathcal{W}_0^{(n_0, n_1)} = a_{D, \text{pert}}^{(n_0, n_1)} + a_{D, \text{inst}}^{(n_0, n_1)} \quad (10.68)$$

denote the dual period obtained from the resummed twisted superpotential, including the instanton contribution $a_{D, \text{inst}}^{(n_0, n_1)}$ obtained from the a -derivative of (10.8), together with the classical and one-loop terms:

$$a_{D, \text{pert}}^{(n_0, n_1)} = -4a \log \mathbf{t} + 2 \log \frac{\Gamma(1+2a)}{\Gamma(1-2a)} - \sum_{i_0=1}^{n_0} \log \frac{\Gamma(1+m_{i_0+2}+a)}{\Gamma(1+m_{i_0+2}-a)} - \sum_{i_1=1}^{n_1} \log \frac{\Gamma(1+m_{i_1}+a)}{\Gamma(1+m_{i_1}-a)}. \quad (10.69)$$

We conjecture that, at every resonant point,

$$\lim_{a \rightarrow \frac{n}{2}} \exp \left(a_D^{(n_0, n_1)} \right) = 1, \quad n \in \mathbb{Z}. \quad (10.70)$$

Since the limiting value of $a_D^{(n_0, n_1)}$ is independent of $\mathbf{t}$, this implies that

$$\lim_{a \rightarrow n/2} \mathbf{t} \partial_{\mathbf{t}} a_D^{(n_0, n_1)} = 0, \quad (10.71)$$

and therefore (10.64). We tested this conjecture in the $N_f = (2, 2)$, $N_f = (1, 1)$ and $N_f = (0, 0)$ theories by computing the resummed instanton NS prepotential to the highest accessible order in the resummation functions in each case. Moreover, the explicit form of the leading resummation functions in (10.9) is consistent with this conjecture. In particular, these functions cancel the singularities arising from the Gamma-function contributions in (10.69) at the resonant points.

10.3 Special mass loci and semisimple resonant monodromy

Let us now consider the loci in the $\mathbf{m}$ -parameter space on which the resonant A-cycle monodromy may become semisimple, working at the level of series expansions around $t = 0$, and setting again $\hbar = 1$ in order to simplify the notation. The $\hbar$ -dependence can be restored by rescaling all parameters appropriately. We consider a nonzero resonant value $2a = n$, with $n \in \mathbb{Z} \setminus \{0\}$, and write $N := |n|$. The case $n = 0$ will not be discussed here since, as mentioned before, the instanton functions considered below are regular at $a = 0$, so the resonant limit can already be taken term by term in the instanton expansion.

As already discussed above, for a fixed resonant branch $\sigma \in \{+, -\}$, the coefficient $K_\sigma^{(n)}$ is a local representative of the nilpotent part of the resonant A-cycle monodromy. Its value depends on the chosen normalization of the resonant Floquet solution, but its vanishing is invariant under regular changes of local frame. Thus, the intrinsic semisimplicity condition is

$$K_\sigma^{(n)} = 0, \quad (10.72)$$

together with the requirement that, at the zeros of $K_\sigma^{(n)}$, the corresponding resonant accessory parameter, as well as the (anti-)periodic and logarithmic solutions, remain regular and that their limits continue to form a basis of solutions.

The problem of determining this locus can already be approached at the level of the instanton expansions of the NS prepotentials $\mathcal{W}_0$, $\mathcal{W}_1$, and $\hat{\mathcal{W}}_\beta$. In what follows, all statements refer to the standard small- t instanton expansion and have been checked only to finite order. We start from the twisted effective superpotential $\mathcal{W}_0$, which determines the accessory parameter through (9.20). Since this NS function is even in a , it is enough to study the positive resonant limit $2a \rightarrow N$. At all orders checked, the coefficients of the highest-order pole contributions at $2a = N$ are proportional to the mass factor

$$w_N^{(n_0, n_1)} = \prod_{i=1}^{n_0} \mathbf{p}_N(m_{i+2}) \prod_{j=1}^{n_1} \mathbf{p}_N(m_j), \quad (10.73)$$

where the polynomial $\mathbf{p}_N$ is defined in (10.34). Thus the vanishing of one factor in $w_N^{(n_0, n_1)}$ removes the leading source of singularities but it does not in general make the instanton expansion regular in the limit $a \rightarrow N/2$.

For example, in the $N_f = (2, 2)$ theory⁸, if m_1 is specialized so that $\mathbf{p}_N(m_1) = 0$, the residues of the remaining poles at $a = N/2$ are found to contain the common factor

$$\tilde{w}_N^{(2,2)} = \mathbf{p}_N(m_2)\mathbf{p}_N(m_3)\mathbf{p}_N(m_4). \quad (10.74)$$

Thus, a further specialization of one of the remaining mass parameters completely removes the remaining poles at $a = N/2$ in $\mathcal{W}_{0, \text{inst}}$. An analogous pole-cancellation pattern holds for $\mathcal{W}_{1, \text{inst}}$ in the $N_f = (2, 2)$ theory and, by taking appropriate decoupling limits, for the bulk NS functions of theories with fewer hypermultiplets.

Let $I_{n_0, n_1} := \{1, \dots, n_1\} \cup \{3, \dots, n_0 + 2\}$ denote the set of labels of the hypermultiplet masses present in the $N_f = (n_0, n_1)$ theory. We find that, on the locus

$$\mathcal{D}_{\text{bulk}}^{(N)} := \bigcup_{\substack{i, j \in I_{n_0, n_1} \\ i < j}} \{\mathbf{m} \mid \mathbf{p}_N(m_i) = 0, \quad \mathbf{p}_N(m_j) = 0\}, \quad (10.75)$$

⁸We recall that the bulk instanton prepotentials $\mathcal{W}_0$ and $\mathcal{W}_1$ are symmetric under permutations of the mass parameters $\mathbf{m} = (m_1, m_2, m_3, m_4)$, whereas the functions $\hat{\mathcal{W}}_\beta$ are symmetric only under separate permutations of the pairs (m_1, m_2) and (m_3, m_4) .

the small- $\mathfrak{t}$ expansions of both bulk instanton prepotentials $\mathcal{W}_{0,\text{inst}}$ and $\mathcal{W}_{1,\text{inst}}$ admit a regular resonant limit at $2a = n$. Consequently, the same holds for the accessory parameter $u^{(n_0, n_1)}$. Complete cancellation of the bulk poles therefore requires the specialization of at least two distinct mass parameters. In particular

$$\mathcal{D}_{\text{bulk}}^{(N)} = \emptyset \quad \text{if} \quad n_0 + n_1 \leq 1. \quad (10.76)$$

At the level of the small- $\mathfrak{t}$ instanton expansion, $\mathcal{D}_{\text{bulk}}^{(N)}$ is precisely the locus on which the N -th spectral gap closes. Indeed, for $\mathbf{m}_* \in \mathcal{D}_{\text{bulk}}^{(N)}$, the two lateral accessory parameters coincide order by order in $\mathfrak{t}$,

$$u^{(n)}(\mathbf{m}_*; \mathfrak{t}) := u_+^{(n)}(\mathbf{m}_*; \mathfrak{t}) = u_-^{(n)}(\mathbf{m}_*; \mathfrak{t}), \quad N = |n|. \quad (10.77)$$

Furthermore, these gap-closing loci are nested separately for odd and even values of N . Indeed, $\mathfrak{p}_N(m)$ divides $\mathfrak{p}_{N+2r}(m)$ for $r \in \mathbb{Z}_{\geq 0}$, and therefore

$$\mathcal{D}_{\text{bulk}}^{(N)} \subseteq \mathcal{D}_{\text{bulk}}^{(N+2r)}, \quad r \in \mathbb{Z}_{\geq 0}. \quad (10.78)$$

Thus, a mass specialization belonging to $\mathcal{D}_{\text{bulk}}^{(N)}$ closes all gaps indexed by $N + 2r$, $r \in \mathbb{Z}_{\geq 0}$. This does not in general define a finite-gap potential, since the gaps of the opposite parity may remain open.

We now turn to the instanton defect prepotentials $\hat{\mathcal{W}}_{\beta, \text{inst}}$. Since the two choices of defect NS functions are related by $a \rightarrow -a$, it is enough to discuss $\beta = 0$. In this case, the coefficients of the leading poles at the negative and positive resonances, $2a = -N$ and $2a = N$, are proportional, respectively, to

$$v_{0,N}^{(n_0)} = \prod_{i=1}^{n_0} \mathfrak{p}_N(m_{i+2}), \quad v_{1,N}^{(n_1)} = \prod_{i=1}^{n_1} \mathfrak{p}_N(m_i), \quad (10.79)$$

as defined in (10.33). For $\beta = 1$ the roles of these two factors are exchanged, as a consequence of the relation $a \rightarrow -a$. Once one mass has been tuned to cancel the leading-order poles, for instance at the negative resonance $2a = -N$, the residues of all remaining poles at the same resonance vanish upon specializing any additional mass parameter to a zero of $\mathfrak{p}_N(m)$. The same holds for positive resonances.

Not every component of the bulk locus $\mathcal{D}_{\text{bulk}}^{(N)}$ is therefore compatible with the regularity of the defect prepotentials, at least at the level of their instanton expansion. By requiring $\mathcal{W}_{0,\text{inst}}$, $\mathcal{W}_{1,\text{inst}}$, and both defect prepotentials $\hat{\mathcal{W}}_{\beta, \text{inst}}$ to have a finite limit as $a \rightarrow n/2$, the instanton expansion singles out the smaller locus

$$\mathcal{D}_{\text{defect}}^{(N)} := \bigcup_{i=1}^{n_1} \bigcup_{j=1}^{n_0} \{\mathbf{m} \mid \mathfrak{p}_N(m_i) = 0, \mathfrak{p}_N(m_{j+2}) = 0\} \subseteq \mathcal{D}_{\text{bulk}}^{(N)}. \quad (10.80)$$

Equivalently, one must require the vanishing of at least one factor in $v_{1,N}^{(n_1)}$ and at least one factor in $v_{0,N}^{(n_0)}$. In particular

$$\mathcal{D}_{\text{defect}}^{(N)} = \emptyset \quad \text{if} \quad n_0 = 0 \text{ or } n_1 = 0, \quad (10.81)$$

and, similarly to the bulk case,

$$\mathcal{D}_{\text{defect}}^{(N)} \subseteq \mathcal{D}_{\text{defect}}^{(N+2r)}, \quad r \in \mathbb{Z}_{\geq 0}. \quad (10.82)$$

For example, in the $N_f = (2, 2)$ theory this means that one mass from the pair (m_1, m_2) and one mass from the pair (m_3, m_4) must be specialized to zeros of $\mathfrak{p}_N(m)$. On this locus, the pole cancellations of the bulk and defect instanton prepotentials give two regular and linearly independent resonant limits, one obtained from each defect function. Hence $\mathcal{D}_{\text{defect}}^{(N)}$ is contained in the semisimple locus of the resonant A-cycle monodromy.

Conversely, since semisimplicity requires the corresponding spectral gap to close, any additional component of the semisimple locus detected by the small- t expansion would have to lie in $\mathcal{D}_{\text{bulk}}^{(N)} \setminus \mathcal{D}_{\text{defect}}^{(N)}$. For example, suppose that the masses are chosen so that the poles of $\hat{\mathcal{W}}_0$ at the negative resonance $2a = -N$ are cancelled. One then obtains the regular accessory parameter $u^{(n)}(\mathbf{m}; t)$ in (10.77), together with a resonant (anti-)periodic solution from the corresponding limit of the wavefunction constructed from $\hat{\mathcal{W}}_0$. The wavefunction constructed from $\hat{\mathcal{W}}_1$ instead diverges in the same resonant limit. Multiplying it by a suitable z -independent normalization factor, which vanishes as $2a \rightarrow -N$, produces a finite limit proportional to the solution obtained from $\hat{\mathcal{W}}_0$. It therefore does not provide a second linearly independent resonant Floquet solution and, within the small- t expansion, the semisimple locus coincides with $\mathcal{D}_{\text{defect}}^{(N)}$. We will provide further evidence for this statement below.

From the point of view of the instanton expansion, the natural order of operations is to first specialize the masses to a point $\mathbf{m}_* \in \mathcal{D}_{\text{defect}}^{(N)}$ and only then take the resonant limit $2a \rightarrow n$. The resummation procedure gives access to the complementary order of limits. At generic masses one first resums the singular instanton series and then takes the one-sided limits $2a \rightarrow n^\pm$. This produces, for example, the two resonant solutions $\tilde{\phi}_\pm^{(n)}$ in (10.38), which are in general associated with two different limiting accessory parameters $u_\pm^{(n)}$ and therefore with two different resonant Heun equations. However, it is already clear from explicit examples of resummation functions, such as $g_{2,1}$ in (C.89) and $\tilde{h}_{2,1}^\pm$ in (C.102), that, in general, the resonant limit does not commute with the limit $\mathbf{m} \rightarrow \mathbf{m}_* \in \mathcal{D}_{\text{defect}}^{(N)}$. Therefore, to recover the regular resonant solutions obtained from the mass-specialized instanton expansion, one should impose the mass specialization before taking the resonant limit. Nevertheless, since each resummation function captures infinitely many terms in the instanton expansion, they still provide non-trivial information on the semisimple mass loci of interest.

Indeed, the leading resummation functions show the same mass factors that appeared in the previous instanton analysis. For the bulk twisted superpotential $\mathcal{W}_0$, the leading function $g_{1,N}$ depends on $w_N^{(n_0, n_1)}$ which is precisely the factor controlling the leading bulk poles. Similarly, the leading singular part of $\mathcal{W}_{1, \text{inst}}$ is governed by $\tilde{f}_{1,N}$ and contains the same factor $w_N^{(n_0, n_1)}$. For the defect prepotentials, the leading functions $h_{1,N}^\pm$ and $\tilde{h}_{1,N}^\pm$ distinguish the two factors $v_{0,N}^{(n_0)}$ and $v_{1,N}^{(n_1)}$, in agreement with the fact that the two defect functions distinguish between positive and negative resonant values of a . More precisely, when $v_{0,N}^{(n_0)} = 0$, the functions $h_{1,N}^+$ and $\tilde{h}_{1,N}^+$ vanish, whereas

$$h_{1,N}^- = \tilde{h}_{1,N}^- = \log \left(1 + \frac{z^N v_{1,N}^{(n_1)}}{\zeta_N} x \right). \quad (10.83)$$

Conversely, when $v_{1,N}^{(n_1)} = 0$, the functions $h_{1,N}^-$ and $\tilde{h}_{1,N}^-$ vanish, while

$$h_{1,N}^+ = \tilde{h}_{1,N}^+ = \log \left(1 + \frac{v_{0,N}^{(n_0)}}{z^N \zeta_N} x \right). \quad (10.84)$$

As discussed in Section 10.2.1, $w_N^{(n_0, n_1)}$ is the only factor controlling the square-root branch cuts responsible for the splitting of the resummed accessory parameter and wavefunctions in the resonant limit. When $w_N^{(n_0, n_1)} = 0$, the square-root branch cuts disappear from the bulk and defect resummation functions, although these functions may still fail to admit a finite resonant limit if all poles in the instanton expansion have not been cancelled. This is manifest in the examples given in (10.83) and (10.84).

Thus, the leading resummation functions are consistent with the instanton-level pole-cancellation pattern described above. On the locus $\mathcal{D}_{\text{defect}}^{(N)}$, where at least one factor in $v_{0,N}^{(n_0)}$ and one factor in $v_{1,N}^{(n_1)}$ vanish, the branch-point singularities of the resummed NS functions are expected to disappear. Since $\mathcal{D}_{\text{defect}}^{(N)} \subseteq \mathcal{D}_{\text{bulk}}^{(N)}$, the accessory parameters already coincide as in (10.77). Correspondingly, by first specializing the mass parameters and only then taking the resonant limit in the two resummed defect solutions, one obtains two resonant wavefunctions that are expected to give linearly independent (anti-)periodic solutions of the same resonant Heun equation.

Let us instead consider the case in which the mass parameters $\mathbf{m}$ belong to $\mathcal{D}_{\text{bulk}}^{(N)} \setminus \mathcal{D}_{\text{defect}}^{(N)}$. Then $w_N^{(n_0, n_1)} = 0$, while exactly one of the two factors $v_{0,N}^{(n_0)}$ and $v_{1,N}^{(n_1)}$ vanishes. Let us assume that $v_{1,N}^{(n_1)} = 0$, the other case being analogous. The resummed accessory parameter then admits a finite and unambiguous resonant limit as $2a \rightarrow N$, and the same holds for one of the resummed Floquet solutions, say $\tilde{\phi}_0$ in (10.37). By contrast, the corresponding limit of the second resummed solution $\tilde{\phi}_1$ diverges, as is clear from (10.84). Since the accessory parameter remains regular, this divergence reflects a singular choice of normalization of $\tilde{\phi}_1$. It must therefore be removable, at the level of the small- $\mathbf{t}$ expansion, by multiplying $\tilde{\phi}_1$ by a suitable z -independent normalization factor. The leading behavior in (10.84) then shows that the renormalized limit is not linearly independent of the limit of $\tilde{\phi}_0$. If the same proportionality persists order by order after including the subleading resummation functions, the two resummed Floquet solutions yield only one resonant (anti-)periodic solution, with the second linearly independent solution being logarithmic. Then, within the small- $\mathbf{t}$ expansion, the semisimple locus coincides with $\mathcal{D}_{\text{defect}}^{(N)}$.

10.4 Example: $N_f = (1, 1)$

In Appendix C.5 we illustrate the resummation procedure explicitly in the $N_f = (1, 1)$ theory, deriving the resummation functions $\tilde{h}$ appearing in (10.25). The relation (10.26) then gives the corresponding h -functions entering (10.24), as shown at the end of Appendix C.5. In this subsection we use these results to display the first few terms in the $\mathbf{t}$ -expansion of certain resonant solutions together with their logarithmic companions. We begin with the $N_f = (1, 1)$ Heun equation,

$$\left[\partial_z^2 + \left(\mathbf{t} + \frac{1}{z} - \frac{\mathbf{t}}{z^2} \right) \partial_z - \left(\frac{m_1 \mathbf{t}}{z} - \frac{u^{(1,1)}}{z^2} + \frac{m_3 \mathbf{t}}{z^3} \right) \right] \psi(z) = 0, \quad \hbar = 1, \quad (10.85)$$

where the accessory parameter is determined by the twisted effective superpotential as

$$u^{(1,1)} = \frac{1}{2} \mathbf{t} \partial_{\mathbf{t}} \mathcal{W}_0^{(1,1)}. \quad (10.86)$$

Using the resummed expression (10.8) for the instanton part of $\mathcal{W}_0^{(1,1)}$, we obtain the resonant accessory parameters by

$$u_{\pm}^{(n)} = \lim_{2a \rightarrow n \pm} u_{\text{res}}^{(1,1)}. \quad (10.87)$$

It is enough to consider positive resonances, $n > 0$. Indeed, since the accessory parameter is an even function of a , the negative resonances are related to the positive ones by

$$\lim_{2a \rightarrow n^\pm} u_{\text{res}}^{(1,1)} = \lim_{2a \rightarrow (-n)^\mp} u_{\text{res}}^{(1,1)}. \quad (10.88)$$

For the first two resonances this gives

$$u_{\pm}^{(1)} = -\frac{1}{4} \mp \frac{1}{2} \sqrt{(2m_1 + 1)(2m_3 + 1)} \mathfrak{t} + \frac{1}{8} (4m_1 m_3 + 2m_1 + 2m_3 - 3) \mathfrak{t}^2 + \mathcal{O}(\mathfrak{t}^3), \quad (10.89a)$$

$$u_{\pm}^{(2)} = -1 - \frac{1}{3} \left(2 + m_1 + m_3 + 2m_1 m_3 \pm 3\sqrt{m_1(m_1 + 1)m_3(m_3 + 1)} \right) \mathfrak{t}^2 + \mathcal{O}(\mathfrak{t}^4). \quad (10.89b)$$

Here, and in the examples below, we take $m_1, m_3 > 0$ for simplicity. The corresponding resonant solutions can be obtained, for example, by taking the same limits of $\tilde{\phi}_0$ in (10.37). For the first two positive resonances one finds

$$\begin{aligned} \tilde{\phi}_{\pm}^{(1)} = & \frac{1}{2} \left(z^{-\frac{1}{2}} \mp \sqrt{\frac{1+2m_1}{1+2m_3}} z^{\frac{1}{2}} \right) + \frac{1}{4} \left[-\frac{1-2m_3}{2} z^{-\frac{3}{2}} \mp \frac{1-4m_1 m_3}{\sqrt{(1+2m_1)(1+2m_3)}} z^{-\frac{1}{2}} \right. \\ & \left. + \frac{(1+2m_1)(1-2m_3)}{1+2m_3} z^{\frac{1}{2}} \pm \frac{(1-2m_1)(1+2m_1)}{2\sqrt{(1+2m_1)(1+2m_3)}} z^{\frac{3}{2}} \right] \mathfrak{t} + \mathcal{O}(\mathfrak{t}^2), \end{aligned} \quad (10.90)$$

$$\begin{aligned} \tilde{\phi}_{\pm}^{(2)} = & z^{-1} \pm \sqrt{\frac{m_1(m_1+1)}{m_3(m_3+1)}} z + \left[\frac{m_3-1}{3} z^{-2} - 1 - m_1 \right. \\ & \left. \mp \sqrt{\frac{m_1(m_1+1)(m_3+1)}{m_3}} \pm \frac{m_1(m_1^2-1)}{3\sqrt{m_1(m_1+1)m_3(m_3+1)}} z^2 \right] \mathfrak{t} + \mathcal{O}(\mathfrak{t}^2). \end{aligned} \quad (10.91)$$

Equivalently, one may take the same resonant limits at the level of $\tilde{\phi}_1$ in (10.37). The resulting solutions agree with those above up to an overall normalization independent of $\mathfrak{t}$, at least to the order displayed.

Following the discussion of the previous sections, the resonant solutions above determine the $\mathfrak{t}$ -expansion of their canonical logarithmic companions. Using (10.58), we find

$$\begin{aligned} \tilde{\phi}_{\pm, \log}^{(1)} = & \pm 2 \sqrt{\frac{1+2m_3}{1+2m_1}} z^{-\frac{1}{2}} + \left[\pm \frac{4m_3^2-1}{2\sqrt{(1+2m_1)(1+2m_3)}} z^{-\frac{3}{2}} \right. \\ & - \frac{28m_1 m_3 + 6m_1 + 10m_3 + 1}{2(1+2m_1)} z^{-\frac{1}{2}} \pm \frac{2m_1-1}{2} \sqrt{\frac{1+2m_3}{1+2m_1}} z^{\frac{1}{2}} \\ & \left. + \left(-(1+2m_3) z^{-\frac{1}{2}} \pm \sqrt{(1+2m_1)(1+2m_3)} z^{\frac{1}{2}} \right) \log z \right] \mathfrak{t} + \mathcal{O}(\mathfrak{t}^2), \end{aligned} \quad (10.92)$$

and

$$\begin{aligned} \tilde{\phi}_{\pm, \log}^{(2)} = & \mp \frac{1}{2} \sqrt{\frac{m_3(1+m_3)}{m_1(1+m_1)}} z^{-1} \\ & \mp \frac{1}{6} \sqrt{\frac{m_3(1+m_3)}{m_1(1+m_1)}} \left[(m_3-1) z^{-2} - 3(1+m_1) \right] \mathfrak{t} + \mathcal{O}(\mathfrak{t}^2). \end{aligned} \quad (10.93)$$

For the second resonance the logarithmic term first appears at order $\mathfrak{t}^2$, and therefore requires carrying the resummation procedure one order further. At the order displayed

here, the nontrivial resonant A-cycle monodromy coefficients associated with the $n = 1$ resonant solutions are

$$K_{\pm}^{(1)} = -2(2m_3 + 1)t + \mathcal{O}(t^2), \quad (10.94)$$

and contributions proportional to $2m_1 + 1$ are expected to appear at higher orders in the t -expansion.

Concluding remarks

In this chapter, we have extended the resummation procedure developed in the previous chapter to the Floquet solutions of the Heun equation and its confluent limits. This provides further evidence that the blow-up equations might offer a systematic framework for the resummation of both accessory parameters and solutions, and suggests possible extensions to spectral problems associated with more general gauge theories. We have also studied the semisimple locus of the resonant A-cycle monodromy for these equations, proposing a characterization within the instanton expansion and its resummation, supported by finite-order checks. Rigorous proofs of the general structure of the resummation functions and of the existence and uniqueness of a solution to the NS blow-up equations within the proposed resummation framework remain open. Nevertheless, the blow-up equations provide a promising starting point for addressing these questions.

In the next chapter, we continue to explore how constructions inspired by physics can be used to study spectral problems, turning to finite-difference equations associated with quantum mirror curves in topological string theory. Our main aim is to construct exact solutions to the corresponding eigenvalue problems. Guided by the open TS/ST correspondence, we propose explicit expressions for these solutions and test them both analytically and numerically.

Chapter IV

Quantum Mirror Curves

In this final chapter, we shift our focus slightly and explore methods inspired by topological string theory and supersymmetric gauge theory for the study of spectral problems associated with quantum mirror curves. While the previous chapters concerned differential equations arising from four-dimensional gauge theories, the Seiberg–Witten curves of five-dimensional theories lead instead to finite-difference (or q -difference) equations.

The main framework is the topological string/spectral theory correspondence, in particular its open variant, reviewed in Subsection 4.2. We apply this construction to the family of $Y^{N,0}$ toric Calabi–Yau geometries, which geometrically engineer five-dimensional $\mathcal{N} = 1$ $SU(N)$ gauge theories. The results presented in this chapter are based on [156].

However, the main focus of the chapter is the spectral problem obtained from a particular four-dimensional limit of these quantum mirror curves. In this limit, the operator of interest is

$$H_N = 2\Lambda^N \cosh(\mathbf{p}) + V_N(\mathbf{x}), \quad [\mathbf{x}, \mathbf{p}] = i\hbar,$$

where

$$V_N(x) = \sum_{k=0}^{N-1} (-1)^k h_k x^{N-k}, \quad h_0 = 1, \quad h_1 = 0,$$

is an arbitrary monic polynomial potential of degree N . The parameters h_k with $2 \leq k \leq N-1$ characterize the potential, while the remaining modulus is identified with the energy through

$$h_N = (-1)^{N-1} E.$$

The operator H_N can be understood as a deformation of the standard polynomial Schrödinger operator

$$\hat{H}_N = \mathbf{p}^2 + V_N(\mathbf{x}).$$

Furthermore, the operator H_N is obtained by quantizing the Seiberg–Witten curve of four-dimensional $\mathcal{N} = 2$ $SU(N)$ supersymmetric Yang–Mills theory [20, 29, 115, 157]. In the choice of canonically conjugate variables adopted here, which differs from the conventional one by a Fourier transform, the quantized curve becomes a finite-difference equation rather than a higher-order differential equation [158, 159]. The corresponding eigenvalue equation reads

$$\Lambda^N [\psi(x + i\hbar, \mathbf{h}) + \psi(x - i\hbar, \mathbf{h})] + V_N(x)\psi(x, \mathbf{h}) = E\psi(x, \mathbf{h}).$$

The Hamiltonian H_N was first studied in [44] and subsequently in [160], where its Fredholm determinant and exact quantization conditions were obtained. It also appears in the context of $T\bar{T}$ deformations [161]. Through the four-dimensional limit, we construct explicit analytic solutions of the difference equation above for arbitrary N and arbitrary

polynomial potentials $V_N(x)$. These solutions, given in (11.25) and (11.31) for even and odd N , respectively, describe both bound and resonant states.

For generic values of the parameters h_k and the energy E , the functions we construct solve the difference equation and are entire in x . This analyticity property is highly non-trivial: unlike an ordinary differential equation, a finite-difference equation generally admits meromorphic solutions with arbitrary periodic factors. In the present construction, the wavefunction is expressed as a particular combination of two open-string contributions, or saddles. Each saddle is individually meromorphic and possesses an infinite set of poles, but these poles cancel in their sum. From the topological-string perspective, this cancellation is a consequence of background independence in the open-string modulus.

We refer to the resulting functions at generic values of E as *generalized* or *off-shell* eigenfunctions. They are generally not square-integrable and therefore do not define physical states of the operator. They become *on-shell* eigenfunctions only when E satisfies the exact quantization conditions derived from the closed TS/ST correspondence [44]. At these discrete values, the growing asymptotic contributions cancel and the wavefunctions acquire the decay properties required by the relevant spectral problem.

The spectral interpretation depends substantially on the parity of N . When N is even, the polynomial potential is confining, and the operator admits a canonical self-adjoint extension with a real and discrete spectrum. In this case, the on-shell solutions obtained from the open TS/ST correspondence are genuine bound-state eigenfunctions. When N is odd, the potential is unbounded from below in one direction, and the relevant spectral data generally correspond to resonant states with complex energies. The analytic expressions derived here describe both situations within the same gauge-theoretic framework.

The deformed Hamiltonian also exhibits several phenomena with no direct counterpart in ordinary quantum mechanics. For even N , degenerate ground states may occur at special values of the moduli. Moreover, the standard oscillation theorem for one-dimensional Schrödinger operators does not apply: even for a confining potential, the n -th eigenfunction need not have precisely n zeros. For odd N , there are special values of the moduli at which square-integrable eigenfunctions with real energies exist despite the fact that the potential is unbounded from below. Our explicit expressions make it possible to study these properties directly at the level of the wavefunctions.

This exact solvability is ultimately made possible by the Lagrangian gauge-theory description underlying H_N . The quantum mirror map expresses the parameters h_k in terms of the Coulomb moduli of the $SU(N)$ theory, while its four-dimensional limit gives the generalized Matone relations. The wavefunctions themselves are constructed from NS free energies and surface-defect partition functions. By comparison, the ordinary anharmonic oscillator $\hat{H}_N$ is associated with Argyres–Douglas theories [162–165], which do not admit a conventional Lagrangian description. No equally explicit analogue of the gauge-theoretic NS functions is currently available in that setting, and closed analytic expressions for generic anharmonic-oscillator eigenfunctions remain unknown.

Starting from the solutions of the eigenvalue equation for H_N , we also construct solutions to several related finite-difference equations. These include operators obtained by replacing the potential with $-V_N(x)$, the kinetic term $\cosh(\mathbf{p})$ with $\sinh(\mathbf{p})$, or both. Depending on the parity of N , the choice of kinetic term, and the sign of the potential, these equations display different asymptotic and square-integrability properties. The resulting solutions include bound states, resonances, and square-integrable families that are not subject to an energy quantization condition, see Table IV.1 for a summary.

The difference equation above is also closely related to the Baxter equation of the N -particle closed quantum Toda lattice [166–170]. The Toda spectral problem imposes

additional boundary conditions beyond the square-integrability requirements considered here. Consequently, the Baxter solutions exist only at distinguished values of all the moduli h_k , determined by $N - 1$ simultaneous quantization conditions. We refer to these special points in moduli space as the *Toda points*. We analyze how our generalized eigenfunctions behave at these points and clarify the relation between the open TS/ST solutions and the conventional Toda wavefunctions.

Finally, we derive the four-dimensional eigenfunctions directly from the open TS/ST correspondence for the $Y^{N,0}$ geometries. Taking the four-dimensional limit of the resulting background-independent open partition function reproduces the explicit solutions (11.25) and (11.31), thereby establishing their topological-string origin.

Structure of the chapter. In Section 11, we formulate the four-dimensional spectral problem associated with H_N . After introducing the gauge-theoretic building blocks and the generalized Matone relations, we present the explicit off-shell eigenfunctions for even and odd N , derive the corresponding quantization conditions, and analyze their asymptotic and analytic properties. We then compare these expressions with numerical results for cubic and quartic polynomial potentials.

In Section 12, we construct solutions for the related operators with an inverted potential and with a $\sinh(\mathbf{p})$ kinetic term. Section 13 discusses the relation between our solutions and the Baxter equation of the closed quantum Toda lattice, with particular emphasis on the distinguished Toda points.

In Section 14, we return to the five-dimensional origin of the construction. We formulate the open TS/ST correspondence for the $Y^{N,0}$ geometries, construct the corresponding background-independent solution of the quantum mirror curve, and take its four-dimensional limit, thereby recovering the eigenfunctions introduced in Section 11.

Appendix A.4 summarizes our conventions for the $SU(N)$ root system. Appendix A.1.2 reviews the Faddeev quantum dilogarithm, while Appendix A.5 collects the gauge-theory special functions entering the construction. Finally, Appendix D presents additional numerical evidence for the analytic eigenfunctions and their pole-cancellation properties.

N even	$+V_N$	$-V_N$
$\cosh(p)$	Quantization condition: (11.29) <i>Eigenvalues:</i> $E_n \in \mathbb{R}$, $E_n \leq E_{n+1}$. <i>Solutions:</i> $\psi_n \in L^2(\mathbb{R} + i\alpha)$, $\alpha \in \mathbb{R}$. <i>Decay:</i> exponential at $\pm\infty$.	No quantization condition <i>Eigenvalues:</i> any $E \in \mathbb{C}$, see (12.6). <i>Solutions:</i> $\psi_E \in L^2(\mathbb{R} + i\alpha)$, $ \alpha < \alpha_N$. <i>Decay:</i> power-law at $\pm\infty$. Quantization condition: (11.29) <i>Eigenvalues:</i> $E_n \in \mathbb{R}$, $E_n \geq E_{n+1}$. <i>Solutions:</i> $\psi_n^\pm \in L^2(\mathbb{R} + i\alpha)$, $ \alpha < \alpha_N$. <i>Decay:</i> exponential at $\mp\infty$, power-law at $\pm\infty$.
$\sinh(p)$	Quantization condition: (11.29) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi_n \in L^2(\mathbb{R} + i\alpha)$, $\alpha > -\alpha_N$. <i>Decay:</i> power-law at $-\infty$, exponential at $+\infty$.	Quantization condition: (11.29) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi_n \in L^2(\mathbb{R} + i\alpha)$, $\alpha < \alpha_N$. <i>Decay:</i> exponential at $-\infty$, power-law at $+\infty$. Quantization condition: (12.20) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi_n \in L^2(\mathbb{R} + i\alpha)$, $\alpha < \alpha_N$. <i>Decay:</i> power-law at $\pm\infty$.
N odd	$+V_N$	$-V_N$
$\cosh(p)$	Quantization condition: (11.35) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $\alpha > -\alpha_N$. <i>Decay:</i> power-law at $-\infty$, exponential at $+\infty$. Quantization condition: (11.38) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $ \alpha < \alpha_N$. <i>Decay:</i> power-law at $-\infty$, exponential at $+\infty$.	Quantization condition: (11.35) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $ \alpha < \alpha_N$. <i>Decay:</i> exponential at $-\infty$, power-law at $+\infty$. Quantization condition: (11.38) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $\alpha < \alpha_N$. <i>Decay:</i> exponential at $-\infty$, power-law at $+\infty$.
$\sinh(p)$	No solutions in $L^2(\mathbb{R})$ generically Quantization conditions: (11.35) & (11.38) <i>Note:</i> only possible for special V_N . <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $\alpha \in \mathbb{R}$. <i>Decay:</i> exponential at $\pm\infty$.	Quantization condition: (11.35) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $ \alpha < \alpha_N$. <i>Decay:</i> power-law at $\pm\infty$. Quantization condition: (11.38) <i>Eigenvalues:</i> $E_n \in \mathbb{C}$. <i>Solutions:</i> $\psi \in L^2(\mathbb{R} + i\alpha)$, $ \alpha < \alpha_N$. <i>Decay:</i> power-law at $\pm\infty$.

Table IV.1: Quantization conditions and square-integrable solutions for even and odd N . Throughout this table we assume that the parameters h_k entering the potentials $\pm V_N$ are real and generic, so that the solutions exhibit no accidental enhancement of their decay. The parameter α_N is defined by $\alpha_N = \frac{\hbar}{2} \left(1 - \frac{1}{N}\right)$.

11 The spectral problem and its solution

11.1 The spectral problem

We consider the quantized Seiberg-Witten curve

$$\mathbf{H}_N = \Lambda^N (e^{\mathbf{p}} + e^{-\mathbf{p}}) + V_N(\mathbf{x}), \quad [\mathbf{x}, \mathbf{p}] = i\hbar, \quad \Lambda, \hbar \in \mathbb{R}_{>0}, \quad (11.1)$$

$$V_N(x) = \sum_{k=0}^{N-1} (-1)^k h_k x^{N-k}, \quad h_k \in \mathbb{R}, \quad (11.2)$$

where we set $h_0 = 1$ and $h_1 = 0$, without loss of generality.¹

A natural domain $\mathcal{D}$ on which the operator $\mathbf{H}_N : \mathcal{D} \rightarrow L^2(\mathbb{R})$ is symmetric is given by

$$\mathcal{D} = (\mathcal{D}_1 \cap \mathcal{D}_2) \subset L^2(\mathbb{R}), \quad \mathcal{D}_1 = \left\{ \psi(x) \in L^2(\mathbb{R}) \mid V_N(x)\psi(x) \in L^2(\mathbb{R}) \right\}, \quad (11.3)$$

while $\mathcal{D}_2$ consists of all $\psi(x+iy)$ that admit an analytic continuation to the strip $-\hbar < y < \hbar$ such that $\psi(x+iy) \in L^2(\mathbb{R})$ for all fixed $-\hbar < y < \hbar$, and for which the limits

$$\lim_{y \rightarrow \hbar^-} \psi(x \pm iy) \quad (11.4)$$

exist in the sense of convergence in $L^2(\mathbb{R})$. For N even, one can in fact consider a canonical self-adjoint extension of this operator, see below.

The eigenvalue equation associated with (11.1) is a difference equation and reads²

$$\Lambda^N (\psi(x+i\hbar, \mathbf{h}) + \psi(x-i\hbar, \mathbf{h})) + V_N(x)\psi(x, \mathbf{h}) = E\psi(x, \mathbf{h}), \quad (11.5)$$

where $\mathbf{h}$ contains $h_2, \dots, h_N$ with

$$h_N = (-1)^{N-1} E, \quad (11.6)$$

and E plays the role of energy. Note that the spectral properties of $\mathbf{H}_N$ depend on whether N is even or odd.

N even: confining potential. When N is even, the potential $V_N(x)$ is confining. One can consider a canonical self-adjoint extension of the operator $\mathbf{H}_N$, the so-called Friedrichs extension.³ It was shown in [171], based on [172], that this extension has a purely discrete spectrum corresponding to bound states, and has a trace-class inverse. Numerically, the spectrum and eigenfunctions can be computed via Hamiltonian truncation in the harmonic oscillator basis.

N odd: unbounded potential. When N is odd, the potential $V_N(x)$ is confining as $x \rightarrow +\infty$ but unbounded from below as $x \rightarrow -\infty$. One therefore expects the existence of resonant states with complex energies.⁴

¹The restriction $\Lambda, \hbar \in \mathbb{R}_{>0}$ and $h_k \in \mathbb{R}$ is not essential: the solutions we construct remain well-defined even for other choices of the parameters. However, this assumption simplifies the discussion of the operator's spectral properties, and we will adopt this convention.

²To simplify notation, we suppress the explicit dependence of the eigenfunctions on Λ and $\hbar$.

³We will not explicitly construct this extension, and all our on-shell eigenfunctions will in fact turn out to be in the domain $\mathcal{D}$ in (11.3).

⁴Note that while the existence of a self-adjoint extension of $\mathbf{H}_N$ is guaranteed by von Neumann's theorem [173, thm. X.3] [174, prop. 13.25], there is no clear *canonical* choice for N odd. In fact, it is known that for the Schrödinger case $\mathbf{p}^2 + V_N(\mathbf{x})$ with N odd there is an infinite family of self-adjoint extensions, all of which have a discrete spectrum [175, lemma. 2.2].

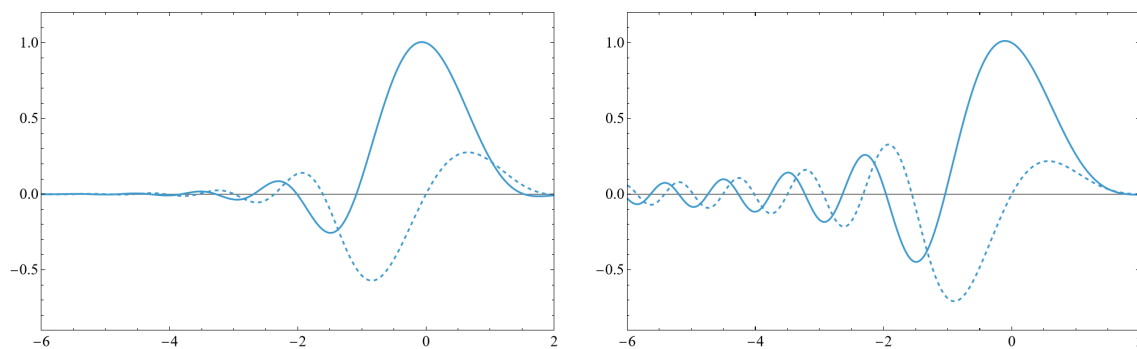


Figure IV.1: Numerical ground state eigenfunction ($E_0 \approx 0.54151 + i0.25905$) of (11.1) with $V_3(x) = x^3$, obtained via complex dilation. Left: rotated eigenfunction $\psi_0^\theta(x, \hbar)$. Right: true eigenfunction $\psi_0(x, \hbar) = \psi_0^\theta(e^{-i\theta}x, \hbar)$. Here we set $\Lambda = 1/2$, $\hbar = 1$ and $\theta = -1/10$. The eigenfunction is normalized so that $\psi_0(0, \hbar) = 1$. Solid lines denote the real part; dashed lines denote the imaginary part.

Such potentials have been studied numerically for the standard Schrödinger equation using the method of complex dilation [176], and rigorous analyses of resonances are available in [175, 177, 178]. As emphasized in [44], complex dilation can be applied directly to the deformed Hamiltonian (11.1); this is the numerical method we adopt here. Let us briefly recall the main idea. For N odd, the eigenfunctions of (11.1) decay only as a power-law for $x \rightarrow -\infty$ and numerical diagonalization in the harmonic oscillator basis does not seem to be convergent. To circumvent this problem, we perform a small complex rotation of the position and momentum operators by an angle θ . This yields the rotated Hamiltonian

$$H_{N,\theta} = 2\Lambda^N \cosh(e^{-i\theta}\mathbf{p}) + V_N(e^{i\theta}\mathbf{x}) \quad (11.7)$$

which admits exponentially decaying, square-integrable eigenfunctions

$$\psi_n^{(\theta)}(x, \hbar) \quad (11.8)$$

with complex eigenvalues. Therefore, $\psi_n^{(\theta)}(x, \hbar)$ can be computed efficiently by diagonalizing (11.7) in the harmonic oscillator basis. The eigenfunctions of the original Hamiltonian (11.1) are then obtained by analytic continuation:

$$\psi_n(x, \hbar) = \psi_n^{(\theta)}(e^{-i\theta}x, \hbar). \quad (11.9)$$

An explicit example is shown in Figure IV.1 for the cubic potential $V_3(x) = x^3$. One observes the exponential decay of the rotated eigenfunctions (11.8) (left), in contrast with the slower power-law decay of the true eigenfunctions (11.9) (right).

In the next section, we construct explicit solutions to (11.5) for arbitrary N . These solutions are entire in x for *all* values of E and h_k , $k = 2, \dots, N-1$, and we refer to them as *off-shell* eigenfunctions. As we will see, such solutions become square-integrable wavefunctions only when E satisfies an appropriate quantization condition. The corresponding eigenfunctions, evaluated at these discrete energy values, will be denoted as *on-shell* eigenfunctions.

11.2 Gauge theoretic building blocks

The explicit construction of our eigenfunctions is based on the open TS/ST correspondence [17, 26, 28]. A key lesson from studying spectral problems through the lens of

supersymmetric gauge theory and topological strings is that eigenfunctions — and spectral quantities more generally — take a simpler form when written in terms of the Coulomb branch (or Kähler) parameters $\{a_I\}_{I=1}^N$, rather than the complex moduli $\{h_k\}_{k=2}^N$ that appear in the difference equation (11.5). The map relating the two sets of parameters has a precise geometrical meaning: it is known as the quantum mirror map [115, 157, 179]. In the framework of five-dimensional, $\mathcal{N} = 1$ supersymmetric gauge theory, the inverses of these maps are interpreted as Wilson loop expectation values [95, 122], and in the four-dimensional limit they reduce to the generalized Matone relations [39, 40, 44]. At leading order in Λ , we have

$$h_k = s_{(1^k)}(\mathbf{a}) + \mathcal{O}(\Lambda^{2N}) = \sum_{1 \leq I_1 < I_2 < \dots < I_k \leq N} a_{I_1} a_{I_2} \dots a_{I_k} + \mathcal{O}(\Lambda^{2N}), \quad (11.10)$$

where s_R denotes the Schur polynomial associated to the Young tableau R , and (1^k) refers to the tableau consisting of one column with k boxes. Thus, $s_{(1^k)}$ is an elementary symmetric polynomial, and the roots of $V_N(x) - E$, with the potential (11.2), coincide with the Coulomb branch parameters a_I up to corrections of order Λ^{2N} . We refer to (11.10) as the generalized Matone relation. Explicit examples are

$$\begin{aligned} h_2 &= -\frac{1}{2} \sum_{I=1}^N a_I^2 + \mathcal{O}(\Lambda^{2N}), & h_3 &= \frac{1}{3} \sum_{I=1}^N a_I^3 + \mathcal{O}(\Lambda^{2N}), \\ h_4 &= \frac{1}{8} \left(\left(\sum_{I=1}^N a_I^2 \right)^2 - 2 \sum_{I=1}^N a_I^4 \right) + \mathcal{O}(\Lambda^{2N}), \end{aligned} \quad (11.11)$$

recalling that the Coulomb branch parameters a_I sum to zero in the $\mathrm{SU}(N)$ case,

$$\sum_{I=1}^N a_I = 0. \quad (11.12)$$

The complete expression for the map, including all orders in Λ and $\hbar$ dependence, is reviewed in subsection A.5.3; see in particular equation (A.82). One can invert (11.10) and (A.82) to obtain an expression for a_I as a function of h_k , Λ and $\hbar$. Geometrically, the quantities $a_I(\mathbf{h}, \Lambda, \hbar)$, for $I = 1, \dots, N-1$, represent the quantum A-periods associated with the SW curve [115, 157, 179]

$$2\Lambda^N \cosh(p) + V_N(x) = E, \quad x, p \in \mathbb{C}. \quad (11.13)$$

As noted in the introduction, (11.5) coincides with a quantization of the SW curve (11.13) describing the low-energy dynamics of four-dimensional, $\mathcal{N} = 2$ $\mathrm{SU}(N)$ SYM. It is therefore natural to expect that the eigenfunctions inherit an $\mathrm{SU}(N)$ structure. We thus begin by recalling a few representation-theoretic notions related to $\mathrm{SU}(N)$.

Let $\mathcal{W}_N$ denote the Weyl group of $\mathrm{SU}(N)$. Consider a vector

$$\mathbf{v} = \sum_{I=1}^N v_I \mathbf{e}_I, \quad (11.14)$$

where $\mathbf{e}_I$ are the weights of the fundamental representation, see Section A.4 for our conventions. The Weyl orbit of $\mathbf{v}$ is defined by

$$\mathcal{W}_N \cdot \mathbf{v} = \{w(\mathbf{v}) \mid w \in \mathcal{W}_N\} = \left\{ \sum_{I=1}^N v_{\sigma(I)} \mathbf{e}_I \mid \sigma \in \mathcal{S}_N \right\}, \quad (11.15)$$

where $\mathcal{S}_N$ is the permutation group of N elements. We define

$$\gamma = \frac{1}{2} \sum_{I=1}^N (-1)^{I-1} \mathbf{e}_I = \sum_{k=1}^{N-1} (-1)^{k-1} \boldsymbol{\lambda}_k, \quad (11.16)$$

where $\boldsymbol{\lambda}_k$ is a fundamental weight (A.60), and we organize the Coulomb branch parameters a_I of $\mathcal{N} = 2$ $\text{SU}(N)$ SYM into the vector

$$\mathbf{a} = \sum_{I=1}^N a_I \mathbf{e}_I, \quad \sum_{I=1}^N a_I = 0. \quad (11.17)$$

It is also useful to introduce the linear involution

$$\mathbf{f}(\mathbf{a}) = - \sum_{I=1}^N a_{N-I+1} \mathbf{e}_I, \quad (11.18)$$

which acts on the parameters h_k according to $h_k(\mathbf{f}(\mathbf{a})) = h_k(-\mathbf{a}) = (-1)^k h_k(\mathbf{a})$.

The starting point of our construction is the following defect partition function:

$$Z_D(x, \mathbf{a}, \Lambda, \hbar) = \left(\frac{\Lambda}{\hbar} \right)^{iN \frac{x}{\hbar}} e^{\pi(1 + \frac{N}{2}) \frac{x}{\hbar}} \left\{ \prod_{I=1}^N \Gamma\left(\frac{i \mathbf{e}_I \cdot \mathbf{a} - i x}{\hbar} \right) \right\} Z_D^{\text{inst}}(x, \mathbf{a}, \Lambda, \hbar), \quad (11.19)$$

where the instanton part Z_D^{inst} is defined in (A.74). It is known [91, 122, 147, 170, 180, 181] that this function provides a formal solution to the difference equation (11.5). However, Z_D has poles at

$$x = a_I + i\hbar k, \quad k \in \mathbb{Z}, \quad I \in \{1, \dots, N\}. \quad (11.20)$$

Hence, the function (11.19) alone is insufficient to construct entire, square-integrable eigenfunctions. In what follows, we will combine two copies of (11.19) with an additional function in such a way that the resulting expression becomes entire in x for *any* values of a_I , and hence *any* value of h_k . The function we shall use is

$$\mathcal{P}_{\mathbf{n}}(x, \mathbf{a}, \Lambda, \hbar) = \frac{\exp\left(\left(\frac{i}{\hbar} \partial_{\mathbf{a}} F_{\text{NS}} - \frac{\pi \mathbf{a}}{\hbar} \mathbb{1}_{2\mathbb{N}+1}(N)\right) \cdot \mathbf{n}\right)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi \alpha \cdot \mathbf{a}}{\hbar}))^{(\alpha \cdot \mathbf{n})^2}} \prod_{I=1}^N (1 - e^{\frac{2\pi}{\hbar} (e_I \cdot \mathbf{a} - x)})^{\frac{1}{2} - n_I}, \quad (11.21)$$

$$\mathbf{n} = \sum_{I=1}^N n_I \mathbf{e}_I, \quad n_I \in \left\{ -\frac{1}{2}, +\frac{1}{2} \right\}, \quad (11.22)$$

where $\mathbb{1}_{2\mathbb{N}+1}(N)$ is the indicator function vanishing for N even, Δ_+ is the set of positive roots (A.62), the $\mathbf{e}_I$ are the weights of the fundamental representation in (A.60), and $\mathbf{n}$ is a particular element in the weight lattice, see later. The dual quantum period is given by⁵

$$\begin{aligned} \frac{i}{\hbar} \partial_{\mathbf{a}} F_{\text{NS}} &= \sum_{I=1}^N \frac{i}{\hbar} \frac{\partial F_{\text{NS}}^{\text{inst}}}{\partial a_I} \mathbf{e}_I \\ &\quad - \sum_{\alpha \in \Delta_+} \left[i \frac{\pi}{2} + i 2 \left(\boldsymbol{\alpha} \cdot \frac{\mathbf{a}}{\hbar} \right) \log\left(\frac{\Lambda}{\hbar} \right) + \log\left(\frac{\Gamma(1 - i(\boldsymbol{\alpha} \cdot \frac{\mathbf{a}}{\hbar}))}{\Gamma(1 + i(\boldsymbol{\alpha} \cdot \frac{\mathbf{a}}{\hbar}))} \right) \right] \boldsymbol{\alpha}, \end{aligned} \quad (11.23)$$

⁵The derivatives with respect to a_I are taken before imposing the constraint $\sum_{I=1}^N a_I = 0$.

where $F_{\text{NS}}^{\text{inst}}$ is defined in (A.71). Note that the functions $\mathcal{P}_n$ are invariant under the shifts appearing in the difference operator in (11.5).

Finally, to analyse the asymptotic behaviour of the resulting solutions, it is convenient to introduce the shorthand notation

$$\begin{aligned}\mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar) &= \frac{\exp\left(\left(\frac{i}{\hbar}\partial_{\mathbf{a}}F_{\text{NS}} - \frac{\pi\mathbf{a}}{\hbar}\mathbb{1}_{2N+1}(N)\right) \cdot \mathbf{n}\right)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi\alpha \cdot \mathbf{a}}{\hbar}))^{(\alpha \cdot \mathbf{n})^2}}, \\ \mathcal{Q}_n^1(\mathbf{a}, \Lambda, \hbar) &= -\mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar) \sum_{I=1}^N \left(\frac{1}{2} - n_I\right) \exp\left(2\pi \mathbf{e}_I \cdot \frac{\mathbf{a}}{\hbar}\right), \\ \mathcal{Q}_n^M(\mathbf{a}, \Lambda, \hbar) &= (-1)^M \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar) \exp\left(\sum_{I=1}^N \left(\frac{1}{2} - n_I\right) 2\pi \mathbf{e}_I \cdot \frac{\mathbf{a}}{\hbar}\right),\end{aligned}\tag{11.24}$$

where $M = \sum_{I=1}^N (1/2 - n_I) \in \mathbb{N}$. These are the coefficients of the leading, subleading and most suppressed contributions to the large x expansion of $\mathcal{P}_n$ (11.21). Using the building blocks (11.19) and (11.21), we then obtain the following entire, off-shell solutions to the difference equation (11.5).

11.3 The eigenfunctions

N even: confining potential. For N even, we have:

$$\begin{aligned}\psi_N^{\text{even}}(x, \mathbf{h}) &= Z_D(x, \mathbf{a}, \Lambda, \hbar) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{P}_{\mathbf{n}}(x, \mathbf{a}, \Lambda, \hbar) \\ &\quad + i e^{\frac{2\pi x}{\hbar}} Z_D(-x, \mathbf{f}(\mathbf{a}), \Lambda, \hbar) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{P}_{\mathbf{n}}(-x, \mathbf{f}(\mathbf{a}), \Lambda, \hbar),\end{aligned}\tag{11.25}$$

where we recall that $\mathbf{f}(\mathbf{a})$ is defined in (11.18), and $\mathbf{a}$ is related to $\mathbf{h}$ via the generalized Matone relations; see (11.10) and (A.82).⁶ We will refer to the first and second terms in (11.25) as saddles, reflecting their origin in topological string theory [17, 26, 28]; see Section 14.

Let us now analyse the asymptotic behaviour of (11.25). For N even, the Weyl orbit of γ contains $\binom{N}{N/2}$ elements, while the orbit of $\gamma + \mathbf{e}_N$ contains $\binom{N}{N/2-1}$ elements. Moreover, in this case the number of non-trivial factors appearing in the product in (11.21) equals $N/2$ for elements in the orbit of γ , and $N/2 - 1$ for those in the orbit of $\gamma + \mathbf{e}_N$. This leads to the following asymptotic behaviour as $\text{Re}(x) \rightarrow \pm\infty$, with $\text{Im}(x)$ held fixed:

$$\begin{aligned}\psi_N^{\text{even}}(x, \mathbf{h}) &\simeq \\ e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|}\right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|}{\Lambda}\right)^{N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty,\end{aligned}\tag{11.26}$$

$$\begin{aligned}\psi_N^{\text{even}}(x, \mathbf{h}) &\simeq e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|}\right)^{\frac{N}{2}} \left\{ \left(\frac{|\text{Re}(x)|}{\Lambda}\right)^{N \frac{\text{Im}(x)}{\hbar}} u_N^{-1}(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^{\frac{N}{2}}(\mathbf{a}, \Lambda, \hbar) \right. \\ &\quad \left. + i \left(\frac{|\text{Re}(x)|}{\Lambda}\right)^{-N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) \right\}, \quad \text{Re}(x) \rightarrow -\infty,\end{aligned}\tag{11.27}$$

⁶Note that $\mathcal{W}_N \cdot \gamma = \mathcal{W}_N \cdot \lambda_{N/2}$ and $\mathcal{W}_N \cdot (\gamma + \mathbf{e}_N) = \mathcal{W}_N \cdot \lambda_{(N/2)+1}$ for N even, with $\lambda_{N/2}, \lambda_{(N/2)+1} \in \Lambda_w$ fundamental weights.

where $u_N(x)$ is an x -dependent factor of unit modulus,

$$u_N(x) := e^{i\frac{\pi}{4}N} \left(\frac{e\Lambda}{|\operatorname{Re}(x)|} \right)^{iN \frac{|\operatorname{Re}(x)|}{\hbar}}. \quad (11.28)$$

Due to the exponential growth near positive infinity, (11.25) is not square-integrable unless we impose

$$\sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar) = \sum_{n \in \mathcal{W}_N \cdot \gamma} \frac{\exp\left(\frac{i}{\hbar} \partial_{\mathbf{a}} F_{\text{NS}} \cdot \mathbf{n}\right)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi \alpha \cdot \mathbf{a}}{\hbar}))^{(\alpha \cdot \mathbf{n})^2}} = 0, \quad (11.29)$$

which is precisely the quantization condition for the energy E obtained in [44] by explicitly constructing the Fredholm determinant. Note that we are implicitly using the generalized Matone relations (11.10) to write $\mathbf{a} = \mathbf{a}(\mathbf{h})$. After imposing this condition, one has

$$\begin{aligned} \psi_N^{\text{even}}(x, \mathbf{h}) \Big|_{(11.29)} &\simeq e^{-\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{\operatorname{Re}(x)} \right)^{\frac{N}{2}} \left\{ \left(\frac{\operatorname{Re}(x)}{\Lambda} \right)^{N \frac{\operatorname{Im}(x)}{\hbar}} u_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^1(\mathbf{a}, \Lambda, \hbar) \right. \\ &\left. + i \left(\frac{\operatorname{Re}(x)}{\Lambda} \right)^{-N \frac{\operatorname{Im}(x)}{\hbar}} u_N^{-1}(x) \sum_{n \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{Q}_n^{\frac{N}{2}-1}(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) \right\}, \quad \operatorname{Re}(x) \rightarrow +\infty, \quad (11.30) \end{aligned}$$

so the on-shell eigenfunctions are decaying exponentially also for $\operatorname{Re}(x) \rightarrow +\infty$.

In summary, the functions (11.25) are entire solutions of the difference equation (11.5), and they are square-integrable and belong to the domain $\mathcal{D}$ in (11.3) if and only if the quantization condition (11.29) is satisfied. At these quantized values of E , they thus define genuine eigenfunctions of $\mathbf{H}_N$ in (11.1).

N odd: unbounded potential. For N odd, the potential is unbounded from below as $x \rightarrow -\infty$, and we find the following off-shell solution of (11.5):

$$\begin{aligned} \psi_N^{\text{odd}}(x, \mathbf{h}) &= Z_D(x, \mathbf{a}, \Lambda, \hbar) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{P}_n(x, \mathbf{a}, \Lambda, \hbar) \\ &\quad + e^{\frac{\pi x}{\hbar}} Z_D(-x, \mathbf{f}(\mathbf{a}), \Lambda, \hbar) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{P}_n(-x, \mathbf{f}(\mathbf{a}), \Lambda, \hbar), \end{aligned} \quad (11.31)$$

where, again, we recall that $\mathbf{f}(\mathbf{a})$ is defined in (11.18) and $\mathbf{a}$ is related to $\mathbf{h}$ via the generalized Matone relations; see (11.10).⁷ Once more, we will refer to the first and second terms in (11.31) as saddles. Let us now analyse the asymptotic behaviour of this solution. For N odd, the Weyl orbit of γ contains $\binom{N}{(N-1)/2}$ elements. In this case, the number of non-trivial factors appearing in the product in equation (11.21) is $(N-1)/2$. Hence, for $\operatorname{Re}(x) \rightarrow \pm\infty$ with $\operatorname{Im}(x)$ held fixed, we find:

$$\begin{aligned} \psi_N^{\text{odd}}(x, \mathbf{h}) &\simeq \\ e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{\operatorname{Re}(x)} \right)^{\frac{N}{2}} \left(\frac{\operatorname{Re}(x)}{\Lambda} \right)^{N \frac{\operatorname{Im}(x)}{\hbar}} u_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar), \quad \operatorname{Re}(x) \rightarrow +\infty, \end{aligned} \quad (11.32)$$

⁷Note that $\mathcal{W}_N \cdot \gamma = \mathcal{W}_N \cdot \boldsymbol{\lambda}_{(N+1)/2}$ for N odd, with $\boldsymbol{\lambda}_{(N+1)/2} \in \Lambda_w$ a fundamental weight.

$$\psi_N^{\text{odd}}(x, \hbar) \simeq \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{-N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar), \quad \text{Re}(x) \rightarrow -\infty. \quad (11.33)$$

Let us emphasize that, although at $\text{Re}(x) \rightarrow -\infty$ there is only power-law decay, the solution remains square-integrable in this direction as long as

$$\text{Im}(x) > -\frac{\hbar}{2} \left(1 - \frac{1}{N} \right). \quad (11.34)$$

In contrast, the exponential growth at $\text{Re}(x) \rightarrow +\infty$ must be eliminated to achieve square-integrability, which requires imposing

$$\sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar) = \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \frac{\exp\left(\left(\frac{i}{\hbar} \partial_{\mathbf{a}} F_{\text{NS}} - \frac{\pi \mathbf{a}}{\hbar}\right) \cdot \mathbf{n}\right)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi \alpha \cdot \mathbf{a}}{\hbar}))^{(\alpha \cdot \mathbf{n})^2}} = 0. \quad (11.35)$$

Once again, this is precisely the quantization condition found in [44]. After imposing this condition, the solution decays exponentially as $\text{Re}(x) \rightarrow +\infty$,

$$\begin{aligned} \psi_N^{\text{odd}}(x, \hbar) \Big|_{(11.35)} &\simeq e^{-\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{\text{Re}(x)} \right)^{\frac{N}{2}} \left\{ \left(\frac{\text{Re}(x)}{\Lambda} \right)^{N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^1(\mathbf{a}, \Lambda, \hbar) \right. \\ &\quad \left. + \left(\frac{\text{Re}(x)}{\Lambda} \right)^{-N \frac{\text{Im}(x)}{\hbar}} u_N^{-1}(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^{\frac{N-1}{2}}(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) \right\}, \quad \text{Re}(x) \rightarrow +\infty. \end{aligned} \quad (11.36)$$

and becomes a square-integrable solution to the difference equation (11.5). As we will discuss below, the same square-integrable solutions also appear to be captured by the complex dilation method.

However, the quantization condition (11.35) is not the only one that gives rise to square-integrable solutions of the difference equation (11.5). Indeed, introducing

$$e^{-\frac{2\pi x}{\hbar}} \psi_N^{\text{odd}}(x, \hbar), \quad (11.37)$$

one obtains a new family of square-integrable solutions upon imposing the alternative quantization condition

$$\sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) = \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \frac{\exp\left(\left(\frac{i}{\hbar} \partial_{\mathbf{f}(\mathbf{a})} F_{\text{NS}} - \frac{\pi \mathbf{f}(\mathbf{a})}{\hbar}\right) \cdot \mathbf{n}\right)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi \alpha \cdot \mathbf{f}(\mathbf{a})}{\hbar}))^{(\alpha \cdot \mathbf{n})^2}} = 0. \quad (11.38)$$

This second family of solutions once again decays exponentially as $\text{Re}(x) \rightarrow +\infty$, as is clear from (11.32). For $\text{Re}(x) \rightarrow -\infty$, on the other hand, the solutions exhibit power-law decay,

$$\begin{aligned} e^{-\frac{2\pi x}{\hbar}} \psi_N^{\text{odd}}(x, \hbar) \Big|_{(11.38)} &\simeq \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left\{ \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{N \frac{\text{Im}(x)}{\hbar}} u_N^{-1}(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^{\frac{N-1}{2}}(\mathbf{a}, \Lambda, \hbar) \right. \\ &\quad \left. + \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{-N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^1(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) \right\} \quad \text{Re}(x) \rightarrow -\infty, \end{aligned} \quad (11.39)$$

and are therefore square-integrable along lines parallel to the real axis provided that

$$|\operatorname{Im}(x)| < \frac{\hbar}{2} \left(1 - \frac{1}{N}\right). \quad (11.40)$$

In summary, (11.31) are entire solutions of the difference equation (11.5), and are square-integrable in the shifted upper half plane (11.34) if and only if (11.35) holds. On the other hand, the solutions (11.37) are square-integrable in the strip (11.40) if and only if (11.38) holds. However, neither of them is in the domain $\mathcal{D}$ in (11.3), and the eigenvalues solving the quantization conditions (11.35) and (11.38) are generically complex.⁸ Thus, these square-integrable solutions do not belong to the domain of any symmetric or self-adjoint realization of the difference operator.⁹ Moreover, the solutions to the quantization condition (11.38) do not appear to be captured by the complex dilation method.

Discussion. We conclude this section with a few remarks on the expressions (11.25) and (11.31).

1. Whereas the Schrödinger equation $(-\partial_x^2 + V_N(x) - E)\psi = 0$ admits only entire solutions, this is not the case for the difference equations associated with the deformed operator (11.1). For instance, each of the two terms in (11.25) and (11.31) is, by itself, a solution of (11.5), but each has poles at

$$x = a_I + i\hbar n, \quad n \in \mathbb{Z}. \quad (11.41)$$

Only for the special linear combinations (11.25) and (11.31) do these poles cancel, yielding a function that is entire in x for any value of the parameters h_k , $k = 2, \dots, N$.¹⁰

2. Eigenfunctions are defined up to an overall normalization. In our conventions, we choose the normalization so that the asymptotic behaviour (11.26) and (11.32) reproduces the Fredholm determinant of the corresponding spectral problem as computed in [44, 160]. This choice is natural from the perspective of the TS/ST correspondence, where eigenfunctions are required to be entire both in the open and closed string moduli. However, as we discuss in Section 13, this normalization is not convenient for studying the Toda points.
3. A noteworthy property of the deformed Hamiltonian (11.5) is that its bound states violate the standard oscillation theorem [182, p. 66, thm. 3.5]. For the deformed Schrödinger equation (11.5) it is no longer true that the n^{th} eigenfunction has n zeroes, even if the potential is confining. This behaviour occurs even for the simplest potential $V_2(x) = x^2$, see [28].¹¹ This violation is evident from the oscillatory asymptotics (11.27) and (11.30), see also Figure IV.4 for a graphical representation.

⁸As discussed in Section 13, the eigenvalues are real if we take the parameters of the potential to be at the Toda points $\{h_k^T\}$. At these special points, the eigenfunctions are expected to belong to $\mathcal{D}$, suggesting that the corresponding difference operator has qualitatively different spectral properties and may even be essentially self-adjoint.

⁹Since they are square-integrable solutions of the difference equation, one may speculate that they are instead eigenfunctions of the adjoint H_N^* . This is indeed what happens for the usual Schrödinger equation, but at present it is not clear whether the same holds true for difference equations. We thank Lukas Schimmer for an illuminating discussion on this point.

¹⁰Although we do not have a rigorous proof of this statement, we have checked the pole cancellation for $N = 2, 3, 4, 5, 6$ up to three orders in the Λ^{2N} expansion. Further numerical evidence is provided in Appendix D.

¹¹A similar phenomenon has been observed for difference equations arising in the context of the open TS/ST correspondence [17, 26, 28].

4. For N odd, the asymptotic expression (11.33) exhibits an outgoing oscillatory behaviour for $\text{Re}(x) \rightarrow -\infty$, characteristic of resonant states. See also Figure IV.2.
5. As noted in [44], there are special values of the moduli h_k – those corresponding to eigenvalues of the quantum Toda Hamiltonians – at which the spectrum and eigenfunctions exhibit special features. For N even, certain eigenstates, including the ground state, become degenerate, i.e. tunnelling is suppressed at these special points. A similar phenomenon occurs for N odd. For generic values of h_k , the energies associated with square-integrable solutions are complex. Nevertheless, at these special values of h_k , the eigenfunctions exhibit enhanced exponential decay as $\text{Re}(x) \rightarrow -\infty$, and the corresponding energies are real, despite the potential being unbounded from below.¹² We will return to these special points in Section 13.
6. We recover [28, eq. (4.16)] by evaluating (11.25) for $N = 2$.

11.4 Examples

11.4.1 Cubic potentials

Let us consider the $N = 3$ case:

$$H_3 = 2\Lambda^3 \cosh(p) + x^3 + h_2 x. \quad (11.42)$$

The potential is unbounded from below, and for generic values of h_k the spectrum contains resonant states with complex energies. Such resonances always occur in complex-conjugate pairs, and throughout this Chapter we focus on those having a positive imaginary part of the energy. For $N = 3$, the Weyl orbit of γ is given by:

$$\mathcal{W}_N \cdot \gamma = \left\{ \frac{1}{2}(e_1 - e_2 + e_3), \frac{1}{2}(e_1 + e_2 - e_3), \frac{1}{2}(-e_1 + e_2 + e_3) \right\}. \quad (11.43)$$

Examples of on-shell eigenfunctions are shown in Figures IV.2, D.1, D.2, and D.3. In Figure IV.2, we consider the case $h_2 = 0$ and focus on the third excited state, with complex energy:

$$E_3 \approx 2.511486 + i 2.512027. \quad (11.44)$$

Through the generalized Matone relations (11.10), this corresponds to:

$$a_1 \approx 1.473672 + i 0.395307, \quad a_2 \approx -0.394390 - i 1.473919. \quad (11.45)$$

The power-law decay of the on-shell wavefunctions as $x \rightarrow -\infty$ is particularly evident in this figure. Similarly, Figures D.1, D.2, and D.3 illustrate the ground, first, and second excited state for $h_2 = -5$, $h_2 = -3$ and $h_2 = 2$ respectively.

Each figure shows excellent agreement between the analytic expression (11.31) and the numerical results obtained via complex dilation. In addition, Figure D.2 clearly shows the cancellation of poles between the two terms in (11.31). Finally, an example of off-shell eigenfunction is shown in Figure IV.3. The figure also clearly shows that each individual saddle in (11.31) develops poles, whereas the complete sum is free of singularities. Since we are off-shell, the resulting off-shell eigenfunction does not belong to $L^2(\mathbb{R})$.

11.4.2 Quartic potentials

Let us consider the $N = 4$ Hamiltonian:

$$H_4 = 2\Lambda^4 \cosh(p) + x^4 + h_2 x^2 - h_3 x. \quad (11.46)$$

¹²Recently, these phenomena were also studied from the point of view of WKB in [45].

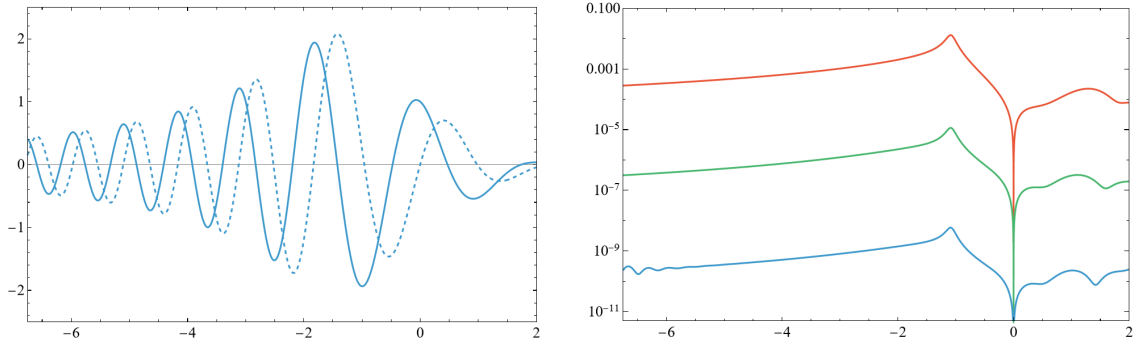


Figure IV.2: Left: third excited state of the $V_3(x)$ potential with $h_2 = 0$, $\hbar = 1$, and $\Lambda = \frac{1}{2}$. Dashed lines denote the imaginary part of the eigenfunction, while solid lines denote the real part. Right: difference between the numerical eigenfunction and the analytic expression from (11.31). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

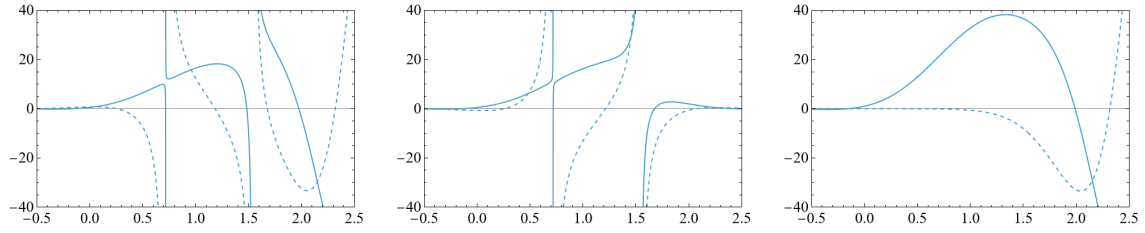


Figure IV.3: Off-shell eigenfunction of the potential $V_3(x)$ with $h_2 = -4$, $h_3 = -2.5$, $\hbar = 1$ and $\Lambda = 2/5$. From left to right: first saddle, second saddle and the full eigenfunction given by (11.31). Dashed lines denote the imaginary part, while solid lines denote the real part.

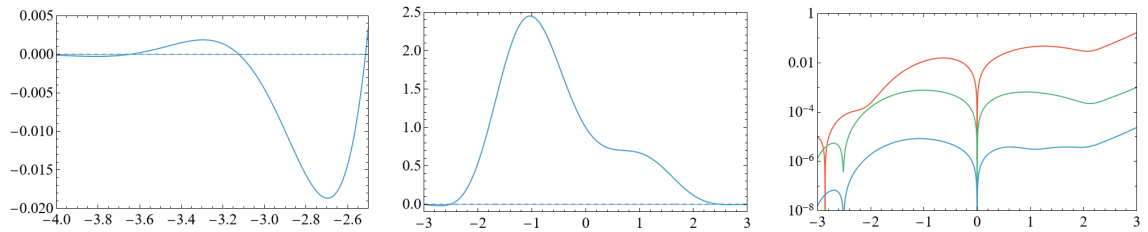


Figure IV.4: The ground state of the $V_4(x)$ potential with $h_2 = -3$, $h_3 = -0.1$, $\Lambda = (1/2)^{1/2}$ and $\hbar = 1$; the corresponding energy is $E_0 \approx -0.527521$. Dashed lines represent the imaginary part of the eigenfunction, while solid lines represent the real part. Left: illustration of the failure of the oscillation theorem. Centre: eigenfunction obtained from (11.25). Right: difference between the numerical eigenfunction and the analytic expression from (11.25). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

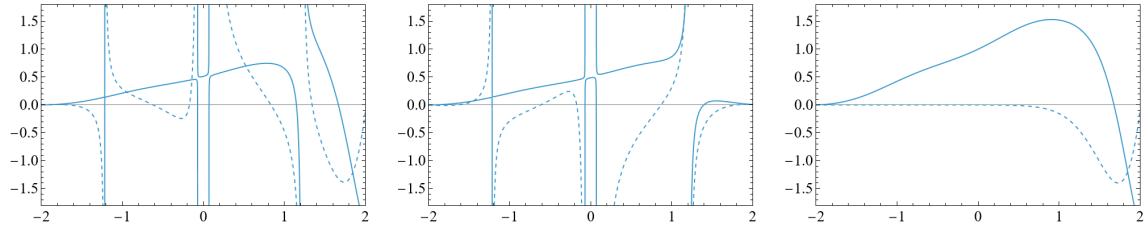


Figure IV.5: Off-shell eigenfunction of the potential $V_4(x)$ with $h_2 = -1.6$, $h_3 = 0.45$, $h_4 = 0.06$, $\hbar = 1$ and $\Lambda = 13/23$. From left to right: first saddle, second saddle and the full eigenfunction given by (11.25). Dashed lines represent the imaginary part, while solid lines the real part.

In this case, the potential is confining and the operator has a purely real and discrete spectrum, corresponding to bound states. The Weyl orbits of γ and $\gamma + e_N$ are, respectively:

$$\begin{aligned} \mathcal{W}_N \cdot \gamma = & \left\{ \frac{1}{2}(e_1 - e_2 + e_3 - e_4), \frac{1}{2}(e_1 - e_2 - e_3 + e_4), \right. \\ & \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \frac{1}{2}(-e_1 + e_2 + e_3 - e_4), \\ & \left. \frac{1}{2}(-e_1 + e_2 - e_3 + e_4), \frac{1}{2}(-e_1 - e_2 + e_3 + e_4) \right\}, \end{aligned} \quad (11.47)$$

$$\begin{aligned} \mathcal{W}_N \cdot (\gamma + e_N) = & \left\{ \frac{1}{2}(e_1 - e_2 + e_3 + e_4), \frac{1}{2}(e_1 + e_2 - e_3 + e_4), \right. \\ & \left. \frac{1}{2}(e_1 + e_2 + e_3 - e_4), \frac{1}{2}(-e_1 + e_2 + e_3 + e_4) \right\}. \end{aligned} \quad (11.48)$$

Examples of on-shell solutions are shown in Figures IV.4, D.4, and D.5. In Figure IV.4, we exhibit the ground state eigenfunction for $h_2 = -3$ and $h_3 = -0.1$, corresponding to the eigenvalue:

$$E_0 \approx -0.527521. \quad (11.49)$$

Through the generalized Matone relations, this corresponds to:

$$a_1 \approx 0.446118, \quad a_2 \approx 1.656437, \quad a_3 \approx -1.694744. \quad (11.50)$$

In this figure, the quartic potential is a slight deformation of the symmetric double well (the symmetric case has $h_2 < 0$ and $h_3 = 0$), and the corresponding ground state clearly reflects this small asymmetry. Similarly, Figures D.4 and D.5 show the first and second excited state for $h_2 = 3.5$, $h_3 = 0$ and $h_2 = 1.2$, $h_3 = -4.7$, respectively. Again, each figure shows excellent agreement between the analytic expression (11.25) and the results obtained by numerically diagonalizing the Hamiltonian (11.46). The cancellation of poles is clearly visible for the on-shell eigenfunction in Figure D.4 and in the off-shell eigenfunction in Figure IV.5. In the latter case, the resulting eigenfunction does not belong to $L^2(\mathbb{R})$, being off-shell.

12 Related spectral problems

One can obtain solutions to some variants of (11.5) from (11.25) and (11.31) by multiplying them by appropriate exponential factors. In this section, we will discuss the solutions to the case where the variant of $\mathbf{H}_N$ in (11.1) has an inverted potential $-V_N(x)$ and/or a $\sinh(\mathbf{p})$ kinetic term. We will not go into the details of these related spectral problems, but simply discuss the solutions we obtain and their properties.

12.1 Inverted potential

Given a solution ψ of (11.5), for either even or odd N , one also has a solution of the corresponding equation with an inverted potential,

$$\Lambda^N \left(\tilde{\psi}_\pm(x + i\hbar, \mathbf{h}) + \tilde{\psi}_\pm(x - i\hbar, \mathbf{h}) \right) - V_N(x) \tilde{\psi}_\pm(x, \mathbf{h}) = \tilde{E} \tilde{\psi}_\pm(x, \mathbf{h}), \quad (12.1)$$

$$\tilde{\psi}_\pm(x) := \exp\left(\pm \frac{\pi x}{\hbar}\right) \psi(x), \quad \tilde{E} := -E, \quad (12.2)$$

where the associated symmetric operator again has domain $\mathcal{D}$, defined in (11.3).¹³ Here, the subscript $\pm$ labels two distinct solutions corresponding to the same eigenvalue $\tilde{E}$. The analytic properties of $\tilde{\psi}_\pm$ are inherited directly from those of ψ , as is evident from their definition.

Inverted potential for N even. The two choices of sign for $\tilde{\psi}_{\pm, N}^{\text{even}}$ lead to qualitatively different classes of solutions. For $\tilde{\psi}_{+, N}^{\text{even}}$ one finds

$$\tilde{\psi}_{+, N}^{\text{even}}(x, \mathbf{h}) \simeq e^{\frac{2\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty, \quad (12.3)$$

$$\begin{aligned} \tilde{\psi}_{+, N}^{\text{even}}(x, \mathbf{h}) \simeq & e^{\frac{2\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left\{ \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{N \frac{\text{Im}(x)}{\hbar}} u_N^{-1}(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^{\frac{N}{2}}(\mathbf{a}, \Lambda, \hbar) \right. \\ & \left. + i \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{-N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) \right\}, \quad \text{Re}(x) \rightarrow -\infty, \end{aligned} \quad (12.4)$$

which follows from (11.26) and (11.27). Thus, $\tilde{\psi}_{+, N}^{\text{even}}$ is square-integrable parallel to the real line if and only if the quantization condition (11.29) is satisfied and

$$|\text{Im}(x)| < \frac{\hbar}{2} \left(1 - \frac{1}{N} \right). \quad (12.5)$$

It then decays exponentially for $\text{Re}(x) \rightarrow -\infty$, while it exhibits a power-law decay for $\text{Re}(x) \rightarrow +\infty$, which can be read off from (11.30). We therefore obtain square-integrable solutions of the difference equation (12.1) for a discrete set of *real* eigenvalues bounded from *above*. These eigenvalues are precisely the negatives of those of the corresponding operator with a non-inverted potential.

On the other hand, it is evident from (11.26) and (11.27) that $\tilde{\psi}_{-, N}^{\text{even}}$ has a power-law decay for both $\text{Re}(x) \rightarrow \pm\infty$, and is square-integrable parallel to the real line inside the strip (12.5). This holds for *any* value of

$$E \in \mathbb{C}. \quad (12.6)$$

We therefore obtain a one-parameter family of square-integrable solutions of the difference equation. Note that this also means that we have two linearly independent, square-integrable

¹³See footnote 4 for a comment on the existence of self-adjoint extensions, which also holds for the case of the inverted potential for both even and odd N .

solutions when the quantization condition (11.29) holds, which are simply related by the factor $\exp(2\pi x/\hbar)$.

It is important to note that even when $\tilde{\psi}_{\pm,N}^{\text{even}}$ are entire and square-integrable solutions to the difference equation (12.1), neither of them is in the domain $\mathcal{D}$ of the original quantized curve as defined around (11.3).¹⁴

Inverted potential for N odd. For $\tilde{\psi}_{+,N}^{\text{odd}}$ the asymptotics is simply

$$\tilde{\psi}_{+,N}^{\text{odd}}(x, \mathbf{h}) \simeq e^{\frac{2\pi x}{\hbar}} \left(\frac{2\pi\hbar}{\text{Re}(x)} \right)^{\frac{N}{2}} \left(\frac{\text{Re}(x)}{\Lambda} \right)^{N \frac{\text{Im}(x)}{\hbar}} \times u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty, \quad (12.7)$$

$$\tilde{\psi}_{+,N}^{\text{odd}}(x, \mathbf{h}) \simeq e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|}{\Lambda} \right)^{-N \frac{\text{Im}(x)}{\hbar}} \times u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar), \quad \text{Re}(x) \rightarrow -\infty, \quad (12.8)$$

following from (11.32) and (11.33). To find square-integrable solutions we need once more to impose the quantization condition (11.35) so that the asymptotics becomes

$$\begin{aligned} \tilde{\psi}_{+,N}^{\text{odd}}(x, \mathbf{h}) \Big|_{(11.35)} &\simeq \left(\frac{2\pi\hbar}{\text{Re}(x)} \right)^{\frac{N}{2}} \left\{ \left(\frac{\text{Re}(x)}{\Lambda} \right)^{N \frac{\text{Im}(x)}{\hbar}} u_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^1(\mathbf{a}, \Lambda, \hbar) \right. \\ &\quad \left. + \left(\frac{\text{Re}(x)}{\Lambda} \right)^{-N \frac{\text{Im}(x)}{\hbar}} u_N^{-1}(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^{\frac{N-1}{2}}(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) \right\} \quad \text{Re}(x) \rightarrow +\infty. \end{aligned} \quad (12.9)$$

Thus $\tilde{\psi}_{+,N}^{\text{odd}}$ is square-integrable parallel to the real line if and only if the quantization condition (11.35) is satisfied and (12.5) holds. Note that this gives square-integrable solutions of the difference equation for a discrete set of complex eigenvalues. These eigenvalues are simply minus the eigenvalues of the non-inverted potential. On the other hand, it is evident that $\tilde{\psi}_{-,N}^{\text{odd}}$ is square-integrable parallel to the real line if and only if the alternative quantization condition (11.38) is satisfied and $\text{Im}(x) < (\hbar/2)(1 - 1/N)$.

It is important to note that even when $\tilde{\psi}_{\pm,N}^{\text{odd}}$ are entire and square-integrable solutions to the difference equation (12.1), neither of them is generically in the domain $\mathcal{D}$ in (11.3), or even in the domain of any symmetric or self-adjoint realization of the difference operator.

12.2 Kinetic term with sinh(p)

Similarly, one can construct solutions of the difference equation obtained by replacing the kinetic term $\cosh(\mathbf{p})$ with $\sinh(\mathbf{p})$,

$$i\Lambda^N(\omega_{\pm}(x + i\hbar, \mathbf{h}) - \omega_{\pm}(x - i\hbar, \mathbf{h})) \pm V_N(x)\omega_{\pm}(x, \mathbf{h}) = \mathcal{E}_{\pm}\omega_{\pm}(x, \mathbf{h}), \quad (12.10)$$

$$\omega_{\pm}(x) := \exp\left(\mp \frac{\pi x}{2\hbar}\right) \psi(x), \quad \mathcal{E}_{\pm} := \pm E, \quad (12.11)$$

¹⁴ That is for generic choices of the coefficients h_k . For special values of h_k the decay of the solutions may be faster. This is what happens at the Toda points h_k^T , for example.

with the corresponding operator having again the domain $\mathcal{D}$ in (11.3). In this case, the subscript $\pm$ labels solutions of two different equations, with potential $\pm V_N(x)$, respectively. If one requires the operators to be symmetric, the appropriate condition in this case is $i\Lambda^N > 0$, rather than $\Lambda^N > 0$. As before, the analytic properties of $\omega_{\pm}$ are inherited directly from those of ψ . However, the spectral problems related to (11.5) and (12.10) are different.

Kinetic term with $\sinh(\mathbf{p})$ and confining potential for N even. Let us look at the equation (12.10) for the confining potential $+V_N$ and N even. The asymptotic behaviour for $\operatorname{Re}(x) \rightarrow \pm\infty$, with $\operatorname{Im}(x)$ held fixed, is now

$$\omega_{+,N}^{\text{even}}(x, \hbar) \simeq e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\operatorname{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\operatorname{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\operatorname{Im}(x)}{\hbar}} \times v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar), \quad \operatorname{Re}(x) \rightarrow +\infty, \quad (12.12)$$

$$\omega_{+,N}^{\text{even}}(x, \hbar) \simeq \left(\frac{2\pi\hbar}{|\operatorname{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\operatorname{Re}(x)|^N}{i\Lambda^N} \right)^{-\frac{\operatorname{Im}(x)}{\hbar}} \times i v_N(x) \sum_{n \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{Q}_n^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar), \quad \operatorname{Re}(x) \rightarrow -\infty, \quad (12.13)$$

where v_N is an x -dependent factor of unit modulus, similar to u_N in equation (11.28),

$$v_N(x) := e^{i\frac{\pi}{4}N} \left(\frac{e^N i\Lambda^N}{|\operatorname{Re}(x)|^N} \right)^{i\frac{|\operatorname{Re}(x)|}{\hbar}}. \quad (12.14)$$

Looking for square-integrable solutions leads once again to the quantization condition (11.29), and the asymptotics becomes

$$\omega_{+,N}^{\text{even}}(x, \hbar) \Big|_{(11.29)} \simeq e^{-\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\operatorname{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\operatorname{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\operatorname{Im}(x)}{\hbar}} \times v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^1(\mathbf{a}, \Lambda, \hbar), \quad \operatorname{Re}(x) \rightarrow +\infty. \quad (12.15)$$

Hence, for a discrete set of eigenvalues satisfying¹⁵ (11.29) we have solutions to the difference equation (12.10) which are square-integrable parallel to the real line if

$$\operatorname{Im}(x) > -\frac{\hbar}{2} \left(1 - \frac{1}{N} \right). \quad (12.16)$$

It is important to note that even when $\omega_{+,N}^{\text{even}}$ is an entire and square-integrable solution to the difference equation (12.10), it is generically¹⁶ not in the original domain $\mathcal{D}$ of the quantized curve as defined around (11.3).

¹⁵However, as said, we now have $i\Lambda^N > 0$ rather than $\Lambda^N > 0$, meaning that we will get different $E \in \mathbb{C}$ satisfying (11.29).

¹⁶That is for generic choices of the parameters h_k , see footnote 14.

Kinetic term with $\sinh(\mathbf{p})$ and inverted potential for N even. Let us look at the equation (12.10) for the inverted potential $-V_N$ and N even. The asymptotics of $\omega_{-,N}^{\text{even}}$ is simply

$$\omega_{-,N}^{\text{even}}(x, \hbar) \simeq e^{\frac{2\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\text{Im}(x)}{\hbar}} \times v_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty, \quad (12.17)$$

$$\omega_{-,N}^{\text{even}}(x, \hbar) \simeq e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{-\frac{\text{Im}(x)}{\hbar}} \times i v_N(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar), \quad \text{Re}(x) \rightarrow -\infty, \quad (12.18)$$

following (12.12) and (12.13). This leads once again to the quantization condition (11.29), and for a discrete set of eigenvalues satisfying (11.29) we have solutions to the difference equation (12.10) which are square-integrable parallel to the real line if

$$\text{Im}(x) < \frac{\hbar}{2} \left(1 - \frac{1}{N} \right). \quad (12.19)$$

However, in this case, we also find a second quantization condition. One should note that $\exp(-2\pi x/\hbar) \omega_{-,N}^{\text{even}}(x, \hbar)$ is again a solution with now a power law decay near positive real infinity and an exponential growth near negative real infinity, which can be read off from (12.12) and (12.13). To get square-integrable solutions we need

$$\sum_{\mathbf{n} \in \mathcal{W}_N \cdot (\gamma + \mathbf{e}_N)} \mathcal{Q}_{\mathbf{n}}^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar) = 0. \quad (12.20)$$

If this quantization condition is satisfied, the asymptotics near negative infinity is

$$\exp(-2\pi x/\hbar) \omega_{-,N}^{\text{even}}(x, \hbar) \simeq \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\text{Im}(x)}{\hbar}} v_N^{-1}(x) \sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_{\mathbf{n}}^{\frac{N}{2}}(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow -\infty, \quad (12.21)$$

so once more we have a discrete set of solutions to the difference equation (12.10) which are square-integrable parallel to the real line if

$$\text{Im}(x) < \frac{\hbar}{2} \left(1 - \frac{1}{N} \right). \quad (12.22)$$

Note the power law decay in both directions.

It is important to stress that even when $\omega_{-,N}^{\text{even}}$ is an entire and square-integrable solution to the difference equation (12.10), it is generically not in the original domain $\mathcal{D}$ of the quantized curve as defined around (11.3).

Kinetic term with $\sinh(\mathbf{p})$ and unbounded potential for N odd. Let us look at the equation (12.10) for the unbounded potential $+V_N$ and N odd. The asymptotic for $\text{Re}(x) \rightarrow \pm\infty$ for fixed $\text{Im}(x)$ is now

$$\omega_{+,N}^{\text{odd}}(x, \hbar) \simeq e^{\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\text{Im}(x)}{\hbar}} \times v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty, \quad (12.23)$$

$$\omega_{+,N}^{\text{odd}}(x, \hbar) \simeq e^{-\frac{\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{-\frac{\text{Im}(x)}{\hbar}} \times v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar), \quad \text{Re}(x) \rightarrow -\infty. \quad (12.24)$$

Hence, there is no single quantization condition that gives rise to square-integrable solutions, and the two quantization conditions (11.35) and (11.38) must be imposed together. However, they do not admit simultaneous solutions for arbitrary values of the coefficients h_k , but only for a special subset thereof. It is noteworthy that the Toda points satisfying (13.4) are a part of this subset. This is not surprising, since for odd N the kinetic term in the Baxter equation (13.1) is given by $\sinh(\mathbf{p})$.

Kinetic term with $\sinh(\mathbf{p})$ and inverted unbounded potential for N odd. Let us look at the equation (12.10) for the inverted unbounded potential $-V_N$ and N odd. The asymptotics for $\text{Re}(x) \rightarrow \pm\infty$ with fixed $\text{Im}(x)$ is now

$$\omega_{-,N}^{\text{odd}}(x, \hbar) \simeq e^{\frac{2\pi x}{\hbar}} \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\text{Im}(x)}{\hbar}} v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty, \quad (12.25)$$

$$\omega_{-,N}^{\text{odd}}(x, \hbar) \simeq \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{-\frac{\text{Im}(x)}{\hbar}} v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^0(\mathbf{f}(\mathbf{a}), \Lambda, \hbar), \quad \text{Re}(x) \rightarrow -\infty. \quad (12.26)$$

The quantization condition giving rise to square integrable solutions is once again (11.35), but where we should now take $i\Lambda^N > 0$ rather than $\Lambda^N > 0$. If (11.35) is satisfied, one finds

$$\omega_{-,N}^{\text{odd}}(x, \hbar) \Big|_{(11.35)} \simeq \left(\frac{2\pi\hbar}{|\text{Re}(x)|} \right)^{\frac{N}{2}} \left(\frac{|\text{Re}(x)|^N}{i\Lambda^N} \right)^{\frac{\text{Im}(x)}{\hbar}} v_N(x) \sum_{n \in \mathcal{W}_N \cdot \gamma} \mathcal{Q}_n^1(\mathbf{a}, \Lambda, \hbar), \quad \text{Re}(x) \rightarrow +\infty. \quad (12.27)$$

So we have solutions to the difference equation (12.10) which are square-integrable parallel to the real line if and only if (11.35) holds and $|\text{Im}(x)| < (\hbar/2)(1 - 1/N)$. We have power law-decay both at positive and negative infinity in that case.

However, in this case, we can find a second quantization condition. One can see that $\exp(-2\pi x/\hbar)\omega_{-,N}^{\text{odd}}(x, \mathbf{h})$ is again a solution to the difference equation (12.10) with now exponential growth at negative infinity and power-law decay at positive infinity. Hence, to get square-integrability, we should satisfy the alternative quantization condition (11.38) and $|\text{Im}(x)| < (\hbar/2)(1 - 1/N)$.

So we have solutions to the difference equation (12.10) which are square-integrable parallel to the real line if and only if (11.35) or (11.38) holds. It is important to note that even when $\omega_{-,N}^{\text{odd}}(x, \mathbf{h})$ or $\exp(-2\pi x/\hbar)\omega_{-,N}^{\text{odd}}(x, \mathbf{h})$ are entire and square-integrable solutions to the difference equation (12.10), they are generically not in the original domain $\mathcal{D}$ of the quantized curve as defined around (11.3).

13 Comments on the Toda points

In subsection 11.1 we have discussed the spectral problem (11.5) from the standpoint of standard quantum mechanics, wherein one first constructs entire off-shell solutions and then imposes square-integrability as a boundary condition. However, there is another way to view equation (11.5), namely from the perspective of integrability. Indeed, the difference equation (11.5) is closely related to the Baxter equation for the quantum $\text{SU}(N)$ Toda lattice [166–169], namely

$$(i\Lambda)^N Q(x + i\hbar) + (-i\Lambda)^N Q(x - i\hbar) - V_N(x)Q(x) = -EQ(x). \quad (13.1)$$

In this setting, one looks for entire solutions to (13.1) such that:

$$Q(x) \in L^2(\mathbb{R}) \quad \text{with} \quad Q(x) \simeq e^{-\frac{N\pi}{2\hbar}|x|}, \quad x \rightarrow \pm\infty. \quad (13.2)$$

By defining

$$Q(x) = \left(\sum_{n_i \in S} d_i e^{-\frac{2\pi x n_i}{\hbar}} \right) e^{-\frac{\pi(N+2)}{2\hbar}x} \psi(x), \quad S \subset \mathbb{Z}, \quad d_i \in \mathbb{C}, \quad (13.3)$$

it is easy to show that $Q(x)$ solves (13.1) for any choice of $S \subset \mathbb{Z}$. However, the mapping between the Baxter solution (13.2) and our solutions (11.25), (11.31) requires a particular choice of S . This question will be addressed in future work.

Nevertheless, even after taking (13.3) into account, one can see that for generic values of the potential parameters h_k ($k = 2, \dots, N-1$), the on-shell eigenfunctions (11.25) and (11.31) do not reproduce (13.2). For odd N , this follows directly from the fact that the asymptotics of (11.31) are outgoing with power-law decay on one side and exponentially decaying on the other, with a rate which is independent of N . Likewise, for N even, this follows from the fact that the decay behaviour of (11.25) as $x \rightarrow \pm\infty$ is the same for both plus and minus infinity, and it is independent of N . This is in line with the works of [166–169], where they found that eigenfunctions satisfying (13.2) exist only for special values of the moduli h_k , $k = 2, \dots, N$. We refer to these distinguished values as *Toda points* and denote them by h_k^T , $k = 2, \dots, N$. This, in turn, means that for (13.2) to hold, one must impose simultaneously $N-1$ quantization conditions, thereby quantizing all the h_k , $k = 2, \dots, N$. This was nicely illustrated in [169]. From the gauge theory perspective, it was found in [29, 170, 179] that the quantization conditions determining the Toda points are (following the notation of [44]):

$$\frac{i}{\hbar} \frac{\partial F_{\text{NS}}}{\partial \mathbf{a}} = i2\pi\ell + i\pi\boldsymbol{\rho}, \quad (13.4)$$

where ℓ encodes the quantum numbers,

$$\ell = \sum_{k=1}^{N-1} \ell_k \lambda_k, \quad \ell_k \in \mathbb{N}, \quad \rho = \sum_{I=1}^N (N-I) e_I, \quad (13.5)$$

and ρ is a constant vector. It is important to note that (13.4) is a vector equation: it consists of $N-1$ quantization conditions for the a_I , the corresponding solutions being denoted by $\mathbf{a}(\ell)$. After using the generalized Matone relations (11.10) and (A.82), these translate into $N-1$ quantization conditions for the complex moduli h_k , $k=2, \dots, N$. In other words, it constrains the h_k to lie at the Toda points, which we denote by $h_k^T(\ell)$. This stands in sharp contrast with standard quantum mechanics, where square-integrability imposes only a single quantization condition, namely (11.29) for N even and (11.35) for N odd. This condition quantises the energy E , while the other parameters h_k , $k=2, \dots, N-1$, in the potential remain free. However, as shown in [44], if the h_k satisfy the Toda quantization conditions (13.4), they automatically satisfy the quantum mechanical quantization conditions (11.29), (11.35). Hence, the Toda points correspond to special loci in our quantum mechanical problem, where eigenfunctions have an enhanced decay. As discussed above, this enhancement also comes together with some interesting phenomena, such as degeneracies among bound states. It is then natural to ask how these phenomena are captured by our explicit eigenfunctions (11.25) and (11.31).

Let us first review what is known about the Toda eigenfunction (13.2) from a gauge-theoretic perspective. It was found in [170, 180] that, in gauge-theoretic language, the eigenfunctions of the Toda Baxter equation (13.1) take the form:

$$Q(x, \mathbf{h}^T(\ell)) = e^{-\frac{\pi(N+2)}{2\hbar}x} Z_D(x, \mathbf{a}(\ell), \Lambda, \hbar) + \xi e^{\frac{\pi(N+2)}{2\hbar}x} Z_D(-x, -\mathbf{a}(\ell), \Lambda, \hbar), \quad (13.6)$$

where $\mathbf{a}(\ell)$ are solutions to the $N-1$ quantization conditions (13.4). The factor ξ in (13.6) is an x -independent function that should ensure the cancellation of poles between the two terms in (13.6). It was shown in [170] that the existence of such x -independent function is guaranteed only if $\mathbf{a} = \mathbf{a}(\ell)$, where $\mathbf{a}(\ell)$ are solutions to (13.4). For example, in the $SU(2)$ case there is only a single quantum number ℓ , and one finds:

$$\xi = (-1)^\ell. \quad (13.7)$$

Hence, we can see another important difference between (13.6) and (11.25), (11.31):

- Our eigenfunctions (11.25) and (11.31) are entire in x for generic values of $\mathbf{a}$, i.e. generic values of h_k , $k=2, \dots, N$. In this case, no quantization condition is required to guarantee pole cancellation. This is made possible by the x -dependent factor appearing inside the sums over the Weyl orbit in (11.25) and (11.31).
- The eigenfunctions constructed in [170, 180] are pole-free only if $\mathbf{a} \equiv \mathbf{a}(\ell)$, where the $\mathbf{a}(\ell)$ are fixed by the $N-1$ quantization conditions (13.4). This is because ξ in (13.6) is assumed to be x -independent.

Given these considerations, let us examine what happens to the eigenfunctions (11.25) and (11.31) once we restrict to the Toda points, that is, upon imposing the $N-1$ quantization conditions (13.4). Let us first consider the case of even $N > 2$.¹⁷ We find that the following special combinations vanish identically¹⁸:

$$\sum_{\mathbf{n} \in \mathcal{W}_N \cdot \gamma} \frac{\exp(i(2\pi\ell + \pi\rho) \cdot \mathbf{n})}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi\alpha \cdot \alpha}{\hbar}))^{(\mathbf{n} \cdot \alpha)^2}} \prod_{I=1}^N (e^{\frac{2\pi x}{\hbar}} - e^{\frac{2\pi \mathbf{a} \cdot \mathbf{e}_I}{\hbar}})^{\frac{1}{2} - n_I} = 0, \quad (13.8)$$

¹⁷For $N=2$ these sums do not vanish, see [28].

¹⁸These identities were rigorously proved by Karim Aziz as part of an unpublished bachelor's project [183].

$$\sum_{n \in \mathcal{W}_N \cdot (\gamma + e_N)} \frac{\exp(i(2\pi\ell + \pi\rho) \cdot n)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi\alpha \cdot \alpha}{\hbar}))^{(n \cdot \alpha)^2}} \prod_{I=1}^N (e^{\frac{2\pi x}{\hbar}} - e^{\frac{2\pi\alpha \cdot e_I}{\hbar}})^{\frac{1}{2} - n_I} = 0. \quad (13.9)$$

Likewise, for N odd we have:

$$\sum_{n \in \mathcal{W}_N \cdot \gamma} \frac{\exp(i(2\pi\ell + \pi\rho + i\frac{\pi}{\hbar}\mathbf{a}) \cdot n)}{\prod_{\alpha \in \Delta_+} (2 \sinh(\frac{\pi\alpha \cdot \alpha}{\hbar}))^{(n \cdot \alpha)^2}} \prod_{I=1}^N (e^{\frac{2\pi x}{\hbar}} - e^{\frac{2\pi\alpha \cdot e_I}{\hbar}})^{\frac{1}{2} - n_I} = 0. \quad (13.10)$$

This means that, strictly speaking, the eigenfunctions (11.25), (11.31) vanish at the Toda points. However, since the two terms in the sums (11.25) and (11.31) go to zero at the same rate, an appropriate normalization can be introduced to obtain a non-vanishing result. A more detailed analysis will be the subject of future work.

14 Derivation from the TS/ST correspondence

In this section, we derive (11.25) and (11.31) from the TS/ST correspondence [15–17, 26, 28].

14.1 Open TS/ST for the $Y^{N,0}$ geometries

We consider the toric Calabi–Yau threefolds obtained as crepant resolutions of the $Y^{N,0}$ singularities, which geometrically engineer $\mathcal{N} = 1$ $SU(N)$ SYM in five dimensions or $\mathcal{N} = 2$ $SU(N)$ SYM in four dimensions [19, 184]. The associated mirror curve [19] has the form:

$$\Lambda^N (e^y + e^{-y}) + \sum_{\ell=0}^N \kappa_\ell e^{(\frac{N}{2} - \ell)x} = 0 \quad (14.1)$$

where Λ and κ_ℓ are the complex moduli, and we can set without loss of generality $\kappa_0 = 1 = \kappa_N$. It is also useful to introduce the chemical potentials μ_ℓ by

$$\kappa_\ell = e^{\mu_\ell}, \quad \ell \in \{1, \dots, N-1\}, \quad (14.2)$$

and the Batyrev coordinates as

$$z_i = \frac{\kappa_{i-1}\kappa_{i+1}}{\kappa_i^2}, \quad i \in \{1, \dots, N-1\}, \quad z_N = \frac{\Lambda^{2N}}{\kappa_0\kappa_N}. \quad (14.3)$$

The complex moduli κ_ℓ are related to the Kähler parameters t_k through the mirror maps [185]. Our conventions for the classical mirror maps are as follows:

$$t_k = \sum_{\ell=1}^{N-1} C_{k\ell} \mu_\ell + \mathcal{O}(z_i), \quad k \in \{1, \dots, N-1\}, \quad t_N = -\log(z_N), \quad (14.4)$$

where $C_{k\ell}$ is the Cartan matrix of $SU(N)$.

Within the TS/ST correspondence, we consider the quantization of the mirror curve (14.1). Upon quantization, this curve is promoted to a difference equation [20, 29, 115, 157]:

$$\Lambda^N (\varphi(x + i\hbar, \boldsymbol{\kappa}) + \varphi(x - i\hbar, \boldsymbol{\kappa})) + \sum_{\ell=0}^N \kappa_\ell e^{(\frac{N}{2} - \ell)x} \varphi(x, \boldsymbol{\kappa}) = 0, \quad (14.5)$$

and, simultaneously, the classical Kähler parameters t_k are replaced by their quantum counterparts, giving rise to the quantum mirror maps [115, 157], which we denote by

$$t_k(\hbar), \quad k \in \{1, \dots, N-1\}. \quad (14.6)$$

We refer to [103, 186] for explicit expressions of the quantum mirror maps for the $Y^{N,0}$ geometries. In the limit $\hbar \rightarrow 0$, (14.6) reduces to the classical mirror map (14.4). For the geometries under consideration, the quantum mirror maps can also be interpreted as the inverses of Wilson loop VEVs of five-dimensional $\mathcal{N} = 1$ SYM in the NS limit [95, 122]. Explicit expressions for the relevant Wilson loop VEVs in the case of $Y^{N,0}$ geometries can be found in [186, app. D]. In this context, it is often convenient to introduce the 5d Coulomb branch parameters¹⁹:

$$a_I, \quad I \in \{1, \dots, N\}, \quad \sum_{I=1}^N a_I = 0, \quad (14.7)$$

which parametrize the VEVs of the complexified scalars in the vector multiplet of 5d SYM, compactified on $\mathbb{R}^4 \times \mathbb{S}^1$. These are related to the quantum mirror maps by

$$t_k(\hbar) = a_k - a_{k+1}, \quad k \in \{1, \dots, N-1\}, \quad (14.8)$$

or equivalently, inverting the relations, by

$$a_I = \sum_{\ell=1}^{I-1} \left(-\frac{\ell}{N}\right) t_\ell(\hbar) + \sum_{\ell=I}^{N-1} \left(\frac{N-\ell}{N}\right) t_\ell(\hbar). \quad (14.9)$$

Given the underlying $SU(N)$ structure, it is convenient to organize these parameters as

$$\mathbf{t} = \sum_{I=1}^N a_I \mathbf{e}_I = \sum_{k=1}^{N-1} t_k(\hbar) \boldsymbol{\lambda}_k, \quad (14.10)$$

where $\mathbf{e}_I$ are the weights of the fundamental representation and $\boldsymbol{\lambda}_k$ denote the fundamental weights, see Section A.4 for details on the conventions.

The TS/ST correspondence for the $Y^{N,0}$ Calabi–Yau geometries has been studied in the closed string sector in [44, 103, 186, 187]. In this Chapter, we also consider the open sector, extending the results of [17, 26, 28] to this class of geometries. A central object in the correspondence is the full grand potential, which we denote by

$$\mathbf{J}(x, \mathbf{t}, \Lambda, \hbar) = \mathbf{J}^{\text{closed}}(\mathbf{t}, \Lambda, \hbar) + \mathbf{J}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar). \quad (14.11)$$

The closed topological string grand potential $\mathbf{J}^{\text{closed}}(\mathbf{t}, \Lambda, \hbar)$ is constructed from a specific combination of the refined topological string partition function in the Gopakumar–Vafa (GV) limit and in the NS limit [15, 16, 188]. For the $Y^{N,0}$ geometries, the explicit expression is given in [44, p. 19]. The open topological string grand potential $\mathbf{J}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar)$ has an analogous structure, but it involves instead the open string partition functions [17, 26, 28]. More concretely, for the $Y^{N,0}$ geometries we have:

$$\mathbf{J}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar) = \mathbf{J}_p^{\text{open}}(x, \Lambda, \hbar) + \mathbf{J}_{1\text{-loop}}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar) + \mathbf{J}_{\text{inst}}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar). \quad (14.12)$$

Here, the polynomial part is given by

$$\mathbf{J}_p^{\text{open}}(x, \Lambda, \hbar) = i \log(\Lambda^N) \frac{x}{\hbar} + \pi \frac{x}{\hbar} - \frac{N}{4} x \left(1 + i \frac{x}{\hbar}\right). \quad (14.13)$$

This term is obtained by analysing the large- x behaviour of (14.5) and demanding that $\exp \mathbf{J}_p^{\text{open}}$ provides a formal solution in the large- x asymptotic regime. This procedure

¹⁹We are using the same notation for the 5d and 4d Coulomb branch parameters. The distinction should be clear from the context.

fixes the form of (14.13) up to an $i\hbar$ periodic factor $\sim \exp\left(\frac{2\pi}{\hbar}x\right)$, which is then fixed with hindsight by requiring the cancellation of poles between the two terms in (14.15). The one-loop contribution takes the form [28, 180]

$$\exp\left(J_{1\text{-loop}}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar)\right) = \prod_{I=1}^N \Phi_b\left(\frac{-x + \mathbf{e}_I \cdot \mathbf{t} + i\pi b^2}{2\pi b}\right), \quad \hbar = 2\pi b^2, \quad (14.14)$$

where Φ_b is Faddeev's non-compact quantum dilogarithm, see subsection A.1.2. Finally, the instanton contribution $J_{\text{inst}}^{\text{open}}(x, \mathbf{t}, \Lambda, \hbar)$ is built from a specific combination of the refined open topological string partition function on the $Y^{N,0}$ geometry, taken in the GV and NS limits, see [17, sec. 4.1], [26, sec. 2.2] and [28, sec. 3.3].

From the topological string perspective, the open string grand potential (14.12) provides a non-perturbative completion of the open topological string free energy in the large-radius frame. This quantity, however, is not background independent and moreover develops poles at finite values of x , which is identified with the open string modulus. To remedy this, it was shown in [17, 26, 28] that one must consider a particular linear combination of the form:

$$\varphi(x, \boldsymbol{\kappa}) = \sum_{s \in \{-1, +1\}} \sum_{\mathbf{n} \in Q_{N-1}} \exp[J_s(x, \mathbf{t} + i2\pi \mathbf{n}, \Lambda, \hbar)], \quad (14.15)$$

where Q_{N-1} is the root lattice defined in (A.63) and $J_+(x, \mathbf{t}, \Lambda, \hbar) = J(x, \mathbf{t}, \Lambda, \hbar)$, while J_- is obtained from J_+ by a suitable transformation, as we will see later. Note that in (14.15) we are implicitly using the quantum mirror maps (14.6) relating $\boldsymbol{\kappa}$ to the quantum Kähler parameters $\mathbf{t}$. In the topological string theory, both summations in (14.15) are essential to achieve background independence in the open and closed string moduli. Indeed, the sum over $\mathbf{n}$ renders the expression entire in the closed string moduli $\boldsymbol{\kappa}$, while the sum over s ensures entireness in the open modulus x . From the perspective of spectral theory, equation (14.15) is particularly significant, as it provides an exact entire solution to the quantum mirror curve associated with the underlying Calabi-Yau geometry. Although formal solutions to the quantum mirror curve can be constructed in various ways, (14.15) stands out due to its analyticity in both the open modulus x and the closed moduli $\boldsymbol{\kappa}$. Moreover, when evaluated on-shell, it yields genuine square-integrable eigenfunctions, see [17, 26, 28] for more details.

An important open question in the context of the open TS/ST correspondence is the precise characterization of the second term in (14.15). In [17, 26] it was suggested that intuitively this has to do with moving to a different sheet of the mirror curve. In practice, however, it is challenging to implement this explicitly, that is, to find a simple explicit relation between J_+ and J_- . For local $\mathbb{F}_0$, this problem was solved in [28], where, using insights from Painlevé equations [27], it was shown that the two terms are simply related by shifts of the complex moduli and of the variable x . At the level of the mirror curve, these shifts are characterized by the fact that they leave the curve itself invariant. Let us now apply the same idea to (14.1). We consider transformations of the form

$$x \rightarrow s x - i2\pi \frac{k_x}{N}, \quad y \rightarrow \frac{1}{s} y - i\pi k_y, \quad k_x, k_y \in \mathbb{Z}, \quad k_x + k_y \in 2\mathbb{Z}, \quad (14.16)$$

combined with a corresponding transformation of the moduli μ_ℓ , which we will introduce below. Drawing inspiration from the example of local $\mathbb{F}_0$ [28] and the Toda lattice [170, 180], it is natural to choose $s = -1$. The classical curve is then invariant if and only if the closed moduli transform as

$$\mu_\ell \rightarrow \mu_{N-\ell} - i2\pi \frac{k_x \ell}{N}. \quad (14.17)$$

Following again the analogy with local $\mathbb{F}_0$ [28], we set $k_x = 1 = k_y$. This gives

$$\varphi(x, \boldsymbol{\kappa}) = \varphi_+(x, \boldsymbol{\kappa}) + \exp\left(\frac{i}{\hbar} \frac{\pi^2}{N} + \pi \frac{x}{\hbar}\right) \varphi_+\left(-x - i \frac{2\pi}{N}, \tilde{\boldsymbol{\kappa}}\right), \quad (14.18)$$

$$\varphi_+(x, \boldsymbol{\kappa}) = \sum_{\mathbf{n} \in Q_{N-1}} e^{J(x, \mathbf{t} + i2\pi \mathbf{n}, \Lambda, \hbar)}, \quad (14.19)$$

where $\boldsymbol{\kappa} = \{\kappa_1, \dots, \kappa_{N-1}\}$, $\tilde{\boldsymbol{\kappa}} = \{\tilde{\kappa}_1, \dots, \tilde{\kappa}_{N-1}\}$ with $\tilde{\kappa}_\ell = \kappa_{N-\ell} e^{-i \frac{2\pi}{N} \ell}$. Let us also note that, after applying the quantum mirror map, the transformation (14.17) is equivalent to

$$\mathbf{t} \rightarrow \mathbf{f}(\mathbf{t}) + i2\pi k_x \mathbf{e}_N, \quad \mathbf{f}(\mathbf{t}) = \sum_{k=1}^{N-1} t_{N-k} \boldsymbol{\lambda}_k = - \sum_{I=1}^N a_{N-I+1} \mathbf{e}_I. \quad (14.20)$$

We have carried out preliminary tests of (14.18) for $N = 4$. The next section provides further evidence in favour of (14.18) for generic N and is the origin of our proposal in subsection 11.3.

14.2 The four-dimensional limit

Topological string theory on the $Y^{N,0}$ geometries provides a framework to engineer four-dimensional $\mathcal{N} = 2$ $SU(N)$ SYM [19, 184]. To realize this setup, one performs a specific geometric limit, usually called the four-dimensional limit. In this section, we show that, after carrying out this limit, the eigenfunctions (14.18) reduce to (11.25) and (11.31), respectively. The computation of the four-dimensional limit of the eigenfunctions mirrors that for the spectral determinant in [44, sec. 5], which we follow closely. To take the limit on the mirror curve, we need a different parametrization from the one used in (14.1). The reparametrization is implemented by some simple shifts, see also the discussion in [44, sec. 5]. Combining these shifts with the natural scaling one finds that the four-dimensional limit is

$$x \rightarrow Rx - i \frac{\pi}{N} \mathbb{1}_{2N+1}(N), \quad \mathbf{t} \rightarrow R\mathbf{a} + i2\pi\boldsymbol{\gamma}, \quad \Lambda^{2N} \rightarrow (-1)^N (R\Lambda)^{2N}, \quad \hbar \rightarrow R\hbar, \quad (14.21)$$

and then $R \rightarrow 0$ from above, with $\boldsymbol{\gamma}$ defined in (11.16). One can check that, by taking this limit, the quantum mirror curve (14.5) reduces to the quantum Seiberg-Witten curve of four-dimensional $\mathcal{N} = 2$ $SU(N)$ SYM (11.5).

For simplicity, we set the four-dimensional reduced Planck constant $\hbar$ to 1. We can reintroduce it by sending $(x, \mathbf{a}, \Lambda) \rightarrow (x/\hbar, \mathbf{a}/\hbar, \Lambda/\hbar)$ at the end. The four-dimensional limit acts similarly on both terms in the right-hand side of (14.18). Hence, it is convenient to introduce the variables $s \in \{-1, +1\}$ and $k_x = k_y = k \in \{0, 1\}$, and look at the $R \rightarrow 0$ limit of

$$\begin{aligned} \sum_{\mathbf{n} \in Q_{N-1}} \exp \left[-J^{\text{closed}} \left(R\mathbf{a} + i2\pi\boldsymbol{\gamma}, e^{-i \frac{\pi}{2} \frac{\mathbb{1}_{2N+1}(N)}{N}} R\Lambda, R \right) + i \frac{\pi^2}{N} \frac{k^2}{R} - sk\pi \left(x - i \frac{\pi}{N} \frac{\mathbb{1}_{2N+1}(N)}{R} \right) \right. \\ \left. + J \left(sRx - i \frac{\pi}{N} (2k + s \mathbb{1}_{2N+1}(N)), R\mathbf{f}_s(\mathbf{a}) + i2\pi\mathbf{w}, e^{-i \frac{\pi}{2} \frac{\mathbb{1}_{2N+1}(N)}{N}} R\Lambda, R \right) \right], \quad (14.22) \end{aligned}$$

where we introduced the convenient notation

$$\mathbf{f}_s(\mathbf{t}) = \begin{cases} \mathbf{f}(\mathbf{t}) & s = -1 \\ \mathbf{t} & s = +1 \end{cases}, \quad \mathbf{w} = \boldsymbol{\gamma} + \left(\frac{1-s}{2} \right) \mathbb{1}_{2N}(N) \mathbf{e}_N + \mathbf{n}. \quad (14.23)$$

The first term on the right-hand side of (14.18) is then given by $s = 1$ and $k = 0$, while the second term corresponds to $s = -1$ and $k = 1$. Note that we normalized the eigenfunctions by the x -independent constant $\exp(\mathbf{J}^{\text{closed}})$, which is convenient in the four-dimensional limit [44]. In (14.23), we have already used the fact that

$$\mathbf{f}(\gamma) + \mathbf{e}_N + \mathbf{m} = (-1)^N \gamma + \mathbf{e}_N + \mathbf{m} = \gamma + \mathbb{1}_{2\mathbb{N}}(N) \mathbf{e}_N + \mathbf{n}, \quad (14.24)$$

where $\mathbf{n}, \mathbf{m} \in Q_{N-1}$ are related by a simple shift of the origin of the root lattice Q_{N-1} when N is odd, and are instead equal when N is even.

14.2.1 The leading terms and the closed part

The key observation in the four-dimensional limit is that only finitely many of the $\mathbf{n} \in Q_{N-1}$ appearing in the sum (14.22) contribute. As discussed below, which $\mathbf{n}$ dominate is determined by the polynomial part of the closed grand potential.

Let us first consider the shifts of $\mathbf{f}_s(\mathbf{t})$ by $i2\pi\mathbf{w}$ in (14.22). We introduce β for future convenience,

$$\beta = \gamma + \left(\frac{1-s}{2}\right) \mathbb{1}_{2\mathbb{N}}(N) \mathbf{e}_N, \quad \mathbf{w} = \beta + \mathbf{n}. \quad (14.25)$$

The polynomial part of the closed grand potential then scales as

$$\begin{aligned} \mathbf{J}_p^{\text{closed}} \left(R \mathbf{f}_s(\mathbf{a}) + i2\pi\mathbf{w}, e^{-i\frac{\pi}{2} \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{N}} R\Lambda, R \right) &= 2\pi N \mathbf{w}^2 \frac{\log(R)}{R} \\ &+ \left[\pi \mathbf{w}^2 \log(\Lambda^{2N}) + i\pi^2 \left(\frac{1}{3} \sum_{\alpha \in \Delta_+} [\alpha \cdot \mathbf{w} - 2(\alpha \cdot \mathbf{w})^3] + \mathbf{w}^2 \mathbb{1}_{2\mathbb{N}+1}(N) \right) \right] \frac{1}{R} \\ &- i2N(\mathbf{w} \cdot \mathbf{f}_s(\mathbf{a})) \log(R) + \mathcal{O}(R^0). \end{aligned} \quad (14.26)$$

This is the only part of the total grand potential in (14.22) with an $\mathbf{n}$ -dependent divergence of order $\log(R)/R$; all other contributions are subleading. Therefore, this part determines which terms in the sum (14.22) survive. The $R \rightarrow 0$ limit of (14.22) is then dominated by the minima of the positive definite form

$$\mathbf{w}^2 = (\mathbf{n} + \beta)^2. \quad (14.27)$$

These are precisely the elements of the Weyl orbit of β [44], that is we have minima for²⁰

$$\mathbf{w} \in \mathcal{W}_N \cdot \beta. \quad (14.28)$$

For the rest of the closed part, the limit works exactly as outlined in [44]. From the closed 1-loop part we have

$$\begin{aligned} \mathbf{J}_{1\text{-loop}}^{\text{closed}} \left(R \mathbf{f}_s(\mathbf{a}) + i2\pi\mathbf{w}, e^{-i\frac{\pi}{2} \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{N}} R\Lambda, R \right) &= \\ i\frac{\pi^2}{3} \left(\sum_{\alpha \in \Delta_+} \alpha \cdot \mathbf{w} \right) \frac{1}{R} &+ i2N(\mathbf{w} \cdot \mathbf{f}_s(\mathbf{a})) \log(R) + \mathcal{O}(R^0), \end{aligned} \quad (14.29)$$

²⁰Note that with $\mathbf{w} = \sum_I w_I \mathbf{e}_I$ and $\beta = \sum_I \beta_I \mathbf{e}_I$ we have the constraint $\sum_I w_I = \sum_I \beta_I$. Let us now take $\beta_I \in \{\pm 1/2\}$ and hence $w_I \in \mathbb{Z} + 1/2$ for convenience. One can see that $\mathbf{w}^2 = \sum_I w_I^2 - (\sum_I \beta_I)^2/N$, which gets minimized for $w_I \in \{\pm 1/2\}$, and since $\sum_I w_I = \sum_I \beta_I$ we find that the w_I 's should be a permutation of the β_I 's. This means exactly that $\mathbf{w} \in \mathcal{W}_N \cdot \beta$.

while the instanton part is regular. This means that for the second term in (14.18), i.e. for $s = -1$, we get an extra divergent factor which is the exponential of

$$\begin{aligned} & \left[J^{\text{closed}} \left(R\mathbf{f}(\mathbf{a}) + i2\pi(\gamma + \mathbb{1}_{2\mathbb{N}}(N)\mathbf{e}_N), e^{-i\frac{\pi}{2}\frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{N}} R\Lambda, R \right) - \right. \\ & \left. J^{\text{closed}} \left(R\mathbf{a} + i2\pi\gamma, e^{-i\frac{\pi}{2}\frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{N}} R\Lambda, R \right) \right] = -2\pi\mathbb{1}_{2\mathbb{N}}(N) \frac{\log(R)}{R} - \frac{\pi}{N}\mathbb{1}_{2\mathbb{N}}(N) \frac{\log(\Lambda^{2N})}{R} \\ & \quad + \mathcal{O}(R^0). \end{aligned} \quad (14.30)$$

The $\mathcal{O}(R^0)$ contribution coming from the closed sector can then be obtained directly from [44, p. 34], and the standard four-dimensional limit becomes

$$\begin{aligned} \lim_{R \rightarrow 0} (14.22) & \propto \sum_{\mathbf{w} \in \mathcal{W}_N \cdot \beta} \frac{\exp((i\partial_{\mathbf{a}} F_{\text{NS}} - \pi\mathbf{a}\mathbb{1}_{2\mathbb{N}+1}(N)) \cdot \mathbf{w})}{\prod_{\alpha \in \Delta_+} (2 \sinh(\pi\mathbf{a} \cdot \alpha))^{(\mathbf{w} \cdot \alpha)^2}} \cdot \lim_{R \rightarrow 0} e^{(\frac{1-s}{2})(14.30) + i\frac{\pi^2}{N} \frac{k^2}{R}} \\ & \cdot \lim_{R \rightarrow 0} e^{-sk\pi \left(x - i\frac{\pi}{N} \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{R} \right) + J^{\text{open}}(sRx - i\frac{\pi}{N}(2k + s\mathbb{1}_{2\mathbb{N}+1}(N)), R\mathbf{f}_s(\mathbf{a}) + i2\pi\mathbf{w}, e^{-i\frac{\pi}{2}\frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{N}} R\Lambda, R)}. \end{aligned} \quad (14.31)$$

It should be noted that the proportionality constant we are neglecting here is finite and independent of x , s and k . Hence, it is an overall constant of the eigenfunctions and the same for both saddles. One can compare with [44, p. 34].

14.2.2 The open part

One finds for the open polynomial part:

$$\begin{aligned} & i\frac{\pi^2}{N} \frac{k^2}{R} - sk\pi \left(x - i\frac{\pi}{N} \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{R} \right) + J^{\text{open}}_p \left(sRx - i\frac{\pi}{N}(2k + s\mathbb{1}_{2\mathbb{N}+1}(N)), e^{-i\frac{\pi}{2}\frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{N}} R\Lambda, R \right) \\ & = \begin{cases} 2\pi k \frac{\log(R)}{R} + 2\pi \frac{k}{N} \frac{\log(\Lambda^N)}{R} + isNx \log(R) + i\frac{\pi}{2}k + isx \log(\Lambda^N) + s\pi(1 - 2k)x \\ \pi \frac{\log(R)}{R} + \frac{\pi}{N} \frac{\log(\Lambda^N) - i\frac{5}{4}\frac{\pi^2}{N}}{R} + isNx \log(R) + i\frac{\pi}{4} + isx \log(\Lambda^N) + s\pi(1 - k)x \end{cases} \end{aligned} \quad (14.32)$$

up to $\mathcal{O}(R)$ corrections, and where the first case is for N even and the second for N odd. We will come back to the divergent terms near the end of the section.

For the open 1-loop part, we need the $b \rightarrow 0$ limit of

$$\begin{aligned} & \Phi_b \left(-b(sx - \mathbf{e}_I \cdot \mathbf{f}_s(\mathbf{a})) + i\frac{b}{2} + i \left(\mathbf{e}_I \cdot \mathbf{w} + \frac{k}{N} + s \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{2N} \right) b^{-1} \right) \\ & = \Phi_b \left(-i \left(\frac{1}{2} - w_I \right) b^{-1} + ib + i \frac{b^{-1}}{2} - ibz_I \right), \end{aligned} \quad (14.33)$$

where $b = \sqrt{R/2\pi}$ and we used that

$$\mathbf{e}_I \cdot \mathbf{w} - \frac{1}{2} + \frac{k}{N} + s \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{2N} = - \left(\frac{1}{2} - w_I \right), \quad z_I = \frac{1}{2} - i(sx - \mathbf{e}_I \cdot \mathbf{f}_s(\mathbf{a})), \quad (14.34)$$

with $\mathbf{w} = \sum_I w_I \mathbf{e}_I$. It should be noted that $1/2 - w_I$ is either 0 or 1. Using the quasi-periodicity of the quantum dilogarithm (A.18) under shifts of ib and $-ib^{-1}$ gives

$$\Phi_b \left(-i \left(\frac{1}{2} - w_I \right) b^{-1} + ib + i \frac{b^{-1}}{2} - ibz_I \right)$$

$$= \frac{\left(1 + \left(\frac{1}{2} - w_I\right) \exp(-i2\pi z_I)\right)}{1 - \exp\left(-i2\pi b^2\left(z_I - \frac{1}{2}\right)\right)} \Phi_b\left(i\frac{b^{-1}}{2} - ibz_I\right). \quad (14.35)$$

Expanding around $b \rightarrow 0$ and using [28, eq. (4.29)], we find

$$\begin{aligned} & \frac{\left(1 + \left(\frac{1}{2} - w_I\right) \exp(-i2\pi z_I)\right)}{1 - \exp\left(-i2\pi b^2\left(z_I - \frac{1}{2}\right)\right)} \Phi_b\left(i\frac{b^{-1}}{2} - ibz_I\right) = \exp\left(-i\frac{\pi}{12}b^{-2} + (z_I - 1) \log(2\pi b^2)\right) \\ & \left(-\frac{i}{\sqrt{2\pi}}\right) \left(1 + \left(\frac{1}{2} - w_I\right) \exp(-i2\pi z_I)\right) \exp\left(i\frac{\pi}{2}z_I\right) \Gamma\left(z_I - \frac{1}{2}\right) (1 + \mathcal{O}(b^2)) \end{aligned} \quad (14.36)$$

where we also used the integral representations of the quantum dilogarithm (A.19) and of the log gamma function (A.20). The leading contribution from the 1-loop part in the four-dimensional limit is then

$$\begin{aligned} & \lim_{R \rightarrow 0} \exp\left[J_{1\text{-loop}}^{\text{open}}\left(sRx - i\frac{\pi}{N}(2k + s\mathbb{1}_{2N+1}(N)), R\mathbf{f}_s(\mathbf{a}) + i2\pi\mathbf{w}, R\right)\right] \\ & = \left(\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi}}\right)^N \exp\left(-i\frac{\pi^2}{6}\frac{N}{R} - N\left(\frac{1}{2} + isx\right) \log(R)\right) \exp\left(s\frac{\pi}{2}Nx\right) \\ & \prod_{I=1}^N \left\{1 - \left(\frac{1}{2} - w_I\right) \exp(-2\pi[sx - \mathbf{e}_I \cdot \mathbf{f}_s(\mathbf{a})])\right\} \Gamma(-i[sx - \mathbf{e}_I \cdot \mathbf{f}_s(\mathbf{a})]) \end{aligned} \quad (14.37)$$

Regarding the instanton part, the contributions from the 5d GV free energy vanish in the 4d limit, while the contributions from the 5d NS free energy lead directly to their 4d counterparts [14, 189]. This is exactly as noted in [44] for the closed sector and in [28] for the open sector in the $N = 2$ case. That is

$$\begin{aligned} & \lim_{R \rightarrow 0} \exp\left[J_{\text{inst}}^{\text{open}}\left(sRx - i\frac{\pi}{N}(2k + s\mathbb{1}_{2N+1}(N)), R\mathbf{f}_s(\mathbf{a}) + i2\pi\mathbf{w}, e^{-i\frac{\pi}{2}\frac{\mathbb{1}_{2N+1}(N)}{N}}R\Lambda, R\right)\right] \\ & = Z_D^{\text{inst}}(sx, \mathbf{f}_s(\mathbf{a}), \Lambda, 1), \end{aligned} \quad (14.38)$$

and there are no divergences coming from the instanton part.

Let us briefly come back to the divergent factors. For N odd, we have no divergences coming from the closed part of the grand potential, and only an overall, x -independent divergence coming from the open part. This can be dealt with by introducing an appropriate, overall normalization. For N even on the other hand, the polynomial part of the closed grand potential gives rise to a divergent constant for the second saddle (14.30) that is not present for the first saddle. However, this gets exactly cancelled by the divergences from the polynomial part of the open grand potential (14.32), leaving only an x -independent, overall divergent factor. This can again be dealt with by an appropriate overall normalization.

14.2.3 The result of the four-dimensional limit

Hence, up to an x -independent, divergent overall factor, we obtain for both saddles

$$e^{i\frac{\pi}{2}k\mathbb{1}_{2N}(N)} e^{-s\pi(1+\mathbb{1}_{2N}(N))kx} Z_D(sx, \mathbf{f}_s(\mathbf{a}), \Lambda, 1) \sum_{\mathbf{w} \in \mathcal{W}_N \cdot \beta} \mathcal{P}_{\mathbf{w}}(sx, \mathbf{f}_s(\mathbf{a}), \Lambda, 1) \quad (14.39)$$

where we have $s = 1$ and $k = k_x = k_y = 0$ for the first saddle, while $s = -1$ and $k = k_x = k_y = 1$ for the second saddle. We used the expressions²¹

$$Z_D(x, \mathbf{a}, \Lambda, 1) = e^{i \log(\Lambda^N) x + \pi(\frac{N}{2} + 1)x} \left(\prod_{I=1}^N \Gamma(i[e_I \cdot \mathbf{a} - x]) \right) Z_D^{\text{inst}}(x, \mathbf{a}, \Lambda, 1), \quad (14.40)$$

$$\mathcal{P}_{\mathbf{w}}(x, \mathbf{a}, \Lambda, 1) = \frac{e^{i \partial_{\mathbf{a}} F_{\text{NS}} \cdot \mathbf{w} - \pi \mathbf{a} \cdot \mathbf{w} \mathbb{1}_{2\mathbb{N}+1}(N)}}{\prod_{\alpha \in \Delta_+} (2 \sinh(\pi \mathbf{a} \cdot \alpha))^{(\mathbf{w} \cdot \alpha)^2}} \prod_{I=1}^N \left(1 - \left(\frac{1}{2} - w_I \right) e^{2\pi(e_I \cdot \mathbf{a} - x)} \right), \quad (14.41)$$

$$\beta = \gamma + \left(\frac{1-s}{2} \right) \mathbb{1}_{2\mathbb{N}}(N) \mathbf{e}_N. \quad (14.42)$$

The number of non-trivial factors in (14.41) is $N/2$ for $s = 1$ and $(N-2)/2$ for $s = -1$ when N is even, and $(N-1)/2$ when N is odd. This can be verified directly from the expression of β in (14.42). Let us end with the comment that

$$\left(\frac{1}{2} - w_I \right) = \frac{1}{2} - \mathbf{e}_I \cdot \mathbf{w} - \left(\frac{k}{N} + s \frac{\mathbb{1}_{2\mathbb{N}+1}(N)}{2N} \right) = \frac{1}{2} - \mathbf{e}_I \cdot \mathbf{w} - \left(\frac{1-s\mathbb{1}_{2\mathbb{N}}(N)}{2N} \right) \quad (14.43)$$

where the latter two expressions are coordinate invariant and independent of the choice $w_I \in \{\pm 1/2\}$, and they are always 0 or 1.

Concluding remarks

In this final chapter, we studied the spectral problems associated with a family of quantum mirror curves and formulated a precise conjecture for the eigenfunctions of the corresponding difference operators. We mainly tested this conjecture in the four-dimensional limit, both analytically, at finite orders in the instanton expansion, and numerically. A rigorous proof of the conjectured properties of these solutions remains an open problem, as does a deeper understanding of the spectral theory of some of the difference operators considered. The explicit formulas obtained here might provide a starting point for such an analysis.

We now turn to the conclusions of this thesis, starting with a brief overview of the main results and then discussing some of the open questions that emerged from this work.

²¹Note that $\mathcal{P}_{\mathbf{w}}$ in (14.41) is indeed equal to (11.21), since $1/2 - w_I$ is either 0 or 1.

Conclusions and Outlook

In this thesis, we developed and applied methods inspired by two-dimensional conformal field theory, supersymmetric gauge theory and topological string theory to the study of spectral problems. Our analysis followed two main directions: periodic spectral problems governed by second-order differential equations and spectral problems associated with second-order finite-difference equations. Both directions give rise to several open questions and natural avenues for further research.

In the first part of the thesis, we solved the connection problem for the Lamé equation using the analytic-continuation properties of degenerate Virasoro conformal blocks. We also developed a systematic resummation procedure for the bulk NS functions associated with the Lamé and Heun equations and for the orbifold surface-defect NS functions associated with the Heun equation and its confluent limits. This procedure resolves the singular instanton expansions at resonant values of the Floquet exponent and provides access to the corresponding periodic and anti-periodic spectral data. For the Heun equation, we also constructed the logarithmic companions of the resonant solutions and distinguished the loci where spectral gaps close from the smaller loci where the resonant monodromy becomes semisimple.

In the second part, we studied finite-difference equations associated with the $Y^{N,0}$ quantum mirror curves. Within the open TS/ST correspondence, we proposed non-perturbative wavefunctions whose four-dimensional limit gives explicit analytic solutions for the finite-difference Hamiltonians $2\Lambda^N \cosh(\mathbf{p}) + V_N(\mathbf{x})$. We derived their quantization conditions, described bound and resonant states for even and odd N , and discussed related operators with an inverted potential and/or a $\sinh(\mathbf{p})$ kinetic term.

Concerning the first direction of the thesis, a central role was played by the new blow-up equations satisfied by the NS functions encoding both the accessory parameters and the solutions of the associated differential equations. These results suggest several directions for further investigation:

- From a practical standpoint, solving the bulk and defect NS blow-up equations derived in Chapters II and III is computationally demanding. It would therefore be useful to develop a more efficient method for extracting their solutions. In particular, it would be desirable to identify recursion relations among the resummation functions, although it is presently unclear whether such relations can exist.
- Another possible direction is to extend the resummation procedure from Floquet solutions to local solutions of the associated differential equations. In gauge-theoretic terms, this would involve the resummation of vortex-string partition functions [32, 52]. Although preliminary evidence suggests that such an extension is possible, this direction requires further investigation. In particular, fixing the regular part of the corresponding defect NS prepotentials appears to be less straightforward than for orbifold surface-defect partition functions.

- It would be interesting to determine whether the conjectures regarding the vanishing of the resonant resummed dual quantum periods in footnote 18 and equation (10.70) hold and, if so, to clarify their physical interpretation in both the four-dimensional gauge theories and the corresponding two-dimensional effective gauge theories, for example in light of the discussion in [38]. From a mathematical perspective, these conjectures may also have implications for the geometry of the space of monodromy data, on which a and a_D serve locally as Darboux coordinates.
- It would also be interesting to determine, or at least characterize, the full $\mathfrak{t}$ -dependent semisimple locus of the resonant A-cycle monodromy beyond the expansion around $\mathfrak{t} = 0$ discussed in Section 10.3. An analogous analysis could be carried out for the Lamé equation and the A-cycle monodromy of its solutions.
- It would also be interesting to extend the resummation procedure to other gauge theories and investigate its implications for their associated quantum Seiberg–Witten curves and quantum integrable systems. The most immediate step would be to study the surface-defect NS functions of the $\mathcal{N} = 2^*$ SU(2) theory, thereby obtaining resummed solutions of the Lamé equation. The relevant defect blow-up equations have already been proved in [150]. A subsequent generalization would be to study both bulk and defect NS functions in broader classes of four-dimensional $\mathcal{N} = 2$ gauge theories, including linear, circular, and more general quiver theories, as well as theories with higher-rank unitary gauge groups.
- The blow-up-equation framework developed in this thesis may provide a useful tool for studying the analytic structure of classical conformal blocks in more general settings. Indeed, the AGT dual of SU(2) gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$ is known to be $\mathcal{N} = 1$ super-Liouville CFT [134, 135, 190, 191], so our results may constitute a first step towards understanding the analytic structure of classical super-Virasoro conformal blocks. More generally, it would be interesting to investigate whether blow-up equations can be used to study classical conformal blocks in strong-coupling regimes, including at Argyres–Douglas points [140, 192–201], as well as in $\mathcal{N} = 1$ and $\mathcal{N} = 1^*$ vacua [202, 203].
- It would be interesting to clarify the implications of the resummation procedure for the associated isomonodromic deformation problems. The procedure developed here appears to have a counterpart in the study of Painlevé tau functions with resonant initial conditions, see Subsection 8.1. Such initial conditions can be reached through a suitable double-scaling limit [143–145], which produces logarithmic terms in the time expansion of the tau function and offers an alternative perspective on the periodic spectral problems considered in this thesis. Indeed, resonant values of the A-cycle monodromy define loci where the usual coordinates become singular or the monodromy becomes non-semisimple. Near these loci, the resummed NS functions provide well-behaved gauge-theoretic quantities, while the split eigenfunctions distinguish the resonant solution from its logarithmic companion. This suggests that the resummation procedure may provide a natural regularization of the isomonodromic tau function, or equivalently a preferred local description of the flat connection, near resonant monodromy strata. Along the lines of [52], understanding this relation could also clarify how defect blow-up equations encode the non-semisimple and logarithmic sectors of the corresponding isomonodromic systems. A further generalization would be to study theories with arbitrary simple gauge groups and their associated isomonodromic deformation problems [204, 205].

- It would also be interesting to explore applications of the results obtained here to physical settings in which the Lamé and Heun equations naturally arise. The band-gap structure of the Lamé equation, and more generally of Hill equations, is phenomenologically relevant in inflationary cosmology, particularly in the preheating problem [206] and in models of the early-universe cosmology [207]. Moreover, classical Virasoro conformal blocks themselves have intriguing applications to holographic descriptions of three-dimensional gravitational systems. It would therefore be interesting to identify the gravitational counterpart of the analytic structure uncovered in this thesis [208–210].

The Heun equation and its confluent limits, together with their gauge-theoretic and CFT avatars, $SU(2)$ NS functions and, through the AGT correspondence, classical Virasoro conformal blocks, have likewise found a growing range of applications to black-hole quasinormal modes and other observables [126, 211–218], compact horizonless gravitational objects [219, 220], holographic correlation functions [221], and holographic geodesic networks [208, 209]. These developments have been facilitated by the gauge-theoretic derivation of the Heun connection formulas [32] and their CFT counterparts [25]. Since related resummation procedures have already been used to investigate pole structures and gravitational response functions in some of these settings [210, 222], it would be natural to determine whether the resonant solutions and logarithmic companions constructed here admit applications to these and other physical problems governed by the Heun equation.

Turning to the second direction pursued in this thesis, the framework of the TS/ST correspondence allowed us to construct explicit analytic solutions and determine the corresponding quantization conditions for several families of finite-difference equations. For some of these equations, however, their spectral-theoretic interpretation remains unclear. Natural directions for further investigation therefore include:

- Within the open TS/ST correspondence, the structure of the non-perturbative open partition function still poses some interesting challenges. As shown in [17, 26, 28] and summarized in equation (14.18), this function can be expressed as a sum of two contributions, or saddles. Although in some cases we have a technical prescription to construct the second saddle from the first, its geometrical and physical meaning remains elusive and calls for a deeper understanding.
- It would be interesting to investigate how our solutions (14.18), (11.25) and (11.31) emerge from a resurgent analysis of the open-string wavefunction and an exact WKB analysis; see for instance [223–229].
- The deformed Hamiltonian (11.1) exhibits several novel spectral features absent in standard Schrödinger operators. Due to its difference-equation nature, the usual oscillation theorem [182, p. 66, thm. 3.5] does not apply. Moreover, the system can display ground-state degeneracies and, remarkably, square integrable solutions with real energies even in potentials unbounded from below. These phenomena reveal a rich and unconventional spectral structure whose physical and mathematical interpretation deserves further study.
- It is important to undertake a rigorous study of the spectral properties of the operators considered in this work. The case with a $\cosh(p)$ kinetic term and a confining potential was analysed in [171], where suitable self-adjoint realizations with purely discrete

spectra were established.²² However, many of the spectral problems discussed here fall outside this framework, including operators with potentials unbounded from below (see Subsections 11.1 and 12.1), as well as those with a $\sinh(p)$ kinetic term (see Subsection 12.2). While similar questions have been addressed for standard Schrödinger operators with unbounded potentials (see [175, 177]), an analogous understanding for the finite-difference operators studied here is still missing, and questions regarding the natural choice of operator domains and the characterization of the spectra remain open. The explicit solutions obtained in this work may provide a possible starting point.

- It was pointed out in [187, 231] that there exists another four-dimensional limit, the so-called dual 4d limit, which can be implemented on the mirror curve (14.1) and leads to a family of integral operators. The corresponding spectral problems make contact with the 4d gauge theory in the self-dual phase of the Ω -background, rather than the NS phase, as is the case for the spectral problems studied in this thesis. For $N = 2$, this limit on the eigenfunctions was analysed in [27, 28], where a new functional relation between the modified Mathieu and the McCoy–Tracy–Wu operators was found. It would be interesting to extend this analysis to $N > 2$ in light of our new results on the eigenfunctions.
- It would also be interesting to generalize our construction of the eigenfunctions of the deformed Hamiltonian (11.1) to other four-dimensional quantum mirror curves, including those associated with SW theories with matter or with different gauge groups. Such extensions could reveal new classes of solvable quantum systems and provide further insights into the open TS/ST correspondence.

Finally, several further directions arise at the intersection of the two main themes explored in this thesis, including:

- Connection formulae for the q -difference equations arising from the quantum Seiberg–Witten curves of five-dimensional gauge theories can be studied using degenerate q -Virasoro conformal blocks; see [232] for the case of q -Painlevé III₃. On the other hand, TS/ST spectral determinants naturally appear as connection coefficients between distinguished solutions of the same quantum curves. It would therefore be interesting to clarify the relation between these two descriptions and determine how insights from each can inform the other.
- Another natural direction is to develop a five-dimensional analogue of the resummation procedure introduced in this thesis. Five-dimensional gauge-theory partition functions exhibit a pole structure similar to that encountered in four dimensions, suggesting that they may admit an analogous resummation. For five-dimensional $\mathcal{N} = 1$ SU(2) gauge theories, the quantum Seiberg–Witten curves are q -difference equations, or equivalently finite-difference equations in logarithmic variables. Their spectral data are encoded in expectation values of BPS Wilson loops, while their solutions are engineered by codimension-two defect partition functions. Blow-up equations are known for five-dimensional partition functions [4], as are their connections with Painlevé tau functions [233–237], and analogous equations are available for Wilson loops and defect observables [238, 239]. It would therefore be interesting to determine whether these equations can be used to resum both bulk and defect quantities and thereby study periodic spectral problems governed by q -difference equations. More

²²See also [230] for extensions to certain non-self-adjoint operators with complex potentials.

generally, one could extend this construction to five-dimensional gauge theories with higher-rank gauge groups, as well as to quiver gauge theories and their associated higher-order quantum curves.

Appendix A

Special Functions and Conventions

A.1 Special functions

In this section, we summarize the conventions used for the special functions appearing throughout the main text.

A.1.1 Elliptic functions

The Weierstrass $\wp$ -function, defined by

$$\wp(z; 2\omega_1, 2\omega_2) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right), \quad (\text{A.1})$$

admits the following alternative representations:

$$\wp(z) = \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3} + \pi^2 \sum_{n \neq 0} \left(\frac{1}{\sin^2 \pi(z - n\tau)} - \frac{1}{\sin^2 \pi n\tau} \right), \quad (\text{A.2a})$$

$$\wp(z) = \frac{\pi^2/\tau^2}{\sin^2 \frac{\pi z}{\tau}} - \frac{\pi^2}{3\tau^2} + \frac{\pi^2}{\tau^2} \sum_{m \neq 0} \left(\frac{1}{\sin^2 \frac{\pi(z-m)}{\tau}} - \frac{1}{\sin^2 \frac{\pi m}{\tau}} \right), \quad (\text{A.2b})$$

where $\wp(z)$ always denotes $\wp(z; 1, \tau)$, $\tau = \omega_2/\omega_1$. In particular, in the limit $\tau \rightarrow +i\infty$, we find

$$\wp(z) = \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3} + 16\pi^2 \mathfrak{q} \sin^2 \pi z + \mathcal{O}(\mathfrak{q}^2), \quad (\text{A.3})$$

where $\mathfrak{q} = e^{2\pi i \tau}$. Moreover, the values of the $\wp$ -function at the half-periods are denoted respectively by $e_1 = \wp(\omega_1; 2\omega_1, 2\omega_2)$, $e_2 = \wp(\omega_1 + \omega_2; 2\omega_1, 2\omega_2)$ and $e_3 = \wp(\omega_2; 2\omega_1, 2\omega_2)$. The Weierstrass ζ -function, defined as

$$\zeta(z; 2\omega_1, 2\omega_2) = \frac{1}{z} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{z - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{z}{(2m\omega_1 + 2n\omega_2)^2} \right), \quad (\text{A.4})$$

is related to the Weierstrass $\wp$ -function by

$$\wp(z; 2\omega_1, 2\omega_2) = -\zeta'(z; 2\omega_1, 2\omega_2), \quad (\text{A.5})$$

and, once again, $\zeta(z)$ always denotes $\zeta(z; 1, \tau)$. For the Jacobi theta functions, we use the following conventions:

$$\theta_1(z|\tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n e^{i\pi\tau(n+\frac{1}{2})^2} e^{2\pi i z(n+\frac{1}{2})}, \quad (\text{A.6a})$$

$$\theta_2(z|\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n+\frac{1}{2})^2} e^{2\pi iz(n+\frac{1}{2})}, \quad (\text{A.6b})$$

$$\theta_3(z|\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2} e^{2\pi izn}, \quad (\text{A.6c})$$

$$\theta_4(z|\tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{i\pi\tau n^2} e^{2\pi izn}. \quad (\text{A.6d})$$

The Euler φ -function is defined by

$$\varphi(\mathfrak{q}) = \prod_{n \geq 1} (1 - \mathfrak{q}^n), \quad (\text{A.7})$$

while the Dedekind η -function as

$$\eta(\tau) = \mathfrak{q}^{1/24} \varphi(\mathfrak{q}). \quad (\text{A.8})$$

In terms of Jacobi theta functions, the Weierstrass elliptic functions can be represented as follows:

$$\wp(z) = -\partial_z^2 \log \theta_1(z|\tau) - 2\eta_1(\tau), \quad (\text{A.9a})$$

$$\zeta(z) = \partial_z \log \theta_1(z|\tau) + 2z\eta_1(\tau), \quad (\text{A.9b})$$

where

$$\eta_1(\tau) = -2\pi i \partial_\tau \log \eta(\tau) = 4\pi^2 \left(\frac{1}{24} - \sum_{n=1}^{\infty} \frac{n\mathfrak{q}^n}{1 - \mathfrak{q}^n} \right). \quad (\text{A.10})$$

Moreover, we have the relation:

$$E_2(\tau) = \frac{6}{\pi^2} \eta_1(\tau). \quad (\text{A.11})$$

Finally, the Jacobi elliptic sine function is given by:

$$\text{sn}(z; \kappa^2) = \frac{\theta_3(0|\tau)\theta_1(u(z)|\tau)}{\theta_2(0|\tau)\theta_4(u(z)|\tau)}, \quad u(z) = \frac{z}{2K(\kappa^2)}, \quad (\text{A.12})$$

where

$$K(\kappa^2) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}, \quad (\text{A.13})$$

is the complete elliptic integral of the first kind. The function $\text{sn}(z; \kappa^2)$ is an odd, doubly periodic function of periods $4K(\kappa^2)$ and $2iK(1-\kappa^2)$, related to the Weierstrass $\wp$ -function by:

$$\wp(z) = e_3 + \frac{e_1 - e_3}{\text{sn}^2(z\sqrt{e_1 - e_3}; \kappa^2)}, \quad (\text{A.14})$$

with the following formulas relating modular parameters:

$$\kappa^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\theta_2(0|\tau)^4}{\theta_3(0|\tau)^4}, \quad (\text{A.15a})$$

$$\tau = i \frac{K(1-\kappa^2)}{K(\kappa^2)}. \quad (\text{A.15b})$$

A.1.2 Faddeev's non-compact quantum dilogarithm

The defining representation of the quantum dilogarithm is often taken to be [240, eq. (42)]

$$\Phi_b(z) = \exp \left(\frac{1}{4} \int_{\mathbb{R}+i0} \frac{e^{-i2zu}}{\sinh(bu)\sinh(b^{-1}u)} \frac{du}{u} \right), \quad 2|\operatorname{Im}(z)| < |\operatorname{Re}(b+b^{-1})|. \quad (\text{A.16})$$

It can be analytically continued to a meromorphic function of z on the whole complex plane with poles and roots at [240, eq. (45)]

$$z = \begin{cases} +i \left[\left(k + \frac{1}{2}\right)b + \left(\ell + \frac{1}{2}\right)b^{-1} \right] & \text{poles} \\ -i \left[\left(k + \frac{1}{2}\right)b + \left(\ell + \frac{1}{2}\right)b^{-1} \right] & \text{roots} \end{cases}, \quad k, \ell \in \mathbb{N}, \quad (\text{A.17})$$

and with an essential singularity at complex infinity [240, p. 34]. One can determine the order of the poles and roots when $b^2 \in \mathbb{Q}_{>0}$, based on [241, eq. (21)], see [28, eq. (A.10)]. The parameter b is in general such that $b^2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, but we are mostly interested in $b^2 > 0$, since this corresponds to $\hbar, g_s > 0$. The quantum dilogarithm satisfies the following quasi-periodicity relations in z [240, eq. (48)] [241, eq. (77)],

$$\Phi_b(z + sib^\pm) = \left(1 + e^{s\pi b^\pm} e^{2\pi b^\pm z}\right)^{-s} \Phi_b(z) \quad s \in \{-1, +1\}. \quad (\text{A.18})$$

When taking the 4d limits the following representation of Faddeev's quantum dilogarithm comes in useful [242, eqs. (3.2)-(3.8)]¹

$$\log \Phi_b(z) = -\frac{i}{2\pi b^2} \operatorname{Li}_2 \left(-e^{2\pi b z} \right) - i \int_0^{+\infty} \frac{du}{1 + e^{2\pi u}} \log \left(\frac{1 + e^{2\pi b z - 2\pi b^2 u}}{1 + e^{2\pi b z + 2\pi b^2 u}} \right), \quad (\text{A.19})$$

under the condition that $2|\operatorname{Im}(z)| < b^{-1}$ when $b > 0$.² In taking the standard or dual 4d limit on the integral representation above, one may use the following integral representation of the log gamma function [243, p. 8],

$$\log \Gamma \left(z + \frac{1}{2} \right) = \frac{\log(2\pi)}{2} - z + z \log(z) - 2 \int_0^{+\infty} \frac{du}{1 + e^{2\pi u}} \arctan \left(\frac{u}{z} \right), \quad \operatorname{Re}(z) > 0, \quad (\text{A.20})$$

which can be obtained from Binet's second formula for the log gamma function.

A.2 Nekrasov functions and AGT dictionary for Chapter II

Starting from the Nekrasov partition function of the $SU(2)$ gauge theory with N_f fundamental hypermultiplets, we adopt the conventions of [140] and write

$$\mathcal{Z}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = \mathcal{Z}_{\text{cl}}^{[N_f]}(a; \varepsilon_1, \varepsilon_2; t) \mathcal{Z}_{\text{1-loop}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2) \mathcal{Z}_{\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t), \quad (\text{A.21})$$

where

$$\mathcal{Z}_{\text{cl}}^{[N_f]}(a; \varepsilon_1, \varepsilon_2; t) = t^{\frac{(\varepsilon_1 + \varepsilon_2)^2/4 - a^2}{\varepsilon_1 \varepsilon_2}}, \quad (\text{A.22a})$$

¹There appears to be a constant term missing in [242, eqs. (3.2)-(3.8)].

²One can note from the behaviour under shifts of z by ib^{-1} that the representation above has a limited domain of validity: the quantum dilogarithm is quasi-periodic under such shifts while the same shifts act trivially on the right-hand side above. Some numerical checks seem to suggest that the representation above is only valid for $2|\operatorname{Im}(z)| < b^{-1}$ when $\operatorname{Re}(z) \geq 0$.

$$\mathcal{Z}_{\text{1-loop}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2) = \prod_{\pm} \frac{\prod_{i=1}^{N_f} \exp(\gamma_{\varepsilon_1, \varepsilon_2}(\mu_i \pm a - (\varepsilon_1 + \varepsilon_2)/2))}{\exp(\gamma_{\varepsilon_1, \varepsilon_2}(\pm 2a))}, \quad (\text{A.22b})$$

$$\mathcal{Z}_{\text{inst}}^{[N_f]}(a, \boldsymbol{\mu}; \varepsilon_1, \varepsilon_2; t) = \sum_{\mathbf{Y}=(Y_+, Y_-)} \frac{\prod_{i=1}^{N_f} z_{\text{inst}}^{\text{fund}}(a, \mu_i, \mathbf{Y})}{z_{\text{inst}}^{\text{vec}}(a, \mathbf{Y})} t^{|\mathbf{Y}|}, \quad (\text{A.22c})$$

and $\mathbf{Y} = (Y_+, Y_-)$ denotes a couple of Young tableaux, $|\mathbf{Y}|$ being the total number of boxes of the pair. The function $\gamma_{\varepsilon_1, \varepsilon_2}$ is defined as

$$\gamma_{\varepsilon_1, \varepsilon_2}(x) = \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dz}{z} z^s \frac{e^{-xz}}{(e^{\varepsilon_1 z} - 1)(e^{\varepsilon_2 z} - 1)} \Big|_{s=0}, \quad \text{Re}(x), \text{Re}(\varepsilon_1), \text{Re}(\varepsilon_2) > 0, \quad (\text{A.23})$$

and satisfies the difference equation:

$$\gamma_{\varepsilon_1, \varepsilon_2}(x - \varepsilon_1) + \gamma_{\varepsilon_1, \varepsilon_2}(x - \varepsilon_2) - \gamma_{\varepsilon_1, \varepsilon_2}(x - \varepsilon_1 - \varepsilon_2) - \gamma_{\varepsilon_1, \varepsilon_2}(x) = \log x. \quad (\text{A.24})$$

Many useful properties of this function, including its shift relations, can be found e.g. in the appendices of [140] and [244]. Coming back to the instanton part of the partition function³, we have [80, 245]:

$$z_{\text{inst}}^{\text{fund}}(a, \mu, \mathbf{Y}) = \prod_{\pm} \prod_{(i,j) \in Y_{\pm}} (\mu \pm a + (i - 1/2)\varepsilon_1 + (j - 1/2)\varepsilon_2), \quad (\text{A.25})$$

while, in order to define $z_{\text{inst}}^{\text{vec}}$, we first define:

$$z_{\text{inst}}^{\text{bif}}(\vec{a}, \vec{b}, m, \mathbf{Y}, \mathbf{W}) = \prod_{\alpha, \beta = \pm} \prod_{s_1 \in Y_{\alpha}} \left[a_{\alpha} - b_{\beta} + \varepsilon_2 (A_{Y_{\alpha}}(s_1) + 1) - \varepsilon_1 L_{W_{\beta}}(s_1) - m \right] \prod_{s_2 \in W_{\beta}} \left[a_{\alpha} - b_{\beta} + \varepsilon_1 (L_{Y_{\alpha}}(s_2) + 1) - \varepsilon_2 A_{W_{\beta}}(s_2) - m \right], \quad (\text{A.26})$$

where, following [22], $\vec{a} = (a_+, a_-)$ and $\vec{b} = (b_+, b_-)$, s_1 and s_2 label a box in the corresponding Young tableau, A_Y and L_Y are defined as:

$$A_Y(s) = \lambda_i - j, \quad L_Y(s) = \lambda'_j - i, \quad s = (i, j) \in Y, \quad (\text{A.27})$$

and the Young tableau Y is identified with the corresponding partition $Y = \{\lambda_1 \geq \dots \geq \lambda_n > 0\}$, $Y^T = \{\lambda'_1 \geq \dots \geq \lambda'_m > 0\}$ being its transpose. Then, we have:

$$z_{\text{inst}}^{\text{adj}}(a, m, \mathbf{Y}) = z_{\text{inst}}^{\text{bif}}(\vec{a}, \vec{a}, m, \mathbf{Y}, \mathbf{Y}), \quad (\text{A.28})$$

where $\vec{a} = (a, -a)$, and finally:

$$z_{\text{inst}}^{\text{vec}}(a, \mathbf{Y}) = z_{\text{inst}}^{\text{adj}}(a, 0, \mathbf{Y}). \quad (\text{A.29})$$

Moving on to the partition function of the $\mathcal{N} = 2^* \text{SU}(2)$ gauge theory, we have

$$\mathcal{Z}(a, m; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \mathcal{Z}_{\text{cl}}(a; \varepsilon_1, \varepsilon_2; \mathbf{q}) \mathcal{Z}_{\text{1-loop}}(a, m; \varepsilon_1, \varepsilon_2) \mathcal{Z}_{\text{inst}}(a, m; \varepsilon_1, \varepsilon_2; \mathbf{q}), \quad (\text{A.30})$$

where

$$\mathcal{Z}_{\text{cl}}(a; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \mathbf{q}^{-\frac{a^2}{\varepsilon_1 \varepsilon_2}}, \quad (\text{A.31a})$$

³The convention for the fundamental masses adopted here differs from those used in Section 3 and Chapter III by an overall sign and a simple shift depending on ε_1 and ε_2 . Furthermore, the vector-multiplet instanton contribution $z_{\text{inst}}^{\text{vec}}$ appearing here is the inverse of that appearing in Section 3.

$$\mathcal{Z}_{1\text{-loop}}(a, m; \varepsilon_1, \varepsilon_2) = \prod_{\pm} \frac{\exp(\gamma_{\varepsilon_1, \varepsilon_2}(\pm 2a - m))}{\exp(\gamma_{\varepsilon_1, \varepsilon_2}(\pm 2a))}, \quad (\text{A.31b})$$

$$\mathcal{Z}_{\text{inst}}(a, m; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \varphi(\mathbf{q})^{1 - \frac{2m(\varepsilon_1 + \varepsilon_2 - m)}{\varepsilon_1 \varepsilon_2}} \mathcal{Z}_{\text{inst}}^{\text{U}(2)}(a, m; \varepsilon_1, \varepsilon_2; \mathbf{q}), \quad (\text{A.31c})$$

and finally:

$$\mathcal{Z}_{\text{inst}}^{\text{U}(2)}(a, m; \varepsilon_1, \varepsilon_2; \mathbf{q}) = \sum_{\mathbf{Y}} \mathbf{q}^{|\mathbf{Y}|} \frac{z_{\text{inst}}^{\text{adj}}(a, m, \mathbf{Y})}{z_{\text{inst}}^{\text{vec}}(a, \mathbf{Y})}. \quad (\text{A.32})$$

In conclusion, we describe how the classical limit $\psi_s^{(n)}$ of the degenerate torus blocks in the necklace basis $\Phi_s^{(n)}$, as in equation (5.29), can be expressed in terms of surface defect partition functions of the $\mathcal{N} = 2^*$ SU(2) gauge theory. We consider the quiver gauge theory [74, 84, 246] associated to the circular quiver with two SU(2) gauge nodes, one of which is degenerate [146], and compute its instanton partition function as

$$\begin{aligned} \mathcal{Z}_{s, \text{inst}}^{\text{def}}(\zeta, \sigma, m; \varepsilon_1, \varepsilon_2; \mathbf{q}) &= \sum_{\mathbf{Y}, \mathbf{W}} \zeta^{|\mathbf{Y}| - |\mathbf{W}|} \mathbf{q}^{\frac{|\mathbf{Y}| + |\mathbf{W}|}{2}} \frac{z_{\text{inst}}^{\text{bif}}(\vec{\sigma}_1, \vec{\sigma}_2, m, \mathbf{Y}, \mathbf{W})}{z_{\text{inst}}^{\text{vec}}(\sigma, \mathbf{Y})} \\ &\cdot \frac{z_{\text{inst}}^{\text{bif}}(\vec{\sigma}_2, \vec{\sigma}_1, -\varepsilon_2/2, \mathbf{W}, \mathbf{Y})}{z_{\text{inst}}^{\text{vec}}(\sigma + s\varepsilon_2/2, \mathbf{W})}, \end{aligned} \quad (\text{A.33})$$

where $\vec{\sigma}_1 = (\sigma, -\sigma)$ and $\vec{\sigma}_2 = (\sigma + s\varepsilon_2/2, -\sigma - s\varepsilon_2/2)$, $s = \pm$. With respect to the variables used in Section 5, we have

$$\zeta = e^{2\pi i x}, \quad z = x + \tau/2, \quad y = e^{2\pi i z} = \mathbf{q}^{\frac{1}{2}} \zeta. \quad (\text{A.34})$$

The AGT correspondence implies the relation

$$\Phi_s^{(n)}(x + \tau/2, a, m; \mathbf{q}) = e^{2\pi i(\Delta_{a_s} - \Delta_a)x} \mathbf{q}^{\frac{\Delta_{a_s} + \Delta_a}{2} - \frac{c}{24}} \frac{\mathcal{Z}_{s, \text{inst}}^{\text{def}}(e^{2i\pi x}, a - Q/2, m; b^{-1}, b; \mathbf{q})}{\mathcal{Z}^{\text{U}(1)}(e^{2i\pi x}, m; \mathbf{q})}, \quad (\text{A.35})$$

where $a_s = a - sb/2$, $c = 1 + 6Q^2$ and $Q = b + b^{-1}$, while

$$\begin{aligned} \mathcal{Z}^{\text{U}(1)}(\zeta, m; \mathbf{q}) &= \prod_{l \geq 0} \left(1 - \mathbf{q}^{l+1}\right)^{2m(Q-m) - 3b^2/2 - 2} \left(1 - \zeta \mathbf{q}^{l+1/2}\right)^{-b(Q-m)} \\ &\cdot \left(1 - \zeta^{-1} \mathbf{q}^{l+1/2}\right)^{2m(Q+b/2)}. \end{aligned} \quad (\text{A.36})$$

Thus, taking the classical limit, we obtain the following representation for a pair of linearly independent Floquet solutions of the Lamé equation (5.30):

$$\psi_s^{(n)}(x + \tau/2, \nu, \mu; \mathbf{q}) = e^{\pi i s \nu x} \mathbf{q}^{\frac{s\nu}{4}} \lim_{b \rightarrow 0} \frac{\mathcal{Z}_{s, \text{inst}}^{\text{def}}(e^{2i\pi x}, \frac{\nu}{2b} - \frac{b}{2}, \frac{\mu}{b}; b^{-1}, b; \mathbf{q})}{\mathcal{Z}^{\text{U}(1)}(e^{2i\pi x}, \frac{\mu}{b}; \mathbf{q}) \mathcal{Z}_{\text{inst}}(\frac{\nu}{2b} - \frac{b}{2}, \frac{\mu}{b}; b^{-1}, b; \mathbf{q})}. \quad (\text{A.37})$$

A.3 Nekrasov functions for Chapter III

The Nekrasov partition function $\mathcal{Z}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{q})$ of the $\mathcal{N} = 2$ SU(2) gauge theory with $N_f = (n_0, n_1)$ hypermultiplets, with $n_0, n_1 \leq 2$, depends on the Coulomb branch parameter a , the $n_0 + n_1$ hypermultiplet masses $\mathbf{m}$, the two equivariant parameters ε_i , and the instanton counting parameter $\mathbf{q}$. It factorizes into classical, one-loop, and instanton contributions,

$$\mathcal{Z}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{q}) = \mathcal{Z}_{\text{cl}}(a, \varepsilon_1, \varepsilon_2; \mathbf{q}) \mathcal{Z}_{1\text{-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2) \mathcal{Z}_{\text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{q}), \quad (\text{A.38})$$

where the classical contribution is

$$\mathcal{Z}_{\text{cl}}(a, \varepsilon_1, \varepsilon_2; \mathbf{q}) = \mathbf{q}^{-\frac{a^2}{\varepsilon_1 \varepsilon_2}}. \quad (\text{A.39})$$

The one-loop contribution can be conveniently expressed in terms of the Barnes double gamma function⁴,

$$\Gamma_2(x; \varepsilon_1, \varepsilon_2) = \exp \left[-\frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \frac{e^{-\beta x}}{(1 - e^{-\beta \varepsilon_1})(1 - e^{-\beta \varepsilon_2})} \right], \quad (\text{A.40})$$

as

$$\mathcal{Z}_{1\text{-loop}}^{(n_0, n_1)} = \prod_{\pm} \frac{\Gamma_2(0; \varepsilon_1, \varepsilon_2) \Gamma_2(\pm 2a; \varepsilon_1, \varepsilon_2)}{\prod_{i_0=1}^{n_0} \Gamma_2(\pm a - m_{2+i_0}; \varepsilon_1, \varepsilon_2) \prod_{i_1=1}^{n_1} \Gamma_2(\pm a - m_{i_1}; \varepsilon_1, \varepsilon_2)}. \quad (\text{A.41})$$

The instanton contribution is written as a sum over pairs of Young diagrams $\lambda = (\lambda^{(0)}, \lambda^{(1)})$,

$$\mathcal{Z}_{\text{inst}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{q}) = \sum_{\lambda} \mathbf{q}^{|\lambda|} \mathcal{E}(-T[\lambda]^{(n_0, n_1)}), \quad (\text{A.42})$$

where the full character of the equivariant torus action is

$$T[\lambda]^{(n_0, n_1)} = NK^* + q_1 q_2 N^* K - (1 - q_1)(1 - q_2) K K^* - M^{(n_0, n_1)} K^*, \quad (\text{A.43})$$

with elementary characters

$$N = \sum_{\alpha=0,1} e^{a_\alpha}, \quad (\text{A.44a})$$

$$K = \sum_{\alpha=0,1} \sum_{(i,j) \in \lambda^{(\alpha)}} e^{a_\alpha + \varepsilon_1(i-1) + \varepsilon_2(j-1)}, \quad (\text{A.44b})$$

$$M^{(n_0, n_1)} = \sum_{\alpha=1}^{n_0} e^{m_{2+\alpha}} + \sum_{\alpha=1}^{n_1} e^{m_\alpha}. \quad (\text{A.44c})$$

To write the one-loop contribution to the orbifold surface defect partition functions considered in this work, we also introduce the single gamma function

$$\Gamma_1(x; \varepsilon_1) = \exp \left[-\frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \frac{e^{-\beta x}}{1 - e^{-\beta \varepsilon_1}} \right] = \frac{\sqrt{2\pi\varepsilon_1}}{\varepsilon_1^{\frac{x}{\varepsilon_1}} \Gamma\left(\frac{x}{\varepsilon_1}\right)}. \quad (\text{A.45})$$

The one-loop part of the orbifold surface defect partition function in (9.14) is then

$$\Psi_{\beta, 1\text{-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2) = \mathcal{Z}_{1\text{-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2) \hat{\mathcal{Z}}_{\beta, 1\text{-loop}}^{(n_1)}(a, \mathbf{m}, \varepsilon_1), \quad (\text{A.46})$$

where

$$\hat{\mathcal{Z}}_{\beta, 1\text{-loop}}^{(n_1)}(a, \mathbf{m}, \varepsilon_1) = \frac{\prod_{i_1=1}^{n_1} \Gamma_1(a_\beta - m_{i_1}; \varepsilon_1)}{\Gamma_1(2a_\beta; \varepsilon_1)}, \quad (\text{A.47})$$

with $a_0 = a$ and $a_1 = -a$. The one-loop bulk and defect Nekrasov functions admit an ε_2 -expansion analogous to the instanton contributions in (9.17). Defining

$$\begin{aligned} \gamma_2(x; \varepsilon_1) = \varepsilon_1 \left[\frac{1}{12} + \left(\frac{x}{\varepsilon_1} - 1 \right) \log \Gamma\left(\frac{x}{\varepsilon_1}\right) - \log G\left(\frac{x}{\varepsilon_1}\right) \right. \\ \left. - \log A + \frac{\log \varepsilon_1 - 1}{2} B_2\left(\frac{x}{\varepsilon_1}\right) \right], \end{aligned} \quad (\text{A.48})$$

⁴Throughout this section, we mostly follow the conventions of [52].

where A is Glaisher's constant, G is the Barnes G -function and

$$B_2(z) = z^2 - z + \frac{1}{6}, \quad (\text{A.49})$$

is the second Bernoulli polynomial, one finds formally

$$\log \Gamma_2(x; \varepsilon_1, \varepsilon_2) = \frac{1}{\varepsilon_2} \gamma_2(x; \varepsilon_1) + \frac{1}{2} \log \Gamma_1(x; \varepsilon_1) + \mathcal{O}(\varepsilon_2). \quad (\text{A.50})$$

Thus, for the 1-loop part of the twisted effective superpotential we find

$$\begin{aligned} \mathcal{W}_{0, \text{1-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1) &= 2\gamma_2(0; \varepsilon_1) + \sum_{\pm} \left(\gamma_2(\pm 2a; \varepsilon_1) \right. \\ &\quad \left. - \sum_{i_0=1}^{n_0} \gamma_2(\pm a - m_{2+i_0}; \varepsilon_1) - \sum_{i_1=1}^{n_1} \gamma_2(\pm a - m_{i_1}; \varepsilon_1) \right), \end{aligned} \quad (\text{A.51})$$

while, for its first ε_2 -correction,

$$\mathcal{W}_{1, \text{1-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1) = \frac{1}{2} \log \prod_{\pm} \frac{\prod_{i_0=1}^{n_0} \Gamma\left(\pm \frac{a}{\varepsilon_1} - \frac{m_{2+i_0}}{\varepsilon_1}\right) \prod_{i_1=1}^{n_1} \Gamma\left(\pm \frac{a}{\varepsilon_1} - \frac{m_{i_1}}{\varepsilon_1}\right)}{\Gamma\left(\pm \frac{2a}{\varepsilon_1}\right)}, \quad (\text{A.52})$$

Meanwhile, for the defect prepotential, the one-loop part reads:

$$\hat{\mathcal{W}}_{\beta, \text{1-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1) = \mathcal{W}_{1, \text{1-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1) + \log \frac{\Gamma\left(\frac{2a_{\beta}}{\varepsilon_1}\right)}{\prod_{i_1=1}^{n_1} \Gamma\left(\frac{a_{\beta}}{\varepsilon_1} - \frac{m_{i_1}}{\varepsilon_1}\right)}. \quad (\text{A.53})$$

In both expressions (A.52) and (A.53), we have omitted constant terms that are irrelevant for our purposes.

To obtain the ℓ -factors appearing in (10.27), we introduce the L -factors

$$L_n^{(n_0, n_1)} = \frac{\mathcal{Z}_{1\text{-loop}}^{(n_0, n_1)}(a + n\varepsilon_1, \mathbf{m} + \xi_n \varepsilon_1, \varepsilon_1, \varepsilon_2 - \varepsilon_1) \mathcal{Z}_{1\text{-loop}}^{(n_0, n_1)}(a + n\varepsilon_2, \mathbf{m} + \xi_n \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2)}{\mathcal{Z}_{1\text{-loop}}^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2)}, \quad (\text{A.54})$$

where ξ_n vanishes for $n \in \mathbb{Z}$, and equals $\frac{1}{2}$ for $n \in \mathbb{Z} + \frac{1}{2}$. Using the shift properties of the double gamma function (A.40), these reduce to rational functions of the equivariant parameters,

$$L_n^{(n_0, n_1)}(a, \mathbf{m}, \varepsilon_1, \varepsilon_2) = \frac{\prod_{i_0=1}^{n_0} \prod_{\pm} s_{\pm n - \xi_n}(m_{2+i_0} \mp a) \prod_{i_1=1}^{n_1} \prod_{\pm} s_{\pm n - \xi_n}(m_{i_1} \mp a)}{\prod_{\pm} s_{\pm 2n}(\mp 2a)} \quad (\text{A.55})$$

where:

$$s_n(x) = \begin{cases} \prod_{i+j \leq n-1}^{i, j \geq 0} (x - i\varepsilon_1 - j\varepsilon_2), & n > 0, \\ \prod_{i+j \leq -n-2}^{i, j \geq 0} (x + (i+1)\varepsilon_1 + (j+1)\varepsilon_2), & n < -1, \\ 1, & n = 0 \text{ or } n = -1. \end{cases} \quad (\text{A.56})$$

The ℓ -factors are then given by

$$\ell_n^{(n_0, n_1)}(a, \mathbf{m}) = L_n^{(n_0, n_1)}(a, \mathbf{m}, 1, 1) \frac{\hat{\mathcal{Z}}_{0, \text{1-loop}}^{(n_1)}(a + n, \mathbf{m} + \xi_n, 1)}{\hat{\mathcal{Z}}_{0, \text{1-loop}}^{(n_1)}(a, \mathbf{m}, 1)} \quad (\text{A.57})$$

which, again using the shift properties of the gamma function in (A.45), can be expressed as rational functions of the equivariant parameters as in equation (10.29).

The regular parts of the resummation ansatz appearing in (10.8), (10.16) and (10.24) for the $N_f = (2, 2)$ theory can be obtained by studying the large- a behavior of the corresponding instanton prepotentials, and read respectively

$$R_0(a, \mathbf{m}, \hbar; \mathbf{t}) = \frac{a^2}{\hbar} \left[\frac{\pi K(1 - \mathbf{t}^2)}{K(\mathbf{t}^2)} + \log\left(\frac{\mathbf{t}^2}{16}\right) \right] + \log\left(\frac{2K(\mathbf{t}^2)}{\pi}\right) \left[\sum_{i=1}^4 m_i + \frac{1}{\hbar} \sum_{i=1}^4 m_i^2 + \frac{\hbar}{2} \right] + \frac{1}{4\hbar} \log(1 - \mathbf{t}^2) \left(\sum_{i=1}^4 m_i + \hbar \right)^2, \quad (\text{A.58a})$$

$$R_1(a, \mathbf{m}, \hbar; \mathbf{t}) = \frac{1}{\hbar} \left(\sum_{i=1}^4 m_i + \hbar \right) \left[\log\left(\frac{2K(\mathbf{t}^2)}{\pi}\right) + \frac{1}{2} \log(1 - \mathbf{t}^2) \right], \quad (\text{A.58b})$$

$$\begin{aligned} \hat{R}_0(a, \mathbf{m}, \hbar; z, \mathbf{t}) &= \frac{a}{2\hbar} \left[\frac{\pi K(1 - \mathbf{t}^2)}{K(\mathbf{t}^2)} + \log\left(\frac{\mathbf{t}^2}{16}\right) + 2\hat{s}(z, \mathbf{t}) \right] \\ &\quad + \frac{2(m_1 + m_2) + \hbar}{4\hbar} \log(1 - \mathbf{t}z) \\ &\quad + \frac{2(m_3 + m_4) + \hbar}{4\hbar} \left[2 \log\left(\frac{2K(\mathbf{t}^2)}{\pi}\right) + \log\left(1 - \frac{\mathbf{t}}{z}\right) \right], \end{aligned} \quad (\text{A.58c})$$

where the non-trivial function $\hat{s}(z, \mathbf{t})$ can be determined from a large- a WKB analysis of the Heun equation (10.1) and is given by

$$\hat{s}(z, \mathbf{t}) = \int^z \left(\frac{1}{\zeta} - \frac{\pi}{2\zeta K(\mathbf{t}^2) \sqrt{(1 - \mathbf{t}\zeta)(1 - \mathbf{t}/\zeta)}} \right) d\zeta. \quad (\text{A.59})$$

The regular parts for the other cases of $N_f = (n_0, n_1)$ can be obtained from these by taking suitable decoupling limits, as described in Section 9.3. In the previous equations, $K(k^2)$ denotes the complete elliptic integral of the first kind.

A.4 $\text{SU}(N)$ root-system conventions for Chapter IV

Let us introduce some objects from the Lie algebra $\mathfrak{su}(N)$:

$$\begin{cases} \{\mathbf{b}_I\}_{I \in \{1, \dots, N\}} & \text{standard Euclidean orthonormal basis;} \\ \{\mathbf{e}_I\}_{I \in \{1, \dots, N\}} & \text{weights of the fundamental representation;} \\ \{\boldsymbol{\lambda}_k\}_{k \in \{1, \dots, N-1\}} & \text{fundamental weights, basis of the weight lattice;} \\ \{\boldsymbol{\alpha}_\ell\}_{\ell \in \{1, \dots, N-1\}} & \text{simple roots, dual to the fundamental weights;} \end{cases} \quad (\text{A.60})$$

These quantities are related to each other by:

$$\mathbf{e}_I = \mathbf{b}_I - \frac{1}{N} \sum_{J=1}^N \mathbf{b}_J, \quad \boldsymbol{\alpha}_k = \mathbf{e}_k - \mathbf{e}_{k+1} = \mathbf{b}_k - \mathbf{b}_{k+1}, \quad \boldsymbol{\lambda}_k = \sum_{I=1}^k \mathbf{e}_I, \quad (\text{A.61a})$$

$$\mathbf{b}_I \cdot \mathbf{b}_J = \delta_{I,J}, \quad \mathbf{e}_I \cdot \mathbf{e}_J = \delta_{I,J} - \frac{1}{N}, \quad (\text{A.61b})$$

$$\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_\ell = C_{k,\ell} = -\delta_{k,\ell-1} + 2\delta_{k,\ell} - \delta_{k,\ell+1}, \quad \boldsymbol{\lambda}_k \cdot \boldsymbol{\alpha}_\ell = \delta_{k,\ell}, \quad (\text{A.61c})$$

where $C_{k,\ell}$ is the $SU(N)$ Cartan matrix. The set of positive roots is given by:

$$\Delta^+ = \left\{ \alpha_{k,\ell} = e_k - e_\ell \mid 1 \leq k < \ell \leq N \right\}, \quad (\text{A.62})$$

while the root lattice is denoted by:

$$Q_{N-1} = \left\{ \sum_{\ell=1}^{N-1} m_\ell \alpha_\ell \mid m_\ell \in \mathbb{Z} \right\} = \left\{ \sum_{I=1}^N m_I e_I \mid m_I \in \mathbb{Z} \wedge \sum_{I=1}^N m_I = 0 \right\}. \quad (\text{A.63})$$

In our construction of the eigenfunctions, we also need

$$\gamma = \frac{1}{2} \sum_{I=1}^N (-1)^{I-1} e_I = \sum_{k=1}^{N-1} (-1)^{k-1} \lambda_k = \frac{1}{2} \sum_{\ell=1}^{N-1} \left(\mathbb{1}_{2N+1}(\ell) - \frac{\ell}{N} \mathbb{1}_{2N+1}(N) \right) \alpha_\ell. \quad (\text{A.64})$$

A.5 Gauge-theory special functions for Chapter IV

The Nekrasov–Shatashvili (NS) functions capture physical information about supersymmetric gauge theories in the NS limit of the gravitational Ω -background, with the case relevant here being four-dimensional $\mathcal{N} = 2$ $SU(N)$ SYM. It is convenient to organize the Coulomb branch parameters a_I into the vector⁵

$$\mathbf{a} = \sum_{I=1}^N a_I e_I, \quad \sum_{I=1}^N a_I = 0, \quad (\text{A.65})$$

where e_I are the weights of the fundamental representation of $SU(N)$ (see Section A.4 for our conventions).

Note on convention: when dealing with Nekrasov functions, the term “instanton partition function” refers to the instantons of the supersymmetric gauge theory that appear in localization computations [1, 80, 245, 247, 248]. These should not be confused with quantum-mechanical instantons.

A.5.1 The NS free energy

Let $\mathbf{Y} = (Y_1, \dots, Y_N)$ denote an N -tuple of Young diagrams, each one being equivalently specified by an ordered partition $Y = (y_1, y_2, \dots)$, $y_1 \geq y_2 \geq \dots \geq 0$. The total size of the N -tuple is given by

$$\ell(\mathbf{Y}) = \sum_{I=1}^N |Y_I|, \quad |Y| = \sum_{i \geq 1} y_i. \quad (\text{A.66})$$

For a box $s = (i, j) \in Y$, we define the arm and leg lengths as

$$A_Y(s) = y_i - j, \quad L_Y(s) = y_j^T - i, \quad (\text{A.67})$$

where $Y^T = (y_1^T, y_2^T, \dots)$ denotes the transposed partition. For two Young diagrams Y and W , we introduce the building block

$$N_{Y,W}(z; \varepsilon_1, \varepsilon_2) = \prod_{s \in Y} (z - \varepsilon_1 L_W(s) + \varepsilon_2 (A_Y(s) + 1)) \prod_{t \in W} (z + \varepsilon_1 (L_Y(t) + 1) - \varepsilon_2 A_W(t)). \quad (\text{A.68})$$

⁵The a_I parameters here are denoted by α_I in [44].

The 4d instanton Nekrasov partition function for this gauge theory is given by [1, 80, 245, 247, 248]

$$Z^{\text{inst}}(\mathbf{a}, \Lambda, \varepsilon_1, \varepsilon_2) = \sum_{\mathbf{Y}} ((-1)^N \Lambda^{2N})^{\ell(\mathbf{Y})} \mathcal{Z}_{\mathbf{Y}}(\mathbf{a}, \varepsilon_1, \varepsilon_2), \quad (\text{A.69})$$

with

$$\mathcal{Z}_{\mathbf{Y}}(\mathbf{a}, \varepsilon_1, \varepsilon_2) = \prod_{I,J=1}^N \frac{1}{N_{Y_I, Y_J}(a_I - a_J; \varepsilon_1, \varepsilon_2)}. \quad (\text{A.70})$$

The instanton part of the NS free energy is then defined as

$$F_{\text{NS}}^{\text{inst}}(\mathbf{a}, \Lambda, \hbar) = i\hbar \lim_{\varepsilon_2 \rightarrow 0} \varepsilon_2 \log Z^{\text{inst}}(\mathbf{a}, \Lambda, i\hbar, \varepsilon_2). \quad (\text{A.71})$$

Note that the functions (A.69) and (A.71) are expressed as series in Λ . These series are, however, convergent; see, e.g. [131, 249, 250].

A.5.2 The NS partition function in presence of a surface defect

In the presence of a type-II chiral defect, the partition function was computed in [95, 122] and is nicely summarized in [180, app. A]. By evaluating the residues in these expressions explicitly, the partition function can be written as

$$Z_{2\text{d}/4\text{d}}^{(c),\text{inst}}(x, \mathbf{a}, \Lambda, \varepsilon_1, \varepsilon_2) = \sum_{\mathbf{Y}} ((-1)^N \Lambda^{2N})^{\ell(\mathbf{Y})} \mathcal{Z}_{\mathbf{Y}}(\mathbf{a}, \varepsilon_1, \varepsilon_2) \mathcal{Z}_{\mathbf{Y}}^{(c)}(x, \mathbf{a}, \varepsilon_1, \varepsilon_2), \quad (\text{A.72})$$

where the defect factor is

$$\mathcal{Z}_{\mathbf{Y}}^{(c)}(x, \mathbf{a}, \varepsilon_1, \varepsilon_2) = \prod_{I=1}^N \prod_{(i,j) \in Y_I} \frac{a_I - x + i\varepsilon_1 + j\varepsilon_2}{a_I - x + i\varepsilon_1 + (j-1)\varepsilon_2}. \quad (\text{A.73})$$

Here x denotes the two-dimensional twisted mass parameter. We then define the full instanton defect partition function in the NS limit as

$$Z_{\text{D}}^{\text{inst}}(x, \mathbf{a}, \Lambda, \hbar) = \lim_{\varepsilon_2 \rightarrow 0} \frac{Z_{2\text{d}/4\text{d}}^{(c),\text{inst}}(x, \mathbf{a}, \Lambda, i\hbar, \varepsilon_2)}{Z^{\text{inst}}(\mathbf{a}, \Lambda, i\hbar, \varepsilon_2)}. \quad (\text{A.74})$$

A.5.3 The Wilson loops

Another ingredient we need is the four-dimensional limit of Wilson-loop expectation values [95, 122]; here we closely follow [44]. As before, let $\mathbf{Y} = (Y_1, \dots, Y_N)$ denote an N -tuple of Young diagrams. We define the equivariant Chern character as

$$\text{Ch}_{\mathbf{Y}}(\mathbf{a}, \varepsilon_1, \varepsilon_2) = \mathcal{W} - (1 - e^{R\varepsilon_1})(1 - e^{R\varepsilon_2})\mathcal{V}_{\mathbf{Y}}, \quad (\text{A.75})$$

$$\mathcal{W} = \sum_{I=1}^N e^{Ra_I}, \quad \mathcal{V}_{\mathbf{Y}} = \sum_{I=1}^N e^{Ra_I} \sum_{(k,l) \in Y_I} e^{(k-1)R\varepsilon_1 + (l-1)R\varepsilon_2}. \quad (\text{A.76})$$

In the $R \rightarrow 0$ limit, the Chern character behaves as

$$\text{Ch}_{\mathbf{Y}}(\mathbf{a}, \varepsilon_1, \varepsilon_2) = N + \sum_{k \geq 2} \mathcal{C}_{\mathbf{Y}}^{(k)}(\mathbf{a}, \varepsilon_1, \varepsilon_2) R^k, \quad (\text{A.77})$$

where, for example:

$$\begin{aligned}\mathcal{C}_Y^{(2)}(\mathbf{a}, \varepsilon_1, \varepsilon_2) &= \frac{1}{2} \sum_{I=1}^N a_I^2 - \varepsilon_1 \varepsilon_2 \ell(\mathbf{Y}), \\ \mathcal{C}_Y^{(3)}(\mathbf{a}, \varepsilon_1, \varepsilon_2) &= \frac{1}{6} \sum_{I=1}^N a_I^3 - \varepsilon_1 \varepsilon_2 \left(\frac{\varepsilon_1 + \varepsilon_2}{2} \ell(\mathbf{Y}) + \varepsilon_1 \sum_{I=1}^N c_2(Y_I^T) \right. \\ &\quad \left. + \varepsilon_2 \sum_{I=1}^N c_2(Y_I) + \sum_{I=1}^N a_I |Y_I| \right),\end{aligned}\tag{A.78}$$

while

$$c_2(Y) = \frac{1}{2} \sum_{i \geq 1} y_i (y_i - 1).\tag{A.79}$$

We define the “4d Wilson loops” as

$$W^{(k)}(\mathbf{a}, \Lambda, \varepsilon_1, \varepsilon_2) = \frac{1}{Z_{\text{inst}}} \sum_{\mathbf{Y}} ((-1)^N \Lambda^{2N})^{\ell(\mathbf{Y})} \mathcal{C}_Y^{(k)}(\mathbf{a}, \varepsilon_1, \varepsilon_2) \mathcal{Z}_Y(\mathbf{a}, \varepsilon_1, \varepsilon_2).\tag{A.80}$$

Here we are interested in their NS limit, which we denote by

$$W_{\text{NS}}^{(k)}(\mathbf{a}, \Lambda, \hbar) = \lim_{\varepsilon_2 \rightarrow 0} W^{(k)}(\mathbf{a}, \Lambda, i\hbar, \varepsilon_2), \quad k = 2, \dots, N.\tag{A.81}$$

These observables are directly related to the complex moduli parameters h_k appearing in the mirror curve, and they generalise the Matone relation to higher rank gauge theories. More precisely, we have

$$h_j = \sum_{\mathbf{k}} \frac{(-1)^{|\mathbf{k}| - \ell(\mathbf{k})}}{z_{\mathbf{k}}} W_{\text{NS}}^{\mathbf{k}}(\mathbf{a}, \Lambda, \hbar), \quad j \in \{2, \dots, N\},\tag{A.82}$$

where $\mathbf{k} = (0, k_2, \dots)$ satisfies

$$\ell(\mathbf{k}) = \sum_{m \geq 2} m k_m = j,\tag{A.83}$$

and we used

$$W_{\text{NS}}^{\mathbf{k}}(\mathbf{a}, \Lambda, \hbar) = \prod_{j \geq 1} (j! W_{\text{NS}}^{(j)}(\mathbf{a}, \Lambda, \hbar))^{k_j}, \quad z_{\mathbf{k}} = \prod_{j \geq 1} k_j! j^{k_j}, \quad |\mathbf{k}| = \sum_m k_m.\tag{A.84}$$

Hence, for instance:⁶

$$\begin{aligned}h_2 &= -W_{\text{NS}}^{(2)}, \\ h_3 &= 2W_{\text{NS}}^{(3)}, \\ h_4 &= \frac{1}{2} (W_{\text{NS}}^{(2)})^2 - 6W_{\text{NS}}^{(4)}, \\ h_5 &= -2W_{\text{NS}}^{(2)}W_{\text{NS}}^{(3)} + 24W_{\text{NS}}^{(5)}, \\ h_6 &= -\frac{1}{6} (W_{\text{NS}}^{(2)})^3 + 6W_{\text{NS}}^{(2)}W_{\text{NS}}^{(4)} + 2(W_{\text{NS}}^{(3)})^2 - 120W_{\text{NS}}^{(6)}.\end{aligned}\tag{A.85}$$

One can further check that

$$W_{\text{NS}}^{(2)} = -\frac{1}{2N} \Lambda \partial_{\Lambda} F_{\text{NS}},\tag{A.86}$$

giving rise to the usual quantum Matone relation [39, 40].

⁶We omit the dependence on $(\mathbf{a}, \Lambda, \hbar)$ in $W_{\text{NS}}^{(2)}$ for brevity.

Appendix B

Checks for Chapter II

B.1 Comparison with Hill determinant method

In this section, we describe how the expansions of the eigenvalues and eigenfunctions of the Lamé equation can be computed using standard techniques. The first method we present concerns the computation of eigenvalues and is known as the Hill determinant method [47]. Consider the following complex Hill equation:

$$-\partial_x^2 \psi(x) + V(x)\psi(x) = 0, \quad (\text{B.1})$$

with $V(x)$ an even, periodic potential of period π . Its Fourier expansion reads:

$$V(x) = a_0 + \sum_{k \neq 0} a_k e^{2ikx}, \quad (\text{B.2})$$

where $a_{-k} = a_k$, due to the parity of the potential, $a_k \in \mathbb{C}$ for any $k \in \mathbb{Z}$, and a_0 can play the role of eigenvalue. We search for a Floquet solution $\psi(x)$, meaning a quasi-periodic function such that $\psi(x + \pi) = e^{i\nu\pi}\psi(x)$, where $\nu \in \mathbb{C}$ is the Floquet exponent. We can then write $\psi(x)$ in the form $\psi(x) = e^{i\nu x}\phi(x)$, where $\phi(x)$ is a periodic function of period π , and expand $\phi(x)$ in Fourier series, obtaining:

$$\psi(x) = e^{i\nu x} \sum_{n=-\infty}^{\infty} b_n e^{2inx}, \quad (\text{B.3})$$

where $b_n \in \mathbb{C}$ for any $n \in \mathbb{Z}$. Inserting this expansion into equation (B.1), we find:

$$\sum_{n=-\infty}^{\infty} b_n (\nu + 2n)^2 e^{i(\nu+2n)x} + \sum_{k=-\infty}^{\infty} a_k e^{2ikx} \sum_{n=-\infty}^{\infty} b_n e^{i(\nu+2n)x} = 0, \quad (\text{B.4})$$

and thus an infinite number of linear equations of the form:

$$(\nu + 2n)^2 b_n + \sum_{k=-\infty}^{\infty} a_k b_{n-k} = 0, \quad \forall n \in \mathbb{Z}. \quad (\text{B.5})$$

This system can be reformulated as the following matrix equation:

$$\sum_{m \in \mathbb{Z}} B_{nm} b_m = 0, \quad B_{nn} = a_0 + (\nu + 2n)^2, \quad B_{nm} = a_{n-m}, \quad m \neq n. \quad (\text{B.6})$$

We can then truncate the matrix to $|n|, |m| \leq N$ and impose $\det B^{(N)} = 0$. Increasing N allows us to determine the perturbative coefficients of the eigenvalues of the periodic

spectral problem in (B.1). In the case of equation (8.1), the relevant Fourier expansion of the potential is given by:

$$\wp(x + \pi\tau/2; \pi, \pi\tau) = -2\frac{\eta_1}{\pi^2} - 4 \sum_{k \geq 1} 2k \cos 2kx \frac{q^{k/2}}{1 - q^k}, \quad (\text{B.7})$$

which leads to the identification:

$$a_0 = -\mathfrak{e} - 2\mu(\mu - 1)\frac{\eta_1}{\pi^2}, \quad a_{\pm k} = -4\mu(\mu - 1)k \frac{q^{k/2}}{1 - q^k}, \quad k \geq 1.$$

We can then solve the condition $\det B^{(N)} = 0$ as an equation for $\mathfrak{e} = \nu^2 - \frac{1}{3}\mu(\mu - 1) + \dots$, obtaining a power series expansion in $q^{1/2}$. In particular, solving the equation for generic values of ν we recover the expansion (5.33). Meanwhile, fixing ν to a specific integer value $|\nu| \geq 1$ from the outset yields the expansion for the corresponding periodic or anti-periodic eigenvalue. The two distinct eigenvalues corresponding to the same value of $|\nu|$ arise from the two distinct solutions of a quadratic equation satisfied by the first non-trivial coefficient in the expansion. An equivalent approach for computing the eigenvalues is provided by a continued fraction expansion, which is also available in the literature [251]. Alternatively, one can directly solve equation (8.1) as a power series in $q^{1/2}$ by expanding both the eigenfunction and the eigenvalue accordingly. Indeed, considering the Fourier expansion (B.7), the equation takes the form:

$$\left(\partial_x^2 + 4\mu(\mu - 1) \sum_{k \geq 1} 2k \cos 2kx \frac{q^{k/2}}{1 - q^k} + \mathfrak{e} + 2\mu(\mu - 1)\frac{\eta_1}{\pi^2} \right) \psi = 0, \quad (\text{B.8})$$

where the eigenvalue is expanded as:

$$\mathfrak{e}(\nu, \mu; q) = \sum_{n \geq 0} e_n(\nu, \mu) q^{n/2}. \quad (\text{B.9})$$

To proceed, we must specify the form of the eigenfunction $\psi(x)$, distinguishing between the case of non-integer and integer ν , which correspond respectively to quasi-periodic and (anti-)periodic solutions. In the first case, we can write:

$$\psi_{\pm}(x, \nu, \mu; q) = e^{\pm i\nu x} \sum_{n \geq 0} \psi_n(x, \pm\nu, \mu) q^{n/2}, \quad (\text{B.10})$$

where the coefficients $\psi_n(x, \nu, \mu)$ are periodic functions of period π , more specifically Laurent polynomials in e^{2ix} . Equivalently, we can consider the even and odd combinations:

$$\psi_{\text{even}}(x, \nu, \mu; q) = \psi_+(x, \nu, \mu; q) + \psi_-(x, \nu, \mu; q), \quad (\text{B.11a})$$

$$\psi_{\text{odd}}(x, \nu, \mu; q) = \psi_+(x, \nu, \mu; q) - \psi_-(x, \nu, \mu; q). \quad (\text{B.11b})$$

Meanwhile, for a fixed integer value of ν , $|\nu| \geq 1$, we can write:

$$\psi_{\text{even}}^{(\nu)}(x, \mu; q) \propto \cos(|\nu|x) + \sum_{n \geq 1} \psi_{\text{even}, n}^{(\nu)}(x, \mu) q^{n/2}, \quad (\text{B.12a})$$

$$\psi_{\text{odd}}^{(\nu)}(x, \mu; q) \propto \sin(|\nu|x) + \sum_{n \geq 1} \psi_{\text{odd}, n}^{(\nu)}(x, \mu) q^{n/2}, \quad (\text{B.12b})$$

where each coefficient $\psi_{\text{even},n}^{(\nu)}$ and $\psi_{\text{odd},n}^{(\nu)}$ is a linear combination of elementary trigonometric functions. For example, the solution associated with a vanishing Floquet exponent $\nu = 0$ takes the form:

$$\psi^{(0)}(x) \propto 1 + 2\mu(\mu - 1) \cos 2x \sqrt{q} + \frac{1}{2}\mu(-2 + 3\mu - 2\mu^2 + \mu^3) \cos 4x q + O(q^{3/2}), \quad (\text{B.13})$$

up to a choice of initial condition.

B.2 Consistency check at $\tau \rightarrow +i\infty$

In this section we perform a consistency check of the quantization condition (8.46) in the $\tau \rightarrow +i\infty$ limit, where the problem reduces to the analysis of the hyperbolic Pöschl–Teller potential [252]. Indeed, consider the Lamé equation:

$$-\partial_x^2 \psi + \mu(\mu - 1) \wp(x; 1, \tau) \psi = E \psi, \quad (\text{B.14})$$

Using the transformation properties of the Weierstrass $\wp$ -function, we can write:

$$\wp(x; 1, \tau) = t^{-2} \wp(x/t; 1/t, i), \quad (\text{B.15})$$

where $t = \text{Im}(\tau)$, so that, performing the change of variable $z = x/t$, the equation becomes:

$$-\partial_z^2 \psi + \mu(\mu - 1) \wp(z; 1/t, i) \psi = t^2 E \psi. \quad (\text{B.16})$$

The Weierstrass $\wp$ -function can be represented as:

$$\wp(z; 1/t, i) = t^2 \left(\frac{\pi^2}{\sin^2 \pi z t} - \frac{\pi^2}{3} + \sum_{k \neq 0} \frac{\pi^2}{\sin^2 \pi t(z - ki)} - \frac{\pi^2}{\sin^2 \pi t ki} \right), \quad (\text{B.17})$$

and in the $t \rightarrow +\infty$ limit we can approximate it by:

$$\wp(z; 1/t, i) \simeq t^2 \left(\frac{\pi^2}{\sin^2 \pi z t} - \frac{\pi^2}{3} \right). \quad (\text{B.18})$$

Setting now $z = iv + 1/2t$, so as to lie along the vertical line at the center of the torus, we find:

$$\left[-\partial_v^2 - \mu(\mu - 1) \frac{t^2 \pi^2}{\cosh^2 \pi t v} + t^2 \left(E + \frac{\pi^2}{3} \mu(\mu - 1) \right) \right] \psi = 0. \quad (\text{B.19})$$

Up to exponentially small corrections, we can approximate the potential on the B-cycle as an infinite sum of such localized contributions centered at $v \in \mathbb{Z}$, with the previous equation serving as the effective description in the vicinity of $v = 0$. To solve it, we make the substitutions:

$$y = \cosh^2 \pi t v, \quad E = -\kappa^2 \pi^2 - \frac{\pi^2}{3} \mu(\mu - 1), \quad (\text{B.20})$$

obtaining the hypergeometric equation:

$$y(1 - y) \psi'' + \left(\frac{1}{2} - y \right) \psi' - \frac{1}{4} \left(\frac{\mu(\mu - 1)}{y} + \kappa^2 \right) \psi = 0.$$

Setting then $\psi = y^{\mu/2} \phi$, we find:

$$y(1 - y) \phi'' + \left(\frac{1}{2} + \mu - y(1 + \mu) \right) \phi' - \frac{1}{4} (\kappa^2 + \mu^2) \phi = 0,$$

which we can easily solve near $y = 1$, or $v = 0$, in terms of hypergeometric functions. Expressing the solutions in terms of the variable v , we find:

$$\psi_{\text{even}}(v) = \cosh^\mu \pi t v {}_2F_1 \left(\frac{\mu + i\kappa}{2}, \frac{\mu - i\kappa}{2}, \frac{1}{2}; -\sinh^2 \pi t v \right), \quad (\text{B.21a})$$

$$\psi_{\text{odd}}(v) = \sinh \pi t v \cosh^\mu \pi t v {}_2F_1 \left(\frac{1 + \mu + i\kappa}{2}, \frac{1 + \mu - i\kappa}{2}, \frac{3}{2}; -\sinh^2 \pi t v \right), \quad (\text{B.21b})$$

distinguishing the two linearly independent solutions by parity. Then, in the vicinity of the other integer v points, we will have similar local solutions. For example, near $v = 1$, we would find: $\tilde{\psi}_{\text{even/odd}}(v) = \psi_{\text{even/odd}}(1 - v)$. To obtain the desired quantization condition, we must impose the periodicity of the solutions, which can be achieved by matching their asymptotic behaviors. Indeed, assuming $t \gg 1$, for $0 < v < 1$, we have:

$$\begin{aligned} \psi_{\text{even}}(v) &\simeq \frac{1}{2} \frac{\Gamma\left(\frac{i\kappa}{2}\right) \Gamma\left(\frac{1}{2} + \frac{i\kappa}{2}\right)}{\Gamma\left(\frac{i\kappa}{2} + \frac{\mu}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\mu}{2} + \frac{i\kappa}{2}\right)} e^{i\kappa\pi t v} + \frac{1}{2} \frac{\Gamma\left(-\frac{i\kappa}{2}\right) \Gamma\left(\frac{1}{2} - \frac{i\kappa}{2}\right)}{\Gamma\left(-\frac{i\kappa}{2} + \frac{\mu}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\mu}{2} - \frac{i\kappa}{2}\right)} e^{-i\kappa\pi t v} \\ &= \Omega_{\text{even}} e^{i\kappa\pi t v} + \tilde{\Omega}_{\text{even}} e^{-i\kappa\pi t v}, \end{aligned} \quad (\text{B.22})$$

and imposing (anti-)periodicity, namely $\psi_{\text{even}}(v) = \pm \psi_{\text{even}}(v + 1)$, we obtain:

$$e^{i\kappa\pi t} \frac{\Omega_{\text{even}}}{\tilde{\Omega}_{\text{even}}} = \pm 1. \quad (\text{B.23})$$

Similarly, for the odd solution we have:

$$\begin{aligned} \psi_{\text{odd}}(v) &\simeq \frac{1}{4} \frac{\Gamma\left(\frac{i\kappa}{2}\right) \Gamma\left(\frac{1}{2} + \frac{i\kappa}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\kappa}{2} + \frac{\mu}{2}\right) \Gamma\left(1 - \frac{\mu}{2} + \frac{i\kappa}{2}\right)} e^{i\kappa\pi t v} + \frac{1}{4} \frac{\Gamma\left(-\frac{i\kappa}{2}\right) \Gamma\left(\frac{1}{2} - \frac{i\kappa}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{i\kappa}{2} + \frac{\mu}{2}\right) \Gamma\left(1 - \frac{\mu}{2} - \frac{i\kappa}{2}\right)} e^{-i\kappa\pi t v} \\ &= \Omega_{\text{odd}} e^{i\kappa\pi t v} + \tilde{\Omega}_{\text{odd}} e^{-i\kappa\pi t v}, \end{aligned} \quad (\text{B.24})$$

and imposing $\psi_{\text{odd}}(v) = \pm \psi_{\text{odd}}(v + 1)$, we find the quantization condition:

$$e^{i\kappa\pi t} \frac{\Omega_{\text{odd}}}{\tilde{\Omega}_{\text{odd}}} = \mp 1. \quad (\text{B.25})$$

By solving equation (8.46) for $e^{\eta/2}$ in the case of periodic or anti-periodic solutions, corresponding to $\cos(\pi\nu_D) = \pm 1$, we obtain:

$$e^{\eta/2} = \mp \frac{\sin 2\pi a - \sin \pi \mu}{\sin \pi(2a - \mu)}, \quad (\text{B.26a})$$

$$e^{\eta/2} = \mp \frac{\sin 2\pi a + \sin \pi \mu}{\sin \pi(2a - \mu)}, \quad (\text{B.26b})$$

where

$$\eta = 4\pi a t + 2 \log \frac{\Gamma(\mu - 2a) \Gamma(1 + 2a)}{\Gamma(\mu + 2a) \Gamma(1 - 2a)}, \quad (\text{B.27})$$

neglecting corrections that are exponentially suppressed in the limit $t \rightarrow +\infty$. Upon identifying $\kappa = 2a/i$, we find that the quantization condition (B.23) matches (B.26a), while the quantization condition (B.25) corresponds to (B.26b).

Appendix C

Blow-up equations

C.1 Useful formulae for $N_f > 0$

In this section, we collect some lengthy formulas relevant to the $\mathbb{C}^2$ and $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations for generic $N_f \leq 4$. First, the regular parts of the resummation ansatz for $N_f = 4$ are given by [253]:

$$R_0^{[4]}(a, \boldsymbol{\mu}, \hbar; t) = \frac{a^2}{\hbar} \left(\frac{\pi K(1-t)}{K(t)} + \log \left(\frac{t}{16} \right) \right) + \frac{1}{4\hbar} \left(\sum_{i=1}^4 \mu_i \right)^2 \log(1-t) \\ + \frac{1}{\hbar} \left(\sum_{i=1}^4 \mu_i^2 \right) \log \left(\frac{2}{\pi} K(t) \right) + \frac{1}{2} \left(\sum_{i=1}^4 \mu_i \right) \log(1-t) + \frac{\hbar}{4} \left(\log(1-t) - 2 \log \left(\frac{2K(t)}{\pi} \right) \right) \quad (\text{C.1})$$

$$R_1^{[4]}(\boldsymbol{\mu}, \hbar; t) = \frac{1}{2} \left(\log(1-t) - 2 \log \left(\frac{2K(t)}{\pi} \right) \right) + \frac{1}{2\hbar} \left(\sum_{i=1}^4 \mu_i \right) \log(1-t). \quad (\text{C.2})$$

From these expressions, it is possible to derive all other regular parts for $N_f \leq 3$ by decoupling the appropriate number of hypermultiplets. The function $R_1^{[4]}(\boldsymbol{\mu}, \hbar; t)$ is determined in terms of $R_0^{[4]}(\boldsymbol{\mu}, \hbar; t)$, for example by taking the large a limit of the blow-up equation (7.4). Indeed, in this limit, the equation reduces to:

$$t \partial_t \left(\mathcal{W}_{1, \text{inst}}(a, \boldsymbol{\mu}, \hbar; t) - \left(\partial_\hbar + \frac{1}{\hbar} \right) \mathcal{W}_{0, \text{inst}}(a, \boldsymbol{\mu}, \hbar; t) \right) \simeq 0 \quad (\text{C.3})$$

thus fixing $R_1^{[4]}$ up to a t -independent term, which can then be seen to vanish. Then, writing:

$$\tilde{\mathcal{G}}_{\text{exp/inst}}^{\mp, [N_f]} = \tilde{\mathcal{G}}_{\text{exp/inst}}^{[N_f]} \left(\frac{m_1 x + t^{m_2/2}}{2x}, \{\mu_i\}, \mp j - m_1, t \right)$$

the expressions necessary to determine the differential equations satisfied by the resummation functions read:

$$\log \tilde{\mathcal{G}}_{\text{exp}}^{\mp, [N_f]} = 2 \sum_{k \geq 1} t^{k-1} \left\{ \tilde{f}_{k,j} \mp m_1 y^2 \partial_y g_{k,j} \mp t^{j/2} \left([1 - (4 - N_f)(k + j/2 - 1)] g_{k,j} \right. \right. \\ \left. \left. + [1 - (4 - N_f)j/2] y \partial_y g_{k,j} - \tilde{g}_{k,j} - \sum_{i=1}^{N_f} \mu_i \partial_{\mu_i} g_{k,j} \right) \right\} + \sigma_{\text{exp}}^{\mp, [N_f]}(x, \{\mu_i\}, t), \quad (\text{C.4})$$

$$\begin{aligned}
 \tilde{\mathcal{G}}_{\text{inst}}^{\mp, [N_f]} &= \frac{j}{2}(j \pm m_1) + \sum_{k \geq 1} t^{k-1} \left\{ (k-1) \tilde{f}_{k,j} + j \frac{y}{2} \partial_y \tilde{f}_{k,j} \right. \\
 &\mp m_1 y^2 \left[(j+k-1) \partial_y g_{k,j} + j \frac{y}{2} \partial_y^2 g_{k,j} \right] \pm t^{j/2} [(k+j/2-1) \tilde{g}_{k,j} \\
 &+ j \frac{y}{2} \partial_y \tilde{g}_{k,j} - [2 - (4 - N_f)(k+j/2-1)](k+j/2-1) g_{k,j} \\
 &- [k+2j-1-j(4-N_f)(k+3j/4-1)] y \partial_y g_{k,j} - j(1 - \frac{j}{2}(4-N_f)) \frac{y^2}{2} \partial_y^2 g_{k,j} \\
 &\left. + \sum_{i=1}^{N_f} \mu_i \left((k+j/2-1) \partial_{\mu_i} g_{k,j} + j \frac{y}{2} \partial_{\mu_i} \partial_y g_{k,j} \right) \right\} + \sigma_{\text{inst}}^{\mp, [N_f]}(x, \{\mu_i\}, t),
 \end{aligned} \tag{C.5}$$

where $y = xt^{(j-m_2)/2}$. Assuming the asymptotic behavior of the resummation functions as in Section 6.4, we find again that the expansions of the blow-up equations under consideration are proper series expansions. Finally, if $\varsigma_{m_2} = \frac{1-(-1)^{m_2}}{2}$ and we define:

$$\tilde{c}_{m_1, m_2}(\mu) = \mu^{m_1 \varsigma_{m_2}} \prod_{j=\frac{1}{2}+\frac{\varsigma_{m_2}}{2}}^{\frac{m_2-1}{2}} (\mu-j)^{m_1} (\mu+j)^{m_1}, \tag{C.6}$$

we have:

$$d_{m_1, m_2}(\{\mu_i\}) = -\frac{\prod_{i=1}^{N_f} \tilde{c}_{m_1, m_2}^2(\mu_i)}{m_2^{4m_1} [(m_2-1)!]^{8m_1}}. \tag{C.7}$$

C.2 Example: $\mathbb{C}^2$ blow-up equations, $\mathcal{N} = 2^*$ theory

In this section, we present and solve the first few equations derived from the expansion of the $\mathbb{C}^2$ blow-up equation (6.33). We focus on this case because the presence of both integer and half-integer shifts allows us to work with lower values of m_1 , thereby reducing the complexity introduced by the Taylor coefficients of the resummation functions appearing in the equations. Specifically, we analyze the equations for $m_2 = 1$ and $m_1 = 1, 2$. The first two of these read, respectively:

$$e^{2\tilde{f}_{1,1}(\mu, x) + x^2 \partial_x g_{1,1}(\mu, x)} + \mu^2 (\mu-1)^2 x^2 e^{2\tilde{f}_{1,1}(\mu, x) - x^2 \partial_x g_{1,1}(\mu, x)} - 1 = 0, \tag{C.8a}$$

$$e^{2\tilde{f}_{1,1}(\mu, x) + 2x^2 \partial_x g_{1,1}(\mu, x)} - \mu^4 (\mu-1)^4 x^4 e^{2\tilde{f}_{1,1}(\mu, x) - 2x^2 \partial_x g_{1,1}(\mu, x)} - 1 = 0. \tag{C.8b}$$

Performing the substitutions $\tilde{f}_{1,1} = \frac{1}{2} \log u_1$ and $g'_{1,1} = \frac{1}{x^2} \log(xv_1)$, and solving for u_1 , we obtain:

$$u_1(\mu, x) = \frac{1}{x(v_1(\mu, x) + \mu^2(-1+\mu)^2 v_1(\mu, x)^{-1})}, \tag{C.9a}$$

$$u_1(\mu, x) = \frac{1}{x^2(v_1(\mu, x)^2 - \mu^4(-1+\mu)^4 v_1(\mu, x)^{-2})} \tag{C.9b}$$

so that, by equating the two expressions for u_1 , we find:

$$v_1(\mu, x) = \frac{1 \pm \sqrt{1 + 4\mu^2(\mu-1)^2 x^2}}{2x}, \tag{C.10}$$

where only the choice of the $+$ sign in front of the square root is compatible with the boundary condition $g_{1,1}(\mu, 0) = 0$. Thus, we find the following expressions for the first two resummation functions:

$$\tilde{f}_{1,1}(\mu, x) = -\frac{1}{4} \log(1 + 4\mu^2(\mu-1)^2 x^2) \tag{C.11a}$$

$$g_{1,1}(\mu, x) = \frac{\sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} - \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\mu^2(\mu - 1)^2 x^2}\right) - 1}{x}. \quad (\text{C.11b})$$

Using the explicit forms of $\tilde{f}_{1,1}$ and $g_{1,1}$, and after some simplification, the second equation for $m_1 = 1$ takes the form:

$$\left(1 + \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2}\right) \tilde{g}_{1,1}(\mu, x) + 2\mu(\mu^3 - 2\mu + 1)x = 0 \quad (\text{C.12})$$

which can easily be solved to obtain:

$$\tilde{g}_{1,1}(\mu, x) = \frac{(-1 + \mu + \mu^2) \left(1 - \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2}\right)}{2\mu(\mu - 1)x}. \quad (\text{C.13})$$

Then, the second equation for $m_1 = 2$ is identically satisfied. Moving on to the third equation for $m_1 = 1$, and expanding it to the first few orders around $x = 0$, one obtains the following relation:

$$\begin{aligned} \partial_x^2 \tilde{f}_{1,2}(\mu, 0) &= \frac{1}{4(\mu + 1)} \left[-\mu^2(\mu - 1)^2(-20 + 56\mu - 27\mu^2 - 45\mu^3 + 19\mu^4 + \mu^5) \right. \\ &\quad \left. + 8\partial_x^2 \tilde{f}_{2,1}(\mu, 0) - 4(\mu + 1)\partial_x g_{1,2}(\mu, 0) + 8\partial_x g_{2,1}(\mu, 0) \right] \end{aligned} \quad (\text{C.14})$$

which simplifies the equation and allows it to be easily solved for $\tilde{f}_{2,1}$, yielding:

$$\tilde{f}_{2,1}(\mu, x) = -\frac{x^2}{2\sqrt{1 + 4\mu^2(\mu - 1)^2 x^2}} \partial_x g_{2,1}(\mu, x) + \dots \quad (\text{C.15})$$

where the remaining terms form a lengthy expression in x and μ , involving only $\partial_x^2 \tilde{f}_{2,1}(\mu, 0)$ and $\partial_x g_{2,1}(\mu, 0)$. Similarly, expanding the third equation for $m_1 = 2$ around $x = 0$, we find the following relation:

$$\begin{aligned} \partial_x g_{1,2}(\mu, 0) &= \frac{1}{4(\mu + 1)} \left[(\mu - 1)^2 (2\mu^6 - 5\mu^5 + 17\mu^4 - 31\mu^3 - 29\mu^2 + 50\mu - 16) \mu^2 \right. \\ &\quad \left. + (-\mu^2 + \mu - 6) \partial_x^2 \tilde{f}_{2,1}(\mu, 0) - 2(\mu^2 - \mu + 2) \partial_x g_{2,1}(\mu, 0) \right] \end{aligned} \quad (\text{C.16})$$

which can be used to simplify it. Substituting the expression of $\tilde{f}_{2,1}$ as a function of $g_{2,1}$, one can easily solve the equation to find:

$$\begin{aligned} \partial_x g_{2,1}(\mu, x) &= \frac{1}{2} \left[\mu^4(\mu - 1)^4 (5\mu^4 - 10\mu^3 - 7\mu^2 + 12\mu - 6) x^2 + 2\partial_x g_{2,1}(\mu, 0) \right. \\ &\quad \left. + x^2 (6\mu^2(\mu - 1)^2 \partial_x g_{2,1}(\mu, 0)) + \left(2\partial_x g_{2,1}(\mu, 0) + 2(\mu - 1)^2 \mu^2 \partial_x g_{2,1}(\mu, 0) x^2 \right. \right. \\ &\quad \left. \left. + (\mu - 1)^4 \mu^4 (\mu^4 - 2\mu^3 - 3\mu^2 + 4\mu - 2) x^2 \right) \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} \right] \\ &\quad \left[1 + 5\mu^2(\mu - 1)^2 x^2 + 4\mu^4(\mu - 1)^4 x^4 + \left(1 + 3\mu^2(\mu - 1)^2 x^2 \right) \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} \right]^{-1}. \end{aligned} \quad (\text{C.17})$$

Expanding the previous function for $x \rightarrow +\infty$, we find at leading order:

$$\partial_x g_{2,1}(\mu, x) \simeq \frac{1}{4\mu(\mu - 1)} \left(2\partial_x g_{2,1}(\mu, 0) + \mu^2(\mu - 1)^2 (\mu^4 - 2\mu^3 - 3\mu^2 + 4\mu - 2) \right) x^{-1} \quad (\text{C.18})$$

which is clearly incompatible with the expansion (6.25), thereby forcing $\partial_x g_{2,1}(\mu, 0)$ to take the form:

$$\partial_x g_{2,1}(\mu, 0) = -\frac{1}{2}(\mu - 1)^2 \mu^2 (\mu^4 - 2\mu^3 - 3\mu^2 + 4\mu - 2). \quad (\text{C.19})$$

Thus, we obtain:

$$\begin{aligned} g_{2,1}(\mu, x) = & -\frac{2(\mu - 2)(\mu - 1)(\mu + 1)\mu - 2 + 3(\mu - 1)^4 \mu^4 x^2}{12(\mu - 1)^2 \mu^2 x^3} \\ & - \frac{\sqrt{1 + 4(\mu - 1)^2 \mu^2 x^2}}{12(\mu - 1)^2 \mu^2 x^3} \left[2 - 2\mu^4 + 4\mu^3 + 2\mu^2 - 4\mu \right. \\ & \left. + (\mu^8 - 4\mu^7 - 2\mu^6 + 20\mu^5 - 27\mu^4 + 16\mu^3 - 4\mu^2) x^2 \right], \end{aligned} \quad (\text{C.20})$$

$$\tilde{f}_{2,1}(\mu, x) = \frac{(3\mu^8 - 12\mu^7 + 10\mu^6 + 12\mu^5 - 25\mu^4 + 16\mu^3 - 4\mu^2) x^2}{2(1 + 4(\mu - 1)^2 \mu^2 x^2)}. \quad (\text{C.21})$$

Moving on to the next equation for $m_1 = 1$, and expanding around $x = 0$ to the first few orders, we find the relations:

$$\partial_x^2 \tilde{f}_{1,2}(\mu, 0) = -\frac{1}{2} \mu^2 (\mu + 1)^2 (\mu^2 - 3\mu + 2)^2, \quad (\text{C.22a})$$

$$\partial_x g_{1,2}(\mu, 0) = \frac{1}{4} \mu^2 (\mu + 1)^2 (\mu^2 - 3\mu + 2)^2, \quad (\text{C.22b})$$

and

$$\begin{aligned} \partial_\mu \partial_x g_{1,2}(\mu, 0) = & \frac{-8(\mu + 1) \partial_x \tilde{g}_{1,2}(\mu, 0) + 16 \partial_x \tilde{g}_{2,1}(\mu, 0)}{4\mu(\mu + 1)} \\ & + \frac{(\mu - 1)(7\mu^7 - 42\mu^6 + 28\mu^5 + 118\mu^4 - 67\mu^3 - 68\mu^2 + 48\mu - 8)}{4(\mu + 1)}, \end{aligned} \quad (\text{C.23})$$

which allow us to simplify the equation. Then, the equation, now involving only the remaining Taylor coefficient $\partial_x \tilde{g}_{2,1}(\mu, 0)$, can be readily solved for $\tilde{g}_{2,1}$, which at leading order as $x \rightarrow +\infty$ reads:

$$\begin{aligned} \tilde{g}_{2,1}(\mu, x) \simeq & -\mu(\mu - 1) \left(\mu(\mu - 1) \left(2 - 6\mu + 4\mu^2 + 9\mu^3 - 9\mu^4 - 5\mu^5 + 3\mu^6 \right) \right. \\ & \left. - 2 \partial_x \tilde{g}_{2,1}(\mu, 0) \right) x^2 + \mathcal{O}(1). \end{aligned} \quad (\text{C.24})$$

Again, this is clearly incompatible with the expansion (6.25), and thus we find:

$$\partial_x \tilde{g}_{2,1}(\mu, 0) = \frac{1}{2} \mu(\mu - 1) \left(2 - 6\mu + 4\mu^2 + 9\mu^3 - 9\mu^4 - 5\mu^5 + 3\mu^6 \right). \quad (\text{C.25})$$

which, when substituted into $\tilde{g}_{2,1}(\mu, x)$, yields:

$$\begin{aligned} \tilde{g}_{2,1}(\mu, x) = & (12(\mu - 1)^3 \mu^3 x^3)^{-1} \left[2\mu^6 - 6\mu^5 + 6\mu^4 + 2\mu^3 - 6\mu^2 + 6\mu - 2 \right. \\ & \left. + (6\mu^{10} - 24\mu^9 + 33\mu^8 - 12\mu^7 - 12\mu^6 + 12\mu^5 - 3\mu^4) x^2 \right] \\ & + \left(12(\mu - 1)^3 \mu^3 x^3 \sqrt{1 + 4(\mu - 1)^2 \mu^2 x^2} \right)^{-1} \left[-2\mu^6 + 6\mu^5 - 6\mu^4 - 2\mu^3 \right. \\ & \left. + 6\mu^2 - 6\mu + 2 + (10\mu^{14} - 58\mu^{13} + 84\mu^{12} + 90\mu^{11} - 372\mu^{10} + 354\mu^9 \right. \\ & \left. - 20\mu^8 - 202\mu^7 + 162\mu^6 - 56\mu^5 + 8\mu^4) x^4 \right. \\ & \left. + (-10\mu^{10} + 44\mu^9 - 73\mu^8 + 44\mu^7 + 20\mu^6 - 52\mu^5 + 43\mu^4 - 20\mu^3 + 4\mu^2) x^2 \right], \end{aligned} \quad (\text{C.26})$$

while the corresponding equation for $m_1 = 2$ vanishes identically. This procedure can be straightforwardly extended to higher orders in the expansion, and to different values of m_1 and m_2 , with no essential change in the steps involved.

C.3 Example: $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations, $N_f = 0$ theory

In this section we show and solve the first few differential equations obtained from the expansion of the blow-up equation (7.6). This is instructive in order to discuss a couple of subtleties which are not manifest from the general form of the equations presented in the main part of this work. In particular, we consider the expansions for $m_2 = 1$ and $m_1 = 1, 2$, up to the first six equations. The very first equation for $m_1 = 1$ reads:

$$\begin{aligned} e^{-2x^2 g'_{1,1}(x)} x^3 \left(-2 - x \tilde{f}'_{1,1}(x) + 2x^2 g'_{1,1}(x) + x^3 g''_{1,1}(x) \right) \\ + e^{2x^2 g'_{1,1}(x)} \left(\tilde{f}'_{1,1}(x) + 2x g'_{1,1}(x) + x^2 g''_{1,1}(x) \right) = 0, \end{aligned} \quad (\text{C.27})$$

and can be solved algebraically for $\tilde{f}'_{1,1}$. Integrating the solution and imposing the vanishing of the function at $x = 0$, we find:

$$\tilde{f}_{1,1}(x) = -\frac{1}{2} \log(e^{2x^2 g'_{1,1}(x)} - x^4 e^{-2x^2 g'_{1,1}(x)}), \quad (\text{C.28})$$

or equivalently, expressing the solution in terms of $u_1(x)$ and $\tilde{v}_1(x)$, where $\tilde{f}_{1,1} = \frac{1}{2} \log u_1$ and $g'_{1,1} = \frac{1}{2x^2} \log(x^2 \tilde{v}_1)$, we can write:

$$u_1(x) = \frac{1}{x^2} \frac{1}{\tilde{v}_1 - \tilde{v}_1^{-1}}. \quad (\text{C.29})$$

Considering now the first equation for $m_1 = 2$, we find:

$$\begin{aligned} 1 + e^{2\tilde{f}_{1,1}(x) - 4x^2 g'_{1,1}(x)} \cdot x^8 \left(-3 - x \tilde{f}'_{1,1}(x) + 4x^2 g'_{1,1}(x) + 2x^3 g''_{1,1}(x) \right) \\ + e^{2\tilde{f}_{1,1}(x) + 4x^2 g'_{1,1}(x)} \cdot \left(-1 + x \tilde{f}'_{1,1}(x) + 4x^2 g'_{1,1}(x) + 2x^3 g''_{1,1}(x) \right) = 0. \end{aligned} \quad (\text{C.30})$$

Writing again everything in terms of u_1 , $\tilde{v}_1$ and solving for the former, we find:

$$u_1(x) = \frac{1 - c_1 x^2}{x^4} \frac{1}{\tilde{v}_1^2 - \tilde{v}_1^{-2}}, \quad (\text{C.31})$$

where now the integration constant c_1 cannot be fixed by requiring $u_1(0) = 1$ and $\tilde{v}_1(x) \simeq 1/x^2$, i.e. $\tilde{f}_{1,1}(0) = g_{1,1}(0) = 0$. Solving the quadratic equation obtained from (C.29) and (C.31), we find:

$$\tilde{v}_1(x) = \frac{1}{2x^2} \left(1 - x^2 c_1 \pm \sqrt{(x^2 c_1 - 1)^2 - 4x^4} \right). \quad (\text{C.32})$$

In order to satisfy $\tilde{v}_1(x) \simeq 1/x^2$, we must choose the plus sign in front of the square root. Then, c_1 is related to the value of $g'_{1,1}$ at zero by $c_1 = -2g'_{1,1}(0)$, and by consequence of $\tilde{f}_{1,1}''(0) = -2g'_{1,1}(0)$, also to the value of $\tilde{f}_{1,1}''$ at zero. To determine c_1 , we can consider the second equation for $m_1 = 1$. Writing $\tilde{f}_{1,1}(x)$ as in equation (C.28) and $g'_{1,1}(x)$ in terms of $\tilde{v}_1(x)$, using (C.32), the resulting equation reads:

$$\begin{aligned} \left(1 - 2x^2 c_1 + x^4 (c_1^2 - 4) \right)^{-2} \left\{ x c_1 + x^3 (4 - 3c_1^2) + 3x^5 c_1 (c_1^2 - 4) - x^7 (c_1^2 - 4)^2 \right. \\ + \sqrt{1 - 2x^2 c_1 + x^4 (c_1^2 - 4)} \cdot \left[-4x^3 \left(1 + \log \frac{1}{2} \left(1 - x^2 c_1 + \sqrt{(x^2 c_1 - 1)^2 - 4x^4} \right) \right) \right. \\ - 8x^4 g_{1,1}(x) + \left(-1 + 3x^2 c_1 + x^6 c_1 (c_1^2 - 4) - x^4 (4 + 3c_1^2) \right) \tilde{g}_{1,1}(x) \\ \left. \left. + x(-1 + x^2 (c_1 - 2))(x^2 c_1 - 1)(x^2 (2 + c_1) - 1) \tilde{g}'_{1,1}(x) \right] \right\} = 0. \end{aligned} \quad (\text{C.33})$$

The integration constant c_1 can be fixed by also considering the first equation obtained from the $m_1 = 0$, $m_2 = 1$ expansion. Using the solutions u_1 and $\tilde{v}_1$ of the first two equations, together with the expansion of $\tilde{g}_{1,1}$ obtained from (C.33), the expansion of the first $m_1 = 0$ equation around $x = 0$ reads

$$0 = \frac{4 - c_1^2}{3} x^4 + O(x^6), \quad (\text{C.34})$$

which implies $c_1^2 = 4$. Requiring $g_{1,1}(x)$ to be real along the real x -axis then selects $c_1 = -2$, since $c_1 = 2$ would produce the square root $\sqrt{1 - 4x^2}$. In view of the next section, we note here that the correct choice is the one for which the coefficient of x^4 inside the square root in (C.32) vanishes, while also maintaining the function real on the real axis. With this choice, we find:

$$\tilde{f}_{1,1}(x) = -\frac{1}{4} \log(1 + 4x^2), \quad g_{1,1}(x) = -\frac{\log\left(1/2 + \sqrt{1/4 + x^2}\right) + 1 - \sqrt{1 + 4x^2}}{x}, \quad (\text{C.35})$$

and equation (C.33) for $\tilde{g}_{1,1}(x)$ simplifies to:

$$4x^3 - 2\sqrt{1 + 4x^2}(x + 4x^3) - (1 + 6x^2 + 16x^4) \tilde{g}_{1,1}(x) - (x + 6x^3 + 8x^5) \tilde{g}'_{1,1}(x) = 0, \quad (\text{C.36})$$

which we can solve, to obtain:

$$\tilde{g}_{1,1}(x) = \frac{1 - \sqrt{1 + 4x^2}}{2x}, \quad (\text{C.37})$$

where the integration constant has been fixed by enforcing $\tilde{g}_{1,1}(0) = 0$. Then, having already found $\tilde{g}_{1,1}$, the second equation for $m_1 = 2$ becomes a simple algebraic relation between the Taylor coefficients of the functions for $m_2 = 2$, namely:

$$8g'_{1,2}(0) + 2\tilde{f}''_{1,2}(0) - 1 = 0 \implies \tilde{f}''_{1,2}(0) = \frac{1}{2} - 4g'_{1,2}(0), \quad (\text{C.38})$$

which we can use to simplify the next equations. The third equation for $m_1 = 1$ reads:

$$\begin{aligned} 0 = & \sqrt{1 + 4x^2} - 1 - 2x^2 \left(7 + 6x^2 - 2\sqrt{1 + 4x^2}\right) + 4(1 + 4x^2)^{3/2} \tilde{f}_{2,1}(x) \\ & + 2x(1 + 4x^2)^{3/2} \tilde{f}'_{2,1}(x) + 8x^2(1 + 6x^2 + 10x^4) g'_{2,1}(x) + 2x^3(1 + 6x^2 + 8x^4) g''_{2,1}(x), \end{aligned} \quad (\text{C.39})$$

which we can solve to find:

$$\tilde{f}_{2,1}(x) = -\frac{(2 + x^2) \left(-1 - 2x^2 + \sqrt{1 + 4x^2}\right) + 4x^4(1 + 2x^2) g'_{2,1}(x)}{4x^2 \sqrt{1 + 4x^2}}, \quad (\text{C.40})$$

where the integration constant has been fixed by requiring $\tilde{f}_{2,1}(0) = 0$. This solution can then be substituted into the third equation for $m_1 = 2$, which simplifies to:

$$\begin{aligned} -1 - 2x^2 + x^4 - 2(x^4 + 6x^6) g'_{2,1}(x) - (x^5 + 4x^7) g''_{2,1}(x) \\ + \sqrt{1 + 4x^2} \left(1 + x^3 \left(9 - 40g'_{1,2}(0) - 8\tilde{g}'_{1,2}(0)\right)\right) = 0. \end{aligned} \quad (\text{C.41})$$

Expanding this equation at leading order around $x = 0$, we find:

$$(9 - 40g'_{1,2}(0) - 8\tilde{g}'_{1,2}(0))x^3 + \mathcal{O}(x^4) \implies \tilde{g}'_{1,2}(0) = \frac{9}{8} - 5g'_{1,2}(0), \quad (\text{C.42})$$

which gives us another relation between Taylor coefficients. Using this relation and solving for $g_{2,1}(x)$, we obtain:

$$g_{2,1}(x) = \frac{(2-x^2)\sqrt{1+4x^2} - (2+3x^2)}{12x^3} \implies \tilde{f}_{2,1}(x) = \frac{3}{2} \frac{x^2}{1+4x^2}, \quad (\text{C.43})$$

where now both integration constants of $g_{2,1}$ are fixed by enforcing its vanishing at zero. Then, the next equation for $m_1 = 1$ reads:

$$\begin{aligned} & 3 + x^2 \left(10 + x(3 - 10x + 24x^2 - 80x^3 + 48x^4 - 12(1+4x^2)^2 g'_{1,2}(0)) \right) + \sqrt{1+4x^2} \\ & \cdot \left(3x^2(1+6x^2+8x^4) \tilde{g}'_{2,1}(x) + x(9+54x^2+96x^4) \tilde{g}_{2,1}(x) - 3 - 22x^2 - 36x^4 \right) = 0. \end{aligned} \quad (\text{C.44})$$

Expanding this equation near $x = 0$, we find:

$$\left(18 - 12 \tilde{g}'_{2,1}(0) \right) x^2 + \frac{3}{2} \left(-2 + 8 g'_{1,2}(0) \right) x^3 + \mathcal{O}(x^4) = 0, \quad (\text{C.45})$$

obtaining:

$$g'_{1,2}(0) = 1/4 \implies \tilde{f}''_{1,2}(0) = -1/2, \quad \tilde{g}'_{1,2}(0) = -1/8. \quad (\text{C.46})$$

We notice here that the blow-up equation already determines at this stage the analogue of the previous integration constant c_1 , for the $m_2 = 2$ functions. Using the value of $g'_{1,2}(0)$ to simplify the equation, we can solve it for $\tilde{g}_{2,1}(x)$ obtaining:

$$\tilde{g}_{2,1}(x) = \frac{(5x^4 - 5x^2 - 1)(1+4x^2)^{-1/2} + 3x^2 + 1}{6x^3}. \quad (\text{C.47})$$

Then, the fourth equation for $m_1 = 2$ reads:

$$2 \left(8 + 36 g'_{2,2}(0) + 9 \tilde{f}''_{2,2}(0) \right) + x^2 \left(3 + 48 g^{(3)}_{1,2}(0) + 2 \tilde{f}^{(4)}_{1,2}(0) \right) = 0, \quad (\text{C.48})$$

from which we find the following relations:

$$\tilde{f}''_{2,2}(0) = -\frac{4}{9}(2 + 9g'_{2,2}(0)), \quad \tilde{f}^{(4)}_{1,2}(0) = -\frac{3}{2}(1 + 16g^{(3)}_{1,2}(0)). \quad (\text{C.49})$$

Carrying on, the fifth equation for $m_1 = 1$ reads:

$$\begin{aligned} & 27 + 341x^2 + 2796x^4 + 9888x^6 + 12320x^8 + 72x^2(1+4x^2)^3 \left(4\tilde{f}_{3,1}(x) + x\tilde{f}'_{3,1}(x) \right) \\ & + (1+4x^2)^{3/2} \left(-27 - 179x^2 + 438x^4 + 1270x^6 \right. \\ & \left. + 72x^4 \left[(6+36x^2+56x^4) g'_{3,1}(x) + x(1+6x^2+8x^4) g''_{3,1}(x) \right] \right) = 0, \end{aligned} \quad (\text{C.50})$$

which can be solved to find:

$$\begin{aligned} \tilde{f}_{3,1}(x) &= \frac{-72 - x^2(630 + 1601x^2 + 1864x^4 + 3080x^6)}{288x^4(1+4x^2)^2} \\ &+ \frac{(1+2x^2)(72 + 54x^2 - 127x^4 - 288x^6 g'_{3,1}(x))}{288x^4(1+4x^2)^{1/2}}. \end{aligned} \quad (\text{C.51})$$

Using this expression, the fifth equation for $m_1 = 2$ simplifies to:

$$\begin{aligned}
 & 488 - \frac{162}{(1+4x^2)^2} - \frac{809}{\sqrt{1+4x^2}} - 381\sqrt{1+4x^2} - \frac{1152(x^2+5x^4)g'_{3,1}(x)}{\sqrt{1+4x^2}} \\
 & - 288x^3\sqrt{1+4x^2}g''_{3,1}(x) + \frac{8}{x^4} \left[18 - 18\sqrt{1+4x^2} + 36x^2\sqrt{1+4x^2} \right. \\
 & - 16x^5 \left(118 + 243g'_{2,2}(0) + 27\tilde{g}'_{2,2}(0) \right) + x^6 \left(1 - 288g'_{1,3}(0) - 72\tilde{f}''_{1,3}(0) \right) \\
 & \left. - 12x^7 \left(15 + 88g^{(3)}_{1,2}(0) + 8\tilde{g}^{(3)}_{1,2}(0) \right) \right] = 0.
 \end{aligned} \tag{C.52}$$

Expanding the equation around $x = 0$, we have:

$$\begin{aligned}
 & \left(-\frac{236}{9} - 54g'_{2,2}(0) - 6\tilde{g}'_{2,2}(0) \right) x + \left(\frac{7}{4} - 4g'_{1,3}(0) - 2g'_{3,1}(0) - \tilde{f}''_{1,3}(0) \right) x^2 \\
 & + \frac{1}{6} \left(-15 - 88g^{(3)}_{1,2}(0) - 8\tilde{g}^{(3)}_{1,2}(0) \right) x^3 + \mathcal{O}(x^4) = 0,
 \end{aligned} \tag{C.53}$$

from which we find:

$$\tilde{g}'_{2,2}(0) = -\frac{118}{27} - 9g'_{2,2}(0), \quad \tilde{g}^{(3)}_{1,2}(0) = -\frac{15}{8} - 11g^{(3)}_{1,2}(0). \tag{C.54}$$

Using these relations in the previous equation and solving for $g_{3,1}(x)$, we obtain:

$$\begin{aligned}
 g_{3,1}(x) &= \frac{-72 - 90x^2 + 635x^4 + \frac{72 + 234x^2 - 599x^4 + 88x^6}{\sqrt{1+4x^2}}}{1440x^5} \\
 &- \frac{1}{288} \left(-1 + 288g'_{1,3}(0) + 72\tilde{f}''_{1,3}(0) \right) \operatorname{arcsinh}(2x)
 \end{aligned} \tag{C.55}$$

where the $\operatorname{arcsinh}(2x)$ term is clearly incompatible with the asymptotics of the function near infinity. Indeed, (C.55) behaves as:

$$g_{3,1}(x) \simeq \pm \frac{11}{360} + \left(\frac{1}{576} - \frac{1}{2}g'_{1,3}(0) - \frac{1}{8}\tilde{f}''_{1,3}(0) \right) \log 16x^2 + \mathcal{O}(x^{-1}). \tag{C.56}$$

Requiring the vanishing of this term leads to another relation between Taylor coefficients of resummation functions:

$$\tilde{f}''_{1,3}(0) = \frac{1}{72} - 4g'_{1,3}(0). \tag{C.57}$$

Finally, the sixth equation for $m_1 = 1$ reads:

$$\begin{aligned}
 & -810 + 3932x^2 - 2(1+4x^2)^{3/2} \left(-405 + 4396x^2 + 41280x^4 + 76900x^6 \right. \\
 & + 1080x^3 \left[(5 + 30x^2 + 48x^4)\tilde{g}_{3,1}(x) + x(1 + 6x^2 + 8x^4)\tilde{g}'_{3,1}(x) \right] \Big) \\
 & + x^4 \left[74622 + 280152x^2 + 324992x^4 - 87168x^6 + 1200x \left(4 + 9g'_{2,2}(0) \right) \right. \\
 & + 45x^3 \left(1307 + 2880g'_{2,2}(0) + 144g^{(3)}_{1,2}(0) \right) + 180x^5 \left(1361 + 2880g'_{2,2}(0) + 432g^{(3)}_{1,2}(0) \right) \\
 & \left. + 240x^7 \left(1523 + 2880g'_{2,2}(0) + 1296g^{(3)}_{1,2}(0) \right) + 25920x^9 \left(3 + 16g^{(3)}_{1,2}(0) \right) \right] = 0.
 \end{aligned} \tag{C.58}$$

Considering its expansion around $x = 0$, the first few orders are:

$$\begin{aligned}
 & -30x^4 \left(1861 + 432 \tilde{g}'_{3,1}(0)\right) + 1200x^5 \left(4 + 9 g'_{2,2}(0)\right) \\
 & -40x^6 \left(10625 + 3888 \tilde{g}'_{3,1}(0) + 72 \tilde{g}^{(3)}_{3,1}(0)\right) \\
 & + 45x^7 \left(1307 + 2880 g'_{2,2}(0) + 144 g^{(3)}_{1,2}(0)\right) + \mathcal{O}(x^8) = 0,
 \end{aligned} \tag{C.59}$$

from which we obtain:

$$g'_{2,2}(0) = -\frac{4}{9}, \quad g^{(3)}_{1,2}(0) = -\frac{3}{16}. \tag{C.60}$$

Using these values to simplify the previous equation, we find:

$$\begin{aligned}
 & -810 + 3932x^2 + 74622x^4 + 280152x^6 + 324992x^8 - 87168x^{10} \\
 & -2(1 + 4x^2)^{3/2} \left(-405 + 4396x^2 + 41280x^4 + 76900x^6 \right. \\
 & \left. + 1080x^3 \left[(5 + 30x^2 + 48x^4) \tilde{g}_{3,1}(x) + x(1 + 6x^2 + 8x^4) \tilde{g}'_{3,1}(x) \right] \right) = 0,
 \end{aligned} \tag{C.61}$$

which we can easily solve, obtaining:

$$\tilde{g}_{3,1}(x) = \frac{216 + 405x^2 - 3845x^4}{2160x^5} + \frac{-216 - 1701x^2 + 119x^4 + 12199x^6 - 1816x^8}{2160x^5(1 + 4x^2)^{3/2}}. \tag{C.62}$$

Then, the sixth equation for $m_1 = 2$ reads:

$$\begin{aligned}
 & \frac{1}{300} \left(-143 + 21312 g'_{1,3}(0) + 2160 \tilde{g}'_{1,3}(0) \right) + \left(-\frac{55}{432} + 19 g'_{1,3}(0) + 2 \tilde{g}'_{1,3}(0) \right) x \\
 & + \left(-\frac{14161}{1296} + 16 g'_{3,2}(0) + 4 \tilde{f}''_{3,2}(0) \right) x^2 + \frac{5}{36} \left(-16 + 72 g^{(3)}_{2,2}(0) + 3 \tilde{f}^{(4)}_{2,2}(0) \right) x^4 \\
 & + \left(g^{(5)}_{1,2}(0) + \frac{1}{60} \left(-15 + \tilde{f}^{(6)}_{1,2}(0) \right) \right) x^6 = 0,
 \end{aligned} \tag{C.63}$$

from which we find:

$$\begin{aligned}
 g'_{1,3}(0) &= \frac{1}{144}, \quad \tilde{g}'_{1,3}(0) = -\frac{1}{432}, \quad \tilde{f}''_{3,2}(0) = \frac{14161}{5184} - 4 g'_{3,2}(0), \\
 \tilde{f}^{(4)}_{2,2}(0) &= \frac{16}{3} - 24 g^{(3)}_{2,2}(0), \quad \tilde{f}^{(6)}_{1,2}(0) = 15 - 60 g^{(5)}_{1,2}(0).
 \end{aligned} \tag{C.64}$$

As a final summary, the relations we found up until this order read:

$$\begin{aligned}
 g'_{1,2}(0) &= \frac{1}{4}, \quad \tilde{g}'_{1,2}(0) = -\frac{1}{8}, \quad \tilde{f}''_{1,2}(0) = -\frac{1}{2}, \\
 g^{(3)}_{1,2}(0) &= -\frac{3}{16}, \quad \tilde{g}^{(3)}_{1,2}(0) = \frac{3}{16}, \quad \tilde{f}^{(4)}_{1,2}(0) = 3, \\
 g'_{1,3}(0) &= \frac{1}{144}, \quad \tilde{g}'_{1,3}(0) = -\frac{1}{432}, \quad \tilde{f}''_{1,3}(0) = -\frac{1}{72}, \\
 g'_{2,2}(0) &= -\frac{4}{9}, \quad \tilde{g}'_{2,2}(0) = -\frac{10}{27}, \quad \tilde{f}''_{2,2}(0) = \frac{8}{9}, \\
 \tilde{f}^{(6)}_{1,2}(0) &= 15 - 60 g^{(5)}_{1,2}(0), \quad \tilde{f}^{(4)}_{2,2}(0) = \frac{16}{3} - 24 g^{(3)}_{2,2}(0), \\
 \tilde{f}''_{3,2}(0) &= \frac{14161}{5184} - 4 g'_{3,2}(0).
 \end{aligned} \tag{C.65}$$

Going to higher order in the expansion or to higher values of m_1 and m_2 does not introduce any new feature in the equations, which can be solved straightforwardly. As we have

seen, the basic strategy to obtain the desired solution is to impose the proper parity of the functions and the correct behavior at zero and at infinity. Expanding the equations around zero or infinity before solving them often helps identify combinations of Taylor coefficients of the resummation functions that vanish, which in turn makes the integration of the equations easier.

C.4 Example: $\mathbb{C}^2/\mathbb{Z}_2$ blow-up equations, $\mathcal{N} = 2^*$ theory

In this section we follow the same steps as in the previous one, showing and solving the differential equations for the resummation functions in the $\mathcal{N} = 2^*$ case. We suppress the arguments of the functions and denote by $\tilde{f}'_{1,1}$ the derivative of $\tilde{f}_{1,1}(\mu, x)$ with respect to x , unless the functions are valued at some particular point. The first equation for $m_1 = 1$ reads:

$$\begin{aligned} & (\mu - 1)^4 \mu^4 x^4 e^{-2x^2 g'_{1,1}} \left(-x \tilde{f}'_{1,1} + x^3 g''_{1,1} + 2x^2 g'_{1,1} - 2 \right) \\ & + x e^{2x^2 g'_{1,1}} \left(\tilde{f}'_{1,1} + 2x g'_{1,1} + x^2 g''_{1,1} \right) = 0, \end{aligned} \quad (\text{C.66})$$

and depends algebraically on $\tilde{f}'_{1,1}$. Solving it and integrating the result, we find:

$$\tilde{f}_{1,1}(\mu, x) = -\frac{1}{2} \log \left(e^{2x^2 g'_{1,1}} - (\mu - 1)^4 \mu^4 x^4 e^{-2x^2 g'_{1,1}} \right), \quad (\text{C.67})$$

where the integration constant has been determined by imposing the vanishing of the function $\tilde{f}_{1,1}$ at $x = 0$. Writing $\tilde{f}_{1,1} = \frac{1}{2} \log u_1$ and $g'_{1,1} = \frac{1}{2x^2} \log(x^2 \tilde{v}_1)$, we have:

$$u_1(\mu, x) = \frac{1}{x^2} \frac{1}{\tilde{v}_1 - \mu^4 (\mu - 1)^4 \tilde{v}_1^{-1}}. \quad (\text{C.68})$$

Meanwhile, the first equation for $m_1 = 2$ reads:

$$\begin{aligned} 0 = & (\mu - 1)^8 \mu^8 x^8 \left(x \tilde{f}'_{1,1} - 2x^3 g''_{1,1} - 4x^2 g'_{1,1} + 3 \right) e^{2\tilde{f}_{1,1} - 4x^2 g'_{1,1}} \\ & + \left(1 - x \tilde{f}'_{1,1} - 4x^2 g'_{1,1} - 2x^3 g''_{1,1} \right) e^{2\tilde{f}_{1,1} + 4x^2 g'_{1,1}} - 1, \end{aligned} \quad (\text{C.69})$$

which we can solve for u_1 , to obtain:

$$u_1(\mu, x) = \frac{1 - c_1 x^2}{x^4} \frac{1}{\tilde{v}_1^2 - \mu^8 (\mu - 1)^8 \tilde{v}_1^{-2}}. \quad (\text{C.70})$$

Then, equating (C.68) and (C.70) leads to:

$$\tilde{v}_1(\mu, x) = \frac{1 - c_1 x^2 \pm \sqrt{1 - 2c_1 x^2 + x^4 (c_1^2 - 4\mu^4 (\mu - 1)^4)}}{2x^2}, \quad (\text{C.71})$$

where, by looking at the behavior near $x = 0$, we have to select the positive relative sign and we have to choose $c_1 = -2\mu^2(\mu - 1)^2$ to cancel the coefficient of x^4 inside the square root, as discussed in the previous section around equation (C.34). We thus find:

$$\begin{aligned} \tilde{f}_{1,1}(\mu, x) &= -\frac{1}{4} \log(1 + 4\mu^2(\mu - 1)^2 x^2), \\ g_{1,1}(\mu, x) &= \frac{\sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} + \log 2 - \log \left(1 + \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2} \right) - 1}{x}, \end{aligned} \quad (\text{C.72})$$

where the remaining integration constant for $g_{1,1}$ is fixed by $g_{1,1}(\mu, 0) = 0$. We can then look at the second equation for $m_1 = 1$, which reads:

$$\begin{aligned} & \left[1 + 8(\mu - 1)^4 \mu^4 x^4 + 6(\mu - 1)^2 \mu^2 x^2\right] x \tilde{g}'_{1,1} \\ & + \left[1 + 16(\mu - 1)^4 \mu^4 x^4 + 6(\mu - 1)^2 \mu^2 x^2\right] \tilde{g}_{1,1} \\ & - 4(\mu - 1)^3 \mu^3 (\mu^2 + \mu - 1) x^3 + 2\mu(\mu^3 - 2\mu + 1) x \left[1 + 4\mu^2(\mu - 1)^2 x^2\right]^{3/2} = 0. \end{aligned} \quad (\text{C.73})$$

The equation can be easily solved, obtaining:

$$\tilde{g}_{1,1}(\mu, x) = \frac{(\mu^2 + \mu - 1) \left(1 - \sqrt{1 + 4\mu^2(\mu - 1)^2 x^2}\right)}{2\mu(\mu - 1)x}, \quad (\text{C.74})$$

where the integration constant has been fixed by $\tilde{g}_{1,1}(\mu, 0) = 0$. Then, the second equation for $m_1 = 2$ simplifies to:

$$2\tilde{f}''_{1,2}(\mu, 0) + 8g'_{1,2}(\mu, 0) - (\mu - 2)^2(\mu - 1)^2\mu^2(\mu + 1)^2 = 0, \quad (\text{C.75})$$

from which we find:

$$\tilde{f}''_{1,2}(\mu, 0) = \frac{1}{2}(\mu - 2)^2(\mu - 1)^2\mu^2(\mu + 1)^2 - 4g'_{1,2}(\mu, 0). \quad (\text{C.76})$$

Moving on, the third equation for $m_1 = 1$ reads:

$$\begin{aligned} & \left[1 + 4(\mu - 1)^2 \mu^2 x^2\right]^{3/2} \left[4\tilde{f}_{2,1} + 2x\tilde{f}'_{2,1} + (\mu - 1)^2 \mu^2\right] \\ & + 8 \left[10(\mu - 1)^4 \mu^4 x^6 + 6(\mu - 1)^2 \mu^2 x^4 + x^2\right] g'_{2,1} \\ & + 2 \left[8(\mu - 1)^4 \mu^4 x^7 + 6(\mu - 1)^2 \mu^2 x^5 + x^3\right] g''_{2,1} \\ & - (\mu - 1)^2 \mu^2 \left[12(\mu - 1)^4 \mu^4 x^4 + 2(7\mu^4 - 14\mu^3 - \mu^2 + 8\mu - 4)x^2 + 1\right] = 0, \end{aligned} \quad (\text{C.77})$$

which can be solved to obtain:

$$\begin{aligned} \tilde{f}_{2,1}(\mu, x) &= \frac{-2\mu^4 + 4\mu^3 + 2\mu^2 - 4\mu + 2 + (-\mu^8 + 4\mu^7 - 6\mu^6 + 4\mu^5 - \mu^4)x^2}{4(\mu - 1)^2 \mu^2 x^2} \\ &+ \frac{1 + 2(\mu - 1)^2 \mu^2 x^2}{4(\mu - 1)^2 \mu^2 x^2 \sqrt{1 + 4(\mu - 1)^2 \mu^2 x^2}} \left[2\mu^4 - 4\mu^3 - 2\mu^2 + 4\mu - 2\right. \\ &\left. + (\mu^8 - 4\mu^7 + 6\mu^6 - 4\mu^5 + \mu^4)x^2 - 4(\mu - 1)^2 \mu^2 x^4 g'_{2,1}\right], \end{aligned} \quad (\text{C.78})$$

where the integration constant is determined by requiring $\tilde{f}_{2,1}(\mu, 0) = 0$. Then, the third equation for $m_1 = 2$ reads:

$$\begin{aligned} & \sqrt{1 + 4(\mu - 1)^2 \mu^2 x^2} \left\{ (\mu - 2)(\mu + 1) \left[\mu^4 - 2\mu^3 - \mu^2 + 2\mu - 1 \right. \right. \\ & + \left. \left(9\mu^{10} - 45\mu^9 + 92\mu^8 - 98\mu^7 + 53\mu^6 - 5\mu^5 - 10\mu^4 + 4\mu^3 \right) x^3 \right] \\ & - 8(\mu - 1)^2 \mu^2 x^3 \left(-3g'_{1,2}(\mu, 0) + \mu \partial_\mu g'_{1,2}(\mu, 0) + \tilde{g}'_{1,2}(\mu, 0) \right) \left. \right\} \\ & + (\mu - 2)(\mu + 1) \left\{ (\mu - 1)^2 \mu^2 x^4 \left[-2 \left(1 + 6(\mu - 1)^2 \mu^2 x^2 \right) g'_{2,1} \right. \right. \\ & - x \left(1 + 4(\mu - 1)^2 \mu^2 x^2 \right) g''_{2,1} \left. \right] - \mu^4 + 2\mu^3 + \mu^2 - 2\mu + 1 \\ & + \left(\mu^{12} - 6\mu^{11} + 15\mu^{10} - 20\mu^9 + 15\mu^8 - 6\mu^7 + \mu^6 \right) x^4 \\ & + \left. \left(-2\mu^8 + 8\mu^7 - 8\mu^6 - 4\mu^5 + 12\mu^4 - 8\mu^3 + 2\mu^2 \right) x^2 \right\} = 0. \end{aligned}$$

Expanding the equation around $x = 0$, the first non-zero coefficient gives us:

$$\begin{aligned} \partial_\mu g'_{1,2}(\mu, 0) &= \frac{1}{8\mu} (24g'_{1,2}(\mu, 0) - 8\tilde{g}'_{1,2}(\mu, 0) \\ &+ 9\mu^8 - 36\mu^7 + 38\mu^6 + 12\mu^5 - 47\mu^4 + 32\mu^3 - 8\mu), \end{aligned} \quad (\text{C.79})$$

which we can use to simplify and solve the equation, obtaining:

$$\begin{aligned} g_{2,1}(\mu, x) &= -\frac{2(\mu - 2)(\mu - 1)(\mu + 1)\mu - 2 + 3(\mu - 1)^4 \mu^4 x^2}{12(\mu - 1)^2 \mu^2 x^3} \\ &- \frac{\sqrt{1 + 4(\mu - 1)^2 \mu^2 x^2}}{12(\mu - 1)^2 \mu^2 x^3} \left[2 - 2\mu^4 + 4\mu^3 + 2\mu^2 - 4\mu \right. \\ &+ \left. \left(\mu^8 - 4\mu^7 - 2\mu^6 + 20\mu^5 - 27\mu^4 + 16\mu^3 - 4\mu^2 \right) x^2 \right], \end{aligned} \quad (\text{C.80})$$

$$\tilde{f}_{2,1}(\mu, x) = \frac{(3\mu^8 - 12\mu^7 + 10\mu^6 + 12\mu^5 - 25\mu^4 + 16\mu^3 - 4\mu^2) x^2}{2(1 + 4(\mu - 1)^2 \mu^2 x^2)}. \quad (\text{C.81})$$

fixing the integration constants via $g_{2,1}(\mu, 0) = 0$. Then, the fourth equation for $m_1 = 1$ reads:

$$\begin{aligned}
 & 24(\mu - 2)(\mu + 1)x^3 \left[1 + 4(\mu - 1)^2\mu^2x^2 \right]^2 g'_{1,2}(\mu, 0) \\
 & + \sqrt{4(\mu - 1)^2\mu^2x^2 + 1} \left\{ -6x \left[32(\mu - 1)^4\mu^4x^4 + 18(\mu - 1)^2\mu^2x^2 + 3 \right] \tilde{g}_{2,1} \right. \\
 & - 6 \left[8(\mu - 1)^4\mu^4x^6 + 6(\mu - 1)^2\mu^2x^4 + x^2 \right] \tilde{g}'_{2,1} \\
 & + (\mu - 1)\mu \left[6\mu^2 - 3 + 36(\mu - 1)^4(2\mu^2 - 1)\mu^4x^4 \right. \\
 & \left. \left. + \left(44\mu^6 - 96\mu^5 + 42\mu^4 + 44\mu^3 - 42\mu^2 + 24\mu - 8 \right) x^2 \right] \right\} \\
 & + (\mu - 1)\mu \cdot \left[3 - 6\mu^2 - 96(\mu - 2)^3(\mu - 1)^5\mu^5(\mu + 1)^3x^7 \right. \\
 & + 32(\mu - 1)^4\mu^4 \left(5\mu^6 - 9\mu^5 - 24\mu^4 + 23\mu^3 + 9\mu^2 - 12\mu + 4 \right) x^6 \\
 & - 48(\mu - 2)^3(\mu - 1)^3\mu^3(\mu + 1)^3x^5 \\
 & + 4(\mu - 1)^2\mu^2 \left(5\mu^6 + 3\mu^5 - 150\mu^4 + 95\mu^3 + 75\mu^2 - 84\mu + 28 \right) x^4 \\
 & - 6(\mu - 2)^3(\mu - 1)\mu(\mu + 1)^3x^3 \\
 & \left. \left. + 4 \left(-5\mu^6 + 15\mu^5 - 39\mu^4 + 13\mu^3 + 24\mu^2 - 24\mu + 8 \right) x^2 \right] \right\} = 0.
 \end{aligned} \tag{C.82}$$

Expanding it around $x = 0$, we find:

$$g'_{1,2}(\mu, 0) = \frac{1}{4}(\mu - 2)^2(\mu - 1)^2\mu^2(\mu + 1)^2, \tag{C.83}$$

which, due to (C.76), implies:

$$\tilde{f}''_{1,2}(\mu, 0) = -\frac{1}{2}(\mu - 2)^2(\mu - 1)^2\mu^2(\mu + 1)^2, \tag{C.84}$$

thus determining the equivalent of the integration constant c_1 for the $m_2 = 2$ resummation functions. Then, solving the equation we obtain:

$$\begin{aligned}
 \tilde{g}_{2,1}(\mu, x) &= (12(\mu - 1)^3\mu^3x^3)^{-1} \left[2\mu^6 - 6\mu^5 + 6\mu^4 + 2\mu^3 - 6\mu^2 + 6\mu - 2 \right. \\
 & + \left(6\mu^{10} - 24\mu^9 + 33\mu^8 - 12\mu^7 - 12\mu^6 + 12\mu^5 - 3\mu^4 \right) x^2 \Big] \\
 & + \left(12(\mu - 1)^3\mu^3x^3\sqrt{1 + 4(\mu - 1)^2\mu^2x^2} \right)^{-1} \left[-2\mu^6 + 6\mu^5 - 6\mu^4 - 2\mu^3 \right. \\
 & + 6\mu^2 - 6\mu + 2 + (10\mu^{14} - 58\mu^{13} + 84\mu^{12} + 90\mu^{11} - 372\mu^{10} + 354\mu^9 \\
 & - 20\mu^8 - 202\mu^7 + 162\mu^6 - 56\mu^5 + 8\mu^4)x^4 \\
 & \left. \left. + \left(-10\mu^{10} + 44\mu^9 - 73\mu^8 + 44\mu^7 + 20\mu^6 - 52\mu^5 + 43\mu^4 - 20\mu^3 + 4\mu^2 \right) x^2 \right] \right],
 \end{aligned} \tag{C.85}$$

so that the fourth equation for $m_1 = 2$ becomes:

$$\begin{aligned}
 & \frac{1}{6}x^2 \left(9\tilde{f}''_{2,2}(\mu, 0) + 36g'_{2,2}(\mu, 0) + 8(\mu - 1)^4(-\mu^2 + \mu + 2)^2\mu^4 \right) \\
 & + x^4 \left(\frac{1}{6}\tilde{f}^{(4)}_{1,2}(\mu, 0) + 4g^{(3)}_{1,2}(\mu, 0) + \frac{1}{4}(\mu - 2)^4(\mu - 1)^4\mu^4(\mu + 1)^4 \right) = 0.
 \end{aligned} \tag{C.86}$$

Solving it, we find two relations between Taylor coefficients at $x = 0$ of the $m_2 = 2$ resummation functions:

$$\begin{aligned}\tilde{f}_{2,2}''(\mu, 0) &= -\frac{8}{9}(\mu-1)^4\mu^4(\mu-2)^2(\mu+1)^2 - 4g_{2,2}'(\mu, 0), \\ \tilde{f}_{1,2}^{(4)}(\mu, 0) &= -\frac{3}{2}(\mu-2)^4(\mu-1)^4\mu^4(\mu+1)^4 - 24g_{1,2}^{(3)}(\mu, 0).\end{aligned}\tag{C.87}$$

In summary, it is clear that the only difference with respect to the pure gauge theory case is the cumbersomeness of the equations and of the resummation functions. Thus, in principle one can proceed to higher orders in the expansion, and to different values of m_1 and m_2 , without any qualitative difference with what we have seen so far.

C.5 Solving the $N_f = (1, 1)$ blow-up equation

In this section, we present the first few equations obtained by expanding the blow-up equation (10.27), and solve them explicitly to illustrate the resummation procedure in the $N_f = (1, 1)$ case for the resummation ansatz (10.25). At the end, we list the corresponding h -functions, obtained through the relation (10.26). Throughout this section we set $\hbar = 1$. Before turning to the expansion, we also list the resummation functions $g_{k,j}$ needed in the derivation. Let us define for convenience the function

$$\gamma_j(\mathbf{m}, x) = \sqrt{1 + 4w_j^{(1,1)}x^2/\zeta_j^2}.\tag{C.88}$$

For notational simplicity, we will suppress one or both of its arguments in lengthy expressions. As already discussed in Section 10, the resummation functions $g_{k,j}$ take the form [125]

$$g_{1,j}(\mathbf{m}, x) = -\frac{1}{x}\left(\log\left[\frac{1}{2} + \frac{1}{2}\gamma_j(\mathbf{m}, x)\right] + 1 - \gamma_j(\mathbf{m}, x)\right),\tag{C.89a}$$

$$g_{k\geq 2,j}(\mathbf{m}, x) = \frac{1}{x^{2k-1}}\left[\gamma_j(\mathbf{m}, x)^{5-2k}Q_{k,j}(\mathbf{m}, x^2) + P_{k,j}(\mathbf{m}, x^2)\right]\tag{C.89b}$$

where $Q_{k,j}$ and $P_{k,j}$ are polynomials in x^2 of degrees $2k-3$ and $k-1$, respectively:

$$Q_{k,j}(\mathbf{m}, x^2) = \sum_{\alpha=0}^{2k-3} q_{k,j,\alpha}(\mathbf{m})x^{2\alpha}, \quad P_{k,j}(\mathbf{m}, x^2) = \sum_{\alpha=0}^{k-1} p_{k,j,\alpha}(\mathbf{m})x^{2\alpha}.\tag{C.90}$$

The coefficients $p_{k,j,\alpha}$ are fixed in terms of the $q_{k,j,\alpha}$ by imposing the condition

$$g_{k,j}(\mathbf{m}, x) = \mathcal{O}(x), \quad x \rightarrow 0,\tag{C.91}$$

while the relevant q -coefficients are:

$$q_{2,1,0} = \frac{1}{12}\left(2 - \frac{1}{(2m_1+1)^2} - \frac{1}{(2m_3+1)^2}\right),\tag{C.92a}$$

$$q_{2,1,1} = \frac{3 + 4(m_1 + m_1^2 + m_3 + m_3^2) - 16(m_1m_3 + m_1^2m_3 + m_1m_3^2 + m_1^2m_3^2)}{48(2m_1+1)(2m_3+1)},\tag{C.92b}$$

$$q_{2,2,0} = 0,\tag{C.92c}$$

$$q_{2,2,1} = \frac{2}{9}(2m_1+1)(2m_3+1),\tag{C.92d}$$

$$q_{3,1,0} = \frac{1}{40}\left(-\frac{1}{(2m_1+1)^4} - \frac{1}{(2m_3+1)^4} + 2\right),\tag{C.93a}$$

$$q_{3,1,1} = \frac{1}{80(2m_1+1)^3(2m_3+1)^3} \left[2m_1(m_1+1)(1+2m_1+2m_1^2) \right. \\ \left. \times (104m_3(m_3+1)(1+2m_3+2m_3^2)+9)-1 \right] + \frac{9(2m_3+1)}{320(2m_1+1)^3}, \quad (\text{C.93b})$$

$$q_{3,1,2} = \frac{1}{23040(2m_1+1)^2(2m_3+1)^2} \left(225 + 504m_1 + 504m_3 + 72m_1^2 \right. \\ - 11456m_1m_3 + 72m_3^2 - 864m_1^3 - 56128m_1^2m_3 - 56128m_1m_3^2 - 864m_3^3 \\ - 432m_1^4 - 89344m_1^3m_3 - 254144m_1^2m_3^2 - 89344m_1m_3^3 - 432m_3^4 \\ - 44672m_1^4m_3 - 396032m_1^3m_3^2 - 396032m_1^2m_3^3 - 44672m_1m_3^4 \\ - 198016m_1^4m_3^2 - 613376m_1^3m_3^3 - 198016m_1^2m_3^4 - 306688m_1^4m_3^3 \\ \left. - 306688m_1^3m_3^4 - 153344m_1^4m_3^4 \right), \quad (\text{C.93c})$$

$$q_{3,1,3} = \frac{1}{11520(1+2m_1)(1+2m_3)} \left(45 + 216m_1 + 216m_3 + 648m_1^2 \right. \\ + 256m_1m_3 + 648m_3^2 + 864m_1^3 + 128m_1^2m_3 + 128m_1m_3^2 + 864m_3^3 \\ + 432m_1^4 - 256m_1^3m_3 - 2816m_1^2m_3^2 - 256m_1m_3^3 + 432m_3^4 \\ - 128m_1^4m_3 - 5888m_1^3m_3^2 - 5888m_1^2m_3^3 - 128m_1m_3^4 - 2944m_1^4m_3^2 \\ \left. - 11264m_1^3m_3^3 - 2944m_1^2m_3^4 - 5632m_1^4m_3^3 - 5632m_1^3m_3^4 - 2816m_1^4m_3^4 \right). \quad (\text{C.93d})$$

We now turn to the equations for the resummation functions $\tilde{h}_{k,j}^\pm$ obtained from the blow-up equation (10.27). We will mostly focus on the equations obtained for $\sigma_1 = -1$ and $\sigma_1 = 0$. In particular, for $\sigma_2 = 1$ and at order $\mathfrak{t}^0$ one finds

$$1 + \frac{(1+2m_3)x\tilde{H}_{1,1}^-(\mathbf{m}, z, -x)}{(1+\gamma_1(\mathbf{m}, x))z} - \frac{1}{2}(1+\gamma_1(\mathbf{m}, x))\tilde{H}_{1,1}^+(\mathbf{m}, z, x) = 0, \quad (\text{C.94a})$$

$$1 - \tilde{H}_{1,1}^-(\mathbf{m}, z, -x) - \left(\frac{1}{2} + m_1\right)xz\tilde{H}_{1,1}^+(\mathbf{m}, z, x) = 0, \quad (\text{C.94b})$$

where we define $\tilde{h}_{1,1}^\pm = \log \tilde{H}_{1,1}^\pm$ in order to linearize the equations. Solving this system we obtain the leading resummation functions

$$\tilde{h}_{1,1}^+(\mathbf{m}, z, x) = \log \left(\frac{\gamma_1(\mathbf{m}, x) + xz(1+2m_1)-1}{xz(1+2m_1)\gamma_1(\mathbf{m}, x)} \right), \quad (\text{C.95a})$$

$$\tilde{h}_{1,1}^-(\mathbf{m}, z, x) = \log \left(\frac{\gamma_1(\mathbf{m}, x) + xz(1+2m_1)+1}{2\gamma_1(\mathbf{m}, x)} \right), \quad (\text{C.95b})$$

which are responsible for the highest-order poles at $2a = \mp 1$, appearing at each order in the $\mathfrak{t}$ -expansion of (10.25). In the limit $x \rightarrow \pm\infty$ one finds

$$\tilde{h}_{1,1}^+(\mathbf{m}, z, x) = \log \left(\frac{1+2m_3 \pm \sqrt{(1+2m_1)(1+2m_3)}z}{xz(1+2m_1)(1+2m_3)} \right) \quad (\text{C.96a})$$

$$\mp \frac{1}{\left(\sqrt{(1+2m_1)(1+2m_3)} \pm z(1+2m_1)\right)x} + \mathcal{O}(x^{-2}), \\ \tilde{h}_{1,1}^-(\mathbf{m}, z, x) = \log \left(\frac{1}{2} \pm \frac{\sqrt{(1+2m_1)(1+2m_3)}z}{2+4m_3} \right) \quad (\text{C.96b})$$

$$\pm \frac{1}{\left(\sqrt{(1+2m_1)(1+2m_3)} \pm z(1+2m_1)\right)x} + \mathcal{O}(x^{-2}),$$

which matches the behavior discussed in Section 10.2.1 of the main text. We can then move on to the equations at order $\mathfrak{t}$, which take the form

$$\frac{(1+2m_3)x(-1+xz(1+2m_1)-\gamma_1(x))}{2z\gamma_1(x)(1+\gamma_1(x))}\tilde{h}_{2,1}^-(z,-x) - \frac{x(r_0^{(-1)}+r_1^{(-1)}x+r_2^{(-1)}x^2)}{8z^2\gamma_1(x)(1+\gamma_1(x))^2} \quad (\text{C.97a})$$

$$+ \frac{(1+\gamma_1(x))(-1+xz(1+2m_1)+\gamma_1(x))}{2(1+2m_1)xz\gamma_1(x)}\tilde{h}_{2,1}^+(z,x) = 0,$$

$$4(1+2m_1)xz((1+2m_3)x+z(1+\gamma_1(x)))\tilde{h}_{2,1}^+(z,x) + x(r_0^{(0)}+r_1^{(0)}x) \quad (\text{C.97b})$$

$$+ (8z(1+\gamma_1(x)) + 4(1+2m_1)xz((1+2m_3)x-z(1+\gamma_1(x))))\tilde{h}_{2,1}^-(z,-x) = 0,$$

where the coefficients $r_i^{(\sigma_1)}$ are given by

$$r_0^{(-1)} = -2\left((1+2m_3)^2 + (3+2m_1-2m_3-12m_1m_3)z^2\right)(1+\gamma_1(x)), \quad (\text{C.98a})$$

$$r_1^{(-1)} = -z(1+2m_3)\left(7+6m_1-2m_3-20m_1m_3+(1+2m_1)^2z^2\right)(1+\gamma_1(x)), \quad (\text{C.98b})$$

$$r_2^{(-1)} = -(1+2m_1)(1+2m_3)^2\left(1+2m_3-(1+2m_1)z^2\right), \quad (\text{C.98c})$$

$$r_0^{(0)} = -z\left(7+6m_1-2m_3-20m_1m_3+(1+2m_1)^2z^2\right. \quad (\text{C.98d})$$

$$\left.- (1+2m_1)(1+2m_3-(1+2m_1)z^2)\gamma_1(x)\right),$$

$$r_1^{(0)} = (1+2m_1)\left((1+2m_3)^2 + (7+m_1(6-20m_3)-2m_3)z^2\right). \quad (\text{C.98e})$$

Solving the equations we find

$$\tilde{h}_{2,1}^\pm(\mathbf{m}, z, x) = \frac{\hat{Q}_{2,1}^\pm(\mathbf{m}, z, x) + \hat{P}_{2,1}^\pm(\mathbf{m}, z, x)\sqrt{1+4w_1^{(1,1)}x^2}}{\hat{D}_{2,1}^\pm(\mathbf{m}, z, x)}, \quad (\text{C.99})$$

where the functions $\hat{Q}_{2,1}^\pm$ and $\hat{P}_{2,1}^\pm$ are polynomials in x of degree two and one respectively, with coefficients

$$\hat{q}_{2,1,0}^+ = 8(1+m_1-2m_1m_3)z^2, \quad (\text{C.100a})$$

$$\hat{q}_{2,1,1}^+ = (1+2m_3)(3+2m_1-2m_3-12m_1m_3)z \quad (\text{C.100b})$$

$$- (1+2m_1)(3+2m_1-2m_3-12m_1m_3)z^3,$$

$$\hat{q}_{2,1,2}^+ = -(1+2m_1)(1+2m_3)^3 + (1+2m_1)^2(1+2m_3)^2z^2, \quad (\text{C.100c})$$

$$\hat{q}_{2,1,0}^- = \hat{q}_{2,1,0}^+z^{-1}, \quad (\text{C.100d})$$

$$\hat{q}_{2,1,1}^- = -\hat{q}_{2,1,1}^+z^{-1}, \quad (\text{C.100e})$$

$$\hat{q}_{2,1,2}^- = (1+2m_1)^2(1+2m_3)^2z - (1+2m_1)^3(1+2m_3)z^3, \quad (\text{C.100f})$$

$$\hat{p}_{2,1,\alpha}^\pm = -\hat{q}_{2,1,\alpha}^\pm, \quad \alpha = 0, 1, \quad (\text{C.100g})$$

while the denominators read

$$\hat{D}_{2,1}^+(\mathbf{m}, z, x) = -4xz(1+2m_1)(1+2m_3)(2z+x((1+2m_3)-(1+2m_1)z^2)), \quad (\text{C.101a})$$

$$\hat{D}_{2,1}^-(\mathbf{m}, z, x) = 4x(1 + 2m_1)(1 + 2m_3)((1 + 2m_3)x - 2z - (1 + 2m_1)xz^2). \quad (\text{C.101b})$$

As $x \rightarrow \pm\infty$, the two resummation functions above behave as

$$\tilde{h}_{2,1}^+(\mathbf{m}, z, x) \simeq \frac{2m_3 + 1}{4z} \pm \frac{3 + 2m_1 - 2m_3 - 12m_1m_3}{4\sqrt{(2m_1 + 1)(2m_3 + 1)}} \quad (\text{C.102a})$$

$$+ \left(\frac{3}{4} - \frac{1}{2(1 + 2m_1)} - \frac{1}{1 + 2m_3} - \frac{1}{2} \frac{1}{1 \pm \sqrt{\frac{2m_1 + 1}{2m_3 + 1}}} z \right) x^{-1},$$

$$\tilde{h}_{2,1}^-(\mathbf{m}, z, x) \simeq \frac{1}{4}(2m_1 + 1)z \pm \frac{3 + 2m_1 - 2m_3 - 12m_1m_3}{4\sqrt{(2m_1 + 1)(2m_3 + 1)}} \quad (\text{C.102b})$$

$$+ \left(\frac{1}{4} - \frac{1}{2(1 + 2m_1)} - \frac{1}{1 + 2m_3} + \frac{1}{2} \frac{1}{1 \pm \sqrt{\frac{2m_1 + 1}{2m_3 + 1}}} z \right) x^{-1}, \quad (\text{C.102c})$$

again confirming the expected behavior discussed in Section 10.2.1. Before turning to the order $\mathfrak{t}^2$ equations, it is useful to consider the order $\mathfrak{t}$ expansions of the blow-up equation (10.27) for $\sigma_1 = -2$ and $\sigma_1 = 0$. Using the explicit expressions for all resummation functions determined so far, these give

$$\partial_x \tilde{h}_{1,2}^+(m_1, m_3, z, 0) = -\frac{m_3(m_3 + 1)}{2z^2}, \quad (\text{C.103a})$$

$$\partial_x \tilde{h}_{1,2}^-(m_1, m_3, z, 0) = -\frac{1}{2}m_1(m_1 + 1)z^2, \quad (\text{C.103b})$$

which will simplify the $\sigma_1 = -1$ and $\sigma_1 = 0$ equations at the next order. Those order- $\mathfrak{t}^2$ equations then determine the subleading resummation functions $\tilde{h}_{3,1}^\pm$ which we do not report here, as their explicit expressions are rather cumbersome. Nevertheless, they are needed in order to determine the next two resummation functions, $\tilde{h}_{1,2}^\pm$ and $\tilde{h}_{2,2}^\pm$.

Rather than extracting the Taylor coefficients in (C.103) from additional expansion points, one could proceed differently: solve the order- $\mathfrak{t}^2$ equations for $\tilde{h}_{3,1}^\pm$ while leaving these coefficients as free parameters, and then fix them by imposing the required analytic behavior of the resummation functions (in particular, regularity as $x \rightarrow \infty$). This prescription fixes precisely the values in (C.103). More generally, similar undetermined Taylor coefficients may appear at intermediate steps and they can be eliminated either by imposing suitable boundary/regularity conditions (e.g. vanishing at $x = 0$ or regularity at $x \rightarrow \infty$), or equivalently by considering further expansion points of the blow-up equation (10.27), as in the previous examples.

Turning to the $\sigma_2 = 2$ equations, and again considering the expansions for $\sigma_1 = -1$ and $\sigma_1 = 0$, we find at leading order (i.e. at order $\mathfrak{t}$)

$$\frac{m_3(1 + m_3)x}{1 + \gamma_2(\mathbf{m}, x)} \tilde{H}_{1,2}^-(m_1, m_3, z, -x) + \frac{1 + \gamma_2(\mathbf{m}, x)}{2} z^2 \tilde{H}_{1,2}^+(m_1, m_3, z, x) - z^2 = 0 \quad (\text{C.104a})$$

$$2 - 2\tilde{H}_{1,2}^-(m_1, m_3, z, -x) + m_1(1 + m_1)xz^2 \tilde{H}_{1,2}^+(m_1, m_3, z, x) = 0 \quad (\text{C.104b})$$

where we again perform the substitution $\tilde{h}_{1,j}^\pm = \log \tilde{H}_{1,j}^\pm$. The system can be solved to give

$$\tilde{h}_{1,2}^+(m_1, m_3, z, x) = \log \left(\frac{-\gamma_2(\mathbf{m}, x) + m_1(m_1 + 1)xz^2 + 1}{m_1(m_1 + 1)xz^2\gamma_2(\mathbf{m}, x)} \right), \quad (\text{C.105a})$$

$$\tilde{h}_{1,2}^-(m_1, m_3, z, x) = \log \left(\frac{\gamma_2(\mathbf{m}, x) - m_1(m_1 + 1)xz^2 + 1}{2\gamma_2(\mathbf{m}, x)} \right), \quad (\text{C.105b})$$

and these functions account for the highest-order poles at $a = \mp 1$ appearing at every order in the $\mathbf{t}$ -expansion of (10.25). Expanding them for $x \rightarrow \pm\infty$, one finds

$$\tilde{h}_{1,2}^+(m_1, m_3, z, x) = \log \left(\frac{\pm \sqrt{m_1(m_1+1)m_3(m_3+1)}z^2 - m_3(m_3+1)}{m_1(m_1+1)m_3(m_3+1)xz^2} \right) \quad (\text{C.106a})$$

$$+ \frac{1}{x} \frac{1}{m_1(m_1+1)z^2 \mp \sqrt{m_1(m_1+1)m_3(m_3+1)}} + \mathcal{O}(x^{-2}),$$

$$\tilde{h}_{1,2}^-(m_1, m_3, z, x) = \log \left(\frac{1}{2} \mp \frac{1}{2} \frac{m_1(m_1+1)z^2}{\sqrt{m_1(m_1+1)m_3(m_3+1)}} \right) \quad (\text{C.106b})$$

$$- \frac{1}{x} \frac{1}{m_1(m_1+1)z^2 \mp \sqrt{m_1(m_1+1)m_3(m_3+1)}} + \mathcal{O}(x^{-2}).$$

Turning to the next set of equations at order $\mathbf{t}$, we find:

$$2m_3(1+m_3)x(1+2m_3-(1+2m_1)z^2)(1+\gamma_2(\mathbf{m}, x) + m_1(1+m_1)xz^2) \quad (\text{C.107a})$$

$$+ 3m_3(1+m_3)xz(1+\gamma_2(\mathbf{m}, x) + m_1(1+m_1)xz^2)\tilde{h}_{2,2}^-(\mathbf{m}, z, -x)$$

$$+ 3z(2(1+\gamma_2(\mathbf{m}, x))z^2 + m_3x(-1-\gamma_2(\mathbf{m}, x) + m_1(1+m_1)xz^2))$$

$$+ m_3^2x(-1-\gamma_2(\mathbf{m}, x) + m_1(1+m_1)xz^2)\tilde{h}_{2,2}^+(\mathbf{m}, z, x) = 0,$$

$$(-1+\gamma_2(\mathbf{m}, x) - xz^2m_1(1+m_1))(2+4m_3-(2+4m_1)z^2 - 3z\tilde{h}_{2,2}^+(\mathbf{m}, z, x)) \quad (\text{C.107b})$$

$$- 3z(1+\gamma_2(\mathbf{m}, x) + xz^2m_1(1+m_1))\tilde{h}_{2,2}^-(\mathbf{m}, z, -x) = 0,$$

which can again be solved straightforwardly, giving:

$$\tilde{h}_{2,2}^+(\mathbf{m}, z, x) = \frac{2(1+2m_3-(1+2m_1)z^2)(z^2(1-\gamma_2(\mathbf{m}, x)) - m_3(m_3+1)x)}{3m_1(m_1+1)xz^5 - 3m_3(m_3+1)xz + 6z^3}, \quad (\text{C.108a})$$

$$\tilde{h}_{2,2}^-(\mathbf{m}, z, x) = \frac{2z((2m_1+1)z^2 - 2m_3 - 1)(\gamma_2(\mathbf{m}, x) + m_1(m_1+1)xz^2 - 1)}{3m_1(m_1+1)xz^4 - 3m_3(m_3+1)x - 6z^2}. \quad (\text{C.108b})$$

As $x \rightarrow \pm\infty$, the two functions behave as

$$\tilde{h}_{2,2}^+(\mathbf{m}, z, x) = \frac{2}{3z} \frac{(1+2m_3-(1+2m_1)z^2)}{m_3(m_3+1) - m_1(m_1+1)z^4} \quad (\text{C.109a})$$

$$\times \left(\pm z^2 \sqrt{m_1(m_1+1)m_3(m_3+1)} + m_3(1+m_3) \right) + \mathcal{O}(x^{-1}),$$

$$\tilde{h}_{2,2}^-(\mathbf{m}, z, x) = \frac{2z}{3} \frac{((2m_1+1)z^2 - 1 - 2m_3)}{m_1(m_1+1)z^4 - m_3(m_3+1)} \quad (\text{C.109b})$$

$$\times \left(\pm \sqrt{m_1(m_1+1)m_3(m_3+1)} + m_1(m_1+1)z^2 \right) + \mathcal{O}(x^{-1}), \quad (\text{C.109c})$$

and one can also notice that they are related by

$$\tilde{h}_{2,2}^+(m_1, m_3, z, x) = \tilde{h}_{2,2}^-(m_3, m_1, z^{-1}, x). \quad (\text{C.110})$$

Finally, we can determine the corresponding h -functions from the relation (10.26). As input, we use the functions $\tilde{f}_{1,1}$, $\tilde{f}_{1,2}$, $\tilde{f}_{2,1}$, and $\tilde{g}_{1,1}$, where

$$\tilde{f}_{2,1}(\mathbf{m}, x) = \frac{3(2m_1+1)^2(2m_3+1)^2 - 2(2m_1+1)^2 - 2(2m_3+1)^2}{32(1+(2m_1+1)(2m_3+1)x^2)}x^2, \quad (\text{C.111a})$$

$$\tilde{g}_{1,1}(\mathbf{m}, x) = -\frac{(1-4m_1m_3)(1-\sqrt{1+(2m_1+1)(2m_3+1)x^2})}{2(2m_1+1)(2m_3+1)x}, \quad (\text{C.111b})$$

while the remaining two functions can be found in (10.18). In this way, we recover $h_{1,1}^\pm$ and $h_{1,2}^\pm$ in the form given in (10.31), and

$$h_{2,1}^\pm(\mathbf{m}, z, x) = \tilde{h}_{2,1}^\pm(\mathbf{m}, z, x) - \tilde{g}_{1,1}(\mathbf{m}, x), \quad (\text{C.112a})$$

$$h_{3,1}^\pm(\mathbf{m}, z, x) = \tilde{h}_{3,1}^\pm(\mathbf{m}, z, x) - \tilde{f}_{2,1}(\mathbf{m}, x), \quad (\text{C.112b})$$

$$h_{2,2}^\pm(\mathbf{m}, z, x) = \tilde{h}_{2,2}^\pm(\mathbf{m}, z, x). \quad (\text{C.112c})$$

Appendix D

Figures and numerical evidence

In this appendix, we present additional numerical evidence supporting our results. We focus on the cubic and quartic potentials, (11.42) and (11.46) respectively, and display, for various choices of parameters, the ground, first, and second excited states for the cubic case, the first and second excited states for the quartic case.

In Figure D.1 we show the ground-state wavefunction for the cubic potential for $h_2 = -5$. Due to the deep local minimum of the potential, the wavefunction closely resembles that of a bound state. In Figure D.2 we show the first excited state for $h_2 = -3$, together with the two singular saddles, making the cancellation of poles in their sum evident. Finally, for the cubic case, Figure D.3 shows the wavefunction corresponding to the second excited state for the cubic potential with $h_2 = 2$. In this case, the potential has no real stationary points, yet resonant eigenfunctions still exist. Moving on to the quartic case, in Figure D.4 we show the wavefunction of the first excited state for the symmetric ($h_3 = 0$) single well potential with $h_2 = 3.5$, together with the two singular saddle contributions. The on-shell solution is antisymmetric and normalized to 1 at $x = 1/2$. Finally, Figure D.5 displays the wavefunction corresponding to the second excited state of the quartic potential with $h_2 = 1.2$ and $h_3 = -4.7$. The corresponding potential is an asymmetric quartic single well with a unique minimum.

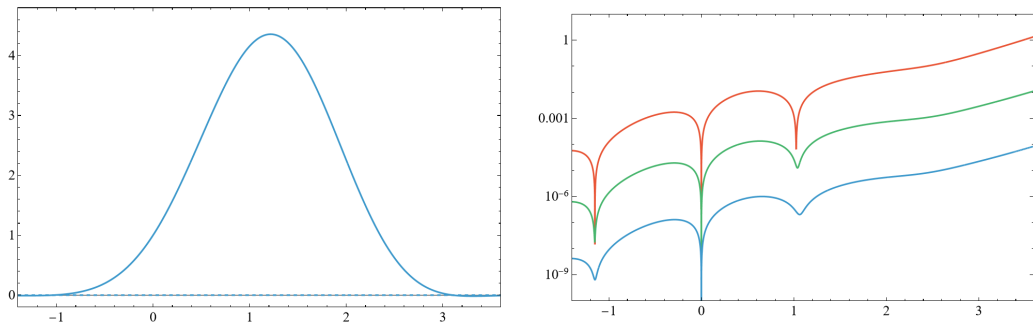


Figure D.1: Left: ground state of the $V_3(x)$ potential with $h_2 = -5$, $\hbar = 1$ and $\Lambda = (3/5)^{1/2}$; the corresponding complex energy is $E_0 \approx -1.855383 + i 5.697572 \cdot 10^{-9}$. Dashed lines denote the imaginary part of the eigenfunction, while solid lines denote the real part. Right: difference between the numerical eigenfunction and the analytic expression from (11.31). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

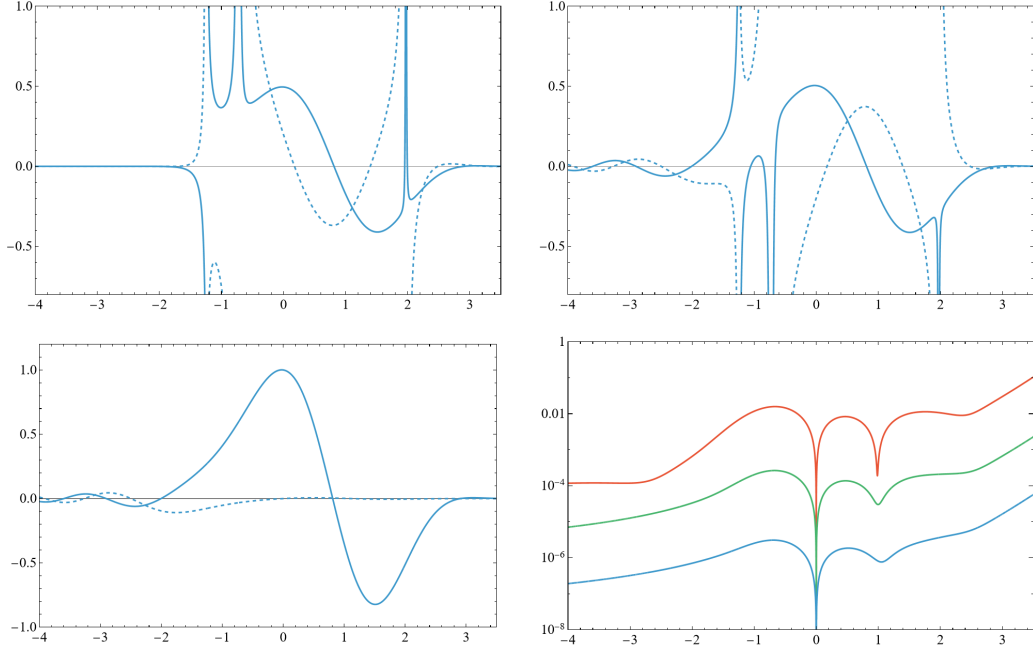


Figure D.2: The first excited state of the $V_3(x)$ potential with $h_2 = -3$, $\hbar = 1$, and $\Lambda = \frac{2}{3}$; the corresponding complex energy is $E_1 \approx 1.758084 + i0.010614$. Dashed lines denote the imaginary part of the eigenfunction, while solid lines denote the real part. Top panels: first (left) and second (right) saddle contributions in the analytic expression (11.31), evaluated on-shell. The two functions develop poles individually, which cancel in the sum. Bottom left: full on-shell eigenfunction from (11.31). Bottom right: difference between the numerical eigenfunction and the analytic result from (11.31). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

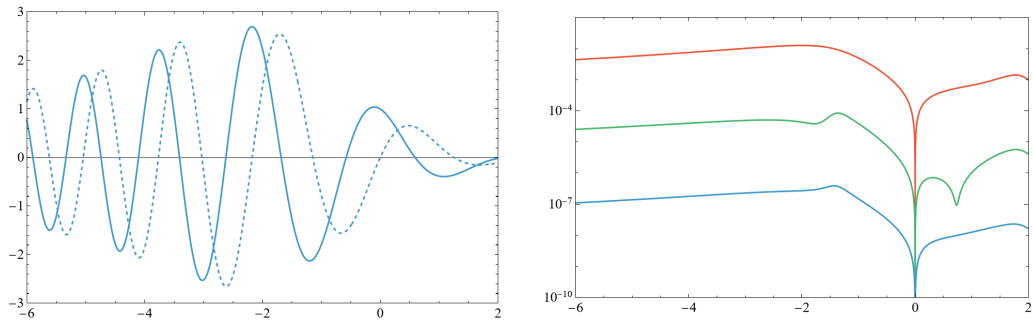


Figure D.3: Left: second excited state of the $V_3(x)$ potential with $h_2 = 2$, $\hbar = 1$ and $\Lambda = \frac{9}{10}$; the corresponding complex energy is $E_2 \approx 7.842147 + i8.389762$. Dashed lines denote the imaginary part of the eigenfunction, while solid lines denote the real part. Right: difference between the numerical eigenfunction and the analytic expression from (11.31). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

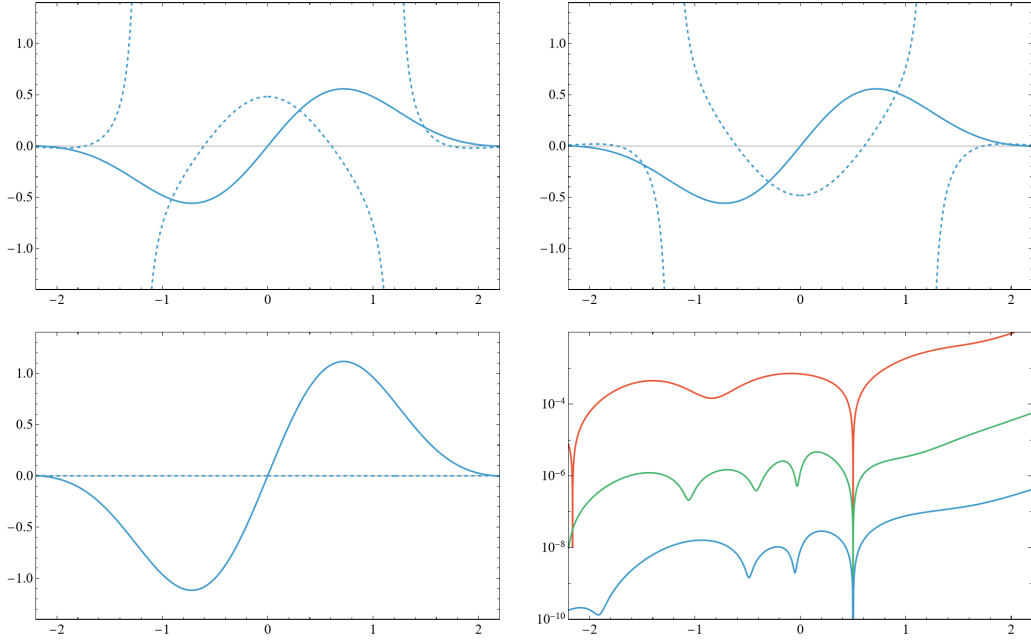


Figure D.4: First excited state of the symmetric $V_4(x)$ potential with $h_2 = 3.5$, $h_3 = 0$, $\hbar = 1$ and $\Lambda = \frac{7}{8}$; the corresponding energy is $E_1 \approx 7.324917$. Dashed lines denote the imaginary part of the eigenfunction, while solid lines denote the real part. Top panels: first (left) and second (right) saddle contributions in the analytic expression (11.25), evaluated on-shell. The two functions develop poles individually, which cancel in the sum. Bottom left: full on-shell eigenfunction from (11.25). Bottom right: difference between the numerical eigenfunction and the analytic result from (11.25). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

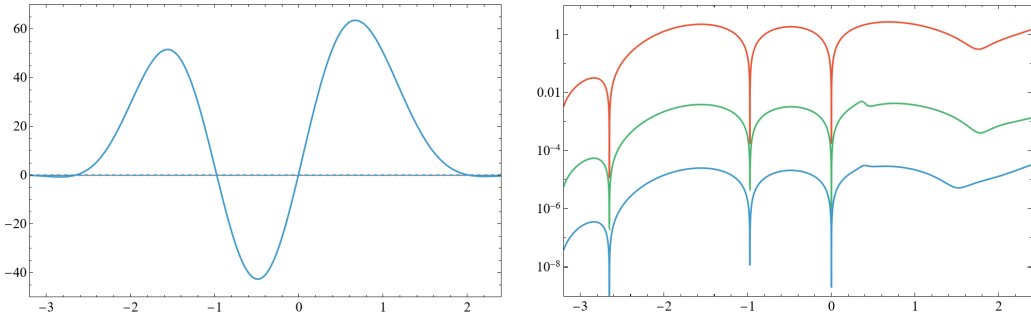


Figure D.5: Left: second excited state of the $V_4(x)$ potential with $h_2 = 1.2$, $h_3 = -4.7$, $\hbar = 1$ and $\Lambda = \frac{4}{5}$; the corresponding energy is $E_2 \approx 8.094909$. Dashed lines represent the imaginary part of the eigenfunction, while solid lines represent the real part. Right: difference between the numerical eigenfunction and the analytic expression from (11.25). The coloured curves show the effect of including an increasing number of terms in the Λ -expansion of the eigenfunction: red (0 terms), green (1 term), blue (2 terms).

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