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A Non-Commutative Approach to Rigid Analytic Geometry

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Alla mia famiglia.

Abstract

This thesis consists mainly of three chapters, based respectively on [Mon25; Mon26; MP25].

In Chapter 1, we extend a theorem of Toën and Vaquié to the non-Archimedean and formal settings. More precisely, we prove that a smooth and proper rigid analytic variety is algebraizable if and only if its category of perfect complexes is smooth and proper. We will then use this result to prove an analog statement for formal schemes. In particular, in this chapter we highlight that the notion of smooth and proper categories is, classically, intrinsically algebraic.

In Chapter 2, we introduce a notion of smooth and proper categories adapted to the analytic setting. We achieve this by first studying the notion of smooth categories within the framework of a six-functor formalism. Subsequently, using the theory of condensed mathematics and analytic stacks, we demonstrate that a rigid analytic variety is smooth if and only if its associated category of nuclear sheaves is smooth. Furthermore, we relate the atomic generation of the category of nuclear sheaves to the algebraization of the rigid analytic variety. We then employ these results to obtain an example of a non-atomically generated but internally smooth category.

In Chapter 3, which is joint work with Emanuele Pavia, we develop a general framework in which to study the notion of 1-affineness, originally introduced by Gaitsgory in [Gai15]. We then apply this framework to establish 1-affineness for sheaves of nuclear categories over rigid analytic varieties, for the Betti stack (as studied in [Sch24]) of a finite-dimensional metrizable compact Hausdorff space, and, as a corollary, for the analytic de Rham stack of a compact complex manifold.

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Introduction

0.1 A quick overview of the thesis

Over the last few decades, algebraic geometry has undergone a profound paradigm shift, evolving toward the intrinsic analysis of varieties via their associated categories. In the modern development of non-commutative algebraic geometry, highly structured categories (such as stable ∞ -categories) are treated as geometric spaces. Within this landscape, the precise translation of geometric properties, such as a variety being smooth or proper, into purely categorical definitions over $\mathrm{QCoh}(X)$ or $\mathrm{Perf}(X)$ has become a foundational tool. A natural ambition is to transport this powerful categorical machinery into the realm of (rigid) analytic geometry.

Another major direction is the comparison between algebraic and analytic spaces, as illustrated by GAGA-style theorems and the algebraization problem: understanding the precise conditions under which a rigid analytic variety X arises as the analytification of a scheme.

As explicitly formalized in Definition 0.2.1, the standard notion of smooth and proper ∞ -categories crucially relies on the tensor product of stable presentable ∞ -categories introduced in [Lur17]. This tensor product is intrinsically an “algebraic” operation and behaves poorly in the analytic setting. For example, the categorical Künneth formula

fails for Tate algebras over a non-Archimedean field k , in the sense that

$$\mathrm{Perf}(k\langle t \rangle) \otimes_k \mathrm{Perf}(k\langle u \rangle) \neq \mathrm{Perf}(k\langle t, u \rangle). \quad (1)$$

Consequently, the classical definition of a smooth and proper ∞ -category is essentially algebraic in nature and ill-behaved in the analytic framework.

Tate algebras are the fundamental building blocks of rigid analytic geometry. They are utilized to construct rigid analytic varieties in much the same way that affine schemes are used to construct schemes in algebraic geometry (see [Bos14] for a comprehensive introduction). To a first approximation, rigid analytic varieties can intuitively be viewed as analogues of complex analytic varieties, suitably adapted to the setting of non-Archimedean norms.

Throughout this paper, we consider rigid analytic varieties over a complete non-Archimedean field k of characteristic 0, and we assume all such varieties are locally of finite type.

Following [Bos14, Section 5.4], it is possible to define the *analytification functor* $(-)^{\mathrm{an}}$ from the category of schemes locally of finite type over k to the category of rigid analytic varieties over k .

One of the primary aims of this Ph.D. thesis is to characterize the essential image of this functor using categorical properties of certain categories of sheaves. Along this direction, one of the main theorems that we will prove in Chapter 1 is the following:

Theorem 0.1.1 ([Mon25, Theorem 3.1]). *Let X be a smooth, proper, quasi-compact, and separated rigid analytic variety over k . Then $\mathrm{Perf}(X)$ is smooth and proper as a k -linear ∞ -category if and only if X is algebraizable (i.e., there exists an algebraic space Y such that $Y^{\mathrm{an}} \simeq X$).*

Although this theorem provides a rigorous characterization of the essential image of the analytification functor, it reveals a fundamental limitation of the notion of smooth ∞ -category. Indeed, this theorem is a clear manifestation of the fact that the notion of a smooth ∞ -category in Definition 0.2.1 is intrinsically algebraic and is not suitable for characterizing smooth analytic varieties. We refer to [TVo8a] for a closely related theorem in complex analytic geometry.

In Chapter 2, we propose an approach to this problem based on the theory of condensed mathematics and analytic stacks developed by Clausen–Scholze. In particular, we will introduce a categorical criterion to detect the smoothness of rigid analytic varieties. We will achieve this result by first studying the notion of smooth ∞ -categories in a general six-functor formalism. Subsequently, by studying the nuclear ∞ -categories (Definition 2.2.6) associated with rigid analytic varieties, we will describe a six-functor formalism for this sheaf of ∞ -categories over rigid analytic varieties. Combining these

results, we will prove the following:

Theorem 0.1.2 ([Mon26, Theorem 3.4.1]). *Let $f: X \rightarrow \mathrm{Sp}(k)$ be a (classical) rigid analytic variety over k . Then X is smooth if and only if the ∞ -category $\mathrm{Nuc}(X)$ is internally smooth in $\mathrm{Pr}_{\mathrm{Nuc}(k)}^{\mathrm{L}}$.*

In this way, we also obtain a notion of categorical smoothness that is well-defined within the analytic setting.

Furthermore, we will use this result to obtain another categorical characterization of the essential image of the analytification functor, thereby introducing a categorical criterion to detect the algebraizability of a rigid analytic variety. In particular, we will prove the following:

Theorem 0.1.3 ([Mon26, Theorem 3.4.9]). *Let $p: X \rightarrow \mathrm{Sp}(k)$ be a smooth, proper, quasi-compact, and separated rigid analytic variety over $\mathrm{Sp}(k)$. Then X is algebraizable if and only if $\mathrm{Nuc}(X)$ admits a compact (equivalently, $\mathrm{Nuc}(k)$ -atomic) generator.*

Finally, we apply these results to construct an interesting class of examples of internally smooth enriched ∞ -categories that are not atomically generated. This counterexample is motivated by the work of Maxime Ramzi [Ram24a; Ram24b]. Indeed, Stefanich proved that a smooth, presentable ∞ -category linear over a bounded $\mathbb{E}_{\infty}$ -ring has a single compact generator.

In *loc. cit.*, the author introduces the notion of atomic objects; this concept is a suitable adaptation of the standard notion of compact objects to the setting of enriched categories. Working over a rigid base, the notions of atomic and compact objects coincide. It is therefore a natural question, already proposed in [Ram24a], whether the existence of a single atomic generator holds for every smooth enriched ∞ -category :

Question 0.1.4. Let $\mathcal{V}$ be a commutative algebra in Pr^{L} , and let $\mathcal{C}$ be a presentable ∞ -category linear over $\mathcal{V}$ and smooth in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$. Is it true that $\mathcal{C}$ has a single atomic generator?

Our result provides a natural counterexample to this question. It can also be interpreted as evidence that the existence of a global compact (or atomic) generator is a peculiar feature of algebraic geometry and cannot be naturally extended to the analytic setting. For the ∞ -category of quasi-coherent sheaves on schemes, the existence of a compact generator has been established in [Toë12b] and [BB03].

The third part of this thesis is based on a joint work with Emanuele Pavia, [MP25]. The notion of 1-affineness for (pre)stack in derived algebraic geometry has been introduced by Gaitsgory in [Gai15].

Just as quasi-coherent sheaves on an affine scheme $\mathrm{Spec}(A)$ are completely determined by the A -module structure on their global sections, this property actually characterizes affine schemes among all quasi-separated schemes. A 1-affine (pre)stack is defined as a (pre)stack for which any quasi-coherent sheaf of categories is completely determined by some $\mathrm{QCoh}(X)$ -linear ∞ -category. More precisely, we have a functor

$$\Gamma_X: \mathrm{ShvCat}(X) \longrightarrow \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathrm{Pr}^{\mathrm{L}})$$

sending a sheaf of categories $\mathcal{C} \in \mathrm{ShvCat}(X)$ to its ∞ -category of global sections $\Gamma_X(\mathcal{C})$, which carries a natural action of the monoidal ∞ -category $\mathrm{QCoh}(X)$. The prestack X is then said to be 1-affine if this functor is an equivalence of ∞ -categories.

Establishing whether a given prestack is 1-affine is a highly nontrivial problem that carries deep geometric and categorical implications. It allows one to study categorified sheaf-theoretic phenomena by reducing them to the study of a single ∞ -category of modules over a symmetric monoidal ∞ -category.

The main aim of Chapter 3 is to prove this property for analytic stacks (as introduced by Clausen-Scholze [CS24]) and for rigid analytic varieties. In particular, by developing an abstract setting in which to study the notion of 1-affineness, we will prove the following results.

Theorem 0.1.5 ([MP25, Theorem 3.1.2]). *Let X be an analytic stack admitting an affine $!$ -cover $\{\mathrm{AnSpec}(A_i) \rightarrow X\}_{i \in I}$ such that all the fiber products $\mathrm{AnSpec}(A_{i_1}) \times_X \cdots \times_X \mathrm{AnSpec}(A_{i_n})$ are affine. Then X is 1-affine.*

This theorem has also been independently proved in [Kes25], which was written and appeared on arXiv during the final stages of preparation of [MP25]. Although the underlying approach is very similar, we obtain a slightly more general 1-affineness result (see Theorem 3.1.35). Nevertheless, for the applications described later in this paper, such as the 1-affineness of the Betti stack, the formulation of Theorem 3.3.2 suffices.

Using different methods, we will prove the following 1-affineness result for rigid analytic varieties, considering nuclear ∞ -categories.

Theorem 0.1.6 ([MP25, Theorem 3.3.3]). *Let X be a quasi-compact and semi-separated derived rigid analytic space over k . Then X is 1-affine with respect to the sheaf of ∞ -categories $\mathrm{Nuc}(-)$. More precisely, there is an equivalence of ∞ -categories*

$$\mathrm{Pr}_{\mathrm{Nuc}(X)}^{\mathrm{L}} \simeq \lim_{\mathrm{dSp}(A) \subset X} \mathrm{Pr}_{\mathrm{Nuc}(A)}^{\mathrm{L}},$$

where the limit is taken over an affinoid cover of X .

0.2 A detailed overview of the thesis

Since approximately two-thirds of this thesis is devoted to the notion of smooth categories, we begin this overview by reviewing this concept, introduced by Kontsevich [KS09], and examining how it has been studied within algebraic geometry, along with the outcomes it has produced. Subsequently, we will detail the main results of this thesis, which are primarily based on the application of smooth categories within the framework of rigid analytic geometry. The third part of the thesis deals with the notion of 1-affineness in the analytic setting. This topic reveals an important property of specific categories of sheaves on certain analytic stacks (as introduced by Clausen–Scholze) and on rigid analytic varieties.

0.2.1 Non-commutative smoothness in algebraic geometry

The investigation of an algebraic variety X via its abelian category of quasi-coherent sheaves, $\mathrm{QCoh}(X)$, or its subcategory of coherent sheaves, $\mathrm{Coh}(X)$, perfectly captures its essential geometric properties; classical reconstruction theorems (such as those of Rosenberg [Bra18]) even demonstrate that these categories can fully recover the underlying space. Nevertheless, this abelian framework ultimately proved too rigid for broader geometric and categorical manipulations.

To obtain a more flexible and homologically robust framework, the focus naturally shifted toward the associated triangulated categories. Following the seminal work of Bondal and Orlov [BO01], it became evident that the bounded derived category of coherent sheaves, $\mathrm{D}^b\mathrm{Coh}(X)$, encodes a wealth of deep geometric information about X . However, classical triangulated categories are known to suffer from technical shortcomings, most notably the lack of functorial cones and the absence of well-behaved limits and colimits.

These technical limitations necessitated a further conceptual leap: lifting these triangulated structures to appropriate homotopical enhancements. By employing *differential graded (dg) categories* or, more broadly, *∞ -categories*, one can retain the higher homotopical data that is inevitably lost upon passing to the standard homotopy category. Consequently, the field has increasingly focused on the ∞ -category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ and the category of perfect complexes $\mathrm{Perf}(X)$ (the ∞ -categorical generalization of vector bundles).

Building on the pioneering work of Orlov [Orl16] in the triangulated setting, the subsequent evolution of categorical enhancements paved the way for the modern development of *non-commutative algebraic geometry*. In this framework, highly structured categories (such as dg-categories or stable ∞ -categories) are no longer viewed merely

as invariants of spaces, but are treated as geometric spaces in their own right. Within this landscape, the notion of a *smooth category* (respectively, a *proper category*) emerges as the natural and necessary generalization of a classical smooth (respectively, proper) algebraic variety.

Formally, following the philosophy introduced by Kontsevich [Kon98], a dg-category $\mathcal{C}$ is defined as *smooth* if the diagonal bimodule represented by $\mathcal{C}$ in $\mathcal{C} \otimes \mathcal{C}^{\text{op}}$ is a compact object. In the ∞ -categorical setting, the definitions of smooth and proper ∞ -categories can be rephrased as follows:

Definition 0.2.1. Let $\mathcal{C}$ be a stable presentable ∞ -category, and $\mathcal{V}$ a symmetric monoidal, stable, presentable ∞ -category.

1. We say that $\mathcal{C}$ is dualizable if there exists a stable presentable ∞ -category $\mathcal{C}^\vee$, together with two functors in $\text{Pr}_{\mathcal{V}}^{\text{L}}$, called the evaluation and coevaluation functors:

$$\text{ev} : \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^\vee \rightarrow \mathcal{V}$$

$$\text{coev} : \mathcal{V} \rightarrow \mathcal{C}^\vee \otimes_{\mathcal{V}} \mathcal{C},$$

such that the composite maps

$$\mathcal{C} \xrightarrow{\text{id} \boxtimes \text{coev}} \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^\vee \otimes_{\mathcal{V}} \mathcal{C} \xrightarrow{\text{ev} \boxtimes \text{id}} \mathcal{C} \quad (2)$$

$$\mathcal{C}^\vee \xrightarrow{\text{coev} \boxtimes \text{id}} \mathcal{C}^\vee \otimes_{\mathcal{V}} \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^\vee \xrightarrow{\text{id} \boxtimes \text{ev}} \mathcal{C}^\vee \quad (3)$$

are equivalent to the respective identity functors.

2. We say that $\mathcal{C}$ is *smooth* if the coevaluation map admits a right adjoint that preserves colimits.
3. We say that $\mathcal{C}$ is *proper* if the evaluation map admits a right adjoint that preserves colimits.

Similarly, if $\mathcal{C}$ is a small, idempotent complete ∞ -category, we say that $\mathcal{C}$ is smooth (respectively, proper) if its ind-completion is smooth (respectively, proper) as in Definition 0.2.1.

When $\mathcal{C}$ is the ∞ -category of quasi-coherent sheaves associated with an algebraic variety X , the condition that $\mathcal{C}$ is smooth (respectively, proper) is strictly equivalent to X being a smooth (respectively, proper) variety in the classical sense (see [Lun10] and [BB03]). We will provide an alternative proof of these known results in a more general setting in Section 2.5.1. However, the true power of this definition lies in its applicability

to purely non-commutative contexts, decoupling the condition of regularity from the existence of an underlying topological space.

Building on these ideas, Toën and Vaquié demonstrated that smooth and proper dg-categories $\mathcal{C}$ admit well-behaved moduli stacks of objects [TV07], denoted with $\mathcal{M}_{\mathcal{C}}$. This foundational result provides a systematic method for constructing stacks of geometric interest from purely categorical data. This construction provides a natural bridge between non-commutative and derived algebraic geometry and provides an interesting way of constructing moduli spaces (or better stacks) in algebraic geometry. Indeed when $\mathcal{C} = \text{Perf}(X)$ for a smooth and proper variety X , one recovers the moduli stack of perfect complexes on X , together with substacks defined by imposing stability conditions. The geometry of such moduli spaces is rich and well-studied, with significant and recent contributions in the literature, such as in [BLM+21] and [HR25].

Currently, categorical smoothness is not merely a technical artifact, but a foundational requirement driving several major research programs in algebraic geometry. For instance, following the seminal works of Lunts [Lun10] and Kuznetsov [KL15], the concept of smooth categories allows for the introduction of *categorical resolutions of singularities*. A categorical resolution consists of a fully faithful embedding of the category of perfect complexes, $\text{Perf}(X)$, into a smooth and proper dg-category.

In the realm of *non-commutative algebraic geometry*, the concept of smooth and proper categories is now a central pillar. It facilitates the extension of fundamental algebraic geometric results to the non-commutative setting, while simultaneously presenting new challenges at the interface of the commutative and non-commutative worlds (see, for example, [Per19] and [BP24]).

Extending these ideas to the context of analytic geometry is a natural question. In this thesis, we explore several concepts related to this inquiry. In particular, we discuss the challenges that arise and present a possible approach to this problem, yielding some positive results in this direction.

0.2.2 Smooth and proper categories of perfect complexes in analytic geometry

This Chapter is based on [Mon25].

The main aim of this Chapter is to study the category of perfect complexes on a rigid analytic variety and on a formal scheme, along with the notions of smooth and proper categories. We will demonstrate that in both settings, the situation differs significantly from the algebraic setting. To develop these ideas in the formal and analytic settings, we encounter various difficulties. First, in derived analytic geometry, the definition of

quasi-coherent sheaves is not straightforward. The recent work of Clausen and Scholze addresses this problem by developing the theory of *Condensed Mathematics* [Sch26b] and using the ∞ -category of solid quasi-coherent sheaves (see [Sch26a; CS26]). An alternative approach can be found in [BKK24; BB16], where the authors develop the theory of *Bornological Mathematics* and propose defining the ∞ -category of “analytic quasi-coherent sheaves” as the ∞ -category of Ind-Banach modules.

For our purposes, it suffices to work with the category of perfect complexes on an analytic variety. This provides a much simpler framework where the subtleties inherent in developing a full theory of “analytic quasi-coherent sheaves” are avoided. In particular, we define perfect complexes as in [PY21, Definition 7.5]. We utilize the foundations of derived non-Archimedean analytic geometry as developed in [HP25; PY21; PY18; PY16], and derived formal geometry as developed in Chapter 8 of [Lur18] and [Ant19]. We remark that the category of perfect complexes on an analytic variety can also be studied within the settings of Condensed and Bornological mathematics: indeed, as explained in [And21], perfect complexes can be recovered as a full subcategory of the ∞ -categories of “analytic quasi-coherent sheaves.” However, unlike the usual setting of algebraic geometry, perfect complexes are no longer compact generators of these ∞ -categories. We will show that the ∞ -category of perfect complexes in the non-Archimedean analytic and formal settings is not as well-behaved as it is in the algebraic setting.

Throughout this paper, we consider rigid analytic varieties over a complete non-Archimedean field k of characteristic 0, and we assume all such varieties are locally of finite type. A standard reference for this topic is [Bos14]. Furthermore, [Bos14, Section 5.4] describes the analytification functor $(-)^{\text{an}}$ from the category of schemes locally of finite type over k to the category of rigid analytic varieties over k . In our main result, we characterize the essential image of this functor in certain favorable cases via the ∞ -category of perfect complexes. Indeed, our main theorem is the following:

Theorem 0.2.2 (Theorem 1.3.1). *Let X be a smooth and proper, quasi-compact, and separated rigid analytic variety over k . Then $\text{Perf}(X)$ is smooth and proper as a k -linear ∞ -category if and only if X is algebraizable (i.e., there exists an algebraic space Y such that $Y^{\text{an}} \simeq X$).*

From this theorem, we deduce that, in general, the category of perfect complexes of a smooth and proper rigid analytic variety is not smooth and proper, standing in stark contrast to the situation in algebraic geometry [Orl16]. This suggests that the notion of smooth and proper categories is intrinsically algebraic, and is not inherently well-adapted to the non-Archimedean and formal settings.

One of the main technical ingredients in our argument is an extension of the notion of simple objects from [TV07] to derived rings and rigid analytic varieties. In [TV07], Toën and Vaquié define, for a compactly generated k -linear dg-category $\mathcal{C}$ of finite type,

the derived moduli stack of objects

$$\mathcal{M}_{\mathcal{C}}: \mathbf{CAlg}_k \rightarrow \mathcal{S}$$

from the ∞ -category of derived commutative algebras to the ∞ -category of spaces. The ∞ -categorical analogue of this functor is informally described by sending a derived commutative algebra A over k to the maximal groupoid of the ∞ -category of k -linear stable functors $\mathrm{Fun}_k^{\mathrm{st}}((\mathcal{C}^\omega)^{\mathrm{op}}, \mathrm{Perf}(A))$. In [TV07], the authors also introduce the (underived) substack of simple objects $t_0\mathcal{M}_{\mathcal{C}}^{\mathrm{sim}} \subset t_0\mathcal{M}_{\mathcal{C}}$.

We extend both constructions to the derived analytic setting by introducing the (rigid) analytic derived stack of objects $\mathcal{An}\mathcal{M}_{\mathcal{C}}$, where $\mathcal{C}$ is a smooth and proper ∞ -category, and the analytic derived substack of simple objects $\mathcal{An}\mathcal{M}_{\mathcal{C}}^{\mathrm{sim}}$. In Definition 1.2.5, we also introduce a general notion of simple objects in a linear ∞ -category, which allows us to construct the derived stack of simple objects $\mathcal{M}_{\mathcal{C}}^{\mathrm{sim}}$. Furthermore, within this general framework, we introduce the analytic derived stack of simple perfect complexes $\mathrm{Perf}_X^{\mathrm{sim}}$ for a rigid analytic variety X .

We then establish an equivalence between the analytic derived stack of simple objects $\mathcal{An}\mathcal{M}_{\mathcal{C}}^{\mathrm{sim}}$ and the analytification of the algebraic stack of simple objects $(\mathcal{M}_{\mathcal{C}}^{\mathrm{sim}})^{\mathrm{an}}$ (Proposition 1.2.45). Finally, if X is a proper rigid analytic variety such that $\mathrm{Perf}(X)$ is smooth and proper, we show that $\mathrm{Perf}_X^{\mathrm{sim}}$ is equivalent to the analytic derived stack $\mathcal{An}\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}}$ (Proposition 1.2.47). These technical developments are detailed in Section 2 and constitute a key ingredient in the proof of our main theorem. Indeed, by employing the theory of analytic stacks developed in Section 2, we show in the final section that there exists an embedding of X into the coarse moduli space of $(t_0\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}})^{\mathrm{an}}$ (see Proposition 1.3.4). This embedding, together with the concept of non-Archimedean Moishezon spaces introduced in [Con07], provides the main tools used in the proof of Theorem 1.3.1.

As a consequence of our main theorem, we prove two related results on algebraizability for formal schemes. In this setting, we consider formal schemes over $\mathrm{Spf}(\mathbb{O}_k)$, where $\mathbb{O}_k$ is a DVR with maximal ideal $\mathfrak{m}$ and fraction field k . We assume that $\mathbb{O}_k$ is complete with respect to the $\mathfrak{m}$ -adic topology. All formal schemes are assumed to be separated, quasi-compact, topologically almost of finite presentation, and locally Noetherian.

Theorem 0.2.3 (Theorem 1.3.13). *Let $\mathcal{X}$ be a smooth and proper formal scheme over $\mathrm{Spf}(\mathbb{O}_k)$. Then $\mathrm{Perf}(\mathcal{X})$ is smooth and proper as an $\mathbb{O}_k$ -linear stable ∞ -category if and only if $\mathcal{X}$ is algebraizable.*

Theorem 0.2.4 (Theorem 1.3.23). *Let $\mathcal{X}$ be a proper formal scheme such that $\mathrm{Coh}^b(\mathcal{X})$ is smooth as a ∞ -category in $\mathrm{Cat}_{\mathbb{O}_k}^{\mathrm{Perf}}$, and $\mathcal{X}^{\mathrm{rig}}$ is smooth. Then $\mathcal{X}$ is algebraizable.*

These theorems are, of course, closely related. We prove both theorems by using

Proposition 1.3.21 to reduce these statements to Theorem 1.3.1.

Theorem 1.3.23 shows that, in general, for a proper non-algebraizable formal scheme $\mathcal{X}$, we cannot expect $\mathrm{Coh}^b(\mathcal{X})$ to be smooth. On the other hand, for a separated scheme X over a perfect field, Lunts demonstrated in [Lun10] that $\mathrm{Coh}^b(X)$ is smooth. More generally, Efimov showed in [Efi18] that under mild conditions on X , the category $\mathrm{Coh}^b(X)$ is of finite type. Since being of finite type implies smoothness, the above theorem confirms that these classical algebraic statements cannot be broadly generalized to the setting of formal schemes.

Theorem 1.3.1 is a generalization of the main result of [TVo8a] to the non-Archimedean setting. In [TVo8a], the authors prove that a complex analytic, connected, smooth, and proper variety is algebraizable if and only if its category of perfect complexes is smooth and proper as a $\mathbb{C}$ -linear DG-category. We remark that our proof strategy is modelled after the arguments in [TVo8a], although we have to deal with many new subtleties that are absent in the complex setting. We can interpret our result and the results from [TVo8a] as evidence of the fact that the category of perfect complexes in the analytic and formal settings is not as well-behaved as in the algebraic setting. The key issue is that the definition of smooth and proper categories is not well adapted to the analytic setting. In Chapter 2 we will see how to adapt the notion of smooth categories in the analytic setting.

0.2.3 A new framework for non-commutative analytic geometry

This chapter is based on [Mon26].

Derived analytic geometry was initially developed by Porta and Porta–Yu (see, for example, [PY20; Por19; PY18; PY22]), utilizing an approach similar to the one employed by Lurie in [Lur18]. Another approach to derived analytic geometry, closer in style to the framework developed for derived algebraic geometry in [TVo8b], has recently been introduced and utilized, for instance, in [BKK24; BBK18]. However, the setting for derived analytic geometry employed in this paper relies on the recent work of Clausen and Scholze on *condensed mathematics* and their theory of *analytic stacks*, as developed in [Sch26a; Sch26b; CS24; CS26]. Within this framework, one can study rigid analytic varieties, as demonstrated in [Mik23; And21], and even construct fundamental new analytic stacks in rigid geometry, as explained in [Cam24]. One of the powerful features of this approach is that it admits a robust six-functor formalism and a “categorical completed tensor product”.

Throughout this paper, we work over a non-Archimedean field k of characteristic zero. We adopt the definition of *animated affinoid algebras* introduced in [Mik23, Definition 2.11] (see Definition 2.2.7). Derived rigid analytic spaces are then defined by “gluing animated

affinoid algebras along analytic open immersions" (see Definition 2.3.11). We introduce and study the sheaf of nuclear ∞ -categories over derived rigid analytic varieties; in Section 2.3, we will demonstrate its favorable properties.

The notions of *nuclear* objects and *nuclear subcategories* were introduced in [CS26]. These concepts have subsequently been employed for instance, in conjunction with Efimov's seminal work on the K-theory of dualizable ∞ -categories [Efi25a] to study the *algebraic K-theory* of rigid analytic varieties [And23]. We utilize nuclear ∞ -categories in this paper because they are rigid in our setting (see Proposition 2.3.13). This rigidity, in particular, provides a simple description of $\mathrm{Nuc}(k)$ -atomic objects in $\mathrm{Nuc}(X)$ in terms of perfect complexes over X (see Lemma 2.4.8). As explained in the following section, this is the key property that allows us to relate the smoothness of X to the categorical notion of smoothness for $\mathrm{Nuc}(X)$ (see Theorem 2.4.1). Combined with Theorem 0.1.1, this property will be crucial in relating the algebraization of a rigid analytic variety X to the atomic generation of $\mathrm{Nuc}(X)$.

To obtain this result, another fundamental step consists of relating the geometric definition of smoothness for a rigid analytic variety X to a categorical definition. The idea is to work relative to a base ∞ -category $\mathcal{C}$ (in this case, $\mathrm{Nuc}(k)$), where the tensor product in $\mathrm{Pr}_{\mathcal{C}}^{\mathrm{L}}$ satisfies a Künneth formula for rigid varieties. In our setting, this property is proven in Proposition 2.3.14. This motivates us to consider presentable ∞ -categories that are linear with respect to the action of a base symmetric monoidal, stable, presentable ∞ -category $\mathcal{V}$. In Section 2.1, we review the necessary theory to work with these objects, primarily following [Ben24; Ram24a; Hin21; GH15], which provide a rigorous treatment of the subject. Furthermore, we generalize the notion of smooth ∞ -categories in this setting via the following definition, which is also considered in [Ram24a].

Definition 0.2.5. Let $\mathcal{C}$ be a presentable and dualizable ∞ -category over $\mathcal{V}$. We say that $\mathcal{C}$ is *internally smooth over $\mathcal{V}$* if the coevaluation map admits a $\mathcal{V}$ -linear right adjoint that preserves colimits.

Within the analytic geometry framework described in this paper, we can now relate the categorical notion of smoothness to its geometric counterpart.

Theorem 0.2.6 (Theorem 2.4.1). *Let $f: X \rightarrow \mathrm{Sp}(k)$ be a (classical) rigid analytic variety over k . Then X is smooth if and only if the ∞ -category $\mathrm{Nuc}(X)$ is internally smooth in $\mathrm{Pr}_{\mathrm{Nuc}(k)}^{\mathrm{L}}$.*

This theorem follows as a consequence of Corollary 2.4.4, which demonstrates that X is smooth if and only if $\Delta_* \mathcal{O}_X$ is a perfect complex on $X \hat{\times} X$, where $\Delta: X \rightarrow X \hat{\times} X$ is the diagonal map. These results are obtained by combining the Künneth formula (Proposition 2.3.14) and the results for smooth ∞ -category in a six-functor formalism developed in Section 2.1. In our setting, the construction of an appropriate six-functor

formalism for nuclear ∞ -categories is described in Theorem 2.3.15. Specifically, this six-functor formalism is constructed using the theory developed in [MAN22] and revisited in [HM24]. Furthermore, it is inspired by the formalism described in [Sch25, Lecture 8] within the setting of derived algebraic geometry. We observe that, although this theorem ultimately concerns the study of a classical property of rigid analytic varieties, obtaining the categorical (or non-commutative) characterization of smoothness requires entering the realm of condensed mathematics and the setting of derived rigid analytic geometry described herein. Indeed, these methods are also essential for obtaining the requisite six-functor formalism.

As previously explained, this theorem plays a crucial role in relating the atomic generation of $\mathrm{Nuc}(X)$ to the algebraization of the rigid analytic variety X .

Together with the notion of smooth and proper ∞ -categories, which are closely related to their geometric counterparts, one of the main features of non-commutative algebraic geometry is the existence of a single compact generator for the ∞ -category of quasi-coherent sheaves on a quasi-compact and quasi-separated scheme X over a field k . In particular, there exists a non-commutative k -algebra A such that we have an equivalence of ∞ -categories

$$\mathrm{QCoh}(X) \simeq \mathrm{LMod}_A.$$

These results have been established, for example, in [BB03] and, within the setting of derived algebraic geometry, in [Toë12b]. Moreover, assuming that the variety X is smooth and proper, one can deduce that the algebra A is also smooth and proper. We will demonstrate how to reprove a similar result in Section 2.5.1 using the theory of six-functor formalisms.

More generally, the existence of a compact generator was proved by Toën in the setting of dg-categories; the analogous result in the setting of ∞ -categories is the following.

Theorem 0.2.7 ([Lur18, Proposition 11.3.2.4]). *Let R be an $\mathbb{E}_\infty$ -ring spectrum and $\mathcal{C}$ be a stable, R -linear, smooth and compactly generated ∞ -category. Then $\mathcal{C}$ has a single compact generator.*

Without the assumption of compact generation, whether the previous result holds remains an open conjecture, which can be stated as follows:

Conjecture 0.2.8 ([Ram24a, Conjecture 0.6]). *Let R be an $\mathbb{E}_\infty$ -ring spectrum and $\mathcal{C}$ be a smooth, stable, presentable R -linear ∞ -category. Then $\mathcal{C}$ admits a single compact generator.*

This conjecture has been verified in some particular cases in [Ste25]. However, in the generality presented above, the conjecture remains open to the author's knowledge.

In [Ram24a; Ram24b], Maxime Ramzi developed the theory of atomic objects and dualizable ∞ -categories; see also [Efi25a; Efi25b] for closely related results and their

applications to the theory of localizing invariants. In this context, atomic objects can be viewed as a variant of compact objects suitably adapted to the setting of *enriched ∞ -categories*.

Building upon the notion of a $\mathcal{V}$ -atomically generated ∞ -category, as detailed in Definition 2.1.8, Maxime Ramzi proposed a generalization of the preceding conjecture to the setting of $\mathcal{V}$ -enriched ∞ -categories.

Conjecture 0.2.9. *Let $\mathcal{V}$ be a presentable symmetric monoidal ∞ -category and let $\mathcal{C}$ be a $\mathcal{V}$ -linear presentable ∞ -category, internally smooth over $\mathcal{V}$. Then $\mathcal{C}$ is atomically generated.*

In this paper, we obtain a counterexample to this conjecture by studying the nuclear ∞ -category of a smooth and proper rigid analytic variety. Indeed, we prove the following result.

Theorem 0.2.10 (Theorem 2.4.9). *Let $p: X \rightarrow \mathrm{Sp}(k)$ be a smooth and proper, quasi-compact, and separated rigid analytic variety over $\mathrm{Sp}(k)$. Then X is algebraizable if and only if $\mathrm{Nuc}(X)$ admits a single $\mathrm{Nuc}(k)$ -atomic (equivalently, compact) generator.*

If we assume X to be algebraizable (i.e., $X \simeq Y^{\mathrm{an}}$ for some scheme Y), we can prove that $\mathrm{Nuc}(X)$ is $\mathrm{Nuc}(k)$ -atomically generated. Building on the relative GAGA theorem of [Wan26], we establish the following equivalence in Corollary 2.4.7:

$$\mathrm{Nuc}(Y^{\mathrm{an}}) \simeq \mathrm{QCoh}(Y) \otimes_{\mathrm{Mod}_k} \mathrm{Nuc}(k).$$

We then deduce our statement regarding the compact generation of $\mathrm{QCoh}(Y)$. Conversely, if we assume that $\mathrm{Nuc}(X)$ admits a single $\mathrm{Nuc}(k)$ -atomic generator, we can use Lemma 2.4.8 to deduce that this generator must be a perfect complex. Furthermore, applying Theorem 0.2.6, we infer that $\mathrm{Perf}(X)$ must also be smooth; in this way, we reduce our algebraization result to Theorem 0.1.1.

To obtain the announced counterexample, we will also utilize the following result, which can be viewed as a generalization of Theorem 0.2.7 to the enriched setting.

Proposition 0.2.11 (Proposition 2.1.27). *Let $\mathcal{V}$ be a commutative algebra in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ with compact unit, and let $\mathcal{C}$ be a smooth ∞ -category over $\mathcal{V}$ which is $\mathcal{V}$ -atomically generated. Then $\mathcal{C}$ admits a single $\mathcal{V}$ -atomic (equivalently, compact) generator.*

With these results, we construct a counterexample to Conjecture 0.2.9 and demonstrate that it is not reasonable to extend Conjecture 0.2.8 over a generic rigid base different from Mod_R . Indeed, if we consider a non-algebraizable smooth and proper rigid analytic variety X , see for example the p -adic Hopf Surface studied in [Mus77; Vos91], combining Theorem 0.2.6 and Theorem 0.2.10 reveals that $\mathrm{Nuc}(X)$ cannot admit a single

$\text{Nuc}(k)$ -atomic generator. Thus, by Proposition 2.1.27, it cannot be $\text{Nuc}(k)$ -atomically generated.

We have been informed that recently Maxime Ramzi has also constructed a counterexample to Conjecture 0.2.9 in the case where $\mathcal{V}$ is an ∞ -category of the form $\text{Shv}(X, \text{Sp})$.

We conclude the introduction observing that Theorem 0.2.6 implies that it is not, in general, possible to glue local, relatively compact generators of a sheaf of categories in the analytic (or fpqc) topology to obtain a global compact generator. Locally in the analytic topology over X , the sheaf $\text{Nuc}(-)$ is of the form $\text{Mod}_A(\text{Nuc}(k))$ for affinoid algebras A . However, Theorem 2.4.9 demonstrates that, globally, such a generator algebra does not necessarily exist.

This property of gluing local compact generators is one of the foundational features of *Non-Commutative Algebraic Geometry*, and it highlights a key distinction between algebraic and analytic topologies. This gluing procedure for compact generators has been shown to hold for various topologies utilized in algebraic geometry; see, for example, [TV07] and [BB03].

0.2.4 Analytic 1-affineness

The last part of this thesis is based on a joint work with Emanuele Pavia, [MP25]. In this appendix we will study the notion of sheaves of categories and 1-affineness in the setting of analytic geometry, we moreover prove some technical lemma, such as Proposition 3.1.48 crucially used in Chapter 2.

The notion of sheaves of categories provides a categorification of the theory of quasi-coherent sheaves from ordinary algebraic geometry. This is the suitable technical framework where to study higher categorical structures which vary "continuously" over some geometric object (with respect to some Grothendieck topology).

This notion has been first developed in the seminal work [Gai15]: in *loc. cit.*, Gaitsgory defines a functor

$$\text{ShvCat}: \text{PSt}_{\mathbb{k}} \longrightarrow \widehat{\text{Cat}}^{\text{rex}}$$

which assigns to any derived prestack X (defined over a ground ring $\mathbb{k}$) a cocomplete ∞ -category consisting of quasi-coherent sheaves of categories on X .

Roughly speaking, an object of $\text{ShvCat}(X)$ is a rule that assigns to each affine test scheme $\text{Spec}(A) \rightarrow X$ a presentable $\text{QCoh}(A)$ -linear ∞ -category, together with a homotopy-coherent system of descent data that allows to glue such ∞ -categories along any fiber product of affine schemes over X . In this way, we are able to work with higher categorical data varying over a base X in a way which is conceptually very close to what one is used to in ordinary algebraic geometry when one works with quasi-coherent sheaves of

modules.

In particular, for a quasi-coherent sheaf of categories we can define operations which generalize familiar notions such as pullbacks, pushforwards, relative tensor products, which remain sensitive to the geometry of our base prestack.

The key concept of [Gai15] is the notion of *1-affineness*. Just like quasi-coherent sheaves on an affine scheme $\mathrm{Spec}(A)$ are completely determined by the A -module structure on their global sections, this property actually characterizes affine schemes among all quasi-separated schemes. A 1-affine (pre)stack is defined to be a (pre)stack for which any quasi-coherent sheaf of categories is completely determined by some $\mathrm{QCoh}(X)$ -linear ∞ -category. More precisely, we have a functor

$$\Gamma_X: \mathrm{ShvCat}(X) \longrightarrow \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathrm{Pr}^{\mathrm{L}})$$

sending a sheaf of categories $\mathcal{C} \in \mathrm{ShvCat}(X)$ to its ∞ -category of global sections $\Gamma_X(\mathcal{C})$ which carries a natural action of the monoidal ∞ -category $\mathrm{QCoh}(X)$. Then, the prestack X is said to be *1-affine* if this functor is an equivalence of ∞ -categories.

Establishing whether a given prestack is 1-affine is a highly nontrivial problem and carries deep geometric and categorical implications. It allows to study categorified sheaf-theoretic phenomena by reducing to the study of a single ∞ -category of modules over a symmetric monoidal ∞ -category.

The aim of this Chapter is to introduce the notions of sheaves of categories and 1-affineness in the setting of rigid and analytic geometry. Our trivial observation is that the problem of 1-affineness can be posed in a very abstract generality, and some affirmative results do not really depend on the fact that we are dealing with algebraic or analytic stacks but are formal consequences of the theory. For this reason, we study the notion of 1-affineness from a somewhat axiomatic perspective: we allow $\mathcal{C}$ to be a general ∞ -site, with some reasonable assumptions on the ∞ -category and on the Grothendieck topology (see Assumptions 3.1.1). We then consider a sheaf of $\mathcal{V}$ -linear presentable symmetric monoidal ∞ -categories on $\mathcal{C}$

$$\mathcal{D}: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})),$$

which again satisfy some reasonable assumptions. We can thus define the "categorification of $\mathcal{D}$ " to be the functor

$$\begin{aligned} \mathcal{D}^{\mathrm{cat}}: \mathcal{C}^{\mathrm{op}} &\longrightarrow \widehat{\mathrm{Cat}}^{\mathrm{rex}} \\ U &\mapsto \mathrm{Pr}_{\mathcal{D}(U)}^{\mathrm{L}}. \end{aligned}$$

The first problem we study is to find reasonable conditions on $\mathcal{C}$ and on $\mathcal{D}$ for which the functor $\mathcal{D}^{\text{cat}}(-)$ is a sheaf for the Grothendieck topology on $\mathcal{C}$. In particular, we show that this holds in the following cases.

1. The sheaf $\mathcal{D}$ is already the categorification of a sheaf of commutative algebras

$$D: \mathcal{C}^{\text{op}} \longrightarrow \text{CAlg}(\mathcal{V}),$$

in the sense that we have a natural equivalence of functors

$$\mathcal{D} \simeq \text{Mod}_{D(-)}(\mathcal{V}),$$

and that moreover all the maps in S are descendable, as in Definition 3.1.37. (The problem of 1-affineness studied by Gaitsgory falls in this setting.)

2. The sheaf of ∞ -categories $\mathcal{D}$ is obtained as one operation in a strongly monoidal six-functor formalism

$$(\mathcal{D}^*, \mathcal{D}!): \text{Corr}(\mathcal{C})_{S', \text{all}}^{\otimes} \longrightarrow \text{Pr}_{\mathcal{V}}^{\text{L}, \otimes}$$

in the sense of [HM24], while the topology on $\mathcal{C}$ is the universal $!$ -able topology (see Definition 3.1.18).

When we have a fully faithful dense functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, we have two different ways to extend $\mathcal{D}^{\text{cat}}$ to an assignment on all of $\widehat{\mathcal{C}}$: we can either take the right Kan extension of $\mathcal{D}^{\text{cat}}$ along ι ($\text{Ran}_{\iota} \mathcal{D}^{\text{cat}}(-)$), or we can take categorical modules over the right Kan extension of $\mathcal{D}$ along ι ($\text{Mod}_{\text{Ran}_{\iota} \mathcal{D}(-)}(\text{Pr}^{\text{L}})$). These two operations are *a priori* very different, but for any object Y of $\widehat{\mathcal{C}}$, we have a natural comparison functor

$$\text{Mod}_{\text{Ran}_{\iota} \mathcal{D}(Y)}(\text{Pr}^{\text{L}}) \longrightarrow \text{Ran}_{\iota} \mathcal{D}^{\text{cat}}(Y). \quad (4)$$

induced by the universal property of limits.

Definition 0.2.12. Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumptions 3.1.1, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor. We say that an object Y in $\widehat{\mathcal{C}}$ is *1-affine with respect to ι* if the comparison functor (4) is an equivalence.

We put ourselves this general setup motivated by two different problems, both concerning 1-affineness in rigid or analytic geometry. In Section 3.1.3) we study the problem of 1-affineness in the setting of rigid analytic geometry. In this setting, the site $\mathcal{C}$ is then defined to be the ∞ -category of affinoid spaces, while the ∞ -category $\widehat{\mathcal{C}}$ is the ∞ -category

of rigid analytic varieties, and the sheaf $\mathcal{D}(-)$ is just the functor $\text{Nuc}(-)$. In this way, we can prove the following theorem.

Theorem 0.2.13 (Theorem 3.3.26). *Let X be a quasi-compact and separated rigid analytic variety. Then X is 1-affine, i.e., there is an equivalence of categories*

$$\text{Mod}_{\text{Nuc}(X)}(\text{Pr}_{\text{st}}^{\text{L}}) \simeq \lim_{\text{Spa}(A) \subset X} \text{Mod}_{\text{Nuc}(A)}(\text{Pr}_{\text{st}}^{\text{L}}),$$

where the limit ranges over an affinoid open cover of X .

The second problem which we want to investigate is 1-affineness for *analytic stacks*, as defined in [CS24]: this fits in the abstract setting we study in Section 3.1.2. Analytic stacks were recently introduced by Clausen and Scholze using the novel theoretic framework of *condensed mathematics* ([Sch26b; Sch26a; CS22; CS24]). Using this formalism they manage to capture a wide array of spaces appearing in complex geometry, non-archimedean geometry, algebraic geometry, and topology. They also come with a well-behaved notion of quasi-coherent sheaves, which enjoy all the functorial properties of a *six-functor formalism* ([Sch25; HM24]). We also remark that analytic geometry can be studied also following different approaches, which have been currently developed in [BKK24; BB16; Soo25; KM25]. This parallel line of work still exhibits notable structural affinities with the condensed approach.

To study the property of 1-affineness for analytic stacks, we can consider $\mathcal{C}$ to be the ∞ -category of affine analytic stacks (modeled by analytic rings in the sense of Definition 2.2.1), and $\widehat{\mathcal{C}}$ to be the ∞ -category of all analytic stacks. The sheaf of ∞ -categories $\mathcal{D}(-)$ is precisely the sheaf sending an analytic stack to its derived ∞ -category, hence our choice of notation.

However, our assumptions are sufficiently general that we expect that this setting can be used to successfully explore the notion of 1-affineness in the context of motivic six-functor formalisms for schemes; we leave this application for future work.

Our main result in this context is the following.

Theorem 0.2.14 (Theorem 3.3.2). *Let X be an analytic stack admitting an affine universal $!$ -cover $\{\text{AnSpec}(A_i) \rightarrow X\}_{i \in I}$ such that all fiber products $\text{AnSpec}(A_{i_1}) \times_X \cdots \times_X \text{AnSpec}(A_{i_n})$ are again affine. Then X is 1-affine.*

We need to stress that Theorem 0.2.14 can also be obtained using some results contained in [Kes25], which was written and appeared on ArXiv at the final stages of preparation of [MP25]. While the stream of ideas is very similar, we obtain a slightly more

general 1-affineness results that holds under the hypotheses of Assumption 3.1.13, see Theorem 3.1.35.

The latter is one of the main motivations for this paper. In [PPS25a] the authors studied 1-affineness of a different incarnation of the Betti stack associated to topological cases – that we call *homotopical* Betti stack, since it captures only the underlying anima of a topological space. There, it is shown that the homotopical Betti stack often fails to be 1-affine, and that a somewhat surprising topological obstruction to 1-affineness lies in any non-trivial element of the second homotopy group.

On the other hand, in analytic geometry we have an analytic incarnation of the Betti stack of a topological space X , which is sensitive to the whole topology of X (indeed, its derived ∞ -category agrees with the derived ∞ -category of sheaves over X with values in derived condensed abelian groups). This was recently introduced in [Sch24].

We discover a fundamental difference between the analytic Betti stack and the "homotopical" Betti stack from algebraic geometry from the perspective of 1-affineness, in the sense that the topological constraints are far less strict.

Theorem 0.2.15 (Theorem 3.3.12). *Let X be a finite dimensional, metrizable, compact Hausdorff space. Then the analytic Betti stack associated to X is 1-affine.*

Using the Riemann–Hilbert correspondence of [Sch24], we can relate the analytic Betti stack with the analytic De Rham stack over the complex numbers when X is a compact complex manifold. In this way, we obtain an analytic analogue of the 1-affineness for the de Rham stack in derived algebraic geometry from [Gai15].

Theorem 0.2.16 (Corollary 3.3.14). *Let X be a compact complex manifold. Then the analytic De Rham stack $X_{\mathrm{dR}}^{\mathrm{an}}$ is 1-affine.*

0.3 Notation and conventions

In this thesis, we work with a fixed base field k of characteristic 0. In the context of formal schemes, we further assume that k is the field of fractions of a discrete valuation ring (DVR) $\mathcal{O}_k$, complete with respect to the $\mathfrak{m}$ -adic topology. Additionally, in the non-Archimedean setting, we require the field k to be complete.

Throughout this paper, we freely employ the language of derived geometry, ∞ -categories, and higher algebra. In this setting, we follow the conventions of [Luro9]; in particular, we denote by $\mathcal{S}$ or Ani the ∞ -category of spaces (also called anima). All rings and modules considered are derived unless explicitly stated otherwise. In particular, a module over an $\mathbb{E}_\infty$ -algebra A will be an object in its derived ∞ -category, as

explained in [Lur17, Chapter 7]. We denote this category by Mod_A , its subcategory of algebra objects by CAlg_A , and the opposite category of CAlg_A by Aff_A .

In Chapter 1 when discussing (derived) algebraic stacks, we equip the category Aff_k (or dAff_k) with the étale topology. In the rigid analytic setting, we denote by dAfd_k the ∞ -category of derived affinoid spaces, as defined in [PY18, Definition 7.3]. Similarly, in the underived setting, we denote by Afd_k the category of affinoid spaces over k . When working with derived analytic stacks, we consider the étale topology on the category dAfd_k . For derived geometry, we follow the notation of [Lur18] in the algebraic and formal settings, while in the rigid analytic setting, our main references are [HP25] and [PY21]. Furthermore, all formal schemes and rigid analytic varieties are assumed to be topologically of finite type, quasi-compact, and separated.

In Chapter 2 and Chapter 3 For any presentably symmetric monoidal ∞ -category $\mathcal{V}$, we write $\text{Pr}_{\mathcal{V}}^{\text{L}}$ for the ∞ -category $\text{Mod}_{\mathcal{V}}(\text{Pr}^{\text{L}})$ of *presentably $\mathcal{V}$ -linear ∞ -categories*. When $\mathcal{V} = \text{Sp}$ is the ∞ -category of spectra, we shall simply write $\text{Pr}_{\text{st}}^{\text{L}}$ since this is the ∞ -category of stable presentable ∞ -categories. Again, the relative tensor product over $\mathcal{V}$ yields a symmetric monoidal structure on $\text{Pr}_{\mathcal{V}}^{\text{L}}$, so we can consider algebra objects in it. We call a commutative algebra in $\text{Pr}_{\mathcal{V}}^{\text{L}}$ a *presentably $\mathcal{V}$ -linear symmetric monoidal ∞ -category*. If $\mathcal{C}$ is enriched over an ∞ -category $\mathcal{V}$ and moreover symmetric monoidal, we adopt the following notation.

- We let $\text{Map}_{\mathcal{C}}(-, -)$ denote the underlying anima of maps between two objects in $\mathcal{C}$;
- We let $\underline{\text{Hom}}_{\mathcal{C}}(-, -)$ denote the internal hom functor.
- We denote with $\text{hom}^{\mathcal{V}}(-, -)$ the $\mathcal{V}$ -enriched hom functor as defined in [Ben24, Definition 4.20].
- If $\mathcal{C}$ is stable, we denote the mapping spectrum by $\text{map}_{\mathcal{C}}(-, -)$.

This convention allows us to distinguish whether we are viewing a morphism object as a mere anima or as a more structured object. Regarding the theory of condensed mathematics and analytic stacks, we follow the formalism developed in [CS24].



An algebraicity criterion via the category of perfect complexes

1.1 Moduli of objects for k -linear stable ∞ -categories

In this section, we revisit the construction of the moduli stack of objects in a DG category as described in [TV07], adopting the language of ∞ -categories. Since we shall employ this construction both for ∞ -categories in $\mathrm{Pr}^{\mathrm{L},\omega}$ and for small ∞ -categories, we begin with some remarks on stable Morita theory, which facilitates the transition between these two settings.

1.1.1 A perspective on Morita theory via Ind-completion

Here we describe Morita theory for stable ∞ -categories. Building upon unstable Morita theory, we show how to extend this correspondence to the stable setting. We denote by $\mathrm{Cat}^{\mathrm{Perf},\kappa}$, where κ is a regular cardinal, the ∞ -category that has as objects all small ∞ -categories with κ -small colimits, which are idempotent complete and whose morphisms preserve κ -small filtered colimits (see [Luro9] and [BGT13]). Additionally, $\mathrm{Pr}^{\mathrm{L},\kappa}$ denotes the ∞ -category of presentable ∞ -categories generated by κ -compact objects, where morphisms preserve κ -compact objects.

The functor $\mathrm{Ind}_{\kappa}(-)$ is the formal completion with respect to κ -filtered colimits; the

functor $(-)^{\kappa}$ sends a category to the full subcategory of its κ -compact objects. By the definition of an accessible ∞ -category, this subcategory is small and κ -small cocomplete, as κ -small limits commute with κ -filtered colimits. Furthermore, it is idempotent complete, since compact objects are closed under retracts.

The main result in the unstable setting is the following theorem.

Theorem 1.1.1. *Let κ be a regular cardinal. Then the following hold:*

1. *There is an equivalence of ∞ -categories given by the following pair of adjoint functors:*

$$\mathrm{Cat}^{\mathrm{Perf}, \kappa} \begin{array}{c} \xrightarrow{\mathrm{Ind}_{\kappa}} \\ \xleftarrow[(-)^{\kappa}]{\tau} \end{array} \mathrm{Pr}^{\mathrm{L}, \kappa}.$$

2. *The ∞ -category $\mathrm{Pr}^{\mathrm{L}, \kappa}$ is presentable.*
3. *The obvious functor $\mathrm{Pr}^{\mathrm{L}, \kappa} \rightarrow \mathrm{Pr}^{\mathrm{L}}$ commutes with small colimits.*

Proof. The first statement is exactly [Luro9, Proposition 5.5.7.8], and the second follows from [Aok25, Proposition 2.3]. The third statement can be deduced using [Luro9, Proposition 5.5.7.6, and Theorem 5.5.3.18] and by considering that, from [Luro9, Corollary 5.5.3.4], we have an equivalence

$$(\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{L}}.$$

Using [Luro9, Proposition 5.5.7.2], we can deduce that this equivalence restricts to an equivalence

$$(\mathrm{Pr}^{\mathrm{R}, \kappa})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{L}, \kappa}.$$

□

We can apply the first part of the theorem to the case where $\kappa = \omega$. In this setting, we denote the ∞ -category $\mathrm{Cat}^{\mathrm{Perf}, \omega}$ simply by $\mathrm{Cat}^{\mathrm{Perf}}$, which is the ∞ -category of small, idempotent complete ∞ -categories with finite colimits. This gives us the following equivalence:

$$\mathrm{Cat}^{\mathrm{Perf}} \simeq \mathrm{Pr}^{\mathrm{L}, \omega}.$$

By restricting to the subcategory of stable objects $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{Perf}}$ with exact functors, we obtain the following version of stable Morita theory:

Proposition 1.1.2. *There is an equivalence of ∞ -categories*

$$\mathrm{Cat}_{\mathrm{st}}^{\mathrm{Perf}} \simeq \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}, \omega}.$$

Proof. This essentially follows from the above equivalence, noting that stable ∞ -categories are closed under Ind_ω (usually denoted simply by Ind) [Lur17, Proposition 1.1.3.6]. Additionally, the subcategory of compact generators of a stable ∞ -category is itself stable. $\square$

We note that we can also start with a stable, small, ∞ -category $\mathcal{C}$ that is not idempotent complete and obtain the same result as above by replacing $\mathcal{C}$ with $\mathcal{C}^{\text{idem}}$, here, as explained in [Luro9], $\mathcal{C}^{\text{idem}}$ denotes the idempotent completion of $\mathcal{C}$. Moreover, $\mathcal{C}^{\text{idem}}$ remains small [Luro9, Proposition 5.4.2.4] and stable [Lur17, Corollary 1.1.3.7]. We will now extend the above stable Morita theory to k -linear ∞ -categories. We begin by noting that, as described in [BGT13, Theorem 3.1], there is a symmetric monoidal structure on $\text{Cat}_{\text{st}}^{\text{Perf}}$. This provides the context for the following definition.

Definition 1.1.3. We define the ∞ -category of stable, idempotent complete, k -linear ∞ -categories as

$$\text{Cat}_k^{\text{Perf}} := \text{Mod}_{\text{Perf}(k)} \text{Cat}_{\text{st}}^{\text{Perf}}.$$

The ∞ -category of stable, presentable, compactly generated k -linear ∞ -categories is:

$$\text{Pr}_k^{\text{L},\omega} := \text{Mod}_{\text{Mod}_k} \text{Pr}^{\text{L},\omega}.$$

Remark 1.1.4. Note that it is natural to consider, in Cat^{Perf} , modules over $\text{Perf}(k)$ rather than over Mod_k , since Mod_k is not a small category, and in $\text{Cat}_{\infty,k}$ we only deal with small ∞ -categories. However, under the equivalence from Proposition 1.1.2, Mod_k corresponds precisely to $\text{Perf}(k)$.

Lemma 1.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two symmetric monoidal ∞ -categories, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric monoidal equivalence. If A is an algebra object in $\mathcal{C}$, then F induces an equivalence:

$$\text{Mod}_A \mathcal{C} \simeq \text{Mod}_{F(A)} \mathcal{D}.$$

Proof. The proof is straightforward. $\square$

From the above lemma we can deduce the following characterization for the k -linear stable Morita ∞ -category.

Proposition 1.1.6. The pair of adjoint functors

$$\text{Cat}_k^{\text{Perf}} \begin{array}{c} \xrightarrow{\text{Ind}_\omega} \\ \xleftarrow{(-)^\omega} \end{array} \text{Pr}_k^{\text{L},\omega}$$

forms an equivalence.

1.1.2 Smooth and proper ∞ -categories and algebras

In this subsection, we begin by reviewing several established results concerning smooth and proper algebras and ∞ -categories.

Let A be an $\mathbb{E}_\infty$ -ring and B an $\mathbb{E}_1$ -algebra in Mod_A .

Definition 1.1.7. We say that B is *proper* over k if it is a perfect complex in Mod_A .

Definition 1.1.8. We say that B is *smooth* over k if it is a compact object in $\text{Mod}_{B \otimes_k B^{\text{op}}}$.

We have the following useful characterization:

Proposition 1.1.9 ([Lur17, 4.6.4.4 and 4.6.4.12]).

1. An algebra B is proper if and only if for every $M \in \text{LMod}_B$, if M is dualizable in $\text{LMod}_B(\text{Mod}_A)$, then M is dualizable in Mod_A .
2. An algebra B is smooth if and only if for every $M \in \text{LMod}_B(\text{Mod}_A)$, if M is dualizable as an A -module, then M is dualizable in $\text{LMod}_B(\text{Mod}_A)$.

Lemma 1.1.10. If B is smooth and proper, then $\text{LMod}_B(\text{Perf}(A)) \simeq \text{Perf}(B)$.

Proof. This follows directly from the characterization of smooth and proper algebras, noting that dualizable objects coincide with perfect complexes in Mod_A . $\square$

In the setting of presentable, compactly generated ∞ -categories, we define smoothness and properness as follows. Let $\mathcal{C}$ be an ∞ -category in Pr_k^L , and let $\mathcal{C}^\vee = \text{Fun}_k^L(\mathcal{C}, \text{Mod}_k)$ be its dual category, equipped with the evaluation map $\text{ev}: \mathcal{C} \otimes \mathcal{C}^\vee \rightarrow \text{Mod}_k$.

Definition 1.1.11. A ∞ -category $\mathcal{C} \in \text{Pr}_A^L$ is *dualizable* if there exists a coevaluation map

$$\text{coev}: \text{Mod}_A \rightarrow \mathcal{C} \otimes \mathcal{C}^\vee$$

classifying $\mathcal{C}$ as a module over $\mathcal{C} \otimes \mathcal{C}^\vee$, such that both compositions induced by evaluation and coevaluation:

$$\begin{aligned} \mathcal{C} &\xrightarrow{\text{id} \otimes \text{coev}} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\text{ev} \otimes \text{id}} \mathcal{C} \\ \mathcal{C}^\vee &\xrightarrow{\text{coev} \otimes \text{id}} \mathcal{C}^\vee \otimes \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\text{id} \otimes \text{ev}} \mathcal{C}^\vee \end{aligned}$$

are homotopic to the identity.

Definition 1.1.12. We say that $\mathcal{C} \in \mathrm{Pr}_A^{\mathrm{L},\omega}$ is *proper* if the evaluation map is a morphism in $\mathrm{Pr}_A^{\mathrm{L},\omega}$ (i.e., it admits a right adjoint and preserves compact objects).

Definition 1.1.13. We say that $\mathcal{C} \in \mathrm{Pr}_A^{\mathrm{L},\omega}$ is *smooth* if it is dualizable and the coevaluation map is a morphism in $\mathrm{Pr}_A^{\mathrm{L},\omega}$.

Definition 1.1.14. We say that $\mathcal{C} \in \mathrm{Pr}_A^{\mathrm{L},\omega}$ is of *finite type* if there exists a compact A -algebra C such that $\mathcal{C}$ is isomorphic to Mod_C .

We also define the notion of smoothness (respectively, properness) for a small, A -linear, idempotent complete ∞ -category by requiring that its idempotent completion is smooth (respectively, proper) as in Definition 1.1.13 (respectively, Definition 1.1.12). In this context, a small, k -linear, idempotent complete ∞ -category $\mathcal{B}$ is proper if and only if, for every pair of objects b, b' in $\mathcal{B}$, the A -module $\mathrm{Hom}_{\mathcal{B}}(b, b')$ is a perfect complex.

Lemma 1.1.15. *Let $\mathcal{C}$ be a smooth A -linear ∞ -category in $\mathrm{Pr}_A^{\mathrm{L},\omega}$; then $\mathcal{C}$ admits a compact generator.*

Proof. See the proof of [Lur18, Proposition 11.3.2.4]. □

Corollary 1.1.16. *An A -linear ∞ -category in $\mathrm{Pr}_A^{\mathrm{L},\omega}$ is smooth and proper if and only if it is equivalent to LMod_B for some smooth and proper algebra B . In this setting, B can be taken to be the algebra of endomorphisms of a compact generator.*

Proof. If $\mathcal{C}$ is a smooth and proper A -linear presentable ∞ -category, using [Lur17, Theorem 7.1.2.1] and Lemma 1.1.15, we can deduce that there exists an algebra B such that $\mathcal{C}$ is equivalent to LMod_B . The statement is then clear, as the categorical definitions of smoothness and properness directly correspond to their algebraic counterparts. □

Lemma 1.1.17 ([TV07, Lemma 2.6]). *An ∞ -category $\mathcal{C}$ in $\mathrm{Pr}_A^{\mathrm{L},\omega}$ is smooth and proper if and only if its subcategory of compact objects $\mathcal{C}^\omega$ is smooth and proper.*

1.1.3 The stack of pseudo-perfect objects

Here, we review the main construction of [TV07] using the language of ∞ -categories. Although the DG category setting is used in [TV07], similar ideas can be applied in the ∞ -categorical setting, as demonstrated in [AG14]. For a comprehensive overview of this construction in the language of ∞ -categories, see also [Lam25].

Let k be a discrete field and let $\mathcal{C}$ be a stable, presentable, k -linear, compactly generated ∞ -category (i.e., $\mathcal{C} \in \mathrm{Pr}_k^{\mathrm{L},\omega}$).

Definition 1.1.18. A functor from $\mathcal{C}$ to Mod_A is called *pseudo-perfect* if it sends compact objects of $\mathcal{C}$ to perfect A -modules.

The following functor, $\mathcal{M}_{\mathcal{C}}$, was constructed in [TV07] and can be described as the ∞ -functor:

$$\begin{aligned} \mathcal{M}_{\mathcal{C}} : \mathrm{CAlg}_k &\longrightarrow \mathcal{S} \\ A &\longmapsto (\mathrm{Fun}_{\mathrm{ps},k}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_A))^{\simeq} \end{aligned}$$

where the subscript “ps” denotes pseudo-perfect functors.

Informally, the functor

$$\mathrm{Mod}_- : \mathrm{CAlg}_k \rightarrow \mathrm{Pr}^{\mathrm{L}}$$

is defined by sending a morphism $A \rightarrow B$ to the base change functor $- \otimes_A B : \mathrm{Mod}_A \rightarrow \mathrm{Mod}_B$. We then use the functoriality of $\mathrm{Fun}_{\mathrm{ps},k}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, -)$, which relies on the fact that $\mathrm{Fun}_k^{\mathrm{R}}(-, -)$ is already known to be a functor, and that the tensor product preserves perfect objects.

We can also describe $\mathrm{Fun}_{\mathrm{ps},k}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_A)$ equivalently as follows:

$$\begin{aligned} \mathrm{Fun}_{\mathrm{ps},k}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_A) &\simeq \mathrm{Fun}_k^{\mathrm{R},\omega}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_A) \\ &\simeq \mathrm{Fun}_k^{\mathrm{R},\omega}((\mathrm{Ind}(\mathcal{C}^{\omega}))^{\mathrm{op}}, \mathrm{Mod}_A) \\ &\simeq \mathrm{Fun}_k^{\mathrm{L},\omega}(\mathrm{Ind}(\mathcal{C}^{\omega}), \mathrm{Mod}_A^{\mathrm{op}})^{\mathrm{op}} \text{ }^{\textcolor{red}{1}} \\ &\simeq \mathrm{Fun}_k^{\mathrm{st}}(\mathcal{C}^{\omega}, \mathrm{Perf}(A)^{\mathrm{op}})^{\mathrm{op}} \\ &\simeq \mathrm{Fun}_k^{\mathrm{st}}((\mathcal{C}^{\omega})^{\mathrm{op}}, \mathrm{Perf}(A)). \end{aligned} \tag{1.1}$$

The last equivalence provides the ∞ -categorical analog of the one used in [TV07].

Remark 1.1.19. We observe that the ∞ -category $\mathcal{M}_{\mathrm{LMod}_B}(A)$ is equivalent to $\mathrm{LMod}_B(\mathrm{Perf}(A))$. Indeed, a pseudo-perfect, k -linear functor from $\mathrm{LMod}_B^{\mathrm{op}}$ to Mod_A corresponds to a perfect complex F on A and a map

$$\mathrm{End}_{\mathrm{LMod}_B^{\mathrm{op}}}(B, B) \rightarrow \mathrm{End}_{\mathrm{Mod}_A}(F).$$

It suffices to observe that, as an $\mathbb{E}_1$ -algebra, $\mathrm{End}_{\mathrm{LMod}_B^{\mathrm{op}}}(B, B)$ is equivalent to B .

Theorem 1.1.20 ([TV07]). *The functor $\mathcal{M}_{\mathcal{C}} : \mathrm{CAlg}_k \rightarrow \mathcal{S}$ is a derived stack, and if $\mathcal{C}$ is of finite type, $\mathcal{M}_{\mathcal{C}}$ is locally geometric.*

Definition 1.1.21. We call the functor $\mathcal{M}_{\mathcal{C}}$ the moduli stack of objects of $\mathcal{C}$.

Now we can view $\mathcal{M}_-$ as a functor from $(\mathrm{Pr}_k^{\mathrm{L},\omega})^{\mathrm{op}} \rightarrow \mathrm{dSt}$ (where dSt denotes the ∞ -category of derived stacks).

¹This follows from the universal property of Ind-completion (i.e., the ∞ -category of functors preserving filtered colimits from $\mathrm{Ind}(\mathcal{C})$ to $\mathcal{D}$ is equivalent to the ∞ -category of functors from $\mathcal{C}$ to $\mathcal{D}$) and the fact that all colimits can be obtained as a composition of finite colimits and filtered colimits.

1.2 Simple objects in an ∞ -category and moduli of objects in rigid geometry

In this section, we define several analytic stacks. In particular, we construct a rigid analytic version of the derived moduli stack of objects introduced in [TV07]. Given a smooth and proper k -linear ∞ -category $\mathcal{C}$, we define the analytic stack of objects in $\mathcal{C}$, denoted by $\mathcal{ANM}_{\mathcal{C}}$. We adapt the notion of simple objects from [TV07] to define an analytic variant of the moduli stack of simple objects, as in [TV07, Definition 3.21]. Specifically, for a smooth and proper k -linear ∞ -category $\mathcal{C}$, we construct the analytic substack $\mathcal{ANM}_{\mathcal{C}}^{\text{sim}} \subset \mathcal{ANM}_{\mathcal{C}}$. Furthermore, we introduce the analytic stack of simple perfect complexes over a rigid analytic variety X , denoted by $\underline{\text{Perf}}_X^{\text{sim}}$. The goal of this section is to compare these analytic stacks with the analytification of the algebraic stack $\mathcal{M}_{\text{Perf}(X)}^{\text{sim}}$. In particular, we will prove the following result, which plays a crucial role in the proof of Theorem 1.3.1.

Theorem 1.2.1. *Let X be a proper, quasi-compact, and separated rigid analytic variety over a non-Archimedean field k , such that $\text{Perf}(X)$ is a smooth and proper k -linear ∞ -category. Then, there is an equivalence of analytic stacks between $\underline{\text{Perf}}_X$ and the analytification of the algebraic stack $\mathcal{M}_{\text{Perf}(X)}$. Moreover, this equivalence restricts to simple objects, and we obtain an equivalence of 1-stacks between $t_0 \underline{\text{Perf}}_X^{\text{sim}}$ and $(t_0 \mathcal{M}_{\text{Perf}(X)}^{\text{sim}})^{\text{an}}$.*

We obtain this theorem as a consequence of Proposition 1.2.45, Proposition 1.2.47, and Corollary 1.2.53.

We begin by defining the general notion of a simple object in Section 1.2.1. Then, in Section 1.2.2, Section 1.2.3, and Section 1.2.4, we introduce the main stacks considered in this paper and adapt the concept of simple objects to this setting. Finally, we prove Proposition 1.2.45 in Section 1.2.5, followed by the proofs of Proposition 1.2.47 and Corollary 1.2.53 in Section 1.2.6.

We denote by Afd_k the category of affinoid spaces (as defined in [Bos14]) over a fixed complete non-Archimedean field k , of characteristic 0, endowed with a non-trivial valuation. Affinoid spaces play the same role in rigid geometry as affine schemes do in algebraic geometry. In our context, the most relevant examples of affinoid spaces for constructing rigid analytic varieties are those whose algebra of functions is a quotient of the Tate algebra $k\langle t_1, \dots, t_n \rangle$ by a finitely generated ideal; we will refer to these as affinoid spaces of finite type. For precise definitions of affinoid spaces and rigid analytic varieties, we refer to [Bos14]. We implicitly assume that all rigid analytic varieties are quasi-compact, separated, and of finite type.

To properly formulate the notion of derived stacks in rigid analytic geometry, we

adopt the framework developed in [PY21; PY18; PY16]. Throughout our discussion of stacks, we consistently work with the étale topology.

We denote by dAfd_k the ∞ -category of derived affinoid spaces over k , as described in [PY18, Definition 7.3]. Additionally, we denote by $\mathrm{Sp}(A)$ the underived affinoid space associated with an underived affinoid algebra A . It is important to note that, following the conventions of [PY21], derived affinoids do not strictly take the form $\mathrm{Sp}(A)$ as they do in the classical setting. However, if S is a derived affinoid, its algebra of functions is a derived affinoid algebra (i.e., a connective $\mathbb{E}_\infty$ -algebra A in Mod_k such that $\pi_0(A)$ is an underived affinoid algebra and the $\pi_i(A)$ are finitely generated modules over $\pi_0(A)$).

1.2.1 Simple objects in ∞ -categories

Let $\mathcal{V}$ be a commutative algebra object in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ equipped with a t-structure, and let $\mathcal{C}$ be a presentable $\mathcal{V}$ -linear ∞ -category (i.e., an object in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$). Since $\mathcal{C}$ is a module over $\mathcal{V}$, we denote by

$$\mathcal{V} \otimes \mathcal{C} \rightarrow \mathcal{C}$$

the canonical action of $\mathcal{V}$ on $\mathcal{C}$. Let F be an object in $\mathcal{C}$. Since the functor $- \otimes F$ commutes with colimits, it admits a right adjoint, which we denote by $\mathrm{Hom}_{\mathcal{V} \otimes \mathcal{C}}(F, -)$.

$$\mathcal{C} \begin{array}{c} \xleftarrow{- \otimes F} \\ \xrightarrow[\mathrm{Hom}_{\mathcal{V} \otimes \mathcal{C}}(F, -)]{\perp} \end{array} \mathcal{V} \quad (1.2)$$

Remark 1.2.2. Note that for every object $F \in \mathcal{C}$, the action of $\mathcal{V}$ induces a map:

$$\mathbf{1}_{\mathcal{V}} \otimes F \xrightarrow{id} F$$

By adjunction, there is also a canonical map:

$$\mathbf{1}_{\mathcal{V}} \rightarrow \mathrm{Hom}_{\mathcal{V} \otimes \mathcal{C}}(F, F).$$

We now use this map to define simple objects, and in the subsequent subsections, we will apply this definition within various settings. Note that this map depends only on the action and the adjunction, rather than on the relative tensor product.

Remark 1.2.3. Even in the noncommutative setting, within the ∞ -category $\mathrm{LMod}_B(\mathcal{C})$ where B is an associative algebra in a symmetric monoidal ∞ -category $\mathcal{C}$, it is possible to give a meaningful definition of dualizable object. The definitions and main properties are essentially the same as in the commutative setting, albeit with a few differences. For instance, if $F \in \mathrm{LMod}_B$, then its left dual $F^\vee$, if it exists, belongs to RMod_B . Moreover,

the relative tensor product $F^\vee \otimes_B F$ is an object of $\mathcal{C}$ equipped with a canonical map from the monoidal unit $\mathbf{1}_{\mathcal{C}}$, induced by the identity morphism. Indeed, by [Lur17, Proposition 4.6.2.18], we have

$$\mathrm{Hom}_{\mathrm{RMod}_B}(F^\vee \otimes \mathbf{1}_{\mathcal{C}}, F^\vee) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, F^\vee \otimes_{\mathcal{C}} F).$$

Note that the first tensor product is associated with the action of $\mathcal{C}$ on F , whereas the latter is the exterior tensor product (i.e., $- \otimes_B - : \mathrm{RMod}_B(\mathcal{C}) \times \mathrm{LMod}_B(\mathcal{C}) \rightarrow \mathcal{C}$). For precise definitions and the main properties of the relative tensor product, we refer the reader to [Lur17, Section 4.6.2].

Definition 1.2.4. Let $\mathcal{V}$ be a symmetric monoidal ∞ -category equipped with a t-structure. We say that an object $C \in \mathcal{V}$ has Tor-amplitude strictly smaller than zero if the homological amplitude of the tensor product $C \otimes D$ is strictly smaller than zero for every D in the heart of $\mathcal{V}$.

Definition 1.2.5. Let $\mathcal{V}$ be a presentable symmetric monoidal ∞ -category equipped with a t-structure, and let $\mathcal{C}$ be a $\mathcal{V}$ -linear ∞ -category. We say that an object F in $\mathcal{C}$ is *simple* if

$$\mathrm{cofib}(\mathbf{1}_{\mathcal{V}} \rightarrow \mathrm{Hom}_{\mathcal{V} \otimes \mathcal{C}}(F, F))$$

has Tor-amplitude strictly smaller than zero.

In the following subsections, we analyze this definition within the algebraic and rigid analytic settings. In particular, we will show that this general definition recovers the non-Archimedean analogues of the definitions of simple objects given in [TVo8a] for the stacks $\mathcal{M}_{\mathcal{C}}$, $\mathrm{An}\mathcal{M}_{\mathcal{C}}$, and Perf_X , where $\mathcal{C}$ is a smooth and proper ∞ -category and X is a smooth and proper rigid analytic variety. Furthermore, using Corollary 1.1.16, we can reduce to the case where $\mathcal{C}$ is equivalent to LMod_B for a smooth and proper algebra B . This is precisely the case we will begin studying.

1.2.2 Simple objects in $\mathcal{M}_B$ with B a smooth and proper $\mathbb{E}_1$ -algebra

To define the substack of simple objects of $\mathcal{M}_B$, we fix a connective $\mathbb{E}_{\infty}$ -algebra A in Mod_k and a smooth and proper $\mathbb{E}_1$ -algebra B in Mod_k . Based on the general discussion above, we now define simple objects in $\mathcal{M}_B(A)$.

Remark 1.2.6. We can apply Definition 1.2.5 by taking $\mathcal{A} = \mathrm{Pr}^{\mathrm{L}}$, $\mathcal{V} = \mathrm{Mod}_A$, and $\mathcal{C} = \mathrm{LMod}_B \otimes \mathrm{Mod}_A$. Here, the action is informally given by sending $(M, M' \otimes N)$ to $M \otimes_A$

$M' \otimes N$. This induces an adjunction:

$$\mathrm{LMod}_B \otimes \mathrm{Mod}_A \begin{array}{c} \xleftarrow{F \otimes_A -} \\ \xrightarrow[\mathrm{Hom}_{B \otimes A}(F, -)]{\perp} \end{array} \mathrm{Mod}_A$$

with $F \in \mathrm{LMod}_B \otimes \mathrm{Mod}_A$, where we denote by $\mathrm{Hom}_{B \otimes A}$ the external Hom constructed in (1.2).

We will now prove that this induces an adjunction:

$$\mathrm{Perf}(B \otimes A) \begin{array}{c} \xleftarrow{F \otimes_A -} \\ \xrightarrow[\mathrm{Hom}_{B \otimes A}(F, -)]{\perp} \end{array} \mathrm{Perf}(A)$$

with $F \in \mathrm{Perf}(B \otimes A)$. Note that the tensor product used here is the usual derived tensor product of A -modules, since the tensor product of perfect complexes is again perfect.

Proposition 1.2.7. *Under the hypotheses described above, the functor*

$$F \otimes_A - : \mathrm{Perf}(A) \rightarrow \mathrm{Perf}(B \otimes A)$$

admits a right adjoint, which we denote by $\mathrm{Hom}_{B \otimes A}(F, -)$.

Proof. Consider the following diagram, in which the vertical maps are fully faithful. Note that at the level of module categories, we already know that the adjunction exists:

$$\begin{array}{ccc} \mathrm{LMod}_B \otimes \mathrm{Mod}_A & \begin{array}{c} \xleftarrow{F \otimes_A -} \\ \xrightarrow[\mathrm{Hom}_{B \otimes A}(F, -)]{} \end{array} & \mathrm{Mod}_A \\ \uparrow & & \uparrow \\ \mathrm{Perf}(B \otimes A) & \xleftarrow{F \otimes_A -} & \mathrm{Perf}(A) \end{array}$$

Since the tensor product functors coincide, it suffices to prove that if $F, G \in \mathrm{Perf}(B \otimes A)$, then $\mathrm{Hom}_{B \otimes A}(F, G)$ remains a perfect complex. To this end, let $\mathcal{W}$ be the smallest stable full subcategory of $\mathrm{LMod}_B \otimes \mathrm{Mod}_A \simeq \mathrm{LMod}_{B \otimes A}(\mathrm{Mod}_A)$ containing all objects F such that $\mathrm{Hom}_{B \otimes A}(F, G)$ is perfect. To prove our statement, it is enough to show that $\mathrm{Perf}(B \otimes A)$ is contained in $\mathcal{W}$. For this, it suffices to show that $B \otimes A$ belongs to $\mathcal{W}$.

The functor $\mathrm{Hom}_{B \otimes A}(B \otimes A, -)$ is precisely the forgetful functor (i.e., the right adjoint to $B \otimes A \otimes_A -$). Since B is smooth and proper, the forgetful functor preserves and reflects dualizable objects (see [Lur17, 4.6.4.4 and 4.6.4.12]), which in this setting coincide with perfect complexes over $B \otimes A$. $\square$

Remark 1.2.8. We can identify the ∞ -category $\mathcal{M}_B(A)$ with the ∞ -category of perfect A -modules equipped with an action of B , which we denote by $\mathrm{LMod}_B(\mathrm{Perf}(A))$; this is, in turn, equivalent to $\mathrm{LMod}_{B \otimes A}(\mathrm{Perf}(A))$. When B is smooth and proper, $B \otimes A$ is smooth and proper over A . Therefore, in this case, by Lemma 1.1.10, the ∞ -category $\mathrm{LMod}_B(\mathrm{Perf}(A))$ coincides with $\mathrm{Perf}(B \otimes A)$.

In this setting, Definition 1.2.5 translates to the following:

Definition 1.2.9. We say that $F \in \mathrm{LMod}_{B \otimes A}$ is *simple* if it is a perfect complex and the canonical map

$$\alpha: A \rightarrow \mathrm{Hom}_{B \otimes A}(F, F)$$

constructed as in Remark 1.2.2, is such that its cofiber $\mathrm{cofib}(\alpha)$ has Tor-amplitude strictly less than 0.

Remark 1.2.10. Let F and G be perfect complexes in $\mathrm{LMod}_{B \otimes A}$, and let A' be in Mod_A . Then

$$\mathrm{Hom}_{B \otimes A}(F, G) \otimes_A A' \simeq \mathrm{Hom}_{B \otimes A'}(F \otimes_A A', G \otimes_A A'). \quad (1.3)$$

By adjunction, providing the map in (1.3) is equivalent to providing a map:

$$(\mathrm{Hom}_{B \otimes A}(F, G) \otimes_A A' \otimes_{A'} F) \otimes_A A' \rightarrow G \otimes_A A'$$

which we can take to be the counit. To prove that the map in (1.3) is an equivalence, we consider the smallest full stable subcategory of $\mathrm{LMod}_{B \otimes A}$ containing modules F for which (1.3) is an equivalence. Since this category is stable under finite limits, finite colimits, and retracts, it must coincide with the category of perfect objects in $\mathrm{LMod}_{B \otimes A}$ (see [Lur17, Definition 7.2.4.1]). Thus, (1.3) is an equivalence for every perfect complex.

Lemma 1.2.11. *The following statements are equivalent:*

1. $F \in \mathrm{Perf}(B \otimes A)$ is simple.
2. For every morphism $A \rightarrow A'$ of derived commutative algebras, $F \otimes_A A'$ is simple; i.e., the cofiber of the canonical map induced by taking the tensor product in Definition 1.2.9,

$$\mathrm{cofib}(A' \rightarrow \mathrm{Hom}_{B \otimes A'}(F \otimes_A A', F \otimes_A A'))$$

has Tor-amplitude strictly less than 0.

3. The same statement holds as above, but with A' underived.

Proof. (1) $\Rightarrow$ (2): This follows from the fact that Tor-amplitude is stable under base change. Indeed, by Remark 1.2.10, we have:

$$\begin{aligned} \operatorname{cofib}(A \rightarrow \operatorname{Hom}_{B \otimes A}(F, F)) \otimes_A A' &\simeq \operatorname{cofib}(A \otimes_A A' \rightarrow \operatorname{Hom}_{B \otimes A}(F, F) \otimes_A A') \\ &\simeq \operatorname{cofib}(A' \rightarrow \operatorname{Hom}_{B \otimes A'}(F \otimes_A A', F \otimes_A A')). \end{aligned}$$

The converse immediately follows by taking $A' = A$ and the identity map.

(3) $\Rightarrow$ (2): We observe that $M \in \operatorname{Mod}_{A'}$ has Tor-amplitude strictly less than 0 if and only if this property holds for $M \otimes_{A'} \pi_0 A'$. We apply this to $M = \operatorname{cofib}(A' \rightarrow \operatorname{Hom}_{B \otimes A'}(F \otimes_A A', F \otimes_A A'))$. The opposite direction is trivial. $\square$

Lemma 1.2.12. *Let $F \in \operatorname{Perf}(B \otimes A)$. Then F is simple if and only if for every map $A \rightarrow A'$ of commutative algebras with A' underived, we have:*

$$\pi_i \operatorname{Hom}_{B \otimes A'}(F \otimes_A A', F \otimes_A A') = 0 \quad \forall i \geq 1$$

and

$$\pi_0 \operatorname{Hom}_{B \otimes A'}(F \otimes_A A', F \otimes_A A') \simeq A'.$$

Proof. The forward implication ($\Rightarrow$) follows from statement (3) in Lemma 1.2.11, via the exact sequence induced by the fiber sequence:

$$A' \rightarrow \operatorname{Hom}_{B \otimes A'}(F \otimes_A A', F \otimes_A A') \rightarrow \operatorname{cofib}.$$

For the converse implication ($\Leftarrow$), by statement (3) in Lemma 1.2.11, we may assume that A is underived. By hypothesis, the cofiber has homological amplitude concentrated in degrees strictly less than 0. Consequently, for every map $A \rightarrow k$, where k is a field, the cofiber

$$\operatorname{cofib}(k \rightarrow \operatorname{Hom}_{B \otimes k}(F \otimes_A k, F \otimes_A k))$$

has homological amplitude in degrees strictly less than 0 (which, over a field, coincides with Tor-amplitude). We then conclude by applying the local criterion of flatness (see [Lur18, Corollary 6.1.4.7]) to $M = \operatorname{cofib}(A \rightarrow \operatorname{Hom}_{B \otimes A}(F, F))$. $\square$

Remark 1.2.13. Note that the same proof works if we only consider maps from A to the residue fields of $\pi_0 A$.

As an immediate corollary, we deduce the following:

Corollary 1.2.14. *If A is an underived commutative algebra, then $F \in \operatorname{Perf}(B \otimes A)$ is simple if and only if the canonical map*

$$A \rightarrow \tau_{\geq 0} \operatorname{Hom}_{B \otimes A}(F, F)$$

is an equivalence.

This corollary establishes the equivalence between [TVo8a, Definition 4.1] and our Definition 1.2.9. We also emphasize that when A is a derived commutative algebra, the characterization presented in the preceding corollary is not appropriate. A naive definition of a simple perfect complex might stipulate that the canonical map

$$A \rightarrow \tau_{\geq 0} \mathrm{Hom}_{B \otimes A}(F, F)$$

is an equivalence. However, when A is derived, this condition is not stable under base change, and Lemma 1.2.12 fails, as demonstrated by the following example.

Example 1.2.15. Let us consider Mod_k equipped with an action of $k[T]$ induced by the natural map $k[T] \rightarrow k$. We define $k[\epsilon]$ to be $k \oplus k[1]$, and we consider $\mathrm{Mod}_{k[\epsilon]}$ equipped with an action of $k[T, \epsilon] = k[T] \oplus k[T][1]$. The inclusion map induces a base-change functor $f^*: \mathrm{Mod}_k \rightarrow \mathrm{Mod}_{k[\epsilon]}$ given by $- \otimes_k k[\epsilon]$. Note that $k \in \mathrm{Mod}_k$ is simple with respect to the action of $k[T]$, since $\mathrm{Hom}_{k[T]}(k, k) \simeq k \oplus k[-1]$. However, $f^*(k) = k[\epsilon]$ fails to be simple under the naive definition because $\mathrm{Hom}_{k[T, \epsilon]}(f^*k, f^*k) \simeq \mathrm{Hom}_{k[T]}(k, k) \otimes_k k[\epsilon] \simeq k[1] \oplus k^2 \oplus k[-1]$. In particular, the degree 0 part does not reduce merely to k .

1.2.3 Simple perfect complexes over a rigid analytic variety

We define the sheaf of small ∞ -categories

$$\mathrm{Perf}(-): \mathrm{dAfd}_k \rightarrow \mathrm{Cat}_\infty \tag{1.4}$$

to be the functor that sends a derived affinoid S to the ∞ -category of perfect complexes over S , denoted by $\mathrm{Perf}(S)$ (see [PY21, Definition 7.5]). We denote by $\underline{\mathrm{Perf}}$ the associated derived analytic stack.

We begin by recalling [PY21, Theorem 3.4]. We fix a derived affinoid space S with A its algebra of global sections and we consider the following diagram.

$$\begin{array}{ccc} \mathrm{Mod}_A & \xrightarrow{\Psi} & \mathrm{Mod}_{\mathcal{O}_S} \\ \uparrow & & \uparrow \\ \mathrm{Perf}(A) & \longrightarrow & \mathrm{Perf}(S). \end{array}$$

Here the functor Ψ sends an A -module M to the sheaf $\Psi(M)$, which, when evaluated on

an étale affinoid $\hat{S} \rightarrow S$ with $B := \Gamma(\hat{S}, \mathcal{O}_{\hat{S}})$, yields the module $M \hat{\otimes}_A B$, where $\hat{\otimes}$ denotes the completed tensor product. The functor Ψ is symmetric monoidal and its inverse is given by the global sections functor. Notably, it follows from [PY21, Theorem 3.4 and Lemma 7.6] that the functor Ψ restricts to perfect complexes; in this case, the completion is unnecessary as these modules are finitely generated. Using the results of [PY21], we moreover deduce the following:

Theorem 1.2.16 ([PY21, Theorem 3.4], Tate acyclicity). *The functor Ψ is t -exact and induces a symmetric monoidal equivalence between the ∞ -categories of perfect complexes, with its inverse given by the global sections functor $\Gamma(-, S)$.*

Proof. We can deduce that Ψ is an equivalence using [PY21, Theorem 3.4 and Lemma 7.6]. To prove that Ψ is symmetric monoidal, we observe that in the proof of [PY21, Theorem 3.4], for $M \in \text{Perf}(A)$, $\Psi(M)$ is identified with the presheaf

$$\mathcal{O}_S(-) \otimes M: \text{dAfd}/_S \rightarrow \text{Mod}_{\mathcal{O}_S},$$

which is, moreover, a hypersheaf. This implies that $\Psi(-)$ is symmetric monoidal. $\square$

Proposition 1.2.17 ([Mat22]). *The functor in (1.4) satisfies descent for the flat topology.*

Proof. This follows from [Mat22, Theorem 7.8] and Theorem 1.2.16. $\square$

Remark 1.2.18. Note that the functor $\text{Perf}(-)$ is also a sheaf of stable, idempotent-complete ∞ -categories. This follows from [Luro9, Proposition 5.1.4.9] and [Lur17, Theorem 1.1.4.4].

Definition 1.2.19. We define the analytic stack of perfect complexes over a derived analytic stack X , denoted by $\underline{\text{Perf}}_X$, as:

$$\underline{\text{Perf}}_X := \underline{\text{Map}}(X, \underline{\text{Perf}}),$$

where $\underline{\text{Map}}$ denotes the analytic mapping stack as defined in [PY21].

Remark 1.2.20. We note that the functor of points of $\underline{\text{Perf}}_X$ can be described explicitly by sending an affinoid space S to the ∞ -category $\text{Perf}(X \times S)^\simeq$.

We will now formulate the notion of a simple object in $\underline{\text{Perf}}_X(S)$, where X is a smooth and proper rigid analytic variety and S is a derived affinoid with $A = \Gamma(S, \mathcal{O}_S)$. To do this, we apply Remark 1.2.2 with $\mathcal{V} = \text{Mod}_{\mathcal{O}_S}$ and $\mathcal{C} = \text{Mod}_{\mathcal{O}_{X \times S}}$, where the action is induced by sending the pair $(\mathcal{F} \in \text{Mod}_{\mathcal{O}_S}, \mathcal{G} \in \text{Mod}_{\mathcal{O}_{X \times S}})$ to $p^* \mathcal{F} \otimes_{X \times S} \mathcal{G}$. Here, $p: X \times S \rightarrow S$ denotes the projection. In this case, the adjunction can be written more

explicitly. Indeed, by composing the tensor-Hom adjunction for $\mathcal{O}_{X \times S}$ -modules with the adjunction between p^* and p_* , we obtain the following adjunction:

$$\mathrm{Mod}_{\mathcal{O}_{X \times S}} \begin{array}{c} \xrightarrow{p_* \mathcal{H}om_{X \times S}(\mathcal{G}, -)} \\ \xleftarrow[p^*(-) \otimes_{X \times S} \mathcal{G}]{\top} \end{array} \mathrm{Mod}_{\mathcal{O}_S}$$

where we denote by $\mathcal{H}om(-, -)$ the internal Hom sheaf in $\mathrm{Mod}_{\mathcal{O}_S}$.

Lemma 1.2.21. *Let Y be a rigid analytic variety and let $\mathcal{F}, \mathcal{G}$ be perfect complexes over Y . Then $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})$ is also a perfect complex.*

Proof. The statement is local on Y . Locally, we may assume that $\mathcal{F}$ is a retract of a finite colimit of copies of $\mathcal{O}_Y$. This implies that, locally, $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})$ is a retract of a finite colimit of copies of $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{G}) \simeq \mathcal{G}$, which is a perfect complex. Therefore, $\mathcal{H}om_{X \times S}(\mathcal{F}, \mathcal{G})$ is also a perfect complex. $\square$

Corollary 1.2.22. *Let $\mathcal{F} \in \mathrm{Perf}(X \times S)$ and let $p_S: X \times S \rightarrow S$ be the projection map. Then there is an adjunction:*

$$\mathrm{Perf}(X \times S) \begin{array}{c} \xrightarrow{p_{S*} \mathcal{H}om_{X \times S}(\mathcal{F}, -)} \\ \xleftarrow[\mathcal{F} \otimes_{X \times S} p_S^*(-)]{\top} \end{array} \mathrm{Perf}(S).$$

Proof. Starting from the adjunction between $\mathrm{Mod}_{\mathcal{O}_{X \times S}}$ and $\mathrm{Mod}_{\mathcal{O}_S}$, we observe that since X is smooth and proper, the functors p_S^* and p_{S*} preserve perfect objects [PY21, Proposition 7.8]. Furthermore, the tensor product of two perfect complexes is again a perfect complex. Thus, we only need to prove that $\mathcal{H}om_{X \times S}(\mathcal{F}, \mathcal{G})$ is a perfect complex whenever $\mathcal{G}$ is perfect. This follows immediately from Lemma 1.2.21. $\square$

We denote the right adjoint in Corollary 1.2.22 by $\mathcal{H}om_{X \times S}(\mathcal{F}, -)$ and use it to describe simple objects as in Definition 1.2.5. In particular, we have the following definition:

Definition 1.2.23. An object $\mathcal{F} \in \mathrm{Perf}(X \times S)$ is *simple* if the cofiber of the canonical map (as constructed in Remark 1.2.2),

$$\mathcal{O}_S \rightarrow \mathcal{H}om_{X \times S}(\mathcal{F}, \mathcal{F}),$$

has Tor-amplitude in strictly negative degrees.

Remark 1.2.24. Let S be a derived affinoid and s a closed point in S (i.e., a closed point in the associated underived affinoid $\pi_0 S$) classified by a map $s: \mathrm{Sp}(K) \rightarrow S$. Consider

the following Cartesian diagram:

$$\begin{array}{ccc} X \times \mathrm{Sp}(K) & \xrightarrow{i_s := id \times s} & X \times S \\ \downarrow q & & \downarrow p \\ \mathrm{Sp}(K) & \xrightarrow{s} & S \end{array}$$

where p and q are the projections. We have that

$$s^* \underline{\mathcal{H}om}_{X \times S}(\mathcal{F}, \mathcal{F}) := s^* p_* \mathcal{H}om_{X \times S}(\mathcal{F}, \mathcal{F}) \simeq q_* i_s^* \mathcal{H}om_{X \times S}(\mathcal{F}, \mathcal{F}).$$

Now, since a perfect complex is dualizable and the functor i_s^* is symmetric monoidal, we obtain:

$$s^* \underline{\mathcal{H}om}_{X \times S}(\mathcal{F}, \mathcal{F}) \simeq q_* \mathcal{H}om_{X \times \mathrm{Sp}(K)}(i_s^* \mathcal{F}, i_s^* \mathcal{F}) := \underline{\mathcal{H}om}_{X \times \mathrm{Sp}(K)}(i_s^* \mathcal{F}, i_s^* \mathcal{F}).$$

In particular, taking global sections of the map in Definition 1.2.23, we obtain a map:

$$K \rightarrow \mathrm{Hom}_{X \times \mathrm{Sp}(K)}(i_s^* \mathcal{F}, i_s^* \mathcal{F}), \quad (1.5)$$

where K is the finite field extension of k corresponding to the point s .

Lemma 1.2.25. *Let $\mathcal{F} \in \mathrm{Perf}(X \times S)$. Using the notation from Remark 1.2.24, $\mathcal{F}$ is simple if and only if for every closed point $s: \mathrm{Sp}(K) \rightarrow S$, the sheaf $i_s^* \mathcal{F}$ is simple (i.e., the canonical map*

$$\mathbb{O}_{\mathrm{Sp}(K)} \rightarrow q_* \mathcal{H}om_{X \times \mathrm{Sp}(K)}(i_s^* \mathcal{F}, i_s^* \mathcal{F}) := \underline{\mathcal{H}om}_{X \times \mathrm{Sp}(K)}(i_s^* \mathcal{F}, i_s^* \mathcal{F}) \quad (1.6)$$

has its cofiber in strictly negative homological degrees). This is equivalent to requiring that the map constructed in (1.5),

$$K \rightarrow \mathrm{Hom}_{X \times \mathrm{Sp}(K)}(i_s^* \mathcal{F}, i_s^* \mathcal{F}),$$

has its cofiber in strictly negative homological degrees.

Proof. We first observe that applying Theorem 1.2.16 to the cofiber of the map in Definition 1.2.23 yields an object in $\mathrm{Perf}(A)$ given by

$$\mathrm{cofib}(A \rightarrow \mathrm{Hom}_{X \times S}(\mathcal{F}, \mathcal{F})).$$

By Theorem 1.2.16, we deduce that this object has Tor-amplitude in strictly negative

degrees if and only if $\mathcal{F}$ is simple. Consequently, for every closed point

$$s: \mathrm{Sp}(K) \rightarrow S,$$

the cofiber of the map in (1.6) has Tor-amplitude in strictly negative degrees if and only if the same holds for the map in (1.5). Therefore, showing that $\mathcal{F}$ is simple if and only if $i_s^* \mathcal{F}$ is simple reduces to Lemma 1.2.11. Indeed, upon taking global sections (which preserves simple objects), the pullback s^* can be identified with the functor $- \otimes_A K$. This yields exactly the map in (1.5). $\square$

In this way, we can see that our Definition 1.2.23 is exactly the non-Archimedean analogue of [TVo8a, Definition 5.2].

Corollary 1.2.26. *If S is underived, then $\mathcal{F}$ is simple if and only if the map from Definition 1.2.23 induces an equivalence:*

$$\mathcal{O}_S \simeq \tau_{\geq 0} \mathcal{H}om_{X \times S}(\mathcal{F}, \mathcal{F}).$$

Proof. If S is underived, then $\mathcal{O}_S$ is concentrated in degree 0. The result now follows from Lemma 1.2.25 and Lemma 1.2.12. $\square$

1.2.4 The rigid analytic stack of objects in an ∞ -category

We begin this section by defining the (rigid) derived analytic stack of objects in an ∞ -category.

Definition 1.2.27. The functor of points $\mathcal{A}n\mathcal{M}_{\mathcal{C}}: \mathrm{dAfd}_k^{\mathrm{op}} \rightarrow \mathcal{S}$ is defined by sending a derived affinoid S to the space of functors $\mathrm{Fun}_k^{\mathrm{st}}(\mathcal{C}^\omega, \mathrm{Perf}(S))^\simeq$.

As a corollary of Proposition 1.2.17, we obtain the following result:

Corollary 1.2.28. *The functor $\mathcal{A}n\mathcal{M}_{\mathcal{C}}$ is a derived analytic stack.*

We will now define the substack of simple objects.

Consider a derived affinoid space S and let $A := \Gamma(S, \mathcal{O}_S)$. To define simple objects in $\mathcal{A}n\mathcal{M}_{\mathrm{Mod}_B}(S)$, we will utilize the definition of simple objects established above for the algebraic case, combined with Tate acyclicity, as follows.

Remark 1.2.29. When $\mathcal{C}$ is equivalent to LMod_B for a smooth and proper $\mathbb{E}_1$ -algebra B , we have

$$\mathcal{A}n\mathcal{M}_{\mathcal{C}}(S) := \mathrm{Fun}_k^{\mathrm{st}}(\mathrm{Perf}(B), \mathrm{Perf}(S)),$$

which classifies perfect $\mathcal{O}_S$ -modules equipped with an action of B . For this reason, we can identify this ∞ -category with $\mathrm{LMod}_B(\mathrm{Perf}(S))$.

Corollary 1.2.30. *The functor $\Psi(-)$ from Theorem 1.2.16 induces an equivalence between $\mathrm{LMod}_B(\mathrm{Perf}(A))$ and $\mathrm{LMod}_B(\mathrm{Perf}(S))$.*

Proof. This is a consequence of Theorem 1.2.16 and Lemma 1.1.5, taking into account Remark 1.2.29 and Remark 1.1.19. $\square$

We observe that since $\Psi(-)$ is symmetric monoidal, by using Lemma 1.1.5 we also deduce that the ∞ -category $\mathcal{A}n\mathcal{M}_{\mathcal{C}}(S)$ is equivalent to $\mathrm{LMod}_{\Psi(B \otimes A)}(\mathrm{Perf}(S))$.

Proposition 1.2.31. *For every object $\mathcal{F}$ in $\mathrm{LMod}_B(\mathrm{Perf}(S))$, the functor*

$$\mathcal{F} \otimes_{\mathcal{O}_S} - : \mathrm{Perf}(S) \rightarrow \mathrm{LMod}_B(\mathrm{Perf}(S))$$

admits a right adjoint.

Proof. Using Theorem 1.2.16, Corollary 1.2.30, and Proposition 1.2.7, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{LMod}_B(\mathrm{Perf}(S)) & \xleftarrow{\simeq} & \mathrm{LMod}_B(\mathrm{Perf}(A)) \\ \mathcal{F} \otimes - \uparrow & & \Gamma(\mathcal{F}, S) \otimes - \uparrow \downarrow \mathrm{Hom}_{B \otimes A}(\Gamma(\mathcal{F}, S), -) \\ \mathrm{Perf}(S) & \xleftarrow{\simeq} & \mathrm{Perf}(A). \end{array} \quad (1.7)$$

From (1.7), we deduce the existence of the required right adjoint. $\square$

Definition 1.2.32. For $\mathcal{F} \in \mathrm{Mod}_B(\mathrm{Perf}(S))$, we denote by $\underline{\mathrm{Hom}}_B(\mathcal{F}, -)$ the right adjoint to $\mathcal{F} \otimes_{\mathcal{O}_S} -$.

Definition 1.2.33. We say that $\mathcal{F} \in \mathrm{LMod}_B(\mathrm{Perf}(S))$ is *simple* if the cofiber of the canonical map

$$\mathcal{O}_S \rightarrow \underline{\mathrm{Hom}}_B(\mathcal{F}, \mathcal{F})$$

has Tor-amplitude concentrated in homological degrees strictly less than 0.

Proposition 1.2.34. *Let $\mathcal{F}$ be an object in $\mathrm{LMod}_B(\mathrm{Perf}(S))$, and let*

$$F := \Gamma(\mathcal{F}, S)$$

be the corresponding A -module of global sections. Then F is simple in the sense of Definition 1.2.9 if and only if $\mathcal{F}$ is simple in the sense of Definition 1.2.33.

Proof. By Theorem 1.2.16, we can assume without loss of generality that $\mathcal{F} \simeq \Psi(F)$. Consider the canonical map from Definition 1.2.9:

$$A \xrightarrow{\alpha} \mathrm{Hom}_{B \otimes A}(F, F)$$

This induces the following diagram in $\mathrm{Mod}_{\mathcal{O}_S}$, where β is the map introduced in Definition 1.2.33:

$$\begin{array}{ccc} \Psi(A) & \xrightarrow{\Psi(\alpha)} & \Psi(\mathrm{Hom}_{B \otimes A}(F, F)) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathcal{O}_S & \xrightarrow{\beta} & \underline{\mathrm{Hom}}_B(\mathcal{F}, \mathcal{F}) \end{array}$$

The vertical map on the left is an equivalence (see Definition 1.2.33 and (1.7)). Since Ψ is a t-exact symmetric monoidal equivalence, the diagram above identifies $\Psi(\mathrm{cofib}(\alpha))$ with $\mathrm{cofib}(\beta)$; moreover Ψ preserves and reflects Tor-amplitude. Hence $\mathrm{cofib}(\alpha)$ has Tor-amplitude strictly less than 0 if and only if $\mathrm{cofib}(\beta)$ does. $\square$

Lemma 1.2.35. *Let $\mathcal{F}$ be an object in $\mathrm{LMod}_B(\mathrm{Perf}(S))$. Then, $\mathcal{F}$ is simple if and only if for every map of derived affinoids*

$$i: S' \rightarrow S,$$

$i^\mathcal{F}$ is simple.*

Proof. In light of Proposition 1.2.34, this lemma can be deduced directly from Lemma 1.2.11. $\square$

Lemma 1.2.36. *Let $\mathcal{F} \in \mathcal{A}n\mathcal{M}_B(S)$, where B is a smooth and proper algebra. Then $\mathcal{F}$ is simple if and only if for every closed point $i: s \rightarrow S$ associated to a field K , the cofiber of the canonical map*

$$K \rightarrow \mathrm{Hom}_{B \otimes K}(i^*\mathcal{F}, i^*\mathcal{F})$$

(as in Remark 1.2.2) induces the following equivalences:

$$\pi_0 \mathrm{Hom}_{B \otimes K}(i^*\mathcal{F}, i^*\mathcal{F}) \simeq K,$$

$$\pi_j \mathrm{Hom}_{B \otimes K}(i^*\mathcal{F}, i^*\mathcal{F}) \simeq 0 \quad \text{for every } j > 0.$$

Proof. By passing to global sections and using Proposition 1.2.34, we can reduce to the algebraic setting. Our statement then corresponds to Lemma 1.2.12 applied to every map $A \rightarrow K$, where K is a finite field extension (see also Remark 1.2.13). $\square$

This proposition demonstrates that Definition 1.2.33 is the non-Archimedean derived analogue of [TVo8a, Definition 4.9].

Corollary 1.2.37. *If S is an underived affinoid, then $\mathcal{F} \in \mathrm{Mod}_B(\mathrm{Perf}(S))$ is simple if and only if the canonical map from Definition 1.2.33 induces an equivalence:*

$$\mathcal{O}_S \simeq \tau_{\geq 0} \underline{\mathrm{Hom}}_B(\mathcal{F}, \mathcal{F}).$$

Proof. If S is underived, then $\mathcal{O}_S$ is concentrated in degree 0. Our statement now follows from Lemma 1.2.36. $\square$

1.2.5 Comparison between the stacks $\mathcal{M}_B$ and $\mathcal{AnM}_B$

The aim of this subsection is to prove that if $\mathcal{C}$ is a smooth and proper ∞ -category, then there is an equivalence between $\mathcal{AnM}_{\mathcal{C}}$ and $(\mathcal{M}_{\mathcal{C}})^{\mathrm{an}}$, and that this equivalence moreover preserves simple objects (see Proposition 1.2.45). The proof is based on comparing the functors of points of the two stacks. We begin by introducing the main tool we need (i.e., [HP25, Theorem 3.10]).

Let Y be a derived algebraic stack over a non-Archimedean field k , and let Y^{an} be its analytification as in [HP25]. We consider the following functor:

Definition 1.2.38. We denote by $G(Y)$ the functor:

$$\begin{array}{ccc} G_*(Y): \mathrm{dAfd} & \longrightarrow & \mathcal{S} \\ S & \longrightarrow & Y(\mathrm{Spec}(\Gamma(S, \mathcal{O}_S))) \end{array}$$

In the notation of [HP25], this presheaf is denoted by $G_*^p(Y)$, and its sheafification is denoted by $G_*^s(Y)$, where $*$ is the final object in dAnSt_k .

Theorem 1.2.39 ([HP25, Theorem 3.10]). *Let Y be a locally geometric derived stack locally almost of finite presentation. Then there is an equivalence of presheaves:*

$$G_*(Y) \simeq Y^{\mathrm{an}}. \tag{1.8}$$

Using Theorem 1.2.39, it is straightforward to see that the functor of points of $(\mathcal{M}_{\mathcal{C}})^{\mathrm{an}}$ coincides with that of $\mathcal{AnM}_{\mathcal{C}}$. We will first construct a map between these two derived stacks, and subsequently prove that it is an equivalence.

Construction 1.2.40. Let $\mathcal{C}$ be a smooth and proper ∞ -category in $\mathrm{Pr}^{\mathrm{L}, \omega}$. We consider

the composition:

$$\mathcal{A}n\mathcal{M}_{\mathcal{C}} \longrightarrow G(\mathcal{M}_{\mathcal{C}}) \xrightarrow{\alpha} (\mathcal{M}_{\mathcal{C}})^{\text{an}} \quad (1.9)$$

where α is the map considered in [HP25, Theorem 3.10], and the first map is induced by the global sections functor, as in Theorem 1.2.16.

Proposition 1.2.41. *If $\mathcal{C}$ is a smooth and proper ∞ -category in $\text{Pr}_{\text{st}}^{\text{L},\omega}$, then the map in (1.9) is an equivalence.*

Proof. We will prove that both maps

$$(\mathcal{M}_{\mathcal{C}})^{\text{an}} \rightarrow G(\mathcal{M}_{\mathcal{C}})$$

and

$$\mathcal{A}n\mathcal{M}_{\mathcal{C}} \rightarrow G(\mathcal{M}_{\mathcal{C}})$$

are equivalences. To prove that the first map is an equivalence, we observe that [TV07, Corollary 3.15] (since $\mathcal{C}$ is smooth and proper) states that the functor $\mathcal{M}_{\mathcal{C}}$ is a derived geometric stack locally of finite presentation; therefore, we can apply Theorem 1.2.39. Now we move to the second equivalence, where we can then easily compare the functors of points of $G(\mathcal{M}_{\mathcal{C}})$ and $\mathcal{A}n\mathcal{M}_{\mathcal{C}}$. Indeed, they are equivalent by Tate acyclicity (Theorem 1.2.16). $\square$

Note that this result allows us to describe the functor of points of the analytification of $\mathcal{M}_{\mathcal{C}}$, which is otherwise only defined as a left Kan extension. Indeed, we now have an explicit description for the functor of points via the stack $\mathcal{A}n\mathcal{M}_{\mathcal{C}}$. Since we always assume $\mathcal{C}$ to be a smooth and proper ∞ -category in $\text{Pr}_{\text{st}}^{\text{L},\omega}$, by Corollary 1.1.16 we may assume without loss of generality that $\mathcal{C}$ is of the form LMod_B , where B is a smooth and proper $\mathbb{E}_1$ -algebra. We now aim to prove that the equivalence in Proposition 1.2.41 restricts to simple objects.

Definition 1.2.42. We define $\mathcal{M}_B^{\text{sim}}$ as the substack of $\mathcal{M}_B$ that sends a derived commutative algebra A to the groupoid of simple objects $\mathcal{M}_B^{\text{sim}}(A)^{\simeq}$, as defined in Definition 1.2.9.

Using Lemma 1.2.11, we can readily see that $\mathcal{M}_B^{\text{sim}}$ is a substack of $\mathcal{M}_B$.

Remark 1.2.43. Using Corollary 1.2.14, we deduce that the functor $\mathcal{M}_B^{\text{sim}}$, when restricted to underived commutative algebras (denoted by $t_0\mathcal{M}_B^{\text{sim}}$), is naturally a 1-stack. Moreover, by [TV07, Corollary 3.22], it is an Artin 1-stack. We also note that $t_0\mathcal{M}_B^{\text{sim}}$ corresponds to the Artin 1-stack of simple objects defined in [TV07].

Definition 1.2.44. We define $\mathcal{A}n\mathcal{M}_B^{\text{sim}}$ as the functor that sends a derived affinoid space S to the groupoid of simple objects $\mathcal{A}n\mathcal{M}_B^{\text{sim}}(S)^\simeq$, as defined in Definition 1.2.33.

The functoriality of the above definition follows from Lemma 1.2.35. Using Corollary 1.2.37, we can see that the restriction of $\mathcal{A}n\mathcal{M}_B^{\text{sim}}$ to underived affinoids, denoted by $t_0\mathcal{A}n\mathcal{M}_B^{\text{sim}}$, is naturally a 1-stack.

Proposition 1.2.45. *If B is a smooth and proper algebra, then the equivalence of Proposition 1.2.41 restricts to an equivalence of functors $(\mathcal{M}_B^{\text{sim}})^{\text{an}} \simeq \mathcal{A}n\mathcal{M}_B^{\text{sim}}$.*

Proof. We begin by observing that Proposition 1.2.41, applied with $\mathcal{C} = \text{Mod}_B$, yields the equivalence between $(\mathcal{M}_B)^{\text{an}}$ and $\mathcal{A}n\mathcal{M}_B$. To establish the equivalence between the subfunctors of simple objects, we apply the same argument utilizing the functor G ; we merely need to check that the functor $\Psi(-)$ preserves and reflects simple objects. This has been proven in Proposition 1.2.34. $\square$

In particular, this proposition implies that $\mathcal{A}n\mathcal{M}_B$ is also a locally geometric derived analytic stack.

1.2.6 Comparison between $\underline{\text{Perf}}_X$ and $\mathcal{A}n\mathcal{M}_{\text{Perf}(X)}$

Definition 1.2.46. We define the functor $\underline{\text{Perf}}_X^{\text{sim}}$ as the subfunctor of $\underline{\text{Perf}}_X$ defined by sending a derived affinoid space S to the groupoid of simple objects $\text{Perf}(X \times S)^\simeq$.

We observe that when X is a proper (underived) rigid analytic variety, Definition 1.2.46 is well-posed since the pullback of a simple object remains simple. This follows from the base change property explained in [PY21, Theorem 6.8].

The aim of this section is to prove the following proposition.

Proposition 1.2.47. *Let $X \rightarrow \text{Sp}(k)$ be a proper, quasi-compact, and separated rigid analytic variety such that $\text{Perf}(X)$ is a smooth and proper ∞ -category over $\text{Perf}(k)$. Then there is an equivalence*

$$\underline{\text{Perf}}_X \simeq \mathcal{A}n\mathcal{M}_{\text{Perf}(X)}.$$

Furthermore, this equivalence restricts to an equivalence between simple objects (i.e., $\underline{\text{Perf}}_X^{\text{sim}} \simeq \mathcal{A}n\mathcal{M}_{\text{Perf}(X)}^{\text{sim}}$).

To prove this proposition, we require several lemmas.

Lemma 1.2.48. *Let $p: X \rightarrow \text{Sp}(k)$ be a proper rigid analytic variety. If $\mathcal{F}$ is a perfect complex, then $p_*(\mathcal{F})$ is also a perfect complex in Mod_k .*

Proof. This lemma follows from [PY21, Lemma 7.6]. We merely need to show that p has finite coherent cohomological dimension and finite Tor-amplitude. The morphism p has finite coherent cohomological dimension because X is of finite type. It also has finite Tor-amplitude because the target is a field and X is underived. $\square$

Lemma 1.2.49. *Let $p: X \rightarrow \mathrm{Sp}(K)$ be a proper rigid analytic variety over a non-Archimedean field K , such that $\mathrm{Perf}(X)$ is a smooth and proper, K -linear, idempotent-complete ∞ -category. Then $\mathrm{Perf}(X)$ admits a generator $\mathcal{E}$. Moreover, the pair of adjoint functors*

$$\mathrm{Perf}(X) \begin{array}{c} \xleftarrow{p^*(-) \otimes_{\mathcal{O}_X} \mathcal{E}} \\ \xrightarrow[\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, -)]{\perp} \end{array} \mathrm{Perf}(K)$$

induces an equivalence between $\mathrm{Perf}(X)$ and $\mathrm{LMod}_B(\mathrm{Perf}(K))$, where B is the $\mathbb{E}_1$ -algebra of endomorphisms of the generator $\mathcal{E}$.

Proof. We begin by observing that the adjunction is well-defined since X is proper over $\mathrm{Sp}(K)$. By definition, $\mathrm{Ind}(\mathrm{Perf}(X))$ is a smooth and proper, presentable, and compactly generated ∞ -category. The existence of a compact generator for $\mathrm{Ind}(\mathrm{Perf}(X))$ follows from Lemma 1.1.15. The remainder of the proof relies on the Barr–Beck–Lurie Theorem [Lur17, Theorem 4.7.3.5]. Let $\mathcal{E}$ be the compact generator of $\mathrm{Ind}(\mathrm{Perf}(X))$. The functor $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{Perf}(X))}(\mathcal{E}, -)$ is conservative (since $\mathcal{E}$ is a generator) and commutes with colimits (since $\mathcal{E}$ is compact and the categories are stable). By the aforementioned theorem, the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, -)$ induces an equivalence

$$\mathrm{Ind}(\mathrm{Perf}(X)) \simeq \mathrm{LMod}_T(\mathrm{Mod}_K),$$

where T is the monad $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} p^*(-))$. We will now identify the monad T with $B \otimes_K -$. For every K -module M , there is a canonical map:

$$\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) \otimes_K M \rightarrow \mathrm{Hom}_X(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{O}_X} p^*M). \quad (1.10)$$

Let $\mathcal{W}$ be the full subcategory of Mod_K consisting of all objects for which (1.10) is an equivalence. We observe that $\mathcal{W}$ contains K and is closed under colimits, implying that it is the entire ∞ -category Mod_K . In this way, we obtain an equivalence:

$$\mathrm{LMod}_T(\mathrm{Mod}_K) \simeq \mathrm{LMod}_B.$$

Since B is a smooth and proper algebra in Mod_K , we can identify the compact objects in LMod_B with $\mathrm{LMod}_B(\mathrm{Perf}(K))$. Because X is proper, the functor $\mathrm{Hom}_X(\mathcal{E}, -)$ pre-

serves perfect complexes, and thus it restricts to an equivalence between $\mathrm{Perf}(X)$ and $\mathrm{LMod}_B(\mathrm{Perf}(K))$. $\square$

Lemma 1.2.50. *Let $p: X \rightarrow \mathrm{Sp}(k)$ be a separated rigid analytic variety such that $\mathrm{Perf}(X)$ is smooth and proper over $\mathrm{Perf}(k)$. Let K be a finite field extension of k . We denote by X_K the fiber product rigid analytic variety in the following pullback diagram:*

$$\begin{array}{ccc} X_K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Sp}(K) & \longrightarrow & \mathrm{Sp}(k). \end{array}$$

Then we have an equivalence:

$$\mathrm{Perf}(X_K) \simeq \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(K).$$

Consequently, if $\mathrm{Perf}(X)$ is smooth and proper over $\mathrm{Perf}(k)$, then $\mathrm{Perf}(X_K)$ is smooth and proper over $\mathrm{Perf}(K)$.

Proof. We begin by noting that $\mathrm{Perf}(K)$ is a dualizable ∞ -category over $\mathrm{Perf}(k)$; by [BGT13, Theorem 3.4], this is equivalent to saying that $\mathrm{Perf}(K)$ is smooth and proper over $\mathrm{Perf}(k)$. Indeed, K is a finite separable field extension of k (which is a field of characteristic 0), implying that the map

$$\mathrm{Spec}(K) \rightarrow \mathrm{Spec}(k)$$

is étale. This ensures that $\mathrm{Perf}(K)$ is smooth over $\mathrm{Perf}(k)$. It is also proper because K is a finitely generated projective k -module. When X is isomorphic to an affinoid space $\mathrm{Sp}(A)$, we have:

$$\begin{aligned} \mathrm{Perf}(\mathrm{Sp}(A)) \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(K) &\simeq \mathrm{Perf}(A) \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(K) \\ &\simeq \mathrm{Perf}(A \otimes_k K) \\ &\simeq \mathrm{Perf}(\mathrm{Sp}(A) \times_{\mathrm{Sp}(k)} \mathrm{Sp}(K)). \end{aligned}$$

The first equivalence follows from Theorem 1.2.16, and the last follows from the fact that, in this case, the tensor product $A \otimes_k K$ can be identified with the completed tensor product. Since X is separated, we can take a cover by affinoid spaces and express $\mathrm{Perf}(X)$ as a limit of ∞ -categories of the form $\mathrm{Perf}(\mathrm{Sp}(A_i))$. We can then compute the tensor

product as follows:

$$\begin{aligned} \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(K) &\simeq \lim \left(\mathrm{Perf}(\mathrm{Sp}(A_i)) \otimes_{\mathrm{Perf}(k)} \mathrm{Perf}(K) \right) \\ &\simeq \lim \mathrm{Perf}(\mathrm{Sp}(A_i) \times_{\mathrm{Sp}(k)} \mathrm{Sp}(K)) \\ &\simeq \mathrm{Perf}(X_K). \end{aligned}$$

The final statement follows from the fact that smooth and proper ∞ -categories are stable under base change. $\square$

Lemma 1.2.51. *Let $p: X \rightarrow \mathrm{Sp}(k)$ be a quasi-compact and separated rigid analytic variety such that $\mathrm{Ind}(\mathrm{Perf}(X))$ admits a compact generator $\mathcal{E}$. Let K be a finite field extension of k . We denote by X_K the corresponding pullback rigid analytic variety:*

$$\begin{array}{ccc} X_K & \xrightarrow{i_X} & X \\ \downarrow & & \downarrow \\ \mathrm{Sp}(K) & \xrightarrow{i} & \mathrm{Sp}(k). \end{array}$$

Then $i_X^(\mathcal{E})$ is a compact generator of $\mathrm{Ind}(\mathrm{Perf}(X_K))$. Moreover, if $\mathrm{Perf}(X)$ is smooth and proper over k and B is the $\mathbb{E}_1$ -algebra of endomorphisms of $\mathcal{E}$, then $\mathrm{Perf}(X_K)$ is equivalent to $\mathrm{LMod}_{B \otimes_k K}(\mathrm{Perf}(K))$. In particular, the K -algebra of endomorphisms of $i_X^*(\mathcal{E})$ is Morita equivalent to the K -algebra $B \otimes_k K$.*

Proof. We observe that i_X^* preserves perfect complexes, so $i_X^*(\mathcal{E})$ is a compact object in $\mathrm{Ind}(\mathrm{Perf}(X_K))$. To show that $i_X^*(\mathcal{E})$ is a generator, it suffices to prove that the right adjoint of $\mathrm{Ind}(i_X^*)$ is conservative. We observe that $i_{X,*}(-)$ preserves perfect complexes. Indeed, being a perfect complex is a local property, allowing us to assume that X is an affinoid space $\mathrm{Sp}(A)$. In this case, X_K is identified with $\mathrm{Sp}(A \otimes_k K)$. Using Theorem 1.2.16, restricted to perfect complexes, $i_{X,*}$ is identified with the forgetful functor, which preserves perfect complexes because K is a finite extension of k . Thus, we can identify the right adjoint of $\mathrm{Ind}(i_X^*)$ with $\mathrm{Ind}(i_{X,*})$. Furthermore, locally on X , we can identify $\mathrm{Ind}(i_{X,*})$ with the forgetful functor between module ∞ -categories, which is conservative. This implies that $i_X^*\mathcal{E}$ is a generator. The final statement is deduced by first applying Lemma 1.2.50 to conclude that $\mathrm{Perf}(X_K)$ is smooth and proper over $\mathrm{Perf}(K)$, and subsequently applying Lemma 1.2.49. $\square$

Lemma 1.2.52. *Consider the following diagram of algebras over k , where $B' = A' \otimes_A B$:*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

For every $M \in \text{Mod}_{A'}$, we can construct the following induced diagram:

$$\begin{array}{ccc} \text{Mod}_A & \begin{array}{c} \xleftarrow{\quad \tau \quad} \\ \xrightarrow{\quad M \otimes_A - \quad} \end{array} & \text{Mod}_{A'} \\ \downarrow B \otimes_A - & & \downarrow B' \otimes_{A'} - \\ \text{Mod}_B & \begin{array}{c} \xleftarrow{\quad \tau \quad} \\ \xrightarrow{\quad B' \otimes_B - \quad} \end{array} & \text{Mod}_{B'} \end{array}$$

In this situation, the diagram formed by the left adjoints commutes. Furthermore, let BC_M denote the Beck-Chevalley transformation associated with this diagram, and let $\mathcal{C}$ be the full subcategory of $\text{Mod}_{A'}$ consisting of modules M for which BC_M is an equivalence. Then $\text{Perf}(A') \subseteq \mathcal{C}$.

Proof. We first note that A' belongs to $\mathcal{C}$. In this case, the right adjoint to the tensor product is simply the forgetful functor, and the Beck-Chevalley transformation is trivially an equivalence because for every module $N \in \text{Mod}_{A'}$, the relation $B' = A' \otimes_A B$ implies:

$$B \otimes_A N \simeq B' \otimes_{A'} N.$$

To complete the proof, we note that if M is a perfect complex over A' , it can be obtained as a finite colimit of retracts of finite direct sums of A' . Since the subcategory $\mathcal{C}$ is closed under retracts and finite colimits, it follows that $M \in \mathcal{C}$. $\square$

Proof of Proposition 1.2.47. Fix a derived affinoid space S with global sections algebra A . Our goal is to demonstrate a functorial equivalence of ∞ -categories between $\underline{\text{Perf}}_X(S)$ and $\mathcal{A}n\mathcal{M}_{\text{Perf}(X)}(S)$. Recall that since $\text{Perf}(X)$ is smooth and proper, by Lemma 1.2.49 it admits a compact generator $\mathcal{E}$ and can be identified with $\text{Perf}(B)$, where B is the $\mathbb{E}_1$ -algebra of endomorphisms of $\mathcal{E}$. Using Remark 1.2.29, we obtain an equivalence:

$$\mathcal{A}n\mathcal{M}_{\text{Perf}(X)}(S) := \text{Fun}_k^{\text{st}}(\text{Perf}(X)^{\text{op}}, \text{Perf}(S)) \simeq \text{LMod}_B(\text{Perf}(S)).$$

Furthermore, by definition:

$$\underline{\mathrm{Perf}}_X(S) := \mathrm{Perf}(X \times S).$$

Consider the projection diagram:

$$\begin{array}{ccc} & X \times S & \\ p_1 \swarrow & & \searrow p_2 \\ X & & S \end{array}$$

Using the compact generator, we can establish an adjunction:

$$\mathrm{Perf}(X \times S) \begin{array}{c} \xleftarrow{\phi := p_2^*(-) \otimes p_1^* \mathcal{E}} \\ \perp \\ \xrightarrow{\psi := p_{2,*} \mathcal{H} \circ m_{X \times S}(p_1^* \mathcal{E}, -)} \end{array} \mathrm{LMod}_B(\mathrm{Perf}(S)). \quad (1.11)$$

By Lemma 1.2.49, the adjunction in (1.11) is an equivalence when $S \simeq \mathrm{Sp}(k)$, as these are exactly the functors producing the adjoint equivalence between $\mathrm{Perf}(X)$ and $\mathrm{Perf}(B)$. Using Lemma 1.2.49, Lemma 1.2.50, and Lemma 1.2.51, we deduce that this adjunction is also an equivalence when $S \simeq \mathrm{Sp}(K)$ for every finite field extension K of k .

The strategy is to deduce the general case from the fields $S \simeq \mathrm{Sp}(K)$. For every closed point $s: \mathrm{Sp}(K) \rightarrow S$ with residue field K , we consider the Cartesian diagram:

$$\begin{array}{ccc} X_K := X \times \mathrm{Sp}(K) & \xrightarrow{i_s} & X \times S \\ \downarrow p_s & & \downarrow p_2 \\ \mathrm{Sp}(K) & \xrightarrow{s} & S \end{array} \quad (1.12)$$

This induces the following diagram:

$$\begin{array}{ccc} \mathrm{Perf}(X \times S) & \begin{array}{c} \xleftarrow{\psi} \\ \perp \\ \xrightarrow{\phi} \end{array} & \mathrm{LMod}_B(\mathrm{Perf}(S)) \\ i_s^* \downarrow & & \downarrow s^* \\ \mathrm{Perf}(X_K) & \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} & \mathrm{LMod}_B(\mathrm{Perf}(K)) \simeq \mathrm{Perf}(B \otimes_k K) \end{array} \quad (1.13)$$

We observe that this diagram commutes when traversing the left adjoints. Because derived affinoids are Jacobson, the family of pullback functors $\{i_s^*\}_{s \in S}$ is jointly conserva-

tive. Furthermore, we know that the bottom adjunction is an equivalence. Therefore, to prove that the unit η and counit ϵ of the adjunction between ϕ and ψ are equivalences, it suffices to show that the map i_s^* maps the unit of the top adjunction to the unit of the bottom adjunction, and similarly for the counit via s^* . This reduces to verifying that the diagram is right adjointable [Lur17, Definition 4.7.4.13].

By Lemma 1.2.50, the object $i_s^* p_1^*(\mathcal{E})$ is a generator of $\text{Perf}(X_K)$. Since the forgetful functor from $\text{LMod}_B(\text{Perf}(K))$ to $\text{Perf}(K)$ is conservative, the right adjointability of the diagram in (1.13) is equivalent to the right adjointability of the following expanded diagram:

$$\begin{array}{ccccc}
 \text{Perf}(X \times S) & \xleftarrow[-\otimes p_1^* \mathcal{E}]{\perp} & \text{Perf}(X \times S) & \xleftarrow[p_2^*]{\perp} & \text{Perf}(S) \\
 \downarrow i_s^* & \xrightarrow[\mathcal{H}om_{X \times S}(p_1^* \mathcal{E}, -)]{} & \downarrow i_s^* & \xrightarrow[p_{2,*}]{\perp} & \downarrow s^* \\
 \text{Perf}(X_K) & \xleftarrow[-\otimes i_s^* p_1^* \mathcal{E}]{\perp} & \text{Perf}(X_K) & \xleftarrow[p_{s,*}]{\perp} & \text{Perf}(K)
 \end{array} \quad (1.14)$$

We prove this by showing that both the left and right squares are individually right adjointable. The right adjointability of the square on the right follows from the base change property applied to the diagram in (1.12), which was proven in [PY21, Lemma 6.4]. The right adjointability of the square on the left follows utilizing the fact that perfect complexes are dualizable and the functor i_s^* is symmetric monoidal.

To prove that this equivalence restricts to simple objects, for every $\mathcal{F} \in \text{Perf}(X \times S)$ we consider the following diagram:

$$\begin{array}{ccc}
 & \text{Perf}(S) & \\
 \mathcal{H}om_{X \times S}(\mathcal{F}, -) \swarrow & & \searrow \underline{\text{Hom}}_B(\psi(\mathcal{F}), -) \\
 & \text{Perf}(X \times S) & \text{LMod}_B(\text{Perf}(S)) \\
 \mathcal{F} \otimes - \swarrow & \xleftarrow[\psi := p_{1*} \mathcal{H}om_{X \times S}(p_1^* \mathcal{E}, -)]{\phi := p_1^* \mathcal{E} \otimes p_2^*(-)} & \searrow \psi(\mathcal{F}) \otimes - \\
 & \text{Perf}(X \times S) & \text{LMod}_B(\text{Perf}(S))
 \end{array} \quad (1.15)$$

We adopt the following notation:

$$L := \psi(\mathcal{F}) \otimes - ; L' := \mathcal{F} \otimes - ; R := \underline{\text{Hom}}_B(\psi(\mathcal{F}), -) ; R' := \mathcal{H}om_{X \times S}(\mathcal{F}, -).$$

The diagram commutes on the left adjoints. Indeed, we have:

$$\begin{aligned}
\phi \circ L &:= \phi(\psi(\mathcal{F}) \otimes -) \\
&\simeq p_1^* \mathcal{E} \otimes p_2^*(\psi(\mathcal{F}) \otimes -) \\
&\simeq p_1^*(\mathcal{E}) \otimes p_2^*(\psi(\mathcal{F}) \otimes p_2^*(-)) \\
&\simeq \phi(\psi(\mathcal{F})) \otimes p_2^*(-) \\
&\simeq \mathcal{F} \otimes p_2^*(-) := L'.
\end{aligned}$$

Since the adjunction between ϕ and ψ is an equivalence, we deduce that

$$L \simeq \psi \circ L'.$$

Passing to the right adjoints implies that

$$R \simeq R' \circ \phi.$$

Evaluating at the respective identity objects, we obtain:

$$\underline{\mathrm{Hom}}_B(\psi(\mathcal{F}), \psi(\mathcal{F})) \simeq \underline{\mathrm{Hom}}_{X \times S}(\mathcal{F}, \mathcal{F}).$$

Consequently, $\mathcal{F}$ is simple in the sense of Definition 1.2.23 if and only if $\psi(\mathcal{F})$ is simple in the sense of Definition 1.2.33. This proves that the equivalence ψ restricts to an equivalence between simple objects. $\square$

In the following, we denote by $t_0 \underline{\mathrm{Perf}}_X^{\mathrm{sim}}$ the restriction of $\underline{\mathrm{Perf}}_X^{\mathrm{sim}}$ to underived affinoids. Note that by Corollary 1.2.26, it is naturally a rigid analytic 1-stack.

Corollary 1.2.53. *Let $X \rightarrow \mathrm{Sp}(k)$ be a proper, quasi-compact, and separated rigid analytic variety such that $\mathrm{Perf}(X)$ is a smooth and proper ∞ -category over $\mathrm{Perf}(k)$. Then the analytic 1-stacks $t_0 \underline{\mathrm{Perf}}_X^{\mathrm{sim}}$ and $(t_0 \mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}})^{\mathrm{an}}$ are equivalent.*

Proof. Combining Proposition 1.2.45 and Proposition 1.2.47, we obtain an equivalence between $\underline{\mathrm{Perf}}_X^{\mathrm{sim}}$ and $(\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}})^{\mathrm{an}}$. Moreover, since the analytification functor preserves geometric stacks (see [PY16, Lemma 2.24]), we deduce that $(t_0 \mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}})^{\mathrm{an}}$ is a rigid analytic geometric 1-stack. Using Theorem 1.2.39 and Theorem 1.2.16, the equivalence in Proposition 1.2.45 restricts to an equivalence between the rigid analytic 1-stacks $(t_0 \mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}})^{\mathrm{an}}$ and $t_0 \mathcal{A}n \mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}}$. We conclude by applying Proposition 1.2.47, observing that all the stacks considered are naturally analytic 1-stacks, as explained above. $\square$

1.3 Main theorem

The aim of this section is to prove the following main result:

Theorem 1.3.1. *Let $X \rightarrow \mathrm{Sp}(k)$ be a smooth and proper, quasi-compact, and separated rigid analytic variety. Then $\mathrm{Perf}(X)$ is smooth and proper over $\mathrm{Perf}(k)$ if and only if X is algebraizable (i.e., there exists an algebraic space Y and an equivalence $Y^{\mathrm{an}} \simeq X$).*

In the final subsection, we shall also deduce an analogous statement for formal schemes.

One direction of the proof is straightforward, as an algebraic space Y is smooth and proper if and only if $\mathrm{Perf}(Y)$ is smooth and proper ([Lur18, Theorem 11.4.0.3]). Furthermore, by GAGA theorems (see [PY16]), we obtain the same result for $\mathrm{Perf}(Y^{\mathrm{an}})$. In the formal setting, an analogous argument applies using the Grothendieck Existence Theorem ([Lur18, Theorem 8.5.0.3]).

The converse direction is more involved. We provide here an overview of the proof strategy, with the technical details deferred to the subsequent subsections. We begin by observing that, by Lemma 1.2.49, there exists an equivalence between $\mathrm{Perf}(X)$ and $\mathrm{Perf}(B)$, where B is the k -algebra of endomorphisms of a compact generator of $\mathrm{Perf}(X)$. Moreover, by Corollary 1.1.16, B is a smooth and proper k -algebra.

We consider the truncation of the derived algebraic stack $\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}}$, denoted by $t_0\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}}$. According to [TV07], this is an Artin 1-stack that admits a good moduli space M_X . The projection map

$$t_0\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}} \rightarrow M_X \tag{1.16}$$

is a $\mathbb{G}_m$ -gerbe. In particular, M_X is an algebraic space locally of finite type, and its analytification M_X^{an} is an underived rigid analytic space.

The proof then proceeds via the following main steps:

1. We prove that X can be embedded as an open subspace into M_X^{an} .
2. We establish an algebraization result for such an embedding.

The proofs of these two points will be detailed in the next subsections, completing the proof of the main theorem. Before proceeding, we verify that the properness of $\mathrm{Perf}(X)$ holds whenever X is proper; however, as Theorem 1.3.1 suggests, this correspondence does not extend to smoothness in the same way.

Corollary 1.3.2. *Let X be a proper, quasi-compact rigid analytic variety. Then $\mathrm{Perf}(X)$ is a proper k -linear ∞ -category.*

Proof. We must show that $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ is a perfect k -module for any pair of perfect complexes $\mathcal{F}$ and $\mathcal{G}$. This follows directly from Lemma 1.2.48 and Lemma 1.2.21. $\square$

Remark 1.3.3. Note that by the previous corollary, to prove algebraizability it suffices to verify the smoothness of $\mathrm{Perf}(X)$. This property represents the key categorical distinction between the algebraic and analytic settings.

1.3.1 Embedding into the analytification of the algebraic space

Let $X \rightarrow \mathrm{Sp}(k)$ be a smooth and proper, quasi-compact, and separated rigid analytic variety, such that $\mathrm{Perf}(X)$ is smooth and proper over $\mathrm{Perf}(k)$. We denote by M_X the coarse moduli space of $t_0\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}}$, as established in (1.16).

In this subsection, we construct a map

$$\alpha: X \times \mathrm{BG}_m^{\mathrm{an}} \rightarrow t_0\mathrm{Perf}_X^{\mathrm{sim}} \simeq (t_0\mathcal{M}_{\mathrm{Perf}(X)})^{\mathrm{an}}. \quad (1.17)$$

Subsequently, we will prove the following proposition:

Proposition 1.3.4. *Under the assumptions of Theorem 1.3.1, the map constructed in (1.17) induces an open immersion of X into M_X^{an} .*

To construct the map in Equation (1.17), we begin with the following setup. The functor $\mathcal{X}$ introduced below can be viewed as the rigid analytic analogue of the algebraic stack of coherent sheaves of length 1, as studied in [AT07].

Construction 1.3.5. Consider the following (underived) functor:

$$\mathcal{X}: \mathrm{Afd}_k \rightarrow \mathcal{S}$$

which assigns to an affinoid space S the core of the full subcategory of $\mathrm{Perf}(X \times S)$ spanned by objects $\mathcal{F}$ having Tor-amplitude $[0, 0]$ relative to S (i.e., those which are flat over S , and whose only non-zero cohomology sheaf is coherent and concentrated in degree 0). Furthermore, for every closed point $s \in S$, we require the fiber $\mathcal{F}_s$ to be the skyscraper sheaf of length 1 supported on a single point of the fiber X_s .

Observe that since the objects in $\mathcal{X}(S)$ lie in the heart of the t -structure on $\mathrm{Mod}_{\mathcal{O}_{X \times S}}$, their mapping spaces are 0-truncated [Lur17, Remark 1.2.1.12]. In particular, $\mathcal{X}$ can be viewed as a functor taking values in 1-groupoids.

Lemma 1.3.6. *There is an equivalence of presheaves between $\mathcal{X}$ and $X \times \mathrm{BG}_m^{\mathrm{an}}$.*

Proof. Recall that the functor $\mathcal{X}$ takes values in 1-groupoids, and the same holds for the product stack $X \times \mathrm{BG}_m^{\mathrm{an}}$. It therefore suffices to fix an arbitrary (classical) affinoid space S and construct an equivalence of 1-groupoids evaluated at S .

We begin by constructing the functor

$$\Psi: (X \times \mathrm{BG}_m^{\mathrm{an}})(S) \rightarrow \mathcal{X}(S).$$

An object in the domain is a pair $(f, \mathcal{L})$, where $f: S \rightarrow X$ is a morphism of analytic spaces and $\mathcal{L}$ is a line bundle on S . Considering the graph morphism $\Gamma_f: S \rightarrow X \times S$, we define:

$$\Psi(f, \mathcal{L}) := \Gamma_{f*} \mathcal{L}.$$

Observe that, since X is separated, Γ_f is a closed immersion and $\mathcal{L}$ is locally free over S . We verify the relative Tor-amplitude condition via the projection formula [PY21, Theorem 5.7]. For any discrete $\mathcal{O}_S$ -module N , we compute the derived tensor product:

$$(\Gamma_f)_* \mathcal{L} \otimes_{\mathcal{O}_{X \times S}} p_S^* N \simeq \Gamma_{f*} \left(\mathcal{L} \otimes_{\mathcal{O}_S} \Gamma_f^* p_S^* N \right).$$

Because $p_S \circ \Gamma_f \simeq \mathrm{id}_S$, we have $\Gamma_f^* p_S^* N \simeq N$, and the expression simplifies to $\Gamma_{f*} (\mathcal{L} \otimes_{\mathcal{O}_S} N)$. Since $\mathcal{L}$ is a line bundle on S (in particular, it is locally free and concentrated in degree 0), the derived tensor product $\mathcal{L} \otimes_{\mathcal{O}_S}^{\mathbb{L}} N$ is also concentrated in degree 0. Finally, since Γ_f is a closed immersion, its pushforward is t-exact by [PY21, Lemma 6.2]. Thus, the resulting complex $\Gamma_{f*} \mathcal{L}$ has Tor-amplitude $[0, 0]$ relative to S .

Furthermore, we must verify the condition on the fibers. Let $s \in S$ be a closed point, and consider the inclusion of the fiber $j_s: X_s \rightarrow X \times S$. To compute $j_s^* \Gamma_{f*} \mathcal{L}$, we consider the Cartesian square:

$$\begin{array}{ccc} \mathrm{Sp}(k(s)) & \xrightarrow{i_s} & S \\ i_{\tilde{x}} \downarrow & & \downarrow \Gamma_f \\ X_s & \xrightarrow{j_s} & X \times S \end{array} \quad (1.18)$$

where i_s is the standard inclusion of the point s , and the vertical morphism $i_{\tilde{x}}$ is the inclusion of a $k(s)$ -rational point $\tilde{x} \in X_s$. We observe that $\tilde{x}$ is a lift of the point $f(s) \in X$ to X_s .

Since Γ_f is a proper morphism (being a closed immersion), the derived base change theorem [PY21, Theorem 6.8] yields a canonical isomorphism:

$$j_s^* \Gamma_{f*} \mathcal{L} \simeq i_{\tilde{x}*} i_s^* \mathcal{L}.$$

Since $\mathcal{L}$ is a line bundle, its pullback $i_s^* \mathcal{L}$ to the point s is a 1-dimensional vector space over $k(s)$ concentrated in degree 0. Moreover, since $i_{\tilde{x}}$ is a closed immersion, $i_{\tilde{x}*} i_s^* \mathcal{L}$ is a discrete skyscraper sheaf on X_s supported at the point $\tilde{x}$ of length 1, perfectly matching

the functor's requirements.

Regarding morphisms, since $\mathcal{X}(S)$ is a 1-groupoid, a morphism

$$\alpha: (f, \mathcal{L}) \rightarrow (f', \mathcal{L}')$$

is given by a map $f \rightarrow f'$ and an isomorphism of line bundles $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$. Because the mapping space $\mathrm{Map}(S, X)$ is 0-truncated, any such map implies $f \simeq f'$. Thus, it suffices to consider the case where $f = f'$, allowing us to define

$$\Psi(\alpha) := \Gamma_{f*}(\alpha).$$

We now prove that Ψ is fully faithful and essentially surjective. We begin by demonstrating fully faithfulness. Let $(f_1, \mathcal{L}_1)$ and $(f_2, \mathcal{L}_2)$ be two objects in $(X \times \mathrm{BG}_m^{\mathrm{an}})(S)$. We must show that Ψ induces a bijection on the associated mapping spaces. As explained above, the mapping space in the domain is empty if $f_1 \neq f_2$, and it naturally identifies with $\mathrm{Isom}_S(\mathcal{L}_1, \mathcal{L}_2)$ if $f_1 = f_2 = f$.

Suppose first that $f_1 \neq f_2$. The sheaves $\Gamma_{f_1*} \mathcal{L}_1$ and $\Gamma_{f_2*} \mathcal{L}_2$ are topologically supported on the closed subsets $\Gamma_{f_1}(S)$ and $\Gamma_{f_2}(S)$ in $X \times S$. Since X is separated, the graph of any morphism is a closed immersion, and for distinct morphisms $f_1 \neq f_2$, their graphs are distinct subspaces. This implies

$$\mathrm{Isom}_{\mathcal{X}(S)}(\Gamma_{f_1*} \mathcal{L}_1, \Gamma_{f_2*} \mathcal{L}_2) = \emptyset,$$

which exactly matches the empty mapping space of the domain.

Suppose now that $f_1 = f_2 = f$. We must verify that the pushforward functor along the graph induces a bijection:

$$\Gamma_{f*}: \mathrm{Isom}_S(\mathcal{L}_1, \mathcal{L}_2) \xrightarrow{\sim} \mathrm{Isom}_{\mathcal{X}(S)}(\Gamma_{f*} \mathcal{L}_1, \Gamma_{f*} \mathcal{L}_2).$$

Since $p_S \circ \Gamma_f = \mathrm{id}_S$, injectivity is immediate: if $\Gamma_{f*}(\gamma_1) = \Gamma_{f*}(\gamma_2)$, applying p_{S*} to both sides yields $p_{S*} \Gamma_{f*}(\gamma_1) = p_{S*} \Gamma_{f*}(\gamma_2)$, which simplifies directly to $\gamma_1 = \gamma_2$.

To demonstrate surjectivity, let $\beta: \Gamma_{f*} \mathcal{L}_1 \xrightarrow{\sim} \Gamma_{f*} \mathcal{L}_2$ be an arbitrary isomorphism in $\mathcal{X}(S)$. We define a candidate isomorphism in the domain as $\gamma := p_{S*}(\beta): \mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$. We must show that $\Gamma_{f*}(\gamma) \simeq \beta$. Observe that both β and $\Gamma_{f*}(\gamma)$ are supported on the graph of f . Furthermore, the restriction of $\Gamma_f \circ p_S$ to the graph of f is equivalent to the identity functor. This implies that

$$\Gamma_{f*} \circ p_{S*}(\beta) \simeq \beta,$$

concluding the proof that Ψ is fully faithful.

We now turn to the essential surjectivity of Ψ . Let $\mathcal{F}$ be an object in $\mathcal{X}(S)$; we will construct a pair $(f, \mathcal{L})$ in $(X \times \mathrm{BG}_m^{\mathrm{an}})(S)$ such that $\Psi(f, \mathcal{L}) \simeq \mathcal{F}$.

By definition, the sheaf $\mathcal{F}$ in $\mathcal{X}(S)$ corresponds to an object in $\mathrm{Perf}(X \times S)$ such that for every closed point s in S , the fiber $\mathcal{F}_s$ is supported at a single point in X_s . Consequently, the support of $\mathcal{F}$ can be described as the graph of the function

$$\mathrm{supp}(\mathcal{F})(-): S \rightarrow X$$

sending a point s to the projection onto X of the single point in the fiber X_s that supports $\mathcal{F}_s$. Moreover, since $\mathcal{F}$ has Tor-amplitude $[0, 0]$ relative to S and X is proper, we deduce that

$$\mathcal{L} := p_{S*} \mathcal{F}$$

is a vector bundle on S . The specific requirement on the fiber $\mathcal{F}_s$ ensures that $\mathcal{L}$ is, in fact, a line bundle on S .

We need to show that

$$\Psi(\mathrm{supp}(\mathcal{F})(-), \mathcal{L}) \simeq \mathcal{F}.$$

We observe that the support of $\mathcal{F}$ coincides with the graph of the function $\mathrm{supp}(\mathcal{F})(-)$. In particular, the composition $\Gamma_{\mathrm{supp}(\mathcal{F})(-)} \circ p_S$, when restricted to this graph, is equivalent to the identity functor. Therefore, we obtain

$$\Psi(\mathrm{supp}(\mathcal{F})(-), \mathcal{L}) := \Gamma_{\mathrm{supp}(\mathcal{F})(-)*} (p_{S*}(\mathcal{F})) \simeq \mathcal{F}.$$

This completes the proof. □

As a consequence of Lemma 1.3.6, $\mathcal{X}$ is a stack. Furthermore, we observe that the objects of $\mathcal{X}(S)$ are simple sheaves over S in the sense of Definition 1.2.23. In particular, this induces a canonical map:

$$\tilde{\alpha}: \mathcal{X} \rightarrow t_0 \underline{\mathrm{Perf}}_X^{\mathrm{sim}}. \quad (1.19)$$

Lemma 1.3.7. *The map constructed in (1.19) is an open immersion.*

Proof. We will prove that the canonical map (1.19) is a monomorphism and formally étale. Note that, by Corollary 1.2.53, these stacks are locally of finite presentation, so this will imply it is an open immersion.

By definition, for every affinoid space S , the ∞ -category $\mathcal{X}(S)$ is a full subcategory of $\mathrm{Perf}(X \times S)^{\mathrm{sim}}$. This immediately implies that the diagonal map

$$\mathcal{X} \rightarrow \mathcal{X} \times_{t_0 \underline{\mathrm{Perf}}_X^{\mathrm{sim}}} \mathcal{X}$$

is an equivalence, hence $\tilde{\alpha}$ is a monomorphism.

We now prove that $\tilde{\alpha}$ is formally étale. Let $\mathrm{Sp}(A)$ be an affinoid space, I a square-zero ideal in A , and consider the following lifting problem:

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\quad} & \mathrm{Sp}(A/I) \\ \downarrow \tilde{\alpha} & \nwarrow \text{dotted} & \downarrow \\ t_0\mathrm{Perf}_X^{\mathrm{sim}} & \xleftarrow{\quad} & \mathrm{Sp}(A). \end{array}$$

We must prove the existence of the dotted arrow. The above diagram is specified by a perfect complex $\mathcal{E}$ on $X \times \mathrm{Sp}(A)$ such that, under the canonical inclusion

$$j: X \times \mathrm{Sp}(A/I) \rightarrow X \times \mathrm{Sp}(A),$$

the pullback $j^*\mathcal{E}$ belongs to the subcategory $\mathcal{X}(\mathrm{Sp}(A/I))$. We need to show that $\mathcal{E}$ itself lies in $\mathcal{X}(\mathrm{Sp}(A))$.

To prove that $\mathcal{E}$ has Tor-amplitude $[0, 0]$ relative to $\mathrm{Sp}(A)$, we can work locally and assume $X = \mathrm{Sp}(R)$ for some affinoid algebra R . Using Theorem 1.2.16 and [Lur18, Corollary 2.7.3.2], we deduce that $\mathcal{E}$ indeed has Tor-amplitude $[0, 0]$ relative to $\mathrm{Sp}(A)$. Finally, the condition that the restriction of $\mathcal{E}$ to the fiber over every closed point in $\mathrm{Sp}(A)$ is a skyscraper sheaf is automatically satisfied: since I is a nilpotent ideal, the affinoid spaces $\mathrm{Sp}(A)$ and $\mathrm{Sp}(A/I)$ share the exact same underlying topological space and closed points. $\square$

According to [TV07, Corollary 3.22], there exists an algebraic space locally of finite type, M_X , such that the Artin stack $t_0\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}}$ admits it as a good moduli space:

$$p: t_0\mathcal{M}_{\mathrm{Perf}(X)}^{\mathrm{sim}} \rightarrow M_X. \quad (1.20)$$

Moreover, the map p is a $\mathbb{G}_m$ -gerbe. By analyzing the analytification of this map, we can conclude the proof of Proposition 1.3.4.

Proof of Proposition 1.3.4. Taking the analytification of the map in (1.20) and invoking the equivalence established in Corollary 1.2.53, we obtain the following $\mathbb{G}_m^{\mathrm{an}}$ -gerbe projection:

$$p^{\mathrm{an}}: t_0\mathrm{Perf}_X^{\mathrm{sim}} \rightarrow M_X^{\mathrm{an}}.$$

Using Lemma 1.3.6 and Lemma 1.3.7, we can identify $X \times \mathrm{BG}_m^{\mathrm{an}}$ as an open substack of $t_0\mathrm{Perf}_X^{\mathrm{sim}}$. Since p^{an} is a $\mathbb{G}_m^{\mathrm{an}}$ -gerbe, taking the quotient of this open substack by the

gerbe structure exactly yields X . Because the projection map of a gerbe is smooth (and thus open), the image of this substack under p^{an} is canonically isomorphic to X , and it embeds as an open analytic subspace of M_X^{an} . $\square$

1.3.2 Non-Archimedean Moishezon spaces

The notion of non-Archimedean Moishezon spaces was developed and studied in [Cono7]. In this section, we recall the definition of non-Archimedean Moishezon spaces and state the main theorem characterizing them, from which we will deduce the algebraization result needed to complete the proof in the non-Archimedean case.

Definition 1.3.8 ([Cono7]). A rigid analytic variety X is *Moishezon* if it is proper and each irreducible component X_i of X , endowed with its reduced structure, satisfies $\text{trdeg}_k(K(X_i)) = \dim X_i$, where $K(X_i)$ is the field of meromorphic functions and trdeg_k denotes its transcendence degree over k .

Theorem 1.3.9 ([Cono7]). *The full subcategory of Moishezon spaces inside the category of rigid analytic varieties coincides with the essential image of the analytification functor from the category of proper algebraic spaces over k .*

The proof of the following corollary is inspired by [TVo8a, Proposition 5.7].

Corollary 1.3.10. *Let M be a quasi-separated algebraic space, and let X be a rigid analytic variety satisfying the assumptions of Theorem 1.3.1. If $j: X \rightarrow M^{\text{an}}$ is an open immersion, then X is algebraizable.*

Proof. We will prove that X is a Moishezon space.

Since X is quasi-compact, its image $j(X) \subset M^{\text{an}}$ is also quasi-compact. Because M is an algebraic space, it admits an étale covering by affine schemes. We can select a finite number of these affine schemes whose images cover $j(X)$. The union of these images forms a quasi-compact, open algebraic subspace $M' \subset M$ of finite type over k . The map j then factors through an open immersion $X \rightarrow (M')^{\text{an}}$. Consequently, we may assume without loss of generality that M is quasi-compact and locally of finite type over k .

To check the Moishezon condition, we evaluate X component by component; thus, we may assume that X is irreducible. The open immersion j then factors through an open immersion $i: X \rightarrow Z$, where Z is an irreducible component of M^{an} . Furthermore, there exists an irreducible component $M_0 \subset M$ such that $Z = M_0^{\text{an}}$. Therefore, we may also assume that M is irreducible.

Since X and M^{an} are irreducible, their rings of meromorphic functions $K(M^{\text{an}})$ and $K(X)$ are fields. The open immersion j induces an injective field extension:

$$K(M^{\text{an}}) \hookrightarrow K(X).$$

Let n be the dimension of X . Because j is an open immersion, we have

$$\dim M^{\text{an}} = \dim M = n.$$

Since M is an irreducible algebraic space of finite type, it contains a dense open affine subscheme U of dimension n . We therefore have the field inclusions:

$$K(U) \simeq K(M) \subset K(M^{\text{an}}) \subset K(X).$$

Since U is an affine variety of dimension n over k , its function field satisfies $\text{trdeg}_k(K(U)) = n$. This implies that

$$\text{trdeg}_k(K(X)) \geq n.$$

On the other hand, by [Con07, Theorem 2.2.1], for any proper rigid analytic variety, we naturally have $\text{trdeg}_k(K(X)) \leq \dim X = n$. We conclude that $\text{trdeg}_k(K(X)) = n$, which means X is a Moishezon space. The algebraizability of X now follows directly from Theorem 1.3.9. $\square$

For completeness, we now assemble the complete proof of the main theorem.

Proof of Theorem 1.3.1. One direction is straightforward: an algebraic space Y is smooth and proper if and only if $\text{Perf}(Y)$ is smooth and proper ([Lur18, Theorem 11.4.0.3]). Additionally, by GAGA ([PY16, Corollary 7.5], see also [HP25, Example 5.3]), we obtain the corresponding result for $\text{Perf}(Y^{\text{an}})$. The converse direction follows by combining the open immersion from Proposition 1.3.4 with the algebraization result of Corollary 1.3.10. $\square$

1.3.3 Formal setting

In this section, we consider formal schemes over $\text{Spf}(\mathcal{O}_k)$, where $\mathcal{O}_k$ is a discrete valuation ring (DVR) with maximal ideal $\mathfrak{m}$ and fraction field k . We assume that $\mathcal{O}_k$ is complete with respect to the $\mathfrak{m}$ -adic topology. All formal schemes are assumed to be separated, quasi-compact, topologically almost of finite presentation, and locally Noetherian. Under these hypotheses, we rely on [Lur18] and [Ant19] as the main references for the theory of formal schemes in the derived setting.

Given a formal scheme $\mathfrak{X}$ as above, we can consider its ∞ -category of perfect complexes $\mathrm{Perf}(\mathfrak{X})$. This is an idempotent-complete ∞ -category, and by the Grothendieck Existence Theorem (see [Lur18, Theorem 8.5.0.3]), it is also $\mathbb{O}_k$ -linear (i.e., it belongs to $\mathrm{Cat}_{\mathbb{O}_k}^{\mathrm{Perf}}$).

Definition 1.3.11. Let X be a (derived) stack over $\mathrm{Spec}(\mathbb{O}_k)$. Its *formalization* (also called formal completion) is defined as the following pullback:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\quad} & X \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spf}(\mathbb{O}_k) & \longrightarrow & \mathrm{Spec}(\mathbb{O}_k) \end{array}$$

In this section, we will prove an analogue of Theorem 1.3.1 in the setting of formal schemes.

Definition 1.3.12. Given a formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathbb{O}_k)$, we say that $\mathfrak{X}$ is *algebraizable* if there exists an algebraic space X over $\mathrm{Spec}(\mathbb{O}_k)$ and an equivalence $\mathfrak{X} \simeq \widehat{X}$. In this case, we say that $\widehat{X}$ is the formal completion of X .

The aim of this section is to prove the following theorem.

Theorem 1.3.13. *Let $\mathfrak{X}$ be a smooth and proper formal scheme over $\mathrm{Spf}(\mathbb{O}_k)$. Then $\mathrm{Perf}(\mathfrak{X})$ is smooth and proper as an $\mathbb{O}_k$ -linear stable ∞ -category if and only if $\mathfrak{X}$ is algebraizable.*

In order to prove this theorem, we require some preliminary results. At the end of the section, we will also describe related results that do not rely on the smoothness hypothesis for $\mathfrak{X}$.

We begin by proving the following lemma, which extends [LX15, Proposition 2.5] to the case of algebraic spaces (which we always assume to be separated and of finite type).

Lemma 1.3.14. *Let X_k be an algebraic space over k . Then X_k admits a model over $\mathbb{O}_k$. Furthermore, if X_k is proper, we can choose the model to be proper.*

Proof. We observe that, since X_k is of finite type, we can extend it to a model X over an open subset $V \subseteq \mathrm{Spec}(\mathbb{O}_k)$, such that X is again separated and of finite type. Consider an atlas $U_k \rightarrow X_k$. By [LX15, Proposition 2.5], we can extend the scheme U_k to a model U that is faithfully flat over $\mathrm{Spec}(\mathbb{O}_k)$. By suitably shrinking V , we may assume that the

restriction $U|_V \rightarrow X$ is a surjective étale map. Consider the following pushout diagram:

$$\begin{array}{ccc} U|_V & \twoheadrightarrow & X \\ \downarrow & \lrcorner & \downarrow j \\ U & \xrightarrow{\alpha} & Y \end{array}$$

By [CLO12, Lemma 3.1.6], we deduce that j is injective, α is an atlas, and Y is an algebraic space of finite type, faithfully flat over $\mathrm{Spec}(\mathcal{O}_k)$. By construction, Y is a model of X_k , though it may not be proper. To resolve this, we invoke [CLO12, Theorem 1.2.1] to find a proper algebraic space $\tilde{Y}$ equipped with an open immersion $Y \rightarrow \tilde{Y}$. The closure of X inside $\tilde{Y}$ then provides the required proper model, by [Sta24, Tag oCMK]. $\square$

We denote by $(-)^{\mathrm{rig}}$ the generic fiber functor for derived formal schemes and $\mathcal{O}_{\mathfrak{X}}$ -modules, as constructed in [Ant20]. In *loc. cit.*, the authors establish the existence of a functor $(-)^{\mathrm{rig}}$ from the ∞ -category of derived formal schemes to the ∞ -category of k -analytic spaces, which, at the underived level, recovers the classical generic fiber functor of [Bos14]. This functor naturally extends to categories of sheaves, leading to the following theorem:

Theorem 1.3.15. *Let $\mathfrak{X}$ be a formal scheme. The generic fiber functor induces a functor between the ∞ -categories of almost perfect complexes (still denoted by $(-)^{\mathrm{rig}}$)*

$$(-)^{\mathrm{rig}}: \mathrm{Coh}^+(\mathfrak{X}) \rightarrow \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$$

which preserves perfect complexes and is t -exact. Furthermore, for every $\mathcal{F} \in \mathrm{Coh}^b(\mathfrak{X}^{\mathrm{rig}})$, there exists a formal model $\mathcal{G} \in \mathrm{Coh}^b(\mathfrak{X})$ (i.e., $(\mathcal{G})^{\mathrm{rig}} \simeq \mathcal{F}$), and the category of formal models for $\mathcal{F}$ is filtered.

Proof. This is essentially the content of [Ant20, Appendix A.2.1]. The proof demonstrates that if $\mathcal{F}$ is bounded, we can choose $\mathcal{G}$ to be bounded as well. The only remaining assertion is that the rigidification functor preserves perfect complexes. Working locally, we may assume $\mathfrak{X} = \mathrm{Spf}(A)$ for an $\mathcal{O}_k$ -algebra A . By [PY21, Theorem 3.1], we can identify $\mathrm{Coh}^+((\mathrm{Spf}(A))^{\mathrm{rig}})$ with $\mathrm{Coh}^+(A \otimes_{\mathcal{O}_k} k)$. Additionally, using [Lur18, Theorem 8.5.0.3], we can identify $\mathrm{Coh}^+(\mathrm{Spf}(A))$ with $\mathrm{Coh}^+(A)$. In this affine setting, the rigidification functor for sheaves coincides with the ordinary tensor product of modules:

$$- \otimes_{\mathcal{O}_k} k: \mathrm{Mod}_A \rightarrow \mathrm{Mod}_{A \otimes_{\mathcal{O}_k} k},$$

which inherently preserves perfect complexes. $\square$

Lemma 1.3.16. *Let $\mathcal{X}$ be a formal scheme that is smooth over $\mathrm{Spf}(\mathbb{O}_k)$. Then $\mathrm{Perf}(\mathcal{X})$ is equivalent to $\mathrm{Coh}^b(\mathcal{X})$.*

Proof. Note that both ∞ -categories satisfy étale descent, so we may locally assume $\mathcal{X} = \mathrm{Spf}(A)$. Furthermore, since $\mathcal{X}$ is smooth, we may assume étale-locally that $A = \mathbb{O}_k\langle T_1, \dots, T_n \rangle$, implying that $\mathrm{Spec}(A)$ is regular over $\mathrm{Spec}(\mathbb{O}_k)$ (see [GR02, Proposition 7.3.23]). In the locally Noetherian setting, the equivalence established in [Lur18, Proposition 8.4.1.2] restricts to an equivalence of perfect objects and preserves t-structures (see [Lur18, Remark 8.4.2.6]). In particular, this yields an equivalence between $\mathrm{Coh}^b(\mathrm{Spf}(A))$ and $\mathrm{Coh}^b(\mathrm{Spec}(A))$. Since $\mathrm{Coh}^b(\mathrm{Spec}(A))$ is equivalent to $\mathrm{Perf}(\mathrm{Spec}(A))$, it follows that $\mathrm{Perf}(\mathrm{Spf}(A))$ is equivalent to $\mathrm{Coh}^b(\mathrm{Spf}(A))$. $\square$

Corollary 1.3.17. *If $\mathcal{X}$ is smooth over $\mathrm{Spf}(\mathbb{O}_k)$, then $\mathrm{Coh}^b(\mathcal{X}^{\mathrm{rig}})$ is equivalent to $\mathrm{Perf}(\mathcal{X}^{\mathrm{rig}})$.*

Proof. If $\mathcal{F}$ is a bounded coherent complex over $\mathcal{X}^{\mathrm{rig}}$, then by Theorem 1.3.15 it admits a bounded formal model. This formal model is a perfect complex by Lemma 1.3.16. Consequently, its rigidification $\mathcal{F}$ is also a perfect complex, again by Theorem 1.3.15. $\square$

Definition 1.3.18 ([Lur18, Definition 7.1.1.6]). Let $\mathcal{C}$ be an $\mathbb{O}_k$ -linear ∞ -category. We say that an object $C \in \mathcal{C}$ is *nilpotent relative to $\mathfrak{m}$* if, for every element $x \in \mathfrak{m}$, the relative tensor product

$$\mathbb{O}_k[x^{-1}] \otimes_{\mathbb{O}_k} C$$

vanishes. We denote by $\mathcal{C}_{\mathrm{nil}}$ the full subcategory spanned by nilpotent objects.

Informally, we are requiring that $\mathcal{C}$ is supported set-theoretically at the closed point associated with the maximal ideal $\mathfrak{m}$. When $\mathcal{C}$ is the ∞ -category $\mathrm{Coh}^b(\mathcal{X})$ for a formal scheme $\mathcal{X}$, our definition coincides with [AP20, Definition 3.1] (see [AP20, Lemma 3.2] and [Lur18, Example 7.1.1.2]).

In the following lemma, we utilize one of the main theorems of [TT07]. It states that there is a fiber-cofiber sequence in $\mathrm{Cat}^{\mathrm{Perf}}$ of the form:

$$\mathrm{Perf}_{\mathfrak{m}}(\mathbb{O}_k) \rightarrow \mathrm{Perf}(\mathbb{O}_k) \rightarrow \mathrm{Perf}(k). \quad (1.21)$$

Here, $\mathrm{Perf}_{\mathfrak{m}}(\mathbb{O}_k)$ denotes the ∞ -category of perfect complexes supported on the closed point associated with the maximal ideal $\mathfrak{m}$ (i.e., complexes that vanish on the open complement of the point).

Lemma 1.3.19. *Let $\mathcal{A}$ be an ∞ -category in $\mathrm{Cat}_{\mathbb{O}_k}^{\mathrm{Perf}}$. Then the map $\mathrm{Perf}_{\mathfrak{m}}(\mathbb{O}_k) \otimes_{\mathbb{O}_k} \mathcal{A} \rightarrow \mathcal{A}$, obtained by tensoring the first map of (1.21) with $\mathcal{A}$, is fully faithful, and its essential image*

coincides with $\mathcal{A}_{\text{nil}}$. Furthermore, there is a cofiber sequence in Cat^{Perf} of the form:

$$\text{Perf}_{\mathfrak{m}}(\mathcal{O}_k) \otimes_{\mathcal{O}_k} \mathcal{A} \rightarrow \mathcal{A} \rightarrow \text{Perf}(k) \otimes_{\mathcal{O}_k} \mathcal{A}.$$

Proof. By [Lur18, Proposition 7.1.5.3], we can identify $\text{Perf}_{\mathfrak{m}}(\mathcal{O}_k)$ with $\text{Perf}(\mathcal{O}_k)_{\text{nil}}$. The first statement for the ind-completion of our ∞ -categories follows from [Lur18, Corollary 7.1.2.11]. By [BP23, Lemma 3.10], we know that this map preserves compact objects, thus allowing us to extend the result to idempotent-complete ∞ -categories. To obtain the cofiber sequence, we simply tensor (1.21) by $-\otimes_{\mathcal{O}_k} \mathcal{A}$. $\square$

Proposition 1.3.20. *Let $\mathcal{X}$ be a smooth and proper formal scheme over $\text{Spf}(\mathcal{O}_k)$. Then there is an equivalence of k -linear ∞ -categories between $\text{Perf}(\mathcal{X}) \otimes_{\mathcal{O}_k} \text{Perf}(k)$ and $\text{Perf}(\mathcal{X}^{\text{rig}})$.*

Proof. By the preceding lemma, we have a cofiber sequence:

$$\text{Perf}_{\mathfrak{m}}(\mathcal{O}_k) \otimes_{\mathcal{O}_k} \text{Perf}(\mathcal{X}) \rightarrow \text{Perf}(\mathcal{X}) \rightarrow \text{Perf}(k) \otimes_{\mathcal{O}_k} \text{Perf}(\mathcal{X}).$$

We can identify $\text{Perf}_{\mathfrak{m}}(\mathcal{O}_k) \otimes_{\mathcal{O}_k} \text{Perf}(\mathcal{X})$ with $\text{Perf}(\mathcal{X})_{\text{nil}}$. [AP20, Definition 3.5] introduces the exact sequence:

$$\text{Coh}^b(\mathcal{X})_{\text{nil}} \rightarrow \text{Coh}^b(\mathcal{X}) \rightarrow \text{Coh}^b(\mathcal{X})_{\text{loc}}.$$

[AP20, Corollary 3.8] proves that the right-hand side is equivalent to $\text{Coh}^b(\mathcal{X}^{\text{rig}})$ (note that all functors here are t-exact). By invoking Lemma 1.3.16 and Corollary 1.3.17, we obtain the equivalence stated in the proposition. $\square$

Proposition 1.3.21. *Let $\mathcal{X}$ be a proper formal scheme such that $\mathcal{X}^{\text{rig}}$ is smooth and algebraizable. Then $\mathcal{X}$ is algebraizable.*

Proof. By Theorem 1.3.1, there exists a smooth and proper algebraic space Y over $\text{Spec}(k)$ such that $Y^{\text{an}} \simeq \mathcal{X}^{\text{rig}}$. Using Lemma 1.3.14, we can find a proper algebraic space W over $\mathcal{O}_k$ such that $\widehat{W}$ is a formal model for $\mathcal{X}^{\text{rig}}$. We now possess two formal models for $\mathcal{X}^{\text{rig}}$. By [BL93, Theorem 4.1], there exists another formal scheme $\tilde{Y}$ fitting into a zig-zag of formal schemes:

$$\begin{array}{ccc} & \tilde{Y} & \\ \swarrow & & \searrow \\ \mathcal{X} & & \widehat{W} \end{array} \tag{1.22}$$

where we may assume both maps are formal blow-ups. Since the map $\tilde{Y} \rightarrow \widehat{W}$ is a formal blow-up and W is algebraic, we deduce by [Art70, Theorem 3.1] that $\tilde{Y}$ is also algebraic.

Finally, since $\tilde{Y}$ is algebraic, applying [Art70, Theorem 3.2] allows us to conclude that $\mathcal{X}$ must be algebraic as well. $\square$

Proof of Theorem 1.3.13. Assuming that $\mathrm{Perf}(\mathcal{X})$ is smooth and proper, Proposition 1.3.20 implies that $\mathrm{Perf}(\mathcal{X}^{\mathrm{rig}})$ is smooth and proper over k , as it is the base change of a smooth and proper ∞ -category. Using Theorem 1.3.1, we see that $\mathcal{X}^{\mathrm{rig}}$ is algebraizable. Proposition 1.3.21 then guarantees that $\mathcal{X}$ is algebraizable. The converse direction follows immediately from [Lur18, Theorem 8.5.0.3]. $\square$

Remark 1.3.22. Note that to prove that $\mathcal{X}$ is algebraizable, it is sufficient to assume that $\mathcal{X}$ is proper, $\mathcal{X}^{\mathrm{rig}}$ is smooth, and $\mathrm{Perf}(\mathcal{X})$ is smooth and proper. Under these relaxed assumptions, the proof proceeds in exactly the same manner.

Using these same ideas, we can also prove the following result:

Theorem 1.3.23. *Let $\mathcal{X}$ be a proper formal scheme such that $\mathrm{Coh}^b(\mathcal{X})$ is smooth as a ∞ -category in $\mathrm{Cat}_{\mathbb{O}_k}^{\mathrm{Perf}}$, and $\mathcal{X}^{\mathrm{rig}}$ is smooth. Then $\mathcal{X}$ is algebraizable.*

Proof. Applying Lemma 1.3.19 with $\mathcal{A} = \mathrm{Coh}^b(\mathcal{X})$, we obtain the cofiber sequence:

$$\mathrm{Perf}_m(\mathbb{O}_k) \otimes_{\mathbb{O}_k} \mathrm{Coh}^b(\mathcal{X}) \rightarrow \mathrm{Coh}^b(\mathcal{X}) \rightarrow \mathrm{Perf}(k) \otimes_{\mathbb{O}_k} \mathrm{Coh}^b(\mathcal{X}),$$

along with an equivalence between $\mathrm{Perf}_m(\mathbb{O}_k) \otimes_{\mathbb{O}_k} \mathrm{Coh}^b(\mathcal{X})$ and $\mathrm{Coh}^b(\mathcal{X})_{\mathrm{nil}}$. Exactly as in Proposition 1.3.20, we obtain an equivalence between $\mathrm{Coh}^b(\mathcal{X}^{\mathrm{rig}})$ and $\mathrm{Coh}^b(\mathcal{X}) \otimes_{\mathrm{Perf}(\mathbb{O}_k)} \mathrm{Perf}(k)$. Since $\mathcal{X}^{\mathrm{rig}}$ is smooth, Corollary 1.3.17 allows us to identify $\mathrm{Coh}^b(\mathcal{X}^{\mathrm{rig}})$ with $\mathrm{Perf}(\mathcal{X}^{\mathrm{rig}})$. In particular, $\mathrm{Coh}^b(\mathcal{X}^{\mathrm{rig}})$ is smooth, and it is also proper by Corollary 1.3.2. Thus, by Theorem 1.3.1 and Proposition 1.3.21, we conclude that $\mathcal{X}$ is algebraizable. $\square$

Remark 1.3.24. In particular, this theorem indicates that for a general proper formal scheme $\mathcal{X}$, we do not expect $\mathrm{Coh}^b(\mathcal{X})$ to be smooth. For instance, consider a proper, non-algebraizable formal scheme $\mathcal{X}$ whose rigid generic fiber is smooth (but not algebraizable). The preceding theorem forces us to conclude that $\mathrm{Coh}^b(\mathcal{X})$ cannot be smooth.

By contrast, for a separated scheme X over a perfect field, Lunts demonstrated in [Lun10] that $\mathrm{Coh}^b(X)$ is always smooth. More generally, Efimov proved in [Efi18] that under mild conditions on X , the ∞ -category $\mathrm{Coh}^b(X)$ is of finite type. Because being of finite type implies smoothness, the above proposition highlights a sharp contrast: neither of these statements can be generalized unconditionally to the setting of formal schemes.

A non-commutative approach to rigid analytic geometry

2.1 Smooth and proper categories in a six-functor formalism

The notion of smooth and proper triangulated categories was introduced in [KS09]. Furthermore, in the setting of dg-categories (respectively, stable ∞ -categories), an analogous notion of smooth and proper categories has been studied, for example, in [TV07; Efi18] (respectively, in [Lur18; AG14]).

In this section, we work over a stable, presentable, symmetric monoidal ∞ -category $\mathcal{V}$. We will employ a modification of the standard notion of smooth and proper ∞ -categories, adapted to the setting of $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$. We will compare these definitions with the notion of smooth and proper algebras defined in [Lur17, Definitions 4.6.4.2 and 4.6.4.13]. Finally, at the end of the section, we explore how to obtain smooth ∞ -categories in the context of a six-functor formalism, with the aim of relating these results to the usual notion of smoothness in analytic geometry in subsequent sections.

2.1.1 Remarks on presentable tensored categories

In this section, we consider a stable, presentable, symmetric monoidal ∞ -category $\mathcal{V}$, and we denote its tensor product by $\otimes$. We let $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ denote the $(\infty, 2)$ -category $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$ and refer to an object in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ as a $\mathcal{V}$ -linear ∞ -category. We now review some known facts and definitions in the setting of $\mathcal{V}$ -linear ∞ -categories. The main references for this topic are [Ben24; Ram24a; Hin21; Hei23; GH15], which provide a rigorous treatment of the subject.

We begin by observing that the $(\infty, 2)$ -category $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ admits a symmetric monoidal structure [Lur17, Theorem 4.5.2.1]; we denote its tensor product (induced by that of $\mathcal{V}$) by $\otimes_{\mathcal{V}}$. Furthermore, any object $\mathcal{C}$ in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ is equipped with a $\mathcal{V}$ -enriched internal hom functor $\mathrm{hom}_{\mathcal{C}}^{\mathcal{V}}(-, -)$, as defined in [Ben24, Definition 4.20]. We note that, for an object $x \in \mathcal{C}$, the functor $\mathrm{hom}_{\mathcal{C}}^{\mathcal{V}}(x, -)$ is, in general, only lax $\mathcal{V}$ -linear. Indeed, it is defined as the right adjoint to the $\mathcal{V}$ -linear functor

$$- \otimes_{\mathcal{V}} x: \mathcal{V} \rightarrow \mathcal{C},$$

given by the action of $\mathcal{V}$ on $\mathcal{C}$. It follows from [Ben24, Proposition 4.17] that the right adjoint of a $\mathcal{V}$ -linear left adjoint functor is lax $\mathcal{V}$ -linear. When the ∞ -category under consideration is clear from the context, we will simplify the notation and denote the $\mathcal{V}$ -enriched internal hom by $\mathrm{hom}^{\mathcal{V}}(-, -)$.

Proposition 2.1.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable $\mathcal{V}$ -tensored ∞ -categories, and let $L: \mathcal{C} \rightarrow \mathcal{D}$ be a $\mathcal{V}$ -linear left adjoint. Then $R: \mathcal{D} \rightarrow \mathcal{C}$ is the right adjoint of L if and only if there is a natural isomorphism*

$$\mathrm{hom}^{\mathcal{V}}(L(-), -) \xrightarrow{\sim} \mathrm{hom}^{\mathcal{V}}(-, R(-)).$$

Proof. One direction is proved in [Ben24, Proposition 4.21]. Conversely, suppose that the canonical map

$$\mathrm{hom}^{\mathcal{V}}(L(-), -) \rightarrow \mathrm{hom}^{\mathcal{V}}(-, R(-))$$

is a natural isomorphism. Applying the mapping space functor $\mathrm{Map}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, -)$ yields a natural isomorphism of spaces

$$\mathrm{Map}(L(-), -) \simeq \mathrm{Map}(-, R(-))$$

(see [GH15, Remark 7.2.12]). □

Definition 2.1.2 ([Ben24, Proposition 5.2]). We say that a $\mathcal{V}$ -linear left adjoint functor $L: \mathcal{C} \rightarrow \mathcal{D}$ is *internally left adjoint* if its lax $\mathcal{V}$ -linear right adjoint $R: \mathcal{D} \rightarrow \mathcal{C}$ is $\mathcal{V}$ -linear and is itself a left adjoint.

Definition 2.1.3. Let $\mathcal{C}$ be a ∞ -category in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ and let x be an object of $\mathcal{C}$. We say that x is *compact relative to $\mathcal{V}$* if the functor $\mathrm{hom}_{\mathcal{C}}^{\mathcal{V}}(x, -)$ preserves colimits.

Note that since we always work with stable ∞ -categories, the previous definition is equivalent to requiring that $\mathrm{hom}_{\mathcal{C}}^{\mathcal{V}}(x, -)$ commutes with filtered colimits.

Definition 2.1.4. Let $\mathcal{C}$ be an $\mathbb{E}_{\infty}$ -algebra in $\mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$ and let x be an object of $\mathcal{C}$. We say that x is *$\mathcal{V}$ -atomic* if it is compact relative to $\mathcal{V}$ and the functor $\mathrm{hom}^{\mathcal{V}}(x, -)$ is $\mathcal{V}$ -linear. Equivalently, the functor $- \otimes_{\mathcal{V}} x$ is internally left adjoint.

Lemma 2.1.5. *Let $\mathcal{V}$ be generated under colimits by dualizable objects and let $\mathcal{C}$ be a symmetric monoidal presentable $\mathcal{V}$ -tensored ∞ -category. Then an object $x \in \mathcal{C}$ is compact relative to $\mathcal{V}$ if and only if it is $\mathcal{V}$ -atomic.*

Proof. Every $\mathcal{V}$ -atomic object is compact relative to $\mathcal{V}$ by definition. Conversely, assume x is compact relative to $\mathcal{V}$. By [Ram24b, Corollary 3.8], for any $x, y \in \mathcal{C}$ and any dualizable object $v \in \mathcal{V}$, the canonical map

$$v \otimes \mathrm{hom}^{\mathcal{V}}(x, y) \rightarrow \mathrm{hom}^{\mathcal{V}}(x, v \otimes y)$$

is an equivalence. Since $\mathcal{V}$ is generated under colimits by dualizable objects, any object $w \in \mathcal{V}$ can be written as a colimit of dualizable objects, say $w \simeq \mathrm{colim}_{i \in I} v_i$. Using the assumption that x is compact relative to $\mathcal{V}$ (i.e., that $\mathrm{hom}^{\mathcal{V}}(x, -)$ preserves colimits), we obtain the following equivalences:

$$\begin{aligned} \mathrm{hom}^{\mathcal{V}}(x, w \otimes y) &\simeq \mathrm{hom}^{\mathcal{V}}(x, (\mathrm{colim}_{i \in I} v_i) \otimes y) \\ &\simeq \mathrm{hom}^{\mathcal{V}}(x, \mathrm{colim}_{i \in I} (v_i \otimes y)) \\ &\simeq \mathrm{colim}_{i \in I} \mathrm{hom}^{\mathcal{V}}(x, v_i \otimes y) \\ &\simeq \mathrm{colim}_{i \in I} (v_i \otimes \mathrm{hom}^{\mathcal{V}}(x, y)) \\ &\simeq (\mathrm{colim}_{i \in I} v_i) \otimes \mathrm{hom}^{\mathcal{V}}(x, y) \\ &\simeq w \otimes \mathrm{hom}^{\mathcal{V}}(x, y). \end{aligned}$$

This shows that $\mathrm{hom}^{\mathcal{V}}(x, -)$ is $\mathcal{V}$ -linear, concluding the proof. $\square$

Remark 2.1.6. We observe that the hypotheses of Lemma 2.1.5 are satisfied when $\mathcal{V}$ is equivalent to Mod_A for an $\mathbb{E}_{\infty}$ -algebra A in Sp

Moreover, when $\mathcal{V}$ is the ∞ -category of spectra (denoted by Sp) or the ∞ -category Mod_R of modules over an $\mathbb{E}_{\infty}$ -ring R , we have the following characterization of $\mathcal{V}$ -atomic objects.

Proposition 2.1.7. *Let $\mathcal{V}$ be the ∞ -category of spectra or Mod_R for an $\mathbb{E}_\infty$ -ring R . Let $\mathcal{C}$ be a ∞ -category in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ and let x be an object of $\mathcal{C}$. Then the following are equivalent:*

1. *The object x is compact in $\mathcal{C}$.*
2. *The object x is compact relative to $\mathcal{V}$.*
3. *The object x is $\mathcal{V}$ -atomic.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from Lemma 2.1.5. We now prove the equivalence (1) $\Leftrightarrow$ (2) in the case where $\mathcal{V}$ is the ∞ -category of spectra. An analogous argument is used in [BS23, Proposition 2.8].

Assume that x is compact relative to Sp . The mapping space $\mathrm{Map}_{\mathcal{C}}(x, -)$ can be written as the delooping of the mapping spectrum $\Omega^\infty \mathrm{map}_{\mathcal{C}}(x, -)$. Since the functor Ω^∞ preserves filtered colimits, this implies that x is compact.

Conversely, suppose that x is compact in $\mathcal{C}$. Observe that the shift functor $\Sigma^n : \mathrm{Sp} \rightarrow \mathrm{Sp}$ preserves colimits for every integer n . Consequently, the functor

$$\Omega^\infty \Sigma^n \mathrm{map}_{\mathcal{C}}(x, -) \simeq \mathrm{Map}_{\mathcal{C}}(x, \Sigma^n -)$$

preserves colimits. The statement now follows from the fact that the functors

$$\Omega^\infty \Sigma^n : \mathrm{Sp} \rightarrow \mathrm{Ani}$$

are jointly conservative.

The equivalence (1) $\Leftrightarrow$ (2) when $\mathcal{V} = \mathrm{Mod}_R$ follows from the case where $\mathcal{V} = \mathrm{Sp}$, using the fact that for every objects c and c' in $\mathcal{C}$ we have a functorial equivalence

$$\mathrm{map}_{\mathcal{C}}(c, c') \simeq \mathrm{map}_{\mathrm{Mod}_R}(R, \mathrm{hom}^{\mathrm{Mod}_R}(c, c'))$$

and R is a compact generator of Mod_R . □

Henceforth, for a stable ∞ -category, implicitly using Proposition 2.1.7, we will no longer distinguish between compact, Sp -compact, and Sp -atomic objects.

Definition 2.1.8. Let $\mathcal{C}$ be a ∞ -category in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$, and S a subset of objects in $\mathcal{C}$. We denote by $\mathcal{C}_S$ the full subcategory of $\mathcal{C}$ spanned by the objects in S .

1. We say that $\mathcal{C}$ is $\mathcal{V}$ -relatively generated by S if $\mathcal{C}$ coincides with the smallest full subcategory of itself that contains $\mathcal{C}_S$ and is closed under colimits and the action of $\mathcal{V}$.

2. We say that $\mathcal{C}$ is $\mathcal{V}$ -atomically generated if it is $\mathcal{V}$ -relatively generated by its subset of $\mathcal{V}$ -atomic objects.
3. We say that $\mathcal{C}$ is $\mathcal{V}$ -relatively compactly generated if it is $\mathcal{V}$ -relatively generated by its subset of $\mathcal{V}$ -compact objects.

We note that our definition of a $\mathcal{V}$ -atomically generated ∞ -category coincides with the notion of a $\mathcal{V}$ -molecular ∞ -category introduced in [Ben24, Definition 2.10].

Remark 2.1.9. We also observe that the set of $\mathcal{V}$ -atomic objects is small [Ben24, Proposition 5.9]. This allows us to “control” a $\mathcal{V}$ -atomically generated ∞ -category by a small subcategory. Indeed, by applying [Ram24a, Observation 1.28], it follows that if $\mathcal{M} \in \mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ is $\mathcal{V}$ -atomically generated, there is an equivalence

$$\mathcal{M} \simeq \mathcal{P}_{\mathcal{V}}((\mathcal{M})^{\mathrm{at}}),$$

where $\mathcal{P}_{\mathcal{V}}(-)$ denotes the category of $\mathcal{V}$ -enriched presheaves as defined in [Hei23], and $(\mathcal{M})^{\mathrm{at}}$ denotes the full subcategory of atomic objects in $\mathcal{M}$. Moreover, it follows from [Ben24, Section 6] that $\mathcal{P}_{\mathcal{V}}(-)$ commutes with colimits.

Remark 2.1.10. It follows from [RZ25, Corollary 3.13] that $\mathcal{C}$ is $\mathcal{V}$ -relatively generated by S as in Definition 2.1.8 if and only if the family of functors

$$\{\mathrm{hom}_{\mathcal{C}}^{\mathcal{V}}(s, -) : \mathcal{C} \rightarrow \mathcal{V}\}_{s \in S}$$

is jointly conservative.

We will now consider the case where $\mathcal{V}$ is a rigid ∞ -category, in which the various definitions introduced above coincide.

In this paper, we extensively use the notion of rigid ∞ -categories as introduced in [GR17]. The main feature is that when $\mathcal{V}$ is rigid, $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ enjoys several favorable properties (see, for example, [GR17, Chapter 1, Sec 9] and [AM24, Lemma 3.33]). Another motivation for introducing rigid ∞ -categories comes from rigid analytic geometry; indeed, the ∞ -category of nuclear sheaves in this setting is rigid. This ∞ -category will be studied in detail later in this paper.

Proposition 2.1.11. *Let $\mathcal{V}$ be a stable, presentable, symmetric monoidal, rigid ∞ -category, let $\mathcal{C}$ be a ∞ -category in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$, and let x be an object in $\mathcal{C}$. Then the following are equivalent:*

1. *The object x in $\mathcal{C}$ is compact.*
2. *The object x in $\mathcal{C}$ is compact relative to Sp .*

3. The object x in $\mathcal{C}$ is compact relative to $\mathcal{V}$.
4. The object x in $\mathcal{C}$ is $\mathcal{V}$ -atomic.

Proof. This follows by combining Proposition 2.1.7 and [GR17, Corollary 9.3.4, Lemma 9.3.6]. $\square$

From the above proposition, we can immediately deduce the following.

Corollary 2.1.12. *If $\mathcal{V}$ is rigid, then the definitions of a $\mathcal{V}$ -atomically generated ∞ -category and a $\mathcal{V}$ -compactly generated ∞ -category coincide.*

For this reason, we will not further distinguish between the two notions when working over a rigid base.

2.1.2 Reminder on smooth and proper categories and algebras

This subsection introduces the notions of smooth and proper enriched ∞ -categories that will be utilized throughout this paper. We also prove the main result of this section, Proposition 2.1.27, which plays a central role in refuting Conjecture 0.2.9.

Definition 2.1.13. Let A be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. We say that A is *smooth* if the diagonal module is dualizable as a module over $A \otimes_{\mathcal{V}} A^{\text{op}}$.

Definition 2.1.14 ([Lur17, Definition 4.6.4.2]). Let A be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. We say that A is *proper* if it is dualizable as an object of $\mathcal{V}$.

Definition 2.1.15. Let $\mathcal{C}$ be a $\mathcal{V}$ -linear ∞ -category. We say that $\mathcal{C}$ is *dualizable* over $\mathcal{V}$ if there exists a ∞ -category $\mathcal{C}^{\vee}$ in $\text{Pr}_{\mathcal{V}}^{\text{L}}$, together with two functors in $\text{Pr}_{\mathcal{V}}^{\text{L}}$, called the evaluation and coevaluation functors:

$$\text{ev} : \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^{\vee} \rightarrow \mathcal{V}$$

$$\text{coev} : \mathcal{V} \rightarrow \mathcal{C}^{\vee} \otimes_{\mathcal{V}} \mathcal{C},$$

such that the composite maps

$$\mathcal{C} \xrightarrow{\text{id} \boxtimes \text{coev}} \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^{\vee} \otimes_{\mathcal{V}} \mathcal{C} \xrightarrow{\text{ev} \boxtimes \text{id}} \mathcal{C} \quad (2.1)$$

$$\mathcal{C}^{\vee} \xrightarrow{\text{coev} \boxtimes \text{id}} \mathcal{C}^{\vee} \otimes_{\mathcal{V}} \mathcal{C} \otimes_{\mathcal{V}} \mathcal{C}^{\vee} \xrightarrow{\text{id} \boxtimes \text{ev}} \mathcal{C}^{\vee} \quad (2.2)$$

are equivalent to the respective identity functors.

Remark 2.1.16. It is a general fact that in a closed symmetric monoidal ∞ -category, the dual of a dualizable object X can be identified with the internal hom $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbb{1})$. In particular, in our setting, we can identify $\mathcal{C}^\vee$ with $\mathrm{Fun}_{\mathcal{V}}^L(\mathcal{C}, \mathcal{V})$ (see [GR17, p. 4.3.3]).

Remark 2.1.17. This definition is well-posed since $\mathcal{V}$ is a symmetric monoidal ∞ -category. If we only assume $\mathcal{V}$ to be an $\mathbb{E}_1$ -algebra object in $\mathrm{Pr}_{\mathcal{V}}^L$, then one can employ the definition of a dualizable ∞ -category as explained in [Efi25b, Definition 1.10]. We will not discuss the non-symmetric monoidal case further, as we assume throughout that our base $\mathcal{V}$ is symmetric monoidal.

Remark 2.1.18. We note that Equation (2.1) implies that $\mathrm{coev}(\mathbb{1}_{\mathcal{V}})$ classifies the identity functor under the equivalence

$$\mathcal{C}^\vee \otimes_{\mathcal{V}} \mathcal{C} \simeq \mathrm{Fun}_{\mathcal{V}}^L(\mathcal{C}, \mathcal{C}).$$

Definition 2.1.19. Let $\mathcal{C}$ be a presentable and dualizable ∞ -category over $\mathcal{V}$. We say that $\mathcal{C}$ is *smooth over $\mathcal{V}$* if the coevaluation map admits a right adjoint that preserves colimits.

Definition 2.1.20. Let $\mathcal{C}$ be a presentable and dualizable ∞ -category over $\mathcal{V}$. We say that $\mathcal{C}$ is *proper over $\mathcal{V}$* if the evaluation map admits a right adjoint that preserves colimits.

Note that, in general, the right adjoints of the evaluation and coevaluation maps are only lax $\mathcal{V}$ -linear. This limitation motivates the introduction of the following variants of the notions of smooth and proper $\mathcal{V}$ -linear ∞ -categories.

Definition 2.1.21. Let $\mathcal{C}$ be a presentable and dualizable ∞ -category over $\mathcal{V}$. We say that $\mathcal{C}$ is *internally smooth over $\mathcal{V}$* if the coevaluation map admits a $\mathcal{V}$ -linear right adjoint that preserves colimits.

Definition 2.1.22. Let $\mathcal{C}$ be a presentable and dualizable ∞ -category over $\mathcal{V}$. We say that $\mathcal{C}$ is *internally proper over $\mathcal{V}$* if the evaluation map admits a $\mathcal{V}$ -linear right adjoint that preserves colimits.

Definition 2.1.23. Let $\mathcal{C}$ be a dualizable ∞ -category over $\mathcal{V}$. We say that $\mathcal{C}$ is *fully dualizable over $\mathcal{V}$* if it is internally smooth and internally proper; that is, if both the evaluation and coevaluation maps admit a $\mathcal{V}$ -linear right adjoint that preserves colimits.

Remark 2.1.24. Observe that $\mathcal{C}$ satisfies Definition 2.1.19 and Definition 2.1.20 if and only if it is dualizable in the ∞ -category $\mathrm{Pr}_{\mathcal{V}}^{L, \mathrm{dual}}$, which consists of dualizable ∞ -categories over $\mathcal{V}$ and $\mathcal{V}$ -linear strongly continuous functors. Indeed, this condition is equivalent to requiring that both the coevaluation and evaluation functors admit continuous right adjoints.

Remark 2.1.25. We observe that $\mathcal{C}$ is smooth if and only if $\mathrm{coev}(\mathbb{1}_{\mathcal{V}})$ is $\mathcal{V}$ -atomic, in particular if $\mathbb{1}_{\mathcal{V}}$ is compact in $\mathcal{V}$ then also $\mathrm{coev}(\mathbb{1}_{\mathcal{V}})$ is compact. When $\mathcal{V}$ is rigid, this is equivalent to requiring that $\mathrm{coev}(\mathbb{1}_{\mathcal{V}})$ is compact (see Proposition 2.1.11).

Note that, *a priori*, the right adjoint of a $\mathcal{V}$ -linear functor is not necessarily $\mathcal{V}$ -linear. However, this property holds in specific cases; for example, if $\mathcal{V}$ is the ∞ -category of spectra or, more generally, a *mode* as explained in [Ram24a, Example 1.13]. In such cases, being fully dualizable over $\mathcal{V}$ is strictly equivalent to being smooth and proper over $\mathcal{V}$. We observe that this equivalence also holds when $\mathcal{V}$ is rigid.

Lemma 2.1.26. *Let $\mathcal{V}$ be rigid and let $\mathcal{C}$ be an object in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$. Then $\mathcal{C}$ is internally smooth (respectively, internally proper) if and only if it is smooth (respectively, proper).*

Proof. It suffices to verify that the right adjoint of a $\mathcal{V}$ -linear map is itself $\mathcal{V}$ -linear. This follows from the fact that $\mathcal{V}$ is rigid (see [GR17, Lemma 9.3.6]). $\square$

Proposition 2.1.27. *Let $\mathcal{V}$ be a commutative algebra in $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ with compact unit and let $\mathcal{C}$ be a smooth ∞ -category over $\mathcal{V}$ which is $\mathcal{V}$ -atomically generated (equivalently, $\mathcal{V}$ -compactly generated). Then $\mathcal{C}$ admits a single $\mathcal{V}$ -atomic (equivalently, compact) generator.*

Proof. Let S denote the set of $\mathcal{V}$ -atomic objects of $\mathcal{C}$. For every subset $T \subset S$, let $\mathcal{C}_T$ be the full subcategory of $\mathcal{C}$ spanned by T , and let $\widehat{\mathcal{C}}_T$ be the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{C}_T$ that is closed under colimits and the action of $\mathcal{V}$. Since S is the set of $\mathcal{V}$ -atomic generators, we have a canonical equivalence

$$\widehat{\mathcal{C}}_S \simeq \mathcal{C}.$$

We claim that the canonical map

$$\mathrm{colim}_{T \subset S} \widehat{\mathcal{C}}_T \rightarrow \mathcal{C} \tag{2.3}$$

is an equivalence, where the colimit is taken over the filtered system of finite subsets of S . By Remark 2.1.9, each $\widehat{\mathcal{C}}_T$ is $\mathcal{V}$ -atomically generated by T and can be identified with the ∞ -category of $\mathcal{V}$ -enriched presheaves $\mathcal{P}_{\mathcal{V}}(\mathcal{C}_T)$. Similarly, $\mathcal{C}$ is identified with $\mathcal{P}_{\mathcal{V}}(\mathcal{C}_S)$. Since $\mathcal{P}_{\mathcal{V}}(-)$ is a left adjoint (see Remark 2.1.9), it preserves colimits; thus, to show that (2.3) is an equivalence, it suffices to observe that the canonical map

$$\mathrm{colim}_{T \subset S} \mathcal{C}_T \rightarrow \mathcal{C}_S$$

is a tautological equivalence.

We now observe that the coevaluation functor

$$\mathrm{coev}: \mathcal{V} \rightarrow \mathcal{C}^\vee \otimes_{\mathcal{V}} \mathcal{C}$$

classifies the identity functor $\mathrm{id}_{\mathcal{C}}$. Since $\mathcal{C}$ is smooth and $\mathbb{1}_{\mathcal{V}}$ is a compact object, $\mathrm{id}_{\mathcal{C}}$ is a compact object in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ (see Remark 2.1.25). Using (2.3), we can express the identity as a filtered colimit

$$\mathrm{id}_{\mathcal{C}} \simeq \mathrm{colim}_{T \subset S} f_T,$$

where $f_T: \widehat{\mathcal{C}}_T \rightarrow \mathcal{C}$ is the natural inclusion. The compactness of $\mathrm{id}_{\mathcal{C}}$ implies that the identity natural transformation

$$\mathbb{1}_{\mathrm{id}_{\mathcal{C}}}: \mathrm{id}_{\mathcal{C}} \rightarrow \mathrm{id}_{\mathcal{C}}$$

factors through f_T for some finite subset $T \subset S$. This implies that $\mathrm{id}_{\mathcal{C}}$ factors through $\widehat{\mathcal{C}}_T$, and in particular, that $\mathcal{C}$ coincides with $\widehat{\mathcal{C}}_T$. It follows that $\mathcal{C}$ is generated by the single compact object $\bigoplus_{a_i \in T} a_i$. $\square$

We now turn to a detailed study of the case where the ∞ -category is $\mathrm{LMod}_A(\mathcal{V})$, for A an algebra object in $\mathcal{V}$.

2.1.3 Modules over an algebra in $\mathcal{V}$

In this subsection, we study the ∞ -category $\mathrm{LMod}_A(\mathcal{V})$ in detail. In particular, we show that it is generated under colimits and the action of $\mathcal{V}$. We also analyze its dualizable and compact objects. Finally, we compare the notion of internal smoothness for $\mathrm{LMod}_A(\mathcal{V})$ to the smoothness of A as an algebra in $\mathcal{V}$ (see Proposition 2.1.33). This subsection, along with its main result (Proposition 2.1.33), will be particularly relevant when we study the compact generation of the ∞ -category of nuclear sheaves, specifically in the proof of Theorem 2.4.9.

Definition 2.1.28. Let R be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$, and let X be an object in $\mathrm{LMod}_R(\mathcal{V})$. An object Y in $\mathrm{RMod}_R(\mathcal{V})$ is said to be *left dual* to X if there exists an adjunction

$$X \otimes -: \mathcal{V} \xrightleftharpoons[\perp]{} \mathrm{LMod}_R(\mathcal{V}) : Y \otimes_R -,$$

where the tensor product on the left is given by the action of $\mathcal{V}$ on $\mathrm{LMod}_R(\mathcal{V})$, and the one on the right is described in Equation (2.5).

Lemma 2.1.29. *Let R be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. Then R is $\mathcal{V}$ -atomic in $\mathrm{LMod}_R(\mathcal{V})$. Furthermore, for every $\mathcal{V}$ -atomic object v in $\mathcal{V}$, the object $R \otimes v$ is $\mathcal{V}$ -atomic in $\mathrm{LMod}_R(\mathcal{V})$.*

Proof. The first assertion follows from the observation that $\mathrm{hom}_{\mathrm{LMod}_R}^{\mathcal{V}}(R, -)$ can be identified with the forgetful functor

$$\mathrm{LMod}_R(\mathcal{V}) \rightarrow \mathcal{V}.$$

This functor preserves colimits and is $\mathcal{V}$ -linear (see [Lur17, p. 4.2.3.7]). Moreover, since the free module functor

$$R \otimes -: \mathcal{V} \rightarrow \mathrm{LMod}_R(\mathcal{V})$$

is an internal left adjoint, it preserves atomic objects. $\square$

Proposition 2.1.30. *Let R be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. Then $\mathrm{LMod}_R(\mathcal{V})$ is $\mathcal{V}$ -atomically generated. Moreover, if an ∞ -category $\mathcal{C}$ in $\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}}$ is generated by a single $\mathcal{V}$ -atomic object X , then $\mathcal{C}$ is equivalent to $\mathrm{LMod}_R(\mathcal{V})$, where R is the $\mathbb{E}_1$ -algebra $\mathrm{hom}^{\mathcal{V}}(x, x)$.*

Proof. It suffices to show that $\mathrm{LMod}_R(\mathcal{V})$ is generated under colimits by objects of the form $R \otimes v$ with $v \in \mathcal{V}$. This follows from [Lur17, p. 4.7.3.14] by observing that $\mathrm{LMod}_R(\mathcal{V})$ is the category of modules over the monad $R \otimes -$. The latter statement follows from an application of the Barr–Beck–Lurie theorem. $\square$

Corollary 2.1.31. *Let R be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. Then an object X in $\mathrm{LMod}_R(\mathcal{V})$ is dualizable if and only if it is $\mathcal{V}$ -atomic.*

Proof. Suppose that X is dualizable in $\mathrm{LMod}_R(\mathcal{V})$. Then there exists an object $Y \in \mathrm{RMod}_R(\mathcal{V})$ and an equivalence

$$\mathrm{hom}_{\mathrm{LMod}_R}^{\mathcal{V}}(X, -) \simeq Y \otimes_R -.$$

Since the functor $Y \otimes_R -$ preserves colimits and is $\mathcal{V}$ -linear (see [Lur17, p. 4.4.3.14]), it follows that X is $\mathcal{V}$ -atomic. Conversely, suppose that X is $\mathcal{V}$ -atomic in $\mathrm{LMod}_R(\mathcal{V})$. We define

$$X^{\vee} := \mathrm{hom}_{\mathrm{LMod}_R}^{\mathcal{V}}(X, R).$$

This is canonically an object in $\mathrm{RMod}_R(\mathcal{V})$. To prove that $X^{\vee}$ is the dual of X , it suffices to show that the functor $X^{\vee} \otimes_R -$ is equivalent to $\mathrm{hom}_{\mathrm{LMod}_R}^{\mathcal{V}}(X, -)$. This follows from Proposition 2.1.30: since both functors preserve colimits and are $\mathcal{V}$ -linear, it is enough to observe that they coincide when evaluated on R . Indeed, we have

$$X^{\vee} \otimes_R R \simeq X^{\vee} = \mathrm{hom}_{\mathrm{LMod}_R}^{\mathcal{V}}(X, R).$$

$\square$

We now turn to the notion of internal smoothness for the ∞ -category $\mathrm{LMod}_A(\mathcal{V})$ and compare it to the smoothness of the algebra A . For instance, if R is an algebra object in the ∞ -category of spectra, then LMod_R is right dualizable with right dual $\mathrm{LMod}_{R^{op}} \simeq \mathrm{RMod}_R$. We will demonstrate that such results remain valid when working over our symmetric monoidal base $\mathcal{V}$ instead of spectra. In particular, we prove that if A is an $\mathbb{E}_1$ -algebra in $\mathcal{V}$, then the ∞ -category of left A -modules, $\mathrm{LMod}_A(\mathcal{V})$, is always dualizable over $\mathcal{V}$. Furthermore, if $\mathcal{V}$ is rigid, then $\mathrm{LMod}_A(\mathcal{V})$ is also rigid.

We begin by observing that, as explained in [Lur17, p. 4.1.1.7], an involution can be defined on the ∞ -operad **Assoc**, which allows for the definition of the opposite algebra for every algebra object in $\mathcal{V}$. We identify $\mathrm{RMod}_A(\mathcal{V})$ with $\mathrm{LMod}_{A^{op}}(\mathcal{V})$ and show that this ∞ -category is the dual of $\mathrm{LMod}_A(\mathcal{V})$. To achieve this, we explicitly describe the coevaluation and evaluation functors and verify that they constitute a duality datum.

The coevaluation functor is obtained as a $\mathcal{V}$ -linear map by sending the unit $\mathbb{1}_{\mathcal{V}}$ to A equipped with the diagonal bimodule structure:

$$\begin{aligned} \mathrm{coev}: \mathcal{V} &\longrightarrow \mathrm{LMod}_A(\mathcal{V}) \otimes \mathrm{RMod}_A(\mathcal{V}) \simeq \mathrm{Mod}_{A \otimes_{\mathcal{V}} A^{op}}(\mathcal{V}) \\ \mathbb{1}_{\mathcal{V}} &\longmapsto A_{\Delta}. \end{aligned} \tag{2.4}$$

The diagonal bimodule structure on A is obtained via the forgetful functor $m_*: \mathrm{Mod}_A \rightarrow \mathrm{Mod}_{A \otimes_{\mathcal{V}} A^{op}}$ induced by the multiplication map $m: A \otimes A^{op} \rightarrow A$. It is straightforward to verify that the coevaluation functor has a right adjoint given by $\mathrm{hom}_{A \otimes_{\mathcal{V}} A^{op}}^{\mathcal{V}}(A_{\Delta}, -)$.

The evaluation functor is defined via the relative tensor product; explicitly, it is given by the map:

$$\begin{aligned} \mathrm{ev}: \mathrm{RMod}_A(\mathcal{V}) \otimes \mathrm{LMod}_A(\mathcal{V}) &\longrightarrow \mathcal{V} \\ M \boxtimes N &\longmapsto M \otimes_A N. \end{aligned} \tag{2.5}$$

In particular, this functor sends $A^{op} \boxtimes A$ to A (see [Lur17, Sec. 4.4] and [GR17, p. 4.2.1]).

Proposition 2.1.32. *Let $\mathcal{V}$ be a stable, presentable, symmetric monoidal ∞ -category and let A be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. Then $\mathrm{LMod}_A(\mathcal{V})$ is dualizable over $\mathcal{V}$ with dual $\mathrm{LMod}_{A^{op}}(\mathcal{V})$.*

Proof. We need to check that the coevaluation and evaluation functors defined in Equation (2.4) and Equation (2.5) constitute a duality datum. We observe that the analogue of Equation (2.1) is given by the composition

$$\mathrm{LMod}_A \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} \mathrm{LMod}_A \otimes \mathrm{RMod}_A \otimes \mathrm{LMod}_A \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \mathrm{LMod}_A$$

which sends a module M to $A_\Delta \otimes_A M$. By [Lur17, Proposition 4.6.2.17], this composition is homotopic to the identity. Similarly, one can show that the analogue of the map in Equation (2.2) is homotopic to the identity. $\square$

Proposition 2.1.33. *Let A be an $\mathbb{E}_1$ -algebra in $\mathcal{V}$. Then A is smooth as an algebra if and only if the category of modules $\mathrm{LMod}_A(\mathcal{V})$ is internally smooth as a $\mathcal{V}$ -linear ∞ -category. In particular, the notion of an internally smooth ∞ -category is Morita invariant.*

Proof. By definition, A is smooth as an algebra in $\mathcal{V}$ if and only if $\mathrm{coev}(\mathbb{1}_{\mathcal{V}}) \simeq A_\Delta$ is dualizable in $\mathrm{LMod}_{A \otimes_{\mathcal{V}} A^{op}}(\mathcal{V})$. By invoking Corollary 2.1.31, we deduce that this holds if and only if A_Δ is $\mathcal{V}$ -atomic. In particular, this is equivalent to the condition that the right adjoint of the coevaluation map, given by

$$\mathrm{hom}_{\mathrm{LMod}_{A \otimes_{\mathcal{V}} A^{op}}}^{\mathcal{V}}(A_\Delta, -),$$

preserves colimits and is $\mathcal{V}$ -linear. $\square$

2.1.4 Smooth categories in a six-functor formalism

The notion of a *six-functor formalism* has been extensively studied in recent years, particularly following the works of [HM24; Sch25; LZ24; Zav23; Kes25]. These ideas have found significant applications in rigid analytic geometry and condensed mathematics, as illustrated, for instance, in [MAN22; Cam24; Soo25]. Motivated by the objective of utilizing an appropriate 6-functor formalism in analytic geometry to characterize smooth varieties, we first investigate the conditions under which smooth ∞ -category arise within such a framework. This is established in Proposition 2.1.35 and its subsequent corollaries. Specifically, in this section, we consider the following.

Setup 2.1.34. *Let $\mathcal{C}$ be an ∞ -category arising from a geometric setup as defined in [HM24, Definition 2.1.1], equipped with a class E of $!$ -able maps. Furthermore, let*

$$\mathcal{D}: \mathcal{C} \rightarrow \mathrm{Pr}_{\mathcal{D}(X)}^{\mathrm{L}}$$

be a symmetric monoidal, stable, and presentable 6-functor formalism, as in [HM24, Definition 3.2.1].

In this setup, employing the same argument as in [MP25, Corollary 1.2.10], we observe that for every $!$ -able map $Y \rightarrow X$, the category $\mathcal{D}(Y)$ is dualizable over $\mathcal{D}(X)$.

Indeed, utilizing the span diagrams

$$\begin{array}{ccc} & Y & \\ f \swarrow & & \searrow \Delta \\ X & & Y \times_X Y \end{array} \quad \text{and} \quad \begin{array}{ccc} & Y & \\ \Delta \swarrow & & \searrow f \\ Y \times_X Y & & X \end{array}$$

we can explicitly express the evaluation and coevaluation maps as follows:

$$\text{coev}: \mathcal{D}(X) \xrightarrow{f^*} \mathcal{D}(Y) \xrightarrow{\Delta_!} \mathcal{D}(Y \times_X Y) \quad (2.6)$$

$$\text{ev}: \mathcal{D}(Y \times_X Y) \xrightarrow{\Delta^*} \mathcal{D}(Y) \xrightarrow{f_!} \mathcal{D}(X) \quad (2.7)$$

Observe that the pullback functors are inherently linear. Moreover, the projection formula implies that $f_!$ is $\mathcal{D}(X)$ -linear, and Equation (2.8) demonstrates that $\Delta_!$ is also $\mathcal{D}(X)$ -linear. Consequently, both the evaluation and coevaluation functors are $\mathcal{D}(X)$ -linear.

Their right adjoints admit explicit descriptions: the right adjoint to the evaluation functor is given by

$$\text{ev}^R \simeq \Delta_* \circ f^!,$$

while the right adjoint to the coevaluation functor is given by

$$\text{coev}^R \simeq f_* \circ \Delta^!.$$

To determine when the ∞ -category $\mathcal{D}(Y)$ is smooth over $\mathcal{D}(X)$, we must investigate the conditions under which the functor coev^R preserves colimits.

Proposition 2.1.35. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. Then the category $\mathcal{D}(Y)$ is smooth over $\mathcal{D}(X)$ (internally smooth) provided the following conditions are satisfied:*

1. *The functor f_* preserves colimits (and, moreover, is $\mathcal{D}(X)$ -linear).*
2. *The object $\Delta_! \mathbb{1}_Y$ is internally compact in $\mathcal{D}(Y \times_X Y)$, i.e. its internal hom functor $\underline{\text{Hom}}(\Delta_! \mathbb{1}_Y, -)$ preserves filtered colimits (it is dualizable in $\mathcal{D}(Y \times_X Y)$).*
3. *Letting $\text{pr}: Y \times_X Y \rightarrow Y$ denote one of the projections, we require pr_* to preserve colimits (and, moreover, to be $\mathcal{D}(X)$ -linear).*

Proof. We prove only the smoothness of $\mathcal{D}(Y)$ over $\mathcal{D}(X)$; internal smoothness follows by an analogous argument. Using the projection formula, we observe that $\Delta_!(-)$ acts

as a $\mathcal{D}(Y)$ -linear functor. Indeed, for all M, N in $\mathcal{D}(Y)$, we have the following natural equivalences:

$$\Delta_!(M \otimes N) \simeq \Delta_!(M \otimes \Delta^* \text{pr}^* N) \simeq \Delta_!(M) \otimes \text{pr}^*(N). \quad (2.8)$$

In particular, for every M in $\mathcal{D}(Y)$, we can express $\Delta_!(M)$ as follows:

$$\Delta_!(M) \simeq \Delta_!(\mathbb{1}_{\mathcal{D}(Y)} \otimes M) \simeq \Delta_!\mathbb{1}_{\mathcal{D}(Y)} \otimes \text{pr}^*(M).$$

It follows that its right adjoint can be explicitly written as

$$\text{pr}_* \underline{\text{Hom}}_{\mathcal{D}(Y \times_X Y)}(\Delta_!\mathbb{1}_{\mathcal{D}(Y)}, (-)).$$

Consequently, by invoking the hypotheses, this right adjoint preserves colimits, thus defining a functor in Pr^{L} . By condition (1), f_* also belongs to Pr^{L} . $\square$

Remark 2.1.36. Note that conditions 2. and 3. in Proposition 2.1.35 are equivalent to requiring $\Delta_!\mathbb{1}_Y$ to be compact (atomic) relative to $\mathcal{D}(Y)$.

We can specialize Proposition 2.1.35 to two cases of particular interest. To do this, we first recall the definitions of suave and prim maps.

Definition 2.1.37. [HM24, Lemma 4.4.6] Let $\mathcal{D}$ be a 6-functor formalism as in Setup 2.1.34, let $f: X \rightarrow S$ be a $!$ -able map, and let $P \in \mathcal{D}(X)$. Then P is said to be f -prim if and only if the natural map

$$f_!(\pi_{2*} \underline{\text{Hom}}(\pi_1^* P, \Delta_!\mathbb{1}) \otimes P) \longrightarrow f_* \underline{\text{Hom}}(P, P)$$

becomes an isomorphism after applying $\text{Hom}(\mathbb{1}, -)$. Here, $\pi_i: X \times_S X \rightarrow X$ denote the two projections, and $\Delta: X \rightarrow X \times_S X$ is the diagonal map. Moreover, if P is f -prim, we denote by $\mathbb{D}_f(P)$ the object $\pi_{2*} \underline{\text{Hom}}(\pi_1^* P, \Delta_!\mathbb{1})$.

We say that f is $\mathcal{D}$ -prim if $\mathbb{1}_X$ is f -prim. In this case, we denote by δ_f the object $\mathbb{D}_f(\mathbb{1}_X)$.

Definition 2.1.38. [HM24, Lemma 4.4.5] Let $\mathcal{D}$ be a 6-functor formalism as in Setup 2.1.34, let $f: X \rightarrow S$ be a $!$ -able map, and let $P \in \mathcal{D}(X)$. Then P is said to be f -suave if and only if the natural map

$$\pi_1^* \underline{\text{Hom}}(P, f^! \mathbb{1}) \otimes \pi_2^* P \longrightarrow \underline{\text{Hom}}(\pi_1^* P, \pi_2^* P)$$

becomes an isomorphism after applying $\text{Hom}(\mathbb{1}, \Delta^!(-))$. In this case, we define $\text{SID}_f(P)$ to be $\underline{\text{Hom}}(P, f^! \mathbb{1})$.

We say that f is $\mathcal{D}$ -suave if $\mathbb{1}_X$ is f -suave. In this case, we denote by ω_f the object $\text{SID}_f(\mathbb{1})$.

Proposition 2.1.39. [HM24, Corollary 4.5.11] Let $\mathcal{D}$ be a 6-functor formalism as in Setup 2.1.34, and let $f: X \rightarrow Y$ be a $!$ -able map.

1. If f is $\mathcal{D}$ -suave, then the natural maps

$$\omega_f \otimes f^*(-) \xrightarrow{\sim} f^!(-) \quad \text{and} \quad f^*(-) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\omega_f, f^!(-))$$

are isomorphisms of functors $\mathcal{D}(Y) \rightarrow \mathcal{D}(X)$.

2. If f is $\mathcal{D}$ -prim, then the natural maps

$$f_!(\delta_f \otimes -) \xrightarrow{\sim} f_*(-) \quad \text{and} \quad f_!(-) \xrightarrow{\sim} f_*\underline{\mathrm{Hom}}(\delta_f, -)$$

are isomorphisms of functors $\mathcal{D}(X) \rightarrow \mathcal{D}(Y)$.

Corollary 2.1.40. Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. Then the category $\mathcal{D}(Y)$ is internally smooth over $\mathcal{D}(X)$ provided that f is $\mathcal{D}$ -prim and Δ_f is $\mathcal{D}$ -suave.

Proof. We verify the hypotheses of Proposition 2.1.35. Since f is $\mathcal{D}$ -prim, pr is also $\mathcal{D}$ -prim (see [HM24, Lemma 4.4.8]). Invoking Proposition 2.1.39, we obtain two equivalences:

$$\mathrm{pr}_!(\delta_{\mathrm{pr}} \otimes -) \simeq \mathrm{pr}_*(-) \quad \text{and} \quad f_!(\delta_f \otimes -) \simeq f_*(-).$$

Therefore, both $f_*(-)$ and $\mathrm{pr}_*(-)$ preserve colimits and are $\mathcal{D}(X)$ -linear by the projection formula. Using the $\mathcal{D}(Y)$ -linearity of $\Delta_{f,!}$, we obtain an equivalence

$$\mathrm{hom}_{\mathcal{D}(Y \times_X Y)}^{\mathcal{D}(Y)}(\Delta_{f,!}\mathbb{1}, -) \simeq \Delta_f^!(-),$$

since both functors are right adjoints to $\Delta_{f,!}(-)$. Furthermore, by Proposition 2.1.39, we can identify the internal hom functor with $\omega_{\Delta_f} \otimes \Delta_f^*(-)$, which is a colimit-preserving $\mathcal{D}(Y)$ -linear functor. This implies that $\Delta_{f,!}\mathbb{1}$ is $\mathcal{D}(Y)$ -atomic. $\square$

Remark 2.1.41. Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map. As observed earlier via Equation (2.8), the coevaluation functor is inherently $\mathcal{D}(X)$ -linear. Consequently, the right adjoint to coev can be identified with the $\mathcal{D}(X)$ -enriched hom functor:

$$\mathrm{coev}^R(-) \simeq \mathrm{hom}_{\mathcal{D}(Y \times_X Y)}^{\mathcal{D}(X)}(\mathrm{coev}(\mathbb{1}_X), -). \quad (2.9)$$

It follows that $\mathcal{D}(Y)$ is internally smooth over $\mathcal{D}(X)$ if and only if $\mathrm{coev}(\mathbb{1}_X)$ is $\mathcal{D}(X)$ -atomic.

In the rigid case, this remark specializes to the following result.

Lemma 2.1.42. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34. Let $Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. If $\mathcal{D}(X)$ is rigid, then the category $\mathcal{D}(Y)$ is internally smooth over $\mathcal{D}(X)$ if and only if $\Delta_!(\mathbb{1}_Y)$ is compact.*

Proof. This follows directly by combining Remark 2.1.41 and Proposition 2.1.11. $\square$

2.1.5 Proper categories in a six-functor formalism

By analogy with the previous section, we investigate here how to obtain a proper ∞ -category from a six-functor formalism, as defined in Setup 2.1.34. We conclude by proving a criterion for obtaining $\mathcal{D}$ -prim covers, which will be used in Section 2.5.2.

Proposition 2.1.43. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. Then the category $\mathcal{D}(Y)$ is proper over $\mathcal{D}(X)$ (and internally proper) provided the following conditions are satisfied:*

1. *The functor $\Delta_{f,*}$ preserves colimits (and, moreover, is $\mathcal{D}(X)$ -linear).*
2. *The functor $f^!$ preserves colimits (and, moreover, is $\mathcal{D}(X)$ -linear).*

Proof. This statement follows immediately from the explicit descriptions of the right adjoints to the evaluation and coevaluation functors provided in Equation (2.7) and Equation (2.6). $\square$

Nevertheless, from the preceding proposition, we can deduce reasonable conditions under which proper ∞ -category arise within a six-functor formalism.

Corollary 2.1.44. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. Then the category $\mathcal{D}(Y)$ is internally proper over $\mathcal{D}(X)$ provided that f is $\mathcal{D}$ -suave and Δ_f is $\mathcal{D}$ -prim.*

Proof. This follows from the natural equivalences

$$\omega_f \otimes f^*(-) \xrightarrow{\sim} f^!(-) \quad \text{and} \quad \Delta_{f,!}(\delta_{\Delta_f} \otimes -) \xrightarrow{\sim} \Delta_{f,*}(-),$$

combined with the $\mathcal{D}(Y)$ -linearity of $\Delta_{f,!}(-)$, as established in Equation (2.8). $\square$

Remark 2.1.45. Another instance where the hypotheses of Proposition 2.1.43 are satisfied occurs when considering rigid categories. Indeed, let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. If $\mathcal{D}(X)$ is rigid, then for $\mathcal{D}(Y)$ to be internally proper over $\mathcal{D}(X)$, it suffices for the right adjoint $\text{ev}^R(-)$ to preserve colimits.

To conclude this section, we combine the obtained results to provide an explicit criterion for a ∞ -category to be smooth and proper within a six-functor formalism.

Proposition 2.1.46. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34, and let $f: Y \rightarrow X$ be a $!$ -able map in $\mathcal{C}$. Then the ∞ -category $\mathcal{D}(Y)$ is internally smooth and proper over $\mathcal{D}(X)$ if f is both $\mathcal{D}$ -suave and $\mathcal{D}$ -prim.*

Proof. By [HM24, Lemma 4.5.9 (i)], if f is $\mathcal{D}$ -suave (respectively, $\mathcal{D}$ -prim), then Δ_f is $\mathcal{D}$ -suave (respectively, $\mathcal{D}$ -prim). The statement then follows from Corollary 2.1.40 and Corollary 2.1.44. $\square$

We now describe a criterion for obtaining $\mathcal{D}$ -prim covers, which will be used in Section 2.5.2. We refer the reader to [HM24] for the standard definition of the $\mathcal{D}$ -topology (also called the $!$ -topology) and for the notion of a $\mathcal{D}$ -cover. In particular, a map $W \rightarrow X$ is a $\mathcal{D}$ -cover if the functor $\mathcal{D}(-)$ satisfies universal $*$ - and $!$ -descent along $W \rightarrow X$.

From now on, we assume the following:

Assumptions 2.1.47. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Setup 2.1.34. Assume, moreover, that $\mathcal{C}$ is the ∞ -category of sheaves of $\mathbf{Ani}$ on an ∞ -site $\mathcal{C}_0$, and that the six-functor formalism $\mathcal{D}$ arises as the extension of a stable, presentable six-functor formalism on $\mathcal{C}_0$, as explained in [HM24].*

Proposition 2.1.48. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Assumptions 2.1.47, and let X be an object in $\mathcal{C}$ such that:*

- (1) *There exists an affine object W and an effective epimorphism $g: W \rightarrow X$ along which $\mathcal{D}(-)$ satisfies $*$ -descent, and there exists an integer n such that the Čech nerve of g is an n -truncated diagram (i.e., the canonical map*

$$X \simeq \operatorname{colim}_{[m] \in \Delta_{\leq n}} (W \rightrightarrows W \times_X W \rightrightarrows W \times_X W \times_X W \rightrightarrows \cdots)$$

is an equivalence).

- (2) *The projection maps $W \times_X W \rightarrow W$ are $\mathcal{D}$ -prim.*

Then $g: W \rightarrow X$ is a descendable $\mathcal{D}$ -cover (i.e., $g_\mathbb{1}$ is descendable). Moreover, g is $\mathcal{D}$ -prim.*

Proof. We begin by observing that since being prim is local on the target [HM24, Lemma 4.5.7], it follows from hypothesis (2) that g is $\mathcal{D}$ -prim.

First, we prove that g is a descendable map. We need to verify that $g: W \rightarrow X$ is such that $g_*\mathbb{1} \in \mathcal{D}(X)$ is descendable (i.e., $\mathbb{1}_X$ belongs to the thick tensor ideal $\langle g_*\mathbb{1}_W \rangle$).

Observe that by denoting the n -fold fiber product $W \times_X W \times_X \cdots \times_X W$ by W_n and considering the maps

$$g_n: W_n \rightarrow W \rightarrow X,$$

we obtain

$$\mathbb{1}_X \simeq \lim_{[m] \in \Delta_{\leq n}} g_{m,*} \mathbb{1}_{W_m}. \quad (2.10)$$

We now show that $g_{m,*} \mathbb{1}_{W_m} \simeq g_* \mathbb{1}_W \otimes_X \cdots \otimes_X g_* \mathbb{1}_W$. This follows by induction, starting with the base case $m = 2$. We consider the following diagram:

$$\begin{array}{ccc} & W \times_X W & \\ pr_2 \swarrow & & \searrow pr_1 \\ W & & W \\ g \searrow & & \swarrow g \\ & X & \end{array}$$

Since g is prim, we have the following equivalences:

$$\begin{aligned} g_{2,*} \mathbb{1}_{W_2} &\simeq g_* pr_{2,*} \mathbb{1}_{W \times_X W} \\ &\simeq g_* (pr_{2,*} pr_1^* \mathbb{1}_W) \\ &\simeq g_* (\mathbb{1}_W \otimes pr_{2,*} pr_1^* \mathbb{1}_W) \\ &\simeq g_* (\mathbb{1}_W \otimes g^* g_* \mathbb{1}_W) \\ &\simeq g_* \mathbb{1}_W \otimes g_* \mathbb{1}_W. \end{aligned}$$

Here, the fourth equivalence follows from the base change property, and the last one from the projection formula. This computation implies that the finite limit in Equation (2.10) belongs to the thick tensor ideal $\langle g_* \mathbb{1}_W \rangle$. Now, from [HM24, Lemma 4.7.4], we deduce that g is a $\mathcal{D}$ -cover. $\square$

Corollary 2.1.49. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Assumptions 2.1.47, and let $f: X \rightarrow Y$ be a map in $\mathcal{C}$. Assume the following conditions:*

- (1) *The induced map $W \rightarrow Y$ is $\mathcal{D}$ -prim.*
- (2) *There exists an affine object W and an effective epimorphism $g: W \rightarrow X$ along which $\mathcal{D}(-)$ satisfies $*$ -descent, and there exists an integer n such that the Čech nerve of g is an n -truncated diagram (i.e., the canonical map*

$$X \simeq \operatorname{colim}_{[m] \in \Delta_{\leq n}} (W \rightrightarrows W \times_X W \rightrightarrows W \times_X W \times_X W \rightrightarrows \cdots)$$

is an equivalence).

- (3) *The projection maps $W \times_X W \rightarrow W$ are $\mathcal{D}$ -prim.*

Then f is $\mathcal{D}$ -prim.

Proof. The condition that $f: X \rightarrow Y$ is $\mathcal{D}$ -prim can be checked locally on Y (see [HM24, Lemma 4.5.7]); in particular, we may assume Y is affine. Moreover, by Proposition 2.1.48 and [HM24, Corollary 4.7.5], it suffices to show that the composite map $W \rightarrow Y$ is $\mathcal{D}$ -prim. This follows directly from condition (1). $\square$

Remark 2.1.50. We observe that condition (1) in Corollary 2.1.49 is implied by the following condition:

- (1') For every affine object T equipped with a map $T \rightarrow Y$, the pullback map $W \times_Y T \rightarrow T$ is $\mathcal{D}$ -prim, and Y is semi-separated (i.e., it has an affine diagonal).

We note that although the hypotheses of Corollary 2.1.49, especially condition (2), appear stringent, they are satisfied in

2.2 Some Remarks on Condensed Mathematics

2.2.1 Analytic rings

In this subsection, we review the definition of analytic rings as presented in [CS24], with a primary focus on those arising from rigid geometry. Throughout this section, we work over a complete non-archimedean base field k of characteristic 0, with a pseudo-uniformizer ω and ring of integers $\mathbb{O}_k$.

The main objective of this section is to introduce the notation used in subsequent parts of this paper, such as the definition of *animated affinoid k -algebras* as given in [Mik23], and to describe their tensor product (see Proposition 2.2.9). We begin by introducing some basic notions related to condensed mathematics and analytic stacks, as explained in [CS24].

We denote by $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cond}}$ the unbounded derived ∞ -category of condensed abelian groups. We say that $\mathcal{A}$ is a *condensed $\mathbb{E}_\infty$ -algebra* if $\mathcal{A}$ is an $\mathbb{E}_\infty$ -algebra in $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cond}}$. In this case, we denote by $\mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}}$ the ∞ -category of $\mathcal{A}$ -modules in $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cond}}$.

Definition 2.2.1. An analytic ring is a pair $A = (\mathcal{A}, \mathcal{D}(A))$, where $\mathcal{A}$ is a condensed $\mathbb{E}_\infty$ -algebra and $\mathcal{D}(A)$ is a full subcategory of $\mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}}$ satisfying the following properties:

1. The ∞ -category $\mathcal{D}(A)$ contains the object $\mathcal{A}$.
2. Let $\iota_A: \mathcal{D}(A) \hookrightarrow \mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}}$ denote the natural inclusion functor. Then ι_A commutes with all limits and colimits, and it admits a left adjoint (called the *completion functor along A*), which we denote by $- \otimes_{\mathcal{A}} A$. The composition $\iota_A \circ (- \otimes_{\mathcal{A}} A)$ preserves connective objects.

3. For every object M in $\mathcal{D}(A)$ and every N in $\text{Mod}_{\mathbb{Z}}^{\text{cond}}$, the internal mapping object $\underline{\text{Hom}}_{\mathbb{Z}}(N, M)$ lies in $\mathcal{D}(A)$.

Remark 2.2.2. [MP25, Remark 2.2.3] By definition, the essential image of ι_A is a localization of $\text{Mod}_{\mathcal{A}}^{\text{cond}}$. In particular, since $\text{Mod}_{\mathcal{A}}^{\text{cond}}$ is presentable and stable, it follows that $\mathcal{D}(A)$ is presentable and stable as well. Moreover, $\mathcal{D}(A)$ admits a symmetric monoidal structure obtained by completing the natural tensor product of $\text{Mod}_{\mathcal{A}}^{\text{cond}}$ along A .

In this paper, we are mainly interested in analytic rings arising from rigid analytic geometry and affinoid algebras (i.e., quotients of Tate algebras by a finitely generated ideal). Recall that to every complete Huber pair (R, R^+) , we can associate an analytic ring

$$(R, R^+)_{\blacksquare} := (R^{\flat}, \mathcal{D}(R)_{\blacksquare})$$

defined as follows.

We define $R^{\flat}$ to be the condensed ring associated with R via its given topology. We then consider the following construction, as explained in [And21]. For $f \in R^+$ and $M \in \text{Mod}_R^{\text{cond}}$, we say that M is *f-solid* if

$$1 - f\sigma: \underline{\text{Hom}}_{\mathbb{Z}}(\text{Null}_{\mathbb{Z}}, M) \longrightarrow \underline{\text{Hom}}_{\mathbb{Z}}(\text{Null}_{\mathbb{Z}}, M)$$

is an equivalence. The inclusion of the full sub- ∞ -category of *f-solid* R -modules admits a left adjoint $(-)^{f\blacksquare}$, called *f-solidification*. We define

$$\mathcal{D}(R)_{\blacksquare} \subseteq \text{Mod}_R^{\text{cond}}$$

to be the full sub- ∞ -category of condensed R -modules that are *f-solid* for all $f \in R^+$. This construction defines an analytic ring, as shown in [And21, Theorem 3.28]. Furthermore, using [And21, Theorem 3.33], we see that this construction determines a fully faithful embedding from the category of complete Huber pairs into the category of analytic rings. In particular, when the Huber pair arises from an affinoid algebra A , we denote the associated analytic ring by

$$A_{\blacksquare} := (A, A^{\circ})_{\blacksquare} := (A^{\flat}, \mathcal{D}(A)_{\blacksquare}).$$

The objects in $\mathcal{D}(A)_{\blacksquare}$ are called *A° -solid modules*. We denote the symmetric monoidal structure of $\mathcal{D}(A)_{\blacksquare}$ by $- \otimes_A^{\blacksquare} -$.

Remark 2.2.3. We note that, by [Mik23, Proposition 2.3], Tate algebras over k and, more generally, affinoid k -algebras lie in $\mathcal{D}(k)_{\blacksquare}$.

The following is an analogue of [And21, Definition 5.8].

Definition 2.2.4. We define the *k-solidification functor* to be the functor

$$\underline{(-)}: \text{Mod}_k \rightarrow \mathcal{D}(k)_{\blacksquare}$$

determined by mapping k to $k^{\triangleright}$ (endowed with its non-archimedean topology) and requiring that it commutes with colimits. The objects in the image of this functor are called *discrete k-solid modules*.

Note that, although the term “discrete” appears in the nomenclature, the objects in the image of the functor defined in Definition 2.2.4 do not necessarily carry the discrete topology. Furthermore, this functor is by definition a left adjoint; we denote its right adjoint by

$$(-)(*) : \mathcal{D}(k)_{\blacksquare} \rightarrow \text{Mod}_k.$$

Similarly, for every affinoid k -algebra A , we can define an adjunction

$$\text{Mod}_{A^{\triangleright}(*)} \begin{array}{c} \xleftarrow{(-)(*)} \\ \xrightarrow[\underline{(-)}]{\top} \end{array} \mathcal{D}(A)_{\blacksquare}$$

The objects in the image of $\underline{(-)}$ are referred to as *discrete A-solid modules*.

Lemma 2.2.5. Let $\underline{A}$ and $\underline{B}$ be two discrete k -solid modules. Then the tensor product $\underline{A} \otimes_k^{\blacksquare} \underline{B}$ is equivalent to the discrete k -solid module associated with the k -linear tensor product of A and B .

Proof. It suffices to observe that every object in Mod_k can be written as a colimit of copies of k . Since the k -solidification functor commutes with colimits and sends k to $k^{\triangleright}$ in $\mathcal{D}(k)_{\blacksquare}$, we deduce that

$$\underline{A} \otimes_k^{\blacksquare} \underline{B} \simeq \underline{A \otimes_k B}.$$

□

2.2.2 Animated affinoid algebras

In this section, we review the definition of condensed derived affinoid k -algebras, following [Mik23]. Throughout this section, we work over the analytic ring $(k, \mathbb{Z})_{\blacksquare}$, as is also done in [Mik23]. However, the essential requirement for the arguments in the subsequent subsection is a tensor product that recovers the classical completed tensor product for affinoid algebras, as explained in Remark 2.2.10. Any other choice of base analytic ring satisfying this property, such as $k_{\blacksquare}$, functions equally well.

In this subsection, we denote by $\otimes_k^{\blacksquare}$ the tensor product in $\mathcal{D}(k, \mathbb{Z})_{\blacksquare}$.

Definition 2.2.6. Let $\mathcal{C}$ be a stable closed symmetric monoidal ∞ -category. An object $X \in \mathcal{C}$ is called *nuclear* if, for every compact object $P \in \mathcal{C}$, the canonical map

$$\mathrm{Map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \underline{\mathrm{Hom}}_{\mathcal{C}}(P, \mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{C}} X) \longrightarrow \mathrm{Map}_{\mathcal{C}}(P, X)$$

is an equivalence of anima. The full subcategory of $\mathcal{C}$ spanned by nuclear objects is denoted by $\mathrm{Nuc}(\mathcal{C}) \subseteq \mathcal{C}$.

Let A be an analytic ring and let $\mathcal{D}(A)$ be its associated ∞ -category. The ∞ -category $\mathrm{Nuc}(A)$ is a presentable, stable, full sub- ∞ -category of $\mathcal{D}(A)$ that is closed under colimits and tensor products [CS26, Theorem 8.6]. This closure property equips $\mathrm{Nuc}(A)$ with a symmetric monoidal structure inherited from $\mathcal{D}(A)$.

Definition 2.2.7. [Mik23, Definition 2.11] An *animated affinoid k -algebra* A is a condensed animated k -algebra such that:

1. A is a $\mathbb{Z}$ -solid k -algebra and, moreover, A is nuclear over $(k, \mathbb{Z})_{\blacksquare}$.
2. $\pi_0(A)$ is an underived condensed affinoid algebra over k (i.e., there exists a classical affinoid k -algebra B such that, if we denote by $\underline{B}$ the algebra B regarded as a condensed k -module, then there is an equivalence $\pi_0(A) \simeq \underline{B}$).

Remark 2.2.8. We observe that the first condition of Definition 2.2.7 is equivalent to requiring A to be nuclear over $k_{\blacksquare}$. Indeed, by [Mik23, Corollary 2.7], an animated affinoid k -algebra is nuclear over $k_{\blacksquare}$ if and only if it is nuclear over $(k, \mathbb{Z})_{\blacksquare}$.

We denote by $\mathrm{dAfdAlg}_k$ the ∞ -category of animated affinoid k -algebras, regarded as a full subcategory of $\mathcal{D}(k, \mathbb{Z})_{\blacksquare}$.

Proposition 2.2.9. *The ∞ -category $\mathrm{dAfdAlg}_k$ is stable under finite products and tensor products.*

Proof. The fact that the tensor product of animated affinoid k -algebras yields an animated affinoid k -algebra is proven in [Mik23, Proposition 2.14]. To verify that the ∞ -category $\mathrm{dAfdAlg}_k$ is stable under finite products, we observe that $\pi_0(-)$ commutes with products, and that nuclear categories are stable under finite limits. $\square$

Remark 2.2.10. As explained in the proof of [Mik23, Proposition 2.14], the tensor product of derived affinoid k -algebras recovers the completed tensor product of classical affinoid algebras. In particular, we have

$$\pi_0(A \otimes_B^{\blacksquare} C) \simeq \pi_0 A \hat{\otimes}_{\pi_0 B} \pi_0 C.$$

Motivated by this property, we denote the tensor product in $\mathrm{dAfdAlg}_k$ by $-\widehat{\otimes}-$. In particular, it can be regarded as a tensor product between animated affinoid algebras in $\mathrm{Nuc}(k)$.

2.2.3 Nuclear categories for animated affinoid algebras

By [And23, Kollar 3.18], we deduce an equivalence between $\mathrm{Nuc}(\mathcal{D}(k)_{\blacksquare})$ and $\mathrm{Nuc}(\mathcal{D}(k, \mathbb{Z})_{\blacksquare})$. For this reason, hereafter we will simply write $\mathrm{Nuc}(k)$.

For animated affinoid k -algebras in $\mathrm{Nuc}(k)$, we consider the tensor product described above, denoted by $-\widehat{\otimes}-$.

If A is an animated affinoid k -algebra, by [Sch26a, Proposition 12.21] and using the analytic ring structure on $\pi_0(A)$ (as described in [And21, Theorem 3.28]), we can associate to it an analytic ring

$$A_{\blacksquare} := (A, \mathcal{D}(A)_{\blacksquare}). \quad (2.11)$$

In particular, an object $M \in \mathrm{Mod}_A^{\mathrm{cond}}$ lies in $\mathcal{D}(A)_{\blacksquare}$ if and only if $\pi_i(M)$ belongs to $\mathcal{D}(\pi_0(A))_{\blacksquare}$ for every integer i .

In this way, as explained in [Mik23, Proposition 2.18], we obtain a fully faithful embedding:

$$\begin{array}{ccc} \mathrm{dAfdAlg}_k & \longrightarrow & \mathrm{AnRing}_{(k, \mathbb{Z})_{\blacksquare}} \\ A & \longmapsto & A_{\blacksquare}. \end{array}$$

For an animated affinoid k -algebra A , we will hereafter denote by $\mathrm{Nuc}(A)$ the ∞ -category $\mathrm{Nuc}(\mathcal{D}(A)_{\blacksquare})$ of nuclear objects in $\mathcal{D}(A)_{\blacksquare}$. The following proposition provides an explicit description of this ∞ -category.

Proposition 2.2.11. *Let A be an animated affinoid k -algebra. Then there is an equivalence*

$$\mathrm{Nuc}(A) \simeq \mathrm{Mod}_A(\mathrm{Nuc}(k)).$$

In particular, $\mathrm{Nuc}(A)$ is rigid.

Proof. This is a special case of [Wan26, Lemma 4.4.7]. We apply the cited lemma to the canonical map of analytic rings

$$(k, \mathbb{Z})_{\blacksquare} \rightarrow A_{\blacksquare}.$$

Indeed, by Definition 2.2.7 and Remark 2.2.8, A belongs to $\mathrm{Nuc}(k)$. Moreover, for every profinite set S , the condensed k -module of continuous functions $\mathcal{C}(S, k)$ lies in $\mathrm{Nuc}(k)$ by [And23, Kollar 3.14]. $\square$

From the above proposition, we immediately deduce the following corollary.

Corollary 2.2.12. *If $A \rightarrow B$ is a map of animated affinoid k -algebras, then B is nuclear over A .*

Proof. Since B can be viewed as an A -module, the statement reduces, by Proposition 2.2.11, to showing that B is nuclear over k . This follows directly from Remark 2.2.8. $\square$

Corollary 2.2.13. *Let A , B , and C be animated affinoid k -algebras. Then there is an equivalence*

$$\mathrm{Nuc}(A) \otimes_{\mathrm{Nuc}(B)} \mathrm{Nuc}(C) \simeq \mathrm{Nuc}(A \widehat{\otimes}_B C).$$

Proof. By Proposition 2.2.9, the tensor product $A \widehat{\otimes}_B C$ is an animated affinoid algebra. The statement then easily follows from the explicit description of nuclear ∞ -categories provided in Proposition 2.2.11. $\square$

Lemma 2.2.14. *Let $f: A \rightarrow B$ be a map of animated affinoid k -algebras. Then the pullback functor*

$$f^*: \mathcal{D}(A)_{\blacksquare} \rightarrow \mathcal{D}(B)_{\blacksquare}$$

preserves nuclear modules. Moreover, under the equivalence of Proposition 2.2.11, the induced map

$$f^*: \mathrm{Nuc}(A) \rightarrow \mathrm{Nuc}(B)$$

can be identified with the tensor product $-\widehat{\otimes}_A B$.

Proof. From [MW25, Theorem 2.9 (c)], we deduce that the pullback functor f^* restricts to the nuclear subcategories. The latter statement follows from the exact same argument used in [MP25, Proposition 3.2.8], which extends verbatim to our setting. $\square$

2.3 Nuclear categories for derived rigid analytic spaces

In this section, we introduce the notion of derived rigid analytic spaces, which will be employed in the final section. A derived rigid space will be described via the gluing of animated affinoid k -algebras along rational localizations. We will then consider quasi-compact semi-separated derived rigid analytic spaces and their nuclear ∞ -categories proving a Künneth formula (Proposition 2.3.14) and establishing the existence of a six-functor formalism (Theorem 2.3.15).

2.3.1 Rational localization and derived rigid analytic spaces

In this subsection, we define condensed derived rigid analytic varieties as the gluing of animated affinoid k -algebras along rational localizations. A similar approach has been carried out in the bornological setting by Aron Soor in [Soo25].

As explained in [Mik23, Lemma 2.24], let A be a classical affinoid algebra, and let $f_1, \dots, f_n, g$ be elements that generate the unit ideal of A . We can view A as a $\mathbb{Z}[T]$ -algebra via the map sending T to g . Under this $\mathbb{Z}[T]$ -module structure, we have the equivalence:

$$A \otimes_{(\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}} (\mathbb{Z}[T, T^{-1}], \mathbb{Z})_{\blacksquare} \simeq A[g^{-1}].$$

Moreover, we can consider $A[g^{-1}]$ as an algebra over $\mathbb{Z}[U_1, \dots, U_n]$ via the map sending U_i to f_i/g . This yields an isomorphism:

$$(A \otimes_{(\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}} (\mathbb{Z}[T, T^{-1}], \mathbb{Z})_{\blacksquare}) \otimes_{(\mathbb{Z}[U_1, \dots, U_n], \mathbb{Z})_{\blacksquare}} \mathbb{Z}[U_1, \dots, U_n]_{\blacksquare} \simeq A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle.$$

Here, $A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$ denotes the usual rational localization of A .

When A is a derived affinoid k -algebra, we define:

$$A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle := (A \otimes_{(\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}} (\mathbb{Z}[T, T^{-1}], \mathbb{Z})_{\blacksquare}) \otimes_{(\mathbb{Z}[U_1, \dots, U_n], \mathbb{Z})_{\blacksquare}} \mathbb{Z}[U_1, \dots, U_n]_{\blacksquare}. \quad (2.12)$$

This leads to the following proposition:

Proposition 2.3.1. [Mik23, Proposition 2.25] *If A is an animated affinoid k -algebra, then $A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$ is also an animated affinoid k -algebra such that*

$$\pi_0 \left(A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle \right) \simeq (\pi_0 A) \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle.$$

Definition 2.3.2. [Mik23, Definition 2.26] Let A be a derived affinoid k -algebra. A rational localization of A is a map of animated affinoid k -algebras

$$A \rightarrow A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

induced by Equation (2.12).

If A is an animated affinoid k -algebra, we denote by Rat_A the ∞ -category of rational localizations of A .

Remark 2.3.3. By [Mik23, Proposition 2.29], there is an equivalence of categories from Rat_A to $\text{Rat}_{\pi_0(A)}$. This category, as explained above, coincides with the category of usual rational localizations of the affinoid k -algebra $\pi_0 A(*)$.

Definition 2.3.4. We say that a map $A \rightarrow B$ of animated affinoid k -algebras is a *homotopy epimorphism* if the induced map

$$B \widehat{\otimes}_A B \rightarrow B$$

is an equivalence.

Lemma 2.3.5. *The class of rational localizations is stable under pushouts and compositions. Moreover, every rational localization is a homotopy epimorphism.*

Proof. Let $A \rightarrow B$ and $A \rightarrow C$ be maps of derived affinoid k -algebras, with $A \rightarrow B$ a rational localization. To prove that $C \rightarrow B \widehat{\otimes}_A C$ is a rational localization, using [Mik23, Proposition 2.29], it suffices to show that the induced map on π_0 is a classical rational localization.

Using the equivalence

$$\pi_0(B \otimes_A^{\blacksquare} C) \simeq \pi_0(B) \otimes_{\pi_0(A)}^{\blacksquare} \pi_0(C),$$

and the fact that the tensor product can be identified with the completed tensor product, we deduce that the induced map

$$\pi_0(C) \rightarrow \pi_0(B \otimes_A^{\blacksquare} C) \simeq \pi_0(B) \otimes_{\pi_0(A)}^{\blacksquare} \pi_0(C)$$

can be identified with a rational localization.

The fact that a rational localization is a homotopy epimorphism follows from [Mik23, Lemma 2.27] and [Sch26a, Exercise 12.17].

Remark 2.3.6. In particular, this lemma implies that a rational localization is a Zariski morphism in the opposite ∞ -category $\mathrm{dAfdAlg}^{\mathrm{op}}$, as defined in [MP25, Definition 1.3.4]. This property will be utilized later.

To conclude the proof, by invoking [Mik23, Proposition 2.29], we deduce that rational localizations are stable under compositions, since classical rational localizations are stable under compositions. $\square$

We denote by dAfd_k the opposite ∞ -category of $\mathrm{dAfdAlg}_k$, and we denote by $\mathrm{dSp}(A)$ the object in dAfd_k associated with the animated affinoid k -algebra A . We refer to the objects in the ∞ -category dAfd_k as *derived k -affinoid spaces*.

Definition 2.3.7. [Soo25, Definition 4.12] Let $X := \mathrm{dSp}(A)$ be a derived k -affinoid space.

(i) The *small analytic site* of X is the ∞ -site with:

- (a) underlying ∞ -category given by the full subcategory Rat_X of $\mathrm{dAfd}/_X$ spanned by rational subdomains $\mathrm{Sp}(B) \rightarrow \mathrm{Sp}(A)$;
- (b) covering sieves generated by finite families of rational subdomains

$$\{\mathrm{dSp}(B_i) \rightarrow \mathrm{dSp}(B)\}_{i=1}^n$$

such that

$$\{\mathrm{Sp}(\pi_0(B_i)(*)) \rightarrow \mathrm{Sp}(\pi_0(B)(*))\}_{i=1}^n$$

is an analytic cover of the classical rigid space $\mathrm{Sp}(\pi_0(B)(*))$.

(ii) The *big analytic site* on dAfd is the ∞ -site with:

- (a) underlying ∞ -category given by dAfd ;
- (b) covering sieves generated by finite families of rational subdomains

$$\{\mathrm{dSp}(B_i) \rightarrow \mathrm{dSp}(B)\}_{i=1}^n$$

such that

$$\{\mathrm{Sp}(\pi_0(B_i)(*)) \rightarrow \mathrm{Sp}(\pi_0(B)(*))\}_{i=1}^n$$

is an analytic cover of the classical rigid space $\mathrm{Sp}(\pi_0(B)(*))$.

Definition 2.3.8. We define

$$\mathrm{Shv}(\mathrm{dAfd}) := \mathrm{Shv}(\mathrm{dAfd}, \mathrm{Ani}) \quad (2.13)$$

to be the ∞ -category of sheaves on the ∞ -site dAfd equipped with the analytic topology.

Remark 2.3.9. Since every classical affinoid space $\mathrm{Sp}(A)$ is quasi-compact, from Definition 2.3.7 and Remark 2.3.3 we immediately deduce that every analytic cover of $\mathrm{Sp}(A)$ can be refined to a finite cover.

Lemma 2.3.10. *The following properties hold.*

- (i) *The functor $\mathrm{Nuc}: \mathrm{dAfd}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ is a sheaf on dAfd in the analytic topology.*
- (ii) *For every animated affinoid k -algebra A , the functor represented by $\mathrm{dSp}(A)$,*

$$\mathrm{dSp}(A): \mathrm{dAfd}^{\mathrm{op}} \rightarrow \mathrm{Ani},$$

is a sheaf in the analytic topology.

Proof. To prove (i), we observe that by Remark 2.3.9 we can reduce the problem to considering a finite cover

$$\{B \rightarrow A_i\}_{i \in I}$$

of animated affinoid k -algebras. By Proposition 2.2.11 and [Lur18, Lemma D.3.5.5], the functor

$$\mathrm{Nuc}(-) \simeq \mathrm{Mod}_{(-)}(\mathrm{Nuc}(k)): \mathrm{dAfd}^{\mathrm{op}} \rightarrow \mathrm{Pr}_{\mathrm{st}}^L$$

commutes with finite products. Under this observation, using [Mat16, Proposition 3.22], we can reduce the problem to showing that

$$C := \prod_{i \in I} A_i$$

is a descendable algebra in $\text{Nuc}(B)$. First, we observe that by Proposition 2.2.9, C is an animated affinoid k -algebra. The statement then follows from [Mik23, Theorem 4.15], observing that by Corollary 2.2.12, C lies in $\text{Nuc}(B)$.

We now prove that the functor

$$\text{Map}_{\text{dAfdAlg}_k}(A, -) : \text{dAfdAlg}_k \rightarrow \text{Ani}$$

is a sheaf in the analytic topology. Consider an animated affinoid k -algebra B and a cover as described above. As already explained, C is descendable in $\text{Nuc}(B)$; therefore, by [Mat16, Proposition 3.20], we can realize B as the following limit in $\text{Nuc}(B) \simeq \text{Mod}_B(\text{Nuc}(k))$:

$$B \xrightarrow{\simeq} \lim_{[n] \in \Delta} \left(C \rightrightarrows C \otimes_{\text{Nuc}(B)} C \rightrightarrows C \otimes_{\text{Nuc}(B)} C \otimes_{\text{Nuc}(B)} C \rightrightarrows \cdots \right).$$

Now, since $\text{Map}_{\text{dAfdAlg}_k}(A, -)$ commutes with limits, the statement follows from [Lur17, Proposition 4.6.2.17]. $\square$

Definition 2.3.11. [Soo25, Definition 4.19]

- (i) A *derived k -affinoid space* is an object of $\text{Shv}(\text{dAfd})$ that is isomorphic to $\text{dSp}(A)$ for some $A \in \text{dAfdAlg}_k$.
- (ii) Let $X := \text{dSp}(A)$ be a derived k -affinoid space. An *analytic subspace* $U \hookrightarrow X$ is a subsheaf U of X such that:
 - ★ There exists a small collection $\{\text{dSp}(A_i)\}_{i \in I}$ of derived affinoid spaces and an effective epimorphism $\coprod_{i \in I} \text{dSp}(A_i) \rightarrow U$ in $\text{Shv}_{\text{weak}}(\text{dAfd})$ such that each map $\text{dSp}(A_i) \rightarrow \text{dSp}(A)$ is a rational subdomain.
- (iii) Let $X \in \text{Shv}(\text{dAfd})$. An *analytic subspace* $Y \hookrightarrow X$ is a subsheaf such that for every derived affinoid space $\text{dSp}(A) \rightarrow X$ mapping to X , the pullback $Y \times_X \text{dSp}(A) \hookrightarrow \text{dSp}(A)$ is an analytic subspace in the sense of (ii). The morphism $Y \rightarrow X$ is then termed an *open immersion*.
- (iv) A *derived rigid analytic space* is an object $X \in \text{Shv}(\text{dAfd})$ such that there exists a small collection $\{\text{dSp}(A_i)\}_{i \in I}$ of derived affinoid subspaces for which

$\coprod_{i \in I} \mathrm{dSp}(A_i) \rightarrow X$ is an effective epimorphism. We denote the full subcategory of $\mathrm{Shv}(\mathrm{dAfd})$ consisting of derived rigid spaces by dRig .

We remark that the category dRig admits all fiber products and colimits. The terminal object is $\mathrm{Sp}(k)$. Moreover, every classical rigid analytic variety can be viewed as a derived rigid analytic space; we reserve the term *rigid analytic varieties* exclusively for those derived rigid analytic spaces that arise from classical rigid analytic varieties.

Definition 2.3.12. Let $f: X \rightarrow Y$ be a map of derived rigid analytic spaces.

1. We say that f is *affine* if for every derived affinoid map $\mathrm{dSp}(A) \rightarrow Y$, the pullback $X \times_Y \mathrm{dSp}(A)$ is a derived affinoid space.
2. We say that X is *quasi-compact* if every affinoid cover of X can be refined to a finite cover.
3. We say that X is *semi-separated* if the diagonal map

$$X \rightarrow X \times_{\mathrm{dSp}(k)} X$$

is affine.

2.3.2 Nuclear categories over derived rigid analytic spaces

Let X be a derived rigid analytic space over $\mathrm{Sp}(k)$. We define the *nuclear category* of X , via Kan extension using Lemma 2.3.10, as the following limit over a cover of X consisting of derived affinoid spaces:

$$\mathrm{Nuc}(X) := \lim_{\mathrm{dSp}(A_i) \rightarrow X} \mathrm{Nuc}(\mathrm{dSp}(A_i)).$$

In this way, for every map $f: X \rightarrow Y$ of derived rigid analytic varieties, we obtain a pullback functor for nuclear categories

$$f^*: \mathrm{Nuc}(Y) \rightarrow \mathrm{Nuc}(X) \tag{2.14}$$

which extends the one introduced in Lemma 2.2.14.

In particular, given a derived rigid analytic space X , its nuclear category $\mathrm{Nuc}(X)$ is a presentable ∞ -category, tensored over $\mathrm{Nuc}(k)$.

Consequently, given two rigid analytic varieties, we can define the tensor product of their associated nuclear categories as the tensor product in $\mathrm{Mod}_{\mathrm{Nuc}(k)}(\mathrm{Pr}^{\mathrm{L}})$. We will

employ the following notation: let X and Y be two rigid analytic varieties over $\mathrm{Sp}(k)$; we define the tensor product of their nuclear categories, denoted by $-\hat{\otimes}-$, as follows:

$$\mathrm{Nuc}(X) \hat{\otimes} \mathrm{Nuc}(Y) := \mathrm{Nuc}(X) \otimes_{\mathrm{Nuc}(k)} \mathrm{Nuc}(Y),$$

where the term on the right-hand side denotes the tensor product in $\mathrm{Mod}_{\mathrm{Nuc}(k)}(\mathrm{Pr}^{\mathrm{L}})$.

Proposition 2.3.13. *Let X be a quasi-compact and semi-separated derived rigid analytic space. Then $\mathrm{Nuc}(X)$ is a rigid ∞ -category .*

Proof. Observe that our setup satisfies [MP25, Assumption 1.3.10] by taking $\mathcal{C}$ to be the ∞ -category of derived affinoid spaces, and $\hat{\mathcal{C}}$ to be the ∞ -category $\mathrm{Shv}(\mathrm{dAdf})$. Therefore, this is a special case of [MP25, Proposition 1.3.17]. $\square$

Using this tensor product, we obtain a Künneth formula for nuclear ∞ -categories .

Proposition 2.3.14. *Let X and Y be two quasi-compact, semi-separated derived rigid analytic spaces over $\mathrm{Sp}(k)$. Then we have the following equivalence of categories:*

$$\mathrm{Nuc}(X) \hat{\otimes} \mathrm{Nuc}(Y) \simeq \mathrm{Nuc}(X \times Y).$$

Proof. Consider an affinoid cover $\{\mathrm{dSp}(A_i)\}_i$ of the derived rigid analytic space X . We first prove that

$$\mathrm{Nuc}(X \times \mathrm{dSp}(B)) \simeq \mathrm{Nuc}(X) \hat{\otimes} \mathrm{Nuc}(\mathrm{dSp}(B)). \quad (2.15)$$

By descent, we can express $\mathrm{Nuc}(X)$ as $\lim_i \mathrm{Nuc}(\mathrm{dSp}(A_i))$. Since $\mathrm{Nuc}(\mathrm{dSp}(B))$ is rigid, it is dualizable as a $\mathrm{Nuc}(k)$ -linear ∞ -category ; hence, the functor $-\hat{\otimes} \mathrm{Nuc}(\mathrm{dSp}(B))$ commutes with limits. In particular, we obtain

$$\lim_i (\mathrm{Nuc}(\mathrm{dSp}(A_i)) \hat{\otimes} \mathrm{Nuc}(\mathrm{dSp}(B))) \simeq (\lim_i \mathrm{Nuc}(\mathrm{dSp}(A_i))) \hat{\otimes} \mathrm{Nuc}(\mathrm{dSp}(B)).$$

Since $\{\mathrm{dSp}(A_i) \times \mathrm{dSp}(B)\}_i$ constitutes a cover of $X \times \mathrm{dSp}(B)$, we can again use descent for nuclear categories and the completed tensor product to identify $\lim_i (\mathrm{Nuc}(\mathrm{dSp}(A_i)) \hat{\otimes} \mathrm{Nuc}(\mathrm{dSp}(B)))$ with $\mathrm{Nuc}(X \times \mathrm{dSp}(B))$.

To conclude, we observe that the same argument applies using an affinoid cover of Y instead of $\mathrm{dSp}(B)$, utilizing the fact that $\mathrm{Nuc}(X)$ is also rigid. $\square$

2.3.3 six-functor formalism for nuclear categories over rigid analytic spaces

As explained in Equation (2.14), to every map of derived rigid analytic spaces $f: Y \rightarrow X$, we can associate a pullback functor at the level of nuclear ∞ -categories :

$$f^*: \text{Nuc}(X) \rightarrow \text{Nuc}(Y).$$

This functor is a left adjoint, and we denote its right adjoint by f_* . When X and Y are derived affinoid spaces, f_* can be identified with the forgetful functor.

Theorem 2.3.15. *The assignment that sends a derived rigid analytic space X over $\text{Sp}(k)$ to its ∞ -category $\text{Nuc}(X)$ of nuclear sheaves can be extended to a 6-functor formalism with values in stable, presentable ∞ -categories :*

$$\text{Nuc}(-): \text{Corr}(\text{dRig}_k)_{\text{qcqs,all}} \rightarrow \text{Pr}_{\text{st}}^{\text{L}},$$

where, for every map of quasi-compact and semi-separated derived rigid analytic spaces, we define the lower shriek functor $f_!$ to be the pushforward f_* .

In order to prove this theorem, we require the following lemma.

Lemma 2.3.16. *Let $\alpha: Y \rightarrow X$ be a map of quasi-compact, semi-separated derived rigid analytic spaces. Then the following properties hold:*

- 1 (Base change) *Let $\beta: V \rightarrow X$ be a morphism of quasi-compact and semi-separated derived rigid analytic spaces, and consider the diagram*

$$\begin{array}{ccc} Y \times_X V & \xrightarrow{\alpha'} & V \\ \beta' \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\alpha} & X. \end{array}$$

Then, for every object M in $\text{Nuc}(Y)$, the canonical map

$$\beta^* \alpha_* M \longrightarrow \alpha'_* \beta'^* M \tag{2.16}$$

is an equivalence.

- 2 (Projection formula) *For every N in $\text{Nuc}(Y)$ and M in $\text{Nuc}(X)$, the natural map*

$$\alpha_*(N \otimes_{\text{Nuc}(Y)} \alpha^* M) \longrightarrow \alpha_* N \otimes_{\text{Nuc}(X)} M \tag{2.17}$$

is an equivalence.

3 The functor

$$\alpha_* : \mathrm{Nuc}(Y) \rightarrow \mathrm{Nuc}(X)$$

commutes with colimits; in particular, it admits a right adjoint.

Proof. We begin the proof by observing that our setup satisfies [MP25, Assumption 1.3.10], taking $\mathcal{C}$ to be the ∞ -category of derived affinoid spaces and $\widehat{\mathcal{C}}$ to be the ∞ -category $\mathrm{Shv}(\mathrm{dAdf})$.

To prove (1), suppose V is a derived rigid analytic variety. We can use a derived affinoid cover $\coprod_{i \in I} \mathrm{dSp}(A_i)$ of V and Lemma 2.3.10 to verify Equation (2.16) after restriction to each derived affinoid space $\mathrm{dSp}(A_i)$. In this way, we can reduce the verification of Equation (2.16) to the case where V is a derived affinoid space. In this case, the result follows directly from Proposition 3.3.27.(1).

We observe that (2) is a special case of Proposition 3.3.27.(5), and (3) follows from Proposition 3.3.27.(2). $\square$

Proof of Theorem 2.3.15. This follows from Lemma 2.3.16 and [MAN22, Proposition A.5.10], taking I to be the class of equivalences and P to be the class of maps between quasi-compact and semi-separated derived rigid analytic spaces. $\square$

Remark 2.3.17. We observe that in the 6-functor formalism of Theorem 2.3.15, every $!$ -able map (i.e., every map between quasi-compact and semi-separated derived rigid analytic spaces) is prim. This follows from the fact that for all such maps, the lower star and lower shriek functors coincide, by [HM24, Lemma 4.5.5].

2.4 Smooth categories and compact generation in rigid analytic geometry

In this section, we study the compact generation of the ∞ -category of nuclear modules over rigid analytic varieties. In particular, we will show that this ∞ -category is compactly (equivalently, atomically) generated if and only if the rigid analytic variety is algebraizable (see Theorem 2.4.9). We then use this result to falsify Conjecture 0.2.9. Two important steps in the proof are a version of the GAGA theorem, as described in [Wan26] in the setting of nuclear ∞ -categories, and the comparison between the geometric notion of smooth rigid varieties and the categorical definition of smoothness (see Theorem 2.4.1).

2.4.1 Non-commutative smoothness and classical smoothness

We begin by comparing the classical notion of smoothness in rigid analytic geometry with the notion of a smooth ∞ -category introduced in Section 2.1. In particular, we will prove the following theorem:

Theorem 2.4.1. *Let $f: X \rightarrow \mathrm{Sp}(k)$ be a (classical) rigid analytic variety over k . Then X is smooth if and only if the ∞ -category $\mathrm{Nuc}(X)$ is internally smooth in $\mathrm{Pr}_{\mathrm{Nuc}(k)}^{\mathrm{L}}$.*

We observe that an analogous theorem has been proved for schemes in [Orl16] and in [Lun10] for the category of perfect complexes. However, the generalization to rigid analytic varieties is not straightforward, and we cannot reproduce the same proof as in [Orl16]. Indeed, the analogue of Theorem 2.4.1 does not hold for the ∞ -category of perfect complexes in rigid analytic geometry. It has been shown in [Mon25] that the notion of smoothness for the ∞ -category of perfect complexes over a rigid analytic variety is intrinsically algebraic and does not reflect the smoothness of the variety itself. To obtain a correspondence between smooth “categories of sheaves” and smooth varieties, it is necessary to use a “completed tensor product of ∞ -categories” with the property that, for a variety X ,

$$\mathrm{Shv}(X \times X) \simeq \mathrm{Shv}(X) \widehat{\otimes} \mathrm{Shv}(X).$$

This can be achieved, for example, in the framework of analytic geometry developed by Clausen and Scholze [CS24], or in the setting developed by Ben-Bassat, Kelly, and Kremnizer [BKK24]. Here, we have chosen to explore these ideas within the setting of analytic stacks as in [CS24], focusing on the sheaf of nuclear ∞ -categories. However, we believe that it would not be difficult to adapt some of the ideas presented here to the setting of [BKK24], utilizing the 6-functor formalism described in [Soo25], or to other “analytic sheaf theories,” such as the $\mathcal{D}$ -solid sheaves described in [Cam24].

We will prove Theorem 2.4.1 at the end of this section as a corollary of Proposition 2.4.3.

Definition 2.4.2. An affinoid algebra A over k is called *smooth* if, for every finite field extension K of k , the tensor product $A \otimes_k K$ is a regular ring (i.e., for every maximal ideal $\mathfrak{m}$ in $A \otimes_k K$, the local ring $(A \otimes_k K)_{\mathfrak{m}}$ is regular).

We now relate this classical notion of smoothness to the notion of a smooth algebra in $\mathrm{Nuc}(k)$, as defined in Definition 2.1.13.

Proposition 2.4.3. *Let A be a (classical) affinoid algebra over k , and let $A^{\triangleright}$ be the associated condensed ring. Then $A^{\triangleright}$ is a smooth algebra in $\mathrm{Nuc}(k)$ in the sense of Definition 2.1.13 if and only if A is smooth over k in the sense of Definition 2.4.2.*

Proof. We begin by assuming that $A^\triangleright$ is smooth in $\text{Nuc}(k)$. To prove that A is smooth, after replacing k with a finite field extension K and A with its base change to K , it suffices to show that for every maximal ideal $\mathfrak{m}$ in A , the localization $A_{\mathfrak{m}}$ is regular. By Serre's criterion [Lur18, Lemma 11.3.3.3], it is enough to show that the residue field $\tilde{K} := A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$ is a perfect complex when regarded as a module over $A_{\mathfrak{m}}$. First, we observe that the following diagram is a pullback of derived affinoid spaces (where all tensor products are taken relative to k):

$$\begin{array}{ccc} \text{Sp}(\tilde{K}) & \xrightarrow{x} & \text{Sp}(A) \\ \downarrow x & & \downarrow \Delta \\ \text{Sp}(A \otimes \tilde{K}) & \xrightarrow{\text{id} \times x} & \text{Sp}(A \hat{\otimes} A). \end{array} \quad (2.18)$$

This follows from the following standard computation:

$$\begin{aligned} A \hat{\otimes}_{A \hat{\otimes} A} (A \hat{\otimes} \tilde{K}) &\simeq A \hat{\otimes}_{A \hat{\otimes} A} (A \hat{\otimes} A \hat{\otimes}_A \tilde{K}) \\ &\simeq (A \hat{\otimes}_{A \hat{\otimes} A} A \hat{\otimes} A) \hat{\otimes}_A \tilde{K} \\ &\simeq A \hat{\otimes}_A \tilde{K} \simeq \tilde{K}. \end{aligned}$$

Note that since $\tilde{K}$ is a finite field extension, the tensor product $A \otimes_k \tilde{K}$ does not need to be completed. Since $A^\triangleright$ is smooth, we have that $\Delta_* A^\triangleright$ is dualizable in $\text{Nuc}(A \hat{\otimes} A)$.

Using the base change property of Lemma 2.3.16, we see that $\tilde{K}$, viewed as a module over $A \otimes_k \tilde{K}$, can be written as

$$x_* \tilde{K}^\triangleright \simeq x_* x^* (A \otimes_k \tilde{K})^\triangleright \simeq (\text{id} \times x)^* \Delta_* A^\triangleright.$$

In particular, since $\Delta_* A^\triangleright$ is dualizable over $\text{Nuc}(A \hat{\otimes} A)$, it follows that $\tilde{K}^\triangleright$ is also dualizable over $\text{Nuc}(A \otimes_k \tilde{K})$. By invoking [And21, Theorem 5.50] and [And21, Lemma 5.47], we deduce that $\tilde{K}$ is dualizable over $\text{Mod}_{A \otimes_k \tilde{K}}$. Since dualizable objects in $\text{Mod}_{A \otimes_k \tilde{K}}$ are perfect complexes and $\tilde{K}$ is a finite field extension of k , we conclude that $\tilde{K}$ is dualizable in Mod_A . To conclude, we observe that

$$\tilde{K} \simeq \tilde{K} \otimes_A A_{\mathfrak{m}}.$$

This implies, in particular, that $\tilde{K}$ is dualizable as a module over $A_{\mathfrak{m}}$.

To conclude the proof, we assume that A is smooth, and we need to prove that $A^\triangleright$ is dualizable as a diagonal module in $\text{Nuc}(A \hat{\otimes}_k A)$. By [And21, Corollary 5.51.1],

it is enough to show that A is dualizable in $\mathrm{Mod}_{A \hat{\otimes}_k A}$. The diagonal map is a closed immersion and is, moreover, a local complete intersection (lci) by [Zav25, Corollary 5.11]. Thus, we can locally assume it to be a regular immersion; using the Koszul resolution, this implies that $\Delta_* A$ is dualizable. $\square$

Corollary 2.4.4. *Let X be a rigid analytic variety over $\mathrm{Sp}(k)$. Then X is smooth if and only if $\Delta_* \mathcal{O}_X$ is a compact object in $\mathrm{Nuc}(X \hat{\times} X)$.*

Proof. Both the smoothness of X and the compactness of $\Delta_* \mathcal{O}_X$ are local properties on X in the analytic topology (see [And21, Theorem 5.41 and Theorem 5.42]). Consequently, we can reduce the proof to the affine case treated in Proposition 2.4.3. $\square$

We conclude this subsection by proving its main result.

Proof of Theorem 2.4.1. Since in our 6-functor formalism $\Delta_! := \Delta_*$, and the ∞ -categories considered here are rigid, we can deduce from Lemma 2.1.42 that $\mathrm{Nuc}(X)$ is internally smooth over $\mathrm{Nuc}(k)$ if and only if $\Delta_*(\mathcal{O}_X)$ is compact. This reduces the statement to Corollary 2.4.4. $\square$

2.4.2 GAGA for nuclear categories in rigid analytic geometry

In this subsection, we will describe a GAGA theorem for nuclear ∞ -categories, utilizing some ideas presented in [Wan26]. In *loc. cit.*, the author introduces two analytic stacks associated with an algebraic stack X , called respectively the *algebraic realization* $X^{k\text{-alg}}$ and the *analytification* X^{an} . When X is a proper scheme, the author proves a GAGA-type theorem relating the solid derived ∞ -categories of $X^{k\text{-alg}}$ and X^{an} (see [Wan26, Theorem 4.1.1]).

We begin by reviewing the *algebraic realization* construction as introduced in [Wan26]; we refer to *loc. cit.* for a comprehensive treatment. We will not review the *analytification* construction described in [Wan26] here. Indeed, we will only consider the case where X is an underived scheme of finite type. In this scenario, the analytification functor described in [Wan26] produces exactly the analytic stack associated to the rigid analytic variety X^{an} (see [Wan26, Lemma 3.2.9 and Lemma 3.2.12]). For this reason, by a slight abuse of notation, we will still denote this analytic stack by X^{an} .

If R is an animated ring of finite type over k , we can associate to it an analytic ring $R^{\mathrm{triv}} := (\underline{R}, \mathrm{Mod}_{\underline{R}}^{\mathrm{cond}})$, where $\underline{R}$ is the discrete condensed ring associated with R . We can also define the *algebraic realization* of R , following [Wan26, Definition 3.2.1], to be the

following pushout of analytic rings:

$$\begin{array}{ccc} k^{\text{triv}} & \longrightarrow & R^{\text{triv}} \\ \downarrow & \lrcorner & \downarrow \\ k_{\blacksquare} & \longrightarrow & R^{k\text{-alg}} \end{array}$$

In particular, using [Sch26a, Proposition 12.12], we can identify $R^{k\text{-alg}}$ with the analytic ring given by $(\underline{R}, \mathcal{D}(R^{k\text{-alg}}))$, where $\underline{R}$ is the condensed derived ring defined in Definition 2.2.4, and we have an equivalence:

$$\mathcal{D}(R^{k\text{-alg}}) \simeq \text{Mod}_{\underline{R}}(\mathcal{D}(k)_{\blacksquare}).$$

Moreover, since $\underline{R}$ is discrete relative to k (and in particular nuclear over k), and $R^{k\text{-alg}}$ carries the induced analytic ring structure from $k_{\blacksquare}$, we obtain the following equivalence via [Mik23, Lemma 2.9]:

$$\text{Nuc}(R^{k\text{-alg}}) \simeq \text{Mod}_{\underline{R}}(\text{Nuc}(k)).$$

Similarly, if X is a derived scheme of finite type, we can regard it as an analytic stack X^{triv} , obtained by gluing analytic rings of the form R^{triv} in the Zariski topology. Note that since we are considering the trivial analytic ring structure on R^{triv} , all transition maps in the gluing process to obtain X^{triv} are !-able. This ensures that X^{triv} is a well-defined analytic stack over $\text{AnSpec}(k^{\text{triv}})$.

We define $X^{k\text{-alg}}$ via the following pullback of analytic stacks:

$$\begin{array}{ccc} X^{k\text{-alg}} & \longrightarrow & X^{\text{triv}} \\ \downarrow & & \downarrow \\ \text{AnSpec}(k_{\blacksquare}) & \longrightarrow & \text{AnSpec}(k^{\text{triv}}). \end{array}$$

In particular, $X^{k\text{-alg}}$ is obtained by gluing the analytic stacks associated with analytic rings of the form $R^{k\text{-alg}}$.

Lemma 2.4.5. *The functor*

$$\text{CAlg}(\text{Mod}_k) \xrightarrow{(-)^{k\text{-alg}}} \text{AnRing} \xrightarrow{\text{Nuc}(-)} \text{Pr}^{\text{L}} \quad (2.19)$$

defined by sending a commutative algebra R to the ∞ -category of nuclear modules over the analytic ring $R^{k\text{-alg}}$, satisfies descent for the fppf topology.

Proof. Let $S \rightarrow R$ be an fppf cover of commutative algebras in Mod_k . By [Mat16, Corollary 3.33], R is a descendable algebra in Mod_S . Since $\underline{R}$ is nuclear over k and the k -solidification functor in Definition 2.2.4 is symmetric monoidal (see Lemma 2.2.5), it induces a functor

$$\text{Mod}_S \rightarrow \text{Nuc}(S^{k\text{-alg}}) \simeq \text{Mod}_{\underline{S}}(\text{Nuc}(k)).$$

Applying [Mat16, Corollary 3.21] to this functor, we deduce that $\underline{R}$ is a descendable algebra in $\text{Nuc}(S^{k\text{-alg}}) \simeq \text{Mod}_{\underline{S}}(\text{Nuc}(k))$. Consequently, by [Mat16, Proposition 3.22], we deduce that the functor in Equation (2.19) is a sheaf for the fppf topology. $\square$

If $\{\text{Spec}(A_i)\}_{i \in I}$ is a Zariski cover of a separated scheme X over $\text{Spec}(k)$, we can observe that $\{\text{AnSpec}(A_i^{k\text{-alg}})\}_{i \in I}$ is a cover of $X^{k\text{-alg}}$. Using Lemma 2.4.5, we obtain an equivalence:

$$\text{Nuc}(X^{k\text{-alg}}) \simeq \lim_{[n] \in \Delta^{\text{op}}} \prod_{\{i_1, \dots, i_n\} \subseteq I} \text{Nuc}(A_{i_1, \dots, i_n}^{k\text{-alg}})$$

where $\text{Spec}(A_{i_1, \dots, i_n})$ denotes the intersection $\text{Spec}(A_{i_1}) \times_X \cdots \times_X \text{Spec}(A_{i_n})$.

Proposition 2.4.6. *Let X be a connective derived scheme, separated and of finite type over k . Then there is an equivalence of categories between $\text{Nuc}(X^{k\text{-alg}})$ and $\text{QCoh}(X) \otimes_{\text{Mod}_k} \text{Nuc}(k)$.*

Proof. As above, let $\{\text{Spec}(A_i)\}_{i \in I}$ be a cover of X , and let $\text{Spec}(A_{i_1, \dots, i_n})$ denote the intersection $\text{Spec}(A_{i_1}) \times_X \cdots \times_X \text{Spec}(A_{i_n})$. Note that since X is separated, $\text{QCoh}(X)$ can be written as a limit of quasi-coherent sheaves on affine schemes. In particular, using the same notation as above, we have that

$$\text{QCoh}(X) \simeq \lim_{[n] \in \Delta^{\text{op}}} \prod_{\{i_1, \dots, i_n\} \subseteq I} \text{Mod}_{A_{i_1, \dots, i_n}}.$$

Thus, $\text{QCoh}(X)$ can be written as a limit of Mod_A for affine opens $\text{Spec}(A)$ in X . Now, since both $\text{Nuc}(k)$ and Mod_k are rigid, it follows that $\text{Nuc}(k)$ is dualizable over Mod_k ; in particular, the functor $- \otimes_{\text{Mod}_k} \text{Nuc}(k)$ commutes with limits. Therefore, we can write

$$\text{QCoh}(X) \otimes_{\text{Mod}_k} \text{Nuc}(k) \simeq \lim_{[n] \in \Delta^{\text{op}}} \prod_{\{i_1, \dots, i_n\} \subseteq I} \left(\text{Mod}_{A_{i_1, \dots, i_n}} \otimes_{\text{Mod}_k} \text{Nuc}(k) \right). \quad (2.20)$$

The right-hand side can be identified with

$$\lim_{[n] \in \Delta^{\text{op}}} \prod_{\{i_1, \dots, i_n\} \subseteq I} \text{Mod}_{A_{i_1, \dots, i_n}}(\text{Nuc}(k)).$$

To conclude, it is enough to observe that $\mathrm{Mod}_{\underline{A}}(\mathrm{Nuc}(k))$ can be identified with $\mathrm{Nuc}(A^{k\text{-alg}})$, and that $\{\mathrm{AnSpec}(A_i^{k\text{-alg}})\}_{i \in I}$ defines a cover of $X^{k\text{-alg}}$. Hence, using descent for $\mathrm{Nuc}(-)$ (Lemma 2.4.5), we can identify the limit in Equation (2.20) with $\mathrm{Nuc}(X^{k\text{-alg}})$. $\square$

Corollary 2.4.7. *Let X be a proper scheme of finite type, quasi-compact and separated, over $\mathrm{Spec}(k)$, and let X^{an} be its analytification as a rigid analytic variety. Then we have an equivalence*

$$\mathrm{Nuc}(X^{\mathrm{an}}) \simeq \mathrm{QCoh}(X) \otimes_{\mathrm{Mod}_k} \mathrm{Nuc}(k).$$

Proof. This is a consequence of [Wan26, Theorem 4.1.1] and Proposition 2.4.6. Indeed, as explained in the proof of [Wan26, Theorem 4.1.1], the pullback along the inclusion of the categorical locales associated with X^{an} and $X^{k\text{-alg}}$ yields an equivalence of solid derived ∞ -categories :

$$\mathcal{D}(X^{k\text{-alg}})_{\blacksquare} \simeq \mathcal{D}(X^{\mathrm{an}})_{\blacksquare}.$$

Since this pullback is a symmetric monoidal functor, it preserves the nuclear subcategories (see [MW25, Theorem 2.4 (c)]). We can therefore conclude by applying Proposition 2.4.6. $\square$

2.4.3 Compact generation of nuclear categories in rigid analytic geometry

Lemma 2.4.8. *Let X be a rigid analytic variety over k , and let $\mathcal{F}$ be an object in $\mathrm{Nuc}(X)$. Then the following properties are equivalent:*

1. $\mathcal{F}$ is a dualizable object in $\mathrm{Nuc}(X)$.
2. $\mathcal{F}$ is a compact object in $\mathrm{Nuc}(X)$.
3. $\mathcal{F}$ is a $\mathrm{Nuc}(k)$ -compact (equivalently, $\mathrm{Nuc}(k)$ -atomic) object in $\mathrm{Nuc}(X)$.
4. $\mathcal{F}$ is a perfect complex over X .

Proof. The equivalence between (2) and (3) follows from Proposition 2.1.11.

We also observe that all these properties are local on X (see [And21]), so we may assume X to be an affinoid space. Since $\mathrm{Nuc}(k)$ is rigid, and locally we can assume $\mathrm{Nuc}(X)$ to be of the form $\mathrm{Mod}_A(\mathrm{Nuc}(k))$, the equivalence between (1) and (3) follows from Corollary 2.1.31.

Moreover, we note that by [And21, Corollary 5.51.1], we can identify dualizable objects in $\mathcal{D}(X)_{\blacksquare}$ with perfect complexes over X . To conclude, we observe that by [And21,

Lemma 5.45 and Theorem 5.50], we can also identify dualizable objects in $\mathcal{D}(X)_{\blacksquare}$ with dualizable objects in $\text{Nuc}(X)$. $\square$

Theorem 2.4.9. *Let $p: X \rightarrow \text{Sp}(k)$ be a smooth, proper, quasi-compact, and separated rigid analytic variety over $\text{Sp}(k)$. Then X is algebraizable if and only if $\text{Nuc}(X)$ admits a compact (equivalently, $\text{Nuc}(k)$ -atomic) generator.*

Proof. The strategy of the proof is to show that $\text{Perf}(X)$ is a smooth, proper, idempotent-complete k -linear ∞ -category, and then apply [Mon25, Theorem 3.1].

Suppose there exists an object $\mathcal{F}$ which is a $\text{Nuc}(k)$ -compact generator of $\text{Nuc}(X)$. Since $\mathcal{F}$ is a $\text{Nuc}(k)$ -compact generator, we can apply the Barr–Beck–Lurie theorem ([Lur17, Proposition 4.8.5.8]) to the functor

$$\text{hom}^{\text{Nuc}(k)}(\mathcal{F}, -): \text{Nuc}(X) \rightarrow \text{Nuc}(k).$$

In this way, we can identify $\text{Nuc}(X)$ with $\text{LMod}_A(\text{Nuc}(k))$, where A is the $\mathbb{E}_1$ -algebra of endomorphisms of the $\text{Nuc}(k)$ -compact generator $\mathcal{F}$. Explicitly,

$$A := p_* \underline{\text{Hom}}_{\text{Nuc}(X)}(\mathcal{F}, \mathcal{F}) \simeq \text{hom}^{\text{Nuc}(k)}(\mathcal{F}, \mathcal{F})$$

as an $\mathbb{E}_1$ -algebra in $\text{Nuc}(k)$.

Since dualizable objects in $\text{Nuc}(X)$ (respectively, in $\text{Nuc}(k)$) correspond to perfect complexes on X (respectively, on $\text{Sp}(k)$) by Lemma 2.4.8, and since X is proper, [PY21, Proposition 7.8] implies that A is a perfect complex of k -modules. In particular, it is a dualizable object in $\text{Nuc}(k)$ and thus a proper algebra in $\text{Nuc}(k)$. Because X is smooth, Corollary 2.4.4 implies that $\text{Nuc}(X)$ is smooth over $\text{Nuc}(k)$. Therefore, by Proposition 2.1.33, A is a smooth algebra over $\text{Nuc}(k)$. Using [Lur17, Proposition 4.6.4.4 and Proposition 4.6.4.12], we obtain

$$\text{Perf}(X) \simeq \text{Nuc}(X)^{\text{dual}} \simeq \text{LMod}_A(\text{Nuc}(k))^{\text{dual}} \simeq \text{LMod}_A(\text{Perf}(k)) \simeq \text{Perf}(A).$$

Now, to prove that $\text{Perf}(X)$ is a smooth and proper idempotent-complete k -linear ∞ -category, it suffices to show that A is a smooth and proper k -algebra. We observe that, since A is a perfect complex over k , the completed tensor product $A \widehat{\otimes}_k A$ coincides with the uncompleted tensor product $A \otimes_k A$. This implies that A is a smooth algebra over Mod_k . Furthermore, since A is a perfect complex of k -modules, it is dualizable in Mod_k , meaning it is also a proper algebra in Mod_k . Hence using [Mon25, Theorem 3.1] we deduce that X is algebraizable.

The converse follows from Corollary 2.4.7. Indeed, we have an equivalence

$$\mathrm{Nuc}(X^{\mathrm{an}}) \simeq \mathrm{QCoh}(X) \otimes_{\mathrm{Mod}_k} \mathrm{Nuc}(k).$$

In particular, using Corollary 2.5.6, we deduce the existence of a discrete, smooth, and proper algebra A in Mod_k such that

$$\mathrm{QCoh}(X) \simeq \mathrm{LMod}_A.$$

This implies that

$$\mathrm{Nuc}(X) \simeq \mathrm{LMod}_{\underline{A}}(\mathrm{Nuc}(k)).$$

Since A is a perfect complex over k , the same argument explained above shows that $\underline{A}$ is a smooth and proper algebra in $\mathrm{Nuc}(k)$. $\square$

Corollary 2.4.10. *Let $p: X \rightarrow \mathrm{Sp}(k)$ be a smooth, proper, quasi-compact, separated and non-algebraizable rigid analytic variety over $\mathrm{Sp}(k)$. Then $\mathrm{Nuc}(X)$ is internally smooth over $\mathrm{Nuc}(k)$, but it does not have a compact (or, equivalently, a $\mathrm{Nuc}(k)$ -atomic) generator.*

Proof. In this case, Corollary 2.4.4 shows that $\mathrm{Nuc}(X)$ is smooth over $\mathrm{Nuc}(k)$, while Theorem 2.4.9 demonstrates that $\mathrm{Nuc}(X)$ does not admit a compact generator. $\square$

Remark 2.4.11. Using Corollary 2.4.10 we are able to produce an example of a category that is internally smooth over a rigid base but fails to have a compact generator. It suffices to consider the category of nuclear modules over a non-algebraizable, smooth, and proper rigid analytic variety, such as the p -adic Hopf surface studied in [Mus77; Vos91]. This provides a counterexample to Conjecture 0.2.9, illustrating that Stefanich's theorem ([Ste25]) cannot be extended to arbitrary rigid symmetric monoidal bases.

Remark 2.4.12. Our previous theorem implies that it is not, in general, possible to glue local, relatively compact generators of a sheaf of categories in the analytic topology to obtain a global compact generator. Locally in the analytic topology over X , the sheaf $\mathrm{Nuc}(-)$ is of the form $\mathrm{Mod}_A(\mathrm{Nuc}(k))$ for affinoid algebras A . However, Theorem 2.4.9 demonstrates that, globally, such a generator algebra does not necessarily exist.

This property of gluing local compact generators is one of the foundational features of Non-Commutative Algebraic Geometry, and it highlights a key distinction between algebraic and analytic topologies. This gluing procedure for compact generators has been shown to hold for various topologies utilized in algebraic geometry; see, for example, [TV07] and [BB03].

2.5 Other examples of smooth and proper categories in a six-functor formalism

We conclude this chapter applying the general theory developed in Section 2.1 to the setting of derived algebraic geometry to obtain a proof of known results and to the setting of analytic stacks.

2.5.1 Smooth and proper categories in derived algebraic geometry

In this section, we will always work over a discrete field k of characteristic 0, and we will employ the foundations for derived geometry as described in [Lur18] and [TVo8b]. We emphasize that the results presented here are known to experts; however, we provide an alternative proof based on the six-functor formalism and the theory of smooth ∞ -categories as developed in Section 2.1. In particular, we will prove the following theorem.

Theorem 2.5.1. *Let $f: X \rightarrow Y$ be a morphism of finite Tor-amplitude between quasi-compact, semi-separated derived schemes of finite type. Then the following hold:*

1. *The map f is smooth if and only if $\mathrm{QCoh}(X)$ is smooth over $\mathrm{QCoh}(Y)$.*
2. *If $\mathrm{QCoh}(X)$ is proper over $\mathrm{QCoh}(Y)$, then f is universally closed (i.e., for every derived scheme W and for every map $W \rightarrow Y$, the induced map of topological spaces $|X \times_Y W| \rightarrow |W|$ is closed).*
3. *If f is proper, then $\mathrm{QCoh}(X)$ is proper over $\mathrm{QCoh}(Y)$.*

Remark 2.5.2. In particular, under the hypotheses of Theorem 2.5.1, if f is separated, then f is proper if and only if $\mathrm{QCoh}(X)$ is proper over $\mathrm{QCoh}(Y)$. Moreover, as explained in [Lur18, Theorem 11.4.2.1], if we assume $\mathrm{QCoh}(X)$ to be smooth, then X is automatically separated (since the diagonal is a closed immersion; see [Lur18, Lemma 11.3.6.2]). This implies that, under the hypotheses of Theorem 2.5.1, X is smooth and proper if and only if $\mathrm{QCoh}(X)$ is smooth and proper over $\mathrm{QCoh}(Y)$.

As a corollary, we immediately deduce the following, which is the algebraic geometry analogue of Theorem 2.4.1.

Corollary 2.5.3. *Let $X \rightarrow \mathrm{Spec}(k)$ be a quasi-compact and separated underived scheme of finite type over k . Then X is smooth (respectively, proper) if and only if $\mathrm{QCoh}(X)$ is smooth (respectively, proper).*

When f is the structure map $X \rightarrow \mathrm{Spec}(k)$ from a (classical) scheme to a field, a similar statement has been proved in [Orl16, Proposition 3.30] regarding properness, and in [Lun10, Proposition 3.13] regarding smoothness, by considering the DG category of perfect complexes over X . Our corollary is also closely related to [Lur18, Theorem 11.4.2.1] (see also [Lur18, Proposition 11.1.4.3 and Theorem 11.3.6.1] for results similar to those obtained in Theorem 2.5.1).

However, the techniques we employ in the proof of Theorem 2.5.1 are different and rely on a specific six-functor formalism for quasi-compact and semi-separated schemes.

We begin by reviewing some classical definitions to clarify our statements.

We consider derived schemes to be those derived stacks that admit a Zariski cover by affine derived schemes of finite type over k , glued together along Zariski open immersions.

We recall the following definitions of quasi-compact, semi-separated, and separated maps.

Definition 2.5.4. Let $f: X \rightarrow Y$ be a map of derived schemes of finite type over k .

1. We say that f is quasi-compact if, for every map $\mathrm{Spec}(A) \rightarrow Y$, the pullback $\mathrm{Spec}(A) \times_Y X$ admits a finite cover by affine schemes.
2. We say that f is *semi-separated* if the diagonal Δ_f is affine.
3. We say that f is *separated* if the diagonal Δ_f is a closed immersion.
4. We say that a derived stack X over $\mathrm{Spec}(k)$ is quasi-compact (respectively, semi-separated) if the canonical map $X \rightarrow \mathrm{Spec}(k)$ is quasi-compact (respectively, semi-separated). In particular, in this case, X admits a finite atlas of affine derived schemes whose intersections are again affine.

Theorem 2.5.5 ([Sch25, Theorem 8.11]). *There exists a symmetric monoidal six-functor formalism on derived schemes*

$$\mathrm{QCoh}(-): \mathrm{Corr}(\mathrm{Sch}_k)_{\mathrm{qc}, \mathrm{ss}, \mathrm{all}} \rightarrow \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$$

with the following properties. A derived stack X is assigned to the ∞ -category $\mathrm{QCoh}(X)$, and for every map of schemes $f: X \rightarrow Y$, we consider the usual pullback and pushforward functors. We take the class of $!$ -able maps to be the maps between quasi-compact and semi-separated derived schemes, for which we define $f_! := f_$.*

We remark here that this six-functor formalism is not the one commonly used in algebraic geometry; in particular, our upper shriek functors are not the ones used in

Grothendieck–Serre duality. However, for our purposes, this six-functor formalism is sufficient.

Proof. This statement follows from [MP25, Proposition 1.3.13] and [MAN22, Proposition A.5.10] by taking I to be the class of identity maps and P to be the class of maps between quasi-compact, semi-separated derived schemes. The fact that this six-functor formalism is symmetric monoidal easily follows from the fact that, for a quasi-compact and semi-separated derived scheme X , the ∞ -category $\mathrm{QCoh}(X)$ is rigid (see [MP25, Proposition 1.3.17] for a proof of this fact). $\square$

Proof of Theorem 2.5.1. We begin by proving (1). We consider the coevaluation functor obtained as in Equation (2.6):

$$\mathrm{coev}: \mathrm{QCoh}(Y) \xrightarrow{f^*} \mathrm{QCoh}(X) \xrightarrow{\Delta_{f!}} \mathrm{QCoh}(X \times_Y X).$$

By Lemma 2.1.26, we deduce that $\mathrm{QCoh}(X)$ is smooth over $\mathrm{QCoh}(Y)$ if and only if $\Delta_* \mathcal{O}_X$ is compact in $\mathrm{QCoh}(X \times_Y X)$. Since both conditions are local with respect to the Zariski topology, we may assume for the remainder of the proof that X and Y are affine. Suppose now that $\mathrm{QCoh}(X)$ is smooth. To prove that f is also smooth, we may assume, by virtue of [Lur18, Proposition 11.2.3.6], that $Y \simeq \mathrm{Spec}(k)$ for some field k . In this setting, the claim follows by adapting the argument of Proposition 2.4.3. Indeed, that proof relies on the base-change property and on the fact that compact and dualizable objects coincide in $\mathrm{Nuc}(A)$; both of these properties hold true in the setting considered here. Conversely, if f is smooth, then the diagonal morphism Δ_f is a quasi-smooth (i.e., local complete intersection) closed immersion. These properties ensure that the pushforward along Δ_f preserves perfect complexes (see, e.g., [Toë12a]).

To prove (3), we assume f is proper, and we need to prove that the evaluation functor

$$\mathrm{ev}: \mathrm{QCoh}(X \otimes_Y X) \xrightarrow{\Delta^*} \mathrm{QCoh}(X) \xrightarrow{f_!} \mathrm{QCoh}(Y)$$

has a continuous right adjoint. Since $\Delta_* \simeq \Delta_!$, it is enough to show that $f^!$ is continuous, which follows from [Sch25, Theorem 8.13].

To prove statement (2), let W be a derived scheme over k and consider a map $W \rightarrow Y$, we must prove that the induced map $X \times_Y W \rightarrow W$ is closed. As also explained in [Sta24, Tag 0CL3] by covering W with affine derived schemes, we can reduce to prove the statement when W is affine. Now in this setting the ∞ -categories of quasi-coherent sheaves are rigid, this easily implies the following Künneth formula (see, for example,

the proof of Proposition 2.3.14, or [BFN10]):

$$\mathrm{QCoh}(X \times_Y W) \simeq \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(W).$$

Since proper ∞ -categories are stable under base change (see [Lur18, Proposition 11.1.1.5]), we deduce that $\mathrm{QCoh}(X \times_Y W)$ is proper over $\mathrm{QCoh}(W)$. The statement now follows from [Lur18, Lemma 11.1.4.6]. $\square$

As a corollary of Theorem 2.5.1, we obtain the following.

Corollary 2.5.6 ([BB03; Toë12b]). *Let $p: X \rightarrow \mathrm{Spec}(k)$ be a smooth, proper, quasi-compact, and semi-separated scheme over $\mathrm{Spec}(k)$. Then there is a smooth and proper $\mathbb{E}_1$ -algebra A in Mod_k such that $\mathrm{QCoh}(X)$ is equivalent to RMod_A .*

Proof. It is a standard result that $\mathrm{QCoh}(X)$ is compactly generated by its full subcategory of perfect complexes (see, for example, [Sch25, Theorem 8.3]). By Theorem 2.5.1, $\mathrm{QCoh}(X)$ is also smooth; using [Lur18, Proposition 11.3.2.4], we can deduce that it admits a single compact generator, say $\mathcal{G}$.

Using the same argument as in Proposition 2.1.7, we can see that $\mathcal{G}$ is also a compact generator relative to Mod_k . To show the existence of the algebra A , we will use the Barr–Beck–Lurie theorem ([Lur17, Proposition 4.8.5.8]). To apply *loc. cit.*, we only need to show that for every $M \in \mathrm{Mod}_k$ and $\mathcal{F} \in \mathrm{QCoh}(X)$, the natural map

$$\mathrm{hom}^{\mathrm{Mod}_k}(\mathcal{G}, \mathcal{F}) \otimes_k M \rightarrow \mathrm{hom}^{\mathrm{Mod}_k}(\mathcal{G}, p^*(M) \otimes_{\mathcal{O}_X} \mathcal{F}) \quad (2.21)$$

is an equivalence. To prove this, we consider $\mathcal{X}$, the stable full subcategory of Mod_k spanned by all objects $M \in \mathrm{Mod}_k$ such that the map in Equation (2.21) is an equivalence. Obviously, k is in $\mathcal{X}$, and $\mathcal{X}$ is closed under colimits since $\mathcal{G}$ is compact relative to Mod_k . This implies that $\mathcal{X}$ is the entirety of Mod_k , and Equation (2.21) is an equivalence for all M . Defining A as the $\mathbb{E}_1$ -algebra

$$A := p_* \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{G}) \simeq \mathrm{hom}^{\mathrm{Mod}_k}(\mathcal{G}, \mathcal{G}),$$

we thus obtain an equivalence between $\mathrm{QCoh}(X)$ and RMod_A . Finally, we need to show that A is a smooth and proper algebra. The fact that A is smooth follows from Proposition 2.1.33 and Theorem 2.5.1. Moreover, A is proper since X is proper, and in this case p_* preserves perfect complexes (see [Sch25, Theorem 8.10]). $\square$

We conclude by recalling that, for a quasi-compact and quasi-separated scheme, the existence of a compact generator was proved in [BB03], and in a more general setting in [Toë12b].

2.5.2 Smooth and proper categories in analytic stacks

In this section, we apply the results obtained in Section 2.1 to provide examples of smooth and proper ∞ -categories for analytic stacks, as introduced by Clausen and Scholze. We employ the theory of analytic stacks developed in [CS24]. For a comprehensive overview, we refer the reader to [Kes25]. In what follows, we consider the ∞ -category of analytic stacks defined as the ∞ -category of sheaves with values in $\mathbf{Anima}$ with respect to the $\mathcal{D}$ -topology on the ∞ -category of analytic rings.

In this appendix, we focus on analytic stacks arising from rigid analytic geometry. However, it should be straightforward to adapt these ideas to other contexts, for example, by viewing schemes as analytic stacks endowed with the trivial analytic ring structure, or complex manifolds equipped with the liquid (or gaseous) analytic ring structure.

In this section, we fix a non-Archimedean field k with a pseudo-uniformizer ω and a ring of power-bounded elements k° . We will prove the following result.

Proposition 2.5.7. *Let $f: X \rightarrow \mathrm{Sp}(k)$ be a smooth, proper, quasi-compact, and separated rigid analytic variety. Then, considering the induced map of analytic stacks $f_\blacksquare: X_\blacksquare \rightarrow \mathrm{AnSpec}(k_\blacksquare)$, the ∞ -category $\mathcal{D}(X)_\blacksquare$ is internally smooth and proper over $\mathcal{D}(k)_\blacksquare$.*

Remark 2.5.8. We can interpret the preceding proposition as stating that the coevaluation functor

$$\mathrm{coev}: \mathcal{D}(k)_\blacksquare \xrightarrow{f_\blacksquare^*} \mathcal{D}(X) \xrightarrow{\Delta_{f_\blacksquare,!}} \mathcal{D}(X \hat{\times} X)$$

admits a continuous and linear right adjoint. By the results in Section 2.1, we deduce that this implies $\Delta_{f_\blacksquare,!}(\mathbb{1}_X)$ is atomic relative to $\mathcal{D}(k)_\blacksquare$. However, the precise relationship between $\mathcal{D}(k)_\blacksquare$ -atomic objects in $\mathcal{D}(X \hat{\times} X)$ and perfect complexes over $X \hat{\times} X$ remains unclear to the author. This prevents us from deducing the smoothness of the variety from the smoothness of the ∞ -category $\mathcal{D}(X)_\blacksquare$ and the analogous results established in Section 2.4.

We begin by explaining how to construct a functor from rigid analytic varieties to analytic stacks. For this reason, throughout this appendix, we restrict our attention to Huber pairs weakly of finite type. Related ideas have also been discussed in [Mik26] and [And21].

We first recall the following result.

Lemma 2.5.9 ([PY16, Lemma 2.14]). *Let $\mathcal{C}$ and $\mathcal{X}$ be ∞ -sites, and let $F: \mathcal{C} \rightarrow \mathcal{X}$ be a functor that preserves coverings and finite limits. Then F can be extended to a functor $\mathrm{Shv}(\mathcal{C}) \rightarrow \mathrm{Shv}(\mathcal{X})$ that preserves finite limits and colimits.*

To every analytic Huber pair (A, A^+) we associate an analytic ring $(A, A^+)_\blacksquare$, see [And21, Theorem 3.28].

Proposition 2.5.10. *The functor from the category of analytic Huber pairs weakly of finite type over k to analytic stacks*

$$(-)_{\blacksquare} : \text{AnHuber}_k^{\text{wft}} \rightarrow \text{AnRing}_{k_{\blacksquare}}$$

defined by sending (A, A^+) to $\text{AnSpec}(A, A^+)_{\blacksquare}$ is a fully faithful embedding that commutes with finite limits. Moreover, it sends covers in the analytic topology to $\mathcal{D}$ -covers.

Proof. The fact that the embedding is fully faithful was proved in [And21, Theorem 3.34]. The final assertion follows from [Cam24, Lemma 3.2.9]. Since the functor $(-)_{\blacksquare}$ preserves the initial object it is enough to show that the functor $(-)_{\blacksquare}$ commutes with tensor products. Indeed, given a pushout diagram of Huber pairs weakly of finite type

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \downarrow & & \downarrow \\ (C, C^+) & \longrightarrow & (D, D^+), \end{array}$$

we can identify the ring D with the completed tensor product of Tate algebras $B \widehat{\otimes}_A C$, and D^+ with the minimal open and integrally closed subring containing the images of B^+ and C^+ . The analogous tensor product in the ∞ -category of analytic rings yields an analytic ring $E := (E^{\triangleright}, \mathcal{D}(E))$, where $E^{\triangleright}$ coincides with the condensed ring associated to the completed tensor product $B \widehat{\otimes}_A C$ (see Remark 2.2.10). By [Sch26a, Proposition 12.12], the analytic ring structure on E is obtained by completing the analytic ring structure induced by $(B, B^+)_{\blacksquare}$ and $(C, C^+)_{\blacksquare}$. Since $E^{\triangleright}$ is an affinoid algebra, it is nuclear over $(B, B^+)_{\blacksquare}$ and $(C, C^+)_{\blacksquare}$; in particular, the induced analytic ring structure is already complete. Applying [And21, Proposition 3.32], we deduce that $(D, D^+)_{\blacksquare}$ coincides with E . $\square$

Applying Lemma 2.5.9, we obtain the following colimit-preserving functor:

$$\begin{array}{ccc} \text{AnHuber}_k^{\text{wft}} & \xrightarrow{(-)_{\blacksquare}} & \text{AnStack}_{k_{\blacksquare}} \\ \downarrow & \nearrow & \\ \text{Shv}(\text{AnHuber}_k^{\text{wft}}) & & \end{array}$$

In particular, by composing this functor with the natural map $\text{AnAdic} \rightarrow \text{Shv}_{\text{an}}(\text{AnHuber})$, we obtain the required functor:

$$\text{RigVar} \rightarrow \text{AnStacks}.$$

Explicitly, for a rigid analytic variety X , we can consider the associated analytic stack, defined as:

$$X_{\blacksquare} := \varinjlim_{\mathrm{Sp}(A) \hookrightarrow X} \mathrm{AnSpec}(A, A^{\circ})_{\blacksquare}. \quad (2.22)$$

For a rigid analytic variety X , we denote by $\mathcal{D}(X)_{\blacksquare}$ the ∞ -category of “analytic quasi-coherent sheaves” on X , defined via descent using Equation (2.22). Moreover, if $f: \mathrm{Sp}(A) \rightarrow X$ is a cover, we can identify the analytic stack $X_{\blacksquare}$ with the Čech nerve of $f_{\blacksquare}$.

Lemma 2.5.11. *Let $f: X \rightarrow \mathrm{Spa}(k, k^{\circ})$ be a proper and quasi-compact rigid analytic variety. Then there exists a descendable $\mathcal{D}$ -prim cover of analytic stacks of the form $f_{k_{\blacksquare}}: \mathrm{AnSpec}(A, k^{\circ})_{\blacksquare} \rightarrow X_{\blacksquare}$.*

Proof. We observe that since X is quasi-compact, there exists a cover $f: \mathrm{Sp}(A) \rightarrow X$ consisting of the disjoint union of finitely many open affinoids in X . Because X is separated, the Čech nerve of f is n -truncated for some n , taking the form:

$$X \simeq \mathrm{colim}_{[m] \in \Delta_{\leq n}} (\mathrm{Sp}(A) \rightrightarrows \mathrm{Sp}(A_2) \rightrightarrows \cdots)$$

where we denote by $\mathrm{Sp}(A_n)$ the n -fold intersection $\mathrm{Sp}(A) \times_X \cdots \times_X \mathrm{Sp}(A)$. Since X is proper and separated, this colimit can be equivalently described in the category of adic spaces using the analogous diagram with the adic compactifications $\mathrm{Spa}(A_n, k^{\circ})$. Applying Lemma 2.5.9, we obtain an effective epimorphism:

$$f_{k_{\blacksquare}}: \mathrm{AnSpec}(A, k^{\circ})_{\blacksquare} \rightarrow X_{\blacksquare}.$$

This map satisfies the assumptions of Proposition 2.1.48; it follows that $f_{k_{\blacksquare}}$ is a descendable $\mathcal{D}$ -prim cover. $\square$

Proof of Proposition 2.5.7. The proof relies on Proposition 2.1.46. To apply loc. cit., we must verify that $f_{\blacksquare}$ is $\mathcal{D}$ -prim and $\mathcal{D}$ -suave. The fact that $f_{\blacksquare}$ is $\mathcal{D}$ -suave follows from [Mik26, Theorem 1.62]. Furthermore, using Lemma 2.5.11 and Corollary 2.1.49, we deduce that $f_{\blacksquare}$ is indeed $\mathcal{D}$ -prim. $\square$

An axiomatic approach to analytic 1-affineness

joint with Emanuele Pavia

3.1 Categorized sheaves over ∞ -sites

3.1.1 The general setup

Standing assumption 3.1.1. Throughout this section, we fix the following.

- (1) A small ∞ -site $\mathcal{C}$ admitting all finite coproducts, binary products, and pullbacks. We call objects in $\mathcal{C}$ *test spaces*, and denote them as $U \rightarrow X$: they have to be thought as open quasi-compact subobjects of some geometric object X (which may, or may not, belong to the ∞ -site $\mathcal{C}$ itself). When dealing with morphisms of test spaces, we shall trim our notations and just write $\varphi: U \rightarrow V$.
- (2) A stable and presentably symmetric monoidal ∞ -category $\mathcal{V}$.
- (3) A sheaf of presentably $\mathcal{V}$ -linear symmetric monoidal ∞ -categories $\mathcal{D}: \mathcal{C}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\mathcal{V}}^{\text{L}})$ which sends finite limits existing in $\mathcal{C}$ to finite colimits in $\text{CAlg}(\text{Pr}_{\mathcal{V}}^{\text{L}})$.

We moreover assume the following.

- (4) Finite coproducts in $\mathcal{C}$ are *disjoint* and *universal*: let $\emptyset$ denote the initial object of $\mathcal{C}$. Then for any finite collection of objects $\{U_i \rightarrow X\}_{i \in I}$, we have that

$$U_{i_1} \times_{U_{i_1}} \coprod_{U_{i_2}} U_{i_2} \simeq \emptyset.$$

Moreover, for any diagram of test spaces $U_i \rightarrow V \rightarrow W$, the natural map

$$\coprod_{i \in I} (U_i \times_W V) \longrightarrow \left(\coprod_{i \in I} U_i \right) \times_W V$$

is an equivalence.

- (5) The Grothendieck topology on $\mathcal{C}$ is generated by a collection of maps S which contains all equivalences and is stable under compositions, pullbacks and finite coproducts, in the sense of [Lur18, Proposition A.3.2.1].

Remark 3.1.2. Let us say something more on the set of conditions in Assumptions 3.1.1.

1. The ∞ -site admits all finite coproducts, but the condition of having all *binary* products in Assumption 3.1.1.(1) means only that whenever $U \rightarrow X$ and $V \rightarrow X$ are test spaces, then there exists a test space $U \times_X V$ in $\mathcal{C}$. In particular, $\mathcal{C}$ can fail to have a terminal object: for example, this happens when $\mathcal{C}$ is the little site spanned by a base of quasi-compact open subsets of a topological space X which is not quasi-compact itself.
2. The assumptions 3.1.1.(4) and 3.1.1.(5) identify the Grothendieck topology on $\mathcal{C}$ as the one described as follows: a collection of maps $\{U_i \rightarrow V\}_{i \in I}$ is a covering if and only if there exists a *finite* subset $J \subseteq I$ such that $\coprod_{j \in J} U_j \rightarrow V$ is a morphism in S . In particular, each test space is tautologically quasi-compact, in the sense that any τ -covering can be refined to a finite one.
3. Since $\mathcal{V}$ is assumed to be stable and presentable, it is a Sp -module in Pr^{L} ([Lur17, Proposition 4.8.2.18]). Being $\mathcal{V}$ symmetric monoidal, [Lur17, Corollary 4.8.2.19] implies that there exists a symmetric monoidal functor $\mathrm{Sp} \rightarrow \mathcal{V}$, hence we obtain a forgetful functor

$$\mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}} \longrightarrow \mathrm{Pr}_{\mathrm{Sp}}^{\mathrm{L}} \simeq \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}},$$

where $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ is the ∞ -category of presentable and stable ∞ -categories. In particular, Assumption 3.1.1.(3) implies that for all test spaces $U \rightarrow X$ the ∞ -category $\mathcal{D}(U)$ is stable.

Definition 3.1.3. Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumptions 3.1.1. The *categorification* of $\mathcal{D}$ is the functor

$$\begin{aligned} \mathcal{D}^{\text{cat}}: \mathcal{C}^{\text{op}} &\longrightarrow \widehat{\text{Cat}}^{\text{rex}} \\ U &\mapsto \text{Pr}_{\mathcal{D}(U)}^{\text{L}}. \end{aligned} \quad (3.1)$$

We are interested in extending the categorification of the sheaf $\mathcal{D}$ to some (possibly large) ∞ -category $\widehat{\mathcal{C}}$ containing $\mathcal{C}$, in such a way that objects in $\widehat{\mathcal{C}}$ are still governed by the ones in $\mathcal{C}$. In order to formalize this idea, we need to fix some notations.

Definition 3.1.4. Let $\mathcal{C}$ be a ∞ -category, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a functor. We say that ι is *dense* if the identity functor on $\widehat{\mathcal{C}}$ is a left Kan extension of ι along itself. Equivalently, this means that for all objects D of $\widehat{\mathcal{C}}$ the functor

$$\mathcal{C} \times_{\widehat{\mathcal{C}}} \widehat{\mathcal{C}}_{/D} \longrightarrow \mathcal{C} \xrightarrow{\iota} \widehat{\mathcal{C}}$$

admits a colimit, which is equivalent to D . If ι is moreover fully faithful, we shall say that ι is a *dense embedding* and that $\mathcal{C}$ is a *dense sub- ∞ -category*.

Remark 3.1.5. As explained in [Luro9, ArXiv v4, Remark 5.2.9.4], if $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is a functor and $\widehat{\mathcal{C}}$ admits all small colimits, then ι is dense if and only if the induced functor $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ani}) \rightarrow \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a localization of the ∞ -category of presheaves over $\mathcal{C}$. If we drop the assumption that $\mathcal{C}$ is small, the same holds if we accept to enlarge our universe and consider presheaves of *large* anima $\text{Fun}(\mathcal{C}^{\text{op}}, \widehat{\text{Ani}})$.

Example 3.1.6. Let $\mathcal{C}$ be a Grothendieck site. Then the Yoneda embedding $\mathfrak{y}: \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Ani})$ is always a dense embedding, and composing with the natural sheafification functor $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ani}) \rightarrow \text{Shv}_{\tau_{\mathcal{C}}}(\mathcal{C})$ one obtains a dense functor $\mathfrak{y}: \mathcal{C} \rightarrow \text{Shv}_{\tau_{\mathcal{C}}}(\mathcal{C})$. This is a dense embedding if and only if the Grothendieck topology $\tau_{\mathcal{C}}$ is sub-canonical.

Construction 3.1.7. Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumptions 3.1.1, let $\mathcal{D}^{\text{cat}}: \mathcal{C}^{\text{op}} \rightarrow \widehat{\text{Cat}}^{\text{rex}}$ be the categorification of $\mathcal{D}$ as in Definition 3.1.3, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor. We have two ways to extend $\mathcal{D}^{\text{cat}}$ to an assignment to all $\widehat{\mathcal{C}}$.

1. We can either consider the right Kan extension $\text{Ran}_{\iota} \mathcal{D}(Y)$ of $\mathcal{D}$ along (the opposite functor to) $\iota: \mathcal{C} \subseteq \widehat{\mathcal{C}}$, which is given by the assignment

$$\mathcal{D}(Y) \simeq \lim_{\substack{\iota(X) \rightarrow Y \\ X \in \mathcal{C}}} \mathcal{D}(X),$$

and then consider its categorification

$$\begin{aligned}\widehat{\mathcal{C}} &\longrightarrow \widehat{\mathbf{Cat}}^{\text{rex}} \\ Y &\mapsto \text{Pr}_{\text{Ran}_! \mathcal{D}(Y)}^{\text{L}}.\end{aligned}$$

2. Or we can consider the right Kan extension of $\mathcal{D}^{\text{cat}}$ along $\iota: \mathcal{C} \subseteq \widehat{\mathcal{C}}$, obtaining the functor $\text{Ran}_! \mathcal{D}^{\text{cat}}$. On an object Y of $\widehat{\mathcal{C}}$, it is explicitly described by the assignment

$$Y \mapsto \lim_{\substack{\iota(X) \rightarrow Y \\ X \in \mathcal{C}}} \text{Pr}_{\mathcal{D}(X)}^{\text{L}}.$$

An object in $\text{Pr}_{\text{Ran}_! \mathcal{D}(Y)}^{\text{L}}$ is a *categorical $\mathcal{D}(Y)$ -module*, while an object in $\text{Ran}_! \mathcal{D}^{\text{cat}}(Y)$ can be loosely described as a *sheaf of ∞ -categories* described by a rule

$$\begin{aligned}\mathcal{C} \times_{\widehat{\mathcal{C}}} \widehat{\mathcal{C}}_{/Y} &\longrightarrow \widehat{\mathbf{Cat}}^{\text{rex}} \\ \{\alpha: \iota(U) \rightarrow Y\} &\mapsto \mathcal{E}_U,\end{aligned}$$

where $\mathcal{E}_U$ is a categorical $\mathcal{D}(U)$ -module, together with suitable homotopy coherent co-cycle conditions. For any object Y of $\widehat{\mathcal{C}}$, we have a natural comparison functor

$$\text{Pr}_{\text{Ran}_! \mathcal{D}(Y)}^{\text{L}} \longrightarrow \text{Ran}_! \mathcal{D}^{\text{cat}}(Y). \quad (3.2)$$

induced by the universal property of limits.

Definition 3.1.8. Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumptions 3.1.1, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor. We say that an object Y in $\widehat{\mathcal{C}}$ is *1-affine with respect to ι* if the comparison functor (3.2) is an equivalence.

Notation 3.1.9. For the remainder of the paper, we will frequently abuse notation and denote the right Kan extension $\text{Ran}_! \mathcal{D}(-)$ simply by $\mathcal{D}(-)$; the same applies to $\mathcal{D}^{\text{cat}}(-)$.

Remark 3.1.10. If $\widehat{\mathcal{C}}$ admits all colimits, then Remark 3.1.5 implies that it is a full sub- ∞ -category of $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ani})$. In this case, if Y is an object of $\widehat{\mathcal{C}} \subseteq \text{Fun}(\mathcal{C}^{\text{op}}, \text{Ani})$ we shall simply say that Y is *1-affine*. This choice of terminology agrees with the one used of [Gai15].

The fundamental question of this paper is the following.

Question 3.1.11. Let $\mathcal{C}$, $\mathcal{V}$, and $\mathcal{D}$ be as in Assumptions 3.1.1, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor. Which objects of $\widehat{\mathcal{C}}$ can be proved to be 1-affine?

3.1.12. The quest for 1-affineness with respect to a dense functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ can be rephrased in more explicit terms. Recall the functor (3.2): this is the left adjoint in the adjunction

$$\widehat{(-)}: \mathbf{Pr}_{\mathcal{D}(Y)}^L \xrightleftharpoons[\perp]{} \mathcal{D}^{\text{cat}}(Y) : \Gamma(Y, -). \quad (3.3)$$

The left adjoint is a categorified version of the sheafification of modules on a locally ringed space. Given a categorical $\mathcal{D}(Y)$ -module $\mathcal{E}$, its sheafification is described by the rule

$$\{\alpha: \iota(U) \rightarrow Y\} \mapsto \mathcal{E} \otimes_{\mathcal{D}(Y)} \mathcal{D}(U).$$

This formula makes sense, since $\mathcal{D}(Y)$ is a limit of symmetric monoidal ∞ -categories (hence it is naturally symmetric monoidal), so $\mathcal{D}(U)$ becomes a categorical $\mathcal{D}(Y)$ -module via pullback.

On the other hand, the right adjoint $\Gamma(Y, -)$ is a categorified version of the global sections functor. For an object in $\mathcal{D}^{\text{cat}}(Y)$, described by the rule

$$\{\alpha: \iota(U) \rightarrow Y\} \mapsto \mathcal{E}_U,$$

we have an associated diagram

$$\begin{aligned} \mathcal{C} \times_{\widehat{\mathcal{C}}} \widehat{\mathcal{C}}_Y &\longrightarrow \mathbf{Pr}_{\mathcal{D}(Y)}^L \\ \{\alpha: \iota(U) \rightarrow Y\} &\mapsto \alpha_* \mathcal{E}_U. \end{aligned}$$

Then we have

$$\Gamma((\mathcal{E}_U)_\alpha, Y) \simeq \lim_{\substack{\alpha: \iota(U) \rightarrow Y \\ U \in \mathcal{C}}} \alpha_* \mathcal{E}_U.$$

This discussion makes sense also in the case when $Y = \iota(U)$ lies in $\mathcal{C}$. Of course, since $\mathcal{D}$ is assumed to be a sheaf for the Grothendieck topology on $\mathcal{C}$, the discussion in this case greatly simplifies: for any test space U in $\mathcal{C}$ we have equivalences

$$\text{Ran}_i \mathcal{D}(\iota(U)) \simeq \mathcal{D}(U) \quad \text{and} \quad \mathcal{D}^{\text{cat}}(\iota(U)) \simeq \mathcal{D}^{\text{cat}}(U),$$

so the adjunction (3.3) actually boils down to an adjunction

$$\widehat{(-)}: \mathbf{Pr}_{\mathcal{D}(U)}^L \xrightleftharpoons[\perp]{} \mathcal{D}^{\text{cat}}(U) : \Gamma(X, -), \quad (3.4)$$

which is trivially an adjoint equivalence. In general, for all objects Y in $\widehat{\mathcal{C}}$ and for any $\mathcal{E}$

in $\mathrm{Pr}_{\mathcal{D}(Y)}^{\mathrm{L}}$ we have a unit functor

$$\eta: \mathcal{E} \longrightarrow \Gamma(Y, \widehat{\mathcal{E}}) \quad (3.5)$$

and for any $(\mathcal{G}_X)_{\alpha: \iota(X) \rightarrow Y}$ in $\mathcal{D}^{\mathrm{cat}}(Y)$ we have counit functor

$$\epsilon: (\mathcal{G}_X)_{\alpha: \iota(X) \rightarrow Y} \longrightarrow \left(\lim_{\alpha: \iota(U) \rightarrow Y} \alpha_* \mathcal{G} \otimes_{\mathrm{Ran}_i \mathcal{D}(Y)} \mathcal{D}(X) \right)_X. \quad (3.6)$$

An object is 1-affine if and only if both functors (3.5) and (3.6) are equivalences.

In general, it is difficult to characterize 1-affine objects for an arbitrary choice of $\mathcal{E}$ and $\mathcal{D}$. For example, it is not clear at all that the categorification $\mathcal{D}^{\mathrm{cat}}$ is again a sheaf over $\mathcal{E}$, even if $\mathcal{D}$ was a sheaf in the first place. In the following, we propose two suitably general frameworks in which $\mathcal{D}^{\mathrm{cat}}$ can be proved to be still a sheaf.

Assumption 3.1.13. *Let $\mathcal{E}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumptions 3.1.1. Suppose that one of the following two conditions hold.*

- (1) *The sheaf of ∞ -categories $\mathcal{D}$ is obtained from a strongly monoidal six-functor formalism*

$$(\mathcal{D}^*, \mathcal{D}_!) : \mathrm{Corr}(\mathcal{E})_{E, \mathrm{all}}^{\otimes} \longrightarrow \mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}, \otimes}$$

in the sense of [MAN22], and the topology on $\mathcal{E}$ is the universal $!$ -able topology (see Definition 3.1.18).

- (2) *The sheaf $\mathcal{D}$ already is the categorification of a sheaf of commutative algebras*

$$D: \mathcal{E}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{V}),$$

in the sense that we have a natural equivalence of functors

$$\mathcal{D} \simeq \mathrm{Mod}_{D(-)}(\mathcal{V}).$$

While the meaning of Assumption 3.1.13.(2) is quite self-evident, Assumption 3.1.13.(1) is a compact way to formulate many different sub-conditions: we shall describe in detail what we mean in Section 3.1.2. We finish this section by collecting some pieces of terminology that we will use later (essentially to describe non-trivial 1-affine object).

Definition 3.1.14. Let $\mathcal{E}$, $\mathcal{V}$, and $\mathcal{D}$ be as in Assumptions 3.1.1. Let $\iota: \mathcal{E} \rightarrow \widehat{\mathcal{E}}$ be a dense functor.

- (1) An object X of $\widehat{\mathcal{C}}$ is $\mathcal{C}$ -affine if it belongs to the essential image of ι .
- (2) A morphism $\alpha: Y \rightarrow X$ is $\mathcal{C}$ -affine if for any $\mathcal{C}$ -affine object $\iota(U) \rightarrow X$ the map $Y \times_X \iota(U)$ is $\mathcal{C}$ -affine and the projection $Y \times_X \iota(U) \rightarrow \iota(U)$ is the image of a map of $\mathcal{C}$ under ι .
- (3) We say that a collection of maps $\{\varphi_i: W_i \rightarrow X\}$ is a $\mathcal{C}$ -affine atlas (or affine $\mathcal{C}$ -atlas) if each W_i is $\mathcal{C}$ -affine and for each $\mathcal{C}$ -affine $\iota(U) \rightarrow X$ the collection of maps $\{W_i \times_X \iota(U) \rightarrow \iota(U)\}$ is the image under ι of a covering of $\iota(U)$ for the Grothendieck topology on $\mathcal{C}$.

Remark 3.1.15. Let $\mathcal{C}$ be as in Assumptions 3.1.1, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor in a ∞ -category $\widehat{\mathcal{C}}$ where colimits are universal. Let $\{\iota(W_i) \rightarrow X\}$ be a $\mathcal{C}$ -affine atlas of an object X in $\widehat{\mathcal{C}}$. Then we can write

$$X \simeq \operatorname{colim}_{\iota(U) \rightarrow X} \iota(U).$$

Since $\{\iota(W_i) \rightarrow X\}$ is a $\mathcal{C}$ -affine atlas, for any $\iota(U) \rightarrow X$ we let

$$\pi_U: \coprod_{i=1}^n \iota(W_i) \times_X \iota(U) \rightarrow \iota(U)$$

be the induced covering of $\iota(U)$, and let $\check{C}(\pi_U)$ denote its Čech cover, so as to write

$$\iota(U) \simeq \operatorname{colim}_{\Delta^{\text{op}}} \check{C}(\pi_U).$$

For each $[n] \in \Delta$, we have

$$\check{C}(\pi_U)^n \simeq \coprod_{\{i_1, \dots, i_n\} \subseteq \{1, \dots, m\}} \iota(W_{i_1}) \times_X \cdots \times_X \iota(W_{i_n}) \times_X \iota(U),$$

and so using the fact that colimits in $\widehat{\mathcal{C}}$ are universal we have

$$\operatorname{colim}_{\iota(U) \rightarrow X} \check{C}(\pi_U)^n \simeq \coprod_{\{i_1, \dots, i_n\} \subseteq \{1, \dots, m\}} \iota(W_{i_1}) \times_X \cdots \times_X \iota(W_{i_n}) \simeq \check{C}(\varphi)^n.$$

So we have

$$X \simeq \operatorname{colim}_{\Delta^{\text{op}}} \operatorname{colim}_{\iota(U) \rightarrow X} \check{C}(\pi_U) \simeq \operatorname{colim}_{\Delta^{\text{op}}} \check{C}(\varphi).$$

In particular, it is still true that if X admits $\mathcal{C}$ -affine atlas, then it is realized as the colimit of its associated Čech nerve.

3.1.2 Categorized sheaves under Assumption 3.1.13.(1)

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumptions 3.1.1. We first spend some words on what the technical machinery of six-functor formalisms, established in [MAN22, §A.5], specializes to in our setting, and explain clearly what we mean in Assumption 3.1.13.(1).

Fact 3.1.16.

- (1) *The pair $(\mathcal{C}, E)$ forms a geometric setup in the sense of [MAN22, Definition A.5.1]. In particular, E is a class of morphisms which contains all equivalences and is closed under pullbacks.*
- (2) *Let $\mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}$ denote the sub- ∞ -category of the ∞ -category of correspondences of $\mathcal{C}$, whose objects are the same as the objects of $\mathcal{C}$ but where we allow a span $V \leftarrow U \rightarrow W$ to be a morphism in $\mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}$ only if $U \rightarrow W$ belongs to E . The ∞ -category $\mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}$ can be regarded as the underlying ∞ -category of a ∞ -operad*

$$\mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}^{\otimes} := \mathrm{Corr}(((\mathcal{C}^{\mathrm{op}})^{\mathrm{II}})^{\mathrm{op}})_{S^-, \mathrm{all}}$$

using the formalism of coCartesian ∞ -operads (see [Lur17, Proposition 2.4.3.3]).

- (3) *We have a map of ∞ -operads*

$$(\mathcal{D}^*, \mathcal{D}_!) : \mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}^{\otimes} \longrightarrow \mathrm{Pr}_{\mathcal{V}}^{\mathrm{L}, \otimes},$$

such that for all morphisms $\varphi: U \rightarrow V$ we have a symmetric monoidal functor between presentably $\mathcal{V}$ -linear symmetric monoidal ∞ -categories $\varphi^: \mathcal{D}(V) \rightarrow \mathcal{D}(U)$. If $\psi: W \rightarrow Y$ lies in E , then we have a functor $\psi_!: \mathcal{D}(W) \rightarrow \mathcal{D}(Y)$.*

- (4) *Both φ^* and $\psi_!$ admit right adjoints for abstract reasons. At the same time, the symmetric monoidal structure on each $\mathcal{D}(U)$ is closed.*

Remark 3.1.17. If $\mathcal{C}$ admits a terminal object, and thus all limits, then $\mathcal{C}^{\mathrm{op}}$ admits all finite coproducts, hence $((\mathcal{C}^{\mathrm{op}})^{\mathrm{II}})^{\mathrm{op}}$ can be regarded as a twist of the Cartesian monoidal structure on $\mathcal{C}$. Concretely, the fiber over a map of finite pointed sets $\alpha: J \rightarrow I$ of finite sets along the coCartesian fibration of ∞ -operads $((\mathcal{C}^{\mathrm{op}})^{\mathrm{II}})^{\mathrm{op}} \rightarrow \mathrm{Fin}_*$ consists of maps

$$X_i \longrightarrow \prod_{j \in \alpha^{-1}(i)} Y_j,$$

rather than maps in the opposite direction. Still, this defines a symmetric monoidal ∞ -category ([Lur17, Remark 2.4.3.4]) and so $\mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}^{\otimes}$ is a symmetric monoidal ∞ -category as well ([LZ24, Proposition 6.1.3]): its symmetric monoidal structure is inherited

from such twisted Cartesian monoidal structure of $\mathcal{C}$. The ∞ -operad $\text{Corr}(\mathcal{C})_{E, \text{all}}^{\otimes}$ in Fact 3.1.16.(2) is just the underlying ∞ -operad of such symmetric monoidal structure. Thus, the map of ∞ -operads in Fact 3.1.16.(3) boils down to a lax monoidal structure on the functor $(\mathcal{D}^*, \mathcal{D}!)$.

Let us now explain what the universal $!$ -able Grothendieck topology on $\mathcal{C}$ is.

Definition 3.1.18 ([Sch25, Definition 4.14]). Let $\mathcal{C}$ be any ∞ -category, and let $(\mathcal{D}^*, \mathcal{D}!)$ a six-functor formalism on $(\mathcal{C}, E)$. We define a Grothendieck topology $\tau_{\mathcal{C}}^!$ on $\mathcal{C}$ by saying that a collection of maps $\{\varphi_i: U_i \rightarrow V\}$ is a *universal $!$ -able cover* if all maps φ_i lie in E and the following conditions hold.

- (1) (The topology is universally sub-canonical.) For any object Z in $\mathcal{C}$ and for any map $W \rightarrow V$ from an object in $\mathcal{C}$, the functor $\text{Map}_{\mathcal{C}}(-, Z)$ exhibits $\text{Map}_{\mathcal{C}}(W, Z)$ as the limit for the diagram $\text{Map}_{\mathcal{C}}(-, Z)|_{U_i \times_V W \rightarrow W}$.
- (2) (The topology satisfies universal $*$ -descent.) For any map $W \rightarrow V$ from an object in $\mathcal{C}$, the functor $\mathcal{D}$ exhibits $\mathcal{D}(W)$ as the limit for the diagram $\mathcal{D}|_{U_i \times_V W \rightarrow W}$ computed along the functors φ^* .
- (3) (The topology satisfies universal $!$ -descent.) For any map $X \rightarrow \mathfrak{z}(V)$ in $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ani})$, the functor $\mathcal{D}$ exhibits $\mathcal{D}(X)$ as the limit for the diagram $\mathcal{D}|_{\mathfrak{z}(U_i) \times_{\mathfrak{z}(V)} X \rightarrow X}$ computed along the functors $\varphi^!$.

We say that the Grothendieck topology $\tau_{\mathcal{C}}^!$ is the *universal $!$ -able topology* on $\mathcal{C}$.

All together, Assumption 3.1.13.(1) means that the functor $\mathcal{D}$ is formally obtained as the restriction of the functor $(\mathcal{D}^*, \mathcal{D}!)$ along the natural inclusion of ∞ -operads $(\mathcal{C}^{\text{op}})^{\text{II}} \subseteq \text{Corr}(\mathcal{C})_{E, \text{all}}^{\otimes}$. Moreover, the functor $\mathcal{D}$ is a sheaf for the universal $!$ -able topology on $\mathcal{C}$.

3.1.19. The assumption that $\mathcal{D}$ turns finite limits of $\mathcal{C}$ into finite colimits of $\text{CAlg}(\text{Pr}_{\mathcal{V}}^{\text{L}})$ (Assumption 3.1.1.(3)) actually implies that the six-functor formalism

$$(\mathcal{D}^*, \mathcal{D}!): \text{Corr}(\mathcal{C})_{E, \text{all}}^{\otimes} \longrightarrow \text{Pr}_{\mathcal{V}}^{\text{L}, \otimes}$$

is *locally strongly monoidal*, in the following sense. For any test space V , the ∞ -category $\mathcal{C}_{/V}$ admits all finite limits (all finite limits but the terminal object are inherited by limits in $\mathcal{C}$, while the terminal object is realized by V), so the procedure described in Fact 3.1.16.(2) equips $\text{Corr}(\mathcal{C}_{/V})_{E, \text{all}}$ with a symmetric monoidal structure. Then the induced functor

$$(\mathcal{D}^*, \mathcal{D}!)|_V: \text{Corr}(\mathcal{C}_{/V})_{E, \text{all}}^{\otimes} \longrightarrow \text{Pr}_{\mathcal{D}(V)}^{\text{L}, \otimes}$$

is a symmetric monoidal functor of ∞ -categories. This is true because the product in $\mathcal{C}_{/V}$ of two objects $U \rightarrow V$ and $W \rightarrow V$ (which is the pullback $U \times_V W$ in $\mathcal{C}$) is sent to the pushout of the diagram $\mathcal{D}(U) \leftarrow \mathcal{D}(V) \rightarrow \mathcal{D}(W)$ in $\mathbf{CAlg}(\mathbf{Pr}_{\mathcal{C}}^{\mathbf{L}})$, which is precisely the relative tensor product $\mathcal{D}(U) \otimes_{\mathcal{D}(V)} \mathcal{D}(W)$.

We now explain how Assumption 3.1.13.(1) implies that we can deduce a strong dualizability condition at the categorified level, whenever a map of test spaces $\varphi: U \rightarrow V$ is a universal $!$ -able cover.

Construction 3.1.20. Let V be any test space in $\mathcal{C}$, let $\mathcal{C}_{/V}$ denote the slice ∞ -category over V . Let $\mathcal{C}_{/V}^S$ denote the sub- ∞ -category of $\mathcal{C}_{/V}$ described as follows.

1. Objects are maps $U \rightarrow V$ which belong to E .
2. A morphism between two arrows $\varphi: U \rightarrow V$ and $\psi: W \rightarrow V$ is a map in $\mathcal{C}_{/V}$.

Notice that [HM24, Lemma 2.1.5.iii] implies that whenever $U \rightarrow V$ and $W \rightarrow V$ belong to E , then any morphism $U \rightarrow W$ commuting with the maps to V is forced to lie in E as well. This is easily seen to imply the following.

Proposition 3.1.21. *Let $\mathcal{C}$ and E be as in Assumption 3.1.13.(1). Then $\mathcal{C}_{/V}^E \subseteq \mathcal{C}_{/V}$ is a full sub- ∞ -category closed under finite limits.*

As a consequence of Proposition 3.1.21, we have a strongly monoidal inclusion

$$\mathrm{Corr}(\mathcal{C}_{/V}^E)^{\otimes} := \mathrm{Corr}(\mathcal{C}_{/V}^E)_{\mathrm{all}, \mathrm{all}}^{\otimes} \subseteq \mathrm{Corr}(\mathcal{C}_{/V})_{E, \mathrm{all}}^{\otimes}.$$

In particular, the restriction

$$(\mathcal{D}^*, \mathcal{D}!)^E \Big|_V : \mathrm{Corr}(\mathcal{C}_{/V}^E)^{\otimes} \subseteq \mathrm{Corr}(\mathcal{C}_{/V})_{E, \mathrm{all}}^{\otimes} \longrightarrow \mathbf{Pr}_{\mathcal{D}(V)}^{\mathbf{L}, \otimes} \quad (3.7)$$

remains strongly monoidal.

We want to show how Proposition 3.1.21 implies that $\mathcal{D}(U)$ is $\mathcal{D}(V)$ -dualizable, when $\varphi: U \rightarrow V$ lies in E . In order to do so, recall the following fundamental property of (higher) ∞ -categories of correspondences.

Proposition 3.1.22 ([Hau18, Theorem 1.4]). *If $\mathcal{C}$ is a ∞ -category admitting all finite limits, then for all $n \geq 1$ every object in the (∞, n) -category $\mathrm{Corr}_n(\mathcal{C})$ of iterated spans is fully dualizable.*

Remark 3.1.23. Actually, the proof of Proposition 3.1.22 in [Hau18] shows something more: each object is self dual. The evaluation and coevaluation correspondences

$$\begin{array}{ccc} & X & \\ \swarrow \Delta & & \searrow \Delta \\ \{*\} & & X \times X \end{array} \quad \text{and} \quad \begin{array}{ccc} & X & \\ \swarrow \Delta & & \searrow \Delta \\ X \times X & & \{*\} \end{array}$$

are one the transpose of the other.

Corollary 3.1.24 ([CS24, Lecture 17, around 1:19:00]). *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{V}$ be as in Assumption 3.1.13.(1). Then for any map of test spaces $\varphi: U \rightarrow V$ lying in E the pullback $\varphi^*: \mathcal{D}(V) \rightarrow \mathcal{D}(U)$ turns $\mathcal{D}(U)$ into a dualizable $\mathcal{D}(V)$ -module. In fact, this $\mathcal{D}(V)$ -module is self-dual, and for any map of test spaces $\alpha: U \rightarrow W$ over V which belongs to E the dual of the functor $\alpha^*: \mathcal{D}(W) \rightarrow \mathcal{D}(U)$ corresponds to $\alpha_!: \mathcal{D}(U) \rightarrow \mathcal{D}(W)$ under the equivalences*

$$\mathcal{D}(U) \simeq \mathcal{D}(U)^\vee \quad \text{and} \quad \mathcal{D}(W) \simeq \mathcal{D}(W)^\vee.$$

Proof. Just notice that, being symmetric monoidal, the functor (3.7) preserves duals. Then everything is a formal consequence of Proposition 3.1.22 and Remark 3.1.23. $\square$

We are now ready to prove that the functor (3.1) is a sheaf for the Grothendieck topology on $\mathcal{C}$.

Proposition 3.1.25. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumptions 3.1.1. Assume Assumption 3.1.13.(1) to hold. Then the functor (3.1) is a sheaf for the universal !-able Grothendieck topology on $\mathcal{C}$.*

Proof. The assumption on our Grothendieck topology $\tau_{\mathcal{C}}$ allows us to reduce the above claim to the following (see also [Lur18, Proposition A.3.3.1]).

1. The functor $\mathcal{D}^{\text{cat}}$ preserves finite products.
2. For any map of test spaces $\varphi: V \rightarrow U$ which is a universal !-able cover, the functor

$$\mathcal{D}^{\text{cat}}(U) \longrightarrow \lim_{[n] \in \Delta} \left(\mathcal{D}^{\text{cat}}(V) \rightleftharpoons \mathcal{D}^{\text{cat}}(V \times_U V) \rightleftharpoons \mathcal{D}^{\text{cat}}(V \times_U V \times_U V) \rightleftharpoons \cdots \right)$$

is an equivalence.

The first claim follows from an easy check of the Barr–Beck–Lurie’s monadicity theorem, using the fact that for *any* collection of presentably symmetric monoidal ∞ -categories $\mathcal{V}_i$ one has

$$\prod_{i \in I} \text{Pr}_{\mathcal{V}_i}^{\text{L}} \simeq \text{Pr}_{\prod_i \mathcal{V}_i}^{\text{L}},$$

see for example [PPS25b, Proposition 3.2.32]. In order to prove the second claim, we argue as follows: since all objects involved belong to $\mathcal{C}$, they are clearly 1-affine with respect to ι . So the claim reduces to proving that the functor

$$\mathrm{Pr}_{\mathcal{D}(U)}^L \longrightarrow \lim_{[n] \in \Delta} \mathrm{Pr}_{\mathcal{D}(V^{\times_U(n+1)})}^L$$

is an equivalence. Once again, this is a left adjoint functor which sends a categorical $\mathcal{D}(U)$ -module $\mathcal{E}$ to the collection of $\mathcal{D}(V^{\times_U(n+1)})$ -modules $\mathcal{E} \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(n+1)})$. Its right adjoint simply takes a compatible collection of $\mathcal{D}(V^{\times_U(n+1)})$ -modules $\mathcal{E}_n$ and computes the totalization of the diagram $\varphi_* \mathcal{E}_n$ in $\mathrm{Pr}_{\mathcal{D}(U)}^L$. The fact that the unit functor

$$\mathcal{E} \longrightarrow \lim_{[n] \in \Delta} \mathcal{E} \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(n+1)})$$

is an equivalence is essentially due to the fact that $\mathcal{D}$ is a sheaf for the universal $!$ -able topology. Indeed, we argue as follows: we have an equivalence

$$\mathcal{D}(U) \simeq \lim_{[n] \in \Delta}^* \mathcal{D}(V^{\times_U(n+1)}) \simeq \lim_{[n] \in \Delta}^! \mathcal{D}(V^{\times_U(n+1)}),$$

where the superscripts $*$ and $!$ denote the functors with respect to which the limits are computed. Since each $\mathcal{D}(V^{\times_U(n+1)})$ is self-dual over $\mathcal{D}(U)$ (Corollary 3.1.24), we have that

$$\mathcal{E} \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(n+1)}) \simeq \mathrm{Fun}_{\mathcal{D}(U)}^L(\mathcal{D}(V^{\times_U(n+1)}), \mathcal{E})$$

so we have

$$\begin{aligned} \lim_{[n] \in \Delta}^* \mathcal{E} \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(n+1)}) &\simeq \lim_{[n] \in \Delta}^* \mathrm{Fun}_{\mathcal{D}(U)}^L(\mathcal{D}(V^{\times_U(n+1)}), \mathcal{E}) \\ &\simeq \mathrm{Fun}_{\mathcal{D}(U)}^L\left(\mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}}^{*, \vee} \mathcal{D}(V^{\times_U(n+1)}), \mathcal{E}\right) \\ &\simeq \mathrm{Fun}_{\mathcal{D}(U)}^L\left(\mathrm{colim}_{[n] \in \Delta^{\mathrm{op}}}^! \mathcal{D}(V^{\times_U(n+1)}), \mathcal{E}\right) \\ &\simeq \mathrm{Fun}_{\mathcal{D}(U)}^L\left(\lim_{[n] \in \Delta}^! \mathcal{D}(V^{\times_U(n+1)}), \mathcal{E}\right) \\ &\simeq \mathrm{Fun}_{\mathcal{D}(U)}^L(\mathcal{D}(U), \mathcal{E}) \simeq \mathcal{E}. \end{aligned}$$

The fact that for an object $\{\mathcal{E}_n\}_{[n]}$ in $\lim_{[n]} \mathrm{Pr}_{\mathcal{D}(V^{\times_U(n+1)})}^L$ the counit

$$\left\{ \left(\lim_{[m] \in \Delta} \mathcal{E}_m \right) \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(n+1)}) \right\}_{[n] \in \Delta} \longrightarrow \{\mathcal{E}_n\}_{[n] \in \Delta}$$

is an equivalence is a consequence of the fact that each $\mathcal{D}(V^{\times_U(n+1)})$ is dualizable over $\mathcal{D}(U)$. Indeed, it is sufficient that the above functor is an equivalence at the $n = 0$ stage, since for $n \geq 1$ the functor at stage n is obtained by tensoring with $\mathcal{D}(V^{\times_U(n+1)})$. In this case, we have

$$\lim_{[m] \in \Delta} \mathcal{E}_0 \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(m+1)}) \longrightarrow \mathcal{E}_0.$$

But the cosimplicial object $\mathcal{E}_0 \otimes_{\mathcal{D}(U)} \mathcal{D}(V^{\times_U(m+1)})$ is cosplit via the diagonal, with coaugmentation given precisely by $\mathcal{E}_0 \rightarrow \mathcal{E}_0 \otimes_{\mathcal{D}(U)} \mathcal{D}(V)$. Therefore, $\mathcal{E}_0$ is the (universal) limit of such coaugmented cosimplicial diagram, and this finishes the proof. $\square$

In particular from the above proposition we can deduce the following corollary

Corollary 3.1.26. *Every object in $\mathcal{C}$ is 1-affine.*

Remark 3.1.27. In Proposition 3.1.25, it is not strictly necessary to consider the Čech nerve of a covering map. The same proof shows that $\mathcal{D}^{\mathrm{cat}}$ satisfies descent along any simplicial $\mathcal{C}$ -affine diagram whose colimit is U , provided that $\mathcal{D}$ satisfies universal $*$ -descent and universal $!$ -descent along it.

Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumptions 3.1.1, and suppose that Assumption 3.1.13.(1) holds. In order to study the problem of 1-affineness relatively to a dense functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, we shall assume the following.

Assumption 3.1.28.

- (1) *The dense functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is fully faithful.*
- (2) *The ∞ -category $\widehat{\mathcal{C}}$ admits all finite limits, and the dense functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ preserves pullbacks of $\mathcal{C}$ along all maps in E .*
- (3) *If the pullback of a morphism $\alpha: X \rightarrow \iota(U)$ along a map $\iota(W) \rightarrow \iota(U)$ lies in E , then $X \simeq \iota(V)$ and α is the image under ι of a morphism of E .*

We start by extending the six-functor formalism $(\mathcal{D}^*, \mathcal{D}_!)$ from $\mathcal{C}$ to $\widehat{\mathcal{C}}$ along ι , following a recipe due to Heyer and Mann [HM24]

Proposition 3.1.29 ([HM24, Proposition 3.4.2]). *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumption 3.1.13.(1), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor as in Assumption 3.1.28. Let $\bar{E}$ be the class of morphisms $\varphi: X \rightarrow Y$ of $\widehat{\mathcal{C}}$ satisfying the following: for all test spaces U in $\mathcal{C}$ and for all maps $\iota(U) \rightarrow Y$, the pullback $\iota(U) \times_Y X$ is equivalent to $\iota(V)$ for some object V in $\mathcal{C}$ and $\iota(U) \times_Y X \rightarrow \iota(U)$ is the image of a morphism of E under ι . Then we have the following.*

- (1) *The pair $(\widehat{\mathcal{C}}, \bar{E})$ is a geometric setup.*
- (2) *The functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is a morphism of geometric setups ([HM24, §2.1]).*
- (3) *The six-functor formalism $(\mathcal{D}^*, \mathcal{D}_!)$ extends uniquely to a six-functor formalism $(\bar{\mathcal{D}}^*, \bar{\mathcal{D}}_!)$ on $\widehat{\mathcal{C}}$, via a left Kan extension*

$$\bar{\mathcal{D}}(X) := \lim_{\iota(U) \rightarrow X} \mathcal{D}(U).$$

Proof. The first two claims can clearly be proved in the same way as in [HM24, Proposition 3.4.2], while the third claim is just [MAN22, Proposition A.5.16]. $\square$

Proposition 3.1.30 ([HM24, Theorem 3.4.11]). *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumption 3.1.13.(1), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor satisfying the assumptions of Proposition 3.1.29. Let $(\widehat{\mathcal{C}}, \bar{E})$ denote the geometric setup obtained from the geometric $(\mathcal{C}, E)$ as in Proposition 3.1.29. Then there exists a collection of morphisms $\hat{E}$ in $\widehat{\mathcal{C}}$ satisfying the following.*

- (1) *The functor ι defines a morphism of geometric setups $(\mathcal{C}, E) \rightarrow (\widehat{\mathcal{C}}, \bar{E})$, and the six-functor formalism $(\bar{\mathcal{D}}^*, \bar{\mathcal{D}}_!)$ on the source extends uniquely to a six-functor formalism $(\hat{\mathcal{D}}^*, \hat{\mathcal{D}}_!)$ on the target.*
- (2) *The class $\hat{E}$ is $*$ -local on the target: if a morphism $\alpha: X \rightarrow Y$ is such that for every morphism $\iota(U) \rightarrow Y$ the pullback $\iota(U) \times_Y X \rightarrow \iota(U)$ lies in $\hat{E}$, then α is in $\hat{E}$.*
- (3) *The class $\hat{E}$ is $!$ -local on both source and target: if a morphism $\alpha: X \rightarrow Y$ is such that there exists either a small universal $!$ -cover $\{\iota(U_i) \rightarrow Y\}$ such that all morphisms $\iota(U_i) \times_Y X \rightarrow \iota(U_i)$ are in $\hat{E}$, or a small universal $!$ -cover $\{\iota(V_j) \rightarrow X\}$ such that all compositions $\iota(V_j) \rightarrow Y$ are in $\hat{E}$, then α lies in $\hat{E}$.*
- (4) *The class $\hat{E}$ is tame: every morphism $Y \rightarrow \iota(U)$ which lies in $\hat{E}$ is $!$ -locally on the source in E .*

Moreover, such choice can be chosen to be minimal with respect to the inclusion order.

Remark 3.1.31. The recipe in Proposition 3.1.30 extends in a non-trivial way the class of morphisms α for which the functors $\alpha_!$ and $\alpha^!$ are defined, but it does not alter the stable

and presentable ∞ -category $\overline{\mathcal{D}}(X)$ associated to an object X in $\widehat{\mathcal{C}}$. In particular, the stable and presentable ∞ -category $\widehat{\mathcal{D}}(X)$ is once again $\overline{\mathcal{D}}(X)$, which in turn is just $\mathcal{D}(X)$. All in all, Proposition 3.1.30 tells us that the left Kan extension of one of the operations the six-functor formalism is itself part of a six-functor formalism, and there exists a clever way to add maps to the collection $\overline{E}$ in order to keep all the good properties of the six-functor formalism.

Proof. While the statement in [HM24, Theorem 3.4.11] only considers the case where $\widehat{\mathcal{C}}$ is the ∞ -category of sheaves for the universal $!$ -able topology, the proof applies in our generality as well. Indeed, the only point where the proof explicitly relies on the fact that $\widehat{\mathcal{C}} = \mathrm{Shv}_{\tau_{\widehat{\mathcal{C}}}^!}(\mathcal{C})$ is to ensure that the class $\mathcal{S}$ of all possible collections of morphisms $\widehat{E}$ which satisfy Proposition 3.1.30.(1) and Proposition 3.1.30.(4) is not empty. However, as already observed in Proposition 3.1.29, this is true also in our case since $\mathcal{S}$ contains at least $\overline{E}$. $\square$

Remark 3.1.32. If ι satisfies all conditions in Assumption 3.1.28 except for the fully faithfulness, it should still be feasible to adapt the proofs of [HM24, Proposition 3.4.2 and Theorem 3.4.11] obtaining a six-functor formalism $(\widehat{\mathcal{D}}^*, \widehat{\mathcal{D}}_!)$ on $\widehat{\mathcal{C}}$. Although this will not be an extension of the six-functor formalism $(\mathcal{D}^*, \mathcal{D}_!)$ anymore, it still is the universal six-functor formalism making the diagram of ∞ -operads

$$\begin{array}{ccc} \mathrm{Corr}(\mathcal{C})_{E, \mathrm{all}}^{\otimes} & \xrightarrow{(\mathcal{D}^*, \mathcal{D}_!)} & \mathrm{Pr}_{\mathcal{V}}^{L, \otimes} \\ \downarrow \iota & \nearrow & \\ \mathrm{Corr}(\widehat{\mathcal{C}})_{\overline{E}, \mathrm{all}}^{\otimes} & & (\widehat{\mathcal{D}}^*, \widehat{\mathcal{D}}_!) \end{array}$$

commute. However, since the Grothendieck topology $\tau_{\widehat{\mathcal{C}}}^!$ is required to be sub-canonical, and since we want $\widehat{\mathcal{D}}(\iota(U))$ to be the same as $\mathcal{D}(U)$ for any test space U , we do not investigate the matter further.

Remark 3.1.33. With the general recipe of Proposition 3.1.30, the six-functor formalism can be extended to the geometric setup $(\widehat{\mathcal{C}}, \widehat{E})$. So it makes sense to define universal $!$ -able covers also in $\widehat{\mathcal{C}}$.

The following proposition is close in spirit to the results studied in [Kes25] concerning categorical Künneth formulas for analytic stacks, and especially to [Kes25, Theorem 3.26]. The proof is almost the same as the one in *loc. cit.*, but we still provide a proof for the reader's convenience.

Proposition 3.1.34. *Let $\mathcal{C}$, $\mathcal{V}$, and $\mathcal{D}$ be as in Assumption 3.1.13.(1), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense embedding as in Assumption 3.1.28. Let Y be an object in $\widehat{\mathcal{C}}$ that can be realized as the colimit $\operatorname{colim}_{[n] \in \Delta^{\text{op}}} W_{i_1 \dots i_n}$ of a simplicial diagram in which each $W_{i_1 \dots i_n}$ is $\mathcal{C}$ -affine, and along which $\mathcal{D}(-)$ satisfies universal $*$ -descent and universal $!$ -descent. Then, for every pair of indices α, β in the simplicial diagram, the natural functor*

$$\mathcal{D}(\iota(W_\alpha) \times_Y \iota(W_\beta)) \longrightarrow \mathcal{D}(W_\alpha) \otimes_{\mathcal{D}(Y)} \mathcal{D}(W_\beta)$$

is an equivalence.

Proof. Let us fix some notations first. For α and β as in the statement, write

$$\begin{aligned} W_{\alpha_1 \dots \alpha_n} & \text{ for } W_\alpha \times_Y W_{i_1 \dots i_n}, \\ W_{\beta_1 \dots \beta_n} & \text{ for } \coprod_{\{i_1, \dots, i_n\} \subseteq I} W_\beta \times_Y W_{i_1 \dots i_n}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{D}(\iota(W_\alpha)) \otimes_{\mathcal{D}(Y)} \mathcal{D}(\iota(W_\beta)) & \simeq \mathcal{D}(W_\alpha) \otimes_{\mathcal{D}(Y)} \mathcal{D}(W_\beta) \\ & \simeq \operatorname{colim}_{[n] \in \Delta^{\text{op}}} \mathcal{D}(W_\alpha) \otimes_{\mathcal{V}} \mathcal{D}(Y)^{\otimes n} \otimes_{\mathcal{V}} \mathcal{D}(W_\beta). \end{aligned}$$

Using descent along the universal $!$ -able covers $\{W_i \rightarrow Y\}$, $\{W_\alpha \times_Y W_i \rightarrow W_\alpha\}$ and $\{W_\beta \times_Y W_i \rightarrow W_\beta\}$ we deduce

$$\begin{aligned} \mathcal{D}(W_\alpha) & \simeq \lim_{[m] \in \Delta}^* \mathcal{D}(W_{\alpha_1 \dots \alpha_m}) \simeq \lim_{[m] \in \Delta}^! \mathcal{D}(W_{\alpha_1 \dots \alpha_m}) \simeq \operatorname{colim}_{[m] \in \Delta^{\text{op}}}^! \mathcal{D}(W_{\alpha_1 \dots \alpha_m}), \\ \mathcal{D}(Y) & \simeq \lim_{[m] \in \Delta}^* \mathcal{D}(W_{i_1 \dots i_m}) \simeq \lim_{[m] \in \Delta}^! \mathcal{D}(W_{i_1 \dots i_m}) \simeq \operatorname{colim}_{[m] \in \Delta^{\text{op}}}^! \mathcal{D}(W_{i_1 \dots i_m}), \\ \mathcal{D}(W_\beta) & \simeq \lim_{[m] \in \Delta}^* \mathcal{D}(W_{\beta_1 \dots \beta_m}) \simeq \lim_{[m] \in \Delta}^! \mathcal{D}(W_{\beta_1 \dots \beta_m}) \simeq \operatorname{colim}_{[m] \in \Delta^{\text{op}}}^! \mathcal{D}(W_{\beta_1 \dots \beta_m}). \end{aligned}$$

Moreover, since Δ^{op} is sifted, $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}, \times n}$ is a final functor for all integers n , thus we have

$$\mathcal{D}(Y)^{\otimes n} \simeq \left(\operatorname{colim}_{[m] \in \Delta^{\text{op}}}^! \mathcal{D}(W_{i_1 \dots i_m}) \right)^{\otimes n} \simeq \operatorname{colim}_{[m] \in \Delta^{\text{op}}}^! (\mathcal{D}(W_{i_1 \dots i_m})^{\otimes n}).$$

All things considered, we can write

$$\begin{aligned}
\operatorname{colim}_{[n] \in \Delta^{\text{op}}} \mathcal{D}(W_\alpha) \otimes_{\mathcal{V}} \mathcal{D}(Y)^{\otimes n} \otimes_{\mathcal{V}} \mathcal{D}(W_\beta) &\simeq \operatorname{colim}_{\Delta^{\text{op}}, \times 4} \mathcal{D}(W_{\alpha_1 \dots \alpha_{m_1}}) \otimes_{\mathcal{V}} \mathcal{D}(W_{i_1 \dots i_{m_2}})^{\otimes n} \otimes_{\mathcal{V}} \mathcal{D}(W_{\beta_1 \dots \beta_{m_3}}) \\
&\simeq \operatorname{colim}_{\Delta^{\text{op}}, \times 2} \mathcal{D}(W_{\alpha_1 \dots \alpha_m}) \otimes_{\mathcal{V}} \mathcal{D}(W_{i_1 \dots i_m})^{\otimes n} \otimes_{\mathcal{V}} \mathcal{D}(W_{\beta_1 \dots \beta_m}) \\
&\simeq \operatorname{colim}_{[m] \in \Delta^{\text{op}}} \mathcal{D}(W_{\alpha_1 \dots \alpha_m}) \otimes_{\mathcal{D}(W_{i_1 \dots i_m})} \mathcal{D}(W_{\beta_1 \dots \beta_m}) \\
&\simeq \operatorname{colim}_{[m] \in \Delta^{\text{op}}} \mathcal{D} \left(W_{\alpha_1 \dots \alpha_m} \times_{W_{i_1 \dots i_m}} W_{\beta_1 \dots \beta_m} \right).
\end{aligned}$$

Notice that in the second equivalence we used once again that Δ^{op} is sifted, while in the last equivalence we used the fact that the six-functor formalism $(\mathcal{D}^*, \mathcal{D}_!)$ is assumed to be strongly monoidal, and thus $\mathcal{D}$ sends fiber products of objects in $\mathcal{C}$ to relative tensor products of presentably symmetric monoidal ∞ -categories (see discussion in §3.1.19). We now observe that

$$\begin{aligned}
W_{\alpha_1 \dots \alpha_m} \times_{W_{i_1 \dots i_m}} W_{\beta_1 \dots \beta_m} &:= W_\alpha \times_Y W_{i_1 \dots i_n} \times_{W_{i_1 \dots i_m}} W_\beta \times_Y W_{i_1 \dots i_m} \\
&\simeq W_\alpha \times_Y W_\beta \times_Y W_{i_1 \dots i_m}.
\end{aligned}$$

Since $\mathcal{D}(-)$ satisfies universal $!$ -descent along the diagram $\operatorname{colim}_{[n] \in \Delta^{\text{op}}} W_{i_1 \dots i_n}$ using $!$ -descent we conclude as desired. $\square$

Proposition 3.1.34 is the main stepping stone in proving the following theorem.

Theorem 3.1.35. *Let $\mathcal{C}$, $\mathcal{V}$, $\mathcal{D}$ be as in Assumption 3.1.13.(1), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense embedding as in Assumption 3.1.28. Let Y be an object in $\widehat{\mathcal{C}}$, and suppose that Y satisfies the hypothesis in Proposition 3.1.34. Then Y is 1-affine relatively to ι .*

Remark 3.1.36. We observe that the hypotheses of Proposition 3.1.34 are satisfied, in particular, when Y admits a universal $!$ -able cover of the form $\{\iota(W_i) \rightarrow Y\}_{i \in I}$ such that all n -fold fiber products $\iota(W_{i_1}) \times_Y \dots \times_Y \iota(W_{i_n})$ are again $\mathcal{C}$ -affine. In this case, the simplicial diagram under consideration is the Čech nerve of this cover.

Proof. We endow the ∞ -category $\widehat{\mathcal{C}}$ of the universal $!$ -able topology for the six-functor formalism established in Proposition 3.1.30. As explained in [HM24, Theorem 3.4.11], the functor $\mathcal{D}$ is again a sheaf for the universal $!$ -able topology on $\widehat{\mathcal{C}}$, since $\mathcal{D}$ was a sheaf in the first place (Proposition 3.1.25).

In order to prove the 1-affineness of Y , we simply need to prove that the functor $\operatorname{Pr}_{\mathcal{D}(-)}^{\text{L}}$ satisfies descent along the cover $\{\iota(W_i) \rightarrow Y\}$. Once we know this, since all $\mathcal{C}$ -

affine objects are trivially 1-affine relatively to ι , we would have a chain of equivalences

$$\begin{aligned} \mathcal{D}^{\text{cat}}(Y) &:= \lim_{[n] \in \Delta^{\text{op}}} \mathcal{D}^{\text{cat}}(W_{i_1 \dots i_n}) \\ &\simeq \lim_{[n] \in \Delta^{\text{op}}} \text{Pr}_{\mathcal{D}(W_{i_1 \dots i_n})}^{\text{L}} \\ &\simeq \text{Pr}_{\mathcal{D}(Y)}^{\text{L}}. \end{aligned}$$

Notice that, while it is true that we have at our disposal a six-functor formalism $(\widehat{\mathcal{D}}^*, \widehat{\mathcal{D}}_!)$ on all $\widehat{\mathcal{C}}$ which extends the six-functor formalism $(\mathcal{D}^*, \mathcal{D}_!)$ over $\mathcal{C}$, it is not true that the six-functor formalism $(\widehat{\mathcal{D}}^*, \widehat{\mathcal{D}}_!)$ is locally strongly monoidal (in the sense of Paragraph 3.1.19) anymore. It follows that, in general, for a map $Z \rightarrow Y$ lying in $\widehat{E}$ the categorical $\mathcal{D}(Y)$ -module $\mathcal{D}(Z)$ will not be self-dual anymore. But Proposition 3.1.34 is precisely the ingredient that allows us to deduce that each $\mathcal{D}(W_i)$ is self-dual as a categorical $\mathcal{D}(Y)$ -module. Indeed, let $\langle \iota(W_i) \rangle_{/Y}$ be the smallest sub- ∞ -category of $\widehat{\mathcal{C}}_{/Y}$ containing all maps $\iota(W_{i_1 \dots i_n}) \rightarrow Y$ and the identity $Y = Y$. Then Proposition 3.1.34 means that the composition

$$\text{Corr}(\langle \iota(W_i) \rangle_{/Y})^{\otimes} \longrightarrow \text{Corr}(\widehat{\mathcal{C}}_{/V})_{\widehat{E}, \text{all}}^{\otimes} \xrightarrow{(\widehat{\mathcal{D}}^*, \widehat{\mathcal{D}}_!)} \text{Pr}_{\mathcal{D}(Y)}^{\text{L}, \otimes}$$

is now strongly monoidal, hence it preserves duals, and so we can safely apply Proposition 3.1.22 to establish that each $\mathcal{D}(W_i)$ is self-dual as a categorical $\mathcal{D}(Y)$ -module. At this point, we can follow the same strategy of proof of Proposition 3.1.25 (see also Remark 3.1.27) to deduce that the natural functor

$$\text{Pr}_{\mathcal{D}(Y)}^{\text{L}} \longrightarrow \lim_{\iota(W_i) \rightarrow Y} \text{Pr}_{\mathcal{D}(W_i)}^{\text{L}}$$

is an equivalence, and so we conclude. $\square$

3.1.3 Categoricalized sheaves under Assumption 3.1.13.(2)

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumptions 3.1.1. Assume moreover that Assumption 3.1.13.(2) holds: this means that for any morphism of test spaces $\varphi: U \rightarrow V$ the adjunction $\varphi^* \dashv \varphi_*$ can be interpreted as a free-forgetful adjunction of the form

$$- \otimes_{D(V)} D(U): \text{Mod}_{D(V)}(\mathcal{V}) \xrightleftharpoons{\perp} \text{Mod}_{D(U)}(\mathcal{V}) : \text{oblv}_{D(U)}.$$

In particular, the right adjoint commutes with *all* limits and colimits and it is lax monoidal ([HHLN23, Proposition A]). It follows that $\text{oblv}_{D(U)} D(U)$ is still a commu-

tative algebra inside $\mathrm{Mod}_{D(V)}(\mathcal{V})$ (equivalently, it is a commutative $D(V)$ -algebra). By abuse of notation, we shall still denote it as $D(U)$.

Definition 3.1.37 ([Mat16, Definition 3.18]). Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumptions 3.1.1. We say that a morphism of test spaces $\varphi: U \rightarrow V$ is *descendable* if the thick $\otimes$ -ideal generated by $f_*\mathbb{1}_{\mathcal{D}(U)}$ in $\mathrm{Mod}_{D(V)}$ coincides with the whole $\mathrm{Mod}_{D(V)}$.

The following is a straight-forward generalization of [Mat16, Proposition 3.45].

Proposition 3.1.38. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{V}$ be as in Assumptions 3.1.1, and assume that Assumption 3.1.13.(2) holds. Assume moreover that any morphism of test spaces $\varphi: U \rightarrow V$ belonging to S is descendable. Then the functor (3.1) is a sheaf for the Grothendieck topology on $\mathcal{C}$.*

Proof. This is a formal consequence of descent for ∞ -categories of modules, as showed in [Mat16, Proposition 3.45]. There, this assertion is stated in the particular setting when $\mathcal{V} = \mathrm{Sp}$ is the ∞ -category of spectra. However, its proof only relies on the notion of descendability (that makes sense also in our more general setting), on Barr–Beck–Lurie’s (co)monadicity theorem (which holds for any comonad over any ∞ -category), and on the fact that given a map of test spaces $\varphi: U \rightarrow V$ which is descendable the functor $\mathcal{D}(-)$ satisfies descent along φ (which is a piece of datum already in Assumptions 3.1.1).

In particular, that proof can be carried out *verbatim* replacing Sp with $\mathcal{V}$. $\square$

Remark 3.1.39. Assuming that $\mathcal{V}$ admits a left complete Y -structure, one could also expand the result of Proposition 3.1.38 using the theory of *faithfully flat monads*. Namely, for $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ as in Assumptions 3.1.1, with $\mathcal{D}$ further satisfying Assumption 3.1.13.(2), assume that for any morphism of test spaces $\varphi: U \rightarrow V$ the monadic adjunction $\varphi^* \dashv \varphi_*$ is faithfully flat in the sense of [Lur18, Definition D.6.4.1]. Then the functor (3.1) is a sheaf. Already when $\mathcal{V}$ is the ∞ -category of spectra, descendable morphisms and faithfully flat monads are not known to be compatible in general: for example, faithfully flat morphisms of commutative ring spectra always yield faithfully flat monads, but without some finiteness assumptions they are still not known to be descendable in the sense of Definition 3.1.37. On the converse, descendable morphisms of commutative ring spectra need not to be faithfully flat ([Lur18, Remark D.3.3.3]).

In this level of generality, studying the property of 1-affineness for objects in $\widehat{\mathcal{C}}$ is very cumbersome. It turns out that both in the algebro-geometric and in the analytic settings the key property that we need is that, among all maps that form coverings for the topology on affine schemes, we can single out some particularly nice class of maps (the Zariski open immersions). Analytic open immersions are defined for analytic stacks generalizing some homological properties of pullback functors along finite open

immersions (Definition 3.2.13.(2)). We use instead a different notion, already appeared in the literature as *homotopy monomorphism* ([Sav24; BKK24]), which asks instead for some good property with respect to self-intersections.

Definition 3.1.40. Let $\mathcal{C}$ be an ∞ -category, and let $\alpha: U \rightarrow X$ be a morphism. We say that α is a *Zariski morphism* if the diagonal map $U \rightarrow U \times_X U$ is an equivalence.

Remark 3.1.41.

- (1) Let $\mathcal{C}$ be a ∞ -category. By a straightforward argument using the pasting law for pullbacks, it is easily seen that Zariski morphisms are stable under pullbacks and enjoy the following property: if $\varphi: Y \rightarrow X$ and $\psi: Z \rightarrow Y$ are morphisms of $\mathcal{C}$ such that φ is a Zariski morphism, then $\varphi \circ \psi$ is Zariski if and only if ψ is Zariski. In particular, Zariski morphisms are closed under composition, are right cancellative, and if $\varphi: Y \rightarrow X$ and $\psi: Z \rightarrow Y$ are Zariski morphisms then we have

$$Z \simeq Z \times_Y Z \simeq Z \times_X Z. \quad (3.8)$$

- (2) Let $\mathcal{C}$ be a ∞ -category, and let $\alpha: U \rightarrow X$ and $\beta: V \rightarrow X$ be two Zariski morphism. One can define the union of U and V as

$$U \cup V := U \coprod_{U \times_X V} V.$$

It comes provided by a natural map $\gamma: U \cup V \rightarrow X$. If colimits in $\mathcal{C}$ are universal, we have that γ is Zariski itself: indeed, in this case we have

$$\begin{aligned} (U \cup V) \times_X (U \cup V) &:= \left(U \coprod_{U \times_X V} V \right) \times_X \left(U \coprod_{U \times_X V} V \right) \\ &\simeq \left((U \times_X U) \coprod_{U \times_X V} (V \times_X U) \right) \coprod_{U \times_X V} \left((U \times_X V) \coprod_{U \times_X V} (V \times_X V) \right) \\ &\simeq U \coprod_{U \times_X V} V. \end{aligned}$$

It follows from Remark 3.1.41.(1) that also the inclusions $U \rightarrow U \cup V$ and $V \rightarrow U \cup V$ are Zariski morphisms.

- (3) Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.13.(2), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor such that colimits in $\widehat{\mathcal{C}}$ are universal. If $\alpha: \iota(U) \rightarrow X$ and $\beta: \iota(V) \rightarrow X$ are Zariski morphisms which form a $\mathcal{C}$ -affine atlas for X (Definition 3.1.14.(3)) then $\iota(U) \rightarrow \iota(U) \cup \iota(V)$ and $\iota(V) \rightarrow \iota(U) \cup \iota(V)$ form a $\mathcal{C}$ -affine atlas for $\iota(U) \cup \iota(V)$

as well: this is a clear consequence of the fact that $\iota(U) \rightarrow \iota(U) \cup \iota(V)$ and $\iota(V) \rightarrow \iota(U) \cup \iota(V)$ are Zariski morphisms, together with the fact that all squares in the following diagram

$$\begin{array}{ccccc}
 \iota(U) \times_X \iota(W) & \longrightarrow & \iota(W) & \longrightarrow & \iota(W) \\
 \downarrow & & \downarrow & & \downarrow \\
 \iota(U) & \longrightarrow & \iota(U) \cup \iota(V) & \longrightarrow & \iota(U) \cup \iota(V) \\
 \downarrow & & \downarrow & & \downarrow \\
 \iota(U) & \longrightarrow & \iota(U) \cup \iota(V) & \longrightarrow & X
 \end{array}$$

are pullbacks. By an easy inductive argument (using the local property of covering sieves in ∞ -sites) it follows that if $\{\iota(U_i) \rightarrow X\}_{i=1}^k$ is a $\mathcal{C}$ -affine atlas of Zariski morphisms, and W_i denotes the union

$$\iota(U_1) \cup \cdots \cup \widehat{\iota(U_i)} \cup \cdots \cup \iota(U_k),$$

where the i -th element is omitted, then $\iota(U_i) \rightarrow X$ and $W_i \rightarrow X$ still form a $\mathcal{C}$ -affine atlas of Zariski morphisms for X .

- (4) If $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ are as in Assumption 3.1.13.(2), and if the functor

$$D(-): \mathcal{C}^{\text{op}} \longrightarrow \text{CAlg}(\mathcal{V})$$

sends finite limits of $\mathcal{C}^{\text{op}}$ to finite colimits of $\text{CAlg}(\mathcal{V})$, then for any Zariski morphism of test spaces $\varphi: U \rightarrow V$ we have that the forgetful functor in the adjunction

$$- \otimes_{D(V)} D(U): \text{Mod}_{D(V)}(\mathcal{V}) \xrightleftharpoons{\perp} \text{Mod}_{D(U)}(\mathcal{V}) : \text{oblv}_{D(U)}$$

is fully faithful. Indeed, if $(X_i)_{i \in I}$ is a set of compact generators of $\mathcal{V}$, the compact generators of $\text{Mod}_{D(U)}(\mathcal{V})$ is given by $(X_i \otimes_{\mathcal{V}} D(U))_{i \in I}$: in particular, on any generator of this form the counit of the adjunction $- \otimes_{D(V)} D(U) \dashv \text{oblv}_{D(U)}$

$$\text{oblv}_{D(U)}(X_i \otimes_{\mathcal{V}} D(U)) \otimes_{D(V)} D(U) \longrightarrow X_i \otimes_{\mathcal{V}} D(U)$$

is readily seen to be an equivalence because $D(U) \otimes_{D(V)} D(U) \simeq D(U \times_V U) \simeq D(U)$. The equivalence on *all* objects of $\text{Mod}_{D(U)}(\mathcal{V})$ follows from the fact that both functors commute with colimits.

The fundamental property of Zariski morphisms is that if an object X in $\widehat{\mathcal{C}}$ admits a

$\mathcal{C}$ -affine atlas comprised of a single map $\iota(W) \rightarrow X$ which is a Zariski morphism, then X is itself $\mathcal{C}$ -affine. Moreover, if X admits a finite $\mathcal{C}$ -affine atlas $\{\alpha_i: \iota(W_i) \rightarrow X\}$ where each α_i is a Zariski morphism, then X can be expressed as a *finite* colimit of objects in the essential image of ι .

Proposition 3.1.42. *Let $\mathcal{C}$ be as in Assumptions 3.1.1, and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor. Assume that colimits in $\widehat{\mathcal{C}}$ are universal.*

Consider an object X in $\widehat{\mathcal{C}}$ admitting a $\mathcal{C}$ -affine atlas $\{\alpha_i: \iota(W_i) \rightarrow X\}_{i=1}^k$ where each α_i is a Zariski morphism. Let $\varphi: \coprod_i \iota(W_i) \rightarrow X$ be the induced map from the disjoint union, and let $\check{C}(\varphi) \rightarrow X$ be its Čech nerve. Then the maps in the natural composition

$$\operatorname{colim}_{\Delta^{\text{op}}} \check{C}(\varphi) \longrightarrow \operatorname{colim}_{\Delta_{\leq k-1}^{\text{op}}} \check{C}(\varphi) \longrightarrow X$$

are equivalences. In particular, if $k = 1$ then $X \simeq \iota(W_1)$.

Proof. Since each α_i is a Zariski morphism, for all $h \geq k$ we have an equivalence

$$\iota(W_{i_1}) \times_X \cdots \times_X \iota(W_{i_h}) \simeq \iota(W_1) \times_X \cdots \times_X \iota(W_k).$$

Thus, the simplicial diagram corresponding to the Čech nerve $\check{C}(\varphi)$ is constant for all $k \geq m - 1$. In particular, it is $(m - 1)$ -skeletal, in the sense that the simplicial diagram $\Delta^{\text{op}} \rightarrow \widehat{\mathcal{C}}$ is a left Kan extension of its restriction to $\Delta_{\leq k}^{\text{op}}$. Since left Kan extensions along fully faithful functors do not alter colimits, the first map is naturally an equivalence. So, it is sufficient to notice that this colimit is just X because of Remark 3.1.15. $\square$

Proposition 3.1.42 offers a key technical advantage in working with objects of $\widehat{\mathcal{C}}$: when an object X of $\widehat{\mathcal{C}}$ admits a *finite* $\mathcal{C}$ -affine atlas consisting only of Zariski morphisms, we are often allowed to run an inductive strategy on the cardinality of the $\mathcal{C}$ -affine cover, starting by proving our statements when objects are $\mathcal{C}$ -affine. We therefore introduce the following piece of terminology.

Definition 3.1.43. Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.13.(2), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor.

- (1) We say that an object X in $\widehat{\mathcal{C}}$ is *quasi-compact* if it admits a *finite* affine $\mathcal{C}$ -atlas $\{\alpha_i: \iota(W_i) \rightarrow X\}$ such that each α_i is a Zariski morphism.
- (2) We say that an object X in $\widehat{\mathcal{C}}$ is *semi-separated* if all Zariski morphisms $\iota(U) \rightarrow X$ are $\mathcal{C}$ -affine.
- (3) A map $\alpha: X \rightarrow Y$ is called $\mathcal{C}$ -affine if it is representable by $\mathcal{C}$ -affine objects (i.e. every pullback of an $\mathcal{C}$ -affine object is still $\mathcal{C}$ -affine).

Remark 3.1.44.

- (1) Given a dense functor $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, then any $\mathcal{C}$ -affine object is always quasi-compact. If ι preserves pullbacks, then any $\mathcal{C}$ -affine object is also semi-separated.
- (2) If X is semi-separated and $\alpha: Y \rightarrow X$ is a Zariski morphism, then Y is semi-separated as well. Indeed, if $\iota(U) \rightarrow Y$ is a Zariski morphism, then $\iota(U) \rightarrow X$ is a Zariski morphism and $\iota(U) \times_X Y \simeq \iota(U)$. So, for any map $\iota(W) \rightarrow Y$ we have an equivalence

$$\iota(W) \times_Y \iota(U) \simeq \iota(W) \times_Y (Y \times_X \iota(U)) \simeq \iota(W) \times_X \iota(U),$$

which is $\mathcal{C}$ -affine.

- (3) If X is quasi-compact and $\alpha: Y \rightarrow X$ is a $\mathcal{C}$ -affine Zariski morphism, then Y is quasi-compact as well. Indeed, pulling back any finite $\mathcal{C}$ -affine atlas of Zariski morphism for X yields a finite $\mathcal{C}$ -affine atlas of Zariski morphisms for Y .

In order to obtain the best behavior of Zariski morphisms, for the rest of this section we add the following assumptions to our setting.

Assumption 3.1.45. Let $\mathcal{C}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.13.(2), and let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ be a dense functor. We assume that ι preserves finite limits existing in $\mathcal{C}$, that colimits in $\widehat{\mathcal{C}}$ are universal, and that the sheaf of commutative algebras

$$D(-): \mathcal{C}^{\text{op}} \longrightarrow \text{CAlg}(\mathcal{V})$$

sends finite limits of $\mathcal{C}$ to finite colimits in $\text{CAlg}(\mathcal{V})$.

Our main 1-affineness result of this section can then be stated as follows.

Theorem 3.1.46. Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45, and assume that $\mathcal{V}$ is rigid. Let X be an object in $\widehat{\mathcal{C}}$ admitting a finite affine $\mathcal{C}$ -atlas $\{\varphi_i: \iota(W_i) \rightarrow X\}_{i=1}^k$ such that each φ_i is a Zariski morphism, and such that for all subsets of indices $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, k\}$ the functor

$$\varphi_I^*: \mathcal{D}(\iota(W_{i_1}) \times_X \cdots \times_X \iota(W_{i_n})) \longrightarrow \mathcal{D}(X)$$

admits a left adjoint $\varphi_{I, \natural}$. Then X is 1-affine.

Remark 3.1.47. The condition on the existence of left adjoints for pullback functors seems a bit unnatural. However, this is what happens for finite flat morphisms in algebraic

geometry, for analytic open immersions in rigid analytic geometry (Proposition 3.3.28), and in general for open immersions of affine analytic stacks (Definition 3.2.13.(2)).

The proof of Theorem 3.1.46 is quite complex and long, and relies on many auxiliary results that allow us to extend the good properties of Zariski open immersions in algebraic geometry to our abstract framework. We prove them following the same stream of ideas of [MAN22, Lemma 2.4.16] and of [Soo25, Lemma 3.2].

Proposition 3.1.48. *Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45. Let $\alpha: Y \rightarrow X$ be a morphism between quasi-compact and semi-separated objects of $\widehat{\mathcal{C}}$.*

(1) (Base change) *Let $\beta: \iota(V) \rightarrow X$ be a morphism, and consider the diagram*

$$\begin{array}{ccc} Y \times_X \iota(V) & \xrightarrow{\alpha'} & \iota(V) \\ \beta' \downarrow & & \downarrow \beta \\ Y & \xrightarrow{\alpha} & X. \end{array}$$

Then for every object M in $\mathcal{D}(Y)$ the canonical map

$$\beta^* \alpha_* M \longrightarrow \alpha'_* \beta'^* M \quad (3.9)$$

is an equivalence.

(2) *The functor α_* commutes with colimits. If α is Zariski, then α_* is fully faithful. Moreover if α is a $\mathcal{C}$ -affine morphism then α_* is conservative. In particular, it is both monadic and comonadic functor.*

We begin with the following lemma

Lemma 3.1.49. *Let $\alpha: Y \rightarrow X$ a map in $\mathcal{C}$ and $\{\varphi: \coprod U_i \rightarrow X\}$ a finite $\mathcal{C}$ -affine cover. For every finite subset $J = \{i_1, \dots, i_m\}$ of $\{1, \dots, n\}$ we denote with U_I the intersection $U_{i_1} \times_X \dots \times_X U_{i_m}$ and we denote with V_I the following pullback*

$$\begin{array}{ccc} V_I & \xrightarrow{\alpha_I} & U_I \\ \bar{\varphi}_I \downarrow & & \downarrow \varphi_I \\ Y & \xrightarrow{\alpha} & X \end{array} \quad (3.10)$$

(1) *For every M in $\mathcal{D}(Y)$ $\alpha_* M$ is equivalent to the limit $\lim_I \varphi_{I,*} \alpha_{J,*} \bar{\varphi}_I^* M$.*

(2) *The base change property holds for the diagram in Equation (3.10)*

Proof. To prove the first part we observe that every M in $\mathcal{D}(Y)$ can be written as a finite limit over the Čech nerve of the $\mathcal{C}$ -affine cover $\{\coprod V_i \rightarrow Y\}$, $\lim_I \bar{\varphi}_{I,*} \bar{\varphi}_I^* M$. For every N in $\mathcal{D}(X)$ we have the following equivalences

$$\begin{aligned} \mathrm{Map}(N, \alpha_* M) &\simeq \mathrm{Map}(\alpha^* N, \lim_I \bar{\varphi}_{I,*} \bar{\varphi}_I^* M) \\ &\simeq \lim_I \mathrm{Map}(\bar{\varphi}_I^* \alpha^* N, \bar{\varphi}_I^* M) \\ &\simeq \lim_I \mathrm{Map}(\alpha_I^* \varphi_I^* N, \bar{\varphi}_I^* M) \\ &\simeq \lim_I \mathrm{Map}(N, \varphi_{I,*} \alpha_{I,*} \bar{\varphi}_I^* M) \\ &\simeq \mathrm{Map}(N, \lim_I \varphi_{I,*} \alpha_{I,*} \bar{\varphi}_I^* M). \end{aligned}$$

The Yoneda embedding implies the equivalence in Lemma 3.1.49.(1). To prove the second part of the lemma, for every M in $\mathcal{D}(Y)$, we use the first part and Proposition 3.1.42, to identify $\alpha_*(M)$ as the object in the projective system given by $\lim_I \alpha_{I,*} \bar{\varphi}_I^*(M)$. This implies the base-change property. $\square$

Proof of Proposition 3.1.48. We first observe that we can reduce to the case where Y is $\mathcal{C}$ -affine. Indeed we can cover Y with a finite $\mathcal{C}$ -affine cover $\{W_i \rightarrow Y\}_{i \in \{1, \dots, n\}}$. For every finite subset $I = \{i_1, \dots, i_m\}$ of $\{1, \dots, n\}$ we denote with W_I the intersection $W_{i_1} \times_X \cdots \times_X W_{i_m}$. We moreover consider the following pullback diagram

$$\begin{array}{ccccc} W_I \times_X \iota(V) & \longrightarrow & Y \times_X \iota(V) & \longrightarrow & \iota(V) \\ \downarrow & & \downarrow & & \downarrow \beta \\ W_I & \longrightarrow & Y & \xrightarrow{\alpha} & X \end{array}$$

Using Proposition 3.1.42 we deduce that α_* can be described as a finite limit $\lim_I \alpha_{I,*}$ where α_I denotes the composition $W_I \rightarrow Y \rightarrow X$.

We observe that in this case the pullback $W_i \times_X \iota(V)$ is $\mathcal{C}$ -affine cover of $Y \times_X \iota(V)$. Since we are working in a stable setting the functors considered in the base-change formula commute with finite limits, in this way we can reduce to verify the formula in the case where Y is $\mathcal{C}$ -affine.

Now assuming that Y is $\mathcal{C}$ -affine and using the notations as in Lemma 3.1.49 we

consider the following diagram

$$\begin{array}{ccccc}
 V_I \times_X \iota(V) & \xrightarrow{\quad} & U_I \times_X \iota(V) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & \iota(V) \times_X Y & \xrightarrow{\alpha'} & \iota(V) & \\
 & \downarrow \beta' & & \downarrow \beta & \\
 V_I & \xrightarrow{\alpha_I} & U_I & & \\
 \searrow \bar{\varphi}_I & & \searrow \varphi_I & & \\
 & Y & \xrightarrow{\alpha} & X &
 \end{array}$$

Since U_I is part of the Čech nerve of a cover of X , Equation (1.18) can be proved after pullback to every $U_I \times_X \iota(V)$. Observing that X is semi-separated and Y is assumed to be $\mathcal{C}$ -affine we obtain that every cartesian product and also every U_I and V_I are $\mathcal{C}$ -affine. In this way using Lemma 3.1.49 we can reduce to the case where both X and Y are $\mathcal{C}$ -affine. When $\alpha: \iota(W) \rightarrow \iota(U)$ is a morphism between $\mathcal{C}$ -affine objects, the claim is trivial because $Y \times_X \iota(V)$ is just the object $\iota(W \times_U V)$ and so the base change equivalence

$$M \otimes_{D(W)} D(W \times_U V) \simeq M \otimes_{D(W)} D(W) \otimes_{D(U)} D(V) \longrightarrow M \otimes_{D(U)} D(V)$$

reduces to the associativity of relative tensor products.

To prove Proposition 3.1.48.(2), we start by proving that α_* commutes with colimits. Using the notations in Lemma 3.1.49 we reduce to show that for every $I = \{i_1, \dots, i_n\}$ $\varphi^* \alpha_*$ commutes with colimits, in this way we reduce to the case where X is $\mathcal{C}$ -affine. Using the base change property in Lemma 3.1.49 we are reduced to check the analogous statement when $Y \simeq \iota(W)$ is $\mathcal{C}$ -affine: in this case, the claim is obvious since α_* is a forgetful functor between module ∞ -categories (Remark 3.1.41.(4)).

We now prove the fully faithfulness of α_* in the case $\alpha: Y \rightarrow X$ is a Zariski morphism and $\{j_i: W_i \rightarrow Y\}_{i \in \{1, \dots, n\}}$ a $\mathcal{C}$ -affine cover of Y . For any M in $\mathcal{D}(Y)$ write

$$M \simeq \lim_{[n] \in \Delta_{\leq k}} j_{i_1 \dots i_n}^* j_{i_1 \dots i_n}^* M.$$

So, in the counit morphism $\iota^* \iota_* M \rightarrow M$ for the adjunction $\alpha^* \dashv \alpha_*$, the source can be

written as

$$\begin{aligned}
\alpha^* \alpha_* M &\simeq \alpha^* \alpha_* \left(\lim_{[n] \in \Delta} j_{i_1 \dots i_n, *} j_{i_1 \dots i_n}^* M \right) \\
&\simeq \lim_{[n] \in \Delta} \left(\alpha^* \alpha_* j_{i_1 \dots i_n, *} j_{i_1 \dots i_n}^* M \right) \\
&\simeq \lim_{[n] \in \Delta_{\leq k}} \left(\alpha^* \alpha_{i_1 \dots i_n, *} j_{i_1 \dots i_n}^* M \right) \\
&\simeq \lim_{[n] \in \Delta_{\leq k}} j_{i_1 \dots i_n, *} j_{i_1 \dots i_n}^* M \simeq M.
\end{aligned}$$

Notice that in the equivalence in the last row we used both Proposition 3.1.48.(1), and the fact that $\iota(W_i) \times_X Y \simeq \iota(W_i)$ (Remark 3.1.41.(1)).

We finish the proof proving that if $\alpha: Y \rightarrow X$ is $\mathcal{C}$ -affine then α_* is conservatives. Considering an affine cover of X and using the fact that α is $\mathcal{C}$ -affine we can reduce to the case where both Y and X are affines. In this case the statement is obvious since the functor α_* is the forgetful functor between modules categories. $\square$

Proposition 3.1.50. *Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45. Let $\alpha: Y \rightarrow X$ be a morphism between quasi-compact and semi-separated objects of $\widehat{\mathcal{C}}$. Then for every N in $\mathcal{D}(Y)$ and M in $\mathcal{D}(X)$ the natural map*

$$\alpha_*(N \otimes_{\mathcal{D}(Y)} \alpha^* M) \longrightarrow \alpha_* N \otimes_{\mathcal{D}(X)} M \quad (3.11)$$

is an equivalence.

Proof. First, assume that Y is itself $\mathcal{C}$ -affine: write it $Y \simeq \iota(V)$. Since X is quasi-compact, we can consider a finite $\mathcal{C}$ -affine atlas $\{\varphi_i: \iota(W_i) \rightarrow X\}_{i=1}^k$ consisting of Zariski morphisms; let us denote with $j_i: \iota(W_i) \rightarrow X$ the inclusion. Arguing as in the previous proof, by descent along the $\mathcal{C}$ -affine atlas we can check whether (3.11) is an equivalence locally on the target. So, consider the pullback diagram

$$\begin{array}{ccc}
\iota(V) \times_X \iota(W_i) & \xrightarrow{\alpha'} & \iota(W_i) \\
j'_i \downarrow & & \downarrow j_i \\
\iota(V) & \xrightarrow{\alpha} & X.
\end{array}$$

Since X is semi-separated, $\iota(W) \times_X \iota(W_i)$ is $\mathcal{C}$ -affine: let us denote it by $\iota(U_i)$. We

now have

$$\begin{aligned}
 j_i^* \alpha_* (N \otimes_{D(V)} \alpha^* M) &\simeq \alpha'_* j_i'^* (N \otimes_{D(V)} \alpha^* M) \\
 &\simeq \alpha'_* (j_i'^* N \otimes_{D(V)} j_i'^* \alpha^* M) \\
 &\simeq \alpha'_* j_i'^* M \otimes_{D(V)} j_i^* N \\
 &\simeq j_i^* (\alpha_* M \otimes_{\mathcal{D}(X)} N).
 \end{aligned}$$

We highlight that we used the base change property of Proposition 3.1.48.(1), and the fact that for $\mathcal{C}$ -affine objects the projection equivalence in (3.11) holds trivially (it boils down to the fact that $N \otimes_{D(V)} (D(V) \otimes_{D(U)} M)$ is equivalent to $N \otimes_{D(U)} M$). For a general Y , we consider a finite $\mathcal{C}$ -affine atlas of Y given by Zariski morphisms, and write each N as a finite limit along such cover; thus, we reduce ourselves to the case when Y is $\mathcal{C}$ -affine, which we have just proved. $\square$

Concatenating Propositions 3.1.48 and 3.1.50 we immediately obtain the following.

Corollary 3.1.51. *Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45. Let $\alpha: \iota(U) \rightarrow X$ be a morphism to a quasi-compact and semi-separated object of $\widehat{\mathcal{C}}$. We have an equivalence of ∞ -categories*

$$\mathcal{D}(U) \simeq \text{Mod}_{\alpha_* D(U)}(\mathcal{D}(X)).$$

Proof. The proof is a standard argument based on Barr–Beck–Lurie’s monadicity theorem. Indeed, by Proposition 3.1.48.(2) the functor ι_* is monadic, and one simply needs to do a routine computation to show that this monad corresponds to the monad $-\otimes_{\mathcal{D}(X)} \alpha_* D(U)$, using the projection formula. $\square$

Proposition 3.1.52. *Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45. If X is a quasi-compact and semi-separated object of $\widehat{\mathcal{C}}$ and $\mathcal{V}$ is a rigid symmetric monoidal ∞ -category, then $\mathcal{D}(X)$ is rigid as well.*

Proof. We argue inductively on the cardinality of $\mathcal{C}$ -affine atlas $\{\varphi_i: \iota(W_i) \rightarrow X\}_{i=1}^k$, where each φ_i is a Zariski morphism.

If $k = 1$, then X is $\mathcal{C}$ -affine and in this case the assertion follows from the more general fact that if $\mathcal{V}$ is rigid and A is a commutative algebra object in $\mathcal{V}$, then $\text{Mod}_A(\mathcal{V})$ is rigid. Let now $k > 1$, and assume that $\mathcal{D}(Y)$ is rigid whenever Y admits a $\mathcal{C}$ -affine atlas consisting of $k - 1$ Zariski morphisms. Let us write

$$Z := \iota(W_1) \cup \cdots \cup \iota(W_{k-1}).$$

By the inductive hypothesis (together with Remark 3.1.41.(2)) both $\mathcal{D}(Z)$ and $\mathcal{D}(Z \times_X \iota(W_k))$ are rigid. By descent along the $\mathcal{C}$ -affine atlas, $\mathcal{D}(X)$ sits in a pullback diagram

$$\begin{array}{ccc} \mathcal{D}(X) & \longrightarrow & \mathcal{D}(Z) \\ \downarrow & & \downarrow \\ \mathcal{D}(W_k) & \longrightarrow & \mathcal{D}(Z \times_X \iota(W_k)). \end{array} \quad (3.12)$$

Propositions 3.1.48 and 3.1.50 imply that the pullback functors $\mathcal{D}(X) \rightarrow \mathcal{D}(Z)$ and $\mathcal{D}(X) \rightarrow \mathcal{D}(Z \times_X \iota(W_k))$ are internally left adjoints, in the sense of [Ram24a, Definition 1.9]. So we can simply apply [Ram24a, Corollary 4.5] and obtain that $\mathcal{D}(X)$ is a pullback also inside the full sub- ∞ -category $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}, \mathrm{dbl}} \subseteq \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ of *dualizable* presentable stable ∞ -categories. Since the inclusion of rigid presentable symmetric monoidal ∞ -categories into dualizable presentable ∞ -categories preserves limits ([Ram24b, Corollary 4.85]), we can view the pullback in (3.12) as a limit of rigid ∞ -categories. In particular $\mathcal{D}(X)$ is rigid. $\square$

The last result that we need in the proof of Theorem 3.1.46 is the following lemma, due to Gaitsgory.

Lemma 3.1.53. *Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45. Let $\alpha: Y \rightarrow X$ be a morphism in $\widehat{\mathcal{C}}$. Assume that there is a $\mathcal{C}$ -affine atlas $\{\iota(W_i) \rightarrow X\}_{i \in I}$ of X such that for every index i the canonical map*

$$\mathcal{D}(W_i) \otimes_{\mathcal{D}(X)} \mathcal{D}(Y) \longrightarrow \mathcal{D}(\iota(W_i) \times_X Y)$$

is an equivalence. Assume moreover that $\mathcal{D}(Y)$ is dualizable as a categorical $\mathcal{D}(X)$ -module. Then the diagram

$$\begin{array}{ccc} \mathrm{Pr}_{\mathcal{D}(X)}^{\mathrm{L}} & \xrightarrow{(\widehat{-})_X} & \mathcal{D}^{\mathrm{cat}}(X) \\ \alpha^* \downarrow & & \downarrow \tilde{\alpha}^* \\ \mathrm{Pr}_{\mathcal{D}(Y)}^{\mathrm{L}} & \xrightarrow{(\widehat{-})_Y} & \mathcal{D}^{\mathrm{cat}}(Y) \end{array}$$

is vertically right adjointable.

Proof. Completely analogous to the proof of [Gai15, Proposition 3.2.6 (ii)]. $\square$

However, in practice, often one would wish for some concrete way to check whether the Künneth formula in the hypotheses of Lemma 3.1.53 is satisfied. The following offers a nice criterion when $\mathcal{V}$ is a rigid symmetric monoidal ∞ -category.

Proposition 3.1.54. *Let $\iota: \mathcal{C} \rightarrow \widehat{\mathcal{C}}$, $\mathcal{V}$ and $\mathcal{D}$ be as in Assumption 3.1.45, and assume $\mathcal{V}$ to be rigid as a symmetric monoidal ∞ -category. Let $\alpha: Y \rightarrow X$ be a morphism between quasi-compact and semi-separated objects of $\widehat{\mathcal{C}}$. For every morphism $\varphi: \iota(V) \rightarrow X$, the canonical functor*

$$\mathcal{D}(Y) \otimes_{\mathcal{D}(X)} \mathcal{D}(V) \longrightarrow \mathcal{D}(Y \times_X \iota(V))$$

is an equivalence.

Proof. First assume that $Y \simeq \iota(U)$ is itself $\mathcal{C}$ -affine. Now $\iota(U) \times_X \iota(V)$ is $\mathcal{C}$ -affine because X is semi-separated: write $\iota(W)$ for such pullback, and let $\alpha': \iota(W) \rightarrow \iota(V)$ and $\varphi': \iota(W) \rightarrow \iota(U)$ denote the corresponding morphisms. Using Corollary 3.1.51, we can write

$$\mathcal{D}(U) \simeq \text{Mod}_{\alpha_* D(U)}(\mathcal{D}(X))$$

and so the tensor product $\mathcal{D}(U) \otimes_{\mathcal{D}(X)} \mathcal{D}(V)$ is equivalent to $\text{Mod}_{\alpha_* D(U)}(\mathcal{D}(V))$ via [Lur17, Theorem 4.8.4.6]. This in turn is by definition $\text{Mod}_{\varphi^* \alpha_* D(U)}(\mathcal{D}(V))$, since $\mathcal{D}(V)$ is tensored over $\mathcal{D}(X)$ via the symmetric monoidal pullback φ^* .

On the other hand, using the base change property (Proposition 3.1.48.(1)), we can write

$$\begin{aligned} \mathcal{D}(\iota(U) \times_X \iota(V)) &\simeq \text{Mod}_{\alpha'_* D(W)}(\mathcal{D}(V)) \\ &\simeq \text{Mod}_{\alpha'_* \varphi'^* D(U)}(\mathcal{D}(V)) \\ &\simeq \text{Mod}_{\varphi^* \alpha_* D(U)}(\mathcal{D}(V)). \end{aligned}$$

For the general case, consider a $\mathcal{C}$ -affine atlas $\{\iota(U_i) \rightarrow Y\}_{i \in I}$ consisting of Zariski morphisms. Since $\mathcal{D}(V)$ and $\mathcal{D}(X)$ are rigid (Proposition 3.1.52) it follows that $\mathcal{D}(V)$ is dualizable as a categorical $\mathcal{D}(X)$ -module, so we have

$$\begin{aligned} \mathcal{D}(Y) \otimes_{\mathcal{D}(X)} \mathcal{D}(V) &\simeq \left(\lim_{i \in I} \mathcal{D}(U_i) \right) \otimes_{\mathcal{D}(X)} \mathcal{D}(V) \\ &\simeq \lim_{i \in I} \left(\mathcal{D}(U_i) \otimes_{\mathcal{D}(X)} \mathcal{D}(V) \right) \\ &\simeq \lim_{i \in I} \mathcal{D}(\iota(U_i) \times_X \iota(V)) \\ &\simeq \mathcal{D}(Y \times_X \iota(V)). \end{aligned}$$

□

Proof of Theorem 3.1.46. Let X be quasi-compact and semi-separated, and let $\{\varphi_i: \iota(W_i) \rightarrow X\}_{i=1}^k$ be a finite $\mathcal{C}$ -affine atlas as in the statement. We will prove the theorem by induction on the cardinality k of the affine $\mathcal{C}$ -atlas.

If $k = 1$, X is $\mathcal{C}$ -affine so there is nothing to prove. For $k > 1$, let

$$Y := \iota(W_2) \cup \cdots \cup \iota(W_k)$$

denote the union of the last $k - 1$ elements of the atlas; let $\psi: Y \rightarrow X$, $\alpha_1: Y \times_X \iota(W_1) \rightarrow \iota(W_1)$ and $\beta: Y \times_X \iota(W_1) \rightarrow Y$ denote the natural maps (which are all Zariski, in virtue of Remark 3.1.41.(2)). Consider the following diagram.

$$\begin{array}{ccccc}
 & \text{Pr}_{\mathcal{D}(X)}^L & \xrightarrow{\widehat{(-)}_X} & \mathcal{D}^{\text{cat}}(X) & \\
 \psi^* \swarrow & \downarrow \varphi_1^* & & \swarrow \tilde{\psi}^* & \\
 \text{Pr}_{\mathcal{D}(Y)}^L & \xrightarrow{\quad \approx \quad} & \mathcal{D}^{\text{cat}}(Y) & & \\
 \downarrow \beta^* & \downarrow \alpha_1^* & \downarrow \tilde{\beta}^* & \downarrow \tilde{\alpha}_1^* & \\
 \text{Pr}_{\mathcal{D}(Y \times_X \iota(W_1))}^L & \xrightarrow{\quad \approx \quad} & \mathcal{D}^{\text{cat}}(Y \times_X \iota(W_1)) & &
 \end{array} \tag{3.13}$$

Everything is naturally commutative, and the equivalences are all determined by the fact that the adjunction (3.3) is an equivalence for $\iota(W_1)$ (being $\mathcal{C}$ -affine) and for Y and $Y \times_X \iota(W_1)$ (by the induction hypothesis). We want to prove that $\widehat{(-)}_X$ is an equivalence: for this purpose, we will prove that for all objects $\mathcal{E}$ in $\text{Pr}_{\mathcal{D}(X)}^L$ and for all objects $(\mathcal{E}_U)_\gamma: \iota(U) \rightarrow X$ the unit (3.5) and the counit (3.6) are equivalences.

Notice that both the top and the bottom faces of the diagram (3.13) are right adjointable, because of Lemma 3.1.53 (the hypotheses are satisfied thanks to Proposition 3.1.54). In particular we have the following.

1. The functor ψ^* sends the unit functor for the adjunction $\widehat{(-)}_X \dashv \Gamma(X, -)$ to the unit functor for the adjunction $\widehat{(-)}_Y \dashv \Gamma(Y, -)$, which is an equivalence.
2. The functor φ_1^* sends the unit functor for the adjunction $\widehat{(-)}_X \dashv \Gamma(X, -)$ to the unit functor for the adjunction $\widehat{(-)}_{\iota(W_1)} \dashv \Gamma(\iota(W_1), -)$, which is an equivalence.
3. The functor $\tilde{\psi}^*$ sends the counit functor for the adjunction $\widehat{(-)}_X \dashv \Gamma(X, -)$ to the counit functor for the adjunction $\widehat{(-)}_Y \dashv \Gamma(Y, -)$, which is an equivalence.
4. The functor $\tilde{\varphi}_1^*$ sends the counit functor for the adjunction $\widehat{(-)}_X \dashv \Gamma(X, -)$ to the unit functor for the adjunction $\widehat{(-)}_{\iota(W_1)} \dashv \Gamma(\iota(W_1), -)$, which is an equivalence.

So it is sufficient to prove that the functors ψ^* , and φ_1^* (respectively $\tilde{\psi}^*$ and $\tilde{\varphi}_1^*$) are jointly conservative functors. For $\tilde{\psi}^*$ and $\tilde{\varphi}_1^*$, this is a obvious consequence of the fact that Y and $\iota(U_1)$ form a $\mathcal{C}$ -affine atlas for X , with respect to which $\mathcal{D}^{\text{cat}}$ satisfies descent. For ψ^* and φ_1^* , it is enough to show that for each categorical $\mathcal{D}(X)$ -module $\mathcal{E}$ the diagram

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{\text{id}_{\mathcal{E}} \otimes \psi^*} & \mathcal{E} \otimes_{\mathcal{D}(X)} \mathcal{D}(Y) \\
 \text{id}_{\mathcal{E}} \otimes \varphi_1^* \downarrow & & \downarrow \text{id}_{\mathcal{E}} \otimes \alpha_1^* \\
 \mathcal{E} \otimes_{\mathcal{D}(X)} \mathcal{D}(W_1) & \xrightarrow{\text{id}_{\mathcal{E}} \otimes \beta^*} & \mathcal{E} \otimes_{\mathcal{D}(X)} \mathcal{D}(Y \times_X \iota(W_1))
 \end{array} \tag{3.14}$$

is Cartesian. For this, we will need the following lemma.

Lemma 3.1.55. *Let $\iota: \mathcal{C} \rightarrow \hat{\mathcal{C}}, \mathcal{V}, \mathcal{D}$ and X be as in the statement of Theorem 3.1.46. Let*

$$Y := \bigcup_{i_1, \dots, i_n} \iota(W_i)$$

and the diagram

$$\begin{array}{ccc}
 \iota(V) \times_X Y & \xrightarrow{\alpha_1} & \iota(W_1) \\
 \beta \downarrow & & \downarrow \varphi_1 \\
 Y & \xrightarrow{\psi} & X
 \end{array}$$

as in the proof of Theorem 3.1.46. Then, the functors ψ^* , α_1^* and β^* admit left adjoints, denoted respectively with $\alpha_{\natural}, \alpha_{1, \natural}$ and $\beta^{\natural}$.

Proof. To prove that ψ^* has a left adjoint we will show that it commutes with limits. We consider an diagram in $\mathcal{D}(X)$ and the canonical map

$$\psi^* \lim M_k \rightarrow \lim \psi^* M_k.$$

If we denote with j_i the canonical map $\iota(W_i) \rightarrow Y$ it is enough to show that the map

$$j_i^* \psi^* \lim M_k \rightarrow j_i^* \lim \psi^* M_k$$

is an equivalence.

The composition $j_i \circ \psi$ is canonically φ_i , this implies that the left hand side commutes with colimits. For the right hand side we observe that j_i^* has a left adjoint given by $\psi^* \circ \varphi_{i, \natural}$. Indeed since $\psi_*: \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$ is fully faithful (Proposition 3.1.48.(2)) we can

write

$$\begin{aligned}
\mathrm{Map}_{\mathcal{D}(W_i)}(M, j_i^* N) &\simeq \mathrm{Map}_{\mathcal{D}(W_i)}(M, j_i^* \psi^* \psi_* N) \\
&\simeq \mathrm{Map}_{\mathcal{D}(W_i)}(M, \varphi_i^* \psi_* N) \\
&\simeq \mathrm{Map}_{\mathcal{D}(X)}(\varphi_{i,\natural} M, \psi_* N) \\
&\simeq \mathrm{Map}_{\mathcal{D}(W_i)}(\psi^* \varphi_{i,\natural} M, N).
\end{aligned}$$

In particular also the left hand side commutes with limits.

We now prove that $\alpha_{1,*}^*$ and β^* have left adjoints. We can use the maps $\iota(W_1) \times_X \iota(W_i) \rightarrow Y$ whose composition with ψ is $\varphi_{1,i}$ and apply the same argument as above to show the existence of the left adjoint for $(\psi \circ \beta)^*$, denoted with ε . Again with the same argument as above we can identify $\beta_{\natural}$ with $\psi^* \circ \varepsilon$. We now prove the existence of the left adjoint for α_1^* , using Proposition 3.1.48 we deduce that α_1^* can be written as $\beta^* \circ \alpha^* \circ \varphi_*$. We now have the following equivalences that prove the existence of the left adjoint

$$\begin{aligned}
\mathrm{Map}_{\mathcal{D}(W_i)}(M, \alpha_1^* N) &\simeq \mathrm{Map}_{\mathcal{D}(W_i)}(M, \beta^* \psi^* \varphi_{1,*} N) \\
&\simeq \mathrm{Map}_{\mathcal{D}(W_i)}(\varepsilon M, \varphi_{1,*} N) \\
&\simeq \mathrm{Map}_{\mathcal{D}(X)}(\varphi_1^* \varepsilon M, N).
\end{aligned}$$

□

Using Lemma 3.1.55, we see that both β^* and α_1^* admit left adjoints $\alpha_{1,\natural}$ and $\beta_{\natural}$. Passing to the diagram of left adjoints (and using the compatibility of colimits and tensor products), from the diagram (3.14) we obtain a pushout square

$$\begin{array}{ccc}
\mathcal{E} \otimes_{\mathcal{D}(X)} \left(\mathcal{D}(W_1) \coprod_{\mathcal{D}(Y \times_X \iota(W_1))} \mathcal{D}(Y) \right) & \longleftarrow & \mathcal{E} \otimes_{\mathcal{D}(X)} \mathcal{D}(Y) \\
\uparrow & & \uparrow \mathrm{id}_{\mathcal{E}} \otimes \alpha_{1,\natural} \\
\mathcal{E} \otimes_{\mathcal{D}(X)} \mathcal{D}(W_1) & \longleftarrow \mathrm{id}_{\mathcal{E}} \otimes \beta_{\natural} & \mathcal{E} \otimes_{\mathcal{D}(X)} \mathcal{D}(Y \times_X \iota(W_1)),
\end{array}$$

so our claim follows if

$$\mathcal{D}(W_1) \coprod_{\mathcal{D}(Y \times_X \iota(W_1))} \mathcal{D}(Y) \simeq \mathcal{D}(X).$$

Again, by passing to the left adjoints, this is equivalent to saying that

$$\mathcal{D}(X) \simeq \mathcal{D}(W_1) \times_{\mathcal{D}(Y \times_X \iota(W_1))} \mathcal{D}(Y),$$

which is true because the maps $\iota(W_1) \rightarrow X$ and $Y \rightarrow X$ form a $\mathcal{C}$ -affine Zariski atlas of X .

□

3.2 Recollections on analytic geometry after Clausen–Scholze

Our main goal is to apply the general formalism established in Section 3.1 to the setting of rigid and analytic geometry. More precisely:

1. The site $\mathcal{C}$ will be either the opposite of the ∞ -category AnRing equipped with the universal $!$ -able topology, or the ∞ -category dAfd_k equipped with the analytic topology. The former is the opposite of the ∞ -category of condensed analytic rings (equivalently: the ∞ -category of affine analytic stacks), while the latter is the opposite ∞ -category of animated affinoid spaces over a complete non-archimedean field k .
2. The sheaf of presentably symmetric monoidal and stable ∞ -categories $\mathcal{D}$ will be either the functor $\mathcal{D}(-)$ or its sub-functor $\text{Nuc}(-)$. The former assigns to an analytic ring A its derived ∞ -category $\mathcal{D}(A)$ of complete modules, while the latter is defined for analytic rings coming from affinoid algebras and selects the sub- ∞ -category $\text{Nuc}(A) \subseteq \mathcal{D}(A)$ ■ spanned by *nuclear* modules.
3. Accordingly, the stable and presentably symmetric monoidal ∞ -category $\mathcal{V}$ will be either the derived ∞ -category $\mathcal{D}(\mathbb{Z}^{\text{triv}})$, where $\mathbb{Z}$ is equipped with the trivial analytic structure, or the ∞ -category $\text{Nuc}(k)$.
4. The dense functor $\iota: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ will be either the Yoneda embedding (in the analytic case) or the inclusion of affinoid spaces in the ∞ -category of all rigid varieties.

While the theory of rigid varieties is classical (see for example [Bos14] for an introduction), the theory of analytic stacks has been only recently developed in the framework of condensed mathematics. Thus, before proceeding in studying the problem of 1-affineness in the context of analytic stacks, we start by reviewing some fundamental definitions and fixing the notations that will be used in Section 3.3.1. We stress that this section does not bear any addition or improvement to the existing literature, but we include it in the paper for the reader's convenience: if they are already acquainted with the language of condensed mathematics, they may safely skip it.

We mainly adapt the conventions and terminology from [CS24]; further references are provided in the text when needed.

3.2.1 Condensed objects in ∞ -categories

Recall that a *pro-finite set* is by definition a pro-finite object in the category $\mathbf{Fin}$ of finite sets. More explicitly, a pro-finite set is a set X which can be realized as a projective limit of finite sets, i.e., it is a set admitting a presentation

$$X \cong \lim_{i \in I} X_i$$

where I is a cofiltered diagram and each X_i is finite. We denote the category of pro-finite sets as $\mathbf{ProFin}$.

Definition 3.2.1. A pro-finite set X is *light* if it can be presented as a limit of finite sets over a *countable* cofiltered diagram.

Remark 3.2.2. Equipping finite sets with the discrete topology, every pro-finite comes equipped with the inverse limit topology. Indeed, by Stone duality, pro-finite sets are precisely totally disconnected compact Hausdorff topological spaces. In Section 3.3, we shall often blur this distinction and consider pro-finite sets as objects inside the category of topological spaces.

Under this equivalence, light pro-finite sets correspond to those totally disconnected compact Hausdorff topological spaces which are moreover metrizable [CS24, Lecture 2].

Light pro-finite sets define a full sub-category of $\mathbf{ProFin}$, that we denote as $\mathbf{ProFin}^{\text{light}}$. This category naturally admits a Grothendieck topology, for which a cover is a finite collection of jointly surjective maps.

Definition 3.2.3. Let $\mathcal{C}$ be any ∞ -category. A *condensed object* in $\mathcal{C}$ is a sheaf from the category of light pro-finite sets to $\mathcal{C}$. We denote the ∞ -category of condensed objects in $\mathcal{C}$ as $\mathcal{C}^{\text{cond}}$.

Remark 3.2.4. If $\mathcal{C}$ is presentable, then $\mathcal{C}^{\text{cond}}$ is presentable as well. This follows from the fact that $\mathbf{ProFin}^{\text{light}}$ is small enough to let $\text{Fun}((\mathbf{ProFin}^{\text{light}})^{\text{op}}, \mathcal{C})$ be presentable itself – light pro-finite sets are singled out precisely for this reason – and so $\mathcal{C}^{\text{cond}}$ defines a topological localization of $\text{Fun}((\mathbf{ProFin}^{\text{light}})^{\text{op}}, \mathcal{C})$.

At the same time, if $\mathcal{C}$ is stable then so is $\mathcal{C}^{\text{cond}}$. Again, this follows from the fact that $\mathcal{C}^{\text{cond}}$ is a topological localization of the stable ∞ -category $\text{Fun}((\mathbf{ProFin}^{\text{light}})^{\text{op}}, \mathcal{C})$.

Let $\mathcal{C}$ be a presentably symmetric monoidal ∞ -category. Then the ∞ -category of $\mathcal{C}$ -valued presheaves over $\mathbf{ProFin}^{\text{light}}$ admits a point-wise tensor product turning it into a presentably symmetric monoidal ∞ -category as well. Hence, one can define a presentably symmetric monoidal structure also on $\mathcal{C}^{\text{cond}}$, given by sheafifying the point-wise monoidal structure on $\text{Fun}((\mathbf{ProFin}^{\text{light}})^{\text{op}}, \mathcal{C})$.

Thus, it is natural to ask what is the relationship between condensed commutative algebras in $\mathcal{C}$, and commutative algebras in $\mathcal{C}^{\text{cond}}$. The following proposition shows that there is no ambiguity whatsoever.

Proposition 3.2.5. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞ -category. Then there is an equivalence*

$$\text{CAlg}(\mathcal{C})^{\text{cond}} \simeq \text{CAlg}(\mathcal{C}^{\text{cond}})$$

between condensed $\mathbb{E}_\infty$ -algebras in $\mathcal{C}$, and $\mathbb{E}_\infty$ -algebras in the ∞ -category of condensed objects of $\mathcal{C}$.

Remark 3.2.6. Proposition 3.2.5 applies in particular to the case when $\mathcal{C}$ is either the ∞ -category of derived k -modules, or the ∞ -category of *connective* derived k -modules.

Proof of Proposition 3.2.5. Since both $\mathcal{C}$ and $\text{CAlg}(\mathcal{C})$ admit all limits, [Lur18, Proposition 1.3.1.7] yields equivalences

$$\mathcal{C}^{\text{cond}} := \text{Sh}(\text{ProFin}^{\text{light}}, \mathcal{C}) \simeq \text{Fun}^{\text{R}}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right)$$

and

$$\text{CAlg}(\mathcal{C})^{\text{cond}} := \text{Sh}(\text{ProFin}^{\text{light}}, \text{CAlg}(\mathcal{C})) \simeq \text{Fun}^{\text{R}}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \text{CAlg}(\mathcal{C})\right),$$

where the decoration R denotes those functors which preserve limits. So we can define a natural functor

$$\tilde{\phi}: \text{CAlg}(\mathcal{C})^{\text{cond}} \longrightarrow \text{CAlg}(\text{Fun}(\text{ProFin}^{\text{light}}, \mathcal{C}))$$

as the composition

$$\begin{array}{ccc} \text{Fun}^{\text{R}}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \text{CAlg}(\mathcal{C})\right) & \hookrightarrow & \text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \text{CAlg}(\mathcal{C})\right) \\ \wr \uparrow & & \downarrow \wr \\ \text{CAlg}(\mathcal{C})^{\text{cond}} & \xrightarrow{\tilde{\phi}} & \text{CAlg}\left(\text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right)\right). \end{array}$$

In the above diagram, the vertical map on the right is an equivalence by [Lur17, Remark 2.1.3.4], and the top horizontal arrow is a fully faithful embedding. In particular it follows that $\tilde{\phi}$ is a fully faithful embedding as well.

To prove that its essential image is the whole $\text{CAlg}(\mathcal{C}^{\text{cond}})$, it is enough to show the following: given a commutative algebra object F in $\text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right)$ such that the

underlying functor $(\text{Ani}^{\text{cond}})^{\text{op}} \rightarrow \mathcal{C}$ commutes with limits, then F is in the image of $\tilde{\phi}$. But under the equivalence

$$\text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \text{CAlg}(\mathcal{C})\right) \simeq \text{CAlg}\left(\text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right)\right),$$

the forgetful functor $\text{CAlg}\left(\text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right)\right) \rightarrow \text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right)$ corresponds to the functor

$$\text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \text{CAlg}(\mathcal{C})\right) \xrightarrow{\text{oblv}_{\text{CAlg}} \circ -} \text{Fun}\left((\text{Ani}^{\text{cond}})^{\text{op}}, \mathcal{C}\right).$$

Since the forgetful functor $\text{CAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ is conservative and commutes with limits ([Lur17, Corollary 3.2.2.4 and Lemma 3.2.2.6],) our claim follows tautologically. $\square$

Remark 3.2.7. Let $\mathcal{C}$ be a presentably symmetric monoidal ∞ -category, and let $\mathcal{A}$ be a condensed commutative algebra in $\mathcal{C}$. In virtue of Proposition 3.2.5 $\mathcal{A}$ corresponds to a commutative algebra in the presentably symmetric monoidal ∞ -category $\mathcal{C}^{\text{cond}}$. In particular, [Lur17, §4.8] yields a well-defined functor

$$\text{Mod}_{(-)}(\mathcal{C}^{\text{cond}}): \text{CAlg}(\mathcal{C})^{\text{cond}} \simeq \text{CAlg}(\mathcal{C}^{\text{cond}}) \longrightarrow \text{CAlg}(\text{Pr}_{\mathcal{C}^{\text{cond}}}^{\text{L}})$$

sending a condensed $\mathbb{E}_{\infty}$ -algebra in $\mathcal{C}$ to its (symmetric monoidal) derived ∞ -category of modules inside the ∞ -category of condensed objects of $\mathcal{C}$. When $\mathcal{C} = \text{Mod}_{\mathbb{Z}}$, $\mathcal{C}^{\text{cond}}$ is derived ∞ -category of condensed abelian groups, this category is sometimes also denoted by $\mathcal{D}(\mathbb{Z}^{\text{triv}})$, as will be explained in Example 3.2.11.(1). In this case, given a condensed $\mathbb{E}_{\infty}$ -ring $\mathcal{A}$, we shall denote the ∞ -category $\text{Mod}_{\mathcal{A}}(\text{Mod}_{\mathbb{Z}}^{\text{cond}})$ as $\text{Mod}_{\mathcal{A}}$.

3.2.2 Analytic rings and stacks

In this subsection we review the six-functor formalism for analytic rings as defined by Clausen-Scholze in [CS24], and we show how to extend it to the context of analytic stacks. A nice overview of the following definitions and results can also be found in [Kes25].

First, let us recall Definition 2.2.1.

Definition 3.2.8. An analytic ring is a pair $A = (\mathcal{A}, \mathcal{D}(A))$, where $\mathcal{A}$ is a condensed $\mathbb{E}_{\infty}$ -algebra, and $\mathcal{D}(A)$ is a full subcategory of $\text{Mod}_{\mathcal{A}}^{\text{cond}}$ enjoying the following properties.

- (1) The ∞ -category $\mathcal{D}(A)$ contains the object $\mathcal{A}$.
- (2) Let $\iota_A: \mathcal{D}(A) \hookrightarrow \text{Mod}_{\mathcal{A}}$ denote the natural inclusion functor. Then ι_A commutes with all limits and colimits, and it has a left adjoint (called the *completion along*

A functor) that we denote with $- \otimes_{\mathcal{A}} A$. The composition $\iota_A \circ (- \otimes_{\mathcal{A}} A)$ sends connective objects to connective objects.

- (3) For every object M in $\mathcal{D}(A)$ and every N in $\text{Mod}_{\mathbb{Z}}^{\text{cond}}$, the internal mapping object $\underline{\text{Hom}}_{\mathbb{Z}}(N, M)$ lies in $\mathcal{D}(A)$.

A morphism between analytic rings $f: A \rightarrow B$ is a morphism of the underlying condensed $\mathbb{E}_{\infty}$ -rings $\ell: \mathcal{A} \rightarrow \mathcal{B}$ such that the induced forgetful functor $\text{Mod}_{\mathcal{B}} \rightarrow \text{Mod}_{\mathcal{A}}$ sends $\mathcal{D}(B)$ to $\mathcal{D}(A)$. Analytic rings are naturally gathered in a ∞ -category AnRing .

Remark 3.2.9. At a heuristic level, objects in $\text{Mod}_{\mathcal{A}}^{\text{cond}}$ play the role of topological $\mathcal{A}$ -modules. Following this analogy, objects in $\mathcal{D}(A)$ correspond to those topological $\mathcal{A}$ -modules which are complete. For this reason, this ∞ -category was originally denoted as $\mathcal{D}(\mathcal{A}, m)$ in [Sch26a], where m should suggest a space of measures. This justifies the name of the left adjoint appearing in Definition 3.2.8.(2).

Remark 3.2.10. By definition, ι_A is a localization functor. In particular, since $\text{Mod}_{\mathcal{A}}$ is presentable and stable, it follows that $\mathcal{D}(A)$ is presentable and stable as well. Moreover, it admits a symmetric monoidal structure given by completing the natural tensor product of $\text{Mod}_{\mathcal{A}}$ along A . We denote such tensor product as $\otimes_A$.

Example 3.2.11.

- (1) As explained in [CS24, Lecture 19], to every $\mathbb{E}_{\infty}$ - $\mathbb{Z}$ -algebra R one can associate an analytic ring R^{triv} , in the following way. Via the functor

$$\text{CAlg}_{\mathbb{Z}} \xrightarrow{\text{const}} \text{Fun}\left((\text{ProFin}^{\text{light}})^{\text{op}}, \text{CAlg}_{\mathbb{Z}}\right) \xrightarrow{(-)^+} \text{CAlg}_{\mathbb{Z}}^{\text{cond}}$$

we can associate a discrete condensed ring $\underline{R}$ to any $\mathbb{E}_{\infty}$ - $\mathbb{Z}$ -algebra R . So the analytic ring R^{triv} is explicitly given by the pair $(\underline{R}, \text{Mod}_{\underline{R}})$.

- (2) Another source of analytic rings comes from rigid analytic geometry. As explained in [And21, Proposition 3.34], there is a fully faithful functor from the category of affinoid rings (and more generally from the category of complete Huber pairs) to the category of analytic rings. By abuse of notation we continue to denote by A the analytic ring associated to an affinoid ring A , and we denote with $\mathcal{D}(A)$ the associated ∞ -category of "complete modules".

We are interested in (derived) analytic stacks, which are locally modeled by analytic rings. As customary in algebraic geometry, we shall denote the opposite ∞ -category of AnRing as Aff , and call it the ∞ -category of affine analytic stacks. Given an analytic ring A , its associated affine stack will be denoted as $\text{AnSpec}(A)$.

3.2.12. Let $f: \text{AnSpec}(A) \rightarrow \text{AnSpec}(B)$ be a morphism of affine analytic rings. Let us denote by f_* the restriction of the forgetful functor $\text{Mod}_B \rightarrow \text{Mod}_A$ to the sub- ∞ -category $\mathcal{D}(A)$: by the definition of morphism of analytic rings, this is a functor $f_*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$, which we can write as a composition of right adjoints

$$\mathcal{D}(B) \xrightarrow{\iota_B} \text{Mod}_B \longrightarrow \text{Mod}_A \xrightarrow{f^R} \mathcal{D}(A).$$

The last functor is the right adjoint of ι_A , which exists because $\mathcal{D}(A)$ is a presentable ∞ -category closed under all colimits inside Mod_A . The fact that the functor acts as the identity on the image of $\mathcal{D}(B)$ follows from the facts that the forgetful functor preserves complete modules and f^R is a colocalization functor.

It follows that the functor f_* admits a left adjoint $f^*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ given by the composition

$$\mathcal{D}(B) \xrightarrow{\iota_B} \text{Mod}_B \xrightarrow{-\otimes_B A} \text{Mod}_A \xrightarrow{-\otimes_A A} \mathcal{D}(A).$$

In virtue of [Kes25, Proposition 4.11] the assignment

$$A \mapsto \mathcal{D}(A), \quad f \mapsto f^*$$

can be promoted to a symmetric monoidal functor

$$\mathcal{D}: \text{AnRing} \simeq \text{Aff}^{\text{op}} \longrightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}), \quad (3.15)$$

which generalizes the usual functor sending a commutative ring to its stable ∞ -category of derived modules.

Definition 3.2.13. Let $f: \text{AnSpec}(A) \rightarrow \text{AnSpec}(B)$ be a map of affine analytic stacks.

- (1) We say that f is *proper* if f^* induces an equivalence $\mathcal{D}(A) \simeq \text{Mod}_A(\mathcal{D}(B))$. This means that B has the analytic ring structure induced by the one of A .
- (2) We say that f is an *open immersion* if f^* has a left adjoint $f_!$ that satisfies the projection formula. This means that there is a canonical equivalence of functors

$$f_!(-) \otimes_B (-) \simeq f_!((-) \otimes_A f^*(-))$$

from $\mathcal{D}(A) \otimes \mathcal{D}(B)$ to $\mathcal{D}(B)$.

- (3) We say that f is *!-able* if can be written as a composition $f \simeq p \circ j$, where j is an open immersion and p is proper.

Remark 3.2.14. If f is a $!$ -able map with given decomposition $f \simeq p \circ j$, then we can define a functor $f_! : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ to be $p_* \circ j_!$. It is indeed a left adjoint: this follows from the fact that $j_!$ is a left adjoint by definition, while p_* (which is normally only a right adjoint) is also a *left* adjoint when p is proper. Indeed, in this case, p_* identifies with the forgetful functor $\mathrm{Mod}_A(\mathcal{D}(B)) \rightarrow \mathcal{D}(A)$, which preserves all limits and colimits. In particular, $f_!$ admits a right adjoint $f^!$.

The following theorem appears in the video series on analytic stacks by Clausen and Scholze. For completeness, we include a proof here, drawing on results from the existing literature.

Theorem 3.2.15 ([CS24]). *Let E denote the class of $!$ -able morphisms, let I denote the class of open immersions, and let P denote the class of proper maps in Aff .*

- (1) *The pair (Aff, E) defines a geometric setup, in the sense of [HM24, Definition 2.2.1].*
- (2) *The pair (P, I) is a suitable decomposition of E , in the sense of [HM24, Definition 3.3.2]*
- (3) *The functor (3.15) is a sheaf for the universal $!$ -topology, and it is one of the operations in a symmetric monoidal six-functor formalism over (Aff, E) .*

Proof. Theorem 3.2.15.(1) and Theorem 3.2.15.(2) are proved in [Kes25, Lemmas 4.6 and 4.7]. For Theorem 3.2.15.(3), we note that the fact that $\mathcal{D}(-)$ is a sheaf for the universal $!$ -able topology follows directly from the definition of universal $!$ -able topology (Definition 3.1.18), while the fact that $\mathcal{D}(-)$ can be promoted to a six-functor formalism follows from [HM24, Proposition 3.3.3]. The fact that $\mathcal{D}(-)$ is a symmetric monoidal functor follows from [Kes25, Proposition 4.11]. $\square$

Example 3.2.16. Let R be an affinoid ring, seen as an analytic ring as in Example 3.2.11.(2). Then, [CS24] shows that the functor from affinoid rings to analytic rings sends analytic open immersions of affinoid rings to universal $!$ -able maps of analytic rings. This makes it possible to regard rigid analytic varieties as analytic stacks. Again, when X is a rigid analytic variety we shall abuse our notations and still denote its associated analytic stack by X .

3.2.3 Proper objects and descent

In this subsection we will briefly describe the notion of $\mathcal{D}$ -proper maps, as defined in [HM24] (where they are called prim maps). For the reader's convenience, we also recall some properties and results borrowed from [Sch25] (where they are called "cohomologically proper") and [Cam24] (where they are called "co-smooth") that we will need in the rest of the paper.

We refer the reader to the above references for a complete treatment of this topic and all the relevant proofs.

Notation 3.2.17. Let $\mathcal{C}$ be a ∞ -category with finite limits. Fix a class of morphisms E such that the pair $(\mathcal{C}, E)$ is a geometric setup, and let $(\mathcal{D}^*, \mathcal{D}_!)$ be a six-functor formalism over $(\mathcal{C}, E)$. Given a map $f: X \rightarrow S$, we denote with $p_{1,2}: X \times_S X \rightarrow X$ the two projection maps, and with $\Delta: X \rightarrow X \times_S X$ the diagonal. If P is an object in $\mathcal{D}(X)$ we denote with $\mathcal{P}_f(P)$ the object of $\mathcal{D}(X)$ given by $p_{2,*}\underline{\mathrm{Hom}}_{\mathcal{D}(X \times_S X)}(p_1^*(P), \Delta_!(\mathbb{1}_{\mathcal{D}(X)}))$.

Definition 3.2.18 ([Cam24, Proposition 3.1.11],[HM24, Lemma 4.4.6]). Let $f: X \rightarrow S$ be a map lying in E .

- (1) We say that an object $P \in \mathcal{D}(X)$ is *f-proper* if the natural map

$$f_!(P \otimes \mathcal{P}_f(P)) \xrightarrow{f_*} f_*\underline{\mathrm{Hom}}_{\mathcal{D}(X)}(P, P)$$

is an equivalence.

- (2) We say that f is *$\mathcal{D}$ -proper* if $\mathbb{1}_{\mathcal{D}(X)}$ is *f-proper* and $\mathcal{P}_f(\mathbb{1}_{\mathcal{D}(X)})$ is invertible with respect to the symmetric monoidal structure of $\mathcal{D}(X)$.

Remark 3.2.19. Note that in [HM24] the terminology *f-prim* is used for what we call *f-proper*. In that reference, the term *f-proper* is reserved for a stronger condition.

The following result is the key tool that allows us to establish whether a morphism constitutes a universal $!$ -able cover of the target.

Theorem 3.2.20 ([Sch25, Theorem 6.19]). Let $f: X \rightarrow S$ be a morphism in E . If f is descendable (Definition 3.1.37) and $\mathbb{1}_{\mathcal{D}(X)}$ is *f-proper*, then $\mathcal{D}(-)$ satisfies universal $*$ - and $!$ -descent along f .

It follows that in order to understand whether a map is a universal $!$ -able cover, we have to be able to detect whether an object is proper. The next proposition allows us to check this condition $*$ -locally on the target.

Proposition 3.2.21 ([HM24, Lemma 4.4.8]). Let $f: X \rightarrow S$ be a morphism in E . Suppose that S admits a universal $*$ -cover $\{g_i: T_i \rightarrow S\}$, and denote with $f_i: X \times_S T_i \rightarrow T_i$ and $g_{i,X}: X \times_S T_i \rightarrow X$ the obvious projection maps. An object P of $\mathcal{D}(X)$ is *f-proper* if the objects $g_{i,X}^*(P)$ are *f_i-proper* for all i .

In order to show that $\mathbb{1}_{\mathcal{D}(X)}$ is *f-proper* and apply Theorem 3.2.20, in many cases we can use the following weaker definition of properness.

Definition 3.2.22 ([Cam24, Definition 3.1.19]). Let $f: X \rightarrow S$ be a morphism of $\mathcal{C}$. We say that $f: X \rightarrow S$ is *weakly cohomologically proper* if the following conditions hold.

- (1) The object $\mathbb{1}_{\mathcal{D}(X)}$ is Δ_f -proper.
- (2) There is an equivalence $\mathcal{P}_{\Delta_f}(\mathbb{1}_{\mathcal{D}(X)}) \xrightarrow{\simeq} \mathbb{1}_{\mathcal{D}(X)}$.
- (3) The object $\mathbb{1}_{\mathcal{D}(X)}$ is f -proper.

Remark 3.2.23. Definition 3.2.22 bears an important advantage over Definition 3.2.13.(1). Indeed, when the six-functor formalism we are considering admits a suitable decomposition (I, P) then all proper maps are also weakly cohomologically proper, as proved in [Cam24, Lemma 3.1.21]; in particular, in this case, given any proper map $f: X \rightarrow S$ the object $\mathbb{1}_{\mathcal{D}(X)}$ is f -proper. Thanks to Theorem 3.2.15.(2), this discussion applies to the case of the six-functor formalism over AnRing as well.

3.3 1-affineness in analytic geometry

Finally, we are ready to prove our main 1-affineness results in the context of rigid and analytic geometry, highlighting a class of analytic stacks and rigid varieties which are 1-affine. While the final results that we obtain are similar in both the analytic and rigid cases, the strategy of the proofs and the reason why certain objects are 1-affine are different, so we split our presentation in two parts.

3.3.1 1-affineness for derived categories of analytic stacks

Consider the ∞ -category AnRing of analytic rings (in the sense of Definition 2.2.1), and consider the sheaf of presentably symmetric monoidal ∞ -categories $\mathcal{D}(-): \text{AnRing} \rightarrow \text{CAlg}(\text{Pr}^{\text{L}})$. Endow AnRing with the universal $!$ -able topology. Then, Theorem 3.2.15.(3) guarantees that this piece of data fits in the setting of Assumption 3.1.13.(1). In particular, Proposition 3.1.25 tells us the following.

Proposition 3.3.1. *The functor*

$$\text{Mod}_{\mathcal{D}(-)}: \text{AnRing}_{\mathbb{Z}^{\text{triv}}} \longrightarrow \text{Pr}_{\mathcal{D}(\mathbb{Z}^{\text{triv}})}^{\text{L}}$$

is a sheaf for the universal $!$ -able topology.

Our goal is to single out a class of analytic stacks that are 1-affine. This problem can be seen as a particular instance of Question 3.1.11. Indeed, in this case Theorem 3.1.35 immediately specializes to the following theorem.

Theorem 3.3.2. *Let X be an analytic stack admitting an affine $!$ -cover $\{\mathrm{AnSpec}(A_i) \rightarrow X\}_{i \in I}$ such that all the fiber products $\mathrm{AnSpec}(A_{i_1}) \times_X \cdots \times_X \mathrm{AnSpec}(A_{i_n})$ are affine. Then X is 1-affine.*

Corollary 3.3.3. *Let X be an analytic stack admitting an affine universal $!$ -cover $\{\mathrm{AnSpec}(A_i) \rightarrow X\}_{i \in I}$ such that all the fiber products $\mathrm{AnSpec}(A_{i_1}) \times_X \cdots \times_X \mathrm{AnSpec}(A_{i_n})$ are affine. Then the functor*

$$\mathrm{Mod}_{\mathcal{D}(-)} : \mathrm{AnStacks} \rightarrow \mathrm{Cat}^{\mathrm{rex}} \quad (3.16)$$

satisfies descent along the cover $\{\mathrm{AnSpec}(A_i) \rightarrow X\}_{i \in I}$.

Proof. This follows using Theorem 3.3.2 since by definition $\mathcal{D}^{\mathrm{cat}}(-) := \mathrm{Mod}_{\mathcal{D}(-)}$ satisfies descent. $\square$

Theorem 3.3.2 has also been independently proved in [Kes25] while this paper was in preparation. However, the formulation of our general result, Theorem 3.1.35, *a priori* allows us to obtain a more general statement. For instance, it enables us to consider the 1-affineness property with respect to certain $!$ -hypercovers, rather than restricting exclusively to Čech covers. Indeed, the definition of analytic stacks in [CS24] lies between the notions of a sheaf and a hypersheaf. Nevertheless, for the applications described later in this paper, such as the Betti stack, the formulation of Theorem 3.3.2 suffices.

Notation 3.3.4. *Let $\mathrm{CondAni}$ denote the ∞ -category of condensed anima. It is defined in [CS24] as the ∞ -category of hypercomplete sheaves on ProFin with values in Ani . We also note that for the 1-category condensed sets there is no difference in considering sheaves or hypesheaves on ProFin , see [Luro9, Lemma 6.5.2.9]. The fact that the ∞ -category of analytic stacks also features hypersheaves (such as condensed anima) rather than merely sheaves provides one of the primary motivations for developing the theory in Section 3.3.1, which applies to any localization of the ∞ -category of sheaves.*

Inside the ∞ -category $\mathrm{CondAni}$, there are two classes of objects coming from topology.

1. *Homotopy types or anima:* they are the objects in the essential image of the constant functor $\mathrm{Ani} \rightarrow \mathrm{CondAni}$.
2. *Topological spaces (not up to homotopy):* all topological spaces can be understood as condensed sets ([Sch26b, Example 1.5]). The functor $\mathrm{Top} \rightarrow \mathrm{Set}^{\mathrm{cond}}$ is always faithful, and moreover fully faithful on the subcategory of κ -compactly generated topological spaces ([Sch26b, Proposition 1.7]). So, post-composing with the limit-preserving inclusion $\iota_0 : \mathrm{Set} \subseteq \mathrm{Ani}$ we obtain a faithful functor $\mathrm{Top} \rightarrow \mathrm{CondAni}$.

If we denote with $|-|$ the functor sending a topological space to its underlying anima, and with $\underline{(-)}$ the functor from topological spaces to condensed sets as described above, we obtain the diagram

$$\begin{array}{ccc}
 & \text{CondAni} & \\
 \lrcorner \nearrow & & \nwarrow \text{const} \\
 \text{Set}^{\text{cond}} & & \text{Ani} \\
 \nwarrow \underline{(-)} & & \nearrow |-| \\
 & \text{Top.} &
 \end{array} \tag{3.17}$$

which however is *not* commutative.

In particular we can see two different realizations of a topological space in the category CondAni . Using the right hand side of the diagram (3.17) we can assign to a topological space the sheafification of the constant presheaf with values in its underlying homotopy type. On the other side, we can assign to a topological space a 0-truncated object in the ∞ -topos of condensed anima (in the sense of [Luro9, Definition 5.5.6.1]). This means that we have a collection of homotopy *condensed* groups, and all the higher homotopy condensed groups are trivial.

In this section we will view topological spaces as objects in condensed anima using the composition on the left hand side of the diagram (3.17). This functor is actually very well behaved.

Lemma 3.3.5. *The functor*

$$\text{Top} \xrightarrow{\underline{(-)}} \text{Set}^{\text{cond}} \xrightarrow{\iota_0} \text{CondAni}$$

appearing in the left hand side of the diagram (3.17) commutes with all limits. Moreover, it is fully faithful when restricted to metrizable compact Hausdorff topological spaces.

Proof of Lemma 3.3.5. We begin observing that Set^{cond} can be identified with $\tau_{\leq 0}\text{CondAni}$, this follows from [Luro9, Lemma 6.5.2.9].

The first assertions follows from the fact that both the functor $\text{Top} \rightarrow \text{Set}^{\text{cond}}$ and the functor $\text{Set}^{\text{cond}} \rightarrow \text{CondAni}$ are right adjoints. For the first functor, this is proved in [Cam26, Proposition 2.2.4] (see also [Sch26b, Proposition 1.7]); the latter follows from [Luro9, Proposition 5.5.6.18].

The second statement follows again from [Cam26, Proposition 2.2.4] since the second is always fully faithful. $\square$

The above observations allow us to define the *condensed Betti stack of a topological space*, following [Sch24]. We start by defining this functor on light pro-finite sets, and

then we extend it to every condensed anima (hence, to every topological space in virtue of Lemma 3.3.5).

Construction 3.3.6. Let $(-)_\text{Betti}$ denote the functor

$$\text{ProFin}^{\text{light}} \longrightarrow \text{AnStacks} \quad (3.18)$$

defined by sending a pro-finite set T to the affine analytic stack $\text{AnSpec}(\mathcal{C}^0(T, \mathbb{Z})^{\text{triv}})$. Here, $\mathcal{C}^0(T, \mathbb{Z})^{\text{triv}}$ denotes the trivial analytic ring on the $\mathbb{E}_\infty$ -ring of continuous functions from T (equipped with its inverse limit topology) to $\mathbb{Z}$, as described in Example 3.2.11.(1). In other words, the class of complete condensed modules over $\mathcal{C}^0(T, \mathbb{Z})$ coincides with the whole $\text{Mod}_{\mathcal{C}^0(T, \mathbb{Z})}$.

Proposition 3.3.7 ([CS24, Lecture 20]). *The functor (3.18) preserves finite limits and sends hypercovers of pro-finite sets to universal $!$ -covers.*

Sketch of proof. Since each pro-finite set is an inverse limit of finite sets one reduces to prove the claim for finite sets, for which the proofs are easy. $\square$

Corollary 3.3.8. *The functor (3.18) admits a unique left exact and colimit-preserving extension to a functor $\text{CondAni} \rightarrow \text{AnStacks}$. We still denote such extension as $(-)_\text{Betti}$, and call this functor the extended Betti stack functor.*

Proof. This is a direct consequence of Proposition 3.3.7 applying [Luro9, Proposition 6.2.3.20]. $\square$

Let now X be a topological space and let T be a light pro-finite set such that there exists a surjection $f: T \rightarrow X$ (here we are viewing light pro-finite sets inside Top in virtue of Remark 3.2.2). For any $i \geq 0$, let us denote with T_i the i -th object in the simplicial hypercover associated to Čech nerve of f . Since $(-)_\text{Betti}$ commutes with colimits and finite limits, we can describe X_Betti as the geometric realization of the simplicial diagram

$$(T_1)_\text{Betti} \rightrightarrows (T_2)_\text{Betti} \rightrightarrows (T_3)_\text{Betti} \rightrightarrows \cdots$$

In general, the $(T_i)_\text{Betti}$'s are not affine analytic stacks. The goal of the following lemmas is to show that, at least when X is a finite dimensional, metrizable and compact topological space, then the above simplicial diagram arises as the Čech nerve of a universal $!$ -cover of analytic stacks, and at each step such simplicial diagram consists of affine analytic stacks. In particular, this applies to the case when X is a compact topological manifold.

Lemma 3.3.9. *Let X be a Hausdorff topological space and let T_1 and T_2 two light pro-finite sets mapping into X . Then the pullback $T_1 \times_X T_2$ is still a light pro-finite set.*

Proof. Let $f_i: T_i \rightarrow X$ be the two maps in the statement. We have a natural map $\langle f_1, f_2 \rangle: T_1 \times T_2 \rightarrow X \times X$. Taking the pullback of this map along the diagonal $\Delta: X \rightarrow X \times X$ yields a natural map

$$\alpha: (T_1 \times T_2) \times_{X \times X} X \longrightarrow T_1 \times T_2.$$

Since X is Hausdorff, the diagonal $X \rightarrow X \times X$ is a closed embedding, and so is α . However, the source of α is homeomorphic to $T_1 \times_X T_2$. In particular, it is a closed subset of the compact metrizable totally disconnected Hausdorff topological space $T_1 \times T_2$, so it is compact, metrizable, totally disconnected and Hausdorff itself. $\square$

Lemma 3.3.10. *Let X be a metrizable and compact topological space. Then there exist a light pro-finite set K and a surjection $f: K \rightarrow X$. Its Čech nerve yields a simplicial diagram, in CondAni whose colimit is X , which at each step consists only of light pro-finite sets. Furthermore $f_{\text{Betti}}: K_{\text{Betti}} \rightarrow X_{\text{Betti}}$ is a universal $*$ -cover.*

Proof. Since X is a metrizable compact topological space, we have a continuous surjection from the Cantor set $K := \{0, 1\}^{\mathbb{N}}$ to X . Let f denote such cover map.

Using Lemma 3.3.9, we have that every pullback $K \times_X \cdots \times_X K$ is again a light pro-finite set. Now the Čech nerve of f is the required hypercover of X consisting of light pro-finite sets. To prove that f_{Betti} is a $*$ -cover we need to show that the functor $\mathcal{D}(-)$ satisfies descent along f_{Betti}^* and along every fiber product of f_{Betti}^* . By construction f_{Betti} is an effective epimorphism, so it naturally satisfies $*$ -descent. Universal $*$ -descent follows from the fact that effective epimorphisms are stable under pullbacks ([Luro9, Proposition 6.2.3.15]). $\square$

Lemma 3.3.11. *Let X be a finite dimensional, metrizable and compact topological space. Then there is a universal $!$ -cover of $f_{\text{Betti}}: K_{\text{Betti}} \rightarrow X_{\text{Betti}}$ where K is a light pro-finite sets and the Čech nerve of f_{Betti} is made of affine analytic stacks.*

Proof. Arguing as in the above proof we can find a light pro-finite set K and a surjection $K \rightarrow X$ such that the associated Čech nerve is made of light pro-finite sets. In virtue of [Sch24, Proposition II.1.1] the object $f_{\text{Betti}}(\mathbb{1})$ is descendable, so if $\mathbb{1}_{K_{\text{Betti}}}$ was also f_{Betti} -proper we could apply [HM24, Lemma 4.7.4] to deduce that f_{Betti} yields a universal $!$ -cover.

Using [HM24, Lemma 4.4.8] and Lemma 3.3.9 we can assume that f is a map of light pro-finite sets. In this case, f_{Betti} is actually weakly cohomologically proper in the sense

of Definition 3.2.22. Indeed, the analytic ring structure on $\mathcal{C}^0(T, \mathbb{Z})$ is always induced from the one of $\mathbb{Z}^{\text{triv}}$ for all light pro-finite sets T , and so we simply apply Remark 3.2.23 to get our desired assertion.

The fact that the Čech nerve of f_{Betti} is made of affine analytic stacks follows directly from Corollary 3.3.8 and Lemma 3.3.9. $\square$

Theorem 3.3.12. *Let X be a finite dimensional, metrizable, compact Hausdorff space. Then the analytic stack X_{Betti} is 1-affine. Moreover the functor*

$$\text{Mod}_{\mathcal{D}(-)} : \text{AnStacks} \longrightarrow \widehat{\text{Cat}}^{\text{rex}}$$

satisfies descent along the universal $!$ -cover described in Lemma 3.3.11.

Proof. Applying Lemma 3.3.11, we see that the hypotheses of Theorem 3.3.2 are evidently satisfied, so X_{Betti} is indeed 1-affine. The last part follows from Corollary 3.3.3. $\square$

Remark 3.3.13. In derived algebraic geometry, one can define another Betti stack of a topological space X . This is the sheafification of the constant prestack with values in the underlying anima $|X|$.

In [PPS25a], it is proved that this "homotopical" Betti stack is never 1-affine when the second homotopy group of X is non-trivial. However, there are plenty metrizable and compact finite dimensional topological manifolds which admit non-trivial π_2 : one example is provided by the real projective space $\mathbb{RP}^2$. Thus, Theorem 3.3.12 shows a curious discrepancy between the analytic and the homotopical Betti stacks.

We collect now some consequences of Theorem 3.3.2. First, suppose that X is a compact complex manifold. Using the analytic Riemann–Hilbert correspondence as stated in [Sch24, Theorem II.3.1] and our Theorem 3.3.12 we can immediately prove the following result, that can be understood as the "analytic version of a theorem of Gaitsgory (see [Gai15, Theorem 2.6.3]).

Corollary 3.3.14. *Let X be a compact complex manifold, Then the analytic de Rham stack, $X_{\text{dR}}^{\text{an}}$ as defined in [Sch24] is 1-affine.*

Another immediate consequence of our result is 2-descent for derived categories of sheaves over well-behaved topological spaces.

Corollary 3.3.15. *Let X be a finite dimensional, metrizable, compact Hausdorff space. Fix a surjection $f: T \rightarrow X$ from a light pro-finite set T . Then the functor $\text{Sh}(-, \mathcal{D}(\mathbb{Z}^{\text{triv}}))$ sending a topological space X to its derived ∞ -category of sheaves with values in $\mathcal{D}(\mathbb{Z}^{\text{triv}}) \simeq \text{Mod}_{\mathbb{Z}}$ satisfies 2-descent along f . In other words, the functor*

$$\text{Pr}_{\text{Sh}(-, \mathcal{D}(\mathbb{Z}^{\text{triv}}))}^{\text{L}} : \text{Top} \longrightarrow \widehat{\text{Cat}}^{\text{rex}}$$

satisfies descent along f .

Proof. This follows from Theorem 3.3.12, once we observe that for any topological space X the ∞ -category $\mathcal{D}(X_{\text{Betti}})$ agrees with the derived ∞ -category $\text{Sh}(X, \mathcal{D}(\mathbb{Z}^{\text{triv}}))$ ([Sch24, Corollary II.1.2]). $\square$

Warning 3.3.16. Note that it is not reasonable to extend the above corollary to every compact Hausdorff space. Indeed, in [Hai22] it was proved that for a general presentable ∞ -category $\mathcal{C}$ the functor

$$\text{Sh}(-, \mathcal{C}) : \text{CHaus} \longrightarrow \widehat{\text{Cat}}_{\infty}$$

from compact Hausdorff spaces does not satisfy descent along a surjection $f : T \rightarrow X$ from a light pro-finite set. It does, however, satisfy descent up to Postnikov completion ([Hai22, Corollary 2.8]). In particular, for $\text{Sh}(-, \mathcal{C})$ to satisfy descent along f it is necessary that $\text{Sh}(X, \mathcal{C})$ is itself Postnikov complete. This is the case for example when X is a pro-finite set or is a CW-complex, see [Hai22, Examples 1.28 and 1.29].

3.3.2 Nuclear categories in rigid geometry

We want to provide an answer to Question 3.1.11 in the setting of rigid analytic geometry. In order to do so, we need to turn to the setting of ∞ -categories of nuclear modules.

In this section we recall the relevant definitions, and we prove that this problem fits in the general framework of categorified sheaves described in Section 3.1.3 (see Proposition 3.3.18).

Let $\mathcal{C}$ be a symmetric monoidal ∞ -category. Using the definition of nuclear objects in an ∞ -category, see Definition 2.2.6, the full subcategory of $\mathcal{C}$ spanned by nuclear objects will be denoted as $\text{Nuc}(\mathcal{C}) \subseteq \mathcal{C}$. Given an analytic ring A , the ∞ -category $\text{Nuc}(A)$ is a presentable full sub- ∞ -category of $\mathcal{D}(A)$ which is closed under colimits and tensor product ([Sch26b, Theorem 8.6]). This equips $\text{Nuc}(A)$ of a symmetric monoidal tensor product that we denote as $\widehat{\otimes}_A$.

For the remainder of this section we will use the setup introduced in [Mon26], in particular we will work with animated affinoid k -algebras and derived rigid analytic spaces defined over a complete non-archimedean base field k , as defined in Definition 2.3.11. We denote with dAfd_k the ∞ -category of derived k -affinoid spaces (i.e. the opposite ∞ -category of the ∞ -category of animated affinoid k -algebras) and with dRig_k the ∞ -category of derived rigid analytic spaces over k . Following the notation used in Equation (2.11) when A is an animated affinoid k -algebra we denote with $A_{\blacksquare}$ the associated analytic ring and with $\mathcal{D}(A)_{\blacksquare}$ the solid ∞ -category associated to A . As in the previous chapter when $\mathcal{C}$ is the derived ∞ -category $\mathcal{D}(A)$ of an analytic ring A , we shall simply write $\text{Nuc}(A)$ instead of $\text{Nuc}(\mathcal{D}(A)_{\blacksquare})$.

We also remind here that as already explained in the previous chapter the map $A \mapsto \text{Nuc}(A)$ defines a sheaf for the analytic topology on animated affinoid k -algebras (Lemma 2.3.10). Moreover given a map of animated affinoid k -algebras $f: A \rightarrow B$, the functor $\text{Nuc}(A) \rightarrow \text{Nuc}(B)$ is the restriction of the functor $\mathcal{D}(A) \rightarrow \mathcal{D}(B)$ to the full sub- ∞ -categories of nuclear objects ([MW25, Theorem 2.9 (c)]).

In this subsection we want to study a categorified version of the sheaf of nuclear modules. More precisely, we will study the functor

$$\begin{aligned} \text{dAfd}_k &\longrightarrow \widehat{\text{Cat}}^{\text{rex}} \\ A &\mapsto \text{Pr}_{\text{Nuc}(A)}^{\text{L}}. \end{aligned} \tag{3.19}$$

We will prove that such functor is a sheaf for the analytic topology, (Proposition 3.3.22.(2)).

We start remind to the reader some results proved in the previous chapter.

Proposition 3.3.17. *Let A be an animated affinoid k -algebra. Then A is nuclear over k .*

Proof. This follows from Remark 2.2.8. □

Proposition 3.3.17 tells us that given any animated affinoid k -algebra A , we can view it as a $\mathbb{E}_\infty$ -algebra object inside $\text{Nuc}(k)$. So we can consider two *a priori* distinct symmetric monoidal ∞ -categories associated to an animated affinoid k -algebra.

1. The ∞ -category $\text{Nuc}(A)$, with tensor product given by the restriction of the tensor product of $\mathcal{D}(A)$ ■.
2. The ∞ -category $\text{Mod}_A(\text{Nuc}(k))$, with tensor product given by the usual relative tensor product over A .

We now show that these two ∞ -categories are actually symmetric monoidally equivalent.

Proposition 3.3.18. *Let A be an animated affinoid algebra over k . Then there is an equivalence of ∞ -categories between $\text{Mod}_A(\text{Nuc}(k))$ and $\text{Nuc}(A)$. In particular $\text{Nuc}(A)$ is a rigid ∞ -category.*

Proof. This is exactly Proposition 2.2.11. □

Remark 3.3.19. Combining Proposition 3.3.17 with Proposition 3.3.18, we can see that for any map of animated affinoid k -algebras $A \rightarrow B$ we can consider B as an object in $\text{Nuc}(A)$. Indeed, $A \rightarrow B$ is in particular a map of commutative algebras inside $\text{Nuc}(k)$, so B inherits a natural A -module structure and so corresponds to an object in $\text{Mod}_A(\text{Nuc}(k))$. This ∞ -category is the same as the ∞ -category $\text{Nuc}(A)$, so our claim follows.

Proposition 3.3.20. *Let $A \rightarrow B$ a map of animated affinoid algebras over k . Under the identification $\mathrm{Mod}_A(\mathrm{Nuc}(k)) \simeq \mathrm{Nuc}(A)$ of Proposition 3.3.18, the pullback functor $\mathrm{Nuc}(A) \rightarrow \mathrm{Nuc}(B)$ can be identified with the usual base-change functor $-\widehat{\otimes}_A B$.*

Proof. Using the categorical Eilenberg–Watts theorem ([Lur17, Theorem 4.8.4.1]), we can identify the ∞ -category of $\mathrm{Nuc}(k)$ -linear, colimit-preserving functors from $\mathrm{Nuc}(A) \simeq \mathrm{Mod}_A(\mathrm{Nuc}(k))$ to $\mathrm{Nuc}(B) \simeq \mathrm{Mod}_B(\mathrm{Nuc}(k))$ with the ∞ -category of (A, B) -bimodules in $\mathrm{Nuc}(k)$. More explicitly, this equivalence is given by

$$\begin{aligned} \mathrm{Fun}_{\mathrm{Nuc}(k)}^L(\mathrm{Nuc}(A), \mathrm{Nuc}(B)) &\xrightarrow{\simeq} {}_A\mathrm{BMod}_B(\mathrm{Nuc}(k)) \\ F &\mapsto F(A). \end{aligned}$$

In particular the diagram

$$\begin{array}{ccc} \mathrm{Nuc}(A) & \xrightarrow{f^*} & \mathrm{Nuc}(B) \\ \wr \uparrow & & \wr \uparrow \\ \mathrm{Mod}_A(\mathrm{Nuc}(k)) & \xrightarrow{-\widehat{\otimes}_A B} & \mathrm{Mod}_B(\mathrm{Nuc}(k)) \end{array}$$

yields two functors $\mathrm{Mod}_A(\mathrm{Nuc}(k)) \rightarrow \mathrm{Mod}_B(\mathrm{Nuc}(k))$, both mapping the object A to the object B with the (A, B) -bimodule structure induced by the map $A \rightarrow B$. It follows that the two functors must be equivalent; in particular, the above diagram is commutative up to natural equivalence. $\square$

Remark 3.3.21. Applying Proposition 3.3.20 to the case $A = B$, we see that the endofunctor $\mathrm{Nuc}(A) \rightarrow \mathrm{Nuc}(A)$ given by the tensor product with a nuclear A -module M corresponds to the functor $-\widehat{\otimes}_A M: \mathrm{Mod}_A(\mathrm{Nuc}(k)) \rightarrow \mathrm{Mod}_A(\mathrm{Nuc}(k))$. In particular, the equivalence of Proposition 3.3.18 is symmetric monoidal.

In the following, we will blur the distinction between the functor $\mathrm{Nuc}(-)$ and the functor $\mathrm{Mod}_{(-)}(\mathrm{Nuc}(k))$, since the previous statements imply that they are naturally equivalent.

We now endow the ∞ -category of derived k -affinoid spaces with the Grothendieck topology described in Definition 2.3.7.

Proposition 3.3.22. *Let $\varphi: \mathrm{dSp}(B) \rightarrow \mathrm{dSp}(A)$ be a cover of animated affinoid spaces in the analytic topology of Definition 2.3.11.*

- (1) *We have that B is descendable as an algebra in $\mathrm{Nuc}(A)$.*

(2) *The functor*

$$\mathrm{Nuc}(-): \mathrm{Afd}_k \longrightarrow \mathrm{Pr}_{\mathrm{Nuc}(k)}^{\mathrm{L}}$$

satisfies descent along φ .

Proof. This follows using the same proof as in Lemma 2.3.10. $\square$

The previous corollary was the last brick of the theory that we needed in order to apply the general machinery developed in Section 3.1.3. In particular, we obtain the following.

Theorem 3.3.23. *Let $\mathrm{Spa}(B) \rightarrow \mathrm{Spa}(A)$ be a cover map of derived k -affinoid spaces. Then the adjunction*

$$- \otimes_{\mathrm{Nuc}(A)} \mathrm{Nuc}(B): \mathrm{Pr}_{\mathrm{Nuc}(A)}^{\mathrm{L}} \xrightleftharpoons[\perp]{} \mathrm{Pr}_{\mathrm{Nuc}(B)}^{\mathrm{L}} : \text{forget}$$

is comonadic. In particular the functor (3.19) is a sheaf for the analytic topology described in Definition 2.3.7.

Proof. This follows from Propositions 3.3.22.(1) and 3.1.38, after the identification

$$\mathrm{Nuc}(-) \simeq \mathrm{Mod}_{\mathrm{Nuc}(-)}(\mathrm{Nuc}(k))$$

of Proposition 3.3.18. $\square$

3.3.3 1-affineness for categories of nuclear modules over rigid varieties

After proving that the functor $\mathrm{Pr}_{\mathrm{Nuc}(-)}^{\mathrm{L}}$ is a sheaf for the analytic topology on animated affinoid k -algebras, we are ready to tackle Question 3.1.11 for more general derived rigid analytic spaces. We first set some piece of terminology.

Notation 3.3.24. *Let dRig_k denote the ∞ -category of derived rigid analytic spaces over our complete non-archimedean base field k . Since $\mathrm{Nuc}(-)$ is a sheaf for the analytic topology on animated affinoid k -algebras, it can be extended to a sheaf for the analytic topology on all dRig_k , i.e., we have*

$$\mathrm{Nuc}(-): \mathrm{dRig}_k \longrightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{Nuc}(k)}^{\mathrm{L}}).$$

We shall denote with $\mathrm{Nuc}^{\mathrm{cat}}$ the categorification of the above functor, as in Definition 3.1.3. More precisely, this is the right Kan extension along the inclusion $\mathrm{dAfd}_k \subseteq \mathrm{dRig}_k$ of the functor

$$\begin{aligned} \mathrm{Nuc}^{\mathrm{cat}}: \mathrm{dAfd}_k &\longrightarrow \widehat{\mathrm{Cat}}^{\mathrm{rex}} \\ \mathrm{Spa}(A) &\mapsto \mathrm{Pr}_{\mathrm{Nuc}(A)}^{\mathrm{L}}. \end{aligned}$$

In virtue of Theorem 3.3.23 and Proposition 3.1.38, $\mathrm{Nuc}^{\mathrm{cat}}$ is a sheaf for the analytic topology on rigid analytic varieties.

Warning 3.3.25. Assumption 3.1.45 almost holds for the case of classical rigid analytic varieties and nuclear modules, but not quite. The deal-breaker of the theory is that the functor $\mathrm{Nuc}(-)$ will not send finite limits of affinoid spaces to colimits of $\mathrm{Nuc}(k)$ -linear presentably symmetric monoidal ∞ -categories: indeed, in general the pullback $X \times_Y Z$ of affinoid spaces needs to be derived. The theory then can work in the context of derived rigid geometry as we will explain here.

Still, pullbacks are sent to tensor products when we restrict to flat morphisms of affinoid spaces (indeed, in this case we do not have to derive the tensor product of their coordinate animated affinoid k -algebras); in particular, pullbacks along analytic open immersions are sent to tensor products of ∞ -categories of nuclear modules. So, the results in Section 3.1.3 still hold under these stricter assumptions as explained in the original version of the paper [MP25].

We can now state our main result.

Theorem 3.3.26. *Let X be a quasi-compact and semi-separated derived rigid analytic space over k . Then X is 1-affine with respect to the sheaf of ∞ -categories $\mathrm{Nuc}^{\mathrm{cat}}(-)$. More precisely, there is an equivalence of ∞ -categories*

$$\mathrm{Pr}_{\mathrm{Nuc}(X)}^{\mathrm{L}} \simeq \lim_{\mathrm{dSp}(A) \subset X} \mathrm{Pr}_{\mathrm{Nuc}(A)}^{\mathrm{L}},$$

where the limit is taken over an affinoid cover of X .

Theorem 3.3.26 is a particular instance of Theorem 3.1.46. Indeed, the proof of Theorem 3.3.26 boils down to proving that the assumptions of Theorem 3.1.46 are satisfied.

Recall that the proof of Theorem 3.1.46 relies on various auxiliary results, among which some variants of categorical Künneth formulas. Since they might be of independent interest, we explicitly write the statements down when applying the formalism of Section 3.1.3 to the particular case of nuclear modules and rigid analytic varieties.

Proposition 3.3.27. *Let $\iota: Y \rightarrow X$ and $j: \mathrm{dSp}(B) \rightarrow X$ be morphisms of derived rigid analytic spaces, and assume that both Y and X are quasi-compact and semi-separated.*

(1) (Base change, Proposition 3.1.48.(1)) Consider the diagram

$$\begin{array}{ccc} \mathrm{dSp}(A) \times_X Y & \xrightarrow{\iota'} & \mathrm{dSp}(A) \\ j' \downarrow & & \downarrow j \\ Y & \xrightarrow{\iota} & X. \end{array}$$

Then for every nuclear sheaf M over Y the canonical map

$$j^* \iota_* M \longrightarrow \iota'_* j'^* M$$

is an equivalence.

- (2) (Proposition 3.1.48.(2)) When restricted to the nuclear ∞ -category, the functors ι_* and j_* commute with colimits. Moreover if j is an open immersion j_* is conservative. In particular, it both monadic and comonadic functors. The monad $j_* j^*$ on $\text{Nuc}(X)$ is naturally identified with the monad $-\widehat{\otimes}_{\text{Nuc}(X)} j_* A$, and thus we have an equivalence of ∞ -categories

$$\text{Mod}_{j_* A}(\text{Nuc}(X)) \simeq \text{Nuc}(A).$$

- (3) If ι is an analytic open immersion, then ι_* is moreover fully faithful.

- (4) (Categorical Künneth formula, Proposition 3.1.54) The canonical functor

$$\text{Nuc}(Y) \otimes_{\text{Nuc}(X)} \text{Nuc}(B) \longrightarrow \text{Nuc}(Y \times_X \text{Spa}(B))$$

is an equivalence.

- (5) (Projection formula, Proposition 3.1.50) For every N in $\text{Nuc}(Y)$ and M in $\text{Nuc}(X)$, the natural map

$$\iota_*(N \otimes_{\text{Nuc}(Y)} \iota^* M) \longrightarrow \iota_* N \otimes_{\text{Nuc}(X)} M$$

is an equivalence.

- (6) (Proposition 3.1.52) The ∞ -category $\text{Nuc}(X)$ is rigid.

In order to apply Theorem 3.1.46, the only thing that is not a formal consequence of the theory is that analytic open immersions of affinoid subsets $\iota: \text{dSp}(A) \rightarrow X$ provide an adjunction $\iota_{\natural} \dashv \iota^*$. This is the content of the following proposition. To avoid confusion with the lower shriek functor used in the previous chapter here we will denote the left adjoint to i^* by $i_{\natural}$.

Proposition 3.3.28. *Let $\iota: U \rightarrow X$ be an analytic open immersion into a semi-separated and quasi-compact derived rigid analytic space. Then the functor*

$$\iota^*: \text{Nuc}(X) \longrightarrow \text{Nuc}(U)$$

admits a left adjoint $\iota_{\natural}$.

In the proof we will use the following lemma.

Lemma 3.3.29. *Let $\iota: U \rightarrow \mathrm{dSp}(A)$ be a rational localization of a derived k -affinoid space, as in Definition 2.3.2, then the functor*

$$\iota^*: \mathcal{D}(A)_{\blacksquare} \longrightarrow \mathcal{D}(U)_{\blacksquare}$$

admits a fully faithful left adjoint $\iota_{\natural}$ that satisfies projection formula (i.e. is a categorical open immersion).

Proof. Since categorical open immersions are stable under compositions and base change, using the same argument in [Cam26, Lemma 7.2.3] we can reduce to prove that the map

$$(\mathbb{Z}[U], \mathbb{Z})_{\blacksquare} \rightarrow \mathbb{Z}[U]_{\blacksquare}$$

is a categorical open immersion. This follows from [Cam26, Lemma 5.4.13]. $\square$

Proof of Proposition 3.3.28. We will show that ι^* commutes with limits, i.e., that for any diagram $I \rightarrow \mathrm{Nuc}(X)$ selecting objects $(M_i)_{i \in I}$ the canonical map

$$\lim_{i \in I} \iota^* M_i \longrightarrow \iota^* \left(\lim_{i \in I} M_i \right)$$

is an equivalence. If X is not affinoid, we can still consider a finite affinoid cover $\{V_j \rightarrow X\}_{j \in J}$. Using descent of $\mathrm{Nuc}(-)$ we write

$$\mathrm{Nuc}(X) \simeq \lim_{j \in J} \mathrm{Nuc}(V_j),$$

so each M_i can be realized as a finite limit of a cosimplicial diagram $M_i^{\bullet}$, associated to the Čech nerve of the cover $\{V_j \rightarrow X\}$. In this way using the base-change property and the fact that ι^* commutes with finite limits we can reduce to prove the statement in the case $U \rightarrow X$ is an affinoid subdomain into a derived affinoid space.

So that we can assume X to be a derived affinoid and we can cover U with rational subdomains of X . Using the same argument at the beginning of proof of Lemma 3.1.55 we can moreover assume that U is a rational subset. Using Lemma 3.3.29 the inclusion of a rational open subset $U \hookrightarrow X$ is also an open immersion in the sense of Definition 3.2.13.(2), so we obtain an adjunction at the level of solid ∞ -categories

$$\iota_{\natural}: \mathcal{D}(U)_{\blacksquare} \xrightleftharpoons[\perp]{} \mathcal{D}(X)_{\blacksquare} : \iota^*$$

where $\iota_{\natural}$ is moreover fully faithful and satisfies the projection formula. In particular, for

every N and M in $\mathcal{D}(U)$ we have that

$$\iota_{\natural}(N \otimes_{\mathcal{D}(U)} M) \simeq \iota_{\natural}(N \otimes_{\mathcal{D}(U)} \iota^* \iota_{\natural} M) \simeq \iota_{\natural} N \otimes_{\mathcal{D}(X)} \iota_{\natural} M.$$

So $\iota_{\natural}$ is actually a strongly monoidal colimit-preserving functor, and thus using [MW25, Theorem 2.9 (c)] we can conclude that $\iota_{\natural}$ preserves nuclear subcategories. In particular, $\iota_{\natural}: \mathcal{D}(X)_{\blacksquare} \rightarrow \mathcal{D}(U)_{\blacksquare}$ restricts to a right adjoint for $\iota^*: \text{Nuc}(X) \rightarrow \text{Nuc}(U)$ that thus obviously commutes with limits. $\square$

Remark 3.3.30. In the context of derived rigid analytic varieties developed in [Soo25], we still have that an analogue of Proposition 3.3.28 holds ([Soo25, Proposition 4.5]). In particular, one could get a 1-affineness result also for quasi-compact and semi-separated *derived* rigid analytic varieties with respect to the sheaf of presentably symmetric monoidal ∞ -categories $\text{Nuc}(-)$ on derived rigid analytic varieties: indeed, $\text{Nuc}(k)$ is rigid thanks to [KM25, Corollary 6.38].

We conclude by presenting some consequences of Theorem 3.3.26. In particular we will obtain a relative version Proposition 2.3.14.

Corollary 3.3.31. *Let $X \rightarrow Z$ and $Y \rightarrow Z$ be two maps of quasi-compact and semi-separated derived rigid analytic spaces. Then the diagram of ∞ -categories*

$$\begin{array}{ccc} \text{Pr}_{\text{Nuc}(Z)}^L & \xrightarrow{- \otimes_{\text{Nuc}(Z)} \text{Nuc}(Y)} & \text{Pr}_{\text{Nuc}(Y)}^L \\ \downarrow - \otimes_{\text{Nuc}(Z)} \text{Nuc}(X) & & \downarrow - \otimes_{\text{Nuc}(Y)} \text{Nuc}(X \times_Z Y) \\ \text{Pr}_{\text{Nuc}(X)}^L & \xrightarrow{- \otimes_{\text{Nuc}(X)} \text{Nuc}(X \times_Z Y)} & \text{Pr}_{\text{Nuc}(X \times_Z Y)}^L \end{array} \quad (3.20)$$

is vertically right adjointable.

Proof. The proof can be obtained by adapting the argument of [Ste23, Proposition 3.2.14]. We want to show that for every $\mathcal{C}$ in $\text{Pr}_{\text{Nuc}(X)}^L$ the canonical functor

$$\mathcal{C} \otimes_{\text{Nuc}(Z)} \text{Nuc}(Y) \longrightarrow \mathcal{C} \otimes_{\text{Nuc}(X)} \text{Nuc}(X \times_Z Y)$$

is an equivalence. Consider an affinoid open cover of X , so as to write

$$\text{Pr}_{\text{Nuc}(X)}^L \simeq \lim_{\text{dSp}(A_i) \subseteq X} \text{Pr}_{\text{Nuc}(A_i)}^L$$

using Theorem 3.3.26. Notice that all the forgetful functors and the functors $- \otimes_{\text{Nuc}(X)} \text{Nuc}(X \times_Z Y)$ and $- \otimes_{\text{Nuc}(Z)} \text{Nuc}(Y)$ commute with limits (the latter in virtue of Propo-

sition 3.3.27.(6)), so we can reduce ourselves to the case when X is affinoid. Now we apply (the dual of) [Ste23, Lemma 2.2.7] as in the proof of [Ste23, Proposition 3.2.14] to reduce to the case where both Z and Y are affinoid as well. In this case, the statement follows from the fact that the diagram

$$\begin{array}{ccc} \mathrm{Nuc}(Z) & \longrightarrow & \mathrm{Nuc}(Y) \\ \downarrow & & \downarrow \\ \mathrm{Nuc}(X) & \longrightarrow & \mathrm{Nuc}(X \times_Z Y) \end{array}$$

is a pushout diagram of commutative algebras in Pr^{L} (Proposition 3.3.27.(4)). $\square$

Remark 3.3.32. Applying Corollary 3.3.31 to the categorical $\mathrm{Nuc}(X)$ -module $\mathcal{C} \simeq \mathrm{Nuc}(X)$, we obtain an equivalence of categorical $\mathrm{Nuc}(Y)$ -modules

$$\mathrm{Nuc}(X \times_Z Y) \simeq \mathcal{C} \otimes_{\mathrm{Nuc}(Z)} \mathrm{Nuc}(Y),$$

which represents a stronger variant of the categorical Künneth formula for nuclear modules of Proposition 2.3.14

Bibliography

- [And21] Andreychev, G. *Pseudocoherent and Perfect Complexes and Vector Bundles on Analytic Adic Spaces*. 2021. arXiv: [2105.12591](https://arxiv.org/abs/2105.12591) [math.AG]. URL: <https://arxiv.org/abs/2105.12591>.
- [And23] Andreychev, G. *K-Theorie adischer Räume*. 2023. arXiv: [2311.04394](https://arxiv.org/abs/2311.04394) [math.KT]. URL: <https://arxiv.org/abs/2311.04394>.
- [ATo7] Anel, M. and Toen, B. *On derived equivalence classes of algebraic varieties*. 2007. arXiv: [math/0611545](https://arxiv.org/abs/math/0611545) [math.AG]. URL: <https://arxiv.org/abs/math/0611545>.
- [AM24] Anschütz, J. and Mann, L. *Descent for solid quasi-coherent sheaves on perfectoid spaces*. 2024. arXiv: [2403.01951](https://arxiv.org/abs/2403.01951) [math.AG]. URL: <https://arxiv.org/abs/2403.01951>.
- [AG14] Antieau, B. and Gepner, D. “Brauer groups and étale cohomology in derived algebraic geometry”. In: *Geometry and; Topology* 18.2 (Apr. 2014), pp. 1149–1244. ISSN: 1465-3060. DOI: [10.2140/gt.2014.18.1149](https://doi.org/10.2140/gt.2014.18.1149). URL: <http://dx.doi.org/10.2140/gt.2014.18.1149>.
- [Ant19] Antonio, J. F. “Derived integral models and its applications to the study of certain derived rigid analytic moduli spaces”. PhD thesis. Université Paul Sabatier-Toulouse III, 2019.
- [Ant20] António, J. *Derived $\mathcal{O}_k$ -adic geometry and derived Raynaud localization theorem*. 2020. arXiv: [1805.03302](https://arxiv.org/abs/1805.03302) [math.AG]. URL: <https://arxiv.org/abs/1805.03302>.
- [AP20] António, J. and Porta, M. “DERIVED NON-ARCHIMEDEAN ANALYTIC HILBERT SPACE”. In: *Journal of the Institute of Mathematics of Jussieu* 21.1 (Aug. 2020), pp. 335–366. ISSN: 1475-3030. DOI: [10.1017/S1474748020000092](https://doi.org/10.1017/S1474748020000092). URL: <http://dx.doi.org/10.1017/S1474748020000092>.
- [Aok25] Aoki, K. *Higher presentable categories and limits*. 2025. arXiv: [2510.13503](https://arxiv.org/abs/2510.13503) [math.CT]. URL: <https://arxiv.org/abs/2510.13503>.

- [Art70] Artin, M. “Algebraization of formal moduli. II. Existence of modifications”. In: *Ann. of Math.* (2) 91 (1970), pp. 88–135. ISSN: 0003-486X. DOI: [10.2307/1970602](https://doi.org/10.2307/1970602). URL: <https://doi.org/10.2307/1970602>.
- [BB16] Bambozzi, F. and Ben-Bassat, O. “Dagger geometry as Banach algebraic geometry”. In: *J. Number Theory* 162 (2016), pp. 391–462. ISSN: 0022-314X, 1096-1658. DOI: [10.1016/j.jnt.2015.10.023](https://doi.org/10.1016/j.jnt.2015.10.023). URL: <https://doi.org/10.1016/j.jnt.2015.10.023>.
- [BBK18] Bambozzi, F., Ben-Bassat, O., and Kremnizer, K. “Stein domains in Banach algebraic geometry”. In: *Journal of Functional Analysis* 274.7 (2018), pp. 1865–1927. ISSN: 0022-1236. DOI: <https://doi.org/10.1016/j.jfa.2018.01.003>. URL: <https://www.sciencedirect.com/science/article/pii/S0022123618300168>.
- [BLM+21] Bayer, A., Lahoz, M., Macrì, E., Nuer, H., Perry, A., and Stellari, P. “Stability conditions in families”. In: *Publ. Math. Inst. Hautes Études Sci.* 133 (2021), pp. 157–325. ISSN: 0073-8301, 1618-1913. DOI: [10.1007/s10240-021-00124-6](https://doi.org/10.1007/s10240-021-00124-6). URL: <https://doi.org/10.1007/s10240-021-00124-6>.
- [BKK24] Ben-Bassat, O., Kelly, J., and Kremnizer, K. *A Perspective on the Foundations of Derived Analytic Geometry*. 2024. arXiv: [2405.07936](https://arxiv.org/abs/2405.07936) [math.AG]. URL: <https://arxiv.org/abs/2405.07936>.
- [Ben24] Ben-Moshe, S. “Naturality of the infinity-categorical enriched Yoneda embedding”. In: *Journal of Pure and Applied Algebra* 228.6 (2024), p. 107625. ISSN: 0022-4049. DOI: [10.1016/j.jpaa.2024.107625](https://doi.org/10.1016/j.jpaa.2024.107625). URL: <http://dx.doi.org/10.1016/j.jpaa.2024.107625>.
- [BS23] Ben-Moshe, S. and Schlank, T. M. “Higher semiadditive algebraic K-theory and redshift”. In: *Compositio Mathematica* 160.2 (Dec. 2023), pp. 237–287. ISSN: 1570-5846. DOI: [10.1112/S0010437X23007595](https://doi.org/10.1112/S0010437X23007595). URL: <http://dx.doi.org/10.1112/S0010437X23007595>.
- [BFN10] Ben-Zvi, D., Francis, J., and Nadler, D. “Integral transforms and Drinfeld centers in derived algebraic geometry”. In: *J. Amer. Math. Soc.* 23.4 (2010), pp. 909–966. ISSN: 0894-0347, 1088-6834. DOI: [10.1090/S0894-0347-10-00669-7](https://doi.org/10.1090/S0894-0347-10-00669-7). URL: <https://doi.org/10.1090/S0894-0347-10-00669-7>.

- [BP24] Beraldo, D. and Pippi, M. *Proof of the Deligne-Milnor conjecture*. 2024. arXiv: 2410.02327 [math.AG]. URL: <https://arxiv.org/abs/2410.02327>.
- [BP23] Binda, F. and Porta, M. *GAGA problems for the Brauer group via derived geometry*. 2023. arXiv: 2107.03914 [math.AG]. URL: <https://arxiv.org/abs/2107.03914>.
- [BGT13] Blumberg, A. J., Gepner, D., and Tabuada, G. “A universal characterization of higher algebraic K-theory”. In: *Geometry & Topology* 17.2 (Apr. 2013), pp. 733–838. ISSN: 1465-3060. DOI: 10.2140/gt.2013.17.733. URL: <http://dx.doi.org/10.2140/gt.2013.17.733>.
- [BB03] Bondal, A. and Bergh, M. van den. “Generators and representability of functors in commutative and noncommutative geometry”. In: *Mosc. Math. J.* 3.1 (2003), pp. 1–36, 258. ISSN: 1609-3321,1609-4514. DOI: 10.17323/1609-4514-2003-3-1-1-36. URL: <https://doi.org/10.17323/1609-4514-2003-3-1-1-36>.
- [BO01] Bondal, A. and Orlov, D. “Reconstruction of a variety from the derived category and groups of autoequivalences”. In: *Compositio Math.* 125.3 (2001), pp. 327–344. ISSN: 0010-437X,1570-5846. DOI: 10.1023/A:1002470302976. URL: <https://doi.org/10.1023/A:1002470302976>.
- [Bos14] Bosch, S. *Lectures on formal and rigid geometry*. Vol. 2105. Lecture Notes in Mathematics. Springer, Cham, 2014, pp. viii+254. DOI: 10.1007/978-3-319-04417-0. URL: <https://doi.org/10.1007/978-3-319-04417-0>.
- [BL93] Bosch, S. and Lütkebohmert, W. “Formal and rigid geometry. I. Rigid spaces.” In: *Mathematische Annalen* 295.2 (1993), pp. 291–318. URL: <http://eudml.org/doc/165044>.
- [Bra18] Brandenburg, M. “Rosenberg’s reconstruction theorem”. In: *Expo. Math.* 36.1 (2018), pp. 98–117. ISSN: 0723-0869,1878-0792. DOI: 10.1016/j.exmath.2017.08.005. URL: <https://doi.org/10.1016/j.exmath.2017.08.005>.
- [Cam24] Camargo, J. E. R. *The analytic de Rham stack in rigid geometry*. 2024. arXiv: 2401.07738 [math.AG]. URL: <https://arxiv.org/abs/2401.07738>.
- [Cam26] Camargo, J. E. R. “Notes on solid geometry”. In: *arXiv preprint arXiv:2603.03012* (2026).

- [CS22] Clausen, D. and Scholze, P. “Condensed mathematics and complex geometry”. In: *Lecture notes, Bonn–Copenhagen* (2022).
- [CS24] Clausen, D. and Scholze, P. *Analytic stacks*. <https://www.youtube.com/watch?v=YxSZ1mTIpA>. Video course. 2024.
- [CS26] Clausen, D. and Scholze, P. *Condensed Mathematics and Complex Geometry*. 2026. arXiv: 2605.11731 [math.CV]. URL: <https://arxiv.org/abs/2605.11731>.
- [Cono7] Conrad, B. *Moishezon spaces in rigid geometry*. Preprint, Stanford University. 2007. URL: <https://math.stanford.edu/~conrad/papers/moish.pdf>.
- [CLO12] Conrad, B., Lieblich, M., and Olsson, M. “Nagata compactification for algebraic spaces”. In: *J. Inst. Math. Jussieu* 11.4 (2012), pp. 747–814. ISSN: 1474-7480,1475-3030. DOI: 10.1017/S1474748011000223. URL: <https://doi.org/10.1017/S1474748011000223>.
- [Efi18] Efimov, A. I. *Homotopy finiteness of some DG categories from algebraic geometry*. 2018. arXiv: 1308.0135 [math.AG]. URL: <https://arxiv.org/abs/1308.0135>.
- [Efi25a] Efimov, A. I. *K-theory and localizing invariants of large categories*. 2025. arXiv: 2405.12169 [math.KT]. URL: <https://arxiv.org/abs/2405.12169>.
- [Efi25b] Efimov, A. I. *Localizing invariants of inverse limits*. 2025. arXiv: 2502.04123 [math.KT]. URL: <https://arxiv.org/abs/2502.04123>.
- [GRo2] Gabber, O. and Ramero, L. *Almost ring theory - sixth release*. 2002. arXiv: math/0201175 [math.AG]. URL: <https://arxiv.org/abs/math/0201175>.
- [Gai15] Gaitsgory, D. “Sheaves of categories and the notion of $\mathfrak{t}$ -affineness”. In: *Stacks and categories in geometry, topology, and algebra*. Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 127–225. DOI: 10.1090/conm/643/12899. URL: <https://doi.org/10.1090/conm/643/12899>.
- [GR17] Gaitsgory, D. and Rozenblyum, N. *A study in derived algebraic geometry*. Vol. I. *Correspondences and duality*. Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017, pp. xl+533. ISBN: 978-1-4704-3569-1. DOI: 10.1090/surv/221.1. URL: <https://doi.org/10.1090/surv/221.1>.

- [GH15] Gepner, D. and Haugseng, R. “Enriched ∞ -categories via non-symmetric ∞ -operads”. In: *Advances in Mathematics* 279 (July 2015), pp. 575–716. ISSN: 0001-8708. DOI: [10.1016/j.aim.2015.02.007](https://doi.org/10.1016/j.aim.2015.02.007). URL: <http://dx.doi.org/10.1016/j.aim.2015.02.007>.
- [Hai22] Haine, P. J. *Descent for sheaves on compact Hausdorff spaces*. 2022. arXiv: [2210.00186](https://arxiv.org/abs/2210.00186) [math.AT]. URL: <https://arxiv.org/abs/2210.00186>.
- [HR25] Halpern-Leistner, D. and Robotis, A.-A. *The space of augmented stability conditions*. 2025. arXiv: [2501.00710](https://arxiv.org/abs/2501.00710) [math.AG]. URL: <https://arxiv.org/abs/2501.00710>.
- [Hau18] Haugseng, R. “Iterated spans and classical topological field theories”. In: *Math. Z.* 289.3-4 (2018), pp. 1427–1488. ISSN: 0025-5874. DOI: [10.1007/s00209-017-2005-x](https://doi.org/10.1007/s00209-017-2005-x). URL: <https://doi.org/10.1007/s00209-017-2005-x>.
- [HHLN23] Haugseng, R., Hebestreit, F., Linskens, S., and Nuiten, J. “Lax monoidal adjunctions, two-variable fibrations and the calculus of mates”. In: *Proc. Lond. Math. Soc.* (3) 127.4 (2023), pp. 889–957. ISSN: 0024-6115,1460-244X. DOI: [10.1112/plms.12548](https://doi.org/10.1112/plms.12548). URL: <https://doi.org/10.1112/plms.12548>.
- [Hei23] Heine, H. *An equivalence between enriched ∞ -categories and ∞ -categories with weak action*. 2023. arXiv: [2009.02428](https://arxiv.org/abs/2009.02428) [math.AT]. URL: <https://arxiv.org/abs/2009.02428>.
- [HM24] Heyer, C. and Mann, L. *6-Functor Formalisms and Smooth Representations*. 2024. arXiv: [2410.13038](https://arxiv.org/abs/2410.13038) [math.CT]. URL: <https://arxiv.org/abs/2410.13038>.
- [Hin21] Hinich, V. *Yoneda lemma for enriched infinity categories*. 2021. arXiv: [1805.07635](https://arxiv.org/abs/1805.07635) [math.CT]. URL: <https://arxiv.org/abs/1805.07635>.
- [HP25] Holstein, J. and Porta, M. “Analytification of mapping stacks”. In: *Algebr. Geom.* 12.1 (2025), pp. 13–83. ISSN: 2313-1691,2214-2584. DOI: [10.14231/ag-2025-002](https://doi.org/10.14231/ag-2025-002). URL: <https://doi.org/10.14231/ag-2025-002>.
- [KM25] Kelly, J. and Mukherjee, D. *Localising invariants in derived bornological geometry*. 2025. arXiv: [2505.15750](https://arxiv.org/abs/2505.15750) [math.KT]. URL: <https://arxiv.org/abs/2505.15750>.
- [Kes25] Kesting, Y. *Categorical Künneth formulas for analytic stacks*. 2025. arXiv: [2507.08566](https://arxiv.org/abs/2507.08566) [math.AG]. URL: <https://arxiv.org/abs/2507.08566>.

- [KS09] Kontsevich, M. and Soibelman, Y. “Notes on A_∞ -algebras, A_∞ -categories and non-commutative geometry”. In: *Homological mirror symmetry*. Vol. 757. Lecture Notes in Phys. Springer, Berlin, 2009, pp. 153–219. ISBN: 978-3-540-86374-8.
- [Kon98] Kontsevich, M. “Triangulated categories and geometry”. In: *Course at the École Normale Supérieure* (1998).
- [KL15] Kuznetsov, A. and Lunts, V. A. “Categorical resolutions of irrational singularities”. In: *Int. Math. Res. Not. IMRN* 13 (2015), pp. 4536–4625. ISSN: 1073-7928,1687-0247. DOI: [10.1093/imrn/rnu072](https://doi.org/10.1093/imrn/rnu072). URL: <https://doi.org/10.1093/imrn/rnu072>.
- [Lam25] Lampetti, E. *Good moduli space for constructible sheaves and Stokes functors*. 2025. arXiv: [2510.23850](https://arxiv.org/abs/2510.23850) [math.AG]. URL: <https://arxiv.org/abs/2510.23850>.
- [LX15] Liu, Q. and Xu, F. “Very strong approximation for certain algebraic varieties”. In: *Mathematische Annalen* 363.3-4 (Feb. 2015), pp. 701–731. ISSN: 1432-1807. DOI: [10.1007/s00208-014-1107-6](https://doi.org/10.1007/s00208-014-1107-6). URL: <http://dx.doi.org/10.1007/s00208-014-1107-6>.
- [LZ24] Liu, Y. and Zheng, W. *Enhanced six operations and base change theorem for higher Artin stacks*. 2024. arXiv: [1211.5948](https://arxiv.org/abs/1211.5948) [math.AG]. URL: <https://arxiv.org/abs/1211.5948>.
- [Lun10] Lunts, V. A. “Categorical resolution of singularities”. In: *J. Algebra* 323.10 (2010), pp. 2977–3003. ISSN: 0021-8693,1090-266X. DOI: [10.1016/j.jalgebra.2009.12.023](https://doi.org/10.1016/j.jalgebra.2009.12.023). URL: <https://doi.org/10.1016/j.jalgebra.2009.12.023>.
- [Lur09] Lurie, J. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. DOI: [10.1515/9781400830558](https://doi.org/10.1515/9781400830558). URL: <https://doi.org/10.1515/9781400830558>.
- [Lur17] Lurie, J. *Higher Algebra*. Preprint. 2017. URL: <https://people.math.harvard.edu/~lurie/papers/HA.pdf>.
- [Lur18] Lurie, J. *Spectral algebraic geometry*. Draft. 2018. URL: <https://people.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf>.

- [MAN22] MANN, L. *A p -Adic 6-Functor Formalism in Rigid-Analytic Geometry*. Thesis (Ph.D.)–Rheinische Friedrich-Wilhelms-Universitaet Bonn (Germany). ProQuest LLC, Ann Arbor, MI, 2022, p. 290. URL: https://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:31909601.
- [Mat16] Mathew, A. “The Galois group of a stable homotopy theory”. In: *Adv. Math.* 291 (2016), pp. 403–541. ISSN: 0001-8708,1090-2082. DOI: [10.1016/j.aim.2015.12.017](https://doi.org/10.1016/j.aim.2015.12.017). URL: <https://doi.org/10.1016/j.aim.2015.12.017>.
- [Mat22] Mathew, A. “Faithfully flat descent of almost perfect complexes in rigid geometry”. In: *J. Pure Appl. Algebra* 226.5 (2022), Paper No. 106938, 31. ISSN: 0022-4049,1873-1376. DOI: [10.1016/j.jpaa.2021.106938](https://doi.org/10.1016/j.jpaa.2021.106938). URL: <https://doi.org/10.1016/j.jpaa.2021.106938>.
- [MW25] Meyer, S. and Wagner, F. *q -Hodge complexes and refined TC^-* . 2025. arXiv: [2410.23115](https://arxiv.org/abs/2410.23115) [math.AT]. URL: <https://arxiv.org/abs/2410.23115>.
- [Mik23] Mikami, Y. *Fppf-descent for condensed animated rings*. 2023. arXiv: [2311.13408](https://arxiv.org/abs/2311.13408) [math.AG]. URL: <https://arxiv.org/abs/2311.13408>.
- [Mik26] Mikami, Y. *Finiteness and duality of cohomology of (φ, Γ) -modules and the 6-functor formalism of locally analytic representations*. 2026. arXiv: [2504.01780](https://arxiv.org/abs/2504.01780) [math.NT]. URL: <https://arxiv.org/abs/2504.01780>.
- [Mon25] Montagnani, M. *Algebraization of rigid analytic varieties and formal schemes via perfect complexes*. 2025. arXiv: [2501.13911](https://arxiv.org/abs/2501.13911) [math.AG]. URL: <https://arxiv.org/abs/2501.13911>.
- [Mon26] Montagnani, M. *Smooth categories in a 6 functor formalism and compact generation for nuclear categories in analytic geometry*. 2026. arXiv: [2605.21024](https://arxiv.org/abs/2605.21024) [math.AG]. URL: <https://arxiv.org/abs/2605.21024>.
- [MP25] Montagnani, M. and Pavia, E. *An axiomatic approach to analytic 1-affineness*. 2025. arXiv: [2509.04341](https://arxiv.org/abs/2509.04341) [math.AG]. URL: <https://arxiv.org/abs/2509.04341>.
- [Mus77] Mustafin, G. A. “ p -adic Hopf varieties”. In: *Functional Analysis and Its Applications* 11.3 (1977), pp. 234–235.

- [Orl16] Orlov, D. “Smooth and proper noncommutative schemes and gluing of DG categories”. In: *Advances in Mathematics* 302 (Oct. 2016), pp. 59–105. ISSN: 0001-8708. DOI: [10.1016/j.aim.2016.07.014](https://doi.org/10.1016/j.aim.2016.07.014). URL: <http://dx.doi.org/10.1016/j.aim.2016.07.014>.
- [PPS25a] Pascaleff, J., Pavia, E., and Sibilla, N. *Higher Koszul duality and n -affineness*. 2025. arXiv: [2504.16935](https://arxiv.org/abs/2504.16935) [math.AG]. URL: <https://arxiv.org/abs/2504.16935>.
- [PPS25b] Pascaleff, J., Pavia, E., and Sibilla, N. *Higher local systems and the categorified monodromy equivalence*. 2025. arXiv: [2501.10241](https://arxiv.org/abs/2501.10241) [math.AT]. URL: <https://arxiv.org/abs/2501.10241>.
- [Per19] Perry, A. “Noncommutative homological projective duality”. In: *Adv. Math.* 350 (2019), pp. 877–972. ISSN: 0001-8708,1090-2082. DOI: [10.1016/j.aim.2019.04.052](https://doi.org/10.1016/j.aim.2019.04.052). URL: <https://doi.org/10.1016/j.aim.2019.04.052>.
- [Por19] Porta, M. “GAGA theorems in derived complex geometry”. In: *J. Algebraic Geom.* 28.3 (2019), pp. 519–565. ISSN: 1056-3911,1534-7486. DOI: [10.1090/jag/716](https://doi.org/10.1090/jag/716). URL: <https://doi.org/10.1090/jag/716>.
- [PY22] Porta, M. and YU, T. Y. *Non-archimedean Gromov-Witten invariants*. 2022. arXiv: [2209.13176](https://arxiv.org/abs/2209.13176) [math.AG]. URL: <https://arxiv.org/abs/2209.13176>.
- [PY16] Porta, M. and Yu, T. Y. “Higher analytic stacks and GAGA theorems”. In: *Adv. Math.* 302 (2016), pp. 351–409. ISSN: 0001-8708,1090-2082. DOI: [10.1016/j.aim.2016.07.017](https://doi.org/10.1016/j.aim.2016.07.017). URL: <https://doi.org/10.1016/j.aim.2016.07.017>.
- [PY18] Porta, M. and Yu, T. Y. “Derived non-archimedean analytic spaces”. In: *Selecta Math. (N.S.)* 24.2 (2018), pp. 609–665. ISSN: 1022-1824,1420-9020. DOI: [10.1007/s00029-017-0310-1](https://doi.org/10.1007/s00029-017-0310-1). URL: <https://doi.org/10.1007/s00029-017-0310-1>.
- [PY20] Porta, M. and Yu, T. Y. “Representability theorem in derived analytic geometry”. In: *J. Eur. Math. Soc. (JEMS)* 22.12 (2020), pp. 3867–3951. ISSN: 1435-9855,1435-9863. DOI: [10.4171/jems/998](https://doi.org/10.4171/jems/998). URL: <https://doi.org/10.4171/jems/998>.

- [PY21] Porta, M. and Yu, T. Y. “Derived Hom spaces in rigid analytic geometry”. In: *Publ. Res. Inst. Math. Sci.* 57.3-4 (2021), pp. 921–958. ISSN: 0034-5318,1663-4926. DOI: [10.4171/prims/57-3-7](https://doi.org/10.4171/prims/57-3-7). URL: <https://doi.org/10.4171/prims/57-3-7>.
- [Ram24a] Ramzi, M. *Dualizable presentable ∞ -categories*. 2024. arXiv: [2410.21537](https://arxiv.org/abs/2410.21537) [math.CT]. URL: <https://arxiv.org/abs/2410.21537>.
- [Ram24b] Ramzi, M. *Locally rigid ∞ -categories*. 2024. arXiv: [2410.21524](https://arxiv.org/abs/2410.21524) [math.CT]. URL: <https://arxiv.org/abs/2410.21524>.
- [RZ25] Reutter, D. and Zetto, M. *Enriched ∞ -categories as marked module categories*. 2025. arXiv: [2501.07697](https://arxiv.org/abs/2501.07697) [math.AT]. URL: <https://arxiv.org/abs/2501.07697>.
- [Sav24] Savage, R. *A Representability Theorem for Stacks in Derived Geometry Contexts*. 2024. arXiv: [2405.08361](https://arxiv.org/abs/2405.08361) [math.AG]. URL: <https://arxiv.org/abs/2405.08361>.
- [Sch24] Scholze, P. *Geometrization of the real local Langlands correspondence*. 2024. URL: <https://people.mpim-bonn.mpg.de/scholze/RealLocalLanglands.pdf>.
- [Sch25] Scholze, P. *Six-Functor Formalisms*. 2025. arXiv: [2510.26269](https://arxiv.org/abs/2510.26269) [math.AG]. URL: <https://arxiv.org/abs/2510.26269>.
- [Sch26a] Scholze, P. *Lectures on Analytic Geometry*. 2026. arXiv: [2605.03655](https://arxiv.org/abs/2605.03655) [math.AG]. URL: <https://arxiv.org/abs/2605.03655>.
- [Sch26b] Scholze, P. *Lectures on Condensed Mathematics*. 2026. arXiv: [2605.03658](https://arxiv.org/abs/2605.03658) [math.NT]. URL: <https://arxiv.org/abs/2605.03658>.
- [Soo25] Soor, A. *A six-functor formalism for quasi-coherent sheaves and stratifications on rigid-analytic varieties*. 2025. arXiv: [2409.07592](https://arxiv.org/abs/2409.07592) [math.NT]. URL: <https://arxiv.org/abs/2409.07592>.
- [Sta24] Stacks project authors, T. *The Stacks project*. <https://stacks.math.columbia.edu>. 2024.
- [Ste23] Stefanich, G. *Tannaka duality and 1-affineness*. 2023. arXiv: [2311.04515](https://arxiv.org/abs/2311.04515) [math.AG]. URL: <https://arxiv.org/abs/2311.04515>.
- [Ste25] Stefanich, G. *Classification of fully dualizable linear categories*. 2025. arXiv: [2307.16337](https://arxiv.org/abs/2307.16337) [math.CT]. URL: <https://arxiv.org/abs/2307.16337>.

- [TT07] Thomason, R. W. and Trobaugh, T. “Higher algebraic K-theory of schemes and of derived categories”. In: *The Grothendieck Festschrift: a collection of articles written in honor of the 60th birthday of Alexander Grothendieck* (2007), pp. 247–435.
- [Toë12a] Toën, B. *Proper local complete intersection morphisms preserve perfect complexes*. 2012. arXiv: 1210.2827 [math.AG]. URL: <https://arxiv.org/abs/1210.2827>.
- [Toë12b] Toën, B. “Derived Azumaya algebras and generators for twisted derived categories”. In: *Invent. Math.* 189.3 (2012), pp. 581–652. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-011-0372-1. URL: <https://doi.org/10.1007/s00222-011-0372-1>.
- [TV07] Toën, B. and Vaquié, M. “Moduli of objects in dg-categories”. In: *Ann. Sci. École Norm. Sup. (4)* 40.3 (2007), pp. 387–444. ISSN: 0012-9593. DOI: 10.1016/j.ansens.2007.05.001. URL: <https://doi.org/10.1016/j.ansens.2007.05.001>.
- [TV08a] Toën, B. and Vaquié, M. “Algébrisation des variétés analytiques complexes et catégories dérivées”. In: *Math. Ann.* 342.4 (2008), pp. 789–831. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-008-0257-9. URL: <https://doi.org/10.1007/s00208-008-0257-9>.
- [TV08b] Toën, B. and Vezzosi, G. “Homotopical algebraic geometry. II. Geometric stacks and applications”. In: *Mem. Amer. Math. Soc.* 193.902 (2008), pp. x+224. ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/0902. URL: <https://doi.org/10.1090/memo/0902>.
- [Vos91] Voskuil, H. “Non archimedean Hopf surfaces”. In: *Séminaire de Théorie des Nombres de Bordeaux* 3.2 (1991), pp. 405–466. URL: <https://www.jstor.org/stable/44165666>.
- [Wan26] Wang, Q. *The relative GAGA Theorem and an application to the analytic mapping stacks*. 2026. arXiv: 2601.12299 [math.AG]. URL: <https://arxiv.org/abs/2601.12299>.
- [Zav23] Zavyalov, B. *Poincaré Duality in abstract 6-functor formalisms*. 2023. arXiv: 2301.03821 [math.AG]. URL: <https://arxiv.org/abs/2301.03821>.
- [Zav25] Zavyalov, B. *Some Foundational Results In Adic Geometry*. 2025. arXiv: 2409.15516 [math.AG]. URL: <https://arxiv.org/abs/2409.15516>.