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There and back again: a holographic journey towards black-hole microstates

Candidate:
Saman Soltani

Advisor:
Francesco Benini

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Aujourd'hui, maman est morte. Ou peut-être hier, je ne sais pas.

Albert Camus

Abstract

The microscopic interpretation of the Bekenstein-Hawking entropy is still an open challenge in theoretical physics. Supersymmetric indices and holography, turn out to be major tools to improve our knowledge.

In the first part of this thesis, using a Bethe Ansatz formulation, we compute the large N limit of the superconformal index with arbitrary chemical potentials for all charges and angular momenta, for generic 4d $\mathcal{N} = 1$ conformal theories with a holographic dual. We conjecture and bring evidence that a particular universal Bethe vacuum dominates the index at large N. For $\mathcal{N} = 4$ super-Yang-Mills, this contribution correctly leads to the entropy of BPS Kerr-Newman black holes in $AdS_5 \times S^5$ for arbitrary values of the conserved charges, completing the derivation of their microstates. We also consider theories dual to $AdS_5 \times SE_5$, where SE_5 is a Sasaki-Einstein manifold. We first check our results against the so-called universal black hole. We then explicitly construct the near-horizon geometry of BPS Kerr-Newman black holes in $AdS_5 \times T^{1,1}$, charged under the baryonic symmetry of the conifold theory, and with equal angular momenta. We compute their entropy using the attractor mechanism and find complete agreement with the field theory predictions.

For BPS black holes with an AdS_2 factor at the horizon, the black-hole microstates can be seen as ground states of a dual 1d theory. In the second part of this dissertation, we construct an $\mathcal{N} = 2$ supersymmetric gauged 1d model by starting from the 3d $\mathcal{N} = 2$ Chern-Simons matter theory holographically dual to massive type IIA string theory on $AdS_4 \times S^6$, and Kaluza-Klein reducing it on S^2 with a background dual to the asymptotics of static dyonic BPS black holes in AdS_4. The background involves a choice of gauge fluxes, that we fix via a saddle-point analysis of the 3d topologically twisted index at large N. The ground-state degeneracy of the effective quantum mechanics reproduces the entropy of BPS black holes, and we expect its low-lying spectrum to contain information about near-extremal horizons. Interestingly, the model has a large number of statistically-distributed couplings, reminiscent of SYK-like models.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following published works and preprints.

[1] F. Benini, E. Colombo, S. Soltani, A. Zaffaroni, and Z. Zhang, "Superconformal indices at large # and the entropy of AdS$_5$ SE$_5$ black holes," *Class. Quant. Grav.* **37** no. 21, (2020) 215021, `arXiv:2005.12308 [hep-th]`

[2] F. Benini, S. Soltani, and Z. Zhang, "A quantum mechanics for magnetic horizons," `arXiv:2212.00672 [hep-th]`

Saman Soltani
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Chapter 1

Introduction

The formulation of a complete quantum gravity theory is still an open problem in modern theoretical physics. The extreme regimes at which quantum corrections become relevant in the treatment of gravitational systems make experimental tests of quantum gravity theories very difficult.[1] Because of this, one has often to resort to alternative methods to probe promising candidates. In this respect, black holes turn out to be a privileged theoretical laboratory where to put our favorite theory under investigation. Black holes turn out to exhibit a macroscopic entropy [3–6], which can be computed and motivated semi-classically. It is known as Bekenstein-Hawking entropy and it is proportional to the area of the black-hole horizon

$$(\quad = \frac{:\ 2^3}{\backslash} \frac{}{4 \quad _{\#}} \bullet \tag{1.1}$$

Notice that the presence of $\backslash$ at the denominator tells us that this quantity has an intrinsically quantum origin. It is difficult to think of a purely classical way to identify microstates $3 \quad _{\text{-micro}}$ accounting for such an entropy via the usual statistical expression

$$(\quad = :\ \log 3 \quad _{\text{-micro}} \bullet \tag{1.2}$$

This is because a classical black hole is a terribly simple system, due to the presence of the event horizon which prevents access to the black hole's interior. From here on, we will use a system of units in which $2 = \backslash = : \quad = 1$.

A good quantum gravity candidate should be able to reproduce and motivate the Bekenstein-Hawking entropy from a microscopic perspective (1.2). The most concrete candidate we have up to date is string theory. In this framework, much work has been done and the discovery of D-branes [7] allowed people, starting from the seminal work [8] for BPS Reissner-Nordström

1The advent of the gravitational-wave astronomy era and advances in experimental cosmology made realistic quantum gravity tests nearer than they were some 10 years ago.

black holes in type IIB on $(^1$ K3, to model asymptotically-flat black holes as D-brane systems. By counting the degeneracy of these systems the Bekenstein-Hawking entropy was reproduced, and it has also been possible to compute quantum corrections to it [9–30] (see [31] and [32] for a recent account and further references).

Another class of black holes that one might be interested in studying is the one of asymptotically-AdS black holes. For these black holes, a brane picture is not available, and one has to resort to alternative descriptions. In this respect holography, under the disguise of the AdS/CFT correspondence [33, 34], is a tremendously powerful tool. The AdS/CFT correspondence can be taken as a complete, non-perturbative definition of quantum gravity via the dual lower-dimensional (non-gravitational) field theory, leaving at the conformal boundary of AdS. This concrete take on the holographic principle [35–37] has been so powerful that people are actively working to extend the correspondence to different setups; the celestial holography program (see [38] for a recent account), for example, aims at a formulation of holography for asymptotically-flat gravitational systems.

In this thesis, the main focus will be on how the AdS/CFT correspondence accounts for the asymptotically-AdS black-hole microstates. The analysis for a generic black hole, though, is complicated: the computations in the dual field theory have to be carried out at strong coupling, and, in these regimes, there are usually few handles one can rely on. A simplifying setup is the one of supersymmetric (BPS) black holes: in this case, instead of computing a partition function, one can compute an index, for which supersymmetric localization [39] often offers exact results.

1.1 Supersymmetric localization

Supersymmetric localization is a powerful technique in supersymmetric quantum field theories, which allows one to "localize" a complicated path integral to a simpler finite dimensional regular integral over BPS configurations. This technique takes inspiration from equivariant localization in mathematics. Rigorous definitions go beyond the scope of this dissertation and we refer to [39, 40] for further details and a broad list of references.

Let us sketch what is the idea behind this technique. We will focus on the following Euclidean path integral

$$I_O = {}^{1}\; D\, i\, O\, 4^{\,(\,»\, i\, ¼}\,. \tag{1.1.1}$$

Let us assume that the action $($ is annihilated by a fermionic symmetry generator Q, which we will refer to as supersymmetry. Let us also assume that the operator O and the measure

$D\Phi$ are annihilated by Q. We can now consider the following modification of (1.1.1):

$$I_O(t) = \int D\Phi\, O\, e^{-S[\Phi] - tQV[\Phi]}, \tag{1.1.2}$$

where QV is a positive semi-definite functional of the fields and $t \geq 0$ a real parameter. The functional V has to be chosen in such a way that $Q^2 V = 0$. One can now explicitly prove the invariance of (1.1.2) with respect to t

$$\begin{aligned} \frac{d}{dt} I_O(t) &= -\int D\Phi\, O\, QV[\Phi]\, e^{-S[\Phi] - tQV[\Phi]} \\ &= -\int D\Phi\, Q\left(O\, V[\Phi]\, e^{-S[\Phi] - tQV[\Phi]}\right) = 0\,. \end{aligned} \tag{1.1.3}$$

Notice here how crucial the invariance of the measure concerning Q is. In particular, when computing thermal correlators, this forces fermions to have periodic boundary conditions. From an operatorial perspective, this corresponds to $(-1)^F$ insertions in the correlator, F being the fermion parity operator. One often refers to these quantities as indices.

Given (1.1.3), one has the following relation:

$$I_O = I_O(0) = I_O(t) = \lim_{t \to \infty} I_O(t)\,. \tag{1.1.4}$$

The last equality is the essence of supersymmetric localization. At this point, the integral can be evaluated by the saddle-point method. Let Φ_0 be the BPS configurations, *i.e.*, the configurations satisfying

$$\text{BPS} \equiv \{\Phi_0 \mid QV[\Phi_0] = 0\}\,. \tag{1.1.5}$$

Notice that, since QV is semi-positive definite, Φ_0, is an extremum of QV. By splitting the field integration variable as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\hat{\Phi}, \tag{1.1.6}$$

one gets, in the strict limit $t \to \infty$, that

$$\begin{aligned} I_O &= \int_{\text{BPS}} d\Phi_0\, O\, e^{-S[\Phi_0]} \int D\hat{\Phi}\, e^{-\frac{\delta^2}{\delta\Phi^2} QV\big|_{\Phi_0} \hat{\Phi}^2} \\ &\equiv \int_{\text{BPS}} d\Phi_0\, O\, e^{-S[\Phi_0]}\, Z_{\text{1-loop}}(\Phi_0), \end{aligned} \tag{1.1.7}$$

where we have defined

$$Z_{\text{1-loop}}(\Phi_0) \equiv \int D\hat{\Phi}\, e^{-\frac{\delta^2}{\delta\Phi^2} QV\big|_{\Phi_0} \hat{\Phi}^2}\,. \tag{1.1.8}$$

This quantity is called 1-loop determinant and it is a product of determinants of the quadratic forms at the exponent. These determinants appear at the numerator when coming from fermionic fields and at the denominator when coming from bosonic ones.

This trick is particularly useful when the BPS locus (1.1.5) is a finite-dimensional manifold. In this case, one has reduced a complicated path integral down to a more tractable finite-dimensional integral, which can be evaluated either analytically or numerically.

1.2 Holographic microstate counting

Starting from [41,42], the possibility of capturing the microstates of BPS asymptotically-AdS black holes using indices became apparent in the case of static magnetically-charged black holes in AdS$_4$ $(^{7}. This result was subsequently generalized in many respects [43–71], for example extending it to different classes of black holes or by computing quantum corrections to the leading Bekenstein-Hawking term. Assuming that the cancellations due to the ^{1} 1^o insertion are optimally obstructed, the entropy is obtained as follows. The index I^{1} ^{o} is computed in the grand-canonical ensemble, in the presence of fugacities for the global symmetries commuting with the supercharges, and the entropy is obtained by taking its Fourier transform

$$4^{(\ ^{1\&o}} = {}^{1}\ \ddot{O}\ 3\ {}^{!}\ I^{1}\ {}^{o}4^{\ 2c8\ ^{\acute{i}}\ \&} \,\bullet \tag{1.2.1}$$

By evaluating this expression at large $\#$ for large black holes, we get at leading order the entropy as Legendre transform of the index

$$(\ \ {}^{1\&o} = I^{1b o}\ \ 2c8\ \tilde{O}\ \&\ b\ -\ \ b \quad \text{s.t.} \quad \frac{mI}{m\ \ _{b}} = 2c8\& \,\bullet \tag{1.2.2}$$

This procedure goes under the name of I-extremization [72].

Another important step in this direction was made when the entropy of BPS Kerr-Newman AdS$_5$ $(^{5} black holes was holographically reproduced in [73–75] by computing the superconformal index of the dual 4d $N = 4$ super-Yang-Mills (SYM) theory, *i.e.*, by counting its $1\bullet 16$-BPS states on $(^1$ $(^3$. Also in this case, there has been a lot of effort in the last few years to generalize and understand these results [76–132]. In this context, our work [1][2] generalized the results obtained in [73] in the so-called Bethe ansatz formulation of the superconformal index to the case of generic angular and flavor chemical potentials and a broader class of $N = 1$ holographic quiver models, including toric quiver models.

[2] While this work was ready to be posted on the arXiv, the preprint [93] appeared, which discusses the index in the particular case $g = f$ using a different approach.

The results in the AdS$_5$ case have been particularly relevant. The superconformal index was first formulated [133, 134] to capture the entropy of Gutowski-Reall black holes [135]. The large N computation performed in [134], though, did not agree with this expectation, suggesting a large cancellation between bosonic and fermionic BPS states. A computation with real flavor fugacities was performed, which gave an order $O(1)$ entropy function, incapable of reproducing the $O(N^2)$ black-hole entropy. This remained a puzzle, until in [73–75], taking inspiration from the results for 4d black holes [41, 42], the importance of using complex flavor fugacities was understood, and the correct result was obtained.

Why an index should capture the entropy of BPS asymptotically-AdS black holes was nicely argued in [41, 42]. Taking inspiration from a similar argument [23, 28, 31] in the case of asymptotically-flat black holes, the black-hole solution is interpreted as a renormalization-group flow across dimensions. The asymptotically-AdS$_d$ region is dual to the UV CFT$_{d-1}$ whose index captures the entropy. The near horizon AdS$_2 \times M_{d-2}$ region, where M_{d-2} is some (d-2)-dimensional manifold, is supposedly dual to an IR CFT$_1$ which captures the near-horizon physics of the black hole. In the case of single-center black holes without hair, the entropy information should only be encoded in this theory. In particular, the presence of a superconformal R-symmetry in the CFT$_1$ motivates the complete obstruction to the boson-fermion cancellations provided by the $(-1)^F$ insertion in the index, and thus the equivalence (up to a sign) between the zero temperature partition function, capturing the black-hole entropy, and the index.

1.3 Leaving the BPS safe harbor

The idea of capturing the near-horizon physics of black holes is one of the driving forces in the study of AdS$_2$/CFT$_1$ correspondence. Due to the low dimensionality of the theories involved in this duality, it can offer a unique opportunity to clarify aspects of holography that are too complicated to be addressed in higher dimensions.

Starting from [136–142], people have studied and generalized the correspondence between the Sachdev-Ye-Kitaev (SYK) model [143–145] and Jackiw-Teitelboim (JT) gravity [146, 147]. This correspondence exhibits many peculiar features; for example, the gravity theory is conjecturally dual not to a single quantum mechanical model, but to a random average of models, instead. For a detailed treatment and further references, we refer to [148–151].

In the series of papers [152–158] this low-dimensional holographic setup was used to clarify many aspects in black-hole thermodynamics. The supergravity zero modes around a black-hole background in the near-horizon region were matched by a particular JT-gravity effective theory. In particular, the contribution to the density of states coming from near-extremal black holes was predicted at low temperatures by using this effective theory: the large order $O(N^2)$ degeneracy of AdS$_5$ BPS black holes at $T = 0$ was reproduced, and strong suppression of the density of states for extremal black holes in the non-BPS case was obtained; the analysis also revealed an order $O(N^{-2})$ gap above extremality for BPS black holes. This being said, a first principle derivation of the holographic dual 1d quantum-mechanical model, describing the near-horizon degrees of freedom, was still missing.

In [2] we fill in this gap by explicitly reducing the boundary CFT$_3$ dual to a class of asymptotically-AdS$_4$ magnetic black holes on a sphere S^2, and obtaining a 1d quantum-mechanical model putatively dual to the black-hole near-horizon physics. This model correctly reproduces the entropy of BPS black holes and it should match the gravitational results in the near-BPS case. Having a tool to study near-BPS black holes is a major result since the absence of supersymmetry is most of the time prohibitive in the quantitative analysis of a system. Moreover, by studying this model, one can probe basic features of the AdS$_2$/CFT$_1$ correspondence: for example in our model (in the spirit of what was done in [159] in the context of asymptotically-flat black holes in string theory), disorder averages are only introduced as a simplifying tool while performing the computations, being the couplings "statistically distributed", but fixed; in the SYK model, instead, disorder averages are a structural ingredient of the theory. We leave further analysis of this model for future works.

1.4 Outline

We will now outline the structure of this thesis, by going into some more detail on the two original works [1, 2] on which our discussion is based. They will be the focus of Chapter 2 and Chapter 3, respectively.

Chapter 2. The family of AdS$_5 \times S^5$ supersymmetric black holes found in [135, 160–163] depends on three charges Q_I associated with the Cartan subgroup of the internal isometry SO(6), and two angular momenta J_i in AdS$_5$, subject to a non-linear constraint.[3] The entropy can be written as the value at the critical point of the entropy function [118]

$$S(\Delta_I, \sigma, \tau) = -i\pi N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\sigma\tau} - 2\pi i \left(\sum_{I=1}^{3} \Delta_I Q_I + \sigma J_1 + \tau J_2 \right) \tag{1.4.1}$$

with the constraint $\Delta_1 + \Delta_2 + \Delta_3 - \sigma - \tau = \pm 1$, where N is the number of colors of the dual 4d $\mathcal{N} = 4$ SU(N) SYM theory. The same entropy function can also be obtained by computing the

[3] Supersymmetric hairy black holes depending on all charges have been recently found in [164, 165], but their entropy seems to be parametrically smaller in the range of parameters where our considerations apply.

zero-temperature limit of the on-shell action of a class of supersymmetric but non-extremal complexified Euclidean black holes [74, 96]. The two constraints with sign lead to the same value for the entropy, which is real precisely when the non-linear constraint on the black-hole charges is imposed. The parameters $-_0$, g, and f are chemical potentials for the conserved charges $\&_0$ and $_8$ and can also be identified with the parameters the superconformal index depends on. With this identification, we expect that the entropy $(^1\&_0 - {}_1 - {}_2{}^o$ is just the constrained Legendre transform of $\log I^1 -_0 - g - f^o$, where $I^1 -_0 - g - f^o$ is the superconformal index.

Initially, the entropy of AdS_5 $(^5$ Kerr-Newman black holes has been derived from the superconformal index and shown to agree with (1.4.1) only in particular limits. In [75], the entropy was derived for large black holes (whose size is much larger than the AdS radius) using a Cardy limit of the superconformal index where $\text{Im}^1 -_0{}^o - g - f$ 1. In [73], the entropy was instead derived in the large # limit in the case of black holes with equal angular momenta, $_1 = {}_2$.[4] The large # limit has been evaluated by writing the index as a sum over Bethe vacua [85], an approach that has been successful for AdS black holes in many other contexts.

It is one of the purposes of this Chapter to extend the derivation of [73] to the case of unequal angular momenta, thus providing a large # microscopic counting of the microstates of BPS Kerr-Newman black holes in AdS_5 $(^5$ for arbitrary values of the conserved charges. We will make use of the Bethe ansatz formulation of the superconformal index derived for $g = f$ in [166] and generalized to unequal angular chemical potentials in [85]. This formulation allows us to write the index as a sum over the solutions to a set of Bethe Ansatz Equations (BAEs) — whose explicit form and solutions have been studied in [62, 73, 81, 111, 116, 125] — and over some auxiliary integer parameters $<_8$. We expect that, in the large # limit, one particular solution dominates the sum.[5] In Section 2.1 we will show that the "basic solution" to the BAEs, already used in [73], correctly reproduces the entropy of black holes in the form (1.4.1) for a choice of integers $<_8$. We stress that our result comes from a single contribution to the index, which is an infinite sum. Such a contribution might not be the dominant one — and so our estimate of the index might be incorrect — in some regions of the space of chemical potentials. It is known from the analysis in [73] that when the charges become smaller than a given threshold, new solutions take over and dominate the asymptotic behavior of the index. This suggests the existence of a rich structure to which other black holes might also contribute. However, we conjecture and will bring some evidence that the contribution

4The same result has been later reproduced with a different approach in [90].

5It is argued in [81] that there exist families of continuous solutions. This does not affect our argument provided the corresponding contribution to the index is subleading.

of the basic solution is the dominant one in the region of the space of chemical potentials corresponding to sufficiently large charges.

In Section 2.2 we will also extend the large N computation of the index to a general class of superconformal theories dual to $AdS_5 \times SE_5$, where SE_5 is a 5d Sasaki-Einstein manifold. The analysis for $\omega_1 = \omega_2$ was already performed in [125]. For toric holographic quiver gauge theories, we find a prediction for the entropy of black holes in $AdS_5 \times SE_5$ in the form of the entropy function

$$S(\Delta_a, \omega_1, \omega_2) = \frac{\pi i N^2}{6} \sum_{a,b,c} C_{abc} \frac{\Delta_a \Delta_b \Delta_c}{\omega_1 \omega_2} - 2\pi i \Lambda \left(\sum_{a=1} \Delta_a - \omega_1 - \omega_2 \mp 1 \right) \tag{1.4.2}$$

with the constraint $\sum_{a=1} \Delta_a - \omega_1 - \omega_2 = \pm 1$, in terms of chemical potentials Δ_a for a basis of independent R-symmetries R_a. The coefficients $C_{abc} N^2 = \frac{1}{4} \mathrm{Tr}\, R_a R_b R_c$ are the 't Hooft anomaly coefficients for this basis of R-symmetries. The form of the entropy function (1.4.2) was conjectured in [167] and reproduced for various toric models in the special case $\omega_1 = \omega_2$ in [125]. We will give a general derivation, valid for all toric quivers and even more. We will also show that both constraints in (1.4.2), which lead to the same value for the entropy, naturally arise from the index in different regions of the space of chemical potentials. The function (1.4.2) was also derived in the Cardy limit in [77].

In the last part of the Chapter, we will provide some evidence that (1.4.2) correctly reproduces the entropy of black holes in $AdS_5 \times SE_5$. In Section 2.3 we first check that our formula correctly reproduces the entropy of the universal black hole that arises as a solution in 5d minimal gauged supergravity, and, as such, can be embedded in any $AdS_5 \times SE_5$ compactification. It corresponds to a black hole with electric charges aligned with the exact R-symmetry of the dual superconformal field theory and with arbitrary angular momenta J_1 and J_2. Since the solution is universal, the computation can be reduced to that of $\mathcal{N} = 4$ SYM and it is almost trivial. More interesting are black holes with general electric charges. Unfortunately, to the best of our knowledge, there are no available such black hole solutions in compactifications based on Sasaki-Einstein manifolds SE_5 other than S^5. To overcome this obstacle, in Section 2.4 we will explicitly construct the near-horizon geometry of supersymmetric black holes in $AdS_5 \times T^{1,1}$ with equal angular momenta and charged under the baryonic symmetry of the dual Klebanov-Witten theory [168]. Luckily, the background $AdS_5 \times T^{1,1}$ admits a consistent truncation to a 5d gauged supergravity containing the massless gauge field associated with the baryonic symmetry [169–171]. We then use the strategy suggested in [118]: a rotating black hole in 5d with $J_1 = J_2$ can be dimensionally reduced along the Hopf fiber of the horizon three-sphere to a static solution of 4d $\mathcal{N} = 2$ gauged supergravity. We will explicitly solve the BPS equations [172–174] for the horizon of static black holes with the appropriate electric and magnetic charges in $N = 2$ gauged supergravity in 4d. The main complication is the presence of hypermultiplets. By solving the hyperino equations at the horizon, we will be able to recast all other supersymmetric conditions as a set of attractor equations, and we will show that these are equivalent to the extremization of (1.4.2) for the Klebanov-Witten theory with $g = f$. This provides a highly non-trivial check of our result and the conjecture that the basic solution to the BAEs dominates the index.

Chapter 3. In this Chapter we construct a supersymmetric gauged quantum mechanics (QM) that we expect to capture information about near-extremal black-hole horizons. We work in a very specific setup: massive Type IIA string theory on $(^6$, which is dual to the 3d $N = 2$ $\mathrm{SU}^{1}\#^{o}_{:}$ Chern-Simons-matter (CS-matter) theory [175] in Section 3.1.2. The supergravity admits asymptotically-AdS$_4$ static magnetic (or topologically twisted) BPS black holes [176–178], that we aim to describe. The quantum mechanics is then obtained by reducing the dual 3d field theory on $(^2$, with a specific background that corresponds to the black-hole asymptotics.[6]

More specifically, the entropy of static magnetically-charged BPS black holes in AdS$_4$ is captured by the topologically-twisted (TT) index [72, 179] of the dual 3d boundary theory [41, 42, 63, 67, 180–182], see in particular [43, 64] for the specific example in massive Type IIA studied here. In the Lagrangian formulation, the topologically-twisted index is the Euclidean partition function of the theory on $(^2 \quad (^1$, in the presence of a supersymmetric background that holographically reflects the asymptotics of the BPS black hole. The background can be thought of as a topological twist on $(^2$ that preserves two supercharges, or equivalently as an external magnetic flux for the R-symmetry. In Section 3.1.1 we observe that the topologically-twisted index takes the form of the Witten index of a quantum mechanics, obtained by reducing the 3d theory on $(^2$ with the twisted background. Up to exponentially small corrections at large $\#$, the index is the grand-canonical partition function for the BPS ground states of that quantum mechanics. In other words, the ground states of that quantum mechanics are the microstates of a BPS black hole with given charges, and one expects the excited states to describe near-extremal black holes. The goal of this Chapter is to construct such quantum mechanics.

The procedure we outlined has a technical complication: the formula for the topologically-twisted index — schematically in (3.1.1) — has an infinite sum over gauge fluxes on $(^2$. For each term in the sum, one obtains different quantum mechanics upon reduction. Thus it appears that, even at finite $\#$, one has to deal with a quantum mechanical model with

[6] The background is dual to the black-hole chemical potentials, or charges, depending on the ensemble.

an infinite number of sectors, over which we do not have good control.[7] Nevertheless, in the large $\#$ limit we expect one sector to dominate the entropy[8] and thus contribute to the majority of the states. In Section 3.1 we determine such a sector by performing a saddle-point evaluation of the index in the sum over fluxes. This gives us an $N = 2$ supersymmetric gauged quantum mechanics with a finite number of fields (at finite $\#$).

The resulting $N = 2$ quantum mechanics, that we exhibit in Section 3.3 and on which we discuss the stability in Section 3.4, has some interesting features. It has $\mathrm{U}^{1}1^{\mathrm{o}\#}$ gauge group and a number of fields that scales as $\#^{\frac{7}{3}}$. It has an $\mathrm{SU}^{1}2^{\mathrm{o}}$ global symmetry, dual to the isometry of the $(^2$ black-hole horizon. More importantly, it has a large number of couplings among the fields, expressed in terms of Clebsch-Gordan coefficients (arising in the reduction from the overlap of Landau-level wave functions on $(^2$). Therefore, although the quantum mechanics is specific and well-defined, at large $\#$ its couplings can be approximated by random variables following a statistical distribution. This makes us hopeful that the IR dynamics might have some traits in common with supersymmetric SYK models [154, 183].

In the large $\#$ saddle-point evaluation of the topologically-twisted index, we noticed that there is actually a series of saddle points — one of which dominates the large $\#$ expansion. These saddle points are labeled by shifts of the chemical potentials by 1 and likely correspond to a series of complex supergravity solutions with the very same boundary conditions, as in [184, 185].

Appendices. Technical computations as well as some review material can be found in several appendices. In Appendix A we report all the details of the large $\#$ computations performed at various stages of the thesis. Appendix B and Appendix C set our supergravity conventions and display all the details of the Scherk-Schwarz reduction, respectively. Appendix D contains details on the expansion in monopole harmonics on $(^2$. Finally, Appendix E contains a brief overview of 1d $N = 2$ supersymmetry.

7This is partially because the reduction is in the grand-canonical ensemble for the electric charges (though it is micro-canonical for the magnetic charges), with fixed chemical potentials. Therefore, the states of all BPS and near-BPS black holes are mixed up together.

8We are grateful to Juan M. Maldacena for suggesting this possibility to us years ago.

Chapter 2

Superconformal indices at large N and the entropy of $\mathrm{AdS}_5 \times \mathrm{SE}_5$ black holes

In this Chapter, we holographically compute the entropy of a broad class of BPS asymptotically-AdS_5 black holes. It is organized as follows. In Section 2.1 we review the setting introduced in [73] and we evaluate the large N contribution of the “basic solution" to the BAEs to the superconformal index for generic angular fugacities. We show that it correctly captures the semi-classical Bekenstein-Hawking entropy of BPS black holes in $\mathrm{AdS}_5 \times S^5$. In Section 2.2 we discuss the generalization of this result to general toric quiver gauge theories and find agreement with the entropy function prediction (1.4.2) in certain corners of the space of chemical potentials. In Section 2.3 we discuss the particular case of the universal black hole, which can be embedded in all string and M-theory supersymmetric compactifications with an AdS_5 factor. In Section 2.4 we match formula (1.4.2) with the entropy of a supersymmetric black hole in $\mathrm{AdS}_5 \times Y^{1,1}$, whose near-horizon geometry we explicitly construct.

2.1 The SCI of $\mathcal{N} = 4$ SYM at large N

We are interested in evaluating the large N limit of the superconformal index of 4d $\mathcal{N} = 1$ holographic theories. We will consider in this Section the simplest example, namely $\mathcal{N} = 4$ $SU(N)$ SYM. The superconformal index counts (with sign) the 1/16-BPS states of the theory on $\mathbb{R} \times S^3$ that preserve one complex supercharge $\mathcal{Q}$. These states are characterized by two angular momenta $J_{1,2}$ on S^3 and three R-charges for $U(1)^3 \subset SO(6)_R$. We write $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ notation in terms of a vector multiplet and three chiral multiplets Φ_a and introduce a symmetric basis of R-symmetry generators $R_{1,2,3}$ such that $R_a(\Phi_b) = 2\delta_{ab}$. The index is defined by the trace [133, 134]

$$\mathcal{I}_{sc}(p,q,y_1,y_2) = \operatorname{Tr}(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} y_1^{q_1} y_2^{q_2}\,, \tag{2.1.1}$$

in terms of two flavor generators $q_{1,2} = (Q_{1,2} - Q_3)/2$ commuting with $\mathcal{Q}$, and the R-charge $r = (Q_1 + Q_2 + Q_3)/3$. Notice that $(-1)^F = e^{2\pi i J_{1,2}} = e^{\pi i Q_{1,2,3}}$. Here p, q, y_a with $a = 1,2$ are complex fugacities associated with the various quantum numbers, while the corresponding chemical potentials τ, σ, ξ_a are defined by

$$p = e^{2\pi i\tau}\,, \qquad q = e^{2\pi i\sigma}\,, \qquad y_a = e^{2\pi i\xi_a}\,. \tag{2.1.2}$$

The index is well-defined for $|p|, |q| < 1$. It is convenient to redefine the flavor chemical potentials in terms of

$$\Delta_a = \xi_a + \frac{\tau+\sigma}{3} \qquad \text{for} \qquad a = 1,2\,. \tag{2.1.3}$$

It is also convenient to introduce an auxiliary chemical potential Δ_3 such that

$$\tau + \sigma - \Delta_1 - \Delta_2 - \Delta_3 \in 2\mathbb{Z} + 1\,, \tag{2.1.4}$$

and use the corresponding fugacities

$$y_a = e^{2\pi i\Delta_a}\,. \tag{2.1.5}$$

The index then takes the more transparent form

$$\mathcal{I}_{sc} = \operatorname{Tr}_{\text{BPS}}\, p^{J_1} q^{J_2} y_1^{Q_1/2} y_2^{Q_2/2} y_3^{Q_3/2}\,. \tag{2.1.6}$$

It shows that the constrained fugacities p, q, y_a with $a = 1,2,3$ are associated with the angular momenta $J_{1,2}$ and the charges $Q_a/2$.

Our starting point is the so-called Bethe ansatz formulation of the superconformal index [85, 166]. The special case that the two angular chemical potentials are equal, $\tau = \sigma$, was already studied in [73] (see also [81]). Here we take them to be unequal. The formula of [85] can be applied when the ratio between the two angular chemical potentials is a rational number.[1] We thus set

$$\sigma = a\tau\,, \qquad \omega = b\tau \qquad \text{with} \qquad \mathrm{Im}\,\tau > 0 \tag{2.1.7}$$

and with $a,b \in \mathbb{N}$ coprime positive integers. We call $\mathbb{H} = \{\tau \,|\, \mathrm{Im}\,\tau > 0\}$ the upper half-plane. We then have the fugacities

$$h = e^{2\pi i\tau}\,, \qquad p = h^a = e^{2\pi i\sigma}\,, \qquad q = h^b = e^{2\pi i\omega} \qquad \text{with} \qquad |h|,|p|,|q| < 1\,. \tag{2.1.8}$$

The formula in [85] allows us to write the superconformal index as a sum over the solutions to a set of Bethe Ansatz Equations (BAEs). Explicitly, the index reads

$$\mathcal{I}_{ed} = \kappa_N \sum_{u \in \text{BAE}} \mathcal{Z}_{\text{tot}}\, H^{-1}\Big|_u\,. \tag{2.1.9}$$

The expressions of κ_N, H and $\mathcal{Z}_{\text{tot}}$ for a generic $\mathcal{N}=1$ theory are given in [85]. Here, we specialize them to $\mathcal{N}=4$ $SU(N)$ SYM. The quantity

$$\kappa_N = \frac{1}{N!}\left(\frac{(p;p)_\infty (q;q)_\infty\, \widetilde{\Gamma}(\Delta_1;\sigma,\omega)\,\widetilde{\Gamma}(\Delta_2;\sigma,\omega)}{\widetilde{\Gamma}(\Delta_1+\Delta_2;\sigma,\omega)}\right)^{N-1} \tag{2.1.10}$$

is a pre-factor written in terms of the elliptic gamma function $\widetilde{\Gamma}$ and the Pochhammer symbol defined in (A.1.6) and (A.1.1), respectively. The sum in (2.1.9) is over the solution set to the following BAEs[2]

$$1 = Q_i(u;\Delta,\tau) \equiv e^{2\pi i(\lambda + 3\sum_j u_{ij})} \prod_{j=1}^{N} \frac{\theta_0(u_{ji}+\Delta_1;\tau)\,\theta_0(u_{ji}+\Delta_2;\tau)\,\theta_0(u_{ji}-\Delta_1-\Delta_2;\tau)}{\theta_0(u_{ij}+\Delta_1;\tau)\,\theta_0(u_{ij}+\Delta_2;\tau)\,\theta_0(u_{ij}-\Delta_1-\Delta_2;\tau)}\,, \tag{2.1.11}$$

written in terms of $u_{ij} = u_i - u_j$ with $i,j = 1,\ldots,N$ and the theta function defined in (A.1.3). The unknowns are the "complexified $SU(N)$ holonomies", which are expressed here in terms of $U(N)$ holonomies u_i further constrained by

$$\sum_{i=1}^{N} u_i = 0 \pmod{\mathbb{Z}}\,. \tag{2.1.12}$$

[1] This might sound like a strong limitation. However, the index (2.1.6) is invariant under integer shifts of σ and ω compatible with (2.1.4). As proven in [85], the set of complex number pairs $\{\sigma,\omega\} \in \mathbb{H}^2$ (two copies of the upper half-plane) whose ratio becomes a (real) rational number after some integer shifts of σ and ω, is dense in $\mathbb{H}^2$. Thus, by continuity, the formula of [85] fixes the large N limit of the superconformal index for generic complex chemical potentials.

[2] The Bethe operators Q_i should not be confused with the charges Q introduced before.

as well as a “Lagrange multiplier” $\_$. The $\mathrm{SU}^1\#^{\mathrm{o}}$ holonomies are to be identified with the first $\#\quad 1$ variables $D_{8=1,\dots,\#\ 1}$. As unknowns in the BAEs, they are subject to the identification

$$D_8 \qquad D_8 \text{¸} 1 \qquad D_8 \text{¸} / \text{-} \tag{2.1.13}$$

meaning that each one of them naturally lives on a torus of modular parameter $/$. Instead, the last holonomy $D_\#$ is determined by the constraint (2.1.12). The relation between $\mathrm{SU}^1\#^{\mathrm{o}}$ and $\mathrm{U}^1\#^{\mathrm{o}}$ holonomies will be further clarified in Appendix A.2.2. The prescription in (2.1.9) is to sum over all the inequivalent solutions on the torus [85]. The function is the Jacobian

$$= \det \frac{1}{2c8} \frac{m^1\&_1\text{-}\bullet\bullet\bullet\text{-}\&_\#{}^{\mathrm{o}}}{m^1D_1\text{-}\bullet\bullet\bullet\text{-}D_\#\ {}_{1\text{-}\_}{}^{\mathrm{o}}} \bullet \tag{2.1.14}$$

Finally, the function Z_{tot} is the following sum over a set of integers $<_8 = 1\text{-}\bullet\bullet\bullet\text{-}01$:

$$Z_{\mathrm{tot}} = \sum_{\mathrm{f}<_8\mathrm{g}=1} Z\ D \quad < / ; g\text{-}\ f \quad \text{-} \tag{2.1.15}$$

where Z, for $N = 4$ $\mathrm{SU}^1\#^{\mathrm{o}}$ SYM, reads

$$Z = \prod_{\substack{8\text{-}9=1\\ 8<9}} \frac{\mathrm{e}^1 D_{89} \text{¸} \quad 1; g\text{-}\ f^{\mathrm{o}}\, \mathrm{e}^1 D_{89} \text{¸} \quad 2; g\text{-}\ f^{\mathrm{o}}}{\mathrm{e}^1 D_{89} \text{¸} \quad 1 \text{¸} \quad 2; g\text{-}\ f^{\mathrm{o}}\, \mathrm{e}^1 D_{89}; g\text{-}\ f^{\mathrm{o}}} \bullet \tag{2.1.16}$$

The sum in (2.1.15) freely varies over the first $\#\quad 1$ integers $<_{8=1,\dots,\#\ 1}$ as indicated, while $<_\#$ is determined by the constraint

$$\sum_{8=1} <_8 = 0 \bullet \tag{2.1.17}$$

More details can be found in [73, 85]. In the following, when a double sum starts from 1 we will leave it implicit.

2.1.1 The SCI building block

We will show that one particular contribution to the sums in (2.1.9) and (2.1.15) alone reproduces the entropy function of [118], and therefore it captures the Bekenstein-Hawking entropy of BPS black holes in AdS$_5$ $(^5$. To that aim, we are interested in the contribution from the so-called "basic solution" to the BAEs [62, 73, 116], namely

$$u_i = \frac{N-i}{N}\tau + \bar{u}\,, \qquad u_{ij} \equiv u_i - u_j = \frac{j-i}{N}\tau\,, \qquad \_ = \frac{N-1}{2}\,. \tag{2.1.18}$$

Here $\bar{u}$ is fixed by enforcing the constraint (2.1.12). We also consider the contribution from a particular choice for the integers $\{m_j\}$:

$$m_j \in \{1,\ldots,ab\} \qquad \text{such that} \qquad m_j = j \mod ab\,. \tag{2.1.19}$$

Note that this choice for $\{m_j\}$ does not satisfy the constraint (2.1.17). Nevertheless, we show in Appendix A.2.2 that this does not affect the contribution to the leading order in N, in the sense that changing the single entry m_N has a subleading effect.

Now, the crucial technical point is to evaluate the following basic building block

$$_{\text{SCI}} = \sum_{i\neq j}^{N} \log \Gamma_e\Big(+ \tau\frac{j-i}{N} + \tau\,(m_j - m_i);\, a\tau, b\tau\Big)\,, \tag{2.1.20}$$

for $N \to \infty$. Here plays the role of an electric chemical potential. To simplify the discussion, we assume that N is a multiple of ab, *i.e.*, we take $N = ab\bar{N}$. As we show in Appendix A.2.3, this assumption can be removed without affecting the leading behavior at large N. By making use of the identity (A.1.11) we can rewrite it as

$$_{\text{SCI}} = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{i\neq j}^{N} \log \Gamma_e\Big(+ \tau\frac{j-i}{N} + \tau\,(m_j - m_i) + as + br;\, ab\tau, ab\tau\Big)\,. \tag{2.1.21}$$

Let us set $i = \alpha ab + c$, $j = \beta ab + d$ with $\alpha, \beta = 0,\ldots,\bar{N}-1$ and $c, d = 1,\ldots,ab$. Then

$$_{\text{SCI}} = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\underbrace{\sum_{\alpha,\beta=0}^{\bar{N}-1}\sum_{c,d=1}^{ab}}_{\text{s.t. } i\neq j} \log \Gamma_e\Big(+ \tau\frac{\beta-\alpha}{\bar{N}} + \tau\frac{d-c}{N} + \tau\,(d-c) + as + br;\, ab\tau, ab\tau\Big)\,. \tag{2.1.22}$$

We will now perform two simplifications and prove in Appendix A.2.1 that their effect is of subleading order at large N. More precisely, $_{\text{SCI}}$ is of order N^2 while the two simplifications modify it at most at order N if $\operatorname{Im}(/\tau) \notin \mathbb{Z}\operatorname{Im}(1/\tau)$, or at most at order $N\log N$ if $= 0$. First, we substitute the condition $i \neq j$ with the condition $\alpha \neq \beta$ in the summation. Second and more importantly, we drop the term $\tau(d-c)/N$ in the argument. We then redefine

$2 \; ! \; 01 \quad 2, 3 \; ! \; 3 \; ¸ \; 1, W \; ! \; W \quad 1, X \; ! \; X \quad 1$ and obtain

$$\mathrm{SCI} \; ' \sum_{A=0}^{0Õ1} \sum_{B=0}^{1Õ1} \sum_{W<X}^{\tilde{\#}} \sum_{2\text{-}3=0}^{0Õ1} \log e \quad ¸ \; l \frac{X \quad W}{\#} \; ¸ \; l \; 3 \; ¸ \; 2 \; ¸ \; 1 \quad 01 \; ¸ \; 0B \; ¸ \; 1A \; ; 01\,l \text{-} 01\,l \tag{2.1.23}$$

where $'$ means equality at leading order in $\#$. At this point we can resum over $2\text{-}3$ using (A.1.10) (with $g\text{-} f \; ! \; l$ and $0\text{-}1 \; ! \; 01$):

$$\mathrm{SCI} \; ' \sum_{A=0}^{0Õ1} \sum_{B=0}^{1Õ1} \sum_{W<X}^{\tilde{\#}} \log e \quad ¸ \; l \frac{X \quad W}{\#} \; ¸ \; l \; 1 \quad 01 \; ¸ \; 0B \; ¸ \; 1A \; ; l \text{-} l \quad • \tag{2.1.24}$$

We can now recall the large $\#$ limit computed in [73]

$$\sum_{8<9}^{\tilde{\#}} \log e \quad ¸ \; l \frac{9 \quad 8}{\#} ; l \text{-} l \;=\; c8\#^2 \frac{3 \; » \; ¼^0_l \quad l}{3\,l^2} \; ¸ \; O^1\#^o \text{-} \tag{2.1.25}$$

valid for $\mathrm{Im} \quad • l \quad 8 \; Z \quad \mathrm{Im} \; 1 • l$. Here 3^1G^o is a Bernoulli polynomial, defined in (A.1.13) and satisfying (2.1.28). The function $» \; ¼^0_l$ is defined in the following way:

$$» \; ¼^0_l = \; l \quad l = \quad \mathrm{mod}\, 1 \text{-} \quad 0 \; ¡ \; \mathrm{Im} \; \frac{l}{l} \quad ¡ \; \mathrm{Im} \; \frac{1}{l} \quad • \tag{2.1.26}$$

This function is only defined for $\mathrm{Im} \quad • l \quad 8 \; Z \quad \mathrm{Im} \; 1 • l$, it is continuous in each open connected domain, and it is periodic by construction under $\; ! \quad ¸ \; 1$. In the following we will also use the function $» \; ¼_l = » \; ¼^0_l \quad 1$, that is

$$» \; ¼_l = \; l \quad l = \quad \mathrm{mod}\, 1 \text{-} \quad \mathrm{Im} \quad \frac{1}{l} \; ¡ \; \mathrm{Im} \; \frac{l}{l} \; ¡ \; 0 \quad • \tag{2.1.27}$$

The functions $» \; ¼_l$ and $» \; ¼^0_l$ are the mod 1 reductions of to the fundamental strips shown in Figure 2.1. Then, one can explicitly prove the following formula

$$\begin{aligned} \frac{1}{01} \sum_{A=0}^{0Õ1} \sum_{B=0}^{1Õ1} \; {}_3\,G \; ¸ \; l \, {}^1 0B \; ¸ \; 1A \quad 01^o \; &= \\ &= \; {}_3 \; G \quad \frac{0 \; ¸ \; 1}{2} \, l \quad ¸ \; \frac{20^2 1^2 \quad 0^2 \quad 1^2}{4} \, l^2 \quad {}_1 \; G \quad \frac{0 \; ¸ \; 1}{2} \, l \quad \text{-} \end{aligned} \tag{2.1.28}$$

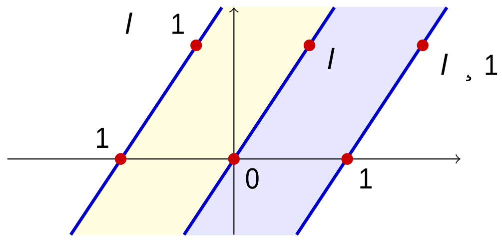

Figure 2.1 Fundamental strips for » ¼ $_l$ and » ¼^0_l. The function » ¼ $_l$ is the restriction of mod 1 to the region $\mathrm{Im}^1\ 1\bullet l^o$ ¡ $\mathrm{Im}^1\ \bullet l^o$ ¡ 0 (in yellow, on the left), while » ¼^0_l is the restriction of mod 1 to the region 0 ¡ $\mathrm{Im}^1\ \bullet l^o$ ¡ $\mathrm{Im}^1 1\bullet l^o$ (in blue, on the right).

where $_1{}^1G^o$ is another Bernoulli polynomial defined in (A.1.13). Thus

$$\mathrm{SCI} = \frac{c8\#^2}{3gf}\ 3\ » ¼^0_l\ \frac{g\ ¸\ f}{2}\quad \frac{c8\#^2}{12}\ 201\ \frac{0}{1}\ \frac{1}{0}\quad 1\ » ¼^0_l\ \frac{g\ ¸\ f}{2}\ ¸\ O^1\#^o \tag{2.1.29}$$

for $\mathrm{Im}\ \ \bullet l\ \ 8\ Z\ \ \mathrm{Im}\ 1\bullet l$. As a check, notice that

$$g\ ¸\ f\qquad {}^0_l = g\ ¸\ f\ ¸\ 1\quad » ¼^0_l\ \bullet \tag{2.1.30}$$

From the properties of $_{1\text{-}3}{}^1G^o$ noticed in (2.1.28), it follows that

$$\mathrm{SCI}^1 g\ ¸\ f\quad {}^o\ {}'\quad \mathrm{SCI}^1\ {}^o \tag{2.1.31}$$

at leading order in #. This is in accordance with the inversion formula of the elliptic gamma function (A.1.9).

The case = 0 requires some care, because »0¼ $_l$ is undefined. Taking the limit of SCI as ! 0 from the left or the right, one obtains two values that differ by an imaginary quantity. The limit from the right corresponds to taking » ¼^0_l ! 0 in (2.1.29), while the limit from the left corresponds to » ¼ $_l$! 0 (*i.e.*, » ¼^0_l ! 1). The difference is

$$\mathrm{SCI}_{» ¼^0_l\ !\ 0}\quad \mathrm{SCI}_{» ¼_l\ !\ 0} = \frac{8c\#^2}{6}\ 3\ ¸\ 01\ ¸\ \frac{0}{1}\ ¸\ \frac{1}{0}\ \bullet \tag{2.1.32}$$

Since SCI is, in any case, ambiguous by shifts of $2c8$ because it is a logarithm, only the remainder modulo $2c8$ is meaningful but this is an order 1 quantity which can be neglected. It turns out that, with $\# = 01\#$, the quantity on the right-hand-side of (2.1.32) is always an integer multiple of $8c\#$, and so its exponential is a sign. We should also notice that, for = 0, our approximation gets corrections at order # log #.

2.1.2 The SCI entropy function

We are now ready to put all the ingredients together. Our working assumption is that, in the large N limit, the index (2.1.9) is dominated by the basic solution (2.1.18) and the choice of integers (2.1.19). Some evidence that the basic solution dominates the index for $g = f$ has been given in [73] (see also [81]).

The leading contribution to (2.1.9) originates from Z_{tot} that can be evaluated using (2.1.29). Indeed, the term $\wedge_N$ is manifestly sub-leading. That the contribution of is also subleading follows from the analysis in [73] for $g = f$, since only depends on the solutions to the BAEs and not explicitly on g and f. The large N limit of the index at leading order is then

$$\log I_{\infty} = {}_{\text{SCI}}(\Delta_1) + {}_{\text{SCI}}(\Delta_2) - {}_{\text{SCI}}(\Delta_1+\Delta_2) - {}_{\text{SCI}}(0), \tag{2.1.33}$$

where the definition of the last term has an ambiguity of order 1.

Recall that in (2.1.4) we introduced the auxiliary chemical potential Δ_3. Notice, in particular, that the chemical potentials are defined modulo 1. Using the basic properties

$$[\Delta+1]_\tau = [\Delta]_\tau, \qquad [\Delta+\tau]_\tau = [\Delta]_\tau + \tau, \qquad [-\Delta]_\tau = -[\Delta]_\tau - 1, \tag{2.1.34}$$

we find

$$[\Delta_3]_\tau = g + f - 1 - [\Delta_1+\Delta_2]_\tau\,. \tag{2.1.35}$$

It follows from the definition of the function $[\Delta]_\tau$ that

$$[\Delta_1+\Delta_2]_\tau = [\Delta_1]_\tau + [\Delta_2]_\tau + \eta \tag{2.1.36}$$

where $\eta = 0$ or $\eta = 1$. The result then breaks into two cases. If $[\Delta_1+\Delta_2]_\tau = [\Delta_1]_\tau + [\Delta_2]_\tau$ then

$$[\Delta_1]_\tau + [\Delta_2]_\tau + [\Delta_3]_\tau - g - f = -1, \tag{2.1.37}$$

and, using (2.1.33) and (2.1.29),

$$\begin{aligned}\log I_{\infty} &= -\pi i N^2 \frac{[\Delta_1]_\tau [\Delta_2]_\tau \left(g + f - 1 - [\Delta_1]_\tau - [\Delta_2]_\tau\right)}{g f} \\ &= -i\pi N^2 \frac{[\Delta_1]_\tau [\Delta_2]_\tau [\Delta_3]_\tau}{g f}\,.\end{aligned} \tag{2.1.38}$$

To obtain this formula we used ${}_{\text{SCI}}(0) = {}_{\text{SCI}}([\Delta]_\tau \to 0)$. Notice that the contributions from $_1$ cancel out. As we will see in Section 2.2, this is a consequence of the holographic relation $a = c$ among the two 4d central charges in the large N limit. If $[\Delta_1+\Delta_2]_\tau = [\Delta_1]_\tau + [\Delta_2]_\tau + 1$, namely » 1 ¸ 2¼0/ = » 1¼0/ ¸ » 2¼0/, then

$$» 1¼0/ ¸ » 2¼0/ ¸ » 3¼0/ \quad g \quad f = 1 - \tag{2.1.39}$$

and

$$\begin{aligned} \log I_{43} &= c8\#^2 \frac{» 1¼0/ » 2¼0/ \; g ¸ f ¸ 1 \quad » 1¼0/ \quad » 2¼0/}{g\, f} \\ &= 8c\#^2 \frac{» 1¼0/ » 2¼0/ » 3¼0/}{g\, f} \, . \end{aligned} \tag{2.1.40}$$

This time we used SCI¹0º = SCI » ¼0/ ! 0·

As in [73], we can extract the entropy of the dual black holes by taking the Legendre transform of the logarithm of the index. The precise identification of the charges associated with the chemical potentials follows from (2.1.6). The prediction for the entropy can then be combined into two constrained entropy functions

$$S\ ^1- -g- f- \ ^º = 8c\#^2 \frac{-1 -2 -3}{g\, f} \quad 2c8 \quad \tilde{③}_{=1} \; - \& \; ¸ g\ 1 ¸ f\ 2 \; ¸ \; ! \\ 2c8 \quad -1 ¸ -2 ¸ -3 \quad g \quad f \quad 1 - \tag{2.1.41}$$

where we used a neutral variable - to denote either » ¼/ or » ¼0/, we introduced a Lagrange multiplier to enforce the constraint, and we recall that & = ' •2. This completes our derivation of the entropy of supersymmetric black holes in AdS5 (⁵ for generic angular momenta and electric charges. The expression (2.1.41) represents indeed the two entropy functions derived in [118], where it was shown that the (constrained) extremization of (2.1.41) reproduces the entropy of a black hole of angular momenta 1 and 2 and charges & . The two results correspond to the two entropy functions that reproduce the same black-hole entropy and are associated with two Euclidean complex solutions that regularize the black-hole horizon [74].

2.2 The SCI of quiver theories with a holographic dual

We want now to generalize the large # computation of the superconformal index to theories dual to AdS5 SE5 compactifications, where SE5 is a 5d Sasaki-Einstein manifold. We can write general formulae with very few assumptions. We consider 4d $N = 1$ theories with SU¹#º gauge groups, as well as adjoint and bi-fundamental chiral multiplet fields. To cancel gauge anomalies, the total number of fields transforming in the fundamental representation of a group must be the same as the number of anti-fundamentals. We also require equality of the conformal central charges $2 = 0$ in the large # limit, as dictated by holography. Our analysis extends the results found in [125] for equal angular momenta.

We then assume that in the large # limit, as for $N = 4$ SYM, the leading contribution to the superconformal index comes from the basic solution with the choice of integers f$<_8$g discussed in (2.1.19). As already shown in [111, 125], the basic solution to the BAEs for $N = 4$ SYM [62, 73, 116] can easily be extended to quiver gauge theories by setting

$$D_{89}^{UV} \quad D_8^U \quad D_9^V = \frac{9 \quad 8}{\#} \, I \qquad\qquad U\text{-}\, V = 1\text{-}\bullet\bullet\bullet\text{-} \quad \text{-} \tag{2.2.1}$$

where U- V run over the various gauge groups in the theory and is the number of gauge groups. Similarly, we choose the integers

$$<_9^U \; 2 \; \text{f}1\text{-}\bullet\bullet\bullet\text{-}01\text{g} \qquad \text{such that} \qquad <_9^U = 9 \quad \text{mod } 01 \, \bullet \tag{2.2.2}$$

Notice in particular that neither D_{89}^{UV} nor $<_9^U$ depend on U- V. As for $N = 4$ SYM, the contribution of the determinant to the Bethe ansatz expansion (2.1.9) is subleading [125].

Using the general expressions given in [85] and following the logic of Section 2.1, it is easy to write the large # limit of the leading contribution to the superconformal index of a holographic theory, with adjoint and bi-fundamental chiral fields. We find

$$\log I_{43} \;' \; \tilde{\textcircled{\#}} \;"\; \sum_{8<9} \; \sum_{UV} \log \text{e} \; D_{89}^{UV} \quad I \; <_8^U \quad <_9^V \; \text{¸} \quad {}_{UV}; g\text{-} \, f \quad \sum_{U=1} \log \text{e} \; D_{89}^{UU} \quad I \; <_8^U \quad <_9^V \; ; g\text{-} \, f \;\; \# \tag{2.2.3}$$

where $I_8^U = 4^{2c8D_8^U}$ are the gauge fugacities, D_8^U represent the basic solution (2.2.1) and $<_8^U$ are given in (2.2.2). The sum over ${}_{UV}$ is over all adjoint (if $U = V$) and bi-fundamental (if $U < V$) chiral multiplets in the theory. The second sum is the contribution of vector multiplets. When no confusion is possible, we will keep the gauge group indices implicit and just write ${}_{UV}$. In the previous formula,

$$= b \quad \text{¸} \, A \; \frac{g \; \text{¸} \; f}{2} \; \text{-} \tag{2.2.4}$$

where A is the exact R-charge of the field and b are the flavor chemical potentials. The R-charges satisfy

$$\sum_{2 \,,} A \; = 2 \tag{2.2.5}$$

for each superpotential term $,$ in the Lagrangian. In this notation, the index $,$ runs over the monomials in the superpotential, while $2\,,$ indicates all chiral fields appearing in a given monomial. Using that each superpotential term must be invariant under the flavor symmetries, but chemical potentials are only defined up to integers, we also require

$$\tilde{O}_{2,}\, b = =_, \qquad \text{for some} \qquad =_, 2 Z \bullet \tag{2.2.6}$$

The values $=_, \quad =_0 = \quad 1$ have been used in [91, 123] to study the Cardy limit. As a consequence of the previous formulae, for each superpotential term, we have

$$\tilde{O}_{2,} \quad = g \,\text{¸}\, f \,\text{¸}\, =_, \bullet \tag{2.2.7}$$

Hence, we stress that the chemical potentials are not independent. Notice that the expression (2.2.3) correctly reduces to the one for $N = 4$ SYM (2.1.16), once we use the definition (2.1.4) as well as the inversion formula for the elliptic gamma function (A.1.9). We also need to use the exact R-charges $A = 2\bullet 3$ of the chiral fields .

Applying (2.1.29), we can evaluate the large # limit of (2.2.3) and obtain

$$\log I_{43} \;' \; \frac{c8\#^2}{3gf} \tilde{O} \quad 3 \,\text{»}\, \text{¼}\, I \,\text{¸}\, 1 \quad \frac{g\,\text{¸}\,f}{2} \;\text{¸}\; \frac{gf}{4}\; 201 \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad 1 \,\text{»}\, \text{¼}\, I \,\text{¸}\, 1 \quad \frac{g\,\text{¸}\,f}{2} \quad \text{¸}$$
$$\text{¸}\; \frac{c8 \quad \#^2}{3gf} \quad 3 \; 1 \quad \frac{g\,\text{¸}\,f}{2} \;\text{¸}\; \frac{gf}{4}\; 201 \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad 1 \; 1 \quad \frac{g\,\text{¸}\,f}{2} \quad \bullet \tag{2.2.8}$$

The corrections are of order # log # or smaller. The formula is obtained by summing (2.1.29) for each chiral multiplet, as well as (2.1.29) with » ¼ I ! 0 (and opposite sign) for each vector multiplet. We stress that (2.2.8) comes from a single contribution — in the Bethe ansatz expansion — to the index. Such a contribution might not be the dominant one, and so our estimate of the index might be incorrect, in some regions of the space of chemical potentials. However, we conjecture and will bring some evidence that this contribution always captures the semi-classical Bekenstein-Hawking entropy of BPS black holes.

Due to the presence of the brackets » ¼ I, the expression (2.2.8) assumes different analytic forms in different regions of the space of chemical potentials . There are two regions where the expression greatly simplifies. They correspond to the natural generalization of the two regions for $N = 4$ SYM discussed in Section 2.1.2 and are expected to lead to the correct black-hole entropy. In particular, they smoothly reduce to the results obtained in the Cardy limit [77, 91, 123] and match the previous analysis done for equal angular momenta [125].

The first region corresponds to chemical potentials satisfying

$$\tilde{O}_{2\,,} \;» \;¼_I = g \,¸\, f \quad 1 \,• \tag{2.2.9}$$

As we will discuss later, many models — in particular all toric ones — exhibit a corner in the space of chemical potentials where this constraint is satisfied. We can define the rescaled variables

$$\mathrm{b} = 2\,\frac{» \;¼_I}{g \,¸\, f \quad 1} \tag{2.2.10}$$

which, under the assumption (2.2.9), satisfy

$$\tilde{O}_{2\,,}\; \mathrm{b} = 2 \tag{2.2.11}$$

and can be interpreted as an assignment of R-charges to the chiral fields in the theory. In terms of b the contributions in (2.2.8) combine into

$$\begin{aligned} \log I_{43} \;\;' \;\; & \frac{c8\#^2}{24}\,\frac{{}^1g \,¸\, f \quad 1^{o3}}{g\,f}\;\tilde{O}\;\; \mathrm{b} \quad 1^{\;3} \,¸ \quad ¸ \\ ¸\; & \frac{c8\#^2}{24}\,\frac{{}^1g \,¸\, f \quad 1^{o}}{g\,f}\;\; 1 \quad g\,f \;\; 201\;\; \frac{0}{1}\;\;\frac{1}{0}\;\;\; \tilde{O}\;\; \mathrm{b} \quad 1 \,¸ \quad • \end{aligned} \tag{2.2.12}$$

Introducing the charge operator $'\,{}^1\mathrm{b}^o$ of R-charges parameterized by b and indicating with Tr the sum over all fermions in the theory, we can also write

$$\log I_{43} \;\;' \;\; \frac{c8}{24}\,\text{"}\;\frac{{}^1g \,¸\, f \quad 1^{o3}}{g\,f}\;\mathrm{Tr}\; '\,{}^1\mathrm{b}^{o3} \;\; \frac{{}^1g \,¸\, f \quad 1^{o}}{g\,f}\;\; 1 \quad g\,f\;\; 201\;\;\frac{0}{1}\;\;\frac{1}{0}\;\;\; \mathrm{Tr}\; '\,{}^1\mathrm{b}^{o}\;\#\; - \tag{2.2.13}$$

valid at leading order in #.

In the large # limit, theories with a holographic dual satisfy $2 = 0$. Using standard formulae for the central charges 0 and 2 in terms of the fermion R-charges [186], one finds

$$\mathrm{Tr}\; ' = O{}^1 1^o\, - \qquad \text{and} \qquad 0 = \frac{9}{32}\,\mathrm{Tr}\; '^{\,3} \,¸\, O{}^1 1^o\, - \tag{2.2.14}$$

from which we obtain the final expression

$$\log I_{43} \;\;' \;\; \frac{4c8}{27}\,\frac{{}^1g \,¸\, f \quad 1^{o3}}{g\,f}\; 0{}^1\mathrm{b}^o\, - \tag{2.2.15}$$

where

$$0 = \frac{9}{32} \#^2 \quad \tilde{\text{O}} \quad \text{b} \quad 1^{\,3} \,¸ \quad - \tag{2.2.16}$$

at leading order in #. The result (2.2.15) was conjectured in [167] — see (A.7) there. It is also compatible with the Cardy limit performed in [91, 123].

We can find an analogous result in a second region of chemical potentials where

$$\underset{2\,,}{\tilde{\text{O}}} \quad » \quad ¼^0_I = g \,¸\, f \,¸\, 1 \, - \tag{2.2.17}$$

written in terms of the primed bracket » ¼^0_I = » ¼$_I$ ¸ 1. As discussed at the end of Section 2.1, the contribution of vector multiplets can be written, up to subleading terms, as minus the contribution of a chiral multiplet with » ¼^0_I ! 0. After defining another set of normalized R-charges,

$$\text{b}^0 = 2 \frac{» \quad ¼^0_I}{g \,¸\, f \,¸\, 1} \tag{2.2.18}$$

which satisfy

$$\underset{2\,,}{\tilde{\text{O}}} \quad \text{b}^0 = 2 \tag{2.2.19}$$

under the assumption (2.2.17), we can rewrite the index as

$$\log I_{43} \quad ' \quad " \quad \frac{c8}{24} \frac{{}^1g \,¸\, f \,¸\, 1^{o3}}{g\,f} \,\text{Tr}\; '\,{}_1\text{b}^0{}_o{}^3 \quad \frac{{}^1g \,¸\, f \,¸\, 1^{o}}{g\,f} \quad 1 \quad g\,f \;\; 201 \quad \frac{0}{1} \quad \frac{1}{0} \quad \text{Tr}\; '\,{}_1\text{b}^0{}_o \quad \# \tag{2.2.20}$$

at leading order in #. This reduces to the simple expression

$$\log I_{43} \quad ' \quad \frac{4c8}{27} \frac{{}^1g \,¸\, f \,¸\, 1^{o3}}{g\,f} \, 0_1\text{b}^0{}_o \tag{2.2.21}$$

for holographic theories.

In the remainder of this Section, we will interpret the general results (2.2.15) and (2.2.21) and provide examples. In particular, we will show that both regions (2.2.9) and (2.2.17) in the space of chemical potentials always exist in toric quiver gauge theories. We will also see that the two expressions (2.2.15) and (2.2.21) lead to the very same result for the semi-classical entropy of dual black holes, generalizing what happens for N = 4 SYM.

2.2.1 Example: the conifold

We start with the example of the Klebanov-Witten theory dual to AdS$_5$) $^{1\text{-}1}$, the near-horizon limit of a set of # D3-branes sitting at a conifold singularity [168]. This example was already

Field	A	$\&_1$	$\&_2$	$\&$	$'_1$	$'_2$	$'_3$	$'_4$
$_1$	$\frac{1}{2}$	1	0	1	2	0	0	0
$_2$	$\frac{1}{2}$	1	0	1	0	2	0	0
$_1$	$\frac{1}{2}$	0	1	1	0	0	2	0
$_2$	$\frac{1}{2}$	0	1	1	0	0	0	2

Table 2.1 Charges of chiral multiplets in the Klebanov-Witten theory, under the maximal torus of the global symmetry $\mathrm{U}^1 1^\circ$, $\mathrm{SU}^1 2^\circ{}_1$ $\mathrm{SU}^1 2^\circ{}_2$ $\mathrm{U}^1 1^\circ$. In the table, we indicate two useful bases. Notice that A and $'$ are R-charges, while $\&_{1\text{-}2}$ and $\&$ are flavor charges.

studied for equal angular momenta in [125] and our results are consistent with those found there when we set $g = f$.

The theory has gauge group $\mathrm{SU}^1 \#^\circ$ $\mathrm{SU}^1 \#^\circ$, bi-fundamental chiral multiplets $_{1\text{-}}$ $_2$ transforming in the representation $^1\#\text{-}\overline{\#}^\circ$ and $_{1\text{-}}$ $_2$ transforming in the representation $^1\overline{\#}\text{-}\#^\circ$, and a superpotential

$$, = \mathrm{Tr} \quad {}_1 \; {}_1 \; {}_2 \; {}_2 \qquad {}_1 \; {}_2 \; {}_2 \; {}_1 \; \bullet \tag{2.2.22}$$

The global symmetry of the theory is $\mathrm{U}^1 1^\circ$, $\mathrm{SU}^1 2^\circ{}_1$ $\mathrm{SU}^1 2^\circ{}_2$ $\mathrm{U}^1 1^\circ$, where the first factor is the superconformal R-symmetry with charge A, while the other three factors are flavor symmetries. The charge assignments of chiral multiplets under the maximal torus are in Table 2.1. The index is defined as

$$I_{43} = \mathrm{Tr}^1 \; 1^\circ \; 4^{\,VfQ\text{-}Q^Yg} \, ?^{\,1,A\bullet 2} \, @^{\,2,A\bullet 2} \, E_1^{\&_1} \, E_2^{\&_2} \, E^{\&} \; \bullet \tag{2.2.23}$$

It is convenient to introduce an alternative basis of R-charges $'$ with $= 1\text{-}2\text{-}3\text{-}4$, such that each of them assigns R-charge 2 to one of the chiral multiplets and zero to the other ones. Correspondingly, we associate a variable to each chiral multiplet. Notice that $^1 \; 1^\circ = 4^{2c8 \; 1\text{-}2} = 4^{c8 \, ' 1\text{-}2\text{-}3\text{-}4}$. According to (2.2.4) and up to integer ambiguities, the variables are related to the chemical potentials for the charges in Table 2.1 by

$$
\begin{aligned}
{}_1 &= b_1 \,, b \,, \frac{g \,, f}{4} \,\text{-} \qquad & {}_3 &= b_2 \quad b \,, \frac{g \,, f}{4} \,\text{-} \\
{}_2 &= \;\; b_1 \,, b \,, \frac{g \,, f}{4} \,\text{-} \qquad & {}_4 &= \;\; b_2 \quad b \,, \frac{g \,, f}{4} \,, {}^1 2Z \,, 1^\circ \,\text{-}
\end{aligned}
\tag{2.2.24}
$$

where the chemical potentials b are related to the fugacities a by (2.1.2). Then, the constraint (2.2.7) reads

$$1 \text{¸} \quad 2 \text{¸} \quad 3 \text{¸} \quad 4 = g \text{¸} f \text{¸} =, \tag{2.2.25}$$

and the index takes the more transparent form

$$I_{43} = \mathrm{Tr}_{\mathrm{BPS}} ?\ ^1 @\ ^2 H_1^{'1\bullet 2} H_2^{'2\bullet 2} H_3^{'3\bullet 2} H_4^{'4\bullet 2} - \tag{2.2.26}$$

where H is defined as in (2.1.5). This shows that are the chemical potentials associated with the charges $\&$ $'$ $\bullet 2$.

We select three independent variables, say $_1$- $_2$, and $_3$. Then, using (2.1.34) we find

$$» \ 4¼ I = g \text{¸} f \quad 1 \quad » \ 1 \text{¸} \quad 2 \text{¸} \quad 3¼ I \bullet \tag{2.2.27}$$

In general, there are three possible cases:

$$» \ 1 \text{¸} \quad 2 \text{¸} \quad 3¼ I = » \ 1¼ I \text{¸} » \ 2¼ I \text{¸} » \ 3¼ I \text{¸} = \qquad \text{with} \qquad = = 0\text{-}1\text{-}2 \tag{2.2.28}$$

that we call Case I, II, and III, respectively.[3] Case I corresponds to the corner of moduli space (2.2.9), where

$$» \ 1¼ I \text{¸} » \ 2¼ I \text{¸} » \ 3¼ I \text{¸} » \ 4¼ I = g \text{¸} f \quad 1 \bullet \tag{2.2.29}$$

In this corner, we can use (2.2.15). One can explicitly compute, at leading order in $\#$,

$$\mathrm{Tr}\ '1 b o^3\ '\ \#^2\ 2 \text{¸} \sum_{=1}^{4} b \quad 1^{\ 3} = 3\#^2\ b_1 b_2 b_3 \text{¸} b_1 b_2 b_4 \text{¸} b_1 b_3 b_4 \text{¸} b_2 b_3 b_4 \tag{2.2.30}$$

imposing $\sum_{=1}^{4} b = 2$. Using (2.2.10), we can write the index (2.2.15) as

$$\begin{aligned} \log I_{43} \ ' \ \frac{c8\#^2}{g f} \ & » \ 1¼ I » \ 2¼ I » \ 3¼ I \text{¸} » \ 1¼ I » \ 2¼ I » \ 4¼ I \text{¸} \\ & \text{¸} » \ 1¼ I » \ 3¼ I » \ 4¼ I \text{¸} » \ 2¼ I » \ 3¼ I » \ 4¼ I \end{aligned} \tag{2.2.31}$$

with the constraint (2.2.29).[4] Case III corresponds to the corner of moduli space (2.2.17). Indeed we have

$$» \ 1¼^0 I \text{¸} » \ 2¼^0 I \text{¸} » \ 3¼^0 I \text{¸} » \ 4¼^0 I = g \text{¸} f \text{¸} 1 \bullet \tag{2.2.32}$$

---

[3] For the sake of comparison, the notation is the same as in [125].

[4] This result is a special case of the one for toric models, discussed in detail in Section 2.2.2.

In this corner, we can use (2.2.21) and (2.2.18) and find

$$\log I_{43} \ ' \ \frac{c8\#^2}{g f} \ » \ 1¼^0_I » \ 2¼^0_I » \ 3¼^0_I \ ¸ » \ 1¼^0_I » \ 2¼^0_I » \ 4¼^0_I \ ¸$$
$$¸ » \ 1¼^0_I » \ 3¼^0_I » \ 4¼^0_I \ ¸ » \ 2¼^0_I » \ 3¼^0_I » \ 4¼^0_I \qquad (2.2.33)$$

with the constraint (2.2.32).

The entropy, which is the logarithm of the number of states, is given by the Legendre transform of the index, *i.e.*, by the critical value of the entropy function

$$S = \frac{c8\#^2}{g f} \ \text{-}_1\text{-}_2\text{-}_3 \ ¸ \ \text{-}_1\text{-}_2\text{-}_4 \ ¸ \ \text{-}_1\text{-}_3\text{-}_4 \ ¸ \ \text{-}_2\text{-}_3\text{-}_4 \ ¸$$
$$2c8 \ g \ _1 ¸ \ f \ _2 ¸ \ \sum_{=1}^{4} \text{-} \ \& \ 2c8 \sum_{=1}^{4} \text{-} \ \ g \ \ f \ \ 1 \ • \qquad (2.2.34)$$

Here the variables - stand for » ¼ $_I$ or » ¼^0_I depending on whether we are in case I or III, respectively, and the sign is chosen accordingly. One can check that the two signs lead to the same entropy. We will give a general argument in Section 2.2.3.

In Section 2.4 we will compare the field theory result (2.2.34) with the entropy of black holes in AdS5) $^{1\text{-}1}$, in the special case that $_1 = \ _2$ and the SU¹2º $_1$ SU¹2º $_2$ symmetry is unbroken. To that purpose, let us specialize the index to the case that $g = f$ and $b_1 = b_2 = 0$, which corresponds to $\text{-}_1 = \text{-}_2$ and $\text{-}_3 = \text{-}_4$. It is then useful to define the new variables

$$\text{-}_{'} = \text{-}_1 ¸ \ \text{-}_3 \text{-} \qquad \qquad \text{-} \ = \frac{\text{-}_1 \quad \text{-}_3}{2} \ \text{-} \qquad (2.2.35)$$

associated with R-symmetry and baryonic symmetry, respectively. The entropy function takes the simplified form

$$S = \frac{c8\#^2}{2g^2} \ \text{-}_{'} \ \text{-}^2_{'} \ \ 4\text{-}^2 \ \ 2c8 \ 2g \ ¸ \ \text{-}_{'} A \ ¸ \ \text{-} \ \& \ \ 2c8 \ \ 2\text{-}_{'} \ \ 2g \ \ 1 \ • \qquad (2.2.36)$$

2.2.2 Example: toric models

In this Section, we consider the gauge theory dual to an AdS5 SE5 geometry, where SE5 is a toric Sasaki-Einstein manifold. The theory lives on a stack of # D3-branes sitting at the toric Calabi-Yau singularity ¹SE5º obtained by taking the cone over SE5 [168, 187]. There is a general construction to extract gauge theory data from the geometry of the Calabi-Yau singularity [188–191]. The main complication compared to the $\mathbb{C}^3$ and the conifold cases are that there is no one-to-one correspondence between bi-fundamental fields (and associated variables) and R-symmetries R_a. However, we will argue in general that there always exist two corners of the space of chemical potentials where (2.2.9) and (2.2.17) are satisfied and the results (2.2.15) and (2.2.21) are valid. Other corners should be analyzed separately for every specific model. Our findings are consistent with the case-by-case analysis performed in [125] for equal angular momenta.

We first need to understand how to write the trial central charges $a(\Delta)$ and $a(\Delta')$ that enter in the expressions (2.2.15) and (2.2.21). Since the quantities Δ and Δ' satisfy the constraints (2.2.11), they can be interpreted as a set of trial R-charges for the chiral fields in the quiver. In the toric case, we can find an efficient parameterization of the trial R-charges of fields using the data of the toric diagram. Let us review how this is done.

A toric Calabi-Yau threefold singularity can be specified by a fan, *i.e.*, a convex cone in $\mathbb{R}^3$ defined by d integer vectors $v_a = (1, \vec{w}_a)$, with ends on the plane $z = 1$. The restrictions $\vec{w}_a$ of those vectors to the plane define a regular convex polygon with integer vertices called the toric diagram. In the list $\{v_a\}$ we should include all integer vectors such that $\vec{w}_a$ is along the perimeter of the polygon, *i.e.*, we should include all integer points along the edges of the toric diagram. Moreover, we take the points $\vec{w}_a$ to be ordered in a counterclockwise fashion. The number of vectors in the fan is associated with the total rank of the global symmetry of the dual field theory [190]: for a toric model with d vectors in the fan (including integer points along the edges of the toric diagram) there is a flavor symmetry of rank $d-1$, besides the R-symmetry $U(1)_R$.[5] This allows us to parameterize flavor and R-symmetries in terms of variables associated with the vertices of (and integer points along) the toric diagram. In particular, the possible R-charges of fields in a toric theory can be parameterized using variables Δ_a satisfying the constraint

$$\sum_{a=1}^{d} \Delta_a = 2 \tag{2.2.37}$$

and the corresponding R-charge can be written as

$$R(\Delta) = \sum_{a=1}^{d} \frac{\Delta_a}{2} R_a \tag{2.2.38}$$

[5] The distinction between R- and flavor symmetries changes in the case of extended supersymmetry.

in terms of a basis $\{R_a\}$. This is done as follows [192]. In a minimal toric phase,[6] the theory contains a number G of gauge group factors $SU(N)$ equal to twice the area of the toric diagram. Moreover, defining the vectors $v_a = V_{a+1} - V_a$ lying in the plane (we identify indices modulo d, so that, for example, $v_{d+1} \equiv v_1$), for each pair (a,b) such that v_a can be rotated counterclockwise into v_b in the plane with an angle smaller than π, there are precisely[7] $\det\{v_a, v_b\}$ bi-fundamental chiral fields Φ_{ab} with R-charge

$$R[\Phi_{ab}] = X_{a+1} + X_{a+2} + \dots + X_b . \tag{2.2.39}$$

Interestingly, for all toric models the trial central charge $a(X)$ is a homogeneous function of degree three at large N:

$$a(X) = \frac{9}{32}\operatorname{Tr} R(X)^3 = \frac{9N^2}{64}\sum_{a,b,c=1}^{d} C_{abc}\, X_a X_b X_c . \tag{2.2.40}$$

Here C_{abc} are the ’t Hooft anomaly coefficients, which can be read from the toric data through $C_{abc} = |\det\{V_a, V_b, V_c\}|$, see [193], and in field theory are defined by

$$N^2 C_{abc} = \frac{1}{4}\operatorname{Tr} R_a R_b R_c . \tag{2.2.41}$$

Another important property of toric models that we will use in the following is that the constraints

$$\sum_{\Phi \in W} R[\Phi] = 2 , \tag{2.2.42}$$

that must be satisfied for each monomial term W in the superpotential, always reduce to (2.2.37). Indeed, it follows from tiling techniques [188–192] that the R-charges $R[\Phi]$, $\Phi \in W$, of the chiral fields entering in a superpotential monomial W correspond to a partition of the d elementary R-charges $\{X_1, \dots, X_d\}$ into sums of the form (2.2.39), with each X_a entering in just one $R[\Phi]$.

We can similarly parameterize the chemical potentials $\Delta[\Phi]$ entering the superconformal index in terms of d basic quantities Δ_a, $a = 1, \dots, d$. For the chiral fields Φ_{ab} we have

$$\Delta[\Phi_{ab}] = \Delta_{a+1} + \Delta_{a+2} + \dots + \Delta_b . \tag{2.2.43}$$

6 Many different quiver theories describe the same IR SCFT. They are called “phases”, and are related by Seiberg dualities. The toric phases are the quiver theories where all gauge groups are $SU(N)$ with the same rank N. It turns out that all toric phases have the same number G of gauge groups, but have different matter content. The “minimal” phases correspond to the quivers with the smallest number of chiral fields. There could be one or more minimal toric phases, for a given IR SCFT.

7 The condition on the angle guarantees that the formula for the number of fields gives a non-negative integer.

The conditions

$$\text{Õ}_{2\,,} \; » \; ¼ = g \,¸\, f \,¸\, =_{,} \; - \tag{2.2.44}$$

to be imposed for each monomial term $,$ in the superpotential (and where $=_{,}$ is the same for all monomial terms), are then equivalent to

$$\text{Õ}_{0=1} \; {}_{0} = g \,¸\, f \,¸\, =_{,} \; • \tag{2.2.45}$$

Independently of the value of $=_{,}$, we have

$$» \; ¼_{l} = g \,¸\, f \quad 1 \quad \text{Õ}_{0=1}^{1} \; {}_{0} \; {}_{l} \; • \tag{2.2.46}$$

In general

$$\text{Õ}_{0=1}^{1} \; {}_{0} \; {}_{l} = \text{Õ}_{0=1}^{1} \; » \; 0¼_{l} \,¸\, = \tag{2.2.47}$$

where $= = 0 - • • • -$ 2, thus dividing the space of parameters into 1 regions.

Two regions are particularly important for our analysis. The region $= = 0$ corresponds to

$$\text{Õ}_{0=1} \; » \; 0¼_{l} = g \,¸\, f \quad 1 \, - \tag{2.2.48}$$

while $= =$ 2 corresponds to

$$\text{Õ}_{0=1} \; » \; 0¼^{0}_{l} = g \,¸\, f \,¸\, 1 \; • \tag{2.2.49}$$

We can argue that the two regions (2.2.48) and (2.2.49) are always realized somewhere in the space of parameters. For example, we can choose one elementary variable, say $_{1}$, to live in the fundamental strip $\mathrm{Im}^{1} \; 1 • l^{o} \; ¡ \; \mathrm{Im} \; {}_{1} • l \; ¡ \; 0$ (see Fig. 2.1) and slightly on the right of the vertical line passing through $g \,¸\, f \quad 1$, while all the other $_{0}$ to live in the fundamental strip and slightly on the left of the vertical line passing through zero. One easily verifies that they can be arranged to satisfy (2.2.48). A similar construction gives parameters satisfying (2.2.49). We now argue that (2.2.48) and (2.2.49) imply (2.2.9) and (2.2.17), respectively. We start noticing that

$$\text{Õ}_{0=1} \; » \; 0¼_{l} = g \,¸\, f \quad 1 \qquad) \qquad \mathrm{Im} \; \frac{1}{l} \; \text{Õ}_{0=1} \; » \; 0¼_{l} \; = \mathrm{Im} \; \frac{1}{l} \; • \tag{2.2.50}$$

Since each of the » 0¼ I lives in the fundamental strip Im¹ 1• I º ¡ Im » 0¼ I • I ¡ 0, the previous equation implies that Im¹ 1• I º ¡ Im Í 02(» 0¼ I • I ¡ 0 for any proper subset (of the indices f1-•••- g. Thus (2.2.48) implies that

$$\text{h Õ}_{02(}\ {}^{0}{}_{I}\ \text{i} = \text{Õ}_{02(}\ » 0¼ I \tag{2.2.51}$$

for any proper subset ((f1-•••- g. This implies that all charges in (2.2.43) split, in the sense that 0¸1 ¸ ••• ¸ 1 I = » 0¸1¼ I ¸ ••• ¸ » 1¼ I . At this point, since all » ¼ I split and each 0 enters precisely once in every superpotential constraint, the condition (2.2.9) is a consequence of (2.2.48).[8] A similar argument shows that (2.2.49) implies (2.2.17). Notice that the region specified by (2.2.9) can be larger than (2.2.48) and, similarly, the region specified by (2.2.17) can be larger than (2.2.49). This, in particular, happens for Calabi-Yau cones with codimension-one orbifold singularities. This is the case of the models SPP and dP_4 discussed in [125].[9] For all the cones without orbifold singularities that we checked, the two regions (2.2.9) and (2.2.48) coincide. It would be interesting to see if this is a general result.

We are now ready to evaluate the index. Consider region (2.2.9) first. Since the chemical potentials » ¼ I split, the rescaled quantities

$$\text{b}_0 = 2\,\frac{» 0¼ I}{g ¸ f \quad 1} \qquad \text{with} \qquad \sum_{0=1} \text{b}_0 = 2 \tag{2.2.52}$$

8There is an alternative algorithm that produces potentials satisfying (2.2.9). Choose a perfect matching ?U of the dimer model of the theory [190]. It divides the chiral fields into two groups: those P appearing in the perfect matching, and those NP not doing so. Choose the potentials NP to be in the fundamental strip and slightly on the left of the origin. Each superpotential term , contains one and only one of the fields P (by definition of perfect matching): choose the corresponding P to be in the fundamental strip and slightly on the right of the point g ¸ f 1, in such a way that (2.2.9) for that particular , is satisfied. The drawback of this construction is that it does not tell us what the independent variables 0 are.

9Models with codimension-one orbifold singularities are characterized by toric diagrams where at least one vector E0 lies in the interior of an edge. The parameters X0 associated with integer points lying in the interior of an edge of the polygon enter in the parameterization (2.2.39) of the R-charges of chiral fields, but no elementary field carries precisely charge X0. To recover the region (2.2.9), we can require the following. Construct a set " by grouping the points f1-•••- g along the toric diagram in the following way: break each edge in two pieces at a non-integer point, and then for each vertex form a group (that will be an element of ") that contains the vertex itself and all other integer points (if any) along the two pieces of edges on the two sides. (In the absence of orbifold singularity, " necessarily coincides with f1-•••- g.) Then require that the sums split over the groups in " for every proper subgroup (0 (" , and every possible choice of " . This region is typically larger than (2.2.48).

provide a parameterization of the R-charges of chiral fields in the quiver in the sense discussed above. Using the general formula (2.2.40) we can then write

$$0^1 b^o = \frac{9\#^2}{64} \tilde{O}_{0\text{-}1\text{-}2=1} \;{}_{012}\, b_0\, b_1\, b_2 \bullet \tag{2.2.53}$$

Plugging it into (2.2.15) and re-expressing the result in terms of the chemical potentials $» \;{}_{0}¼_{I}$, we find the large $\#$ limit of the superconformal index in region (2.2.9):[10]

$$\log I_{43} \;' \;\; c8\#^2 \tilde{O}_{0\text{-}1\text{-}2=1} \frac{{}_{012}}{6} \frac{» \;{}_{0}¼_{I} » \;{}_{1}¼_{I} » \;{}_{2}¼_{I}}{g\,f} \;- \;\; \tilde{O}_{0=1} \; » \;{}_{0}¼_{I} = g \,¸\, f \;\; 1 \bullet \tag{2.2.54}$$

A similar argument shows that, in region (2.2.17),

$$\log I_{43} \;' \;\; c8\#^2 \tilde{O}_{0\text{-}1\text{-}2=1} \frac{{}_{012}}{6} \frac{» \;{}_{0}¼^{0}_{I} » \;{}_{1}¼^{0}_{I} » \;{}_{2}¼^{0}_{I}}{g\,f} \;- \;\; \tilde{O}_{0=1} \; » \;{}_{0}¼^{0}_{I} = g \,¸\, f \,¸\, 1 \bullet \tag{2.2.55}$$

We will show in the next Section that both (2.2.54) and (2.2.55) lead to the same entropy.

2.2.3 The toric entropy function

For toric holographic quivers, we have found two different expressions, (2.2.54) and (2.2.55), for the large $\#$ limit of the superconformal index, valid in two different regions in the space of chemical potentials. The two expressions differ only for the constraint and give rise to the very same entropy. This generalizes an observation made in [74] for $N = 4$ SYM and holds for general quivers.

To show that, we define two entropy functions

$$\begin{aligned} S \;=\; c8\#^2 \tilde{O}_{0\text{-}1\text{-}2=1} \frac{{}_{012}}{6} \frac{-_0 -_1 -_2}{g\,f} \;\; 2c8\; g \;{}_1 \,¸\, f \;{}_2 \,¸\; \tilde{O}_{0=1} \; -_0 \&_0 \;¸ \\ 2c8 \;\; \tilde{O}_{0=1} \;\; -_0 \;\; g \;\; f \;\; 1 \;- \end{aligned} \tag{2.2.56}$$

where is a Lagrange multiplier and we used neutral variables $-_0$ to denote either $» \;{}_{0}¼_{I}$ or $» \;{}_{0}¼^{0}_{I}$. Each of the electric charges $\&_0$ $'_0 \bullet 2$ is defined in terms of an R-charge $'_0$ that assigns charge 2 to all chiral multiplets ${}_{01}$ such that X_0 appears in the decomposition

[10] For the conifold: $E_1 = {}^1 1\text{-}0\text{-}0^o\text{-}\, E_2 = {}^1 1\text{-}1\text{-}0^o\text{-}\, E_3 = {}^1 1\text{-}1\text{-}1^o\text{-}E_4 = {}^1 1\text{-}0\text{-}1^o$ and thus ${}_{123} = {}_{124} = {}_{134} = {}_{234} = 1$ (and symmetrizations), recovering (2.2.31).

(2.2.39), and zero to all the other ones. The 't Hooft anomaly coefficients are defined by (2.2.41). Above, $\mathcal{S}_+$ is the prediction for the entropy of the dual black hole based on the superconformal index in the region of parameters (2.2.9) while $\mathcal{S}_-$ in the region (2.2.17). The form of the entropy function (2.2.56) was first conjectured in [167].

Observe that since $\mathcal{E}_\pm$ and $2\pi i\Lambda$ are homogeneous functions of degree one in $\{Q_a, J_1, J_2\}$, the values of the functions $\mathcal{E}_\pm(Q_a, J_1, J_2)$ at the critical point are related to the Lagrange multiplier by

$$\mathcal{E}_\pm\big|_{\text{crit}} = 2\pi i\Lambda_\pm \,. \tag{2.2.57}$$

Observe also that, if Q_a, J_i are real (as charges should be), then the two functions are related by $\overline{\mathcal{E}_+(Q_a, J_1, J_2)} = \mathcal{E}_-(\bar{Q}_a, \bar{J}_1, \bar{J}_2)$. Hence, if $\{Q_a, J_1, J_2\}$ is a critical point of $\mathcal{E}_+$, then $\{\bar{Q}_a, \bar{J}_1, \bar{J}_2\}$ is a critical point of $\mathcal{E}_-$ with critical value

$$\mathcal{E}_-\big|_{\text{crit}} = \overline{\mathcal{E}_+\big|_{\text{crit}}} \,. \tag{2.2.58}$$

For arbitrary and general real charges Q_a and J_i, the critical value of $\mathcal{E}_+$ is not real. For $\mathcal{N} = 4$ SYM, however, it becomes real and equal to the entropy when imposing the non-linear constraint on conserved charges that characterizes supersymmetric black holes [74, 118]. The same phenomenon was already observed in AdS$_4$ in [42]. We expect the same to be true for general black holes in Sasaki-Einstein compactifications. Even if this were wrong and $\mathcal{E}_+$ were not real, it would still make sense to identify the entropy with $\mathrm{Re}\,\mathcal{E}_+$. In all cases, we see from (2.2.58) that both constraints in (2.2.56) lead to the very same result for the entropy.

The entropy functions (2.2.56) give our result for the entropy of generic black holes in AdS$_5 \times$ SE$_5$. We derived it for toric quiver gauge theories, but the very same argument can be extended to a class of more general non-toric quivers. In particular, the expression (2.2.56) only depends on the 't Hooft anomaly coefficients C_{abc} for a basis of R-symmetries and, as such, we expect that it is the correct result for generic holographic quiver theories.

2.3 The universal AdS$_5$ rotating black hole

In this Section, we discuss the case of the universal rotating black hole which has an electric charge aligned with the exact R-symmetry of the theory. This black hole arises as a solution of minimal gauged supergravity in 5d and, as such, it can be embedded in any AdS$_5 \times$ SE$_5$ compactification of type IIB and, more generally, in any AdS$_5$ solution of type II or M-theory.[11]

[11] It is believed and checked in many cases that the effective theory for all such compactifications can be consistently truncated to minimal gauged supergravity.

Due to its universal character, most of the analysis is identical to the one for AdS$_5\times S^5$. It is however interesting to see how the details work.

The universal black hole in AdS$_5$ was found in [162] in minimal gauged supergravity in 5d. It has charge Q_R under the gravi-photon and angular momenta J_1 and J_2 in AdS$_5$.[12] The entropy can be compactly written as [194]

$$S(Q_R, J_i) = 2\pi\sqrt{3Q_R^2 - 2a(J_1+J_2)} \tag{2.3.1}$$

where we introduced the quantity

$$a = \frac{\pi\ell_5^3}{8G_N^{(5)}}\,, \tag{2.3.2}$$

where $G_N^{(5)}$ is the 5d Newton constant and ℓ_5 is the radius of AdS$_5$. The conserved charges must satisfy the nonlinear constraint

$$8Q_R^3 + 6aQ_R^2 - 6a(J_1+J_2)Q_R - 2aJ_1J_2 - 4a^2(J_1+J_2) = 0 \tag{2.3.3}$$

for the BPS black hole to have a smooth horizon.

Consider now the uplift of the universal black hole to AdS$_5\times SE_5$, where SE_5 is a Sasaki-Einstein manifold. In such an embedding, the standard holographic dictionary identifies a with the central charge of the dual CFT$_4$. The black hole carries angular momenta J_1 and J_2 and an electric charge aligned with the exact R-symmetry of the dual CFT$_4$. We need to check that its entropy is reproduced by our result (2.2.15) (the same result can be similarly obtained using (2.2.21), instead). It is convenient to parameterize the chemical potentials as

$$\Delta_a = \frac{\omega_1+\omega_2-2\pi i}{2}\left(\hat{\Delta}_a^{(0)} + \delta_a\right), \tag{2.3.4}$$

where $\hat{\Delta}_a^{(0)}$ is the exact superconformal R-symmetry of the dual CFT$_4$ while δ_a parameterize a basis of flavor symmetries. These quantities satisfy

$$\sum_{a=1}^{d}\hat{\Delta}_a^{(0)} = 2\,, \qquad \sum_{a=1}^{d}\delta_a = 0\,. \tag{2.3.5}$$

[12] To compare with the notations of [162]: $Q_{\text{there}} = \sqrt{3}\,Q_{R,\text{here}}$, $G_N^{(5)} = 1$, and $\ell_5 = 1/g$.

The entropy of the universal black hole is given by the Legendre transform of (2.2.15). Using (2.2.48) we can write the entropy function as

$$S = \frac{4\pi i}{27}\frac{(\sigma+\tau-1)^3}{\sigma\tau}\, a\left(\Delta^{(0)} + \delta\Delta\right) - 2\pi i\left((\sigma+\tau-1)Q_R + \sigma J_1 + \tau J_2\right) \tag{2.3.6}$$

where we introduced a charge

$$Q_R = \frac{1}{2}\sum_{a=1} \Delta_a^{(0)} Q_a \tag{2.3.7}$$

in the direction of the exact R-symmetry, and set all other charges to zero. We need to extremize the function S with respect to σ, τ, and $\delta\Delta_a$ subject to the constraint (2.3.5). By a-maximization, since $\Delta_a^{(0)}$ is the exact R-symmetry, the function is extremized at $\delta\Delta_a = 0$. We can then restrict the entropy function to

$$S = \frac{4\pi i a}{27}\frac{(\sigma+\tau-1)^3}{\sigma\tau} - 2\pi i\left((\sigma+\tau-1)Q_R + \sigma J_1 + \tau J_2\right) \tag{2.3.8}$$

where $a \equiv a(\Delta^{(0)})$ is the central charge of the CFT_4, or, introducing a Lagrange multiplier Λ,

$$S = 4\pi i a\frac{\varphi^3}{\sigma\tau} - 2\pi i\left(3\varphi Q_R + \sigma J_1 + \tau J_2\right) - 2\pi i\Lambda\left(3\varphi - \sigma - \tau + 1\right). \tag{2.3.9}$$

If we set $a = a_{\mathcal{N}=4} = N^2/4$, the function (2.3.9) becomes identical to the entropy function of $\mathcal{N}=4$ SYM for equal charges $Q_1 = Q_2 = Q_3 \equiv Q_R$, which is known to correctly reproduce (2.3.1) [118]. An analytic derivation of (2.3.1) and (2.3.3) for $\mathcal{N}=4$ SYM is explicitly discussed in [74] and for equal angular momenta in [73]. The charge constraint (2.3.3) is obtained as the requirement that the extremum of S be real.

At this point, the result for the universal black hole simply follows from the homogeneity properties of (2.3.9):

$$\mathcal{S}(Q_R, J_1, J_2) = \frac{a}{a_{\mathcal{N}=4}}\,\mathcal{S}_{\mathcal{N}=4}\left(\frac{a_{\mathcal{N}=4}}{a}Q_R, \frac{a_{\mathcal{N}=4}}{a}J_1, \frac{a_{\mathcal{N}=4}}{a}J_2\right). \tag{2.3.10}$$

It is then immediate to derive the relations (2.3.1) and (2.3.3), thus completing our derivation.

2.4 AdS_5 Kerr-Newman black holes in $T^{1,1}$

We would like to compare the entropy function we obtained in Section 2.2 from the large N limit of the superconformal index of generic (toric) quiver gauge theories, with the Bekenstein-Hawking entropy of BPS black holes in the corresponding 5d gauged supergravities. In particular, the setup we would like to analyze is that of type IIB supergravity on asymptotically-$AdS_5 \times SE_5$ space-times, where SE_5 is a toric Sasaki-Einstein manifold,[13] reduced and truncated to a 5d $\mathcal{N} = 2$ gauged supergravity on AdS_5. Unfortunately, except for the case of S^5 truncated to the so-called 5d STU model, and the case of any SE_5 truncated to minimal $\mathcal{N} = 2$ gauged supergravity (that we analyzed in Section 2.3), all other known consistent truncations are to gauged supergravities with hypermultiplets (besides vector multiplets), and no supersymmetric black-hole solutions have been constructed in such theories to date.

The strategy we propose to perform a test of our field theory results is as in [118]. We assume that a 5d BPS rotating black-hole solution exists. Such a solution has the topology of a fibration of AdS_2 over S^3 (the three-sphere being the topology of the event horizon), and thus we can reduce it along the Hopf fiber of S^3. This gives a (putative) 4d BPS rotating black-hole solution, with the same entropy.[14] The reduction generates an extra vector field A^0, corresponding to the isometry along the Hopf fiber. The 4d black hole has one unit of magnetic charge under A^0, corresponding to the first Chern class of the Hopf fibration. Calling J_1 and J_2 the 5d angular momenta along two orthogonal planes, the quantity $J_1 + J_2$ appears in 4d as the electric charge under A^0, while $J_1 - J_2$ becomes the angular momentum of the 4d black hole. Constructing such a 4d rotating black-hole solution is still a difficult task, and an attractor mechanism is not known in general.[15] However, if we restrict to 5d black holes with two equal angular momenta $J_1 = J_2$ (so that the isometry of the squashed S^3 is enhanced from $U(1)^2$ to $U(1) \times SU(2)$), then the 4d black hole is static: in this case, we can determine its entropy by exploiting the attractor mechanism in the near-horizon geometry [172–174], without actually constructing the whole solution.

The simplest non-trivial example is when SE_5 is $T^{1,1}$, the base of the conifold Calabi-Yau threefold, whose holographic dual is the Klebanov-Witten gauge theory [168]. We already presented the field theory analysis in Section 2.2.1. On the other hand, starting from 10d type IIB supergravity on $T^{1,1}$, we can exploit a consistent truncation that preserves $SU(2)^2 \times U(1)$ isometry, down to a 5d $\mathcal{N} = 2$ gauged supergravity with the graviton multiplet, two vector multiplets, and two hypermultiplets. This is the second truncation presented in Section 7 of [169] (see also [170, 171]). On the AdS_5 vacuum, one vector multiplet (sometimes called "Betti multiplet") is massless and is associated with the baryonic symmetry, while the other vector multiplet is massive.

13 More precisely, the cone over SE_5 is a toric Calabi-Yau threefold.

14 The 4d solution has an exotic asymptotic behavior, that follows from the reduction of AdS_5 [195]. Nonetheless, it has a regular extremal horizon, whose area determines the entropy.

15 There are however some general results for theories with vector multiplets [196, 197].

Hence, with the simplification that $_1 = \ _2$ and only the R-symmetry and baryonic symmetry charges are turned on (while the $\mathrm{SU}^1 2^{o2}$ isometry of $)^{1\text{-}1}$ is unbroken), we will be able to match the Legendre transform of the superconformal index at large # with the extremization problem that comes from the attractor mechanism in supergravity. It follows that the bulk and boundary computations of the entropy exactly match.

2.4.1 Reduction from 5d to 4d and the attractor mechanism

A 5d $N = 2$ Abelian gauged supergravity with $=_+$ vector multiplets and $=$ hypermultiplets — whose main building blocks we summarize in Appendix B.1 — is specified by the following data [198–200].

1. A very special real manifold SM of real dimension $=_+$, specified by a symmetric tensor of Chern-Simons couplings with - - $= 1\text{-}\bullet\bullet\bullet\text{-}=_+ \ _{\text{¸}} 1$. The coordinates are with the cubic constraint

$$V^1 \ ^o \ \frac{1}{6} \qquad = 1 \bullet \tag{2.4.1}$$

2. A quaternionic-Kähler manifold QM of real dimension $4=$ with coordinates $@^D$.

3. A set of $=_+ \ _{\text{¸}} 1$ Killing vectors $:^D$ (that could be linearly dependent, or vanish) on QM, compatible with the quaternionic-Kähler structure, representing the isometries to be gauged by the vector fields . Each Killing vector comes equipped with a triplet of moment maps ®.[16]

On the other hand, a 4d $N = 2$ Abelian gauged supergravity with $=_+ \ _{\text{¸}} 1$ vector multiplets and $=$ hypermultiplets — that we summarize in Appendix B.2 — is specified by the following data (see for instance [201, 202]).

1. A special Kähler manifold KM of complex dimension $=_+ \ _{\text{¸}} 1$, with coordinates I and $= 1\text{-}\bullet\bullet\bullet\text{-}=_+ \ _{\text{¸}} 1$. We will work in a duality frame in which the geometry is specified by holomorphic sections - $^1 I^o$, with $= 0\text{-}\bullet\bullet\bullet\text{-}=_+ \ _{\text{¸}} 1$, and a holomorphic prepotential $^1\text{-}^o$, homogeneous of degree two.

2. A quaternionic-Kähler manifold QM of real dimension $4=$ with coordinates $@^D$.

3. In duality frames in which all gaugings are purely electric, a set of $=_+ \ _{\text{¸}} 2$ Killing vectors $:^D$ (that could be linearly dependent, or vanish) on QM, compatible with the

[16] If $= \ = 0$, instead, one has to specify $=_+ \ _{\text{¸}} 1$ Fayet-Iliopoulos parameters Z , not all vanishing.

quaternionic-Kähler structure, representing the isometries to be electrically gauged by the vector fields . Each Killing vector comes equipped with a triplet of moment maps (see footnote 16).

We reduce the 5d theory on a circle, which will eventually be the Hopf fiber of $(^3$. Following [118, 203–207] we use the ansatz

$$\begin{aligned} 3B^2_{15^o} &= 4^{2} 3B^2_{14^o} \,_{\text{¸}}\, 4^{4} \; 3H \quad {}^{0}_{14^o} \quad {}^{2} \\ &= \quad 4^{2} \operatorname{Im} I \;\bullet \end{aligned} \tag{2.4.2}$$

Here H is the direction of the circular fiber, that we take with range $4c\bullet 6$ in terms of the coupling $6 = \;_5{}^1$, inversely proportional to the AdS$_5$ radius $_5$, therefore the size of the circle is $4^{\;2} 4c\bullet 6$. Because of the constraint $V^1 \;\; {}^o = 1$ in (2.4.1), the field is redundant and can be eliminated with

$$4^{\;6} = \;\; V^1 \operatorname{Im} I \;{}^o \bullet \tag{2.4.3}$$

On the other hand, ${}^{0}_{14^o}$ is the Kaluza-Klein (KK) vector. As noted in [118, 208], a Scherk-Schwarz (SS) twist for the gravitino as in [206] is necessary to satisfy the BPS conditions in 4d. We prefer to work in a gauge in which all bosonic fields are periodic around the circle, but there are flat gauge connections b turned on along H. This corresponds to the ansatz

$$_{15^o} = \;\; _{14^o} \,_{\text{¸}}\, \operatorname{Re} I \quad 3H \quad {}^{0}_{14^o} \,_{\text{¸}}\, b \;\, 3H \,\text{-} \tag{2.4.4}$$

together with no H-dependence for any field. Notice that this ansatz is invariant under the redefinitions

$$I \;\; ! \;\; I \;\; _{\text{¸}} Xb \;\; \text{-} \qquad _{14^o} \;\; ! \quad _{14^o} \,_{\text{¸}}\, Xb \quad {}^{0}_{14^o} \,\text{-} \qquad b \;\; ! \;\; b \qquad Xb \tag{2.4.5}$$

where Xb are real parameters. We will fix this redundancy below. The reduction of the 5d theory can be found in Appendix C.1. The 4d data in terms of 5d ones are as follows.

1. The special Kähler manifold in 4d is described by the prepotential

$$_1 \text{-} {}^o = \frac{1}{6} \qquad \frac{\check{-}\;\check{-}\;\check{-}}{\text{-}\,{}^0} \qquad \text{with} \qquad \check{-} \;=\; \text{-} \;\; _{\text{¸}}\, b \;\; \text{-}^{\,0} \bullet \tag{2.4.6}$$

The holomorphic sections - can be used as homogeneous coordinates, and the physical scalars are identified with the special coordinates $I \;\; = \text{-} \;\; \bullet \text{-}^{0}$.

2. The quaternionic-Kähler manifold in 4d is the same as in 5d.

3. The 4d Killing vectors $:^D$ are inherited from 5d, while the additional Killing vector is

$$:^D_0 = b \; :^D \qquad) \qquad \text{\%}_0 = b \; \text{\%} \; - \tag{2.4.7}$$

   and is gauged by the KK vector field $^{0}_{14^o}$.

Next, we study the attractor equations for the near-horizon limit of 4d BPS static black-hole solutions [172–174]. Our goal is to use the BPS equations to fix the VEVs in massive vector multiplets and hypermultiplets, and be left with an extremization principle for the scalars in massless vector multiplets, similarly to [43, 64]. We consider the near-horizon geometry AdS$_2$ $(^{2}:

$$3B^2_{\text{near-horizon}} = \frac{A^2}{!^2_{\mathrm{A}}} 3C^2 \; \text{¸} \; \frac{!^2_{\mathrm{A}}}{A^2} 3A^2 \; \text{¸} \; !^2_{\mathrm{S}} \, 3B^2_{(^2} \; - \tag{2.4.8}$$

where $!_{\mathrm{A}}$ and $!_{\mathrm{S}}$ are the radii of AdS$_2$ and $(^2$, respectively. Electric and magnetic charges are defined as appropriate integrals over $(^2$ in the near-horizon region, respectively:

$$@ \; = \frac{6}{4c} \int^{1}_{(^2} 16c \; {}^{14^o}_{\mathrm{N}} \frac{X(_{4\mathrm{d}}}{X} \; - \qquad\qquad ? \; = \frac{6}{4c} \int^{1}_{(^2} \; \bullet \tag{2.4.9}$$

Here ${}^{14^o}_{\mathrm{N}}$ is the 4d Newton constant, related to the 5d one by

$$\frac{4c}{6 \; {}^{15^o}_{\mathrm{N}}} = \frac{1}{{}^{14^o}_{\mathrm{N}}} \; - \tag{2.4.10}$$

while $(_{4\mathrm{d}}$ is the 4d supergravity action. The 4d black holes we are interested in have both electric and magnetic charges. The magnetic charge $?^0 = 1$ is equal to the first Chern class of the Hopf fibration. On the other hand, we fix the redundancy (2.4.5) by setting the remaining magnetic charges to zero. In Appendix C.2 we compute the relation of the 5d charges $\&$ and angular momentum measured at infinity, with the 4d charges measured at the horizon. We should be careful that only massless vector fields are associated with conserved charges. We indicate as B the matrix of linear redefinitions such that B ` are the 5d mass eigenstates in the AdS$_5$ vacuum, and we take the index T to run only over the massless vectors B$^{\mathrm{T}}$ `. The corresponding conserved charges are $\&_{\mathrm{T}}$ $\&$ ^{1}B ^{1o} $_{\mathrm{T}}$. We find

$$\begin{aligned} ?^0 &= 1 \, - \qquad & @_0 &= 4 \; {}^{14^o}_{\mathrm{N}} 6^2 \; \text{¸} \; \frac{1}{3} \quad b \; b \; b \; - \\ ? \; &= 0 \, - \qquad & @_{\mathrm{T}} &= 4 \; {}^{14^o}_{\mathrm{N}} 6^2 \&_{\mathrm{T}} \; \text{¸} \; \frac{1}{2} \; \mathrm{T} \quad b \; b \; - \end{aligned} \tag{2.4.11}$$

where $_1 = \; _2$, while the "non-conserved charges" $@_{<T}$ will be fixed by the equations of motion. Notice that the conserved charges $\&_T$ are the same, but possibly expressed in a different basis, as the charges $\&_0$ introduced in Sections 2.2.2 and 2.2.3.[17]

Using a symplectic covariant notation, electric and magnetic charges form a symplectic vector

$$Q = {}^1? \; -@ \; {}^0 \bullet \tag{2.4.12}$$

One also defines

$$\mathcal{P} = {}^10\text{-}\,\mathcal{G}\; {}^0\,- \qquad\qquad \mathcal{Q} = \langle \mathcal{P}\text{-}\, Q\rangle\, - \tag{2.4.13}$$

where vectors are triplets and $\langle + \text{-} \,,\, \rangle = + \;\; , \qquad + \;\; ,$ is the symplectic-invariant antisymmetric form.

To find covariantly-constant spinors, we impose the following twisting ansatz:

$$n_8 = \; \mathcal{Q} \;\; \mathcal{T}_8{}^9 \;\; {}^{CA} n_9 \, - \tag{2.4.14}$$

whose square gives $\mathcal{Q} \;\; \mathcal{Q} = 1$. Here ^{CA} is the antisymmetric product of two gamma matrices with flat indices C and A. We choose a gauge in which $Q^1 = Q^2 = 0$ and

$$Q^3 = \;\; 1 \tag{2.4.15}$$

at the horizon, as in [43].

The remaining BPS conditions are in general complicated, but they simplify at the horizon [172–174]. First, Maxwell's equations give

$$K^D \;\; {}_{DE} \langle K^E\text{-}\, Q \rangle = 0\, - \tag{2.4.16}$$

where we defined

$$K^D = {}^10\text{-}\, :^D {}^0 \tag{2.4.17}$$

because we work in a duality frame with purely electric gaugings. In fact, (2.4.16) in this case is equivalent to

$$? \;\; :^D = 0\, - \tag{2.4.18}$$

that must hold in the full solution simply because of spherical symmetry (see Appendix C.2). Second, the vanishing of the hyperino variation implies

$$\langle K^D\text{-}\, V \rangle = 0\, - \tag{2.4.19}$$

[17] Similarly, the restriction of $\quad$ to $_{TJK}$ with curly indices is the same, but possibly in a different basis, as the 't Hooft anomaly coefficients $\;\; _{012}$ previously defined.

where $V^1 I\text{-}\bar{I}^0 = 4^{K \bullet 2 1}$ - - ^{0} is the covariantly-holomorphic section defined in (B.2.3) and $= m$ ^{1} - ^{0}. Third, we have the attractor equations[18]

$$\frac{m}{m I} \frac{Z}{L} = 0 \text{-} \qquad \frac{Z}{L} = 2 8 6^2 !^2_S \text{-} \tag{2.4.20}$$

where the derivatives are with respect to the physical scalars I and we defined

$$Z = \text{h}Q\text{-}V\text{i} \text{-} \qquad L = \text{h}P^3\text{-}V\text{i} \bullet \tag{2.4.21}$$

The equation on the right in (2.4.20) determines $!_S$, and thus the horizon area.

2.4.2 Example: the conifold

We apply the general strategy to the case of the conifold. We start with the 5d $N = 2$ gauged supergravity with $=_+ = 2$ vector multiplets and $= \ = 2$ hypermultiplets constructed in Section 7 of [169] (called the "second model" in that paper), obtained as a consistent truncation of 10d type IIB supergravity on $)^{1\text{-}1}$ that preserves the $\mathrm{SU}^1 2^{02}$ $\mathrm{U}^1 1^0$ isometry. In Appendix B.1.1 we have recast its action as in the general formalism, and in Appendix C.1.1 we have reduced it down to 4d $N = 2$ gauged supergravity. We are now ready to look for BPS near-horizon black-hole solutions.

Using (B.1.46) and (B.1.47), the conditions (2.4.15) and (2.4.18) take the form:

$$\left(\begin{array}{l} \%^3_0 = 1 \\ :^D_0 = 0 \end{array} \right) \qquad 1_{1\text{-}2} = 2_{1\text{-}2} = 0 \text{-} \qquad b^1 = b^2 = \frac{1}{3} \text{-} \tag{2.4.22}$$

where $1_{1\text{-}2}\text{-}2_{1\text{-}2}\text{-}0\text{-}q\text{-} \ _0\text{-}D$ are the scalar fields in hypermultiplets. In fact, since (2.4.18) must hold in the whole solution, so (2.4.22) does. Using the form (B.1.47) of the moment maps, this is consistent with $Q^1 = Q^2 = 0$. The hyperino condition (2.4.19) then gives

$$\text{-}^1 \ _\text{¸} \ \text{-}^2 = 0 \tag{2.4.23}$$

at the horizon, where - are the holomorphic sections. The fields $_0$ and q are not fixed by the equations of motion. However, together they form the axio-dilaton of type IIB supergravity and are thus fixed by the boundary conditions that set them in terms of the complexified gauge coupling of the boundary theory. As apparent from the expression of $:^D_2$ in (B.1.46), 0 is a

18There is an extra factor of 2 in front of $!^2_S$ compared to [42, 43, 201] due to the different normalization of kinetic terms in the Lagrangian (B.2.2): this is noticed in footnote 4 of [206] and in footnote 10 of [118].

Stückelberg field that breaks an Abelian gauge symmetry and is eaten up as the corresponding gauge field becomes massive via the Higgs mechanism.

The remaining BPS conditions are the attractor equations (2.4.20). Given in (B.1.42), the prepotential is

$$1 - 0 = \frac{\check{-}1 \; 1 \check{-} 2o2 \quad 1 \check{-} 3o2}{-0} \qquad \text{where} \qquad \check{-} = - \;_{\text{¸}} b \; -^{0} \bullet \tag{2.4.24}$$

Using special coordinates $I = - \bullet -^{0}$ as well as homogeneity of the prepotential $1 - 0$, one can easily show that the two equations in (2.4.20) are equivalent to

$$m \; 4^{K\bullet 2} \Big[Z 1 - 0 \quad 2 8 6^2 !^2_S L 1 - 0 \Big] = 0 - \tag{2.4.25}$$

where the derivatives are with respect to independent sections $-$. In these equations $!_S$ should be regarded as one of the unknowns. Notice that (2.4.20) or (2.4.25) give, in general, isolated solutions in terms of $1 I - !_S 0$, however the sections $-$ are only fixed up to the "gauge" redundancy (related to Kähler transformations on $K M$) $-$ $!$ $4^5 -$. To remove the redundancy, we choose to fix $L 1 - 0$ to a constant, which can elegantly be imposed by taking a derivative of the square bracket in (2.4.25) with respect to $!^2_S$ as well. More precisely, expanding Z and L using (B.1.47), we consider the following set of equations:

$$\begin{aligned} m & \; \frac{-1 \; 1 - 2o2 \quad 1 - 3o2}{1 - 0o2} \;_{\text{¸}} \mathbb{D} - \quad 2 8 6^2 !^2_S \; 3 -^1 \quad -^0 \quad 24^{\;4D1 - 1} \;_{\text{¸}} -^{2o} \quad U = 0 \\ \frac{m}{m !^2_S} & \; \frac{-1 \; 1 - 2o2 \quad 1 - 3o2}{1 - 0o2} \;_{\text{¸}} \mathbb{D} - \quad 2 8 6^2 !^2_S \; 3 -^1 \quad -^0 \quad 24^{\;4D1 - 1} \;_{\text{¸}} -^{2o} \quad U = 0 \end{aligned} \tag{2.4.26}$$

where

$$\mathbb{D} = @ \; \frac{1}{2} \; b \; b \; - \qquad \mathbb{D}_0 = @_0 \; \frac{1}{3} \; b \; b \; b \; \bullet \tag{2.4.27}$$

The first line is the same as (2.4.25), except for the addition of the constant U that does not affect the equations. The second line fixes the gauge $L = U$. Notice that (2.4.23) should be imposed after solving (2.4.26).

From the point of view of AdS/CFT, only massless vector fields correspond to symmetries of the boundary theory and only their charges are conserved and fixed by the boundary conditions. On the contrary, the "charges" under massive vector fields are not conserved, and their radial profile should be determined by the equations of motion. The spectrum of the 5d supergravity under consideration around its supersymmetric AdS$_5$ vacuum was computed in [169] and we report it in our conventions in (B.1.49). In the basis

$$: ' \quad \tfrac{1}{3}{}^{1} :_1 \quad :_2{}^{0} - \qquad :_3 - \qquad : , \quad \tfrac{1}{3}{}^{1} 2 :_1 \, ¸ \, :_2{}^{0} - \tag{2.4.28}$$

it turns out that ' (corresponding to the R-symmetry) and ^{3} are massless, while ' is massive because of Higgs mechanism eating up the Stückelberg field 0. In (2.4.28) we have indicated also the Killing vectors of the corresponding gauged isometries. On the black-hole background, the mass eigenstates may change (because the gauge kinetic functions have a non-trivial radial profile), however the fact that

$$:' = :_3 = 0 \tag{2.4.29}$$

everywhere — which follows from (2.4.22) — guarantees that there is no hypermultiplet source in the 5d Maxwell equations (C.2.4) and thus the Page charges $\&'$ and $\&_3$ are conserved (while $\&_,$ is not).

Indeed, the variation in (2.4.26) with respect to $-^2$ gives the complex equation

$$2 \frac{-^1 -^2}{1 - 0o2} \, ¸ \, @_2 \, ¸ \, 4 8 6^2 \, !^2_S \, 4^{4D} = 0 \tag{2.4.30}$$

that fixes D and the "non-conserved charge" $@_2$ in terms of the sections and $!_S$. We can then use the hyperino condition (2.4.23) to eliminate $-^2$ as well. Notice that the second condition in (2.4.22) implies that in 5d we cannot turn on a "flat connection" for ' along the circle.

We are left with the unknowns $-^0 - -^1 - -^3 - !^2_S$. One can check that, when (2.4.23) and (2.4.30) are in place, the remaining equations in (2.4.26) are equivalent to the conditions of extremization of the function

$$S = V \left[\frac{-^1 \, 1 - 1o2 \quad 1 - 3o2}{1 - 0o2} \, ¸ \, @_0 -^0 \, ¸ \, 3 @' -^1 \, ¸ \, @_3 -^3 \quad 2 8 6^2 \, !^2_S \, 3 -^1 \quad -^0 \quad U \right] \tag{2.4.31}$$

with respect to the variables $-^0 - -^1 - -^3 - !^2_S$. Here V is a constant included for later convenience, while $@'$ is the charge with respect to the massless vector $_R$:

$$@' = \frac{@_1 \quad @_2}{3} = \frac{6}{4c} \, (^2 \, 16c \, {}^{14^o}_N \frac{X(_{4d}}{X \, '} \; \frac{1}{6} \quad 1 \quad 2 \quad b \; b \; = 46^2 \, {}^{14^o}_N \& ' \, \bullet \tag{2.4.32}$$

It is encouraging that we find an extremization problem in which only conserved charges appear. Since S is homogeneous in $-$ of degree 1 except for the term involving U, it follows that $S_{\text{crit}} = 28UV6^2\,!^2_S$ at the critical point. With the choice

$$UV = \frac{c}{28 \quad {}^{14^o}_{N} 6^2} \tag{2.4.33}$$

we obtain that S_{crit} is the black-hole entropy:

$$S_{\text{crit}} = \frac{4c\,!^2_S}{4 \quad {}^{14^o}_{N}} = (_{BH} \text{-} \tag{2.4.34}$$

and therefore S is the entropy function. Using (2.4.11) and (2.4.27) we can express the 4d charges $\mathbb{D}_0$, $\mathbb{D}_T$ computed at the horizon in terms of the 5d black-hole charges , $\&_T$ computed at infinity:

$$S = \frac{1}{U}\frac{c}{28 \quad {}^{14^o}_{N} 6^2}\frac{1 \text{-} 1o3 \quad \text{-} 11 \text{-} 3o2}{1 \text{-} 0o2} \quad 2c8 \quad \text{-}^0 \,{}_{\text{¸}}\, 3\&\,{}_{\text{،}}\, \text{-}^1 \,{}_{\text{¸}}\, \&_3 \text{-}^3 \,{}_{\text{¸}}$$

$$2c8 \quad 3\text{-}^1 \quad \text{-}^0 \quad U \quad \text{-} \tag{2.4.35}$$

where we redefined the Lagrange multiplier $!^2_S = 28 \quad {}^{14^o}_{N}$ for convenience.

It remains to spell out the AdS/CFT dictionary between gravity and field theory charges. First, the gauge group ranks in field theory are determined by (see Appendix C.2.1)

$$\#^2 = \frac{8c}{27 \quad {}^{15^o}_{N} 6^3} = \frac{2}{27 \quad {}^{14^o}_{N} 6^2} \bullet \tag{2.4.36}$$

This is in agreement with (2.3.2) using $0 = 27\#^2 \bullet 64$ for the Klebanov-Witten theory. Second, the angular momentum is the same in gravity and field theory. Third, the electric charges are identified as

$$A = 2\&\,{}_{\text{،}}\, \text{-} \qquad \& \;= \frac{4}{3}\&_3 \bullet \tag{2.4.37}$$

This is determined as follows. From (B.1.36) we infer that the gravitino components have charge $\&\,{}_{\text{،}} = \;\; 1\bullet 2$. In the boundary field theory, the corresponding operators are of the schematic form $\text{Tr}^1 \; \grave{}_a \; {}^a\_{}^o$ (where is a field strength and _ a gaugino) and have charge $A = \;\; 1$ under $\text{U}^1 1^o{}_{\text{،}}$. We deduce the first relation in (2.4.37). Obtaining the second relation is more subtle because no supergravity field is charged under ^{3}: what is charged are massive particles obtained from D3-branes wrapped on the 3-cycle of $)^{1\text{-}1}$, corresponding to dibaryon operators $\#_{1\text{-}2}$ or $\#_{1\text{-}2}$ in field theory. The 5d supergravity gauge field ^{3} comes from the reduction of the Ramond-Ramond field strength $_5^{\mathrm{RR}}$ of 10d type IIB supergravity on $)^{1\text{-}1}$. Therefore, from the 10d flux quantization condition we can deduce the 5d charge quantization condition $4\&_3 \bullet 3\# \; 2 \; Z$ (see the details in Appendix C.2.1). In field theory the dibaryon operators have charge $\& \;\; = \;\; \#$, implying the second relation in (2.4.37). Alternatively, we could compare the Chern-Simons terms restricted to massless vector fields in the 5d Lagrangian with the 't Hooft anomalies of the boundary theory. Taking into account the 't Hooft anomalies

$$\mathrm{Tr}^1 A^{3\mathrm{o}} = \frac{3}{2}\#^2 - \qquad\qquad \mathrm{Tr}^1 A\&^{2\,\mathrm{o}} = \;\; 2\#^2 - \tag{2.4.38}$$

at leading order in $\#$, the restriction of the 5d Chern-Simons action in (B.1.2) to $' \;\; ! \;\; 0$ matches the general expression

$$(_{\mathrm{CS}} = \frac{6^3}{24c^2}^{\;\;1} \;\; \mathrm{Tr}^1\&_0\&_1\&_2{}^{\mathrm{o}} \;\; {}^0\wedge \;\; {}^1\wedge \;\; {}^2 \tag{2.4.39}$$

after setting $' \;\; ! \;\; 2 \;\; {}_A$ and $^3 \;\; ! \;\; 4 \;\; \bullet 3$. These correspond to (2.4.37).

Rewriting the entropy function (2.4.35) in terms of field theory charges, we find

$$S = \frac{1}{U}\;" \;\; \frac{27c8\#^2}{4} \, \frac{{}^1 - {}^1\mathrm{o}3 \quad - {}^1{}^1 - {}^3\mathrm{o}2}{{}^1 - {}^0\mathrm{o}2} \quad 2c8 \quad -{}^0 \, , \, \frac{3}{2}A - {}^1 \, , \, \frac{3}{4}\& \;\; -{}^3 \, ,$$

$$\# \qquad 2c8 \quad 3 - {}^1 \quad -{}^0 \quad U \quad \bullet \tag{2.4.40}$$

This exactly matches the entropy function (2.2.36) we found in field theory from the large $\#$ limit of the superconformal index of the Klebanov-Witten theory, after the change of coordinates $-{}^0 \; ! \; 2Ug$, $-{}^1 \; ! \; 2U - {}' \bullet 3$, $-{}^3 \; ! \; 4U - \; \bullet 3$.

Chapter 3

A quantum mechanics for magnetic horizons

In this Chapter, we try to construct an effective 1d theory describing the black-hole near-horizon degrees of freedom. It is organized as follows. In Section 3.1 we re-examine the large $\#$ limit of the topologically twisted index by performing a saddle-point approximation, both in the integration variables as well as in the sum over fluxes. This analysis appeared recently in [65]. Section 3.2, which is the most technical one, is devoted to the dimensional reduction of the 3d theory on $(^2$ in the presence of gauge magnetic fluxes. This reduction involves a judicious choice of gauge fixing. In Section 3.3 we exhibit the effective $N = 2$ gauged quantum mechanics we are looking for; the hurried reader who is only interested in the final result can directly jump there. Finally, in Section 3.4 we comment on which type of classical and quantum corrections to our analysis one might expect.

3.1 Saddle-point approach to the TTI

We begin by re-examining the evaluation of the topologically twisted index of 3d $N = 2$ gauge theories at large $\#$. The localization formula for the index found in [72] involves a sum over gauge fluxes $\mathfrak{m}$ on $(^2$, as well as a contour integral in the space of complexified gauge connections D on $(^1$. At large $\#$, we apply a saddle-point approximation both to the integral over D as well as to the sum over fluxes, treated as a continuous variable $\mathfrak{m}$. The idea to compute a supersymmetric index in this way was put forward, for instance, in [57, 209] (see also [65, 210, 211]).[1] The upshot is to identify a specific gauge flux sector that dominates the

[1] In particular, the evaluation of the (refined) topologically twisted index of the specific model studied here, through a saddle-point approximation of the sum over fluxes, has recently already appeared in [65].

index and, via holography, the BPS black-hole entropy. In Section 3.2 we will use that flux sector to perform a reduction of the 3d theory on S^2 down to a quantum mechanics.

The analysis in this and the following sections are performed in a specific (and simple) model, presented in Section 3.1.2. This choice is made for the sake of concreteness, but other theories (for instance ABJM [212]) could be studied similarly.

3.1.1 The basic idea

We are interested in the topologically twisted index [72] of the theory, because this quantity is known to reproduce the entropy of a class of BPS AdS$_4$ magnetic black holes [43, 64]. The localization formula for the index can be written schematically as

$$I_{S^2\times S^1} = \frac{1}{|W|}\sum_{\mathfrak{m}\in\Gamma_\mathfrak{h}} \oint_{\mathcal{C}} \prod_{i=1}^{N} du_i\, e^{\mathfrak{m}\, iB(u) + \cdots(u)}\,. \tag{3.1.1}$$

Here $|W|$ is the order of the Weyl group, $\Gamma_\mathfrak{h}$ is the co-root lattice, N is the rank of the gauge group, and $\mathcal{C}$ is an appropriate integration contour for the complexified Cartan-subalgebra-valued holonomies $\{u_i\}\in\mathfrak{h}_\mathbb{C}/\Gamma_\mathfrak{h}$. Let us outline three different approaches to this expression at large N.

1. The approach developed in [72] was to resum over $\mathfrak{m}$, schematically

$$I_{S^2\times S^1} = \frac{1}{|W|}\oint_{\mathcal{C}} \prod_{i=1}^{N} du_i\, \frac{e^{\cdots(u)}}{1-e^{iB(u)}}\,, \tag{3.1.2}$$

then determine the positions $\bar u$ of the poles by solving the "Bethe Ansatz Equations" (BAEs)

$$e^{iB(\bar u)} = 1\,, \tag{3.1.3}$$

and, finally, take the residues

$$I^{\text{BAE}}_{S^2\times S^1} = \frac{1}{|W|}\sum_{\bar u\in\text{BAE}} \frac{(2\pi)^N e^{\cdots(\bar u)}}{i^N\, \partial B(\bar u)}\,. \tag{3.1.4}$$

2. Alternatively, we can evaluate both the sum over $\mathfrak{m}$ and the integral over u in (3.1.1) in the saddle-point approximation, treating $\mathfrak{m}$ as a continuous variable. The simultaneous

saddle-point equations for m and D are, schematically:

$$\begin{cases} 0 = +^{0}1\bar{D}^{o} \\ 0 = \bar{\mathsf{m}}+^{00}1\bar{D}^{o} \,¸\, ^{0}1\bar{D}^{o} \bullet \end{cases} \tag{3.1.5}$$

Taking into account that $+^{0}1D^{o}$ in (3.1.1) is defined up to integer shifts by $2c8$, the first set of equations is exactly the set of BAEs, while the second set of equations uniquely fixes $\bar{\mathsf{m}}$ in terms of $\bar{D}$. The Jacobian at the saddle point is

$$^{3d}1\mathsf{m}\text{-}D^{o} = \det \begin{pmatrix} 0 & +^{00}1D^{o} \\ +^{00}1D^{o} & \mathsf{m}+^{000}1D^{o} \,¸\, ^{00}1D^{o} \end{pmatrix} = \left(+^{00}1D^{o}\right)^{2} \bullet \tag{3.1.6}$$

Therefore, in the saddle-point approximation:

$$I^{\text{saddle}}_{(^2\ (^1} \;' \; \frac{1}{jWj} \sum_{\bar{D}2\text{saddles}} {}^{1}2c^{o\#} \frac{4^{\,1\bar{D}^{o}}}{\sqrt{3d}} = \frac{1}{jWj} \sum_{\bar{D}2\text{BAEs}} \frac{{}^{1}2c^{o\#}4^{\,1\bar{D}^{o}}}{8^{\#}+^{00}1\bar{D}^{o}} \bullet \tag{3.1.7}$$

This method gives the same answer as the previous method.

3. A more rough approximation is to fix m in (3.1.1) to the value determined by the equations (3.1.5),

$$I^{\text{fix}\,\bar{\mathsf{m}}}_{(^2\ (^1} \;' \; I_{(^1} \quad \frac{1}{jWj} \int_{C} \prod_{8=1}^{¼} 3D_8\, 4^{\bar{\mathsf{m}}+^{0}1D^{o}\,¸\,\, 1D^{o}} \text{-} \tag{3.1.8}$$

and then solve the integral in D in the saddle-point approximation. The saddle-point equations are $\bar{\mathsf{m}}+^{00}1D^{o} \,¸\, ^{0}1D^{o} = 0$, therefore all solutions $\bar{D}$ of (3.1.5) are also saddle points of (3.1.8). Assuming that there are no other solutions, we find

$$I_{(^1} \;' \; \frac{1}{jWj} \sum_{\bar{D}2\text{BAEs}} {}^{1}2c^{o\#} \frac{4^{\,1\bar{D}^{o}}}{\sqrt{1d}} \bullet \tag{3.1.9}$$

The Jacobian in this case is

$$^{1d} = \bar{\mathsf{m}}+^{000}1\bar{D}^{o} \,¸\, ^{00}1\bar{D}^{o} = +^{00} \frac{0\ \ 0}{+^{00}}\, 1\bar{D}^{o} \text{-} \tag{3.1.10}$$

and is different from before, however, as long as the Jacobian is subleading with respect to the exponential contribution, this approach captures the leading behavior.

In our setup we will find a series of saddle points ¹$\bar{D}$- $\bar{\mathrm{m}}$º, and the expression $I_{(^1}$ in (3.1.8) evaluated on the dominant one will turn out to be the Witten index of an effective quantum mechanics we will construct. To do that, we will first have to find the saddle-point flux $\bar{\mathrm{m}}$ and then reduce the 3d theory on $(^2$ in the presence of such a flux.

3.1.2 The 3d CS-matter model

We consider the AdS/CFT pair discovered in [175], that was used in [43, 64] to study certain magnetic black holes in massive type IIA on AdS$_4$ $(^6$ [176–178]. The field theory is a 3d $N = 2$ Chern-Simons-matter theory with gauge group SU¹#º$_:$, coupled to three chiral multiplets $_{=1\text{-}2\text{-}3}$ in the adjoint representation. We can simplify the computation by considering a U¹#º$_:$ gauge theory, with no sources for the new topological symmetry. No field is charged under U¹1º U¹#º and thus the only effect of this is to introduce a decoupled sector, whose Hilbert space on $_g$ consists of $:^g$ states, which is a single one in the case of $(^2$. The theory has a superpotential[2]

$$, = \_{3\mathrm{d}}\,\mathrm{Tr}\quad {}_1 » \;\; {}_2\text{-}\;\; {}_3¼ • \tag{3.1.11}$$

The global symmetry is SU¹3º U¹1º$_'$. We parameterize its Cartan subalgebra with three R-charges ', characterized by the charge assignment ' ¹ º = 2X . We choose the Cartan generators of the flavor symmetry to be $@_{1\text{-}2}$ = ¹ '$_{1\text{-}2}$ '$_3$º•2. In this basis, all fields have integer charges. Notice that $4^{8c'}$ = ¹ 1º for = 1- 2- 3.

To study AdS$_4$ BPS magnetic black holes, we place the theory on[3] $(^2$ R using a topological twist on $(^2$, so that one complex supercharge is preserved [213]. This is precisely the background of the topologically twisted index in [72]. In other words, there is a background gauge field $_'$ corresponding to an R-symmetry that is equal and opposite to the spin connection when acting on the top component of the supersymmetry parameter n:

$$\frac{1}{2c}\int^{1}_{(^2} 3 \;_' = \; 1• \tag{3.1.12}$$

The R-symmetry used for the twist must have integer charge assignments, and a generic such R-charge can be written as

$$@_' = '_3 \quad \mathrm{n}_1 @_1 \quad \mathrm{n}_2 @_2\, \text{-} \qquad \mathrm{n}_{1\text{-}2}\ 2\ \mathbb{Z}\,;• \tag{3.1.13}$$

2Notice that this theory is just the dimensional reduction on $(^1$ of the 4d $N = 4$ SYM theory we studied in Section 2.1, deformed by an $N = 2$ supersymmetric Chern-Simons term.

3One could also study the theory on a Riemann surface $_g$ [179, 180], but here we will focus on the sphere.

Note that $\sum$ ¹@ ' º = 2 and the superpotential correctly has R-charge 2. Under these inequivalent twists, the scalar component of experiences a flux

$$n = {}^1@\,{}'\,{}^{\circ} \int_{(^2} \frac{3\ '}{2c} = {}^1\,'_3{}^{\circ} \ ¸\ n_1{}^1@_1{}^{\circ} \ ¸\ n_2{}^1@_2{}^{\circ} \ ;\bullet \tag{3.1.14}$$

This formula defines n_3 2 n_1 n_2. Thus, twisting by a generic R-symmetry with integer charge assignments is the same as twisting with respect to $'_3$ and simultaneously turning on background gauge fields $_{1\text{-}2}$ coupled to the flavor charges $@_{1\text{-}2}$ with

$$\frac{1}{2c} \int_{(^2} 3\ _{1\text{-}2} = n_{1\text{-}2}\,\bullet \tag{3.1.15}$$

The theory that we are considering has a UV Lagrangian consisting of various building blocks which are individually supersymmetric off-shell. The vector multiplet + (in Wess-Zumino gauge) contains the adjoint-valued fields ¹ f -_-_- `- º, where f is a dynamical real scalar field and a real auxiliary field. We consider a supersymmetrized Chern-Simons Lagrangian for it, but we also add the SYM Lagrangian as a regulator. The chiral multiplets contain the adjoint-valued fields ¹q - - 5 º, for which we consider the kinetic Lagrangian and the superpotential term. These Lagrangians, in Lorentzian signature and Wess-Zumino gauge, are:

$$L_{\mathrm{YM}} = \frac{1}{24^2_{\mathrm{3d}}} \mathrm{Tr} \ \frac{1}{2}\ `_a\ `^a \quad `f\ \ `f\ ¸\ \ ^2\ \ 8\_\ /\ \ \ f\ \_\ - \tag{3.1.16}$$

$$L_{\mathrm{CS}} = \frac{:}{4c} \mathrm{Tr} \ \ n^{`ad} \ \ `m_a\ \ d \ \ \frac{28}{3}\ \ `\ \ a\ \ d \ \ 8\_\ \ \ 2\ \ f\ \ -$$

$$L_{\mathrm{chiral}} = \ `q^y\ \ `q \ \ \ q^y\ f^2\ ¸\ \ \ q\ ¸\ 5^y 5 \ \ \ 8\ \ \ /\ ¸\ f \ \ \ ¸\ 8\ \ \_q\ ¸\ 8q^y\_ \ \ \ -$$

$$L_{\mathrm{W}} = \frac{m\ ,}{mq}\ 5\ ¸\ \frac{1}{2}\frac{m^2\ ,}{mq\ mq}\ \ \overline{\ \ 2} \quad ¸\ \text{c.c.}\ -$$

where we used the convention ^{2} $8 f_1$ for the conjugated spinor. The superpotential must be a gauge-invariant holomorphic function of R-charge 2. The supersymmetry variations preserved by these Lagrangians, in terms of a single Dirac spinor n, are:

$$\begin{array}{lll} \& q = 0 & \& \ = 8W^{`}\ \ `q \ \ 8fq\ \ n & \& \ = n^2\, 5 \\ \& q = \ \bar{n} & \& \ \bar{} = \ \bar{n}\ \ 8W^{`}\ \ `q^y\ ¸\ 8q^y f & \& \bar{} \ = \ \overline{n^2}\, 5^y \\ \& q^y = \bar{}\ \ n & \& 5 = \ \overline{n^2}\ \ 8W^{`}\ \ `\ \ ¸\ 8f \ \ \ 8\_q & \& 5 = 0 \\ \& q^y = 0 & \& 5^y = \ 8\ \ `\bar{}W^{`}\ \ 8\bar{}\ f\ ¸\ 8q^y\_\ \ n^2 & \& 5^y = 0 \end{array} \tag{3.1.17}$$

and

$$\begin{aligned}
&\& \;` = \frac{8}{2}\bar{\ }W`n && \&_ = \frac{1}{2}W`^{a}\;`a\,¸\,8\quad 8W`\;`f\;n && \&_ = 0\\
&\& \;` = \frac{8}{2}\bar{n}W`_ && \bar{\&}_ = \bar{n}\;\frac{1}{2}W`^{a}\;`a\,¸\,8\;¸\,8W`\;`f && \bar{\&}_ = 0\\
&\& f = \frac{1}{2}\bar{\ }n && \& = \frac{1}{2}\;`\bar{\ }W`\quad f\bar{\ }\,n &&\\
&\& f = \frac{1}{2}\bar{n}_ && \& = \frac{1}{2}\bar{n}\;W`\quad `_\quad f_\;\bullet &&
\end{aligned} \tag{3.1.18}$$

To obtain a microscopic description of the BPS black-hole entropy, one counts the ground states of this theory. It is convenient to work in the grand canonical ensemble, in which one introduces a set of chemical potentials $\;,\; = 1\text{-}2$ for each flavor Cartan generator. As for the fluxes (and as done in Section 2.1), it is useful to introduce a third chemical potential $_3$, constrained because of supersymmetry [42], such that

$$_1\,¸\;\; _2\,¸\;\; _3\; 2\; \mathbb{Z}\,\bullet \tag{3.1.19}$$

All chemical potentials are only defined modulo 1. Computing the thermal partition function is hard because the theory is strongly coupled in the IR, therefore one can start from a quantity protected by supersymmetry: the topologically twisted index

$$I_{3\mathrm{d}}{}^{1}n\text{-}\;{}^{o} = \mathrm{Tr}\,{}^{1}\;1^{o}\;4^{\;V}\;4^{2c8\;\acute{\imath}\;@}\;\;\text{-} \tag{3.1.20}$$

where is the Fermion number, the Hamiltonian on the sphere $(^2$ in the presence of the magnetic fluxes (3.1.12)-(3.1.15), and the trace is over the Hilbert space of states. This quantity only gets contributions from the ground states of the theory. It was argued in [41], exploiting the $\mathrm{su}^{1}1\text{-}1|1^{o}$ superconformal symmetry algebra expected to emerge from the AdS_2 $(^2$ near-horizon region in gravity, that the BPS states of a pure single-center black hole have constant statistics $^{1}\;1^{o}$ in each charge sector, meaning that the index gets non-interfering contributions (at least at leading order in $\#$) and can account for the black-hole entropy.[4]

[4] This expectation was confirmed for rotating black holes in AdS_5 in [152]. For asymptotically-flat black holes, the argument was first formulated in [18, 28].

The Topologically twisted index (3.1.20) can be computed exactly using supersymmetric localization techniques [72, 179], and for the model considered here one obtains [43, 64]:

$$Z_{3d}(\mathfrak{n},\Delta)=\frac{(-1)^N}{N!}\prod_{a=1}\left(\frac{y^{N^2(\mathfrak{n}+1)/2}}{(1-y)^{N(\mathfrak{n}+1)}}\right)\sum_{\mathfrak{m}\in\mathbb{Z}^N}\oint_{JK}\prod_{i=1}^{N}\frac{dx_i}{2\pi i x_i}x_i^{k\mathfrak{m}_i}$$
$$\times\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)\prod_{a=1}\prod_{i\neq j}\left(\frac{x_i-y\,x_j}{x_j-y\,x_i}\right)^{\mathfrak{m}_i}\left(1-y\frac{x_i}{x_j}\right)^{\mathfrak{n}-1}. \qquad (3.1.21)$$

Here $x_i \equiv e^{2\pi i u_i}$ and $y \equiv e^{2\pi i \Delta}$. This expression can be conveniently compiled into the same form as (3.1.1):

$$Z_{3d}(\mathfrak{n},\Delta)=\frac{1}{N!}\sum_{\mathfrak{m}\in\mathbb{Z}^N}\oint_{JK}\prod_{i=1}^{N}du_i\; e^{i\left(\sum_i \mathfrak{m}_i B'_i(u,\Delta)+V(u,\mathfrak{n},\Delta)\right)}. \qquad (3.1.22)$$

The two functions appearing in the exponent are

$$\sum_{i=1}^{N}\mathfrak{m}_i B'_i(u,\Delta)=\sum_{i=1}^{N}\mathfrak{m}_i\left(\sum_{j=1}^{N}\sum_{a=1}\left[\mathrm{Li}_1\left(e^{2\pi i(u_{ji}-\Delta)}\right)-\mathrm{Li}_1\left(e^{2\pi i(u_{ji}+\Delta)}\right)\right]\right.$$
$$\left.+\,2\pi i k u_i+i\pi N-2\pi n_i\right), \qquad (3.1.23)$$

$$V(u,\mathfrak{n},\Delta)=\sum_{a=1}(1+\mathfrak{n})\sum_{i\neq j}\mathrm{Li}_1\left(e^{2\pi i(u_{ji}+\Delta)}\right)-\sum_{i<j}\mathrm{Li}_1\left(e^{2\pi i u_{ji}}\right) \qquad (3.1.24)$$
$$+\,i\pi N^2\sum_{a=1}(1+\mathfrak{n})+\pi i(2\tilde{n}+N).$$

where $u_{ji}=u_j-u_i$ whilst n_i and $\tilde{n}$ are integer ambiguities. The JK integration contour is the so-called Jeffrey-Kirwan residue [214]. We used the poly-logarithm function

$$\mathrm{Li}_1(x)=-\log(1-x). \qquad (3.1.25)$$

while more properties can be found in Appendix A.1.

3.1.3 The large *N* limit of the TTI

To obtain the saddle-point equations, we first formulate (3.1.22) in a large N continuum description as in [215] and subsequently take functional derivatives. The Weyl symmetry permuting the discrete Cartan subalgebra index i can be used to order the holonomies u_i such that $\mathrm{Im}\, u_i$ increases with i. The discrete index i is then substituted with a continuous variable $t \in [t_-, t_+]$, after which u and the flux $\mathfrak{m}$ become functions of t. The reparameterization symmetry in t is fixed by identifying, up to normalization, t with $\mathrm{Im}\, u(t)$:

$$u(t) = N^{\alpha}\, i t + v(t) \,. \tag{3.1.26}$$

This introduces the density

$$\rho(t) = \frac{1}{N}\frac{di}{dt}\,, \qquad \sum_i \to N \int^{\infty} dt\, \rho(t)\,, \tag{3.1.27}$$

in terms of which any sum will be replaced by an integral. The density ρ must be real, positive, and integrate to 1 in the defining range.

We perform the large N computation in Appendix A.3.2. In (A.3.19) and (A.3.20) we find:

$$\begin{aligned} \int^{\infty} dt\, \mathfrak{m}\, \Delta' &= 2\pi i k N \int^{\infty} dt\, \rho\, \mathfrak{m}\, u + 2\pi i N^{2-2\alpha} g(\Delta) \int^{\infty} dt\, \frac{\mathfrak{m}'\rho^2}{(1 - i v')^2} + O\left(\mathfrak{m} N^{2-3\alpha}\right), \\ &= 2\pi N^{2-\alpha} f_+(\mathfrak{n}, \Delta) \int^{\infty} dt\, \frac{\rho^2}{1 - i v'} + O\left(N^{2-2\alpha}\right), \end{aligned} \tag{3.1.28}$$

where we introduced the functions

$$g(\Delta) = \frac{1}{6}\sum_{a=1}^{3} B_3(\Delta_a)\,, \qquad f_+(\mathfrak{n}, \Delta) = \frac{1}{2}\sum_{a=1}^{3} (1 + \mathfrak{n}_a)\left(B_2(\Delta_a) - B_2(0)\right), \tag{3.1.29}$$

where B_2 and B_3 are the Bernoulli polynomials defined in (A.1.13). The entire exponent in the integrand of (3.1.22) is the functional:

$$\begin{aligned} V = {} & 2\pi i k N^{1+\alpha} \int^{\infty} dt\, \rho\, \mathfrak{m}\, (it + v) + 2\pi i N^{2-2\alpha} g(\Delta) \int^{\infty} dt\, \frac{\mathfrak{m}'\rho^2}{(1 - i v')^2} + {} \\ & 2\pi N^{2-\alpha} f_+(\mathfrak{n}, \Delta) \int^{\infty} dt\, \frac{\rho^2}{1 - i v'} + 2\pi N^{2-\alpha} \mu \left(\int^{\infty} dt\, \rho - 1 \right), \end{aligned} \tag{3.1.30}$$

where we added a Lagrange multiplier μ to enforce the normalization of ρ. For the terms in V to compete, we need $\alpha = \frac{1}{3}$ and $\mathfrak{m} \propto N^{\frac{1}{3}}$.

To find the saddle-point configurations at large $\#$, we extremize V with respect to d, E, m and $`$. After some massaging, the saddle-point equations are:

$$0 = \frac{3}{3C} \; 2 \; \frac{\mathrm{m}\, d}{1 \quad 8E} \quad \#^{\frac{1}{3}} \, ` \; {}^{1}8C \, {}_{\text{¸}} \, E^{\text{o}} \quad {}_{\text{¸}} \, 28\#^{\frac{1}{3}} \, 5_{\text{¸}} \; d \, - \tag{3.1.31}$$

$$0 = d\,\mathrm{m} \quad \frac{28}{:} \; \frac{3}{3C} \; \frac{\mathrm{m}\, d^2}{{}^{1}1 \quad 8E^{\text{o}3}} \quad {}_{\text{¸}} \; \frac{5_{\text{¸}}}{} \, d\, D \, - \tag{3.1.32}$$

$$0 = \frac{3}{3C} \; : \; {}^{1}8C \, {}_{\text{¸}} \, E^{\text{o}2} \quad 48 \; \frac{d}{1 \quad 8E} \quad - \tag{3.1.33}$$

together with $3C\, d = 1$. One can check that the functional V is invariant under reparametrizations of C that preserve the scaling ansatz (3.1.26) for the holonomies. Such reparameterizations act as:

$$\begin{aligned} C &= C^{1}C^{0\text{o}} \, - & E^{1}C^{\text{o}} &= 8 \; C^{0} \quad C^{1}C^{0\text{o}} \quad {}_{\text{¸}} \, E^{0\,1}C^{0\text{o}} \, - \\ d^{1}C^{\text{o}} &= \frac{3C^{1}C^{0\text{o}}}{3C^{0}} \;{}^{1}\; d^{0\,1}C^{0\text{o}} \, - & \mathrm{m}^{1}C^{\text{o}} &= \mathrm{m}^{0\,1}C^{0\text{o}} \, \bullet \end{aligned} \tag{3.1.34}$$

Notice in particular that E^{0} becomes complex after the transformation.

As we review in Appendix A.3.3, the equations (3.1.31)–(3.1.33) can be solved, yielding:

$$D^{1}C^{\text{o}} = \frac{3\#}{:}^{\frac{1}{3}} \, C \, - \quad \mathrm{m}^{1}C^{\text{o}} = \frac{\#}{9: \quad 2}^{\frac{1}{3}} \, 5_{\text{¸}} \, C \, - \quad d^{1}C^{\text{o}} = \frac{3}{4} \; 1 \quad C^{2} \quad - \quad C \; 2 \; » \; 1\text{-}1¼ \, \bullet \tag{3.1.35}$$

This solution is obtained after making use of the reparameterization symmetry, so, in particular, $E^{1}C^{\text{o}}$ is complex. The value of the functional V at the saddle point for d, E and m — which reproduces the logarithm of the index at leading order — is

$$V = \frac{2c8\#^{\frac{5}{3}}}{5} \; \frac{9:}{1 \quad \text{o}}^{\frac{1}{3}} \; 5_{\text{¸}} \, {}^{1}\mathrm{n}\text{-} \quad {}^{\text{o}} \, \bullet \tag{3.1.36}$$

If $\acute{\imath}$ $= 1$, the two functions and $5_{\text{¸}}$ take the particularly simple form

$$1 \quad {}^{\text{o}} = \frac{1}{2} \; 1 \;\; 2 \;\; 3 \, - \qquad 5_{\text{¸}} \, {}^{1}\mathrm{n}\text{-} \quad {}^{\text{o}} = \frac{1}{2} \; 1 \;\; 2 \;\; 3 \; \sum_{=1}^{\text{③}} \frac{\mathrm{n}}{} \, \bullet \tag{3.1.37}$$

In this case, the saddle-point value of the logarithm of the index is

$$V = \frac{2c8\#^{\frac{5}{3}}}{5} \; \frac{9:}{4}^{\frac{1}{3}} \; 1 \;\; 2 \;\; 3^{\frac{2}{3}} \; \sum_{=1}^{\text{③}} \frac{\mathrm{n}}{} \, \bullet \tag{3.1.38}$$

When the ’s are real this expression matches the result of [43, 64], which reproduces the black-hole entropy upon performing a Legendre transform.

As mentioned above, the chemical potentials are defined modulo 1. The expression for V in (3.1.36), however, is not periodic under ! ¸ 1. This means that we have found an infinite number of saddle points, parameterized by the shifts.[5] This suggests that — as in AdS_3 [184] and AdS_5 [185] — there might be an infinite number of complex BPS black-hole-like supergravity solutions dual to the semi-classical expansion of the topologically twisted index. This issue deserves more study. In the following, we will assume that we have identified the dominant saddle point, and we will work with it.

3.2 Reduction on a flux background

The next step is to perform a Kaluza-Klein (KK) reduction of the 3d $N = 2$ gauge theory on the sphere $(^2$, in the presence of the flux background m (3.1.35) determined as the saddle point of the topologically twisted index. By keeping only the light modes, we will obtain a 1d quantum mechanical model which we expect to contain information about the near-horizon degrees of freedom of the magnetic AdS_4 black holes we are interested in. This section is rather technical, and the reader only interested in the final result can directly jump to Section 3.3.

Here we will first show how the full twisted theory can be seen as a gauged $N = 2$ quantum mechanics. Afterward, we will introduce the background of the reduction and review the standard procedure to fix the 3d gauge group down to the 1d gauge group. We will then explain why complications arise when computing the KK spectrum of the vector multiplet, and how they can be resolved by a further modification of the gauge-fixing Lagrangian. Lastly, we will exhibit the KK spectra of the vector and chiral multiplets.

3.2.1 Decomposing 3d multiplets into 1d multiplets

After the topological twist, the theory exactly fits into the framework of a gauged $N = 2$ quantum mechanics, and we perform various changes of variables in this section to make it explicit. A similar discussion can be found in [216]. We give a brief review of 1d $N = 2$ supersymmetry in Appendix E, adapted from [217], but in E.2.3 and E.2.4 we also present new supersymmetric Lagrangians peculiar to our 3d theory.

5 In general, only a subset of the complex saddle points contribute to the contour integral: which ones do (depending on the contour) should be determined with the steepest descent.

We shall write the supersymmetry transformations in terms of anticommuting generators $\&$ and $\overline{\&}$, with the understanding that generators should be multiplied by a complex anti-commuting parameter to produce a generic supersymmetry transformation. With $n = {}^1 1 \text{-} 0^{o\mathsf{T}}$, $\&$ is obtained from $\&_{3\text{d}}$ while $\overline{\&}$ is obtained from $\&_{3\text{d}}$ in (3.1.17) and (3.1.18). Note that $\&$ and $\overline{\&}$ are related by Hermitian conjugation, that is $\overline{{}^1\& \text{-} {}^o} = {}^1\ 1^o\ \ \overline{\&}\ \overline{\text{-}}$. The supersymmetry algebra is

$$\&^2 = \overline{\&}^2 = 0\ \text{-} \qquad \mathrm{f}\&\text{-}\overline{\&}\mathrm{g} = 8\ m_{\mathrm{C}}\ \ X_{\text{gauge}}{}^1\ {}_{\mathrm{C}}\ {}_{\text{¸}}\ f^{\,o}\ \ \text{-} \tag{3.2.1}$$

where $X_{\text{gauge}}{}^1 U^o$ is a gauge transformation with parameter U. We will use frame fields $4^{\bar{1}}$, $4^{\bar{\bar{1}}}$ on $(^2$, which we introduce in Appendix D, and write differential forms on $(^2$ with flat indices $1\text{-}\bar{1}$. From now on, $\overline{\text{-}}$ will denote the Hermitian conjugate of - (since Dirac conjugates are no longer present anyway). After this rewriting, the supersymmetry variations and supersymmetric Lagrangians are described below.

Vector multiplet. In the Wess-Zumino gauge, the 3d vector multiplet consists of the gauge field $\ ^{\backprime}$, a real scalar f, a real auxiliary scalar $\ $, and a Dirac spinor $\_$. The bosonic components are R-neutral while $\_$ has R-charge $\ 1$. We decompose $\_$ in components as

$$\_ = \begin{matrix} \overline{\ \ }_{\mathrm{C}} \\ \bar{1} \end{matrix}\ \text{-} \tag{3.2.2}$$

and redefine with a shift

$$^0 = \quad 28\ {}_{1\bar{1}}\ \bullet \tag{3.2.3}$$

Now, $\ _{\bar{1}}$ has R-charge $\ 1$ whereas $\ _{\mathrm{C}}$ has R-charge $\text{¸}1$. These field redefinitions have trivial Jacobian. Under the supercharges preserved by the twist, the supersymmetry variations of the vector multiplet split into 2 sets of variations. The first set (Hermitian conjugate relations being implied) is:

$$\begin{aligned} &\&\ {}_{\mathrm{C}} = \ \&\,f = \frac{8}{2}\overline{\ \ }_{\mathrm{C}}\ \text{-} \qquad &&\&\ {}_{\mathrm{C}} = \ {}_{\mathrm{C}}f\ \ 8\ \ \text{-} \\ &\&\ \ = \frac{1}{2}{}^1\ {}_{\mathrm{C}}\ \ 8f^{\,o}\overline{\ \ }_{\mathrm{C}}\ \text{-} \qquad &&\overline{\&}\ {}_{\mathrm{C}} = 0\ \bullet \end{aligned} \tag{3.2.4}$$

These coincide with the supersymmetry variations (E.2.32) of a 1d $\mathrm{U}^1\#^o$ vector multiplet in the Wess-Zumino gauge. Note that here the fields and gauge transformations are also functions on $(^2$. The second set is:

$$\&\ {}_{\bar{1}} = \frac{1}{2}\ {}_{\bar{1}}\ \text{-} \qquad \overline{\&}\ {}_{\bar{1}} = 0\ \text{-} \qquad \&\ {}_{\bar{1}} = 0\ \text{-} \qquad \overline{\&}\ {}_{\bar{1}} = 28\ m_{\mathrm{C}}\ {}_{\bar{1}}\ \ \ {}_{\bar{1}}{}^1\ {}_{\mathrm{C}}\ {}_{\text{¸}}\ f^{\,o}\ \bullet \tag{3.2.5}$$

These coincide with the supersymmetry variations (E.2.34) of a chiral multiplet $\bar{1} \text{-} \frac{1}{2} \; \bar{1}$ in Wess-Zumino gauge, provided that the corresponding superfields

$$\bar{1}\text{-} = \bar{1} \text{¸} \frac{\backslash}{2} \; \bar{1} \quad \frac{8}{2} \backslash \bar{\backslash} m_c \; \bar{1} \text{-} \qquad 1\text{-}^{-} \quad \overline{\bar{1}\text{-}} = 1 \quad \frac{\bar{\backslash}\,-}{2} \, 1 \text{¸} \frac{8}{2} \backslash \bar{\backslash} m_c \; 1 \text{-} \tag{3.2.6}$$

satisfying $\overline{} \; \bar{1}\text{-} = \quad 1\text{-}^{-} = 0$, transform as connections under super-gauge transformations:

$$\bar{1}\text{-} \; ! \quad \bar{1}\text{-} \text{¸} 8m_{\bar{1}} \; ^{1} \text{-} \qquad 1\text{-}^{-} \; ! \; ^{-\,1} \; 1\text{-}^{-} \text{¸} 8m_1 \; ^{-} \text{-} \tag{3.2.7}$$

with $= 4^j$ and $\overline{} j = 0$. We indicated as $\overline{}_1$ the complex conjugate to $\bar{1}$.

The Yang-Mills Lagrangian is composed of two pieces, independently supersymmetric:

$$24^2_{3d} L_{\rm YM} = \text{Tr} \; 4 \; c\bar{1}^{\,2} \text{¸} 48 \; 1\bar{1} \; 4 \; \bar{1} f^{\,2} \text{¸} 8 \overline{} 1^1 \; c \text{¸} 8 f^{\circ} \; \bar{1} \text{¸} 2 \; c \; 1 \; \bar{1} \; 2 \overline{} 1 \; \bar{1} \; \bar{c} \text{¸}$$
$$\text{¸} \, \text{Tr} \; ^1 \; c f^{\circ 2} \text{¸} \; ^2 \text{¸} 8 \overline{} c^1 \; c \; 8 f^{\circ} \; c \; \bullet \tag{3.2.8}$$

Notice also that

$$24^2_{3d} L_{\rm YM} = \&\bar{\&}\,\text{Tr} \; 48 \; 1 m_c \; \bar{1} \text{¸} 48^1 \; c \; f^{\circ} \; 1\bar{1} \text{¸} \&\bar{\&}\,\text{Tr} \; \bar{c} \; c \text{-} \tag{3.2.9}$$

so both terms are exact. The first piece is the appropriate kinetic term for a chiral transforming as a connection and its superspace expression is in (E.2.52). The second piece is the standard 1d gauge kinetic term (E.2.42). Likewise, the Chern-Simons Lagrangian splits into two pieces which are separately supersymmetric:

$$\frac{4c}{:} L_{\rm CS} = \text{Tr} \, \text{h} \, 48 \; 1 m_c \; \bar{1} \; 48^1 \; c \text{¸} f^{\circ} \; 1\bar{1} \text{¸} \overline{} 1 \; \bar{1} \, \text{i} \text{¸} \text{Tr} \, \text{h} \overline{} c \; c \; 2 \; f \, \text{i} \bullet \tag{3.2.10}$$

The superspace expression of the first piece is given in (E.2.60), whereas the second piece matches (E.2.46).

Chiral multiplet. A 3d chiral multiplet consists of a complex scalar q and a Dirac spinor . We split into components as

$$= 8 \; \begin{matrix} k \\ [\end{matrix} \; \bullet \tag{3.2.11}$$

Their R-charges are $'^1 k^{\circ} = '^1 [^{\circ} = '^1 q^{\circ} \quad 1$. Under the supercharges preserved by the twist, the supersymmetry variations of the 3d chiral multiplet can also be organized into two sets.

The first set (Hermitian conjugate relations are again implicit) is:

$$Q\phi = \psi\,, \qquad \bar{Q}\phi = 0\,, \qquad Q\psi = 0\,, \qquad \bar{Q}\psi = i(D_t - i\sigma)\phi\,. \tag{3.2.12}$$

They coincide with the supersymmetry variations (E.2.34) of a 1d chiral multiplet (ϕ, ψ) in Wess-Zumino gauge, with corresponding superfield

$$\Phi = \phi + \theta\psi - \frac{i}{2}\theta\bar{\theta}D_t\phi\,. \tag{3.2.13}$$

The second set is instead:

$$Q\eta = F\,, \quad \bar{Q}\eta = -2D_{\bar{1}}\phi\,, \quad QF = 0\,, \quad \bar{Q}F = -i(D_t - i\sigma)\eta - 2D_{\bar{1}}\psi + iD_{\bar{1}}\phi\,. \tag{3.2.14}$$

They match the variations (E.2.36) of a 1d Fermi multiplet (η, F) in Wess-Zumino gauge, whose corresponding superfield

$$Y = \eta - \theta F + 2\bar{\theta}D_{\bar{1}}\phi + \theta\bar{\theta}\left(-\frac{i}{2}D_t\eta - 2D_{\bar{1}}\psi + iD_{\bar{1}}\phi\right) \tag{3.2.15}$$

satisfies

$$\bar{D}Y = E_{\bar{1}}\,, \qquad E_{\bar{1}} = 2D_{\bar{1}}\Phi\,. \tag{3.2.16}$$

Here $D_{\bar{1}}$ contains the background $U(1)'$ connection. In the language of 1d supersymmetry, there is an E-term superpotential for Y. After the shift (3.2.3), the kinetic term of a 3d chiral multiplet also splits into two separately supersymmetric pieces, *i.e.*, the kinetic terms of the 1d chiral (E.2.47) and of the 1d Fermi (E.2.50):

$$\begin{aligned} L_{\text{chiral}} = {} & \Big[|D_t\phi|^2 - |\sigma\phi|^2 - \bar{\phi}D\phi + i\bar{\psi}(D_t + i\sigma)\psi - i\bar{\psi}\bar{\lambda}\phi + i\bar{\phi}\lambda\psi \Big] \\ & + \Big[i\bar{\eta}(D_t - i\sigma)\eta + \bar{F}F - |2D_{\bar{1}}\phi|^2 - 2\bar{\psi}D_{\bar{1}}\eta + 2\bar{\eta}D_{\bar{1}}\psi - i\bar{\eta}\lambda_{\bar{1}}\phi + i\bar{\phi}\bar{\lambda}_{\bar{1}}\eta \Big]\,. \end{aligned} \tag{3.2.17}$$

Note that one has

$$L_{\text{chiral}} = Q\bar{Q}\left(i\bar{\phi}(D_t - i\sigma)\phi\right) + Q\bar{Q}\left(\bar{\eta}\eta\right)\,, \tag{3.2.18}$$

so that both terms are exact.

The superpotential terms can be written as

$$L_W = -Q\left(\eta\frac{\partial W}{\partial\phi} + \bar{Q}\left(\bar{\eta}\frac{\partial\bar{W}}{\partial\bar{\phi}}\right)\right)\,, \tag{3.2.19}$$

which in the language of 1d supersymmetry are J-terms for the Fermi multiplets $[$ with $= \frac{m\,,}{mq}$. Supersymmetry of the first term under $\&$, and the second term under $\overline{\&}$, are obvious. When $\overline{\&}$ acts on the first term we get, up to a total time derivative,

$$\&\overline{\&}\ [\ \frac{m\,,}{mq} \;=\; 2\&\quad {}_{\bar{1}}q\ \frac{m\,,}{mq} \;=\; 2\&^{1}m_{\bar{1}}\,,^{\circ} = \;2m_{\bar{1}}\&\,,\; - \qquad (3.2.20)$$

which is another total derivative. Thus the superpotential terms are $\&\ {}_{\text{¸}}\ \overline{\&}$ -exact. The supersymmetric Chern-Simons Lagrangian is the only piece that is not exact under any supercharge.

3.2.2 Reduction background

As mentioned at the beginning of this section, we want to reduce the theory in the presence of background fluxes for the global symmetries. In particular, we turn on a (negative) unit flux for the R-symmetry $@\,'$. Since it is a background for a non-dynamical field, it can be off-shell without any consequences. The presence of this background, under which the chiral multiplets are differently charged, generically breaks the $\mathrm{SU}^{1}3^{\circ}$ flavor symmetry down to its diagonal subgroup $\mathrm{U}^{1}1^{\circ 2}$. We also single out a configuration of fluxes for the dynamical gauge fields:

$$_{1\bar{1}} = \frac{8\,\mathrm{m}}{4\,'^{2}} - \qquad \text{where } \mathrm{m} \text{ is a constant in the Cartan subalgebra.} \qquad (3.2.21)$$

The choice of m will eventually be the one dictated by the saddle-point approximation to the topologically twisted index, discussed in Section 3.1. Since $_{1\bar{1}}$ couples to the auxiliary field in (3.2.8) like a FI parameter, the D-term equation for supersymmetric vacua is:

$$\frac{28}{4^{2}_{3\mathrm{d}}}\ {}_{1\bar{1}}\ {}_{\text{¸}}\ \tilde{\mathrm{O}}\ \text{»}\bar{q}\ \text{-}\ q\ \text{¼}\ \ \frac{:}{2c}\,f\ = 0\ \bullet \qquad (3.2.22)$$

The background should satisfy the D-term equation to be supersymmetric, and it is simplest to turn on a background for f to cancel the background flux. This falls into the class of "topological" vacua discussed in [218]. Moreover, since $_{c}\ {}_{\text{¸}}\ f$ appears in the algebra (3.2.1), we also find it appropriate to turn on a background for $_{c}$, opposite to that of f, so that the background of $_{c}\ {}_{\text{¸}}\ f$ is zero. This ensures that BPS states have zero energy even before projecting onto gauge singlets. Thus, the background we use for the reduction is:

$$_{1\bar{1}} = \frac{8\mathrm{m}}{4\,'^{2}} - \qquad f\ = \frac{\mathrm{m}}{2<_{:}\,'^{2}} - \qquad _{c} = \frac{\mathrm{m}}{2<_{:}\,'^{2}} - \qquad \text{where} \quad <_{:}\quad \frac{:\,4^{2}_{3\mathrm{d}}}{2c}\ \bullet \qquad (3.2.23)$$

One can check that all the equations of motion are satisfied on this background, except for that of $c ¸ f$, unless $\mathsf{m} = 0$. Consequently, when expanding the action, there will be a Lagrangian term linear in $c ¸ f$, that is

$$\mathrm{Tr}\ \frac{:\mathsf{m}}{4c\,'^2}\ ¹\ c\ ¸\ f^{o}\ • \tag{3.2.24}$$

In other words, background fluxes produce a background electric charge in the presence of Chern-Simons terms. As we will discuss later, the presence of this linear term is crucial and it is the main source of complications when computing the vector multiplet spectrum.

We parameterize the Lie algebra $\mathrm{su}¹\#^{o}$ by $\#\ \ \#$ matrices $_{89}$ (8- 9 = 1-•••-$\#$) which have a single nonzero entry 1 in row 8 and column 9: $¹\ \ _{89}{}^{o}{}_{:;} = X_{8:}X_{9;}$. Elements with $8 = 9$ are a basis for the Cartan subalgebra, while those with $8 < 9$ correspond to roots with root vector $¹U_{89}{}^{o}{}_{:} = X_{:8}\ \ X_{:9}$. The commutation relations in this basis are

$$»\ \ _{89}-\ \ _{:;}¼ = X_{9:}\ \ _{8;}\ \ \ X_{8;}\ \ _{:9}\ • \tag{3.2.25}$$

Note also that $\overline{\ \ _{89}} = \ \ _{98}$ and

$$\mathrm{Tr}\ \ _{89}\ \ _{:;} = X_{9:}X_{8;}\ -\qquad \mathrm{Tr}\ \ _{89}»\ \ _{:;}-\ \ _{<=}¼ = X_{9:}X_{;<}X_{=8}\ \ \ X_{8;}X_{9<}X_{:=}\ • \tag{3.2.26}$$

We write the expansion of adjoint fields in this basis as $- = -^{89}\ \ _{89}$. Note that $\overline{-^{89}} = \overline{-^{98}}$. The Cartan components will sometimes be written as $-^{8}\ \ \ -^{88}$ for simplicity.

In the presence of global and gauge fluxes, the Lie algebra components of various fields in the vector multiplet and chiral multiplets are $\mathrm{U}¹1^{o}{}_{\text{spin}}$ sections with different monopole charges $@$ (see Appendix D for details). A field $j_{@}¹C- \backslash- i^{o}$ with monopole charge $@$ can then be expanded in a complete set of monopole harmonics $._{@-;-<}¹\backslash- i^{o}$, and the time-dependent expansion coefficients $j_{@-;-<}¹C^{o}$ are the 1d fields after the reduction:

$$j_{@}¹C- \backslash- i^{o} = \sum_{;\ \ j@j}\ \sum_{j<j\ \ ;} j_{@-;-<}¹C^{o}\ ._{@-;-<}¹\backslash- i^{o}\ • \tag{3.2.27}$$

Defining the quantities

$$@_{89}\ \ \ \frac{\mathsf{m}_8\ \ \ \mathsf{m}_9}{2}\ -\qquad @_{89}\ \ \ \frac{\mathsf{m}_8\ \ \ \mathsf{m}_9\ ¸\ \mathsf{n}}{2}\ - \tag{3.2.28}$$

the monopole charges of the fields and their charges under the global symmetries of the theory are summarized in Table 3.1.

VM	f^{89}- $^{89}_{C}$- ^{89}	$^{89}_{C}$	$\overline{^{89}_{1}}$	$^{89}_{1}$	$\overline{^{89}_{1}}$	$\overline{^{89}}_{1}$
@	$@_{89}$	$@_{89}$	$@_{89}$ ¸ 1	$@_{89}$ 1	$@_{89}$ ¸ 1	$@_{89}$ 1
@ '	0	1	0	0	1	1
$@_1$	0	0	0	0	0	0
$@_2$	0	0	0	0	0	0

CM	q^{89}	k^{89}	$[^{89}$	5^{89}
@	$@_{89}$	$@_{89}$	$@_{89}$ ¸ 1	$@_{89}$ ¸ 1
@ '	n	n 1	n 1	n 2
$@_1$	X_1 X_3	X_1 X_3	X_1 X_3	X_1 X_3
$@_2$	X_2 X_3	X_2 X_3	X_2 X_3	X_2 X_3

Table 3.1 Monopole and global charges of all fields. The R-charge is @ ', while $@_{1\text{-}2}$ are flavor charges. Above: modes from 3d vector multiplets. The modes are defined for pairs 8- 9 such that $@_{89}$ ¡ 0. Below: modes from 3d chiral multiplets, defined for any 89. In both cases, the modes are in SU¹2º representations with ; j@j and ; = @ mod 1.

We assume that $m_8 < m_9$, 8 8 < 9, since this is true for the saddle-point flux, and thus $@_{89} < 0$ for 8 < 9. Given a Hermitian adjoint field - = -^{89} $_{89}$ = $\overline{-}$ in a vector multiplet (*i.e.*, $_C$, f,), its components satisfy -^{98} = $\overline{-^{89}}$. We parameterize the off-diagonal components in terms of complex fields -^{89} with 89 such that $@_{89}$ ¡ 0. For complex adjoint fields . = .^{89} $_{89}$ in vector multiplets (*i.e.*, $\overline{_1}$, $_1$, $\overline{_1}$, $\overline{\ }_1$), we initially parameterize the off-diagonal components in terms of complex fields .^{89}, $\overline{.}^{89}$ with 89 such that $@_{89}$ ¡ 0. For complex adjoint fields in chiral multiplets, instead, we simply use all components .^{89}.

The flux breaks the gauge group U¹#º to its maximal torus U¹1º$^{\#}$, and the 1d gauge group will consequently be U¹1º$^{\#}$. Indeed, the generators of 1d gauge transformations have to be constant on $(^2$, however the components n^{89} of the gauge-transformation parameter have monopole charges $@_{89}$, and since ; j$@_{89}$j, only those in the Cartan subalgebra have an ; = 0 mode which is constant on $(^2$.

3.2.3 Partial gauge fixing

To reduce to a gauged quantum mechanics, we need to fix the 3d gauge group to the unbroken 1d gauge group, consisting of time-dependent transformations that are constant on $(^2$. A systematic procedure to achieve this, which we review here, was presented in [219] to which we refer for more details. Let G be the infinite-dimensional group of gauge transformations, and f4 g a Hermitian basis for its algebra g. Denote the structure constants of g as »4 -4 ¼ = 8 5 4 . The basis f4 g is also chosen such that it is orthonormal under the inner product

$$\text{Tr}\,{}^{1}4\;4\;{}^{\text{o}} = X \quad • \tag{3.2.29}$$

Let R G be a subgroup, which will be the group of residual gauge transformations after partial gauge fixing. We call its algebra r g (r stands for residual). We split the basis as f4 g = f4$_8$-4$_0$g, where f4$_8$g is a basis for r whereas f4$_0$g is a basis for f g•r (f stands for gauge fixed).[6] Since R is a subgroup, r is a subalgebra and »r-r¼ r, or $5_{89O} = 0$. By anti-symmetry of the structure constants this implies $5_{8O9} = 0$, or »r-f¼ f. In summary, the algebra of g decomposes as

$$»4_8\text{-}4_9¼ = 8\,5_{89:}\,4_:\,\text{-} \qquad »4_8\text{-}4_O¼ = 8\,5_{8O1}\,4_1\,\text{-} \qquad »4_O\text{-}4_1¼ = 8\,5_{O18}\,4_8\;¸\;8\,5_{O12}\,4_2\;• \tag{3.2.30}$$

In particular, this implies that the 4_O's transform under the adjoint action in a real orthogonal representation of R, which we call $'_5$.

To fix G to R, we need to choose as many gauge-fixing conditions as there are generators in f. In other words, we need to choose gauge-fixing functions $^{O}_{\text{gf}}{}^{1}\text{-}^{\text{o}}$, where - collectively denotes the physical fields in chiral and vector multiplets. Notice that $^{O}_{\text{gf}}{}^{1}\text{-}^{\text{o}}$ should transform in $'_5$ under R. This is true for all the gauge-fixing functions we can think of. The first step in the gauge-fixing procedure is to integrate in an adjoint scalar 2 g, and add $\frac{1}{2}\,\text{Tr}\;\;{}^{2}$ to the action. Notice that will have mass dimension » ¼ = 3•2. Since is completely decoupled from everything else, introducing it does not change the path integral. We then insert 1 in the path integral, written as

$$1 = {}^{1}\text{-}\text{-}\;{}^{\text{o}}\int_G^{1} D6 \prod_O X\;{}^{O}_{\text{gf}}{}^{1}\text{-}^{6}{}^{\text{o}}\;{}^{1}\;{}^{6}{}^{\text{o}}{}^{O}\;\text{-} \tag{3.2.31}$$

where superscripts $^{1\;\text{o}6}$ denote a finite gauge transformation by 6. Suppose that $6_{\text{-}\text{-}}$ 2 G satisfies

$$^{O}_{\text{gf}}{}^{1}\text{-}^{6_{\text{-}\text{-}}}{}^{\text{o}}\;{}^{1}\;{}^{6_{\text{-}\text{-}}}{}^{\text{o}O} = 0\;\text{-} \tag{3.2.32}$$

then so does $A6_{\text{-}\text{-}}$ for any A 2 R, due to the covariant transformations of $^{O}_{\text{gf}}$ and ^{O} under R. Therefore, R remains as the residual gauge group. Notice that must transform under gauge transformations. This is different from the standard Faddeev-Popov procedure, in which is only integrated over at the very last step. That would have been sufficient if the gauge were completely fixed ($R = 0$). The slightly different procedure described here will

[6] In the Coulomb gauge (3.2.43), r contains diagonal transformations with ; = 0, while f contains diagonal transformations with ; ¡ 0 as well as all off-diagonal transformations.

produce extra interaction terms in the ghost action. Now, as usual, the invariance of $D6$ ensures that the determinant is gauge invariant, and

$$1-- \;o\; 1 = 1-6-- - 6-- o\; 1 = \int_G^1 D6 \; \ddot{O}_0 \; X \; {}^{0}_{\mathrm{gf}} 1 - 66-- o \; 1 \; 66-- o^0 \; \bullet \tag{3.2.33}$$

Assuming no Gribov copies and writing $6 = 1 \,¸\, n\; 4$, $X \quad X_{\mathrm{gauge}} 1 4 \; o$, one can expand the argument of the delta function to linear order in n and obtain

$$n^1 X_1 \quad {}_{\mathrm{gf}} 1 - 6-- o \quad 6-- \; {}^0 \bullet \tag{3.2.34}$$

The fact that the terms with n^8 disappear ensures that Vol$1 R o$ is factorized as an overall factor in the Faddeev-Popov determinant:

$$1-- \; o = \det X_1 \; {}^{0}_{\mathrm{gf}} 1 - 6-- o \; 1 \; 6-- o^0 \bullet \mathrm{Vol} 1 R o \bullet \tag{3.2.35}$$

The determinant can be shown to be well-defined on the coset $R6--$. Having determined $1--\; o$, inserting 1 in the path integral gives

$$\int^1 D-D \; D6 \; 4^{8(1-o \frac{8}{2} \; \mathrm{Tr} \; ^2} \; 1-- \; o \; \ddot{O}_0 \; X \; {}^{0}_{\mathrm{gf}} 1 - 6o \; 1 \; 6o^0 \bullet \tag{3.2.36}$$

Undoing the gauge transformation in the delta function, the integral over the gauge group factorizes and one gets

$$\int^1 D-D \; 4^{8(1-o \frac{8}{2} \; \mathrm{Tr} \; ^2} \det X_1 \; {}^{0}_{\mathrm{gf}} 1 - o \; X_1 \; {}^0 \; \ddot{O}_0 \; X \; {}^{0}_{\mathrm{gf}} 1 - o \; {}^0 \bullet \tag{3.2.37}$$

Using

$$X_1 \; {}^0 = 8 \; »4_1 - 4 \; ¼^0 = \; 5_1 \; {}_0 = \; 5_{018} \; {}^8 \; 5_{012} \; {}^2 \tag{3.2.38}$$

we can explicitly write:

$$\det X_1 \; {}^{0}_{\mathrm{gf}} 1 - o \; X_1 \; {}^0 = \int^1 \ddot{O}_0 \; D\text{æ}^0 D2^0 \; \exp \; \text{æ}^0 X_1 \; {}^{0}_{\mathrm{gf}} 1 - o \,¸\, 5_{018} \; {}^8 \,¸\, 5_{012} \; {}^2 \; 2^1 \; - \tag{3.2.39}$$

where we have introduced the Grassmann scalars 2^0, æ^0. Note that they are valued in f and not in g: modes corresponding to residual gauge transformations are not present. Also note that by dimensional analysis, » æ¼ ¸ »2¼ = » $_{\mathrm{gf}}$¼ = 3•2. Without loss of generality, we can take »2¼ = 0, » æ¼ = 3•2. Integrating out ^{8} and imposing the delta functions for ^{0}, one gets the action:

$$(^{1}\text{-}^{0}\,\text{¸}\quad{}^{1}\quad \mathrm{Tr}\quad \frac{{}^{2}_{\mathrm{gf}}}{2}\,\text{¸}\quad{}_{\mathrm{gf}}\; \text{â-}2\;\;\text{¸}\,8\,\text{â}X_{\mathrm{gauge}}{}^{1}2^{\mathrm{o}}\quad{}_{\mathrm{gf}}\,\text{¸}\,\frac{1}{2}\mathrm{f}\text{â-}2\mathrm{g}_{\mathrm{r}}\mathrm{f}\text{â-}2\mathrm{g}_{\mathrm{r}}\;\bullet \tag{3.2.40}$$

This is equivalent to the following action with extra scalars 1^{0} integrated in:

$$(^{1}\text{-}^{0}\,\text{¸}\quad{}^{1}\quad \mathrm{Tr}\;\frac{1^{2}}{2}\,\text{¸}\,1\quad{}_{\mathrm{gf}}\quad \mathrm{f}\text{â-}2\mathrm{g}\;\;\text{¸}\,8\,\text{â}X_{\mathrm{gauge}}{}^{1}2^{\mathrm{o}}\quad{}_{\mathrm{gf}}\,\text{¸}\,\frac{1}{2}\mathrm{f}\text{â-}2\mathrm{g}^{2}\;\bullet \tag{3.2.41}$$

Notice that 1^{0} have dimension »1¼ $= 3\bullet 2$. One should keep in mind that $2\text{-}\,\text{â-}\,1$ only contain modes in f. We will now rescale

$$_{\mathrm{gf}}\;\;!\;\;4^{1}_{3\mathrm{d}}\;\;{}_{\mathrm{gf}}\qquad 1\;\;!\;\;4^{1}_{3\mathrm{d}}\,1\,\text{-}\qquad 2\;\;!\;\;4^{1}_{3\mathrm{d}}\,2\,\text{-} \tag{3.2.42}$$

after which » $_{\mathrm{gf}}$¼ $= 2$, »2¼ $= 1\bullet 2$, and »1¼ $= 2$. The gauge-fixing action gains an overall factor of $1\bullet 4^{2}_{3\mathrm{d}}$. This is useful because the gauge-fixing function we chose, *i.e.*, the background Coulomb gauge, precisely has dimension 2 (like many other standard choices like Lorenz and background-field gauge). The explicit expression of this gauge-fixing function is

$$_{\mathrm{gf}} = \frac{2}{\mathrm{p}\overline{b}}\quad {}_{1}\;\;\bar{}_{1}\;\text{¸}\;\;\bar{}_{1}\;\;{}_{1}\;\;\text{-} \tag{3.2.43}$$

with »b¼ $= 0$. One can check that it leaves the 1d gauge group unfixed. The covariant derivatives above only contain the spin connection and monopole background. All in all, for any $_{\mathrm{gf}}$, the gauge-fixing procedure adds the following terms to the Lagrangian:

$$\frac{1}{4^{2}_{3\mathrm{d}}}\,\mathrm{Tr}\;\frac{1^{2}}{2}\,\text{¸}\,1\quad{}_{\mathrm{gf}}\quad \mathrm{f}\text{â-}2\mathrm{g}\;\;\text{¸}\,8\,\text{â}X_{\mathrm{gauge}}{}^{1}2^{\mathrm{o}}\quad{}_{\mathrm{gf}}\,\text{¸}\,\frac{1}{2}\,\mathrm{f}\text{â-}2\mathrm{g}^{2}\;\bullet \tag{3.2.44}$$

We can now define a BRST supercharge B as:

$$B\,\text{-}\;=X_{\mathrm{gauge}}{}^{1}2^{\mathrm{o}}\,\text{-}\;\text{-}\qquad B2=\frac{8}{2}\mathrm{f}2\text{-}2\mathrm{g}_{\mathrm{f}}\,\text{-}\qquad B\text{â}=81\,\text{-}\qquad B1=X_{\mathrm{gauge}}{}^{1\,\prime\,\mathrm{o}}\,\text{â-}\qquad{}^{\prime}\qquad\frac{1}{2}\mathrm{f}2\text{-}2\mathrm{g}_{\mathrm{r}}\bullet \tag{3.2.45}$$

One can check that

$$B^{2}=8\,X_{\mathrm{gauge}}{}^{1\,\prime\,\mathrm{o}}\,\text{-}\qquad\qquad B\;{}^{\prime}\;=0\,\bullet \tag{3.2.46}$$

This allows us to define an B-cohomology on invariants of the residual gauge group. The terms produced by gauge fixing can then be written in a BRST-exact form:

$$(3.2.44) = \frac{1}{4^2_{3d}} B\,\mathrm{Tr}\,\bar{2} \quad 8 \quad {}_{\mathrm{gf}} \quad \frac{8}{2} 1 \,\text{¸}\, \frac{8}{2} \mathrm{f}\bar{2}\text{-}2\mathrm{g} \qquad B \quad {}_{\mathrm{gf}} \,\bullet \tag{3.2.47}$$

We note that there is still complete freedom in specifying the inner product in the ghost sector, *i.e.*, the Hermiticity properties of 2 and $\bar{2}$. For the theory to be unitary and have a consistent Hamiltonian formulation [220], one needs that 2 and $\bar{2}$ are Hermitian, so that B is a real supercharge and (3.2.44) is real. With this choice, (3.2.44) is invariant under a ghost-number symmetry valued in R , which acts as:

$$2 \;7\!! \;\; 4^{U} 2\text{-} \qquad \bar{2} \;7\!! \;\; 4^{\;U} \bar{2}\text{-} \qquad B \;7\!! \;\; 4^{U} B\text{-} \tag{3.2.48}$$

with U 2 R. We say that 2 has ghost number $=_6 = 1$ and $\bar{2}$ has $=_6 =\;\; 1$. Physical observables are identified with the B-cohomology at $=_6 = 0$ since external states must be gauge invariant and cannot contain ghosts. Since 2, $\bar{2}$, and 1 are Hermitian, they are neutral under U¹1º ', and (3.2.44) is invariant under U¹1º ', since $\;\;{}_{\mathrm{gf}}$ is R-neutral.

3.2.4 Supersymmetrized gauge fixing

As anticipated, the linear term (3.2.24) causes complications in the computation of the KK spectrum of the vector multiplet, and the following discussion aims to explain why. The standard Faddeev-Popov gauge-fixing procedure we just reviewed generically breaks the supersymmetries that were defined on the original action because of the presence of the BRST-exact term $B \;\; {}_{\mathrm{gf}}$, which might not be supersymmetric. Considering a supercharge $\&$, and assuming that it does not act on the fields in the gauge-fixing complex, the transformation of $B \;\; {}_{\mathrm{gf}}$ is $\;\; B\& \;\; {}_{\mathrm{gf}}$. When computing B-closed (*i.e.*, gauge-invariant) quantities, this is harmless because the potentially violating term is B-exact, and it does not affect the result. For example, supersymmetric Ward identities (as we will show in Section 3.4) can be derived for any observable in the theory, since their correlators do not depend on B-exact terms.

However, the spectrum of the Chern-Simons-matter theory around a monopole background is not gauge invariant, because the quadratic action is not invariant under linearized BRST transformations.[7] This can be seen from the presence of the linear term (3.2.24). Its BRST variation is

$$B(_{1}1^{\circ} = \frac{1}{4c\,'^{2}}\,\mathrm{Tr}\; 8{:}\,\mathrm{m}\,\text{»}2\text{-} \quad {}_{c}\,\text{¸}\;\; f\text{¼} \quad \text{-} \tag{3.2.49}$$

7Although the BRST transformations are non-linear in the fields, to have a gauge-invariant spectrum, it would be enough that the quadratic action be invariant under the linearized transformations.

and it must cancel with the linearized BRST variation of the quadratic action, which is then nonzero. Consequently, there is no guarantee that the spectrum will be supersymmetric, because it is computed from a quadratic action that is not B-closed, and therefore B-exact terms violating supersymmetry cannot be neglected.

A way to resolve this issue takes inspiration from [221]. In addition to adding $B\ _{\text{gf}}$ to gauge fix our path integral, we can further add $Q\ _{\text{gf}}$. The real supercharge Q acts as $Q = \& \ \overline{\&}$ on physical fields, and we choose its action on the gauge-fixing complex such that $X\ \ {}^1B\ Q^o$ closes on symmetries and unfixed gauge transformations. We will show that the further addition of $Q\ _{\text{gf}}$ does not change the expectation value of any (possibly non-supersymmetric) operator O with ghost number $=_6\ \ 0$. In particular, physical observables with $=_6 = 0$ are not affected. At this point, we have added $X\ _{\text{gf}}$ to the original action. The real supercharge X is explicitly preserved because our choice that X^2 contains symmetries and unfixed gauge transformations implies $X^2\ _{\text{gf}} = 0$. With this procedure, the number of preserved supercharges has not changed; while the gauge-fixed action with $B\ _{\text{gf}}$ is invariant under B, the gauge-fixed action with $X\ _{\text{gf}}$ is invariant under X. Its usefulness for computing the spectrum lies in the fact that $\ _C\ f$ can be redefined by shifting with a quadratic combination of ghosts such that $X^1\ {}^0_C\ f^{0o} = 0$, making the linear term (3.2.24) X-closed. By extension, the quadratic action which is modified by the shift is also X-closed, and its spectrum is supersymmetric.

For $X\ _{\text{gf}} = {}^1B\ Q^o\ _{\text{gf}}$ to be invariant under X, X^2 should only contain residual gauge transformations and possibly other symmetries of $\ _{\text{gf}}$. This condition constrains how Q can act on fields in the gauge-fixing complex. The supersymmetry transformations of the physical fields - under Q are given in (3.2.4)-(3.2.5) and (3.2.12)-(3.2.14). Without specifying how Q acts on the fields . in the gauge-fixing complex, we find:

$$\begin{aligned} Q^2 - &= \{\&\text{-}\overline{\&}\}\text{-} = 8\ m_C\ \ X_{\text{gauge}}{}^1\ _C\ f^o\ \text{- -} \qquad \{Q\text{-}B\}\text{-} = X_{\text{gauge}}\ Q2\ \text{- -} \\ X^2 - &= 8\ m_C\ \ X_{\text{gauge}}\ \Big[\ _C\ f\ 8Q2\ \Big]'\ \text{-}\ . \end{aligned} \tag{3.2.50}$$

If we want X to close on time translations and residual gauge transformations, the only possibility is to set

$$Q2 = 8^1\ _C\ f^o{}_f\ . \tag{3.2.51}$$

Hence, physical fields satisfy the algebra:

$$X^2 - = 8\ m_C\ \ X_{\text{gauge}}\ \Big[\ _{C\text{-}r}\ f_r\ \Big]'\ \text{-}\ . \tag{3.2.52}$$

Having fixed $Q2$, we find that 2 also satisfies (3.2.52) and specifically

$$Q^2 2 = 0 - \qquad \mathrm{f} Q - B \mathrm{g} 2 = 8 \ m_{\mathrm{C}} \quad X_{\text{gauge}}{}^1 \ {}_{\mathrm{C}\text{-}\mathrm{r}} \ \text{¸} \ f_{\mathrm{r}}{}^{\mathrm{o}} \ \ 2 - \tag{3.2.53}$$

which imply (3.2.52). For uniformity, we demand that (3.2.53) is satisfied on all fields . in the gauge-fixing complex. Setting $Q\,ȃ = 0$ for simplicity, we find that this fixes $Q1$ and, altogether, Q acts on the fields in the gauge-fixing complex as:

$$Q\,2 = 8^1 \ {}_{\mathrm{C}} \ \text{¸} \ f^{\mathrm{o}}{}_{\mathrm{f}} - \qquad Q\,ȃ = 0 - \qquad Q\,1 = \ m_{\mathrm{C}} \quad X_{\text{gauge}}{}^1 \ {}_{\mathrm{C}\text{-}\mathrm{r}} \ \text{¸} \ f_{\mathrm{r}}{}^{\mathrm{o}} \ ȃ \bullet \tag{3.2.54}$$

Given $\ {}_{\text{gf}}$ that we defined in (3.2.47), we can now determine

$$Q \quad {}_{\text{gf}} = \frac{1}{4^2_{\text{3d}}} \operatorname{Tr} \ 8\,ȃ\,Q \quad {}_{\text{gf}} \ \text{¸} \ \frac{8}{2} ȃ \quad {}_{\mathrm{C}} \quad 8 f \ ȃ \quad - \tag{3.2.55}$$

where f acts in the adjoint representation (namely, $f\,ȃ$ stands for » f - $ȃ$¼ in matrix notation). Hence, collecting the contributions from (3.2.44) and (3.2.55), the supersymmetrized gauge-fixing procedure requires us to add the following terms to the original Lagrangian:

$$X \quad {}_{\text{gf}} = \frac{1}{4^2_{\text{3d}}} \operatorname{Tr} \ \frac{1^2}{2} \ \text{¸}\ 1 \quad {}_{\text{gf}} \quad \mathrm{f}ȃ\text{-}2\mathrm{g} \ \ \text{¸}\ 8\,ȃ \ X_{\text{gauge}}{}^1 2^{\mathrm{o}} \ \text{¸}\ Q \quad {}_{\text{gf}} \ \text{¸} \ \frac{1}{2} \mathrm{f}ȃ\text{-}2\mathrm{g}^2 \ \text{¸} \ \frac{8}{2} ȃ \quad {}_{\mathrm{C}} \quad 8 f \ ȃ \ \bullet \tag{3.2.56}$$

With the choice that 2 and $ȃ$ are Hermitian, $X \quad {}_{\text{gf}}$ is real.

It is important to note (following [221]) that adding $Q \quad {}_{\text{gf}}$ to $B \quad {}_{\text{gf}}$ does not change the expectation values of operators with $=_6 \quad 0$, even if they are not invariant under Q. In particular, it does not change physical observables. This can be shown explicitly for the thermal partition function. We first integrate in an adjoint-valued auxiliary field 0 to rewrite the quartic ghost interactions, after which the gauge-fixing action becomes:

$$X \quad {}_{\text{gf}} = \frac{1}{4^2_{\text{3d}}} \operatorname{Tr} \ \frac{1^2 \quad 0^2}{2} \ \text{¸}\ 1 \quad {}_{\text{gf}}\ \text{¸}\ ȃ \ \ 0 \ \text{¸}\ 1\text{-}2 \ \ \text{¸}\ 8\,ȃ \ X_{\text{gauge}}{}^1 2^{\mathrm{o}} \ \text{¸}\ Q \quad {}_{\text{gf}}\ \text{¸} \ \frac{8}{2} ȃ \quad {}_{\mathrm{C}} \quad 8 f \ ȃ \ \bullet \tag{3.2.57}$$

Note that 0 has both gauge-fixed and residual components. Since the full action is quadratic in the Grassmann fields $\mathrm{f} \quad {}_{\text{phys}}\text{-}\,2\text{-}\,ȃ\mathrm{g}$, where $\ {}_{\text{phys}}$ is the set of physical fermions, we can formally perform the path integral over them, obtaining:

$$\det \begin{pmatrix} (0\mathrm{j} \ - & 0 & Q \quad {}_{\text{gf}}\mathrm{j} \ \text{-}ȃ\mathrm{a} \\ 0 & 0 & B \quad {}_{\text{gf}}\mathrm{j}\,2\text{-}ȃ \\ Q \quad {}_{\text{gf}}\mathrm{j}\,ȃ\text{-} & B \quad {}_{\text{gf}}\mathrm{j}\,ȃ\text{-}2 & Q \quad {}_{\text{gf}}\mathrm{j}\,ȃ\text{-}ȃ \end{pmatrix} \det \begin{pmatrix} 0 & B \quad {}_{\text{gf}}\mathrm{j}\,2\text{-}ȃ \\ B \quad {}_{\text{gf}}\mathrm{j}\,ȃ\text{-}2 & Q \quad {}_{\text{gf}}\mathrm{j}\,ȃ\text{-}ȃ \end{pmatrix} \det \ (0\mathrm{j} \ - \quad \bullet \tag{3.2.58}$$

All entries of the matrix on the LHS are (possibly differential) operators involving the bosons. This proves that the thermal partition function does not depend on the term $Q\,\Psi_{gf}$.

More generally, we prove that the expectation value of any operator O with ghost number $n_6 \neq 0$ is unchanged by the addition of $Q\,\Psi_{gf}$ to the Lagrangian. The key property is that $Q\,\Psi_{gh}$ is the sum of two terms, of ghost number -1 and -2, respectively. Let $\langle\ \rangle_B$ be the path integral with $B\,\Psi_{gf}$ as gauge fixing, and let $\langle\ \rangle_X$ be the path integral with $X\,\Psi_{gf}$ as gauge fixing. We have

$$\langle O\rangle_X = \left\langle O\, e^{-Q\,\Psi_{gf}}\right\rangle_B = \langle O\rangle_B + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left\langle O\,(Q\,\Psi_{gf})^n\right\rangle_B = \langle O\rangle_B\,. \tag{3.2.59}$$

The last equality holds because the ghost number is a symmetry of $\langle\ \rangle_B$, implying a null expectation value for any correlator that has $n_6 < 0$. Since $O\,(Q\,\Psi_{gf})^n$ has $n_6 \leq 0$, one concludes that $\langle O\,(Q\,\Psi_{gf})^n\rangle_B = 0$ for every n. For the restricted set of operators O with $n_6 \neq 0$, one can constrain $\langle\ \rangle_X$ using the symmetries of $\langle\ \rangle_B$. In particular, although both supersymmetry and $U(1)'$ are not symmetries of $\langle\ \rangle_X$ because $Q\,\Psi_{gf}$ breaks them, their Ward identities can still be used to constrain the correlators $\langle O\rangle_X$. This result will play a crucial role in Section 3.4.

We can now show how the linear Lagrangian term containing $\zeta_C \cdot f$ can be made X-invariant using a field redefinition. This is crucial to have a reliable and supersymmetric spectrum. The linear term (3.2.24) only contains modes $(\zeta_C \cdot f)_r$ which are constant on Σ^2, due to the integral over Σ^2. Since $\zeta_{C\text{-}r}\cdot f_r$ appears in (3.2.52) as a central charge, $X(\zeta_{C\text{-}r}\cdot f_r) = 0$. Therefore, by redefining

$$\zeta^0_{C\text{-}r}\cdot f^0_r = \zeta_{C\text{-}r}\cdot f_r \cdot \frac{1}{2}\{2\text{-}2g_r\}\,, \tag{3.2.60}$$

the linear term (3.2.24) becomes (dropping the ^{0} on $\zeta^0_{C\text{-}r}\cdot f^0_r$):

$$\mathcal{L}^{(1)} \supset \frac{\kappa}{4c\,\pi^2}\,\mathrm{Tr}\;\mathfrak{m}\,(\zeta_C\cdot f) + \frac{\sigma_\kappa}{4\,\pi^2\,\ell^2_{3d}}\,\mathrm{Tr}\;2\,\{\mathfrak{m}\text{-}2\}\,, \tag{3.2.61}$$

where $\mathfrak{m}$ is diagonal and σ_κ was defined in (3.2.23). The first term is invariant under X, therefore after adding the second term to the quadratic action, the latter becomes invariant under X as well, and the spectrum has to be supersymmetric (*i.e.*, X-symmetric). Notice that the newly shifted field $\zeta_{C\text{-}r}\cdot f_r$ is still Hermitian because 2 is Hermitian.

3.2.5 Vector multiplet spectrum

We are now ready to compute the spectrum of the (gauge-fixed) vector multiplet action. We start by considering the off-diagonal components. The Yang-Mills, Chern-Simons, and gauge-fixing terms are expanded to quadratic order in fluctuations around (3.2.23). After integrating out the auxiliary fields and 1, the independent components consist of 4 complex bosons $^{89}_{1}$- $^{89}_{C}$- f^{89}- $\bar{^{89}_{1}}$ and 6 complex fermions $\bar{^{89}_{1}}$- $^{89}_{C}$- $\bar{^{89}_{C}}$- 2^{89}- $æ^{89}$- $\bar{^{89}_{1}}$ for every $8 < 9$ such that $@_{89}$ ¡ 0.8 All components are then rescaled by a factor of 4_{3d}• '. Moreover $^{89}_{1}$, $\bar{^{89}_{1}}$ get an extra factor of 1• p 2, while $\bar{^{89}_{1}}$, $\bar{^{89}_{1}}$, $^{89}_{C}$, $\bar{^{89}_{C}}$ get an extra factor of p 2. This is to ensure that the standard 1d kinetic terms are canonically normalized. After expanding in monopole harmonics according to Table 3.1 and integrating over $(^2$, the quadratic action for off-diagonal components in momentum space becomes:

$$\int \frac{3?}{2c} \tilde{O}_{8-9 \, j@_{89}j \, 0} \; \tilde{O}_{;-j<j \, ;} \; \overline{^{89}_{;-<}} \, {}^1?^o \, " \, ^{89}_{;-<} \, {}^1?^o \, \text{¸} \, \overline{^{89}_{;-<}} \, {}^1?^o \, " \, ^{89}_{;-<} \, {}^1?^o \tag{3.2.62}$$

where the vectors of bosonic and fermionic fields are, respectively,

$$\begin{aligned} ^{89}_{;-<} &= \left(^{89}_{1-;-<} - ^{89}_{C-;-<} - f^{89}_{;-<} - \bar{^{89}_{1-;-<}} \right)^T - \\ ^{89}_{;-<} &= \left(\bar{^{89}_{1-;-<}} - ^{89}_{C-;-<} - \bar{^{89}_{C-;-<}} - 2^{89}_{;-<} - æ^{89}_{;-<} - \bar{^{89}_{1-;-<}} \right)^T \bullet \end{aligned} \tag{3.2.63}$$

The operators acting on the bosonic and fermionic fields are:

$$" = \begin{pmatrix} ?^1? \text{¸} <_: \text{¸} 2f_0{}^o \frac{b \text{¸} 1}{b} \frac{B^2}{2'^2} & \frac{8B \; ^1? \text{¸} <_: \text{¸} f_0{}^o}{\text{p}\overline{2}'} & \frac{8f_0B}{\text{p}\overline{2}'} & \frac{1}{b} \frac{b}{2'^2} \frac{B \text{¸} B}{2'^2} \\ \frac{8B \; ^1? \text{¸} <_: \text{¸} f_0{}^o}{\text{p}\overline{2}'} & \frac{B_0^2}{'^2} \text{¸} f_0^2 & f_0{}^1? \text{¸} f_0{}^o & \frac{8B_\text{¸} {}^1? \; <_: \text{¸} f_0{}^o}{\text{p}\overline{2}'} \\ \frac{8f_0B}{\text{p}\overline{2}'} & f_0{}^1? \text{¸} f_0{}^o & {}^1? \text{¸} f_0{}^{o2} \; <_:^2 \; \frac{B_0^2}{'^2} & \frac{8f_0B_\text{¸}}{\text{p}\overline{2}'} \\ \frac{1}{b} \frac{b}{2'^2} \frac{B \text{¸} B}{2'^2} & \frac{8B_\text{¸} {}^1? \; <_: \text{¸} f_0{}^o}{\text{p}\overline{2}'} & \frac{8f_0B_\text{¸}}{\text{p}\overline{2}'} & ?^1? \; <_: \text{¸} 2f_0{}^o \frac{b \text{¸} 1}{b} \frac{B^2_\text{¸}}{2'^2} \end{pmatrix} \tag{3.2.64}$$

with

$$f_0 = \frac{@_{89}}{<_: '^2} - \quad B_0 = \sqrt{;^1; \text{¸} 1^o \quad @_{89}^2} - \quad B \; = \sqrt{;^1; \text{¸} 1^o \quad @_{89}{}^1@_{89} \quad 1^o} = \sqrt{B_0^2 \quad @_{89}} - \tag{3.2.65}$$

8We have chosen to write $^{89}_{1} = \overline{^{98}_{1}}$, $\bar{^{89}_{1}} = \overline{^{98}_{1}}$ and $\bar{^{89}_{C}} = \overline{^{98}_{C}}$ as fundamental fields.

(notice that f_0, B_0, and B depend on 89) and

$$
'' = \begin{pmatrix}
? \; <_: \; 2f_0 & \frac{B}{'} & 0 & 0 & \frac{8B}{p\,2b\,'} & 0 \\
\frac{B}{'} & ? \,¸\, <_: & 0 & 0 & 0 & 0 \\
0 & 0 & ? \; <_: & 0 & 0 & \frac{B_¸}{'} \\
0 & 0 & 0 & \frac{<_: @_{89}}{'^2} & \frac{8B_0^2}{p\,b\,'^2} & 0 \\
\frac{8B}{p\,2b\,'} & 0 & 0 & \frac{8B_0^2}{p\,b\,'^2} & ? & \frac{8B_¸}{p\,2b\,'} \\
0 & 0 & \frac{B_¸}{'} & 0 & \frac{8B_¸}{p\,2b\,'} & ? \,¸\, <_: \; 2f_0
\end{pmatrix} \bullet \qquad (3.2.66)
$$

For $; \quad @_{89} \,¸\, 1$, all modes exist and are massive. Moreover, the masses of the modes[9] from bosons and fermions are paired thanks to the X-invariance of the action, and the ratio of fermionic to bosonic determinants is 1. For $; = @_{89}$, the modes of $\ {}^{89}_{1}$ and $\ {}^{89}_{1}$ do not exist (see Table 3.1), so the rightmost column and the bottom row of the matrices $''$, $''$ should be removed. In this case, there is a massless fermionic mode while the other massive modes are paired between bosons and fermions. The ratio of determinants is $?$. For $; = @_{89} \quad 1$ (this case takes place only if $@_{89} \quad 1$), modes only exist in $\ {}^{89}_{1}$ and $\overline{\ }^{89}_{1}$. The bosonic field $\ {}^{89}_{1}$ has a massless pole, and a massive pole that cancels with that of $\overline{\ }^{89}_{1}$.

The effective degrees of freedom at energies much smaller than $<_:$ and $\frac{1}{'}$ are the massless fermionic modes with $; = @_{89}$ and the massless modes in $\ {}^{89}_{1}$ with $; = @_{89} \quad 1$ (if $@_{89} \quad 1$). The identity of the massless fermionic modes is not immediately clear due to the off-diagonal entries in (3.2.66). We can first rescale the fields $2^{89}_{;\text{-}<} \; ! \; ' \, 2^{89}_{;\text{-}<}$ so that they have the same mass dimension as the other fermions. Defining the dimensionless ratio $U = 1 \bullet ¹<_: \, 'º$ for convenience, the fermionic kinetic operator above becomes:

$$
''_{;=@_{ij}} = \begin{pmatrix}
? \;\; ¹1 \;\; 2@_{89}U^{2}º <_: & \sqrt{2@_{89}}\, U <_: & 0 & 0 & 8\sqrt{\frac{@_{ij}}{b}}\, U <_: \\
\sqrt{2@_{89}}\, U <_: & ? \,¸\, <_: & 0 & 0 & 0 \\
0 & 0 & ? \; <_: & 0 & 0 \\
0 & 0 & 0 & @_{89} <_: & 8\frac{@_{ij}}{\sqrt{b}}\, U <_: \\
8\sqrt{\frac{@_{ij}}{b}}\, U <_: & 0 & 0 & 8\frac{@_{ij}}{\sqrt{b}}\, U <_: & ?
\end{pmatrix} \bullet \qquad (3.2.67)
$$

By introducing a kinetic term $8Y\,\overline{2^{89}}\,m_{\text{C}}2^{89}$ by hand for the fermion 2^{89}, the problem of finding mass eigenstates is reduced to the usual problem of diagonalizing a mass matrix. Taking $Y \; ! \; 0$ at the end of the computation, we obtain the desired $\mathrm{SL}¹5\text{-}\mathrm{C}º$ transformation that

9The counting of modes works as follows. A complex field with a 2-derivative kinetic term gives two modes, with only a 1-derivative kinetic term one mode, whereas with no kinetic term no modes.

diagonalizes (3.2.67):

$$
(= \begin{pmatrix}
\frac{}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & \frac{¸}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & 0 & 0 & \frac{U}{\sqrt{b \,¸\, @_{89}U^2 \,¸\, 2@_{89}^2 U^4}} \\
\frac{}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & \frac{¸}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & 0 & 0 & \frac{\sqrt{2}U^2}{\sqrt{b \,¸\, @_{89}U^2 \,¸\, 2@_{89}^2 U^4}} \\
\frac{2\sqrt{2b}@_{89}U^3 \, 0}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & \frac{2\sqrt{2b}@_{89}U^3 \, 0}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & 1 & 0 & \frac{\sqrt{b}U \, 0}{\sqrt{b \,¸\, @_{89}U^2 \,¸\, 2@_{89}^2 U^4}} \\
\frac{82\sqrt{2}@_{89}U^2}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & \frac{82\sqrt{2}@_{89}U^2}{\sqrt{8@_{89}^2 U^4 b \,¸\, ^2 \,¸\, ^2}} & 0 & 8\sqrt{\frac{b}{@_{89}}} & \frac{8}{\sqrt{b \,¸\, @_{89}U^2 \,¸\, 2@_{89}^2 U^4}}
\end{pmatrix} \tag{3.2.68}
$$

where we have defined

$$
\begin{aligned}
&= \sqrt{2@_{89}}U \quad @_{89}U^2\,{}^1 1 \,¸\, 2b^{\circ} \quad \sqrt{@_{89}^2 U^4\,{}^1 1 \,¸\, 2b^{\circ 2} \,¸\, 4b\,{}^1 @_{89}U^2 \,¸\, b^{\circ}} \\
&= 2b \,¸\, @_{89}U^2\,{}^1 1 \,¸\, 2b^{\circ} \quad \sqrt{@_{89}^2 U^4\,{}^1 1 \,¸\, 2b^{\circ 2} \,¸\, 4b\,{}^1 @_{89}U^2 \,¸\, b^{\circ}} \,•
\end{aligned} \tag{3.2.69}
$$

The resulting fermionic kinetic operator is

$$
(^{y} \;"\; {}_{;=@_{89}} (= \begin{pmatrix}
? \;\_¸ <_: & 0 & 0 & 0 & 0 \\
0 & ? \;\_\; <_: & 0 & 0 & 0 \\
0 & 0 & ? \; <_: & 0 & 0 \\
0 & 0 & 0 & <_: & 0 \\
0 & 0 & 0 & 0 & ?
\end{pmatrix} \tag{3.2.70}
$$

with

$$
\_ \;=\; \frac{@_{89}\, U^2\,{}^1 1 \quad 2b^{\circ} \quad \sqrt{@_{89}^2\, U^4\,{}^1 1 \,¸\, 2b^{\circ 2} \,¸\, 4b \;\; @_{89}\, U^2 \,¸\, b}}{2b} \,• \tag{3.2.71}
$$

Each row of the matrix $($ expresses an original fermion in terms of the mass eigenstates. The linear combinations are generically complicated, but they simplify in the physical regime of interest. Since we want to reduce a Chern-Simons-matter theory on $(^2$, and the Yang-Mills term was only introduced to make propagating gauge degrees of freedom massive, we are motivated to take $<_: \quad 1•\,'$, or $U \;!\; 0$. In this limit, the massless fermion at $; = @_{89}$ is $8\sqrt{b}\,2\!\!\!e$ (last row of $($), and $\_ \quad ! \quad 1$.

The spectrum of the diagonal components can be analyzed in the same way and we will be brief. One finds that every mode is massive for $; \;¡\; 0$. After integrating out the $; = 0$

mode of the auxiliary fields ^{8}, the quadratic Lagrangian (including the linear terms) for the remaining diagonal $; = 0$ modes is:

$$\tilde{O}_{8} : \mathrm{m}_8 \; {}^{8}_{\text{C-0-0}} \,¸\, f^{8}_{0\text{-}0} \,¸\, \frac{4c\,'^2}{4^2_{3d}} \, \frac{1}{2} \, \mathrm{m_C} f^{8}_{0\text{-}0} \, {}^2 \; \frac{1}{2} <^2_: \, f^{8}_{0\text{-}0} \, {}^2 \,¸\, \frac{1}{2} \, \bar{}^{8}_{\text{C-0-0}} \, 8\mathrm{m_C} \,¸\, <_: \; {}^{8}_{\text{C-0-0}} \; • \tag{3.2.72}$$

We observe that $f^{8}_{0\text{-}0}$ and ${}^{8}_{\text{C-0-0}}$ have mass $<_:$ and should be integrated out at low energies $? \quad <_:$. Only the combination ${}^{8}_{\text{C-0-0}} \,¸\, f^{8}_{0\text{-}0}$ remains, which is a 1d gauge field for the gauge group $\mathrm{U}¹1º^{\#}$.[10]

To summarize, we write the quadratic Lagrangian for the modes from the vector multiplet that contain massless poles, including fermionic partners which are necessary for supersymmetry. After having rescaled $\bar{}_1$ and $\bar{}_1$ by $<_:^{1•2}$ we have:

$$: \tilde{O}_{8} \mathrm{m}_8 \, ¹ \, {}^{8}_{\text{C}} \,¸\, f^{8}º \,¸\, \tilde{O}_{8<9} \; ¹@_{89} \; 1º \; \tilde{O}_{j<j \;\; @_{89} \;\; 1} \; \overline{{}^{98}_{1\text{-}@_{89}\;\;1\text{-}<}} \, 8\mathrm{m_C} \, {}^{98}_{1\text{-}@_{89}\;\;1\text{-}<} \,¸\, \overline{{}^{98}_{1\text{-}@_{89}\;\;1\text{-}<}} \; {}^{98}_{1\text{-}@_{89}\;\;1\text{-}<} \,¸$$
$$¸\, \frac{1}{<_:} \, \mathrm{m_C} \; {}^{98}_{1\text{-}@_{89}\;\;1\text{-}<} \; {}^2 \,¸\, \overline{{}^{98}_{1\text{-}@_{89}\;\;1\text{-}<}} \, 8\mathrm{m_C} \; {}^{98}_{1\text{-}@_{89}\;\;1\text{-}<} \;¸\; ¹@_{89}º \, \tilde{O}_{j<j \;\; @_{89}} \; \overline{æ^{89}_{@_{89}\text{-}<}} \, 8\mathrm{m_C} \, æ^{89}_{@_{89}\text{-}<} \tag{3.2.73}$$

where $¹Gº = 1$ for $G \quad 0$ and it vanishes otherwise. Here we have changed notation, and used the fields $\bar{}^{98}_1\text{-} \; \bar{}^{98}_1$ in place of ${}^{89}_1, \; \bar{}^{89}_1$ because the former live in a chiral multiplet, see (3.2.5), while the latter in an anti-chiral multiplet. Besides, notice that there are matching degrees of freedom in $\bar{}^{98}_1$ and $\bar{}^{98}_1$ with mass $<_:$, which should not be included in the effective theory at energies $? \quad <_:$. These modes are encoded in the term proportional to $1•<_:$ and can be integrated out by neglecting that kinetic term. The workings are explained in [222]. The quadratic Lagrangian for the massless modes is then:[11]

$$: \tilde{O}_{8} \mathrm{m}_8 \, ¹ \, {}^{8}_{\text{C}} \,¸\, f^{8}º \,¸\, \tilde{O}_{89} \; ¹@_{89} \; 1º \; \tilde{O}_{j<j \;\; @_{89} \;\; 1} \; \overline{{}^{98}_{1\text{-}@_{89}\;\;1\text{-}<}} \, 8\mathrm{m_C} \, {}^{98}_{1\text{-}@_{89}\;\;1\text{-}<} \,¸$$
$$¸\, \overline{{}^{98}_{1\text{-}@_{89}\;\;1\text{-}<}} \; {}^{98}_{1\text{-}@_{89}\;\;1\text{-}<} \;¸\; ¹@_{89} \; \tfrac{1}{2}º \, \tilde{O}_{j<j \;\; @_{89}} \; \overline{æ^{89}_{@_{89}\text{-}<}} \, 8\mathrm{m_C} \, æ^{89}_{@_{89}\text{-}<} \; • \tag{3.2.74}$$

---

[10] In other words, in the language of Appendix E, we find that the superfield + is massive, while stays light and enforces gauge invariance.

[11] Using the assumption that $@_{89} < 0$ for $8 < 9$, we have substituted $¹@_{89}º \; ! \; ¹@_{89} \; \tfrac{1}{2}º$ in (3.2.73), and consequently we have substituted ${}_{8<9} \; ! \; {}_{89}$.

The bosons $\frac{98}{1}$ and the fermions $\hat{e}^{89}$ have a 1-derivative action, while the fermions $\frac{98}{1}$ are auxiliary.

3.2.6 Matter multiplets spectrum

To find the spectrum of modes coming from the 3d chiral multiplets, we expand the chiral multiplet Lagrangian (3.2.17) to quadratic order in fluctuations around (3.2.23). All fields in the chiral multiplet are rescaled by 1• '. After expanding in monopole harmonics according to Table 3.1 and integrating over $(^2$, the quadratic action in momentum space is:

$$\int \frac{3?}{2c} \sum_{8\text{-}9} \sum_{;\text{-}j<j} \sum_{;} \Bigg(?^1? \,¸\, 2f_0{}^o \frac{B^2_{¸\text{-}}}{'^2}\, q^{89}_{\text{-};\text{-}<}{}^1?^o{}^2 \,¸\, 5^{89}_{\text{-};\text{-}<}{}^1?^o{}^2 \,¸\, \overline{k^{89}_{\text{-};\text{-}<}{}^1?^o} \text{-} \overline{[^{89}_{\text{-};\text{-}<}{}^1?^o} \begin{pmatrix} ? & 2f_0\frac{B_{¸\text{-}}}{'} \\ \frac{B_{¸\text{-}}}{'} & ? \end{pmatrix} \begin{pmatrix} k^{89}_{\text{-};\text{-}<}{}^1?^o \\ [^{89}_{\text{-};\text{-}<}{}^1?^o \end{pmatrix} \Bigg) \tag{3.2.75}$$

where

$$f_0 = @_{89}U^2<; \qquad \frac{<_f}{2}\text{-} \qquad B_{\text{-}} \quad \sqrt{;^1; \,¸\, 1^o \quad @_{89}{}^1@_{89} \quad 1^o}\,• \tag{3.2.76}$$

For $;$ $j@_{89}j$ ¸ 1, all modes exist (see Table 3.1) and are massive. Moreover, the masses of bosons and fermions are paired and the ratio of determinants is 1. The modes with $; = j@_{89}j$ exist in all fields if $@_{89}$ $\frac{1}{2}$, whereas they only exist in q^{89} and k^{89} if $@_{89}$ 0. In the former case, all modes are massive. In the latter case, the field q^{89} has a massless pole and a massive pole that cancels with that of k^{89}. Provided that $@_{89}$ 1, there exist modes with $; = j@_{89}j$ 1 = $@_{89}$ 1 in $[^{89}$ and 5^{89}, such that $[^{89}$ is massless while 5^{89} is auxiliary.

To summarize, the quadratic Lagrangian for modes that contain massless poles, and that of their supersymmetry partners is

$$\sum_{89\text{-}} {}^1@_{89}{}^o \sum_{j<j\ @_{89}} <^{89}_f\, \overline{q^{89}_{\text{-}@_{89}\text{-}<}}\, 8m_c\, q^{89}_{\text{-}@_{89}\text{-}<} \,¸\, \overline{k^{89}_{\text{-}@_{89}\text{-}<}}\, k^{89}_{\text{-}@_{89}\text{-}<} \,¸\, m_c q^{89}_{\text{-}@_{89}\text{-}<}{}^2 \,¸ \tag{3.2.77}$$
$$¸\, \overline{k^{89}_{\text{-}@_{89}\text{-}<}}\, 8m_c\, k^{89}_{\text{-}@_{89}\text{-}<} \,¸\, {}^1 @_{89} \ 1^o \sum_{j<j\ @_{89}\ 1} \overline{[^{89}_{\text{-}\ @_{89}\ 1\text{-}<}}\, 8m_c\, [^{89}_{\text{-}\ @_{89}\ 1\text{-}<} \,¸\, 5^{89}_{\text{-}\ @_{89}\ 1\text{-}<}{}^2\ \text{-}$$

where the 8- 9 dependence of $<_f$ was made explicit. At low energies $?$ $<^{89}_f$, the quadratic kinetic term of $q^{89}_{\text{-}@_{89}\text{-}<}$ and the kinetic term of $k^{89}_{\text{-}@_{89}\text{-}<}$ can again be neglected. Note that $@_{89}$ 0 does not exclude the possibility that $8 = 9$, in which case $<^{89}_f = 0$. We might also have

$<^{89}_{f}$! 0 as U ! 0.[12] In either case, all of $q^{89}_{-@_{89}-<}$ and $k^{89}_{-@_{89}-<}$ would be classically massless. However, quantum effects would still generically generate supersymmetric terms like

$$<^{89}_{f\,(q)} \quad \overline{q^{89}_{-@_{89}-<}}\, 8m_C\, q^{89}_{-@_{89}-<} \; ¸ \; \overline{k^{89}_{-@_{89}-<}}\, k^{89}_{-@_{89}-<} \quad - \tag{3.2.78}$$

whose superspace expression is (E.2.49). At scales $?$ $<^{89}_{f\,(q)}$, the quadratic kinetic term of $q^{89}_{-@_{89}-<}$ and the kinetic term of $k^{89}_{-@_{89}-<}$ would still be negligible. Therefore, rescaling $q^{89}_{-@_{89}-<}$ and $k^{89}_{-@_{89}-<}$ by $1•^1<^{89}_{f}{}^{o1•2}$ (including quantum corrections), the resulting quadratic effective Lagrangian is:

$$\tilde{O}_{89-} \; {}^1@_{89}{}^o \; \tilde{O}_{j<j\ @_{89}} \; \overline{q^{89}_{-@_{89}-<}}\, 8m_C\, q^{89}_{-@_{89}-<} \; ¸ \; \overline{k^{89}_{-@_{89}-<}}\, k^{89}_{-@_{89}-<} \quad ¸ \tag{3.2.79}$$
$$¸ \; {}^1 @_{89} \; 1^o \; \tilde{O}_{j<j\ @_{89}\ 1} \; \overline{[^{89}_{-\ @_{89}\ 1-<}}\, 8m_C\, [^{89}_{-\ @_{89}\ 1-<} \; ¸ \; 5^{89}_{-\ @_{89}\ 1-<}{}^2 \quad •$$

3.3 The effective quantum mechanics

In this Section, we present the proposed low-energy quantum mechanical model, which is the result of setting to zero all massive modes in the gauge-fixed 3d Lagrangian while only keeping the light modes.

The gauge group is $U^1 1^{o\#}$ and the vector multiplet only contains the gauge fields ^8_C ¸ f^8, with $8 = 1 - • • • - \#$.[13] Their role is to impose Gauss's law. Because of the presence of a Wilson line of charges $:m_8$, coming from the 3d Chern-Simons term, Gauss's law projects onto a sector of non-vanishing gauge charges.

The matter content consists of various chiral and Fermi multiplets $-^{89}$ with charges ¸1 under $U^1 1^o{}_8$ $U^1 1^{o\#}$ and 1 under $U^1 1^o{}_9$. They interact with the gauge fields via the covariant derivative

$$\mathcal{C}\, -^{89} = m_C \quad 8 \quad {}^8_C ¸ f^8 \quad {}^9_C \quad f^9 \quad -^{89} • \tag{3.3.1}$$

The matter content depends on the fluxes m_8 — determined in (3.1.35) — and n through the combinations $@_{89}$ and $@_{89}$ defined in (3.2.28). For every pair of indices 89, from the 3d vector multiplet we get the following matter multiplets. If $@_{89}$ 1, there are 1d chiral

[12] Indeed $<_f$ $U^2 <_:$ $U •\,'$, therefore its scaling is not fixed by the choices we already made.

[13] In the Wess-Zumino gauge, the only non-vanishing component of the superfield + (or equivalently of) is $_C$ ¸ f. See Appendix E.2.1.

	$\overline{}^{89}_{1\text{-}<}$ chiral	$\text{æ}^{89}_{<}$ Fermi	$q^{89}_{\text{-}<}$ chiral	$\Gamma^{89}_{\text{-}<}$ Fermi
existence:	$@_{89}$ 1	$@_{89}$ $\frac{1}{2}$	$@_{89}$ 0	$@_{89}$ 1
;	j$@_{89}$j 1	$@_{89}$	$@_{89}$	j$@_{89}$j 1
$'_3$	0	0	$2X_3$	$2X_3$ 1
$@_1$	0	0	X_1 X_3	X_1 X_3
$@_2$	0	0	X_2 X_3	X_2 X_3

Table 3.2 Matter multiplets (we indicate the bottom components) for indices 89 and their representations under the global symmetries. We label the SU¹2º representation by the highest weight ; 2 Z•2. The charges of the lowest components in each multiplet are indicated, while their superpartners have R-charges $'_3$ which are shifted by 1.

multiplets $\overline{}^{89}_{1\text{-}<} = \overline{}^{89}_{1\text{-}<}\text{-}\ \overline{}^{89}_{1\text{-}<}$ in the SU¹2º representation of highest weight ; = $@_{89}$ 1. Otherwise, if $@_{89}$ $\frac{1}{2}$, there are 1d Fermi multiplets $^{89}_{<} = \text{æ}^{89}_{<}\text{-}6^{89}_{<}$ with ; = $@_{89}$. Here we introduced the auxiliary fields $6^{89}_{<}$, even though they are not present in the 3d theory, to make off-shell supersymmetry manifest. From the 3d chiral multiplet with flavor index , we get 1d chiral multiplets $^{89}_{\text{-}<} = q^{89}_{\text{-}<}\text{-}k^{89}_{\text{-}<}$ with ; = $@_{89}$ if $@_{89}$ 0, and otherwise 1d Fermi multiplets $Y^{89}_{\text{-}<} = \Gamma^{89}_{\text{-}<}\text{-}5^{89}_{\text{-}<}$ with ; = $@_{89}$ 1 if $@_{89}$ 1. We summarize this content in Table 3.2, where we also list the representations and charges of each multiplet under the global symmetries SU¹2º, U¹1º² and U¹1º '.

In addition to gauge interactions, other interactions are specified by and superpotentials. We have as many and functions as there are Fermi multiplets. For a given Fermi multiplet Γ, is in the same gauge and flavor representation as Γ, and its R-charge is $'$¹Γº ¸ 1. On the contrary, is in the conjugate gauge and flavor representation with respect to Γ, and its R-charge is $'$¹Γº ¸ 1. We find that the and functions are zero for the Fermi multiplets $\text{æ}^{89}_{<}$. For the Fermi multiplets $\Gamma^{89}_{\text{-}<}$, the and superpotentials are:

$$^{89}_{\text{-}<} = 8 \;\tilde{O}_{:}\; {}^1@_{:9}{}^{\text{o}} \tilde{O}_{j<^0j\ @_{:9}} 4^{:9}_{1d}\, {}^{q}\overline{2@_{:9}\text{¸}1}\ \ ^{j@_{8:}j\ 1\ @_{:9}\ j@_{89}j\ 1}_{<\ <^0\ <^0\ <}\ \ \overline{}^{8:}_{1\text{-}<\ <^0}\, q^{:9}_{\text{-}<^0}\text{¸}$$

$$\qquad {}^1@_{8:}{}^{\text{o}} \tilde{O}_{j<^0j\ @_{8:}} 4^{8:}_{1d}\, {}^{p}\overline{2@_{8:}\text{¸}1}\ \ ^{j@_{:9}j\ 1\ @_{8:}\ j@_{89}j\ 1}_{<\ <^0\ <^0\ <}\ \ q^{8:}_{\text{-}<^0}\ \overline{}^{:9}_{1\text{-}<\ <^0}\ \text{-} \tag{3.3.2}$$

$$ {}^{98}_{-\ <} = \tilde{O}\ n\ {}^{1}@_{9:}{}^{o}\ {}^{1}@_{:8}{}^{o}\ \ {}_{-:}\ \tilde{O}_{\substack{9:8 \\ j<^0 j\ @_{9:} \\ j<,<^0 j\ @_{:8}}} -_{1d}\ r\ \frac{{}^{1}2@_{9:},1^{o}{}^{1}2@_{:8},1^{o}}{2j@_{89}j\ \ 1}\ {}^{1}\ 1^{o}\ {}^{@_{89}\ 1\ <}\ {}^{@_{9:}}_{<^0}\ {}^{@_{:8}}_{<\ <^0}\ {}^{j@_{89}j\ 1}_{<}\ q^{9:}_{-<^0}\ q^{:8}_{-\ <\ <^0}\ - \tag{3.3.3} $$

where $\ ; \ ;0 \ ;00$ $< <0 <00$ are the Clebsch-Gordan coefficients given in (D.23) and we defined

$$ 4^{89}_{1d} = \frac{1}{q\ \ :\ <^{89}_{f}}\ - \qquad {}^{89:}_{-1d} = \frac{-_{3d}}{q\ \ 4c<^{89}_{f}<^{9:}_{f}}\ \bullet \tag{3.3.4} $$

The sign ${}^{1}\ 1^{o}\ {}^{@_{89}\ 1\ <}$ in the J-term is necessary for $SU^{1}2^{o}$ invariance. The term ^{89} in (3.3.2) exists for $@_{89}$ 1, then the condition $@_{:9}$ 0 in the first line guarantees that ${}^{89}_{1}$ and $q^{:9}$ both exist, and the condition $@_{8:}$ 0 in the second line guarantees that $q^{8:}$ and ${}^{:9}_{1}$ both exist. Also the term ^{98} in (3.3.3) exists for $@_{89}$ 1, which is guaranteed by the two conditions $@_{9:}$ 0, $@_{:8}$ 0 on the RHS. The E-term comes from the reduction of (3.2.16) whereas the J-term from the reduction of the 3d superpotential (3.1.11). One can check, by substituting (D.25) and relabeling the indices, that

$$ \tilde{O}_{89-}\ @_{89}\ 1\ \tilde{O}_{j<j\ @_{89}\ 1}\ {}^{89}_{-<}\ {}^{98}_{-\ <} = 0\ - \tag{3.3.5} $$

which is required for supersymmetry. The couplings 4_{1d} and $-_{1d}$ are obtained by tree-level matching.

The complete Lagrangian in terms of the and given above is:

$$ \begin{aligned} L_{QM} = :\ & \tilde{O}_{8}\ m_{8}\ {}^{8}_{c},\ f^{8}\ ,\ \tilde{O}_{89}\ {}^{1}@_{89}\ 1^{o}\ \tilde{O}_{j<j\ @_{89}\ 1}\ \overline{{}^{98}_{1-<}}\ {}^{8}\ c\ {}^{98}_{1-<},\ \overline{{}^{98}_{1-<}}\ {}^{98}_{1-<}\ , \\ & ,\ {}^{1}@_{89}\ \tfrac{1}{2}{}^{o}\ \tilde{O}_{j<j\ @_{89}}\ \overline{\text{æ}^{89}_{<}}\ {}^{8}\ c\ \text{æ}^{89}_{<},\ 6^{89\ 2}_{<}\ ,\ \tilde{O}\ {}^{1}@_{89}{}^{o}\ \tilde{O}_{j<j\ @_{89}}\ \overline{q^{89}_{-<}}\ {}^{8}\ c\ q^{89}_{-<},\ k^{89}_{-<}\ k^{89}_{-<}\ , \\ & ,\ {}^{1}\ @_{89}\ 1^{o}\ \tilde{O}_{j<j\ @_{89}\ 1}\ \overline{\Gamma^{89}_{-<}}\ {}^{8}\ c\ \Gamma^{89}_{-<},\ 5^{89\ 2}_{-<}\ {}^{89\ 2}_{-<}\ \overline{\Gamma^{89}_{-<}}\ \&\ {}^{89}_{-<}\ \overline{\&}\ \overline{{}^{89}_{-<}}\ \Gamma^{89}_{-<}, \\ & \qquad 5^{89}_{-<}\ {}^{98}_{-\ <}\ \overline{{}^{98}_{-\ <}}\ \overline{5^{89}_{-<}}\ \Gamma^{89}_{-<}\ \&\ {}^{98}_{-\ <}\ \overline{\&}\ \overline{{}^{98}_{-\ <}}\ \overline{\Gamma^{89}_{-<}}\ - \end{aligned} \tag{3.3.6} $$

where $8, 9 = 1, \cdots, \#$ whereas $= 1, 2, 3$. Note that both bosons and fermions have at most 1-derivative kinetic terms. The Lagrangian can be more compactly written in superspace:

$$L_{\rm QM} = \int 3\backslash 3\overline{\backslash} \; : \; \tilde{O}_{8} \, \mathrm{m}_8 +^8 \, ¸ \, \tilde{O}_{89} \, {}^1@_{89} \; 1^{\rm o} \, \tilde{O}_{j<j \; @_{89} \; 1} \; \overline{\tfrac{98}{1\text{-}<}} \, \tfrac{98}{1\text{-}<} \, ¸ \; {}^1@_{89} \; \tfrac{1}{2}{}^{\rm o} \, \tilde{O}_{j<j \; @_{89}} \; \overline{\tfrac{89}{<}} \, \tfrac{89}{<} \, ¸$$
$$¸ \, \tilde{O} \, {}^1@_{89}{}^{\rm o} \, \tilde{O}_{j<j \; @_{89}} \; \overline{\tfrac{89}{\text{-}<}} \, \tfrac{89}{\text{-}<} \, ¸ \; {}^1 \, @_{89} \; 1^{\rm o} \, \tilde{O}_{j<j \; @_{89} \; 1} \; \overline{Y^{89}_{\text{-}<}} \, Y^{89}_{\text{-}<} \; ¸$$
$$¸ \, \tilde{O}_{89\text{-}} \, {}^1 \, @_{89} \; 1^{\rm o} \, \tilde{O}_{j<j \; @_{89} \; 1} \; \int 3\backslash \, Y^{89}_{\text{-}<} \, \tfrac{98}{\text{-} <} \, {}^1 \, {}^{\rm o} \, ¸ \int 3\overline{\backslash} \, \overline{Y^{89}_{\text{-}<}} \, \overline{\tfrac{98}{\text{-} <}} \, {}^1 \overline{\,} {}^{\rm o} \, . \tag{3.3.7}$$

Here we promoted the scalar fields in to be chiral superfields.

The observables of the 3d theory include the gauge-invariant operators. After gauge fixing by $B_{\rm gf}$, they are the BRST-closed operators, invariant under the residual gauge symmetry, and with ghost number $=_6 = 0$. The further addition of $Q_{\rm gf}$ to the Lagrangian does not modify their correlators, see (3.2.59). When we go to the effective 1d description (3.3.6), the ghost field 2 is completely integrated out. Any operator containing $ẽ^{89}_<$ should not be regarded as a physical observable, because it will have $=_6 \ddot{Y} 0$. For instance, one might have noticed that the Lagrangian (3.3.6) has a large number of additional global $\mathrm{U}^1 1^{\rm o}$ symmetries that rotate each $ẽ^{89}_<$ independently. However, their currents are not physical observables (because they are constructed with $ẽ^{89}_<$), and indeed the symmetries act trivially on the sector of physical observables.[14] They should not be regarded as emergent symmetries of the physical theory. On the other hand, all operators constructed from the fields of the low-energy 1d description other than $ẽ^{89}_<$ and invariant under $\mathrm{U}^1 1^{\rm o \#}$, are physical observables. This is because the BRST transformations of the physical fields - are $B\text{-} = X_{\rm gauge}{}^1 2^{\rm o}\text{-}$, but 2 is massive and set to zero in the low-energy description.

3.3.1 Quantum mechanics 1-loop determinants and the Witten index

A simple check that we can perform of the proposed 1d quantum mechanics (3.3.7) is that its Witten index matches the topologically twisted index of the 3d theory, at leading order in the large $\#$ expansion. This ensures that its ground-state degeneracy reproduces the entropy of BPS black holes.

[14] In view of holographic applications of the low-energy quantum mechanics, one should not expect the extra symmetries to appear as gauge fields in AdS_2.

The Witten index of $N = 2$ supersymmetric quantum mechanics is defined in the same way as the topologically twisted index in (3.1.20). In the Lagrangian formulation, the chemical potentials are introduced as twisted boundary conditions on the fields. For a class of these models, the Witten index has been computed in [217] (see also [223, 224]), and it takes a Jeffrey-Kirwan contour integral form as in (3.1.21). We want to make sure that the quantum mechanics (3.3.7) reproduces the integrand in (3.1.21) for the value of m_8 singled out by the saddle-point approximation.

After fixing the 1d gauge $m_C \; {}^8_C \text{¸} \; f^8 = 0$, the Wilson line gives a classical contribution $\exp\, 2c8 \;\; {}_8 : \mathrm{m}_8 D_8$, where $2cD$ is the constant mode of the Wick-rotated ${}_C \text{¸} \; f$. The chirals $\bar{1}$ and Fermi's coming from the 3d vector multiplet contribute to the 1-loop determinant as

$$Z_{\bar{1}} = \prod_{8<9} \frac{4^{c8 D_{89}}}{1 \quad 4^{2c8 D_{89}}} {}^{1 \; @_{89} \;\; 1^{o\,1} \;\; 2 @_{89} \;\; 1^o} - \qquad Z \;\; = \prod_{8<9} \frac{4^{2c8 D_{89}} \quad 1}{4^{c8 D_{89}}} {}^{1 @_{89}{}^{o\,1} 2 @_{89} \text{¸} 1^o} - \tag{3.3.8}$$

where $D_{89} = D_8 \quad D_9$. The exponents come from the $2; \text{¸}\ 1$ degeneracy in each $\mathrm{SU}^1 2^o$ representation of highest weight $;$, and the functions ensure that nontrivial contributions only enter when the multiplets exist. Recalling that $@_{89} < 0$ for $8 < 9$, their product simplifies:

$$Z_{\bar{1}} Z \;\; = {}^1 \;\; 1^{o \frac{\#^1\# \;\; 1^o}{2}} \prod_{8<9} 1 \quad \frac{I_8}{I_9} - \tag{3.3.9}$$

where $I_8 = 4^{2c8 D_8}$. The result above matches (up to an inconsequential sign) the 1-loop determinant of a 3d vector multiplet given in [72] and appearing in (3.1.21).[15] As opposed to the indirect Higgsing argument which was used in [72], the result here provides an explicit derivation based on a careful gauge-fixing procedure. This computation shows that the ghost multiplet ^{89} appearing in the quantum mechanics is needed to reproduce the correct degeneracy of BPS states. Lastly, the chirals and Fermi's Y coming from the 3d chiral multiplets contribute to the 1-loop determinant as

$$Z \;\; = \prod_{8-9} \frac{4^{c8^1 D_{89}\text{¸} \quad o}}{1 \quad 4^{2c8^1 D_{89}\text{¸} \quad o}} {}^{1 @_{89}{}^{o\,1} 2 @_{89} \text{¸} 1^o} - \qquad Z_Y = \prod_{8-9} \frac{1 \quad 4^{2c8^1 D_{89}\text{¸} \quad o}}{4^{c8^1 D_{89}\text{¸} \quad o}} {}^{1 \; @_{89} \;\; 1^{o\,1} \;\; 2 @_{89} \;\; 1^o} - \tag{3.3.10}$$

[15] The 1-loop determinant of a Fermi multiplet has a sign ambiguity coming from the assignment of fermion number to states in the fermionic Fock space. We have fixed this ambiguity in a specific way to get (3.3.9), but different conventions are possible. Notice, for example, the different choice made in (3.3.10).

where $2c$ is the (off-shell) background flavor gauge field ¹ c ¸ f° . Their product is

$$
\begin{aligned}
Z \quad Z_Y &= \text{Ö}_{8\text{-}9} \frac{4^{c8^1 D_{89}\,,}\ {}^{\circ}}{1 \quad 4^{2c8^1 D_{89}\,,}\ {}^{\circ}}\ {}^{2@_{89}\,,1} \\
&= \frac{H^{\#^{2}{}^1\text{n}\ ,1^{\circ}\bullet 2}}{{}^1 1 \quad H^{\ \circ\#^1\text{n}\ ,1^{\circ}}} \text{Ö}_{8<9} \frac{I_8 \quad H\, I_9}{I_9 \quad H\, I_8}^{\text{m}_8} 1 \quad H \frac{I_8}{I_9}^{\text{n} \quad 1} \text{-}
\end{aligned}
\tag{3.3.11}
$$

where $H = 4^{2c8}$. The complete integrand is thus

$$
Z_{\text{tot}} = 4^{2c8:} \text{Í}_8 \,\text{m}_8 D_8\, Z \quad Z \quad \text{Ö} \quad Z \quad Z_Y \ \text{-}
\tag{3.3.12}
$$

matching the integrand in (3.1.21). This guarantees that a large # saddle-point computation of the 3d topologically twisted index matches a saddle-point computation of the 1d Witten index.

3.4 Stability under quantum corrections

The gauge-fixing action X_{gf} preserves the real supercharge X, U¹1°^{2}, and SU¹2°. We first use the X invariance of the full action to show that the fermion $\text{ẻ}^{89}_{<}$ only has gauge interactions. This allows us to focus on fields other than $\text{ẻ}^{89}_{<}$. Although the gauge fixing breaks &, $\overline{\&}$, and U¹1°′, we will then give arguments for why they should be preserved in the effective action. The key observation will be (3.2.59). Finally, we will use all the symmetries &, $\overline{\&}$, U¹1°^{2}, U¹1°′ and SU¹2° to discuss which classical and quantum corrections to the quantum mechanics computed in Section 3.3 one could expect.

3.4.1 Interactions involving $\widetilde{c}$

Using the fermionic symmetry X, we can argue that the part of the Lagrangian involving the fermions $\text{ẻ}^{89}_{<}$ cannot be anything other than (3.3.6) at low energies. Let h i$_X$ denote the gauge-fixed path integral, as in (3.2.59). For $8\text{-}9$ such that $@_{89}$ ¡ 0, we consider the quantity

$$
\begin{aligned}
\overline{\text{ẻ}^{89}_{<}}\,{}^1 C^{\circ} \quad \text{ẻ}^{89}_{<}\,{}^1 C^{0}{}^{\circ}\ {}_X &= \overline{\text{ẻ}^{89}_{<}}\,{}^1 C^{\circ}\, X 1^{89}_{<}\,{}^1 C^{0}{}^{\circ}\ {}_X \quad \overline{\text{ẻ}^{89}_{<}}\,{}^1 C^{\circ}\, X_{\text{gauge}}\,{}^{1\ \prime\ \circ}\, \text{ẻ}^{89}_{<}\,{}^1 C^{0}{}^{\circ}\ {}_X \\
&\quad \overline{\text{ẻ}^{89}_{<}}\,{}^1 C^{\circ}\, X 1^{89}_{<}\,{}^1 C^{0}{}^{\circ}\ {}_X \bullet
\end{aligned}
\tag{3.4.1}
$$

Here $1^{89}_{<}$ is the $; = @_{89}$ mode of the auxiliary field 1 in the gauge-fixing complex. In the first equality, we used (3.2.45) and (3.2.54). The approximate equality only holds in the IR limit because the term that was discarded is a correlation function involving massive ghosts 2 in $' = \;\; \mathsf{f}2\text{-}2\mathsf{g}_{\mathsf{r}}\bullet 2$, which is exponentially suppressed at large $C \quad C^0$. We continue using the Leibniz rule on X and the fact that X-exact correlators vanish, to write

$$\overline{\text{ê}^{89}_{<}}1C^o\, X1^{89}_{<}1C^{0}o \;\; _X = \quad X\overline{\text{ê}^{89}_{<}}1C^o\, 1^{89}_{<}1C^{0}o \;\; _X = 8 \;\; \overline{1^{89}_{<}}1C^o\, 1^{89}_{<}1C^{0}o \;\; _X \bullet \tag{3.4.2}$$

The path integral over $1^{89}_{<}$ is quadratic and can be done exactly, yielding

$$\overline{\text{ê}^{89}_{<}}1C^o \quad _{\dot{C}}\, \text{ê}^{89}_{<}1C^{0}o \;\; _X \quad 8 \;\; \overline{1^{89}_{<}}1C^o\, 1^{89}_{<}1C^{0}o \;\; _X = \quad X1C \quad C^{0}o \;\, _{\text{¸}}\, 8 \;\; \overline{O} \;\; 1C^o\, O \;\; 1C^{0}o \;\; _X \qquad X1C \quad C^{0}o\, \text{-} \tag{3.4.3}$$

where

$$O \;\; = \;\; \mathrm{r}\frac{@_{89}}{b\;'\,2} \quad \begin{smallmatrix}89\\ 1\text{-}@_{89}\text{-}<\end{smallmatrix} \qquad \frac{4_{\text{3d}}}{'}\; \mathsf{f}\text{ê-}2\mathsf{g}^{89}_{;=@_{89}\text{-}<} \bullet \tag{3.4.4}$$

The expression $\mathsf{f}\text{ê-}2\mathsf{g}^{89}_{;=@_{89}\text{-}<}$ stands for the $\;;\; = @_{89}\text{-} <\;$ mode of $\mathsf{f}\text{ê-}2\mathsf{g}^{89}$. Both terms inside O contain massive fields only, therefore $\;\overline{O}\;\; 1C^o\, O \;\; 1C^{0}o \;\; _X$ is exponentially suppressed at large distances and the approximation holds to increasing accuracy in the IR. Using only symmetry arguments for X, we have shown that $\text{ê}^{89}_{<}$ must satisfy the Schwinger-Dyson equation derived from (3.3.6) in the IR limit. Any modification of (3.3.6) containing $\text{ê}^{89}_{<}$ would change the Schwinger-Dyson equation, and can thus be excluded.

3.4.2 Presence of $N \; = 2$ supersymmetry and R-symmetry

Having taken care of $\text{ê}^{89}_{<}$, we want to constrain the effective Lagrangian for the remaining fields. Here we show that in the IR it must preserve 1d $N \; = 2$ supersymmetry and $\mathsf{U}^{1}1^{o}\,{}_{'}$, even though these symmetries are broken by the gauge-fixing term $X \;\; _{\text{gf}}$.

First, we show that the Ward identities for the supercharges $\&$ and $\overline{\&}$ are satisfied on correlators O constructed from 1d fields excluding $\text{ê}^{89}_{<}$, which are modes of physical fields in 3d. More precisely, we show that $\mathsf{h}\&O\mathsf{i}_X \quad 0$ (and analogously for $\overline{\&}$). As before, approximate equalities $\quad$ hold in the IR limit. Firstly, since O is constructed from modes of physical fields, it has $=_6 \; = 0$, and the same goes for $\&O$. Then (3.2.59) tells us that $\mathsf{h}\&O\mathsf{i}_X = \mathsf{h}\&O\mathsf{i}_B$. It remains to show that $\mathsf{h}\&O\mathsf{i}_B \quad 0$.

We then follow the standard procedure to derive a Ward identity. In the path integral $\mathsf{h}O\mathsf{i}_B$ we perform a field redefinition $\text{-}^{0} = \text{-} \;_{\text{¸}}\, n\,\&\,\text{-}$ on physical fields - in the form of a supersymmetry transformation, while keeping the fields $\,.\,$ in the gauge-fixing complex unchanged. Let $S_{\rm ph}$ be the original action before gauge fixing. At first order in ϵ we get

$$\begin{aligned}\langle O\rangle_Q &= \int D\phi\, O\, e^{-(S_{\rm ph}+QV_{\rm gf})} = \int D\phi\, (O+\epsilon\,\delta O)\, e^{-(S_{\rm ph}+QV_{\rm gf})-\epsilon Q\delta V_{\rm gf}} \\ &= \langle O\rangle_Q + \epsilon\left(\langle\delta O\rangle_Q - \langle O\, Q\delta V_{\rm gf}\rangle_Q\right) + \cdots\end{aligned} \tag{3.4.5}$$

Suppose that O is fermionic so that $\langle\delta O\rangle_Q \to 0$ is a non-trivial statement. At order ϵ, that equality implies

$$\langle\delta O\rangle_Q = -\langle O\, Q\delta V_{\rm gf}\rangle_Q = -\left\langle Q(O\,\delta V_{\rm gf})\right\rangle_Q = -\left\langle Q_{\rm gauge}(c\,O\,\delta V_{\rm gf})\right\rangle_Q \to 0. \tag{3.4.6}$$

We used that $\langle Q(O\,\delta V_{\rm gf})\rangle_Q = 0$ because the action $S_{\rm ph}+QV_{\rm gf}$ is Q-closed. In the last step, c is massive and therefore its correlators vanish in the IR. We can now use (3.2.59) to conclude that $\langle\delta O\rangle_{\mathcal{Q}} = \langle\delta O\rangle_Q \to 0$.

The Ward identity for $U(1)_R$ can be derived with much less work. Any O built out of 1d fields excluding $\lambda^{\pm}_{\neq}$ has $q_R = 0$, and $\langle O\rangle_{\mathcal{Q}} = \langle O\rangle_Q$ by (3.2.59). Since $QV_{\rm gf}$ is $U(1)_R$ invariant, $\langle O\rangle_Q = 0$ if O has nonzero R-charge. Therefore $\langle O\rangle_{\mathcal{Q}} = 0$ if O has nonzero R-charge.

Given the above Ward identities, any effective action in the IR should have 1d $\mathcal{N}=2$ supersymmetry and $U(1)_R$ symmetry. For $U(1)_R$, we can see this in the following way (the argument for supersymmetry is analogous). Formally, the exact effective action for the fields in the quantum mechanics is given by

$$e^{-(S_0+\sum_{r\neq 0} S_r)} = \int D\phi_h\, e^{-(S_{\rm ph}+\mathcal{Q}V_{\rm gf})}, \tag{3.4.7}$$

where S_r, $r\in\mathbb{Z}$ are pieces of the effective action with R-charge r, and ϕ_h are the massive fields which are integrated out. Note that the $U(1)_R$ violating pieces $S_{r\neq 0}$ can in principle be generated because $\mathcal{Q}V_{\rm gf}$ breaks $U(1)_R$. However, the presence of any $S_{r\neq 0}$ would generically violate the $U(1)_R$ Ward identities. Indeed, suppose S_r is present for some $r\neq 0$ and consider an operator O with R-charge $-r$ which is constructed out of the fields ϕ_ℓ in the quantum mechanics excluding $\lambda^{\pm}_{\neq}$. The Ward identities tell us that $\langle O\rangle_{\mathcal{Q}} = 0$. However, computing $\langle O\rangle_{\mathcal{Q}}$ directly gives:

$$\langle O\rangle_{\mathcal{Q}} = \int D\phi_\ell\, O\, e^{-(S_0+S_r)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int D\phi_\ell\, O\, S_r^n\, e^{-S_0} = -\int D\phi_\ell\, O\, S_r\, e^{-S_0} \neq 0. \tag{3.4.8}$$

We used that S_0 is $U(1)_R$ invariant, while O and $O S_r^{n\geq 2}$ have nonzero R-charge. The operator $O S_r$ has zero R-charge and its expectation value is generically nonzero.

3.4.3 Symmetry constraints

We can use U¹1º ', &, and $\overline{\&}$, together with the other symmetries, to constrain the interactions that could appear in the effective action. We work within the framework of [217] (see also [225]), where the interactions in an $N = 2$ supersymmetric quantum mechanics are specified by and functions, *i.e.*, holomorphic functions of chiral superfields satisfying (3.3.5). The argument in Section 3.4.1 tells us that the and functions corresponding to must vanish in the IR:

$$ {}^{89}_{-<} = 0 - \qquad {}^{98}_{-\ <} = 0 \bullet \tag{3.4.9}$$

Besides, cannot appear in the E- and J-terms of the other Fermi multiplets Y. Since it is already true that $\overline{}Y < 0$ for every Y, one expects that Y's cannot appear in or functions either, because quantum corrections would need to be finely tuned to make them chiral. Therefore, and functions can only be holomorphic functions of and $\bar{1}$.

Let us neglect gauge charges and SU¹2º invariance momentarily, and suppress the corresponding indices. To have the same U¹1º² charges as Y and R-charge '¹Yº ¸ 1, the function corresponding to Y must have the simple form

$$ {}^{1}\ \bar{1}^{\text{º}} - \tag{3.4.10}$$

where is a holomorphic function. Fleshing out the gauge and SU¹2º indices, we enforce that ${}^{89}_{-<}$ have the same gauge charges and be in the same SU¹2º representation as $Y^{89}_{-<}$. Imposing those conditions on the constant term in , we get ${}^{89}_{-<}$ ${}^{89}_{-<}$. However, such a term is impossible because $Y^{89}_{-<}$ (and therefore ${}^{89}_{-<}$) exists when $@_{89}$ 1, while ${}^{89}_{-<}$ exists when $@_{89}$ 0. The two conditions are mutually exclusive.[16] We remain with terms in which are at least linear in $\bar{1}$. Writing the first term explicitly, we find:

$$ {}^{89}_{-<} = \sum_{:} 4^{89}_{-:}\ {}^{1}@_{:9}{}^{\text{º}} \sum_{j<^0 j\ @_{:9}} \ldots \ {}^{8:}_{1-<}\ {}_{<^0}\ {}^{:9}_{-<^0} \,¸ \sum_{:} \not{4}^{89}_{-:}\ {}^{1}@_{8:}{}^{\text{º}} \sum_{j<^0 j\ @_{8:}} \ldots\ {}^{8:}_{-<^0}\ {}^{:9}_{1-<}\ {}_{<^0}\,¸ \cdots \tag{3.4.11}$$

The functions are necessary to ensure that the fields and $\bar{1}$ exist with their corresponding gauge charges. The Clebsch-Gordan coefficients project the product of $\bar{1}$ and to the same SU¹2º representation carried by ${}^{89}_{-<}$, *i.e.*, $; = j@_{89}j \quad 1$. Finally, $4^{89}_{-:}$ and $\not{4}^{89}_{-:}$ are

[16] Because of this, the chirals and Fermi's in the quantum mechanics cannot gap each other out through a dynamically generated E-term.

free coefficients. Analogously, terms of the form ¹ ‾1º⁼ ² should contain a product of = Clebsch-Gordan coefficients and balanced gauge indices.

When constraining the functions corresponding to Y, we again start with U¹1º² and U¹1º ʻ. Now, must have the opposite U¹1º² charges to Y, and R-charge ʻ¹Yº ¸ 1. Thus must have the form

$$ ^1\ \bar{1}^{o}\ - \tag{3.4.12}$$

where and are different flavor indices complementary to . Again, is a holomorphic function. We should impose the gauge and SU¹2º invariance. Expanding as a polynomial in ‾1 and writing the first (constant) term explicitly, we have

$$
\begin{aligned}
{}^{98}_{-\ <} = \tilde{O}\ " \ \frac{\overset{98}{— -:}}{q\overline{2j@_{89}j\ \ 1}}\ {}^1@_{9:}{}^o\ \ {}^1@_{:8}{}^o\ \tilde{O}\ {}^1\ 1^o\ {}^{@_{89}}\ 1\ <\ \ @_{9:}\ <^0\ \ @_{:8}\ <\ <^0\ \ j@_{89}j\ \ 1\ <\ \ {}^{9:}_{-<^0}\ \ {}^{:8}_{-}\ <\ <^0\ ¸ \\
j<^0j\ @_{9:}\quad j<¸<^0j\ @_{:8} \\
¸\ \frac{\overset{98}{e— -:}}{q\overline{2j@_{89}j\ \ 1}}\ {}^1@_{9:}{}^o\ \ {}^1@_{:8}{}^o\ \tilde{O}\ {}^1\ 1^o\ {}^{@_{89}}\ 1\ <\ \ @_{9:}\ <^0\ \ @_{:8}\ <\ <^0\ \ j@_{89}j\ \ 1\ <\ \ {}^{9:}_{-<^0}\ \ {}^{:8}_{-}\ <\ <^0\ ¸\ \cdots \# \\
j<^0j\ @_{9:}\quad j<¸<^0j\ @_{:8}
\end{aligned}
\tag{3.4.13}
$$

The indices and above are chosen such that n = 1, and the factor 1• q‾2j@⁸⁹j 1 was added for later convenience. Similarly to the function, there are two unfixed coefficients ‾⁹⁸₋.: and e‾⁹⁸₋.:. Terms of the form ¹ ‾1º⁼ ¹ should contain a product of = ¸ 1 Clebsch-Gordan coefficients and gauge indices should be balanced.

Lastly, supersymmetry requires (3.3.5). If we restrict ⁸⁹₋< and ⁹⁸₋ < to the terms written explicitly in (3.4.11) and (3.4.13), this condition implies

$$
\begin{aligned}
&4^{89}_{-:}\ \overset{98}{—}_{-;}\ ¸\ 4^{:;}_{\ 8}—^{:;}_{-9} = 0 \quad \text{if} \quad n = 1 \quad \text{and} \quad {}^1@_{:9}{}^o\ \ {}^1@_{9;}{}^o\ \ {}^1@_{;8}{}^o = 1 - \\
&4^{89}_{-:}\ e^{98}_{—\ -;}\ ¸\ 4^{:;}_{\ 8}e^{:;}_{—\ -9} = 0 \quad \text{if} \quad n = 1 \quad \text{and} \quad {}^1@_{:9}{}^o\ \ {}^1@_{9;}{}^o\ \ {}^1@_{;8}{}^o = 1\ •
\end{aligned}
\tag{3.4.14}
$$

Note that none of the indices above are summed over. The coefficients in (3.3.2) and (3.3.3) that we found from the reduction satisfy these equations, but they might not be a unique choice. The constraint (3.3.5) would also have to be enforced on terms with higher powers of ‾1, strongly constraining their coefficients.

From classical scaling arguments, we are not able to rule out the presence in (3.4.11) and (3.4.13) of terms that have higher powers of ‾1. They could be generated both at the tree and at the loop level. It would be consistent to neglect those terms if ‾1, which is classically dimensionless, gained a positive anomalous dimension. This is indeed the case for classically dimensionless fermions in SYK models such as [154, 183], but it remains to be checked in the theory discussed here.

Chapter 4

Conclusions

In this thesis, we investigated how holography can capture and describe the microstates of BPS and near-BPS asymptotically-AdS black holes, by means of a dual field theory analysis.

In Chapter 2, we have evaluated, using a Bethe ansatz approach, the superconformal index of a broad class of 4d $\mathcal{N}=1$ holographic quiver gauge theories at leading order in N, for generic fugacities for the global symmetries commuting with the supercharges. In particular, we focused on the contribution of the so-called "basic solution" [116] to the Bethe ansatz equations. By doing so, we extended previous results for $\mathcal{N}=4$ SYM [85] and toric quiver gauge theories [125]. This result allowed us to match the Bekenstein-Hawking entropy of Gutowski-Reall black holes [135] for generic electric charges and angular momenta. We were also able, in Section 2.3 and 2.4, respectively, to match the entropy of the universal black hole (which can be embedded in any $\mathrm{AdS}_5 \times \mathrm{SE}_5$ compactification of type IIB supergravity) and predict the entropy of putative black holes in a consistent truncation [169] of type IIB supergravity on $\mathrm{AdS}_5 \times T^{1,1}$.

The last result is particularly non-trivial: the matching was done using a purely near-horizon analysis, in the spirit of [118]. A similar computation has been recently carried out for the gravity dual of $\mathcal{N}=1$ theories in [80]. Supersymmetric black holes with generic electric charges have not been constructed yet in the case of type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{SE}_5$, if not in the case of $\mathrm{SE}_5 = S^5$. It would be interesting to construct new supergravity solutions for generic SE_5, and explicitly see the duality at work there.

A better understanding of the solutions to the Bethe ansatz equations and the regimes in which they are dominant is also crucial. First of all, it would be interesting to understand the precise relation between the Bethe ansatz solutions and the saddles one gets from a saddle-point evaluation of the superconformal index [93]. A step in this direction was done in [90, 105]. Secondly, one has to establish a gravity interpretation for the different Bethe ansatz solutions. In [185] a detailed analysis was performed, but a complete picture is still missing. For example, multi-center black-hole solutions could play a role in this business: the existence of multi-center solutions in AdS space is still unsure, but recent analysis [226, 227] point towards a positive answer, and one could thus expect to see their contribution to the index. Other interesting analysis of the gravitational interpretation of the CFT results are [76, 101, 103, 104, 109, 121, 126, 130, 131]. A more basic problem that is relevant for the gravity interpretation is the definition of path-integrals for gravitational systems, for example in light of the new advances [228, 229] regarding the importance of considering complex metrics.

In Chapter 3 we have performed a first principle derivation of the quantum-mechanical theory dual to the horizon of asymptotically-AdS$_4$ $(^6$ static, magnetic black holes in massive type IIA supergravity. To do this, we put the dual 3d $N = 2$ Chern-Simons-matter theory on $(^2$ with fluxes, and we integrated out all the heavy modes. The theory we got in (3.3.7), given (3.3.2) and (3.3.3), is an $N = 2$ gauged quantum mechanics with finitely many dynamical bosons and fermions. All the fields have at most linear kinetic terms, and the E- and J-term couplings are governed by Clebsch-Gordan coefficients.

In the last few years, starting from [139], many 1d models have been studied, aiming at a holographic connection to black-hole physics. In [152–158] the authors were able to derive, by zooming in the near-horizon region of asymptotically-AdS (and asymptotically-flat) black holes, 2d JT-gravity models capturing the near-horizon physics. Unfortunately, first-principle derivations on the dual field theory side were still lacking, but people found different 1d models with promising features; we mention here the supersymmetric models [154, 183]. With respect to them, our model contains not only Fermi but also chiral multiplets. Due to the linear kinetic terms, the chiral multiplets contain a single scalar dynamical degree of freedom. In this respect, they are very similar to the Fermi multiplets. It would be interesting to understand what are the differences between fermionic, bosonic, and mixed models like ours.

Moreover, this model could be used to better understand the basics of the AdS$_2$/CFT$_1$ correspondence. What is the origin of the averages which play a central role in the JT-gravity/SYK-model correspondence, and are they essential? In our theory, the couplings are determined and fixed by the reduction. Nevertheless, they follow a “statistical distribution”: to trade them for a random variable can be seen as a simplifying approximation.

We expect this model to capture the thermodynamics of near-BPS black holes. In the near-BPS sector, one usually does not have enough control to analytically compute quantities of interest. In field theory, for example, one should compute partition functions for a d-dimensional theory at strong coupling, which is known to be a hard task. Our model is particularly useful because it is 1-dimensional: a partition function in 1d can be computed using, for example, the Schwinger-Dyson techniques developed in the last few years in the analysis of the SYK model; in this way, we gained computational power for a problem which was previously difficult to attack.

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Appendix A

Large N computations

In this Appendix, we present the manipulations needed to obtain the large N results which are heavily exploited throughout this dissertation. Appendix A.1 contains the definition of useful special functions. In Appendix A.2 and Appendix A.3, instead, we will focus on the 4d superconformal index and the 3d topologically twisted index, respectively.

A.1 Special functions

It is useful to start by defining some special functions which will turn out useful in what follows. We recall here that $x = e^{2\pi i u}$, $p = e^{2\pi i \sigma}$, $y = e^{2\pi i \Delta}$, and $q = e^{2\pi i \tau}$. We will also list some properties which will play a role in our computations.

Elliptic functions. We begin with the functions which play a role in the evaluation of the superconformal index. Firstly, the h-Pochhammer symbol is defined as

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - x q^n) , \tag{A.1.1}$$

and, for $|x|, |q| < 1$, it admits the plethystic expansion

$$(x; q)_\infty = \exp\left[-\sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1 - q^m} \right] . \tag{A.1.2}$$

The h-theta function is defined as

$$\theta_0(u; \tau) = (x; q)_\infty (q/x; q)_\infty . \tag{A.1.3}$$

with plethystic expansion for $|q| < |z| < 1$

$$\theta_0(u;\tau) = \exp\left[-\sum_{k=1}^{\infty} \frac{1}{k}\frac{z^k + (q/z)^k}{1-q^k}\right]. \tag{A.1.4}$$

It exhibits the reflection relation

$$\theta_0(u;\tau) = -e^{2\pi i u}\,\theta_0(-u;\tau). \tag{A.1.5}$$

The elliptic gamma function $\tilde{\Gamma}$ is defined as

$$\tilde{\Gamma}(u;\sigma,\tau) \equiv \Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}, \tag{A.1.6}$$

with plethystic expansion for $|pq| < |z| < 1$

$$\tilde{\Gamma}(u;\sigma,\tau) = \exp\left[\sum_{k=1}^{\infty} \frac{1}{k}\frac{z^k - (pq/z)^k}{(1-p^k)(1-q^k)}\right]. \tag{A.1.7}$$

It satisfies the following useful shift property

$$\begin{aligned} &\tilde{\Gamma}(u + mab\tau; a\tau, b\tau) = \\ &= \left(-e^{2\pi i u}\right)^{\frac{ab}{2}m^2 + \frac{a+b-1}{2}m} \left(e^{2\pi i\tau}\right)^{\frac{ab}{6}m^3 + \frac{ab(a+b)}{4}m^2 + \frac{a^2+b^2+3ab-1}{12}m}\, \theta_0(u;\tau)^m\, \tilde{\Gamma}(u;a\tau,b\tau), \end{aligned} \tag{A.1.8}$$

where $m \in \mathbb{Z}$ which was proven in [85], the inversion formula

$$\tilde{\Gamma}(u;\sigma,\tau) = 1/\tilde{\Gamma}(\sigma+\tau-u;\sigma,\tau), \tag{A.1.9}$$

and the identity [230]

$$\tilde{\Gamma}(u;\sigma,\tau) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1} \tilde{\Gamma}(u + r\sigma + s\tau; a\sigma, b\tau), \tag{A.1.10}$$

for any $\sigma,\tau \in \mathbb{H}$ and any $a,b \in \mathbb{N}$. This is can be proven by exploiting the infinite product expression of $\tilde{\Gamma}$. Now, exchanging $a \leftrightarrow b$ and $r \leftrightarrow s$ in the formula, and then setting $\sigma \to a\tau$, $\tau \to b\tau$, we obtain the formula [231]

$$\tilde{\Gamma}(u;a\tau,b\tau) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1} \tilde{\Gamma}(u + (as+br)\tau; ab\tau, ab\tau). \tag{A.1.11}$$

Bernoulli polynomials. One can define the Bernoulli polynomials by introducing their generating function

$$B(t,x) = \frac{t e^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} , \qquad B_n(x) = \frac{\partial^n}{\partial t^n} B(t,x)\Big|_{t=0} , \tag{A.1.12}$$

from which we can explicitly extract the first few ones

$$\begin{aligned} B_0(x) &= 1 , & B_1(x) &= x - \frac{1}{2} , \\ B_2(x) &= x^2 - x + \frac{1}{6} , & B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x , \end{aligned} \tag{A.1.13}$$

and check that they satisfy the properties

$$B_n(1-x) = (-1)^n B_n(x) , \qquad B_n'(x) = n B_{n-1}(x) . \tag{A.1.14}$$

Poly-logarithms. For the topologically twisted index is instead useful to introduce the poly-logarithms. They are defined through their Taylor series around $z = 0$

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} , \tag{A.1.15}$$

which is absolutely convergent for $|z| < 1$. This definition can be analytically continued to the whole complex plane, with a branch cut on the real axis from $z = 1$ to $z = \infty$. In particular $\mathrm{Li}_1(z) = -\log(1-z)$, where the principal sheet defined by (A.1.15)) is such that $\operatorname{Im}\log \in (-\pi, \pi]$. The functions $\mathrm{Li}_{k \geq 2}$ have an absolutely convergent series (A.1.15) on the unit circle and are thus continuous at $z = 1$, while the functions $\mathrm{Li}_{k \leq 0}$ have a pole at $z = 1$ but no branch cut, in particular $\mathrm{Li}_0(z) = z(1-z)^{-1}$.

One can define the single-valued analytic functions

$$g_k(u) = \mathrm{Li}_k\left(1 - e^{-2\pi i u}\right) \tag{A.1.16}$$

defined by (A.1.15) in the domain $\left|1 - e^{-2\pi i u}\right| < 1$ with $\operatorname{Re} u \in \left(-\frac{1}{4}, \frac{1}{4}\right)$, implying that $g_k(0) = 0$, and by analytic continuation elsewhere. For instance, we have

$$g_0(u) = e^{2\pi i u} - 1 , \qquad g_1(u) = 2\pi i u . \tag{A.1.17}$$

Whenever the function is differentiable, we have

$$z\, \partial_z \mathrm{Li}_k(z) = \mathrm{Li}_{k-1}(z) \tag{A.1.18}$$

or alternatively

$$m_D \mathrm{Li}_{:}\; 4^{2c8D} = 2c8\,\mathrm{Li}_{:\;1}\; 4^{2c8D} \qquad \text{or} \qquad m_D\;\; {:}\,^1D^o = \frac{2c8}{4^{2c8D} \quad 1}\;\; {:}\;\; {}_1{}^1D^o\,\bullet \tag{A.1.19}$$

The last relation allows one to define

$$:{}^1D^o = \int_0^{1\;D} 3F\,\frac{2c8}{4^{2c8F}\quad 1}\;\; {:}\;\; {}_1{}^1F^o\,- \tag{A.1.20}$$

which is single-valued because the integrand is analytic with no poles. The poly-logarithms satisfy the following identities:

$$\begin{aligned}
\mathrm{Li}_0\; 4^{2c8D} \;{}_{\text{¸}}\; \mathrm{Li}_0\;\; 4\;\; {}^{2c8D} &= \qquad\quad {}_0{}^1D^o\,- \\
\mathrm{Li}_1\; 4^{2c8D} \quad \mathrm{Li}_1\;\; 4\;\; {}^{2c8D} &= \quad {}^12c8^o\;\; {}_1{}^1D^o\,- \\
\mathrm{Li}_2\; 4^{2c8D} \;{}_{\text{¸}}\; \mathrm{Li}_2\;\; 4\;\; {}^{2c8D} &= \frac{1}{2}{}^12c8^{o2}\;\; {}_2{}^1D^o\,- \\
\mathrm{Li}_3\; 4^{2c8D} \quad \mathrm{Li}_3\;\; 4\;\; {}^{2c8D} &= \frac{1}{6}{}^12c8^{o3}\;\; {}_3{}^1D^o\,-
\end{aligned} \tag{A.1.21}$$

where $\;{}_{=}\;{}_3{}^1D^o$ where defined in (A.1.13). These relations are valid for $\mathrm{Re}\,D\;2\;{}^10\text{-}\,1^o$ and the poly-logarithms in their principal determination, and can then be extended to the whole complex plane by analytic continuation (notice that the functions on the RHS are polynomials with no branch cuts).

A.2 The large N SCI

In this Section, we will prove that, up to subleading corrections, the building block (2.1.20) can be written as (2.1.29). This result is the core of our large # computation of the superconformal index.

A.2.1 Simplifications of the SCI building block

We want to show that the terms neglected in passing from (2.1.22) to (2.1.23) are subleading at large #. We will first analyze the effect of dropping the term $I\,{}^13\quad 2^o\bullet\#$ from the arguments of the gamma functions, in all those terms with $W < X$. We will later estimate the contribution from the terms with $W = X$ that were discarded from the sum. Defining

$$5\,{}^1I^o = \sum_{W<X}^{\#} \log e\;\; I\;{}_{\text{¸}}\; I\,\frac{X \quad W}{\#}; 01\,I\text{-}\,01\,I \quad - \tag{A.2.1}$$

we want to show that

$$5\ l \ ¸ \quad l \bullet \# \qquad 5\,¹l^{o} \qquad O¹\#\log\#^{o}\ - \tag{A.2.2}$$

where $= ¹3 \quad 2^{o}\bullet 01$, $l = \quad ¸\ l\ 3 \quad 2\ ¸\ 0B\ ¸\ 1A$ and $2\text{-}3 = 1\text{-}\bullet\bullet\bullet\text{-}01$, $A = 0\text{-}\bullet\bullet\bullet\text{-}0 \quad 1$, $B = 0\text{-}\bullet\bullet\bullet\text{-}1 \quad 1$. Without loss of generality, we can assume $¡\ 0$, because the case $Ÿ\ 0$ is analogous while $= 0$ is trivial. As in [73], we discard the Stokes lines $2\ \mathbb{Z}\ ¸\ \mathbb{R}\ l$ except for the point $= 0$, because the limit we compute would be singular along those lines anyway. If is not on a Stokes line, then the restriction of 5 to the straight line in the complex plane passing through the points l and $l\ ¸\ \ l$ is a ^{1} complex function. In the case $= 0$, instead, we consider the restriction of 5 to the straight closed segment from l to $l\ ¸ \quad l \bullet \#$: one can check that 5 is ^{1} along that segment, because for $W < X$ the segment, suitably shifted, it hits neither zeros nor poles of any of the gamma functions in (A.2.1) (in both cases, 5 is a holomorphic function in a neighborhood of the restricted domain). A complex analog of the Mean Value Theorem (MVT) then states that

$$\begin{aligned} \mathrm{Re}\,\frac{5\ l\ ¸ \quad l\bullet\# \qquad 5\,¹l^{o}}{l} &= \frac{}{\#}\ \mathrm{Re}\ 5^{0}\ \ l\ ¸\ \bar{z}_1\, l\bullet\# \\ \mathrm{Im}\,\frac{5\ l\ ¸ \quad l\bullet\# \qquad 5\,¹l^{o}}{l} &= \frac{}{\#}\ \mathrm{Im}\ 5^{0}\ \ l\ ¸\ \bar{z}_2\, l\bullet\# \end{aligned} \tag{A.2.3}$$

with $\bar{z}_1\text{-}\bar{z}_2\ 2\ ¹0\text{-} \quad ^{o}$. Summing the absolute values follows the bound

$$\frac{5\ l\ ¸ \quad l\bullet\# \qquad 5\,¹l^{o}}{l} \qquad \frac{1}{\#} \quad 5^{0}\ \ l\ ¸\ \bar{z}_1\, l\bullet\# \quad ¸ \quad 5^{0}\ \ l\ ¸\ \bar{z}_2\, l\bullet\# \tag{A.2.4}$$

where we used $\mathrm{j} \quad \mathrm{j} \quad 1 \quad \frac{1}{01}\ Ÿ\ 1$. It is therefore sufficient to show that

$$\frac{1}{\#} \quad 5^{0}\ \ l\ ¸\ \bar{z}\, l\bullet\# \qquad O¹\#\log\#^{o} \tag{A.2.5}$$

for any $\bar{z}\ 2\ ¹0\text{-} \quad ^{o}$. Notice that $0\ Ÿ\ \bar{z}\ Ÿ\ 1 \quad \frac{1}{01}$.

We reason as follows. For $8\ \mathbb{Z}\ ¸\ \mathbb{R}\ l$, the arguments of the elliptic gamma functions in (A.2.1) remain at an $\#$-independent distance from the zeros and poles, that in our case are placed at the points

$$D_{0\text{-}8} = ¹1\ ¸\ 8^{o}01\ \ l\ - \qquad D_{1\text{-}9} = ¹1 \quad 9^{o}01\ \ l \qquad \text{for} \qquad 8\text{-}\ 9\ 2\ \mathbb{Z}\ \ _1\ - \tag{A.2.6}$$

respectively. The orders of the zeros and poles are 8 and 9 respectively. The ratio $\mathrm{e}^0 \bullet \mathrm{e}$ is bounded on the range of possible arguments, therefore

$$\frac{1}{\#} \; 5^0 \; I \,¸\, \overline{2} I \bullet \# \qquad \# \max_{C2» \; 01\text{-}301 \;\; 0 \;\; 1¼} \; \frac{\mathrm{e}^0 \quad ¸ \, C\, I ; 01 \, I \text{-} 01 \, I}{\mathrm{e} \quad ¸ \, C\, I ; 01 \, I \text{-} 01 \, I} = O^1 \#^o \bullet \tag{A.2.7}$$

The case $= 0$ is more subtle since, as $\#$ grows, the arguments of some of the gamma functions can get increasingly close to zeros or poles instead of staying at an $\#$-independent distance, and the $\#$-independent bound above does not apply. This happens when

$$I = \overline{D}_{0\text{-}8} \; 2 \; \mathrm{f} D_{0\text{-}8} \text{-} D_{0\text{-}8} \quad I \, \mathrm{g} \qquad \text{or} \qquad I = \overline{D}_{1\text{-}9} \; 2 \; \mathrm{f} D_{1\text{-}9} \text{-} D_{1\text{-}9} \quad I \, \mathrm{g} \bullet \tag{A.2.8}$$

One can easily see that for $= 0$, I can range from ${}^1 1 \quad 01^o \, I$ to ${}^1 301 \quad 0 \quad 1 \quad 1^o \, I$ so that the problematic points we may approach are the simple zero at $D_{0\text{-}1}$, the simple pole at $D_{1\text{-}1}$, and the double pole at $D_{1\text{-}2}$.

We now introduce a few results for later use. For a meromorphic function 6 whose zeros include $\mathrm{f} I_8 \mathrm{g}$ of order $\mathrm{f} <_8 \mathrm{g}$ and whose poles include $\mathrm{f} ?_9 \mathrm{g}$ of order $\mathrm{f} =_9 \mathrm{g}$, one can write

$$6^1 I^o = \frac{\hat{\mathrm{Î}}_8 \, {}^1 I \quad I_8{}^{o <_8}}{\hat{\mathrm{Î}}_9 \, {}^1 I \quad ?_9{}^{o =_9}} B^1 I^o \text{-} \tag{A.2.9}$$

where $B^1 I^o$ is meromorphic with zeros and poles at the remaining zeros and poles of 6 that were not included in $\mathrm{f} I_8 \mathrm{g}$ and $\mathrm{f} ?_9 \mathrm{g}$. Taking the derivative of this expression and computing $6^0 \bullet 6$, one finds

$$\frac{6^{0\,1} I^o}{6^1 I^o} = \tilde{\mathrm{O}}_8 \frac{<_8}{I \quad I_8} \quad \tilde{\mathrm{O}}_9 \frac{=_9}{I \quad ?_9} \; ¸ \quad {}^1 I^o \text{-} \tag{A.2.10}$$

where ${}^1 I^o = B^{0\,1} I^o \bullet B^1 I^o$ is meromorphic with simple poles at the remaining zeros and poles of 6 that were not included in $\mathrm{f} I_8 \mathrm{g}$ and $\mathrm{f} ?_9 \mathrm{g}$. Therefore we can apply (A.2.10) to the meromorphic function e and say that

$$\frac{1}{\#} \; 5^0 \; I \,¸\, \overline{2} I \bullet \# \qquad \frac{1}{\#} \tilde{\Theta}_{W<X} \; \frac{\mathrm{e}^0 \; I \,¸\, D^2_{W\text{-}X} ; 01 \, I \text{-} 01 \, I}{\mathrm{e} \; I \,¸\, D^2_{W\text{-}X} ; 01 \, I \text{-} 01 \, I}$$

$$\frac{1}{\#} \tilde{\Theta}_{W<X} \left(\frac{1}{j I \,¸\, D^2_{W\text{-}X} \quad 201 \, I \, j} \,¸\, \frac{1}{j I \,¸\, D^2_{W\text{-}X} j} \,¸\, \frac{2}{j I \,¸\, D^2_{W\text{-}X} \,¸\, 01 \, I \, j} \right)! \; ¸ \; {}^1 \# \quad 1^o \tag{A.2.11}$$

where we defined

$$D^{\bar{2}}_{W\text{-}X} = I\,\frac{X \quad W \,¸\, \bar{2}}{\#}\text{-} \qquad = \max_{\text{C2» 01-301 0 1¼}} \text{e}^{1}\text{C}\, I^{\text{o}} \tag{A.2.12}$$

and e is the meromorphic function associated to e in (A.2.10). We can bound its value with an #-independent quantity because it is holomorphic on the range of possible arguments. If $I < \bar{D}_{0\text{-}1}\text{-}\bar{D}_{1\text{-}1}\text{-}\bar{D}_{1\text{-}2}$, the outlying sums in (A.2.11) will be of order $O^1\#^{\text{o}}$ since $I \,¸\, D^{\bar{2}}_{W\text{-}X}$ will be at least at a distance j I j away from the zeros and poles. To complete our proof when $I = \bar{D}_{0\text{-}1}\text{-}\bar{D}_{1\text{-}1}\text{-}\bar{D}_{1\text{-}2}$, we now need to bound the quantities

$$'_G = \frac{1}{\#} \tilde{\bigoplus}_{W<X} \frac{1}{G \,¸\, \frac{X \quad W¸\bar{2}}{\#}} \qquad \text{with} \quad G = 0\text{-} \quad 1\text{-} \tag{A.2.13}$$

where we wrote # in place of # in order not to clutter the formulae. We recall that $0 \,\ddot{Y}\, \bar{2} \,\ddot{Y}\, 1 \quad 1\bullet 01.$ Considering $G = 0$ first, we reparameterize the sum in terms of $X \quad W$ so that, after some manipulations, it becomes

$$'_0 = \tilde{\text{Õ}}^{\#1}_{''=1} \frac{\# \quad ''}{'' \quad ¸\, \bar{2}} \,¸\, \frac{\# \quad ''}{'' \quad \bar{2}} \;\ddot{Y}\; 2 \tilde{\text{Õ}}^{\#1}_{''=1} \frac{\# \quad ''}{'' \quad \bar{2}} \bullet \tag{A.2.14}$$

The summand on the right is a positive decreasing function of $''$, therefore it can be bound by its integral:

$$'_0 \;\ddot{Y}\; \frac{2^1\# \quad 1^{\text{o}}}{1 \quad \bar{2}} \,¸\, 2 \int_1^{\# \; 1} \frac{\# \quad G}{G \quad \bar{2}}\, 3G = O^1\# \log \#^{\text{o}} \bullet \tag{A.2.15}$$

To ensure convergence of sums and integrals it is crucial to recall that $1 \quad \bar{2}\; ¡\; ^1 01^{\text{o}} \; ^1$. In a similar way, for $G = \,¸1$ we can write

$$'_1 = \tilde{\text{Õ}}^{\#1}_{''=1} \frac{\# \quad ''}{\# \,¸\, '' \,¸\, \bar{2}} \,¸\, \frac{\# \quad ''}{\# \quad '' \,¸\, \bar{2}} \;\ddot{Y}\; \tilde{\text{Õ}}^{\#1}_{''=1} 2 = O^1\#^{\text{o}}\text{-} \tag{A.2.16}$$

while for $G = \quad 1$ we can write

$$' \;_1 = \tilde{\text{Õ}}^{\#1}_{''=1} \frac{\# \quad ''}{\# \quad '' \quad \bar{2}} \,¸\, \frac{\# \quad ''}{\# \,¸\, '' \quad \bar{2}} \;\ddot{Y}\; 2 \tilde{\text{Õ}}^{\#1}_{''=1} \frac{\# \quad ''}{\# \quad '' \quad \bar{2}} \;\ddot{Y}\; \frac{2^1\# \quad 1^{\text{o}}}{1 \quad \bar{2}} = O^1\#^{\text{o}} \bullet \tag{A.2.17}$$

It remains to show that the terms we discarded from (2.1.22) when substituting the condition $8 < 9$ with the condition $W < X$ give a subleading contribution. Notice that we now return to the notation in which $\# = 01\#$. These are the terms in (2.1.22) with $W = X$, whose total contribution is

$$\text{SCI} = \# \sum_{A=0}^{\tilde{1}} \sum_{B=0}^{\tilde{1}} \sum_{2<3}^{\tilde{}} \log \mathrm{e}\left(\, \text{¸} \, I \frac{3 \quad 2}{\#} \text{¸} \, I \; 3 \quad 2 \text{¸} \, 0B \, \text{¸} \, 1A \; ; 01 \, I \text{-} 01 \, I \right) \bullet \qquad \text{(A.2.18)}$$

We need to show that this is subleading in the large # limit. We will bound the absolute value of the summand for all possible $2 < 3$, A, B and drop the sums since they give an overall order $O(1)$ factor. After choosing a branch of the logarithm, the phases of e can only give an order # contribution to (the imaginary part of) $_{\text{SCI}}$.

For what concerns the absolute value of e, reasoning in a very similar way to the $W < X$ case discussed above, we see that if is not on a Stokes line then $\log |\mathrm{e}|$ is bounded above by an #-independent quantity and thus $_{\text{SCI}}$ is of order $O(\#)$. When $= 0$, the argument of e can only approach zeros or poles if $I = I \; 3 \quad 2 \text{¸} \, 0B \, \text{¸} \, 1A \; 2 \{D_{0\text{-}1}\text{-}D_{1\text{-}1}\text{-}D_{1\text{-}2}\}$. Using (A.2.9), we can write

$$\log \mathrm{e}\left(I \text{¸} \, I \frac{3 \quad 2}{\#} ; 01 \, I \text{-} 01 \, I \right) = \log \frac{\left(I \text{¸} \, I \frac{3\;2}{\#} \quad D_{0\text{-}1} \right) B_{\mathrm{e}} \left(I \text{¸} \, I \frac{3\;2}{\#} \right)}{\left(I \text{¸} \, I \frac{3\;2}{\#} \quad D_{1\text{-}1} \right) \left(I \text{¸} \, I \frac{3\;2}{\#} \quad D_{1\text{-}2} \right)^{2}} \qquad \text{(A.2.19)}$$

where B_{e} is a function which is regular at $D_{1\text{-}1}\text{-}D_{1\text{-}2}$ and non-zero at $D_{0\text{-}1}$. We can therefore bound $\log |B_{\mathrm{e}}|$ over its possible arguments with an #-independent constant, so that it contributes to $_{\text{SCI}}$ at order $O(\#)$. When $I = D_{0\text{-}1}\text{-}D_{1\text{-}1}\text{-}D_{1\text{-}2}$, only one of the factors multiplying B_{e} is of order $O(\log \#)$ while the other two do not approach zero and can be bounded by an #-independent constant. Explicitly,

$$\# \log \mathrm{e}\left(I \text{¸} \, I \frac{3 \quad 2}{\#} ; 01 \, I \text{-} 01 \, I \right) \quad 2\# \log I \frac{3 \quad 2}{\#} \text{¸} \, O(\#) = O(\# \log \#) \bullet \qquad \text{(A.2.20)}$$

A.2.2 SU(N) *vs.* U(N) holonomies

In what follows, as it is done in Section 2.1 and Section 2.2, in order to parametrize the SU(#) holonomies D^{SU} we introduce U(#) holonomies D^{U}, constrained by

$$\sum_{8=1}^{\tilde{\#}} D_8^{\text{U}} = 0 \bullet \qquad \text{(A.2.21)}$$

With the choice of bases for the Cartan subalgebras of SU(#) and U(#) required to write the BA operators as in (2.1.11), the relation between the two sets of holonomies when expressing a generic element of the Cartan subalgebra of SU¹#º is

$$D_8^{\mathrm{U}} = D_8^{\mathrm{SU}} \quad \text{for } 8 < \# - \qquad\qquad D_\#^{\mathrm{U}} = \sum_{9=1}^{\#\,1} D_9^{\mathrm{SU}} \bullet \tag{A.2.22}$$

Note that the holonomies are only defined modulo Z.

The SU¹#º superconformal index defined by (2.1.9) contains a sum over f$<_8^{\mathrm{SU}}$g that picks up (representatives of) solutions to the BAEs whose residue can contribute to the index, as explained in [85] and made explicit in (2.1.15). Under a shift f$<_8^{\mathrm{SU}}$g of the SU¹#º holonomies, the U¹#º holonomies shift by corresponding amounts given by

$$<_8^{\mathrm{U}} = <_8^{\mathrm{SU}} - \qquad\qquad <_\#^{\mathrm{U}} = \sum_{9=1}^{\#\,1} <_9^{\mathrm{SU}} \bullet \tag{A.2.23}$$

Given these identifications for the holonomies and shifts, the SU¹#º quantities are always equal to the first # 1 U¹#º quantities, so, in what follows, we will drop the superscripts SU and U, remembering that $D_{1-\cdots-\#\ 1}$ and $<_{1-\cdots-\#\ 1}$ are independent while $D_\#$ and $<_\#$ are determined by (A.2.22) and (A.2.23), respectively.

One might then worry that the choice of f$<_9$g given in (2.1.19) is not allowed, since the last integer $<_\#$ there does not satisfy (A.2.23). Specifically, let us choose

$$<_9 \ 2\ \mathrm{f}1-\bullet\bullet\bullet-01\mathrm{g} \qquad \text{such that} \qquad <_9 = 9 \mod 01 - \qquad \text{for } 9 = 1-\bullet\bullet\bullet-\# \quad 1 - \tag{A.2.24}$$

so that $<_\#$ is fixed by (A.2.23) to be a negative integer of O¹#º. To match with the choice in (2.1.19), we want to replace this with $<_\# = \#$ mod 01, belonging to f1-•••-01g. We will show that this replacement does not affect the value of Z to the leading order in #. This will be done in two steps. We will first show that the function Z evaluated on a configuration fD_1-•••-$D_\#$g which is obtained from the basic solution by shifting one or more variables D_8 by multiples of $201\,I$ (or even of $01\,I$, in many cases), is the same as Z evaluated on the basic solution. Using this property, Z is unaltered if evaluated on the following shifted value of $<_\#$:

$$\epsilon_\#\ 2\ \mathrm{f}1-\bullet\bullet\bullet-201\mathrm{g} \qquad \text{such that} \qquad \epsilon_\# = \sum_{8=1}^{\#\,1} <_8 \quad \mod 201 \bullet \tag{A.2.25}$$

We will then show that the contribution to Z of the single holonomy $D_\#$ is subleading, provided $\epsilon_\#$ 2 f1-•••-201g. Therefore, choosing instead $<_\# = \#$ mod 01 and in f1-•••-01g as we did in (2.1.19) does not change Z at leading order in #. This completes the proof.

As shown in [85], when evaluated on solutions to the BAEs, the function Z for a general semi-simple gauge group is invariant under independent shifts of any gauge holonomy by $01/$. This is proven by assuming that gauge and global symmetries are non-anomalous. In our case, this result only allows us to shift the D_8's while preserving the SU¹#º constraint. This property does not allow us to independently shift the last holonomy $D_\#$, since it is always fixed by the SU¹#º constraint. We now show that an independent shift of $D_\#$ by a multiple of $01/$ is also an invariance of Z for $N = 4$ SU¹#º SYM, when this function is evaluated on the basic solution. To prove this, one has to use the property (A.1.8), the fact that the U¹#º BA operators are periodic modulo $/$ in the D_8's, and the explicit form of the basic solution (2.1.18). Applying (A.1.8) for generic < 2 Z, we first have that

$$\prod_{8<9} e\, D_{89} ¸ ¸ <01/ ¹X_{8\#} X_{9\#}º; 0/-1/ \tag{A.2.26}$$

$$= 4^{\,c801<^2¹1¸2\,º¸2c8¹0¸1\,1º<\,\sum_8 D_{8\#}¸c801¹0¸1º<^2/} \prod_8 \frac{\backslash_0¹D_{\#8}¸\ -/º<}{\backslash_0¹D_{8\#}¸\ ;/º<} \prod_{8<9} e¹D_{89}¸\ ;0/-1/º-$$

and so from (2.1.11), (2.1.16) and (2.1.18) one obtains

$$\begin{aligned}
&Z¹D_8¸ <01/X_{8\#};0/-1/-\ º\\
&= \prod_8 \frac{\backslash_0¹D_{\#8}¸\ 1-/º\,\backslash_0¹D_{\#8}¸\ 2-/º\,\backslash_0¹D_{8\#}-/º\,\backslash_0¹D_{8\#}¸\ 1¸\ 2-/º}{\backslash_0¹D_{8\#}¸\ 1-/º\,\backslash_0¹D_{8\#}¸\ 2-/º\,\backslash_0¹D_{\#8}-/º\,\backslash_0¹D_{\#8}¸\ 1¸\ 2-/º}^{<} Z¹D_8;0/-1/-\ º\\
&= ¹\ 1º^{<¹\#\ 1º}\,4^{2c8<_\&_\#^{<}¹D_8;\ /-\ º}\,Z¹D_8;0/-1/-\ º\\
&= Z¹D_8;0/-1/-\ º\,•
\end{aligned} \tag{A.2.27}$$

In the steps above we also used the theta function reflection property (A.1.5).

More generally, we can show that this shift invariance is true for quiver gauge theories when Z is evaluated on the basic solution and the chemical potentials $D^U_\#$ are shifted by a multiple of $201/$ (or even of $01/$, in many cases) simultaneously for all gauge groups SU¹#º$_U$. The steps are the same as in (A.2.27). We should notice that the expression for any particular Lagrange multiplier $\_U$ is more complicated than for $N = 4$ SYM, but the sum of all Lagrange multipliers is simple:

$$4^{2c8\sum_{U=1}\_U} = ¹\ 1º^{=_j¹\#\ 1º}- \tag{A.2.28}$$

where U runs over the SU¹#º gauge group factors and $=_j$ is the number of chiral multiplets in the theory. Performing these steps one obtains

$$Z\ D_8^U\ ¸\ <01\ l\ X_{8\#};0\ l-1\ l- \tag{A.2.29}$$
$$= \frac{4^{2c8<}\ \hat{\mathrm{I}}_{U=1-U\ 1}\ 1^o\ <\ ^1\#\ \ 1^o¸<01^1\#\ \ 1^{o1=_j}\ \ ^o}{\hat{\mathrm{I}}_U\ \&_\#^U\ ^1D_8^U;\ l-\ \ ^o\ ^<}\ Z\ ^1D_8^U;0\ l-1\ l-\ \ ^o$$
$$= ^1\ \ 1^{o<^1}\ \ ^{=_j o^1 01¸1^{o1\#}}\ \ 1^o\ Z\ ^1D_8^0;0\ l-1\ l-\ \ ^o\ •$$

There are now different cases in which the sign in the last line disappears. First, in the case of toric quiver gauge theories, one uses the relation [188, 189]

$$=_j\ ¸\ \#\ ,\ = 0 \tag{A.2.30}$$

between the number of gauge groups, chiral multiplets, and superpotential terms, as well as the fact that the number $\#_,$ of superpotential terms is even, to show that the sign disappears. Second, if # is odd then the sign disappears. Third, if the coprime integers $0-1$ are both odd[1] then the sign disappears. Fourth and most importantly, if we take < even then the sign disappears.

We now proceed to show that the contribution to Z of a single holonomy D_8 is subleading, provided that $<_{8Ÿ\#}$ 2 f1-•••-01g and $<_\#$ 2 f1-•••-201g. In the building block SCI defined in (2.1.20), the contribution of a single holonomy D_8 consists of the two terms

$$\text{SCI-}8\ \ \tilde{\textcircled{\#}}_{9^1<8^o}\ \log\ \mathrm{e}\ \ l\ \ \ l\ \frac{9\ \ 8}{\#};01\ l-01\ l\ \ - \tag{A.2.31}$$

where we have defined

$$l\ \ \ \ \ \ l\ \ <_9\ \ <_8\ ¸\ l\ \ OB\ ¸\ 1A\ \ • \tag{A.2.32}$$

In particular, for the case 8 = # we will use the shift property just proven to substitute $<_\#$ with $\epsilon_\#$ defined in (A.2.25).

We will now show that SCI-8 is subleading. In the case 8 = # this will allow us to choose $\epsilon_\#$ as in (2.1.19). To do this we want to bound the absolute value of the summand log e in SCI-8. What follows will be completely analogous to the argument used to show that (A.2.18) is subleading. After choosing a branch of the logarithm, the phases of e can only contribute

[1] This restriction is quite uninfluential because the set of pairs fg ¸ Z- f ¸ Zg such that g• f = 0•1 2 Q$_{;0}$ with $0-1$ both odd is still dense in H^2.

at order $O¹#º$ to $_{\text{SCI-}8}$. As before, we exclude Stokes lines and note that for < 0 we can bound $\log j\text{e}j$ with an #-independent constant so that $j\ _{\text{SCI-}8}j = O¹#º$. For $= 0$, I have the range

$$I \quad 2\ f \quad 201\ {}_¸ 1\text{-}\bullet\bullet\bullet\text{-}401 \quad 0 \quad 1 \quad 1g\ I\ \text{-} \tag{A.2.33}$$

and the argument of e may approach zeros or poles when $I \ = \bar{D}_{0\text{-}1}\text{-}\bar{D}_{0\text{-}2}\text{-}\bar{D}_{1\text{-}1}\text{-}\bar{D}_{1\text{-}2}\text{-}\bar{D}_{1\text{-}3}$, which are defined in (A.2.8). If this is the case, further inspection is required. Using again (A.2.9), we can write

$$\log \text{e} \quad I \qquad I\frac{9 \quad 8}{\#};01\, I\text{-}01\, I \quad = \log \begin{bmatrix} \dfrac{\hat{I}\,{}^{2}_{<=1} \quad I \qquad I\frac{9\ 8}{\#} \quad D_{0\text{-}<} \quad {}^{<} B_{\text{e}} \quad I \qquad I\frac{9\ 8}{\#}}{\hat{I}\,{}^{3}_{==1} \quad I \qquad I\frac{9\ 8}{\#} \quad D_{1\text{-}=} \quad {}^{=}} \end{bmatrix} \tag{A.2.34}$$

where B_{e} is a function that is regular at $D_{1\text{-}1}$, $D_{1\text{-}2}$, $D_{1\text{-}3}$, and non-zero at $D_{0\text{-}1}$, $D_{0\text{-}2}$. This allows us to bound $\log jB_{\text{e}}j$ with an #-independent constant, and its contribution to $_8$ is of order $O¹#º$. When $I \ = \bar{D}_{0\text{-}1}$, $\bar{D}_{0\text{-}2}$, $\bar{D}_{1\text{-}1}$, $\bar{D}_{1\text{-}2}$, $\bar{D}_{1\text{-}3}$ the logarithms of the other factors are either bounded by an #-independent constant or are of the form

$$\sum_{9<8} \log\ G \quad \frac{9 \quad 8}{\#} \qquad ¹\# \quad 1º \log \# \ \text{-} \tag{A.2.35}$$

where $G = 0\text{-} \ \ 1$. Notice that the use of the shift property previously proved plays a major role here. If we tried to apply this argument directly without first shifting $<_{\#}$, we would have to consider an $O¹#º$ number of poles or zeros whose order is also $O¹#º$. This would lead to an $O¹#^3 \log #º$ bound, which does not help. What we did shows that a single $_8$ is of order $O¹# \log #º$ for any choice of the corresponding $<_8$. In particular, this allows us to choose $\tilde{<}_{\#} = \#$ mod $01\ 2\ f1\text{-}\bullet\bullet\bullet\text{-}01g$ as we do in (2.1.19), without affecting the leading behavior of the building block $_{\text{SCI}}$.

A.2.3 Generic *N*

Here we generalize the computation done in Section 2.1.1 and consider a generic # which is not necessarily a multiple of 01. We will exploit many of the arguments in Section A.2.1. Let $\# = 01\tilde{\#}\ {}_¸ @$, where $@\ 2\ f0\text{-}\bullet\bullet\bullet\text{-}01 \quad 1g$. We need to examine the leading order contribution of the building block

$$_{\text{SCI}} = \sum_{A=0}^{0\tilde{1}\,1}\sum_{B=0}^{\tilde{0}1}\sum_{8<9}^{\tilde{\#}} \log \text{e} \quad {}_¸\ I\ \frac{9 \quad 8}{\#}\ {}_¸\ I\ <_9 \quad <_8\ {}_¸\ 0B\ {}_¸\ 1A\ ;01\, I\text{-}01\, I \quad \bullet \tag{A.2.36}$$

As shown in the final part of Section A.2.2, the contribution to the building block of a single holonomy D_8 is subleading. Therefore the contribution of the last @ holonomies $D_{01\#¸1}-\bullet\bullet\bullet-D_{\#}$ is also subleading and can be discarded. Now, the sum over $8 < 9$ only goes up to $01\#$, and we can decompose the indices as in (2.1.22). Neglecting the $W = X$ terms using the same argument as after (A.2.18), we get

$$\mathrm{SCI} \; ' \; \sum_{A=0}^{0\tilde{O}1} \sum_{B=0}^{1\tilde{O}1} \sum_{W<X=0}^{\#\tilde{O}1} \sum_{2\text{-}3=0}^{0\tilde{O}1} \log \mathrm{e} \left(¸ \; l \, \frac{X \;\; W}{\# ¸ \frac{@}{01}} ¸ \; l \, \frac{3 \;\; 2}{\#} ¸ \; l \;\; 3 \;\;\; 2 ¸ 0B ¸ 1A \; ; 01\, l \text{-} 01\, l \right) \bullet \tag{A.2.37}$$

As in Section A.2.1, we want to drop $l\,{}^1 3 \quad 2^{\circ} \bullet \#$ in the argument of the elliptic gamma function, and we can use the same reasoning given there, with the minor change that (A.2.13) takes the form

$$\mathrm{e}_G = \frac{1}{\# ¸ \frac{@}{01}} \sum_{W<X}^{\tilde{\#}} \frac{1}{G ¸ \frac{X \;\; W ¸ 2}{\# ¸ @ \bullet 01}} \text{-} \qquad G = 0\text{-} \quad 1 \bullet \tag{A.2.38}$$

The same bounds as for $'_G$ can be used here since one can show that

$$\mathrm{e}_0 = \, '_0 \text{-} \qquad\qquad \mathrm{e}_1 \qquad '_1 \bullet \tag{A.2.39}$$

We can then use (A.1.10), as we did in Section 2.1.1, to change the moduli of the elliptic gamma function from ${}^1 01\, l \text{-} 01\, l^{\circ}$ to ${}^1 l \text{-} \, l^{\circ}$:

$$\begin{aligned}
\mathrm{SCI} \; ' & \sum_{A=0}^{0\tilde{O}1} \sum_{B=0}^{1\tilde{O}1} \sum_{W<X=0}^{\#\tilde{O}1} \sum_{2\text{-}3=0}^{0\tilde{O}1} \log \mathrm{e} \left(¸ \; l \, \frac{X \;\; W}{\# ¸ \frac{@}{01}} ¸ \; l \;\; 3 \;\;\; 2 ¸ 0B ¸ 1A \; ; 01\, l \text{-} 01\, l \right) \\
= & \sum_{A=0}^{0\tilde{O}1} \sum_{B=0}^{1\tilde{O}1} \sum_{W<X=0}^{\#\tilde{O}1} \log \mathrm{e} \left(¸ \; l \, \frac{X \;\; W}{\# ¸ \frac{@}{01}} ¸ \; l \;\; 1 \;\;\; 01 ¸ 0B ¸ 1A \; ; \, l \text{-} \, l \right) \\
= & \frac{1}{{}^1 01^{\circ 2}} \sum_{A=0}^{0\tilde{O}1} \sum_{B=0}^{1\tilde{O}1} \sum_{W<X=0}^{\#\tilde{O}1} \sum_{2\text{-}3=0}^{0\tilde{O}1} \log \mathrm{e} \left(¸ \; l \, \frac{X \;\; W}{\# ¸ \frac{@}{01}} ¸ \; l \;\; 1 \;\;\; 01 ¸ 0B ¸ 1A \; ; \, l \text{-} \, l \right) \bullet
\end{aligned} \tag{A.2.40}$$

In the last equality, to make future steps clearer, we added a sum over $2\text{-}\,3$ even though nothing depends on 2 and 3.

Now, to get the desired result we trace our steps backward. First, we will reintroduce the term $l\,{}^1 3 \quad 2^{\circ} \bullet \#$ into the argument of the elliptic gamma functions. Then we will add to the sum in (A.2.40) the $W = X$ terms to form the sum over $8 < 9$ up to $01\#$. Finally, we will add terms containing the last @ holonomies $D_{01\#¸1}-\bullet\bullet\bullet-D_{\#}$ to build the complete sum up to $\#$. These are the same steps we just performed to express SCI as in (A.2.40) up to subleading terms, with the only difference being that the moduli of e are now ¹ l - l º rather than ¹01 l - 01 lº. Therefore the same arguments can be used, with only slight modifications involving the number and order of zeros and poles, but since these are parameterized here by A and B that are #-independent, this is of no consequence. At this point, SCI at leading order is

$$\text{SCI} \; ' \; \frac{1}{{}^1 01^{o2}} \; \tilde{O}1 \; \tilde{O}1 \; \tilde{O} \log e \quad \text{¸} \; l \; \frac{9 \quad 8}{\#} \; \text{¸} \; l \; 1 \quad 01 \; \text{¸} \; 0B \; \text{¸} \; 1A \; ; \; l - l \quad - \qquad \text{(A.2.41)}$$

and using the result of [73] (that is our equation (2.1.25) we obtain

$$\text{SCI} \; ' \; \frac{c8\#^2}{3^1 0\, l^{o1} 1\, l^o} \; \frac{1}{01} \; \tilde{O}1 \; \tilde{O}1 \quad 3 \; » \; ¼^0_l \; \text{¸} \; l \; 0B \; \text{¸} \; 1A \quad 01 \quad \bullet \qquad \text{(A.2.42)}$$

Then, using the property of Bernoulli polynomials (2.1.28), we finally get (2.1.29).

A.3 The large *N* TTI

In this Section, instead, we will first obtain (3.1.28) in a continuum approximation, and then solve the set of differential equations (3.1.31), (3.1.32), (3.1.33) coming from its variations.

A.3.1 Useful integrals

Let us evaluate, at large #, the following useful integrals:

$$\begin{aligned} \text{L-} \; » d ¼ {}^1 C- \quad ^o \qquad & 3C^0 \; d^1 C^{0o} \; 4^{2c8 \; {}^1 D^1 C^{0o} \; D^1 C^{oO}} \; - \\ \text{U-} \; » d ¼ {}^1 C- \quad ^o \qquad & 3C^0 \; d^1 C^{0o} \; 4^{2c8 \; {}^1 D^1 C^o \; D^1 C^{0oO}} \; - \end{aligned} \qquad \text{(A.3.1)}$$

where $D^1 C^o = \#^U \; 8C \; \text{¸} \; E^1 C^o$ and $C \quad C \quad \# \; ^U \mathrm{Im}$ (the subscripts L and U stand for lower and upper, respectively). We Taylor expand part of the integrand around C :

$$\text{L-} \; » d ¼ {}^1 C- \quad ^o = 4 \; {}^{2c8 \; D^1 C^o} \; \tilde{O} \; \frac{1}{<!} \; m^<_G \; \left[d^1 G^o \; 4^{2c8 \; \# ^U E^1 G^o} \right]_{G=C} \; 3C^0 \; 4 \; ^{2c \; \# ^U C^0} \; C^0 \quad C \quad ^< \; \bullet \qquad \text{(A.3.2)}$$

The integral on the RHS can be evaluated integrating by parts:

$$\int_{C}^{1} 3C^{0}\, 4^{\;2c\;\#^{U}C^{0}}\; C^{0}\quad C\quad ^{<} = \sum_{:=0}\frac{<!\,^{1}C_{\text{¸}}\quad C\quad ^{o:}}{:!\,^{1}\#^{U}2c\;^{o<\;:\,¸1}}\,4^{\;2c\;\#^{U}C_{¸}}\;¸\;\frac{<!}{^{1}2c\#^{U}\;^{o<\,¸1}}\,4^{\;2c\;\#^{U}C}\quad - \tag{A.3.3}$$

where $C_{¸}$ is the upper limit of integration. The boundary terms at $C_{¸}$ can be neglected because of an overall factor $4^{\;2c\;\#^{U\,1}C_{¸}\;\;C\;\;^{o}}$, which is exponentially suppressed, with respect to the last term. This gives

$$\int_{C}^{1} 3C^{0}\, 4^{\;2c\;\#^{U}C^{0}}\; C^{0}\quad C\quad ^{<}\;'\;\frac{<!}{^{1}2c\#^{U};^{o<\,¸1}}\,4^{\;2c\;\#^{U}C}\;\bullet \tag{A.3.4}$$

For the derivatives in (A.3.2), the terms up to NLO in the large # expansion are

$$m^{<}\left[d\,4^{2c8\;\#^{U}E}\right]_{G=C} \tag{A.3.5}$$

$$= 4^{2c8\;\#^{U}E}\;^{1}2c8\;\;\#^{U o<\;1}\;\;2c8\;\;\#^{U}\,d\,E^{<}\;¸\;<d\,E^{<\;1}\;¸\;\frac{<^{1}<\;\;1^{o}}{2}\,d\,E^{<\;\;2}E\;¸\;\bullet\bullet\bullet\Big|_{G=C}$$

$$= 4^{82c\;\;1\#^{U}E\;\;\;\mathrm{Im}\;\;\;E^{o}}\;^{1}2c8\;\;\#^{U o<\;1}\;\;2c8\;\;\#^{U}\,d\,E^{<}\;¸\;<d\,E^{<\;1}\;¸\;\frac{<^{1}<\;\;1^{o}}{2}\,d\,E^{<\;\;2}E\;¸$$

$$2c8\;\;\mathrm{Im}\;\;\;d\,E^{<}\;¸\;<d\,E^{<\;1}E\quad\tfrac{1}{2}\,2c8\;\;\mathrm{Im}\;\;\;d\,E^{<}E\;¸\;\bullet\bullet\bullet\;\bullet$$

In the last expression, d and E are functions of C. Other contributions are subleading by powers of $\#\;^{U}$. Plugging this back in (A.3.2), we get

$$\text{L-}\;»d¼^{1}C\text{-}\quad^{o} = 4^{\;2c\;\;\mathrm{Im}\;\;\;^{1}1\;\;8E^{o}}\;\frac{1}{2c\;\;\#^{U}}\,\frac{d}{1\;\;\;8E}\;¸\;\frac{1}{2\#^{2U}}\,^{1}\mathrm{Im}\;\;\;^{o2}\,\frac{8\,d\,E}{1\;\;\;8E}\;¸ \tag{A.3.6}$$

$$¸\;\frac{1}{^{1}2c^{o2}\;\;2\#^{2U}}\;1\;\;\;2c\;\;\;\mathrm{Im}\;\;\;^{1}1\;\;\;8E^{o}\;\;\frac{d}{^{1}1\;\;\;8E^{o2}}\;¸\;\frac{8\,d\,E}{^{1}1\;\;\;8E^{o3}}\;\bullet$$

Repeating the same steps for the other integral we find

$$\text{U-}\;»d¼^{1}C\text{-}\quad^{o} = 4^{\;2c\;\;\mathrm{Im}\;\;\;^{1}1\;\;8E^{o}}\;\frac{1}{2c\;\;\#^{U}}\,\frac{d}{1\;\;\;8E}\quad\frac{1}{2\#^{2U}}\,^{1}\mathrm{Im}\;\;\;^{o2}\,\frac{8\,d\,E}{1\;\;\;8E}\;¸ \tag{A.3.7}$$

$$\frac{1}{^{1}2c^{o2}\;\;2\#^{2U}}\;1\;\;\;2c\;\;\;\mathrm{Im}\;\;\;^{1}1\;\;\;8E^{o}\;\;\frac{d}{^{1}1\;\;\;8E^{o2}}\;¸\;\frac{8\,d\,E}{^{1}1\;\;\;8E^{o3}}\;\bullet$$

A.3.2 Continuum expressions for V^{0} and

Let us start by studying the first line of (3.1.23) and, in particular, the terms involving the Li_1 function, whose definition and properties can be found in Appendix A.1. We first perform the sum over 9 (that becomes an integral over C^0), leaving the sum over 8 (that becomes an integral over C) untouched.

The integral in C^0 has to be broken in two parts, above and below $C \quad C \quad \# \, ^U \mathrm{Im}$. When $\mathrm{Im}^1 D_{98} \quad ^o \, ¡ \, 0$ (for one of the two signs), we can use the series expansion (A.1.15). This allows us to treat the integral above C :

$$\tilde{O}_{9} \; \mathrm{Im}^1 D_{98} \quad ^o \; \mathrm{Li}_1 \; 4^{2c8^1 D_{98}} \quad ^o \quad ! \; \# \int_C^1 3C^0 d^1 C^{0o} \; \tilde{①}_{=1} \frac{1}{} 4^{2c8 \; ^1 D^1 C^{0o} \quad D^1 C^o} \quad ^o$$
$$\# \; \tilde{①}_{=1} \frac{4 \; ^{2c8}}{} \; \mathrm{L}\text{-} \; » d¼^1 C\text{-} \quad ^o \, • \tag{A.3.8}$$

Using the results in Appendix A.3.1 we write (A.3.8) as

$$(A.3.8) = \frac{\#^1 \; ^U d}{2c^1 1 \quad 8Ě^o} \mathrm{Li}_2 \; 4 \; ^{2c8^1 \mathrm{Re} \quad Ě \, \mathrm{Im} \quad ^o} \; ¸ \tag{A.3.9}$$
$$¸ \; \#^{1 \; 2U} \frac{1}{^1 2c^{o2}} \mathrm{Li}_3 \; 4 \; ^{2c8^1 \mathrm{Re} \quad Ě \, \mathrm{Im} \quad ^o} \quad \frac{1}{2c} {}^1 \mathrm{Im} \quad ^{o1} 1 \quad 8Ě^o \, \mathrm{Li}_2 \; 4 \; ^{2c8^1 \mathrm{Re} \quad Ě \, \mathrm{Im} \quad ^o}$$
$$\frac{d}{^1 1 \quad 8Ě^{o2}} \; ¸ \; \frac{8 d \, Ě}{^1 1 \quad 8Ě^{o3}} \quad ¸ \; \frac{8}{2} \#^{1 \; 2U \, 1} \mathrm{Im} \quad ^{o2 \, 1} 1 \quad 8Ě^{o2} \, \mathrm{Li}_1 \; 4 \; ^{2c8^1 \mathrm{Re} \quad Ě \, \mathrm{Im} \quad ^o} \quad \frac{d \, Ě}{^1 1 \quad 8Ě^{o3}} \; ¸$$
$$¸ \; O^1 \#^{1 \; 3U} {}^o \, •$$

When $\mathrm{Im}^1 D_{98} \quad ^o \; Ÿ \, 0$, the steps above are not applicable because the series expansion for Li_1 does not converge, but we can use (A.1.21) so that

$$\mathrm{Li}_1 \; 4^{2c8^1 D_{98}} \quad ^o \; = \mathrm{Li}_1 \; 4^{2c8^1 D_{89}} \quad ^o \quad 2c8 \;\; D_{98} \qquad \frac{1}{2} \; • \tag{A.3.10}$$

Now the Li_1 terms on the RHS can be analyzed in the same way as before

$$\tilde{O}_{9} \; \mathrm{Im}^1 D_{89} \quad ^o \; \mathrm{Li}_1 \; 4^{2c8^1 D_{89}} \quad ^o \quad ! \; \# \int_C^1 3C^0 \, d^1 C^{0o} \; \tilde{①}_{=1} \frac{4^{2c8 \; ^1 D^1 C^o \quad D^1 C^{0o}} \quad ^o}{}$$
$$\# \; \tilde{①}_{=1} \frac{4 \; ^{2c8}}{} \; \mathrm{U}\text{-} \; » d¼ \; \text{-} \tag{A.3.11}$$

and using (A.3.7) we get

$$(\text{A.3.11}) = \#^{1\ U}\,\mathrm{Li}_2\ 4\ ^{2c8^1\mathrm{Re}\quad \mathrm{Im}\ ^\mathrm{o}}\ \frac{d}{2c^1 1\quad 8^\mathrm{o}}\ , \tag{A.3.12}$$

$$\#^{1\ 2U}\ \frac{1}{{}^1 2c^{\mathrm{o}2}}\,\mathrm{Li}_3\ 4\ ^{2c8^1\mathrm{Re}\quad \mathrm{Im}\ ^\mathrm{o}}\quad \frac{1}{2c}{}^1\mathrm{Im}\quad {}^{\mathrm{o}1}1\quad 8^\mathrm{o}\,\mathrm{Li}_2\ 4\ ^{2c8^1\mathrm{Re}\quad \mathrm{Im}\ ^\mathrm{o}}$$

$$\frac{d}{{}^1 1\quad 8^{\mathrm{o}2}}\ ,\ \frac{8d}{{}^1 1\quad 8^{\mathrm{o}3}}\quad \frac{8}{2}\#^{1\ 2U\,1}\mathrm{Im}\quad {}^{\mathrm{o}2\,1}1\quad 8^{\mathrm{o}2}\,\mathrm{Li}_1\ 4\ ^{2c8^1\mathrm{Re}\quad \mathrm{Im}\ ^\mathrm{o}}\quad \frac{d}{{}^1 1\quad 8^{\mathrm{o}3}}\ ,$$

$$,\ O^1\#\ ^{3U}\mathrm{o}\ .$$

To obtain the full integral over C^0, the contributions (A.3.9) and (A.3.12) with upper sign must be summed with minus the ones with the lower sign, and the result can be simplified using (A.1.21). As in (3.1.23), we then integrate over C together with m^1C^o, and sum over $= 1,2,3$. We obtain:

$$2c8\#^{2\ 2U}\ \int^1 3C\ \frac{2\,8\,\mathrm{m}\,d^2}{{}^1 1\quad 8^{\mathrm{o}3}}\ \sum_{=1}^{3}\ \frac{1}{2}{}^1\mathrm{Im}\quad {}^{\mathrm{o}2\,1}8\ ,\ ^{\mathrm{o}2}\ {}_1\ \mathrm{Re}\qquad \mathrm{Im}\qquad , \tag{A.3.13}$$

$$2c8\#^{2\ 2U}\ \int^1 3C\ \mathrm{m}\ \frac{3}{3C}\ \frac{d^2}{{}^1 1\quad 8^{\mathrm{o}2}}\ \sum_{=1}^{3}\ \frac{1}{6}\ {}_3\ \mathrm{Re}\qquad \mathrm{Im}\qquad ,$$

$$,\ \frac{1}{2}{}^1\mathrm{Im}\quad {}^{\mathrm{o}\,1}8\ ,\ ^{\mathrm{o}}\ {}_2\ \mathrm{Re}\qquad \mathrm{Im}\qquad .$$

It remains to add the contribution from the second term on the RHS of (A.3.10). We choose the integer ambiguities $=_8$ in (3.1.23) such that

$$c^1\#\quad 2=_8{}^\mathrm{o} = \quad 2c\sum_{=1}^{3}\sum_{9=1}^{3}\quad \mathrm{Im}^1D_{89}\ ,\quad {}^\mathrm{o}\qquad \mathrm{Im}\,D_{89}\quad ,2\quad {}^1\mathrm{Im}\,D_{89}{}^\mathrm{o}\ \ ,O^1 1^\mathrm{o}\ . \tag{A.3.14}$$

The subleading $O^1 1^\mathrm{o}$ term accounts for the possibility that # might be odd and we would not be able to cancel it completely. The contributions from the second term on the RHS of (A.3.10) and (A.3.14) sum up to

$$2c8\sum_{\text{-}8\text{-}9}\ \mathrm{m}_8\qquad \mathrm{Im}^1D_{89}\ ,\quad {}^\mathrm{o}\qquad {}^1\mathrm{Im}\,D_{89}{}^\mathrm{o}\qquad D_{98}\ ,\qquad \frac{1}{2}\quad , \tag{A.3.15}$$

$$,\qquad \mathrm{Im}^1D_{89}\qquad {}^\mathrm{o}\qquad {}^1\mathrm{Im}\,D_{89}{}^\mathrm{o}\quad D_{98}\ ,\qquad \frac{1}{2}$$

$$= 2c8\#^2\ \sum_{=1}^{3}\ \sum_{,\text{-}}\ \int^1 3C\,\mathrm{m}^1C^\mathrm{o}\,d^1C^\mathrm{o}\ \int_C^{1}{}^{C}\ 3C^0\,d^1C^{0\mathrm{o}}\quad \Big[\ \#^U\ 8C\quad 8C^0\ ,\ E^1C^\mathrm{o}\quad E^1C^{0\mathrm{o}}\quad ,\qquad \frac{1}{2}\Big]\ .$$

In each integral we perform the change of variables $C^0 = C \;\; \#^{U 1}\mathrm{Im} \;\; {}^{o}Y$, obtaining:

$$(\text{A.3.15}) = 2c8\#^{2\;U} \sum_{=1}^{\tilde{3}} \sum_{,-}^{\tilde{O}} \mathrm{Im} \int^{1} 3C\,\mathrm{m}^{1}C^{o}\, d^{1}C^{o} \int_{0}^{1} 3Y \tag{A.3.16}$$

$$d\;C \;\; \#^{U 1}\mathrm{Im} \;\; {}^{o}Y \;\; 8^{1}\mathrm{Im} \;\; {}^{o}Y \;\; \#^{U}E\;C \;\; \#^{U 1}\mathrm{Im} \;\; {}^{o}Y \;\; \#^{U}E^{1}C^{o} \, , \;\; \frac{1}{2} \; .$$

We expand d and E in the Taylor series and keep only the leading terms. Then we integrate in Y and use that $\;_1{}^{1} \;\; {}^{o} = \;\; 1/2$. We obtain the expression:

$$(\text{A.3.15}) = 2c8\#^{2\;2U} \sum_{=1}^{\tilde{3}} {}^{1}\mathrm{Im} \;\; {}^{o2} \int^{1} 3C\,\mathrm{m} \;\; d\,d \;\; {}_{1}\,\mathrm{Re} \;\; E\,\mathrm{Im} \;\; , \tag{A.3.17}$$

$$\frac{\mathrm{Im}}{6} \; \frac{3}{3C} \; \frac{d^2}{{}^{1}1 \;\; 8E^{o2}} \; {}^{1}8 \, , \, E^{o3} \;\; , \; O^{1}\mathrm{m}\#^{2\;3U o} \, .$$

We sum (A.3.13) and (A.3.17). We notice that the various terms can be organized into the Taylor series of $\;_3{}^{1} \;\; {}^{o}/6$ around the point $\mathrm{Re}^{1} \;\; {}^{o} \;\; E\,\mathrm{Im}^{1} \;\; {}^{o}$, which has four terms because $\;_3$ is a cubic polynomial. We obtain the compact expression

$$(\text{A.3.13}) \, , \, (\text{A.3.17}) = \;\; 2c8\#^{2\;2U} \;\; {}_{1}\;{}_{o} \int^{1} 3C\,\mathrm{m} \, \frac{3}{3C} \, \frac{d^2}{{}^{1}1 \;\; 8E^{o2}} \, , \, O\;\, \mathrm{m}\#^{2\;3U}{}^{-}1 \;\; - \tag{A.3.18}$$

where $\;{}^{1} \;\; {}^{o}$ is the function defined in (3.1.29). It remains to add the first term on the RHS of the first line of (3.1.23). We obtain the final expression:

$$\int^{1} 3C\,\mathrm{m}\,+^{0} = 2c8:\# \int^{1} 3C\,d\,\mathrm{m}\,D \, , \, 2c8\#^{2\;2U} \;\; {}_{1}\;{}_{o} \int^{1} 3C \, \frac{\mathrm{m}\, d^2}{{}^{1}1 \;\; 8E^{o2}} \, , \, O\;\, \mathrm{m}\#^{2\;3U} \;\; . \tag{A.3.19}$$

We apply the same steps to obtain the large $\#$ limit of in (3.1.23). To avoid repetition, we only present the result. We set the integer ambiguity $''$ to $\#/2 \, , \, O^{1}1^{o}$. We obtain:

$$= \;\; 2c\#^{2\;U} \, 5_{,}\,{}^{1}\mathrm{n}\text{-} \;\; {}^{o} \int^{1} 3C \, \frac{d^2}{1 \;\; 8E} \, , \, O \;\; \#^{2\;2U} \;\; - \tag{A.3.20}$$

where the function $5_{,}\,{}^{1}\mathrm{n}\text{-} \;\; {}^{o}$ is defined in (3.1.29).

A.3.3 Solutions to the saddle-point equations

In this Appendix, we solve the saddle-point equations (3.1.31)–(3.1.33), in the original parameterization in which $E^{1}C^{o}$ is a real function. Let us first solve (3.1.33). After integrating to

$$: {}^{1}8C \,¸\, E^{o2} \,¸\, \frac{4 \;\; d}{8 \,¸\, Ê} = \;\; 2\,C\,- \qquad \text{(A.3.21)}$$

its real and imaginary parts give

$$4d = \;\; 1 \,¸\, Ê^{2} \;\; \mathrm{Im}^{h} \;\; {}^{1} \;\; : {}^{1}8C \,¸\, E^{o2} \;^{i} \; - \;\; Ê = \frac{\mathrm{Re} \;\; {}^{1} \;\; : {}^{1}8C \,¸\, E^{o2}}{\mathrm{Im} \;\; {}^{1} \;\; : {}^{1}8C \,¸\, E^{o2}} \,\bullet \qquad \text{(A.3.22)}$$

We impose that d is integrable. This necessarily implies that $d \;!\; 0$ as $C \;!\;\; 1$, or that d is defined on compact intervals where d is zero at the endpoints. At infinity, or an endpoint

$$d = 0 \qquad =) \qquad : {}^{1}8C \,¸\, E^{o2} = 0 \,\bullet \qquad \text{(A.3.23)}$$

By considering real and imaginary parts, we see that this equation cannot be satisfied as $C \;!\;\; 1$, and d must have compact support. For d to have two endpoints C and be defined on the interval $»C \; - C_{¸}¼$, cannot be on the positive real axis. Let $\frac{1}{2}$ be the square root whose imaginary part is positive. The boundary conditions are

$$C \;\; = \;\; : \; \tfrac{1}{2} \; \mathrm{Im}^{1} \;\; \tfrac{1}{2}{}^{o} \, - \qquad E^{1}C \;\; {}^{o} = \;\; : \; \tfrac{1}{2} \; \mathrm{Re}^{1} \;\; \tfrac{1}{2}{}^{o} \,\bullet \qquad \text{(A.3.24)}$$

We then solve the equation for $Ê$ in (A.3.22) using (A.3.24) as boundary conditions. The equation can be rewritten and integrated to

$$\mathrm{Im} \;\; {}^{1}\,{}^{1}8C \,¸\, E^{o} \;\; \frac{:}{3}\,{}^{1}8C \,¸\, E^{o2} \;\; = \;\; - \qquad \text{(A.3.25)}$$

where $\;\; 2\,\mathrm{R}$ is an integration constant. The boundary conditions (A.3.24) imply $\; = 0$ and $\mathrm{Im} \;\; {}^{1} \;\; \tfrac{3}{2} \;\; = 0$. Using a real constant to parameterize the real part of ${}^{1} \;\; \tfrac{3}{2}$, we write

$$= : {}^{1} \;\; {}^{o}\tfrac{2}{3} \, - \qquad 2\,\mathrm{R}\, - \qquad \text{(A.3.26)}$$

where $:$ is included for convenience. It is important to keep in mind that there are 3 branches for $\tfrac{1}{3}$ and the same branch is to be used in every expression. There is a triplet of solutions at this point. The equation (A.3.25) can be written as

$$0 = \mathrm{Im} \;\; \tfrac{1}{3}\,{}^{1}8C \,¸\, E^{o} \;\; 3 \;\; \tfrac{2}{3} \,¸\, \;\; \mathrm{Im} \;\; \tfrac{1}{3}\,{}^{1}8C \,¸\, E^{o} \;^{2} \;\; 3 \;\; \mathrm{Re} \;\; \tfrac{1}{3}\,{}^{1}8C \,¸\, E^{o} \;^{2} \,\bullet \qquad \text{(A.3.27)}$$

The solutions obtained by setting to zero the square bracket lead to profiles for d with a single zero, and so they have to be discarded. We remain with

$$\mathrm{Im}\ \ ^{\frac{1}{3}}\, ¹8C\ ¸\ E^{o}\ = 0 \qquad) \qquad E¹C^{o} = \frac{\mathrm{Re}\ \ ^{\frac{1}{3}}}{\mathrm{Im}\ \ ^{\frac{1}{3}}}\, C\ - \tag{A.3.28}$$

which through (A.3.22) gives the following profile for d:

$$d¹C^{o} = \frac{:}{4\ \mathrm{Im}\ \ ^{\frac{1}{3}\ 3}} \Big[\ ^{\frac{2}{3}}\ \mathrm{Im}\ \ ^{\frac{1}{3}\ 2}\ \ C^{2}\Big] \bullet \tag{A.3.29}$$

Requiring that $d\ ¡\ 0$ within $¹C\ \ -C_{¸}{}^{o}$ imposes

$$\mathrm{Im}\ \ ^{\frac{1}{3}}\ ¡\ 0\ - \tag{A.3.30}$$

which restricts the branches we can take for $\ ^{\frac{1}{3}}$. Requiring that $\ \ 3C\ d = 1$ fixes $\ \ = 3\bullet :$ and the final result for D and d is:

$$D¹C^{o} = \#^{\frac{1}{3}} \frac{\ ^{\frac{1}{3}}}{\mathrm{Im}\ \ ^{\frac{1}{3}}}\, C\ - \quad d¹C^{o} = \frac{¹9\ :\ ^{o\frac{1}{3}}}{4\,\mathrm{Im}\ \ ^{\frac{1}{3}}} \quad \frac{:}{4\ \mathrm{Im}\ \ ^{\frac{1}{3}\ 3}}\, C^{2}\ - \quad C\ \ = \ \frac{3}{:}\ ^{\frac{1}{3}}\ \mathrm{Im}\ \ ^{\frac{1}{3}} \bullet \tag{A.3.31}$$

Notice that if $\ $ are real and $\ \ ¡\ 0$, (A.3.30) fixes the branch of the cube root such that $\ ^{\frac{1}{3}}$ has phase $4^{\frac{2c8}{3}}$, and the solutions for D, d reduce to those found in [43]. We can now solve for m using (3.1.32). Inserting (A.3.31) for D and d, the former reduces to:

$$C^{2}\ \ C_{¸}^{2}\ \ \mathrm{m}\!\!\!\!\!\ ¥\ ¸\ 4C\ \mathrm{m}\!\!\!\!\!\ ¶\ ¸\ 2\mathrm{m} = \frac{3^{2}}{3C^{2}}\Big[C^{2}\ \ C_{¸}^{2}\ \ \mathrm{m}\Big] = 2\frac{5_{¸}}{}\, D\ - \tag{A.3.32}$$

whose general solution is

$$\mathrm{m}¹C^{o} = \frac{1}{C^{2}\ \ C_{¸}^{2}}\ \frac{\#^{\frac{1}{3}}\, 5_{¸}}{3}\ \frac{\ ^{\frac{1}{3}}}{\mathrm{Im}\ \ ^{\frac{1}{3}}}\ C^{3}\ ¸\ \ \ C\ ¸\ \ \ \ - \tag{A.3.33}$$

where $\ $ and $\ $ are integration constants. The requirement that m has a compact image, namely that it does not diverge at $C = C_{¸}$, fixes $\ \ = \ C_{¸}^{2}$ and $\ \ = 0$. This leads to the simple solution

$$\mathrm{m}¹C^{o} = \frac{5_{¸}}{3}\, D¹C^{o} \bullet \tag{A.3.34}$$

One can then verify that (3.1.31) is automatically solved, with the following value for the Lagrange multiplier:

$$` = 85_{¸} \frac{:}{3}^{\frac{1}{3}} \bullet \tag{A.3.35}$$

The solution can be expressed more neatly by making use of the reparameterization symmetry (3.1.34), performing the transformation

$$C = \frac{3}{:}^{1\bullet 3} \; \mathrm{Im} \; \frac{1}{3} \; C^{0} \bullet \tag{A.3.36}$$

This brings the solution to the form (3.1.35), in which primes have been omitted.

Appendix B

Supergravity generalities

In this Appendix, we set the supergravity conventions we used for our computations. We do this both in the 5d case, where we also match the supergravity notation of [169] for the conifold consistent truncation we are interested in and in the 4d case.

B.1 5d $\mathcal{N} = 2$ abelian gauged supergravity

We report here the general form of 5d $\mathcal{N} = 2$ Abelian gauged supergravity with n_V vector multiplets and n_H hypermultiplets [198–200].[1] The graviton multiplet contains a graviton, a gravitino, and a vector; each vector multiplet contains a vector, a gaugino, and a real scalar; each hypermultiplet contains four real scalars and a hyperino. All fermions are Dirac, but can conveniently be doubled with a symplectic Majorana condition. We follow the notation of [233, 234]. We use indices

$$I, J = 1, \dots, n_V + 1, \qquad i, j = 1, \dots, n_V, \qquad u, v = 1, \dots, 4n_H \tag{B.1.1}$$

for the gauge fields A^I, for the scalars ϕ^i in vector multiplets, and for the scalars q^u in hypermultiplets, respectively. The data defining the theory are:

1. A very special real manifold $\mathcal{SM}$ of real dimension n_V.

2. A quaternionic-Kähler manifold $\mathcal{QM}$ of real dimension $4n_H$.

3. A set of $n_V + 1$ Killing vectors on $\mathcal{QM}$ compatible with the quaternionic-Kähler structure (if $n_H = 0$, $n_V + 1$ FI parameters not all vanishing).

[1] A more complete discussion was developed in [232].

The Killing vectors could be linearly dependent or vanish.

The bosonic Lagrangian is given by

$$8c \;{}^{15^o}_{\mathrm{N}} 4\;{}^1 L_{5\mathrm{d}} = \frac{'_B}{2} \;\; \frac{1}{2} G_{89}{}^1 q^o\, m_{`} q^8 m^{`} q^9 \;\; \frac{1}{2} \;{}_{DE}{}^1 @^o D_{`} @^D D^{`} @^E \;\; \frac{1}{4} \;\; {}^1 q^o \;\; {}_{`a} \;\; {}^{`a} \,,$$
$$\,,\; \frac{4\;{}^1}{48} \;\; n^{`adf\_} \;\; {}_{`a} \;\; {}_{df} \;\; {}_{\_} \;\; 6^2 + {}^1 q\text{-} @^o \,\bullet \tag{B.1.2}$$

Here ${}^{15^o}_{\mathrm{N}}$ is the 5d Newton constant, $4\,3^5 G$ is the spac-etime volume form, $'_B$ is the scalar curvature, ${}_{`a}$ is the field strength of ${}^{`}$, 6 is a coupling constant, and $+$ is the scalar potential. Let us explain the other terms.

Very special geometry. The scalars q^8 are real coordinates on the very special real manifold $S\,M$ [235]. The latter is specified by the totally symmetric tensor (which, controlling also the Chern-Simons couplings, should be suitably quantized) as the submanifold

$$S\,M = \overset{\mathrm{n}}{\;} V^1 \;{}^o\; \frac{1}{6} \;\; \overset{0}{=1} \;\; \mathbb{R}^{=+,1} \,\bullet \tag{B.1.3}$$

Here are coordinates on $\mathbb{R}^{=+,1}$, and give rise to “sections” ${}^1 q^{8o}$ on $S\,M$. The metrics and G_{89} for vector fields and vector multiplet scalar fields are

$$ {}^1 q^o = \frac{1}{2} \frac{m}{m} \frac{m}{m} \log V \Big|_{V=1} \text{-} \qquad G_{89}{}^1 q^o = m_8 \;\; m_9 \Big|_{V=1} \tag{B.1.4}$$

where $m_8 \;\; m \bullet m q^8$. We recognize that G is the pull-back of from $\mathbb{R}^{=+,1}$ to $S\,M$. From (B.1.3) it immediately follows

$$m_8 \Big|_{V=1} = 0 \,\bullet \tag{B.1.5}$$

With a little bit of algebra one then obtains a more explicit expression for :

$$= \frac{1}{2} \;\; , \frac{1}{8} \;\; ! \;\; "\# \;\; ! \;\; " \;\; \# \Big|_{V=1} \bullet \tag{B.1.6}$$

It follows that the kinetic term for vector multiplet scalars can also be written as

$$\frac{1}{2} G_{89}\, m_{`} q^8 m^{`} q^9 = \frac{1}{4} \;\; m_{`} \;\; m^{`} \Big|_{V=1} \bullet \tag{B.1.7}$$

One can define on $S\,M$ the sections with lower indices:

$$\frac{2}{3} \Big|_{V=1} = \frac{1}{6} \Big|_{V=1} = \frac{1}{3} \frac{mV}{m} \Big|_{V=1} \bullet \tag{B.1.8}$$

With simple algebra one can show the following identities:

$$= 1 - \qquad\qquad = \frac{9}{2} \qquad \frac{1}{2} \qquad - $$
$$m_8 \quad = \frac{2}{3} \quad m_8 \quad - \qquad\qquad m_8 \quad = \quad m_8 \quad = 0 \bullet \tag{B.1.9}$$

In particular, m_8 for $8 = 1 - \bullet\bullet\bullet - =_+$ are the tangent vectors to $S\,M$ in $\mathrm{R}^{=_+,1}$ while is a 1-form orthogonal to $S\,M$. Another identity (and similar ones obtained by lowering one or both of the indices - with the metric) is

$$G^{89}\, m_8 \quad m_9 \quad = \quad \frac{2}{3} \quad - \tag{B.1.10}$$

where is the inverse of . To prove it, one observes that the tensor on the LHS is the projector on $S\,M$, and then verifies that the expression on the RHS has the same property.

When the manifold $S\,M$ is a locally symmetric space, one can find a constant symmetric tensor with upper indices such that [198]

$$\%\& \quad \%1 \quad ! \,"\,{}^{\circ}\& = \frac{4}{3} X_1 \quad ! \,"\,{}^{\circ} \bullet \tag{B.1.11}$$

With some algebra, it follows that

$$= \frac{3}{2} \qquad = \frac{9}{2} \qquad - \qquad\qquad = 2 \qquad 6 \qquad - \tag{B.1.12}$$

as well as

$$= \frac{1}{8} \quad ! \quad " \quad \# \quad ! \,"\,\# \bullet \tag{B.1.13}$$

Quaternionic-Kähler geometry. The scalars $@^D$ are real coordinates on the quaternionic-Kähler manifold $Q\,M$ with metric ${}_{DE}{}^1@^{\circ}$ [236]. For $=$ 2,2 this is a 4= -dimensional Riemannian manifold with holonomy $\mathrm{SU}^1 2^{\circ}$ $\mathrm{Sp}^1 =\ ^{\circ} \bullet \mathrm{Z}_2$. To express this fact, it is convenient to introduce local "vielbeins" $5_D{}^8$ with $8 = 1 - 2$ (not to be confused with the index 8 of very special geometry) in the fundamental of $\mathrm{SU}^1 2^{\circ}$ and $= 1 - \bullet\bullet\bullet - 2=$ in the fundamental of $\mathrm{Sp}^1 =\ ^{\circ}$, such that

$$_{DE} = 5_D{}^8 \quad 5_E{}^9 \quad n_{89} \qquad - \tag{B.1.14}$$

2The case $=$ $= 1$ is special because $\mathrm{SU}^1 2^{\circ 2} \bullet \mathrm{Z}_2$ $\mathrm{SO}^1 4^{\circ}$ and so the holonomy condition does not impose any constraint on (orientable) Riemannian manifolds. However, supersymmetry requires (B.1.25) which we can take as the definition of a quaternionic-Kähler manifold of dimension 4. A 4d space satisfying (B.1.25) is Einstein with self-dual Weyl curvature.

where ϵ_{ij} and C_{AB} are the invariant tensors of SU(2) and Sp(n), respectively. Regarding (iA) as a composite index, the inverse of the matrix $e_a{}^{iA}$ is $e_{iA}{}^{a} = g^{ab} e_b{}^{jB} \epsilon_{ji} C_{BA}$. One can then construct a locally-defined triplet of almost complex structures

$$J^{\alpha}{}_a{}^b \equiv (J^\alpha)_a{}^b = i\, e_a{}^{iA} e_{jA}{}^{b} (\sigma^\alpha)_i{}^j \tag{B.1.15}$$

where $\alpha = 1, 2, 3$ is in the adjoint of SU(2) and σ^α are the Pauli matrices. The derived triplet of almost symplectic forms is $J^\alpha_{ab} = J^\alpha{}_a{}^c g_{cb}$. They are antisymmetric, using that $(\sigma^\alpha)_i{}^j \epsilon_{jk}$ is symmetric.[3] The almost complex structures automatically satisfy the quaternion relation

$$(J^\alpha)_a{}^b (J^\beta)_b{}^c = -\delta^{\alpha\beta}\delta_a^c + \epsilon^{\alpha\beta\gamma}(J^\gamma)_a{}^c . \tag{B.1.17}$$

The Levi-Civita connection takes values in su(2) $\oplus$ sp(n). Calling $\omega_{aj}{}^{i}$ and $\theta_{aA}{}^{B}$ the two projections, respectively, they are determined by the requirement that $e_a{}^{iA}$ be covariantly constant with respect to the full connection:

$$0 = \nabla_b e_a{}^{iA} + e_a{}^{jA}\omega_{bj}{}^{i} + e_a{}^{iB}\theta_{bB}{}^{A} . \tag{B.1.18}$$

We can alternate between the vector and bispinor notations of SU(2) with[4]

$$\omega^\alpha_a = i\,\omega_{ai}{}^{j}(\sigma^\alpha)_j{}^i , \qquad \omega_{ai}{}^{j} = \frac{i}{2}\,\omega^\alpha_a (\sigma^\alpha)_i{}^j . \tag{B.1.20}$$

The two connections are extracted from (B.1.18) through: $\omega_{ai}{}^{j}\delta_A^B + \delta_i^j \theta_{aA}{}^{B} = -e_{iA}{}^{c}\nabla_a e_c{}^{jB}$. From (B.1.18) it immediately follows

$$D_c J^\alpha{}_a{}^b \equiv \nabla_c J^\alpha{}_a{}^b + \epsilon^{\alpha\beta\gamma}\omega^\beta_c J^\gamma{}_a{}^b = 0 . \tag{B.1.21}$$

[3] Using the fact that a 2×2 matrix can be expanded in the basis $\{\mathbb{1}, \sigma^\alpha\}$, we also find

$$2 e_a{}^{iA} e_{jA}{}^{b} = \delta_a^b \delta_j^i + i\, J^\alpha{}_a{}^b (\sigma^\alpha)_j{}^i . \tag{B.1.16}$$

[4] The SU(2) connection satisfies $\epsilon^{jk}\omega_{ak}{}^{l}\epsilon_{li} = \omega_{ai}{}^{j}$, in particular $\omega_{aj}{}^{j} = 0$, and a similar condition is satisfied by θ. This follows from the properties of the Pauli matrices. In going between the vector and bispinor notation one can use the identities

$$(\sigma^\alpha)_k{}^l (\sigma^\alpha)_i{}^j = \delta_k^j \delta_i^l - \epsilon^{lj}\epsilon_{ki} , \qquad (\sigma^\alpha)_i{}^j (\sigma^\alpha)^{kl} = i\,(\sigma^\alpha)_i{}^{k}\delta^{jl} - \delta_i^k (\sigma^\alpha)^{jl} . \tag{B.1.19}$$

In other words, ® is covariantly constant with respect to its natural SU¹2º connection ®. From the integrability condition of (B.1.18) one also obtains (in bispinor and vector notation):

$$'_{DE}{}^{B}{}_{C} = R_{DE8}{}^{9}\, 5_9{}^{B} 5_C{}^{8} \quad ¸ R_{DE} \quad 5_9{}^{B} 5_C{}^{9} \quad = \quad \frac{1}{2} ®_{DE} \quad ®_C{}^{B} ¸ R_{DE} \quad 5_9{}^{B} 5_C{}^{9} \quad - \tag{B.1.22}$$

where $'_{DE}{}^{B}{}_{C}$ is the Riemann tensor of $_{DE}$ and we defined

$$R_{DE8}{}^{9} \quad 2m_{»D}\, l_{E¼8}{}^{9} \quad 2\, l_{»Dj8}{}^{:}\, l_{E¼:}{}^{9} \qquad \text{or} \qquad ®_{DE} \quad 2m_{»D}\, ®_{E¼} ¸ ®_D \quad ®_E$$
$$R_{DE} \quad 2m_{»D} d_{E¼} \quad 2d_{»Dj} \quad d_{E¼} \quad • \tag{B.1.23}$$

In particular

$$'_{DE}{}^{B}{}_{C}\, ®_B{}^{C} = 2= \quad ®_{DE} \; - \tag{B.1.24}$$

i.e., the SU¹2º field strength $®_{DE}$ is the su¹2º projection of the Riemann curvature.

One can prove [237] (see also [236, 238]) that SU¹2º Sp¹= º holonomy manifolds with = 2 are automatically Einstein. They satisfy a stronger property: the Riemann curvature is the sum of the Riemann tensor of HP= and a Weyl part,

$$'_{DEBC} = \frac{'}{8= \;¹= \; ¸ 2º} \quad _{B»D}\ _{E¼C} ¸ ®_{DE} \quad ®_{BC} \quad ®_{B»D} \quad ®_{E¼C} \quad ¸$$
$$¸ \quad 5_D{}^{8}\ 5_E{}^{9}\ n_{89} \quad 5_B{}^{:}\ 5_C \quad n_{:} \quad W \quad • \tag{B.1.25}$$

The tensor W is totally symmetric and controls the Weyl curvature, which is contained in Sp¹= º: it gives rise to a traceless (and thus Ricci flat) contribution to the Riemann curvature. From that expression we obtain

$$'_{EC} = \frac{'}{4=} \quad _{EC} \; - \qquad\qquad ®_{DE} = \frac{'}{4= \;¹= \; ¸ 2º}\, ®_{DE} \; • \tag{B.1.26}$$

The first equation shows that the manifold is Einstein. The second equation shows that the SU¹2º part of the curvature is completely fixed in terms of the triplet of complex structures. The tensor W expresses the freedom in the Sp¹= º part.

While quaternionic-Kähler manifolds can have any size, local supersymmetry requires[5]

$$\_ \quad \frac{'}{4= \;¹= \; ¸ 2º} = \quad 1 \; - \tag{B.1.27}$$

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[5] Had we chosen a canonical normalization for the action of hypermultiplet scalars, the scalar curvature would be fixed in terms of the Planck mass to $\_ = \; <_{\mathrm{Pl}}^{2}$ [236]. This reproduces the fact that the manifold of hypermultiplet scalars is hyper-Kähler in rigid supersymmetry.

fixing the scalar curvature [236]. Hence the manifold of hypermultiplet scalars is a non-trivial quaternionic-Kähler manifold with negative scalar curvature.

Isometries and gauging. We consider gaugings of Abelian isometries of the quaternionic-Kähler manifold $Q\,M$ by the vectors ‵. The isometries are generated by (possibly vanishing or linearly dependent) Killing vectors $:^{D}¹@°$ that also satisfy a quaternionic version of the triholomorphic condition:

$$_{F¹D}\, r_{E°} :^{F} = 0 - \qquad ®_{D}^{F}\, ¹ r_{F} :^{E} ° \quad ¹ r_{D} :^{F} ° ®_{F}^{E} = \_\; ®_{D}^{E} \quad ℛ \;• \tag{B.1.28}$$

The second equation expresses the fact that the derivative of each Killing vector commutes with the triplet of complex structures, up to a rotation parameterized by the SU¹2° sections ℛ . Notice that the LHS can be written, after lowering E, as $2 ⁋_{»D}\; ®_{E¼B} :^{B}$, therefore in the hyper-Kähler case that $\_ = 0$ and the SU¹2° bundle is trivial, this reduces to the familiar condition that the three symplectic forms $®_{DE}$ be preserved by the isometries. By taking the cross product of the second equation in (B.1.28) with $®_{E}^{D}$ we obtain

$$2= \;\_\, ℛ \;=\; ®_{D}^{E}\; r_{E} :^{D} \;• \tag{B.1.29}$$

This shows that on quaternionic Kähler manifolds, the sections ℛ are completely fixed in terms of the Killing vectors. With a little bit of work[6] we obtain

$$⁋_{D} ℛ \;=\; ®_{DF} :^{F} \;• \tag{B.1.30}$$

This shows that ℛ are a triplet of moment maps for the action of $:^{D}$. Taking a derivative and using that $2 ⁋_{»D} ⁋_{E¼} ℛ = ℛ_{DE} \;\; ℛ$ we get back the second equation in (B.1.28), showing that the correction term on the RHS is unavoidable. The divergence of (B.1.30) gives

$$⁋^{D} ⁋_{D} ℛ \;=\; 2= \;\_\, ℛ \;- \tag{B.1.31}$$

showing that the moment maps are eigenfunctions of the Laplacian.

Finally, let us consider for the moment the general case that the Killing vectors might form a non-Abelian group:

$$»: - : ¼^{D} = 2 :^{B}_{»}\; r_{B} :^{D}_{¼} = 5 \quad :^{D} - \tag{B.1.32}$$

---

6 We take the derivative ⁋ of (B.1.29), recalling that ® is covariantly constant. From the algebraic Bianchi identity we have $'_{DEBC}\; ®^{DB} = \frac{1}{2}\, '_{EC}{}^{B}{}_{D}\; ®_{B}^{D} = = \;ℛ_{EC} = = \;\_\; ®_{EC}$. Then we use that the vectors are Killing, as well as the properties of quaternionic-Kähler manifolds.

where on the LHS is the Lie bracket and 5 are the structure constants. Multiplying (B.1.28) by $r_E :^D$ and using (B.1.29), and then exploiting the derivative r_F of (B.1.32), we obtain

$$:^D \,®_{DE}\, :^E = 5 \quad ¸ \_ \quad • \tag{B.1.33}$$

This is called the equivariance relation. In the Abelian case, we just set 5 to zero. In the special case $= \;= 0$ that there are no hypermultiplets, all Killing vectors vanish and the only remnant of the quaternionic-Kähler structure is the condition $= 0$. The solution, up to SU(2) rotations, is $\%^G = X^{G3} Z$ where Z are the so-called Fayet-Iliopoulos (FI) parameters, which in this case are extra parameters one needs to specify.

We now have all the ingredients to write the covariant derivative

$$D_` @^D = m_` @^D \; ¸ \; 6 \quad `:^D \tag{B.1.34}$$

as well as the scalar potential

$$\begin{aligned} + &= \%^G \%^G \;\; \frac{1}{2} G^{89} m_8 \quad m_9 \quad \frac{2}{3} \quad ¸ \; \frac{1}{2} \,{}_{DE}\, :^D :^E \\ &= \%^G \%^G \;\; \frac{1}{2} \quad ¸ \; \frac{1}{2} \,{}_{DE}\, :^D :^E \end{aligned} \tag{B.1.35}$$

that couples the scalars on $S M$ and $Q M$. To go to the second line we used (B.1.10).

The covariant derivative of the supersymmetry parameter n_8^{SUSY} (subject to symplectic-Majorana condition, with $8 = 1, 2$) is

$$`n_8^{\mathrm{SUSY}} = \quad r_` X_8^9 \quad \frac{8}{2} \quad {}_8{}^9 \; n_9^{\mathrm{SUSY}} \tag{B.1.36}$$

with connection

$$\begin{aligned} &= D_` @^D \quad 6 \quad ` \qquad \text{and} \qquad = :^D \quad \_ \\ &= m_` @^D \; ¸ \; 6\_ \quad ` \end{aligned} \tag{B.1.37}$$

where $\_$ is the constant (B.1.27). Under gauge transformations[7]

$$X @^D = 6\, U \; :^D \qquad X \quad ` = \quad m_` U \tag{B.1.38}$$

[7] The covariant derivative transforms as $X D_` @^D = 6\, U \; D_` :^D$.

with parameters U, using (B.1.26), (B.1.30) and (B.1.33) one can show that transforms as an $\mathrm{SU}(2)$ connection:

$$X = m , \quad \text{with} \quad = 6U \,. \tag{B.1.39}$$

Therefore, n_8^{SUSY} is covariant if n_8^{SUSY} transforms as

$$Xn_8^{\text{SUSY}} = \frac{8}{2} \, {}_8{}^9 n_9^{\text{SUSY}} \,. \tag{B.1.40}$$

B.1.1 Conifold truncation

Here we embed the consistent truncation of type IIB supergravity on $)^{1,1}$ to a 5d $N = 2$ gauged supergravity with a so-called "Betti multiplet", described in Section 7 of [169] (called the "second model" in that paper), in the general framework. The model has $=_+ = 2$ and $= = 2$. We identify the fields

$$q^8 = \begin{pmatrix} D, E \\ F \end{pmatrix}_{\text{CF}} = \begin{pmatrix} 4^{4(D,E)\cdot 3} \\ 4^{2(D,E)\cdot 3}\cosh 2F \\ 4^{2(D,E)\cdot 3}\sinh 2F \end{pmatrix}_{\text{CF}} = \begin{pmatrix} 0_1 \\ 0_1 \end{pmatrix}_{\text{CF}} \quad @^D = \begin{pmatrix} 1_1 \\ 1_2 \\ 2_1 \\ 2_2 \\ 0 \\ q \\ 0 \\ D \end{pmatrix}_{\text{CF}} \tag{B.1.41}$$

where "CF" indicates the notation of [169]. The scalar fields $1, 2$ are complex and we used $I_1 = \mathrm{Re}(I)$, $I_2 = \mathrm{Im}(I)$ to indicate their real and imaginary parts, while $D, E, F, 0, q, {}_0$ are real. The hypermultiplet scalars ${}_0$ and q together form the type IIB axio-dilaton ${}_0 , 84^{q}$. Then we identify the Chern-Simons couplings

$${}_{122} = {}_{133} = 2 \tag{B.1.42}$$

and symmetric permutations thereof, while all other components vanish, and the very special geometry of $\mathrm{SO}(1,1) \times \mathrm{SO}(1,1)$:

$$G_{89} = \begin{pmatrix} 4\cdot 3 & 0 \\ 0 & 4 \end{pmatrix} = 4^{\frac{4}{3}(D,E)} \begin{pmatrix} \frac{1}{2}4^{4(D,E)} & 0 & 0 \\ 0 & \cosh(4F) & \sinh(4F) \\ 0 & \sinh(4F) & \cosh(4F) \end{pmatrix} \,. \tag{B.1.43}$$

The tensor has non-vanishing components $^{122} = \quad ^{133} = 1•2$ and permutations.

The quaternionic-Kähler manifold is SO¹4-2º•¹SO¹4º SO¹2ºº. Its metric is

$$_{DE}3@^D3@^E = 4\ ^{4D}\ ^q31\ \ \overline{31}\ \ _¸ 4\ ^{4D¸q}\ 32\ \ \ _031\ \ \ \overline{32}\ \ \ _0\overline{31}\ \ _¸$$
$$_¸\frac{1}{2}4\ ^{8D}\ 230\ _¸\ \mathrm{Re}\ 1\ \ \overline{32}\ \ \ 2\ \ \overline{31}\ ^2\ _¸\frac{1}{2}3q^2\ _¸\frac{1}{2}4^{2q}3\ \ ^2_0\ _¸\ 83D^2\ • \tag{B.1.44}$$

In this normalization $' = \ \ 32$ and thus $\_ = \ \ 1$. The SU¹2º connection is

$$I^1\ \ \ 8\,I^2 = 4\ ^{2D}\ ^{q•2}31\ \ _¸8\,4\ ^{2D¸q•2}\ 32\ \ \ \ _031$$
$$I^3 = \frac{1}{2}4\ ^{4D}\ 230\ _¸\ \mathrm{Re}\ 1\ \ \overline{32}\ \ \ 2\ \ \overline{31}\ \ \ \ \frac{1}{2}4^q3\ \ _0\ • \tag{B.1.45}$$

Finally, we identify the Killing vectors

$$:_1 = 3\ \ \ 1_2\frac{m}{m1_1}\ _¸\ 1_1\frac{m}{m1_2}\ \ \ 2_2\frac{m}{m2_1}\ _¸\ 2_1\frac{m}{m2_2}\ _¸\ 2\frac{m}{m0}\ \ \ \ \ :_2 = 2\frac{m}{m0}\ \ \ \ \ :_3 = 0 \tag{B.1.46}$$

and the corresponding moment maps

$$\%^G_1 = \begin{pmatrix} 34^{q•2\ \ 2D}1\,2_1\ \ \ \ _01_1\ _¸\ 4\ ^q1_2{}^º \\ 34^{q•2\ \ 2D}1\ \ \ _01_2\ \ \ \ 2_2\ _¸\ 4\ ^q1_1{}^º \\ 3\ \ \ 4\ ^{4D}1\,2\ _¸\ 31_2\,2_1\ \ \ \ 31_1\,2_2{}^º \end{pmatrix}\qquad \%^G_2 = \begin{pmatrix} 0 \\ 0 \\ 24\ ^{4D} \end{pmatrix}\qquad \%^G_3 = 0\ • \tag{B.1.47}$$

The SU¹2º connection and the moment maps were given in [171] and can be translated into the notation of [169] (up to a conventional minus sign in the gauge fields) using the identifications

$$q^8 = \begin{pmatrix} 3D_3 \\ D_2 \end{pmatrix}_{\mathrm{HLS}} = \begin{pmatrix} 1 \\ \frac{:_{11}\ \ \ \ :_{12}}{2} \\ \frac{:_{11}\ _¸\ :_{12}}{2} \end{pmatrix}_{\mathrm{HLS}}\ \ @^D = \begin{pmatrix} 2\,\mathrm{Re}\,1^1_0\text{-}\,2\,\mathrm{Im}\,1^1_0\text{-}\,2\,\mathrm{Re}\,1^2_0\text{-}\,2\,\mathrm{Im}\,1^2_0\text{-}\,\frac{:}{2}\text{-}\,q\text{-}\,0\text{-}\,D_1 \end{pmatrix}^{\mathsf{T}}_{\mathrm{HLS}} \tag{B.1.48}$$

where "HLS" indicates the notation of [171].

The theory has a supersymmetric AdS$_5$ vacuum at $D = E = F = 1\ \ \ = 2\ \ \ = 0$ and any value of $0\text{-}\ \ _0\text{-}\,q$ (in particular, the axio-dilaton can take any value). The potential is $+\ _{\mathrm{AdS}} = \ \ 6$ leading to AdS radius $_5 = 6\ ^1$. The spectrum therein was computed in [169] (see its Table 2). We are particularly interested in the spectrum of vector fields and the Killing vectors they couple to:

$$
\begin{aligned}
&' \quad ^1 \quad 2\ ^2 - \quad ^3 : \quad <^2 = 0 - && , \quad ^1 \ \text{¸} \quad ^2 : \quad <^2 = 246^2 \bullet \\
&: \ ' = \tfrac{1}{3}{}^1 :_1 \quad :_2{}^{o} - \quad :_3 && : \ , = \tfrac{1}{3}{}^1 2 :_1 \ \text{¸} \ :_2{}^{o}
\end{aligned}
\tag{B.1.49}
$$

The vector $'$ acquires a mass by the Higgs mechanism, eating the Stückelberg scalar 0. The mass eigenstates are

$$
\mathrm{B} \quad ` \qquad \text{where} \qquad \mathrm{B} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\tag{B.1.50}
$$

is the matrix that diagonalizes them (see also Appendix C.2).

B.2 4d $N = 2$ abelian gauged supergravity

We summarize the salient features of 4d $N = 2$ Abelian gauged supergravity with $=_+$ vector multiplets and $=$ hypermultiplets, following [201, 202, 233]. The graviton multiplet contains a graviton, two gravitini, and a vector; each vector multiplet contains a vector, two gaugini, and a complex scalar; each hypermultiplet contains four real scalars and two hyperini (all fermions can be taken Majorana). We use indices

$$
- \ = 0 - \bullet\bullet\bullet - =_+ - \qquad 8 - 9 = 1 - \bullet\bullet\bullet - =_+ - \qquad D - E = 1 - \bullet\bullet\bullet - 4=
\tag{B.2.1}
$$

for the gauge fields $`$, for the complex scalars I^8 in vector multiplets, and for the real scalars $@^D$ in hypermultiplets, respectively. The data defining the theory are:

1. A special Kähler manifold $K M$ of complex dimension $=_+$.
2. A quaternionic-Kähler manifold $Q M$ of real dimension $4=$.
3. A set of $=_+$ ¸ 1 Killing vectors on $Q M$ compatible with the quaternionic-Kähler structure (if $=$ $= 0$, $=_+$ ¸ 1 FI parameters not all vanishing).

The Killing vectors could be linearly dependent or vanish.

It is always possible to find a duality frame in which all gaugings are purely electric. In such frames the bosonic Lagrangian is

$$
\begin{aligned}
8c\ {}^{14^{o}}_{\mathrm{N}} 4\ {}^1 L_{4\mathrm{d}} &= \frac{'_B}{2} \quad 6_{8\bar{z}}{}^1 I\text{-}T^{o}\, m`\, I^8 m`\, T^{\bar{z}} \quad \frac{1}{2}\ {}_{DE}{}^1 @^{o}\, D`\, @^D D`\, @^E\ \text{¸} \\
&\text{¸}\ \frac{1}{8}\ \mathrm{Im}\, N \quad {}^1 I\text{-}T^{o} \quad `_a \quad `^a\ \text{¸}\ \frac{4\ ^1}{16}\ \mathrm{Re}\, N \quad {}^1 I\text{-}T^{o} \quad `_a \ {}_{df} n^{`adf} \quad 6^2 + {}^1 I\text{-}T\text{-}@^{o} \bullet
\end{aligned}
\tag{B.2.2}
$$

The notation is mostly as in Appendix B.1. Let us explain the other terms.

Special Kähler geometry. The scalars z^i are complex coordinates on the special Kähler manifold $\mathcal{M}$ [202]. This is a Kähler-Hodge manifold — *i.e.*, a Kähler manifold with Kähler potential $\mathcal{K}(z,\bar z)$ and metric $g_{i\bar\jmath}(z,\bar z) = \partial_i\partial_{\bar\jmath}\mathcal{K}$ as well as a line bundle (*i.e.*, a holomorphic vector bundle of rank 1) $\mathcal{L}$ such that its first Chern class coincides (up to a constant) with the Kähler class $J = i\partial\bar\partial\mathcal{K}$ of the manifold[8] — further endowed with a flat $Sp(2n_V+2,\mathbb{R})$ symplectic bundle. The manifold comes equipped with a covariantly-holomorphic section of the tensor product of the symplectic bundle with the $U(1)$-bundle $\mathcal{U}$ associated with $\mathcal{L}$,

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \qquad \text{such that} \qquad \begin{aligned} D_i\mathcal{V} &\equiv \partial_i\mathcal{V} + \tfrac{1}{2}(\partial_i\mathcal{K})\mathcal{V} \\ D_{\bar\imath}\mathcal{V} &\equiv \partial_{\bar\imath}\mathcal{V} - \tfrac{1}{2}(\partial_{\bar\imath}\mathcal{K})\mathcal{V} = 0 \,, \end{aligned} \tag{B.2.3}$$

obeying the constraints

$$\langle \mathcal{V}, \bar{\mathcal{V}} \rangle \equiv M_\Lambda \bar L^\Lambda - L^\Lambda \bar M_\Lambda = -i \tag{B.2.4}$$

and

$$\langle \mathcal{V}, D_i\mathcal{V} \rangle = 0 \,, \tag{B.2.5}$$

where we introduced the Sp-invariant antisymmetric form $i\langle\,\cdot\,,\cdot\,\rangle$. Equivalently, there is a holomorphic section of the tensor product of the symplectic bundle with $\mathcal{L}$,[9]

$$v(z) = e^{-\mathcal{K}/2}\mathcal{V} \equiv \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \qquad \text{such that} \qquad \begin{aligned} D_i v &\equiv \partial_i v + (\partial_i\mathcal{K})\, v \\ D_{\bar\imath} v &\equiv \partial_{\bar\imath} v = 0 \,, \end{aligned} \tag{B.2.6}$$

in terms of which the constraint (B.2.4) reads

$$\mathcal{K} = -\log i\langle v, \bar v\rangle = -\log\left[2\,\mathrm{Im}\left(\bar X^\Lambda F_\Lambda \right) \right] . \tag{B.2.7}$$

while the constraint (B.2.5) becomes $\langle v, D_i v\rangle = \langle v, \partial_i v\rangle = 0$. From (B.2.3)–(B.2.5) it is easy to prove the following properties (or equivalent ones written in terms of v):

$$\begin{aligned} &\langle D_i\mathcal{V}, \bar{\mathcal{V}}\rangle = 0 \,, && D_{\bar\jmath} D_i\mathcal{V} = g_{i\bar\jmath}\mathcal{V} \,, && \langle D_i\mathcal{V}, D_{\bar\jmath}\bar{\mathcal{V}}\rangle = i g_{i\bar\jmath} \\ &\langle D_i\mathcal{V}, D_j\mathcal{V}\rangle = 0 \,, && D_{[i} D_{j]}\mathcal{V} = 0 && \end{aligned} \tag{B.2.8}$$

from which the Kähler metric is extracted in a symplectic-invariant way.

---

[8] Because fermions are sections of the square root of $\mathcal{L}$, the Kähler class of $\mathcal{M}$ equal to the first Chern class of $\mathcal{L}$ is required to be an even integer cohomology class.

[9] In particular, $\theta = \partial\mathcal{K}$ is the Chern connection on $\mathcal{L}$. Moreover, $D_i\mathcal{V} = e^{\mathcal{K}/2} D_i v$ and $D_{\bar\imath}\mathcal{V} = e^{\mathcal{K}/2} D_{\bar\imath} v$.

The rescaling of (X^Λ, F_Λ) under Kähler transformations suggests the use of X^Λ as homogeneous coordinates on KM. It is always possible to find symplectic frames[10] in which the Jacobian matrix $e_i{}^a(z) = \partial_i (X^a/X^0)$ (with $a = 1, \dots, n_V$) is invertible. Notice that

$$\det e_i{}^a = (X^0)^{-n_V-1} \det (X, \partial_i X) = (X^0)^{-n_V-1} \det (X, D_i X) \tag{B.2.9}$$

where the two square matrices on the right have size $n_V + 1$, therefore the matrix $(X^\Lambda, D_i X^\Lambda)$ is invertible as well. Invertibility of the Jacobian ensures that we can use X^Λ as homogeneous coordinates, and regard $F_\Lambda(X)$ as homogeneous functions of degree 1, namely $X^\Lambda \partial_\Lambda F_\Sigma = F_\Sigma$. From (B.2.5) and (B.2.8), written as $\langle V, \partial_i V\rangle = \langle \partial_i V, \partial_j V\rangle = 0$, one obtains the equations

$$(X, \partial_i X)^T (\partial_\Lambda F_\Sigma - \partial_\Sigma F_\Lambda)(X, \partial_j X) = 0 \,. \tag{B.2.10}$$

Invertibility of the matrix implies $\partial_\Lambda F_\Sigma - \partial_\Sigma F_\Lambda = 0$. Hence, in these frames, the sections F_Λ are the derivatives of a holomorphic homogeneous function $F(X)$ of degree 2, called the prepotential, namely $F_\Lambda = \partial_\Lambda F$. In such frames, the Kähler potential (and thus the geometry) is completely specified by the prepotential. The coordinates $z^a \equiv X^a/X^0$ with $a = 1, \dots, n_V$ are called special coordinates.

The couplings of vector fields to the scalars z^a are determined by the $(n_V + 1) \times (n_V + 1)$ period matrix $\mathcal{N}_{\Lambda\Sigma}$, which is uniquely defined by the relations

$$F_\Lambda = \mathcal{N}_{\Lambda\Sigma} X^\Sigma \,, \qquad D_{\bar\imath} \bar F_\Lambda = \mathcal{N}_{\Lambda\Sigma} D_{\bar\imath} \bar X^\Sigma \,. \tag{B.2.11}$$

Explicitly, one needs to invert the matrix relation $(F_\Lambda, D_{\bar\imath} \bar F_\Lambda) = \mathcal{N}_{\Lambda\Sigma} (X^\Sigma, D_{\bar\imath} \bar X^\Sigma)$. The requirement that $g_{i\bar\jmath}$ be positive definite guarantees that the rightmost matrix is invertible [202]. Indeed, introducing the square matrix $\mathcal{L}^\Lambda{}_I = (X^\Lambda, D_{\bar\imath} \bar X^\Lambda)$ of size $n_V + 1$, one can rewrite the scalar products in (B.2.4), (B.2.5) and (B.2.8) as

$$\mathcal{L}^T (\mathcal{N} - \mathcal{N}^T) \mathcal{L} = 0 \,, \qquad \mathcal{L}^\dagger (\mathcal{N} - \mathcal{N}^\dagger) \mathcal{L} = -i \, \mathrm{diag}(1, g_{i\bar\jmath}) \,. \tag{B.2.12}$$

The first equation shows that $\mathcal{N}_{\Lambda\Sigma}$ is a symmetric matrix, given the invertibility of $\mathcal{L}$. The second equation then, assuming that $g_{i\bar\jmath}$ is positive definite, proves that $\mathcal{L}$ is invertible and that $\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}$ is negative definite. It also gives an expression for $\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}$ that, after taking the inverse, reads

$$D_i X^\Lambda \, D_{\bar\jmath} \bar X^\Sigma g^{i\bar\jmath} + \bar X^\Lambda X^\Sigma = -\frac{1}{2} (\mathrm{Im}\,\mathcal{N})^{-1\,\Lambda\Sigma} \,. \tag{B.2.13}$$

[10] See [239] for examples of frames in which, instead, a prepotential does not exist.

This relation, or the equivalent one in terms of the holomorphic section, will be used to rewrite the scalar potential below. When a prepotential exists, $\mathcal{N}$ is obtained from

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\,\frac{(\mathrm{Im}\,F_{IK}X^K)(\mathrm{Im}\,F_{JL}X^L)}{X^M\,\mathrm{Im}\,F_{MN}X^N}\,, \tag{B.2.14}$$

where $F_{IJ} = \partial_I \partial_J F$. In this expression, $\mathcal{N}$ is manifestly symmetric.

Finally, one can define the tensor

$$C_{ijk} = \langle D_i D_j \mathcal{V}, D_k \mathcal{V}\rangle = \langle \mathcal{V}, D_k D_i D_j \mathcal{V}\rangle\,. \tag{B.2.15}$$

Using (B.2.3)–(B.2.8) and the fact that the metric is Kähler, one easily proves that C_{ijk} is totally symmetric and covariantly holomorphic, $D_{\bar\imath} C_{ijk} = 0$ where C has twice the charge of $\mathcal{V}$. One can prove that $(\mathcal{V}, D_i \mathcal{V}, \bar{\mathcal{V}}, D_{\bar\imath}\bar{\mathcal{V}})$ point-wise form a basis for the symplectic bundle [202], hence

$$D_i D_j \mathcal{V} = i\, C_{ijk}\, g^{k\bar k} D_{\bar k} \bar{\mathcal{V}} \tag{B.2.16}$$

follows by taking the product of the LHS with the basis. Among other things, C controls the curvature tensor: $R_{\bar\imath j \bar k l} = g_{j\bar\imath} g_{l\bar k} + g_{j\bar k} g_{l\bar\imath} - C_{jlm} \bar{C}_{\bar\imath \bar k \bar n} g^{m\bar n}$. In special coordinates, the tensor C takes the particularly simple form

$$C_{ijk} = e^{K}\, \partial_i \partial_j \partial_k F(z) \qquad \text{with} \qquad F(z) = (X^0)^{-2} F(X) \tag{B.2.17}$$

and $z^i = X^i / X^0$.

Hypermultiplets and gauging. The part of the action involving the hypermultiplets has the same features as in the 5d case, summarized in Appendix B.1: the hypermultiplet scalars q^u (with $u = 1, \dots, 4n_H$) are coordinates on a quaternionic-Kähler manifold $\mathcal{QM}$ with metric $h_{uv}(q)$. As before, we consider gauging of Abelian isometries of $\mathcal{QM}$, generated by $n_V + 1$ (possibly vanishing or linearly dependent) Killing vectors $k_I^u(q)$ that must be compatible with the quaternionic-Kähler structure, with associated triplets of moment maps $\vec{P}_I(q)$. In full generality, one could consider both electric and magnetic gaugings, described by Killing vectors k_I^u and k^{uI}, respectively, and transforming as a vector under $\mathrm{Sp}(2n_V + 2, \mathbb{R})$ duality transformations. It is always possible to find a duality frame in which all gaugings are purely electric, and we will work in such a frame. Notice that there is no guarantee that in this frame a prepotential exists.

The scalar potential is

$$\begin{aligned} V &= 2 P^x P^x \left(e^K g^{i\bar{j}} D_i L D_{\bar{j}} \bar{L} - 3 L \bar{L} \right) + 4 e^K h_{uv} k^u k^v L \bar{L} \\ &= -P^x P^x \left(\operatorname{Im} \mathcal{N} \right)^{-1} + 8 e^K L \bar{L} + 4 e^K h_{uv} k^u k^v L \bar{L} \, . \end{aligned} \tag{B.2.18}$$

To go to the second line we used (B.2.13).

The covariant derivative of the supersymmetry parameter $\epsilon_i^{\mathrm{SUSY}}$ (subject to symplectic-Majorana condition, with $i = 1, 2$) is

$$D_\mu \epsilon_i^{\mathrm{SUSY}} = \left(\nabla_\mu \delta_i^j + \frac{i}{2} A_\mu \delta_i^j + \frac{i}{2} \omega_\mu^x (\sigma^x)_i{}^j \right) \epsilon_j^{\mathrm{SUSY}} \tag{B.2.19}$$

with connections

$$\begin{aligned} \omega_\mu^x &= \partial_\mu q^u \omega_u^x + g A_\mu P^x \\ A_\mu &= \frac{i}{2} \left[(\partial_U K) \partial_\mu z^U - (\partial_{\bar{U}} K) \partial_\mu \bar{z}^{\bar{U}} \right] . \end{aligned} \tag{B.2.20}$$

Here ω_μ^x is the $\mathrm{SU}(2)$ connection that descends from the quaternionic-Kähler manifold QM, as in the 5d case (B.1.37). Instead A_μ descends from the connection on the $\mathrm{U}(1)$-bundle U on the special Kähler manifold KM.

Appendix C

Scherk-Schwarz reduction

In this Appendix, the focus will be on the Scherk-Schwarz (SS) reduction of a 5d abelian $\mathcal{N} = 2$ supergravity on S^1, down to a 4d abelian $\mathcal{N} = 2$ supergravity. We will first reduce the Lagrangian of the theory, and then we will look at the relation between 5d and 4d charges.

C.1 Reduction with background gauge fields

Following [206] we will now reduce, piece by piece, the bosonic Lagrangian (B.1.2) of 5d $\mathcal{N} = 2$ gauged supergravity down to 4d. We start in 5d with n_V vector multiplets and n hypermultiplets. We use indices

$$I = 1, \dots, n_V + 1, \qquad \Lambda = 0, \dots, n_V + 1, \qquad u, v = 1, \dots, 4n. \tag{C.1.1}$$

We indicate the 5d vector fields as $\hat{A}^I_M$ (where $M, N = 0, \dots, 4$ are space-time indices) and parameterize the vector multiplet scalars in terms of sections h^I subject to the cubic constraint $\mathcal{V}(h) = 1$ in (B.1.3). The hypermultiplet scalars are q^u. We employ the rather standard Kaluza-Klein reduction ansatz (2.4.2) and (2.4.4):

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\phi} g_{\mu\nu} + e^{-4\phi} A^0_\mu A^0_\nu & e^{-4\phi} A^0_\mu \\ e^{-4\phi} A^0_\nu & e^{-4\phi} \end{pmatrix}, \qquad \hat{g}^{MN} = \begin{pmatrix} e^{-2\phi} g^{\mu\nu} & -e^{-2\phi} A^{0\mu} \\ -e^{-2\phi} A^{0\nu} & e^{4\phi} + e^{-2\phi} A^0_\rho A^{0\rho} \end{pmatrix},$$

$$e_{(5)} = e^{2\phi} e_{(4)}, \qquad \cdots = e^{2\phi} \ell_2, \qquad \hat{A}^I_M = \left(A^I_\mu - \ell_1 A^0_\mu,\ \ell_1 + b \right). \tag{C.1.2}$$

The last coordinate, which we call y and whose range Δy we leave generic for now, is compactified on a circle of length $e^{-2\phi}\Delta y$, and no field depends on it. We indicated as $\hat{g}_{MN}$ and $e_{(5)}$ the 5d metric and the square root of its determinant, respectively, and as $g_{\mu\nu}$ and $e_{(4)}$ (with $\mu, \nu = 0, \dots, 3$ space-time indices) their 4d counterparts. In 4d we end up with

$=_{+}$ ¸ 1 vector multiplets, and we indicate as ˋ the vector fields. The physical scalars in 4d vector multiplets are the complex fields I^{8}. With a useful abuse of notation, we utilize the very same index for 5d vector fields and 4d physical scalars, I , because in 4d we have one more vector field than in 5d. We also use the notation

$$I_1 \quad \mathrm{Re}\, I \;- \qquad\qquad I_2 \quad \mathrm{Im}\, I \;• \tag{C.1.3}$$

Notice that the real scalar $\mathfrak{q}$ can be eliminated with the 5d constraint,

$$4^{\;6\mathfrak{q}} = \;\; V^{1} I_2^{\,o} \;- \tag{C.1.4}$$

then the scalars I can be treated as independent. The real parameters b represent background gauge fields along the circle, therefore, up to a gauge transformation, this ansatz is equivalent to a Scherk-Schwarz reduction.

The reduction of the Einstein term gives

$$8c\;{}^{14^o}_{\mathrm{N}}\, L_1 = 4_{15^o}\frac{b_B}{2} = 4_{14^o}\;\frac{'_B}{2}\quad 3\, m^{ˋ}\mathfrak{q}\, m^{ˋ}\mathfrak{q} \quad \frac{4^{\;6\mathfrak{q}}}{8}\;\; {}^{0}_{ˋa}\;\; 0^{ˋa} \;\; ¸ \text{ total derivatives} \;• \tag{C.1.5}$$

Here b_B and $'_B$ are the 5d and 4d Ricci scalars, respectively. The 4d and 5d Newton constants are related by

$$\frac{1}{{}^{14^o}_{\mathrm{N}}} = \frac{H}{{}^{15^o}_{\mathrm{N}}}\;• \tag{C.1.6}$$

In the following, for clarity, we will omit the factor $8c\;{}^{14^o}_{\mathrm{N}}$. The reduction of the kinetic term of vector multiplet scalars gives

$$L_2 = \;\; 4_{15^o}\frac{1}{2}\qquad 6^{\,\prime\prime\,\#}\, m_{\prime\prime} \quad m_{\#} \quad = 4_{14^o}\;\; \frac{4^{4\mathfrak{q}}}{2}\qquad m^{ˋ} I_2\, m^{ˋ} I_2 \;¸\; 3\, m^{ˋ}\mathfrak{q}\, m^{ˋ}\mathfrak{q} \;• \tag{C.1.7}$$

The last term exactly cancels the second term in L_1, therefore

$$L_1 \;¸\; L_2 = 4_{14^o}\;\; \frac{'_B}{2}\;\; \frac{4^{4\mathfrak{q}}}{2}\qquad m^{ˋ} I_2\, m^{ˋ} I_2 \qquad \frac{4^{\;6\mathfrak{q}}}{8}\;\; {}^{0}_{ˋa}\;\; 0^{ˋa} \;\; • \tag{C.1.8}$$

The reduction of the kinetic term of hypermultiplet scalars gives

$$\begin{aligned} L_3 = &\;\; 4_{15^o}\frac{1}{2}\;\; {}_{DE}\, 6^{\,\prime\prime\,\#}\, b_{\prime\prime} @^{D} b_{\#} @^{E} \\ = &\; 4_{14^o}\quad \frac{1}{2}\;\; {}_{DE}\, D^{ˋ} @^{D} D^{ˋ} @^{E} \qquad \frac{6^{2} 4^{6\mathfrak{q}}}{2}\quad :^{D}_{0} \;¸\; I_1 :^{D} \quad {}_{DE}\; :^{E}_{0} \;¸\; I_1 :^{E} \qquad • \end{aligned} \tag{C.1.9}$$

Here $\hat{D}_{"} @^{D} = m_{"} @^{D} ¸ 6 \, b_{"} :^{D}$ is the 5d covariant derivative in (B.1.34), while

$$D_{`} @^{D} = m_{`} @^{D} ¸ 6 \quad _{`} :^{D} ¸ 6 \quad ^{0}_{`} b \ :^{D} = m_{`} @^{D} ¸ 6 \quad _{`} :^{D} \tag{C.1.10}$$

is the 4d covariant derivative, and we defined the new Killing vector

$$:^{D}_{0} \quad b \ :^{D} • \tag{C.1.11}$$

The reduction of the gauge kinetic term gives

$$\begin{aligned} L_{4} &= \ 4_{1}{}_{5º} \frac{1}{4} \quad b_{"\#} \, b^{"\#} \\ &= 4_{1}{}_{4º} \ \frac{4^{2q}}{4} \quad _{`} \quad I_{1} \ ^{0}_{`a} \quad ^{`a} \quad I_{1} \ ^{0`a} \quad \frac{4^{4q}}{2} \quad m_{`} I_{1} m^{`} I_{1} \ - \end{aligned} \tag{C.1.12}$$

where $b_{"\#}$ and $_{`a}$ are the 5d and 4d field strengths, respectively. We used

$$b_{`4} = m_{`} I_{1} - \qquad b_{`a} = \quad _{`a} \quad I_{1} \ ^{0}_{`a} ¸ 2 \ ^{0}_{»`} m_{a¼} I_{1} • \tag{C.1.13}$$

To reduce the Chern-Simons term, we extend the geometry (2.4.2) to a 6d bulk whose boundary is the original 5d space. A convenient way to do that is to complete the circle parameterized by H into a unit disk with radius $d \ 2 \ »0-1¼$. We extend the 5d connections b in (2.4.4) to 6d connections e as follows:

$$e \ = \quad ¸ \, b \quad ^{0} ¸ \, d^{2} \, {}^{1} I_{1} ¸ \, b \ º \ 3H \quad ^{0} • \tag{C.1.14}$$

We then write the Chern-Simons action term as

$$\int_{5d}^{1} L_{5} = \int_{5d}^{1} \frac{1}{12} \quad b \wedge b \wedge b \ = \int_{6d}^{1} \frac{1}{12} \quad e \wedge e \wedge e \ • \tag{C.1.15}$$

Substituting $e \ = 3 \, e$ and performing the integrals over $3d^{2} \wedge {}^{1}3H \quad ^{0}º$, we extract the 4d reduced Lagrangian

$$L_{5} = \frac{1}{16} \quad n^{`adf} \quad I_{1} ¸ \, b \quad _{`a} \ _{df} \quad I_{1} I_{1} \quad b \, b \quad _{`a} \ ^{0}_{df} ¸ \ \frac{I_{1} I_{1} I_{1} ¸ \, b \, b \, b}{3} \quad ^{0}_{`a} \ ^{0}_{df} • \tag{C.1.16}$$

Notice that the contributions containing the b 's are standard theta terms.

Finally, the reduction of the scalar potential gives

$$L_6 = 4_{15^0} 6^2 + = 4_{14^0} 6^2 \; \%^G \%^G \frac{4^{26}}{2} G^{89} m_8 \quad m_9 \qquad \frac{24^{66}}{3} I_2 I_2 \; ¸ \; \frac{4^{66}}{2} \; {}_{DE} :^D :^E I_2 I_2 \; \bullet \tag{C.1.17}$$

We proceed now with recasting the various pieces of the action in the general form (B.2.2) of 4d $N = 2$ gauged supergravity with $=_+ \; ¸ \; 1$ vector multiplets and $=$ hypermultiplets. The Einstein term receives its contribution from L_1:

$$L_1^0 = 4_{14^0} \frac{'_B}{2} \bullet \tag{C.1.18}$$

The kinetic term of vector multiplet scalars gets contributions from L_2 and L_4:

$$L_2^0 = 4_{14^0} \frac{4^{46}}{2} \qquad m^{`} I_2 m^{`} I_2 \; ¸ \; m^{`} I_1 m^{`} I_1 \quad = \quad 4_{14^0} 6 - m I \; m^{`} \bar{T} \; - \tag{C.1.19}$$

where we defined the Hermitian metric

$$6 - = \frac{4^{46}}{2} \qquad - \bullet \tag{C.1.20}$$

The kinetic term of hypermultiplet scalars gets its contribution from L_3,

$$L_3^0 = 4_{14^0} \frac{1}{2} \; {}_{DE} D^{`} @^D D^{`} @^E \; - \tag{C.1.21}$$

with the covariant derivative $D^{`}$ defined in (C.1.10)-(C.1.11). The gauge kinetic term receives contributions from L_1 and L_4:

$$L_4^0 = 4_{14^0} \frac{4 \; ^{66}}{8} \quad {}^0_{`a} \quad {}^{0`a} \; ¸ \; 46 \qquad {}_{`a} \quad I_1 \; {}^0_{`a} \qquad {}^{`a} \quad I_1 \quad {}^{0`a} \quad = 4_{14^0} \frac{1}{8} \operatorname{Im} N \qquad {}_{`a} \quad {}^{`a} \tag{C.1.22}$$

where we defined the field-dependent matrix of gauge couplings

$$\operatorname{Im} N \quad = \quad 4 \; ^{66} \begin{pmatrix} 1 \; ¸ \; 46 \; {}_{!} I_1 I_1^{!} & 46 \quad I_1 \\ 46 \quad I_1 & 46 \end{pmatrix} \tag{C.1.23}$$

in which the indices $-$ run over 0 and then the values of $-$. On the other hand, the field-dependent theta terms are contained in L_5:

$$L_5^0 = L_5 = \frac{1}{16} \operatorname{Re} N \qquad n^{`adf} \qquad {}_{`a} \; {}_{df} \tag{C.1.24}$$

where

$$\mathrm{Re}\,N \quad = \quad \frac{\frac{1}{3} \quad ! \; " \quad l_1\, l_1^{!}\, l_1^{"} \; ¸ \; b \quad b^{!}\, b^{"} \qquad \frac{1}{2} \qquad ! \quad l_1\, l_1^{!} \quad b \quad b^{!}}{\frac{1}{2} \quad ! \quad l_1\, l_1^{!} \quad b \quad b^{!} \qquad\qquad l_1 \; ¸ \; b} \; ! \quad • \tag{C.1.25}$$

It turns out that 6 – and N descend from the following prepotential:

$$\begin{aligned} {}_1 - {}^{\mathrm{o}} &= \frac{1}{6} \quad \frac{\check{-}\;\check{-}\;\check{-}}{-^{0}} \qquad\qquad \text{with } \check{-} \quad - \; ¸ \; b \; -^{0} \\ &= \frac{1}{6} \quad \frac{- \;-\; -}{-^{0}} \; ¸ \; \frac{1}{2} \qquad b \;-\; - \; ¸ \; b\; b \;-\; -^{0} \; ¸ \; \frac{1}{3} b\; b\; b \;\; {}_1 - {}^{0}{}_{\mathrm{o}}{}^{2} \quad • \end{aligned} \tag{C.1.26}$$

The terms in parenthesis involving the b 's only affect standard theta terms, which are topological and thus do not enter into the equations of motion. Indeed, using special coordinates $l \;=\; - \;•\; -^{0}$ and in the Kähler frame $\mathrm{j} - {}^{0}\mathrm{j}^{2} = 1$, one derives the Kähler potential[1]

$$K = \quad \log \frac{1}{68} \qquad l \quad \bar{l} \quad l \quad \bar{l} \quad l \quad \bar{l} \quad = \quad \log 8\,4^{\;6\varphi} \tag{C.1.27}$$

from which the Kähler metric (C.1.20) with (B.1.6) follows. On the other hand,

$$= \frac{\frac{1}{3} \quad ! \; " \quad l \quad l^{!}\, l^{"} \; ¸ \; b \quad b^{!}\, b^{"} \qquad \frac{1}{2} \qquad " \quad l \quad l^{"} \quad b \quad b^{!}}{\frac{1}{2} \quad " \quad l \quad l^{"} \quad b \quad b^{!} \qquad\qquad l \; ¸ \; b} \; ! \tag{C.1.28}$$

from which the matrix N in (C.1.23) and (C.1.25) follows. It might be useful

$$\begin{aligned} {}_1 - {}^{0}{}_{\mathrm{o}}\;{}^{2} - \quad \mathrm{Im} \qquad - &= 4 \qquad \tfrac{1}{3}\,\mathrm{Im}^{1} l \;\; l \;\; l \;\; {}^{\mathrm{o}} \quad \tfrac{1}{2}\,\mathrm{Im}^{1} l \;\; l \; {}^{\mathrm{o}}\,\mathrm{Re}^{1} l \;\; {}^{\mathrm{o}} \\ &= \frac{4}{3} \qquad l_2\, l_2\, l_2 = 4^{\;K} = 8\,4^{\;6\varphi} - \end{aligned} \tag{C.1.29}$$

as well as

$$\mathrm{Im} \qquad - \;•\; -^{0} = 8 \qquad " \; l_2\, l_2^{"} \; • \tag{C.1.30}$$

Finally, the scalar potential gets contributions from L_3 and L_6:

$$\begin{aligned} L_6^{0} = \quad 4_{14^{\mathrm{o}}} 6^{2} \;\; \%^{G} \%^{G} \;\; \frac{4^{2\varphi}}{2} G^{89} m_8 \quad m_9 \qquad \frac{24^{6\varphi}}{3} l_2 l_2 \quad ¸ \\ ¸ \; \frac{4^{6\varphi}}{2} \; {}_{DE} \;\; {:}^{D} {:}^{E} l_2 l_2 \; ¸ \; {}^{1}{:}_{0}^{D} \; ¸ \; l_1 {:}^{D}{}^{\mathrm{o}1}{:}_{0}^{E} \; ¸ \; l_1 {:}^{E}{}^{\mathrm{o}} \quad - \end{aligned} \tag{C.1.31}$$

[1]The completely covariant expression for the Kähler potential is $4^{\;K} = 8\,\mathrm{j} - {}^{0}\mathrm{j}^{2}\, 4^{\;6\varphi}$.

which can be rewritten as

$$L_6^0 = 4_{1}4^{0}6^2 \quad \%^G \%^G \quad \mathrm{Im}\, N^{o\ 1} \quad {}_{¸} 8\, 4^K - \;\bar{-} \quad {}_{¸} 4\, 4^K \;{}_{DE} :^D :^E - \;\bar{-} \quad \bullet \tag{C.1.32}$$

To manipulate the first line we used (B.1.10) as well as

$$\mathrm{Im}\, N^{\ 1} \quad {}_{¸} 8\, 4^K -^{1} \;\bar{-}^{\ o} = 4^{6\varphi} \begin{pmatrix} 0 & 0 \\ 0\ \tfrac{1}{4}6 & I_2 I_2 \end{pmatrix}^{!} - \tag{C.1.33}$$

which immediately follows from (C.1.23). Notice in particular that $\%_0$ drops out of the potential and cannot be extracted from it, but it is still determined as $\%_0 = b\ \%$ from (C.1.11). The action L_6^0 exactly reproduces the potential in (B.2.18).

Summarizing, the compactification gives the following map from 5d to 4d data:

$$\begin{array}{ccc} \text{5d} & & \text{4d} \\ =_{+} \text{ vector multiplets} & & =_{+}\ {}_{¸}\ 1 \text{ vector multiplets} \\ S\,M \text{ with} & \xrightarrow[\text{background fields}]{\text{reduction with } b} & K\,M \text{ with } = \dfrac{1}{6}\ \dfrac{\check{-}\ \check{-}\ \check{-}}{-^{0}} \\ Q\,M \text{ with } {}_{DE}{}^{1}@^{o} & & Q\,M \text{ with } {}_{DE}{}^{1}@^{o} \\ \text{gauging of } :^D & & \text{gauging of } :^D = b\ :^D - :^D \end{array} \tag{C.1.34}$$

where $\check{-} = -\ {}_{¸}\ b\ -^{0}$.

C.1.1 Reduction of the conifold truncation

The reduction of the 5d conifold truncation described in Appendix B.1.1 gives a 4d supergravity with the following data. The prepotential is

$$= \frac{\check{-}^1\ {}_1\check{-}^{2o2}\ \ {}_1\check{-}^{3o2}}{-^{0}} \bullet \tag{C.1.35}$$

It induces the vector multiplet scalar metric

$$6\,- = \frac{1}{2} \begin{pmatrix} \frac{1}{2^1 I_2^{1o2}} & 0 & 0 \\ & \dfrac{{}_1I_2^{2o2}\ {}_{¸}\ {}_1I_2^{3o2}}{\left({}_1I_2^{2o2}\ \ {}_1I_2^{3o2}\right)^2} & \dfrac{2\, I_2^2\, I_2^3}{\left({}_1I_2^{2o2}\ \ {}_1I_2^{3o2}\right)^2} \\ \text{Symmetrized} & & \dfrac{{}_1I_2^{2o2}\ {}_{¸}\ {}_1I_2^{3o2}}{\left({}_1I_2^{2o2}\ \ {}_1I_2^{3o2}\right)^2} \end{pmatrix} \tag{C.1.36}$$

that depends on I_2, the theta terms (C.1.25) that depend on I_1 and b , while the gauge coupling function $\mathrm{Im}\, N$ takes a lengthier expression that depends on I_1 and I_2 and can be easily derived from (C.1.23). Since in 5d $:_3 = 0$, the 4d extra Killing vector is $:_0 = b^1 :_1 \,¸\, b^2 :_2$.

C.2 Reduction of black-hole charges

The electric black-hole charges computed in [163] in our notation read

$$\&_T = \frac{1}{8c\,{}^{15^o}_{N}6} \int^{1}_{(^3_1} {}_T\, ¢_5\, b\ - \tag{C.2.1}$$

where the integral is taken on the three-sphere at infinity, and they are dimensionless. We recall that only a subspace b^T of the vector fields are massless on the AdS$_5$ vacuum and the index T runs over them. The massless vectors are such that the hypermultiplet scalars sit at a fixed point of the gauged isometries, and are thus identified by the conditions

$$:^D_T {}^1@^o = 0\ • \tag{C.2.2}$$

Indeed, let B be a matrix of linear redefinitions such that B $b_{\grave{}}$ are mass eigenstates. Such a matrix is characterized by

$$B \quad {}^{\#} :^D_{\#}\ {}_{DE} :^E_! = \_\,{}_{\#} B^{\#}{}_! - \tag{C.2.3}$$

where _ is the diagonal matrix of squared masses (in units of 6^2). The corresponding linear transformation of charges is $\&$! $\&$ ^{1}B ^{1o} , while the Killing vectors corresponding to the mass eigenstates are $:^{D}{}^1$B ^{1o} . Now consider a massless vector and let the index T be such that $\_^T_T = 0$ (not summed over T). Using non-degeneracy of the metrics and ${}_{DE}$, one easily proves that $:^D{}^1$B ${}^{1o}{}_T = 0$, which is (C.2.2).

Now, the equations of motion for the bosonic fields of 5d gauged supergravity following from (B.1.2) are

$$\begin{aligned} 3 \quad ¢_5\, b &= \frac{1}{4} \quad b \wedge b \quad 6\ {}_{DE} :^D ¢_5\, Ð@^E \\ b_{"\#} &= \quad b_{"\%} b_{\#}{}^{\%} \quad \frac{1}{6} б_{"\#}\, b_{\%\&}\, b^{\%\&} \ ¸ \\ &¸\, G_{89}\, m_{"} q^8 m_{\#} q^9 \ ¸\ {}_{DE}\, Ð_{"} @^D Ð_{\#} @^E \ ¸\ \frac{2}{3} б_{"\#}\, 6^2 +\ • \end{aligned} \tag{C.2.4}$$

Notice that (C.2.2) is just the condition not to have a source in the I-th component of Maxwell's equation from the hypermultiplets. We can express the charges Q_I in terms of integrals at the horizon [240] using the EOMs (C.2.4):

$$Q_I = \frac{1}{8\pi G_N^{(5)}}\left[\int_{S^3_r} {}_I \star F + \int_{S^3_r\times[r,\infty]}\left(\frac{1}{4}\,{}_I\, F\wedge F - 6\, g_{uv}\, k^u_I \star Dq^v\right)\right]. \quad \text{(C.2.5)}$$

The first term is an integral evaluated at radius r, which we will take to be the horizon location. The second term is a correction, integrated on a cylinder $S^3 \times I$ where I is the interval from r to ∞, that leads to a Page charge. Assuming that the condition $k^u_I(q) = 0$ remains true also on the black-hole background,[2] the third term vanishes.

We can apply a similar manipulation to the angular momenta $J_{a=1,2}$. Given the space-time Killing vectors $\xi_a = \xi_a^{\mu}\partial_\mu$, the angular momenta are defined in [163] as

$$J_a = \frac{1}{16\pi G_N^{(5)}}\int_{S^3_\infty} \star d\xi_a \quad \text{(C.2.6)}$$

where we have indicated with the same symbol $\xi_a = \xi_{a\,\mu}dx^\mu$ the 1-forms dual to the Killing vectors, and the integral is evaluated once again at infinity. One can show that the Killing equation implies

$$d\star d\xi = 2R_{\mu\nu}\xi^\mu \star dx^\nu. \quad \text{(C.2.7)}$$

We can then use the EOMs (C.2.4) to replace the Ricci scalar $R_{\mu\nu}$. We assume that S^3 is invariant under the isometries generated by ξ_a, therefore, indicating as ι the interior product, the integral of $R_{\mu\nu}\xi^\mu \star dx^\nu = \iota(\star 1)$ vanishes. We also assume that $\iota\, d\phi^i = 0$. We obtain

$$J_a = \frac{1}{16\pi G_N^{(5)}}\left[\int_{S^3_r}\star d\xi_a + 2\int_{S^3}\left(\iota_a F\wedge\star F + {}_{uv}\,\iota_a Dq^u \star Dq^v\right)\right]. \quad \text{(C.2.8)}$$

Now let us proceed and reduce the charges to 4d imposing the ansatz (C.1.2), in particular

$$\begin{aligned} F &= \;\; + F^0 + (l_1 + F^0)\,dy \\ F &= \;\; l_1 F^0 + d l_1 \wedge dy \end{aligned} \quad \text{(C.2.9)}$$

and performing the integrals along the circle. Notice that because of (C.1.6) and since the horizon areas in 5d and 4d are related by $\mathrm{Area}_{(5)} = H\,\mathrm{Area}_{(4)}$, the black-hole entropy is the

[2] In the case of the conifold compactification discussed in Section 2.4.2, this assumption is true, see (2.4.29). We expect the assumption to be true in all cases.

same in 5d and 4d. We find

$$\int_{(^3}^{1} ¢_5\, b = H \int_{(^2}^{1} 4^{2¢} \quad ¢_4 \quad I_1 \quad ^0 \qquad \int_{(^3}^{1} b \wedge b = H \int_{(^2_A}^{1} 2I_1 \quad I_1 I_1 \quad ^0 \, . \tag{C.2.10}$$

In the second equality, we used that $I_1 \; ! \; 0$ at infinity. The electric charges are thus

$$Q_T = \frac{1}{6} \int_{(^2_A}^{1} \frac{X(_{4d}}{X\; T} \quad \frac{1}{8c\; ^{14^o}_{N} 6} \; T \int_{(^2_A}^{1} \frac{1}{2} b \quad ¸ \frac{1}{4} b\; b \quad ^0 \; - \tag{C.2.11}$$

where

$$\frac{X(_{4d}}{X} = \frac{1}{16c\; ^{14^o}_{N}} \; \mathrm{Im}\, N \quad ¢_4 \quad ¸ \mathrm{Re}\, N \tag{C.2.12}$$

are the derivatives of the action obtained from (B.2.2) with (C.1.23) and (C.1.25).

We define the 4d dimensionless magnetic charges as

$$? \; = \frac{6}{4c} \int_{(^2}^{1} \quad - \tag{C.2.13}$$

where the integral can be done at any radius because of the Bianchi identities. On the other hand, the first Chern class of the circle fibration — that we take to be the Hopf fibration of $(^3$ — is $\frac{1}{H} \; 3 \; ^0 = 1$. Thus, we obtain a properly quantized $?^0 = 1$ if we set

$$H = \frac{4c}{6} \, . \tag{C.2.14}$$

We will use this normalization from now on.

Let us now reduce the angular momentum. We consider the case $_1 = \; _2$, with $_{1\text{-}2}$ normalized such that they generate orbits of length $2c$, and define $\; = {}^1 \; _1 \; ¸ \; _2{}^o {\bullet} 2$. The corresponding Killing vector and dual 1-form are

$$" m " = \frac{H}{4c} \frac{m}{mH} = \frac{1}{6} \frac{m}{mH} \; - \qquad " 3G " = \frac{1}{6} 4^{4¢_1} 3H \quad ^0 o \, . \tag{C.2.15}$$

The first term in (C.2.8) gives

$$\int_{(^3}^{1} ¢_5\, 3 = \frac{H}{6} \int_{(^2}^{1} 4^{6¢} ¢_4 \quad ^0 \, . \tag{C.2.16}$$

To reduce the second term we use i $b = \frac{1}{6} 3 I_1$, integrate by parts, and use the EOMs (C.2.4). To reduce the third term we use i $b @^D = I_1 ¸ b :^D$ and i$_I$ ¹¢ 1º = ¢ I for a 1-form I. Eventually

$$= \frac{1}{8c\ ^{14º}_{N} 6} \int_{(^2_A} \left(1 \right. \left[\frac{1}{2} 4\ ^{6¢} ¢_4\ ^0 ¸ 4\ ^{2¢}\ I_1 ¢_4\ I_1\ ^0 ¸ \right.$$
$$\left. \left. \frac{1}{4} I_1 I_1\ \frac{1}{6} I_1 I_1 I_1\ ^0 \right]_1 ¸ \int_{(^2} ¢_4 6 :^D_0\ _{DE} D@^E \right)$$
$$= \frac{1}{6} \int^1_{(^2_A} \frac{X(_{4d}}{X}\ ^0\ \frac{1}{8c\ ^{14º}_{N} 6} \left[\int^1_{(^2_A} b\ b\ \frac{1}{4}\ ¸ \frac{1}{6} b\ ^0 \right]_1 ¸ \int_{(^2} ¢_4 6 :^D_0\ _{DE} D@^E \ • \tag{C.2.17}$$

The 4d angular momentum of the black-hole solution is proportional to $_1\ _2$, which vanishes in the case under consideration. This implies that we can impose spherical symmetry on $(^2$. The section $D@^E$ is charged under the Abelian vector fields ˓, therefore the magnetic fluxes ? give rise to an effective spin B on $(^2$. However, the spin spherical harmonics [241, 242] have total angular momentum $9\ \ jBj$, which should vanish to have a spherically-symmetric configuration. Since the Abelian symmetries are realized non-linearly on $D@^E$ as soon as $:^D < 0$, we obtain the condition

$$?\ :^D ¹@º = 0 \tag{C.2.18}$$

for spherically-symmetric black-hole solutions. Without loss of generality, in Section 2.4 we have set $? = 0$ which implies $:^D_0 = 0$. We then see that the last term in (C.2.17) vanishes.

The magnetic charges that appear in the attractor equations of [174], in our conventions, are (C.2.13) while the electric charges are

$$@ = \frac{6}{4c} \int^1_{(^2_A} \qquad \text{with} \qquad = 16c\ ^{14º}_{N} \frac{X(_{4d}}{X} \ • \tag{C.2.19}$$

Setting $? = 0$, we obtain the following dictionary between 5d and 4d charges:

$$\begin{aligned} @_0 &= 4\ ^{14º}_{N} 6^2\ ¸ \frac{1}{3}\ b\ b\ b\ ?^0 \\ @_T &= 4\ ^{14º}_{N} 6^2 \&_T ¸ \frac{1}{2}\ _T\ \frac{1}{2} b\ b\ ?^0 \ • \end{aligned} \tag{C.2.20}$$

C.2.1 Baryonic charge quantization in the conifold theory

To fix the exact relation between the supergravity charge &3 and the field theory baryonic charge & , we deduce the Dirac quantization condition satisfied by ^{3} from the consistent reduction of [169].

The metric of $)^{1\text{-}1}$ is

$$3B^2 = \frac{1}{6} \sum_{8=1\text{-}2} 3\backslash_8^2 \,¸\, \sin^2 \backslash_8\, 3i_8^2 \,¸\, [^2 \qquad \text{with} \qquad [= \frac{1}{3}\, 3k \,¸ \sum_{8=1\text{-}2} \cos \backslash_8\, 3i_8 \;• \tag{C.2.21}$$

We define the 2-forms[3]

$$\begin{aligned} &= \frac{1}{6}\ \sin\backslash_1 3\backslash_1 \wedge 3i_1 \,¸\, \sin\backslash_2 3\backslash_2 \wedge 3i_2 \ = \frac{1}{2} 3[\\ &= \frac{1}{6}\ \sin\backslash_1 3\backslash_1 \wedge 3i_1 \quad \sin\backslash_2 3\backslash_2 \wedge 3i_2 \ • \end{aligned} \tag{C.2.22}$$

The expansion of the 10d RR field strength $_5^{RR}$ in [169] around the AdS_5 $)^{1\text{-}1}$ vacuum (where $D = E = F = 1 \ = 2 \ = 0$), keeping only the dependence on the gauge fields and the Stückelberg scalar 0, in our conventions reads

$$\begin{aligned} {}_5^{RR} = 46\ ¢_5 1 \quad 26\ ^1 {}^1¢_5 \quad 0^o \wedge {}^1[\quad 6\, b^{1o} \quad 6\ ^2 {}^1¢_5\, 3\, b^{2o} \wedge \quad ¸\, 6\ ^2 {}^1¢_5\, 3\, b^{3o} \wedge \quad ¸ \\ 6\ ^3\, 3\, b^2 \wedge \quad \wedge {}^1[\quad 6\, b^{1o} \quad 6\ ^3\, 3\, b^3 \wedge \quad \wedge {}^1[\quad 6\, b^{1o}\, ¸ \\ ¸\, 6\ ^4 \quad \wedge \quad \wedge \quad 0\, ¸\, 2^1[\quad 6\, b^{1o} \ - \end{aligned} \tag{C.2.23}$$

where $¢_5$ is the Poincaré dual in AdS_5 while $0 = 30 \,¸\, 26^1 b^1 \,¸\, b^{2o}$. Dirac's quantization condition reads

$$\frac{1}{2\sqrt{c}\,\wedge_{10}} \int_{C_5}^{1} {}_5^{RR}\ 2\ \mathbb{Z} \tag{C.2.24}$$

for any closed 5-cycle C_5. Here $\wedge_{10}$ is the 10d gravitational coupling, related to the 5d Newton constant by

$$\frac{\mathrm{Vol}^1)^{1\text{-}1o}}{6^5 \wedge_{10}^2} = \frac{1}{8c\ {}^{15^o}_{N}} \tag{C.2.25}$$

where $\mathrm{Vol}^1)^{1\text{-}1o} = 16c^3•27$. Applying (C.2.24) to $C_5 =)^{1\text{-}1}$ and imposing that there are # units of 5-form flux, we recover (2.4.36). On the other hand, let us apply (C.2.24) to the 5-cycle $-_2 \quad (^3$, where $-_2$ is the non-trivial 2-cycle of $)^{1\text{-}1}$ while $(^3$ is a spatial 3-sphere in

[3] The 2-form should not be confused with the angular momentum of the black hole.

AdS$_5$. Using $\bar{\ }_{-2} = 0$ and $\bar{\ }_{-2} = 4c\bullet 3$ as well as (2.4.36), we obtain

$$-\frac{1}{2\sqrt{c}\,\hat{\ }_{10}}\int_{-2} (^3\ {}^{\mathrm{RR}}_5 = \frac{1}{6c\ {}^{15^o}_{\mathrm{N}}\ 6\#}\int (^3\ \hat{c}_5\ b^3\ \text{¸}\ b^3 \wedge b^1 \;=\; \frac{4}{3\#}\,\&_3\ 2\ Z\ - \qquad \text{(C.2.26)}$$

where $b^3 = 3\, b^3$. According to (C.2.4) and using (B.1.42) and (B.1.43), the combination in parenthesis gives the Page charge $\&_3$, which is conserved and quantized. Taking the 3-sphere to spatial infinity, it coincides with the charge defined in (C.2.1).

Appendix D

Monopole spherical harmonics on S^2

We use complex coordinates on $(^2$ to perform the reduction in Section 3.2. We define stereographic coordinates

$$I = 4^{8i}\tan\frac{\backslash}{2} \qquad \text{for } \backslash \ \text{Ÿ}\ c - \qquad E = 4^{\ 8i}\cot\frac{\backslash}{2} \qquad \text{for } \backslash\ \text{¡}\ 0 - \tag{D.1}$$

related by $E = 1\bullet I$, which exhibit $(^2$ as CP^1. The round metric with radius $'$ is proportional to the Fubini-Study metric, and the Lorentzian metric on $(^2 \quad \mathrm{R}$ is

$$3B^2 = \frac{4\,'^2}{{}^1 1 \,\text{¸}\, I\bar{I}^{o2}}\, 3I\,3\bar{I} \quad 3C^2 \qquad 6^{\frac{1}{2}} 3I\,3\bar{I} \quad 3C^2 = 4^1 4^{\bar{1}} \quad {}^1 4^{3o2} - \tag{D.2}$$

where we defined the vielbein

$$4^3 = 3C - \qquad 4^1 = 6^{\frac{1}{4}} 3I - \qquad 4^{\bar{1}} = 6^{\frac{1}{4}} 3\bar{I} \bullet \tag{D.3}$$

Here 4^1 and $4^{\bar{1}}$ are complex conjugate of each other, and therefore any real $?$-form expressed in this basis has components satisfying the reality property $-_1 = -_{\bar{1}}$. Flat indices are lowered and raised by the flat metric $[_{01}$ with $[_{1\bar{1}} = [_{\bar{1}1} = \frac{1}{2}$. The volume form has flat components $n_{01\bar{1}} = 8\bullet 2$.

Let us now move to spinors. We choose the set of gamma matrices

$$W_C = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} - \qquad W_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \qquad W_{\bar{1}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \tag{D.4}$$

satisfying $\mathrm{f}W_0 - W_1\mathrm{g} = 2[_{01}\mathbb{1}$. The generators of the Dirac representation are $W_{01} = \frac{1}{2}\text{»}W_0 - W_1\text{¼}$. On $(^2 \quad \mathrm{R}$ the 3d Lorentz group $\mathrm{SO}^1 2 - 1^o$ is broken to the $\mathrm{U}^1 1^o$ generated by $W^{1\bar{1}}$, and fields are characterized by a spin that is the charge under this $\mathrm{U}^1 1^o$. The spin connection, defined by $^1 l^0{}_1{}^o{}^` = 4^0{}_a\ m` 4^a{}_1\ ¸\ {}^a{}_d 4^d{}_1$, has non-zero components

$$^1 l^1{}_1{}^o{}_l = {}^1 l^{\bar{1}}{}_{\bar{1}}{}^o{}_l = \frac{\bar{T}}{1\ ¸\ l\bar{T}} - \qquad {}^1 l^1{}_1{}^o{}_{\bar{T}} = {}^1 l^{\bar{1}}{}_{\bar{1}}{}^o{}_{\bar{T}} = \frac{l}{1\ ¸\ l\bar{T}} • \tag{D.5}$$

The spinor covariant derivative (without gauge connections)

$$` \quad \overset{!}{\frac{k\ ¸}{k}} \quad {}^1 \quad `k\ ¸\ - \quad `k\ {}^{o\mathrm{T}} \tag{D.6}$$

can be written as

$$= 3 \quad 8B\, l \qquad \text{with} \qquad l = 8\,\frac{\bar{T}\,3l \quad l\,3\bar{T}}{1\ ¸\ l\bar{T}} = {}^1\cos\backslash \quad 1^o\,3\,i \tag{D.7}$$

and $B = \quad 1•2$ is the spin. Note that

$$\frac{1}{2c} \int_{(^2}^{1} 3\,l = \quad 2 • \tag{D.8}$$

The components k are sections of the $\mathrm{U}^1 1^o$ bundles associated to the line bundles $K^{\frac{1}{2}}$ $O^1 \quad 1^o$, where K is the canonical bundle. A generic $\mathrm{U}^1 1^o$ bundle is labeled by a half-integer monopole charge $@$, and has covariant derivative $= 3 \quad 8@0$. To conform with the conventions of [242] for the monopole harmonics, we write the connection as a half-integer multiple of $0 = \quad l$.

Similarly, the Levi-Civita connection on 1-forms is a $\mathrm{U}^1 1^o$ connection when projected onto the frame fields:

$$4^l_1\ \mathrm{r}` \quad {}_l = {}^1 m` \quad 8\,l\ `{}^o 4^l_1 \quad {}_l \qquad ` \quad {}_1 - \qquad 4^{\bar{T}}_{\bar{1}}\ \mathrm{r}` \quad {}_{\bar{T}} = {}^1 m`\ ¸\ 8\,l\ `{}^o 4^{\bar{T}}_{\bar{1}} \quad {}_{\bar{T}} \qquad ` \quad {}_{\bar{1}} • \tag{D.9}$$

Thus $_1 = 4^l_1 \quad {}_l$ and $_{\bar{1}} = 4^{\bar{T}}_{\bar{1}} \quad {}_{\bar{T}}$ are sections with $@ = \quad 1$ and $@ = \ ¸1$, respectively. On the other hand, $` \quad {}_3 = m` \quad {}_3$, and thus $_3$, is a section of the trivial bundle, just like a scalar. Defining $_0 = 4_0 \quad `$, one finds

$$^1 3 \quad {}^o{}_{01} = 4_0 4^a_1{}^1\,\mathrm{r}` \quad {}_a \quad \mathrm{r}_a\ `{}^o = \quad {}_0 \quad {}_1 \qquad {}_1 \quad {}_0 • \tag{D.10}$$

If, in addition, the fields are in the adjoint representation of the gauge group and there is a background gauge field with fluxes

$$= \frac{1}{2}\mathrm{m}_8 \quad {}^8 0 \qquad) \qquad \frac{1}{2c}\int_{(^2}^{1} 3 \quad = \mathrm{m}_8 \quad {}^8 - \tag{D.11}$$

then including this background in the covariant derivatives ` shifts the spin $B \; ! \; B \quad \frac{U^1 m^o}{2}$, or equivalently $@ \; ! \; @ \, ¸ \, \frac{U^1 m^o}{2}$, where U are the roots.

The derivatives $_1$ and $_{\bar{1}}$ raise and lower the spin by 1, respectively. This is the opposite in terms of the charge $@$. Their explicit expressions are

$$ {}^{1@^o}_{1} = \frac{1}{2\,'} \; {}^1 1 \, ¸ \, IT^o \, m_I \quad @T \; - \qquad {}^{1@^o}_{\bar{1}} = \frac{1}{2\,'} \; {}^1 1 \, ¸ \, IT^o \, m_T \, ¸ \, @I \; - \tag{D.12} $$

where the superscript indicates the charge of the section they act on, whereas under complex conjugation ${}^{1@^o}_{1} = {}^{1}_{\bar{1}}{}^{@^o}$ and ${}^{1@^o}_{\bar{1}} = {}^{1}_{1}{}^{@^o}$. We define the operators

$$!_¸ = I^2 m_I \, ¸ \, m_T \quad @I \, - \qquad ! \; = \; T^2 m_T \quad m_I \quad @T \, - \qquad !_I = I m_I \quad T m_T \quad @ \, - \tag{D.13} $$

satisfying the $\mathrm{su}^1 2^o$ algebra $» !_I - ! \; ¼ = \; !$ and $» !_¸ - ! \; ¼ = 2 !_I$. The covariant Laplacian is

$$ \begin{aligned} {}^2 \quad !^2 \quad @^2 &= \frac{1}{2} f \, !_¸ - ! \; g \, ¸ \, !_I^2 \quad @^2 = \quad 1 \, ¸ \, IT \; {}^2 m_I m_T \quad @^1 1 \, ¸ \, IT^o \, !_I \quad @^2 \\ &= \quad \frac{1}{\sin \backslash} m_\backslash \;\; \sin \backslash \; m_\backslash \quad ¸ \; \frac{1}{\sin^2 \backslash} \quad 8 m_i \quad @ \, ¸ \, @ \cos \backslash \;\; {}^2 \, - \end{aligned} \tag{D.14} $$

which can be diagonalized simultaneously with $!^2$ and $!_I$. Its eigenfunctions are the monopole spherical harmonics $._{@ - ; - <}$ with $j<j \quad ;$, that we choose to be orthonormal on an $(^2$ of radius 1:

$$ \int_{(^2}^{1} \mathrm{p}\overline{6} \;\; \overline{._{@-;-<}} \; ._{@-;^0-<^0} = X_{;-;^0} \, X_{<-<^0} \, • \tag{D.15} $$

The highest harmonic with $< \, = \, ;$, annihilated by $!_¸$, is

$$._{@-;-;} {}^1 I - T^o \quad / \quad \frac{I^{; ¸ @}}{{}^1 1 \, ¸ \, IT^{o ;}} \, • \tag{D.16} $$

Regularity at the poles implies $; \, ¸ \, @ \; 2 \; \mathrm{Z}_{\;0}$ and $; \quad j@j$.

The Laplacian can be written in terms of the derivatives as

$$ {}^2 = \quad 4\,'^2 \quad {}_1 \quad {}_{\bar{1}} \, ¸ \, @ = \quad 4\,'^2 \quad {}_{\bar{1}} \quad {}_1 \quad @ = \quad 2\,'^2 f \quad {}_1 - \quad {}_{\bar{1}} g \, • \tag{D.17} $$

Besides, one can verify that

$$ » \quad {}_1 - !_I ¼ = » \quad {}_1 - ! \; ¼ = » \quad {}_{\bar{1}} - !_I ¼ = » \quad {}_{\bar{1}} - ! \; ¼ = 0 \, • \tag{D.18} $$

Therefore the derivatives act as bundle-changing operators mapping $._{@-<-;}$ to $._{@ \quad 1-<-;}$. The exact relations can be derived integrating by parts the orthonormality conditions. For a suitable choice of phases one finds [242, 243]:

$$
\begin{aligned}
{}^{1@o}_{1} \cdot @\text{-};\text{-}< &= \frac{B\ {}^{1}@\text{-};{}^{o}}{2\,'} \cdot @\ 1\text{-};\text{-}< &\quad \text{with} \quad B\ {}^{1}@\text{-};{}^{o} = \ ;{}^{1};\ \text{¸}\ 1^{o}\quad @{}^{1}@\quad 1^{o\ \frac{1}{2}}\ \text{-} \\
{}^{1@o}_{1} \cdot @\text{-};\text{-}< &= \frac{B_{\text{¸}}\ {}^{1}@\text{-};{}^{o}}{2\,'} \cdot @\text{¸}1\text{-};\text{-}< &\quad \text{with} \quad B_{\text{¸}}\ {}^{1}@\text{-};{}^{o} = \ ;{}^{1};\ \text{¸}\ 1^{o}\quad @{}^{1}@\ \text{¸}\ 1^{o\ \frac{1}{2}}\ \bullet
\end{aligned}
\tag{D.19}
$$

Following the same conventions as in [243], the monopole harmonics satisfy

$$
\overline{\cdot\, @\text{-};\text{-}<} = {}^{1}\ \ 1^{o@\text{¸}<}\ \cdot\ @\text{-};\text{-}\ < \tag{D.20}
$$

under complex conjugation.

Finally, the triple overlap of harmonics is given in terms of Wigner 3ϑ-symbols:

$$
\begin{aligned}
&{}^{1}\!\!\int_{3} \cdot\, @\text{-};\text{-}< \cdot\, @^{0}\text{-};^{0}\text{-}<^{0} \cdot\, @^{00}\text{-};^{00}\text{-}<^{00} = \\
&= {}^{1}\ \ 1^{o;\text{¸};^{0}\text{¸};^{00}}\ \frac{{}^{1}2;\ \text{¸}\ 1^{o1}2;^{0}\ \text{¸}\ 1^{o1}2;^{00}\ \text{¸}\ 1^{o}}{4c}^{\frac{1}{2}} \begin{pmatrix} ; & ;^{0} & ;^{00} \\ @ & @^{0} & @^{00} \end{pmatrix} \begin{pmatrix} ; & ;^{0} & ;^{00} \\ < & <^{0} & <^{00} \end{pmatrix}\ \text{-}
\end{aligned}
\tag{D.21}
$$

or equivalently

$$
\begin{aligned}
&\cdot\, @\text{-};\text{-}<\ \cdot\, @^{0}\text{-};^{0}\text{-}<^{0} = \\
&\sum_{;^{00}} {}^{1}\ \ 1^{o;\text{¸};^{0}\text{¸};^{00}\text{¸}@^{00}\text{¸}<^{00}}\ \frac{{}^{1}2;\ \text{¸}\ 1^{o1}2;^{0}\ \text{¸}\ 1^{o1}2;^{00}\ \text{¸}\ 1^{o}}{4c}^{\frac{1}{2}} \begin{pmatrix} ; & ;^{0} & ;^{00} \\ @ & @^{0} & @^{00} \end{pmatrix} \begin{pmatrix} ; & ;^{0} & ;^{00} \\ < & <^{0} & <^{00} \end{pmatrix} \cdot\, @^{00}\text{-};^{00}\text{-}\ <^{00}
\end{aligned}
\tag{D.22}
$$

The 3ϑ-symbols are directly related to Clebsch-Gordan coefficients that decompose the angular momentum state $|;^{00}\ <^{00}\rangle$ in terms of $|;\ <\ ;^{0}\ <^{0}\rangle = |;\ <\rangle\ \ |;^{0}\ <^{0}\rangle$:

$$
\begin{pmatrix} ; & ;^{0} & ;^{00} \\ < & <^{0} & <^{00} \end{pmatrix} \quad \langle ;\ <\ ;^{0}\ <^{0} | ;^{00}\ <^{00}\rangle = {}^{1}\ \ 1^{o;\ \ ;^{0}\text{¸}<^{00}} \sqrt{2;^{00}\ \text{¸}\ 1} \begin{pmatrix} ; & ;^{0} & ;^{00} \\ < & <^{0} & <^{00} \end{pmatrix} \bullet \tag{D.23}
$$

In particular, the Clebsh-Gordan coefficients are zero unless $<\ \text{¸}\ <^{0} = <^{00}$, $<^{1\&o}\ \ ;^{1\&o}$ with $<^{1\&o} = ;^{1\&o}$ mod 1, and $;^{1\&o}\ \ ;^{1\vartheta o}\ \text{¸}\ ;^{1:o}$. The 3ϑ-symbol is symmetric under even permutations of its columns and gains a sign ${}^{1}\ \ 1^{o;\text{¸};^{0}\text{¸};^{00}}$ under odd permutations. It also gains a sign ${}^{1}\ \ 1^{o;\text{¸};^{0}\text{¸};^{00}}$ when one changes sign to $<$, $<^{0}$ and $<^{00}$ simultaneously. This implies the following relations among Clebsch-Gordan coefficients:

$$
\begin{aligned}
{}^{;0}_{<0}\;{}^{;00}_{<00}\;{}^{;}_{<} &= {}^{1}\;1^{o; \; ;00 \, ¸ <0} \, \mathrm{h} \, \frac{2; \, ¸ \, 1}{2;00 \, ¸ \, 1} \, \mathrm{i}^{1•2} \quad {}^{;}_{<}\;{}^{;0}_{<0}\;{}^{;00}_{<00} \; - \\
{}^{;00}_{<00}\;{}^{;}_{<}\;{}^{;0}_{<0} &= {}^{1}\;1^{o;00 \; ;0 \, ¸ <} \, \mathrm{h} \, \frac{2;0 \, ¸ \, 1}{2;00 \, ¸ \, 1} \, \mathrm{i}^{1•2} \quad {}^{;}_{<}\;{}^{;0}_{<0}\;{}^{;00}_{<00} \; - \\
{}^{;0}_{<0}\;{}^{;}_{<}\;{}^{;00}_{<00} &= {}^{1}\;1^{o; ¸ ;0 \; ;00} \quad {}^{;}_{<}\;{}^{;0}_{<0}\;{}^{;00}_{<00} \; •
\end{aligned}
\tag{D.24}
$$

In the special case that $;^{00} = ; \; ¸ \; ;^{0} \quad !$ (and $< \; ¸ \; <^{0} = \quad <^{00} \quad "$ as in the general case):

$$
\begin{aligned}
\begin{matrix} ; & ;^0 & ! \\ < & <^0 & " \end{matrix}\;! &= {}^{1}\;1^{o; \; ;0 ¸ "} \, \frac{1}{2! \, ¸ \, 1} \; \frac{2!}{! \; ¸ \; "}^{1} \; \frac{2;}{; \; ¸ \; <} \; \frac{2;^0}{;^0 \; ¸ \; <^0}^{\frac{1}{2}} \; - \\
{}^{;}_{<}\;{}^{;0}_{<0}\;{}^{!}_{"} &= \; \frac{2!}{! \; ¸ \; "}^{1} \; \frac{2;}{; \; ¸ \; <} \; \frac{2;^0}{;^0 \; ¸ \; <^0}^{\frac{1}{2}} \; •
\end{aligned}
\tag{D.25}
$$

Appendix E

1d $\mathcal{N} = 2$ superspace

We review here the 1d $\mathcal{N} = 2$ superspace formalism, drawing from Appendix A of [217]. The $\mathcal{N} = 2$ superspace in quantum mechanics, which we denote as $\mathbb{R}^{1|2}$, has coordinates $(t, \theta, \bar\theta)$, where θ is a complex fermionic coordinate. A supersymmetry transformation is $\delta = \epsilon Q + \bar\epsilon \bar Q$, where ϵ, $\bar\epsilon$ are anticommuting parameters, and Q, $\bar Q$ are anticommuting generators so that δ is commuting. Here Q and $\bar Q$ are defined as differential operators acting on superfields:

$$Q = \partial_\theta + \frac{i}{2}\bar\theta\partial_t, \qquad \bar Q = -\partial_{\bar\theta} - \frac{i}{2}\theta\partial_t. \tag{E.1}$$

They satisfy the algebra $Q^2 = \bar Q^2 = 0$ and $\{Q, \bar Q\} = -i\partial_t$. Moreover, Q and $\bar Q$ anticommute with another set of differential operators

$$D = \partial_\theta - \frac{i}{2}\bar\theta\partial_t, \qquad \bar D = -\partial_{\bar\theta} + \frac{i}{2}\theta\partial_t, \tag{E.2}$$

which satisfy the algebra $D^2 = \bar D^2 = 0$ and $\{D, \bar D\} = i\partial_t$. One has $\overline{[D, \Phi\}} = (-1)^{|\Phi|}[\bar D, \bar\Phi\}$ and $\overline{[\bar D, \Phi\}} = (-1)^{|\Phi|}[D, \bar\Phi\}$.

E.1 Matter multiplets

A chiral superfield Φ is defined by $\bar D\Phi = 0$. Gauge transformations act as

$$\Phi \to g\Phi, \quad g = e^{i\Lambda}, \quad \Lambda : \mathbb{R}^{1|2} \to \mathbb{C}, \quad \bar D\Lambda = 0. \tag{E.1.1}$$

where A is some representation of the gauge group. $\overline{} = 0$ implies that and its complex conjugate anti-chiral superfield $\overline{}$ have expansion:

$$= q \text{¸} \backslash k \quad \frac{8}{2} \backslash \overline{\backslash} m_{c} q \text{-} \qquad \overline{} = \overline{q} \quad \overline{\backslash} \overline{k} \text{¸} \frac{8}{2} \backslash \overline{\backslash} m_{c} \overline{q} \bullet \tag{E.1.2}$$

Acting with (E.1) on and $\overline{}$, we find the following supersymmetry variations:

$$\& q = k \text{-} \qquad \& k = 0 \text{-} \qquad \overline{\&} q = 0 \text{-} \qquad \overline{\&} k = 8 m_{c} q \bullet \tag{E.1.3}$$

Suppose that - are a collection of bosonic chiral superfields. We can also have fermionic Fermi superfields Y, satisfying $\overline{} Y = {}^{1}\ {}^{o}$ for some holomorphic function ${}^{1}\ {}^{o}$, and transforming as $Y\ !\ Y$ under some representation of the gauge group. $\overline{} Y = {}^{1}\ {}^{o}$ implies that Y and its conjugate $\overline{Y}$ have expansion:

$$\begin{aligned} Y &= [\ \backslash 5 \ \overline{\backslash}\ {}^{1}q^{o} \text{¸} \backslash \overline{\backslash}\ m\ {}^{1}q^{o} k \ \ \tfrac{8}{2} m_{c} [\ = [\ \backslash 5 \ \overline{\backslash}\ {}^{1}\ {}^{o} \ \tfrac{8}{2} \backslash \overline{\backslash} m_{c} [\text{-} \\ \overline{Y} &= \overline{[} \ \overline{\backslash}\, \overline{5} \ \backslash \overline{}\, {}^{1}\overline{q}^{o} \text{¸} \backslash \overline{\backslash}\ \overline{k}\ \overline{m}\ \overline{}\, {}^{1}\overline{q}^{o} \text{¸} \tfrac{8}{2} m_{c} \overline{[} \ = \overline{[} \ \overline{\backslash}\, \overline{5} \ \backslash \overline{}\, {}^{1} \overline{}\, {}^{o} \text{¸} \tfrac{8}{2} \backslash \overline{\backslash} m_{c} \overline{[} \bullet \end{aligned} \tag{E.1.4}$$

Acting with (E.1) gives the supersymmetry variations:

$$\& [= \ 5 \text{-} \qquad \& 5 = 0 \text{-} \qquad \overline{\&} [= \ {}^{1}q^{o} \text{-} \qquad \overline{\&} 5 = \ 8 m_{c} [\text{¸} m \ {}^{1}q^{o} k \ \bullet \tag{E.1.5}$$

E.2 Vector multiplet

We assume that the gauge group is semi-simple (inclusion of $U^{1}1^{o}$ factors is trivial) with Lie algebra $\mathfrak{g}$. Denote the complexified algebra as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \ \mathbb{C} = \mathfrak{g} \ {}_{\mathbb{R}} 8\mathfrak{g}$, with the Killing form given by the trace operation Tr. It admits a root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \ {}_{U2} \ !_{U}$, where $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra and is the set of all roots. We can use the Chevalley basis $\mathfrak{g}_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}} f \ {}^{8=1\text{-}\cdots\text{-}\mathrm{rk}} \text{-} \ {}^{U} \ j \ U \ 2 \ \ \mathrm{g}$, where 8 indexes a set of simple roots U^{8} and ^{8} is defined in the following way:

$$9! \ {}^{8}\ 2\, \mathfrak{h}_{\mathbb{C}} \quad U^{8\,1}\ {}^{o} = \mathrm{Tr}^{1}\ {}^{8}\ {}^{o} \text{-}\ 8 \quad 2\, \mathfrak{h}_{\mathbb{C}} \bullet \tag{E.2.1}$$

The element ^{U} is also normalized so that $\mathrm{Tr} \ {}^{U} \ {}^{U} = 1$. The compact real form is

$$\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\ 8 \ {}^{8}\text{-} \ {}^{U} \quad {}^{U}\text{-}8^{1}\ {}^{U} \text{¸} \quad {}^{U o} \quad U\ 2 \quad {}^{›} \ \text{-} \tag{E.2.2}$$

where is the set of positive roots. Using the fact that Tr splits between each summand in $\mathfrak{h}_{\mathbb{C}}$ $U2$, ¹ ! U ! Uº, and that Tr is positive definite on ^{8}, it quickly follows that Tr is negative (positive) definite on $\mathfrak{g}$ ($8\mathfrak{g}$). Any 2 $8\mathfrak{g}$ can be expressed with ^{8}, $^{U}_{1}$, $^{U}_{2}$ 2 R as

$$= \tilde{O} \quad 8 \; 8 \quad \tilde{O} \quad h \; U_1 \; U \quad U_o \quad U_{81} \; U \quad U_o \; i$$
$$= \tilde{O}^{8} \; 8 \; 8 \quad \tilde{O}^{U2} \quad U \; U \; \overline{U} \; U \; - \quad U \quad U_1 \; 8 \; U_2 \, . \tag{E.2.3}$$

Therefore, defining a formal Hermitian conjugation on elements of $\mathfrak{g}_{\mathbb{C}}$ as $\overline{}^{8}$ ^{8}, $\overline{U}$ ^{U}, we can alternatively define $8\mathfrak{g}$ as

$$8\mathfrak{g} = \quad 2\, \mathfrak{g}_{\mathbb{C}} \quad \overline{} = \quad . \tag{E.2.4}$$

A generic group element $: = 4^{8}$ then satisfies $\overline{:} = 4^{\;8} = :^{\;1}$. If $= U¹\#º$, this formal Hermitian conjugation becomes the actual conjugate transpose on # # matrices.

To build gauge interactions, we introduce the independent superfields and + . is valued in $\mathfrak{g}_{\mathbb{C}}$, while + is valued in $8\mathfrak{g}$, *i.e.*, $\overline{+} = +$. One can either use alone or include both and + in the theory. The crucial role played by is to allow for gauge-covariant chiral and Fermi conditions. Under gauge transformations, they transform as:

$$4 \;!\; :4^{\;1} - \quad + \;!\; :+ :^{\;1} \, 8:{}^{1}m_{\mathbb{C}}:{}^{1o} -$$
$$= 4^{j} - \quad j : \mathbb{R}^{1j2} \;!\; \mathfrak{g}_{\mathbb{C}} - \quad \overline{}j = 0 - \tag{E.2.5}$$
$$: = 4^{8} - \quad : \mathbb{R}^{1j2} \;!\; 8\mathfrak{g} - \quad \overline{} = \, .$$

Without loss of generality, + can be expanded as

$$+ = {}_{\mathbb{C}} \; f \quad 8\backslash\overline{} \quad 8\overline{\backslash}\_ \, \backslash\overline{\backslash} \; - \tag{E.2.6}$$

where ¹ $_{\mathbb{C}}$ f- º are valued in $8\mathfrak{g}$ and _ is valued in $\mathfrak{g}_{\mathbb{C}}$. We now define the various ingredients used to construct supersymmetric actions. The gauge-covariant superspace derivatives are defined as

$$D \quad 4^{\overline{}} \quad 4^{\overline{}} - \quad \overline{D} \quad 4 \; \overline{}4 \; - \quad D_{\mathbb{C}} \quad m_{\mathbb{C}} \quad 8+ \; - \tag{E.2.7}$$

which, according to (E.2.5) and using $\overline{} = \overline{} = 0$, transform as

$$D \;!\; :D:^{\;1} - \quad \overline{D} \;!\; :\overline{D}:^{\;1} - \quad D_{\mathbb{C}} \;!\; :D_{\mathbb{C}}:^{\;1} . \tag{E.2.8}$$

They satisfy the algebra

$$D^2 = \overline{D}^2 = 0 - \qquad \mathsf{f} D - \overline{D} \mathsf{g} = 8^1 m_c \quad 8 + {}^{,o} \quad 8 D_{\hat{C}} - \tag{E.2.9}$$

where $+_{,}$ is an $8\mathfrak{g}$-valued superfield constructed out of only:

$$+_{,} \quad 4 \; \overline{}4 \quad {}_{,}\overline{} \; 4^{-} \quad 4^{-} \quad {}_{,} \; 4 \; \overline{}4 \quad -4^{-} \quad 4^{-} \quad \bullet \tag{E.2.10}$$

If the gauge group is Abelian this simplifies to

$$+_{,} = \; » \; - \overline{} ¼ \quad \bullet \tag{E.2.11}$$

As it was for and $\overline{}$, one has $\overline{{}^1 D - {}^o} = {}^1 \; 1^o \; \overline{D} -$ and $\overline{{}^1 \overline{D} - {}^o} = {}^1 \; 1^o \; D -$. One can check that the gauge transformation of $+_{,}$ is identical to that of $+$:

$$+_{,} \; ! \; : +_{,} : {}^1 {}_{,} \, 8 : {}^1 m_c : {}^{1o} - \tag{E.2.12}$$

which is consistent with (E.2.8) and (E.2.9). We will also have occasion to use the field strength superfield

$$» \overline{D} - D_c ¼ = \; 8 \overline{} + \quad m_c \; 4 \quad \overline{}4 \qquad 8 \; 4 \quad \overline{}4 \quad -+ \quad - \tag{E.2.13}$$

which also transforms covariantly as $! \; : \; : {}^1$. From the definition, it follows directly that $\overline{D} \quad = 0$.

Instead of and $+$, we can equivalently use two other superfields $+$ and $+$ defined as

$$4^+ \quad 4 \; \overline{}4 \; - \qquad + \qquad 4^{-} + 4 \; {}_{,} \frac{8}{2} 4^{-} \; m_c 4 \qquad \frac{8}{2} \; m_c 4^{-} \quad 4 \; - \qquad \overline{+} = + \; - \tag{E.2.14}$$

which only transform under the complexified gauge transformations as:

$$4^+ \; ! \; {}^{-1} 4^+ \; {}^1 - \qquad + \; ! \; {}^{-1} + \quad {}^1 {}_{,} \frac{8}{2}{}^{-1} 4^+ m_c \quad {}^1 \quad \frac{8}{2} \; m_c{}^{-1} \; 4^+ \quad {}^1 \bullet \tag{E.2.15}$$

Note that $+$ is constructed solely out of , while $+$ is built out of both $+$ and . In this formulation, the theory might contain $+$ only, or both $+$ and $+$. Analogously to the above, out of $+$ and $+$ we can construct

$$\begin{aligned} +_{,} \quad & \frac{1}{2} 4^{+} \overline{} \; 4 \; {}^{+} \quad 4^+ \; {}_{,} \frac{1}{2} \quad 4^{+} \overline{} 4 \; {}^{+} \quad 4^+ = 4^{-} + {}_{,} 4 \; {}_{,} \frac{8}{2} 4^{-} \; m_c 4 \qquad \frac{8}{2} \; m_c 4^{-} \quad 4 \; - \\ & 8 \, 4^{+} \overline{} \; 4 \; {}^{+} \; + \; {}_{,} \frac{8}{2} m_c 4^+ \quad = 4^{-} \quad 4 \; \bullet \end{aligned} \tag{E.2.16}$$

One can check that $+ ›$ transforms in the same way as $+$, and transforms in the same way as 4^{+}. In an Abelian theory,

$$+ › = \frac{1}{2} 4^{+} \quad — \quad\quad — + • \tag{E.2.17}$$

When writing matter Lagrangians in terms of and Y which transform with chiral gauge transformations , it will be convenient to use $+$ and $+$.

Given any chiral or Fermi superfield, one can define covariantly-chiral counterparts

$$: \quad 4 \quad - \quad Y_{:} \quad 4 \; Y \; - \quad \overline{D} \; _{:} = 0 \; - \quad \overline{D} Y_{:} = \; ^{1} \; _{:}{}^{o} \; - \tag{E.2.18}$$

which transform under the gauge group as $_{:}$! $:$ $_{:}$ and $Y_{:}$! $: Y_{:}$. These fields are useful when one is using and $+$ to describe the vector multiplet.

E.2.1 Wess-Zumino gauge

We can expand and the gauge transformation parameters j, as:

$$= \; _{0} \, ¸ \, \backslash \; _{\backslash} \, ¸ \, \overline{\backslash} \; _{\overline{\backslash}} \, ¸ \, \backslash \overline{\backslash} \; _{\backslash \overline{\backslash}} \; - \quad j = j_{0} \, ¸ \, \backslash \, j_{\backslash} \; \frac{8}{2} \backslash \overline{\backslash} m_{c} \, j_{0} \; - \quad = \; _{0} \, ¸ \, \backslash \; _{\backslash} \; \overline{\backslash} \; \overline{} _{\backslash} \, ¸ \, \backslash \overline{\backslash} \; _{\backslash \overline{\backslash}} \; • \tag{E.2.19}$$

We show that using gauge transformations every component of can be canceled except for $_{\backslash \overline{\backslash}}$, and we can further set $\overline{}_{\backslash \overline{\backslash}} = \; _{\backslash \overline{\backslash}}$, *i.e.*, $_{\backslash \overline{\backslash}}$ is valued in $8\mathfrak{g}$. We shall call this component $^{1} \; _{c} \; ¸ \; f^{o} • 2$, where both $_{c}$ and f are valued in $8\mathfrak{g}$. Due to the relative sign, this is independent of $^{1} \; _{c} \quad f^{o}$ in $+$. In other words, we can bring to the form

$$= \; \frac{1}{2} \backslash \overline{\backslash} \, ^{1} \; _{c} \; ¸ \; f^{o} \; - \tag{E.2.20}$$

that we dub the Wess-Zumino gauge. First, we use the transformation

$$j = \; _{0} \quad \frac{8}{2} \backslash \overline{\backslash} m_{c} \; _{0} \; - \quad\quad = 0 \; - \tag{E.2.21}$$

to set $_{0}$! 0, after which only transformations with $j_{0} = 8 \; _{0}$ preserve $_{0} = 0$ and are allowed. Next, performing the transformation

$$j = \backslash ^{1} \; _{\backslash} \, ¸ \; \overline{}_{\overline{\backslash}}{}^{o} \; - \quad\quad = 8 \backslash \overline{}_{\overline{\backslash}} \, ¸ \, 8 \overline{\backslash} \; _{\overline{\backslash}} \; - \tag{E.2.22}$$

sets $\backslash, \overline{\backslash}$! 0. Further transformation parameters cannot have $\backslash$ or $\overline{\backslash}$ components since otherwise a nonzero $\overline{\backslash}$ would be generated. Lastly, we perform

$$j = 0 - \qquad = \frac{8}{2}\backslash\overline{\backslash}^{1} \quad \backslash\overline{\backslash} \quad \overline{} \quad \backslash\overline{\backslash}^{\circ} - \tag{E.2.23}$$

after which $\backslash\overline{\backslash}$! ^{1} $\backslash\overline{\backslash}$ ¸ $\overline{}$ $\backslash\overline{\backslash}^{\circ}$•2 is valued in $8\mathfrak{g}$. The residual gauge transformations are

$$j = 8 \quad 0 \; ¸ \; \frac{1}{2}\backslash\overline{\backslash}m_{c} \quad 0 - \qquad = \quad 0 - \tag{E.2.24}$$

under which

$$c \; ¸ \; f \quad ! \quad 4^{8} \; {}^{01} \; c \; ¸ \; f^{\circ}4^{\;8\;0} \; ¸ \; 8\,4^{8} \; {}^{0}m_{c}4^{\;8\;0} \; • \tag{E.2.25}$$

These are standard time-dependent gauge transformations, as expected. In this gauge, the gauge-covariant superspace derivatives simplify to

$$D_{c} = \quad c = m_{c} \quad 8^{1} \; c \; ¸ \; f^{\circ} - \qquad D = m_{\backslash} \quad \frac{8}{2}\overline{\backslash} \quad c - \qquad \overline{D} = \quad m_{\overline{\backslash}} \; ¸ \; \frac{8}{2}\backslash \quad c - \tag{E.2.26}$$

and

$$+ \cdot = \quad c \; ¸ \; f - \qquad =\_ \quad \backslash \quad c\,f \; ¸ \; 8 \qquad \frac{8}{2}\backslash\overline{\backslash} \quad c\_ \; • \tag{E.2.27}$$

The action of supersymmetry on , using (E.1), is

$$X \quad = \frac{1}{2}n\overline{\backslash}^{1} \; c \; ¸ \; f^{\circ} \quad \frac{1}{2}\overline{n}\backslash^{1} \; c \; ¸ \; f^{\circ} - \tag{E.2.28}$$

and the Wess-Zumino gauge is not preserved. This can be compensated by an infinitesimal gauge transformation with parameters

$$= \frac{8}{2}n\overline{\backslash}^{1} \; c \; ¸ \; f^{\circ} \; ¸ \; \frac{8}{2}\overline{n}\backslash^{1} \; c \; ¸ \; f^{\circ} \; ¸ \; O^{1}n^{2\circ} - \qquad j = \quad \overline{n}\backslash^{1} \; c \; ¸ \; f^{\circ} \; ¸ \; O^{1}n^{2\circ} \; • \tag{E.2.29}$$

The supersymmetry transformations that preserve the Wess-Zumino gauge are computed using X with the addition of the compensating gauge transformation above. For , its variation under the combined supersymmetry and gauge transformation is $X \;¸\; 8 \quad j = 0$ by construction. The superfields $:$, $Y_{:}$ are only sensitive to the gauge transformations generated by , and not to those generated by j. The addition of the -transformation (E.2.29) to X can be directly absorbed into the supercharges:

$$\&_{\mathrm{WZ}} \quad m_{\backslash} \; ¸ \; \frac{8}{2}\overline{\backslash} \; m_{c} \quad X_{\mathrm{gauge}}{}^{1} \; c \; ¸ \; f^{\circ} \; - \qquad \overline{\&}_{\mathrm{WZ}} \quad m_{\overline{\backslash}} \quad \frac{8}{2}\backslash \; m_{c} \quad X_{\mathrm{gauge}}{}^{1} \; c \; ¸ \; f^{\circ} \; • \tag{E.2.30}$$

Note that $X_{\text{gauge}}{}^1\ {}^o$ acts according to the gauge representation of each superfield, except for $+$, on which $X_{\text{gauge}}{}^1\ {}^o +\ = m_c\ 8 \gg + -$ ¼. The modified supercharges satisfy the algebra

$$\&^2_{\text{WZ}} = \overline{\&}^2_{\text{WZ}} = 0 - \qquad f\&_{\text{WZ}}\text{-}\overline{\&}_{\text{WZ}}g = 8\ m_c\ \ X_{\text{gauge}}{}^1\ c\ \text{¸}\ f^o\ \bullet \tag{E.2.31}$$

E.2.2 Transformations in Wess-Zumino gauge

Acting with (E.2.30) on $+$ and reading off the variations of each component, we find the following supersymmetry variations (and their complex conjugate) for the vector multiplet:

$$\begin{aligned} &\&_{\text{WZ}}\ c = \&_{\text{WZ}}\ f = \frac{8\ \overline{\ }}{2\ \overline{\ }} - \qquad &&\&_{\text{WZ}\_} = c\,f\ \ 8\ - \\ &\&_{\text{WZ}}\ \ = \frac{1}{2}\ c\overline{\_} - &&\overline{\&}_{\text{WZ}\_} = 0\ \bullet \end{aligned} \tag{E.2.32}$$

Note that $\&_{\text{WZ}}{}^1\ c\ \text{¸}\ f^o = \overline{\&}_{\text{WZ}}{}^1\ c\ \text{¸}\ f^o = 0$, consistently with (E.2.31). In Wess-Zumino gauge, $_:$ and its conjugate $\overline{\ }_:$ have expansion:

$$_: = q\ \text{¸}\ \backslash k\ \ \frac{8}{2}\backslash\overline{\backslash}\ \ c\,q - \qquad \overline{\ }_: = \overline{q}\ \ \overline{\backslash}\,\overline{k}\ \text{¸}\ \frac{8}{2}\backslash\overline{\backslash}\ \ c\,\overline{q}\ \bullet \tag{E.2.33}$$

Acting with (E.2.30) on $_:$ we find the following supersymmetry variations:

$$\&_{\text{WZ}}\,q = k - \qquad \&_{\text{WZ}}\,k = 0 - \qquad \overline{\&}_{\text{WZ}}\,q = 0 - \qquad \overline{\&}_{\text{WZ}}\,k = 8\ \ c\,q\ \bullet \tag{E.2.34}$$

Alternatively, we can obtain the same variations by acting with $X\ \text{¸}\ j = n\,\&_{\text{WZ}}\ \text{¸}\ \overline{n}\,\overline{\&}_{\text{WZ}}$ on , with j given in (E.2.29). Analogously, $Y_:$ and its conjugate $\overline{Y}_:$ have the expansions

$$\begin{aligned} Y_: &= [\ \ \backslash 5\ \ \overline{\backslash}\ \ {}^1q^o\ \text{¸}\ \backslash\overline{\backslash}\ m_0\ \ {}^1q^o k_0\ \ \tfrac{8}{2}\ \ c\,[\ \ = [\ \ \backslash 5\ \ \overline{\backslash}\ \ {}^1\ {}^o\ \ \tfrac{8}{2}\backslash\overline{\backslash}\ \ c\,[\\ \overline{Y}_: &= \overline{[}\ \ \overline{\backslash}\,\overline{5}\ \ \backslash\overline{\ }\,{}^1\overline{q}^o\ \text{¸}\ \backslash\overline{\backslash}\ \overline{k}_0\overline{m}_0\overline{\ }\,{}^1\overline{q}^o\ \text{¸}\ \tfrac{8}{2}\ \ c\,\overline{[}\ \ = \overline{[}\ \ \overline{\backslash}\,\overline{5}\ \ \backslash\overline{\ }\,{}^1\overline{\ }{}^o\ \text{¸}\ \tfrac{8}{2}\backslash\overline{\backslash}\ \ c\,\overline{[} - \end{aligned} \tag{E.2.35}$$

and acting with (E.2.30) gives the supersymmetry variations:

$$\&_{\text{WZ}}\,[= 5 - \qquad \&_{\text{WZ}}\,5 = 0 - \qquad \overline{\&}_{\text{WZ}}\,[= {}^1q^o - \qquad \overline{\&}_{\text{WZ}}\,5 = 8\ \ c\,[\ \text{¸}\ m_0\ \ {}^1q^o\,k_0\ \bullet \tag{E.2.36}$$

Again, we can obtain the same variations by acting with $X\ \text{¸}\ j$ on Y .

E.2.3 Supersymmetric Lagrangians

As with the prototypical 4d $N = 1$ supersymmetry, there are two broad classes of supersymmetric terms: D-terms and F-terms. Let - be a bosonic, gauge-invariant, real-valued superfield with expansion

$$- = -_0 \,¸\, \backslash -_\backslash \quad \overline{\backslash}\, \overline{-_\backslash} \,¸\, \backslash\overline{\backslash} -_{\backslash\overline{\backslash}} \,• \tag{E.2.37}$$

Acting with & and $\overline{\&}$, we find that $\& -_{\backslash\overline{\backslash}} = \; 8m_{\complement} -_\backslash •2$ and $\overline{\&} -_{\backslash\overline{\backslash}} = 8m_{\complement} \overline{-_\backslash}•2$ are total derivatives. Moreover, $\&\overline{\&} -_0 = -_{\backslash\overline{\backslash}}$ up to a total derivative. Therefore,

$$\int 3\backslash 3\overline{\backslash}\; - = \; -_{\backslash\overline{\backslash}} = \&\overline{\&}\,¹\; -_0{}^{⁰} \tag{E.2.38}$$

is supersymmetric, and we call such terms D-terms. They are always & and $\overline{\&}$ exact. Conversely, suppose there is a term in the Lagrangian of the form $\&\overline{\&}¹ \; -_0⁰$ where $-_0$ is real and gauge invariant. If there is a real-valued superfield - with bottom component $-_0$, it must have the same expansion (E.2.37). Therefore (E.2.38) holds and this term can be written as a D-term in superspace.

Let . be a fermionic, gauge-invariant, complex-valued chiral superfield, $\overline{D}. = \overline{\;}. = 0$. Its complex conjugate $\overline{.}$ is anti-chiral and satisfies $\overline{.} = 0$. They have expansion:

$$. = ._0 \,¸\, \backslash ._\backslash \quad \frac{8}{2} \backslash\overline{\backslash}\, m_{\complement} ._0 - \qquad \overline{.} = \overline{._0} \,¸\, \overline{\backslash}\, \overline{._\backslash} \,¸\, \frac{8}{2} \backslash\overline{\backslash}\, m_{\complement} \overline{._0} \,• \tag{E.2.39}$$

Acting with & and $\overline{\&}$ on . and $\overline{.}$, one finds that $._\backslash$ and $\overline{._\backslash}$ are separately supersymmetric up to total derivatives. Moreover, $._\backslash = \& ._0$ and $\overline{._\backslash} = \; \overline{\&}\, \overline{._0}$. Therefore:

$$\int 3\backslash\; . \;¸\; \int 3\overline{\backslash}\; \overline{.} = ._\backslash \,¸\, \overline{._\backslash} = \& ._0 \quad \overline{\&}\,\overline{._0} = ¹\& \,¸\, \overline{\&}⁰¹ ._0 \quad \overline{._0}⁰ \tag{E.2.40}$$

is supersymmetric, and we call such terms F-terms. They are always $¹\& \,¸\, \overline{\&}⁰$ exact.

We can now write the following supersymmetric Lagrangians, with component expressions in the Wess-Zumino gauge. In the gauge sector, if the theory only contains or equivalently +, the only term we can think of is a Wilson line in $_{\complement} \,¸\, f$. For a $U¹1⁰$ gauge group, the supersymmetric Wilson loop of charge @ can be written as

$$\exp 8@ \;¼\!\int 3\complement \int 3\backslash 3\overline{\backslash}\; + \overset{\mathrm{WZ}}{=} \exp 8@ \;¼\!\int 3\complement\, ¹ \;_{\complement} \,¸\, f⁰ \; • \tag{E.2.41}$$

If both $+$ and are present, we can write the following terms. The conventional gauge kinetic term is

$$\frac{1}{24^2_{1d}} \int 3\backslash 3\overline{\backslash}\ \mathrm{Tr}\,\overline{\quad} = \frac{1}{24^2_{1d}} \int 3\backslash 3\overline{\backslash}\ \mathrm{Tr}\,\overline{\quad}\, 4^{+}\ 4^{+} \overset{\mathrm{WZ}}{=} \frac{1}{24^2_{1d}} \mathrm{Tr} \Big[1\ \ c f^{o2}\ \, ^{2}\, 8\overline{\_}\ \ c\_ \Big] \bullet \tag{E.2.42}$$

Note that the superfield $+ \quad +$ transforms covariantly, $+ \quad + \ ! \ : {}^{1}+ \quad +{}^{o} : {}^{1}$, under gauge transformations. For an adjoint-invariant form $Z : 8\mathfrak{g} \ ! \ \mathrm{R}$, the Fayet-Iliopoulos term is:

$$\int 3\backslash 3\overline{\backslash}\ Z\ + \quad + = \int 3\backslash 3\overline{\backslash}\ Z \ + \quad +\ 4^{+} \overset{\mathrm{WZ}}{=} Z^{1}\ {}^{o} \bullet \tag{E.2.43}$$

If the gauge group is Abelian

$$+\, 4^{+} = \frac{1}{2}{}^{1}\overline{\quad}\qquad \overline{\quad}{}^{o}+ \tag{E.2.44}$$

becomes a total derivative under the superspace integral. Therefore, FI terms for Abelian gauge groups can be written as

$$\int 3\backslash 3\overline{\backslash}\ Z\ +\ 4^{+} \bullet \tag{E.2.45}$$

We can also write a mass term that gaps $+$ (or equivalently the gaugino and f):

$$\frac{1}{2}\int 3\backslash 3\overline{\backslash}\ \mathrm{Tr}\ + \quad +^{2} = \frac{1}{2}\int 3\backslash 3\overline{\backslash}\ \mathrm{Tr}\ + \quad +\ 4^{+2} \overset{\mathrm{WZ}}{=} \mathrm{Tr}\,\overline{\_\_}\quad 2f \bullet \tag{E.2.46}$$

Moving on to the matter sector, the conventional kinetic term for a chiral multiplet is:

$$8 \int 3\backslash 3\overline{\backslash}\,\overline{\quad} : D_c : = \int 3\backslash 3\overline{\backslash}\ \frac{8}{2}\overline{\quad}\, 4^{+} m_c \quad \frac{8}{2} m_c \overline{\quad}\, 4^{+} \quad \overline{\quad}+ \overset{\mathrm{WZ}}{=} \overline{q}\ \ {}^{2}_{c}\ f^{2}\quad q\ 8\overline{k}\ \ {}_{c}k\ 8\overline{q}\_k\quad 8\overline{k}\_q - \tag{E.2.47}$$

where $c \quad m_c \quad 8^{1}\ c \quad f^{o}$. It requires the presence of both $+$ and . Alternatively, we can write a kinetic term that couples to $+$ in place of $+$, in which case only (or $+$) is required:

$$8 \int 3\backslash 3\overline{\backslash}\,\overline{\quad} : D_c : = \int 3\backslash 3\overline{\backslash}\ \frac{8}{2}\overline{\quad}\, 4^{+} m_c \quad \frac{8}{2} m_c \overline{\quad}\, 4^{+} \quad \overline{\quad}+ \overset{\mathrm{WZ}}{=} {}_{c}\overline{q}\quad {}_{c}q\ 8\overline{k}\quad {}_{c}k \bullet \tag{E.2.48}$$

We can also write a term with a first-order action for q, and it only requires :

$$\int 3\backslash 3\overline{\backslash}\,\overline{}{:}\;{:} = \int 3\backslash 3\overline{\backslash}\,\overline{} 4^{+} \overset{\mathrm{WZ}}{=} 8\overline{q}\;{}_{\mathfrak{c}}q\;{}_{\text{¸}}\,\overline{k}k\,\bullet \tag{E.2.49}$$

The conventional kinetic term for a Fermi multiplet is

$$\int 3\backslash 3\overline{\backslash}\,\overline{Y}_{:}Y_{:} = \int 3\backslash 3\overline{\backslash}\,\overline{Y}\,4^{+}Y \overset{\mathrm{WZ}}{=} 8\overline{\tau}\;{}_{\mathfrak{c}}\tau\;{}_{\text{¸}}\,\overline{5}\,5 \quad {}^{1}q^{\mathrm{o}\,2}\;\overline{\tau}\,m\;{}^{1}q^{\mathrm{o}}\,k \quad \overline{k}\;\overline{m}\;\overline{}{}^{1}\overline{q}^{\mathrm{o}}\,\tau\,- \tag{E.2.50}$$

and it only requires . If present, terms in $^{1}\;^{\mathrm{o}}$ that are linear in the chiral superfields give rise to mass terms that gap out the chiral and Fermi multiplets together. Quadratic or higher-order terms in $^{1}\;^{\mathrm{o}}$ produce cubic or higher-order interactions. We shall call them E-interactions. Suppose now that we have a collection of Fermi superfields Y_8 with $\overline{Y}_8 = {}_8{}^{1}\;{}^{\mathrm{o}}$. In addition to ${}_8$, we associate another holomorphic function ${}_8{}^{1}\;{}^{\mathrm{o}}$ of the chiral superfields to each Fermi such that ${}_8\,{}_8$ (with repeated indices summed) is gauge invariant and ${}_8\,{}_8 = 0$. Then $Y_8\;{}_8{}^{1}\;{}^{\mathrm{o}}$ is a gauge-invariant fermionic chiral superfield. We can therefore write the F-terms:

$$\int 3\backslash\, Y_8\;{}_8{}^{1}\;{}^{\mathrm{o}}\;{}_{\text{¸}} \int 3\overline{\backslash}\,\overline{Y}_8\,\overline{}{}_8{}^{1}\,\overline{}{}^{\mathrm{o}} = 5_8\;{}_8{}^{1}q^{\mathrm{o}}\quad \tau_8\,m\;{}_8{}^{1}q^{\mathrm{o}}\,k \quad \overline{5}_8\,\overline{}{}_8{}^{1}\overline{q}^{\mathrm{o}}\quad \overline{k}\;\overline{m}\;\overline{}{}_8{}^{1}\overline{q}^{\mathrm{o}}\,\overline{\tau}_8\,\bullet \tag{E.2.51}$$

Note that because $Y_8\;{}_8$ is gauge invariant, $Y_{8-}\;{}_8{}^{1}\;{}^{\mathrm{o}} = Y_{8-:}\;{}_8{}^{1}\;{}_{:}{}^{\mathrm{o}}$. We will call interactions that are constructed in this way J-interactions.

E.2.4 Twisted 3d YM and CS terms

In this subsection, we show how the parts of the topologically twisted 3d Yang-Mills and Chern-Simons Lagrangians containing ${}_{\overline{1}}$ can be written in 1d superspace. The terms lie slightly beyond the scope of the exposition above, because ${}_{\overline{1}}$ transforms as a connection on $(^{2}$ under gauge transformations, as reported in (3.2.7).

Yang-Mills. The first line in (3.2.8) can be written in superspace as:

$$\mathrm{Tr}\left[4|{}_{\mathfrak{c}\overline{1}}|^{2}\;{}_{\text{¸}}\,48\;{}_{1\overline{1}}\;4|{}_{\overline{1}}f|^{2}\;{}_{\text{¸}}\,8\,\overline{}{}_{1}{}^{1}\;{}_{\mathfrak{c}}\;{}_{\text{¸}}\,8f\;{}_{\overline{1}}\;{}_{\text{¸}}\,2\;{}_{\mathfrak{c}}\;{}_{1}\;{}_{\overline{1}}\;2\,\overline{}{}_{1}\;{}_{\overline{1}}\,\overline{}{}_{\mathfrak{c}}\right] = \overset{\mathrm{WZ}}{=} 48\int 3\backslash 3\overline{\backslash}\;\mathrm{Tr}\;{}_{1-:}\,m_{\mathfrak{c}}\;{}_{\overline{1}-:}\;\;F_{1\overline{1}-:} + - \tag{E.2.52}$$

where we defined the superfield

$$F_{1\overline{1}-:}\quad m_{1}\;{}_{\overline{1}-:}\quad m_{\overline{1}}\;{}_{1-:}\quad 8\;{}_{1-:}-\;{}_{\overline{1}-:}\;\bullet \tag{E.2.53}$$

Here $F_{1\bar{1}-:}$ transforms covariantly under super-gauge transformations as $F_{1\bar{1}-:}$ 7! $:F_{1\bar{1}-:}:$ ^{1}. Note that the superspace expression has the same form as a Chern-Simons term for superfields, with + playing the role of the connection along C. Therefore, under finite gauge transformations:

$$X_{\text{gauge}}\ 48 \int_1^1 3\backslash 3\bar{\backslash}\ \text{Tr}\ \ 1\text{-}:\ m_C\ \ \bar{1}\text{-}:\ \ \ F_{1\bar{1}-:}\ +$$

$$= 28\ \ 3\backslash 3\bar{\backslash}\ \text{Tr}\ :\ {}^1 m_C :\ \ :\ {}^1 m_1 : \text{-}\ :\ {}^1 m_{\bar{1}} : ¼\ \text{-}$$

$$= 28\ \text{Tr}\ m_C m_\backslash\ \ :\ {}^1 m_{\bar{\backslash}} :\ \ :\ {}^1 m_1 : \text{-}\ :\ {}^1 m_{\bar{1}} :\ \ \ \text{¸ cyclic} \bullet \tag{E.2.54}$$

The omitted terms contain cyclic permutations of ¹C-1-$\bar{1}$º. This gauge variation looks like a winding number for super-gauge transformations. Since we are taking derivatives of the winding number density (albeit with respect to fermionic variables), a total derivative is expected because the winding number is homotopy invariant.

Alternatively, we can use superfields that are only sensitive to complexified gauge transformations. The superspace expression in (E.2.52) can then be written as

$$\text{(E.2.52)} = 48 \int^1 3\backslash 3\bar{\backslash}\ \text{Tr}\ \ 1\text{-}\ \ m_C\ \ \bar{1}\text{-}\ \ \ F_{1\bar{1}-}\ \ 4\ {}^{+}+\ \ \ \text{-} \tag{E.2.55}$$

where total derivatives of the kind (E.2.54) have been neglected. One can check that (E.2.55) is real and gauge invariant up to total derivatives.

Chern-Simons. We now want to write the first piece of (3.2.10) in superspace. To do this, we follow a similar procedure as in [244]. First, the fields - are extended to be functions b of an auxiliary coordinate H 2 ¹0-1º in an arbitrary way (similarly to what we did in C.1), except that they must fulfill boundary conditions

$$\text{b}{}^1\backslash\text{-}\ i\text{-}C\text{-}H = 0^{\text{o}} = 0\ \text{-}\qquad \text{b}{}^1\backslash\text{-}\ i\text{-}C\text{-}H = 1^{\text{o}} = \ \text{-}\ {}^1\backslash\text{-}\ i\text{-}C^{\text{o}} \bullet \tag{E.2.56}$$

Extended quantities will be denoted with a hat. Given (E.2.56), we have:

$$L_{\text{CS-WZ}} = \hat{L}_{\text{CS-WZ}}\ {}^1 H = 1^{\text{o}} = \int^1 \int_0^1 3H\ m_H \hat{L}_{\text{CS-WZ}} \bullet \tag{E.2.57}$$

Now, $m_H \mathcal{L}_{\text{CS-}}$ can be written in superspace as:

$$m_H \mathcal{L}_{\text{CS-}}\Big|_{\text{WZ}} = \text{Tr}\Big[48 m_H{}^1 \hat{}_c \,\text{¸}\, \bar{f}^o \hat{}_{1\bar{1}} \,\text{¸}\, 4 m_H \hat{}_1\, 8 m_c \hat{}_{\bar{1}} \quad 8 \hat{}_{\bar{1}}{}^1 \hat{}_c \,\text{¸}\, \bar{f}^o \,\text{¸}\, m_H \hat{\bar{}}_1 \hat{}_{\bar{1}} \,\text{¸}$$
$$\text{¸}\, 4 m_H \hat{}_{\bar{1}} \quad 8 m_c \hat{}_1 \,\text{¸}\, 8 \hat{}_1{}^1 \hat{}_c \,\text{¸}\, \bar{f}^o \quad m_H \hat{}_{\bar{1}} \hat{\bar{}}_1 \Big]$$
$$= 4 m_H^{\;1} \quad 3 \backslash 3 \bar{\backslash}\ \text{Tr}\Big[\hat{}_{1\text{-}}{}^{-} \hat{}_{\bar{1}\text{-}} \quad 8 \hat{+}\ m_1 \hat{}_{\bar{1}\text{-}} \quad m_{\bar{1}} \hat{}_{1\text{-}}{}^{-} \quad 8 \hat{}_{1\text{-}}{}^{-\text{-}} \hat{}_{\bar{1}\text{-}} \Big] \,\bullet \tag{E.2.58}$$

This superspace expression is only valid in the Wess-Zumino gauge where $+ = \backslash \bar{\backslash}^1{}_c \,\text{¸}\, f^o$, and it is not invariant under super-gauge transformations. Even so, we can take it as a starting point for constructing the gauge-invariant completion. A gauge-invariant expression that reduces to the above in the Wess-Zumino gauge is

$$m_H \mathcal{L}_{\text{CS-}} = 4^{\;1} \quad 3 \backslash 3 \bar{\backslash}\ \text{Tr}\Big[8\, 4^{\,+} m_H\, 4^{\hat{+}}\, \hat{F}_{1\bar{1}\text{-}} \;\text{¸}\; \hat{}_{1\text{-}}\, m_H \hat{}_{\bar{1}\text{-}} \;\text{¸}\; m_H \hat{}_{1\text{-}}{}^{-} \hat{}_{\bar{1}\text{-}}{}^{-} \Big] \bullet \tag{E.2.59}$$

One can check that the first term is Hermitian, while the second and third terms are Hermitian conjugates of each other. Therefore

$$\mathcal{L}_{\text{CS-}} = \text{Tr}\Big[48\ {}_1 m_c\ {}_{\bar{1}} \quad 48^1\ {}_c \,\text{¸}\, f^o\ {}_{1\bar{1}} \,\text{¸}\, \bar{}_1\ {}_{\bar{1}} \Big]$$
$$\overset{\text{WZ}}{=} 4^{\;1}\int_0^1 3H\, 3 \backslash 3 \bar{\backslash}\ \text{Tr}\Big[8\, 4^{\,+} m_H\, 4^{\hat{+}}\, \hat{F}_{1\bar{1}\text{-}} \;\text{¸}\; \hat{}_{1\text{-}}\, m_H \hat{}_{\bar{1}\text{-}} \;\text{¸}\; m_H \hat{}_{1\text{-}}{}^{-} \hat{}_{\bar{1}\text{-}}{}^{-} \Big] \bullet \tag{E.2.60}$$

If the gauge group is Abelian, (E.2.59) is a total derivative in H and the auxiliary coordinate H can be eliminated to give

$$\mathcal{L}_{\text{CS-}} = 4^{\;1} \quad 3 \backslash 3 \bar{\backslash} \quad {}_{1\text{-}}{}^{-}\ {}_{\bar{1}\text{-}} \quad 8+\ m_1\ {}_{\bar{1}\text{-}} \quad m_{\bar{1}}\ {}_{1\text{-}}{}^{-} \;\text{¸}\; \frac{1}{2} m_1 +\, m_{\bar{1}} + \bullet \tag{E.2.61}$$

For non-Abelian gauge groups, there is no compact expression for the integral in H, but we can expand in powers of $+$. Choosing

$$\hat{}_{\bar{1}\text{-}} = H\ {}_{\bar{1}\text{-}}\ \text{-} \qquad\qquad \hat{+} = H + \text{-} \tag{E.2.62}$$

one obtains the following expression up to quadratic terms in $+$:

$$\mathcal{L}_{\text{CS-}} = 4^{\;1} \quad 3 \backslash 3 \bar{\backslash}\ \text{Tr}\quad {}_{1\text{-}}{}^{-}\ {}_{\bar{1}\text{-}} \quad 8+\ m_1\ {}_{\bar{1}\text{-}} \quad m_{\bar{1}}\ {}_{1\text{-}}{}^{-} \quad 8\ {}_{1\text{-}}{}^{-\text{-}}\ {}_{\bar{1}\text{-}} \quad \text{¸}$$
$$\text{¸}\, \frac{1}{2}\, m_1 + \quad 8\ {}_{1\text{-}}{}^{-\text{-}} + \quad m_{\bar{1}} + \quad 8\ {}_{\bar{1}\text{-}}\ \text{-}+ \quad \text{¸}\, O^{1+{}^3 o} \bullet \tag{E.2.63}$$

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