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Some new applications of Γ-convergence to free discontinuity problems, phase transitions, and nonlocal models

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Abstract

In this thesis, we study variational models arising in three different settings: free discontinuity problems, higher-order models for phase transitions, and nonlocal variational problems.

First, we study free discontinuity functionals whose surface terms are of the type commonly used in the variational theory of *cohesive* fracture mechanics. Since these surface terms are, generally, linear near the origin and bounded at infinity, the natural setting of the corresponding energies is that of spaces of generalised functions of bounded variation and generalised functions of bounded deformation. We investigate the fine properties of these spaces and, in particular, contribute to the general picture of generalised functions of bounded deformation by introducing a matrix-valued measure generalising the distributional symmetric gradient.

We also address *homogenisation* problems for free discontinuity functionals. In the full gradient setting, we consider functionals with linear growth in the bulk part, a Cantor contribution, and cohesive surface terms, and prove deterministic and stochastic homogenisation results. In the framework of linearised elasticity, we study instead functionals defined on functions of bounded deformation whose bulk and surface densities have linear growth, and obtain similar deterministic and stochastic homogenisation results.

Finally, we study the relaxation of energies defined on *structured deformations*, where the macroscopic deformation is coupled with a matrix-valued field accounting for the cumulative effect of submacroscopic slips, separations, and other disarrangements. We extend the classical integral representation theory to this cohesive setting, obtaining an explicit representation for the relaxed energies.

The second part of the thesis concerns *higher-order singular perturbation problems* for phase transitions. We consider Modica-Mortola type energies, in which the usual first-order gradient term is replaced by a higher-order perturbation of arbitrarily prescribed order. We show that sequences with equibounded energy converge to sharp interfaces, and that the Γ-limit is given by a perimeter functional. We then study related *anisotropic* models involving general tensor norms and derivatives of intermediate orders, whose coefficients may be negative. For this broader class of energies, we establish equi-coercivity and Γ-convergence to an anisotropic perimeter functional, with an interfacial density determined by the orientation of the limit interface.

The third and final part concerns the passage from *nonlocal* to local models driven by concentration phenomena. We first consider fractional quadratic energies with oscillating coefficients, and hence involving two parameters, one governing the concentration induced by the fractional kernel, while the other determining the frequency of the oscillations. We study the Γ-limit of these energies as the two parameters vary simultaneously and analyse how the interaction between these two scales influences the resulting local energy. We then turn to convolution-type functionals depending on finite differences. In this setting, the concentration of the interaction kernels produces free discontinuity energies with bulk and surface terms, whose densities we identify.

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Introduction

This thesis is devoted to the analysis of the asymptotic behaviour of several classes of variational problems. This analysis is carried out by means of Γ-convergence, a tool that, since its introduction [105], has established itself as the natural notion of convergence for variational problems. The reasons behind its success are clear, as Γ-convergence enjoys all the properties one would desire from a notion of convergence tailored to the study of minimisation problems. Most notably, it is closely related to the convergence of *minima* and *minimisers*. At the same time, it is a rather flexible notion of convergence. Its definition is purely topological and never imposes *a priori* constraints on the form of the limit energies, the space of functions on which these energies are defined, nor the mechanism responsible for such convergence. These features are always determined by the particular form of the energies at hand and, therefore, Γ-convergence constitutes a unifying framework in which different types of variational problems and models can coexist.

This unifying feature is well-reflected in the tripartite structure of this thesis, which treats variational problems of essentially different nature:

(I) the first part is devoted to problems related to the modelling of *cohesive fracture*; in particular, we study the *homogenisation* of free discontinuity functionals, the *relaxation* of bulk-surface energies, and the *fine properties* of functions related to cohesive energies;

(II) the second part focuses on the analysis of *singular perturbations* of non-convex energies, which appear naturally in the variational theory of liquid-liquid phase transitions; in this case, energies defined on Sobolev functions converge to perimeter functionals defined on sharp interfaces;

(III) the third and final part is dedicated to the passage from *nonlocal* models depending on difference quotients to *local* models depending on distributional gradients.

Because the themes treated here arise from different physical and mathematical motivations, it is useful to discuss them separately. For each part, we first recall the underlying variational models and the principal analytical difficulties, and then present the related results obtained in this thesis.

Part I: Free discontinuity problems with cohesive surface terms

Part I pursues two closely related objectives. The first is to study relaxation and homogenisation problems for several classes of free discontinuity functionals arising in fracture mechanics. The second is to develop a functional framework in which these problems can be naturally formulated and analysed. To explain their motivation and highlight the main difficulties they present, we begin with a brief overview of the variational theory of fracture.

Francfort & Marigo's approach to fracture. In his pioneering work [136], Griffith introduced the idea that crack growth in an elastic material is determined by the competition between the elastic energy released in the process of crack opening and the energy dissipated while opening a new portion of the crack. Adopting this point of view, Francfort and Marigo proposed in [126] a variational model to study the *quasistatic* growth of fracture in *brittle* materials. In this model, the evolution in time of a body with reference configuration Ω is described by the deformation field $t \mapsto u(t)$ and the crack set $t \mapsto K(t)$. These two objects are treated as independent variables and the total energy $\mathcal{E}_{\mathrm{br}}(u(t), K(t))$ associated with them at a time t is given by

$$\mathcal{E}_{\mathrm{br}}(u(t), K(t)) \coloneqq \mathcal{W}(u(t), K(t)) + \mathcal{K}(K(t)), \tag{0.0.1}$$

where $\mathcal{W}(u(t), K(t))$ and $\mathcal{K}(K(t))$ represent the bulk elastic energy and the dissipated surface energy.

In the language of Francfort & Marigo, a quasistatic evolution is a pair $(u(t), K(t))$ satisfying three conditions: *global stability*, the *energy balance*, and *irreversibility* of the crack set. The first condition is the one justifying the term quasistatic. At each instant, the system is assumed to be in equilibrium, that is, $(u(t), K(t))$ minimises the total energy $\mathcal{E}_{\mathrm{br}}$ among all admissible configurations at time t. The second condition states that the increment of the internal energy plus the energy dissipated in the evolution process equals the work of the external forces, while the last one states that a brittle material cannot heal. Once a portion of it is broken, it is doomed to always be so. In mathematical terms, $t \mapsto K(t)$ is non-decreasing with respect to set inclusion.

Proving the existence of an evolution $(u(t), K(t))$ is a delicate task. The standard strategy relies on a time-incremental scheme. One first fixes a discrete partition $\{t_i^n\}_i$ of the relevant time interval and then, at each time step, solves the minimisation problem given by the global stability condition, taking into account the cracks accumulated at the preceding steps and thus enforcing irreversibility. Interpolating the pairs (u_i^n, K_i^n) obtained in this fashion and letting the mesh size of the partition tend to zero, one recovers in the limit a quasistatic evolution $(u(t), K(t))$. This passage to the continuous-time limit is, in general, highly nontrivial, since both the deformations and the evolving crack sets must pass to the limit, while simultaneously preserving irreversibility and global minimality.

Although the time-dependent problem is not addressed in this thesis, the preceding discussion highlights the central role of static minimisation problems in the variational theory of fracture. They constitute the building blocks of quasistatic evolutions and, as such, a detailed understanding of the variational properties of the underlying energies is essential. The first part of this thesis is devoted to relaxation and homogenisation problems for functionals arising in this and in related settings. For this reason, we now examine more closely the precise form of the energy $\mathcal{E}_{\mathrm{br}}$ and of its counterpart for *cohesive* fracture.

For simplicity, we restrict our attention to the context of small-strain antiplane shear, adopting the weak formulation of Francfort & Larsen [125]. In this setting, the reference configuration is given by an infinite cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^{d+1}$, whose cross-section is a bounded

open set $\Omega \subset \mathbb{R}^d$ and the displacement is described by a scalar function $u\colon \Omega \to \mathbb{R}$. Rather than being prescribed independently, the cross-section of the crack set K is identified with the jump set J_u of the deformation map u, whose natural ambient space is De Giorgi & Ambrosio's GSBV(Ω) (cf. Section 1.3). In the case of a homogeneous, isotropic material, the energy (0.0.1) to minimise then becomes the Mumford-Shah functional

$$\mathcal{E}_{\mathrm{br}}(u) \coloneqq \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \kappa \mathcal{H}^{d-1}(J_u) \qquad \text{for } u \in \mathrm{GSBV}(\Omega), \tag{0.0.2}$$

where ∇u denotes the approximate gradient of u, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and the constant $\kappa > 0$ is related to the *toughness* of the material. In accordance with Griffith's criterion, the surface contribution is proportional to the $(d-1)$-dimensional measure of the cross section of the crack set.

This weak formulation in GSBV(Ω) avoids imposing closedness on the crack set, a property that is generally difficult to recover through the compactness arguments of the direct method of the calculus of variations. Moreover, it places the problem in the framework of the Mumford-Shah functional, for which both existence and regularity theories are available (see, e.g., [16] and [104]). These results allowed Francfort & Larsen [125] to construct quasistatic evolutions in this weak sense and, later, Babadjan & Giacomini [22] to infer existence of quasistatic evolutions with regular crack sets.

However, it is evident from the definition of $\mathcal{E}_{\mathrm{br}}$ that not all materials can be described by such an energy. If u^+ and u^- denote the traces of the deformation field u on the two sides of J_u, the surface energy is independent of the crack opening $[u] \coloneqq u^+ - u^-$, so that the full fracture cost is paid as soon as a new portion of the crack is created, irrespective of the amplitude of its opening. The brittle model thus describes fracture as an abrupt loss of cohesion and cannot account for the progressive weakening of the interactions between the crack lips before complete separation, as is observed in a wide range of materials (see, e.g., the review paper [159]).

To overcome these limitations, it is convenient to consider *cohesive* zone models in the spirit of the work of Barenblatt [27]. Rather than assuming that the crack lips become completely traction-free as soon as a crack is created, Barenblatt postulated the existence of a small process zone near the crack tip in which the two sides of the crack continue to interact through cohesive forces, which remain active until complete failure occurs. In variational terms, Barenblatt's idea leads to the replacement of $\mathcal{E}_{\mathrm{br}}$ by an energy of the form

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{J_u} g([u]) \,\mathrm{d}\mathcal{H}^{d-1} \qquad \text{for } u \in \mathrm{GSBV}(\Omega), \tag{0.0.3}$$

where g is a suitable non-negative integrand explicitly depending on the crack opening $[u]$. The properties satisfied by the energy density g may vary, but, in accordance with the observations above, it is usually assumed that the function $[0, +\infty) \ni \zeta \mapsto g(\zeta)$ is non-decreasing, satisfies $g(0) = 0$, and is constant at infinity. Moreover, it is often assumed that $g'(0) \in (0, +\infty)$, where $g'(\zeta)$ represents the force per unit length transmitted between the crack lips at an opening ζ, so that g behaves linearly near the origin. In the rest of this preliminary discussion, we shall assume that g coincides with *Dugdale*'s energy density $|\zeta| \wedge 1 \coloneqq \min\{|\zeta|, 1\}$, which, for us, shall be the prototype of cohesive surface density.

From a mathematical point of view, this formulation presents, however, an immediate drawback. Indeed, assumptions $g(0) = 0$ and $g'(0) \in (0, +\infty)$ force a relaxation mechanism in BV(Ω), since, in this case, it is energetically more favourable to replace large gradients by many infinitesimally small jumps, possibly leading in the limit process to the creation of

diffuse cracks of dimension strictly larger than $d-1$. This observation can be made rigorous by relying on results of Bouchitté, Braides, & Buttazzo [38] who showed that the energy in (0.0.3) is not lower semicontinuous and that, for $u \in \mathrm{BV}(\Omega)$, its relaxation should take into account also contributions related to the Cantor part $D^c u$ of the distributional gradient Du. In particular, it should be replaced by an energy which is, morally, of the form

$$\mathcal{V}(u) \coloneqq \int_\Omega |\nabla u| \, \mathrm{d}x + |D^c u|(\Omega) + \int_{J_u} (|[u]| \wedge 1) \, \mathrm{d}\mathcal{H}^{d-1}, \tag{0.0.4}$$

which is indeed lower semicontinuous with respect to the weak*-convergence in $\mathrm{BV}(\Omega)$.

These observations extend, essentially unchanged, to the vectorial setting, where the reference configuration is represented by a general bounded open set $\Omega \subset \mathbb{R}^d$, the deformation field u is vector-valued, and the bulk energy may depend either on the full gradient ∇u or, in the small-strain regime of linearised elasticity, on the symmetrised gradient $\mathcal{E}u$. In this latter case, the correct counterparts of the energies presented in (0.0.2) and (0.0.4) are Griffith energy

$$\int_\Omega |\mathcal{E}u|^2 \, \mathrm{d}x + \kappa \mathcal{H}^{d-1}(J_u), \tag{0.0.5}$$

and the energy $\mathcal{V}_{\mathrm{vec}}$ defined by

$$\mathcal{V}_{\mathrm{vec}}(u) \coloneqq \int_\Omega |\mathcal{E}u| \, \mathrm{d}x + |E^c u|(\Omega) + \int_{J_u} (|[u]| \wedge 1) \, \mathrm{d}\mathcal{H}^{d-1}, \tag{0.0.6}$$

where the displacement u is now allowed to vary in the space of functions of *bounded deformation* $\mathrm{BD}(\Omega)$, and $E^c u$ is the Cantor part of the distributional symmetric gradient Eu (cf. Section 1.4).

Lack of compactness: the spaces $\mathrm{GBV}_\star$ and GBD. The relaxation process described above is not the only issue arising in this context. The choice of the functional domain where to set the study of minimisation problems related to (0.0.4) and (0.0.6) is also a delicate matter. The surface term $|[u]| \wedge 1$ does not control the full jump amplitude $|[u]|$, so that, in general, it is not possible to bound $|Du|(\Omega)$ and $|Eu|(\Omega)$ in terms of $\mathcal{V}(u)$ and $\mathcal{V}_{\mathrm{vec}}(u)$, respectively. Thus, even in the presence of additional lower order terms or Dirichlet boundary conditions, sequences with uniformly bounded $\mathcal{V}$ or $\mathcal{V}_{\mathrm{vec}}$ may not be precompact in $\mathrm{BV}(\Omega)$ or in $\mathrm{BD}(\Omega)$.

An analogous issue occurs in the brittle case (0.0.2) and (0.0.5), where it is solved by considering displacements belonging to the generalised spaces $\mathrm{GSBV}(\Omega)$ and $\mathrm{GSBD}(\Omega)$, introduced by De Giorgi & Ambrosio [103] and Dal Maso [89], respectively. In these larger spaces, it is possible to show coercivity and lower semicontinuity of the Mumford-Shah and Griffith functionals, see, for instance, [16, 119, 127] for the former and [14, 72–74, 129] for the latter.

For the cohesive energy $\mathcal{V}$, this lack of coercivity leads naturally to the space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ (cf. Section 1.3), introduced by Dal Maso & Toader [99] in the scalar case and later extended to the vector-valued setting by the author in [111]. Roughly speaking, $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ consists of all functions $u \in L^0(\Omega;\mathbb{R}^k)$ that are limits, with respect to the convergence in measure, of sequences $\{u_n\}_n \subset \mathrm{BV}(\Omega;\mathbb{R}^k)$ whose components have equibounded energy $\mathcal{V}$. Although the space obtained in this way is far more general than BV, it retains the fine structure needed to study free discontinuity problems. In particular, every such function admits an approximate gradient ∇u, a $(d-1)$-rectifiable jump set J_u, and a matrix-valued measure $D^c u$ generalising the Cantor part of the distributional derivative Du for functions of bounded

variation. Finally, it can be shown that this space enjoys compactness properties (cf. Theorem 2.A.1) that make it the natural functional framework for free discontinuity problems with cohesive-type surface terms depending on the full gradient. A precise description of the main properties of functions in $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ (see Definition 1.3.16), based also on the contents of the author's paper [111] and of the joint paper with Friedrich [112], can be found in Section 1.3.

The corresponding situation in the linearised setting requires instead the space $\mathrm{GBD}(\Omega)$ of *generalised* functions of bounded deformation, whose definition was given by Dal Maso [89] by means of one-dimensional slicing. Remarkably, this slicing-based definition suffices to recover many of the fine properties of $\mathrm{BD}(\Omega)$. In particular, Dal Maso proved in [89] that every such function admits an approximate symmetric gradient $\mathcal{E}u$ and a $(d-1)$-rectifiable jump set J_u, and possesses suitable trace properties. However, while these properties are sufficient for the analysis of the Griffith functional (0.0.5), they are not enough to deal with general cohesive energies. These formulations require, in addition, a diffuse, singular matrix-valued measure that plays the role of the Cantor part $E^c u$ for functions in BD. In Chapter 5, based on the joint work with Dal Maso [91], we shall construct such a measure, thus providing the missing object to study cohesive energies in $\mathrm{GBD}(\Omega)$. Since the compactness results in [14, 73, 74] apply also in this context, $\mathrm{GBD}(\Omega)$ seems a promising ambient space in which to study variational problems for cohesive fracture energies in linearised elasticity. We stress that, however, such problems are not investigated here and questions related to the relaxation and Γ-convergence of cohesive fracture energies on $\mathrm{GBD}(\Omega)$ are left for future work.

Chapter 2

Homogenisation of free discontinuity problems with cohesive-type surface terms: the full gradient case

In Chapter 2, we begin the analysis of variational problems for functionals defined on the space $\mathrm{GBV}_\star$ and address the problem of *homogenisation* of free discontinuity functionals.

Given a bounded open set $\Omega \subset \mathbb{R}^d$ and $k \geq 1$, we shall consider functionals defined for $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ and $\varepsilon > 0$ by

$$F_\varepsilon(u) \coloneqq \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u} g\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \quad (0.0.7)$$

where ν_u is the unit normal at J_u, the integrands $f\colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0,+\infty)$ and $g\colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfy

$$c(|A|-1) \leq f(x,A) \leq C(|A|+1) \quad \text{for all } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{k\times d}, \quad (0.0.8)$$

$$c(|\zeta| \wedge 1) \leq g(x,\zeta,\nu) \leq C(|\zeta| \wedge 1) \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k, \text{ and } \nu \in \mathbb{S}^{d-1}, \quad (0.0.9)$$

and f^∞ is the *recession function* of f, defined by

$$f^\infty(x,A) \coloneqq \limsup_{t\to+\infty} \frac{f(x,tA)}{t} \quad \text{for all } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{k\times d}. \quad (0.0.10)$$

Our aim is to derive necessary and sufficient conditions for the Γ-convergence as $\varepsilon \to 0^+$ of $\{F_\varepsilon\}_\varepsilon$ to a *homogenised* functional of the form

$$F_{\mathrm{hom}}(u) \coloneqq \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u} g_{\mathrm{hom}}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \quad (0.0.11)$$

for suitable integrands f_{hom} and g_{hom} independent of the spatial variable x. As previously discussed, under our growth assumptions on f and g, the Cantor part $D^c u$ and the recession function f^∞ must be taken into account for lower semicontinuity reasons (see Theorem 1.5.1 below).

From a physical viewpoint, the family $\{F_\varepsilon\}_\varepsilon$ describes the behaviour of a heterogeneous material endowed with a fine microstructure whose characteristic length is of order ε. In accordance with the general principles of homogenisation, our aim is to determine an effective functional F_{hom}, associated with a homogeneous medium that captures the properties of F_ε as the microstructure becomes finer and finer and $\varepsilon \to 0^+$.

Homogenisation problems for free discontinuity functionals have been the subject of extensive research in a variety of different contexts. We briefly recall the results that are most closely related to the setting considered here.

The first general homogenisation result in this direction was obtained by Braides, Defranceschi, & Vitali [48], who considered periodic energies for which the underlying functional domain is the space $\mathrm{SBV}^p(\Omega;\mathbb{R}^k)$, for $p > 1$ (see Section 1.3) and no contribution related to the Cantor part needs to be considered. They proved that, in this case, the Γ-limit is a free discontinuity functional of the same type and characterised its effective bulk and surface densities, which are x-independent, through asymptotic cell formulas.

Later, in a series of three works [57–59], Cagnetti, Dal Maso, Scardia, & Zeppieri developed a general theory able to simultaneously handle the homogenisation of non-periodic energies and stochastic homogenisation.

In the first of these papers [57], they study compactness and integral representation properties for sequences of free discontinuity functionals, satisfying growth conditions similar, even though more general, to the ones considered in [48] and in the related work of Giacomini & Ponsiglione [131]. As an application of this general theory, they characterised homogenisation in terms of the asymptotic behaviour of suitable minimisation problems on large cubes.

In the subsequent work [58], the same authors consider random integrands depending on an additional parameter belonging to a probability space. The integrands are assumed to be stationary in the probabilistic variable with respect to a group of measure-preserving transformations. Following the strategy introduced by Dal Maso & Modica [97], based on the Subadditive Ergodic Theorem of Akcoglu & Krengel [2], they proved the almost sure existence of the effective energy densities and, as a consequence, an almost sure Γ-convergence result.

Finally, in [59] they extended their methods to free-discontinuity functionals satisfying linear growth conditions in BV. In this setting, the energy controls the full total variation $|Du|$ so that the integral representation of the limit must also include the Cantor part of Du.

The techniques developed in these three papers have proved to be quite flexible and have since been exploited in a variety of different works and settings. In particular, adopting the strategy of [59], in the recent work [101] Dal Maso & Toader addressed the deterministic and stochastic homogenisation of functionals $\{F_\varepsilon\}_\varepsilon$ as in (0.0.7) with f and g satisfying (0.0.8) and (0.0.9), but in the scalar case $k = 1$. Relying on previous compactness and integral representation results they developed in [100], they studied the problem of deterministic and stochastic homogenisation for such functionals. Although following a strategy similar to that of [59], due to the technical difficulties inherent to the generalised space $\mathrm{GBV}_\star(\Omega)$, their analysis departs substantially from the one developed in [59].

In this chapter, based on the joint paper with Dal Maso [90], we extend the analysis

of [100, 101] to the vector-valued case $k > 1$. This extension is not immediate, since, among other issues, the scalar truncations used in these papers are not expedient in the vector-valued case $k > 1$. This difficulty is overcome by considering a different kind of truncation, which, however, makes the analysis technically more demanding.

In analogy with the previous works, our results go in three directions. We first prove a general compactness and integral representation theorem for functionals satisfying two-sided growth conditions in terms of the functional $\mathcal{V}$ defined in (0.0.4) (see Theorem 2.6.6). This is done by relying on a truncation-perturbation argument which allows us to use previous results of Bouchitté, Fonseca, & Mascarenhas [41]. Note, however, that, due to the application to stochastic homogenisation we have in mind, we cannot apply directly the full integral representation result of [41] (see Theorem 1.5.1), as it relies on a continuity property with respect to translation which is not easily verified in the setting of non-periodic homogenisation.

These results are then used to deduce necessary and sufficient conditions for the functionals in (0.0.7) to Γ-converge to a homogenised energy as in (0.0.11), with f_{hom} and g_{hom} independent of the spatial variable (see Theorem 2.7.1).

Finally, we apply this deterministic characterisation to stationary random integrands. Using the arguments developed in [58, 59], we prove that the corresponding functionals Γ-converge almost surely to a homogenised functional as in (0.0.11) (see Theorem 2.7.9). The periodic homogenisation follows as a particular case (see Corollary 2.7.10).

Chapter 3

Structured deformations for energies with general surface terms

In this chapter, based on the joint paper with Friedrich [112], we present another class of problems for which $\mathrm{GBV}_\star$ appears as the natural domain. Our main goal is to establish a theory for the description of first-order structured deformations for energies with cohesive-type surface densities. Following the convention of the structured deformations literature, we denote deformation fields by g instead of u throughout this chapter.

Structured deformations were introduced by Del Piero & Owen [109] as a mathematical framework to describe multiscale phenomena in materials. In this theory, the description of deformations with discontinuities introduced above, based on a deformation map g defined on a reference configuration Ω and on a crack set $K \subset \Omega$, is replaced by a triple (g, G, K). The map g still represents the macroscopic deformation, whose discontinuities are concentrated on the closed set K, whereas the matrix field G accounts for the contributions at the macroscopical level of smooth sub-macroscopic changes. The tensor $\nabla g - G$ captures those arising from non-smooth sub-macroscopic changes, such as slips and separations (referred to as *disarrangements* [110]). One of the main contributions of [109] consists in an *approximation result* that clarifies this interpretation: every structured deformation (g, G, K) can be seen as the limit of *simple deformations*, i.e., triples of the form $\{(g_n, \nabla g_n, K_n)\}_n$, such that g_n converges to g and the tensor G arises as the weak limit of the sequence $\{\nabla g_n\}_n$. The theory of structured deformations can model a broad class of mechanical phenomena in which the macroscopic response is influenced by microscopic slips, cracks, or other disarrangements, including elasticity, plasticity, fracture, and the mechanics of defective crystalline solids.

Adopting this point of view, Choksi & Fonseca [76] laid the foundations for a variational theory of structured deformations, which presents some similarities with the weak formulation of fracture of Francfort & Larsen [125]. In their formulation, the triple (g, G, K) is

replaced by a pair (g, G), where the deformation map g is a function in $\mathrm{SBV}(\Omega;\mathbb{R}^k)$ and $G \in L^p(\Omega;\mathbb{R}^{k\times d})$ for some $p \geq 1$. The crack site K is no longer prescribed independently, but is instead represented by the jump set J_g of the deformation map. Relying on a previous result of Alberti [3], they were able to recover an analogue of the approximation theorem of Del Piero & Owen. Every pair (g, G) can be approximated, in a suitable sense, by sequences $\{(u_n, \nabla u_n)\}_n$ with $\{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k)$. Starting from a general class of energy functionals with bulk and surface densities of the form

$$\mathcal{E}(u) = \int_\Omega \Psi(x, \nabla u)\,\mathrm{d}x + \int_{J_u} \gamma(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{0.0.12}$$

this result justified the assignment of a *relaxed formulation* I^p representing the energetically most favourable way to approximate a structured deformation, namely

$$I^p(g, G) \coloneqq \inf\Big\{ \liminf_{n\to+\infty} \mathcal{E}(u_n)\colon\ \{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k),\ \{(u_n, \nabla u_n)\}_n \text{ converges to } (g, G)\Big\}.$$

Here, Ψ is a non-negative integrand with p-growth, for $p \geq 1$, and γ satisfies

$$c|\zeta| \leq \gamma(x, \zeta, \nu) \leq C|\zeta| \tag{0.0.13}$$

for suitable positive constants c and C. Under these assumptions, Choksi & Fonseca identified effective bulk and surface energy densities which are defined through cell formulas (computed by looking at simple deformations $(u, \nabla u)$ only) and which allow one to represent I^p as an integral functional.

This variational approach has been developed further over the last years, including higher-order and multi-level theories [30–33, 120], linearised models [128, 151], as well as dimension reduction [62, 152] and homogenisation [15]. For a comprehensive introduction to the theory and for an exhaustive list of contributions on the topic, we refer the interested reader to the recent monograph [149].

We have seen, however, that growth conditions as in (0.0.13) exclude a large plethora of relevant settings, such as models for defects in crystals or cohesive zone models in fracture mechanics. To treat such applications in the framework of structured deformations, it is thus necessary to relax the bounds in (0.0.13) and to consider surface densities satisfying

$$c(|\zeta| \wedge 1) \leq \gamma(x, \zeta, \nu) \leq C(|\zeta| \wedge 1). \tag{0.0.14}$$

Some observations in this direction have already been made, see, e.g., the discussions in [76, Remark 3.3] or [150, Remark 3.1]. However, the approach suggested therein comes at the expense of altering the definition of the relaxation I^p by considering only approximating sequences with *equibounded* BV*-norm*. As the lower bound in (0.0.14) does not guarantee this equiboundedness property, this may lead to an ill-posed existence theory for minimisation problems associated to the relaxed energy I^p. A natural way to circumvent this limitation is to generalise the admissible class of deformation fields g to the natural domain of energies with surface densities as in (0.0.14), namely $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$.

Our goal is to provide a rather complete picture of the relaxation of I^p in the class of admissible deformation fields $g \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, under the growth assumption (0.0.14), without requiring any additional constraint on the norms of the approximating sequences. Our results can be summarised as follows:

(a) an extension of the approximation theorem of Choksi & Fonseca, where we prove that every pair (g, G), with $g \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ and $G \in L^1(\Omega;\mathbb{R}^{k\times d})$, can be approximated by means of a sequence $\{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k)$ (see Theorem 3.1.1);

(b) an integral representation result for abstract functionals by means of the global method for relaxation of Bouchitté, Fonseca, & Mascarenhas [41] (see Theorem 3.1.7);

(c) an explicit integral representation for the relaxation of (0.0.12), under the growth condition (0.0.14) (see Theorems 3.1.13, 3.1.14, and 3.1.16).

Two main difficulties arise in this extension. The approximation theorem must provide estimates in terms of the nonlinear quantity $\mathcal{V}$, rather than the total variation, while the presence of the additional field G prevents a direct application of a perturbation–truncation argument, as we do in Chapter 2. We refer to the introductory section at the beginning of Chapter 3 for a detailed discussion of the new arguments required to overcome these difficulties.

Chapter 4

Γ-convergence and homogenisation of free discontinuity functionals: the symmetric gradient case

We then turn our attention to free discontinuity problems arising in the context of linearised elasticity. In this chapter, based on the submitted paper with Dal Maso [92], we make a first step towards the extension of the homogenisation results for cohesive energies developed in Chapter 2 to cohesive energies depending on symmetric gradients.

Here, we consider an intermediate setting in which the surface densities have linear growth, rather than the cohesive behaviour of (0.0.6). Accordingly, the natural ambient space is $\mathrm{BD}(\Omega)$, a fact that allows us to rely on several results obtained for this space in recent years, while also avoiding the analysis of the fine properties of functions in GBD that the cohesive setting requires. This last problem is instead the subject of Chapter 5.

Given a bounded open set $\Omega \subset \mathbb{R}^d$, we consider sequences of functionals defined on $\mathrm{BD}(\Omega)$ by

$$F_n(u) \coloneqq \int_\Omega f_n(x, \mathcal{E}u)\,\mathrm{d}x + \int_\Omega f_n^\infty\Big(x, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u} g_n(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

where the bulk densities f_n have linear growth, f_n^∞ is the recession function of f_n (see (0.0.10)), and the surface densities g_n are controlled both from above and from below in terms of $\mathbb{R}^d \ni \zeta \mapsto |\zeta|$. The first problem we address is to determine properties on $\{f_n\}_n$ and $\{g_n\}_n$ that make this class of functionals stable under Γ-convergence. This amounts to proving new compactness and integral representation results for functionals defined on $\mathrm{BD}(\Omega)$.

While results concerning the integral representation of abstract functionals with linear growth on BV have been available for quite some time, corresponding results in BD have been established only more recently. Indeed, before the groundbreaking work of De Philippis & Rindler [107] (cf. Theorem 1.4.1) on the rigidity properties of the singular part of Eu, only partial results due to Ebobisse & Toader [115] were available. More recently, Caroccia, Focardi, & Van Goethem [63], performed a refined analysis of the blow-up properties of functions in BD based on [107], which allowed them to prove a general integral representation result in the setting of functions of bounded deformation.

However, in analogy with Chapter 2, due to the applications to the stochastic setting we have in mind, we cannot employ their integral representation result as it is. Indeed, their result relies on a continuity property with respect to translation which is not easily verified in the probabilistic setting. Hence, we shall prove a suitable generalisation of their result,

based on the arguments of Chapter 2. With respect to that chapter, the analysis carried out here is further complicated by the fact that truncation arguments are no longer available in this setting.

Having addressed the stability of sequences $\{F_n\}_n$ as above, we then determine necessary and sufficient conditions for their Γ-convergence. As usual, they are expressed as the existence of limits of suitable minimisation problems on large cubes.

As an application, we then investigate the deterministic and stochastic Γ-convergence of the rescaled functionals

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \mathcal{E}u\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u} \varepsilon g\Big(\frac{x}{\varepsilon}, \frac{[u]}{\varepsilon}, \nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}.$$

In general, the functionals $\{F_\varepsilon\}_\varepsilon$ defined in this way do not coincide with those normally considered in the homogenisation of free discontinuity problems (cf. Chapter 2), unless one assumes that $[0,+\infty) \ni t \mapsto g(x,t\zeta,\nu)$ is positively homogeneous of degree one. The particular choice of the scaling for the surface term of $\{F_\varepsilon\}_\varepsilon$ will allow us to obtain suitable change of variable formulas which are crucial to the study of the asymptotic behaviour of these types of sequences.

We mention that a related homogenisation problem was studied by Caroccia, Focardi, & Van Goethem [63] and by Anza Hafsa & Mandallena [20] for bulk energies of the form

$$\Phi_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \mathcal{E}u\Big)\,\mathrm{d}x & \text{if } Eu \ll \mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

In the first work the case where f is 1-periodic with respect to x and has linear growth is addresse, while in the latter the authors study the stochastically periodic case.

Chapter 5

A matrix-valued measure associated with generalised functions of bounded deformation

The remaining two chapters of Part I are devoted to the study of properties of generalised functions of bounded deformation, with a particular attention to their fine properties. In this chapter, based on the joint paper with Dal Maso [91], we introduce a suitable generalisation of the distributional symmetric gradient Eu for functions in $\mathrm{GBD}(\Omega)$, thus providing a framework for the systematic study of cohesive energies in the setting of linearised elasticity.

Our approach is based on a crucial observation, dating back to the seminal work of Ambrosio, Coscia, & Dal Maso [17], where the fine properties of functions of bounded deformation were thoroughly investigated. The starting point of their analysis is a formula relating the symmetrised gradient Eu of a function $u \in \mathrm{BD}(\Omega)$ with its one-dimensional slices, which, for $\xi \in \mathbb{S}^{d-1}$ and $y \in \Pi^\xi := \{y \cdot \xi = 0\}$, are defined by $u_y^\xi(t) := u(y + t\xi) \cdot \xi$ for all $t \in \Omega_y^\xi$, where we have also set $B_y^\xi := \{t \in \mathbb{R} \colon\ y + t\xi \in B\}$ for any $B \subset \Omega$. One can see that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the function $u_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$ and that

$$(Eu(B)\xi \cdot \xi) = \int_{\Pi^\xi} Du_y^\xi(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) \quad \text{for all Borel sets } B \subset \Omega, \tag{0.0.15}$$

where Du_y^ξ denotes the distributional derivative of the one-dimensional section u_y^ξ. This means that, using the polarisation identity for quadratic forms (see (5.5.1)) it is possible

to reconstruct the matrix $Eu(B) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ just by looking at the right-hand side of (0.0.15) for a finite number of directions. Hence, for functions in $\mathrm{BD}(\Omega)$ it is possible to define Eu purely in terms of the slices of u.

Since functions in GBD are defined by means of one-dimensional slicing (see Definition 1.4.3), this is the natural approach to introduce a generalisation of the symmetric gradient in this context. Indeed, we prove that it is possible to assign to every $u \in \mathrm{GBD}(\Omega)$ a $\mathbb{R}^{d\times d}_{\mathrm{sym}}$-valued bounded Radon measure μ_u such that (0.0.15) holds with Eu replaced by μ_u and for a large class of Borel sets (see Theorem 5.5.1). One can easily see that such formula cannot hold for all Borel subsets of Ω, but it is satisfied, for instance, by all those that satisfy $\mathcal{H}^{d-1}(B \cap J_u) = 0$. The construction of such a measure μ_u is by no means straightforward and is carried out in two steps: we first treat the planar case $d = 2$, relying on a discretisation argument along the slicing directions, and then extend the construction to arbitrary dimension by a standard argument, slicing with two-dimensional planes.

As a byproduct of the result above, we are able to introduce a generalisation of $E^c u$ for generalised functions of bounded deformation. More precisely, letting μ_u^s denote the singular part with respect to the d-dimensional Lebesgue measure $\mathcal{L}^d$, we introduce the Cantor part μ_u^c of μ_u by setting

$$\mu_u^c(B) \coloneqq \mu_u^s(B \setminus J_u) \quad \text{for all Borel sets } B \subset \Omega.$$

Using (0.0.15), it is then easy to see that $\mu_u^c = E_u^c$ whenever $u \in \mathrm{BD}(\Omega)$ and, by standard disintegration arguments, that the following decomposition holds (see Proposition 5.5.5):

$$\mu_u(B) \coloneqq \int_B \mathcal{E}u \,\mathrm{d}x + \mu_u^c(B) + \int_{B\cap J_u} ([u] \odot \nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for all Borel sets } B \subset \Omega \setminus J_u^1,$$

where $\mathcal{E}u$ is the approximate symmetric gradient, $\odot$ the symmetrised tensor product, and $J_u^1 \coloneqq \{x \in J_u \colon |[u](x)| \geq 1\}$. Moreover, in analogy with $E^c u$, we show that μ_u^c gives positive measure only to Cantor-type sets, since it vanishes on every Borel set that is σ-finite with respect to $\mathcal{H}^{d-1}$ (see Proposition 5.5.8).

Unsurprisingly, the introduction of the measure μ_u^c allows us to derive new characterisations of $\mathrm{GSBD}(\Omega)$. Dal Maso introduced this space in [89] through one-dimensional slicing, requiring the slices of a function $u \in \mathrm{GBD}(\Omega)$ to belong to $\mathrm{SBV}_{\mathrm{loc}}$. In the classical setting, instead, the decomposition of the symmetric distributional derivative established in [17] gives $\mathrm{SBD}(\Omega) = \big\{u \in \mathrm{BD}(\Omega) \colon E^c u = 0\big\}$. Since the measure μ_u^c constructed above coincides with $E^c u$ whenever $u \in \mathrm{BD}(\Omega)$, it is natural to ask whether the same characterisation extends to the generalised setting. We prove that this is indeed the case by showing that $\mathrm{GSBD}(\Omega) = \big\{u \in \mathrm{GBD}(\Omega) \colon \mu_u^c = 0\big\}$ (see Theorem 5.5.9). Thus, also in $\mathrm{GBD}(\Omega)$, the special character of a function is equivalent to the absence of a Cantor-type contribution.

Finally, combining this novel characterisation of $\mathrm{GSBD}(\Omega)$ with a recent result by Chambolle & Crismale [74], we are able to show that functions in $\mathrm{GSBD}(\Omega)$ can be characterised by looking only at a finite number of slices (see Theorem 5.5.10). In particular, this establishes a strong version of a result obtained, independently, by Almi, Davoli, Kubin, & Tasso [13, Theorem 3.1].

We conclude mentioning that, with all these results at our disposal, we can finally introduce the natural extension to $\mathrm{GBD}(\Omega)$ of the functional $\mathcal{V}_{\mathrm{vec}}$ (0.0.6) by setting

$$\mathcal{V}_{\mathrm{vec}}(u) \coloneqq \int_\Omega |\mathcal{E}u| \,\mathrm{d}x + |\mu_u^c|(\Omega) + \int_{J_u} (|[u]| \wedge 1)\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for all } u \in \mathrm{GBD}(\Omega).$$

In view of the compactness theorems of Almi & Tasso [14] and of Chambolle & Crismale [73, 74], this functional turns out to be coercive in $\mathrm{GBD}(\Omega)$ and, therefore, μ_u^c appears as

the correct generalisation of $E^c u$ to consider in cohesive energies defined on $\mathrm{GBD}(\Omega)$. We stress once more that properties of energies of this kind are not investigated in this thesis.

Chapter 6

A Korn-Poincaré-Sobolev inequality for generalised functions of bounded deformation

We conclude the first part of this thesis by presenting a Korn-Poincaré-Sobolev inequality for generalised functions of bounded deformation defined over a connected bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary.

For functions $u \in \mathrm{BD}(\Omega)$, the classical Korn-Poincaré-Sobolev inequality proved by Kohn [141] in his PhD thesis ensures the existence of an infinitesimal rigid motion $a(x) = Ax + b$, with A skew symmetric, such that

$$\|u - a\|_{L^{d/(d-1)}(\Omega)} \leq C|Eu|(\Omega). \tag{0.0.16}$$

This inequality is crucial for many applications to variational problems in $\mathrm{BD}(\Omega)$, as it is used to obtain compactness in such spaces. Therefore, it is natural to ask whether a similar estimate, possibly involving the measure μ_u introduced in the previous chapter, holds in $\mathrm{GBD}(\Omega)$. It is immediate to check, however, that a direct analogue of (0.0.16) cannot be expected in $\mathrm{GBD}(\Omega)$, because functions in $\mathrm{GBD}(\Omega)$ are, in general, not even locally integrable, so that the left-hand side of (0.0.16) might be infinite.

To solve this issue, we follow the approach of Chambolle, Conti, & Francfort [68], who, using very sophisticated slicing techniques, proved a related inequality for functions in $\mathrm{SBD}^p(\Omega) := \{u \in \mathrm{SBD}(\Omega) \colon \mathcal{E}u \in L^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \text{ and } \mathcal{H}^{d-1}(J_u) < +\infty\}$. In particular, for $p = 1$, they showed that, if $u \in \mathrm{SBD}^1(\Omega)$ has small jump set, then there is a rigid motion a as above, such that

$$\|u - a\|_{L^{d/(d-1)}(\Omega\setminus\omega)} \leq C \int_\Omega |\mathcal{E}u| \,\mathrm{d}x, \tag{0.0.17}$$

where $\omega \subset \Omega$ is a Borel set whose measure is controlled by $\mathcal{H}^{d-1}(J_u)^{d/(d-1)}$ and C a constant depending only on Ω. This result, together with some related estimates proved in that paper, was initially devised to deal with approximation of brittle energies with non-interpenetration constraints [69], but later became the cornerstone of many regularity, density, and compactness results in $(\mathrm{G})\mathrm{SBD}^p(\Omega)$, a non-exhaustive list including [56, 70–72, 80, 82].

Following the arguments of [68], Chambolle & Crismale proved in [73] that an inequality related to (0.0.17) holds in the context of $\mathrm{GBD}(\Omega)$. More precisely, they showed that, for functions $u \in \mathrm{GBD}(Q)$ with small jump set defined on a cube Q, it holds

$$\|u - a\|_{L^1(Q\setminus\omega)} \leq C\Big(\lambda_u(Q \setminus J_u) + \int_{J_u\setminus J_u^1} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}\Big), \tag{0.0.18}$$

where λ_u, whose technical definition is given in (1.4.15), is a positive measure playing the role of $|Eu|$ for functions in $\mathrm{GBD}(\Omega)$ and $J_u^\sigma := \{x \in J_u \colon |[u](x)| \geq \sigma\}$ for $\sigma > 0$. They later employ this result to prove a new compactness result for functions in $\mathrm{GBD}(\Omega)$.

In this chapter, based on the joint paper with Dal Maso [93] we show how, starting from (0.0.18) and following the strategy of [68], it is possible to refine it, replacing the exponent 1 on the left-hand side with $d/(d-1)$ and the cube Q with a general Lipschitz domain. Similarly to [68], we also obtain related trace estimates.

More precisely, we show in Theorem 6.0.1 that there exist positive constants κ_Ω and K_Ω, depending only on Ω, such that for all $\sigma \geq 0$ and $u \in \mathrm{GBD}(\Omega)$ there exist Borel sets $\omega \subset \Omega$ and $\omega_\partial \subset \partial\Omega$, with

$$\mathcal{L}^d(\omega) \leq \kappa_\Omega \mathcal{H}^{d-1}(J_u^\sigma)^{d/(d-1)} \quad \text{and} \quad \mathcal{H}^{d-1}(\omega_\partial) \leq \kappa_\Omega \mathcal{H}^{d-1}(J_u^\sigma), \tag{0.0.19}$$

and an affine map $a(x) \coloneqq Ax + b$, with A skew-symmetric, such that

$$\Big(\int_{\Omega\setminus\omega} |u-a|^{d/(d-1)} \,\mathrm{d}x \Big)^{(d-1)/d} \leq K_\Omega \Big(\lambda_u(\Omega \setminus J_u) + \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} \Big), \tag{0.0.20}$$

$$\int_{\partial\Omega\setminus\omega_\partial} |u_{\partial\Omega} - a| \,\mathrm{d}\mathcal{H}^{d-1} \leq K_\Omega \Big(\lambda_u(\Omega \setminus J_u) + \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} \Big), \tag{0.0.21}$$

where $u_{\partial\Omega}$ denotes the trace of u on $\partial\Omega$ (see [89, Theorem 5.3]). Since $\mathcal{H}^{d-1}(J_u^\sigma) \to 0$ as $\sigma \to +\infty$ (see (1.4.13)) and the constants κ_Ω and K_Ω do not depend on σ, for every function $u \in \mathrm{GBD}(\Omega)$ the exceptional sets ω and ω_∂ can be made arbitrarily small by taking σ large enough. The choice of a large σ can, however, significantly increase the value of the term

$$\int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}$$

in (0.0.20) and (0.0.21), which might diverge as $\sigma \to +\infty$ (see, for instance, [89, Example 12.3]).

We conclude with a remark on the trace estimates. For a function $u \in \mathrm{GBD}(\Omega)$ the trace on the boundary might not belong to $L^1_{\mathcal{H}^{d-1}}(\partial\Omega;\mathbb{R}^d)$. To the best of our knowledge, the only previously known estimates in this direction were obtained by Tasso in [168]. They concern weighted L^1-norms, with possibly degenerate weights, and assume that $\mathcal{H}^{d-1}(J_u) < +\infty$ and $u \in L^1(\Omega;\mathbb{R}^d)$. By contrast, we here assume no restrictions on the size of J_u nor integrability of u on Ω. This comes, of course, at the expense of removing a small set $\omega_\partial \subset \partial\Omega$ and subtracting an affine function a.

Part II: Singular perturbation problems for phase transitions

The second part of this thesis is devoted to some new results concerning models for phase transitions. We consider variants of the usual Modica-Mortola functionals in which the first-order derivative is replaced by derivatives of arbitrarily high order. We first give physical motivations for the study of these problems and recall some results in the related literature.

Cahn-Hilliard, Gurtin's conjecture, & Modica-Mortola. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set representing a container filled with two immiscible fluids. In the classical description of such a system, the distribution of the two fluids within the container is represented by a function $u\colon \Omega \to \{\alpha, \beta\}$, allowed to take only the two distinct values α, β. The space of admissible configurations of the system is then given by all functions $u\colon \Omega \to \{\alpha, \beta\}$ satisfying suitable volume constraints, which prescribe the amount of the two fluids contained in Ω.

Since the fluids are immiscible, the energy of the system is concentrated on the interface separating the two phases, which is given by the jump set $J_u = \partial^*\{u = \alpha\}$, where $\partial^*\{u = \alpha\}$ denotes the reduced boundary of $\{u = \alpha\}$ (cf. Section 1.3). Neglecting the interaction with the wall of the container, the energy is then given by the perimeter energy

$$E(u) \coloneqq m\mathcal{H}^2(J_u) \quad \text{for } u \in \mathrm{BV}(\Omega; \{\alpha, \beta\}) \text{ with } \int_\Omega u = V, \tag{0.0.22}$$

where $m > 0$ is the *surface tension* and V is a fixed constant. The configurations at equilibrium are then obtained by minimising E among all possible competitors.

This model gives a sharp description of the separation between the two phases, assuming that the density of the mixture changes discontinuously from α to β. Despite its clear geometric interpretation, such a description presents some immediate drawbacks, especially from the computational point of view, since surface energies tend to be rather complicated to handle numerically.

A different approach was proposed instead by Cahn & Hilliard [60], who suggested a description in which the transition between the two phases is regarded as a continuous phenomenon taking place in a thin layer, and a fine mixture between the two fluids is allowed. In their model, a configuration of the system is described by a function $u\colon \Omega \to \mathbb{R}$, representing the local concentration of one of the two components. The values α and β still correspond to the pure phases, while the intermediate values describe mixtures of the two fluids, which occur, ideally, in a transition layer of thickness $\varepsilon > 0$. The energy (0.0.22) is replaced by an energy of the form

$$E_\varepsilon(u) \coloneqq \int_\Omega W(u)\,\mathrm{d}x + \varepsilon^2 \int_\Omega |\nabla u|^2\,\mathrm{d}x. \tag{0.0.23}$$

Here, $W\colon \mathbb{R} \to [0, +\infty)$ is a double-well potential with wells at α and β, that is, a function satisfying $W(\alpha) = W(\beta) = 0$ and $W(u) > 0$ if $u \notin \{\alpha, \beta\}$. For convenience, in what follows we assume that $\alpha = -1$ and $\beta = 1$, the prototypical example of W then being $W(t) = \frac{1}{4}(1 - t^2)^2$.

The two terms in (0.0.23) play two distinct roles. The first term favours configurations whose values are close to the wells -1 and 1, thus promoting separation of the system into two distinct phases. On the other hand, the gradient term penalises rapid transitions from one phase to the other, preventing discontinuous transitions between the phases. Their competition produces thin diffuse transition layers separating regions in which the configuration

is close to either -1 or 1. Finally, the volume constraint prescribes the relative amounts of the two phases and rules out the trivial constant minimisers.

The precise relation between the diffuse-interface model (0.0.23) and the classical sharp-interface one (0.0.22) was investigated by Gurtin [137], who conjectured that, as the thickness ε of the transition layer tends to zero, the equilibrium configurations of (0.0.23) should converge precisely to the equilibrium configurations of (0.0.22) and that the corresponding (rescaled) energies should converge as well.

This conjecture, often referred to as the *minimal-interface criterion*, was later rigorously proved by Modica [153] in terms of Γ-convergence, with a solution based on previous work of Modica & Mortola [154]. In this latter work, in the general d-dimensional case, the authors considered the rescaled energies

$$F_\varepsilon(u) \coloneqq \frac{1}{\varepsilon} E_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega W(u)\,\mathrm{d}x + \varepsilon \int_\Omega |\nabla u|^2\,\mathrm{d}x \quad \text{for } u \in \mathrm{H}^1(\Omega), \tag{0.0.24}$$

the factor $1/\varepsilon$ being necessary to obtain a nontrivial limit as $\varepsilon \to 0^+$. They showed that sequences $\{u_\varepsilon\}_\varepsilon \subset \mathrm{H}^1(\Omega)$ with equibounded energy have sharp interfaces as cluster points and that $\{F_\varepsilon\}_\varepsilon$ Γ-converge, with respect to the $L^1(\Omega)$-topology, to the functional

$$F(u) \coloneqq \begin{cases} m_1 \mathcal{H}^{d-1}(J_u) & \text{if } u \in \mathrm{BV}(\Omega; \{-1, 1\}), \\ +\infty & \text{otherwise.} \end{cases} \tag{0.0.25}$$

The constant m_1 can be characterised as given by an *optimal-profile problem* (see e.g. [5])

$$m_1 = \inf\Big\{ \int_{-\infty}^{+\infty} \big(W(u) + (u')^2\big)\,\mathrm{d}t \colon\ u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}),\ \lim_{t \to \pm\infty} u(t) = \pm 1 \Big\} > 0 \tag{0.0.26}$$

and is a crucial quantity in the proof of equicoercivity of (0.0.24) as it represents the minimum energy required to produce a transition between the two pure phases -1 and 1.

Besides its application to Gurtin's conjecture in [153], this result has found great success in the literature and has been extended in various directions. A non-exhaustive list includes models incorporating anisotropies [28, 37, 77], multiple wells [26], oscillations [85–87, 147], nonlocal perturbations [6, 94, 157, 160, 163], and higher-order singular perturbations [25, 75, 113, 123].

In the second part of this thesis, we focus on the latter type of problems and consider energies obtained by replacing the first-order gradient term in (0.0.24) with terms involving derivatives of higher order. More precisely, we investigate coercivity and Γ-convergence of energies which, in their simplest form, are given by

$$\frac{1}{\varepsilon} \int_\Omega W(u)\,\mathrm{d}x + \varepsilon^{2k-1} \int_\Omega |\nabla^{(k)} u|^2\,\mathrm{d}x \tag{0.0.27}$$

for $k \geq 2$ and $\Omega \subset \mathbb{R}^d$, $d \geq 1$. The case $k = 2$ was previously studied by Fonseca and Mantegazza [123], to which we refer the reader for further details on the model. We here mention that replacing the gradient term by a higher-order perturbation leads to diffuse interface models with better regularity and is also natural in the description of materials whose energy depends on higher-order derivatives (see, for instance, the introduction of [75]). From the mathematical point of view, however, this modification introduces new difficulties. In particular, the higher-order term does not directly control the first derivative of the configuration, and the compactness mechanisms underlying the classical Modica-Mortola

theory can no longer be applied in a straightforward way. Therefore, to deal with this problem a new approach is in order.

We mention that higher-order perturbations like in (0.0.27) have also been used to approximate free discontinuity energies [10, 39, 155, 167], in the study of the Perona-Malik model in visual reconstruction [34, 134], and in phase-field models for free discontinuity problems [23, 122], to improve the regularity of the associated phase-field variable [36] and provide numerical approximation for high-order phase-field models for brittle fracture [156]. In a different but related context, higher-order perturbations have also been used in the study of multi-well problems in elasticity [83, 139] and for higher-order materials [75, 78], and have been examined in relation to other limit procedures, such as dimension reduction [140].

Chapter 7

Higher-order singular perturbation models for phase transitions

In this chapter, based on the joint work with Brusca& Solci [54], we begin the study of coercivity properties and Γ-convergence of the functionals introduced in (0.0.28). We first focus on the one-dimensional case, in which $\Omega = I = (a, b)$ and the energy takes the form

$$F_\varepsilon(u) \coloneqq \frac{1}{\varepsilon}\int_a^b W(u)\,\mathrm{d}x + \varepsilon^{2k-1}\int_a^b |u^{(k)}|^2\,\mathrm{d}x, \qquad u \in \mathrm{H}^k(a,b). \tag{0.0.28}$$

We show that sequences with equibounded energy F_ε converge, up to subsequences, to sharp interfaces and that these functionals Γ-converge to

$$F(u) \coloneqq \begin{cases} m_k \# J_u & \text{if } u \in \mathrm{BV}((a,b);\{-1,1\}), \\ +\infty & \text{otherwise,} \end{cases} \tag{0.0.29}$$

where $\# J_u$ denotes the number of jump points in J_u (see Theorem 7.1.1). Here, the surface tension m_k is given by the optimal-profile problem

$$m_k \coloneqq \inf\Big\{ \int_{-\infty}^{+\infty} \big(W(v) + |v^{(k)}|^2\big)\,\mathrm{d}t\colon\ v \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}), \lim_{t\to\pm\infty} v(t) = \pm 1\Big\} > 0, \tag{0.0.30}$$

in accordance with the case $k = 1$ (0.0.26). As already emerges in the case $k = 2$ studied by Fonseca & Mantegazza, this result cannot be obtained by a straightforward adaptation of the arguments of Modica & Mortola, but requires new *ad hoc* ideas. Similarly to the lower-order case, the most delicate part consists of establishing the equi-coercivity of F_ε, which requires a careful analysis of sequences $\{u_\varepsilon\}_\varepsilon$ with equibounded energy.

In particular, it is necessary to study the sets

$$\{t \in I\colon\ |u_\varepsilon(t) - 1| < \eta\}, \qquad \{t \in I\colon\ |u_\varepsilon(t) + 1| < \eta\}, \tag{0.0.31}$$

that is, the sets of points where u_ε is close to one of the two wells. However, while for $k = 1$ it is sufficient to consider intervals whose endpoints belong to these sets, for $k \geq 2$ the situation is substantially more involved. Indeed, already in the second-order case studied by Fonseca & Mantegazza, in order to compare the energy on such intervals with an optimal transition problem, one must also control the derivatives of u_ε at their endpoints and, more precisely, one should require that u'_ε is small. Unfortunately, their approach, which is based on global interpolation inequalities that allow to control u'_ε in terms of u''_ε and u_ε, does

not extend directly to arbitrarily high orders of derivation and requires a suitable growth assumption on W at infinity.

To solve this issue, we instead introduce a new direct strategy that avoids any growth assumption on W. Its key feature is the use of interpolation estimates localised on intervals contained in the sets appearing in (0.0.31). More precisely, on intervals where u_ε remains close to one of the wells, we obtain simultaneous L^2-estimates for all intermediate derivatives $u_\varepsilon^{(\ell)}$, with $\ell \in \{1, \dots, k-1\}$, by interpolating between the L^2-norm of $u_\varepsilon - 1$, or $u_\varepsilon + 1$, and the L^2-norm of $u_\varepsilon^{(k)}$. This allows us to identify sufficiently many points at which u_ε is close to one of the wells and all its derivatives up to order $k-1$ are small at the natural scales. This, in turn, allows us to compare the asymptotic cost of such intervals with the value of the optimal profile problem (0.0.30).

This argument is an adaptation of a technique introduced by Solci [167] in the approximation of free discontinuity functionals by singular perturbations, extending to higher-order derivatives earlier results of Alicandro, Braides, & Gelli [10] and Alicandro & Gelli [12] (see also [39]). The same strategy was subsequently employed by Solci [166] to treat higher-order fractional singular perturbations and by Braides & Fonseca [49] in the study of the Perona-Malik functional.

Finally, after having settled the one-dimensional case, we turn to the case of higher-dimension $d > 1$. Restricting ourselves to energies (0.0.27) in which the singular perturbation is induced by the operator norm, we focus on their Γ-convergence (see Theorem 7.1.2). The presence of the operator norm in the singular perturbation greatly simplifies the proofs as it allows a more direct comparison with the one-dimensional case. The arguments we employ rely on the blow-up method of Fonseca & Müller [124] for the lower bound, and on a construction in the spirit of [123, 154] for the upper bound.

Chapter 8

Anisotropic higher-order singular perturbation models for phase transitions

In this chapter, based on the joint paper with Brusca & Trifone [55], we continue our study of higher-order singular perturbation problems by considering energies more general than (0.0.27). While the previous chapter focused on isotropic models, we consider energies of the form

$$F_\varepsilon(u) := \int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \Big) \, \mathrm{d}x \quad \text{for } u \in \mathrm{H}^k(\Omega), \tag{0.0.32}$$

with $k \geq 2$, where the coefficients q_ℓ with $\ell \in \{1, \dots, k-1\}$ are allowed to be negative, $q_k > 0$, and $|\cdot|_\ell$ denotes an arbitrary norm on the space of ℓ-tensors. As in the previous chapter, we are interested in understanding coercivity properties and Γ-convergence of such energies.

The difficulties with respect to Chapter 7 are essentially two. First, one needs to understand how to handle the possible anisotropies arising from the general tensor norms, which make slicing techniques inadequate. Secondly, and more importantly, the presence of negative terms prevents us from applying directly the compactness result established in Chapter 7.

To overcome these difficulties, we combine the approaches of Chermisi, Dal Maso, Fonseca, & Leoni [75], and of Cicalese, Spadaro, & Zeppieri in [78], in which, independently, the

case $k = 2$ is treated. In particular, in the first work, the analysis is performed also in higher dimensions, but in the isotropic case, while the latter gives a detailed description of the one dimensional case, including a careful analysis of the optimal-profile problems describing the limit surface energy density.

Adopting their approach, we prove with arguments similar to [78] new nonlinear interpolation inequalities, which make it possible to bound the terms of intermediate order $\ell \in \{1, \dots, k-1\}$ by means of the positive part of the energy

$$\int_\Omega \Big(\frac{1}{\varepsilon} W(u) + q_k \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2\Big) \mathrm{d}x,$$

at least when the coefficients q_ℓ, for $\ell \in \{1, \dots, k-1\}$, are not "too negative"(see Theorem 8.1.1). Under these hypotheses, these inequalities allow us to rely on the compactness results obtained in the previous chapter to obtain that sequences $\{u_\varepsilon\}_\varepsilon$ with equibounded energy still converge, up to subsequences, to a function $u \in \mathrm{BV}(\Omega; \{-1, 1\})$ (see Theorem 8.1.2).

These interpolation inequalities are also fundamental in the proof of the lower bound, which is obtained by means of a blow-up argument similar to the one employed in [75]. Indeed, the crucial part of our argument relies on a lemma that allows us to modify boundary conditions, whose proof requires a careful use of these inequalities to control the error terms. As for the upper bound, the recovery sequence is obtained by performing a direct construction, first carried out for characteristic functions of sets with polygonal boundary and then extended to sets of finite perimeter by density. Combining these two bounds, we show that the functionals in (0.0.32) Γ-converge in the $L^2(\Omega)$-topology to an anisotropic perimeter functional of the form

$$F(u) \coloneqq \begin{cases} \displaystyle\int_{J_u} g(\nu_u)\, \mathrm{d}\mathcal{H}^{d-1} & \text{for } u \in \mathrm{BV}(\Omega; \{-1, 1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where the strictly positive surface density $g \colon \mathbb{S}^{d-1} \to (0, +\infty)$ is determined by an asymptotic cell formula (see (8.1.4) and Theorem 8.1.3).

We conclude with a few remarks on the existing literature. In the case $k = 2$, this asymptotic analysis has already been performed for more general functionals than those considered in [75], [78], and in (0.0.32) allowing, for example, the dependence of the energy densities on the variable x. In particular, in [25], Baía, Barroso, Chermisi, & Matias studied the case $q_1 = 1$, while in the recent paper [113] Donnarumma has addressed the problem of deterministic and stochastic homogenisation of such functionals.

Finally, we mention that, simultaneously and independently, Brazke, Götzmann, & Knüpfer [51] performed the same asymptotic analysis in the one-dimensional case, taking into account a possibly negative singular perturbation described by a derivative of order $k - 1$ and using different arguments for the lower bound.

Part III: nonlocal-to-local variational problems

The last part of this thesis is devoted to the study of the asymptotic behaviour of some nonlocal functionals depending on finite differences. In recent years, these types of functionals have attracted considerable attention and have been investigated in a wide variety of settings. Compared with their local counterparts, they enjoy several advantages. Most notably, since they are formulated in terms of finite differences rather than distributional gradients, they can be defined under very mild regularity assumptions.

A central theme in their analysis is the passage from nonlocal to local models. Under suitable asymptotic regimes, one often expects a family of nonlocal energies to converge to a local functional, typically depending on distributional gradients. The main questions are then to identify the limit energy and to understand how its form is determined by the original nonlocal interactions. The theory of Γ-convergence provides a natural framework for addressing these problems and for rigorously describing the emergence of local continuum models from nonlocal ones.

In the remainder, we shall study two different instances of this phenomenon, leading to two completely different settings in the limit. In Chapter 9, we consider fractional quadratic energies perturbed by an oscillating coefficient. This introduces two different parameters into the problem. The first governs the concentration phenomenon, which is driven by a fractional singular kernel, while the second determines the frequency at which the coefficient oscillates. Our aim is to understand how these two parameters can interact in the limit procedure and how their interplay affects the resulting local energy, which is defined on a Sobolev space.

In Chapter 10, we consider instead convolution-type functionals in the spirit of Gobbino [132]. In this case, the concentration of the interaction kernels gives rise to free-discontinuity energies containing both bulk and surface contributions, whose form we shall characterise.

Chapter 9

Another look at elliptic homogenisation

In their seminal paper [42], Bourgain, Brezis & Mironescu have studied the asymptotic behaviour of Gagliardo seminorms $[u]_{W^{s,p}(\Omega)}$ as $s \to 1^-$, and in particular, in the case $p = 2$, of the seminorm $[u]_{\mathrm{H}^s(\Omega)}$ given by

$$[u]_{\mathrm{H}^s(\Omega)} \coloneqq \Big(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y \Big)^{\frac{1}{2}},$$

where Ω is a bounded open smooth subset of $\mathbb{R}^d$, $s \in (0,1)$, and the fractional Sobolev space $\mathrm{H}^s(\Omega)$ is defined as

$$\mathrm{H}^s(\Omega) \coloneqq \{u \in L^2(\Omega) \colon\ [u]_{\mathrm{H}^s(\Omega)} < +\infty\}.$$

Their results, subsequently extended by Ponce [162], imply that the functionals

$$F_s(u) = (1-s) \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y$$

Γ-converge, with respect to the $L^2(\Omega)$-convergence, to the (multiple of the) Dirichlet integral

$$\frac{\sigma_{d-1}}{2d} \int_\Omega |\nabla u|^2 \,\mathrm{d}x,$$

where σ_{d-1} is the $\mathcal{H}^{d-1}$-dimensional measure of $\mathbb{S}^{d-1}$, with domain the usual Sobolev space $\mathrm{H}^1(\Omega)$. Since the functionals are equi-coercive in $L^2(\Omega)$, by the properties of convergence of minima of Γ-convergence, minimum problems involving the Dirichlet integral can be approximated by problems involving Gagliardo seminorms, upon possibly suitably defining boundary-value problems if necessary.

This fractional-Sobolev space approximation can be extended to other problems involving elliptic integrals, such as homogenisation problems, the most classical of which is the asymptotic analysis of (isotropic) oscillating energies

$$\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2\,\mathrm{d}x, \tag{0.0.33}$$

where a is a 1-periodic function with $0 < \alpha \le a(y) \le \beta < +\infty$. The Γ-limit as $\varepsilon \to 0$ of such energies is the *homogenised* functional

$$F_{\mathrm{hom}}(u) = \int_\Omega A_{\mathrm{hom}}\nabla u \cdot \nabla u\,\mathrm{d}x,$$

where A_{hom} is a symmetric $d \times d$ matrix characterised by the *homogenisation formula*

$$A_{\mathrm{hom}}z \cdot z = \min\Big\{\int_{(0,1)^d} a(y)|z + \nabla\varphi(y)|^2\mathrm{d}y\colon\ \varphi \text{ 1-periodic}\Big\} \tag{0.0.34}$$

(see e.g. [47]). Formally, by the result of Bourgain, Brezis & Mironescu, we can define an approximation of the homogenised functional using the energies

$$F_{\varepsilon,s}(u) = (1-s)\int_\Omega\int_\Omega a\left(\frac{x}{\varepsilon}\right)\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y, \tag{0.0.35}$$

defined for $u \in \mathrm{H}^s(\Omega)$ and $\varepsilon > 0$. Indeed, upon supposing for simplicity that a be continuous, we easily see that for fixed $\varepsilon > 0$ the Γ-limit of $F_{\varepsilon,s}$ as $s \to 1^-$ is indeed

$$\frac{\sigma_{d-1}}{2d}\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2\,\mathrm{d}x,$$

which then Γ-converges to $\frac{\sigma_{d-1}}{2d}F_{\mathrm{hom}}$ as $\varepsilon \to 0^+$. Since all functionals satisfy an equi-coerciveness property in L^2 (see [161, Theorems 1.1 and 1.2]) we can use a diagonal argument (see [88]) and deduce that there exists a choice of $s = s_\varepsilon$ such that $s \to 1^-$ as $\varepsilon \to 0^+$ and the Γ-limit of $F_{\varepsilon,s}$ is still $\frac{\sigma_{d-1}}{2d}F_{\mathrm{hom}}$.

In this chapter, based on the joint paper with Braides & Brusca [46], we investigate the scales $s = s_\varepsilon$ at which this limit holds. If both ε and $1-s$ are allowed to tend to 0 simultaneously, a heuristic argument identifies the critical balance between the two parameters. Testing the Γ-limit of $F_{\varepsilon,s}$ on the natural competitors $u_\varepsilon(x) = \langle z, x\rangle + \varepsilon\varphi(x/\varepsilon)$, with φ as in the formula characterising $A_{\mathrm{hom}}z \cdot z$, one can separate the contribution of long-range interactions, at scale $|x-y| \gg \varepsilon^2$, from that of short-range ones, at scale $|x-y| \lesssim \varepsilon^2$. The contributions related to long-range interactions are asymptotically negligible precisely when $(1-s)|\log\varepsilon| \to 0$, while, under the same assumption, the short-range contributions converge to the homogenisation formula for A_{hom}, up to a small error. This suggests that, when

$$1 - s \ll |\log(\varepsilon)|^{-1} \tag{0.0.36}$$

we may construct recovery sequences optimising the oscillations of a. We will prove in Theorem 9.0.1 that under assumption (0.0.36) a *separation of the scales* ε and s occurs,

and the limit is the one computed above by letting $s \to 1^-$ first and then $\varepsilon \to 0^+$. The main argument in the proof of the lower bound is obtained by a discretization procedure (which as a byproduct also gives a different proof of the results in [162]) based on the use of Kuhn's decomposition [142] and an integration argument on the set of orthogonal bases in $\mathbb{R}^d$ borrowed from a paper by Solci [165]. Thanks to (0.0.36), we can then reduce the computation to studying the limit of oscillating energies (0.0.33) on piecewise-affine interpolations at a scale much smaller than ε, for which the known homogenisation result can be applied.

Chapter 10

Γ-convergence of nonlocal functionals to free discontinuity problems

This chapter is devoted to a class of nonlocal approximations of free discontinuity functionals. As previously mentioned, variational problems involving surface terms are particularly complex to handle numerically. This has motivated the development of a wide array of approximation procedures that allow one to replace the surface part of an energy by means of a computationally more favourable expression.

Among the best known examples there are the elliptic phase-field approximation introduced by Ambrosio & Tortorelli [19] and the nonlocal finite difference approximation proved by Gobbino [132]. In this latter case, in response to a conjecture formulated by De Giorgi in 1996, Gobbino proved that, up to multiplicative constants, the sequence of functionals defined by

$$\frac{1}{\varepsilon^d}\int_\Omega\int_\Omega \min\Big\{\frac{|u(y)-u(x)|^2}{\varepsilon^2},\frac{1}{\varepsilon}\Big\}\rho\Big(\frac{|y-x|}{\varepsilon}\Big)\mathrm{d}x\,\mathrm{d}y, \tag{0.0.37}$$

Γ-converges as $\varepsilon \to 0^+$ in the $L^1_{\mathrm{loc}}(\Omega)$ topology to the Mumford-Shah functional

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^{d-1}(J_u) \quad \text{for } u \in \mathrm{GSBV}(\Omega). \tag{0.0.38}$$

Here, the kernel $\rho\colon \mathbb{R} \to [0,+\infty)$ is any smooth kernel with compact support (see [132, Section 7] and [43, Chapter 5]). This result was later adapted by Chambolle [65], who proved a discrete version of De Giorgi's conjecture.

Gobbino & Mora [133] further discussed a variant of (0.0.37) in which more general convex-concave integrands $\phi_\varepsilon\colon [0,+\infty) \to [0,+\infty)$ are considered and slightly different scaling conditions are taken into account. The choice of the integrands $\{\phi_\varepsilon\}_\varepsilon$ determines the expression of a free discontinuity functional appearing in the limit, which is of the form

$$\int_\Omega f_{\mathrm{bulk}}(|\nabla u|)\,\mathrm{d}x + \int_{J_u} f_{\mathrm{surf}}(|[u]|)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{0.0.39}$$

for suitable integrands f_{bulk} and f_{surf}, where f_{surf} may also be superlinear near 0. Moreover, the authors prove that, for specific choices of f_{bulk} and f_{surf}, the functional (0.0.39) can be approximated via nonlocal energies associated to appropriately chosen integrands $\{\phi_\varepsilon\}_\varepsilon$, for which they provide an explicit expression.

A further variant of Gobbino's result was recently considered in [13] by Almi, Davoli, Kubin, & Tasso, who studied a variant of (0.0.37) in the setting of linearised elasticity and showed its Γ-convergence to the Griffith functional (0.0.5).

The aim of this chapter, based on the joint paper with Brusca, Scalabrino, Trifone, & Voglino [53], is to extend the range of applicability of the result of Gobbino by considering

more general energies of the form

$$F_\varepsilon(u) \coloneqq \frac{1}{\varepsilon^d}\int_\Omega\int_\Omega f_\varepsilon\Big(x, \frac{y-x}{\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}y \quad \text{for } u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k), \tag{0.0.40}$$

where the integrands $f_\varepsilon\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^k \to [0,+\infty)$ satisfy suitable assumptions and growth conditions making the resulting energies comparable with (0.0.37).

To some extent, our energies may be regarded as a continuum counterpart of those considered in [24] (see also the related work [9]), where the authors identify sufficient conditions implying the Γ-convergence of certain discrete energies to free discontinuity functionals as in (0.0.41) below.

Similarly to Chapters 2 and 4, we prove a compactness result, which states that, up to subsequences, every sequence $\{F_\varepsilon\}_\varepsilon$ as in (0.0.40) Γ-converges in the L^1_{loc}-topology to a functional F given by

$$F(u) = \int_\Omega f_{\mathrm{bulk}}(x,\nabla u)\,\mathrm{d}x + \int_{J_u} f_{\mathrm{surf}}(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for } u \in \mathrm{GSBV}^2(\Omega;\mathbb{R}^k), \tag{0.0.41}$$

where the integrands f_{bulk} and f_{surf} are determined by auxiliary minimisation problems on small cubes (see Theorem 10.2.3). This is proved by relying on the localisation method of Γ-convergence and on techniques borrowed from [8], that allow us to restrict our attention to short-range interactions when computing Γ-limits. We later investigate the relation between Γ-convergence and the asymptotic behaviour of the minima of minimisation problems with Dirichlet boundary conditions. Using this relation, we prove necessary and sufficient conditions for the Γ-convergence of sequences of functionals as in (0.0.40) to free discontinuity functionals as in (0.0.41) (see Theorem 10.2.4).

CHAPTER 1

Notation and preliminary results

In this chapter, we introduce the notation used throughout the work and recall some definitions and preliminary results that will be used in what follows. Particular attention is devoted to the various spaces of functions of bounded variation/deformation that are the protagonists of the present work.

The chapter is structured as follows. In Section 1.1, we establish the basic notation used throughout the manuscript. We then recall in Section 1.2 the definition of Γ-convergence (in a metric space) and collect some of the properties of Γ-limits. In Sections 1.3 and 1.4 we introduce various spaces of functions; the first section is devoted to spaces of (generalised) functions of bounded variation, whereas the latter deals with (generalised) functions of bounded deformation. Finally, we conclude with Section 1.5, where some integral representation results for functionals over such spaces are presented.

1.1 Basic notation

We first fix the basic notation that we adopt throughout the work.

Linear Algebra. We fix once and for all two integers $d, k \geq 1$, which are, usually, the dimension of the domain and of the range of the functions we work with. In this section alone, $n, m \in \mathbb{N}$ denote two arbitrary integers that replace d and k when we want to give a definition for arbitrary integer dimensions.

We endow the ambient space $\mathbb{R}^n$ with the Euclidean scalar product, denoted by $\cdot$, while the Euclidean norm of $\mathbb{R}^n$ is denoted by $|\cdot|$. Given $x \in \mathbb{R}^n$, the i-th component of x is denoted by x_i and we denote by e_i the i-th vector of the canonical basis of $\mathbb{R}^n$.

The vector space $\mathbb{R}^{n\times m}$ is identified with the space of $n \times m$ matrices. For every matrix $A \in \mathbb{R}^{n\times m}$, its ij-th component is denoted by A_{ij}. If $A \in \mathbb{R}^{n\times m}$ and $\xi \in \mathbb{R}^m$, the vector $A\xi \in \mathbb{R}^n$ is defined via the standard rules of matrix multiplication. For any two vectors $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ their tensor product $\xi \otimes \eta \in \mathbb{R}^{n\times m}$ is the matrix whose ij component is given by $(\xi \otimes \eta)_{ij} \coloneqq \xi_i \eta_j$. We endow $\mathbb{R}^{n\times m}$ (and any subspace) with the Frobenius norm, which, for $A \in \mathbb{R}^{n\times m}$, is defined by

$$|A| \coloneqq \Big(\sum_{i=1}^{n} \sum_{j=1}^{m} |A_{ij}|^2 \Big)^{1/2}.$$

In the special case $n = m = d$, the symbol $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ (resp. $\mathbb{R}^{d\times d}_{\mathrm{skew}}$) denotes the space of all $d \times d$-symmetric (resp. skew-symmetric) matrices, that is, the space of those matrices $A \in \mathbb{R}^{d\times d}$ such that $A = A^T$ (resp. $A = -A^T$), where A^T is the transpose of A. For any $A \in \mathbb{R}^{d\times d}$ we let $A^{\mathrm{sym}} \coloneqq \frac{1}{2}(A + A^T) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. For any two vectors $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^d$ we let $\xi \odot \eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be the symmetric matrix defined by $\xi \odot \eta \coloneqq (\xi \otimes \eta)^{\mathrm{sym}}$. We also let $\mathrm{SO}(n) \coloneqq \{A \in \mathbb{R}^{n\times n} : AA^T = \mathrm{Id} \text{ and } \det(A) = 1\}$, where Id denotes the identity matrix.

Rectangles, cubes, and spheres. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1} \coloneqq \{\nu \in \mathbb{R}^n \colon |\nu| = 1\}$. We also set $\mathbb{S}^{n-1}_{\pm} \coloneqq \{\nu \in \mathbb{S}^{n-1} \colon \pm\nu_{i(\nu)} > 0\}$, where $i(\nu) \in \{1, \dots, n\}$ is the largest index such that $\nu_{i(\nu)} \neq 0$.

For every $\rho > 0$ and $x \in \mathbb{R}^d$ the open ball in $\mathbb{R}^d$ of centre x and radius ρ is denoted by $B_\rho(x)$, while $Q(x,\rho) \coloneqq x + \big(-\frac{\rho}{2}, \frac{\rho}{2}\big)^d$. Moreover, if $\nu \in \mathbb{S}^{d-1}$, the cube $Q_\nu(x,\rho)$ is a cube obtained by rotating $Q(x,\rho)$ in such a way that two of the faces of $Q_\nu(x,\rho)$ are orthogonal to ν.

For reasons related to the applications to stochastic homogenisation of Chapters 2 and 4, it will be important that the rotation defining the cubes $Q_\nu(x,\rho)$ satisfies certain given properties. In particular, for every $\nu \in \mathbb{S}^{d-1}$, we fix once and for all a rotation $R_\nu \in \mathrm{SO}(d)$ with the following properties:

(I1) $R_\nu(e_d) = \nu$ for all $\nu \in \mathbb{S}^{d-1}$;

(I2) $R_{e_d} = \mathrm{Id}$;

(I3) the restrictions of $\nu \mapsto R_\nu$ to $\mathbb{S}^{d-1}_\pm$ are continuous;

(I4) $R_{-\nu}(Q(0,\rho)) = R_\nu(Q(0,\rho))$ for all $\nu \in \mathbb{S}^{d-1}$ and $\rho > 0$.

We refer to [58, Example A.1] for an example of such a map. In Chapters 2 and 4 we shall assume that for every $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $\rho > 0$, the cube $Q_\nu(x,\rho)$ is given by

$$Q_\nu(x,\rho) \coloneqq x + R_\nu\Big(\Big(-\frac{\rho}{2}, \frac{\rho}{2}\Big)^d\Big). \tag{1.1.1}$$

Given $\lambda \geq 1$, we shall also consider the rectangle $Q^\lambda_\nu(x,\rho)$ defined as

$$Q^\lambda_\nu(x,\rho) \coloneqq x + R_\nu\Big(\Big(-\frac{\lambda\rho}{2}, \frac{\lambda\rho}{2}\Big)^{d-1} \times \Big(-\frac{\rho}{2}, \frac{\rho}{2}\Big)\Big). \tag{1.1.2}$$

Sets and Measure theory. Throughout the work, $\Omega \subset \mathbb{R}^d$ will always be a bounded open set. We stress that no *a priori* assumption on its regularity is made and that, when further properties are assumed, this will be clearly stated.

If $O \subset \mathbb{R}^n$ is an open set, $\mathcal{O}(O)$ (resp. $\mathcal{B}(O)$) denotes the collection of all open (resp. Borel) subsets of O. Given $O_1, O_2 \in \mathcal{O}(\mathbb{R}^n)$, $O_1 \subset\subset O_2$ means that O_1 is relatively compact in O_2. We let $\mathcal{O}_c(O)$ denote the collection of open sets compactly contained in O.

Given a finite dimensional real normed vector space X, $\mathcal{M}_b(\Omega;X)$ is the space of all bounded Radon measures with values in X; the indication of X is omitted if $X = \mathbb{R}$. The symbol $\mathcal{M}^+_b(\Omega)$ denotes the space of all positive bounded Radon measures. Given $\mu \in \mathcal{M}_b(\Omega;X)$ and $\lambda \in \mathcal{M}^+_b(\Omega)$, $\mathrm{d}\mu/\mathrm{d}\lambda$ is the Radon-Nikodým derivative of μ with respect to λ. Given a measure $\mu \in \mathcal{M}_b(\Omega;X)$, the total variation measure of μ with respect to the norm $|\cdot|$ on X, is the Borel measure defined for a Borel set $B \subset \Omega$ by

$$|\mu|(B) \coloneqq \sup \sum_{i\in I} |\mu(B_i)|,$$

where the *supremum* is taken over all finite sets $I \subset \mathbb{N}$ and all Borel partitions $(B_i)_{i\in I}$ of B. Given a measure $\lambda \in \mathcal{M}^+_b(\Omega)$ and a Borel measurable function $f\colon \Omega \to X$, the symbol $f\lambda$ denotes the X-valued measure defined for every $B \in \mathcal{B}(\Omega)$ by

$$f\lambda(B) \coloneqq \int_B f \,\mathrm{d}\lambda.$$

The m-dimensional Lebesgue measure and the m-dimensional Hausdorff measure are denoted by $\mathcal{L}^m$ and $\mathcal{H}^m$, respectively, the dimension of the ambient space being clear from

context. If μ is a measure on Ω and $B \subset \Omega$ is a μ-measurable set, we introduce the restricted measure $\mu \llcorner B$ defined by $(\mu \llcorner B)(E) \coloneqq \mu(B \cap E)$ for every Borel set $E \in \mathcal{B}(\Omega)$. If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, we let χ_E denote the characteristic function of E, that is, the function defined by $\chi(x) = 1$ if $x \in E$ and $\chi(x) = 0$ if $x \in \mathbb{R}^n \setminus E$.

Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, we let $L^0(E;\mathbb{R}^m)$ denote the space of Lebesgue measurable functions from E to $\mathbb{R}^m$, endowed with the topology of the convergence in measure, which we recall being metrisable and separable. In particular, in the case where $\mathcal{L}^n(E) < +\infty$, a distance inducing this topology is given by

$$\mathrm{d}_E(u,v) \coloneqq \int_E |u-v| \wedge 1 \,\mathrm{d}x,$$

where $a \wedge b \coloneqq \min\{a,b\}$ for any $a, b \in \mathbb{R}$.

We say that $E \subset \mathbb{R}^n$ is $(\mathcal{H}^{n-1}, n-1)$-*rectifiable* if there exist a collection of compact sets $(K_i)_{i\in\mathbb{N}}$, a collection of $(n-1)$-dimensional C^1 manifolds $M_i \subset \mathbb{R}^n$, with $K_i \subset M_i$ and $\mathcal{H}^{n-1}(M_i) < +\infty$ for every $i \in \mathbb{N}$, and a set N_0 with $\mathcal{H}^{d-1}(N_0) = 0$ such that

$$E = N_0 \cup \big(\bigcup_{i\in\mathbb{N}} K_i\big). \tag{1.1.3}$$

We refer the reader to [18, Chapter 2] and to [118, Chapter 3] for the properties of these sets.

Slicing. For every $\xi \in \mathbb{R}^d \setminus \{0\}$, $y \in \mathbb{R}^d$, and $E \subset \Omega$ we define

$$E_y^\xi \coloneqq \{t \in \mathbb{R} : y + t\xi \in E\}.$$

Given a function $u\colon E \to \mathbb{R}^k$, we define the slice in direction ξ of u as the function $u_y^\xi\colon \mathbb{R} \to \mathbb{R}^k$ defined by

$$u_y^\xi(t) \coloneqq \begin{cases} u(y+t\xi) & \text{if } t \in E_y^\xi, \\ 0 & \text{otherwise.} \end{cases}$$

When $k = d$, we will also consider the function $\hat{u}_y^\xi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{u}_y^\xi(t) \coloneqq \begin{cases} u(y+t\xi)\cdot\xi & \text{if } t \in E_y^\xi, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.4}$$

Π_x^ξ denotes the hyperplane orthogonal to ξ and passing through $x \in \mathbb{R}^d$, that is,

$$\Pi_x^\xi \coloneqq \{y \in \mathbb{R}^d\colon\ (y-x)\cdot\xi = 0\}, \tag{1.1.5}$$

the notation of x being omitted if $x = 0$. The projection map onto Π^ξ is denoted by $\pi^\xi\colon \mathbb{R}^d \to \Pi^\xi$.

1.2 Γ-convergence

In this section, we give a brief overview of Γ-convergence, using [88] as main reference (see also [44, 45]), to which we refer the interested reader for an exhaustive introduction to the topic. Although the notion of Γ-convergence makes perfect sense for sequences of functionals defined over a topological space X, due to the applications we have in mind,

we avoid presenting it in such generality and limit ourselves to treat the case where X is a metric space.

In the rest of the section, we fix a metric space (X, d) and a sequence of functionals $F_n \colon X \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} =: \overline{\mathbb{R}}$.

Definition 1.2.1 (Γ-convergence)**.** The sequence $\{F_n\}_n$ is said to Γ-converge (with respect to the topology induced by d) to a functional $F \colon X \to \overline{\mathbb{R}}$ if and only if the following two conditions hold simultaneously:

- (Γ-liminf inequality) for every $x \in X$ and for every $\{x_n\}_n \subset X$, with $x_n \to x$, it holds
$$F(x) \leq \liminf_{n \to +\infty} F_n(x_n);$$
- (Γ-limsup inequality) for every $x \in X$ there exists $\{x_n\}_n \subset X$, with $x_n \to x$, such that
$$\limsup_{n \to +\infty} F_n(x_n) \leq F(x).$$

An equivalent characterisation of Γ-convergence can be given in terms of the Γ-liminf and Γ-limsup, which, for $x \in X$, are given by

$$F'(x) \coloneqq \Gamma\text{-}\liminf_{n \to +\infty} F_n(x) \coloneqq \inf \Big\{ \liminf_{n \to +\infty} F_n(x_n) \colon \{x_n\}_n \subset X \text{ with } x_n \to x \Big\},$$
$$F''(x) \coloneqq \Gamma\text{-}\limsup_{n \to +\infty} F_n(x) \coloneqq \inf \Big\{ \limsup_{n \to +\infty} F_n(x_n) \colon \{x_n\}_n \subset X \text{ with } x_n \to x \Big\}.$$

It can be shown (see [88, Proposition 6.8]) that the Γ-liminf and Γ-limsup are always lower semicontinuous with respect to the topology induced by the distance d.

Proposition 1.2.2. *$\{F_n\}$ Γ-converges to a functional F if and only if*

$$F'(x) = F''(x) \quad \text{for every } x \in X.$$

In this case, F coincides with the common value of F' and F''. In particular, F is lower semicontinuous with respect to the topology induced by the distance d.

We now mention two properties of Γ-convergence which will be used at several points of this work. The first is a useful compactness principle, whose proof can be found in [88, Theorem 8.5], which states that, in a separable metric space, every sequence of functionals is pre-compact with respect to Γ-convergence.

Proposition 1.2.3. *Assume that the metric space X is separable. Then there exists a subsequence, not relabelled, such that $\{F_n\}_n$ Γ-converges to some functional F.*

The following result shows that Γ-convergence satisfies the Urysohn property of convergence structures (see [88, Proposition 8.3]). Together with the previous compactness result, this is one of the key tools of the localisation method of Γ-convergence (see [45] and Chapters 2, 4, and 10).

Proposition 1.2.4 (Urysohn property)**.** *The sequence $\{F_n\}_n$ Γ-converges to F if and only if for every subsequence of $\{F_n\}_n$ there exists a further subsequence Γ-converging to F.*

We also state a useful density principle, which enables us to reduce the proof of the Γ-limsup inequality to subsets of X. For a proof of this fact, we refer the reader, for instance, to [45, Remark 2.8].

Proposition 1.2.5. *Let $F\colon X \to \overline{\mathbb{R}}$ and let $X_0 \subset X$ be a subset with the property that, for any $x \in X$, there exists a sequence $\{x_n\} \subset X$ converging to x as $n \to +\infty$ such that*

$$\lim_{n\to+\infty} F(x_n) = F(x).$$

Then,

$$F''(x_0) \leq F(x_0) \quad \text{for every } x_0 \in X_0 \implies F''(x) \leq F(x) \quad \text{for every } x \in X.$$

We conclude this section by stating the so-called *Fundamental Theorem of* Γ*-convergence*, which shows that, under equi-coercivity of the sequence $\{F_n\}_n$, Γ-convergence implies convergence of minima and of minimisers. We recall that $\{F_n\}_n$ is said to be equi-coercive if $\sup_n F_n(u_n) < +\infty$ implies that $\{u_n\}_n$ is pre-compact in X.

Theorem 1.2.6 (Fundamental theorem of Γ-convergence)**.** *Assume that $\{F_n\}_n$ is a sequence of equi-coercive functionals Γ-converging to F as $n \to +\infty$. Then,*

$$\lim_{n\to+\infty} \inf\{F_n(x)\colon\ x \in X\} = \min\{F(x)\colon\ x \in X\}$$

Moreover, if $\{x_n\}_n$ is a precompact sequence such that

$$\lim_{n\to+\infty} F_n(x_n) = \min\{F(x)\colon\ x \in X\},$$

then every limit point of $\{x_n\}_n$ is a minimiser of F.

1.3 Functions of (generalised) bounded variation

In this section, we introduce various spaces of functions of bounded variation and recall some of their main properties, using the monographs [18] and [117] as references. For the properties concerning the space $\mathrm{GBV}_\star$ (see Definition 1.3.16) we refer instead to [99], the author's paper [111], and the joint paper of the author with Friedrich [112].

Functions of bounded variation. The space of functions of bounded variation $\mathrm{BV}(\Omega;\mathbb{R}^k)$ is the space of all functions $u \in L^1(\Omega;\mathbb{R}^k)$ such that their distributional gradient Du belongs to $\mathcal{M}_b(\Omega;\mathbb{R}^{k\times d})$. It is a well-known fact that the measure Du can be decomposed as

$$Du = \nabla u \mathcal{L}^d + D^c u + ([u] \otimes \nu_u)\mathcal{H}^{d-1} \llcorner J_u,$$

where

(i) $\nabla u \in L^1(\Omega;\mathbb{R}^{k\times d})$ is the Radon-Nikodým derivative of Du with respect to $\mathcal{L}^d$,

(ii) $D^c u$, called the *Cantor part* of Du, is a measure singular with respect to $\mathcal{L}^d$ and vanishing on all Borel sets $B \subset \Omega$ that are σ-finite with respect to $\mathcal{H}^{d-1}$,

(iii) the $(d-1)$-rectifiable set J_u is the *jump set* of u, $[u] \coloneqq u^+ - u^-$ is the difference of the unilateral traces u^+, u^- of u on J_u, and ν_u is the unit normal at J_u.

Moreover, ∇u is an approximate gradient of u, that is, for $\mathcal{L}^d$-a.e. $x \in \Omega$ the matrix $\nabla u(x)$ satisfies

$$\lim_{\rho\to0^+} \frac{1}{\rho^d} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{|y-x|}\,\mathrm{d}y = 0. \tag{1.3.1}$$

For the precise definitions of all the above-mentioned objects, we refer the reader to [18, Chapter 3].

The subspace of $\mathrm{BV}(\Omega;\mathbb{R}^k)$ of all functions u with $D^c u = 0$ is denoted by $\mathrm{SBV}(\Omega;\mathbb{R}^k)$. We will sometimes consider the space $\mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ (resp. $\mathrm{SBV}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$) which is defined as the space of those $u \in L^0(\Omega;\mathbb{R}^k)$ such that $u|_O \in \mathrm{BV}(O;\mathbb{R}^k)$ (resp. $\mathrm{SBV}(O;\mathbb{R}^k)$) for any open set $O \subset\subset \Omega$.

The following remarkable rigidity property of $D^c u$ was proved by Alberti [4, Corollary 4.6] (see also [107, Theorem 1.5] and [148]).

Theorem 1.3.1 (Alberti's rank-one Theorem)**.** *Let $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$. Then there exist a Borel set $C(u) \subset \Omega$ with $|D^c u|(\Omega) = |D^c u|(C(u))$ and Borel functions $a_u\colon \Omega \to \mathbb{R}^k$, $b_u\colon \Omega \to \mathbb{R}^d$ such that $|a_u \otimes b_u| = 1$ on $C(u)$ and*

$$\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|} = a_u \otimes b_u \quad \text{on } C(u). \tag{1.3.2}$$

The next proposition characterises the relationship between functions in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ and their one-dimensional slices.

Proposition 1.3.2. *Let $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $\xi \in \mathbb{S}^{d-1}$. Then the following hold*

(a) *$u_y^\xi \in \mathrm{BV}(\Omega_y^\xi;\mathbb{R}^k)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$;*

(b) *for every Borel set $B \subset \Omega$ we have*

$$Du(B)\xi = \int_{\Pi^\xi} Du_y^\xi(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) \quad \text{and} \quad |Du\xi|(B) = \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y);$$

(c) *$\nabla u(y+t\xi)\xi = \nabla u_y^\xi(t)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ and for $\mathcal{L}^1$-a.e. $t \in \Omega_y^\xi$;*

(d) *for every Borel set $B \subset \Omega$ we have*

$$D^c u(B)\xi = \int_{\Pi^\xi} D^c u_y^\xi(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y);$$

(e) *for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have*

$$J_{u_y^\xi} = (J_u)_y^\xi \quad \text{and} \quad (u^\pm)_y^\xi(t) = (u_y^\xi)^\pm(t) \quad \text{for all } t \in (J_u)_y^\xi = J_{u_y^\xi}, \tag{1.3.3}$$

where the normals at J_u and $J_{u_y^\xi}$ are oriented in such a way that $\nu_u \cdot \xi \geq 0$ and $\nu_{u_y^\xi} = 1$.

Moreover, if $u \in L^1(\Omega;\mathbb{R}^k)$ and there exists a basis $\{\xi_1,\dots,\xi_d\}$ of $\mathbb{R}^d$ such that $u_y^{\xi_i} \in \mathrm{BV}(\Omega_y^\xi;\mathbb{R}^k)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^{\xi_i}$ and for all $i \in \{1,\dots,d\}$, and

$$\int_{\Pi^{\xi_i}} |Du_y^{\xi_i}|(\Omega_y^{\xi_i})\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty \quad \text{for all } i \in \{1,\dots,d\},$$

then $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$.

We also recall a Chain Rule for the composition of a smooth and a BV-function (see [18, Proposition 3.96]).

Proposition 1.3.3 (Chain rule). *Let $n \in \mathbb{N}$, let $\Psi \in C^1_c(\mathbb{R}^k;\mathbb{R}^n)$, and let $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$. Then we have $\Psi \circ u \in \mathrm{BV}(\Omega;\mathbb{R}^n)$ and*

$$D(\Psi \circ u) = \nabla\Psi(u)\nabla u\mathcal{L}^d + \nabla\Psi(\widetilde{u})D^c u + [\Psi(u)] \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_{\Psi\circ u} \tag{1.3.4}$$

as Radon measures, where $\widetilde{u}$ denotes the precise representative of u (see (1.3.8) below).

Sets of finite perimeter. We here recall the definition and some relevant properties of sets of finite perimeter. We refer the reader to the monograph [146] for a thorough introduction to the topic.

Given an $\mathcal{L}^d$-measurable set $E \subset \Omega$, we say that E is of *finite perimeter* in Ω if $\chi_E \in \mathrm{BV}(\Omega)$ (recall that Ω is a bounded set). In this case, we call the perimeter of E in Ω, in symbols $P(E,\Omega)$ the quantity

$$P(E,\Omega) \coloneqq |D\chi_E|(\Omega).$$

The *reduced boundary* of E is defined as

$$\partial^* E \coloneqq \Big\{ x \in \mathbb{R}^d \colon \text{ there exists } \nu_E(x) \coloneqq \lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}, \text{ with } |\nu_E(x)| = 1 \Big\},$$

and the unit vector $\nu_E(x)$ stands for the inner unit normal to E at the point x. It can be shown that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E \cap \Omega$, we have that $\nu_E(x)$ coincides with the unit-normal to $J_{\chi_{E\cap\Omega}}$ at x, which, for this reason, we often identify.

We summarise the main properties we need about sets of finite perimeter in the following statements.

Theorem 1.3.4. *If E is a set of finite perimeter in Ω, then $P(E,\Omega) = |D\chi_E|(\Omega) = \mathcal{H}^{d-1}(\partial^* E \cap \Omega)$. Moreover, setting $H^+(x) \coloneqq \{y \in \mathbb{R}^d \colon y \cdot \nu_E(x) > 0\}$, for every point $x \in \partial^* E \cap \Omega$, it holds that*

$$\chi_{\frac{E-x}{\rho}} \text{ converges to } \chi_{H^+(x)} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{as } \rho \to 0^+,$$
$$\frac{\mathcal{H}^{d-1}(\partial^* E \cap Q_{\nu_E(x)}(x,\rho))}{\rho^{d-1}} \text{ converges to } 1 \quad \text{as } \rho \to 0^+.$$

We also recall a well-known result concerning the approximation of sets of finite perimeter via polyhedral sets, see, e.g., [1, Lemma 3.6] and [26, Lemma 3.1].

Theorem 1.3.5. *Let E be a set of finite perimeter in $\Omega \subset \mathbb{R}^d$, $d \geq 2$. There exists a sequence $\{E_n\}_n$ of polyhedral sets satisfying*

(a) $\chi_{E_n} \to \chi_E$ *in* $L^1(\Omega)$ *as* $n \to \infty$,

(b) $P(E_n,\Omega) \to P(E,\Omega)$ *as* $n \to +\infty$,

(c) $\mathcal{H}^{d-1}(\partial^* E_n \cap \partial\Omega) = 0$ *for every* $n \in \mathbb{N}$.

Moreover, for every non-negative continuous function g on $\mathbb{S}^{d-1}$, there holds

$$\lim_{n\to+\infty} \int_{\partial^* E_n \cap \Omega} g(\nu_{E_n}(x))\, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\partial^* E \cap \Omega} g(\nu_E(x))\, \mathrm{d}\mathcal{H}^{d-1}(x). \tag{1.3.5}$$

We also mention that the last part of Proposition 1.3.2 yields a characterization of functions of bounded variation with values in $\{-1,1\}$ in terms of their slices.

Proposition 1.3.6. *A function $u \in L^1(\Omega;\{-1,1\})$ belongs to $\mathrm{BV}(\Omega;\{-1,1\})$ if and only if for every $i \in \{1,\dots,d\}$ the functions $u_y^{e_i}$ belong to $\mathrm{BV}(\Omega_y^{e_i};\{-1,1\})$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^{e_i}$, and*

$$\int_{\Pi^{e_i}} \mathcal{H}^0(J_{u_y^{e_i}} \cap \Omega_y^{e_i})\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty.$$

Generalised functions of bounded variation. We now present the space of *generalised* functions of bounded variation $\mathrm{GBV}(\Omega;\mathbb{R}^k)$ and its subspace of *generalised special* functions of bounded variation $\mathrm{GSBV}(\Omega;\mathbb{R}^k)$, introduced by De Giorgi & Ambrosio in [103] and whose properties have been systematically investigated by Ambrosio in [16].

Definition 1.3.7 ($\mathrm{GBV}(\Omega;\mathbb{R}^k)$)**.** The space $\mathrm{GBV}(\Omega;\mathbb{R}^k)$ (resp. $\mathrm{GSBV}(\Omega;\mathbb{R}^k)$) is the space of $u \in L^0(\Omega;\mathbb{R}^k)$ with the property that $\Phi \circ u \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ (resp. $\mathrm{SBV}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$) for any $\Phi\colon \mathbb{R}^k \to \mathbb{R}^k$ Lipschitz whose gradient has compact support. We recall that $\mathrm{GBV}(\Omega;\mathbb{R}^k)$ and $\mathrm{GSBV}(\Omega;\mathbb{R}^k)$ are not vector spaces, and that they do not coincide with the spaces $\mathrm{GBV}(\Omega)^k$ and $\mathrm{GSBV}(\Omega)^k$ (see, for instance, [18, Example 4.27]).

Moreover (see [158]), in the scalar-valued case $k = 1$ $u \in \mathrm{GBV}(\Omega)$ (resp. $\mathrm{GSBV}(\Omega)$) if and only if $u^{(m)} \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ (resp. $\mathrm{SBV}_{\mathrm{loc}}(\Omega)$) for every $m > 0$, where for $v \in L^0(\Omega;\mathbb{R}^k)$ we have set

$$v^{(m)} \coloneqq \big((v_1 \vee -m) \wedge m, \dots, (v_k \vee -m) \wedge m\big). \tag{1.3.6}$$

It is possible to show that the space $\mathrm{GBV}(\Omega;\mathbb{R}^k)$ retains many of the fine properties of $\mathrm{BV}(\Omega;\mathbb{R}^k)$ necessary to study free discontinuity problems. In particular, functions in this space admit an approximate gradient and satisfy many of the trace properties related to the jump set, although in a slightly weaker sense than for functions of bounded variation. Moreover, it is possible to associate to every such function a (possibly unbounded) positive measure which generalises the total variation of the Cantor part of the derivative of a function of bounded variation. To make all previous claims more precise, we introduce some further notation, related to the definition of approximate gradient and of jump set for Lebesgue measurable functions.

Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ be such that

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(E \cap B_\rho(x))}{\rho^n} > 0. \tag{1.3.7}$$

We say that an $\mathcal{L}^n$-measurable function $u\colon E \to \mathbb{R}^m$ has approximate limit $\widetilde{u}(x) \in \mathbb{R}^m$ at x, in symbols

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in E}} u(y) = \widetilde{u}(x), \tag{1.3.8}$$

if for every $\varepsilon > 0$ we have

$$\lim_{\rho\to 0^+} \frac{\mathcal{L}^n(\{y \in E\colon\ |u(y) - \widetilde{u}(x)| > \varepsilon\} \cap B_\rho(x))}{\rho^n} = 0.$$

Throughout this thesis, the symbol $\widetilde{u}(x)$ is always used to denote the approximate limit of u considered in (1.3.8). By (1.3.7) the vector $\widetilde{u}(x)$ is uniquely defined. The set S_u of *points of approximate discontinuity* is then defined as the complement in E of the set of points where the approximate limit exists. It is well-known that for an $\mathcal{L}^n$-measurable function $u\colon \mathbb{R}^n \to \mathbb{R}^m$ it holds $\mathcal{L}^n(S_u) = 0$.

Definition 1.3.8 (Approximate gradient)**.** If $O \in \mathcal{O}(\mathbb{R}^n)$ and $u \in L^0(O;\mathbb{R}^m)$, we say that u is *approximately differentiable* at $x \in O$, with approximate gradient $\nabla u(x) \in \mathbb{R}^{m\times n}$, if it holds

$$\operatorname{ap}\lim_{\substack{y\to x\\ y\in O}} \frac{u(y)-\widetilde{u}(x)-\nabla u(x)(y-x)}{|y-x|}=0. \tag{1.3.9}$$

An approximate gradient for u is a function $\nabla u \in L^0(O;\mathbb{R}^{m\times n})$ such that (1.3.9) holds for $\mathcal{L}^n$-a.e. $x \in O$, which clearly is uniquely defined up to $\mathcal{L}^n$-negligible sets.

Definition 1.3.9. Let $O \in \mathcal{O}(\mathbb{R}^n)$ and $u \in L^0(O;\mathbb{R}^m)$. The *jump set* J_u is the set of all points $x \in O$ for which there exists $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{n-1}$, with $u^+(x) \neq u^-(x)$, such that

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in H^{\pm}(x)\cap O}} u(y) = u^{\pm}(x),$$

where $H^{\pm}(x) \coloneqq \{y \in \mathbb{R}^n : \pm(y-x)\cdot\nu_u(x) > 0\}$. The triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely defined up to swapping $u^+(x)$ and $u^-(x)$ and changing the sign of $\nu_u(x)$. Given $x \in J_u$, we set $[u](x) \coloneqq u^+(x) - u^-(x)$. For $r > 0$ we also introduce J_u^r as the set defined by

$$J_u^r \coloneqq \{x \in J_u \colon |[u](x)| \geq r\}. \tag{1.3.10}$$

We recall that, remarkably, Del Nin proved in [108] that the jump set J_u of any $u \in L^0(\mathbb{R}^n;\mathbb{R}^m)$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable (see (1.1.3)).

In the following, it will be convenient to consider a particular family of *smooth* truncation functions. For this reason, we let $\Phi \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ be a function satisfying the following conditions (see, for an example, [57, Section 4] or [24, Remark 2.3]):

$$\begin{cases} \Phi(y) = y & \text{for } y \in \{|y| \leq 1\},\\ \Phi(y) = 0 & \text{for } y \in \{|y| \geq \sigma\},\\ \operatorname{Lip}(\Phi) \leq 1, \end{cases} \tag{1.3.11}$$

for a fixed constant $\sigma > 2$ to be chosen precisely later. For every $R > 0$ we set

$$\Phi_R(y) \coloneqq R\Phi\big(\frac{y}{R}\big) \quad \text{for all } y \in \mathbb{R}^k, \tag{1.3.12}$$

which clearly satisfies

$$\begin{cases} \Phi_R(y) = y & \text{for } y \in \{|y| \leq R\},\\ \Phi_R(y) = 0 & \text{for } y \in \{|y| \geq \sigma R\},\\ \operatorname{Lip}(\Phi_R) \leq 1. \end{cases} \tag{1.3.13}$$

The following proposition collects some of the main properties of generalised functions of bounded variation. For this, observe that $\Phi_R \circ u$ lies in $\mathrm{BV}_{\mathrm{loc}}$ and thus admits the usual representation of the distributional derivative $D(\Phi_R \circ u)$.

Proposition 1.3.10. *Let $u \in \mathrm{GBV}(\Omega;\mathbb{R}^k)$. The following properties hold:*

(a) $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$*;*

(b) *u admits an approximate gradient $\nabla u \in L^0(\Omega;\mathbb{R}^{k\times d})$; moreover, $\nabla u(x) = \nabla(\Phi_R\circ u)(x)$ for $\mathcal{L}^d$-a.e. $x \in \{|u| \leq R\}$;*

(c) *for every $R > 0$ we have $J_{\Phi_R \circ u} \subset J_u$ up to a set of $\mathcal{H}^{d-1}$-measure zero and*

$$\mathcal{H}^{d-1}\big(J_u \setminus \bigcup\nolimits_{R>0} J_{\Phi_R \circ u}\big) = 0;$$

moreover, for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ we have $\Phi_R(u)^+(x) = u^+(x)$ and $\Phi_R(u)^-(x) = u^-(x)$, whenever $R > \max\{|u^+(x)|, |u^-(x)|\}$.

Moreover, if $u \in \mathrm{GBV}(\Omega)^k$, then

(d) *there exists a measure $|D^c u| \in \mathcal{M}^+(\Omega)$, possibly unbounded, such that $|D^c u|(B) = |D^c(\Phi_R \circ u)|(B)$ for every Borel set $B \subset \{|\widetilde{u}| \le R\}$ and $R > 0$, and $|D^c u|(B) = 0$ whenever $B \subset \Omega$ is a Borel set σ-finite with respect to $\mathcal{H}^{d-1}$; in addition, we have*

$$|D^c u|(B) = \lim_{R \to +\infty} |D^c(\Phi_R \circ u)|(B) = \sup_{R>0} |D^c(\Phi_R \circ u)|(B) \in [0, \infty]$$

for all Borel sets $B \subset \Omega$.

Remark 1.3.11. Properties (b) and (d) of Proposition 1.3.10 can be strengthened in the scalar case $k = 1$. Indeed, given $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$, we also have

$$|\nabla u^{(m_1)}| \le |\nabla u^{(m_2)}| \quad \mathcal{L}^d\text{-a.e. in } \Omega \quad \text{and} \quad |D^c u^{(m_1)}|(B) \le |D^c u^{(m_2)}|(B),$$
$$J_{u^{(m_1)}} \subset J_{u^{(m_2)}} \quad \text{and} \quad J^1_{u^{(m_1)}} \subset J^1_{u^{(m_2)}} \quad \text{up to } \mathcal{H}^{d-1}\text{-negligible sets},$$
$$|[u^{(m_1)}]| \le |[u^{(m_2)}]| \quad \mathcal{H}^{d-1}\text{-a.e. on } J_{u^{(m_1)}},$$

for all Borel sets $B \subset \Omega$.

The following proposition shows that, with due adaptations, many of the slicing properties described in Proposition 1.3.2 still hold in GBV.

Proposition 1.3.12. *Let $u \in \mathrm{GBV}(\Omega; \mathbb{R}^k)$ and $\xi \in \mathbb{S}^{d-1}$. Then the following hold*

(a) *$u^\xi_y \in \mathrm{GBV}(\Omega^\xi_y; \mathbb{R}^k)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$;*

(b) *$\nabla u(y + t\xi)\xi = \nabla u^\xi_y(t)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ and for $\mathcal{L}^1$-a.e. $t \in \Omega^\xi_y$;*

(c) *for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have*

$$J_{u^\xi_y} = (J_u)^\xi_y \quad \text{and} \quad J^r_{u^\xi_y} = (J^r_u)^\xi_y \quad \text{for all } r \ge 0, \tag{1.3.14}$$
$$(u^\pm)^\xi_y(t) = (u^\xi_y)^\pm(t) \quad \text{for all } t \in (J_u)^\xi_y = J_{u^\xi_y}, \tag{1.3.15}$$

where the normals at J_u and $J_{u^\xi_y}$ are oriented in such a way that $\nu_u \cdot \xi \ge 0$ and $\nu_{u^\xi_y} = 1$.

For future use, we also introduce the space(s) $(\mathrm{G})\mathrm{SBV}^2(\Omega; \mathbb{R}^k)$, whose properties have been systematically investigated in [96, Section 2].

Definition 1.3.13 ($(\mathrm{G})\mathrm{SBV}^2(\Omega; \mathbb{R}^k)$)**.** The space $(\mathrm{G})\mathrm{SBV}^2(\Omega; \mathbb{R}^k)$ consists of all functions $u \in (\mathrm{G})\mathrm{SBV}(\Omega; \mathbb{R}^k)$ with $\nabla u \in L^2(\Omega; \mathbb{R}^{k \times d})$ and $\mathcal{H}^{d-1}(J_u) < +\infty$.

The additional estimates in the definition of $\mathrm{GSBV}^2(\Omega; \mathbb{R}^k)$ allow to improve upon some of the structural shortcomings of the space $\mathrm{GBV}(\Omega; \mathbb{R}^k)$. In particular, it was shown in [96, Proposition 2.3] that it is a vector space and that each component of a function in $\mathrm{GSBV}^2(\Omega; \mathbb{R}^k)$ belongs to $\mathrm{GSBV}^2(\Omega)$.

Lemma 1.3.14. $\mathrm{GSBV}^2(\Omega;\mathbb{R}^k)$ *is a vector space. Moreover, if* $u \in \mathrm{GSBV}^2(\Omega;\mathbb{R}^k)$ *then* $u_i \in \mathrm{GSBV}^2(\Omega)$ *for all* $i \in \{1,\dots,k\}$. *In particular,* $u^{(m)} \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ *for all* $m > 0$ (*see* (1.3.6)).

The space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$**.** We are ready to introduce one of the main characters of this thesis, the space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, introduced by Dal Maso & Toader in [99, Definition 3.1] in the scalar-valued case and then investigated by the author in [111] in the vector-valued case. As we will be mostly interested in the vector-valued setting, we present all definitions and the majority of results in this case, but refer to the original work [99] for some of their proofs, as some of them can be obtained arguing componentwise. Further results about this space will be obtained in Chapters 2 and 3.

In order to define the space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ we introduce a useful functional in some cohesive models for fracture mechanics. In particular, we let $\mathcal{V}\colon L^0(\mathbb{R}^d;\mathbb{R}^k)\times\mathcal{B}(\Omega)\to[0,+\infty]$ be the functional defined for every $O\in\mathcal{O}(\Omega)$ by

$$\mathcal{V}(u,O) \coloneqq \begin{cases} \displaystyle\sum_{i=1}^k \Big(\int_O |\nabla u_i|\,\mathrm{d}x + |D^c u_i|(O) + \int_{J_{u_i}\cap O} |[u_i]|\wedge 1\,\mathrm{d}\mathcal{H}^{d-1}\Big) & \text{if } u\in \mathrm{GBV}(O)^k,\\ +\infty & \text{otherwise} \end{cases} \tag{1.3.16}$$

(see Proposition 1.3.10). The functional $\mathcal{V}$ is then extended to all Borel sets by setting

$$\mathcal{V}(u,B) \coloneqq \inf\big\{\mathcal{V}(u,O)\colon\ O\supset B \text{ with } O\in\mathcal{O}(\Omega)\big\}.$$

Remark 1.3.15. It follows from [99, Theorem 6.1] that $\mathcal{V}$ is lower semicontinuous with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. We do not know whether a similar property would hold replacing $\mathcal{V}$ by the functional defined for $u\in\mathrm{GBV}(\Omega)^k$ and $O\subset\mathbb{R}^d$ open by

$$\mathcal{V}_{\mathrm{vec}}(u,O) \coloneqq \int_O |\nabla u|\,\mathrm{d}x + |D^c u|(O) + \int_{J_u\cap O} |[u]|\wedge 1\,\mathrm{d}\mathcal{H}^{d-1}.$$

Indeed, the standard proof of lower semicontinuity by slicing (see [38, Theorem 2.1]) does not apply to the functional $\mathcal{V}_{\mathrm{vec}}$, unless we use the operator norm in $|\nabla u|$ (see [111, Corollary 4.6]).

Note that the definition of $\mathcal{V}$ is given for functions u defined on the whole $\mathbb{R}^d$. This is particularly convenient for the results presented in Chapter 2. However, due to its local nature, we will often write, with a little abuse of notation, $\mathcal{V}(u,O)$ even when u is a function defined only on the open set O, the definition still being given by (1.3.16).

We are ready to introduce the space $\mathrm{GBV}_\star$.

Definition 1.3.16 ($\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$)**.** The space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ is the space of all functions $u\in L^0(\Omega;\mathbb{R}^k)$ such that one of the following three equivalent conditions holds:

(a) $\mathcal{V}(u,\Omega)<+\infty$, so that $\nabla u\in L^1(\Omega;\mathbb{R}^{k\times d})$, $|D^c u|(\Omega)<+\infty$, $[u]\in L^1_{\mathcal{H}^{d-1}}(J_u\setminus J_u^1;\mathbb{R}^k)$, and $\mathcal{H}^{d-1}(J_u^1)<+\infty$;

(b) $u^{(m)}\in\mathrm{BV}(\Omega;\mathbb{R}^k)$ for all $m>0$ and

$$S_1 \coloneqq \sup_{m>0}\mathcal{V}(u^{(m)},\Omega)<+\infty \tag{1.3.17}$$

and, in this case, $S_1=\mathcal{V}(u,\Omega)$;

(c) $\Phi_R \circ u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ for all $R>0$ and

$$S_2 \coloneqq \sup_{R>0} \mathcal{V}(\Phi_R \circ u, \Omega) < +\infty. \tag{1.3.18}$$

In this case, $S_2 = \mathcal{V}(u,\Omega)$.

We refer the reader to [99, Proposition 3.3] and [111, Proposition 3.7] for the proof of the equivalence of (a) and (b), while the proof of the equivalence of (b) and (c) can be deduced from Proposition 1.3.19 below. The fact that $S_1 = S_2 = \mathcal{V}(u,\Omega)$ is a consequence of the lower semicontinuity of $\mathcal{V}$ (see Remark 1.3.15).

Remark 1.3.17. Given $a,b \in \mathbb{R}^k$ with $a_i \le b_i$ for all $i \in \{1,\dots,k\}$ and $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, it can be shown using Remark 1.3.11 that $v \coloneqq (u \vee a) \wedge b \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, $\mathcal{V}(v,O) \le \mathcal{V}(u,O)$ for all $O \in \mathcal{O}(\Omega)$, as well as $|D^c v|(O) \le |D^c u|(O)$,

$$|Dv|(O \setminus J_u^r) \le (r \vee 1)\mathcal{V}(u, O \setminus J_u^r) \quad \text{ and } |Dv|(J_u^r \cap O) \le k|b-a|\mathcal{H}^{d-1}(J_u^r \cap O),$$

and $J_v^r \subset J_u^r$, up to an $\mathcal{H}^{d-1}$-negligible set, for all $r \ge 0$.

Similarly to $\mathrm{GSBV}^2(\Omega;\mathbb{R}^k)$ (see Lemma 1.3.14), the presence of the additional estimate (1.3.18) allows to prove that $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ is much better behaved than $\mathrm{GBV}(\Omega;\mathbb{R}^k)$. For instance, it is a vector space.

Theorem 1.3.18 ([99, Theorem 3.9])**.** $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ *is a vector space.*

The following proposition shows the equivalence of (b) and (c) of Definition 1.3.16.

Proposition 1.3.19 ([111, Proposition 3.9])**.** *Let $u \in L^0(\Omega;\mathbb{R}^k)$. Then u satisfies* (b) *of Definition 1.3.16 if and only if it satisfies* (c) *of the same definition. In particular,* $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k) = \mathrm{GBV}_\star(\Omega)^k$.

Proof. We begin by proving that if u satisfies (c) of Definition 1.3.16 then $u^{(m)} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ for all $m>0$ and (1.3.17) holds. To this end, for every $m \in \mathbb{N}$ we let $\{\phi_n^m\}_n \subset C^1(\mathbb{R}^k;\mathbb{R}^k)$ be a sequence converging to $y \mapsto y^{(m)}$ uniformly, with the properties that $\mathrm{Lip}(\phi_n^m) \le 1$ and $\|\phi_n^m\|_{L^\infty} \le 2m$ for every $n \in \mathbb{N}$. Furthermore, given $R>0$ and $m,n \in \mathbb{N}$, we set $v_{R,n}^m \coloneqq \phi_n^m \circ (\Phi_R \circ u) \in \mathrm{BV}(\Omega;\mathbb{R}^k)$. It follows immediately from (1.3.13) that

$$\lim_{R\to+\infty} v_{R,n}^m(x) = (\phi_n^m \circ u)(x) \qquad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega$$

for all $m,n \in \mathbb{N}$. Since $\|v_{R,n}^m\|_{L^\infty(\Omega)} \le 2m$, this implies

$$v_{R,n}^m \text{ converges to } (\phi_n^m \circ u) \text{ in } L^1(\Omega;\mathbb{R}^k) \text{ as } R \to +\infty, \tag{1.3.19}$$

which by definition of ϕ_n^m gives

$$\phi_n^m \circ u \text{ converges to } u^{(m)} \text{ in } L^1(\Omega;\mathbb{R}^k) \text{ as } n \to +\infty. \tag{1.3.20}$$

We then apply Proposition 1.3.3 to get

$$\begin{aligned}
&\nabla v_{R,n}^m = \nabla\phi_n^m(\Phi_R \circ u)\nabla(\Phi_R \circ u) \ \mathcal{L}^d\text{-a.e. in } \Omega,\\
&D^c v_{R,n}^m = \nabla\phi_n^m(\Phi_R \circ \widetilde{u})D^c(\Phi_R \circ u) \ \text{ as Borel measures in } \Omega,\\
&[v_{R,n}^m] = \phi_n^m(\Phi_R(u^+)) - \phi_n^m(\Phi_R(u^-)) \ \mathcal{H}^{d-1}\text{-a.e. in } J_{v_{R,n}^m},
\end{aligned}$$

where in the last equality we have also used that $\phi_n^m(\Phi_R(u))^\pm = \phi_n^m(\Phi_R(u^\pm))$ being both ϕ_n^m and Φ_R smooth. As $\mathrm{Lip}(\phi_n^m) \le 1$, this gives

$$\begin{aligned}
&|\nabla v_{R,n}^m| \le |\nabla(\Phi_R \circ u)| \quad \mathcal{L}^d\text{-a.e. in } \Omega,\\
&|D^c v_{R,n}^m|(B) \le |D^c(\Phi_R \circ u)|(B) \quad \text{for every Borel set } B \subset \Omega,\\
&|[v_{R,n}^m]| \le |[\Phi_R \circ u]| \quad \mathcal{H}^{d-1}\text{-a.e. in } J_{v_{R,n}^m}.
\end{aligned}$$

Together with (1.3.18), this implies that, for every $R > 0$ and $m, n \in \mathbb{N}$, we have

$$\int_\Omega |\nabla v_{R,n}^m| \,\mathrm{d}x + |D^c v_{R,n}^m|(\Omega) + \int_{J_{v_{R,n}^m}} |[v_{R,n}^m]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} \le \mathcal{V}(\Phi_R \circ u, \Omega) \le C_u < +\infty, \tag{1.3.21}$$

for a positive constant $C_u > 0$ depending only on u.

We now claim that for every $m > 0$ it holds

$$\sup_{n\in\mathbb{N}} \sup_{R>0} |Dv_{R,n}^m|(\Omega) < +\infty. \tag{1.3.22}$$

To see this, it suffices to note that, by (1.3.21) and $\|v_{R,n}^m\|_{L^\infty(\Omega)} \le 2m$, we have

$$\begin{aligned}
\int_{J_{v_{R,n}^m}} |[v_{R,n}^m]| \,\mathrm{d}\mathcal{H}^{d-1} &\le \int_{J_{v_{R,n}^m} \setminus J^1_{v_{R,n}^m}} |[v_{R,n}^m]| \,\mathrm{d}\mathcal{H}^{d-1} + 4m\mathcal{H}^{d-1}(J^1_{v_{R,n}^m})\\
&\le (4m+1) \int_{J_{v_{R,n}^m}} |[v_{R,n}^m]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1}\\
&\le (4m+1) C_u.
\end{aligned}$$

Recalling (1.3.19) and (1.3.20), inequality (1.3.22) and the lower semicontinuity of the total variation with respect to the $L^1(\Omega;\mathbb{R}^k)$-convergence imply that $u^{(m)} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ for every $m > 0$. Finally, using again (1.3.19)–(1.3.22) and the fact that $\mathrm{Lip}(\phi_n^m) \le 1$, we may apply a result of Bouchitté, Braides, & Buttazzo [38, Theorem 2.1] to each component to obtain that

$$\begin{aligned}
\mathcal{V}(u^{(m)}, \Omega) &\le \sum_{i=1}^k \liminf_{n\to+\infty} \liminf_{R\to+\infty} \Big(\int_\Omega |\nabla (v_{R,n}^m)_i| \,\mathrm{d}x + |D^c (v_{R,n}^m)_i|(\Omega) + \int_{J_{(v_{R,n}^m)_i}} |[(v_{R,n}^m)_i]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} \Big)\\
&\le k \sup_{R>0} \mathcal{V}(\Phi_R \circ u, \Omega) \le C_u,
\end{aligned}$$

which implies (1.3.17).

We now prove that if u satisfies (b) it also satisfies (c), we first observe that it is not restrictive to assume that $k = 1$. We set $v_R = \Phi_R \circ u$ and let $m > \sigma R$ (see (1.3.13)). Clearly, $v_R = \Phi_R \circ u^{(m)}$ $\mathcal{L}^d$-a.e. in Ω. As we assumed that $u^{(m)} \in \mathrm{BV}(\Omega)$, by Proposition 1.3.3 this implies $v_R \in \mathrm{BV}(\Omega)$.

We claim that

$$J_{v_R} \subset J_{u^{(m)}} \quad \text{up to an } \mathcal{H}^{d-1}\text{-negligible set.} \tag{1.3.23}$$

Indeed, by [18, Proposition 1.1 (iii)] every point $x \in \Omega$ of approximate continuity for u is also a point of approximate continuity for v_R, so that $S_{v_R} \subset S_{u^{(m)}}$. As

$$\mathcal{H}^{d-1}(S_{u^{(m)}} \setminus J_{u^{(m)}}) = \mathcal{H}^{d-1}(S_{v_R} \setminus J_{v_R}) = 0$$

by (a) of Proposition 1.3.10, the claim follows.

Using Proposition 1.3.3, (1.3.23), and the fact that $\mathrm{Lip}(\Phi_R) \le 1$ (see (1.3.13)) we see that

$$\begin{aligned}\int_\Omega |\nabla v_R|\,\mathrm{d}x + |D^c v_R|(\Omega) + \int_{J_{v_R}} |[v_R]| \wedge 1\,\mathrm{d}\mathcal{H}^{d-1}\\ \le \int_\Omega |\nabla \Phi_R(u^{(m)})\nabla u^{(m)}|\,\mathrm{d}x + |D^c u^{(m)}|(\Omega) + \int_{J_{\Phi_R(u^{(m)})}} |[\Phi_R(u^{(m)})]| \wedge 1\,\mathrm{d}\mathcal{H}^{d-1}\\ \le \int_\Omega |\nabla u^{(m)}|\,\mathrm{d}x + |D^c u^{(m)}|(\Omega) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

As we assumed (1.3.17), this implies (1.3.18), which gives $u \in \mathrm{GBV}_\star(\Omega)$. This concludes the proof. □

We already observed in Proposition 1.3.10 that it is possible to associate to every $u \in \mathrm{GBV}(\Omega)^k$ a measure $|D^c u|$ generalising the total variation of the Cantor part of the distributional gradient. It was proved by Alicandro & Focardi [11, Lemma 2.10] that if $u \in \mathrm{GBV}(\Omega)^k$ is such that $|D^c u|(\Omega) < +\infty$, then there exists a matrix-valued measure $\lambda \in \mathcal{M}_b(\Omega;\mathbb{R}^{k\times d})$ such that $|\lambda| = |D^c u|$. This is the content of the following proposition. For an alternative proof, we refer the reader to [111, Proposition 3.13] (see also [99, Theorem 2.7 and Proposition 2.9]).

Proposition 1.3.20. *Let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Then there exists a matrix-valued measure $D^c u \in \mathcal{M}_b(\Omega;\mathbb{R}^{k\times d})$ such that $D^c u(B) = D^c(\Phi_R \circ u)(B)$ for every Borel set $B \subset \{|\widetilde{u}| \le R\}$, and $D^c u(B) = 0$ whenever $B \subset \Omega$ is a Borel set σ-finite with respect to $\mathcal{H}^{d-1}$; in addition, we have*

$$D^c u(B) = \lim_{R\to+\infty} D^c(\Phi_R \circ u)(B) \quad \text{for all Borel sets } B \subset \Omega,$$
$$|D^c u|(B) = \lim_{R\to+\infty} |D^c(\Phi_R \circ u)|(B) = \sup_{R>0} |D^c(\Phi_R \circ u)|(B) \quad \text{for all Borel sets } B \subset \Omega.$$

Moreover (see [111, Lemma 3.5]),

$$|D^c u|\Big(\Omega \setminus \bigcup_{R>0} \widetilde{\Omega}_R\Big) = 0, \tag{1.3.24}$$

where $\widetilde{\Omega}_R \coloneqq \{x \in \Omega\colon \widetilde{u}(x)$ exists and $|\widetilde{u}(x)| \le R\}$.

The following proposition describes how $D^c u$ behaves under composition with the truncation function Φ_R.

Proposition 1.3.21 ([111, Proposition 3.13])**.** *Let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ and $R > 0$. Then,*

(i) *for all $\Psi \in C^1_c(\mathbb{R}^k;\mathbb{R}^k)$ we have $D^c(\Psi \circ u) = \nabla\Psi(\widetilde{u})D^c u$ as Radon measures on Ω;*

(ii) *it holds*

$$\frac{\mathrm{d}D^c(\Phi_R \circ u)}{\mathrm{d}|D^c(\Phi_R \circ u)|} = \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|} \quad |D^c u|\text{-a.e. in } \widetilde{\Omega}_R. \tag{1.3.25}$$

As a consequence, we have

$$\lim_{R\to+\infty} \frac{\mathrm{d}D^c(\Phi_R \circ u)}{\mathrm{d}|D^c(\Phi_R \circ u)|} = \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|} \quad |D^c u|\text{-a.e. in } \Omega. \tag{1.3.26}$$

Proof. To prove (i), we need to show that for every Borel set $B \subset \Omega$ we have the equality

$$D^c(\Psi \circ u)(B) = \int_B \nabla\Psi(\widetilde{u})\,\mathrm{d}D^c u. \tag{1.3.27}$$

We claim that it is enough to prove (1.3.27) when B is contained in $\widetilde{\Omega}_m$ for some $m > 0$. To see this, let us fix a Borel set $B \subset \Omega$. We recall that by part (a) of Proposition 1.3.10 we have that $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$, since J_u is σ-finite with respect to $\mathcal{H}^{d-1}$. Moreover, it is easy to check that $\Psi \circ u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$, so that

$$D^c(\Psi \circ u)(S_u) = 0.$$

Since $B \setminus S_u = \bigcup_{m>0} B \cap \widetilde{\Omega}_m$, from the previous equality, (1.3.27) applied with $B \cap \widetilde{\Omega}_m$, and (1.3.20) we obtain

$$\begin{aligned} D^c(\Psi \circ u)(B) &= D^c(\Psi \circ u)(B \setminus S_u) \\ &= \lim_{m\to+\infty} D^c(\Psi \circ u)(B \cap \widetilde{\Omega}_m) \\ &= \lim_{m\to+\infty} \int_{B\cap\widetilde{\Omega}_m} \nabla\Psi(\widetilde{u})\,\mathrm{d}D^c u = \int_B \nabla\Psi(\widetilde{u})\,\mathrm{d}D^c u, \end{aligned}$$

where in the last equality we have also used (1.3.24) and the Dominated Convergence Theorem. This proves the claim.

Let us now fix $m > \max_{y\in\mathrm{supp}(\Psi)} |y|$ and $B \subset \widetilde{\Omega}_m$. Clearly, we have $\Psi \circ u = \Psi \circ u^{(m)}$ $\mathcal{L}^d$-a.e. in Ω, so that

$$D^c(\Psi \circ u)(B) = D^c(\Psi \circ u^{(m)})(B).$$

Since $u^{(m)} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ by (b) of Definition 1.3.16, by Proposition 1.3.3 we have

$$D^c(\Psi \circ u^{(m)})(B) = \int_B \nabla\Psi(\widetilde{u^{(m)}})\,\mathrm{d}D^c u^{(m)} = \int_B \nabla\Psi(\widetilde{u}^{(m)})\,\mathrm{d}D^c u^{(m)}, \tag{1.3.28}$$

where in the second equality we have used that $\widetilde{u^{(m)}} = \widetilde{u}^{(m)}$. As $B \subset \widetilde{\Omega}_m$, from Proposition 1.3.20 it follows that $D^c u(B) = D^c u^{(m)}(B)$. Hence, in view of (1.3.28) and equality $\Psi \circ u = \Psi \circ u^{(m)}$ $\mathcal{L}^d$-a.e. in Ω, we get that

$$D^c(\Psi \circ u)(B) = D^c(\Psi \circ u^{(m)})(B) = \int_B \nabla\Psi(\widetilde{u}^{(m)})\,\mathrm{d}D^c u^{(m)} = \int_B \nabla\Psi(\widetilde{u})\,\mathrm{d}D^c u.$$

This concludes the proof of (i).

We now deal with (ii). To prove this we let $R > 0$ and observe that by (i) applied with $\Psi = \Phi_R$ we have

$$\begin{aligned} D^c(\Phi_R \circ u) \llcorner \widetilde{\Omega}_R = D^c u \llcorner \widetilde{\Omega}_R \quad &\text{as Radon measures on } \Omega, \\ |D^c(\Phi_R \circ u)| \llcorner \widetilde{\Omega}_R = |D^c u| \llcorner \widetilde{\Omega}_R \quad &\text{as Radon measures on } \Omega. \end{aligned}$$

Therefore, for every Borel set $B \subset \Omega$, we have

$$\begin{aligned} \int_{B\cap\widetilde{\Omega}_R} \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\,\mathrm{d}|D^c u| &= D^c u \llcorner \widetilde{\Omega}_R(B) = (D^c(\Phi_R \circ u) \llcorner \widetilde{\Omega}_R)(B) \\ &= \int_{B\cap\widetilde{\Omega}_R} \frac{\mathrm{d}D^c(\Phi_R \circ u)}{\mathrm{d}|D^c(\Phi_R \circ u)|}\,\mathrm{d}|D^c(\Phi_R \circ u)| = \int_{B\cap\widetilde{\Omega}_R} \frac{\mathrm{d}D^c(\Phi_R \circ u)}{\mathrm{d}|D^c(\Phi_R \circ u)|}\,\mathrm{d}|D^c u|. \end{aligned}$$

From the arbitrariness of B, this implies (1.3.25).

To prove (1.3.26), we first note that, since $|D^c u|(\Omega \setminus \bigcup_{R>0} \widetilde{\Omega}_R) = 0$ by (1.3.24), for $|D^c u|$-a.e. $x \in \Omega$, there exists $R_0 = R_0(x)$ such that $x \in \widetilde{\Omega}_{R_0}$. As $\widetilde{\Omega}_{R_0} \subset \widetilde{\Omega}_R$ for all $R > R_0$, the conclusion then follows from (1.3.25). □

An easy consequence of this result is that the Cantor part $D^c u$ satisfies the same rigidity property of Theorem 1.3.1.

Corollary 1.3.22. *Let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Then Theorem 1.3.1 holds true for u.*

Proof. By Proposition 1.3.20 for $|D^c u|$-a.e. x there exists $R_0 = R_0(x)$ such that $x \in \widetilde{\Omega}_{R_0}$. As $\Phi_{R_0} \circ u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$, for $|D^c u|$-a.e. $x \in \widetilde{\Omega}_{R_0}$ the Radon-Nikodým derivative $\mathrm{d}D^c(\Phi_{R_0} \circ u)/\mathrm{d}|D^c(\Phi_{R_0} \circ u)|(x)$ is a rank-one matrix. Thanks to (1.3.25), this implies that the same property holds for $\mathrm{d}D^c u/\mathrm{d}|D^c u|(x)$. The conclusion then follows from (1.3.24). □

Functions in $\mathrm{GBV}_\star$ admit also a weak notion of trace, which has to be intended in the sense of approximate limits.

Proposition 1.3.23. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Then for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ there exists a vector $u_{\partial\Omega}(x) \in \mathbb{R}^k$ such that*

$$u_{\partial\Omega}(x) = \operatorname{ap}\lim_{\substack{y \to x \\ y \in \Omega}} u(y). \tag{1.3.29}$$

Proof. For every $m \in \mathbb{N}$, let $u^{(m)}_{\partial\Omega}$ be the trace of $u^{(m)} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ on $\partial\Omega$. Since $(u^{(n)})^{(m)} = u^{(m)}$ for every $n > m$, by compatibility of the trace of BV functions with composition with Lipschitz functions it follows that

$$u^{(m)}_{\partial\Omega} = (u^{(n)}_{\partial\Omega})^{(m)} \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \partial\Omega, \tag{1.3.30}$$

which implies that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ and for all $i \in \{1,\dots,k\}$ there exists the limit $\lim_{m\to+\infty}(u^{(m)}_{\partial\Omega})_i \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,+\infty\}$.

We set $v_m := u^{(m)}/m$, which converges to 0 $\mathcal{L}^d$-a.e. as $m \to +\infty$ and in $L^1(\Omega;\mathbb{R}^k)$. Moreover, it is easy to check that $|Dv_m|(\Omega) \to 0$ as $m \to +\infty$. Therefore, from the trace estimate for $\mathrm{BV}(\Omega;\mathbb{R}^k)$ functions we have

$$\|(v_m)_{\partial\Omega}\|_{L^1_{\mathcal{H}^{d-1}}(\partial\Omega;\mathbb{R}^k)} \le C\big(\|v_m\|_{L^1(\Omega)} + |Dv_m|(\Omega)\big),$$

for a constant $C > 0$ independent of m. By the previous observations, this implies that

$$(v_m)_{\partial\Omega} = \frac{u^{(m)}_{\partial\Omega}}{m} \to 0 \quad \text{in } L^1_{\mathcal{H}^{d-1}}(\partial\Omega;\mathbb{R}^k) \text{ as } m \to +\infty.$$

By Čebyšëv's inequality and the observation below (1.3.30), this implies that the set N of points $x \in \partial\Omega$ for which there exists $i \in \{1,\dots,k\}$ such that $\lim_{m\to+\infty} |(u^{(m)}_{\partial\Omega})_i| = +\infty$ is $\mathcal{H}^{d-1}$-negligible. Using again the observation below (1.3.30) (1.3.30), we may then define

$$u_{\partial\Omega}(x) := \lim_{m\to+\infty} u^{(m)}_{\partial\Omega} \quad \text{for every } x \in \Omega \setminus N.$$

Observe that we also have

$$u^{(m)}_{\partial\Omega} = (u_{\partial\Omega})^{(m)} \quad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \tag{1.3.31}$$

and for all $m \in \mathbb{N}$.

To prove (1.3.29) it is enough to note that, if $m > |u_{\partial\Omega}(x)|_\infty + \eta$, then

$$\{|u - u_{\partial\Omega}(x)| > \eta\} \subset \{|u^{(m)} - u_{\partial\Omega}(x)| > \eta\},$$

so that the conclusion follows from the classical trace property of $u^{(m)}$ and (1.3.31). This concludes the proof. □

Using the result above, one can show that it is always possible to glue functions in $\mathrm{GBV}_\star$ across a Lipschitz boundary, a fact that will be used at several points in Chapter 3.

Lemma 1.3.24. *Let $O \subset\subset \Omega$ be open sets with Lipschitz boundaries. Let $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$, $v \in \mathrm{GBV}_\star(\Omega \setminus \overline{O};\mathbb{R}^k)$, and consider the function defined for $x \in \Omega \setminus \partial O$ by*

$$w(x) \coloneqq \begin{cases} u(x) & \text{if } x \in O, \\ v(x) & \text{if } x \in \Omega \setminus \overline{O}. \end{cases}$$

Then $w \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Moreover, $|[w]| \coloneqq |u_{\partial O} - v_{\partial O}|$ $\mathcal{H}^{d-1}$-a.e. on ∂O, where $u_{\partial O}$ and $v_{\partial O}$ are the traces of u and v on ∂O in the sense of Proposition 1.3.23.

Proof. It is immediate to check that $w \in \mathrm{GBV}(\Omega;\mathbb{R}^k)$ and that $\mathcal{V}(w,\Omega\setminus\overline{O})+\mathcal{V}(w,O) < +\infty$. Since

$$\sum_{i=1}^k \int_{J_w \cap \partial O} |[w_i]| \wedge 1 \, \mathrm{d}\mathcal{H}^{d-1} \leq k\mathcal{H}^{d-1}(\partial O) < +\infty,$$

we get $\mathcal{V}(w,\Omega) < +\infty$, and hence $w \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. □

We conclude the subsection with a useful compactness result due to Dal Maso & Toader [99, Theorem 3.11] (see also [111, Theorem 4.7]). Stronger compactness results were derived in the same paper(s), but we postpone their presentation to Chapter 2 (see Section 2.A).

Theorem 1.3.25. *Let $\{u_n\}_n \subset \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ be a sequence satisfying $\sup_{n\in\mathbb{N}} \mathcal{V}(u_n,\Omega) < +\infty$. Assume also that there exists an increasing continuous function $h\colon [0,+\infty) \to [0,+\infty)$ with $h(t) \to +\infty$ as $t \to +\infty$, such that*

$$\sup_{n\in\mathbb{N}} \int_\Omega h(|u_n|) \, \mathrm{d}x < +\infty.$$

Then there exist a subsequence, not relabelled, and a function $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ such that $\{u_n\}_n$ converges to u in $L^0(\Omega;\mathbb{R}^k)$.

Remark 1.3.26. An immediate consequence of the previous theorem is the following closure property: if $\{u_n\}_n \subset \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ is a sequence with $\sup_{n\in\mathbb{N}} \mathcal{V}(u_n,\Omega) < +\infty$ and $\{u_n\}_n$ converges in $L^0(\Omega;\mathbb{R}^k)$ to some function u, then $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$.

Slicing properties of $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. In this subsection, we present further results concerning fine properties of $\mathrm{GBV}_\star$-functions. In particular, we show that the space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ can be characterised by means of one-dimensional slicing. This characterisation shows that the space $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ is the space corresponding to Dal Maso's $\mathrm{GBD}(\Omega)$ (see Definition 1.4.3 below) in the context of functions of bounded variation. The results of this subsection are part of the joint paper with Friedrich [112].

The following lemma shows that, for a function $u \in \mathrm{GBV}(\Omega)$, it is always possible to bound the integral of $\mathcal{V}(u_y^\xi, \Omega_y^\xi)$ with respect to $y \in \Pi^\xi$ from above by $\mathcal{V}(u,\Omega)$.

Lemma 1.3.27. *Let $u \in \mathrm{GBV}(\Omega)$ and $\xi \in \mathbb{S}^{d-1}$. Then*

$$\int_{\Pi^\xi} \big(|Du_y^\xi|(\Omega_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{u_y^\xi})\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \mathcal{V}(u,\Omega). \tag{1.3.32}$$

Moreover, if $u \in \mathrm{GBV}_\star(\Omega)$, we have

$$\int_B (\nabla u)\xi\,\mathrm{d}x = \int_{\Pi^\xi}\int_{B_y^\xi} \nabla u_y^\xi(t)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{1.3.33}$$

$$D^c u(B)\xi = \int_{\Pi^\xi} D^c u_y^\xi(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{1.3.34}$$

$$\int_{B\cap(J_u\setminus J_u^1)} [u](\nu_u\cdot\xi)\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\Pi^\xi} Du_y^\xi\big((B\cap(J_u\setminus J_u^1))_y^\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \tag{1.3.35}$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{S}^{d-1}$.

Proof. If $u \notin \mathrm{GBV}_\star(\Omega)$ by the definition (1.3.16) of $\mathcal{V}$ we have $\mathcal{V}(u,\Omega) = +\infty$, so that we may assume that $u \in \mathrm{GBV}_\star(\Omega)$. Let $m \in \mathbb{N}$ and consider the function $u^{(m)} \in \mathrm{BV}(\Omega)$ (see Remark 1.3.11) . By the slicing properties of functions of bounded variation (see [18, Theorems 3.107 and 3.108]) we have

$$\int_\Omega |\nabla u^{(m)}|\,\mathrm{d}x \ge \int_{\Pi^\xi}\Big(\int_{\Omega_y^\xi} |\nabla (u^{(m)})_y^\xi|\,\mathrm{d}t\Big)\mathrm{d}\mathcal{H}^{d-1}(y),$$
$$|D^c u^{(m)}|(\Omega) \ge \int_{\Pi^\xi} |D^c (u^{(m)})_y^\xi|(\Omega_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y),$$

which, thanks to Remark 1.3.11, yields

$$\int_\Omega |\nabla u|\,\mathrm{d}x \ge \int_{\Pi^\xi}\Big(\int_{\Omega_y^\xi} |\nabla (u^{(m)})_y^\xi|\,\mathrm{d}t\Big)\mathrm{d}\mathcal{H}^{d-1}(y), \tag{1.3.36}$$

$$|D^c u|(\Omega) \ge \int_{\Pi^\xi} |D^c (u^{(m)})_y^\xi|(\Omega_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{1.3.37}$$

As $u \in \mathrm{GBV}(\Omega)$, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $u_y^\xi \in \mathrm{GBV}(\Omega_y^\xi)$. Thus, applying Remark 1.3.11 to the slices u_y^ξ, we obtain

$$\int_{\Omega_y^\xi} |\nabla (u^{(m)})_y^\xi|\,\mathrm{d}t \to \int_{\Omega_y^\xi} |\nabla u_y^\xi|\,\mathrm{d}t \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi \text{ as } m \to +\infty,$$
$$|D^c (u^{(m)})_y^\xi|(B_y^\xi) \to |D^c u_y^\xi|(B_y^\xi) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi \text{ as } m \to +\infty,$$

with both convergence being monotone. Therefore, by the Monotone Convergence Theorem and (1.3.36)–(1.3.37) we infer

$$\int_\Omega |\nabla u|\,\mathrm{d}x \ge \int_{\Pi^\xi}\Big(\int_{\Omega_y^\xi} |\nabla u_y^\xi|\,\mathrm{d}t\Big)\mathrm{d}\mathcal{H}^{d-1}(y), \tag{1.3.38}$$

$$|D^c u|(\Omega) \ge \int_{\Pi^\xi} |D^c u_y^\xi|(\Omega_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{1.3.39}$$

As for the jump part, we use the Area Formula (see [18, (2.47)]), (1.3.15), and (1.3.14) to get

$$\begin{aligned}\int_{J_{u^{(m)}}} (|[u^{(m)}]|\wedge 1)\,\mathrm{d}\mathcal{H}^{d-1} &\ge \int_{\Pi^\xi}\Big(\int_{J_{(u^{(m)})_y^\xi}} (|[(u^{(m)})_y^\xi](t)|\wedge 1)\,\mathrm{d}\mathcal{H}^0(t)\Big)\mathrm{d}\mathcal{H}^{d-1}(y)\\ &= \int_{\Pi^\xi}\Big(|D^j (u^{(m)})_y^\xi|(\Omega_y^\xi \setminus J^1_{(u^{(m)})_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{(u^{(m)})_y^\xi})\Big)\mathrm{d}\mathcal{H}^{d-1}(y).\end{aligned}$$

Using again Remark 1.3.11 and the Monotone Convergence Theorem, we obtain

$$\int_{J_u} (|[u]| \wedge 1)\, \mathrm{d}\mathcal{H}^{d-1} \geq \int_{\Pi^\xi} \Big(|D^j u_y^\xi|(\Omega_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{u_y^\xi}) \Big)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{1.3.40}$$

Putting together (1.3.38)–(1.3.40), we obtain (1.3.32).

To prove (1.3.33)–(1.3.35), we use [18, Theorems 3.107 and 3.108] to get

$$\int_B (\nabla u^{(m)})\xi \, \mathrm{d}x = \int_{\Pi^\xi} \Big(\int_{B_y^\xi} \nabla (u^{(m)})_y^\xi(t)\, \mathrm{d}t \Big) \mathrm{d}\mathcal{H}^{d-1}(y), \tag{1.3.41}$$

$$D^c u^{(m)}(B)\xi = \int_{\Pi^\xi} D^c (u^{(m)})_y^\xi(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{1.3.42}$$

Thanks to (1.3.32), for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the function u_y^ξ belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$, so that

$$\int_{B_y^\xi} \nabla (u^{(m)})_y^\xi(t)\, \mathrm{d}t \to \int_{B_y^\xi} \nabla u_y^\xi(t)\, \mathrm{d}t \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi \text{ as } m \to +\infty,$$
$$D^c (u^{(m)})_y^\xi(B_y^\xi) \to D^c u_y^\xi(B_y^\xi) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi \text{ as } m \to +\infty.$$

Thus, we may let $m \to +\infty$ in (1.3.41) and (1.3.42). Then, by Propositions 1.3.10 and 1.3.20, and the Dominated Convergence Theorem we obtain (1.3.33) and (1.3.34).

As for (1.3.35), it is enough to apply the Area Formula and to use (1.3.14) and (1.3.15). □

The following proposition characterises $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ by means of one-dimensional slicing and furnishes an analogue in this generalised setting of the last part of Proposition 1.3.2.

Proposition 1.3.28. *Let $u \in L^0(\Omega;\mathbb{R}^k)$. Then $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ if and only if for every $\xi \in \mathbb{S}^{d-1}$ the following two conditions hold:*

$$u_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi;\mathbb{R}^k) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi, \tag{1.3.43}$$

$$\int_{\Pi^\xi} \big(|Du_y^\xi|(\Omega_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{u_y^\xi}) \big)\, \mathrm{d}\mathcal{H}^{d-1}(y) < +\infty. \tag{1.3.44}$$

Proof. It is enough to prove the statement in the case $k = 1$.

The fact that $u \in \mathrm{GBV}_\star(\Omega)$ implies (1.3.43) and (1.3.44) is a simple consequence of Lemma 1.3.27 and the fact that in dimension 1 the space $\mathrm{GBV}_\star$ is contained in $\mathrm{BV}_{\mathrm{loc}}$ (see [99, Remark 3.2]).

We now prove that (1.3.43) and (1.3.44) imply $u \in \mathrm{GBV}_\star(\Omega)$. To this end, let us fix $\xi \in \mathbb{S}^{d-1}$. By (1.3.43), there exists a Borel set $N_\xi^1 \subset \Pi^\xi$, with $\mathcal{H}^{d-1}(N_\xi^1) = 0$, such that $u_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$ for all $y \in \Pi^\xi \setminus N_\xi^1$. Let us fix $m > 0$. For every $y \in \Pi^\xi \setminus N_\xi^1$, by the Chain Rule for functions of bounded variation (see [18, Theorem 3.99]) we have $(u^{(m)})_y^\xi \in \mathrm{BV}(\Omega_y^\xi)$ and

$$D(u^{(m)})_y^\xi = \chi_{\{|\widetilde{u}_y^\xi| \leq m\}} (D^a u_y^\xi + D^c u_y^\xi) + \big(((u_y^\xi)^+)^{(m)} - ((u_y^\xi)^-)^{(m)} \big) \mathcal{H}^0 \llcorner J_{u_y^\xi} \tag{1.3.45}$$

as Borel measures on Ω_y^ξ. In particular,

$$|D(u^{(m)})_y^\xi|(\Omega_y^\xi) \leq |D^a u_y^\xi|(\Omega_y^\xi) + |D^c u_y^\xi|(\Omega_y^\xi) + |D^j u_y^\xi|(J_{u_y^\xi} \setminus J^1_{u_y^\xi}) + 2m\mathcal{H}^0(J^1_{u_y^\xi})$$

for every $y \in \Pi^\xi \setminus N^1_\xi$. Combining the previous inequality with (1.3.44), we obtain

$$\int_{\Pi^\xi} |D(u^{(m)})^\xi_y|(\Omega^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty.$$

By the arbitrariness of $\xi \in \mathbb{S}^{d-1}$ and [18, Remark 3.104], this implies that $u^{(m)} \in \mathrm{BV}(\Omega)$, whence $u \in \mathrm{GBV}(\Omega)$.

To conclude that $u \in \mathrm{GBV}_\star(\Omega)$, it is then enough to show that

$$\int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1\,\mathrm{d}\mathcal{H}^{d-1} \le C_1, \tag{1.3.46}$$

$$\int_\Omega |(\nabla u^{(m)})e_i|\,\mathrm{d}x + |(D^c u^{(m)})e_i|(\Omega) \le C_2 \quad \text{for all } i \in \{1,\dots,d\}, \tag{1.3.47}$$

for constants C_1, $C_2 > 0$ independent of m.

Let $\xi \in \mathbb{S}^{d-1}$, let $N^1_\xi \subset \Pi^\xi$ be as above, and let $N^2_\xi \subset \Pi^\xi$ be a Borel set, with $\mathcal{H}^{d-1}(N^2_\xi) = 0$, such that (1.3.14) and (1.3.15) hold for every $y \in \Pi^\xi \setminus N^2_\xi$. Resorting to (1.3.45), we get

$$J_{(u^{(m)})^\xi_y} \subset J_{u^\xi_y} = (J_u)^\xi_y \quad \text{and } J^1_{(u^{(m)})^\xi_y} \subset J^1_{u^\xi_y} = (J^1_u)^\xi_y,$$

and

$$\begin{aligned}|D(u^{(m)})^\xi_y|\big(B^\xi_y \setminus J^1_{(u^{(m)})^\xi_y}\big) &\le |D(u^{(m)})^\xi_y|\big(B^\xi_y \setminus J^1_{u^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap (J^1_{u^\xi_y} \setminus J^1_{(u^{(m)})^\xi_y})\big)\\ &\le |Du^\xi_y|\big(B^\xi_y \setminus J^1_{u^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap (J^1_{u^\xi_y} \setminus J^1_{(u^{(m)})^\xi_y})\big)\end{aligned} \tag{1.3.48}$$

for all Borel sets $B \subset \Omega$ and for every $y \in \Pi^\xi \setminus (N^1_\xi \cup N^2_\xi)$.

We now prove (1.3.46). We consider a finite collection of vectors $\{\xi_j\}_{j=1}^N \subset \mathbb{S}^{d-1}$ with the property that the sets $S_j := \{\xi \in \mathbb{S}^{d-1}\colon |\xi - \xi_j| \le 1/2\}$ cover $\mathbb{S}^{d-1}$. We set $B_j := \{x \in J_u\colon \nu_u(x) \in S_j\}$ and note that $J_u = \bigcup_{j=1}^N B_j$. Note also that, if $\xi \in S_j$ for some $j \in \{1,\dots,N\}$, then $\xi \cdot \xi_j = 1 - |\xi - \xi_j|^2/2 \ge 7/8$, so that

$$\nu_u(x) \cdot \xi_j \ge 7/8 \quad \text{for all } x \in B_j. \tag{1.3.49}$$

By the Area Formula (see [18, (2.47)]), (1.3.15), (1.3.48), and (1.3.49), for all $j \in \{1,\dots,N\}$ we have

$$\begin{aligned}\frac{7}{8}\int_{B_j} (|[u^{(m)}]| \wedge 1)\,\mathrm{d}\mathcal{H}^{d-1} &\le \int_{B_j} (|[u^{(m)}]| \wedge 1)(\nu_u \cdot \xi_j)\,\mathrm{d}\mathcal{H}^{d-1}\\ &= \int_{\Pi^{\xi_j}} \Big(\int_{(B_j)^{\xi_j}_y} (|[(u^{(m)})^{\xi_j}_y](t)| \wedge 1)\,\mathrm{d}\mathcal{H}^0(t)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ &= \int_{\Pi^{\xi_j}} \Big(|D(u^{(m)})^{\xi_j}_y|\big((B_j)^{\xi_j}_y \setminus J^1_{(u^{(m)})^{\xi_j}_y}\big)\\ &\quad + \mathcal{H}^0\big((B_j)^{\xi_j}_y \cap J^1_{(u^{(m)})^{\xi_j}_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ &\le \int_{\Pi^{\xi_j}} \Big(|Du^{\xi_j}_y|\big((B_j)^{\xi_j}_y \setminus J^1_{u^{\xi_j}_y}\big) + \mathcal{H}^0\big((B_j)^{\xi_j}_y \cap J^1_{u^{\xi_j}_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y).\end{aligned}$$

Summing over $j \in \{1,\dots,N\}$ the previous inequality, we find by the Area Formula

$$\int_{J_u} (|[u^{(m)}]| \wedge 1)\,\mathrm{d}\mathcal{H}^{d-1} \le \frac{8}{7}\sum_{j=1}^N \int_{\Pi^{\xi_j}} \Big(|Du^{\xi_j}_y|\big((B_j)^{\xi_j}_y \setminus J^1_{u^{\xi_j}_y}\big) + \mathcal{H}^0\big((B_j)^{\xi_j}_y \cap J^1_{u^{\xi_j}_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

Letting C_1 denote the right-hand side of the previous inequality, it follows from (1.3.44)that $C_1 < +\infty$, which proves (1.3.46).

To prove (1.3.47), we use the slicing properties of functions of bounded variation (see [18, Theorem 3.108]) to get

$$\int_\Omega |(\nabla u^{(m)})e_i|\,\mathrm{d}x + |(D^c u^{(m)})e_i|(\Omega) = \int_{\Pi^{e_i}} \Big(\int_{\Omega_y^{e_i}} |\nabla (u^{(m)})_y^{e_i}|\,\mathrm{d}t + |D^c (u^{(m)})_y^{e_i}|(\Omega_y^{e_i}) \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

In view of (1.3.44), of (1.3.48) applied with $B = \Omega \setminus J_u$, and of (1.3.14), the previous equality proves (1.3.47) with

$$C_2 \coloneqq \sum_{i=1}^d \int_{\Pi^{e_i}} \Big(\int_{\Omega_y^{e_i}} |\nabla u_y^{e_i}|\,\mathrm{d}t + |D^c u_y^{e_i}|(\Omega_y^{e_i}) \Big) \mathrm{d}\mathcal{H}^{d-1}(y).$$

This proves that $u \in \mathrm{GBV}_\star(\Omega)$ and concludes the proof. □

Remark 1.3.29. If Ω has Lipschitz boundary and $u \in \mathrm{GBV}(\Omega;\mathbb{R}^k)$, then we may replace $\mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi;\mathbb{R}^k)$ in (1.3.43) by $\mathrm{BV}(\Omega_y^\xi;\mathbb{R}^k)$. Indeed, let $y \in \Pi^\xi$ be such that

$$u_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi;\mathbb{R}^k),\ \mathcal{H}^0(\partial\Omega_y^\xi) < +\infty,\ \text{and } |Du_y^\xi|(\Omega_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{u_y^\xi}) < +\infty. \quad (1.3.50)$$

Since $\mathcal{H}^0(J^1_{u_y^\xi}) < +\infty$, it follows immediately that $|Du_y^\xi|(\Omega_y^\xi) < +\infty$. The assumption $\mathcal{H}^0(\partial\Omega_y^\xi) < +\infty$ implies that Ω_y^ξ is a finite union of open intervals, so that by the Fundamental Theorem of Calculus for one-dimensional BV functions we also get $u_y^\xi \in L^1(\Omega_y^\xi;\mathbb{R}^k)$. As the conditions in (1.3.50) are satisfied by $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, the claim is proved.

We now show that the measure $D^c u$ enjoys some useful disintegration properties.

Lemma 1.3.30. *Let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, let $\varphi\colon \Omega \to \mathbb{R}$ be a bounded Borel function, and let $B \subset \Omega$ be a Borel set. Then for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\Big(\int_B \varphi\,\mathrm{d}D^c u \Big)\xi = \int_{\Pi^\xi} \Big(\int_{B_y^\xi} \varphi_y^\xi\,\mathrm{d}D^c u_y^\xi \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y). \quad (1.3.51)$$

Proof. The case $\varphi = \chi_E$ for some Borel set $E \subset \Omega$ is exactly (1.3.34). This result is clearly extended to simple bounded Borel functions by additivity. The general case can be obtained by approximating each bounded Borel function by a sequence of uniformly bounded simple Borel functions, taking into account (1.3.44), in order to be able to use the Dominated Convergence Theorem on the right-hand side of (1.3.51). □

1.4 Functions of (generalised) bounded deformation

In this section, we recall the main properties of functions of (generalised) bounded deformation, using [169], [17], and [89] as main references.

Functions of bounded deformation. The space of functions of bounded deformation $\mathrm{BD}(\Omega)$ is the collection of all $u \in L^1(\Omega;\mathbb{R}^d)$ such that the symmetric part of their distributional derivative $Eu \coloneqq \frac{1}{2}(Du + Du^T)$ belongs to $\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. We will also consider the space $\mathrm{BD}_{\mathrm{loc}}(\Omega)$, the collection of all $\mathcal{L}^d$-measurable functions $u\colon \Omega \to \mathbb{R}^d$ such that $u|_O \in \mathrm{BD}(O)$ for every open set $O \subset\subset \Omega$.

It was proved in [17] that the measure Eu can always be decomposed as

$$Eu = E^a u + E^c u + E^j u = \mathcal{E}u\mathcal{L}^d + E^c u + [u] \odot \nu_u \mathcal{H}^{d-1} \llcorner J_u, \tag{1.4.1}$$

where:

(i) the *absolutely continuous part* $E^a u$ is the absolutely continuous part of Eu with respect to $\mathcal{L}^d$, whose density is denoted by $\mathcal{E}u \in L^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$,

(ii) the *Cantor part* $E^c u$ is defined as $E^c u := E^s u \llcorner (\Omega \setminus J_u)$, where $E^s u$ is the singular part of Eu with respect to $\mathcal{L}^d$; it is known that $E^c u$ vanishes on all Borel sets $B \subset \Omega$ that are σ-finite with respect to $\mathcal{H}^{d-1}$,

(iii) the *jump part* $E^j u$ is defined by $E^j u := Eu \llcorner J_u = E^s u \llcorner J_u$; it is known that $E^j u = [u] \odot \nu_u \mathcal{H}^{d-1} \llcorner J_u$.

Furthermore, $\mathcal{E}u$ is an *approximate symmetric gradient*, i.e., it satisfies

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^d} \int_{B_\rho(x)} \frac{|\big(u(y) - u(x) - \mathcal{E}u(x)(y-x)\big)\cdot(y-x)|}{|y-x|^2}\,\mathrm{d}y = 0$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$.

We recall that De Philippis & Rindler proved in [107, Theorem 1.18] that the measure $E^c u$ satisfies an analogue of the rigidity property stated in Theorem 1.3.1.

Theorem 1.4.1. *Let $u \in \mathrm{BD}(\Omega)$. Then there exist a Borel set $C(u)$ with $|E^c u|(C(u)) = |E^c u|(\Omega)$ and two Borel maps $a_u, b_u \colon \Omega \to \mathbb{R}^d$ such that*

$$\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|} = a_u \odot b_u \quad \text{and} \quad |a_u \odot b_u| = 1 \quad \text{on } C(u). \tag{1.4.2}$$

The following theorem, whose proof was first obtained in [17, Structure Theorem 4.5], describes the slicing properties of functions in $\mathrm{BD}(\Omega)$.

Theorem 1.4.2. *Let $u \in \mathrm{BD}(\Omega)$ and $\xi \in \mathbb{S}^{d-1}$. Then the following hold*

(a) $\hat{u}^\xi_y \in \mathrm{BV}(\Omega^\xi_y)$ *for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$;*

(b) *for every Borel set $B \subset \Omega$ it holds*

$$Eu(B)\xi\cdot\xi = \int_{\Pi^\xi} D\hat{u}^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) \quad \text{and} \quad |Eu\xi\cdot\xi|(B) = \int_{\Pi^\xi} |D\hat{u}^\xi_y|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y);$$

(c) $\mathcal{E}u(y+t\xi)\xi\cdot\xi = \nabla\hat{u}^\xi_y(t)$ *for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ and for $\mathcal{L}^1$-a.e. $t \in \Omega^\xi_y$;*

(d) *for every Borel set $B \subset \Omega$ we have*

$$E^c u(B)\xi\cdot\xi = \int_{\Pi^\xi} D^c\hat{u}^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y);$$

(e) *for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have*

$$J_{\hat{u}^\xi_y} \subset (J_u)^\xi_y \quad \text{and} \quad u^\pm(y+t\xi)\cdot\xi = (\hat{u}^\xi_y)^\pm(t) \quad \text{for all } t \in (J_u)^\xi_y, \tag{1.4.3}$$

where the normals at J_u and $J_{\hat{u}^\xi_y}$ are oriented in such a way that $\nu_u\cdot\xi \geq 0$ and $\nu_{\hat{u}^\xi_y} = 1$;

Generalised functions of bounded deformation. We recall the definition of the space of *generalised functions of bounded deformation*, introduced in [89, Definition 4.1]. This definition uses the collection $\mathcal{T}$ of regular truncation functions defined by

$$\mathcal{T} \coloneqq \{\tau \in C^1(\mathbb{R})\colon\ -1/2 \le \tau \le 1/2 \text{ and } 0 \le \tau' \le 1\}.$$

Definition 1.4.3. The space $\mathrm{GBD}(\Omega)$ is the space of all $\mathcal{L}^d$-measurable functions $u\colon \Omega \to \mathbb{R}^d$ for which there exists a measure $\lambda \in \mathcal{M}_b^+(\Omega)$ such that the following equivalent conditions are satisfied for every $\xi \in \mathbb{S}^{d-1}$:

(a) for every $\tau \in \mathcal{T}$ we have $D_\xi(\tau(u\cdot\xi)) \in \mathcal{M}_b(\Omega)$ and

$$|D_\xi(\tau(u\cdot\xi))|(B) \le \lambda(B) \quad \text{for all Borel sets } B \subset \Omega; \tag{1.4.4}$$

(b) for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $\hat{u}_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$ and

$$\int_{\Pi^\xi} \big(|D\hat{u}_y^\xi|(B_y^\xi \setminus J^1_{\hat{u}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{u}_y^\xi})\big)\mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda(B) \text{ for all Borel sets } B \subset \Omega. \tag{1.4.5}$$

The space $\mathrm{GSBD}(\Omega)$ is the subspace of $u \in \mathrm{GBD}(\Omega)$ such that for every $\xi \in \mathbb{S}^{d-1}$ one hase $\hat{u}_y^\xi \in \mathrm{SBV}_{\mathrm{loc}}(\Omega_y^\xi)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$.

Remark 1.4.4. If $u \in \mathrm{GBD}(\Omega)$ it follows from (b) of Definition 1.4.3 that for every $\xi \in \mathbb{S}^{d-1}$ there exists a Borel set $N_\xi \subset \Pi^\xi$, with $\mathcal{H}^{d-1}(N_\xi) = 0$, such that $\hat{u}_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$ and $|D\hat{u}_y^\xi|(\Omega_y^\xi) < +\infty$ for every $y \in \Pi^\xi \setminus N_\xi$. In particular, if Ω_y^ξ is an interval and $y \in \Pi^\xi \setminus N_\xi$, then $\hat{u}_y^\xi \in \mathrm{BV}(\Omega_y^\xi)$.

Remark 1.4.5. The previous remark implies that, if $d = 1$, then $\mathrm{GBD}(\Omega) = \{u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)\colon |Du|(\Omega) < +\infty\}$.

Remark 1.4.6. Condition (a) of Definition 1.4.3 can be strengthened by requiring that (1.4.4) holds also for every $\tau \in \mathcal{T}_{\mathrm{Lip}}$, where

$$\mathcal{T}_{\mathrm{Lip}} \coloneqq \{\tau \in \mathrm{Lip}(\mathbb{R})\colon\ -1/2 \le \tau \le 1/2,\ 0 \le \tau' \le 1\ \mathcal{L}^1\text{-a.e., } \tau' \text{ has compact support}\}.$$

Indeed, every function $\tau \in \mathcal{T}_{\mathrm{Lip}}$ can be approximated uniformly on $\mathbb{R}$ by a sequence $\{\tau_n\}_n \subset \mathcal{T}$, so that $\tau_n(u\cdot\xi) \to \tau(u\cdot\xi)$ in $L^1(\Omega)$. Since for every $O \subset \Omega$ open, the function $v \mapsto |D_\xi v|(O)$ is lower semicontinuous with respect to the $L^1(\Omega)$ convergence, by (1.4.4) we obtain that $D_\xi(\tau(u\cdot\xi)) \in \mathcal{M}_b(\Omega)$ and that

$$|D_\xi(\tau(u\cdot\xi))|(O) \le \liminf_{n\to+\infty} |D_\xi(\tau_n(u\cdot\xi))|(O) \le \lambda(O)$$

for every $O \subset \Omega$ open. Inequality (1.4.4) for a general Borel set $B \subset \Omega$ follows by approximation with open sets.

Remark 1.4.7. Inequality (1.4.5) can be extended to non-unit vectors. Elementary arguments show that for every $t \neq 0$, $\xi \in \mathbb{R}^d \setminus \{0\}$, $y \in \Pi^\xi$, and $E \subset \mathbb{R}^d$

$$E_y^\xi = tE_y^{t\xi}. \tag{1.4.6}$$

Since $\hat{u}_y^{t\xi}(s) = t\hat{u}_y^\xi(st)$ for every $s \in \mathbb{R}$, we have

$$\begin{gathered} D\hat{u}_y^{t\xi}(S) = |t|D\hat{u}_y^\xi(tS) \quad \text{and} \quad |D\hat{u}_y^{t\xi}|(S) = |t||D\hat{u}_y^\xi|(tS), \\ tJ_{\hat{u}_y^{t\xi}} = J_{\hat{u}_y^\xi} \quad \text{and} \quad tJ^{|t|}_{\hat{u}_y^{t\xi}} = J^1_{\hat{u}_y^\xi} \end{gathered} \tag{1.4.7}$$

for every Borel set $S \subset \Omega_y^{t\xi}$. In particular, if $B \subset \Omega$ is a Borel set, taking $S = B_y^{t\xi} \setminus J^{|t|}_{\hat{u}_y^{t\xi}}$ and using (1.4.6) and (1.4.7) we get

$$|D\hat{u}_y^{t\xi}|(B_y^{t\xi} \setminus J^{|t|}_{\hat{u}_y^{t\xi}}) = |t||D\hat{u}_y^{\xi}|(B_y^{\xi} \setminus J^{1}_{\hat{u}_y^{\xi}}), \tag{1.4.8}$$

while, taking $S = B_y^{t\xi}$ and using (1.4.6)

$$D\hat{u}_y^{t\xi}(B_y^{t\xi}) = |t|D\hat{u}_y^{\xi}(B_y^{\xi}). \tag{1.4.9}$$

Applying (1.4.7) and (1.4.8) with $t = |\xi|$ and ξ replaced by $\xi/|\xi|$, from (1.4.5) we obtain

$$\begin{gathered}\int_{\Pi^\xi} |\xi||D\hat{u}_y^{\xi}|(B_y^{\xi} \setminus J^{|\xi|}_{\hat{u}_y^{\xi}})\mathrm{d}\mathcal{H}^{d-1}(y) \le |\xi|^2\lambda(B), \\ \int_{\Pi^\xi} \mathcal{H}^0(B_y^{\xi} \cap J^{|\xi|}_{\hat{u}_y^{\xi}})\mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda(B),\end{gathered} \tag{1.4.10}$$

for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$.

Remark 1.4.8. It follows from [14, Remark 3.6] that, if $u \in \mathrm{GBD}(\Omega)$, then $\mathcal{H}^{d-1}(J_u^1) < +\infty$. More precisely, it is shown that, if $\lambda \in \mathcal{M}_b^+(\Omega)$ satisfies (1.4.5), then for every Borel set $B \subset \Omega$ we have the inequality

$$\mathcal{H}^{d-1}(J_u^1 \cap B) \le 4d\lambda(B). \tag{1.4.11}$$

Since by Theorem 1.4.13 below for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have

$$J^{|\xi|}_{\hat{u}_y^{\xi}} \subset (J_u^1)_y^{\xi},$$

it follows from (1.4.7), (1.4.10), and (1.4.11) that

$$\begin{gathered}\int_{\Pi^\xi} |\xi||D\hat{u}_y^{\xi}|((B \setminus J_u^1)_y^{\xi})\mathrm{d}\mathcal{H}^{d-1}(y) \le |\xi|^2\lambda(B), \\ \int_{\Pi^\xi} \mathcal{H}^0((B \cap J_u^1)_y^{\xi})\mathrm{d}\mathcal{H}^{d-1}(y) \le 4d\lambda(B).\end{gathered} \tag{1.4.12}$$

Given $0 < r \le 1$, we can consider the function $v = u/r$, which by [89, Remark 4.6] satisfies (1.4.5) with λ/r. Applying (1.4.11) to this function, we obtain

$$\mathcal{H}^{d-1}(J_u^r \cap B) \le \frac{4d}{r}\lambda(B) \tag{1.4.13}$$

for every Borel set $B \subset \Omega$. In particular, this implies that $[u] \in L^1_{\mathrm{weak}}(J_u, \mathcal{H}^{d-1})$. Moreover, since $J_u = \bigcup_{0<r\le 1} J_u^r$, we have that J_u is σ-finite with respect to $\mathcal{H}^{d-1}$.

Definition 1.4.9. For every $u \in \mathrm{GBD}(\Omega)$ and $\xi \in \mathbb{R}^d \setminus \{0\}$ we introduce the bounded Radon measure λ_u^ξ defined for every Borel set $B \subset \Omega$ by

$$\lambda_u^\xi(B) \coloneqq \int_{\Pi^\xi} \left(|\xi||D\hat{u}_y^{\xi}|(B_y^{\xi} \setminus J^{|\xi|}_{\hat{u}_y^{\xi}}) + \mathcal{H}^0(B_y^{\xi} \cap J^{|\xi|}_{\hat{u}_y^{\xi}})\right) \mathrm{d}\mathcal{H}^{d-1}(y). \tag{1.4.14}$$

The following result was proved in [89, Proposition 4.17] and characterises, in terms of the measures λ_u^ξ, the optimal measure λ for which (1.4.5) holds.

Proposition 1.4.10. *Let $u \in \mathrm{GBD}(\Omega)$ and let $\lambda_u \in \mathcal{M}_b^+(\Omega)$ be the positive measure defined by*

$$\lambda_u(B) \coloneqq \sup \sum_{i=1}^{N} \lambda_u^{\xi_i}(B_i) \quad \text{for every Borel set } B \subset \Omega, \tag{1.4.15}$$

where the supremum is taken over all $N \in \mathbb{N}$, all families $\{\xi_i\}_{i=1}^N$ of vectors of $\mathbb{S}^{d-1}$, and all Borel partitions $\{B_i\}_{i=1}^N$ of B. Then λ_u is the smallest measure that satisfies (1.4.5) *for all $\xi \in \mathbb{S}^{d-1}$.*

Remark 1.4.11. Let $u \in \mathrm{GBD}(\Omega)$ and for every $\rho > 0$ let u_ρ be the function defined for $x \in \rho\Omega$ by $u_\rho(x) \coloneqq u(x/\rho)$. Then for every Borel set $B \subset \Omega \setminus J_u$ we have $\lambda_{u_\rho}(\rho B) = \rho^{d-1}\lambda_u(B)$. Indeed, since $J_{u_\rho} = \rho J_u$ we have $\rho B \cap J_{u_\rho} = \emptyset$ so that, using (1.4.14) and the first inclusion in (1.4.16), a change of coordinates shows that $\lambda_{u_\rho}^\xi(\rho B) = \rho^{d-1}\lambda_u^\xi(B)$ for every $\xi \in \mathbb{S}^{d-1}$. The conclusion then follows from (1.4.15).

Remark 1.4.12. Given $u \in \mathrm{GBD}(\Omega)$, we have $\lambda_u = \lambda_{u+a}$ whenever $a\colon \mathbb{R}^d \to \mathbb{R}^d$ is of the form $a(x) \coloneqq Ax + b$ with $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$. Indeed, for every $\xi \in \mathbb{S}^{d-1}$, $y \in \pi^\xi(\Omega)$ and $t \in \Omega_y^\xi$ it holds $a_y^\xi(t) \coloneqq (A(y + t\xi) + b) \cdot \xi$, so that $Da_y^\xi(t) = A\xi \cdot \xi = 0$, being A skew-symmetric. By (1.4.14) this implies that $\lambda_u^\xi = \lambda_{u+a}^\xi$. The conclusion then follows from (1.4.15).

The following theorem collects some of the fine properties of $\mathrm{GBD}(\Omega)$ functions, proved in [89, Proposition 6.1 and Theorems 6.2, 8.1, 9.1].

Theorem 1.4.13. *Let $u \in \mathrm{GBD}(\Omega)$. Then the following properties hold:*

(a) *(existence of the approximate symmetric gradient) there exists $\mathcal{E}u \in L^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that for $\mathcal{L}^d$-a.e. $x \in \Omega$ we have*

$$\operatorname*{aplim}_{y\to x} \frac{(u(y) - u(x) - \mathcal{E}u(x)(y-x)) \cdot (y-x)}{|x-y|^2} = 0;$$

moreover, for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have

$$\mathcal{E}u(y + t\xi)\xi \cdot \xi = \nabla \hat{u}_y^\xi(t)$$

for $\mathcal{L}^1$-a.e. $t \in \Omega_y^\xi$;

(b) *(J_u and its slices) the jump set J_u is $(\mathcal{H}^{d-1}, d-1)$-rectifiable; for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have*

$$J_{\hat{u}_y^\xi} = \{x \in J_u : [u](x) \cdot \xi \neq 0\}_y^\xi \subset (J_u)_y^\xi, \tag{1.4.16}$$

$$(\hat{u}_y^\xi)^\pm(t) = u^\pm(y + t\xi) \cdot \xi \quad \text{for every } t \in (J_u)_y^\xi, \tag{1.4.17}$$

$$J^{|\xi|}_{\hat{u}_y^\xi} \subset (J_u^1)_y^\xi, \tag{1.4.18}$$

where the normals at J_u and $J_{\hat{u}_y^\xi}$ are oriented in such a way that $\nu_u \cdot \xi \geq 0$ and $\nu_{\hat{u}_y^\xi} = 1$. In particular,

$$|D\widehat{u}_y^\xi|((J_u)_y^\xi \setminus J_{\widehat{u}_y^\xi}) = 0 \tag{1.4.19}$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$.

Moreover, setting

$$\Theta_u \coloneqq \Big\{x \in \Omega\colon \limsup_{\rho\to 0} \frac{\lambda_u(B_\rho(x))}{\rho^{d-1}} > 0\Big\}, \tag{1.4.20}$$

we have that Θ_u is $(\mathcal{H}^{d-1}, d-1)$-rectifiable, $J_u \subset \Theta_u$, and that $\mathcal{H}^{d-1}(\Theta_u \setminus J_u) = 0$.

It is easy to see that for a function $u \in \mathrm{GBD}(\Omega) \setminus \mathrm{BD}(\Omega)$ its jump $[u]$ need not be integrable on J_u with respect to $\mathcal{H}^{d-1}$ (see, for instance, [89, Example 12.3]). More precisely, one can show (see, for instance, [17, Proposition 3.2] or [168, Remark 2.17]) that, if $u \in \mathrm{GBD}(\Omega) \cap L^1(\Omega;\mathbb{R}^d)$ and $[u] \in L^1(J_u, \mathcal{H}^{d-1})$, then u belongs to $\mathrm{BD}(\Omega)$. Nonetheless, we now show that $[u]$ is integrable on $J_u \setminus J_u^1$ with respect to $\mathcal{H}^{d-1}$. This is the content of the following proposition.

Proposition 1.4.14 ([91, Proposition 2.13])**.** *Let $u \in \mathrm{GBD}(\Omega)$. Then*

$$\int_{J_u \setminus J_u^1} |([u] \odot \nu_u)\xi \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} < +\infty \tag{1.4.21}$$

for every $\xi \in \mathbb{R}^d$. This is equivalent to $[u] \in L^1(J_u \setminus J_u^1, \mathcal{H}^{d-1})$.

Proof. By homogeneity it is enough to show (1.4.21) for a fixed $\xi \in \mathbb{S}^{d-1}$. By definition of $\odot$ we have $([u] \odot \nu_u)\xi \cdot \xi = ([u] \cdot \xi)(\nu_u \cdot \xi)$. Therefore, by the Area Formula [164, 12.4], we can write

$$\begin{aligned}
\int_{J_u \setminus J_u^1} |([u] \odot \nu_u)\xi \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} &= \int_{J_u \setminus J_u^1} |([u] \cdot \xi)(\nu_u \cdot \xi)| \, \mathrm{d}\mathcal{H}^{d-1} \\
&= \int_{\Pi^\xi} \Big(\int_{(J_u \setminus J_u^1)_y^\xi} |[u](y + t\xi) \cdot \xi| \, \mathrm{d}\mathcal{H}^0(t) \Big) \mathrm{d}\mathcal{H}^{d-1}(y) \\
&\leq \int_{\Pi^\xi} \Big(\int_{J_{\hat{u}_y^\xi} \setminus J^1_{\hat{u}_y^\xi}} |[\hat{u}_y^\xi](t)| \, \mathrm{d}\mathcal{H}^0(t) \Big) \mathrm{d}\mathcal{H}^{d-1}(y),
\end{aligned} \tag{1.4.22}$$

where in the inequality we have used (1.4.16)–(1.4.18). By (b) of Definition 1.4.3, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $\hat{u}_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$, so that

$$\begin{aligned}
\int_{\Pi^\xi} \Big(\int_{J_{\hat{u}_y^\xi} \setminus J^1_{\hat{u}_y^\xi}} |[\hat{u}_y^\xi](t)| \, \mathrm{d}\mathcal{H}^0(t) \Big) \mathrm{d}\mathcal{H}^{d-1}(y) &= \int_{\Pi^\xi} |D\hat{u}_y^\xi|(J_{\hat{u}_y^\xi} \setminus J^1_{\hat{u}_y^\xi}) \mathrm{d}\mathcal{H}^{d-1}(y) \\
&\leq \int_{\Pi^\xi} |D\hat{u}_y^\xi|(\Omega_y^\xi \setminus J^1_{\hat{u}_y^\xi}) \, \mathrm{d}\mathcal{H}^{d-1}(y) \\
&\leq \lambda(\Omega) < +\infty,
\end{aligned}$$

which, together with (1.4.22), concludes the proof of (1.4.21).

From the polarisation identity it follows that for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$ one has

$$|([u] \odot \nu_u)\xi \cdot \eta| \leq \frac{1}{4}\big(|([u] \odot \nu_u)(\xi + \eta) \cdot (\xi + \eta)| + |([u] \odot \nu_u)(\xi - \eta) \cdot (\xi - \eta)|\big)$$

on J_u. Hence, by (1.4.21) we deduce that $[u] \odot \nu_u \in L^1(J_u \setminus J_u^1, \mathcal{H}^{d-1})$. Since for every $a, b \in \mathbb{R}^d$ we have $|a||b| \leq \sqrt{2}|a \odot b|$, the proof is concluded. □

1.5 Integral representation in BV, SBV^2, and BD

In this section, we collect some results concerning the integral representation of abstract functionals over various spaces of functions of bounded variation/deformation.

In order to present these results, we set first some notation. Here, X always stands for a subspace of $L^1(\Omega;\mathbb{R}^k)$ of functions of bounded variation/deformation, whose nature will be clear from the context. We look at functionals $F\colon X\times\mathcal{B}(\Omega)\to[0,+\infty)$ that satisfy, among others, the following set of assumptions

(X1) (*Measure property*) for every $u\in X$ the set function $F(u,\cdot)$ is a bounded Radon measure on $\mathcal{B}(\Omega)$;

(X2) (*Locality*) for every $O\in\mathcal{O}(\Omega)$ and pairs $u,v\in X$, with $u=v$ $\mathcal{L}^d$-a.e. in O, it holds $F(u,O)=F(v,O)$;

(X3) (*Lower semicontinuity*) for every $O\in\mathcal{O}(\Omega)$ the functional $F(\cdot,O)$ is lower semicontinuous with respect to the $L^1(\Omega;\mathbb{R}^k)$-convergence;

(X4a) (*Invariance under addition of constants*) for every $u\in X$, $a\in\mathbb{R}^k$, and $B\in\mathcal{B}(\Omega)$ it holds
$$F(u+a,B)=F(u,B);$$

(X4b) (*Invariance under addition of rigid motions*) assuming $k=d$, for every $u\in X$, $A\in\mathbb{R}^{d\times d}_{skew}$, $b\in\mathbb{R}^d$, and $B\in\mathcal{B}(\Omega)$, letting $R(y)\coloneqq Ay+b$, it holds
$$F(u+R,B)=F(u,B).$$

Hypotheses (X4a) and (X4b) appear naturally in two different contexts. The first is a natural condition for functionals defined over spaces of functions of bounded variation and it states that, essentially, these depend only on the distributional gradient of such functions. The second is instead natural when the base space is BD, and morally implies that they depend only the symmetric part of the distributional gradient.

Given such functional F satisfying the conditions above, it is convenient to look at some associated auxiliary minimisation problems. Given an open set with Lipschitz boundary $O\subset\Omega$, and $w\in X$, we set
$$\mathfrak{m}^F(w,O)\coloneqq\inf\{F(u,O)\colon\ u\in X\text{ with }u=w\text{ on }\partial O\},\tag{1.5.1}$$
the equality $u=w$ being understood in the sense of traces.

The preferred Dirichlet boundary conditions for the problem above will be given by affine and pure-jump functions. Given $A\in\mathbb{R}^{k\times d}$ we let $\ell_A\colon\mathbb{R}^d\to\mathbb{R}^k$ be the function defined by
$$\ell_A(y)\coloneqq Ay\quad\text{for all }y\in\mathbb{R}^d,\tag{1.5.2}$$
while for $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$ we let $u_{x,\zeta,\nu}\colon\mathbb{R}^d\to\mathbb{R}^k$ be the function defined by
$$u_{x,\zeta,\nu}(y)\coloneqq\begin{cases}\zeta & \text{if }(y-x)\cdot\nu>0,\\ 0 & \text{if }(y-x)\cdot\nu\le 0.\end{cases}\tag{1.5.3}$$

The preferred domains for the minimisation problems (1.5.1) will be given by oriented cubes. We recall that $Q_\nu(x,\rho)$ is the cube given by (1.1.1) while $Q(x,\rho)\coloneqq x+(-\rho/2,\rho/2)^d$.

We are finally ready to introduce the candidate bulk and surface integrands $f_{\mathrm{bulk}}\colon\Omega\times\mathbb{R}^{k\times d}\to[0,+\infty)$ and $f_{\mathrm{surf}}\colon\Omega\times\mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$; they are defined by
$$f_{\mathrm{bulk}}(x,A)\coloneqq\limsup_{\rho\to0^+}\frac{\mathfrak{m}^F(\ell_A,Q(x,\rho))}{\rho^d}\tag{1.5.4}$$

for all $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, and

$$f_{\mathrm{surf}}(x,\zeta,\nu) \coloneqq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \tag{1.5.5}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

Under the hypotheses of the Theorems 1.5.1 and 1.5.4 below, we will always have

$$c(|A|-1) \le f_{\mathrm{bulk}}(x,A) \le C(|A|+1) \quad \text{for } A \in \mathbb{R}^{k\times d}$$

for some constants $c, C > 0$. This condition will allow us to consider f^∞_{bulk}, the *recession function* of f_{bulk}, which is defined by

$$f^\infty_{\mathrm{bulk}}(x,A) \coloneqq \limsup_{t\to+\infty} \frac{f_{\mathrm{bulk}}(x,tA)}{t} \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d} \tag{1.5.6}$$

and that plays a crucial role in some of the results below.

Integral representation in BV. The following integral representation result was proved by Bouchitté, Fonseca, & Mascarenhas [41]. We stress that (1.5.8) could be weakened significantly by requiring a uniform continuity in the x-variable (see [41, (2.4)] for details) instead of independence. As we will not need this more general version, we only state here a simplified statement.

Theorem 1.5.1 ([41, Theorems 3.7 and 3.12])**.** *Let $F\colon \mathrm{BV}(\Omega;\mathbb{R}^k) \times \mathcal{B}(\Omega) \to [0,+\infty)$ be a functional satisfying assumptions* (X1)–(X4a) *with $X = \mathrm{BV}(\Omega;\mathbb{R}^k)$. Assume also that there exist two constants $c, C > 0$ such that*

$$c\big(|Du|(B) - \mathcal{L}^d(B)\big) \le F(u,B) \le C\big(|Du|(B) + \mathcal{L}^d(B)\big) \tag{1.5.7}$$

for every $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$. Then, for every $u \in \mathrm{SBV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ it holds

$$F(u,B) = \int_B f_{\mathrm{bulk}}(x,\nabla u)\,\mathrm{d}x + \int_{J_u\cap B} f_{\mathrm{surf}}(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

where f_{bulk} and f_{surf} are given by (1.5.4) *and* (1.5.5)*, respectively.*

Moreover, if for every $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $x \in \mathbb{R}^d$, it holds

$$F(u(\cdot - x), x + B) = F(u,B) \tag{1.5.8}$$

for every $B \in \mathcal{B}(\Omega)$ with $x + B \subset \Omega$, then f_{bulk} and f_{surf} do not depend on x and (omitting their dependence on this variable) we have

$$F(u,B) = \int_B f_{\mathrm{bulk}}(\nabla u)\,\mathrm{d}x + \int_B f^\infty_{\mathrm{bulk}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap B} f_{\mathrm{surf}}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

for all $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$.

The proof of the second part of the result above is based on the following lemma, concerning the Radon-Nikodým derivative of $F(u,\cdot)$ with respect to $|Du|$ at a point $x \in C(u)$, where $C(u)$ is the set of Theorem 1.3.1. We recall the definition of the rectangles $Q^\lambda_\nu(x,\rho)$ given in (1.1.2).

Lemma 1.5.2. *Let $F\colon \mathrm{BV}(\Omega;\mathbb{R}^k)\times\mathcal{B}(\Omega)\to[0,+\infty)$ be a functional satisfying assumptions* (X1)–(X4a) *with $X=\mathrm{BV}(\Omega;\mathbb{R}^k)$, let $u\in\mathrm{BV}(\Omega;\mathbb{R}^k)$, and let $C(u)$ be the Borel set of Theorem 1.3.1. For every $x\in C(u)$, every $\lambda\geq 1$, and every $\rho>0$ we set*

$$s^\lambda_\rho(x):=\frac{|D^cu|(Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d}\qquad\text{and}\qquad A^\lambda_\rho(x):=s^\lambda_\rho(x)(a_u(x)\otimes b_u(x)),$$

where $(a_u(x),b_u(x))\in\mathbb{R}^k\times\mathbb{S}^{d-1}$ satisfy (1.3.2)*. Then*

$$\lim_{\rho\to0^+}s^\lambda_\rho(x)=+\infty\qquad\text{and}\qquad\lim_{\rho\to0^+}\rho s^\lambda_\rho(x)=0\quad\text{for every }\lambda\geq1,$$
$$\frac{\mathrm{d}F(u,\cdot)}{\mathrm{d}|Du|}(x)=\lim_{\lambda\to+\infty}\limsup_{\rho\to0^+}\frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)},Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^ds^\lambda_\rho(x)},$$

for $|D^cu|$-a.e. $x\in C(u)$.

Integral representation in SBV^2. An analogue of Theorem 1.5.1 was later proved by Bouchitté, Fonseca, Leoni, & Mascarenhas [40] for functionals defined on SBV^2.

Theorem 1.5.3 ([40, Theorem 1])**.** *Let $F\colon \mathrm{SBV}^2(\Omega;\mathbb{R}^k)\times\mathcal{B}(\Omega)\to[0,+\infty)$ be a functional satisfying assumptions* (X1)–(X4a) *with $X=\mathrm{SBV}^2(\Omega;\mathbb{R}^k)$. Assume also that there exist two constants $c,C>0$ such that*

$$c\Big(\int_B(|\nabla u|^2-1)\,\mathrm{d}x+\int_{J_u\cap B}(|[u]|+1)\,\mathrm{d}\mathcal{H}^{d-1}\Big)\leq F(u,B)\leq C\Big(\int_B(|\nabla u|^2+1)\,\mathrm{d}x+\int_{J_u\cap B}(|[u]|+1)\,\mathrm{d}\mathcal{H}^{d-1}\Big)\tag{1.5.9}$$

for every $u\in\mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ and $B\in\mathcal{B}(\Omega)$. Then, for every $u\in\mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ and $B\in\mathcal{B}(\Omega)$ it holds

$$F(u,B)=\int_B f_{\mathrm{bulk}}(x,\nabla u)\,\mathrm{d}x+\int_{J_u\cap B}f_{\mathrm{surf}}(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

where f_{bulk} and f_{surf} are given by (1.5.4) *and* (1.5.5)*, respectively.*

Integral representation in BD. We conclude this section with an analogue of the previous theorems in the setting of functions of bounded deformation. The first part of the statement below was obtained by Ebobisse & Toader in [115], following the lines of the proof [41], while the second part has been recently obtained by Caroccia, Focardi, & Van Goethem [63], by means of a careful blow-up analysis based on Theorem 1.4.1.

Theorem 1.5.4 ([115, Theorem 3.3] and [63, Theorem 2.3 and Corollary 5.5])**.**
Let $F\colon \mathrm{BD}(\Omega)\times\mathcal{B}(\Omega)\to[0,+\infty)$ be a functional satisfying assumptions (X1)–(X3)*, and* (X4b) *with $X=\mathrm{BD}(\Omega)$. Assume also that there exist two constants $c,C>0$ such that*

$$c\big(|Eu|(B)-\mathcal{L}^d(B)\big)\leq F(u,B)\leq C\big(|Eu|(B)+\mathcal{L}^d(B)\big)\tag{1.5.10}$$

for every $u\in\mathrm{BD}(\Omega)$ and $B\in\mathcal{B}(\Omega)$. Then, for every $u\in\mathrm{SBD}(\Omega)$ and $B\in\mathcal{B}(\Omega)$ it holds

$$F(u,B)=\int_B f_{\mathrm{bulk}}(x,\mathcal{E}u)\,\mathrm{d}x+\int_{J_u\cap B}f_{\mathrm{surf}}(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

where f_{bulk} and f_{surf} are given by (1.5.4) *and* (1.5.5)*, respectively.*

Moreover, if for every $u \in \mathrm{BD}(\Omega)$ and $x \in \mathbb{R}^d$, it holds

$$F(u(\cdot - x), x + B) = F(u, B) \tag{1.5.11}$$

for every $B \in \mathcal{B}(\Omega)$ with $x + B \subset \Omega$, then f_{bulk} and f_{surf} do not depend on x and (omitting their dependence on this variable) we have

$$F(u, B) = \int_B f_{\mathrm{bulk}}(\mathcal{E}u)\,\mathrm{d}x + \int_B f^{\infty}_{\mathrm{bulk}}\Big(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u \cap B} f_{\mathrm{surf}}([u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

for all $u \in \mathrm{BD}(\Omega)$ and $B \in \mathcal{B}(\Omega)$.

One of the technical results proved in [63] is a variant in the BD-setting of Lemma 1.5.2, which we will use in Chapter 4. To present its statement we set some further notation. For every $\lambda > 0$ and every pair $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d$ with a, b linearly independent we set

$$P^{a,b}_{\lambda} := \big\{z \in \mathbb{R}^d \colon\ |z \cdot b| < \tfrac{\lambda}{2},\ |z \cdot a| < \tfrac{1}{2},\ |z \cdot \theta_i| < \tfrac{1}{2}\ \text{ for } i \in \{1, \dots, d-2\}\big\},$$

where $\{\theta_i\}_{i=1}^{d-2} \subset \mathbb{S}^{d-1}$ is such that $\{a, b, \theta_1, \dots, \theta_{d-2}\}$ is a basis of $\mathbb{R}^d$, while if a and b are linearly dependent we set

$$P^{a,b}_{\lambda} := \big\{z \in \mathbb{R}^d \colon\ |z \cdot b| < \tfrac{\lambda}{2},\ |z \cdot \theta_i| < \tfrac{1}{2}\ \text{ for } i \in \{1, \dots, d-1\}\big\},$$

where $\{\theta_i\}_{i=1}^{d-1} \subset \mathbb{S}^{d-1}$ is such that $\{b, \theta_1, \dots, \theta_{d-1}\}$ is a basis of $\mathbb{R}^d$. The choice of the vectors $\{\theta_i\}_i$ in the previous definitions is irrelevant for the arguments that follow. Given $x \in \mathbb{R}^d$ and $\rho > 0$, we also set $P^{a,b}_{\lambda}(x, \rho) := x + \rho P^{a,b}_{\lambda}$. Moreover, given $u \in \mathrm{BD}(\Omega)$ and letting $C(u)$, a_u, b_u be the Borel set and the Borel maps of Theorem 1.4.1, for every point $x \in C(u)$ we set $P^x_{\lambda}(x, \rho) := P^{a_u(x), b_u(x)}_{\lambda}(x, \rho)$.

Lemma 1.5.5 ([63, Lemma 5.3])**.** *Let $F \colon \mathrm{BD}(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty)$ be a functional satisfying assumptions* (X1)–(X3)*, and* (X4b) *with $X = \mathrm{BD}(\Omega)$, and let $u \in \mathrm{BD}(\Omega)$. Then there exists $C(u) \in \mathcal{B}(\Omega)$, with $|E^c u|(\Omega) = |E^c u|(C(u))$, such that for every $x \in C(u)$ equality* (1.4.2) *holds and there exist a positive sequence $\{\lambda_j\}_j$, converging to 0 as $j \to +\infty$, and for every $j \in \mathbb{N}$ a positive sequence $\{\rho_{i,j}\}_i$, converging to 0 as $i \to +\infty$, such that, setting*

$$A = A(x) := a_u(x) \odot b_u(x) \quad \text{and} \quad s_{i,j} := \frac{|Eu|(P^x_{\lambda_j}(x, \rho_{i,j}))}{\mathcal{L}^d(P^x_{\lambda_j}(x, \rho_{i,j}))}, \tag{1.5.12}$$

we have

$$\text{for every } j \in \mathbb{N} \text{ the sequence } \{s_{i,j}\}_i \text{ tends to } +\infty \text{ as } i \to +\infty, \tag{1.5.13}$$

$$\frac{\mathrm{d}F(u, \cdot)}{\mathrm{d}|Eu|}(x) = \lim_{j \to +\infty} \limsup_{i \to +\infty} \frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_{\lambda_j}(x, \rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_{\lambda_j}(x, \rho_{i,j}))}. \tag{1.5.14}$$

Proof. This result is proved in [63, Lemma 5.3] under slightly different hypotheses on the functional F. In particular, they assume a condition, which they denote by (H4), that prescribes a joint continuity of the functional F with respect to translations both of the dependent and of the independent variables. Not all functionals F as above satisfy this property. However, examining the proof of [63, Lemma 5.3] one can see that the use of property (H4) can be replaced by the property (X4b). This invariance property also allows us to consider the boundary conditions appearing in (1.5.14) instead of those used in [63, (5.6)], which differ from ours by a rigid motion. □

Part I

Free discontinuity problems with cohesive surface terms

CHAPTER 2

Homogenisation of free discontinuity problems with cohesive surface terms: the full gradient case

This chapter is devoted to the study of homogenisation problems for free discontinuity functionals depending on the full distributional gradient Du, with general nonlinear surface terms. In particular, we investigate via Γ-convergence the asymptotic behaviour as $\varepsilon \to 0^+$ of functionals of the form

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap\Omega} g\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \quad (2.0.1)$$

where Ω is a bounded open subset of $\mathbb{R}^d$, $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ (see Definition 1.3.16), the Borel measurable integrands $f\colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0,+\infty)$ and $g\colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfy

$$c\big(|A|-1\big) \le f(x,A) \le C\big(|A|+1\big) \quad \text{for all } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{k\times d}, \quad (2.0.2)$$

$$c\big(|\zeta|\wedge 1\big) \le g(x,\zeta,\nu) \le C\big(|\zeta|\wedge 1\big) \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k, \text{ and } \nu \in \mathbb{S}^{d-1}, \quad (2.0.3)$$

for suitable positive constants c, C, and the recession function f^∞ is given by (1.5.6), with f_{bulk} replaced by f. The particular form of F_ε is dictated by lower semicontinuity considerations, as under the above growth conditions, one must also take into account contributions related to the Cantor part.

Our goal is twofold. First, without assuming any periodicity in the x-variable, we want to provide necessary and sufficient conditions for the sequence $\{F_\varepsilon\}_\varepsilon$ to Γ-converge, as $\varepsilon \to 0^+$, to a *homogenised* functional

$$F_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap\Omega} g_{\mathrm{hom}}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \quad (2.0.4)$$

where f_{hom} and g_{hom} are x-independent bulk and surface energy densities. This characterisation is then employed to deal with the corresponding problem in the probabilistic setting. More precisely, we assume that the integrands f and g are random functions, and prove that under the usual assumptions of stochastic homogenisation (see, for instance, [58]) the sequence $\{F_\varepsilon\}_\varepsilon$ Γ-converges *almost surely* to F_{hom} as $\varepsilon \to 0^+$.

We here mention that, in order to obtain the above-mentioned characterisation of Γ-convergence of sequences $\{F_\varepsilon\}_\varepsilon$, we develop a robust functional framework able to handle the Γ-convergence of general sequences of integral functionals $\{F_n\}_n$, in which the dependence on the parameter n is general and is not simply given by rescaling as in (2.0.1). The general strategy adopted here relies on the *localisation method* of Γ-convergence (see [45, Chapter 3]), in the variant developed in [101], where the same problem was studied in the scalar case $k = 1$. This strategy allows us to treat rather general sequences $\{F_n\}_n$ and ultimately uses *integral representation results* in $\mathrm{GBV}_\star$ (see Theorems 2.3.16 and 2.5.1), which are of independent interest. As in [101], we make use of truncation techniques at several points. However, due to the vector-valued nature of our setting, a different type of truncation is

required. This renders the work substantially more technical and leads to a non-trivial departure from the scalar case.

We now briefly describe the organisation of the chapter. In Section 2.1 we introduce a general class of functionals defined on $\mathrm{GBV}_\star$, denoted by $\mathfrak{F}$, which includes, in particular, the functionals F_ε defined by (2.0.1) for many choices of f and g. The properties of these general functionals are then investigated in Section 2.2, the main result being a compactness/closure theorem for $\mathfrak{F}$ (see Theorem 2.2.1): every sequence $\{F_n\}_n \subset \mathfrak{F}$ admits a subsequence Γ-converging to a lower semicontinuous functional $F \in \mathfrak{F}$.

We then address the integral representation of possible limit functionals $F \in \mathfrak{F}$. Relying on rather technical truncation arguments based on Lemma 2.3.9 and on the by-now classical integral representation results of Bouchitté, Fonseca, & Mascarenhas [41] (see Section 1.5), we prove in Section 2.3 that the bulk and surface parts of every such F can be represented as integral functionals associated to a suitable choice of integrands f and g satisfying (2.0.2) and (2.0.3) (see Theorem 2.3.16). As customary in integral representation results for free discontinuity problems, f and g are determined by the asymptotic behaviour of certain boundary value problems related to F on small, possibly oriented, cubes.

In order to obtain a complete integral representation for functionals in $\mathfrak{F}$, including their Cantor part, we introduce in Section 2.4 a relevant subclass of $\mathfrak{F}$, denoted by $\mathfrak{F}^{\alpha,\vartheta}$ and we study its closure under Γ-convergence (see Proposition 2.4.12). The additional property satisfied by functionals in this class ensures that the integrands f and g obtained in Theorem 2.3.16 satisfy quantitative estimates relating f to f^∞ and g to the function g^0 defined by $g^0(x,\zeta,\nu) \coloneqq \lim_{t\to 0^+} g(x,t\zeta,\nu)/t$. These estimates are fundamental tools of the analysis carried out in Section 2.5, where we show that, if F is the Γ-limit of a sequence $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\vartheta}$ and the function f given by Theorem 2.3.16 does not depend on x, then F admits a complete integral representation in terms of the integrands f, f^∞, and g obtained in Theorem 2.3.16.

In Section 2.6, we then study the relation between Γ-convergence of a sequence $\{F_n\}_n \subset \mathfrak{F}$ and the asymptotic behaviour of associated boundary value problems. Together with the theory developed in Section 2.5, these results allow us to derive in Section 2.7 the desired necessary and sufficient conditions for the Γ-convergence of the functionals $\{F_\varepsilon\}_\varepsilon$ given by (2.0.1) to a functional F_{hom} as in (2.0.4) (see Theorem 2.7.1). Finally, relying on techniques developed by Cagnetti, Dal Maso, Scardia, & Zeppieri in [58, 59], we use Theorem 2.7.1 to prove the desired homogenisation result in the probabilistic setting (see Theorem 2.7.9).

This chapter is based on the joint work with Dal Maso [90].

2.1 Volume and surface integrands

We begin by introducing the collections of functionals we mainly consider in the following. These collections are rather general and will contain sequences $\{F_\varepsilon\}_\varepsilon$ given in (2.0.1). The study of this particular type of sequences is postponed to Section 2.7.

For the notation used in this chapter, we refer the reader to Sections 1.1–1.3 and 1.5. In particular, we refer to Section 1.3 for the definition and the main properties of the space $\mathrm{GBV}_\star$ and to Section 1.1 for the notation concerning families of open/Borel sets.

Throughout the rest of the chapter, we keep fixed *six* constants $c_1, \dots, c_6 \geq 0$ and a

bounded continuous function $\tau\colon [0,+\infty) \to [0,+\infty)$. We also assume that

$$0 < c_1 \le 1 \le c_3 \le c_5, \quad c_1 k \le c_3, \quad c_5 \ge k^{1/2}, \quad \text{and} \quad c_6 \ge (c_3/c_1)k^{3/2}, \tag{2.1.1}$$
$$\tau(0) = 0 \quad \text{and} \quad \tau(t) \ge (c_1 k^{1/2} t) \wedge c_3 \quad \text{for every } t \ge 0.$$

We introduce the collection of *volume* integrands considered in this chapter.

Definition 2.1.1. Let $\mathscr{F}$ be the collection of all functions $f\colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0,+\infty)$ satisfying the following conditions:

(f1) (*Measurability*) f is Borel measurable;

(f2) (*Lower bound*) $c_1 \sum_{i=1}^k |A_i| - c_2 \le f(x,A)$ for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{k\times d}$;

(f3) (*Upper bound*) $f(x,A) \le c_3 \sum_{i=1}^k |A_i| + c_4$ for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{k\times d}$;

(f4) (*Lipschitz property*) $|f(x,A) - f(x,L)| \le c_5 |A-L|$ for every $x \in \mathbb{R}^d$ and $A, L \in \mathbb{R}^{k\times d}$.

Remark 2.1.2. It follows from the elementary inequalities $|A| \le \sum_{i=1}^k |A_i| \le k^{1/2}|A|$ that

(f2′) $c_1|A| - c_2 \le f(x,A)$ for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{k\times d}$,

(f3′) $f(x,A) \le c_3 k^{1/2}|A| + c_4$ for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{k\times d}$.

The following definition introduces the collection of *surface* integrands considered in this chapter.

Definition 2.1.3. Let $\mathscr{G}$ be the collection of all functions $g : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ that satisfy the following conditions:

(g1) (*Measurability*) g is Borel measurable;

(g2) (*Lower bound*) $c_1 \sum_{i=1}^k (|\zeta_i| \wedge 1) \le g(x,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;

(g3) (*Upper bound*) $g(x,\zeta,\nu) \le c_3 \sum_{i=1}^k (|\zeta_i| \wedge 1)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;

(g4) (*Continuity in* ζ) $|g(x,\zeta,\nu) - g(x,\theta,\nu)| \le \tau(|\zeta-\theta|)$ for every $x \in \mathbb{R}^d$, $\zeta,\theta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;

(g5) (*Symmetry*) $g(x,-\zeta,-\nu) = g(x,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;

(g6) (*Quasi-monotonicity*) for every $\zeta,\theta \in \mathbb{R}^k$ with $c_6 k|\zeta| \le |\theta|$ we have $g(x,\zeta,\nu) \le g(x,\theta,\nu)$ for every $x \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$.

Remark 2.1.4. (A variant of) Property (g6) was already introduced in [57]; its main motivation is that it yields suitable truncation estimates (see the proof of Proposition 2.1.14 below). Combining (g2) and (g3), it is easy to show that

$$g(x,\zeta,\nu) \le \frac{c_3 k}{c_1} g(x,\theta,\nu)$$

whenever $\sum_{i=1}^k |\zeta_i| \le \sum_{i=1}^k |\theta_i|$. Arguing as in [57, Remark 3.2] one can show that this property and (g6) are weaker than monotonicity of g with respect to $|\zeta|$.

We stress that, in the scalar case $k = 1$ treated in [101], the authors assume monotonicity with respect to $|\zeta|$ of the surface integrands they consider. Thus, already in the scalar case, the present framework covers surface integrands that are not included in [101, Definition 2.5].

Remark 2.1.5. Thanks to the lower and upper bounds (f2) and (f3), we can associate to every function $f \in \mathscr{F}$ its recession function f^∞, defined by (1.5.6) with $f_{\rm bulk}$ replaced by f. We observe that, since f satisfies (f2) and (f3), f^∞ satisfies

$$c_1|A| \le c_1 \sum_{i=1}^{k} |A_i| \le f^\infty(x,A) \le c_3 k^{1/2}|A| \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{k\times d}, \tag{2.1.2}$$

while from (f4) it follows that

$$|f^\infty(x,A) - f^\infty(x,L)| \le c_5|A-L| \quad \text{for every } x \in \mathbb{R}^d \text{ and } A, L \in \mathbb{R}^{k\times d}.$$

In Section 2.6, we will also consider integrands g^0 that do not belong to $\mathscr{G}$. For this reason, in the following definition we do not assume $g \in \mathscr{G}$. We refer the reader to Section 1.1 for the notation concerning families of open/Borel sets.

Definition 2.1.6. Given $f \in \mathscr{F}$ and a Borel function $g\colon \mathbb{R}^d\times\mathbb{R}^k\times\mathbb{S}^{d-1} \to [0,+\infty)$ satisfying (g5), $F^{f,g} : L^0(\mathbb{R}^d;\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ is the functional defined, for $O \in \mathcal{O}_c(\mathbb{R}^d)$, by

$$F^{f,g}(u,O) \coloneqq \int_O f(x,\nabla u)\,\mathrm{d}x + \int_O f^\infty\Big(x, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| + \int_{J_u\cap O} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

if $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$, and by $F^{f,g}(u,O) = +\infty$ otherwise. The definition is then extended to $O \in \mathcal{O}(\mathbb{R}^d)$ by setting

$$F^{f,g}(u,O) \coloneqq \sup\{F^{f,g}(u,O')\colon\ O' \subset O,\ O' \in \mathcal{O}_c(\mathbb{R}^d)\},$$

and then to $B \in \mathcal{B}(\mathbb{R}^d)$ by setting

$$F^{f,g}(u,B) \coloneqq \inf\{F^{f,g}(u,O)\colon\ O \supset B,\ O \in \mathcal{O}(\mathbb{R}^d)\}.$$

To study the integral representation of Γ-limits of sequences of functionals of the form $F^{f,g}$ with $f \in \mathscr{F}$ and $g \in \mathscr{G}$, it is convenient to study the properties of functionals $F^{f,g}$ that pass to the Γ-limit. This leads us to define an abstract collection $\mathfrak{F}$ of functionals defined on $L^0(\mathbb{R}^d;\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d)$. We shall prove that the Γ-limit of a sequence of functionals in $\mathfrak{F}$ belongs to $\mathfrak{F}$ (see Theorem 2.2.1) and that for every lower semicontinuous functional in $\mathfrak{F}$ the volume and surface part admit an integral representation of the form

$$\int_A f(x,\nabla u)\,\mathrm{d}x \quad \text{and} \quad \int_{J_u} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

with integrands $f \in \mathscr{F}$ and $g \in \mathscr{G}$ (see Theorem 2.3.16).

For technical reasons, in the definition of the collection $\mathfrak{F}$ we consider the behaviour of the functionals with respect to truncations performed using the functions Φ_R introduced in (1.3.12), for $\sigma = c_6 k + 2$. In particular, given $i \in \mathbb{N}$ and $R > 0$, we consider the function defined by $\Phi_R^i \coloneqq \Phi_{\sigma^{i-1}R}$. We also recall the definitions (1.5.2), (1.5.3) of the auxiliary functions ℓ_A, $u_{x,\zeta,\nu}$, and of the hyperplane Π_x^ν given in (1.1.5).

Definition 2.1.7. Let $\mathfrak{F}$ be the space of functionals $F\colon L^0(\mathbb{R}^d;\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ satisfying the following conditions:

(H1) (*Locality*) for every $O \in \mathcal{O}(\mathbb{R}^d)$ and for every $u, v \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ with $u = v$ $\mathcal{L}^d$-a.e. in O, we have $F(u,O) = F(v,O)$;

(H2) (*Measure property*) for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ the set function $B \mapsto F(u,B)$ is a non-negative Borel measure on $\mathbb{R}^d$ and for every $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u,B) \coloneqq \inf\{F(u,O)\colon\ O \supset B,\ O \in \mathcal{O}(\mathbb{R}^d)\};$$

(H3) (*Lower bound*) for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$c_1\mathcal{V}(u,B) - c_2\mathcal{L}^d(B) \le F(u,B);$$

(H4) (*Upper bound*) for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u,B) \le c_3\mathcal{V}(u,B) + c_4\mathcal{L}^d(B);$$

(H5) (*Invariance under addition of constants*) for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $a \in \mathbb{R}^k$ we have

$$F(u+a,B) = F(u,B);$$

(H6) (*Bulk estimate*) for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $A \in \mathbb{R}^{k\times d}$, $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u+\ell_A,B) \le F(u,B) + c_5|A|\mathcal{L}^d(B);$$

(H7) (*Surface estimate*) for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$F(u+u_{x,\zeta,\nu},B) \le F(u,B) + \tau(|\zeta|)\mathcal{H}^{d-1}(B\cap\Pi^\nu_x),$$

where τ is the function introduced at the beginning of this section;

(H8) (*Truncation property*) for every $m \in \mathbb{N}$, $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $B \in \mathcal{B}(\mathbb{R}^d)$, $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$, and $R > 0$ we have

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m F(w+\Phi^i_R\circ(u-w),B) &\le F(u,B) + c_3k^{1/2}\int_{B^R_{u,w}}|\nabla w|\,\mathrm{d}x + c_4\mathcal{L}^d(B^R_{u,w})\\ &\quad + \frac{C}{m}\Big(F(u,B) + \mathcal{L}^d(B) + \int_B|\nabla w|\,\mathrm{d}x\Big),\end{aligned} \tag{2.1.3}$$

where $B^R_{u,w} \coloneqq \{x\in B\colon\ |u(x)-w(x)|\ge R\}$ and $C \coloneqq \max\{\frac{9c_3k}{c_1}, 2c_3k^{1/2}, c_4+\frac{c_2c_3}{c_1}k^{1/2}\}$;

(H9) (*Estimate for rotations*) for every $0 < \lambda \le 1/(c_6k)$, $R \in \mathrm{SO}(k)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ we have

$$F(\lambda Ru,B) \le F(u,B) + (c_4+c_2)\mathcal{L}^d(B).$$

The subcollection of all functionals $F \in \mathfrak{F}$ with the property that for every $O \in \mathcal{O}(\mathbb{R}^d)$ the functional $F(\cdot,O)$ is lower semicontinuous with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$ is denoted by $\mathfrak{F}_{\mathrm{sc}}$.

Remark 2.1.8. Let $F \in \mathfrak{F}$, $O \in \mathcal{O}(\mathbb{R}^d)$, and $u \in L^0(O;\mathbb{R}^k)$. We can define $F(u,B)$ for every $B \in \mathcal{B}(O)$ by extending the function u to a function $v \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and setting $F(u,B) \coloneqq F(v,B)$. The value of $F(u,B)$ does not depend on the chosen extension thanks to locality property (H1) of Definition 2.1.7.

Remark 2.1.9. Inequalities (H3) and (H4) imply that

$$c_1\mathcal{V}_{\rm vec}(u,B)-c_2\mathcal{L}^d(B)\le F(u,B) \tag{H3$'$}$$

$$F(u,B)\le c_3k\mathcal{V}_{\rm vec}(u,B)+c_4\mathcal{L}^d(B), \tag{H4$'$}$$

for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $B\in\mathcal{B}(\mathbb{R}^d)$. This follows from the elementary inequalities $|A|\le\sum_{i=1}^k|A_i|\le k^{1/2}|A|$ and $|\zeta|\wedge 1\le\sum_{i=1}^k(|\zeta_i|\wedge 1)\le k(|\zeta|\wedge 1)$, where A_i is the i-th row of the matrix A and ζ_i is the i-th component of ζ.

Remark 2.1.10. The functional $\mathcal{V}$ satisfies conditions (H1)–(H7) of Definition 2.1.7. However, it can be shown that, if $k>1$, it does not satisfy (H8) and (H9). On the other hand, the functional $\mathcal{V}_{\rm vec}$ defined in Remark 1.3.15 also satisfies the additional conditions (H8) and (H9). We do not know, however, whether $\mathcal{V}_{\rm vec}$ is lower semicontinuous.

This property is instead satisfied by $\mathcal{V}$, thanks to the same remark. In order not to alter the value of the constant c_1 in the lower bound (H3) when passing to Γ-limits, we give the upper and lower bounds (H3) and (H4) in terms of $\mathcal{V}$, in place of $\mathcal{V}_{\rm vec}$.

Remark 2.1.11. The technical condition (H8) replaces the simpler condition (g) of [100, Definition 3.1], which, in the particular case $w_1=-m$ and $w_2=m$, reads as

$$F(u^{(m)},O)\le F(u,O)+\mathcal{L}^d(\{|u|>m\}). \tag{2.1.4}$$

Although this truncation procedure is still available in the vector-valued case (see Definition 1.3.16 and Proposition 1.3.19), acting component-wise, an estimate of the form (2.1.4) does not hold for $k>1$, even for the prototypical integral functional $\mathcal{V}_{\rm vec}$. This is one of the main difficulties in generalising the results of [101] to the vector-valued case. The new condition (H8) of Definition 2.1.7 is crucial to introduce a different type of truncation operators with good estimates. The presence of an average in the left-hand side of inequality (2.1.3) makes it easy to select a truncation level with the desired properties (see, for instance, the proof of Lemma 2.1.15, where this property is used for the first time). Similar smooth truncations were already considered, for instance, in [48, 57, 64, 124, 130].

Remark 2.1.12. Condition (H9) is of a technical nature and will be used in Proposition 2.3.14 to show that, when $F\in\mathfrak{F}$ admits a (partial) integral representation, its surface integrand always satisfies quasi-monotonicity property (g6).

Remark 2.1.13. It follows directly from (a) of Definition 1.3.16 and from Definition 2.1.7 that if $O\in\mathcal{O}_c(\mathbb{R}^d)$ and $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ then

$$F(u,O)<+\infty \quad\text{if and only if}\quad u\in \mathrm{GBV}_\star(O;\mathbb{R}^k).$$

The following proposition shows that functionals of the type $F^{f,g}$ introduced in Definition 2.1.6 belong to $\mathfrak{F}$.

Proposition 2.1.14. *Let $f\in\mathscr{F}$ and $g\in\mathscr{G}$. Then the functional $F^{f,g}$ belongs to $\mathfrak{F}$ and for every $O\in\mathcal{O}_c(\mathbb{R}^d)$ and $u\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$ we have*

$$F^{f,g}(u,B)=\int_B f(x,\nabla u)\,\mathrm{d}x+\int_B f^\infty\Big(x,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\mathrm{d}|D^cu|+\int_{J_u\cap B}g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.1.5}$$

for every $B\in\mathcal{B}(O)$.

Proof. The proof of equality (2.1.5) and of the fact that $F^{f,g}$ satisfies (H1)–(H7) of Definition 2.1.7 may be deduced from the proof of [100, Proposition 3.11], with minor changes.

We are left with showing that $F^{f,g}$ enjoys properties (H8) and (H9) of Definition 2.1.7. We begin addressing (H8). Since $F^{f,g}$ satisfies properties (H1) and (H2), it is enough to prove (2.1.3) for every $O \in \mathcal{O}_c(\mathbb{R}^d)$. Without loss of generality, we may also assume that $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$.

Let $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$, let $R > 0$, let $m \in \mathbb{N}$, and let $i \in \{1,\dots,m\}$. We set $v^i_R \coloneqq w + \Phi^i_R \circ (u - w)$ and note that $v^i_R \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$, thanks to (c) of Definition 1.3.16 and to the fact that $\mathrm{GBV}_\star(O;\mathbb{R}^k)$ is a vector space.

Consider the set $O_{\mathrm{reg}} \coloneqq \{x \in O\colon\ \widetilde{u}(x) \text{ and } \widetilde{w}(x) \text{ exist}\}$. We recall that O_{reg} and J_u are Borel sets and that, thanks to (a) of Proposition 1.3.10, we have $\mathcal{H}^{d-1}(O \setminus (O_{\mathrm{reg}} \cup J_u)) = 0$. We set

$$\begin{aligned} O^i_{\mathrm{in}} &\coloneqq \{x \in O_{\mathrm{reg}}\colon\ |\widetilde{u}(x) - \widetilde{w}(x)| \le \sigma^{i-1}R\},\\ O^i &\coloneqq \{x \in O_{\mathrm{reg}}\colon\ \sigma^{i-1}R < |\widetilde{u}(x) - \widetilde{w}(x)| < \sigma^i R\},\\ O^i_{\mathrm{out}} &\coloneqq \{x \in O_{\mathrm{reg}}\colon\ |\widetilde{u}(x) - \widetilde{w}(x)| \ge \sigma^i R\}. \end{aligned}$$

In view of (b) of Proposition 1.3.10, for every $i \in \{1,\dots,m\}$ we have that

$$\begin{aligned} &\nabla v^i_R = \nabla u \quad \mathcal{L}^d\text{-a.e. in } O^i_{\mathrm{in}},\\ &\nabla v^i_R = \nabla w + \nabla\Phi^i_R \circ (u - w)(\nabla u - \nabla w) \quad \mathcal{L}^d\text{-a.e. in } O^i,\\ &\nabla v^i_R = \nabla w \quad \mathcal{L}^d\text{-a.e. in } O^i_{\mathrm{out}}. \end{aligned} \tag{2.1.6}$$

Therefore,

$$\int_O f(x,\nabla v^i_R)\,\mathrm{d}x = \int_{O^i_{\mathrm{in}}} f(x,\nabla u)\,\mathrm{d}x + \int_{O^i} f(x,\nabla v^i_R)\,\mathrm{d}x + \int_{O^i_{\mathrm{out}}} f(x,\nabla w)\,\mathrm{d}x. \tag{2.1.7}$$

Exploiting (f2′), (f3′), and (2.1.6), and recalling that $\mathrm{Lip}(\Phi^i_R) \le 1$ by (1.3.13), we also deduce that

$$\begin{aligned} \int_{O^i} f(x,\nabla v^i_R)\,\mathrm{d}x &\le 2c_3k^{1/2}\int_{O^i}|\nabla w|\,\mathrm{d}x + c_3k^{1/2}\int_{O^i}|\nabla u|\,\mathrm{d}x + c_4\mathcal{L}^d(O^i)\\ &\le 2c_3k^{1/2}\int_{O^i}|\nabla w|\,\mathrm{d}x + \frac{c_3k^{1/2}}{c_1}\int_{O^i} f(x,\nabla u)\,\mathrm{d}x + \big(c_4 + \frac{c_2c_3}{c_1}k^{1/2}\big)\mathcal{L}^d(O^i), \end{aligned}$$

and

$$\int_{O^i_{\mathrm{out}}} f(x,\nabla w)\,\mathrm{d}x \le c_3k^{1/2}\int_{O^R_{u,w}}|\nabla w|\,\mathrm{d}x + c_4\mathcal{L}^d(O^R_{u,w}).$$

These inequalities, together with (2.1.7), lead to

$$\begin{aligned} \sum_{i=1}^m \int_O f(x,\nabla v^i_R)\,\mathrm{d}x &\le m\Big(\int_O f(x,\nabla u)\,\mathrm{d}x + c_3k^{1/2}\int_{O^R_{u,w}}|\nabla w|\,\mathrm{d}x + c_4\mathcal{L}^d(O^R_{u,w})\Big)\\ &\quad + \frac{c_3k^{1/2}}{c_1}\int_O f(x,\nabla u)\,\mathrm{d}x + 2c_3k^{1/2}\int_O|\nabla w|\,\mathrm{d}x + \big(c_4 + \frac{c_2c_3}{c_1}k^{1/2}\big)\mathcal{L}^d(O). \end{aligned} \tag{2.1.8}$$

To estimate the term of $F^{f,g}$ depending on $D^c u$, we use (i) of Proposition 1.3.21 to rewrite

$$\begin{aligned}\int_O f^\infty\Big(x, \frac{\mathrm{d}D^c v_R^i}{\mathrm{d}|D^c v_R^i|}\Big)\,\mathrm{d}|D^c v_R^i| &= \int_{O^i_{\mathrm{in}}} f^\infty\Big(x, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| \\ &\quad + \int_{O^i} f^\infty\Big(x, \frac{\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})}{|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|}\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|\mathrm{d}|D^c u| \\ &\quad + \int_{O^i_{\mathrm{out}}} f^\infty(x,0)\,\mathrm{d}|D^c u|,\end{aligned}$$

so that

$$\begin{aligned}\int_O f^\infty\Big(x, \frac{\mathrm{d}D^c v_R^i}{\mathrm{d}|D^c v_R^i|}\Big)\,\mathrm{d}|D^c v_R^i| &\le \int_O f^\infty\Big(x, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\mathrm{d}|D^c u| \\ &\quad + \int_{O^i} f^\infty\Big(x, \frac{\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})}{|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|}\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|\mathrm{d}|D^c u|,\end{aligned}$$

where we have used that $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$ and that $f^\infty(x,0)=0$. Taking advantage of (2.1.2) and of the fact that $\mathrm{Lip}(\Phi_R^i) \le 1$, we then infer that

$$\begin{aligned}\int_{O^i} f^\infty\Big(x, \frac{\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})}{|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|}\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|\mathrm{d}|D^c u| &\le c_3 k^{1/2}\int_{O^i}|\nabla\Phi_R^i(\widetilde{u}-\widetilde{w})|\mathrm{d}|D^c u| \\ &\le \frac{c_3 k^{1/2}}{c_1}\int_{O^i} f^\infty\Big(x, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u|.\end{aligned}$$

Together with the previous inequalities, this last estimate leads to

$$\begin{aligned}\sum_{i=1}^m \int_O f^\infty\Big(x, \frac{\mathrm{d}D^c v_R^i}{\mathrm{d}|D^c v_R^i|}\Big)\,\mathrm{d}|D^c v_R^i| &\le m\int_O f^\infty\Big(x, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| \\ &\quad + \frac{c_3 k^{1/2}}{c_1}\int_O f^\infty\Big(x, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u|. \qquad (2.1.9)\end{aligned}$$

We now estimate the surface term of $F^{f,g}$. To this end, we first remark that we have the inclusion $J_{v_R^i} \subset O \setminus O_{\mathrm{reg}}$, up to an $\mathcal{H}^{d-1}$-negligible set, and that for $\mathcal{H}^{d-1}$-a.e $x \in J_{v_R^i}$ we have $\nu_{v_R^i}(x) = \nu_u(x)$. Moreover, as $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$, for $\mathcal{H}^{d-1}$-a.e. $x \in O$ the approximate limit $\widetilde{w}(x)$ exists, so that we have

$$(v_R^i)^+(x) = \widetilde{w}(x) + \Phi_R^i(u^+(x)-\widetilde{w}(x)), \qquad (2.1.10)$$

$$(v_R^i)^-(x) = \widetilde{w}(x) + \Phi_R^i(u^-(x)-\widetilde{w}(x)), \qquad (2.1.11)$$

for $\mathcal{H}^{d-1}$-a.e. $x \in J_{v_R^i}$.

For $i \in \{1,\dots,m\}$ we introduce the following partition of $\widetilde{J}_{v_R^i} := \{x \in J_{v_R^i} :\ \widetilde{w}(x) \text{ exists}\}$:

$$\begin{aligned}S_1^i &:= \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x)-\widetilde{w}(x)| < \sigma^{i-1}R,\ |u^-(x)-\widetilde{w}(x)| < \sigma^{i-1}R\},\\ S_2^i &:= \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x)-\widetilde{w}(x)|, |u^-(x)-\widetilde{w}(x)| \in [\sigma^{i-1}R, \sigma^i R]\},\\ S_3^i &:= \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x)-\widetilde{w}(x)| > \sigma^i R,\ |u^-(x)-\widetilde{w}(x)| > \sigma^i R\}\},\\ S_4^i &:= \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x)-\widetilde{w}(x)| < \sigma^{i-1}R,\ |u^-(x)-\widetilde{w}(x)| > \sigma^i R\},\end{aligned}$$

$$S_5^i := \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x) - \widetilde{w}(x)| > \sigma^i R,\ |u^-(x) - \widetilde{w}(x)| < \sigma^{i-1}R\},$$
$$S_6^i := \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x) - \widetilde{w}(x)| \in [\sigma^{i-1}R, \sigma^i R],\ |u^-(x) - \widetilde{w}(x)| \notin [\sigma^{i-1}R, \sigma^i R]\},$$
$$S_7^i := \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x) - \widetilde{w}(x)| \notin [\sigma^{i-1}R, \sigma^i R],\ |u^-(x) - \widetilde{w}(x)| \in [\sigma^{i-1}R, \sigma^i R]\}.$$

Note that, for $\ell \in \{2, 6, 7\}$,

$$\text{if } S_\ell^i \cap S_\ell^j \neq \emptyset \ \text{ then } \ |i - j| \leq 1,$$

whence

$$\sum_{i=1}^{m} (\chi_{S_2^i} + \chi_{S_6^i} + \chi_{S_7^i}) \leq 9, \tag{2.1.12}$$

where we recall that $\chi_{S_j^i}$ denotes the characteristic function of S_j^i. Recalling (1.3.13), by definition of S_1^i and S_3^i,(2.1.10), and (2.1.11), we have

$$\int_{S_1^i} g(x, [v_R^i], \nu_{v_R^i})\, \mathrm{d}\mathcal{H}^{d-1} = \int_{S_1^i} g(x, [u], \nu_u)\, \mathrm{d}\mathcal{H}^{d-1}, \tag{2.1.13}$$
$$\int_{S_3^i} g(x, [v_R^i], \nu_{v_R^i})\, \mathrm{d}\mathcal{H}^{d-1} = \int_{S_3^i} g(x, 0, \nu_u)\, \mathrm{d}\mathcal{H}^{d-1} = 0. \tag{2.1.14}$$

Moreover, since $\mathrm{Lip}(\Phi_R^i) \leq 1$ by (1.3.13), we have $|[\Phi_R^i(u - w)]| \leq |[u - w]| = |[u]|$ $\mathcal{H}^{d-1}$-a.e. in $\widetilde{J}_{v_R^i}$. Hence, from (g2) and (g3) we deduce that

$$\begin{aligned}\int_{S_2^i \cup S_6^i \cup S_7^i} g(x, [v_R^i], \nu_{v_R^i})\, \mathrm{d}\mathcal{H}^{d-1} &\leq c_3 k \int_{S_2^i \cup S_6^i \cup S_7^i} (|[\Phi_R^i(u - w)]| \wedge 1)\, \mathrm{d}\mathcal{H}^{d-1} \\ &\leq c_3 k \int_{S_2^i \cup S_6^i \cup S_7^i} (|[u]| \wedge 1)\, \mathrm{d}\mathcal{H}^{d-1} \qquad (2.1.15) \\ &\leq \frac{c_3 k}{c_1} \int_{S_2^i \cup S_6^i \cup S_7^i} g(x, [u], \nu_u)\, \mathrm{d}\mathcal{H}^{d-1}. \qquad (2.1.16)\end{aligned}$$

We are left with estimating the surface integral over S_4^i and S_5^i. By definition, if $x \in S_4^i$, then $|u^+(x) - \widetilde{w}(x)| < \sigma^{i-1}R$ and $|u^-(x) - \widetilde{w}(x)| > \sigma^i R$. Therefore,

$$\begin{aligned}|[u](x)| = |u^+(x) - u^-(x)| &\geq |u^-(x) - \widetilde{w}(x)| - |u^+(x) - \widetilde{w}(x)| \\ &\geq \sigma^i R - \sigma^{i-1}R = \sigma^{i-1}R(\sigma - 1) \geq c_6 k \sigma^{i-1} R, \qquad (2.1.17)\end{aligned}$$

where we have used that, by our choice of σ in the definition of Φ, we have $\sigma > c_6 k + 1$. On the other hand, $[v_R^i](x) = u^+(x) - \widetilde{w}(x)$ for $\mathcal{H}^{d-1}$-a.e. $x \in S_4^i$, so that

$$c_6 k |[v_R^i]| = c_6 k |u^+(x) - \widetilde{w}(x)| \leq c_6 k \sigma^{i-1} R$$

for $\mathcal{H}^{d-1}$-a.e. $x \in S_4^i$. From these inequalities we deduce that $c_6 k |[v_R^i](x)| \leq |[u](x)|$ for $\mathcal{H}^{d-1}$-a.e. $x \in S_4^i$, which by (g6) implies

$$g(x, [v_R^i], \nu_{v_R^i}) \leq g(x, [u], \nu_{v_R^i}) = g(x, [u], \nu_u) \quad \mathcal{H}^{d-1}\text{-a.e. in } S_4^i.$$

The same argument shows that it holds

$$g(x, [v_R^i], \nu_{v_R^i}) \leq g(x, [u], \nu_u) \quad \mathcal{H}^{d-1}\text{-a.e. in } S_5^i.$$

From these last two inequalities we get

$$\int_{S_4^i\cup S_5^i} g(x,[v_R^i],\nu_{v_R^i})\,\mathrm{d}\mathcal{H}^{d-1} \le \int_{S_4^i\cup S_5^i} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}. \tag{2.1.18}$$

Therefore, combining (2.1.13)–(2.1.16) and (2.1.18), we obtain

$$\begin{aligned}\int_{J_{v_R^i}} g(x,[v_R^i],\nu_{v_R^i})\,\mathrm{d}\mathcal{H}^{d-1} \le{}& \int_{J_u} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\\ &+\frac{c_3k}{c_1}\int_{S_2^i\cup S_6^i\cup S_7^i} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},\end{aligned}$$

which, in light of (2.1.12), implies that

$$\sum_{i=1}^m \int_{J_{v_R^i}} g(x,[v_R^i],\nu_{v_R^i})\,\mathrm{d}\mathcal{H}^{d-1} \le \Big(m+\frac{9c_3k}{c_1}\Big)\int_{J_u} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}. \tag{2.1.19}$$

Finally, from (2.1.8), (2.1.9), and (2.1.19) we deduce that

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m F^{f,g}(v_R^i,O) \le{}& F^{f,g}(u,O)+c_3k^{1/2}\int_{O_{u,w}^R}|\nabla w|\,\mathrm{d}x + c_4\mathcal{L}^d(O_{u,w}^R)\\ &+\frac{C}{m}\left(F^{f,g}(u,O)+\mathcal{L}^d(O)+\int_O|\nabla w|\,\mathrm{d}x\right),\end{aligned}$$

where $C\coloneqq\max\{9c_3k/c_1, 2c_3k^{1/2}, c_4+\frac{c_2c_3}{c_1}k^{1/2}\}$. This shows that $F^{f,g}$ satisfies (H8).

To see that $F^{f,g}$ satisfies (H9) it is enough to verify this property for $O\in\mathcal{O}_c(\mathbb{R}^d)$ and for $u\in \mathrm{GBV}_\star(O;\mathbb{R}^k)$. Let $\lambda\le 1/(c_6k)$ and let $R\in\mathrm{SO}(k)$. Note that by (2.1.1) we also have $\lambda\le(c_1/c_3)k^{-1/2}$. We set $v\coloneqq\lambda Ru$, which clearly satisfies $v\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$. Using (f2′), (f3′), (2.1.2), and (g6), we can then estimate

$$\begin{aligned}F^{f,g}(v,O) &= \int_O f(x,\nabla v)\,\mathrm{d}x+\int_O f^\infty\Big(x,\frac{\mathrm{d}D^cv}{\mathrm{d}|D^cv|}\Big)\mathrm{d}|D^cv|+\int_{J_u\cap O} g(x,[v],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\\ &\le c_3k^{1/2}\lambda\Big(\int_O|\nabla u|\,\mathrm{d}x+|D^cu|(O)\Big)+\int_{J_u\cap O} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}+c_4\mathcal{L}^d(O)\\ &\le F^{f,g}(u,O)+(c_4+c_2)\mathcal{L}^d(O),\end{aligned}$$

which proves (H9) and concludes the proof. □

The following lemma provides an approximation result for lower semicontinuous functionals, whose proof heavily relies on property (H8) of Definition 2.1.7. More refined versions of the following result will be presented in the forthcoming sections.

Lemma 2.1.15. *Let $O_1,O_2\in\mathcal{O}_c(\mathbb{R}^d)$, let $u_1\in L^0(O_1;\mathbb{R}^k)$ and $u_2\in L^0(O_2;\mathbb{R}^k)$, and for $j=1,2$ let $F(\cdot,O_j)\colon L^0(O_j;\mathbb{R}^k)\to[0,+\infty]$ be a lower semicontinuous functional satisfying* (H4′) *of Remark 2.1.9 and* (H8) *of Definition 2.1.7 with $B=O_1$ and $B=O_2$. Then there exists a strictly increasing sequence $R_m>0$, with $R_m\to+\infty$ as $m\to+\infty$, such that*

$$\lim_{m\to+\infty} F(\Phi_{R_m}\circ u_j,O_j)=F(u_j,O_j) \tag{2.1.20}$$

for $j=1,2$.

Proof. For every sequence of positive numbers $R_m \to +\infty$ as $m \to +\infty$, we have that $\{\Phi_{R_m} \circ u_j\}_m$ converges to u_j in $L^0(O_j;\mathbb{R}^k)$ as $m \to +\infty$, so that the lower semicontinuity of $F(\cdot, O_j)$ with respect to the topology of $L^0(O_j;\mathbb{R}^k)$ implies that

$$F(u_j, O_j) \le \liminf_{m\to+\infty} F(\Phi_{R_m} \circ u_j, O_j) \tag{2.1.21}$$

for $j = 1, 2$.

Assume now that both $F(u_1, O_1)$ and $F(u_2, O_2)$ are finite. In this case, we choose a sequence $\{r_m\}_m$ converging to $+\infty$ as $m \to +\infty$ such that

$$c_4 \mathcal{L}^d((O_j)^{r_m}_{u_j,0}) \le \frac{1}{m} \quad \text{for every } m \in \mathbb{N} \text{ and for } j = 1, 2,$$

where $(O_j)^{r_m}_{u_j,0} \coloneqq \{x \in O_j \colon |u_j(x)| \ge r_m\}$, in agreement with (H8) of Definition 2.1.7. From that property applied with $w = 0$, we obtain that there exists $i(m) \in \{1, \dots, m\}$ such that

$$\begin{aligned} &F(\Phi^{i(m)}_{r_m} \circ u_1, O_1) + F(\Phi^{i(m)}_{r_m} \circ u_2, O_2) \\ &\qquad \le F(u_1, O_1) + F(u_2, O_2) + \frac{2 + C\big(F(u_1, O_1) + F(u_2, O_2) + \mathcal{L}^d(O_1) + \mathcal{L}^d(O_2)\big)}{m}. \end{aligned}$$

We now set $R_m \coloneqq \sigma^{i(m)-1} r_m$, so that $\Phi^{i(m)}_{r_m} = \Phi_{R_m}$. Up to passing to a subsequence, not relabelled, we may also assume $\{R_m\}_m$ to be strictly increasing. We then observe that the previous inequality gives

$$\limsup_{m\to+\infty} \big(F(\Phi_{R_m} \circ u_1, O_1) + F(\Phi_{R_m} \circ u_2, O_2)\big) \le F(u_1, O_1) + F(u_2, O_2).$$

Combining this last inequality with (2.1.21), we obtain (2.1.20) in the case where both $F(u_1, O_1)$ and $F(u_2, O_2)$ are finite. A simpler proof yields the result in the other cases. □

2.2 A compactness result for $\mathfrak{F}$

In this section, we investigate compactness properties of the collection $\mathfrak{F}$. The main result of this section is the following theorem.

Theorem 2.2.1. *Let $\{F_n\}_n \subset \mathfrak{F}$ be a sequence of functionals. Then there exist a subsequence of $\{F_n\}_n$, not relabelled, and a functional $F \in \mathfrak{F}_{\mathrm{sc}}$ such that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $F_n(\cdot, O)$ Γ-converges to $F(\cdot, O)$ in the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$.*

Proof. The strategy adopted here is that of [100, Lemma 3.24], which is based on the classical localisation method for Γ-convergence (see [88, Chapters 15–19]). As already mentioned, the main difference in the proof of this compactness result in the vectorial case lies in the use of a different type of truncation, which is dealt with in Lemma 2.2.5 below.

Given $O \in \mathcal{O}(\mathbb{R}^d)$, we set

$$F'(\cdot, O) \coloneqq \Gamma\text{-}\liminf_{n\to+\infty} F_n(\cdot, O) \quad \text{and} \quad F''(\cdot, O) \coloneqq \Gamma\text{-}\limsup_{n\to+\infty} F_n(\cdot, O), \tag{2.2.1}$$

$$F'_-(\cdot, O) \coloneqq \sup_{O' \in \mathcal{O}_c(O)} F'(\cdot, O') \quad \text{and} \quad F''_-(\cdot, O) \coloneqq \sup_{O' \in \mathcal{O}_c(O)} F''(\cdot, O'), \tag{2.2.2}$$

where the Γ-lim inf and the Γ-lim sup are computed with respect to the convergence of $L^0(\mathbb{R}^d;\mathbb{R}^k)$ (see Section 1.2 for the definition of Γ-liminf and Γ-limsup). Given $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$,

it is immediate to check that $F'(u,\cdot)$, $F''(u,\cdot)$, $F'_-(u,\cdot)$, and $F''_-(u,\cdot)$ are all increasing set functions.

By a compactness theorem for Γ-convergence of increasing functionals (see [88, Theorem 16.9]), there exists a subsequence of $\{F_n\}_n$, not relabelled, such that

$$F(u,O) \coloneqq F'_-(u,O) = F''_-(u,O) \quad \text{for } u \in L^0(\mathbb{R}^d;\mathbb{R}^k) \text{ and } O \in \mathcal{O}(\mathbb{R}^d). \tag{2.2.3}$$

Thus, we can define $F\colon L^0(\mathbb{R}^d;\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ by setting

$$F(u,B) = \inf\{F(u,O)\colon\ O \supset B,\ O \in \mathcal{O}(\mathbb{R}^d)\} \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d). \tag{2.2.4}$$

In Lemma 2.2.9 we shall prove that

$$F(u,O) \coloneqq F'(u,O) = F''(u,O) \quad \text{for } u \in L^0(\mathbb{R}^d;\mathbb{R}^k) \text{ and } O \in \mathcal{O}_c(\mathbb{R}^d). \tag{2.2.5}$$

This implies that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ in the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. We are left with proving that $F \in \mathfrak{F}_{sc}$.

From some general Γ-convergence results (see [88, Proposition 16.15]), it follows that F satisfies property (H1) of Definition 2.1.7. Property (H2) is much more difficult and will be obtained in Lemma 2.2.8.

Property (H3) can be obtained by using the lower semicontinuity of $\mathcal{V}$, stated in Remark 1.3.15, which gives

$$c_1\mathcal{V}(u,O) - c_2\mathcal{L}^d(O) \le F(u,O) \quad \text{for every } u \in L^0(\mathbb{R}^d;\mathbb{R}^k) \text{ and } O \in \mathcal{O}_c(\mathbb{R}^d).$$

By inner regularity the inequality can be extended to $\mathcal{O}(\mathbb{R}^d)$ and to $\mathcal{B}(\mathbb{R}^d)$, recalling (2.2.4). This completes the proof of (H3). Properties (H4)–(H7) can be derived arguing as in [100, Theorem 3.16].

Property (H8) for F in $\mathcal{O}_c(\mathbb{R}^d)$ is a consequence of Lemmas 2.2.4 and 2.2.9, while (H9) is trivial. The extension of (H8) and (H9) to all Borel sets can easily be obtained using (2.2.4).

Finally, by [88, Proposition 6.8] for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the functionals $F'(\cdot,O)$ and $F''(\cdot,O)$ are lower semicontinuous with respect to the convergence of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Since $F = F' = F''$, we conclude that $F \in \mathfrak{F}_{\rm sc}$. This concludes the proof. □

To conclude the proof of Theorem 2.2.1, it remains to prove Lemmas 2.2.8 and 2.2.9. To deal with the former, we use the De Giorgi-Letta criterion for measures [106] (see also [88, Theorem 14.23]), whose main assumption is that $F(u,\cdot)$ is subadditive. The proof of this property will be obtained in several steps (see Lemmas 2.2.2–2.2.7). The first ingredient is the following version of the *Fundamental Estimate*, where the ambient space $L^0(\mathbb{R}^d;\mathbb{R}^k)$ is replaced by $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$. For the proof we refer to [100, Lemma 3.19], since the arguments in the vectorial case are the same as in the scalar case. This is due to the fact that no truncation is used.

Lemma 2.2.2 ([100, Lemma 3.19])**.** *Let $\{F_n\} \subset \mathfrak{F}$ be a sequence of functionals and let $O', O'', O, U \in \mathcal{O}_c(\mathbb{R}^d)$, with $O' \subset\subset O'' \subset\subset O$. Let $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$ and let $\{w_n\}_n, \{v_n\}_n \subset L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$ be two sequences converging to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$ and such that $\{w_n\}_n \subset \mathrm{BV}(O;\mathbb{R}^k)$, $\{v_n\}_n \subset \mathrm{BV}(U;\mathbb{R}^k)$. Then for every $\eta > 0$ there exists a sequence $\{\varphi_n\}_n \subset C^\infty_c(\mathbb{R}^d,[0,1])$, with $\mathrm{supp}(\varphi_n) \subset O''$ and $\varphi_n = 1$ in a neighbourhood of $\overline{O'}$, such that the functions u_n defined by*

$$u_n \coloneqq \varphi_n w_n + (1-\varphi_n)v_n$$

converge to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$, $u_n \in \mathrm{BV}(O'\cup U;\mathbb{R}^k)$, and

$$\liminf_{n\to+\infty} F_n(u_n, O'\cup U) \le (1+\eta)\liminf_{n\to+\infty}(F_n(w_n,O)+F_n(v_n,U))+\eta,$$
$$\limsup_{n\to+\infty} F_n(u_n, O'\cup U) \le (1+\eta)\limsup_{n\to+\infty}(F_n(w_n,O)+F_n(v_n,U))+\eta.$$

The previous lemma allows us to consider the truncated functions $\Phi_R\circ u$ and to prove that $F'(\Phi_R\circ u,\cdot)$ and $F''(\Phi_R\circ u,\cdot)$ satisfy a sort of weak subadditivity inequality.

Lemma 2.2.3. *Let $\{F_n\}_n \subset \mathfrak{F}$ be a sequence of functionals, let F' and F'' be the functionals defined by (2.2.1), let $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, and let $O', O, U \in \mathcal{O}_c(\mathbb{R}^d)$, with $O'\subset\subset O$. Then for every $R>0$ we have*

$$F'(\Phi_R\circ u, O'\cup U) \le F''(\Phi_R\circ u, O) + F'(\Phi_R\circ u, U), \tag{2.2.6}$$
$$F''(\Phi_R\circ u, O'\cup U) \le F''(\Phi_R\circ u, O) + F''(\Phi_R\circ u, U). \tag{2.2.7}$$

Proof. We now prove (2.2.6). Without loss of generality, we assume that both $F''(\Phi_R\circ u, O)$ and $F'(\Phi_R\circ u, U)$ are finite. Let $\{w_n\}_n, \{v_n\}_n \subset L^0(\mathbb{R}^d;\mathbb{R}^k)$ be two sequences of functions converging in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ to $\Phi_R\circ u$ and such that

$$\limsup_{n\to+\infty} F_n(w_n,O) = F''(\Phi_R\circ u, O) \quad\text{and}\quad \liminf_{n\to+\infty} F_n(v_n,U) = F'(\Phi_R\circ u, U). \tag{2.2.8}$$

We fix a subsequence $\{F_{n_h}\}_h$ of $\{F_n\}_n$ such that

$$\lim_{h\to+\infty} F_{n_h}(v_{n_h},U) = \liminf_{n\to+\infty} F_n(v_n,U) = F'(\Phi_R\circ u, U).$$

Without loss of generality, we may assume that there exists a constant $M>0$ such that $F_{n_h}(w_{n_h},O)<M$ and $F_{n_h}(v_{n_h},U)<M$ for every $h\in\mathbb{N}$. Remark 2.1.13 then implies $\{w_{n_h}\}_h \subset \mathrm{GBV}_\star(O;\mathbb{R}^k)$, $\{v_{n_h}\}_h \subset \mathrm{GBV}_\star(U;\mathbb{R}^k)$. By property (H8) of Definition 2.1.7 for every $h,m\in\mathbb{N}$ we can find $i_{h,m}, j_{h,m}\in\{1,\dots,m\}$ such that

$$F_{n_h}(\Phi^{i_{h,m}}_{\sigma R}\circ w_{n_h}, O) \le F_{n_h}(w_{n_h},O) + c_4\mathcal{L}^d(O^{\sigma R}_{w_{n_h},0}) + C\frac{M+\mathcal{L}^d(O)}{m}, \tag{2.2.9}$$
$$F_{n_h}(\Phi^{j_{h,m}}_{\sigma R}\circ v_{n_h}, U) \le F_{n_h}(v_{n_h},U) + c_4\mathcal{L}^d(U^{\sigma R}_{v_{n_h},0}) + C\frac{M+\mathcal{L}^d(U)}{m}. \tag{2.2.10}$$

Since $\|\Phi^{i_{h,m}}_{\sigma R}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^k)} \le \sigma^{m+1}R$ and $\|\Phi^{j_{h,m}}_{\sigma R}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^k)} \le \sigma^{m+1}R$ for all $h\in\mathbb{N}$ and $m\in\mathbb{N}$, $\|\Phi_R\circ u\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^k)} \le \sigma R/2$, and $\{w_{n_h}\}_h$ and $\{v_{n_h}\}_h$ converge in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ to $\Phi_R\circ u$ as $h\to+\infty$, for every $m\in\mathbb{N}$ we infer that

$$\Phi^{i_{h,m}}_{\sigma R}\circ w_{n_h} \to \Phi_R\circ u \quad \text{in } L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k) \text{ as } h\to+\infty,$$
$$\Phi^{j_{h,m}}_{\sigma R}\circ v_{n_h} \to \Phi_R\circ u \quad \text{in } L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k) \text{ as } h\to+\infty,$$

and

$$\lim_{h\to\infty}\mathcal{L}^d(O^{\sigma R}_{w_{n_h},0}) = \mathcal{L}^d(O^{\sigma R}_{\Phi_R\circ u,0}) = 0, \tag{2.2.11}$$
$$\lim_{h\to\infty}\mathcal{L}^d(U^{\sigma R}_{v_{n_h},0}) = \mathcal{L}^d(U^{\sigma R}_{\Phi_R\circ u,0}) = 0. \tag{2.2.12}$$

For every $V\in\mathcal{O}(\mathbb{R}^d)$ we set

$$\widehat{F}'(\cdot,V) \coloneqq \Gamma\text{-}\liminf_{h\to\infty} F_{n_h}(\cdot,V) \quad\text{and}\quad \widehat{F}''(\cdot,V) \coloneqq \Gamma\text{-}\limsup_{h\to\infty} F_{n_h}(\cdot,V), \tag{2.2.13}$$

and observe that

$$F'(\cdot,V)\le \widehat{F}'(\cdot,V) \quad \text{and} \quad \widehat{F}''(\cdot,V)\le F''(\cdot,V). \tag{2.2.14}$$

By (2.2.8)–(2.2.12), we deduce that

$$\limsup_{h\to\infty} F_{n_h}(\Phi_{\sigma R}^{i_{h,m}}\circ w_{n_h},O)\le F''(\Phi_R\circ u,O)+C\frac{M+\mathcal{L}^d(O)}{m}, \tag{2.2.15}$$

$$\liminf_{h\to\infty} F_{n_h}(\Phi_{\sigma R}^{j_{h,m}}\circ v_{n_h},U)\le F'(\Phi_R\circ u,U)+C\frac{M+\mathcal{L}^d(U)}{m}. \tag{2.2.16}$$

By (b) of Definition 1.3.16, we have that $\Phi_{\sigma R}^{i_{h,m}}\circ w_{n_h}\in \mathrm{BV}(O;\mathbb{R}^k)$ and that $\Phi_{\sigma R}^{j_{h,m}}\circ v_{n_h}\in \mathrm{BV}(U;\mathbb{R}^k)$ for every $h\in\mathbb{N}$. Thus, for every $\eta>0$ and for every $m\in\mathbb{N}$ we may apply Lemma 2.2.2 to obtain a sequence of functions $\{u_h^m\}_h\subset \mathrm{BV}(O'\cup U;\mathbb{R}^k)$ such that $u_h^m\to\Phi_R\circ u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$ as $h\to+\infty$ and such that

$$\begin{aligned}\liminf_{h\to+\infty} F_{n_h}(u_h^m,O'\cup U)&\le (1+\eta)\liminf_{h\to\infty}\Big(F_{n_h}(\Phi_{\sigma R}^{i_{h,m}}\circ w_{n_h},O)+F_{n_h}(\Phi_{\sigma R}^{j_{h,m}}\circ v_{n_h},U)\Big)+\eta\\ &\le (1+\eta)\limsup_{h\to\infty} F_{n_h}(\Phi_{\sigma R}^{i_{h,m}}\circ w_{n_h},O)\\ &\quad+(1+\eta)\liminf_{h\to+\infty} F_{n_h}(\Phi_{\sigma R}^{j_{h,m}}\circ v_{n_h},U)+\eta.\end{aligned}$$

These inequalities, together with (2.2.15) and (2.2.16), yield

$$\liminf_{h\to\infty} F_{n_h}(u_h^m,O'\cup U)\le (1+\eta)\Big(F''(\Phi_R\circ u,O)+F'(\Phi_R\circ u,U)+\frac{S}{m}\Big)+\eta,$$

where S is a positive constant independent of m. Recalling that $u_h^m\to\Phi_R\circ u$ as $h\to+\infty$, by (2.2.13) and (2.2.14) we obtain that

$$\begin{aligned}F'(\Phi_R\circ u,O'\cup U)&\le \widehat{F}'(\Phi_R\circ u,O'\cup U)\\ &\le (1+\eta)\Big(F''(\Phi_R\circ u,O)+F'(\Phi_R\circ u,U)+\frac{S}{m}\Big)+\eta.\end{aligned}$$

Letting $m\to+\infty$ and $\eta\to 0^+$, we finally obtain (2.2.6).

A similar, but easier, argument shows that (2.2.7) holds. □

To extend the previous lemma to $F'(u,\cdot)$ for a general $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ we will approximate $F'(u,O)$ by $F'(\Phi_R\circ u,O)$ for suitable choices of $R>0$. For technical reasons, this approximation result is obtained using property (H8) of Definition 2.1.7 for F', which is proved in the following lemma.

Lemma 2.2.4. *Let $\{F_n\}_n$ be a sequence in $\mathfrak{F}$, let F' be the functional defined by* (2.2.1)*, let $O\in\mathcal{O}_c(\mathbb{R}^d)$, $u\in \mathrm{GBV}_\star(O;\mathbb{R}^k)$, and let $w\in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$. Then for every $R>0$ and $m\in\mathbb{N}$ we have*

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m F'(w+\Phi_R^i\circ(u-w),O)&\le F'(u,O)+c_3k^{1/2}\int_{O^R_{u,w}}|\nabla w|\,\mathrm{d}x+c_4\mathcal{L}^d(O^R_{u,w})\\ &\quad+\frac{C}{m}\left(F'(u,O)+\mathcal{L}^d(O)+\int_O|\nabla w|\,\mathrm{d}x\right),\end{aligned} \tag{2.2.17}$$

where C is the constant appearing in (H8) *of Definition 2.1.7.*

Proof. Let us fix $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$. Without loss of generality, we may assume that $F'(u,O) < +\infty$. Let $\{u_n\}_n \subset L^0(\mathbb{R}^d;\mathbb{R}^k)$ be a sequence of functions converging to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ and such that

$$\liminf_{n\to+\infty} F_n(u_n,O) = F'(u,O). \tag{2.2.18}$$

Fix $R > 0$ and $m \in \mathbb{N}$. For every $i \in \{1,\dots,m\}$, we set $v_n^i := w + \Phi_R^i \circ (u_n - w)$ and $v^i := w + \Phi_R^i \circ (u - w)$. Note that for every $i \in \{1,\dots,m\}$ the sequence $\{v_n^i\}_n$ converges to v^i in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, so that by Definition (2.2.1) of F' we have

$$\frac{1}{m}\sum_{i=1}^m F'(v^i,O) \le \frac{1}{m}\sum_{i=1}^m \liminf_{n\to+\infty} F_n(v_n^i,O) \le \liminf_{n\to+\infty} \frac{1}{m}\sum_{i=1}^m F_n(v_n^i,O). \tag{2.2.19}$$

By property (H8) and by (2.2.18) we get

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m F_n(v_n^i,O) \le F_n(u_n,O) + \frac{C}{m}\Big(F_n(u_n,O) + \mathcal{L}^d(O) + \int_O |\nabla w|\mathrm{d}x\Big)\\ + c_3 k^{1/2}\int_{O^R_{u_n,w}} |\nabla w|\,\mathrm{d}x + c_4\mathcal{L}^d(O^R_{u_n,w}).\end{aligned}$$

By (2.2.18), this inequality gives

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m F'(v^i,O) \le F'(u,O) + \frac{C}{m}\Big(F'(u,O) + \mathcal{L}^d(O) + \int_O |\nabla w|\mathrm{d}x\Big)\\ + \limsup_{n\to+\infty}\Big(c_3 k^{1/2}\int_{O^R_{u_n,w}} |\nabla w|\,\mathrm{d}x + c_4\mathcal{L}^d(O^R_{u_n,w})\Big).\end{aligned} \tag{2.2.20}$$

Since $u_n \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ as $n \to +\infty$, up to passing to a further, not relabelled, subsequence, we have that

$$\limsup_{n\to+\infty} \chi_{O^R_{u_n,w}}(x) \le \chi_{O^R_{u,w}}(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in O.$$

Thus, Fatou's Lemma implies that

$$\limsup_{n\to+\infty}\Big(\int_{O^R_{u_n,w}} |\nabla w|\,\mathrm{d}x + \mathcal{L}^d(O^R_{u_n,w})\Big) \le \int_{O^R_{u,w}} |\nabla w|\,\mathrm{d}x + \mathcal{L}^d(O^R_{u,w}), \tag{2.2.21}$$

which combined with (2.2.20) yields (2.2.17). □

To obtain the weak subadditivity of $F'(u,\cdot)$ we show that $F'(u,O)$ can be approximated by sequences of the form $F'(\Phi_{R_m}\circ u, O)$ for an appropriate choice of $\{R_m\}_m$.

Lemma 2.2.5. *Let $\{F_n\}_n$ be a sequence in $\mathfrak{F}$, let F' be the functional defined by* (2.2.1)*, let $O_1, O_2 \in \mathcal{O}_c(\mathbb{R}^d)$, $u_1 \in L^0(O_1;\mathbb{R}^k)$ and $u_2 \in L^0(O_2;\mathbb{R}^k)$. Then there exists an increasing sequence $R_m > 0$, with $R_m \to +\infty$ as $m \to +\infty$, such that*

$$\lim_{m\to+\infty} F'(\Phi_{R_m}\circ u_j, O_j) = F'(u_j,O_j) \tag{2.2.22}$$

for $j = 1, 2$.

Proof. It is immediate to check that F' satisfies (H4′), while by Lemma 2.2.4 F' satisfies property (H8) for $B = O_1$ and $B = O_2$. Since the functional $F'(\cdot, O_j)$ is lower semicontinuous with respect to the topology of $L^0(O_j;\mathbb{R}^k)$ for $j = 1, 2$ (see [88, Proposition 6.8]), an application of Lemma 2.1.15 then proves the claim. □

We now extend the weak subadditivity inequality proved in Lemma 2.2.3 to $F'(u,\cdot)$, for a general $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$.

Lemma 2.2.6. *Let $\{F_n\}_n$ be a sequence of functionals in $\mathfrak{F}$ and let F', F'' be defined by (2.2.1). Assume that there exists a functional $F\colon L^0(\mathbb{R}^d;\mathbb{R}^k)\times\mathcal{O}(\mathbb{R}^d)\to[0,+\infty]$ such that (2.2.3) holds. Let $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, and $O', O, U \in \mathcal{O}_c(\mathbb{R}^d)$ with $O' \subset\subset O$. Then we have*

$$F'(u, O'\cup U) \le F'(u,O) + F'(u,U). \tag{2.2.23}$$

Proof. By [88, Proposition 15.15] we may choose $O'' \in \mathcal{O}_c(\mathbb{R}^d)$, with $O' \subset\subset O'' \subset\subset O$, such that

$$F''(v,O'') = F'(v,O'') = F(v,O'') \quad \text{for every } v \in L^0(\mathbb{R}^d;\mathbb{R}^k). \tag{2.2.24}$$

Lemma 2.2.5 implies that there exists an increasing sequence $R_m > 0$, with $R_m \to +\infty$ as $m \to +\infty$, such that

$$\lim_{m\to+\infty} F'(\Phi_{R_m}\circ u, O'') = F'(u,O''), \tag{2.2.25}$$

$$\lim_{m\to+\infty} F'(\Phi_{R_m}\circ u, U) = F'(u,U). \tag{2.2.26}$$

Thanks to Lemma 2.2.3, we obtain that

$$\begin{aligned} F'(\Phi_{R_m}\circ u, O'\cup U) &\le F''(\Phi_{R_m}\circ u, O'') + F'(\Phi_{R_m}\circ u, U)\\ &\le F'(\Phi_{R_m}\circ u, O) + F'(\Phi_{R_m}\circ u, U), \end{aligned}$$

where in the second inequality we have used (2.2.24). Since $\Phi_{R_m}\circ u \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ as $m\to+\infty$, by the lower semicontinuity of F', (2.2.25), and (2.2.26) we obtain

$$F'(u,O'\cup U) \le F'(u,O'') + F'(u,U) \le F'(u,O) + F'(u,U),$$

concluding the proof. □

We are now ready to prove the subadditivity of $F(u,\cdot)$ on $\mathcal{O}(\mathbb{R}^d)$.

Lemma 2.2.7. *Let $\{F_n\}_n$ be a sequence in $\mathfrak{F}$ for which (2.2.3) holds for some functional F. Then for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $O, U \in \mathcal{O}(\mathbb{R}^d)$ we have*

$$F(u, O\cup U) \le F(u,O) + F(u,U).$$

Proof. The result follows from Lemma 2.2.6 and by standard arguments (see, for instance, the proof of [88, Lemma 18.4]). □

The following result shows that $B \mapsto F(u,B)$ is a Borel measure.

Lemma 2.2.8. *Let $\{F_n\}_n$ be a sequence in $\mathfrak{F}$ for which (2.2.3) and (2.2.4) hold for some functional F. Then for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ the set function $B \mapsto F(u,B)$ defined by (2.2.3) and (2.2.4) is a non-negative Borel measure on $\mathbb{R}^d$ and for all $B \in \mathcal{B}(\mathbb{R}^d)$ we have*

$$F(u,B) = \inf\{F(u,O)\colon\ O \supset B,\ O \in \mathcal{O}(\mathbb{R}^d)\}.$$

Proof. Let us fix $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$. By the De Giorgi-Letta criterion for measures [106] (see also [88, Theorem 14.23] for the particular version of the theorem employed here) it is enough to show that for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ the function $B \mapsto F(u,B)$ is increasing, inner regular, subadditive, and superadditive. It has already been observed that the set function $F(u,\cdot)$ is increasing, while the inner regularity follows from the definition of $F(u,\cdot)$. The subadditivity is proved in Lemma 2.2.7, whereas the superadditivity is a consequence of [88, Proposition 16.12] and of the fact that F_n satisfies (H2) of Definition 2.1.7. This concludes the proof. □

Finally, we show that $\{F_n(\cdot,O)\}_n$ Γ-converges for every $O \in \mathcal{O}_c(\mathbb{R}^d)$.

Lemma 2.2.9. *Let $\{F_n\}_n$ be a sequence of functionals in $\mathfrak{F}$, let F', F'' be the functionals defined by* (2.2.1)*. Assume that there exists a functional F such that* (2.2.3) *holds. Then $F(u,O) = F'(u,O) = F''(u,O)$ for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $O \in \mathcal{O}_c(\mathbb{R}^d)$.*

Proof. Fix $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $O \in \mathcal{O}_c(\mathbb{R}^d)$. We first show that $F(u,O) = F'(u,O)$. Since $F(u,O) \le F'(u,O)$, it is enough to show the converse inequality. Without loss of generality we may suppose that $F(u,O) < +\infty$. It is immediate to check that F' and F satisfy property (H4) in Definition 2.1.7 on $\mathcal{O}(\mathbb{R}^d)$ and using Remark 1.3.15 we check that F' and F also satisfy (H3) on $\mathcal{O}(\mathbb{R}^d)$, hence $\mathcal{V}(u,O) < +\infty$. This implies that, given $\varepsilon > 0$, there exists a compact set $K \subset O$ such that

$$c_3\mathcal{V}(u, O \setminus K) + c_4\mathcal{L}^d(O \setminus K) \le \varepsilon. \tag{2.2.27}$$

Consider now $O', O'' \in \mathcal{O}_c(O)$ with $K \subset O' \subset\subset O''$ and let $U = O \setminus K$. Note that $O' \cup U = O$. Hence, by Lemma 2.2.6 we obtain

$$F'(u,O) \le F'(u,O'') + F'(u, O \setminus K) \le F(u,O) + \varepsilon, \tag{2.2.28}$$

where in the last inequality we have used (2.2.3), properties (H2), (H4) for F', and (2.2.27). These arguments show that $F(u,O) = F'(u,O)$ for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $O \in \mathcal{O}_c(\mathbb{R}^d)$.

We now prove that $F(u,O) = F''(u,O)$. Since $F(u,O) \le F''(u,O)$ it is enough to show that $F''(u,O) \le F(u,O)$. It is not restrictive to assume that $F(u,O) < +\infty$, so that $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$. Given $R > 0$, we can exploit the same argument used to obtain (2.2.28), replacing (2.2.23) of Lemma 2.2.6 with (2.2.7), to show that for every $\varepsilon > 0$

$$F''(\Phi_R \circ u, O) \le F(\Phi_R \circ u, O) + \varepsilon = F'(\Phi_R \circ u, O) + \varepsilon, \tag{2.2.29}$$

where we have used the equality $F = F'$. By Lemma 2.2.5 there exists an increasing sequence $R_m > 0$, with $R_m \to +\infty$, such that

$$\lim_{m\to+\infty} F'(\Phi_{R_m} \circ u, O) = F'(u,O) = F(u,O).$$

Since $\Phi_{R_m} \circ u \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ as $m \to +\infty$, exploiting the lower semicontinuity of F'' and this last equality, we deduce from (2.2.29) that

$$F''(u,O) \le F(u,O) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $F''(u,O) \le F(u,O)$, concluding the proof. □

2.3 Partial integral representation

In this section, we present and prove a (partial) integral representation result for functionals in $\mathfrak{F}_{\rm sc}$ (see Theorem 2.3.16). We postpone the full representation to Section 2.5, where we will work with functionals in $\mathfrak{F}_{\rm sc}$ satisfying an additional property.

We also mention that the results of this section (whose proof heavily relies on property (H8)) could be obtained without employing any truncation technique. Indeed, in Section 2.5 of Chapter 3, we will show that it is possible to obtain the results of Theorem 2.3.16, by using a different approach based on a careful analysis of the blow-ups of functions in $\mathrm{GBV}_\star$. We refer the reader to Remark 3.3.8 for a more detailed discussion of this point.

To present the above-mentioned result, we first introduce a decomposition of F that mimics the structure of functionals in $F^{f,g}$.

Definition 2.3.1. Let $F\colon L^0(\mathbb{R}^d;\mathbb{R}^k)\times\mathcal{B}(\mathbb{R}^d)\to[0,+\infty]$ be a functional satisfying (H2) and (H4) of Definition 2.1.7, and let $O\in\mathcal{O}_c(\mathbb{R}^d)$. For every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ with $u\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$, we introduce $F^a(u,\cdot)$, $F^s(u,\cdot)$, $F^c(u,\cdot)$, and $F^j(u,\cdot)$ as the Borel measures on $\mathcal{B}(O)$ defined as:

$$\begin{gathered}F^a(u,\cdot)\text{ is the absolutely continuous part of }F(u,\cdot)\text{ with respect to }\mathcal{L}^d,\\ F^s(u,\cdot)\text{ is the singular part of }F(u,\cdot)\text{ with respect to }\mathcal{L}^d,\\ F^j(u,B)\coloneqq F^s(u,B\cap J_u)\quad\text{for every }B\in\mathcal{B}(O),\\ F^c(u,B)\coloneqq F^s(u,B\setminus J_u)\quad\text{for every }B\in\mathcal{B}(O).\end{gathered}$$

Note that the measures $F^j(u,\cdot)$ and $F^c(u,\cdot)$ are the absolutely continuous parts of $F(u,\cdot)$ with respect to $\mathcal{H}^{d-1}\llcorner J_u$ and $|D^cu|$, respectively. Additionally, for every $O\in\mathcal{O}_c(\mathbb{R}^d)$ we have

$$F(u,\cdot)=F^a(u,\cdot)+F^c(u,\cdot)+F^j(u,\cdot)\quad\text{on }\mathcal{B}(O).$$

The aim of this section is to show that for every $O\in\mathcal{O}_c(\mathbb{R}^d)$ and for every $u\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$ we may represent the two measures $F^a(u,\cdot)$ and $F^j(u,\cdot)$ as integrals (see Theorem 2.3.16).

As in [100, Section 6], the idea is to take advantage of the representation results of [41] for functionals on $\mathrm{BV}(O;\mathbb{R}^k)$ with linear growth. These results cannot be directly applied to $F^a(u,\cdot)$ and $F^j(u,\cdot)$, since functionals in $\mathfrak{F}$ are not bounded from below by the total variation measure $|Du|$, which is one of the key assumptions of [41]. This difficulty is treated by restricting our attention first to functions $u\in\mathrm{BV}(O;\mathbb{R}^k)\cap L^\infty(O,\mathbb{R}^k)$, and by considering the perturbed functionals $F_\delta(u,O)\coloneqq F(u,O)+\delta|Du|(O)$. The absolutely continuous part and the jump part of F_δ can then be represented by means of an integral thanks to the results of [41]. We then show that it is possible to pass to the limit as $\delta\to0^+$ and to recover an integral representation of $F^a(u,\cdot)$ and of $F^j(u,\cdot)$ for every $u\in\mathrm{BV}(O;\mathbb{R}^k)$.

The following lemma then allows us to deduce the result for a general $u\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$.

Lemma 2.3.2. *Let $F\in\mathfrak{F}_{\rm sc}$, let $O\in\mathcal{O}_c(\mathbb{R}^d)$, let $u\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$, and let $\{R_m\}_m$ be a sequence with $R_m>0$ and $R_m\to+\infty$. Then*

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^m F(\Phi^i_{R_m}\circ u,B)\Big)=F(u,B),\tag{2.3.1}$$

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^m F^b(\Phi^i_{R_m}\circ u,B)\Big)=F^b(u,B),\tag{2.3.2}$$

for every $B \in \mathcal{B}(O)$, where the letter b stands for one of the letters a, s, j, and c.

Proof. For every $m \in \mathbb{N}$ we set

$$\mu_m(B) \coloneqq \frac{1}{m} \sum_{i=1}^{m} F(\Phi^i_{R_m} \circ u, B) \quad \text{for every } B \in \mathcal{B}(O)$$

and note that, as $u \in \mathrm{GBV}_\star(O; \mathbb{R}^k)$, this defines a finite Radon measure.

Let $O' \in \mathcal{O}(O)$. Thanks to property (H8) of Definition 2.1.7, we have that

$$\mu_m(O') \leq F(u, O') + c_4 \mathcal{L}^d(O^{R_m}_{u,0}) + \frac{C}{m} \left(F(u, O') + \mathcal{L}^d(O') \right),$$

which implies

$$\limsup_{m \to +\infty} \mu_m(O') \leq F(u, O').$$

Then, we observe that, for every $m \in \mathbb{N}$, there exists $i(m) \in \{1, \dots, m\}$ such that

$$F(\Phi^{i(m)}_{R_m} \circ u, O') \leq \mu_m(O').$$

Since $\Phi^{i(m)}_{R_m} \circ u \to u$ in $L^0(\mathbb{R}^d; \mathbb{R}^k)$ as $m \to +\infty$ and $F(\cdot, O')$ is lower semicontinuous with respect to the topology of $L^0(\mathbb{R}^d; \mathbb{R}^k)$, we have

$$F(u, O') \leq \liminf_{m \to +\infty} F(\Phi^{i(m)}_{R_m} \circ u, O') \leq \limsup_{m \to +\infty} \mu_m(O') \leq F(u, O'),$$

so that (2.3.1) holds for every $O' \in \mathcal{O}(O)$. From [100, Lemma 4.4] we then deduce that (2.3.1) holds for every $B \in \mathcal{B}(O)$.

To show that (2.3.2) with $b = a$ holds true, consider a set $N \in \mathcal{B}(O)$, with $\mathcal{L}^d(N) = 0$, such that for every $B \in \mathcal{B}(O)$ we have $F^a(u, B) = F(u, B \setminus N)$ and $\mu^a_m(B) = \mu_m(B \setminus N)$ for every $m \in \mathbb{N}$, where μ^a_m is the absolutely continuous part of μ_m with respect to $\mathcal{L}^d$. By (2.3.1) we have that

$$\lim_{m \to +\infty} \mu^a_m(B) = \lim_{m \to +\infty} \mu_m(B \setminus N) = F(u, B \setminus N) = F^a(u, B),$$

which proves (2.3.2). Taking the difference of (2.3.1) and of (2.3.2) we obtain (2.3.2) with $b = s$.

Finally, equalities (2.3.2) for $b = c$ and $b = j$ can be obtained arguing as in [100, Proposition 4.3], replacing Theorem 2.2(d) with our Proposition 1.3.20. □

The following result shows that the singular part F^s satisfies a simplified version of property (H8) of Definition 2.1.7.

Lemma 2.3.3. *Let $F \in \mathfrak{F}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, $u \in \mathrm{GBV}_\star(O; \mathbb{R}^k)$, $R > 0$, $m \in \mathbb{N}$, $B \in \mathcal{B}(O)$, and $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^k)$. Then*

$$\frac{1}{m} \sum_{i=1}^{m} F^s(w + \Phi^i_R \circ (u - w), B) \leq \Big(1 + \frac{C}{m}\Big) F^s(u, B),$$

where C is the constant of (H8).

Proof. Let $N \in \mathcal{B}(\mathbb{R}^d)$ be a Borel set, with $\mathcal{L}^d(N) = 0$, such that $F^s(u, B) = F(u, B \cap N)$ and $F^s(w + \Phi_R^i \circ (u - w), B) = F(w + \Phi_R^i \circ (u - w), B \cap N)$ for every $i \in \{1, \dots, m\}$ and for every $B \in \mathcal{B}(O)$. Since by (H8) inequality (2.1.3) holds for every Borel set $B \in \mathcal{B}(O)$, we get

$$\begin{aligned}
\frac{1}{m}\sum_{i=1}^m F^s(w + \Phi_R^i \circ (u - w), B) &= \frac{1}{m}\sum_{i=1}^m F(w + \Phi_R^i \circ (u - w), B \cap N) \\
&\le F(u, B \cap N) + \int_{B^R_{u,w} \cap N} (c_3 k^{1/2}|\nabla w| + c_4)\,\mathrm{d}x \\
&\quad + \frac{C}{m}\Big(F(u, B \cap N) + \int_{B \cap N} (|\nabla w| + 1)\,\mathrm{d}x\Big) \\
&= F(u, B \cap N) + \frac{C}{m} F(u, B \cap N) \\
&= \Big(1 + \frac{C}{m}\Big) F^s(u, B),
\end{aligned}$$

which proves the claim. □

We now introduce the perturbed functionals F_δ which will play a fundamental role in the proof of our representation result for functionals in $\mathfrak{F}$.

Definition 2.3.4. Let $F \in \mathfrak{F}$ and let $O \in \mathcal{O}_c(\mathbb{R}^d)$ be given. For every $\delta > 0$ the functional $F_\delta \colon \mathrm{BV}(O; \mathbb{R}^k) \times \mathcal{B}(O) \to [0, +\infty)$ is defined for every $u \in \mathrm{BV}(O; \mathbb{R}^k)$ and $B \in \mathcal{B}(O)$ as

$$F_\delta(u, B) \coloneqq F(u, B) + \delta |Du|(B). \tag{2.3.3}$$

Given $u \in \mathrm{BV}(O; \mathbb{R}^k)$, the measures $F_\delta^a(u, \cdot)$, $F_\delta^s(u, \cdot)$, $F_\delta^j(u, \cdot)$, and $F_\delta^c(u, \cdot)$ are defined as in Definition 2.3.1 with F replaced by F_δ.

Remark 2.3.5. Let $F \in \mathfrak{F}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $\delta > 0$. Thanks to (H3′) and (H4′) of Remark 2.1.9 we have that for every $u \in \mathrm{BV}(O; \mathbb{R}^k)$ we have

$$\delta |Du|(O) - c_2 \mathcal{L}^d(O) \le F_\delta(u, O) \le (c_3 k + \delta)|Du|(O) + c_4 \mathcal{L}^d(O).$$

Definition 2.3.6. Let $O \in \mathcal{O}_c(\mathbb{R}^d)$ and let $F \colon \mathrm{BV}(O; \mathbb{R}^k) \times \mathcal{B}(O) \to [0, +\infty)$. For every $U \in \mathcal{O}(O)$ with Lipschitz boundary and $w \in \mathrm{BV}(U; \mathbb{R}^k)$, we set

$$\mathfrak{m}^F(w, U) \coloneqq \inf\{F(u, U) \colon\ u \in \mathrm{BV}(U; \mathbb{R}^k) \text{ and } u = w \text{ on } \partial U\}, \tag{2.3.4}$$

where the equality $u = w$ has to be understood in the sense of traces. Given $t > 0$, we set

$$\mathfrak{m}_t^F(w, U) \coloneqq \inf\{F(u, U) \colon\ u \in \mathrm{BV}(U; \mathbb{R}^k),\ u = w \text{ on } \partial U,\ \|u - w\|_{L^\infty} \le t\}. \tag{2.3.5}$$

We now introduce some functions which will play a crucial role in the integral representation of the bulk part F^a and the surface part F^j of a functional $F \in \mathfrak{F}_{\mathrm{sc}}$. We recall the definition of the cubes $Q(x, \rho)$ and $Q_\nu(x, \rho)$ given at the beginning of Section 1.1 and of the functions ℓ_A and $u_{x,\zeta,\nu}$ given in (1.5.2) and (1.5.3).

Definition 2.3.7. Let $F \in \mathfrak{F}$. We define the functions $f \colon \mathbb{R}^d \times \mathbb{R}^{k \times d} \to [0, +\infty)$ and $g \colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0, +\infty)$ as

$$f(x, A) \coloneqq \limsup_{\rho \to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x, \rho))}{\rho^d} \quad \text{for } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{k \times d}, \tag{2.3.6}$$

$$g(x, \zeta, \nu) \coloneqq \limsup_{\rho \to 0^+} \frac{\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x, \rho))}{\rho^{d-1}} \quad \text{for } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k, \nu \in \mathbb{S}^{d-1}. \tag{2.3.7}$$

For $\delta > 0$, the functions f_δ and g_δ are defined by replacing F by F_δ in (2.3.6) and (2.3.7).

Remark 2.3.8. It is immediate to check that the functions $\delta \mapsto f_\delta(x,A)$ and $\delta \mapsto g_\delta(x,\zeta,\nu)$ are non-decreasing and that

$$f(x,A) \le f_\delta(x,A) \quad \text{and} \quad g(x,\zeta,\nu) \le g_\delta(x,\zeta,\nu)$$

for every $\delta > 0$, $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

When $F \in \mathfrak{F}_{\rm sc}$, we shall see that the functions f_δ and g_δ will be used in the integral representation of F^a_δ and F^j_δ thanks to the results of [41]. This will lead to an integral representation of F^a and F^j by means of the functions (see the proof of Theorem 2.3.16)

$$\widehat{f}(x,A) \coloneqq \inf_{\delta>0} f_\delta(x,A) \quad \text{for every } x \in \mathbb{R}^d,\ A \in \mathbb{R}^{k\times d}, \tag{2.3.8}$$
$$\widehat{g}(x,\zeta,\nu) \coloneqq \inf_{\delta>0} g_\delta(x,\zeta,\nu) \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k,\ \nu \in \mathbb{S}^{d-1}.$$

For the applications to homogenisation of Section 2.7 it is important to prove that $\widehat{f} = f$ and $\widehat{g} = g$, so that by (2.3.6) and (2.3.7) the integrands used in the bulk and surface parts can be obtained by solving some auxiliary minimisation problems on small cubes.

The proof of the equality $\widehat{f} = f$ is not direct and requires a lot of technical tools, one of which is the following truncation lemma. For future use, we prove the result also for the rectangles $Q^\lambda_\nu(x,\rho)$, which we recall is a rectangle centred at x, with two faces orthogonal to ν at distance ρ and with remaining sides of length $\lambda\rho$. Given $A \in \mathbb{R}^{k\times d}$ and $m \in \mathbb{N}$, we set

$$c_{A,m} \coloneqq (\sigma^m + 1)d^{1/2}\Big(|A| + \frac{c_2 + c_4 + 1}{c_1}\Big). \tag{2.3.9}$$

Lemma 2.3.9. *Let $F \in \mathfrak{F}$. Assume that there exists a function $\widehat{f} \in \mathscr{F}$ such that*

$$F^a(u,O) = \int_O \widehat{f}(x,\nabla u)\,\mathrm{d}x, \tag{2.3.10}$$

for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ and $u \in \mathrm{BV}(O;\mathbb{R}^k)$. Then there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, satisfying the following property: for every $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $A \in \mathbb{R}^{k\times d}$, $\nu \in \mathbb{S}^{d-1}$, and $\lambda \ge 1$ there exists $\rho^{\lambda,\nu}_{m,A}(x) > 0$ such that for every $\rho \in (0, \rho^{\lambda,\nu}_{m,A}(x))$ and $u \in \mathrm{BV}(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $u = \ell_A$ on $\partial Q^\lambda_\nu(x,\rho)$, there exists $v \in \mathrm{BV}(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $v = \ell_A$ on $\partial Q^\lambda_\nu(x,\rho)$ and $\|v - \ell_A\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \le c_{A,m}\lambda\rho$, satisfying the inequality

$$F(v, Q^\lambda_\nu(x,\rho)) \le \Big(1 + \frac{C}{m}\Big)F(u, Q^\lambda_\nu(x,\rho)) + \frac{C}{m}\lambda^{d-1}\rho^d, \tag{2.3.11}$$

where C is a constant depending only on the structural constants c_1, c_2, c_3, c_4, c_5, and k.

Moreover, if $\widehat{f}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^{k\times d}$, then $N = \varnothing$. Finally, if $\widehat{f}$ is independent of x, then $\rho^{\lambda,\nu}_{m,A}(x) = +\infty$.

Proof. Let us fix $m \in \mathbb{N}$. Arguing as in [100, Lemma 4.16], one can construct a Borel function $\omega_m : \mathbb{R}^d \to \mathbb{R}^{k\times d}$ such that

$$\widehat{f}(x,\omega_m(x)) \le \widehat{f}(x,A) + \frac{1}{m} \quad \text{for every } x \in \mathbb{R}^d \text{ and every } A \in \mathbb{R}^{k\times d}. \tag{2.3.12}$$

Note that (f2′), (f3), and (2.3.12) imply that for every $x \in \mathbb{R}^d$ one has

$$|\omega_m(x)| \le \frac{c_2 + c_4 + 1}{c_1}. \tag{2.3.13}$$

Since $\omega_m \in L^\infty(\mathbb{R}^d;\mathbb{R}^{k\times d})$, by the Lebesgue Differentiation Theorem there exists $N_m \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N_m)=0$, such that for every $x\in\mathbb{R}^d\setminus N_m$, $\nu\in\mathbb{S}^{d-1}$, and $\lambda\geq 1$ there exists $\rho^{\lambda,\nu}_{m,A}(x)>0$ with the property that for every $\rho\in(0,\rho^{\lambda,\nu}_{m,A}(x))$ we have

$$\int_{Q^\lambda_\nu(x,\rho)} |\omega_m(x)-\omega_m(y)|\,\mathrm{d}y \leq \frac{1}{m}\rho^d\lambda^{d-1}.$$

We set $N\coloneqq\bigcup_{m\in\mathbb{N}} N_m$ and note that $\mathcal{L}^d(N)=0$. In view of (f4), the previous inequality implies that for every $x\in\mathbb{R}^d\setminus N$, $m\in\mathbb{N}$, $A\in\mathbb{R}^{k\times d}$, $\nu\in\mathbb{S}^{d-1}$, $\lambda\geq 1$, and $\rho\in(0,\rho^{\lambda,\nu}_{m,A}(x))$ we have

$$\int_{Q^\lambda_\nu(x,\rho)} |\widehat{f}\big(y,\omega_m(x)\big)-\widehat{f}\big(y,\omega_m(y)\big)|\,\mathrm{d}y \leq \frac{c_5}{m}\rho^d\lambda^{d-1}. \tag{2.3.14}$$

We also set $R\coloneqq d^{1/2}(|A|+\frac{c_2+c_4+1}{c_1})\lambda\rho$ and

$$w(y)\coloneqq\omega_m(x)(y-x)+\ell_A(x), \tag{2.3.15}$$
$$v^i(y)\coloneqq w(y)+\Phi^i_R(u(y)-w(y)) \tag{2.3.16}$$

for every $y\in Q^\lambda_\nu(x,\rho)$ and $i\in\{1,\dots,m\}$. Observe that with this choice of R we have $|(u-w)|\leq R$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_\nu(x,\rho)$. Moreover, recalling (1.3.13), we get $v^i=\ell_A$ on $\partial Q^\lambda_\nu(x,\rho)$ and

$$\|v^i-\ell_A\|_{L^\infty(Q^\lambda_\nu(x,\rho))}\leq(\sigma^m+1)R=c_{A,m}\lambda\rho.$$

We claim that there exists $i\in\{1,\dots,m\}$ such that (2.3.11) holds with $v=v^i$. To prove this, we consider for every $i\in\{1,\dots,m\}$ the following partition of $Q^\lambda_\nu(x,\rho)$:

$$\begin{aligned}
Q^i_{\mathrm{in}}&\coloneqq\{y\in Q^\lambda_\nu(x,\rho)\colon\ |u(y)-w(y)|\leq\sigma^{i-1}R\},\\
Q^i&\coloneqq\{y\in Q^\lambda_\nu(x,\rho)\colon\ \sigma^{i-1}R<|u(y)-w(y)|<\sigma^iR\},\\
Q^i_{\mathrm{out}}&\coloneqq\{y\in Q^\lambda_\nu(x,\rho)\colon\ |u(y)-w(y)|\geq\sigma^iR\}.
\end{aligned}$$

Using the measure property (H2), we may write

$$F(v^i,Q^\lambda_\nu(x,\rho))=F(v^i,Q^i_{\mathrm{in}})+F(v^i,Q^i)+F(v^i,Q^i_{\mathrm{out}}).$$

Using again (1.3.13) from (2.3.10) and (2.3.16) we infer

$$F^a(v^i,Q^\lambda_\nu(x,\rho))=\int_{Q^i_{\mathrm{in}}}\widehat{f}\big(y,\nabla u(y)\big)\,\mathrm{d}y+\int_{Q^i}\widehat{f}\big(y,\nabla v^i(y)\big)\,\mathrm{d}y+\int_{Q^i_{\mathrm{out}}}\widehat{f}\big(y,\omega_m(x)\big)\,\mathrm{d}y.$$

Taking advantage of (2.3.13)–(2.3.16), by (f2′) and (f3) we may estimate the integral over Q^i as

$$\begin{aligned}
\int_{Q^i}\widehat{f}\big(y,\nabla v^i(y)\big)\,\mathrm{d}y&\leq c_3k^{1/2}\int_{Q^i}|\omega_m(x)+\nabla\Phi^i_R(u(y)-w(y))(\nabla u(y)-\omega_m(x))|\,\mathrm{d}y+c_4\mathcal{L}^d(Q^i)\\
&\leq c_3k^{1/2}\int_{Q^i}|\nabla u(y)|\,\mathrm{d}y+(2c_3k^{1/2}|\omega_m(x)|+c_4)\mathcal{L}^d(Q^i)\\
&\leq\frac{c_3k^{1/2}}{c_1}\int_{Q^i}\widehat{f}\big(y,\nabla u(y)\big)\,\mathrm{d}y+\Big(3c_3k^{1/2}\frac{c_2+c_4+1}{c_1}+c_4\Big)\mathcal{L}^d(Q^i).
\end{aligned}$$

As for the integral over $Q^i_{\rm out}$, we use (2.3.12) and (2.3.14) to get

$$\begin{aligned}\int_{Q^i_{\rm out}} \widehat{f}\big(y,\omega_m(x)\big)\,{\rm d}y &\le \int_{Q^i_{\rm out}} \widehat{f}\big(y,\omega_m(y)\big)\,{\rm d}y + \frac{c_5}{m}\lambda^{d-1}\rho^d\\ &\le \int_{Q^i_{\rm out}} \widehat{f}\big(y,\nabla u(y)\big)\,{\rm d}y + \frac{c_5+1}{m}\lambda^{d-1}\rho^d.\end{aligned}$$

Collecting the previous inequalities, we obtain

$$\begin{aligned}F^a(v^i,Q^\lambda_\nu(x,\rho)) \le \int_{Q^i_{\rm in}} \widehat{f}\big(y,\nabla u(y)\big)\,{\rm d}y + \frac{c_3k^{1/2}}{c_1}\int_{Q^i} \widehat{f}\big(y,\nabla u(y)\big)\,{\rm d}y\\ + C_1\mathcal{L}^d(Q^i) + \int_{Q^i_{\rm out}} \widehat{f}\big(y,\nabla u(y)\big)\,{\rm d}y + \frac{c_5+1}{m}\lambda^{d-1}\rho^d,\end{aligned}$$

for some constant C_1 depending only on c_1, c_2, c_3, c_4, c_5, and k. Summing over $i \in \{1,\dots,m\}$ and dividing by m, we obtain

$$\frac{1}{m}\sum_{i=1}^m F^a(v^i,Q^\lambda_\nu(x,\rho)) \le \Big(1+\frac{C}{m}\Big)F^a(u,Q^\lambda_\nu(x,\rho)) + \frac{C}{m}\lambda^{d-1}\rho^d, \tag{2.3.17}$$

for some constant C depending only on c_1, c_2, c_3, c_4, c_5, and k, which we may assume to be larger than the constant of (H8). Finally, using Lemma 2.3.3, we get

$$\frac{1}{m}\sum_{i=1}^m F^s(v^i,Q^\lambda_\nu(x,\rho)) \le \Big(1+\frac{C}{m}\Big)F^s(u,Q^\lambda_\nu(x,\rho)),$$

which, together with (2.3.17), shows that there exists $i \in \{1,\dots,m\}$ such that (2.3.11) holds with $v = v^i$.

Assume now that $\widehat{f}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^{k\times d}$. Thanks to (f2′) and to (f4), for every $x \in \mathbb{R}^d$ there exists $\omega(x)$ which minimises the continuous function $A \mapsto \widehat{f}(x,A)$. Therefore, by the uniform continuity of $\widehat{f}$ on compact subsets of $\mathbb{R}^d \times \mathbb{R}^{k\times d}$, there exists $\rho^{\lambda,\nu}_m(x) > 0$ such that for every $\rho \in (0,\rho^{\lambda,\nu}_m(x))$ one has

$$|\widehat{f}(y,A) - \widehat{f}(x,A)| \le \frac{1}{m} \tag{2.3.18}$$

for every $y \in Q^\lambda_\nu(x,\rho)$ and $|A| \le (c_2+c_4)/c_1$. Since $|\omega(x)| \le \frac{c_2+c_4}{c_1}$ thanks to (f2) and (f3), the previous displayed inequality implies

$$|\widehat{f}(y,\omega(x)) - \widehat{f}(x,\omega(x))| \le \frac{1}{m}. \tag{2.3.19}$$

Exploiting these inequalities and the minimality of $\omega(x)$, we obtain that

$$\widehat{f}(y,\omega(x)) \le \widehat{f}(y,A) + \frac{2}{m} \tag{2.3.20}$$

for every $y \in Q^\lambda_\nu(x,\rho)$ and $A \in \mathbb{R}^{k\times d}$. Indeed, if $|A| \le (c_2+c_4)/c_1$, we may apply (2.3.18) twice to get

$$\widehat{f}(y,\omega(x)) \le \widehat{f}(x,\omega(x)) + \frac{1}{m} \le \widehat{f}(x,A) + \frac{1}{m} \le \widehat{f}(y,A) + \frac{2}{m}.$$

Conversely, if $|A| > (c_4 + c_2)/c_1$, we may use the minimality of $\omega(x)$, (f2′), and (f3) to get

$$\widehat{f}(x, \omega(x)) \leq \widehat{f}(x, 0) \leq c_4 \leq \widehat{f}(y, A),$$

which, together with (2.3.19), implies (2.3.20). As in (2.3.15), for every $y \in \mathbb{R}^d$ we set $w(y) \coloneqq \omega(x)(y-x)+\ell_A(x)$. Replacing (2.3.14) by (2.3.20) in the argument that we used in the case where $\widehat{f}$ was not assumed to be continuous on $\mathbb{R}^d \times \mathbb{R}^{k\times d}$, we obtain the existence of $i \in \{1, \dots, m\}$ such that

$$F(w + \Phi^i_R \circ (u - w), Q^\lambda_\nu(x, \rho)) \leq \Big(1 + \frac{C}{m}\Big) F(u, Q^\lambda_\nu(x, \rho)) + \frac{C}{m}\lambda^{d-1}\rho^d.$$

In the case where $\widehat{f}$ does not depend on x, the minimising function ω is independent of x, so that (2.3.20) holds for every $y \in \mathbb{R}^d$, since it is a direct consequence of minimality. □

The next result follows immediately from Lemma 2.3.9 and shows that it is always possible to find minimisers to $\mathfrak{m}^F(\ell_A, Q^\lambda_\nu(x, \rho))$ which are bounded in L^∞. Given $A \in \mathbb{R}^{k\times d}$, we set

$$C_A \coloneqq 2c_3 k^{1/2} C|A| + 2C(c_4 + 1), \tag{2.3.21}$$

where C is the constant of Lemma 2.3.9. We also recall that the constant $c_{A,m}$ is given by (2.3.9).

Corollary 2.3.10. *Let $F \in \mathfrak{F}$ and assume that there exists a function $\widehat{f} \in \mathscr{F}$ satisfying* (2.3.10). *Then there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $A \in \mathbb{R}^{k\times d}$, $\nu \in \mathbb{S}^{d-1}$, and $\lambda \geq 1$ there exists $\rho^{\nu,\lambda}_{m,A}(x) > 0$ such that for every $\rho \in (0, \rho^{\nu,\lambda}_{m,A}(x))$ there exists $u \in \mathrm{BV}(Q^\lambda_\nu(x, \rho); \mathbb{R}^k) \cap L^\infty(Q^\lambda_\nu(x, \rho); \mathbb{R}^k)$, with $u = \ell_A$ on $\partial Q^\lambda_\nu(x, \rho)$ and $\|u - \ell_A\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \leq c_{A,m}\lambda\rho$, such that*

$$F(u, Q^\lambda_\nu(x, \rho)) \leq \mathfrak{m}^F(\ell_A, Q^\lambda_\nu(x, \rho)) + \frac{C_A\lambda^{d-1}\rho^d}{m}. \tag{2.3.22}$$

In particular, for $t = c_{A,m}\lambda$ we have

$$\mathfrak{m}^F_{t\rho}(\ell_A, Q^\lambda_\nu(x, \rho)) \leq \mathfrak{m}^F(\ell_A, Q^\lambda_\nu(x, \rho)) + \frac{C_A\lambda^{d-1}\rho^d}{m}. \tag{2.3.23}$$

Moreover, if $\widehat{f}$ is continuous in $\mathbb{R}^d \times \mathbb{R}^{k\times d}$ then $N = \varnothing$. Finally, if $\widehat{f}$ is independent of x then $\rho^{\nu,\lambda}_{m,A}(x) = +\infty$.

Proof. Let $N \in \mathcal{B}(\mathbb{R}^d)$ and $\rho^{\nu,\lambda}_{m,A}(x) > 0$ be as in Lemma 2.3.9. Consider a function $v \in \mathrm{BV}(Q^\lambda_\nu(x, \rho); \mathbb{R}^k)$, with $v = \ell_A$ on $\partial Q^\lambda_\nu(x, \rho)$, such that

$$F(v, Q^\lambda_\nu(x, \rho)) \leq \mathfrak{m}^F(\ell_A, Q^\lambda_\nu(x, \rho)) + \frac{\lambda^{d-1}\rho^d}{m}.$$

Since the function ℓ_A is a competitor for the minimisation problem on the right-hand side of the previous inequality, we also get

$$F(v, Q^\lambda_\nu(x, \rho)) \leq (c_3 k^{1/2}|A| + c_4 + 1)\lambda^{d-1}\rho^d.$$

Thus, we may apply Lemma 2.3.9 to obtain $u \in \mathrm{BV}(Q^\lambda_\nu(x, \rho); \mathbb{R}^k) \cap L^\infty(Q^\lambda_\nu(x, \rho); \mathbb{R}^k)$, with $u = \ell_A$ on $\partial Q^\lambda_\nu(x, \rho)$ and $\|u - \ell_A\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \leq c_{A,m}\lambda\rho$, such that (2.3.22) holds, concluding the proof. □

The following proposition shows that under the hypotheses of Lemma 2.3.9 the functions f_δ and f_δ^∞ converge $\mathcal{L}^d$-a.e. to f and f^∞ as $\delta \to 0^+$ (see Definition 1.5.6).

Proposition 2.3.11. *Let $F \in \mathfrak{F}$. Assume that there exists $\widehat{f} \in \mathscr{F}$ such that* (2.3.10) *holds. Then*

$$f(x,A) = \lim_{\delta\to 0^+} f_\delta(x,A) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and for every } A \in \mathbb{R}^{k\times d}, \tag{2.3.24}$$

$$f^\infty(x,A) = \lim_{\delta\to 0^+} f_\delta^\infty(x,A) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and for every } A \in \mathbb{R}^{k\times d}. \tag{2.3.25}$$

Proof. From Remark 2.3.8, we deduce that the limits in the right-hand side of (2.3.24) and of (2.3.25) exist, $f(x,A) \le \inf_{\delta>0} f_\delta(x,A)$, and $f^\infty(x,A) \le \inf_{\delta>0} f_\delta^\infty(x,A)$.

Hence, we are left with proving that the converse inequalities also hold. Let $N \in \mathcal{B}(\mathbb{R}^d)$ be the $\mathcal{L}^d$-negligible set of Corollary 2.3.10, let $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $A \in \mathbb{R}^{k\times d}$, and $t \ge 1$. By (2.3.6), we have that for $\rho > 0$ small enough

$$\frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{t\rho^d} \le \frac{f(x,tA)}{t} + \frac{1}{tm}. \tag{2.3.26}$$

We can now apply Corollary 2.3.10, with A replaced by tA, to obtain a function $u \in \mathrm{BV}(Q(x,\rho);\mathbb{R}^k)$, with $u = t\ell_A$ on $\partial Q(x,\rho)$ and $\|u - t\ell_A\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le c_{tA,m}\rho$, such that (2.3.22) holds with A replaced by tA. By taking $\rho > 0$ small enough, we may also suppose that $\|u - t\ell_A\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le 1/2$. Inequality (2.3.26) then yields

$$\frac{F(u,Q(x,\rho))}{t\rho^d} \le \frac{f(x,tA)}{t} + \frac{C_{tA}+1}{tm} \le \frac{f(x,tA)}{t} + \frac{C_A+1}{m},$$

where $C_A > 0$ is the constant defined in (2.3.21). Note that in the last inequality we have used the estimate $C_{tA} \le tC_A$ for every $t \ge 1$.

We now compare $F_\delta(u,Q(x,\rho))$ and $F(u,Q(x,\rho))$. Since $\|u - t\ell_A\|_{L^\infty} \le 1/2$ and $[u - t\ell_A] = [u]$, we have $|[u]| \le 1$ $\mathcal{H}^{d-1}$-a.e. on J_u. Therefore, by (H3′) we get

$$\begin{aligned} F_\delta(u,Q(x,\rho)) &= F(u,Q(x,\rho)) + \delta|Du|(Q(x,\rho)) \\ &\le F(u,Q(x,\rho)) + \delta\int_{Q(x,\rho)} |\nabla u|\,\mathrm{d}x + \delta|D^c u|(Q(x,\rho)) + \delta\int_{J_u} (|[u]|\wedge 1)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\le \Big(1+\frac{\delta}{c_1}\Big)F(u,Q(x,\rho)) + \frac{c_2\delta}{c_1}\rho^d. \end{aligned}$$

Letting δ be so small that $\delta/c_1 \le 1/m$ and $\delta c_2/c_1 \le 1/m$, we then obtain

$$\frac{F_\delta(u,Q(x,\rho))}{t\rho^d} \le \Big(1+\frac{1}{m}\Big)\frac{F(u,Q(x,\rho))}{t\rho^d} + \frac{1}{m} \le \Big(1+\frac{1}{m}\Big)\Big(\frac{f(x,tA)}{t} + \frac{C_A+1}{m}\Big)$$

which, recalling that $u = t\ell_A$ on $\partial Q(x,\rho)$, implies

$$\frac{\mathfrak{m}^{F_\delta}(\ell_{tA}, Q(x,\rho))}{t\rho^d} \le \Big(1+\frac{1}{m}\Big)\Big(\frac{f(x,tA)}{t} + \frac{C_A+1}{m}\Big).$$

In particular, using this inequality with $t = 1$ and taking the lim sup as $\rho \to 0^+$, we deduce

$$f_\delta(x,A) \le \Big(1+\frac{1}{m}\Big)\Big(f(x,A) + \frac{C_A+1}{m}\Big),$$

while taking the lim sup as $\rho \to 0^+$ first and letting then $t \to +\infty$, we obtain

$$f_\delta^\infty(x,A) \le \Big(1+\frac{1}{m}\Big)\Big(f^\infty(x,A)+\frac{C_A+1}{m}\Big).$$

Finally, letting $\delta \to 0^+$ first and then taking the limit for $m \to +\infty$, we obtain the inequalities $\inf_{\delta>0} f_\delta(x,A) \le f(x,A)$ and $\inf_{\delta>0} f_\delta^\infty(x,A) \le f^\infty(x,A)$, concluding the proof. □

We now pass to the study of the minimisation problems used to define the integrands for the surface terms.

Lemma 2.3.12. *Let $F \in \mathfrak{F}$, $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $m \in \mathbb{N}$, $\zeta \in \mathbb{R}^k$, and $\rho > 0$. Then there exists $u \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)$, with $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \le \sigma^m|\zeta|$, such that*

$$F(u,Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + \frac{K(|\zeta|\wedge 1)}{m}\rho^{d-1} + K\rho^d, \tag{2.3.27}$$

where $K := \max\{c_3kC, C(1+c_4)+c_4+1\}$ and C is the constant of (H8).

Proof. By (2.3.4) there exists $v \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k)$, with $v = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$, such that

$$F(v,Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + \rho^d \le c_3k(|\zeta|\wedge 1)\rho^{d-1} + c_4\rho^d, \tag{2.3.28}$$

where the second inequality follows from (H4′) and the fact that $u_{x,\zeta,\nu}$ is a competitor for the minimisation problem $\mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,\rho))$.

Let us choose $R = |\zeta|$ and let $u^i := \Phi_R^i \circ v$ for all $i \in \{1,\dots,m\}$. Note that, with this choice of R, it follows from (1.3.13) that $u^i = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and $\|u^i\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \le \sigma^m|\zeta|$. To conclude, it is then enough to observe that by (H8) and (2.3.28) there exists $i \in \{1,\dots,m\}$ such that

$$\begin{aligned} F(u^i,Q_\nu(x,\rho)) &\le F(v,Q_\nu(x,\rho)) + c_4\rho^d + \frac{C}{m}\Big(F(v,Q_\nu(x,\rho)) + \rho^d\Big) \\ &\le \mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + (C(1+c_4)+c_4+1)\rho^d + \frac{c_3kC(|\zeta|\wedge 1)}{m}\rho^{d-1}, \end{aligned}$$

which proves (2.3.27) with $u = u^i$. □

We are now ready to prove that g_δ converges to g.

Proposition 2.3.13. *Let $F \in \mathfrak{F}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$. Then*

$$g(x,\zeta,\nu) = \lim_{\delta\to 0^+} g_\delta(x,\zeta,\nu). \tag{2.3.29}$$

Proof. Thanks to Remark 2.3.8, the limit in the right-hand side of (2.3.29) exists and

$$g(x,\zeta,\nu) \le \lim_{\delta\to 0^+} g_\delta(x,\zeta,\nu). \tag{2.3.30}$$

Hence, we are left with proving that the converse inequality holds true as well. We fix $m \in \mathbb{N}$ and observe that by (2.3.7) we have that

$$\frac{\mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x,\zeta,\nu) + \frac{1}{m} \tag{2.3.31}$$

for $\rho > 0$ small enough. We now apply Lemma 2.3.12 and find $u \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k)$, with $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \le \sigma^m|\zeta|$, such that

$$F(u, Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \frac{K}{m}\rho^{d-1} + K\rho^d.$$

From (2.3.31) we then deduce that

$$\frac{F(u, Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x,\zeta,\nu) + \frac{K+1}{m} + K\rho.$$

To conclude, we need to compare $F_\delta(u, Q_\nu(x,\rho))$ with $F(u, Q_\nu(x,\rho))$. To this end, we first observe that, as $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \le \sigma^m|\zeta|$, we have $|[u]| \le 2\sigma^m(|\zeta| \vee 1)(|[u]| \wedge 1)$. Thanks to this inequality, we may estimate

$$\begin{aligned}
F_\delta(u, Q_\nu(x,\rho)) &= F(u, Q_\nu(x,\rho)) + \delta|Du|(Q_\nu(x,\rho)) \\
&\le F(u, Q_\nu(x,\rho)) + \delta\Big(\int_{Q_\nu(x,\rho)} |\nabla u|\,\mathrm{d}x + |D^c u|(Q_\nu(x,\rho))\Big) \\
&\quad + 2\sigma^m(|\zeta| \vee 1)\delta \int_{J_u \cap Q_\nu(x,\rho)} (|[u]| \wedge 1)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\le \Big(1 + \frac{2\delta\sigma^m(|\zeta| \vee 1)}{c_1}\Big)F(u, Q_\nu(x,\rho)) + \frac{2c_2\delta\sigma^m(|\zeta| \vee 1)}{c_1}\rho^d.
\end{aligned}$$

Letting δ be so small that $2\delta\sigma^m(|\zeta| \vee 1)/c_1 \le 1/m$ and $2c_2\delta\sigma^m(|\zeta| \vee 1)/c_1 \le 1$, we get

$$\frac{F_\delta(u, Q_\nu(x,\rho))}{\rho^{d-1}} \le \Big(1 + \frac{1}{m}\Big)g(x,\zeta,\nu) + \frac{2K+2}{m} + \Big(2K+1\Big)\rho.$$

Recalling the definition of $\mathfrak{m}^{F_\delta}(u, Q_\nu(x,\rho))$, from this last inequality we deduce that

$$\frac{\mathfrak{m}^{F_\delta}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \le \Big(1 + \frac{1}{m}\Big)g(x,\zeta,\nu) + \frac{2K+2}{m} + \Big(2K+1\Big)\rho.$$

Passing to the limsup for $\rho \to 0^+$, we obtain

$$g_\delta(x,\zeta,\nu) \le \Big(1 + \frac{1}{m}\Big)g(x,\zeta,\nu) + \frac{2K+2}{m}.$$

Taking first lim sup to $\delta \to 0^+$, and then $m \to +\infty$, we finally obtain $\lim_{\delta\to 0^+} g_\delta(x,\zeta,\nu) \le g(x,\zeta,\nu)$, which together with (2.3.30) concludes the proof. □

As the following proposition shows, the functions f and g defined by (2.3.6) and (2.3.7) belong to $\mathscr{F}$ and to $\mathscr{G}$, respectively.

Proposition 2.3.14. *Let $F \in \mathfrak{F}_{\mathrm{sc}}$, f, $\widehat{f}$, and g be the functions defined by* (2.3.6), (2.3.8), *and* (2.3.7), *respectively. Then $\widehat{f}, f \in \mathscr{F}$ and $g \in \mathscr{G}$.*

Proof. The proof of the inclusions $\widehat{f}$, $f \in \mathscr{F}$ and of properties (g1)–(g5) for g can be obtained by adapting the arguments of [100, Section 5], with minor adaptations.

We are left with proving that g satisfies (g6). To see this, let $\zeta_1, \zeta_2 \in \mathbb{R}^k \setminus \{0\}$ with $c_6 k|\zeta_1| \le |\zeta_2|$. We set $\lambda := |\zeta_1|/|\zeta_2|$ and note that $\lambda \le 1/(c_6 k)$. Let $R \in \mathrm{SO}(k)$ be

a rotation that maps $\lambda\zeta_2$ to ζ_1, let $x \in \mathbb{R}^d$, and let $\nu \in \mathbb{S}^{d-1}$. By (2.3.7) there exists $u \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k)$, with $u = u_{x,\zeta_2,\nu}$ on $\partial Q_\nu(x,\rho)$, such that

$$F(u, Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \rho^d. \tag{2.3.32}$$

We set $v := \lambda R u$ and observe that $v = u_{x,\zeta_1,\nu}$ on $\partial Q_\nu(x,\rho)$. Then, using (H9), we get

$$F(v, Q_\nu(x,\rho)) \le F(u, Q_\nu(x,\rho)) + (c_4 + c_2)\rho^d,$$

which, by (2.3.32), implies

$$F(v, Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + (c_4 + c_2 + 1)\rho^d. \tag{2.3.33}$$

Dividing this last inequality by ρ^{d-1} and taking the lim sup for $\rho \to 0^+$, we obtain

$$\limsup_{\rho\to 0^+} \frac{F(v, Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x,\zeta_2,\nu),$$

which, in light of the fact that $v = u_{x,\zeta_1,\nu}$ on $\partial Q_\nu(x,\rho)$, implies $g(x,\zeta_1,\nu) \le g(x,\zeta_2,\nu)$, concluding the proof. □

Using the results of [41], we can now establish an integral representation both for the absolutely continuous part and the jump part of the perturbed functionals F_δ. This representation is achieved by means of the functions f_δ and g_δ defined above.

Proposition 2.3.15. *Let $F \in \mathfrak{F}_{\mathrm{sc}}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $\delta > 0$. Let F^a_δ and F^j_δ be the functionals introduced in Definition 2.3.4 and let f_δ and g_δ be as in Definition 2.3.7. Then*

$$F^a_\delta(u,B) = \int_B f_\delta(x,\nabla u)\,\mathrm{d}x, \tag{2.3.34}$$

$$F^j_\delta(u,B) = \int_{J_u\cap B} g_\delta(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.3.35}$$

for every $u \in \mathrm{BV}(O;\mathbb{R}^k)$ and for every $B \in \mathcal{B}(O)$.

Proof. The result follows from [41, Theorem 3.7], using the same arguments as in [100, Theorem 6.1]. □

With this proposition at hand, we are finally ready to prove the main result of the section, i.e., the integral representation of F^a and of F^j.

Theorem 2.3.16. *Let $F \in \mathfrak{F}_{\mathrm{sc}}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and let f and g be the functions defined by* (2.3.6) *and* (2.3.7)*, respectively. Then*

$$F^a(u,B) = \int_B f(x,\nabla u)\,\mathrm{d}x, \tag{2.3.36}$$

$$F^j(u,B) = \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.3.37}$$

for every $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$ and for every $B \in \mathcal{B}(O)$.

Proof. We first show that

$$F^a(u,B)=\int_B f(x,\nabla u)\,\mathrm{d}x \quad \text{for all } u\in \mathrm{BV}(O;\mathbb{R}^k) \text{ and } B\in\mathcal{B}(O). \tag{2.3.38}$$

To this end, we first note that $F^a_\delta(u,B)=F^a(u,B)+\delta\int_B|\nabla u|\mathrm{d}x$ for every $u\in \mathrm{BV}(O;\mathbb{R}^k)$ and every $B\in\mathcal{B}(O)$, so that

$$F^a(u,B)=\inf_{\delta>0}F^a_\delta(u,B)=\lim_{\delta\to 0^+}F^a_\delta(u,B). \tag{2.3.39}$$

Since by definition of f_δ we have that

$$f_\delta(x,A)\le (c_3k^{1/2}+\delta)|A|+c_4 \quad \text{for all } x\in\mathbb{R}^d \text{ and } A\in\mathbb{R}^{k\times d},$$

from (2.3.39), (2.3.34), Proposition 2.3.11, and the Dominated Convergence Theorem we infer (2.3.38).

We now prove that

$$F^j(u,B)=\int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for all } u\in \mathrm{BV}(O;\mathbb{R}^k) \text{ and } B\in\mathcal{B}(O). \tag{2.3.40}$$

To this end, we first note that $F^j_\delta(u,B)=F^j(u,B)+\delta\int_{J_u\cap B}|[u]|\mathrm{d}\mathcal{H}^{d-1}$ for every $u\in \mathrm{BV}(O;\mathbb{R}^k)$ and $B\in\mathcal{B}(O)$; therefore,

$$F^j(u,B)=\inf_{\delta>0}F^j_\delta(u,B)=\lim_{\delta\to 0^+}F^j_\delta(u,B). \tag{2.3.41}$$

It is immediate to check that

$$g_\delta(x,\zeta,\nu)\le (c_3k+\delta)|\zeta| \quad \text{for all } x\in\mathbb{R}^d,\ \zeta\in\mathbb{R}^k, \text{ and } \nu\in\mathbb{S}^{d-1}.$$

Hence, using (2.3.35), Proposition 2.3.13, and the Dominated Convergence Theorem, from (2.3.41) we obtain (2.3.40).

We are left with proving (2.3.36) and (2.3.37) in the general case $u\in \mathrm{GBV}_\star(O;\mathbb{R}^k)$. Let $\{R_m\}_m$ be a positive sequence with $R_m\to+\infty$. Lemma 2.3.2 shows that equality (2.3.2) with $b=a$ and $b=j$ holds for every $B\in\mathcal{B}(O)$. Therefore, to conclude it is enough to show that

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{i=1}^m F^a(\Phi^i_{R_m}\circ u,B)=\int_B f(x,\nabla u)\,\mathrm{d}x, \tag{2.3.42}$$

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{i=1}^m F^j(\Phi^i_{R_m}\circ u,B)=\int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.3.43}$$

for every $B\in\mathcal{B}(O)$.

Since for every $i\in\{1,\dots,m\}$ we have that $\Phi^i_{R_m}\circ u\in \mathrm{BV}(O;\mathbb{R}^k)\cap L^\infty(O;\mathbb{R}^k)$, from (2.3.36) and (2.3.40) we deduce that

$$F^a(\Phi^i_{R_m}\circ u,B)=\int_B f\big(x,\nabla(\Phi^i_{R_m}\circ u)\big)\,\mathrm{d}x,$$
$$F^j(\Phi^i_{R_m}\circ u,B)=\int_{J_{\Phi^i_{R_m}(u)}\cap B} g\big(x,[\Phi^i_{R_m}\circ u],\nu_u\big)\,\mathrm{d}\mathcal{H}^{d-1}.$$

We then observe that for every $m \in \mathbb{N}$ there exist $i(m), j(m) \in \{1, \dots, m\}$ such that

$$\int_B f\big(x, \nabla(\Phi^{i(m)}_{R_m} \circ u)\big)\,\mathrm{d}x \le \frac{1}{m}\sum_{i=1}^{m} F^a(\Phi^i_{R_m} \circ u, B) \le \int_B f\big(x, \nabla(\Phi^{j(m)}_{R_m} \circ u)\big)\,\mathrm{d}x.$$

We observe that both $\nabla(\Phi^{i(m)}_{R_m} \circ u)$ and $\nabla(\Phi^{j(m)}_{R_m} \circ u)$ converge to ∇u pointwise $\mathcal{L}^d$-a.e. in O as $m \to +\infty$ and that $|\nabla(\Phi^i_{R_m} \circ u)| \le |\nabla u|$ for every $i \in \{1, \dots, m\}$, thanks to (1.3.13) and (b) of Proposition 1.3.10. Hence, recalling that by Proposition 2.3.14 the function $A \mapsto f(x, A)$ is continuous (see (f4)), the Dominated Convergence Theorem implies (2.3.42).

To prove (2.3.43), for every $m \in \mathbb{N}$ we choose $i(m), j(m) \in \{1, \dots, m\}$ such that

$$\int_{J_u \cap B} g(x, [\Phi^{i(m)}_{R_m} \circ u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \le \frac{1}{m}\sum_{i=1}^{m} F^j(\Phi^i_{R_m} \circ u, B) \le \int_{J_u \cap B} g(x, [\Phi^{j(m)}_{R_m} \circ u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}.$$

Since by Proposition 2.3.14 the function $\zeta \mapsto g(x, \zeta, \nu)$ is continuous (see (g4)) and for every $i \in \{1, \dots, m\}$ it holds $|[\Phi^i_{R_m} \circ u]| \wedge 1 \le |[u]| \wedge 1$, an application of the Dominated Convergence Theorem and Proposition 1.3.10(c) yield (2.3.43), concluding the proof. □

2.4 A smaller collection of integrands

As the scalar case studied in [101] suggests, to recover a full integral representation for functionals in $\mathfrak{F}_{\mathrm{sc}}$ that is expedient for the study of stochastic homogenisation problems (see Section 2.7) it is convenient to consider a smaller collection of integrands. Indeed, it could be proved (see [100, Theorem 6.7]) that, if $F \in \mathfrak{F}_{\mathrm{sc}}$ satisfies a uniform continuity condition with respect to translations of the independent variable, then also the Cantor part F^c admits an integral representation. However, this uniform continuity property is not easy to verify in non-periodic homogenisation problems. For this reason, it is convenient to introduce suitable subclasses of $\mathfrak{F}_{\mathrm{sc}}$, $\mathscr{F}$, and $\mathscr{G}$.

The aim of this section is to introduce these collections and investigate their main properties.

In the rest of the chapter, we fix two new constants $c_7 > 0$ and $\alpha \in (0, 1)$. Moreover, we fix a continuous non-decreasing function $\vartheta\colon [0, +\infty) \to [0, +\infty)$ such that

$$\vartheta(0) = 0 \quad \text{and} \quad \vartheta(t) \ge \frac{c_1 k^{1/2}}{c_3} t - 1 \quad \text{for every } t \ge 0. \tag{2.4.1}$$

We begin by introducing a smaller collection of volume integrands.

Definition 2.4.1. Let $\mathscr{F}^\alpha$ be the collection of functions $f \in \mathscr{F}$ such that

$$\Big|\frac{f(x, sA)}{s} - \frac{f(x, tA)}{t}\Big| \le \frac{c_7}{s} f(x, sA)^{1-\alpha} + \frac{c_7}{s} + \frac{c_7}{t} f(x, tA)^{1-\alpha} + \frac{c_7}{t} \tag{2.4.2}$$

for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and for every $s, t > 0$ and $A \in \mathbb{R}^{k \times d}$.

Remark 2.4.2. Arguing as in [101, Remark 4.3], one can show that $f \in \mathscr{F}^\alpha$ if and only if for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and for every $A \in \mathbb{R}^{k \times d}$ we have

$$f^\infty(x, A) = \lim_{s \to +\infty} \frac{1}{s} f(x, sA)$$

and

$$\Big|\frac{1}{s}f(x,sA)-f^\infty(x,A)\Big|\le\frac{c_7}{s}+\frac{c_7}{s}f(x,sA)^{1-\alpha} \tag{2.4.3}$$

for $\mathcal{L}^d$-a.e. $x\in\mathbb{R}^d$, for every $A\in\mathbb{R}^{k\times d}$ and for every $s>0$. This is closely related to condition (H4) of [41, Section 4].

We now introduce the smaller collection of surface integrands.

Definition 2.4.3. Let $\mathscr{G}^\vartheta$ be the collection of functions $g\in\mathscr{G}$ such that

$$\Big|\frac{g(x,s\zeta,\nu)}{s}-\frac{g(x,t\zeta,\nu)}{t}\Big|\le\vartheta(s|\zeta|)\frac{g(x,s\zeta,\nu)}{s}+\vartheta(t|\zeta|)\frac{g(x,t\zeta,\nu)}{t}, \tag{2.4.4}$$

for every $s,t>0$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$.

Remark 2.4.4. The arguments of [101, Remark 4.5] show that $g\in\mathscr{G}^\vartheta$ if and only if, for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$ and $\nu\in\mathbb{S}^{d-1}$, we have that the limit

$$g^0(x,\zeta,\nu)\coloneqq\lim_{s\to0^+}\frac{1}{s}g(x,s\zeta,\nu) \tag{2.4.5}$$

exists and

$$\Big|\frac{1}{s}g(x,s\zeta,\nu)-g^0(x,\zeta,\nu)\Big|\le\vartheta(s|\zeta|)\frac{1}{s}g(x,s\zeta,\nu)\le c_3k\vartheta(s|\zeta|)|\zeta| \tag{2.4.6}$$

for every $s>0$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$. By [59, Remark 3.5] these latter two properties are closely related to condition (g5) of that paper.

Remark 2.4.5. Condition (2.4.2) is crucial in Lemma 2.4.13, which is a key instrument in the proof of the integral representation of the Cantor part (see Theorem 2.5.1). Moreover, conditions (2.4.3) and (2.4.6) will be used in Section 2.7 to obtain suitable homogenisation formulas.

Remark 2.4.6. The class $\mathscr{G}^\vartheta$ is non-empty, since the function $(x,\zeta,\nu)\mapsto(c_1k^{1/2}|\zeta|)\wedge c_3$ belongs to $\mathscr{G}^\vartheta$. Arguing as in [101, Remark 4.7], it is possible to show that if $\mathscr{G}^\vartheta\neq\emptyset$ then ϑ satisfies (2.4.1).

In analogy with Section 2.1, we introduce a space of functionals $\mathfrak{F}^{\alpha,\vartheta}$ which contains $F^{f,g}$ for every $f\in\mathscr{F}^\alpha$ and $g\in\mathscr{G}^\vartheta$.

Definition 2.4.7. Let $\mathfrak{F}^{\alpha,\vartheta}$ be the subcollection of functionals $F\in\mathfrak{F}$ satisfying the condition

$$\begin{aligned}\Big|\frac{F(su,O)}{s}-\frac{F(tu,O)}{t}\Big|\le{}&\frac{c_7}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha}+\vartheta(su^O)\frac{F(su,O)}{s}+\frac{c_7}{s}\mathcal{L}^d(O)\\&+\frac{c_7}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha}+\vartheta(tu^O)\frac{F(tu,O)}{t}+\frac{c_7}{t}\mathcal{L}^d(O),\end{aligned} \tag{2.4.7}$$

for every $t,s>0$, $O\in\mathcal{O}_c(\mathbb{R}^d)$, and $u\in\mathrm{BV}(O;\mathbb{R}^k)\cap L^\infty(O;\mathbb{R}^k)$, where $u^O\coloneqq\operatorname{osc}^O u\coloneqq\operatorname{ess\,sup}_{x,y\in O}|u(x)-u(y)|$. We also set $\mathfrak{F}^{\alpha,\vartheta}_{\mathrm{sc}}\coloneqq\mathfrak{F}^{\alpha,\vartheta}\cap\mathfrak{F}_{\mathrm{sc}}$.

Remark 2.4.8. Let $O\in\mathcal{O}_c(\mathbb{R}^d)$ and $u\in\mathrm{BV}(O;\mathbb{R}^k)\cap L^\infty(O;\mathbb{R}^k)$. Then

$$|[u](x)|\le\operatorname{osc}^O u\quad\text{ for every } x\in J_u\cap O. \tag{2.4.8}$$

Indeed, let $x \in J_u \cap O$ and assume, by contradiction, that $|u^+(x) - u^-(x)| > \operatorname{osc}^O u$. Then we can choose $\varepsilon > 0$ such that

$$|u^+(x) - u^-(x)| \geq \operatorname{osc}^O u + 3\varepsilon.$$

By the definition of $u^\pm(x)$, for all sufficiently small $\rho > 0$ the sets $E_\rho^\pm := \{y \in O \cap B_\rho(x) \cap H^\pm(x) \colon\ |u(y) - u^\pm(x)| < \varepsilon\}$ (see Definition 1.3.9) have positive $\mathcal{L}^d$-measure. Hence $E_\rho^+ \times E_\rho^-$ has positive $\mathcal{L}^{2d}$-measure, and for every $(y, z) \in E_\rho^+ \times E_\rho^-$ we have

$$|u(y) - u(z)| \geq |u^+(x) - u^-(x)| - 2\varepsilon \geq \operatorname{osc}^O u + \varepsilon,$$

contradicting the definition of $\operatorname{osc}^O u$.

Proposition 2.4.9. *Let $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\vartheta$. Then the functional $F^{f,g}$ of Definition 2.1.6 belongs to $\mathfrak{F}^{\alpha,\vartheta}$.*

Proof. The result is proved as in [101, Proposition 4.9], replacing Remark 2.9 therein by our Proposition 2.1.14 and using Remark 2.4.8 to obtain the first inequality in (4.16) of that paper. □

To study the Γ-limit of sequences of functionals $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\vartheta}$, given $O \in \mathcal{O}_c(\mathbb{R}^d)$ and a function $u \in \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$, it is important to be able to find approximate recovery sequences for u that are uniformly bounded in $L^\infty(O;\mathbb{R}^k)$. This is dealt with in the next lemma. Note that, for this particular result, there is no need to use condition (2.4.7).

Lemma 2.4.10. *Let $m \in \mathbb{N}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $F \in \mathfrak{F}$. Assume that there exists a sequence $\{F_n\}_n \subset \mathfrak{F}$ with $\{F_n(\cdot, O)\}_n$ Γ-converging to $F(\cdot, O)$ in the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$ as $n \to +\infty$. Then for every $u \in \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$, there exist sequences $\{u_n\}_n \subset \mathrm{BV}(O;\mathbb{R}^k)$ and $\{v_n\}_n \subset \mathrm{BV}(O;\mathbb{R}^k)$ converging to u in $L^1(O;\mathbb{R}^k)$ such that*

$$u_n^O \leq 4\sigma^m u^O \quad \text{and} \quad \|v_n\|_{L^\infty(O;\mathbb{R}^k)} \leq 2\sigma^m \|u\|_{L^\infty(O;\mathbb{R}^k)} \text{ for every } n \in \mathbb{N}, \tag{2.4.9}$$

$$\limsup_{n\to+\infty} F_n(u_n, O) \vee \limsup_{n\to+\infty} F_n(v_n, O) \leq F(u, O) + C\frac{F(u, O) + \mathcal{L}^d(O)}{m}, \tag{2.4.10}$$

where C is the constant of (H8).

Proof. By the invariance property (H5) it is not restrictive to assume that $u^O = \|u\|_{L^\infty(O;\mathbb{R}^k)}$. By definition of Γ-convergence there exists a sequence $\{w_n\}_n \subset L^0(\mathbb{R}^d;\mathbb{R}^k)$ converging to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ such that

$$\lim_{n\to+\infty} F_n(w_n, O) = F(u, O). \tag{2.4.11}$$

Let us fix $R := 2\|u\|_{L^\infty(O;\mathbb{R}^k)}$. By (H8) for every $n \in \mathbb{N}$ we can choose $i(n) \in \{1, \dots, m\}$ such that

$$F_n(\Phi_R^{i(n)} \circ w_n, O) \leq F_n(w_n, O) + C\frac{F_n(w_n, O) + \mathcal{L}^d(O)}{m} + c_4\mathcal{L}^d(O^R_{w_n,0}).$$

Since $\{w_n\}_n$ converges to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, by our choice of R we have that, up to a subsequence, not relabelled, $\mathcal{L}^d(O^R_{w_n,0}) \to 0$ as $n \to +\infty$. Setting $u_n := \Phi_R^{i(n)} \circ w_n$, we deduce from (2.4.11) and the previous inequality that

$$\limsup_{n\to+\infty} F_n(u_n, O) \leq F(u, O) + C\frac{F(u, O) + \mathcal{L}^d(O)}{m}.$$

To conclude, it is then enough to observe that (1.3.13) implies the chain of inequalities

$$u_n^O \le 2\|u_n\|_{L^\infty(O;\mathbb{R}^k)} \le 4\sigma^m \|u\|_{L^\infty(O;\mathbb{R}^k)} = 4\sigma^m u^O$$

and the convergence of $\{u_n\}_n$ to u in $L^1(O;\mathbb{R}^k)$.

The construction of v_n is similar. □

We now want to study Γ-limits of sequences of functionals in $\mathfrak{F}^{\alpha,\vartheta}$. To this aim, it is convenient to introduce a family of subcollections of $\mathfrak{F}$ in which (2.4.7) holds in a weaker form. Given $m \in \mathbb{N}$, we set

$$\vartheta_m(t) := \vartheta(4\sigma^m t) \tag{2.4.12}$$

for every $t \ge 0$.

Definition 2.4.11. Given $m \in \mathbb{N}$, we denote by $\mathfrak{F}^{\alpha,\vartheta_m}$ the collection of functionals $F \in \mathfrak{F}$ satisfying

$$\begin{aligned}\Big|\frac{F(su,O)}{s} - \frac{F(tu,O)}{t}\Big| &\le \frac{c_7}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha} + \vartheta_m(su^O)\frac{F(su,O)}{s} + \frac{c_7}{s}\mathcal{L}^d(O)\\ &+ \frac{c_7}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha} + \vartheta_m(tu^O)\frac{F(tu,O)}{t} + \frac{c_7}{t}\mathcal{L}^d(O) + R_{m,s}(u,O) + R_{m,t}(u,O)\end{aligned}$$

for every $t, s > 0$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $u \in \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O,\mathbb{R}^k)$, where $R_{m,t}(u,O)$ is defined for every $t > 0$ by

$$C\frac{1+\vartheta_m(tu^O)}{t}\frac{F(tu,O)+\mathcal{L}^d(O)}{m} + C^{1-\alpha}\frac{c_7}{t}\Big(\frac{F(tu,O)+\mathcal{L}^d(O)}{m}\Big)^{1-\alpha}\mathcal{L}^d(O)^\alpha, \tag{2.4.13}$$

with C as in (H8). We also set $\mathfrak{F}_w^{\alpha,\vartheta} := \big(\bigcap_{m=1}^\infty \mathfrak{F}^{\alpha,\vartheta_m}\big) \cap \mathfrak{F}_{\mathrm{sc}}$.

While in the scalar case the Γ-limits of sequences in $\mathfrak{F}^{\alpha,\vartheta}$ belong to the same space (see [101, Proposition 6.9]), in the vector-valued case we can only prove that they belong to the larger space $\mathfrak{F}_w^{\alpha,\vartheta}$.

Proposition 2.4.12. *Let $\{F_n\}_n$ be a sequence of functionals in $\mathfrak{F}^{\alpha,\vartheta}$ and let $F \in \mathfrak{F}$. Assume that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then $F \in \mathfrak{F}^{\alpha,\vartheta_m} \cap \mathfrak{F}_{\mathrm{sc}}$ for every $m \in \mathbb{N}$. In particular, $F \in \mathfrak{F}_w^{\alpha,\vartheta}$.*

Proof. By the semicontinuity of Γ-limits we have $F \in \mathfrak{F}_{\mathrm{sc}}$. Thus, we only need to prove that $F \in \mathfrak{F}^{\alpha,\vartheta_m}$ for any $m \in \mathbb{N}$.

Let us fix $m \in \mathbb{N}$. By the continuity of ϑ_m and exchanging the roles of s and t, to conclude it is enough to show that for any $O \in \mathcal{O}_c(\mathbb{R}^d)$, $u \in \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$, and $s, t > 0$ it holds

$$\begin{aligned}&(1-\vartheta_m(su^O))\frac{F(su,O)}{s} - \frac{c_7}{s}\mathcal{L}^d(O) - \frac{c_7}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha}\\ &\le (1+\vartheta_m(tu^O))\frac{F(tu,O)}{t} + \frac{c_7}{t}\mathcal{L}^d(O) + \frac{c_7}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha} + R_{m,t}(u,O),\end{aligned} \tag{2.4.14}$$

assuming that the left-hand side is strictly positive.

Let us fix $O \in \mathcal{O}_c(\mathbb{R}^d)$, $u \in \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$, and $s, t > 0$. By Lemma 2.4.10 there exists a sequence $\{u_n\}_n \subset \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$ such that $u_n \to u$ in $L^1(O;\mathbb{R}^k)$, $u_n^O \le 4\sigma^m u^O$, and

$$\limsup_{n\to+\infty} F_n(tu_n,O) \le F(tu,O) + C\frac{F(tu,O)+\mathcal{L}^d(O)}{m}. \tag{2.4.15}$$

Since $u_n \in \mathrm{BV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$, we may use (2.4.7) to get

$$(1-\vartheta(su_n^O))\frac{F_n(su_n,O)}{s} - \frac{c_7}{s}\mathcal{L}^d(O) - \frac{c_7}{s}\mathcal{L}^d(O)^\alpha F_n(su_n,O)^{1-\alpha} \\ \le (1+\vartheta(tu_n^O))\frac{F_n(tu_n,O)}{t} + \frac{c_7}{t}\mathcal{L}^d(O) + \frac{c_7}{t}\mathcal{L}^d(O)^\alpha F_n(tu_n,O)^{1-\alpha}.$$

Taking the limsup as $n \to +\infty$ and using the monotonicity of ϑ and (2.4.15), we deduce that

$$\begin{aligned}
&\limsup_{n\to+\infty}\Big((1-\vartheta_m(su^O))\frac{F_n(su_n,O)}{s} - \frac{c_7}{s}\mathcal{L}^d(O) - \frac{c_7}{s}\mathcal{L}^d(O)^\alpha F_n(su_n,O)^{1-\alpha}\Big)\\
&\quad\le \limsup_{n\to+\infty}\Big((1-\vartheta(su_n^O))\frac{F_n(su_n,O)}{s} - \frac{c_7}{s}\mathcal{L}^d(O) - \frac{c_7}{s}\mathcal{L}^d(O)^\alpha F_n(su_n,O)^{1-\alpha}\Big)\\
&\quad\le \limsup_{n\to+\infty}\Big(1+\vartheta(tu_n^O))\frac{F_n(tu_n,O)}{t} + \frac{c_7}{t}\mathcal{L}^d(O) + \frac{c_7}{t}\mathcal{L}^d(O)^\alpha F_n(tu_n,O)^{1-\alpha}\Big)\\
&\quad\le (1+\vartheta_m(tu^O))\frac{F(tu,O)}{t} + \frac{c_7}{t}\mathcal{L}^d(O) + \frac{c_7}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha} + R_{m,t}(u,O), \qquad (2.4.16)
\end{aligned}$$

where $R_{m,t}(u,O)$ is defined by (2.4.13). To deal with the first term in the previous chain of inequalities, we introduce the function Ψ defined for every $z \in [0,+\infty)$ as

$$\Psi(z) \coloneqq (1-\vartheta_m(su^O))\frac{z}{s} - \frac{c_7}{s}\mathcal{L}^d(O)^\alpha z^{1-\alpha} - \frac{c_7}{s}\mathcal{L}^d(O).$$

Since the left-hand side of (2.4.14) is assumed to be strictly positive, by definition we have $\Psi(F(su,O)) > 0$, which implies $(1-\vartheta_m(su^O)) > 0$. Let $z_0 \coloneqq c_7^{1/\alpha}(1-\vartheta_m(su^O))^{-1/\alpha}\mathcal{L}^d(O)$ and observe that Ψ is increasing on $(z_0,+\infty)$ and that, if $\Psi(z) > 0$, then $z > z_0$; in particular, $F(su,O) > z_0$. Finally, from the Γ-convergence of $\{F_n(\cdot,O)\}_n$ to $F(\cdot,O)$ and from the convergence of $\{u_n\}_n$ to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, we deduce that

$$F(su,O) \le \liminf_{n\to+\infty} F_n(su_n,O),$$

which, by the monotonicity of Ψ, implies $\Psi(F(su,O)) \le \liminf_{n\to+\infty}\Psi(F_n(su_n,O))$. Recalling the definition of Ψ, from this inequality and (2.4.16) we obtain (2.4.14), concluding the proof. □

The next technical results will be used in the proof of the Representation Theorem 2.5.1. Given $A \in \mathbb{R}^{k\times d}$ and $m \in \mathbb{N}$, we set

$$\kappa_{A,m} \coloneqq 2c_{A,m} + 2d^{1/2}|A|, \qquad (2.4.17)$$

where $c_{A,m}$ is the constant defined by (2.3.9).

Lemma 2.4.13. *For every $A \in \mathbb{R}^{k\times d}$ there exists a constant M_A, depending only on $|A|$ and on the structural constants $c_1,\dots,c_7$, k, and α, such that for every $m \in \mathbb{N}$ and $F \in \mathfrak{F}^{\alpha,\vartheta_m} \cap \mathfrak{F}_{\mathrm{sc}}$ there exists a set $N_m \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N_m) = 0$, independent of A, satisfying the following property: for every $x \in \mathbb{R}^d \setminus N_m$, $\lambda \ge 1$, $\nu \in \mathbb{S}^{d-1}$, and $s,t > 0$ there exist*

$\rho^{\nu,\lambda}_{m,A,t,s}(x) > 0$ *such that for every* $\rho \in (0, \rho^{\nu,\lambda}_{m,A,t,s}(x))$ *we have*

$$
\begin{aligned}
&\Big|\frac{\mathfrak{m}^F(s\ell_A, Q^\lambda_\nu(x,\rho))}{s} - \frac{\mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho))}{t}\Big| \\
&\quad\le \frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{d\alpha}\mathfrak{m}^F(s\ell_A, Q^\lambda_\nu(x,\rho))^{1-\alpha} + \vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{(s\wedge 1)(t\wedge 1)}\Big)\frac{\mathfrak{m}^F(s\ell_A, Q^\lambda_\nu(x,\rho))}{s} \\
&\qquad + \frac{c_7}{t}\lambda^{\alpha(d-1)}\rho^{d\alpha}\mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho))^{1-\alpha} + \vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{(s\wedge 1)(t\wedge 1)}\Big)\frac{\mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho))}{t} \\
&\qquad + \frac{c_7}{s}\lambda^{d-1}\rho^d + \Big(1+\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{(s\wedge 1)(t\wedge 1)}\Big)\Big)\frac{M_A}{(s\wedge 1)m}\lambda^{d-1}\rho^d + \Big(\frac{1}{s^\alpha}+\frac{1}{s}\Big)\frac{M_A}{m^{1-\alpha}}\lambda^{d-1}\rho^d \\
&\qquad + \frac{c_7}{t}\lambda^{d-1}\rho^d + \Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{(s\wedge 1)(t\wedge 1)}\Big)\Big)\frac{M_A}{(t\wedge 1)m}\lambda^{d-1}\rho^d + \Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_A}{m^{1-\alpha}}\lambda^{d-1}\rho^d,
\end{aligned}
$$

where $\kappa_{A,m}$ *is the constant defined by* (2.4.17).

If, in addition, the function f *defined by* (2.3.6) *does not depend on* x, *we have* $N_m = \emptyset$ *and* $\rho^{\nu,\lambda}_{m,A,t,s}(x) = +\infty$.

Proof. By hypothesis $F \in \mathfrak{F}_{\rm sc}$, so that by Theorem 2.3.16, equality (2.3.10) holds with $\widehat{f}$ replaced by f, where f is the function defined by (2.3.6). Hence, we may apply Corollary 2.3.10 to obtain a set $N_m \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N_m) = 0$, satisfying the following property: for every $x \in \mathbb{R}^d \setminus N_m$, $A \in \mathbb{R}^{k\times d}$, $t > 0$, $\nu \in \mathbb{S}^{d-1}$, and $\lambda \ge 1$ there exists $\rho^{\nu,\lambda}_{m,tA}(x)$ such that for every $\rho \in (0, \rho^{\nu,\lambda}_{m,tA}(x))$, there exists $u \in \mathrm{BV}(Q^\lambda_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}u = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)}\ell_A$ and $\|u-\ell_A\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \le \frac{1}{t}c_{tA,m}\lambda\rho \le \frac{1}{t\wedge 1}c_{A,m}\lambda\rho$, such that

$$
\begin{aligned}
F(tu, Q^\lambda_\nu(x,\rho)) &\le \mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho)) + \frac{C_{tA}}{m}\lambda^{d-1}\rho^d \\
&\le \mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho)) + \frac{(t\vee 1)C_A}{m}\lambda^{d-1}\rho^d, \qquad (2.4.18)
\end{aligned}
$$

where C_A is defined by (2.3.21). Note that the oscillation of u on $Q^\lambda_\nu(x,\rho)$ satisfies

$$
u^{Q^\lambda_\nu(x,\rho)} \le \Big(\frac{2}{t\wedge 1}c_{A,m} + \frac{2}{t\wedge 1}d^{1/2}|A|\Big)\lambda\rho = \frac{\kappa_{A,m}}{t\wedge 1}\lambda\rho. \qquad (2.4.19)
$$

We can estimate $\mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho))$ by evaluating $F(\cdot, Q^\lambda_\nu(x,\rho))$ at $t\ell_A$. Recalling (2.4.18) and (H4′), this leads to

$$
F(tu, Q^\lambda_\nu(x,\rho)) \le \big(c_3k^{1/2}t|A| + c_4 + (t\vee 1)C_A\big)\lambda^{d-1}\rho^d. \qquad (2.4.20)
$$

Since $F \in \mathfrak{F}^{\alpha,\vartheta_m}$, by Definition 2.4.11 for every $s > 0$ we have that

$$
\begin{aligned}
&(1-\vartheta_m(su^{Q^\lambda_\nu(x,\rho)}))\frac{F(su, Q^\lambda_\nu(x,\rho))}{s} - \frac{c_7}{s}(\lambda^{d-1}\rho^d)^\alpha F(su, Q^\lambda_\nu(x,\rho))^{1-\alpha} \\
&\qquad -\frac{c_7}{s}\lambda^{d-1}\rho^d - R_{m,s}(u, Q^\lambda_\nu(x,\rho)) \\
&\le (1+\vartheta_m(tu^{Q^\lambda_\nu(x,\rho)}))\frac{F(tu, Q^\lambda_\nu(x,\rho))}{t} + \frac{c_7}{t}(\lambda^{d-1}\rho^d)^\alpha F(tu, Q^\lambda_\nu(x,\rho))^{1-\alpha} \\
&\qquad +\frac{c_7}{t}\lambda^{d-1}\rho^d + R_{m,t}(u, Q^\lambda_\nu(x,\rho)), \qquad (2.4.21)
\end{aligned}
$$

where R is defined by (2.4.13). Using the monotonicity of ϑ_m, (2.4.18)–(2.4.20), and the subadditivity of the function $z \mapsto z^{1-\alpha}$ on $[0,+\infty)$ we see that the left-hand side of (2.4.21) can be bounded from below by

$$\begin{aligned}
&\Big(1-\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{F(su,Q^\lambda_\nu(x,\rho))}{s}-\frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}F(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\\
&\quad-\frac{c_7}{s}\lambda^{d-1}\rho^d-C\Big(1+\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{F(su,Q^\lambda_\nu(x,\rho))+\lambda^{d-1}\rho^d}{sm}\\
&\quad-C^{1-\alpha}\frac{c_7}{s}\frac{F(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}+\lambda^{d-1}\rho^d}{m^{1-\alpha}},
\end{aligned}\tag{2.4.22}$$

while the right-hand side of (2.4.21) can be bounded from above by

$$\begin{aligned}
&\Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{\mathfrak{m}^F(t\ell_A,Q^\lambda_\nu(x,\rho))}{t}+\frac{c_7}{t}\mathfrak{m}^F(t\ell_A,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}\\
&\quad+\frac{c_7}{t}\lambda^{d-1}\rho^d+\Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{C(c_3k^{1/2}|A|+c_4+C_A+1)+C_A}{(t\wedge 1)m}\lambda^{d-1}\rho^d\\
&\quad+\Big(C^{1-\alpha}\frac{c_7}{t^\alpha}+\frac{c_7}{t}\Big)\frac{\big(c_3k^{1/2}|A|+c_4+C_A\big)^{1-\alpha}+1+C_A^{1-\alpha}}{m^{1-\alpha}}\lambda^{d-1}\rho^d.
\end{aligned}\tag{2.4.23}$$

Therefore, by (2.4.21)–(2.4.23) there exists a constant M_A, independent of m, such that

$$\begin{aligned}
&\Big(1-\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{F(su,Q^\lambda_\nu(x,\rho))}{s}-\frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}F(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\\
&\quad-\frac{c_7}{s}\lambda^{d-1}\rho^d-C\Big(1+\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{F(su,Q^\lambda_\nu(x,\rho))+\lambda^{d-1}\rho^d}{sm}\\
&\quad-C^{1-\alpha}\frac{c_7}{s}\frac{F(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}+\lambda^{d-1}\rho^d}{m^{1-\alpha}}\\
&\le\Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{\mathfrak{m}^F(t\ell_A,Q^\lambda_\nu(x,\rho))}{t}+\frac{c_7}{t}\mathfrak{m}^F(t\ell_A,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}\\
&\quad+\frac{c_7}{t}\lambda^{d-1}\rho^d+\Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{M_A}{(t\wedge 1)m}\lambda^{d-1}\rho^d+\Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_A}{m^{1-\alpha}}\lambda^{d-1}\rho^d.
\end{aligned}\tag{2.4.24}$$

We claim that the previous inequality still holds replacing $F(su,Q^\lambda_\nu(x,\rho))$ by the minimum $\mathfrak{m}^F(s\ell_A,Q^\lambda_\nu(x,\rho))$. To prove the claim, for given $s,t>0$ we introduce the function Ψ defined for every $z\in[0,+\infty)$ as

$$\begin{aligned}
\Psi(z)\coloneqq&\Big(1-\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{z}{s}-\frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}z^{1-\alpha}-\frac{c_7}{s}\lambda^{d-1}\rho^d\\
&-C\Big(1+\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda}{t\wedge 1}\rho\Big)\Big)\frac{z+\lambda^{d-1}\rho^d}{sm}-C^{1-\alpha}\frac{c_7}{s}\frac{z^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}+\lambda^{d-1}\rho^d}{m^{1-\alpha}},
\end{aligned}$$

so that the left-hand side of (2.4.24) is equal to $\Psi(F(su,Q^\lambda_\nu(x,\rho)))$ and our claim is that the number $\Psi(\mathfrak{m}^F(s\ell_A,Q^\lambda_\nu(x,\rho)))$ is less than or equal to the right-hand side of (2.4.24). Since this right-hand side is larger than or equal to zero, the claim is trivial if $\Psi(\mathfrak{m}^F(s\ell_A,Q^\lambda_\nu(x,\rho)))$ is not positive. Therefore, to prove the claim we may assume $\Psi(\mathfrak{m}^F(s\ell_A,Q^\lambda_\nu(x,\rho)))>0$. Note that this positivity condition implies that $1-\vartheta_m(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1})>0$.

We now show that

$$\Psi(\mathfrak{m}^F(s\ell_A,Q^\lambda_\nu(x,\rho)))\le\Psi(F(su,Q^\lambda_\nu(x,\rho))),\tag{2.4.25}$$

assuming that $\Psi(\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho))) > 0$. Some straightforward computations show that if $\Psi(z) > 0$ then $z > z_0$, where z_0 is defined by

$$z_0 := c_7^{1/\alpha}\Big(1-\vartheta_m\big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\big)-\frac{C}{m}\Big(1+\vartheta_m\big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\big)\Big)\Big)^{-1/\alpha}\Big(1+\frac{C^{1-\alpha}}{m^{1-\alpha}}\Big)^{1/\alpha}\lambda^{d-1}\rho^d$$

One can also see that Ψ is increasing on $(z_0,+\infty)$. Since $su = s\ell_A$ on $\partial Q_\nu^\lambda(x,\rho)$, we obtain that

$$\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho)) \le F(su, Q_\nu^\lambda(x,\rho)). \tag{2.4.26}$$

Since we assumed that $\Psi(\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho))) > 0$, we have $\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho)) > z_0$. Hence, recalling that Ψ is increasing on $(z_0,+\infty)$, from (2.4.26) we obtain that (2.4.25). Thus, in the left-hand side of (2.4.24) we can substitute $F(su, Q_\nu^\lambda(x,\rho))$ by $\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho))$. This new inequality, together with

$$\frac{\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho))}{s} \le \frac{c_3k^{1/2}|A|+c_4}{s\wedge 1}\lambda^{d-1}\rho^d,$$

implies that

$$\begin{aligned}
&\Big(1-\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho))}{s} - \frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}\mathfrak{m}^F(s\ell_A, Q_\nu^\lambda(x,\rho))^{1-\alpha}\\
&\quad - \frac{c_7}{s}\lambda^{d-1}\rho^d - \Big(1+\vartheta_m\Big(\frac{s\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{M_A}{(s\wedge 1)m}\lambda^{d-1}\rho^d - \Big(\frac{1}{s^\alpha}+\frac{1}{s}\Big)\frac{M_A}{m^{1-\alpha}}\lambda^{d-1}\rho^d\\
&\le \Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{\mathfrak{m}^F(t\ell_A, Q_\nu^\lambda(x,\rho))}{t} + \frac{c_7}{t}\mathfrak{m}^F(t\ell_A, Q_\nu^\lambda(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}\\
&\quad + \frac{c_7}{t}\lambda^{d-1}\rho^d + \Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{M_A}{(t\wedge 1)m}\lambda^{d-1}\rho^d + \Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_A}{m^{1-\alpha}}\lambda^{d-1}\rho^d.
\end{aligned}$$

Exchanging the roles of s and t, this gives the inequality in the statement of the lemma.

If, in addition, the function f defined by (2.3.6) does not depend on x, then in Corollary 2.3.10 we have $N_m = \emptyset$ and $\rho_{m,tA}^{\nu,\lambda}(x) = \rho_{m,sA}^{\nu,\lambda}(x) = +\infty$, concluding the proof in this case. □

The following lemma deals with the case with boundary conditions related to the jump functions $u_{x,\zeta,\nu}$.

Lemma 2.4.14. *There exists a positive constant $M > 0$, depending only on the structural constants $c_1,\dots,c_7$, k, and on α, such that for every $m \in \mathbb{N}$, $F \in \mathfrak{F}^{\alpha,\vartheta_m} \cap \mathfrak{F}_{\rm sc}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$, $s,t > 0$, and $\rho > 0$ we have*

$$\begin{aligned}
&\Big|\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} - \frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t}\Big|\\
&\quad\le \frac{c_7}{s}\rho^{d\alpha}\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \vartheta_{2m+1}(s|\zeta|)\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} + \frac{c_7}{s}\rho^d\\
&\quad+\frac{c_7}{t}\rho^{d\alpha}\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \vartheta_{2m+1}(t|\zeta|)\frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t} + \frac{c_7}{t}\rho^d\\
&\quad+(1+\vartheta_{2m+1}(s|\zeta|))\Big(\frac{M|\zeta|}{m}\rho^{d-1}+\frac{M}{sm}\rho^d\Big) + \frac{1}{s^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha} + \frac{1}{s}\frac{M}{m^{1-\alpha}}\rho^d\\
&\quad+(1+\vartheta_{2m+1}(t|\zeta|))\Big(\frac{M|\zeta|}{m}\rho^{d-1}+\frac{M}{tm}\rho^d\Big) + \frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d.
\end{aligned}\tag{2.4.27}$$

Proof. By Lemma 2.3.12 for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, $t > 0$, $\nu \in \mathbb{S}^{d-1}$, and $\rho > 0$ there exists a function $u \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)$ such that $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$, $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \le \sigma^m|\zeta|$, and

$$F(tu, Q_\nu(x,\rho)) \le \mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \frac{tK|\zeta|}{m}\rho^{d-1} + K\rho^d, \tag{2.4.28}$$

where $K > 0$ is the constant in Lemma 2.3.12. Note that the oscillation of u satisfies $u^{Q_\nu(x,\rho)} \le 2\sigma^m|\zeta|$. We now estimate $\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))$ by evaluating $F(\cdot, Q_\nu(x,\rho))$ at $tu_{x,\zeta,\nu}$ and using (H4) and we get

$$\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le c_3kt|\zeta|\rho^{d-1} + c_4\rho^d,$$

so that from (2.4.28) we deduce

$$F(tu, Q_\nu(x,\rho)) \le t|\zeta|\Big(c_3k + \frac{K}{m}\Big)\rho^{d-1} + (c_4+K)\rho^d. \tag{2.4.29}$$

Since $F \in \mathfrak{F}^{\alpha,\vartheta_m}$, by Definition 2.4.11 for every $s > 0$ we have that

$$\begin{aligned}
&(1-\vartheta_m(su^{Q_\nu(x,\rho)}))\frac{F(su,Q_\nu(x,\rho))}{s} - \frac{c_7}{s}\rho^{\alpha d}F(su,Q_\nu(x,\rho))^{1-\alpha} - \frac{c_7}{s}\rho^d - R_{m,s}(u,Q_\nu(x,\rho))\\
&\le (1+\vartheta_m(tu^{Q_\nu(x,\rho)}))\frac{F(tu,Q_\nu(x,\rho))}{t} + \frac{c_7}{t}\rho^{\alpha d}F(tu,Q_\nu(x,\rho))^{1-\alpha}\\
&\quad + \frac{c_7}{t}\rho^d + R_{m,t}(u,Q_\nu(x,\rho)).
\end{aligned} \tag{2.4.30}$$

Using the inequality $u^{Q_\nu(x,\rho)} \le 2\sigma^m|\zeta|$, the monotonicity of ϑ_m, and the subadditivity of $z \mapsto z^{1-\alpha}$, we deduce that the left-hand side of the last inequality can be estimated from below by

$$\begin{aligned}
&(1-\vartheta_{2m+1}(s|\zeta|))\frac{F(su,Q_\nu(x,\rho))}{s} - \frac{c_7}{s}\rho^{\alpha d}F(su,Q_\nu(x,\rho))^{1-\alpha} - \frac{c_7}{s}\rho^d\\
&-C(1+\vartheta_{2m+1}(s|\zeta|))\frac{F(su,Q_\nu(x,\rho))+\rho^d}{sm} - C^{1-\alpha}\frac{c_7}{s}\frac{F(su,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\rho^d}{m^{1-\alpha}}.
\end{aligned}$$

Moreover, we see that there exists a positive constant $M > 0$, depending only on the structural constants $c_1,\dots,c_7$, k, and on α, such that the right-hand side of (2.4.30) can be bounded from above by

$$\begin{aligned}
&(1+\vartheta_{2m+1}(t|\zeta|))\frac{\mathfrak{m}^F(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t} + \frac{c_7}{t}\mathfrak{m}^F(tu_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + \frac{c_7}{t}\rho^d\\
&\quad + (1+\vartheta_{2m+1}(t|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1} + \frac{(1+\vartheta_{2m+1}(t|\zeta|))}{t}\frac{M}{m}\rho^d + \frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha} + \frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d,
\end{aligned}$$

so that, combining the previous two inequalities we obtain

$$\begin{aligned}
&(1-\vartheta_{2m+1}(s|\zeta|))\frac{F(su,Q_\nu(x,\rho))}{s} - \frac{c_7}{s}\rho^{\alpha d}F(su,Q_\nu(x,\rho))^{1-\alpha} - \frac{c_7}{s}\rho^d\\
&-C(1+\vartheta_{2m+1}(s|\zeta|))\frac{F(su,Q_\nu(x,\rho))+\rho^d}{sm} - C^{1-\alpha}\frac{c_7}{s}\frac{F(su,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\rho^d}{m^{1-\alpha}}\\
&\le (1+\vartheta_{2m+1}(t|\zeta|))\frac{\mathfrak{m}^F(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t} + \frac{c_7}{t}\mathfrak{m}^F(tu_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + \frac{c_7}{t}\rho^d\\
&+(1+\vartheta_{2m+1}(t|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1} + \frac{(1+\vartheta_{2m+1}(t|\zeta|))}{t}\frac{M}{m}\rho^d + \frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha} + \frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d,
\end{aligned}$$

Arguing as in the proof of Lemma 2.4.13 we may write $\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))$ in place of $F(su, Q_\nu(x,\rho))$, so that, taking into account the estimate

$$\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le c_3 k^{1/2} s|\zeta|\rho^{d-1} + c_4\rho^d,$$

we obtain

$$\begin{aligned}
&(1+\vartheta_{2m+1}(s|\zeta|))\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} + \frac{c_7}{s}\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + \frac{c_7}{s}\rho^d\\
&\quad +(1+\vartheta_{2m+1}(s|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1} + \frac{(1+\vartheta_{2m+1}(s|\zeta|))}{s}\frac{M}{m}\rho^d + \frac{1}{s^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha} + \frac{1}{s}\frac{M}{m^{1-\alpha}}\rho^d\\
&\le (1+\vartheta_{2m+1}(t|\zeta|))\frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t} + \frac{c_7}{t}\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + \frac{c_7}{t}\rho^d\\
&\quad +(1+\vartheta_{2m+1}(t|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1} + \frac{(1+\vartheta_{2m+1}(t|\zeta|))}{t}\frac{M}{m}\rho^d + \frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha} + \frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d.
\end{aligned}$$

Exchanging the roles of s, t, we obtain (2.4.27). □

We now investigate the properties of the integrands f and g associated with functionals in $\mathfrak{F}_w^{\alpha,\vartheta}$.

Proposition 2.4.15. *Let $F \in \mathfrak{F}_w^{\alpha,\vartheta}$. Then the function f defined by* (2.3.6) *belongs to $\mathscr{F}^\alpha$.*

Proof. For every $m \in \mathbb{N}$ let $N_m \in \mathcal{B}(\mathbb{R}^d)$ be the $\mathcal{L}^d$-negligible set of Lemma 2.4.13 and let $N := \bigcup_{m\in\mathbb{N}} N_m$. By the same lemma for every $x \in \mathbb{R}^d \setminus N$, $A \in \mathbb{R}^{k\times d}$, $m \in \mathbb{N}$, $s, t > 0$, and every ρ small enough we have

$$\begin{aligned}
&\Big|\frac{\mathfrak{m}^F(s\ell_A, Q(x,\rho))}{s} - \frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{t}\Big|\\
&\le \frac{c_7}{s}\rho^{d\alpha}\mathfrak{m}^F(s\ell_A, Q(x,\rho))^{1-\alpha} + \vartheta_m\Big(\frac{s\kappa_{A,m}\rho}{(s\wedge 1)(t\wedge 1)}\Big)\frac{\mathfrak{m}^F(s\ell_A, Q(x,\rho))}{s} + \frac{c_7}{s}\rho^d\\
&\quad + \frac{c_7}{t}\rho^{d\alpha}\mathfrak{m}^F(t\ell_A, Q(x,\rho))^{1-\alpha} + \vartheta_m\Big(\frac{t\kappa_{A,m}\rho}{(s\wedge 1)(t\wedge 1)}\Big)\frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{t} + \frac{c_7}{t}\rho^d\\
&\quad + \Big(1+\vartheta_m\Big(\frac{s\kappa_{A,m}\rho}{(s\wedge 1)(t\wedge 1)}\Big)\Big)\frac{M_A}{(s\wedge 1)m}\rho^d + \Big(\frac{1}{s^\alpha}+\frac{1}{s}\Big)\frac{M_A}{m^{1-\alpha}}\rho^d\\
&\quad + \Big(1+\vartheta_m\Big(\frac{t\kappa_{A,m}\rho}{(s\wedge 1)(t\wedge 1)}\Big)\Big)\frac{M_A}{(t\wedge 1)m}\rho^d + \Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_A}{m^{1-\alpha}}\rho^d,
\end{aligned} \tag{2.4.31}$$

where $\kappa_{A,m}$ is the constant defined by (2.4.17). Since ϑ_m is continuous and $\vartheta_m(0) = 0$, we have that

$$\limsup_{\rho\to 0^+}\Big(\vartheta_m\Big(\frac{s\kappa_{A,m}\rho}{(s\wedge 1)(t\wedge 1)}\Big)\frac{\mathfrak{m}^F(s\ell_A, Q(x,\rho))}{s\rho^d}\Big) = 0,$$

$$\limsup_{\rho\to 0^+}\Big(\vartheta_m\Big(\frac{t\kappa_{A,m}\rho}{(s\wedge 1)(t\wedge 1)}\Big)\frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{t\rho^d}\Big) = 0.$$

Therefore, dividing (2.4.31) by ρ^d and taking the limsup as $\rho \to 0^+$, we obtain

$$\begin{aligned}
\frac{f(x,sA)}{s} \le{}& \frac{f(x,tA)}{t} + \frac{c_7}{s}f(x,sA)^{1-\alpha} + \frac{c_7}{s} + \frac{c_7}{t}f(x,tA)^{1-\alpha} + \frac{c_7}{t}\\
&+ \frac{M_A}{(s\wedge 1)m} + \Big(\frac{1}{s^\alpha}+\frac{1}{s}\Big)\frac{M_A}{m^{1-\alpha}} + \frac{M_A}{(t\wedge 1)m} + \Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_A}{m^{1-\alpha}}.
\end{aligned}$$

Letting $m \to +\infty$ and exchanging the roles of s and t, we recover (2.4.2), which concludes the proof. □

Proposition 2.4.16. *Let $F \in \mathfrak{F}_w^{\alpha,\vartheta}$, $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$, and $s,t>0$. Then the function g defined by (2.3.7) satisfies*

$$\Big|\frac{g(x,s\zeta,\nu)}{s} - \frac{g(x,t\zeta,\nu)}{t}\Big| \le \vartheta_{2m+1}(s|\zeta|)\frac{g(x,s\zeta,\nu)}{s} + \vartheta_{2m+1}(t|\zeta|)\frac{g(x,t\zeta,\nu)}{t} + (2+\vartheta_{2m+1}(s|\zeta|)+\vartheta_{2m+1}(t|\zeta|))\frac{M|\zeta|}{m},$$

where $M>0$ is the constant of Lemma 2.4.14.

Proof. Thanks to Lemma 2.4.14, inequality (2.4.27) holds. Dividing this inequality by ρ^{d-1} and taking the limsup as $\rho \to 0^+$, we immediately get

$$\frac{g(x,s\zeta,\nu)}{s} \le \frac{g(x,t\zeta,\nu)}{t} + \vartheta_{2m+1}(s|\zeta|)\frac{g(x,s\zeta,\nu)}{s} + \vartheta_{2m+1}(t|\zeta|)\frac{g(x,t\zeta,\nu)}{t} + (1+\vartheta_{2m+1}(s|\zeta|))\frac{M|\zeta|}{m} + (1+\vartheta_{2m+1}(t|\zeta|))\frac{M|\zeta|}{m}.$$

Exchanging the roles of s and t we conclude the proof. □

Remark 2.4.17. In the scalar case treated in [101], due to the different nature of the required vertical truncations, it is possible to prove a stronger version of Propositions 2.4.12 and 2.4.16. First, the space $\mathfrak{F}^{\alpha,\vartheta}$ turns out to be closed under Γ-convergence with respect to the topology of $L^0(\mathbb{R}^d)$. Then, exploiting this stronger version of Proposition 2.4.12, it is shown that the function g associated to F by (2.3.7) belongs to $\mathscr{G}^\vartheta$.

However, due to the presence of error terms in the truncation procedure we employ, which depend on ϑ_m, in our setting one cannot recover that $\mathfrak{F}^{\alpha,\vartheta}$ is closed and subsequently conclude that $g \in \mathscr{G}^\vartheta$.

2.5 Full integral representation

We are finally ready to state and prove the full integral representation result for functionals in the class $\mathfrak{F}_w^{\alpha,\vartheta}$. More precisely, we will show that if $F \in \mathfrak{F}_w^{\alpha,\vartheta}$ and if the bulk integrand f defined by (2.3.6) does not depend on x, then the Cantor part F^c of the functional F can be represented by means of f^∞. Since this result will be employed to obtain an integral representation for functionals arising from homogenisation of functionals in $\mathfrak{F}^{\alpha,\vartheta}$, the hypothesis that f does not depend on x is not restrictive for our purposes.

Theorem 2.5.1. *Let $F \in \mathfrak{F}_w^{\alpha,\vartheta}$ and f and g be defined by (2.3.6) and by (2.3.7), respectively. Assume that there exists a function $\widehat{f}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ such that*

$$f(x,A) = \widehat{f}(A) \quad \text{for every } x \in \mathbb{R}^d \text{ and for every } A \in \mathbb{R}^{k\times d}. \tag{2.5.1}$$

Then $F = F^{f,g}$, where $F^{f,g}$ is the functional introduced in Definition 2.1.6. In particular, for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ we have

$$F(u,B) = \int_B \widehat{f}(\nabla u)\,\mathrm{d}x + \int_B \widehat{f}^\infty\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for every $B \in \mathcal{B}(O)$ and $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$.

To prove this result, we characterise the Radon-Nikodým derivative of the measures $F^c(u,\cdot)$ with respect to $|D^cu|$ for any $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$ with $O \in \mathcal{O}_c(\mathbb{R}^d)$. This is done via a careful truncation procedure and taking advantage of the results of [41]. We recall that Alberti's Rank-One Theorem 1.3.1 ensures that for $|D^cu|$-a.e. $x \in O$ the matrix $\mathrm{d}D^cu/\mathrm{d}|D^cu|(x)$ has rank one, i.e., there exist two Borel functions $a_u\colon O \to \mathbb{S}^{k-1}$ and $b_u\colon O \to \mathbb{S}^{d-1}$ such that for $|D^cu|$-a.e. $x \in O$ we have

$$\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}(x) = a_u(x) \otimes b_u(x). \tag{2.5.2}$$

We also recall the definition of the rectangle $Q^\lambda_\nu(x,\rho)$ given in (1.1.2).

Lemma 2.5.2. *Let $F \in \mathfrak{F}_{\mathrm{sc}}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $u \in \mathrm{BV}(O;\mathbb{R}^k)$. Assume that there exists a function $\widehat{f}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ satisfying (2.5.1). For $|D^cu|$-a.e. $x \in O$, for every $\lambda \geq 1$, and $\rho > 0$ we set*

$$s^\lambda_\rho(x) := \frac{|D^cu|(Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d} \quad \text{and} \quad A^\lambda_\rho(x) := s^\lambda_\rho(x)(a_u(x)\otimes b_u(x)), \tag{2.5.3}$$

where $(a_u(x), b_u(x)) \in \mathbb{S}^{k-1}\times\mathbb{S}^{d-1}$ is given by (2.5.2). Then

$$\lim_{\rho\to 0^+} s^\lambda_\rho(x) = +\infty \quad \text{and} \quad \lim_{\rho\to 0^+} \rho s^\lambda_\rho(x) = 0 \quad \text{for every } \lambda \geq 1, \tag{2.5.4}$$

$$\frac{dF^c(u,\cdot)}{\mathrm{d}|D^cu|}(x) = \lim_{\lambda\to+\infty} \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}, \tag{2.5.5}$$

for $|D^cu|$-a.e. $x \in O$.

Proof. Let us fix $\delta > 0$ and consider the functional F_δ defined by (2.3.3). By Lemma 1.5.2 the equalities in (2.5.4) hold for $|D^cu|$-a.e. $x \in O$ and for any $\delta > 0$ we have

$$\frac{dF^c(u,\cdot)}{\mathrm{d}|D^cu|}(x) + \delta = \frac{dF^c_\delta(u,\cdot)}{\mathrm{d}|D^cu|}(x) = \lim_{\lambda\to+\infty} \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^{F_\delta}(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}.$$

Since $F \leq F_\delta$, we deduce that

$$\frac{dF^c(u,\cdot)}{\mathrm{d}|D^cu|}(x) + \delta \geq \limsup_{\lambda\to+\infty} \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}.$$

Thus, it is enough to show that there exists a constant $K > 0$, depending only on $c_1,\dots,c_4$ and on k, such that for every $\lambda \geq 1$, for every $\delta > 0$, and for $|D^cu|$-a.e. $x \in O$ we have

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^{F_\delta}(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} \leq (1+K\delta) \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}. \tag{2.5.6}$$

We fix $m \in \mathbb{N}$, $\lambda \geq 1$, and consider a point $x \in O$ such that (2.5.2) and (2.5.4) hold. By Theorem 2.3.16 hypothesis (2.3.10) of Corollary 2.3.10 is satisfied. Thus, there exists a function $u^x_\rho \in \mathrm{BV}(Q^\lambda_{b_u(x)}(x,\rho);\mathbb{R}^k)$, with $u^x_\rho = \ell_{A^\lambda_\rho(x)}$ on $\partial Q^\lambda_{b_u(x)}(x,\rho)$ and $\|u^x_\rho - \ell_{A^\lambda_\rho(x)}\|_{L^\infty(Q^\lambda_{b_u(x)}(x,\rho);\mathbb{R}^k)} \leq c_{A^\lambda_\rho(x),m}\lambda\rho$, such that

$$F(u^x_\rho, Q^\lambda_{b_u(x)}(x,\rho)) \leq \mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho)) + \frac{C_{A^\lambda_\rho(x)}}{m}\lambda^{d-1}\rho^d, \tag{2.5.7}$$

where $c_{A^\lambda_\rho(x),m}$ and $C_{A^\lambda_\rho(x)}$ are the positive constants defined by (2.3.9) and (2.3.21), respectively. Recalling that $\|u^x_\rho - \ell_{A^\lambda_\rho(x)}\| \le c_{A^\lambda_\rho(x),m}\lambda\rho$ and observing that $|[u^x_\rho]| = |[u^x_\rho - \ell_{A^\lambda_\rho(x)}]|$, we deduce that $|[u^x_\rho]| \le 2c_{A^\lambda_\rho(x),m}\lambda\rho$. Therefore, we may estimate

$$\begin{aligned}\int_{J_{u^x_\rho}} |[u^x_\rho]|\,\mathrm{d}\mathcal{H}^{d-1} &\le \int_{(J_{u^x_\rho}\setminus J^1_{u^x_\rho})} |[u^x_\rho]|\,\mathrm{d}\mathcal{H}^{d-1} + 2c_{A^\lambda_\rho(x),m}\lambda\rho\mathcal{H}^{d-1}(J^1_{u^x_\rho})\\ &\le (1+2c_{A^\lambda_\rho(x),m}\lambda\rho)\int_{J_{u^x_\rho}} |[u^x_\rho]|\wedge 1\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

From this inequality, by (H3′) and (H4′) of Definition 2.1.7 we see that

$$\begin{aligned}F_\delta(u^x_\rho, Q^\lambda_{b_u(x)}(x,\rho)) &= F(u^x_\rho, Q^\lambda_{b_u(x)}(x,\rho)) + \delta|Du^x_\rho|(Q^\lambda_{b_u(x)}(x,\rho))\\ &\le F(u^x_\rho, Q^\lambda_{b_u(x)}(x,\rho)) + \delta\int_{Q^\lambda_{b_u(x)}(x,\rho)} |\nabla u^x_\rho|\,\mathrm{d}x + \delta|D^c u^x_\rho|(Q^\lambda_{b_u(x)}(x,\rho))\\ &\quad + \delta(1+2c_{A^\lambda_\rho(x),m}\lambda\rho)\int_{J_{u^x_\rho}} |[u^x_\rho]|\wedge 1\,\mathrm{d}\mathcal{H}^{d-1}\\ &\le \Big(1+\delta\frac{1+2c_{A^\lambda_\rho(x),m}\lambda\rho}{c_1}\Big)F(u^x_\rho, Q^\lambda_{b_u(x)}(x,\rho)) + \delta\frac{c_2}{c_1}(1+2c_{A^\lambda_\rho(x),m}\lambda\rho)\lambda^{d-1}\rho^d.\end{aligned}$$

Since $u^x_\rho = \ell_{A^\lambda_\rho(x)}$ on $\partial Q^\lambda_{b_u(x)}(x,\rho)$, combining the previous inequality with (2.5.7), we get

$$\begin{aligned}\frac{\mathfrak{m}^{F_\delta}(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} &\le \Big(1+\delta\frac{1+2c_{A^\lambda_\rho(x),m}\lambda\rho}{c_1}\Big)\Big(\frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} + \frac{C_{A^\lambda_\rho(x)}}{s^\lambda_\rho(x)m}\Big)\\ &\quad + \delta\frac{c_2}{c_1}(1+2c_{A^\lambda_\rho(x),m}\lambda\rho)\frac{1}{s^\lambda_\rho(x)}.\end{aligned} \tag{2.5.8}$$

It is immediate to check that when $s^\lambda_\rho(x) \ge 1$ we have

$$\frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} \le c_3 k^{1/2} + c_4.$$

Hence, from this inequality, (2.3.9), (2.3.21), and (2.5.4), we see that there exist two constants C_1 and C_2 depending only on the structural constants $c_1,\dots,c_4$, k, and d such that

$$\begin{aligned}&\limsup_{\rho\to 0^+} \frac{\delta(1+2c_{A^\lambda_\rho(x),m}\lambda\rho)}{c_1}\Big(\frac{\mathfrak{m}^F(\ell_{A^\lambda_\rho(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} + \frac{C_{A^\lambda_\rho(x)}}{s^\lambda_\rho(x)m}\Big)\\ &\quad\le \limsup_{\rho\to 0^+} \frac{\delta}{c_1}\Big(1+2(\sigma^m+1)\lambda\rho\big(s^\lambda_\rho(x)d^{1/2}+C_1\big)\Big)\Big((c_3k^{1/2}+c_4) + \frac{C_{A^\lambda_\rho(x)}}{s^\lambda_\rho(x)m}\Big)\\ &\quad\le \frac{\delta}{c_1}\Big(c_3k^{1/2}+c_4+\frac{C_2}{m}\Big).\end{aligned} \tag{2.5.9}$$

As for the remaining terms of (2.5.8), by (2.3.21) we see that

$$\limsup_{\rho\to 0^+} \frac{C_{A^\lambda_\rho(x)}}{s^\lambda_\rho(x)m} = \limsup_{\rho\to 0^+} \frac{2c_3k^{1/2}Cs^\lambda_\rho(x) + 2C(c_4+1)}{s^\lambda_\rho(x)m} = \frac{2c_3k^{1/2}C}{m}, \tag{2.5.10}$$

while by (2.3.9) and (2.5.4) we get that

$$\limsup_{\rho\to 0^+} \frac{c_2}{c_1}\delta\big(1+2c_{A^\lambda_\rho(x),m}\lambda\rho\big)\frac{1}{s^\lambda_\rho(x)} = 0. \tag{2.5.11}$$

Finally, using (2.5.9)–(2.5.11) we can take the limsup in (2.5.8) as $\rho \to 0^+$, obtaining an estimate that depends on m and δ. Then, taking the limit as $m \to +\infty$ we get (2.5.6), concluding the proof. □

The next result ensures that estimates on the volume integrand f translate into estimates on the right-hand side of (2.5.5). For the proof of this result we refer the reader to [101, Lemma 5.3].

Lemma 2.5.3. *Let $F \in \mathfrak{F}_{\rm sc}$, $A \in \mathbb{R}^{k\times d}$, $\lambda \geq 1$, $\nu \in \mathbb{S}^{d-1}$, $t > 0$, and $\mu \in [0,+\infty)$. Assume that for every $x \in \mathbb{R}^d$ and $\rho > 0$ we have*

$$\mathfrak{m}^F_{t\rho}(\ell_A, Q(x,\rho)) \leq \mu\rho^d.$$

Then

$$\mathfrak{m}^F_{t\lambda\rho}(\ell_A, Q^\lambda_\nu(x,\rho)) \leq \mu\lambda^{d-1}\rho^d$$

for every $x \in \mathbb{R}^d$ and $\rho > 0$.

If, in addition, there exists some $x_0 \in \mathbb{R}^d$ such that

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F_{t\rho}(\ell_A, Q(x_0,\rho))}{\rho^d} = \mu,$$

then

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F_{t\lambda\rho}(\ell_A, Q^\lambda_\nu(x_0,\rho))}{\lambda^{d-1}\rho^d} = \mu.$$

Moreover, the same conclusions hold replacing the constrained minimisation problems $\mathfrak{m}^F_{t\rho}$ and $\mathfrak{m}^F_{t\lambda\rho}$ by $\mathfrak{m}^F$.

With these two lemmas at hand, we are now ready to prove Theorem 2.5.1.

Proof of Theorem 2.5.1. Thanks to Theorem 2.3.16 and to (2.5.1), for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ we have

$$F^a(u,B) = \int_B \widehat{f}(\nabla u)\,\mathrm{d}x \quad \text{and} \quad F^j(u,B) = \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.5.12}$$

for every $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$ and $B \in \mathcal{B}(O)$.

We are left with proving that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ we can represent the Cantor part F^c as

$$F^c(u,B) = \int_B \widehat{f}^\infty\Big(\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\,\mathrm{d}|D^cu| \tag{2.5.13}$$

for every $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$ and $B \in \mathcal{B}(O)$.

To this end, we begin by observing that, as ℓ_A is an admissible function for the minimisation problem $\mathfrak{m}^F(\ell_A, Q(x,\rho))$, by (2.5.12) we get that

$$\mathfrak{m}^F(\ell_A, Q(x,\rho)) \leq \widehat{f}(A)\rho^d$$

for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{k\times d}$, and $\rho > 0$. Recalling (2.3.6) and (2.5.1), applying Lemma 2.5.3 we obtain

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d} = \widehat{f}(A)$$

for every $\lambda \geq 1$. In particular, for every $t > 0$ we have

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t} = \frac{\widehat{f}(tA)}{t} \tag{2.5.14}$$

so that, taking the lim sup for $t \to +\infty$, we get

$$\limsup_{t\to+\infty}\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(t\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t} = \widehat{f}^\infty(A). \tag{2.5.15}$$

We claim that for every $A \in \mathbb{R}^{k\times d}$, $\lambda \geq 1$, and $x \in \mathbb{R}^d$ we have

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} = \widehat{f}^\infty(A) \tag{2.5.16}$$

whenever $s_\rho \to +\infty$ and $\rho s_\rho \to 0^+$ as $\rho \to 0^+$. To see this, first note that by (2.3.6) and (2.5.1) for every $\rho > 0$ we have $\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho)) \leq \widehat{f}(s_\rho A)\lambda^{d-1}\rho^d$, whence

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} \leq \widehat{f}^\infty(A).$$

Thus, to prove (2.5.16) it is enough to show that

$$\widehat{f}^\infty(A) \leq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}. \tag{2.5.17}$$

Let $\eta > 0$. By definition of $\widehat{f}^\infty$, there exists $t_\eta > 1/\eta$ such that

$$\widehat{f}^\infty(A) - \eta < \frac{\widehat{f}(t_\eta A)}{t_\eta}. \tag{2.5.18}$$

Hence, recalling (2.5.14), we have

$$\widehat{f}^\infty(A) - \eta \leq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}. \tag{2.5.19}$$

Let us fix $m \in \mathbb{N}$. By Lemma 2.4.13 we have

$$\begin{aligned}
\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta} &\leq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} \\
&+ \limsup_{\rho\to 0^+} \Big(\vartheta_m(s_\rho\kappa_{A,m}\lambda\rho)\frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} + \frac{c_7}{s_\rho^\alpha}\Big(\frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha} + \frac{c_7}{s_\rho} \\
&+ (1+\vartheta_m(s_\rho\kappa_{A,m}\lambda\rho))\frac{M_A}{m} + \Big(\frac{1}{s_\rho^\alpha} + \frac{1}{s_\rho}\Big)\frac{M_A}{m^{1-\alpha}}\Big) \\
&+ \limsup_{\rho\to 0^+} \Big(\vartheta_m(t_\eta\kappa_{A,m}\lambda\rho)\frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta} + \frac{c_7}{t_\eta^\alpha}\Big(\frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}\Big)^{1-\alpha} + \frac{c_7}{t_\eta} \\
&+ (1+\vartheta_m(t_\eta\kappa_{A,m}\lambda\rho))\frac{M_A}{m} + \Big(\frac{1}{t_\eta^\alpha} + \frac{1}{t_\eta}\Big)\frac{M_A}{m^{1-\alpha}}\Big),
\end{aligned} \tag{2.5.20}$$

where $M_A > 0$ is the constant of Lemma 2.4.13, which we recall is independent of ρ, s_ρ, t, and m.

We separately study the summands of this last expression. Since ϑ_m is continuous and $\vartheta_m(0) = 0$, while $s_\rho\rho \to 0$ and $s_\rho \to +\infty$ as $\rho \to 0^+$, recalling that $\frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}$ is bounded uniformly with respect to ρ by (H4), we obtain

$$\limsup_{\rho\to 0^+} \Big(\vartheta_m(s_\rho\kappa_{A,m}\lambda\rho)\frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} + \frac{c_7}{s_\rho^\alpha}\Big(\frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha} + \frac{c_7}{s_\rho} + (1+\vartheta_m(s_\rho\kappa_{A,m}\lambda\rho))\frac{M_A}{m} + \Big(\frac{1}{s_\rho^\alpha} + \frac{1}{s_\rho}\Big)\frac{M_A}{m^{1-\alpha}}\Big) = \frac{M_A}{m}. \quad (2.5.21)$$

By (H4) there exists $\Lambda_A > 0$, independent of ρ and t_η, such that

$$\frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta} \le \Lambda_A.$$

Therefore, arguing as in the proof of (2.5.21), by (2.5.18) and (2.5.19) we obtain

$$\begin{aligned}
&\limsup_{\rho\to 0^+} \Big(\vartheta_m(t_\eta\kappa_{A,m}\lambda\rho)\frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta} + \frac{c_7}{t_\eta^\alpha}\Big(\frac{\mathfrak{m}^F(t_\eta\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}\Big)^{1-\alpha} + \frac{c_7}{t_\eta} \\
&\quad + (1+\vartheta_m(t_\eta\kappa_{A,m}\lambda\rho))\frac{M_A}{m} + \Big(\frac{1}{t_\eta^\alpha} + \frac{1}{t_\eta}\Big)\frac{M_A}{m^{1-\alpha}}\Big) \\
&\le \frac{c_7\Lambda_A^{1-\alpha}}{t_\eta^\alpha} + \frac{c_7}{t_\eta} + \frac{M_A}{m} + \Big(\frac{1}{t_\eta^\alpha} + \frac{1}{t_\eta}\Big)\frac{M_A}{m^{1-\alpha}} \\
&< c_7\Lambda_A^{1-\alpha}\eta^\alpha + c_7\eta + \frac{M_A}{m} + (\eta+\eta^\alpha)\frac{M_A}{m^{1-\alpha}}, \qquad (2.5.22)
\end{aligned}$$

where in the last inequality we have used that $t_\eta > 1/\eta$. Finally, combining (2.5.19) and (2.5.22), we get

$$\widehat{f}^\infty(A) - \eta - \frac{2M_A}{m} - c_7\Lambda_A^{1-\alpha}\eta^\alpha - c_7\eta - (\eta+\eta^\alpha)\frac{M_A}{m^{1-\alpha}} \le \limsup_{\rho\to 0^+}\frac{\mathfrak{m}^F(s_\rho\ell_A, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}.$$

Letting $m \to +\infty$ and $\eta \to 0^+$, we obtain (2.5.17) and, as a consequence, (2.5.16).

Consider now a function $u \in \mathrm{BV}(O;\mathbb{R}^k)$. Let $a_u(x)$, $b_u(x)$, and $s^\lambda_\rho(x)$ be as in (2.5.2) and (2.5.3), and set $A(x) \coloneqq a_u(x)\otimes b_u(x)$. By Lemma 2.5.2 for $|D^c u|$-a.e. $x \in O$ we have

$$\frac{dF^c(u,\cdot)}{\mathrm{d}|D^c u|}(x) = \lim_{\lambda\to+\infty}\limsup_{\rho\to 0^+}\frac{\mathfrak{m}^F(s^\lambda_\rho(x)\ell_{A(x)}, Q^\lambda_{b_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}. \quad (2.5.23)$$

Recalling (2.5.4), in light of (2.5.16) and of (2.5.23), we then infer that

$$\frac{dF^c(u,\cdot)}{\mathrm{d}|D^c u|}(x) = \widehat{f}^\infty(a_u(x)\otimes b_u(x)),$$

which by (2.5.2) gives (2.5.13) for every $u \in \mathrm{BV}(O;\mathbb{R}^k)$ and $B \in \mathcal{B}(O)$.

Let us assume now that $u \in \mathrm{GBV}_\star(O;\mathbb{R}^k)$ and for every $R > 0$ we consider the set $\widetilde{O}^R_{u,0} \coloneqq \{x \in O : \widetilde{u}(x) \text{ exists and } |\widetilde{u}(x)| \le R\}$. We claim that it is enough to prove (2.5.13) for every $B \in \mathcal{B}(O)$ for which there exists $R > 0$ such that $B \subset \widetilde{O}^R_{u,0}$. Indeed, by

Proposition 1.3.20 we have $\widetilde{O}^R_{u,0} \nearrow O_{\rm reg} := \{x \in O\colon \widetilde{u}(x) \text{ exists}\}$ and that $|D^c u|(O \setminus O_{\rm reg}) = 0$. Hence, every $B \in \mathcal{B}(O)$ can be written, up to a $|D^c u|$-negligible set, as the increasing union of Borel sets each contained in $\widetilde{O}^R_{u,0}$, for some R. This proves the claim.

Let us fix $R > 0$ and a Borel set $B \subset \widetilde{O}^R_{u,0}$. Let $R_m > R$ be a sequence with $R_m \to +\infty$. Thanks to Lemma 2.3.2, we have

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{i=1}^{m} F^c(\Phi^i_{R_m} \circ u, B) = F^c(u, B). \tag{2.5.24}$$

Since each function $\Phi^i_{R_m} \circ u$ belongs to $\mathrm{BV}(O;\mathbb{R}^k)$ and the integral representation holds in $\mathrm{BV}(O;\mathbb{R}^k)$, we have

$$F^c(\Phi^i_{R_m} \circ u, B) = \int_B \widehat{f}^{\infty}\Big(\frac{\mathrm{d}D^c(\Phi^i_{R_m} \circ u)}{\mathrm{d}|D^c(\Phi^i_{R_m} \circ u)|}\Big)\, \mathrm{d}|D^c(\Phi^i_{R_m} \circ u)| \tag{2.5.25}$$

for every $i \in \{1,\dots,m\}$. Recalling that $B \subset \widetilde{O}^R_{u,0} \subset \widetilde{O}^{R_m}_{u,0}$, from Proposition 1.3.21 and from (1.3.13) we obtain

$$\begin{aligned} \frac{\mathrm{d}D^c(\Phi^i_{R_m} \circ u)}{\mathrm{d}|D^c(\Phi^i_{R_m} \circ u)|} &= \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|} \quad |D^c u|\text{-a.e. in } \widetilde{O}^R_{u,0},\\ |D^c(\Phi^i_{R_m} \circ u)| &= |D^c u| \quad \text{as measures in } \widetilde{O}^R_{u,0}, \end{aligned}$$

for every $i \in \{1,\dots,m\}$. These equalities, together with (2.5.24) and (2.5.25), give (2.5.13), concluding the proof. □

2.6 Integrands of the Γ-limits

In this section we consider a sequence $\{F_n\}_n$ of functionals in $\mathfrak{F}$. In the first part we assume that $\{F_n\}_n$ Γ-converges to some functional $F \in \mathfrak{F}_{\rm sc}$ and we characterise the bulk and surface integrands f and g of F at a point $x \in \mathbb{R}^d$ by taking first the limit as $n \to +\infty$ of the *infima* of suitable minimisation problems for F_n on small cubes and taking then the limit as these cubes shrink to x.

In the last part we assume that each functional F_n belongs to $\mathfrak{F}^{\alpha,\vartheta}$ and prove the converse of the previous results: if the limits mentioned above exist and the *bulk* term is independent of x, then they define two integrands f and g such that the sequence $\{F_n\}_n$ Γ-converges to the functional corresponding to f and g.

We begin this analysis by showing that it is possible to obtain the function f introduced in (2.3.6) by means of limits of constrained minimisation problems.

Lemma 2.6.1. *Let $F \in \mathfrak{F}_{\rm sc}$ and let f be the function defined by* (2.3.6)*. Then there exists a $\mathcal{L}^d$-negligible set $N \in \mathcal{B}(\mathbb{R}^d)$, such that for every $x \in \mathbb{R}^d \setminus N$ and for every $A \in \mathbb{R}^{k\times d}$ we have*

$$f(x,A) = \lim_{m\to+\infty} \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F_{\rho c_{A,m}}(\ell_A, Q(x,\rho))}{\rho^d}, \tag{2.6.1}$$

where $\mathfrak{m}^F_{\rho c_{A,m}}(\ell_A, Q(x,\rho))$ is given by $\mathfrak{m}^F_t(\ell_A, Q(x,\rho))$ (see (2.3.5)*), with $t = \rho c_{A,m}$ (see* (2.3.9) *for the definition of $c_{A,m}$).*

If, in addition, there exists a function $\widehat{f}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ such that $f(x,A) = \widehat{f}(A)$ for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and every $A \in \mathbb{R}^{k\times d}$, then (2.6.1) *holds for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{k\times d}$.*

Proof. It follows immediately from (2.3.4) and (2.3.5) that for every $m \in \mathbb{N}$ and $A \in \mathbb{R}^{k\times d}$ we have $\mathfrak{m}^F \le \mathfrak{m}^F_{c_{A,m}\rho}$. Thus, we only have to show that

$$\lim_{m\to+\infty} \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F_{\rho c_{A,m}}(\ell_A, Q(x,\rho))}{\rho^d} \le \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} \tag{2.6.2}$$

for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and for every $A \in \mathbb{R}^{k\times d}$. Thanks to Theorem 2.3.16, (2.3.10) holds. Therefore, Corollary 2.3.10 implies that there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, satisfying the following property: for every $x \in \mathbb{R}^d \setminus N$, $A \in \mathbb{R}^{k\times d}$, $m \in \mathbb{N}$, and $\rho > 0$ small enough there exists a function $u \in \mathrm{BV}(Q(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q(x,\rho)}u = \mathrm{tr}_{Q(x,\rho)}\ell_A$ and $\|u - \ell_A\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le c_{A,m}\rho$, such that

$$\mathfrak{m}^F_{\rho c_{A,m}}(\ell_A, Q(x,\rho)) \le \mathfrak{m}^F(\ell_A, Q(x,\rho)) + \frac{C_A}{m}\rho^d,$$

where $C_A > 0$ is the constant defined by (2.3.21). Dividing this inequality by ρ^d and letting $\rho \to 0^+$, we conclude that

$$\limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F_{\rho c_{A,m}}(\ell_A, Q(x,\rho))}{\rho^d} \le \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} + \frac{C_A}{m}.$$

Taking the limit for $m \to +\infty$, we obtain (2.6.2).

To conclude the proof, we note that under the additional hypothesis we have $N = \emptyset$ in Corollary 2.3.10. □

The next result is useful to understand the relation between the minima of problems associated with a sequence $\{F_n\}_n$ on an open set O' and the minimum of the problem associated with their Γ-limit F, computed on a larger open set O''.

Lemma 2.6.2. *Let $\{F_n\}_n \subset \mathfrak{F}$, let $F \in \mathfrak{F}_{\mathrm{sc}}$, and let $O', O'' \in \mathcal{O}_c(\mathbb{R}^d)$ have Lipschitz boundaries and $O' \subset\subset O''$. Assume that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$, the sequence $\{F_n(\cdot, O)\}_n$ Γ-converges to $F(\cdot, O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then for every $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$, we have*

$$\mathfrak{m}^F(w, O'') \le \liminf_{n\to+\infty} \mathfrak{m}^{F_n}(w, O') + c_3 k^{1/2} \int_{O''\setminus O'} |\nabla w| \,\mathrm{d}x + c_4 \mathcal{L}^d(O'' \setminus O').$$

Proof. The proof can be obtained by adapting the arguments of [101, Proposition 3.1], replacing [100, Theorem 7.13] by Theorem 2.A.1 below. □

We now prove a result that allows us to compare the limit of the minima of problems associated with a sequence $\{F_n\}_n$ on a cube $Q(x,\rho)$ with the minimum of the problem associated with the Γ-limit F, computed on the same cube.

Lemma 2.6.3. *Let $\{F_n\}_n \subset \mathfrak{F}$, $F \in \mathfrak{F}$, $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{k\times d}$, $m \in \mathbb{N}$, $\rho > 0$, and $s > d^{1/2}|A|\rho$. Assume that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot, O)\}_n$ Γ-converges to $F(\cdot, O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then*

$$\limsup_{n\to+\infty} \mathfrak{m}^{F_n}_t(\ell_A, Q(x,\rho)) \le \mathfrak{m}^F_s(\ell_A, Q(x,\rho)) + \frac{K_A}{m}\rho^d, \tag{2.6.3}$$

where $t := 2\sigma^m(s + d^{1/2}|A|\rho) + d^{1/2}|A|\rho$ and $K_A > 0$ is a constant depending on A, but not on $\{f_n\}_n$, $\{g_n\}_n$, m, s, and ρ.

Proof. Let us fix $0 < \eta < 1$. Consider a function $u \in \mathrm{BV}(Q(x,\rho);\mathbb{R}^k)$, with $\|u - \ell_A\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le s$ and $u = \ell_A$ on $\partial Q(x,\rho)$, such that

$$F(u, Q(x,\rho)) \le \mathfrak{m}_s^F(\ell_A, Q(x,\rho)) + \eta \le (c_3 k^{1/2}|A| + c_4)\rho^d + \eta. \tag{2.6.4}$$

By Lemma 2.4.10 there exists a sequence of functions $(v_n)_n \subset \mathrm{BV}(Q(x,\rho);\mathbb{R}^k)$, with $\|v_n\|_{L^\infty(Q(x,\rho))} \le 2\sigma^m(s + d^{1/2}|A|\rho)$, such that $v_n \to u$ in $L^1(\mathbb{R}^d;\mathbb{R}^k)$

$$\limsup_{n\to+\infty} F_n(v_n, Q(x,\rho)) \le F(u, Q(x,\rho)) + C\frac{F(u,Q(x,\rho)) + \rho^d}{m}. \tag{2.6.5}$$

We fix $0 < r < \rho$. For every $0 < \delta \le \eta$ we apply Lemma 2.2.2 to the open sets $O = Q(x,r)$ and $U = Q(x,\rho) \setminus \overline{Q(x,r)}$ to obtain a sequence $\{u_n\}_n \subset \mathrm{BV}(Q(x,\rho);\mathbb{R}^k)$ converging to u in $L^1(\mathbb{R}^d;\mathbb{R}^k)$, with $u_n = \ell_A$ on $\partial Q(x,\rho)$ and $\|u_n - \ell_A\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le 2\sigma^m(s + d^{1/2}|A|\rho) + d^{1/2}|A|\rho = t$, such that

$$\limsup_{n\to+\infty} F_n(u_n, Q(x,\rho)) \le (1+\delta)\limsup_{n\to+\infty}\Big(F_n(v_n, Q(x,\rho)) + F_n(\ell_A, Q(x,\rho)\setminus\overline{Q(x,r)})\Big) + \eta.$$

Exploiting (2.6.4) and (2.6.5), from this last inequality we deduce that

$$\begin{aligned}\limsup_{n\to+\infty} F_n(u_n, Q(x,\rho)) \le{}& \mathfrak{m}_s^F(\ell_A, Q(x,\rho)) + \eta + \delta\big((c_3k^{1/2}|A| + c_4)\rho^d + \eta\big)\\ &+ (1+\delta)C\frac{(c_3k^{1/2}|A| + c_4 + 1)\rho^d + \eta}{m}\\ &+ (1+\delta)(c_3k^{1/2}|A| + c_4)(\rho^d - r^d) + \eta.\end{aligned} \tag{2.6.6}$$

Choosing r so that $2(c_3k^{1/2}|A|+c_4)(\rho^d - r^d) \le \eta$ and δ such that $\delta((c_3k^{1/2}|A|+c_4)\rho^d+\eta) \le \eta$, recalling that $u_n = \ell_A$ on $\partial Q(x,\rho)$ and $\|u_n - \ell_A\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le t$, from (2.6.6) we obtain

$$\limsup_{n\to+\infty} \mathfrak{m}_t^{F_n}(\ell_A, Q(x,\rho)) \le \mathfrak{m}_s^F(\ell_A, Q(x,\rho)) + 2C\frac{(c_3k^{1/2}|A| + c_4 + 1)\rho^d + \eta}{m} + 4\eta.$$

We conclude the proof by letting $\eta \to 0^+$. □

The next result shows that when F is the Γ-limit of a sequence of functionals $\{F_n\}_n$, the value of its bulk integrand f at (x,A) can be obtained by taking first the limit of $\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))/\rho^d$ as $n \to +\infty$ and then the limit as $\rho \to 0^+$. For technical reasons, we need also a similar result where we replace $\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))$ by its constrained version $\mathfrak{m}_t^{F_n}(\ell_A, Q(x,\rho))$, for a suitable choice of $t > 0$.

Given $m \in \mathbb{N}$ and $A \in \mathbb{R}^{k\times d}$, the constraint t will be given by $\rho\beta_{A,m}$, with

$$\beta_{A,m} \coloneqq 2\sigma^m(c_{A,m} + d^{1/2}|A|) + d^{1/2}|A|, \tag{2.6.7}$$

where $c_{A,m} > 0$ is the constant defined by (2.3.9).

Proposition 2.6.4. *Let $\{F_n\}_n \subset \mathfrak{F}$, $F \in \mathfrak{F}_{\mathrm{sc}}$, and let f be the function defined by* (2.3.6). *Assume that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then there exists an $\mathcal{L}^d$-negligible set $N \in \mathcal{B}(\mathbb{R}^d)$ such that for every $x \in \mathbb{R}^d \setminus N$ and $A \in \mathbb{R}^{k\times d}$ we have*

$$f(x,A) = \limsup_{\rho\to0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} = \limsup_{\rho\to0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d}. \tag{2.6.8}$$

Moreover, for the same x and A, $f(x,A)$ is given also by the common value

$$\lim_{m\to+\infty}\limsup_{\rho\to0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}_{\rho\beta_m}(\ell_A,Q(x,\rho))}{\rho^d}=\lim_{m\to+\infty}\limsup_{\rho\to0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}_{\rho\beta_m}(\ell_A,Q(x,\rho))}{\rho^d},\tag{2.6.9}$$

where β_m stands for the constant $\beta_{A,m}$ defined by (2.6.7).

If, in addition, there exists a function $\widehat{f}\colon\mathbb{R}^{k\times d}\to[0,+\infty)$ such that $f(x,A)=\widehat{f}(A)$ for $\mathcal{L}^d$-a.e. $x\in\mathbb{R}^d$ and for every $A\in\mathbb{R}^{k\times d}$, then (2.6.8) *and* (2.6.9) *hold for every $x\in\mathbb{R}^d$ and for every $A\in\mathbb{R}^{k\times d}$.*

Proof. Let $N\in\mathcal{B}(\mathbb{R}^d)$ be the union of the $\mathcal{L}^d$-negligible sets of Corollary 2.3.10 and Lemma 2.6.1. We fix $m\in\mathbb{N}$, let $\rho>0$, and set $r:=\rho+\rho^2$. Using first Lemma 2.6.2, then Lemma 2.6.3, and finally Corollary 2.3.10, for every $x\in\mathbb{R}^d\setminus N$ and $\rho>0$ small enough we have

$$\begin{aligned}
\mathfrak{m}^F(\ell_A,Q(x,r))&\le\liminf_{n\to+\infty}\mathfrak{m}^{F_n}(\ell_A,Q(x,\rho))+(c_3k^{1/2}|A|+c_4)(r^d-\rho^d)\\
&\le\liminf_{n\to+\infty}\mathfrak{m}^{F_n}_{\rho\beta_m}(\ell_A,Q(x,\rho))+(c_3k^{1/2}|A|+c_4)(r^d-\rho^d)\\
&\le\limsup_{n\to+\infty}\mathfrak{m}^{F_n}_{\rho\beta_m}(\ell_A,Q(x,\rho))+(c_3k^{1/2}|A|+c_4)(r^d-\rho^d)\\
&\le\mathfrak{m}^F_{\rho c_{A,m}}(\ell_A,Q(x,\rho))+(c_3k^{1/2}|A|+c_4)(r^d-\rho^d)+\frac{K_A}{m}\rho^d\\
&\le\mathfrak{m}^F(\ell_A,Q(x,\rho))+(c_3k^{1/2}|A|+c_4)(r^d-\rho^d)+\frac{K_A+C_A}{m}\rho^d,
\end{aligned}$$

where K_A is the constant of Lemma 2.6.3 and C_A is given by (2.3.21). We divide all terms of the previous chain of inequalities by ρ^d, and take first the limsup for $\rho\to0^+$ and then the limit for $m\to+\infty$. By (2.3.6) in this way we obtain (2.6.9), since $(r^d-\rho^d)/\rho^d\to0$ as $\rho\to0^+$.

To prove (2.6.8), one can simply replace the expression in the second line of the previous chain of inequalities by

$$\limsup_{n\to+\infty}\mathfrak{m}^{F_n}(\ell_A,Q(x,\rho))+(c_3k^{1/2}|A|+c_4)(r^d-\rho^d).$$

If the additional hypothesis is satisfied, then the last lines of Corollary 2.3.10 and of Lemma 2.6.1 ensure that $N=\emptyset$, which concludes the proof. □

In the next proposition we show that an equality similar to (2.6.8) holds also for the surface integrand g.

Proposition 2.6.5. *Let $\{F_n\}_n\subset\mathfrak{F}$, let $F\in\mathfrak{F}_{\rm sc}$, and let g be given by* (2.3.7). *Assume that for every $O\in\mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then, for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$, we have*

$$\begin{aligned}
g(x,\zeta,\nu)&=\limsup_{\rho\to0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}\\
&=\limsup_{\rho\to0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}.
\end{aligned}\tag{2.6.10}$$

Proof. The proof can be obtained with the same arguments of [101, Proposition 3.3]. □

We conclude this section by stating a fundamental result for the proof of the homogenisation theorem, which will be the object of the next section. We show that for sequences of functions $\{f_n\}_n \subset \mathscr{F}^\alpha$ and $\{g_n\}_n \subset \mathscr{G}^\vartheta$, a sufficient condition for F^{f_n,g_n} to Γ-converge on every bounded open set is that (2.6.9) and (2.6.10) hold and that the function f is independent of x.

Theorem 2.6.6. *Let $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\vartheta}$. Assume that there exist $\widehat{f}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ and $\widehat{g}\colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that $\widehat{f}(A)$ is given by the common value of*

$$\lim_{m\to+\infty} \limsup_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}_{\rho\beta_m}(\ell_A, Q(x,\rho))}{\rho^d} = \lim_{m\to+\infty} \limsup_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}_{\rho\beta_m}(\ell_A, Q(x,\rho))}{\rho^d}$$

and $\widehat{g}(x,\zeta,\nu)$ is given by the common value of

$$\limsup_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}},$$

for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$, where β_m stands for the constant $\beta_{A,m}$ defined by (2.6.7). Then $\widehat{f} \in \mathscr{F}^\alpha$, $\widehat{g} \in \mathscr{G}$ and for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F^{\widehat{f},\widehat{g}}(\cdot,O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$, where $F^{\widehat{f},\widehat{g}}$ is as in Definition 2.1.6.

Proof. The proof follows closely the lines of [101, Theorem 5.4]. By Theorem 2.2.1 there exists a subsequence, not relabelled, and a functional $F \in \mathfrak{F}$ such that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$ and by Proposition 2.4.12 $F \in \mathfrak{F}^{\alpha,\vartheta}_w$. Thanks to Theorem 2.3.16 and Proposition 2.4.15, the functions f and g defined by (2.3.6) and (2.3.7), respectively, satisfy (2.3.36) and (2.3.37), and $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}$. Additionally, by Proposition 2.6.4 and Proposition 2.6.5, $f = \widehat{f}$ and $g = \widehat{g}$, so that (2.5.1) holds. Hence, by Theorem 2.5.1 we have that $F = F^{\widehat{f},\widehat{g}}$. Since the functions $\widehat{f}$ and $\widehat{g}$ are independent of the chosen subsequence, for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ by the Urysohn property of Γ-convergence (see [88, Proposition 8.3]) the original sequence $\{F_n(\cdot,O)\}$ Γ-converges to $F^{\widehat{f},\widehat{g}}(\cdot,O)$, concluding the proof. □

2.7 Homogenisation of free discontinuity functionals

We are now ready to deal with the homogenisation of functionals in $\mathfrak{F}^{\alpha,\vartheta}$. The arguments that we will make use of are based on the method devised in [101, Section 6]. The main technical difference in the results we obtain, with respect to [101], is the dependence on m of the constant $\beta_{A,m}$ appearing in (2.6.6), which, in the problem of stochastic homogenisation, forces us to use the Subadditive Ergodic Theorem for fixed m and only then to pass to the limit for $m \to +\infty$.

Before introducing the stochastic setting, we state a result which shows that, in the case of homogenisation, the sufficient conditions for Γ-convergence presented in Theorem 2.6.6 can be rewritten in terms of cubes whose side length r tends to $+\infty$. This formulation will be more suitable for the stochastic setting.

Given $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\vartheta$ for every $\varepsilon > 0$ we set $f_\varepsilon(x,A) \coloneqq f(x/\varepsilon, A)$ and $g_\varepsilon(x,\zeta,\nu) \coloneqq g(x/\varepsilon,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$. We observe that $f_\varepsilon \in \mathscr{F}^\alpha$ and $g_\varepsilon \in \mathscr{G}^\vartheta$. We set $F_\varepsilon \coloneqq F^{f_\varepsilon,g_\varepsilon}$ according to Definition 2.1.6. We recall that f^∞ and g^0 are

the functions given by (1.5.6) (with f_{bulk} replaced by f) and (2.4.5), that $f^\infty \in \mathscr{F}^\alpha$, while $g^0 \notin \mathscr{G}$, since it does not satisfy (g3).

The following theorem provides a general condition that guarantees the Γ-convergence of $\{F_\varepsilon\}_\varepsilon$ towards an integral functional whose integrands do not depend on x. We shall see in Corollary 2.7.10 that the hypotheses are satisfied in the case where f and g are periodic with respect to x. The advantage of this formulation is that these hypotheses are satisfied almost surely under the standard hypotheses of stochastic homogenisation.

Theorem 2.7.1. *Assume that there exists a function $g_{\mathrm{hom}}\colon \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ and that for every $m \in \mathbb{N}$ there exists a function $f^m_{\mathrm{hom}}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ such that*

$$f^m_{\mathrm{hom}}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g^0}}_{r\beta_{A,m}}(\ell_A, Q(rx,r))}{r^d} \quad \text{for } x \in \mathbb{R}^d,\ A \in \mathbb{R}^{k\times d}, \tag{2.7.1}$$

$$g_{\mathrm{hom}}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f^\infty,g}}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k, \nu \in \mathbb{S}^{d-1}, \tag{2.7.2}$$

where $\beta_{A,m}$ is the constant defined by (2.6.7). *Let $f_{\mathrm{hom}}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ be the function defined by*

$$f_{\mathrm{hom}}(A) \coloneqq \lim_{m\to+\infty} f^m_{\mathrm{hom}}(A) = \inf_{m\in\mathbb{N}} f^m_{\mathrm{hom}}(A) \tag{2.7.3}$$

for every $A \in \mathbb{R}^{k\times d}$. Then $f_{\mathrm{hom}} \in \mathscr{F}^\alpha$, $g_{\mathrm{hom}} \in \mathscr{G}$, $F^{f_{\mathrm{hom}},g_{\mathrm{hom}}} \in \mathfrak{F}^{\alpha,\vartheta}_w$, and for every $\varepsilon_n \to 0^+$ and for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_{\varepsilon_n}(\cdot,O)\}_n$ Γ-converges to $F^{f_{\mathrm{hom}},g_{\mathrm{hom}}}(\cdot,O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$.

Proof. We first observe that the limit in (2.7.3) exists, since $m \mapsto \mathfrak{m}^{F^{f,g^0}}_{r\beta_{A,m}}$ is non-increasing, hence, the same property holds for $m \mapsto f^m_{\mathrm{hom}}$.

Let us fix a sequence $\varepsilon_n \to 0^+$ as $n \to +\infty$ and set $F_{\varepsilon_n} \coloneqq F^{f_{\varepsilon_n},g_{\varepsilon_n}}$. To prove the result, it is enough to show that the hypotheses of Theorem 2.6.6 are satisfied by f_{hom} and g_{hom}. Since the hypothesis concerning g_{hom} can be checked by repeating verbatim the arguments of [101, Theorem 6.3], we only prove that the hypotheses concerning f_{hom} are satisfied.

To this aim, we fix $m \in \mathbb{N}$, $\rho > 0$ and set $r_n \coloneqq \frac{\rho}{\varepsilon_n}$. The same computations performed in [101, Lemma 6.1] show that for every $\varepsilon \in (0,1)$, $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{k\times d}$ we have

$$\left|\mathfrak{m}^{F_\varepsilon}_{\rho\beta_{A,m}}(\ell_A, Q(x,\rho)) - \varepsilon^d \mathfrak{m}^{F^{f,g^0}}_{\frac{\rho}{\varepsilon}\beta_{A,m}}(\ell_A, Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))\right| \le K_A \vartheta(2\rho\beta_{A,m})\rho^d, \tag{2.7.4}$$

where $K_A > 0$ is a constant depending on A, but not on m, ρ, and x. Let us fix $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{k\times d}$. Using twice the previous inequality with ε replaced by ε_n, by (2.7.1) we get that

$$\begin{aligned}
f^m_{\mathrm{hom}}(A)\rho^d - K_A\vartheta(2\beta_{A,m}\rho)\rho^d &= \lim_{n\to+\infty} (\tfrac{\rho}{r_n})^d \mathfrak{m}^{F^{f,g^0}}_{r_n\beta_{A,m}}(\ell_A, Q(\tfrac{r_n x}{\rho}, r_n)) - K_A\vartheta(2\beta_{A,m}\rho)\rho^d \\
&\le \liminf_{n\to+\infty} \mathfrak{m}^{F_{\varepsilon_n}}_{\rho\beta_{A,m}}(\ell_A, Q(x,\rho)) \\
&\le \limsup_{n\to+\infty} \mathfrak{m}^{F_{\varepsilon_n}}_{\rho\beta_{A,m}}(\ell_A, Q(x,\rho)) \\
&\le \lim_{n\to+\infty} (\tfrac{\rho}{r_n})^d \mathfrak{m}^{F^{f,g^0}}_{r_n\beta_{A,m}}(\ell_A, Q(\tfrac{r_n x}{\rho}, r_n)) + K_A\vartheta(2\beta_{A,m}\rho)\rho^d \\
&= f^m_{\mathrm{hom}}(A)\rho^d + K_A\vartheta(2\beta_{A,m}\rho)\rho^d.
\end{aligned}$$

Since ϑ is continuous and $\vartheta(0) = 0$, dividing by ρ^d and taking the limsup for $\rho \to 0^+$ we obtain

$$f^m_{\mathrm{hom}}(A) = \lim_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_{\varepsilon_n}}_{\rho\beta_{A,m}}(\ell_A, Q(x,\rho))}{\rho^d} = \lim_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_{\varepsilon_n}}_{\rho\beta_{A,m}}(\ell_A, Q(x,\rho))}{\rho^d}.$$

Taking the limit as $m \to +\infty$ we obtain that the hypotheses for f_{hom} in Theorem 2.6.6 are satisfied, so that the proof is concluded. □

The following result shows that hypothesis (2.7.1) can be slightly weakened.

Lemma 2.7.2. *Assume that for every $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $A \in \mathbb{Q}^{k\times d}$, the space of $k \times d$ matrices with rational entries, the limit*

$$f^m_{\mathrm{hom}}(A) \coloneqq \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g^0}}_{r\beta_{A,m}}(\ell_A, Q(rx,r))}{r^d}$$

exists and is independent of x. Then the function f^m_{hom} can be extended to a continuous function, still denoted by f^m_{hom}, defined on the whole $\mathbb{R}^{k\times d}$ and such that (2.7.1) *is satisfied.*

Proof. It is enough to repeat for every $m \in \mathbb{N}$ the arguments of [101, Lemma 6.4]. □

Stochastic homogenisation. We now introduce the stochastic setting in which we are going to deal with the homogenisation problem.

We fix a probability space $(\Theta, \mathcal{T}, P)$ and a group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ of P-preserving transformations on $(\Theta, \mathcal{T}, P)$; that is, a family $\{\tau_z\}_{z\in\mathbb{Z}^d}$ of $\mathcal{T}$-measurable bijective maps $\tau_z\colon \Theta \to \Theta$ such that

(a) $\tau_{z_1} \circ \tau_{z_2} = \tau_{z_1+z_2}$ for every $z_1, z_2 \in \mathbb{Z}^d$;

(b) $P(\tau_z^{-1}(E)) = P(E)$ for all $E \in \mathcal{T}$ and $z \in \mathbb{Z}^d$.

A group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ of P-preserving transformations is said to be ergodic if every set $E \in \mathcal{T}$ with $\tau_z(E) = E$ for every $z \in \mathbb{Z}^d$ has probability either 0 or 1. In analogy with [101], we introduce two classes of stochastic integrands.

Definition 2.7.3. $\mathscr{F}^\alpha_{\mathrm{stoc}}$ is the collection of all $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{k\times d})$-measurable functions $f\colon \Theta \times \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0, +\infty)$ such that for every $\theta \in \Theta$ the function $f(\theta) \coloneqq f(\theta, \cdot, \cdot)$ belongs to $\mathscr{F}^\alpha$ and the following stochastic periodicity condition holds: for every $\theta \in \Theta$, $z \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{k\times d}$ we have

$$f(\theta, x+z, A) = f(\tau_z(\theta), x, A).$$

$\mathscr{G}^\vartheta_{\mathrm{stoc}}$ is the collection of all $\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1})$-measurable functions $g\colon \Theta \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0, +\infty)$ such that for every $\theta \in \Theta$ the function $g(\theta) \coloneqq g(\theta, \cdot, \cdot, \cdot)$ belongs to $\mathscr{G}^\vartheta$, and the following stochastic periodicity condition holds: for every $\theta \in \Theta$, $z \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$g(\theta, x+z, \zeta, \nu) = g(\tau_z(\theta), x, \zeta, \nu).$$

We recall the definition of subadditive process. Let $\mathscr{R}$ be the collection of half-closed rectangles defined by

$$\mathscr{R} := \big\{R \subset \mathbb{R}^d\colon\ R = [a_1, b_1) \times ... \times [a_d, b_d) \text{ with } a_i < b_i \text{ for } i \in \{1, \dots, d\}\big\}. \tag{2.7.5}$$

We also introduce $(\Theta, \widehat{\mathcal{T}}, \widehat{P})$ as the completion of $(\Theta, \mathcal{T}, P)$. It is immediate to see that $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is a group of P-preserving transformations also on $(\Theta, \widehat{\mathcal{T}}, \widehat{P})$.

Definition 2.7.4. A function $\mu\colon \Theta \times \mathscr{R} \to \mathbb{R}$ is said to be a covariant subadditive process with respect to $\{\tau_z\}_{z\in\mathbb{Z}^d}$ if the following properties are satisfied

(a) for every $R \in \mathscr{R}$ the function $\mu(\cdot, R)$ is $\widehat{\mathcal{T}}$-measurable;

(b) for every $\theta \in \Theta$, $R \in \mathscr{R}$, and $z \in \mathbb{Z}^d$ we have $\mu(\theta, R + z) = \mu(\tau_z(\theta), R)$;

(c) given $R \in \mathscr{R}$ and a finite partition $\{R_i\}_{i=1}^n \subset \mathscr{R}$ of R, we have

$$\mu(\theta, R) \le \sum_{i=1}^n \mu(\theta, R_i)$$

for every $\theta \in \Theta$;

(d) there exists $C > 0$ such that $0 \le \mu(\theta, R) \le C\mathcal{L}^d(R)$ for every $\theta \in \Theta$ and $R \in \mathscr{R}$.

We recall the Subadditive Ergodic Theorem of Akcoglu & Krengel [2, Theorem 2.7]. For the particular version used here we refer the reader to [97, Proposition 1] (see also [145]).

Theorem 2.7.5. *Let μ be a subadditive process with respect to the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$. Then there exist a $\mathcal{T}$-measurable set Θ', with $P(\Theta') = 1$, and a function $\varphi\colon \Theta \to [0, +\infty)$ such that*

$$\lim_{r\to+\infty} \frac{\mu(\theta, Q(rx, r))}{r^d} = \varphi(\theta)$$

for every $x \in \mathbb{R}^d$ and every $\theta \in \Theta'$. If the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is also ergodic, then φ is constant P-a.e. on Θ.

Let $f \in \mathscr{F}^\alpha_{\rm stoc}$ and $g \in \mathscr{G}^\vartheta_{\rm stoc}$. For every $\theta \in \Theta$, we set $f^\infty(\theta) := f(\theta)^\infty$ and $g^0(\theta) := g(\theta)^0$. The following lemma shows that it is possible to define a subadditive process closely related to condition (2.7.1).

Lemma 2.7.6. *Let $f \in \mathscr{F}^\alpha_{\rm stoc}$, let $g \in \mathscr{G}^\vartheta_{\rm stoc}$, let $A \in \mathbb{R}^{k\times d}$, and let $m \in \mathbb{N}$. For every $R \in \mathscr{R}$ let $\rho(R)$ be the length of the longest of its sides. Then the function $\Psi_{A,m}\colon \Theta \times \mathscr{R} \to [0, +\infty)$ defined by*

$$\Psi_{A,m}(\theta, R) := \mathfrak{m}^{F^{f(\theta),g^0(\theta)}}_{\rho(R)\beta_{A,m}}(\ell_A, R^\circ) \tag{2.7.6}$$

is a covariant subadditive process.

Proof. The proof can be obtained by arguing exactly as in [101, Lemma 6.9]. □

With this lemma at hand, we are ready to show that condition (2.7.1) with f replaced by $f(\theta)$ and g^0 replaced by $g^0(\theta)$ is satisfied for P-a.e. $\theta \in \Theta$ and for every $m \in \mathbb{N}$.

Proposition 2.7.7. *Let $f \in \mathscr{F}^{\alpha}_{\mathrm{stoc}}$ and $g \in \mathscr{G}^{\vartheta}_{\mathrm{stoc}}$. Then there exist a $\mathcal{T}$-measurable set Θ', with $P(\Theta') = 1$, such that for every $m \in \mathbb{N}$ there exists a function $f^m_{\mathrm{hom}} \colon \Theta \times \mathbb{R}^{k\times d} \to [0, +\infty)$, with $f^m_{\mathrm{hom}}(\cdot, A)$ $\mathcal{T}$-measurable for every $A \in \mathbb{R}^{k\times d}$, such that*

$$\lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f(\theta),g^0(\theta)}}_{r\beta_{A,m}}(\ell_A, Q(rx, r))}{r^d} = f^m_{\mathrm{hom}}(\theta, A)$$

for every $\theta \in \Theta'$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{k\times d}$. Moreover, the function $f_{\mathrm{hom}} \colon \Theta \times \mathbb{R}^{k\times d} \to [0, +\infty)$ defined by

$$f_{\mathrm{hom}}(\theta, A) \coloneqq \lim_{m\to+\infty} f^m_{\mathrm{hom}}(\theta, A) = \inf_{m\in\mathbb{N}} f^m_{\mathrm{hom}}(\theta, A) \tag{2.7.7}$$

belongs to $\mathscr{F}^{\alpha}_{\mathrm{stoc}}$.

If, in addition, $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Θ' appropriately, we have that f^m_{hom} and f_{hom} are independent of θ.

Proof. By Lemma 2.7.6 for every $m \in \mathbb{N}$ and for every $A \in \mathbb{Q}^{k\times d}$ the function $\Psi_{A,m}$ defined by (2.7.6) is a covariant subadditive process. Hence, by the Subadditive Ergodic Theorem 2.7.5, there exists a $\mathcal{T}$-measurable set Θ', with $P(\Theta') = 1$, and for every $m \in \mathbb{N}$ a function $f^m_{\mathrm{hom}} \colon \Theta \times \mathbb{Q}^{k\times d} \to [0, +\infty)$, with $f^m_{\mathrm{hom}}(\cdot, A)$ $\mathcal{T}$-measurable for every $A \in \mathbb{Q}^{k\times d}$, such that

$$\lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f(\theta),g^0(\theta)}}_{r\beta_{A,m}}(\ell_A, Q(rx, r))}{r^d} = f^m_{\mathrm{hom}}(\theta, A) \tag{2.7.8}$$

for every $\theta \in \Theta'$, $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $A \in \mathbb{Q}^{k\times d}$. Thanks to Lemma 2.7.2, the function f^m_{hom} can be extended to a function $f^m_{\mathrm{hom}} \colon \Theta \times \mathbb{R}^{k\times d} \to [0, +\infty)$, $\mathcal{T}$-measurable with respect to θ and continuous with respect to A, such that (2.7.8) holds for every $\theta \in \Theta'$, $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{k\times d}$. We now fix $\theta_0 \in \Theta$ and we redefine f^m_{hom} on $\Theta \setminus \Theta' \times \mathbb{R}^{k\times d}$, by setting $f^m_{\mathrm{hom}}(\theta, A) = f^m_{\mathrm{hom}}(\theta_0, A)$ for every $\theta \in \Theta \setminus \Theta'$ and $A \in \mathbb{R}^{k\times d}$. Note that for every $\theta \in \Theta$ and $A \in \mathbb{R}^{k\times d}$, the sequence $f^m_{\mathrm{hom}}(\theta, A)$ is non-increasing with respect to m, which justifies (2.7.7) and that $f_{\mathrm{hom}}(\cdot, A)$ is $\mathcal{T}$-measurable. By Theorem 2.7.1 for every $\theta \in \Theta$ the function $f_{\mathrm{hom}}(\theta) \in \mathscr{F}^{\alpha}$, hence, $f_{\mathrm{hom}} \in \mathscr{F}^{\alpha}_{\mathrm{stoc}}$.

If, in addition, $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, then $f^m_{\mathrm{hom}}(\cdot, A)$ is constant P-a.e. for every m. This leads to the last sentence of the statement, concluding the proof. □

The following proposition shows that P-a.e. in Θ condition (2.7.2) is also satisfied.

Proposition 2.7.8. *Let $f \in \mathscr{F}^{\alpha}_{\mathrm{stoc}}$, let $g \in \mathscr{G}^{\vartheta}_{\mathrm{stoc}}$. Then there exists a $\mathcal{T}$-measurable set Θ', with $P(\Theta') = 1$, and a $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^k \times \mathbb{S}^{d-1})$-measurable function $g_{\mathrm{hom}} \colon \Theta \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0, +\infty)$ such that*

$$\lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f^\infty(\theta),g(\theta)}}(u_{rx,\zeta,\nu}, Q_\nu(rx, r))}{r^{d-1}} = g_{\mathrm{hom}}(\theta, \zeta, \nu) \tag{2.7.9}$$

for every $\theta \in \Theta'$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

If, in addition, the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Θ' appropriately, we have that the function g_{hom} is independent of θ.

Proof. The result is proved by adapting the same arguments used in [59, Proposition 9.3, Proposition 9.4, Proposition 9.5]. □

Combining Theorem 2.6.6 with Propositions 2.7.7 and 2.7.8, we are finally able to obtain the desired stochastic homogenisation theorem.

Theorem 2.7.9. *Let $f \in \mathscr{F}^{\alpha}_{\mathrm{stoc}}$, let $g \in \mathscr{G}^{\vartheta}_{\mathrm{stoc}}$, and for every $\varepsilon > 0$ and $\theta \in \Theta$ let $F^{\theta}_{\varepsilon} := F^{f_\varepsilon(\theta), g_\varepsilon(\theta)}$, according to Definition 2.1.6. Let $f_{\mathrm{hom}} \colon \Theta \times \mathbb{R}^{k\times d} \to [0,+\infty)$ and $g_{\mathrm{hom}} \colon \Theta \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ be the functions given in Propositions 2.7.7 and 2.7.8. Then $f_{\mathrm{hom}}(\theta,\cdot) \in \mathscr{F}^{\alpha}$ and $g_{\mathrm{hom}}(\theta,\cdot,\cdot) \in \mathscr{G}$ for every $\theta \in \Theta$ and there exists $\mathcal{T}$-measurable set Θ', with $P(\Theta') = 1$, such that for every sequence $\varepsilon_n \to 0^+$, $\theta \in \Theta'$, and $O \in \mathcal{O}_c(\mathbb{R}^d)$, the sequence $\{F^{\theta}_{\varepsilon_n}(\cdot, O)\}_n$ Γ-converges to $F^{f_{\mathrm{hom}}(\theta), g_{\mathrm{hom}}(\theta)}(\cdot, O)$ in the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$.*

If, in addition, the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Θ' appropriately, the functions f_{hom} and g_{hom} are independent of θ.

As a corollary, we deduce the following periodic homogenisation result.

Corollary 2.7.10. *Let $f \in \mathscr{F}^{\alpha}$, $g \in \mathscr{G}^{\vartheta}$, and let $F_\varepsilon := F^{f_\varepsilon, g_\varepsilon}$ be the functional introduced in Definition 2.1.6. Assume that both f and g are 1-periodic in the x variable. Then there exist functions $f_{\mathrm{lim}} \in \mathscr{F}^{\alpha}$ and $g_{\mathrm{lim}} \in \mathscr{G}$, with f_{lim} satisfying (4.7.25), and with g_{lim} satisfying (4.7.4) and (4.7.26), and consequently the following property holds: for every sequence $\{\varepsilon_n\}_n \subset (0,1)$ converging to 0 as $n \to +\infty$ and for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_{\varepsilon_n}(\cdot, O)\}_n$ Γ-converges to $F^{f_{\mathrm{lim}}, g_{\mathrm{lim}}}(\cdot, O)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$.*

Proof. It is enough to apply Theorem 2.7.9 in the case where Θ consists of a single point. □

Appendix to Chapter 2

2.A Compactness in $\mathrm{GBV}_\star$

In this appendix we present a compactness result for functions in $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. This result was obtained in the scalar setting by Dal Maso & Toader in [99, Theorem 5.5]. It was later adapted to the vector-valued setting in [111, Theorem 4.8] and it is strongly based on a technique devised by Friedrich [127]. We omit its proof, as it can be obtained, with relatively simple adaptations, from the ones of [99, Theorem 5.5] (see [111, Theorem 4.8] for details).

In the following, $O, \widetilde{O} \subset \mathbb{R}^d$ are two open sets with Lipschitz boundaries such that $O \subset \widetilde{O}$.

Theorem 2.A.1. *Let $f\colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0,+\infty)$ and $g\colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ be two functions satisfying* (f1)–(f3), (g1)–(g3), (g5), (g6)*, and consider the associated functional $F^{f,g}$ introduced in Definition 2.1.6. Let $w \in \mathrm{W}^{1,1}(\widetilde{O};\mathbb{R}^k)$ and let $\{u_n\}_n \subset \mathrm{GBV}_\star(\widetilde{O};\mathbb{R}^k)$ with $u_n = w$ $\mathcal{L}^d$-a.e. on $\widetilde{O}\setminus O$ and $\sup_n \mathcal{V}(u_n,\widetilde{O}) \le M$ for some $M>0$. Then, for every positive sequence $\{\varepsilon_n\}_n$ with $\varepsilon_n \to 0^+$, there exist a subsequence, not relabelled, modifications $y_n \in \mathrm{GBV}_\star(\widetilde{O};\mathbb{R}^k)$ of u_n, with $y_n = w$ $\mathcal{L}^d$-a.e. on $\widetilde{O}\setminus O$, such that*

$$F^{f,g}(y_n,\widetilde{O}) \le F^{f,g}(u_n,\widetilde{O}) + \varepsilon_n, \tag{2.A.1}$$

and $\{y_n\}$ converges to u in $L^0(\widetilde{O};\mathbb{R}^k)$ for some $u \in \mathrm{GBV}_\star(\widetilde{O};\mathbb{R}^k)$.

Remark 2.A.2. Let f and g be as above and let $w \in \mathrm{W}^{1,1}(\widetilde{O};\mathbb{R}^k)$. Consider a minimising sequence $\{u_n\}_n \subset \mathrm{GBV}_\star(\widetilde{O};\mathbb{R}^k)$ for the problem

$$\inf\{F^{f,g}(u,\widetilde{O})\colon\ u \in \mathrm{GBV}_\star(\widetilde{O};\mathbb{R}^k) \text{ with } \ u = w\ \mathcal{L}^d\text{-a.e. on } \widetilde{O}\setminus O\}. \tag{2.A.2}$$

Thanks to (f2), (f3), (g2), (g3), and (2.1.2), it is easily seen that $\sup_n \mathcal{V}(u_n,\widetilde{O}) \le M$ for some $M>0$. Hence, by Theorem 2.A.1 there exist another sequence $\{y_n\}_n$, still minimising, with $y_n = w$ $\mathcal{L}^d$-a.e. on $\widetilde{O}\setminus O$ satisfying (2.A.1) and such that (a subsequence of) $\{y_n\}_n$ converges to some $u \in \mathrm{GBV}_\star(\widetilde{O};\mathbb{R}^k)$ in measure. If the functional $F^{f,g}$ is lower semicontinuous with respect to the convergence in measure, we conclude that u minimises (2.A.2).

CHAPTER 3

Structured deformations for energies with general surface terms

We now turn to a second variational problem, arising from a different physical model, for which the space $\mathrm{GBV}_\star$ (see Definition 1.3.16) again provides a natural setting.

The purpose of this chapter is to establish a variational theory for the description of first-order structured deformations with general nonlinear surface densities, circumventing some of the limitations of the currently available models (see the Introduction for a more precise discussion on this point and also for details on the general framework of structured deformations). This requires a formulation in the generalised space $\mathrm{GBV}_\star$, which, as we have shown in Chapters 1 and 2, is the natural setting for surface energies that are linear near the origin and bounded at infinity, since compactness can only be achieved in this generalised space. In this novel framework, we prove three main results: an approximation theorem for structured deformations, an integral representation theorem for abstract lower semicontinuous functionals, and an explicit representation formula for relaxed energies.

This study starts with Section 3.1. We present here the variational framework we propose and state our three main results; each of the following three sections is devoted exclusively to the proof of one of these results.

In Section 3.2 we prove the cornerstone of this analysis, Approximation Theorem 3.1.1, a result justifying the assignment to *generalised structured deformations* $\mathrm{SD}^p_\star$ (see (3.1.1)) of an energy $I^p_\star$ defined by relaxation (see (3.1.11)). Although its proof follows, at a broad level, the strategy of earlier results of Choksi & Fonseca [76] and of Šilhavý [171], it requires several substantial new arguments that render the analysis significantly more technical. A crucial ingredient is a new density result for functions of bounded variation (see Theorem 3.2.1 and Remark 3.2.2(i),(iii)), which is of independent interest and may be viewed as an analogue in $\mathrm{GBV}_\star$ of the classical density theorem of Anzellotti & Giacquinta [21]. More precisely, we show that every function $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ can be approximated by a sequence $\{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k) \cap L^\infty(\Omega;\mathbb{R}^k)$ in such a way that $\mathcal{V}(u_n, \Omega \setminus J_{u_n})$ and $\mathcal{V}(u_n, J_{u_n})$ converge to $\mathcal{V}(u, \Omega \setminus J_u)$ and to $\mathcal{V}(u, J_u)$, respectively (see Section 1.3 and in particular (1.3.16) for the definitions of these objects).

In the spirit of the previous chapter, we introduce a general class of abstract functionals $\mathfrak{F}^p_\star$ (see Definition 3.1.3), which includes those defined by relaxation. Using the strategy of the Global Method of Bouchitté, Fonseca, & Mascarenhas [41], we are able to prove in Section 3.3 an integral representation result for functionals in $\mathfrak{F}^p_\star$. We cannot apply *mutatis mutandis* the standard arguments of the global method, nor argue by a *truncation-perturbation* argument as was done in Chapter 2; the functional framework of structured deformations does not allow, in general, for truncation techniques. We resort instead to a different argument, inspired by [81] and [84], which is based on a careful analysis of blow-ups of $\mathrm{GBV}_\star$-functions. This study is carried out in Section 3.C of the Appendix and relies, in turn, on new Poincaré-type estimates for functions in $\mathrm{GBV}_\star$ presented in Section 3.B. This novel approach also allows us to improve some of the results obtained in Chapter 2 (see the discussion in Remark 3.3.8). We stress, however, that Theorem 3.1.7 does not give a complete integral representation of functionals in $\mathfrak{F}^p_\star$, as it neglects the contributions due to

diffuse but singular terms related to the Cantor part. This is due to the fact that our blow-up methods, while being perfectly well-suited to deal with bulk and surface contributions, rely on compactness properties that we are not able to enforce around points of density one for the Cantor part.

Section 3.4 is devoted to the study of the properties of the relaxed energy $I^p_\star$. Here, $p \geq 1$ is fixed and determines the rate of growth at infinity of a certain function related to $I^p_\star$ (see (Ψ2) and (Ψ4)). As a first result, we prove that, for any $p \geq 1$, the functional $I^p_\star$ belongs to $\mathfrak{F}^p_\star$, so that, by Theorem 3.1.7, we immediately deduce an integral representation of its bulk and surface part by means of integrands f_{bulk} and f_{surf}. Then, we specialise to the case $p > 1$, where some truncation techniques are available. In this case, we are able to give more explicit expressions of f_{bulk} and f_{surf} by means of cell formulas consistent with those identified by Choksi & Fonseca (see (3.1.23) and (3.1.24) and compare with [76, Remark 3.3]). Finally, assuming a certain independence of $I^p_\star$ on the spatial variable x (see (3.1.25)), we obtain a complete integral representation of $I^p_\star$, including its Cantor part.

The results contained in this chapter are part of the joint paper with Friedrich [112].

3.1 Setting of the problem and main results

In this section, we present the main results of the chapter. We begin by introducing the class of structured deformations that will be our main object of investigation. We refer the reader to Section 1.1 for all the notation concerning the collections of open/Borel sets and to Section 1.3 for those related to the spaces of functions we work with.

Structured Deformations. In the following, we let $p \geq 1$. We specify only when needed if $p = 1$ or $p > 1$. We recall that $\Omega \subset \mathbb{R}^d$ always denotes a bounded, open set, which we also assume to have Lipschitz boundary. Given $O \in \mathcal{O}(\Omega)$, the class of *generalised structured deformations* $\mathrm{SD}^p_\star(O;\mathbb{R}^k)$ is defined by setting

$$\mathrm{SD}^p_\star(O;\mathbb{R}^k) \coloneqq \big\{(g,G)\colon\ g \in \mathrm{GBV}_\star(O;\mathbb{R}^k)\ \text{ and } G \in L^p(O;\mathbb{R}^{k\times d})\big\}. \tag{3.1.1}$$

We use the term 'generalised' as the deformations g belong to a class of generalised BV-functions. The first main result of this chapter is the following approximation result for structured deformations $\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. Recall the definition of $\mathcal{V}$ given in (1.3.16).

Theorem 3.1.1 (Approximation Theorem)**.** *There exists a constant $C > 0$, depending only on d, such that for all $O \in \mathcal{O}(\Omega)$ with Lipschitz boundary and all $(g,G) \in \mathrm{SD}^p_\star(O;\mathbb{R}^k)$ we can find a sequence $\{u_n\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k)$ such that $\{u_n\}_n$ converges to g in $L^0(O;\mathbb{R}^k)$ as $n \to +\infty$, $\nabla u_n = G$ $\mathcal{L}^d$-a.e. in O for all $n \in \mathbb{N}$, and*

$$\limsup_{n\to+\infty} \mathcal{V}(u_n,O) \leq C\Big(\mathcal{V}(g,O) + \|G\|_{L^1(O)}\Big). \tag{3.1.2}$$

We briefly comment on this result. If (3.1.2) is dropped, the theorem is an immediate consequence of [171, Theorem 1.2] (see also [76, Theorem 2.12]). Indeed, using the truncation properties of $\mathrm{GBV}_\star$-functions described in Definition 1.3.16(c), we can approximate g by means of BV-functions, to which [171, Theorem 1.2] applies. A diagonal argument then gives a sequence converging to g in $L^0(O;\mathbb{R}^k)$ which satisfies the required gradient constraints, but which may fail to satisfy (3.1.2). To overcome this difficulty, we will depart from the results currently available in the literature, and rely on new density results for functions of bounded variation tailored to our functional setting, namely Theorem 3.2.1 and Lemma 3.2.3. In

particular, Theorem 3.2.1 provides an approximation result for BV-functions by means of SBV-functions that applies to a general class of energies, including $\mathcal{V}$ (see Remark 3.2.2(ii)).

We endow $\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ with the following notion of convergence.

Definition 3.1.2. We say that a sequence $\{(g_n,G_n)\}_n \subset \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ converges to $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ as $n\to+\infty$, in symbols $(g_n,G_n)\underset{SD_\star}{\to}(g,G)$ in Ω, if and only if $g_n\to g$ in $L^0(\Omega;\mathbb{R}^k)$ and

$$G_n \rightharpoonup G \text{ in } L^p(\Omega;\mathbb{R}^{k\times d}) \text{ if } p>1 \quad \text{and} \quad G_n \overset{*}{\rightharpoonup} G \text{ weakly}^* \text{ in } \mathcal{M}_b(\Omega;\mathbb{R}^{k\times d}) \text{ if } p=1$$

as $n\to+\infty$.

Given a sequence $\{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k)$, we say that $\{u_n\}_n$ converges to $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, in symbols $u_n \underset{SD_\star}{\to}(g,G)$, if $(u_n,\nabla u_n)\underset{SD_\star}{\to}(g,G)$.

Integral representation for abstract functionals. We present a result concerning the integral representation of functionals defined on $\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. We begin by introducing the class $\mathfrak{F}^p_\star$, which will be one of the main objects of study in this chapter.

Definition 3.1.3. $\mathfrak{F}^p_\star$ is the class of all functionals $\mathcal{F}\colon \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)\times\mathcal{B}(\Omega)\to[0,+\infty)$ satisfying the following conditions:

(H1) (*Measure property*) for every $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ the set function $\mathcal{F}(g,G,\cdot)$ is a Radon measure on $\mathcal{B}(\Omega)$;

(H2) (*Locality*) for every $O\in\mathcal{O}(\Omega)$ and pairs $(g,G),(u,U)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, with $g=u$ and $G=U$ $\mathcal{L}^d$-a.e. in O, it holds $\mathcal{F}(g,G,O)=\mathcal{F}(u,U,O)$;

(H3) (*Lower semicontinuity*) for every $O\in\mathcal{O}(\Omega)$ the functional $\mathcal{F}(\cdot,\cdot,O)$ is lower semicontinuous with respect to the convergence in $\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$;

(H4) (*Growth conditions*) there exist positive constants α,β, with $\alpha\le\beta$, such that

$$\alpha\Big(\mathcal{V}(g,B)+\|G\|^p_{L^p(B)}\Big)\le\mathcal{F}(g,G,B)\le\beta\Big(\mathcal{V}(g,B)+\|G\|^p_{L^p(B)}+\mathcal{L}^d(B)\Big) \tag{3.1.3}$$

for every $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B\in\mathcal{B}(\Omega)$.

Remark 3.1.4. Combining properties (H1) and (H2) we obtain that every functional $\mathcal{F}\in\mathfrak{F}^p_\star$ satisfies the following locality property: if $B\in\mathcal{B}(\Omega)$, $(g,G),(u,U)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, and $g=u$ and $G=U$ $\mathcal{L}^d$-a.e. in a neighbourhood of B, then $\mathcal{F}(g,G,B)=\mathcal{F}(u,U,B)$.

For this reason, given $O\in\mathcal{O}(\Omega)$ with Lipschitz boundary and $(g,G)\in \mathrm{SD}^p_\star(O;\mathbb{R}^k)$, for every $B\in\mathcal{B}(O)$ we can define $\mathcal{F}(g,G,B):=\mathcal{F}(u,U,B)$, where $(u,U)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ is any pair such that $g=u$ and $G=U$ $\mathcal{L}^d$-a.e. on O. For instance, this extension can be performed by using Lemma 1.3.24. The locality property described above implies that the value of $\mathcal{F}(g,G,B)$ does not depend on the chosen extension (u,U).

In Section 3.3 we prove an integral representation result for functionals in $\mathfrak{F}^p_\star$. To give the precise statement of this result, we introduce a decomposition of elements in $\mathfrak{F}^p_\star$ that mimics the usual decomposition of the distributional derivative for functions of bounded variation.

Definition 3.1.5. Let $\mathcal{F} \in \mathfrak{F}^p_\star$ and $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. By (H1) and (H4), $\mathcal{F}(g,G,\cdot)$ is a bounded Radon measure on $\mathcal{B}(\Omega)$. Its absolutely continuous and singular parts with respect to $\mathcal{L}^d$ are denoted by $\mathcal{F}^a(g,G,\cdot)$ and $\mathcal{F}^s(g,G,\cdot)$, respectively. We also introduce the bounded Radon measures $\mathcal{F}^c(g,G,\cdot)$ and $\mathcal{F}^j(g,G,\cdot)$ defined by

$$\mathcal{F}^c(g,G,B) \coloneqq \mathcal{F}^s(g,G,B \setminus J_g) \text{ and } \mathcal{F}^j(g,G,B) \coloneqq \mathcal{F}(g,G,B \cap J_g)$$

for every $B \in \mathcal{B}(\Omega)$. We obtain

$$\mathcal{F}(g,G,B) = \mathcal{F}^a(g,G,B) + \mathcal{F}^c(g,G,B) + \mathcal{F}^j(g,G,B)$$

for every $B \in \mathcal{B}(\Omega)$.

Remark 3.1.6. Note that by Remark 1.3.15 the functional $(g,G,B) \mapsto \mathcal{V}(g,B) + \|G\|^p_{L^p(B)}$ belongs to $\mathfrak{F}^p_\star$. Moreover, if $\mathcal{F} \in \mathfrak{F}^p_\star$ and $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, then

$$\begin{aligned}
&\mathcal{F}^a(g,G,\cdot) \text{ is absolutely continuous with respect to } \mathcal{L}^d,\\
&\mathcal{F}^c(g,G,\cdot) \text{ is absolutely continuous with respect to } |D^c g|,\\
&\mathcal{F}^j(g,G,\cdot) \text{ is absolutely continuous with respect to } \mathcal{V}(g,\cdot) \llcorner J_g.
\end{aligned}$$

Our goal is to represent $\mathcal{F}^a$ and $\mathcal{F}^j$ as integral functionals. As customary in integral representation results for free discontinuity functionals, the bulk and surface integrands are obtained by considering the asymptotic behaviour of the minimum values of certain constrained problems on small cubes. To this end, given $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $O \in \mathcal{O}(\Omega)$ with Lipschitz boundary, we set

$$\mathcal{C}^p_\star(g,G,O) \coloneqq \Big\{(u,U) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)\colon\ u = g \text{ near } \partial O,\ \int_O (G-U)\,\mathrm{d}x = 0\Big\}$$

and consider the auxiliary minimisation problem given by

$$\mathfrak{m}(g,G,O) \coloneqq \inf\Big\{\mathcal{F}(u,U,O)\colon\ (u,U) \in \mathcal{C}^p_\star(g,G,O)\Big\}. \tag{3.1.4}$$

The boundary conditions that we generally use for these minimisation problems are given by affine functions and pure jump functions: for $x \in \Omega$, $a \in \mathbb{R}^k$, and $A \in \mathbb{R}^{k\times d}$ we set $\ell_{x,a,A}(y) \coloneqq a + A(y-x)$, and for $x \in \Omega$, $\zeta,\theta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$ we let

$$u_{x,\zeta,\theta,\nu}(y) \coloneqq \begin{cases} \zeta & \text{if } (y-x)\cdot\nu > 0,\\ \theta & \text{if } (y-x)\cdot\nu \le 0. \end{cases} \tag{3.1.5}$$

Note that these functions are slightly different from those used as boundary conditions in Section 1.5 and in Chapter 2 (see (1.5.2) and (1.5.3)) and in the rest of the work. This is due to the fact, as we do not assume that functionals in $\mathfrak{F}$ satisfy any invariance property in the spirit of (X4a), more complex boundary conditions need to be taken into account. Clearly, this will affect the definition of the candidate bulk and surface integrands below.

Our candidate bulk and surface integrands are given by

$$f_{\mathrm{bulk}}(x,a,A,L) \coloneqq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}(\ell_{x,a,A}, L, Q(x,\rho))}{\rho^d} \tag{3.1.6}$$

for all $x \in \Omega$, $a \in \mathbb{R}^k$, $A, L \in \mathbb{R}^{k\times d}$, and

$$f_{\mathrm{surf}}(x,\zeta,\theta,\nu) \coloneqq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}(u_{x,\zeta,\theta,\nu}, 0, Q_\nu(x,\rho))}{\rho^{d-1}} \tag{3.1.7}$$

for all $x \in \Omega$, $\zeta, \theta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$, where we recall that $Q_\nu(x,\rho)$ is a cube centred at x, of side-length ρ, and with two faces orthogonal to ν, while $Q(x,\rho) \coloneqq x + (-\rho/2, \rho/2)^d$.

The following theorem constitutes our main result concerning the class $\mathfrak{F}^p_\star$. It gives an integral representation of the bulk and surface parts by means of the functions f_{bulk} and f_{surf} above.

Theorem 3.1.7 (Integral representation)**.** *Let $\mathcal{F} \in \mathfrak{F}^p_\star$. Then, for every $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ it holds*

$$\mathcal{F}^a(g,G,B) = \int_B f_{\mathrm{bulk}}\big(x, g, \nabla g, G\big)\,\mathrm{d}x, \tag{3.1.8}$$

$$\mathcal{F}^j(g,G,B) = \int_{J_g\cap B} f_{\mathrm{surf}}\big(x, g^+, g^-, \nu_g\big)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{3.1.9}$$

where f_{bulk} and f_{surf} are given by (3.1.6) *and* (3.1.7)*. In particular, if $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ with $|D^c g|(\Omega) = 0$, then for all $B \in \mathcal{B}(\Omega)$ it holds that*

$$\mathcal{F}(g,G,B) = \int_B f_{\mathrm{bulk}}\big(x, g, \nabla g, G\big)\,\mathrm{d}x + \int_{J_g\cap B} f_{\mathrm{surf}}\big(x, g^+, g^-, \nu_g\big)\,\mathrm{d}\mathcal{H}^{d-1}.$$

Remark 3.1.8. If $\mathcal{F} \in \mathfrak{F}^p_\star$ is invariant under translation, the functions f_{bulk} and f_{surf} have simpler expressions. More precisely, if

$$\mathcal{F}(g+a,G,B) = \mathcal{F}(g,G,B) \tag{3.1.10}$$

for all $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, $a \in \mathbb{R}^k$, and $B \in \mathcal{B}(\Omega)$, then

$$f_{\mathrm{bulk}}(x,a,A,L) = f_{\mathrm{bulk}}(x,0,A,L) \quad \text{and} \quad f_{\mathrm{surf}}(x,\zeta,\theta,\nu) = f_{\mathrm{surf}}(x,\zeta-\theta,0,\nu)$$

for all $x \in \Omega$, $a \in \mathbb{R}^k$, $A, L \in \mathbb{R}^{k\times d}$, $\zeta, \theta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$. To simplify the notation, when the invariance property (3.1.10) holds, we will write $f_{\mathrm{bulk}}(x,A,L)$ and $f_{\mathrm{surf}}(x,\zeta-\theta,\nu)$ in place of $f_{\mathrm{bulk}}(x,0,A,L)$ and $f_{\mathrm{surf}}(x,\zeta-\theta,0,\nu)$.

Relaxation of bulk-surface energies. Our goal is to employ the integral representation result in Theorem 3.1.7 to prove a relaxation result for energies defined on $\mathrm{SBV}(\Omega;\mathbb{R}^k)$, in the spirit of the seminal work [76]. More precisely, we consider a bulk integrand $\Psi\colon \mathbb{R}^d\times\mathbb{R}^{k\times d} \to [0,+\infty)$ and a surface integrand $\gamma\colon \mathbb{R}^d\times\mathbb{R}^k\times\mathbb{S}^{d-1} \to [0,+\infty)$, and the associated integral functional $\mathcal{E}\colon \mathrm{SBV}(\Omega;\mathbb{R}^k)\times\mathcal{B}(\Omega) \to [0,+\infty)$ defined by

$$\mathcal{E}(u,B) \coloneqq \int_B \Psi(x,\nabla u)\,\mathrm{d}x + \int_{J_u\cap B} \gamma(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $(u,B) \in \mathrm{SBV}(\Omega;\mathbb{R}^k)\times\mathcal{B}(\Omega)$.

Under suitable hypotheses on Ψ and γ specified below, we study the functional

$$I^p_\star(g,G) \coloneqq \inf\Big\{ \liminf_{n\to+\infty} \mathcal{E}(u_n,\Omega)\colon\ \{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k),\ u_n \underset{SD_\star}{\to} (g,G) \text{ in } \Omega\Big\} \tag{3.1.11}$$

for all $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. To this end, for $O\in\mathcal{O}(\Omega)$ we consider the localised functional $I^p_\star(\cdot,\cdot,O)$ defined for all $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ by

$$I^p_\star(g,G,O)\coloneqq \inf\Big\{\liminf_{n\to+\infty}\mathcal{E}(u_n,O)\colon\ \{u_n\}_n\subset \mathrm{SBV}(O;\mathbb{R}^k),\ u_n\underset{SD_\star}{\to}(g,G)\text{ in }O\Big\}. \tag{3.1.12}$$

We then extend this definition to every Borel set $B\subset\Omega$ by setting

$$I^p_\star(g,G,B)\coloneqq \inf\big\{I^p_\star(g,G,O)\colon\ O\supset B\text{ with }O\in\mathcal{O}(\Omega)\big\}. \tag{3.1.13}$$

It will be shown in Theorem 3.1.13 that, under suitable hypotheses on Ψ and γ, the set function $O\mapsto I^p_\star(g,G,O)$ given by (3.1.12) coincides with the trace on $\mathcal{O}(\Omega)$ of a bounded Radon measure, so that definitions (3.1.12) and (3.1.13) agree on $\mathcal{O}(\Omega)$.

We now introduce the sets of assumptions we will make on Ψ and γ. Whereas the assumptions on Ψ are modelled after those in [76], for the surface density we fundamentally depart from the usual assumption of 1-homogeneous densities by considering bounded densities which are close to 1-homogeneous functions at the origin. We assume that Ψ satisfies the following properties:

(Ψ1) (measurability) Ψ is a Carathéodory function;

(Ψ2) (Lipschitz continuity) there is a constant $C_\Psi>0$ such that

$$|\Psi(x,A_1)-\Psi(x,A_2)|\le C_\Psi|A_1-A_2|\big(1+|A_1|^{p-1}+|A_2|^{p-1}\big)$$

for $\mathcal{L}^d$-a.e. $x\in\Omega$ and all $A_1,A_2\in\mathbb{R}^{k\times d}$;

(Ψ3) (control from above) there is $A_0\in\mathbb{R}^{k\times d}$ such that $\Psi(\cdot,A_0)\in L^\infty(\Omega)$;

(Ψ4) (lower bound) there is a constant $c_\Psi>0$ such that

$$c_\Psi|A|^p-\frac{1}{c_\Psi}\le\Psi(x,A)$$

for $\mathcal{L}^d$-a.e. $x\in\Omega$ and all $A\in\mathbb{R}^{k\times d}$;

(Ψ5) (continuity in x) there is a continuous function $\omega_\Psi\colon[0,+\infty)\to[0,+\infty)$, with $\omega_\Psi(0)=0$, such that for every $x_1,x_2\in\Omega$ and $A\in\mathbb{R}^{k\times d}$ it holds

$$|\Psi(x_1,A)-\Psi(x_2,A)|\le\omega_\Psi(|x_1-x_2|)\big(\Psi(x_1,A)+\Psi(x_2,A)\big).$$

Remark 3.1.9. Any function Ψ satisfying (Ψ2) and (Ψ3) also satisfies the following property:

(Ψ3') (upper bound) there exists a constant $\overline{C}_\Psi>0$ such that

$$\Psi(x,A)\le\overline{C}_\Psi\big(|A|^p+1\big) \tag{3.1.14}$$

for $\mathcal{L}^d$-a.e. $x\in\Omega$ and all $A\in\mathbb{R}^{k\times d}$.

Indeed, letting A_0 be the matrix of (Ψ3), we may use (Ψ2) to obtain for all $A\in\mathbb{R}^{k\times d}$

$$\Psi(x,A)\le\Psi(x,A_0)+C_\Psi(|A|+|A_0|+|A|^p+|A_0|^p+|A||A_0|^{p-1}+|A_0||A|^{p-1}).$$

In light of (Ψ3), this inequality implies (3.1.14) for $\mathcal{L}^d$-a.e. $x\in\Omega$ and for all $A\in\mathbb{R}^{k\times d}$, for a sufficiently large constant $\overline{C}_\Psi$ depending only on A_0, C_Ψ, Ω, and p.

In the following list we collect the assumptions we make on the surface integrand γ, some of which are closely related to those considered in Definition 2.1.3:

(γ1) (symmetry) for every $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ it holds

$$\gamma(x, \zeta, \nu) = \gamma(x, -\zeta, -\nu);$$

(γ2) (upper and lower bounds) there are two constants $c_\gamma, C_\gamma > 0$ such that for all $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ it holds

$$c_\gamma(|\zeta| \wedge 1) \leq \gamma(x, \zeta, \nu) \leq C_\gamma(|\zeta| \wedge 1);$$

(γ3) (sub-additivity) for all $x \in \Omega$, ζ_1, $\zeta_2 \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ it holds

$$\gamma(x, \zeta_1 + \zeta_2, \nu) \leq \gamma(x, \zeta_1, \nu) + \gamma(x, \zeta_2, \nu);$$

(γ4) (continuity in x) there is a continuous function $\omega_\gamma \colon [0, +\infty) \to [0, +\infty)$, with $\omega_\gamma(0) = 0$, such that for every $x_1, x_2 \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ it holds

$$|\gamma(x_1, \zeta, \nu) - \gamma(x_2, \zeta, \nu)| \leq \omega_\gamma(|x_1 - x_2|)(|\zeta| \wedge 1);$$

(γ5) (quasi-monotonicity) there exists a constant $\beta_\gamma \geq 1$ such that for every $x \in \Omega$ and $\nu \in \mathbb{S}^{d-1}$ it holds

$$\gamma(x, \zeta_1, \nu) \leq \gamma(x, \zeta_2, \nu)$$

whenever $\beta_\gamma|\zeta_1| \leq |\zeta_2|$;

(γ6) (rate of growth at the origin) there is a continuous nondecreasing function $\vartheta \colon [0, +\infty) \to [0, +\infty)$, with $\vartheta(0) = 0$ and $\vartheta(h) \geq \frac{c_\gamma}{C_\gamma} h - 1$ for all $h \geq 0$, with the constants $c_\gamma, C_\gamma > 0$ of (γ2), such that for all $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ the limit

$$\gamma^0(x, \zeta, \nu) \coloneqq \lim_{s \to 0^+} \frac{1}{s} \gamma(x, s\zeta, \nu) \tag{3.1.15}$$

exists and satisfies

$$\Big|\frac{1}{s}\gamma(x, s\zeta, \nu) - \gamma^0(x, \zeta, \nu)\Big| \leq \vartheta(s|\zeta|)\frac{1}{s}\gamma(x, s\zeta, \nu)$$

for all $s > 0$.

Remark 3.1.10 (Assumptions (γ1)–(γ5))**.** Properties (γ1) and (γ3) are standard assumptions on surface densities, while the growth condition (γ2) is a core ingredient in our study, see (0.0.14) in the introduction for the motivation. The continuity condition (γ4) is also a standard assumption, slightly adjusted to our setting by using $(|\zeta| \wedge 1)$ in place of $|\zeta|$. Property (γ5) was introduced in [57] and was later employed in several other works (see, for instance, [58, 99, 127]), as well as in Chapter 2. We recall (see Remark 2.1.4) that it is weaker than monotonicity in $|\zeta|$ of γ.

Remark 3.1.11 (Assumption (γ6))**.** An equivalent formulation of this condition was already considered in Definition 2.4.3. We also observe that γ^0 is positively 1-homogeneous, i.e.,

$$\gamma^0(x,t\zeta,\nu) = t\gamma^0(x,\zeta,\nu) \tag{3.1.16}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$, and $t \geq 0$. Note that, together with (γ2), (3.1.16) implies

$$c_\gamma|\zeta| \leq \gamma^0(x,\zeta,\nu) \leq C_\gamma|\zeta| \tag{3.1.17}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

Remark 3.1.12 (Properties of γ^0)**.** The integrand γ^0 satisfies properties closely related to (γ3)–(γ4). Indeed, it follows directly from (γ3) and (γ6) that γ^0 satisfies (γ3) with γ replaced by γ^0, while (γ4) and (γ6) imply that

$$|\gamma^0(x_1,\zeta,\nu) - \gamma^0(x_2,\zeta,\nu)| \leq \omega_\gamma(|x_1-x_2|)|\zeta| \tag{3.1.18}$$

for all $x_1, x_2 \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

The following is the first of the three results concerning the integral representation of (3.1.11). Note that we do not assume (Ψ5) and (γ5)–(γ6) nor put any restrictions on the value of p.

Theorem 3.1.13 (Relaxation I)**.** *Let Ψ and γ be integrands satisfying assumptions (Ψ1)–(Ψ4) and (γ1)–(γ4). Then (up to a shift) the functional $I^p_\star$ defined by (3.1.13) belongs to $\mathfrak{F}^p_\star$ and for all $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ it holds*

$$(I^p_\star)^a(g,G,B) = \int_B f_{\mathrm{bulk}}\big(x,\nabla g,G\big)\,\mathrm{d}x, \tag{3.1.19}$$

$$(I^p_\star)^j(g,G,B) = \int_{J_g\cap B} f_{\mathrm{surf}}\big(x,[g],\nu_g\big)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{3.1.20}$$

where f_{bulk} and f_{surf} are given by (3.1.6) and (3.1.7), replacing $\mathcal{F}$ with $I^p_\star$ (see also Remark 3.1.8). In particular, if $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ with $|D^c g|(\Omega) = 0$, then for all $B \in \mathcal{B}(\Omega)$ it holds that

$$I^p_\star(g,G,B) = \int_B f_{\mathrm{bulk}}\big(x,\nabla g,G\big)\,\mathrm{d}x + \int_{J_g\cap B} f_{\mathrm{surf}}\big(x,[g],\nu_g\big)\,\mathrm{d}\mathcal{H}^{d-1}.$$

Our second main result gives a more explicit expression for the densities f_{bulk} and f_{surf} appearing in (3.1.19) and (3.1.20) when $p > 1$. To present the result, we first introduce some further notation. For $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$ we set $u_{\zeta,\nu} \coloneqq u_{0,\zeta,0,\nu}$ (see (3.1.5)). Given $A, L \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$, we consider the classes

$$\mathcal{C}^{\mathrm{bulk}}_{p,\star}(A,L) \coloneqq \Big\{u \in \mathrm{SBV}(Q;\mathbb{R}^k)\colon\ u = \ell_{0,0,A} \text{ near } \partial Q, \int_Q \nabla u\,\mathrm{d}x = L,\ \|\nabla u\|^p_{L^p(Q)} < +\infty\Big\} \tag{3.1.21}$$

and

$$\mathcal{C}^{\mathrm{surf}}_{p,\star}(\zeta,\nu) \coloneqq \Big\{u \in \mathrm{SBV}(Q_\nu;\mathbb{R}^k)\colon\ u = u_{\zeta,\nu} \text{ near } \partial Q_\nu,\ \nabla u = 0\ \ \mathcal{L}^d\text{-a.e. on } Q_\nu\Big\}. \tag{3.1.22}$$

We set

$$H_p(x,A,L) \coloneqq \inf_{u\in\mathcal{C}^{\mathrm{bulk}}_{p,\star}(A,L)} \Big\{ \int_Q \Psi\big(x,\nabla u(y)\big)\,\mathrm{d}y + \int_{J_u} \gamma^0\big(x,[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big\} \tag{3.1.23}$$

for all $x\in\Omega$ and $A,L\in\mathbb{R}^{k\times d}$, and

$$h_p(x,\zeta,\nu) \coloneqq \inf_{u\in\mathcal{C}^{\mathrm{surf}}_{p,\star}(\zeta,\nu)} \Big\{ \int_{J_u\cap Q_\nu} \gamma\big(x,[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big\} \tag{3.1.24}$$

for all $x\in\Omega$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$. Note that, in contrast to (3.1.24), in (3.1.23) the density γ^0 appears which is the relevant quantity here as affine boundary conditions are effectively approximated by infinitesimally small jumps, cf. also [76, Remark 3.3].

The following constitutes a characterisation of $(I^p_\star)^a$ and $(I^p_\star)^j$ in terms of the densities H_p and h_p. We remark that the restriction $p>1$ is due to the nature of the truncation techniques we employ (see Lemma 3.4.3), which are ill-behaved when $p=1$. This is due to concentration phenomena that may occur when dealing with the weak*-convergence of measures.

Theorem 3.1.14 (Relaxation II)**.** *Let $p>1$ and let Ψ and γ be integrands satisfying assumptions* (Ψ1)–(Ψ5) *and* (γ1)–(γ6)*. Then, for all $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B\in\mathcal{B}(\Omega)$ equalities* (3.1.19) *and* (3.1.20) *hold with $f_{\mathrm{bulk}}(x,\nabla g,G)$ and $f_{\mathrm{surf}}(x,[g],\nu_g)$ replaced by $H_p(x,\nabla g,G)$ and $h_p(x,[g],\nu_g)$, where H_p and h_p are the functions defined by* (3.1.23) *and* (3.1.24)*, respectively.*

Proposition 3.1.15 (Properties of H_p and h_p)**.** *Let $p>1$ and let Ψ and γ be integrands satisfying assumptions* (Ψ1)–(Ψ5) *and* (γ1)–(γ6)*. Given $L\in\mathbb{R}^{k\times d}$, let $H^L_p\colon\Omega\times\mathbb{R}^{k\times d}\to[0,+\infty)$ be the function defined for $x\in\Omega$ and $A\in\mathbb{R}^{k\times d}$ by $H^L_p(x,A)\coloneqq H_p(x,A,L)$. Then H^L_p satisfies* (Ψ1)–(Ψ5) *with p replaced by 1, while h_p satisfies* (γ1)–(γ5)*.*

Moreover, for every $x\in\Omega$ and $A\in\mathbb{R}^{k\times d}$ there exists a constant $C>0$, depending only on x and on A, such that

$$|H_p(x,A,L_1)-H_p(x,A,L_2)|\le C|L_1-L_2|(1+|L_1|^{p-1}+|L_2|^{p-1}) \quad \text{for all } L_1,L_2\in\mathbb{R}^{k\times d}.$$

Finally, there exist constants $c_H,C_H>0$ such that

$$c_H(|A|+|L|^p)-\frac{1}{c_H}\le H_p(x,A,L)\le C_H(|A|+|L|^p+1)$$

for all $x\in\Omega$ and $A,L\in\mathbb{R}^{k\times d}$.

The final result of this chapter provides a complete integral representation for $I^p_\star$, including its Cantor part $(I^p_\star)^c$. his is obtained under a stronger assumption on Ψ and γ, namely that Ψ and γ are independent of the x-variable. This hypothesis ensures that the densities H_p and h_p are independent of x, and that $I^p_\star$ satisfies the following invariance property: for every $(g,G)\in\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $x\in\mathbb{R}^d$, it holds

$$I^p_\star(g(\cdot-x),G(\cdot-x),x+B)=I^p_\star(g,G,B) \tag{3.1.25}$$

for every $B\in\mathcal{B}(\Omega)$ with $x+B\subset\Omega$. This property makes it possible to rely on the integral representation result of Bouchitté, Fonseca, & Mascarenhas presented in Theorem 1.5.1.

Theorem 3.1.16 (Relaxation III)**.** *Let $p > 1$ and let Ψ and γ be x-independent integrands satisfying assumptions* ($\Psi 1$)–($\Psi 5$) *and* ($\gamma 1$)–($\gamma 6$)*. Then, for every $(g, G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ we have*

$$I^p_\star(g, G, B) = \int_B H_p(\nabla g, G)\,\mathrm{d}x + \int_B H^\infty_p\Big(\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}, 0\Big)\,\mathrm{d}|D^c g| + \int_{J_g\cap B} h_p([g], \nu_g)\,\mathrm{d}\mathcal{H}^{d-1},$$

where H_p and h_p are the functions defined by (3.1.23) *and* (3.1.24) *without x-dependence, and $H^\infty_p(\cdot, 0)$ denotes the recession function of $H_p(\cdot, 0)$, i.e., the function defined for all $A \in \mathbb{R}^{k\times d}$ by*

$$H^\infty_p(A, 0) \coloneqq \limsup_{t\to+\infty} \frac{H_p(tA, 0)}{t}. \tag{3.1.26}$$

Remark 3.1.17 (Compactness)**.** Taking into account some minor modifications due to the presence of the matrix variable G, these compactness result 2.A.1 readily adapts to the functional $I^p_\star$, thanks to Proposition 3.1.15 and Theorem 3.1.16. Thanks to the lower semicontinuity of $I^p_\star$, this guarantees that $I^p_\star$ admits minimisers for minimisation problems with Dirichlet boundary conditions.

3.2 Approximation Theorem in $\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$

This section is devoted to the proof of Theorem 3.1.1. The approach we follow is similar to the one in [171, Theorem 1.2] (see also [76, Theorem 2.12]), but the proof in our case is technically more demanding. To explain why, we recall that the main ingredients of the proof of [171, Theorem 1.2] are essentially three: (1) Anzellotti-Giaquinta's Theorem [21, Theorem 1], (2) [76, Lemma 2.9], which states that every $u \in C^\infty_c(\mathbb{R}^d)$ can be approximated by means of piecewise constant functions whose total variation is controlled by the L^1 norm of ∇u, and (3) Alberti's Theorem (see Theorem 3.2.5).

Proof of the Approximation Theorem. To proceed with this strategy, we need to prove suitable adaptations of the first two results in our functional setting, calling for nontrivial new arguments. We first give the statements of all these results and show how they allow to obtain Theorem 3.1.1, postponing their proofs to the end of the section. The first result, whose proof is based on a classical result of Goffman & Serrin [135, Theorem 4′], shows that, if $O \in \mathcal{O}(\Omega)$ is a Lipschitz set, every $u \in \mathrm{BV}(O;\mathbb{R}^k)$ can be approximated by a sequence of $\mathrm{SBV}(O;\mathbb{R}^k)$-functions $\{u_n\}_n$ in such a way that $\mathcal{V}(u_n, O)$ converges to $\mathcal{V}(u, O)$. The core arguments in the proof of this result were devised by the author together with Dal Maso, as part of a different, unpublished, research project.

Theorem 3.2.1 (Density result in BV)**.** *Let $O', O \in \mathcal{O}(\Omega)$, with $O' \subset\subset O$, and let $u \in \mathrm{BV}(O;\mathbb{R}^k)$. Assume that $\mathcal{H}^{d-1}(J_u \cap \partial O') = 0$. Then there exists a sequence $\{u_n\}_n \subset \mathrm{SBV}(O';\mathbb{R}^k) \cap L^\infty(O';\mathbb{R}^k)$ such that $u_n \to u$ strongly in $L^1(O';\mathbb{R}^k)$ as $n \to +\infty$ and*

$$\limsup_{n\to+\infty} \mathcal{V}(u_n, O') \le \mathcal{V}(u, O). \tag{3.2.1}$$

Moreover, if O has Lipschitz boundary, then we can find another sequence, still denoted by $\{u_n\}_n$, with $\{u_n\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$, such that $u_n \to u$ strongly in $L^1(O;\mathbb{R}^k)$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \mathcal{V}(u_n, O) = \mathcal{V}(u, O). \tag{3.2.2}$$

Remark 3.2.2. (i) Replacing L^1-convergence by L^0-convergence, the statement also holds for $\mathrm{GBV}_\star$-functions by performing an additional truncation procedure, as explained at the beginning of the proof of Theorem 3.1.1 below.

(ii) The theorem holds in a slightly weaker form also for more general energies than $\mathcal{V}$. More precisely, the arguments in the proof can be adapted with no difficulty to energies of the form

$$F(u,O) \coloneqq \int_O f(\nabla u)\,\mathrm{d}x + \int_O f^\infty\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap O} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

with $f\colon \mathbb{R}^{k\times d}\to[0,+\infty)$ convex, $g\colon \mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$ with $\zeta\mapsto g(\zeta,\nu)$ continuous for all $\nu\in\mathbb{S}^{d-1}$, and

$$f(A)\le C(|A|+1) \text{ for all } A\in\mathbb{R}^{k\times d} \quad \text{and} \quad g(\zeta,\nu)\le C|\zeta|\wedge 1 \text{ for all } \zeta\in\mathbb{R}^k \text{ and } \nu\in\mathbb{S}^{d-1}$$

for some constant $C>0$, where the recession function f^∞ is defined as in (3.1.26) with f in place of H_p. Indeed, this is the setting where [135, Theorem 4′] can still be employed. In this case, (3.2.2) has to be replaced by

$$\limsup_{n\to+\infty} F(u_n,O)\le F(u,O),$$

as, in general, F could fail to be $L^1(O;\mathbb{R}^k)$-lower semicontinuous.

(iii) A careful inspection of the proof of Theorem 3.2.1 reveals that the approximating sequence $\{u_n\}_n$ actually satisfies a property stronger than (3.2.2). More precisely, we have

$$\lim_{n\to+\infty}\mathcal{V}(u_n,O\setminus J_{u_n})=\mathcal{V}(u,O\setminus J_u) \quad \text{and} \quad \lim_{n\to+\infty}\mathcal{V}(u_n,J_{u_n})=\mathcal{V}(u,J_u).$$

The same observation applies to the functionals F of (ii), with the caveat that equalities need to be replaced by inequalities if F is not lower semicontinuous.

The general idea is to approximate the sequence obtained in Theorem 3.2.1 by a sequence of piecewise constant functions. The existing results in the literature, see e.g. [76, Lemma 2.9], are not expedient for us as they guarantee convergence of the total variation but not of the functional $\mathcal{V}$, as required in our setting.

Lemma 3.2.3. *Let $O\in\mathcal{O}(\Omega)$ and $u\in\mathrm{BV}(O;\mathbb{R}^k)$. Then there exists a sequence of piecewise constant functions $\{u_n\}_n\subset\mathrm{SBV}(O;\mathbb{R}^k)\cap L^\infty(O;\mathbb{R}^k)$ such that $u_n\to u$ strongly in $L^1(O;\mathbb{R}^k)$ as $n\to+\infty$ and*

$$\lim_{n\to+\infty}\mathcal{V}(u_n,O)=\mathcal{V}(u,O).$$

Remark 3.2.4. Replacing L^1-convergence by L^0-convergence, the statement also holds for $\mathrm{GBV}_\star$-functions by performing an additional truncation procedure, as explained at the beginning of the proof of Theorem 3.1.1 below.

Before proceeding with the proof of Theorem 3.1.1, we recall for the reader's convenience the statement of Alberti's Theorem [3, Theorem 3].

Theorem 3.2.5 (Alberti's Theorem)**.** *There exists a constant $C=C(d)>0$ with the property that for every $O\in\mathcal{O}(\Omega)$ and $G\in L^1(O;\mathbb{R}^{k\times d})$ there exists a function $u\in\mathrm{SBV}(O;\mathbb{R}^k)$ such that $\nabla u=G$ $\mathcal{L}^d$-a.e. in O and*

$$\int_{J_u}|[u]|\,\mathrm{d}\mathcal{H}^{d-1}\le C\|G\|_{L^1(O)}.$$

We are ready to give the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Let us fix $(g,G) \in \mathrm{SD}^p_\star(O;\mathbb{R}^k)$. By (c) of Definition 1.3.16, for every $n \in \mathbb{N}$ the function $g^n := \Phi_n \circ g$ (see (1.3.12)) belongs to $\mathrm{BV}(O;\mathbb{R}^k)$ and

$$\sup_{n\in\mathbb{N}} \mathcal{V}(g^n,O) = \mathcal{V}(g,O), \quad \text{and} \quad \{g^n\}_n \text{ converges to } g \text{ in } L^0(O;\mathbb{R}^k) \text{ as } n \to +\infty.$$

We apply Theorem 3.2.1 to g^n and obtain a sequence $\{u_{\ell,n}\}_\ell \subset \mathrm{SBV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$ converging to g^n strongly in $L^1(O;\mathbb{R}^k)$ as $\ell \to +\infty$ and such that

$$\lim_{\ell\to+\infty} \mathcal{V}(u_{\ell,n},O) = \mathcal{V}(g^n,O) \le \mathcal{V}(g,O). \tag{3.2.3}$$

We can now use Alberti's Theorem 3.2.5 to obtain a function $h \in \mathrm{SBV}(O;\mathbb{R}^k)$ with $\nabla h = G$ $\mathcal{L}^d$-a.e. in O and

$$|Dh|(O) \le C\|G\|_{L^1(O)} \tag{3.2.4}$$

for a suitable constant C depending only on d. Applying Lemma 3.2.3 to $u_{\ell,n} - h$, we find a sequence of piecewise constant functions $\{z^j_{\ell,n}\}_j \subset \mathrm{SBV}(O;\mathbb{R}^k) \cap L^\infty(O;\mathbb{R}^k)$ which converges to $u_{\ell,n} - h$ strongly in $L^1(O;\mathbb{R}^k)$ as $j \to +\infty$ and with

$$\lim_{j\to+\infty} \mathcal{V}(z^j_{\ell,n},O) = \mathcal{V}(u_{\ell,n}-h,O) \le C\big(\mathcal{V}(u_{\ell,n},O) + |Dh|(O)\big) \tag{3.2.5}$$

for a possibly larger constant C. Next, we set $u^j_{\ell,n} := h + z^j_{\ell,n} \in \mathrm{SBV}(O;\mathbb{R}^k)$ and observe that by construction $\nabla u^j_{\ell,n} = G$ $\mathcal{L}^d$-a.e. in O and that $\{u^j_{\ell,n}\}_j$ converges to $u_{\ell,n}$ strongly in $L^1(O;\mathbb{R}^k)$ as $j \to +\infty$. Moreover, by (3.2.3)–(3.2.5) we have, for all $n \in \mathbb{N}$,

$$\limsup_{\ell\to+\infty} \lim_{j\to+\infty} \mathcal{V}(u^j_{\ell,n},O) \le C\big(\mathcal{V}(g,O) + \|G\|_{L^1(O)}\big).$$

Since the L^0-convergence is metrisable, we may use a diagonal argument to obtain sequences $\{\ell_n\}_n$ and $\{j_n\}_n$, with $\ell_n \to +\infty$ and $j_n \to +\infty$ as $n \to +\infty$, so that, setting $u_n := u^{j_n}_{\ell_n,n}$, we have $u_n \to g$ in $L^0(O;\mathbb{R}^k)$ and

$$\limsup_{n\to+\infty} \mathcal{V}(u_n,O) \le C\Big(\mathcal{V}(g,O) + \|G\|_{L^1(O)}\Big).$$

This concludes the proof. □

Proof of the density results. In the rest of the section, we prove Theorem 3.2.1 and Lemma 3.2.3.

Proof of Theorem 3.2.1. Thanks to a truncation argument based on (c) of Definition 1.3.16, we may assume that $u \in L^\infty(O;\mathbb{R}^k)$. Moreover, it is not restrictive to assume $k = 1$ because we can argue componentwise. We begin by proving that there exists $\{u_n\}_n \subset \mathrm{SBV}(O') \cap L^\infty(O')$ converging to u strongly in $L^1(O')$ such that (3.2.1) holds.

Since $J_u \cap O'$ is $(\mathcal{H}^{d-1}, d-1)$-rectifiable, there exist a sequence $\{K_\ell\}_\ell$ of pairwise disjoint compact sets of $J_u \cap O'$, a sequence $\{M_\ell\}_\ell$ of $(d-1)$-dimensional orientable C^1-manifolds, with $K_\ell \subset\subset M_\ell \subset\subset O'$ for every $\ell \in \mathbb{N}$, and a Borel set $N \subset O'$, with $\mathcal{H}^{d-1}(N) = 0$, such that $J_u \cap O' = N \cup \big(\bigcup_{\ell\in\mathbb{N}} K_\ell\big)$. Let ν_{M_ℓ} be a continuous unit normal to M_ℓ. It is not restrictive to assume that for every $\ell \in \mathbb{N}$ there exist $\sigma_\ell > 0$ and $\nu_\ell \in \mathbb{S}^{d-1}$, with $\nu_\ell \cdot \nu_{M_\ell} > 0$ on M_ℓ, such that $y + t\nu_\ell \in O' \setminus M_\ell$ for every $y \in M_\ell$ and for every $t \in (-\sigma_\ell, 0) \cup (0, \sigma_\ell)$. Moreover, we may assume that the sets M_ℓ are pairwise disjoint.

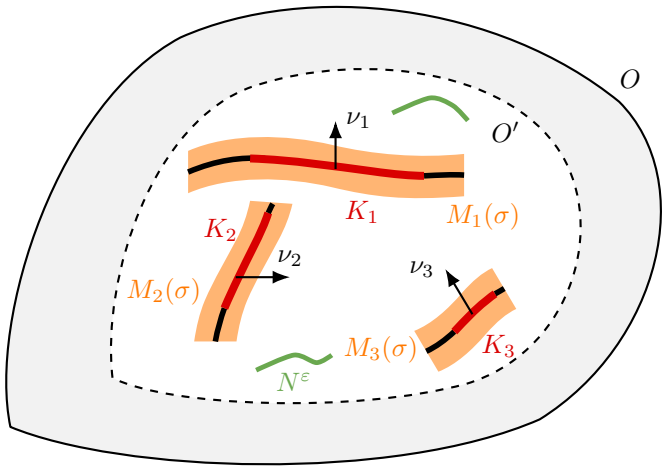


Figure 3.1: The sets $M^\varepsilon(\sigma)$ and N^ε for $\ell_\varepsilon = 3$.

Step 1. (Approximation away from the jump set) Let us fix $\varepsilon > 0$. There exists $\ell_\varepsilon \in \mathbb{N}$ such that, setting $K^\varepsilon := \bigcup_{\ell=1}^{\ell_\varepsilon} K_\ell$ and $N^\varepsilon := \big(\bigcup_{\ell>\ell_\varepsilon} K_\ell\big) \cup N$, we have

$$J_u \cap O' = K^\varepsilon \cup N^\varepsilon \quad \text{and} \quad \int_{N^\varepsilon} |[u]| \, \mathrm{d}\mathcal{H}^{d-1} \le \varepsilon. \tag{3.2.6}$$

Moreover, we may assume that there exists $\eta^\varepsilon > 0$ such that $\operatorname{dist}(M_j, M_\ell) > \eta^\varepsilon$ for every $j, \ell \in \{1, \dots, \ell_\varepsilon\}$ with $j \neq \ell$ and that $\sigma_\ell \le \frac{\eta^\varepsilon}{2}$ for every $\ell \in \{1, \dots, \ell_\varepsilon\}$. For technical reasons, it will be convenient to introduce $(d-1)$-dimensional manifolds M'_ℓ, M''_ℓ such that $K_\ell \subset\subset M''_\ell \subset\subset M'_\ell \subset\subset M_\ell$. For every $\sigma > 0$ we define

$$M_\ell(\sigma) := \big\{ y + t\nu_\ell \colon \ y \in M_\ell \ \text{ and } \ t \in (-\sigma, \sigma) \big\}$$

and give a similar definition for $M'_\ell(\sigma)$ and $M''_\ell(\sigma)$. We also set

$$M^\varepsilon(\sigma) := \bigcup_{\ell=1}^{\ell_\varepsilon} M'_\ell(\sigma) \tag{3.2.7}$$

Later we will send $\sigma \to 0$. For the moment, we choose a fixed σ with

$$0 < \sigma < \sigma^\varepsilon := \min\big\{ \sigma_\ell/4 \colon \ \ell \in \{1, \dots, \ell_\varepsilon\} \big\}, \tag{3.2.8}$$

and observe that with this choice we have $M_\ell(4\sigma) \subset\subset O'$ for every $\ell \in \{1, \dots, \ell_\varepsilon\}$ and $M_j(4\sigma) \cap M_\ell(4\sigma) = \emptyset$ for every $j, \ell \in \{1, \dots, \ell_\varepsilon\}$ with $j \neq \ell$. Moreover, we have that $\tau_\ell := \operatorname{dist}(K_\ell, O' \setminus M'_\ell(\sigma)) > 0$. We refer to Figures 3.1 and 3.2 for an illustration.

We consider a sequence of convolution kernels $\{\rho_n\}_n \subset C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\rho_n) \subset B(0, \frac{1}{n})$ and define $v_n := u * \rho_n$. Observe that, if $O'' \in \mathcal{O}(O)$ is such that $O' \subset\subset O'' \subset\subset O$, $x \in O' \setminus M^\varepsilon(\sigma)$, and if $\frac{1}{n} \le \min\big\{\operatorname{dist}(O', \mathbb{R}^d \setminus O''), \min\{\tau_\ell \colon \ \ell \in \{1, \dots, \ell_\varepsilon\}\}\big\}$, then $B(x, \frac{1}{n}) \subset O'' \setminus K^\varepsilon$. Since $v_n := u * \rho_n$, using a result of Goffman & Serrin [135, Theorem 4′] and (3.2.6)

we may estimate

$$\begin{aligned}\limsup_{n\to+\infty}\mathcal{V}(v_n,O'\setminus M^\varepsilon(\sigma)) &= \limsup_{n\to+\infty}\int_{O'\setminus M^\varepsilon(\sigma)}|\nabla v_n|\,\mathrm{d}x \le \int_{O''}|\nabla u|\,\mathrm{d}x + |D^s u|(O''\setminus K^\varepsilon)\\ &= \mathcal{V}(u,O''\setminus J_u) + \int_{N^\varepsilon\cup(J_u\cap(O''\setminus O'))}|[u]|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\le \mathcal{V}(u,O\setminus J_u) + \int_{J_u\cap(O''\setminus O')}|[u]|\,\mathrm{d}\mathcal{H}^{d-1}+\varepsilon.\end{aligned}$$

As O'' was arbitrary and we assumed that $\mathcal{H}^{d-1}(J_u\cap\partial O')=0$, we may let $O''\searrow O'$ and deduce that

$$\limsup_{n\to+\infty}\mathcal{V}(v_n,O'\setminus M^\varepsilon(\sigma)) \le \mathcal{V}(u,O\setminus J_u)+\varepsilon. \tag{3.2.9}$$

Step 2. (Construction at the jump set) We introduce the function $\phi^\sigma:[-3\sigma,3\sigma]\to[-3\sigma,-\sigma]\cup[\sigma,3\sigma]$ defined by

$$\phi^\sigma(t) \coloneqq \begin{cases}\frac{t}{2}-\frac{3}{2}\sigma & \text{if } -3\sigma\le t\le-\sigma,\\ t-\sigma & \text{if } -\sigma\le t\le 0,\\ t+\sigma & \text{if } 0<t\le\sigma,\\ \frac{t}{2}+\frac{3}{2}\sigma & \text{if } \sigma\le t\le 3\sigma,\end{cases}$$

and consider the function defined on $M'_\ell(3\sigma)$ by

$$v_n^{\sigma,\ell}(y+t\nu_\ell)\coloneqq v_n(y+\phi^\sigma(t)\nu_\ell)\quad \text{for every } y\in M'_\ell \text{ and } t\in[-3\sigma,3\sigma].$$

We observe that $v_n^{\sigma,\ell}=v_n$ on the top and bottom parts of the boundary (see Figure 3.2) of $M'_\ell(3\sigma)$, given by $M'_\ell\pm 3\sigma\nu_\ell$. To deal with the lateral part of the boundary, namely $\partial M'_\ell+[-3\sigma,3\sigma]\nu_\ell$, we consider a further cut-off function $\gamma_\ell\in C^1_c(M'_\ell;[0,1])$ with $\gamma_\ell=1$ in the neighbourhood M''_ℓ (recall that $K_\ell\subset\subset M''_\ell\subset\subset M'_\ell$). We set

$$u_n^{\sigma,\ell}(y+t\nu_\ell)\coloneqq\gamma_\ell(y)v_n^{\sigma,\ell}(y+t\nu_\ell)+(1-\gamma_\ell(y))v_n(y+t\nu_\ell)\quad \text{for all } y\in M'_\ell \text{ and } t\in[-3\sigma,3\sigma]. \tag{3.2.10}$$

Since $u_n^{\sigma,\ell}=v_n$ on the full boundary of $M'_\ell(3\sigma)$, we can define a function $u_n^{\sigma,\varepsilon}$ on O' by setting

$$u_n^{\sigma,\varepsilon}\coloneqq\begin{cases}u_n^{\sigma,\ell} & \text{on } M'_\ell(3\sigma) \text{ for } \ell\in\{1,\dots,\ell_\varepsilon\},\\ v_n & \text{on } O'\setminus M^\varepsilon(3\sigma).\end{cases} \tag{3.2.11}$$

It follows directly from the definition that $u_n^{\sigma,\varepsilon}\in \mathrm{SBV}(O')\cap L^\infty(O')$, that $u_n^{\sigma,\varepsilon}$ is a smooth Lipschitz function on $O'\setminus\big(\bigcup_{\ell=1}^{\ell_\varepsilon}M'_\ell\big)$, and that for every $n\in\mathbb{N}$ we have

$$J_{u_n^{\sigma,\varepsilon}}\subset\bigcup_{\ell=1}^{\ell_\varepsilon}M'_\ell\subset\subset O', \tag{3.2.12}$$

$$\begin{aligned}\nu_{u_n^{\sigma,\varepsilon}}=\nu_{M_\ell}\quad \mathcal{H}^{d-1}\text{-a.e. on } M'_\ell\cap J_{u_n^{\sigma,\varepsilon}},\\ [u_n^{\sigma,\varepsilon}]=\big(v_n(\cdot+\sigma\nu_\ell)-v_n(\cdot-\sigma\nu_\ell)\big)\gamma_\ell\quad \text{on } M'_\ell\cap J_{u_n^{\sigma,\varepsilon}},\end{aligned} \tag{3.2.13}$$

where we have used $M_\ell\subset O'$, the inequality $\nu_\ell\cdot\nu_{M_\ell}>0$, and the definitions of ϕ^σ and $v_n^{\sigma,\ell}$, see Figure 3.2.

Step 3. (L^1-convergence) We estimate the L^1-distance of $u_n^{\sigma,\varepsilon}$ from u on O'. We have

$$\begin{aligned}\int_{O'} |u_n^{\sigma,\varepsilon} - u|\,\mathrm{d}x &\le \int_{O'} |u_n^{\sigma,\varepsilon} - v_n|\,\mathrm{d}x \ + \int_{O'} |v_n - u|\,\mathrm{d}x \\ &= \sum_{\ell=1}^{\ell_\varepsilon} \int_{M'_\ell(3\sigma)} |u_n^{\sigma,\varepsilon} - v_n|\,\mathrm{d}x + \int_{O'} |v_n - u|\,\mathrm{d}x. \end{aligned} \tag{3.2.14}$$

Recalling the definition of $u_n^{\sigma,\ell}$, we observe that

$$\int_{M'_\ell(3\sigma)} |u_n^{\sigma,\varepsilon} - v_n|\,\mathrm{d}x \le \int_{M'_\ell(3\sigma)} \big(|v_n^{\sigma,\ell}| + |v_n|\big)\,\mathrm{d}x.$$

After a change of variables we obtain a constant $C_\ell \ge 1$ such that

$$\int_{M'_\ell(3\sigma)} |v_n^{\sigma,\ell}|\,\mathrm{d}x \le C_\ell \int_{M'_\ell(3\sigma)} |v_n|\,\mathrm{d}x, \tag{3.2.15}$$

hence,

$$\begin{aligned}\int_{M'_\ell(3\sigma)} |u_n^{\sigma,\varepsilon} - v_n|\,\mathrm{d}x &\le 2C_\ell \int_{M'_\ell(3\sigma)} |v_n|\,\mathrm{d}x \\ &\le 2C_\ell \int_{M'_\ell(3\sigma)} |v_n - u|\,\mathrm{d}x + 2C_\ell \int_{M'_\ell(3\sigma)} |u|\,\mathrm{d}x. \end{aligned} \tag{3.2.16}$$

Since the sets $M'_\ell(3\sigma)$ are pairwise disjoint and contained in O' and $\{v_n\}_n$ converges to u strongly in $L^1(O')$, it holds

$$\lim_{\sigma\to 0^+} \sum_{\ell=1}^{\ell_\varepsilon} \omega_\ell(\sigma) = 0, \quad \text{where} \quad \omega_\ell(\sigma) \coloneqq \lim_{n\to+\infty} \int_{M'_\ell(3\sigma)} |v_n - u|\,\mathrm{d}x + \int_{M'_\ell(3\sigma)} |u|\,\mathrm{d}x. \tag{3.2.17}$$

Finally, combining (3.2.14)–(3.2.17) and using again that $\{v_n\}_n$ converges to u strongly in $L^1(O')$, we obtain, for every $0 < \sigma < \sigma^\varepsilon$,

$$\limsup_{n\to+\infty} \int_{O'} |u_n^{\sigma,\varepsilon} - u|\,\mathrm{d}x \le 2 \sum_{\ell=1}^{\ell_\varepsilon} C_\ell \omega_\ell(\sigma). \tag{3.2.18}$$

Step 4. (Estimate on $\mathcal{V}$) We estimate $\mathcal{V}(u_n^{\sigma,\varepsilon}, O')$, given by

$$\mathcal{V}(u_n^{\sigma,\varepsilon}, O') = \int_{O'} |\nabla u_n^{\sigma,\varepsilon}|\,\mathrm{d}x + \int_{O' \cap J_{u_n^{\sigma,\varepsilon}}} |[u_n^{\sigma,\varepsilon}]| \wedge 1\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.2.19}$$

As for the first term, we have by (3.2.9) and (3.2.11)

$$\begin{aligned}\limsup_{n\to+\infty} \int_{O'} |\nabla u_n^{\sigma,\varepsilon}|\,\mathrm{d}x &= \limsup_{n\to+\infty} \Big(\int_{O'\setminus M'(3\sigma)} |\nabla v_n|\,\mathrm{d}x + \sum_{\ell=1}^{\ell_\varepsilon} \int_{M'_\ell(3\sigma)} |\nabla u_n^{\sigma,\varepsilon}|\,\mathrm{d}x \Big) \\ &\le \mathcal{V}(u, O \setminus J_u) + \varepsilon + \limsup_{n\to+\infty} \Big(\sum_{\ell=1}^{\ell_\varepsilon} \int_{M'_\ell(3\sigma)} |\nabla u_n^{\sigma,\varepsilon}|\,\mathrm{d}x \Big). \end{aligned} \tag{3.2.20}$$

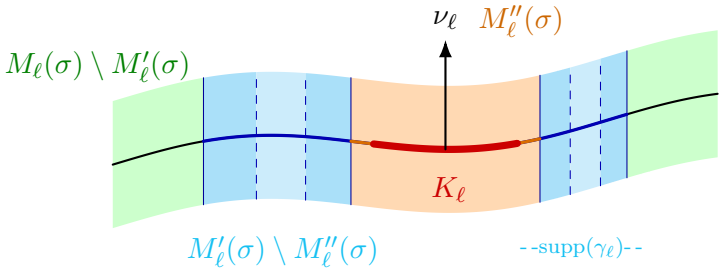


Figure 3.2: Subdivision of $M_\ell(\sigma)$

To estimate the last term in the previous inequality, we note that by (3.2.10) and (3.2.11) we have

$$\nabla u_n^{\sigma,\varepsilon} = \gamma_\ell \nabla v_n^{\sigma,\ell} + (1-\gamma_\ell)\nabla v_n + \nabla\gamma_\ell\big(v_n^{\sigma,\ell} - v_n\big) \quad \mathcal{L}^d\text{-a.e. in } M'_\ell(3\sigma),$$

so that

$$\begin{aligned}\int_{M'_\ell(3\sigma)} |\nabla u_n^{\sigma,\varepsilon}|\,\mathrm{d}x \le \int_{M'_\ell(3\sigma)} |\nabla v_n^{\sigma,\ell}|\,\mathrm{d}x + \int_{M'_\ell(3\sigma)\setminus M''_\ell(3\sigma)} |\nabla v_n|\,\mathrm{d}x \\ + \|\nabla\gamma_\ell\|_{L^\infty(M'_\ell(3\sigma))} \int_{M'_\ell(3\sigma)} |v_n^{\sigma,\ell} - v_n|\,\mathrm{d}x \end{aligned} \tag{3.2.21}$$

for all $\ell \in \{1,\dots,\ell_\varepsilon\}$. We estimate each term separately. For the first term, a change of variables shows that for every $\ell \in \{1,\dots,\ell_\varepsilon\}$ there exists a constant $C_\ell > 0$ such that

$$\int_{M'_\ell(3\sigma)} |\nabla v_n^{\sigma,\ell}|\,\mathrm{d}x \le C_\ell \int_{M'_\ell(3\sigma)\setminus M'_\ell(\sigma)} |\nabla v_n|\,\mathrm{d}x. \tag{3.2.22}$$

We observe that $M'_\ell(3\sigma) \setminus M'_\ell(\sigma) \subset\subset M_\ell(4\sigma) \setminus M_\ell$. Thus, we can apply once again [135, Theorem 4′], and we obtain

$$\limsup_{n\to+\infty} \int_{M'_\ell(3\sigma)\setminus M'_\ell(\sigma)} |\nabla v_n|\,\mathrm{d}x \le \int_{M_\ell(4\sigma)\setminus M_\ell} |\nabla u|\,\mathrm{d}x + |D^s u|(M_\ell(4\sigma)\setminus M_\ell) =: \lambda_\ell(\sigma) \tag{3.2.23}$$

for every $\ell \in \{1,\dots,\ell_\varepsilon\}$. Note that $\lambda_\ell(\sigma) \to 0^+$ as $\sigma \to 0^+$ because $M_\ell(4\sigma) \setminus M_\ell \searrow \emptyset$ as $\sigma \to 0^+$. Similarly, we can use [135, Theorem 4′] to estimate the second term in (3.2.21) as

$$\limsup_{n\to+\infty} \int_{M'_\ell(3\sigma)\setminus M''_\ell(3\sigma)} |\nabla v_n|\,\mathrm{d}x \le \int_{M_\ell(4\sigma)\setminus K_\ell} |\nabla u|\,\mathrm{d}x + |D^s u|\big(M_\ell(4\sigma)\setminus K_\ell\big) =: \mu_\ell(\sigma) \tag{3.2.24}$$

for every $\ell \in \{1,\dots,\ell_\varepsilon\}$. We observe that, since $M_\ell(4\sigma) \setminus K_\ell \searrow M_\ell \setminus K_\ell$, it holds

$$\limsup_{\sigma\to 0^+} \mu_\ell(\sigma) \le \int_{M_\ell\setminus K_\ell} |[u]|\,\mathrm{d}\mathcal{H}^{d-1} = 0, \tag{3.2.25}$$

where we used that $\mathcal{H}^{d-1}(J_u \cap (M_\ell \setminus K_\ell)) = 0$. Finally, for the last term in (3.2.21), we use (3.2.15), the second inequality in (3.2.16), and (3.2.17) to get

$$\limsup_{n\to+\infty} \|\nabla\gamma_\ell\|_{L^\infty(M'_\ell(3\sigma))} \int_{M'_\ell(3\sigma)} |v_n^{\sigma,\ell} - v_n|\,\mathrm{d}x \le C_\ell \omega_\ell(\sigma) \tag{3.2.26}$$

for a constant $C_\ell > 0$ depending only on ℓ. Combining the previous inequality with (3.2.21)–(3.2.24), we obtain, for all $\ell \in \{1, \dots, \ell_\varepsilon\}$,

$$\limsup_{n\to+\infty} \int_{M'_\ell(3\sigma)} |\nabla u_n^{\sigma,\varepsilon}| \,\mathrm{d}x \le C_\ell \lambda_\ell(\sigma) + \mu_\ell(\sigma) + C_\ell \omega_\ell(\sigma), \tag{3.2.27}$$

where $C_\ell > 0$ is the maximum of the constants appearing in (3.2.22) and (3.2.26).

We now estimate the second term on the right-hand side of (3.2.19). To this end, we observe that by (3.2.12) and (3.2.13) we have

$$\int_{J_{u_n^{\sigma,\varepsilon}} \cap M'_\ell} |[u_n^{\sigma,\varepsilon}]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} = \int_{M'_\ell} |(v_n(\cdot + \sigma\nu_\ell) - v_n(\cdot - \sigma\nu_\ell))\gamma_\ell| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.2.28}$$

Let $\partial_\ell^{\pm\sigma} := M_\ell \pm \sigma\nu_\ell$. As $|[u]|\mathcal{H}^{d-1} \llcorner J_u$ is a finite measure and the sets $\partial_\ell^{\pm\sigma}$ are pairwise disjoint, there exists a set $S \subset (0, +\infty)$, with $(0, +\infty) \setminus S$ at most countable, such that $\mathcal{H}^{d-1}(\partial_\ell^{\pm\sigma} \cap J_u) = 0$ for every $\sigma \in S$. By well-known properties of convolutions of BV-functions, this implies that for every $\sigma \in S$ the trace of v_n on $\partial_\ell^{\pm\sigma}$ converges in $L^1_{\mathcal{H}^{d-1}}(\partial_\ell^{\pm\sigma})$ to the trace of u on $\partial_\ell^{\pm\sigma}$, which is well-defined because the traces on both sides coincide. By the Dominated Convergence Theorem, this implies

$$\begin{aligned}\limsup_{n\to+\infty} \int_{M_\ell} |(v_n(\cdot + \sigma\nu_\ell) - v_n(\cdot - \sigma\nu_\ell))\gamma_\ell| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} \\ = \int_{M_\ell} |(u(\cdot + \sigma\nu_\ell) - u(\cdot - \sigma\nu_\ell))\gamma_\ell| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Since $(u(\cdot + \sigma\nu_\ell) - u(\cdot - \sigma\nu_\ell))\gamma_\ell \to [u]\gamma_\ell$ in $L^1_{\mathcal{H}^{d-1}}(M_\ell)$ as $\sigma \to 0^+$ (with $\sigma \in S$), using again the Dominated Convergence Theorem we deduce that

$$\lim_{\substack{\sigma\to 0^+ \\ \sigma\in S}} \int_{M_\ell} |(u(\cdot + \sigma\nu_\ell) - u(\cdot - \sigma\nu_\ell))\gamma_\ell| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} = \int_{M_\ell} |[u]\gamma_\ell| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1}.$$

Therefore, recalling (3.2.28), for every $\ell \in \{1, \dots, \ell_\varepsilon\}$ there exists a function $\eta_\ell \colon S \to [0, +\infty)$, with $\eta_\ell(\sigma) \to 0$ as $\sigma \to 0^+$, such that for every $\sigma \in S$ we have

$$\begin{aligned}\limsup_{n\to+\infty} \int_{M'_\ell} |[u_n^{\sigma,\varepsilon}]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} &\le \int_{K_\ell} |[u]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} + \int_{M_\ell \setminus K_\ell} |[u]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} + \eta_\ell(\sigma), \\ &= \int_{K_\ell} |[u]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} + \eta_\ell(\sigma),\end{aligned} \tag{3.2.29}$$

where in the equality we have used $\mathcal{H}^{d-1}(J_u \cap (M_\ell \setminus K_\ell)) = 0$.

Combining the last inequality with (3.2.19), (3.2.20), and (3.2.27), we obtain

$$\begin{aligned}\limsup_{n\to+\infty} \mathcal{V}(u_n^{\sigma,\varepsilon}, O') &\le \mathcal{V}(u, O \setminus J_u) + \sum_{\ell=1}^{\ell_\varepsilon} \int_{K_\ell} |[u]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} + \varepsilon \\ &\quad + \sum_{\ell=1}^{\ell_\varepsilon} \big(C_\ell \lambda_\ell(\sigma) + \mu_\ell(\sigma) + C_\ell \omega_\ell(\sigma) + \eta_\ell(\sigma)\big) \\ &\le \mathcal{V}(u, O) + \sum_{\ell=1}^{\ell_\varepsilon} \big(C_\ell \lambda_\ell(\sigma) + \mu_\ell(\sigma) + C_\ell \omega_\ell(\sigma) + \eta_\ell(\sigma)\big) + \varepsilon.\end{aligned} \tag{3.2.30}$$

Step 5. (Conclusion of the proof in O') We now fix two sequences of positive numbers $\{\varepsilon_j\}_j$ and $\{\sigma_j\}_j$ converging to 0 such that for every $j \in \mathbb{N}$ we have $0 < \sigma_j < \sigma^{\varepsilon_j}$ (see (3.2.8)) and

$$\sum_{\ell=1}^{\ell_{\varepsilon_j}} \big(C_\ell \lambda_\ell(\sigma_j) + \mu_\ell(\sigma_j) + 2C_\ell \omega_\ell(\sigma_j) + \eta_\ell(\sigma_j)\big) < \varepsilon_j$$

(see (3.2.17)–(3.2.18), (3.2.23), (3.2.25), and (3.2.29)). From this inequality and from (3.2.18) and (3.2.30), we obtain for every $j \in \mathbb{N}$ that

$$\limsup_{n\to+\infty} \int_{O'} |u_n^{\sigma_j,\varepsilon_j} - u| \,\mathrm{d}x < \varepsilon_j \quad \text{and} \quad \limsup_{n\to+\infty} \mathcal{V}(u_n^{\sigma_j,\varepsilon_j}, O') < \mathcal{V}(u, O) + 2\varepsilon_j.$$

Therefore, there exists $n_j \in \mathbb{N}$ such that

$$\int_{O'} |u_n^{\sigma_j,\varepsilon_j} - u| \,\mathrm{d}x \le \varepsilon_j \quad \text{and} \quad \mathcal{V}(u_n^{\sigma_j,\varepsilon_j}, O') \le \mathcal{V}(u, O) + 2\varepsilon_j \tag{3.2.31}$$

for every $n \ge n_j$. It is not restrictive to assume that $n_j < n_{j+1}$ for every $j \in \mathbb{N}$. We set $u_n \coloneqq u_{n_j}^{\sigma_j,\varepsilon_j}$ for $n_j \le n < n_{j+1}$ and conclude observing that by construction $u_n \in \mathrm{SBV}(O') \cap L^\infty(O')$ and that from (3.2.31) it follows that $\{u_n\}_n$ converges strongly in $L^1(O')$ to u as $n \to +\infty$ and that (3.2.1) holds true.

Step 6. (O with Lipschitz boundary.) Assume now that O has Lipschitz boundary and let us prove that there exists $\{u_n\}_n \subset \mathrm{SBV}(O) \cap L^\infty(O)$ converging to u in $L^1(O)$ such that (3.2.2) holds. To this end, we let $\{\varepsilon_j\}_j$ be a sequence of positive numbers converging to 0 as $j \to +\infty$ and let $\{O_j\}_j$ be an increasing sequence of open sets with Lipschitz boundary, compactly contained in O, such that $O_j \nearrow O$ as $j \to +\infty$ and

$$|Du|\big(O \setminus \overline{O_j}\big) \le \varepsilon_j \quad \text{for all } j \in \mathbb{N}. \tag{3.2.32}$$

As J_u is σ-finite with respect to $\mathcal{H}^{d-1}$, we may choose the sequence $\{O_j\}_j$ in such a way that $\mathcal{H}^{d-1}(\partial O_j \cap J_u) = 0$ for all $j \in \mathbb{N}$. We also recall that (see the comment below (3.2.8))

$$\mathrm{dist}(M^{\varepsilon_j}(\sigma^{\varepsilon_j}), \partial O_j) > 0 \tag{3.2.33}$$

for every $j \in \mathbb{N}$, where $M^{\varepsilon_j}(\sigma)$ is given by (3.2.7).

For every $0 < \sigma < \sigma^{\varepsilon_j}$, we consider the functions $u_n^{\sigma,j} \coloneqq u_n^{\sigma,\varepsilon_j}$ constructed in Step 2 and given by (3.2.11). Note that (3.2.33) implies that in a neighbourhood of ∂O_j, independent of n, the function $u_n^{\sigma,j}$ is given by $v_n \coloneqq u * \rho_n$, where ρ_n is a smooth convolution kernel. Therefore, letting $(u_n^{\sigma,j})_{\partial O_j^-}$ (resp. $u_{\partial O_j^-}$) be the trace from the interior of $u_n^{\sigma,j}$ (resp. u) on ∂O_j, recalling that $\mathcal{H}^{d-1}(J_u \cap \partial O_j) = 0$, by the same properties of the convolutions of BV-functions invoked below (3.2.28) we get

$$\{(u_n^{\sigma,j})_{\partial O_j^-}\}_n \text{ converges to } u_{\partial O_j^-} \text{ strongly in } L^1_{\mathcal{H}^{d-1}}(\partial O_j) \text{ as } n \to +\infty. \tag{3.2.34}$$

Next, we apply Anzellotti-Giaquinta's Theorem [21, Theorem 1] on $O \setminus \overline{O_j}$ and obtain a sequence $\{w_n^j\}_n \subset C^\infty(O \setminus \overline{O_j}) \cap L^\infty(O \setminus \overline{O_j})$ (recall that we assumed $u \in L^\infty(O)$) such that

$$\{w_n^j\}_n \text{ converges to } u \text{ strongly in } L^1(O \setminus \overline{O_j}) \text{ as } n \to +\infty \tag{3.2.35}$$

and $|Dw_n^j|(O \setminus \overline{O_j})$ converges to $|Du|(O \setminus \overline{O_j})$ as $n \to +\infty$. Thus, by (3.2.32)

$$\limsup_{n\to+\infty} \mathcal{V}(w_n^j, O \setminus \overline{O_j}) \le \limsup_{n\to+\infty} |Dw_n^j|(O \setminus \overline{O_j}) \le \varepsilon_j \quad \text{for all } j \in \mathbb{N}. \tag{3.2.36}$$

Moreover, letting $(w_n^j)_{\partial O_j^+}$ (resp. $u_{\partial O_j^+}$) be the trace from the interior of w_n^j (resp. u) on ∂O_j^+, by the continuity of the trace operator under strict convergence, we obtain

$$\{(w_n^j)_{\partial O_j^+}\}_n \text{ converges to } u_{\partial O_j^+} \text{ strongly in } L^1_{\mathcal{H}^{d-1}}(\partial O_j) \text{ as } n \to +\infty. \tag{3.2.37}$$

We let $\{z_n^{\sigma,j}\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k)$ be the functions defined by

$$z_n^{\sigma,j} := \begin{cases} u_n^{\sigma,j} & \text{on } O_j, \\ w_n^j & \text{on } O \setminus \overline{O_j}, \end{cases}$$

and observe that, by (3.2.34) and (3.2.37), we have

$$\limsup_{n\to+\infty} \int_{J_{z_n^{\sigma,j}} \cap \partial O_j} |[z_n^{\sigma,j}]| \, \mathrm{d}\mathcal{H}^{d-1} = \limsup_{n\to+\infty} \int_{\partial O_j} |(w_n^j)_{\partial O_j^+} - (u_n^{\sigma,j})_{\partial O_j^-}| \, \mathrm{d}\mathcal{H}^{d-1} = 0 \tag{3.2.38}$$

for all $j \in \mathbb{N}$, where in the last equality we have used that $\mathcal{H}^{d-1}(J_u \cap \partial O_j) = 0$ for all $j \in \mathbb{N}$. Finally, combining (3.2.18), (3.2.30), (3.2.35), (3.2.36), and (3.2.38) the same diagonal argument employed in Step 5 allows to define a sequence $\{u_n\}_n \subset \mathrm{SBV}(O) \cap L^\infty(O)$ converging to u strongly in $L^1(O)$ such that

$$\limsup_{n\to+\infty} \mathcal{V}(u_n, O) \le \mathcal{V}(u, O).$$

In view of the lower semicontinuity of $\mathcal{V}$ (see Remark 1.3.15), this inequality implies (3.2.2), concluding the proof. □

Proof of Lemma 3.2.3. We first give the proof in the scalar case $k = 1$ and for $u \in \mathrm{BV}(O) \cap L^\infty(O)$. Without loss of generality, we may assume that $\mathcal{V}(u, O \setminus J_u^1) > 0$, as otherwise the function u is already piecewise constant, see [18, Theorem 4.23]. Moreover, we can assume that $u \ge 0$.

Step 1. (Scalar case $k = 1$) We let $M := \lceil \|u\|_{L^\infty(O)} \rceil < +\infty$, where for $a \in \mathbb{R}$ the symbol $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. For $t \in (-\infty, +\infty)$ we define $E_t := \{x \in O : u(x) > t\}$. By the coarea formula we have

$$\int_0^M |D\chi_{E_t}|(O \setminus J_u^1) \, \mathrm{d}t = \int_0^M \mathcal{H}^{d-1}\big(\partial^* E_t \cap (O \setminus J_u^1)\big) \, \mathrm{d}t = \mathcal{V}(u, O \setminus J_u^1). \tag{3.2.39}$$

Thanks to the well-known approximation properties of Lebesgue integrals by means of Riemann sums (see, for instance, Lemma 5.A.1 below) we can find a sequence $\{\delta_n\}_n$ converging to 0 as $n \to +\infty$ and $s \in [0,1]$ such that, setting $t_n^j := s + \frac{j}{n}$ for $j \in \mathcal{J}^n := \{j \in \mathbb{Z} : t_n^j \in (0, M)\}$, we have that $E_{t_n^j}$ is of finite perimeter in O and

$$\Big| \sum_{j \in \mathcal{J}^n} \frac{1}{n} |D\chi_{E_{t_n^j}}|(O \setminus J_u^1) - \int_0^M |D\chi_{E_t}|(O \setminus J_u^1) \, \mathrm{d}t \Big| \le \delta_n \tag{3.2.40}$$

for every $n \in \mathbb{N}$. We then define

$$u_n := \frac{1}{n} \sum_{j \in \mathcal{J}^n} \chi_{E^j_{t^j_n}} \in \mathrm{BV}(O) \cap L^\infty(O) \tag{3.2.41}$$

and observe that

$$|u_n(x) - u(x)| \leq \frac{1}{n} \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in O \text{ and for all } n \in \mathbb{N},$$

which implies strong $L^1(O)$-convergence of $\{u_n\}_n$ to u as $n \to +\infty$.

We want to estimate $\mathcal{V}(u_n, O)$ by means of $\mathcal{V}(u, O)$. We write

$$\mathcal{V}(u_n, O) = \mathcal{V}(u_n, J^1_u) + \mathcal{V}(u_n, O \setminus J^1_u). \tag{3.2.42}$$

The first term can be bounded from above as

$$\mathcal{V}(u_n, J^1_u) \leq \mathcal{H}^{d-1}(J^1_u) = \mathcal{V}(u, J^1_u). \tag{3.2.43}$$

As for the second term, we observe that by (3.2.41)

$$\mathcal{V}(u_n, O \setminus J^1_u) \leq \int_{(J_{u_n} \setminus J^1_u) \cap O} |[u_n]| \, \mathrm{d}\mathcal{H}^{d-1} = |Du_n|(O \setminus J^1_u) \leq \sum_{j \in \mathcal{J}^n} \frac{1}{n} |D\chi_{E^j_{t^j_n}}|(O \setminus J^1_u),$$

which by (3.2.40) and (3.2.39) implies

$$\limsup_{n \to +\infty} \mathcal{V}(u_n, O \setminus J^1_u) \leq \mathcal{V}(u, O \setminus J^1_u).$$

Combining the previous inequality with (3.2.42) and (3.2.43), we obtain

$$\limsup_{n \to +\infty} \mathcal{V}(u_n, O) \leq \mathcal{V}(u, O \setminus J^1_u) + \mathcal{V}(u, J^1_u) = \mathcal{V}(u, O).$$

Moreover, as $\{u_n\}_n$ converges to u strongly in $L^1(O)$, the lower semicontinuity of $\mathcal{V}(\cdot, O)$ with respect to the strong L^1-convergence (see Remark 1.3.15) implies $\lim_{n \to +\infty} \mathcal{V}(u_n, O) = \mathcal{V}(u, O)$. This concludes the proof in the case $u \in \mathrm{BV}(O) \cap L^\infty(O)$.

If $u \in \mathrm{BV}(O)$, for every $m \in \mathbb{N}$ we consider the truncation $u^{(m)} := (u \vee -m) \wedge m$ and use the previous step to obtain a sequence $\{u_{m,n}\}_n$ of piecewise constant functions converging to $u^{(m)}$ strongly in $L^1(O)$ as $n \to +\infty$ and such that

$$\lim_{n \to +\infty} \mathcal{V}(u_{m,n}, O) = \mathcal{V}(u^{(m)}, O).$$

Since by (b) of Definition 1.3.16 we have

$$\lim_{m \to +\infty} \mathcal{V}(u^{(m)}, O) = \mathcal{V}(u, O) \quad \text{and} \quad u^{(m)} \to u \text{ strongly in } L^1(O) \text{ as } m \to +\infty,$$

a diagonal argument allows us to conclude.

Step 2. (Vector-valued case $k > 1$) In the vector-valued case $k \geq 2$, we apply the result we have just proved in the scalar case to each component. For each $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$ this yields a finite collection of sets of finite perimeter $\{E^j_{n,i}\}_j$, and numbers $\{t^j_{n,i}\}_j$ such that $u_{n,i} := \sum_j t^j_{n,i} \chi_{E^j_{n,i}} \in \mathrm{BV}(O) \cap L^\infty(O)$ converges to u_i strongly in $L^1(O)$ as $n \to +\infty$ and

$$\lim_{n \to +\infty} \mathcal{V}(u_{n,i}, O) = \mathcal{V}(u_i, O). \tag{3.2.44}$$

For every $n \in \mathbb{N}$, we let $u_n \in \mathrm{BV}(O; \mathbb{R}^k) \cap L^\infty(O; \mathbb{R}^k)$ be the function whose i-th component is given by $u_{n,i}$, which is clearly piecewise constant. By (3.2.44) we then have

$$\lim_{n \to +\infty} \mathcal{V}(u_n, O) = \mathcal{V}(u, O).$$

As $\{u_{n,i}\}_n$ converges to u_i strongly in $L^1(O)$ for all $i \in \{1, \dots, k\}$, the proof is concluded. □

3.3 Integral representation of bulk and surface parts

This section is devoted to the proof of Theorem 3.1.7, that is, the integral representation of the bulk and surface parts of functionals in $\mathfrak{F}^p_\star$. The arguments we employ here are those of the Global Method of Bouchitté, Fonseca, & Mascarenhas [41] (see also [40]), later adapted to the setting of structured deformations in several different works, see for instance [32, 33, 120, 128]. Due to the different growth conditions and underlying function spaces considered here, we cannot always rely on the theory developed in the above mentioned works. Therefore, at several points, significant modifications are necessary and, in particular, Poincaré inequalities and blow-up limits tailor-made for our functional setting are required. These arguments are presented in the appendices, see Lemmas 3.A.1, 3.C.1, and 3.C.2, and Corollary 3.B.7.

The global method. For notational convenience, in this section we depart from the notation introduced in Section 3.1, and will use (u,U), (v,V), and (w,W) to denote structured deformations in place of the usual (g,G). We keep $\mathcal{F}\colon \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)\times\mathcal{B}(\Omega)\to[0,+\infty)$ fixed throughout the section, and we specify at each step the properties we assume on $\mathcal{F}$. We begin by proving that $\mathcal{F}$ satisfies a suitable version of the Fundamental Estimate, which will be used to alter boundary conditions for the displacement fields g in an energetically convenient way. In the following, given $x\in\mathbb{R}^d$ and $\rho>0$, we set

$$\Omega_{\rho,x} \coloneqq x+\rho(\Omega-x). \tag{3.3.1}$$

Lemma 3.3.1 (Fundamental Estimate)**.** *Let $\eta>0$ and let $O,O',O''\in\mathcal{O}(\Omega)$ be with Lipschitz boundary and with $O'\subset\subset O$. Assume that $\mathcal{F}$ satisfies conditions* (H1), (H2), *and* (H4)*. Then, for every $u\in\mathrm{GBV}_\star(O;\mathbb{R}^k)$ and $v\in\mathrm{GBV}_\star(O'';\mathbb{R}^k)$, with $u\in L^1(O\setminus O';\mathbb{R}^k)$ and $v\in L^1((O\setminus O')\cap O'';\mathbb{R}^k)$, there exists a function $\varphi\in C^\infty(\mathbb{R}^d;[0,1])$ such that, setting $w\coloneqq\varphi u+(1-\varphi)v$, we have $w\in\mathrm{GBV}_\star(O'\cup O'';\mathbb{R}^k)$,*

$$w=u \quad \mathcal{L}^d\text{-a.e. in } O' \qquad \text{and} \qquad w=v \quad \mathcal{L}^d\text{-a.e. in } O''\setminus O, \tag{3.3.2}$$

and

$$\begin{aligned}\mathcal{F}(w,W,O'\cup O'') \le{}& (1+\eta)\big(\mathcal{F}(u,W,O)+\mathcal{F}(v,W,O'')\big)\\ &+M\|u-v\|_{L^1((O\setminus O')\cap O'')}+\eta\mathcal{L}^d(O'\cup O'')\end{aligned} \tag{3.3.3}$$

for every $W\in L^p(O\cup O'';\mathbb{R}^{k\times d})$, where $M=M(O,O',O'',d,\alpha,\beta,\eta,k)$ is a positive constant independent of u and v. Moreover, if $\rho>0$ and $x\in\mathbb{R}^d$ are such that $O_{\rho,x},O'_{\rho,x},O''_{\rho,x}\subset\Omega$ (see (3.3.1))*, then*

$$M(O_{\rho,x},O'_{\rho,x},O''_{\rho,x},d,\alpha,\beta,\eta,k)=\frac{1}{\rho}M(O,O',O'',d,\alpha,\beta,\eta,k).$$

Proof. Let us fix $N\in\mathbb{N}$ and consider open subsets $O_1,\dots,O_{N+1}$ of $\mathbb{R}^d$ with

$$O'\subset\subset O_1\subset\subset\cdots\subset\subset O_{N+1}\subset\subset O.$$

For $i\in\{1,\dots,N\}$ we consider functions $\varphi^i\in C^\infty_c(O_{i+1};[0,1])$ with $\varphi^i=1$ in a neighbourhood of $\overline{O_i}$ and with $|\nabla\varphi^i|\le 2N/\delta$, where $\delta\coloneqq\operatorname{dist}(O',\mathbb{R}^d\setminus O)>0$. Thanks to Lemma 3.A.1,

for every $i \in \{1,\dots,N\}$ the function $w^i := \varphi^i u + (1-\varphi^i)v$ belongs to $\mathrm{GBV}_\star(O' \cup O''; \mathbb{R}^k)$, where we have extended u and v arbitrarily outside of O and O'', respectively.

We set $S_i := O'' \cap (O_{i+1} \setminus \overline{O_i})$ and fix $i \in \{1,\dots,N\}$. By the measure property (H1) and locality (H2) we see that

$$\mathcal{F}(w^i, W, O' \cup O'') \le \mathcal{F}(u, W, O) + \mathcal{F}(v, W, O'') + \mathcal{F}(w^i, W, S_i) \tag{3.3.4}$$

for every $W \in L^p(O \cup O''; \mathbb{R}^{k\times d})$. To estimate the term $\mathcal{F}(w^i, W, S_i)$, we exploit the upper bounds in (H4) to get

$$\begin{aligned}\mathcal{F}(w^i, W, S_i) \le \beta \sum_{j=1}^k \Big(\int_{S_i} |\nabla w^i_j| \,\mathrm{d}x + |D^c w^i_j|(S_i) + \int_{J_{w^i_j} \cap S_i} |[w^i_j]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} \Big) \\ + \beta \int_{S_i} (|W|^p + 1) \,\mathrm{d}x.\end{aligned} \tag{3.3.5}$$

By Lemma 3.A.1 we have

$$\begin{aligned}&\nabla w^i = \nabla\varphi^i \otimes (u-v) + \varphi^i \nabla u + (1-\varphi^i)\nabla v \quad \mathcal{L}^d\text{-a.e. in } S_i,\\ &D^c w^i = \varphi^i D^c u + (1-\varphi^i) D^c v \quad \text{as Borel measures on } S_i,\\ &[w^i] = \varphi^i [u] + (1-\varphi^i)[v] \quad \mathcal{H}^{d-1}\text{-a.e. on } S_i \cap J_{w^i},\end{aligned}$$

so that

$$\begin{aligned}&\int_{S_i} |\nabla w^i_j| \,\mathrm{d}x \le \int_{S_i} (|\nabla u_j| + |\nabla v_j|) \,\mathrm{d}x + \|\nabla\varphi^i\|_{L^\infty(\Omega)} \int_{S_i} |u-v| \,\mathrm{d}x,\\ &|D^c w^i_j|(S_i) \le |D^c u_j|(S_i) + |D^c v_j|(S_i),\end{aligned}$$

$$\int_{S_i \cap J_{w^i_j}} |[w^i_j]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} \le \int_{S_i \cap J_{u_j}} |[u_j]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} + \int_{S_i \cap J_{v_j}} |[v_j]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1}$$

for all $j \in \{1,\dots,k\}$. Combining the previous inequalities with (3.3.5), recalling that $|\nabla\varphi^i| \le 2N/\delta$, and using the lower bounds in (H4), we get

$$\begin{aligned}\mathcal{F}(w^i, W, S_i) &\le \beta \sum_{j=1}^k \Big(\int_{S_i} (|\nabla u_j| + |\nabla v_j|) \,\mathrm{d}x + \frac{2N}{\delta} \int_{S_i} |u-v| \,\mathrm{d}x + |D^c u_j|(S_i) + |D^c v_j|(S_i)\\ &\quad + \int_{S_i \cap J_{u_j}} |[u_j]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} + \int_{S_i \cap J_{v_j}} |[v_j]| \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1} \Big) + \beta \int_{S_i} (|W|^p + 1) \,\mathrm{d}x\\ &\le \frac{\beta}{\alpha} \Big(\mathcal{F}(u, W, S_i) + \mathcal{F}(v, W, S_i) \Big) + \frac{2\beta N k}{\delta} \int_{S_i} |u-v| \,\mathrm{d}x + \beta \mathcal{L}^d(S_i).\end{aligned}$$

We set $S := (O \setminus O') \cap O''$, sum the previous inequality over $i \in \{1,\dots,N\}$ and obtain

$$\sum_{i=1}^N \mathcal{F}(w^i, W, S_i) \le \frac{\beta}{\alpha} \Big(\mathcal{F}(u, W, S) + \mathcal{F}(v, W, S) \Big) + \frac{2\beta N k}{\delta} \int_S |u-v| \,\mathrm{d}x + \beta \mathcal{L}^d(S).$$

Therefore, there exists $i^* \in \{1,\dots,N\}$ such that

$$\mathcal{F}(w^{i^*}, W, S_{i^*}) \le \frac{\beta}{\alpha N} \Big(\mathcal{F}(u, W, S) + \mathcal{F}(v, W, S) \Big) + \frac{2\beta k}{\delta} \int_S |u-v| \,\mathrm{d}x + \frac{\beta}{N} \mathcal{L}^d(S).$$

Plugging this inequality back in (3.3.4), we deduce

$$\mathcal{F}(w^{i^*}, W, O' \cup O'') \leq \mathcal{F}(u, W, O) + \mathcal{F}(v, W, O'') + \frac{\beta}{\alpha N}\Big(\mathcal{F}(u, W, S) + \mathcal{F}(v, W, S)\Big) + \frac{2\beta k}{\delta}\int_S |u - v|\,\mathrm{d}x + \frac{\beta}{N}\mathcal{L}^d(S).$$

We now let $N \in \mathbb{N}$ be so large that $\max\{\beta/(\alpha N), \beta/N\} \leq \eta$ and set $M \coloneqq 2\beta k/\delta$, so that the previous inequality yields (3.3.3). Note that by construction w^{i^*} satisfies (3.3.2).

Finally, the remark concerning the behaviour of the constant M under rescaling can be proved by using as cut-off functions $\hat{\varphi}^i(y) \coloneqq \varphi^i(x + \frac{1}{\rho}(y - x))$ for all $y \in (O_{i+1})_{\rho,x}$. □

We now prove a lemma that allows to compare the value of $\mathcal{F}$ on two nested open sets. This observation will be used to modify, in an energetically convenient way, the matrix-valued fields U so as to impose suitable integral constraints.

Lemma 3.3.2. *Let $(u, U) \in \mathrm{SD}^p_\star(\Omega; \mathbb{R}^k)$ and let $O_1, O_2 \in \mathcal{O}(\Omega)$ with $O_1 \subset\subset O_2$. Assume that $\mathcal{F}$ satisfies* (H1) *and* (H4)*. Then*

$$\mathcal{F}(u, U, O_2) \leq \mathcal{F}(u, U, O_1) + \beta\Big(\mathcal{V}(u, O_2 \setminus O_1) + \|U\|^p_{L^p(O_2 \setminus O_1)} + \mathcal{L}^d(O_2 \setminus O_1)\Big). \tag{3.3.6}$$

Proof. It is enough to use properties (H1) and (H4). □

With the previous lemma at our disposal, we can prove a result that allows to compare the value of the minimisation problems $\mathfrak{m}$ defined in (3.1.4) on cubes of different size.

Lemma 3.3.3. *Assume that $\mathcal{F}$ satisfies properties* (H1) *and* (H4)*. Then for every $(u, U) \in \mathrm{SD}^p_\star(\Omega; \mathbb{R}^k)$ the following two conditions hold:*

(*a*) *if $p > 1$ and $Q_\nu(x, \rho) \subset \Omega$, we have*

$$\limsup_{\theta \to 0^+} \mathfrak{m}(u, U, Q_\nu(x, (1 - \theta)\rho)) \leq \mathfrak{m}(u, U, Q_\nu(x, \rho));$$

(*b*) *if $p = 1$ and $O \in \mathcal{O}(\Omega)$ with Lipschitz boundary, we have*

$$\limsup_{\theta \to 0^+} \mathfrak{m}(u, U, O_\theta) \leq \mathfrak{m}(u, U, O),$$

where $O_\theta = \{x \in O\colon \operatorname{dist}(x, \partial O) > \theta\}$.

Proof. It is enough to repeat the arguments of [128, Lemma 3.2], replacing their Remark 3.1 by our Lemma 3.3.2, the total variation of the symmetric distributional gradients with the reference functional $\mathcal{V}$, and the use of [29, Proposition 2.5] by our Lemma 1.3.24. □

To proceed with the Global Method, it is convenient to introduce the collection of cubes

$$\mathcal{Q}(\Omega) \coloneqq \big\{Q_\nu(x, \rho) \subset \Omega\colon\ x \in \Omega,\ \nu \in \mathbb{S}^{d-1},\ \text{and } \rho > 0\big\}.$$

Given $(u, U) \in \mathrm{SD}^p_\star(\Omega; \mathbb{R}^k)$, we introduce the measure

$$\mu \coloneqq \mathcal{L}^d \llcorner \Omega + |D^c u| + \mathcal{V}(u, \cdot) \llcorner J_u, \tag{3.3.7}$$

and consider the minimisation problem $\mathfrak{m}^\delta(u,U,O)$ defined for $\delta>0$ and $O\in\mathcal{O}(\Omega)$ by

$$\mathfrak{m}^\delta(u,U,O) \coloneqq \inf\Big\{ \sum_{i=1}^\infty \mathfrak{m}(u,U,Q_i)\colon\ Q_i\in\mathcal{Q}(\Omega),\ Q_i\subset O,\\ \text{with } Q_i\cap Q_j=\emptyset \text{ if } i\neq j,\ \operatorname{diam}(Q_i)<\delta,\ \text{and }\ \mu\Big(O\setminus\bigcup_{i=1}^\infty Q_i\Big)=0\Big\}.$$

Clearly, the function $\delta\mapsto\mathfrak{m}^\delta(u,U,O)$ is non-increasing, so that the quantity

$$\widetilde{\mathfrak{m}}(u,U,O)\coloneqq\sup_{\delta>0}\mathfrak{m}^\delta(u,U,O)=\lim_{\delta\to0^+}\mathfrak{m}^\delta(u,U,O)$$

is well-defined.

The first step of the Global Method consists in proving that $\mathcal{F}(u,U,O)=\widetilde{\mathfrak{m}}(u,U,O)$. This is accomplished in the following lemma, whose proof is along the lines of [84, Lemma 4.4] and [120, Lemma 4.2].

Lemma 3.3.4. *Assume that $\mathcal{F}\in\mathfrak{F}^p_\star$. Then,*

$$\mathcal{F}(u,U,O)=\widetilde{\mathfrak{m}}(u,U,O)$$

for all $(u,U)\in\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and all $O\in\mathcal{O}(\Omega)$.

Proof. Let us fix $(u,U)\in\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $O\in\mathcal{O}(\Omega)$. Since for each cube $Q\in\mathcal{Q}(\Omega)$ with $Q\subset O$, we have $\mathfrak{m}(u,U,Q)\le\mathcal{F}(u,U,Q)$ by definition, the measure property (H1) then implies that $\mathfrak{m}^\delta(u,U,O)\le\mathcal{F}(u,U,O)$ for all $\delta>0$, and, as a consequence, $\widetilde{\mathfrak{m}}(u,U,O)\le\mathcal{F}(u,U,O)$.

We now prove the converse inequality. To this end, let us fix $\delta>0$ and a family of cubes $\{Q_i^\delta\}_i\subset\mathcal{Q}(\Omega)$ as in the definition of $\mathfrak{m}^\delta(u,U,O)$ such that

$$\sum_{i=1}^\infty\mathfrak{m}(u,U,Q_i^\delta)\le\mathfrak{m}^\delta(u,U,O)+\delta. \tag{3.3.8}$$

By definition of $\mathfrak{m}(u,U,Q_i^\delta)$, for each $i\in\mathbb{N}$ we find $(v_i^\delta,V_i^\delta)\in\mathrm{SD}^p_\star(Q_i^\delta;\mathbb{R}^k)$ such that $v_i^\delta=u$ near ∂Q_i^δ, $\int_{Q_i^\delta}(V_i^\delta-U)\,\mathrm{d}x=0$, and

$$\mathcal{F}(v_i^\delta,V_i^\delta,Q_i^\delta)\le\mathfrak{m}(u,U,Q_i^\delta)+\delta\mathcal{L}^d(Q_i^\delta). \tag{3.3.9}$$

For every $n\in\mathbb{N}$ we then define

$$\begin{aligned} v^{\delta,n}&\coloneqq\sum_{i=1}^n v_i^\delta\chi_{Q_i^\delta}+u\chi_{N^{\delta,n}} \quad\text{and}\quad v^\delta\coloneqq\sum_{i=1}^\infty v_i^\delta\chi_{Q_i^\delta}+u\chi_{N^\delta},\\ V^{\delta,n}&\coloneqq\sum_{i=1}^n V_i^\delta\chi_{Q_i^\delta}+U\chi_{N^{\delta,n}} \quad\text{and}\quad V^\delta\coloneqq\sum_{i=1}^\infty V_i^\delta\chi_{Q_i^\delta}+U\chi_{N^\delta}, \end{aligned} \tag{3.3.10}$$

where we have set $N^{\delta,n}\coloneqq\Omega\setminus\bigcup_{i=1}^n Q_i^\delta$ and $N^\delta\coloneqq\Omega\setminus\bigcup_{i=1}^\infty Q_i^\delta$. Since $v_i^\delta=u$ near ∂Q_i^δ, it follows that $(v^{\delta,n},V^{\delta,n})\in\mathrm{SD}^p_\star(O;\mathbb{R}^k)$ by Lemma 1.3.24. Moreover, by (3.3.8), (3.3.9), the measure property (H1), the locality property (H2), and the upper and lower bounds in (H4)

we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}\Big(\mathcal{V}(v^{\delta,n},\Omega)+\|V^{\delta,n}\|^p_{L^p(\Omega)}\Big)&\le\frac{1}{\alpha}\sup_{n\in\mathbb{N}}\Big(\mathcal{F}(u,U,N^{\delta,n})+\sum_{i=1}^n\mathcal{F}(v_i^{\delta,n},V_i^{\delta,n},Q_i^\delta)\Big)\\&\le\frac{1}{\alpha}\sup_{n\in\mathbb{N}}\Big(\mathcal{F}(u,U,\Omega)+\mathfrak{m}^\delta(u,U,O)+\delta\big(1+\mathcal{L}^d(O)\big)\Big)\\&\le\frac{2\beta}{\alpha}\sup_{n\in\mathbb{N}}\Big(\mathcal{V}(u,\Omega)+\|U\|^p_{L^p(\Omega)}+2+\mathcal{L}^d(\Omega)\Big)=:S<+\infty.\end{aligned}\tag{3.3.11}$$

As $v^{\delta,n}\to v^\delta$ and $V^{\delta,n}\to V^\delta$ pointwise $\mathcal{L}^d$-a.e. in Ω as $n\to+\infty$, by [111, Theorem 4.7] and (3.3.11) we deduce that $\{(v^{\delta,n},V^{\delta,n})\}_n$ converges to (v^δ,V^δ) in $\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ as $n\to+\infty$ if $p>1$. To obtain the same property for $p=1$, it is enough to observe that computations similar to those that led to (3.3.11) give

$$\int_O|V^{\delta,n}-V^\delta|\,\mathrm{d}x=\sum_{i=n+1}^\infty\int_{Q_i^\delta}|V_i^\delta|\,\mathrm{d}x\le\frac{2\beta}{\alpha}\Big(\mathcal{V}(u,N^{\delta,n+1})+\|U\|_{L^1(N^{\delta,n+1})}+\mathcal{L}^d(N^{\delta,n+1})\Big)$$

for every $n\in\mathbb{N}$, where we have also used that (u,U) is a competitor for the minimisation problems $\mathfrak{m}(u,U,Q_i^\delta)$. As $\lim_{n\to\infty}\mu(N^{\delta,n})=\mu(N^\delta)=0$, this implies that $\{V^{\delta,n}\}_n$ converges to V^δ strongly in $L^1(O;\mathbb{R}^{k\times d})$.

Therefore, by locality (H2), the lower semicontinuity property (H3), (3.3.8), and (3.3.9) it follows that

$$\begin{aligned}\mathcal{F}(v^\delta,V^\delta,O)&\le\liminf_{n\to+\infty}\mathcal{F}(v^{\delta,n},V^{\delta,n},O)=\liminf_{n\to+\infty}\Big(\sum_{i=1}^n\mathcal{F}(v_i^\delta,V_i^\delta,Q_i^\delta)+\mathcal{F}(u,U,N^{\delta,n}\cap O)\Big)\\&\le\sum_{i=1}^\infty\mathcal{F}(v_i^\delta,V_i^\delta,Q_i^\delta)+\mathcal{F}(u,U,N^\delta\cap O)\le\sum_{i=1}^\infty\big(\mathfrak{m}(u,U,Q_i^\delta)+\delta\mathcal{L}^d(Q_i^\delta)\big)\\&\le\mathfrak{m}^\delta(u,U,O)+\delta(1+\mathcal{L}^d(O)),\end{aligned}\tag{3.3.12}$$

where in the second step we used $\mu(\partial Q_i^\delta)=0$ for all $i\in\mathbb{N}$ and in the fourth step we have used that $\mu(N^\delta\cap O)=\mathcal{F}(u,U,N^\delta\cap O)=0$, which are a simple consequence of the definition of $\{Q_i^\delta\}_i$ and the upper bound in (H4).

To conclude the proof, it is now enough to show that $\{(v^\delta,V^\delta)\}_\delta$ converges to (u,U) in $\mathrm{SD}^p_\star(O;\mathbb{R}^k)$ as $\delta\to0^+$, since passing to the liminf for $\delta\to0^+$ in (3.3.12) and using the lower semicontinuity (H3) we would then conclude $\mathcal{F}(u,U,O)\le\widetilde{\mathfrak{m}}(u,U,O)$.

We observe that by Remark 1.3.15 and (3.3.11) there exists a subsequence, still denoted by δ for convenience, such that

$$\lim_{\delta\to0^+}\mathcal{V}(u-v^\delta,O)\le2S,\tag{3.3.13}$$

which, up to passing to a further subsequence, implies

$$\sup_{\delta>0}\mathcal{H}^{d-1}(J^1_{u-v^\delta}\cap O)\le3S.\tag{3.3.14}$$

We now show that $\{v^\delta\}_\delta$ converges to u in $L^0(O;\mathbb{R}^k)$ as $\delta\to0^+$. To this end, we apply Corollary 3.B.7 and Remark 3.B.8 on each cube Q_i^δ to the function $u-v_i^\delta$, which equals

zero near ∂Q_i^δ. We get Borel sets $\omega_i^\delta \subset Q_i^\delta$ such that

$$\big(\mathcal{L}^d(\omega_i^\delta)\big)^{(d-1)/d} \le C_{\rm Poin}\mathcal{H}^{d-1}\big(J^1_{u-v^\delta} \cap Q_i^\delta\big), \tag{3.3.15}$$

$$\|u - v_i^\delta\|_{L^1(Q_i^\delta \setminus \omega_i^\delta)} \le C_{\rm Poin}\delta\mathcal{V}(u - v^\delta, Q_i^\delta \setminus J^1_{u-v^\delta}) \tag{3.3.16}$$

for a constant $C_{\rm Poin} > 0$, independent of i and of δ. In light of (3.3.10) and the fact that $\mathcal{L}^d(N^\delta) = 0$, we can estimate

$$\int_O |u - v^\delta| \wedge 1 \,\mathrm{d}x = \sum_{i=1}^\infty \int_{Q_i^\delta} |u - v_i^\delta| \wedge 1 \,\mathrm{d}x \le \sum_{i=1}^\infty \Big(\|u - v_i^\delta\|_{L^1(Q_i^\delta \setminus \omega_i^\delta)} + \mathcal{L}^d(\omega_i^\delta)\Big). \tag{3.3.17}$$

Recalling that the sets Q_i^δ are pairwise disjoint, we may use (3.3.16) and (3.3.13) to get

$$\sum_{i=1}^\infty \|u - v_i^\delta\|_{L^1(Q_i^\delta \setminus \omega_i^\delta)} \le C\delta \tag{3.3.18}$$

for a constant C independent of δ. Finally, since $\omega_i^\delta \subset Q_i^\delta$ and $\mathrm{diam}(Q_i^\delta) \le \delta$, we may use (3.3.14), (3.3.15), and Hölder's inequality to obtain

$$\sum_{i=1}^\infty \mathcal{L}^d(\omega_i^\delta) \le \delta \sum_{i=1}^\infty \big(\mathcal{L}^d(\omega_i^\delta)\big)^{(d-1)/d} \le C_{\rm Poin}\delta\mathcal{H}^{d-1}\big(J^1_{u-v^\delta} \cap O\big) \le C\delta. \tag{3.3.19}$$

Putting together (3.3.17)–(3.3.19), we get

$$\lim_{\delta\to 0^+} \int_O |u - v^\delta| \wedge 1 \,\mathrm{d}x = 0.$$

Since the equality above is equivalent to the $L^0(O;\mathbb{R}^k)$-convergence of $\{v^\delta\}_\delta$ to u and the limit function u does not depend on the chosen subsequence of δ, by Urysohn's principle we get that the entire sequence $\{v^\delta\}_\delta$ converges to u in $L^0(O;\mathbb{R}^k)$ as $\delta \to 0^+$.

To conclude, we are left with showing that $\{V^\delta\}_\delta$ converges to U weakly in $L^p(O;\mathbb{R}^{k\times d})$ if $p > 1$ and weakly* in $\mathcal{M}_b(O;\mathbb{R}^{k\times d})$ if $p = 1$. To prove this, exploiting $\int_{Q_i^\delta}(V_i^\delta - U)\,\mathrm{d}x = 0$, it is enough to use [120, Lemma 3.3] (note that the arguments work also for $\mathbb{R}^{k\times d}$ in place of $\mathbb{R}^{d\times N\times N}$), which implies that $\{V^\delta\}_\delta$ converges to U weakly* in $\mathcal{M}_b(O;\mathbb{R}^{k\times d})$. This gives the desired convergence if $p = 1$. If $p > 1$, it is enough to note that by (3.3.11) the sequence $\{V^\delta\}_\delta$ is uniformly bounded in $L^p(O;\mathbb{R}^{k\times d})$, and thus converges weakly in $L^p(O;\mathbb{R}^{k\times d})$ as well. This concludes the proof of the lemma. □

The following result shows that for μ-a.e. $x \in \Omega$, the Radon-Nikodým derivative of $\mathcal{F}$ with respect to μ can be characterised by means of the minimum values of the problems $\mathfrak{m}$ on small cubes.

Theorem 3.3.5. *Assume that $\mathcal{F} \in \mathfrak{F}^p_\star$. Then, for every $\nu \in \mathbb{S}^{d-1}$ and $(u,U) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, we have*

$$\lim_{\rho\to 0^+} \frac{\mathcal{F}(u,U,Q_\nu(x,\rho))}{\mu(Q_\nu(x,\rho))} = \lim_{\rho\to 0^+} \frac{\mathfrak{m}(u,U,Q_\nu(x,\rho))}{\mu(Q_\nu(x,\rho))}$$

for μ-a.e. $x \in \Omega$, where μ is the measure introduced in (3.3.7).

Proof. The result can be proved by arguing exactly as in [120, Theorem 4.3], replacing [120, Lemmas 4.1 and 4.2] by our Lemmas 3.3.3 and 3.3.4, respectively. Indeed, the proof in [120] relies on measure theoretical arguments alone and does not require the use of the specific structure of the underlying function space. □

Characterisation of bulk and surface densities. In the rest of the section we prove Theorem 3.1.7. We prove separately equalities (3.1.8) and (3.1.9), addressing first the integral representation of the absolutely continuous part of $\mathcal{F}$.

Proposition 3.3.6. *Assume that $\mathcal{F} \in \mathfrak{F}^p_\star$. Then for every $(u,U) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ we have*

$$\mathcal{F}^a(u,U,B) = \int_B f_{\mathrm{bulk}}\big(x,u,\nabla u,U\big)\,\mathrm{d}x, \tag{3.3.20}$$

where f_{bulk} is the function defined by (3.1.6).

Proof. To prove (3.3.20), we will characterise the Radon-Nikodým derivative of $\mathcal{F}$ with respect to the Lebesgue measure, by showing that

$$\frac{\mathrm{d}\mathcal{F}(u,U,\cdot)}{\mathrm{d}\mathcal{L}^d}(x) = f_{\mathrm{bulk}}\big(x,\widetilde{u}(x),\nabla u(x),\widetilde{U}(x)\big) \tag{3.3.21}$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$, where here and in the following $\sim$ refers to the approximate limit, see (1.3.8) below. Let us fix a point $x \in \Omega$ of approximate continuity of u, satisfying

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\int_{Q(x,\rho)} |\nabla u(y) - \nabla u(x)|\,\mathrm{d}y = 0, \tag{3.3.22}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\mathcal{H}^{d-1}(J_u \cap Q(x,\rho)) = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^d}|D^c u|(Q(x,\rho)) = 0, \tag{3.3.23}$$

and

$$\frac{\mathrm{d}\mathcal{F}(u,U,\cdot)}{\mathrm{d}\mathcal{L}^d}(x) = \lim_{\rho\to 0^+} \frac{\mathcal{F}(u,U,Q(x,\rho))}{\rho^d} = \lim_{\rho\to 0^+} \frac{\mathfrak{m}(u,U,Q(x,\rho))}{\rho^d}, \tag{3.3.24}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\int_{Q(x,\rho)} |U(y) - \widetilde{U}(x)|\,\mathrm{d}y = 0, \tag{3.3.25}$$

and such that the statement of Lemma 3.C.1 holds. Note that these conditions are satisfied by $\mathcal{L}^d$-a.e. point of Ω thanks to Proposition 1.3.10, Besicovitch and Lebesgue Derivation Theorems, and Theorem 3.3.5, so that it will be sufficient to show (3.3.21) for such a point x.

Recalling the definition (3.1.6) of the bulk density f_{bulk}, equality (3.3.24) implies that (3.3.21) is equivalent to

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}(u,U,Q(x,\rho))}{\rho^d} = \lim_{\rho\to 0^+} \frac{\mathfrak{m}(\ell_x,\widetilde{U}(x),Q(x,\rho))}{\rho^d}, \tag{3.3.26}$$

where, for brevity, we write $\ell_x \coloneqq \ell_{x,\widetilde{u}(x),\nabla u(x)} = \widetilde{u}(x) + \nabla u(x)(\cdot - x)$. This equality is proved in two steps, dealing with the two inequalities in (3.3.26) separately. Before doing so, we introduce some further notation related to the statement of Lemma 3.C.1. We

let u_ρ and $\overline{u}_\rho$ be the functions defined for $y \in Q(x,\rho)$ by $u_\rho(y) := (u(y) - \widetilde{u}(x))/\rho$ and $\overline{u}_\rho(y) := (u_\rho(y) \vee \tau'(u_\rho, Q(x,\rho))) \wedge \tau''(u_\rho, Q(x,\rho))$. Thanks to the lemma, we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d+1}} \mathcal{L}^d(\{u_\rho \neq \overline{u}_\rho\}) = 0, \tag{3.3.27}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} |\overline{u}_\rho(y) - \nabla u(x) \frac{(y-x)}{\rho}| \, \mathrm{d}y = 0, \tag{3.3.28}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} |\rho \nabla \overline{u}_\rho(y) - \nabla u(x)| \, \mathrm{d}y = 0, \tag{3.3.29}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(J_{\overline{u}_\rho} \cap Q(x,\rho)) = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c \overline{u}_\rho|(Q(x,\rho)) = 0. \tag{3.3.30}$$

Step 1. ($\leq$) In this step we prove that

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}(u, U, Q(x,\rho))}{\rho^d} \leq \lim_{\rho\to 0^+} \frac{\mathfrak{m}(\ell_x, \widetilde{U}(x), Q(x,\rho))}{\rho^d}. \tag{3.3.31}$$

To this end, let us fix $\theta \in (0, 1/4)$, set $Q^{\theta,\rho} := Q(x, (1-3\theta)\rho)$, and consider (v_ρ, V_ρ) belonging to $\mathcal{C}^p_\star(\ell_x, \widetilde{U}(x), Q^{\theta,\rho})$ such that

$$\mathcal{F}\big(v_\rho, V_\rho, Q^{\theta,\rho}\big) \leq \mathfrak{m}\big(\ell_x, \widetilde{U}(x), Q^{\theta,\rho}\big) + \rho^{d+1}. \tag{3.3.32}$$

We extend the function by setting $v_\rho = \ell_x$ on $Q(x,\rho) \setminus Q^{\theta,\rho}$. In view of (3.3.27), Fubini's Theorem implies that for every $\rho > 0$ there exists $s_\rho \in ((1-\theta)\rho, \rho)$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(\partial Q(x, s_\rho) \cap \{\overline{u}_\rho \neq u_\rho\}) = 0. \tag{3.3.33}$$

We introduce the functions $\{z_\rho\}_\rho$ defined by

$$z_\rho(y) := \begin{cases} \overline{u}_\rho(y) & \text{if } y \in Q(x, s_\rho), \\ u_\rho(y) & \text{if } y \in Q(x,\rho) \setminus Q(x, s_\rho), \end{cases} \tag{3.3.34}$$

which, thanks to Lemma 1.3.24, belong to $\mathrm{GBV}_\star(Q(x,\rho); \mathbb{R}^k)$. We note that by (3.3.22)–(3.3.23), (3.3.28)–(3.3.30), and (3.3.33) they satisfy

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,(1-\theta)\rho)} |z_\rho(y) - \nabla u(x) \frac{(y-x)}{\rho}| \, \mathrm{d}y = 0, \tag{3.3.35}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} |\rho \nabla z_\rho(y) - \nabla u(x)| \, \mathrm{d}y = 0, \tag{3.3.36}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(J_{z_\rho} \cap Q(x,\rho)) = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c z_\rho|(Q(x,\rho)) = 0. \tag{3.3.37}$$

Indeed, property (3.3.35) is an immediate consequence of (3.3.28) and of (3.3.34), and (3.3.36) follows from (3.3.22) and (3.3.29). As for the remaining property (3.3.37), we use (3.3.23), (3.3.30), (3.3.33), and

$$J_{z_\rho} \subset \big(J_u \cap Q(x,\rho)\big) \cup \big(\partial Q(x, s_\rho) \cap \{\overline{u}_\rho \neq u_\rho\}\big).$$

We set

$$O' := Q(x, 1-2\theta), \quad O := Q(x, 1-\theta), \quad \text{and } O'' := Q(x,1) \setminus \overline{Q(x, 1-3\theta)}, \tag{3.3.38}$$

and

$$W_\rho(y) := \begin{cases} V_\rho(y) & \text{in } Q^{\theta,\rho}, \\ \dfrac{1}{\mathcal{L}^d(Q(x,\rho)\setminus Q^{\theta,\rho})}\Big(\displaystyle\int_{Q(x,\rho)} U(y)\,\mathrm{d}y - \mathcal{L}^d(Q^{\theta,\rho})\widetilde{U}(x)\Big) & \text{in } Q(x,\rho)\setminus Q^{\theta,\rho}. \end{cases} \tag{3.3.39}$$

Since $(v_\rho, V_\rho) \in \mathcal{C}^p_\star(\ell_x, \widetilde{U}(x), Q^{\theta,\rho})$, we have

$$\int_{Q^{\theta,\rho}} V_\rho(y)\,\mathrm{d}y = \int_{Q^{\theta,\rho}} \widetilde{U}(x)\,\mathrm{d}y = \mathcal{L}^d(Q^{\theta,\rho})\widetilde{U}(x),$$

so that

$$\int_{Q(x,\rho)} W_\rho(y)\,\mathrm{d}y = \int_{Q(x,\rho)} U(y)\,\mathrm{d}y. \tag{3.3.40}$$

We now want to apply Lemma 3.3.1 to v_ρ with u as a boundary datum near $\partial Q(x,\rho)$. We begin by observing that $v_\rho \in L^1(O_{\rho,x}\setminus O'_{\rho,x};\mathbb{R}^k)$ since $v_\rho = \ell_x$ $\mathcal{L}^d$-a.e. in $Q(x,\rho)\setminus Q^{\theta,\rho}$. Moreover, from the definition (3.3.34) of z_ρ, the fact that $s_\rho \geq (1-\theta)\rho$, and the fact that $\bar{u}_\rho$ is bounded it follows immediately that $\widetilde{u}(x)+\rho z_\rho \in L^1(O_{\rho,x}\setminus O'_{\rho,x};\mathbb{R}^k)$. Hence, for fixed $\eta>0$, we may apply Lemma 3.3.1 to v_ρ (in place of u) and to $\widetilde{u}(x)+\rho z_\rho$ (in place of v) and with O', O'', O as in (3.3.38). We obtain a constant $M>0$, depending on θ but independent of ρ, and $w_\rho \in \mathrm{GBV}_\star(Q(x,\rho);\mathbb{R}^k)$ such that

$$\begin{aligned} \mathcal{F}(w_\rho, W_\rho, Q(x,\rho)) \leq{}& (1+\eta)\big(\mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) + \mathcal{F}(\widetilde{u}(x)+\rho z_\rho, W_\rho, O''_{\rho,x})\big) \\ &+ \frac{M}{\rho}\|(\widetilde{u}(x)+\rho z_\rho) - v_\rho\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} + \eta\rho^d. \end{aligned} \tag{3.3.41}$$

Moreover, the same lemma and (3.3.34) ensure that

$$\begin{aligned} w_\rho = v_\rho \quad &\mathcal{L}^d\text{-a.e. in } O'_{\rho,x} \\ w_\rho = \widetilde{u}(x)+\rho z_\rho = u \quad &\mathcal{L}^d\text{-a.e. in } Q(x,\rho)\setminus Q(x,s_\rho). \end{aligned} \tag{3.3.42}$$

From the second equality in (3.3.42) and (3.3.40) we deduce that (w_ρ, W_ρ) belongs to $\mathcal{C}^p_\star(u,U,Q(x,\rho))$. Therefore, by (3.3.41) and the fact that $v_\rho = \ell_x$ $\mathcal{L}^d$-a.e. in $Q(x,\rho)\setminus Q^{\theta,\rho}$ we infer

$$\begin{aligned} \mathfrak{m}(u,U,Q(x,\rho)) \leq{}& (1+\eta)\big(\mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) + \mathcal{F}(\widetilde{u}(x)+\rho z_\rho, W_\rho, O''_{\rho,x})\big) \\ &+ \frac{M}{\rho}\|\rho z_\rho - \nabla u(x)(\cdot - x)\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} + \eta\rho^d. \end{aligned} \tag{3.3.43}$$

We now estimate each term on the right-hand side of (3.3.43). First, note that (3.3.35) implies

$$\lim_{\rho\to 0^+} \frac{M}{\rho^{d+1}}\|\rho z_\rho - \nabla u(x)(\cdot - x)\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} = 0. \tag{3.3.44}$$

We use Lemma 3.3.2, with $O_1 = Q^{\theta,\rho}$ and $O_2 = O_{\rho,x}$, locality (H2), and (3.3.39) to get

$$\begin{aligned} \mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) \leq{}& \mathcal{F}(v_\rho, V_\rho, Q^{\theta,\rho}) + \beta\mathcal{V}(v_\rho, Q(x,\rho)\setminus Q^{\theta,\rho}) \\ &+ \beta\|W_\rho\|^p_{L^p(Q(x,\rho)\setminus Q^{\theta,\rho})} + \beta(1-(1-3\theta)^d)\rho^d. \end{aligned} \tag{3.3.45}$$

Recalling (3.3.32) and that $v_\rho = \ell_x$ on $Q(x,\rho)\setminus Q^{\theta,\rho}$, we then get

$$\mathcal{F}(v_\rho, V_\rho, Q^{\theta,\rho}) \leq \mathfrak{m}(\ell_x, \widetilde{U}(x), Q^{\theta,\rho}) + \rho^{d+1} \leq \beta k(|\nabla u(x)| + |\widetilde{U}(x)|^p + 1)\rho^d + \rho^{d+1}, \tag{3.3.46}$$

$$\mathcal{V}(v_\rho, Q(x,\rho)\setminus Q^{\theta,\rho}) \leq (1-(1-3\theta)^d)k|\nabla u(x)|\rho^d. \tag{3.3.47}$$

Here, in the second inequality of (3.3.46) we used that $(\ell_x, \widetilde{U}(x))$ is a competitor for the minimisation problem $\mathfrak{m}(\ell_x, \widetilde{U}(x), Q^{\theta,\rho})$ and thus $\mathfrak{m}(\ell_x, \widetilde{U}(x), Q^{\theta,\rho}) \le \beta k(|\nabla u(x)|+|\widetilde{U}(x)|^p+1)\rho^d$ by the upper bound in (H4). As for the remaining term in (3.3.45), we observe that

$$\begin{aligned}\int_{Q(x,\rho)\setminus Q^{\theta,\rho}} |W_\rho|^p \,\mathrm{d}y &\le \frac{1}{\rho^{d(p-1)}(1-(1-3\theta)^d)^{p-1}} \Big| \int_{Q(x,\rho)} U(y)\,\mathrm{d}y - \int_{Q^{\theta,\rho}} \widetilde{U}(x)\,\mathrm{d}y\Big|^p \\ &\le \frac{C}{\rho^{d(p-1)}(1-(1-3\theta)^d)^{p-1}} \Big(\Big| \int_{Q(x,\rho)\setminus Q^{\theta,\rho}} U(y)\,\mathrm{d}y\Big|^p + \Big| \int_{Q^{\theta,\rho}} (U(y)-\widetilde{U}(x))\,\mathrm{d}y\Big|^p\Big) \\ &\le \frac{C\rho^{dp}}{\rho^{d(p-1)}(1-(1-3\theta)^d)^{p-1}} \Big(\Big| \fint_{Q(x,\rho)} U(y)\,\mathrm{d}y - (1-3\theta)^d \fint_{Q^{\theta,\rho}} U(y)\,\mathrm{d}y\Big|^p \\ &\qquad + \Big|(1-3\theta)^d \fint_{Q^{\theta,\rho}} (U(y)-\widetilde{U}(x))\,\mathrm{d}y\Big|^p\Big)\end{aligned}$$

for a constant $C>0$ depending only on p. In view of (3.3.25), this implies

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)\setminus Q^{\theta,\rho}} |W_\rho|^p \,\mathrm{d}y \le C|(1-(1-3\theta)^d)|\widetilde{U}(x)|^p. \tag{3.3.48}$$

In a similar fashion, observing that $\mathcal{L}^d(O''_{\rho,x}) = (1-(1-3\theta)^d)\rho^d$ and that $\mathcal{V}$ is invariant by adding constants, we may estimate

$$\mathcal{F}(\widetilde{u}(x)+\rho z_\rho, W_\rho, O''_{\rho,x}) \le \beta\mathcal{V}(\rho z_\rho, O''_{\rho,x}) + \beta\|W_\rho\|^p_{L^p(Q(x,\rho)\setminus Q^{\theta,\rho})} + \beta(1-(1-3\theta)^d)\rho^d, \tag{3.3.49}$$

and observe that

$$\mathcal{V}(\rho z_\rho, O''_{\rho,x}) \le k\Big(\int_{O''_{\rho,x}} |\rho\nabla z_\rho|\,\mathrm{d}x + \rho|D^c z_\rho|(O''_{\rho,x}) + \mathcal{H}^{d-1}(J_{z_\rho}\cap O''_{\rho,x})\Big).$$

Note that by (3.3.36) and (3.3.37) we have

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^d}\mathcal{V}(\rho z_\rho, O''_{\rho,x}) \le k(1-(1-3\theta)^d)|\nabla u(x)|. \tag{3.3.50}$$

We now combine (3.3.43)–(3.3.47) and (3.3.48)–(3.3.50), divide by ρ^d, pass to the limsup as $\rho\to 0^+$, and obtain

$$\begin{aligned}\limsup_{\rho\to 0^+} \frac{\mathfrak{m}(u,U,Q(x,\rho))}{\rho^d} &\le (1-3\theta)^d \limsup_{\rho'\to 0^+} \Big(\frac{\mathfrak{m}(\ell_x,\widetilde{U}(x),Q(x,\rho'))}{(\rho')^d}\Big) \\ &\quad + \eta\beta k(|\nabla u(x)| + |\widetilde{U}(x)|^p + 1) \\ &\quad + (1+\eta)\beta(1-(1-3\theta)^d)(2+2k|\nabla u(x)|) \\ &\quad + (1+\eta)\frac{2C(1-(1-3\theta)^d)^p}{(1-(1-3\theta)^d)^{p-1}}|\widetilde{U}(x)|^p + \eta,\end{aligned}$$

where we have set $\rho' \coloneqq (1-3\theta)\rho$. Letting $\theta\to 0^+$ and $\eta\to 0^+$, we obtain (3.3.31).

Step 2. ($\ge$) The proof of this inequality follows along the same lines of Step 1 by interchanging the roles of (u,U) and $(\ell_x,\widetilde{U}(x))$. The essential difference is that the Fubini-type

argument in (3.3.33) is performed for some $s_\rho \in ((1-4\theta)\rho, (1-3\theta)\rho)$, we choose a competitor $(v_\rho, V_\rho) \in \mathcal{C}^p_\star(u, U, Q(x, s_\rho))$ satisfying

$$\mathcal{F}(v_\rho, V_\rho, Q(x, s_\rho)) \le \mathfrak{m}(u, U, Q(x, s_\rho)) + \rho^{d+1},$$

we extend v_ρ to the whole $Q(x,\rho)$ by setting $v_\rho = \widetilde{u}(x) + \rho\overline{u}_\rho$ outside of $Q(x, s_\rho)$ and we define W_ρ outside of $Q(x, s_\rho)$ by

$$W_\rho = \frac{1}{\mathcal{L}^d(Q(x,\rho) \setminus Q(x, s_\rho))}\Big(\mathcal{L}^d(Q(x,\rho))\widetilde{U}(x) - \int_{Q(x,s_\rho)} U(y)\,\mathrm{d}y\Big)$$

This ensures that the Fundamental Estimate can again be employed to adjust the boundary values. We omit the details. □

We proceed by addressing the integral representation of the jump part of $\mathcal{F}$, concluding the proof of Theorem 3.1.7.

Proposition 3.3.7. *Assume that $\mathcal{F} \in \mathfrak{F}^p_\star$. Then for every $(u, U) \in \mathrm{SD}^p_\star(\Omega; \mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ we have*

$$\mathcal{F}^j(u, U, B) = \int_{J_u \cap B} f_{\mathrm{surf}}\big(x, u^+, u^-, \nu_u\big)\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.3.51}$$

where f_{surf} is the function defined by (3.1.7).

Proof. To prove (3.3.51), we characterise the Radon-Nikodým derivative of $\mathcal{F}$ with respect to the measure $\mathcal{H}^{d-1} \llcorner J_u$ by showing that

$$\frac{\mathrm{d}\mathcal{F}(u, U, \cdot)}{\mathrm{d}\mathcal{H}^{d-1} \llcorner J_u}(x) = f_{\mathrm{surf}}\big(x, u^+(x), u^-(x), \nu_u(x)\big), \tag{3.3.52}$$

for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$. Let us fix a point $x \in J_u$ satisfying

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \int_{Q_{\nu_u(x)}(x,\rho)} |\nabla u|\,\mathrm{d}y = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c u|(Q_{\nu_u(x)}(x,\rho)) = 0, \tag{3.3.53}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \mathcal{H}^{d-1}(J_u \cap E_{\rho,x}) = \mathcal{H}^{d-1}(\Pi^{\nu_u(x)}_x \cap E) \quad \text{for all Borel sets } E \subset Q_{\nu_u(x)}(x, 1), \tag{3.3.54}$$

where $\Pi^\nu_x := \{y \in \mathbb{R}^d \colon (y - x) \cdot \nu = 0\}$ for all $\nu \in \mathbb{S}^{d-1}$, and

$$\frac{\mathrm{d}\mathcal{F}(u, U, \cdot)}{\mathrm{d}\mathcal{H}^{d-1} \llcorner J_u}(x) = \lim_{\rho\to 0^+} \frac{\mathcal{F}(u, U, Q_{\nu_u(x)}(x,\rho))}{\rho^{d-1}} = \lim_{\rho\to 0^+} \frac{\mathfrak{m}(u, U, Q_{\nu_u(x)}(x,\rho))}{\rho^{d-1}}, \tag{3.3.55}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \int_{Q_{\nu_u(x)}(x,\rho)} |U(y)|^p\,\mathrm{d}y = 0, \tag{3.3.56}$$

and such that the statement of Lemma 3.C.2 holds. Note that these conditions are satisfied by $\mathcal{H}^{d-1}$-a.e. point of J_u thanks to Proposition 1.3.10, [18, Theorem 2.56], Besicovitch and Lebesgue Derivation Theorems, and Theorem 3.3.5, where for (3.3.54) we can argue as in (3.C.21)–(3.C.22) below. Thus, it is sufficient to show (3.3.52) for such a point x.

Recalling the definition (3.1.7) of the surface density f_{surf}, equality (3.3.55) implies that (3.3.52) is equivalent to

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}(u, U, Q_\nu(x,\rho))}{\rho^{d-1}} = \lim_{\rho\to 0^+} \frac{\mathfrak{m}(u_x, 0, Q_\nu(x,\rho))}{\rho^{d-1}},$$

where, for brevity, we set $u_x \coloneqq u_{x,u^+(x),u^-(x),\nu_u(x)}$ and $\nu = \nu_u(x)$. This equality is proved in two steps, dealing with the two inequalities separately. To this end, we apply Lemma 3.C.2 to get $u_\rho \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k)$ which satisfy

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\mathcal{L}^d(\{u_\rho \neq u\} \cap Q_\nu(x,\rho)) = 0, \tag{3.3.57}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\int_{Q_\nu(x,\rho)} |u_\rho - u_x|\,\mathrm{d}y = 0, \tag{3.3.58}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\,\mathcal{H}^{d-1}(J_{u_\rho} \cap E_{\rho,x}) \leq \mathcal{H}^{d-1}(\Pi_x^\nu \cap E) \quad \text{for all Borel sets } E \subset Q_\nu(x,1), \tag{3.3.59}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\int_{Q_\nu(x,\rho)} |\nabla u_\rho|\,\mathrm{d}y = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}|D^c u_\rho|(Q_\nu(x,\rho)) = 0. \tag{3.3.60}$$

Step 1. ("$\leq$") In this step we prove that

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}(u,U,Q_\nu(x,\rho))}{\rho^{d-1}} \leq \lim_{\rho\to 0^+} \frac{\mathfrak{m}(u_x,0,Q_\nu(x,\rho))}{\rho^{d-1}}. \tag{3.3.61}$$

To this end, we fix $\theta \in (0,1/4)$, set $Q_\nu^{\theta,\rho} \coloneqq Q_\nu(x,(1-3\theta)\rho)$, and consider $(v_\rho,V_\rho) \in \mathcal{C}^p_\star(u_x,0,Q_\nu^{\theta,\rho})$ such that

$$\mathcal{F}(v_\rho,V_\rho,Q_\nu^{\theta,\rho}) \leq \mathfrak{m}(u_x,0,Q_\nu^{\theta,\rho}) + \rho^d. \tag{3.3.62}$$

We extend v_ρ to the whole cube $Q_\nu(x,\rho)$ by setting $v_\rho = u_x$ on $Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}$. Moreover, we set

$$W_\rho(y) \coloneqq \begin{cases} V_\rho(y) & \text{in } Q_\nu^{\theta,\rho},\\ \dfrac{1}{\mathcal{L}^d(Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho})}\displaystyle\int_{Q_\nu(x,\rho)} U(y)\,\mathrm{d}y & \text{in } Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}. \end{cases} \tag{3.3.63}$$

Since $(v_\rho,V_\rho) \in \mathcal{C}^p_\star(u_x,0,Q_\nu^{\theta,\rho})$, we have

$$\int_{Q_\nu(x,\rho)} W_\rho(y)\,\mathrm{d}y = \int_{Q_\nu(x,\rho)} U(y)\,\mathrm{d}y. \tag{3.3.64}$$

In light of (3.3.57), Fubini's Theorem implies that for every $\rho>0$ there exists $s_\rho \in ((1-\theta)\rho,\rho)$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\mathcal{H}^{d-1}(\partial Q_\nu(x,s_\rho) \cap \{u_\rho \neq u\}) = 0. \tag{3.3.65}$$

We also introduce z_ρ defined by

$$z_\rho(y) \coloneqq \begin{cases} u_\rho(y) & \text{if } y \in Q_\nu(x,s_\rho),\\ u(y) & \text{if } y \in Q_\nu(x,\rho)\setminus Q_\nu(x,s_\rho), \end{cases} \tag{3.3.66}$$

which by Lemma 1.3.24 belong to $\mathrm{GBV}_\star(Q_\nu(x,\rho);\mathbb{R}^k)$. Note that by (3.3.53)–(3.3.54), (3.3.58)–(3.3.60), and (3.3.65) it holds that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\int_{Q_\nu(x,(1-\theta)\rho)} |z_\rho(y) - u_x|\,\mathrm{d}y = 0, \tag{3.3.67}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\mathcal{V}(z_\rho, E_{\rho,x}\cap J_{z_\rho}) \leq k\mathcal{H}^{d-1}(\Pi_x^\nu \cap E) \quad \text{for all Borel sets } E \subset Q_\nu(x,1), \tag{3.3.68}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\int_{Q_\nu(x,\rho)} |\nabla z_\rho(y)|\,\mathrm{d}y = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}|D^c z_\rho|(Q_\nu(x,\rho)) = 0. \tag{3.3.69}$$

We set

$$O' := Q_\nu(x, 1-2\theta), \quad O := Q_\nu(x, 1-\theta), \quad \text{and } O'' := Q_\nu(x,1) \setminus \overline{Q_\nu(x, 1-3\theta)}. \tag{3.3.70}$$

Observe that $v_\rho \in L^1(O_{\rho,x} \setminus O'_{\rho,x}; \mathbb{R}^k)$ since $v_\rho = u_x$ $\mathcal{L}^d$-a.e. in $Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}$. Moreover, from definition (3.3.66) and the fact that $s_\rho \geq (1-\theta)\rho$, it follows that $z_\rho \in L^1(O_{\rho,x} \setminus O'_{\rho,x}; \mathbb{R}^k)$. Hence, for fixed $\eta > 0$, we may apply Lemma 3.3.1 to v_ρ (in place of u) and to z_ρ in place of v and with O', O'', O as in (3.3.70). We obtain a constant $M > 0$ depending on θ but independent of ρ, and functions $w_\rho \in \mathrm{GBV}_\star(Q_\nu(x,\rho); \mathbb{R}^k)$ such that

$$\begin{aligned}
\mathcal{F}(w_\rho, W_\rho, Q_\nu(x,\rho)) &\leq (1+\eta)\big(\mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) + \mathcal{F}(z_\rho, W_\rho, O''_{\rho,x})\big) \\
&\quad + \frac{M}{\rho}\|z_\rho - v_\rho\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} + \eta\rho^d \\
&= (1+\eta)\big(\mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) + \mathcal{F}(z_\rho, W_\rho, O''_{\rho,x})\big) \\
&\quad + \frac{M}{\rho}\|z_\rho - u_x\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} + \eta\rho^d,
\end{aligned} \tag{3.3.71}$$

where in the equality we have used that $v_\rho = u_x$ $\mathcal{L}^d$-a.e. in $Q_\nu(x,\rho) \setminus Q_\nu^{\theta,\rho}$. Moreover, the same lemma and (3.3.66) gives

$$\begin{aligned}
w_\rho &= v_\rho \quad \mathcal{L}^d\text{-a.e. in } O'_{\rho,x} \\
w_\rho &= u \quad \mathcal{L}^d\text{-a.e. in } Q_\nu(x,\rho) \setminus Q_\nu(x, s_\rho).
\end{aligned} \tag{3.3.72}$$

From the second equality in (3.3.72) and (3.3.64) we deduce that $(w_\rho, W_\rho) \in \mathcal{C}^p_\star(u, U, Q_\nu(x,\rho))$. Therefore, by (3.3.71) we infer

$$\mathfrak{m}(u, U, Q_\nu(x,\rho)) \leq (1+\eta)\big(\mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) + \mathcal{F}(z_\rho, W_\rho, O''_{\rho,x})\big) \tag{3.3.73}$$

$$+ \frac{M}{\rho}\|z_\rho - u_x\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} + \eta\rho^d. \tag{3.3.74}$$

We estimate each term on the right-hand side of (3.3.73). First, we note that (3.3.67) implies

$$\lim_{\rho\to 0^+} \frac{M}{\rho^d}\|z_\rho - u_x\|_{L^1(O_{\rho,x}\setminus O'_{\rho,x})} = 0. \tag{3.3.75}$$

We use Lemma 3.3.2, with $O_1 = Q_\nu^{\theta,\rho}$ and $O_2 = O_{\rho,x}$, locality (H2), and (3.3.63) to get

$$\begin{aligned}
\mathcal{F}(v_\rho, W_\rho, O_{\rho,x}) &\leq \mathcal{F}(v_\rho, V_\rho, Q_\nu^{\theta,\rho}) + \beta\mathcal{V}(v_\rho, Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}) \\
&\quad + \beta\|W_\rho\|^p_{L^p(Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho})} + \beta(1-(1-3\theta)^d)\rho^d.
\end{aligned} \tag{3.3.76}$$

Recalling (3.3.62) and that $v_\rho = u_x$ on $Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}$, we have

$$\mathcal{F}(v_\rho, V_\rho, Q_\nu^{\theta,\rho}) \leq \mathfrak{m}(u_x, 0, Q_\nu^{\theta,\rho}) + \rho^d \leq \beta\rho^d + \beta k\rho^{d-1} + \rho^d, \tag{3.3.77}$$

$$\mathcal{V}(v_\rho, Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}) \leq (1-(1-3\theta)^{d-1})k\rho^{d-1}. \tag{3.3.78}$$

Here, in the second inequality of (3.3.77) we used that $(u_x, 0)$ is a competitor for $\mathfrak{m}(u_x, 0, Q_\nu^{\theta,\rho})$ and thus $\mathfrak{m}(u_x, 0, Q_\nu^{\theta,\rho}) \leq \beta\rho^d + \beta k\rho^{d-1}$ by the upper bound in (H4). As for the remaining

term in (3.3.76), using Hölder's inequality, we get

$$\begin{aligned}\int_{Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho}} |W_\rho|^p \,\mathrm{d}y &\le \frac{1}{\rho^{d(p-1)}(1-(1-3\theta)^d)^{p-1}} \Big| \int_{Q_\nu(x,\rho)} U(y)\,\mathrm{d}y\Big|^p \\ &\le \frac{\rho^{d(p-1)}}{\rho^{d(p-1)}(1-(1-3\theta)^d)^{p-1}} \|U\|^p_{L^p(Q_\nu(x,\rho))} \le C\|U\|^p_{L^p(Q_\nu(x,\rho))}\end{aligned} \tag{3.3.79}$$

for some universal $C>0$, where we used $\theta<1/4$. Similarly, we may estimate

$$\mathcal{F}(z_\rho, W_\rho, O''_{\rho,x}) \le \beta\mathcal{V}(z_\rho, O''_{\rho,x}) + \beta\|W_\rho\|^p_{L^p(Q_\nu(x,\rho)\setminus Q_\nu^{\theta,\rho})} + \beta(1-(1-3\theta)^d)\rho^d \tag{3.3.80}$$

and observe that

$$\mathcal{V}(z_\rho, O''_{\rho,x}) \le k\Big(\int_{O''_{\rho,x}} |\nabla z_\rho|\,\mathrm{d}x + |D^c z_\rho|(O''_{\rho,x}) + \mathcal{H}^{d-1}(J_{z_\rho}\cap O''_{\rho,x})\Big). \tag{3.3.81}$$

Hence, using (3.3.68) with $E=O''$ and (3.3.69), we get

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\mathcal{V}(z_\rho, O''_{\rho,x}) \le (1-(1-3\theta)^{d-1})k. \tag{3.3.82}$$

Putting together (3.3.73)–(3.3.82), dividing by ρ^{d-1}, letting $\rho\to 0^+$, and using (3.3.56) we get

$$\begin{aligned}\limsup_{\rho\to 0^+} \frac{\mathfrak{m}(u,U,Q_\nu(x,\rho))}{\rho^{d-1}} &\le (1-3\theta)^{d-1}\limsup_{\rho'\to 0^+}\Big(\frac{\mathfrak{m}(u_x,0,Q_\nu(x,\rho'))}{(\rho')^{d-1}}\Big) + \eta\beta k \\ &\quad + (1+\eta)\Big(\beta(1-(1-3\theta)^{d-1})2k\Big),\end{aligned}$$

where we have set $\rho' \coloneqq (1-3\theta)\rho$. Letting $\theta\to 0^+$ and $\eta\to 0^+$, we obtain (3.3.61).

Step 2. ($\ge$) As in the previous proof, the other inequality can be proven along similar lines. We therefore omit the full proof and indicate only the essential changes. We follow Step 1 by interchanging the roles of (u,U) and $(u_x,0)$. The Fubini-type argument in (3.3.65) is performed for some $s_\rho \in ((1-4\theta)\rho,(1-3\theta)\rho)$, we choose a competitor $(v_\rho,V_\rho)\in\mathcal{C}^p_\star(u,U,Q_\nu(x,s_\rho))$ satisfying

$$\mathcal{F}(v_\rho,V_\rho,Q_\nu(x,s_\rho)) \le \mathfrak{m}(u,U,Q_\nu(x,s_\rho)) + \rho^d,$$

and we extend v_ρ to the whole $Q_\nu(x,\rho)$ by setting $v_\rho = u_\rho$ outside of $Q_\nu(x,s_\rho)$, as well as

$$W_\rho = -\frac{1}{\mathcal{L}^d(Q_\nu(x,\rho)\setminus Q_\nu(x,s_\rho))}\int_{Q_\nu(x,s_\rho)} U(y)\,\mathrm{d}y$$

outside of $Q_\nu(x,s_\rho)$. □

Proof of Theorem 3.1.7. The proof follows by combining Propositions 3.3.6 and 3.3.7. □

Remark 3.3.8. The methods used in this section show that Theorem 2.3.16 of Chapter 2 can be greatly improved. Indeed, since the methods described above do not rely on truncation techniques, it is possible to show that (a suitable version of) the integral representation Theorem 2.3.16 still holds for functionals F that do not satisfy truncation property (H8) of Definition 2.1.7.

3.4 Relaxation of bulk and surface energies

This section is devoted to the proof of the Relaxation Theorems 3.1.13, 3.1.14, and 3.1.16, concerning the integral representation of the relaxed functional $I^p_\star$ defined by (3.1.12) and (3.1.13).

Proof of Theorem 3.1.13 We start with a useful consequence of the Approximation Theorem 3.1.1, which will be exploited in the proof of Theorem 3.1.13. Recall the structural constants c_Ψ, C_Ψ, c_γ, and C_γ in the assumptions on Ψ and γ, as well as $\overline{C}_\Psi$ in (3.1.14).

Lemma 3.4.1. *Let $(g,G)\in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and let $O\in\mathcal{O}(\Omega)$. Assume that Ψ and γ satisfy* (Ψ1)–(Ψ4), (γ1), *and* (γ2). *Then*

$$\alpha\big(\mathcal{V}(g,O)+\|G\|^p_{L^p(O)}-\mathcal{L}^d(O)\big)\le I^p_\star(g,G,O)\le \beta\big(\mathcal{V}(g,O)+\|G\|^p_{L^p(O)}+\mathcal{L}^d(O)\big)$$

for two suitable positive constants α,β, depending only on c_Ψ, C_Ψ, $\overline{C}_\Psi$, c_γ, C_γ, and p.

Proof. Using (Ψ4) we find $\Psi(x,A)\ge \frac{c_\Psi}{2}(|A|^p+|A|)-C'_\Psi$ for $\mathcal{L}^d$-a.e. $x\in\Omega$ and all $A\in\mathbb{R}^{k\times d}$ for some $C'_\Psi>0$ sufficiently large. Then, the lower bound follows by (γ2), together with the lower semicontinuity of $\mathcal{V}$ (see Remark 1.3.15) and of $\|G\|^p_{L^p(O)}$ under weak convergence.

To prove the upper bound, we use the Approximation Theorem 3.1.1 to obtain a sequence $\{u_n\}_n\subset \mathrm{SBV}(O;\mathbb{R}^k)$ with $\nabla u_n=G$ $\mathcal{L}^d$-a.e. in O and such that

$$\limsup_{n\to+\infty}\mathcal{V}(u_n,O)\le C\big(\mathcal{V}(g,O)+\|G\|_{L^1(O)}\big).$$

Thanks to Remark 3.1.9, the previous inequality, and (γ2), we then have

$$\begin{aligned} I^p_\star(g,G,O)\le \liminf_{n\to+\infty}\mathcal{E}(u_n,O)&\le \beta\big(\liminf_{n\to+\infty}\mathcal{V}(u_n,O)+\|G\|^p_{L^p(O)}+\mathcal{L}^d(O)\big)\\ &\le C\beta\big(\mathcal{V}(g,O)+\|G\|_{L^1(O)}+\|G\|^p_{L^p(O)}+\mathcal{L}^d(O)\big).\end{aligned}$$

Using that $|A|\le 1+|A|^p$ for all $A\in\mathbb{R}^{k\times d}$, the proof is concluded. □

We are now ready to prove Theorem 3.1.13.

Proof of Theorem 3.1.13. Thanks to Theorem 3.1.7, to conclude the proof, it is enough to show that $I^p_\star\in\mathfrak{F}^p_\star$.

Let us fix $(g,G)\in\mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. Given $O_1,O_2,O_3\in\mathcal{O}(\Omega)$ with $O_1\subset\subset O_2\subset O_3\subset\Omega$, we claim that

$$I^p_\star(g,G,O_3)\le I^p_\star(g,G,O_2)+I^p_\star(g,G,O_3\setminus\overline{O_1}). \tag{3.4.1}$$

This inequality can be obtained with an argument similar to, for instance, [128, Theorem 2.8], but for the sake of completeness we present here the full proof. Without loss of generality, we prove this inequality only in the case $p=1$ as the case $p>1$ follows by similar, but easier, arguments. Let $\{u_n\}_n\subset\mathrm{SBV}(O_2;\mathbb{R}^k)$ and $\{v_n\}_n\subset\mathrm{SBV}(O_3\setminus\overline{O_1};\mathbb{R}^k)$ be two sequences satisfying the following properties: $u_n\underset{SD_\star}{\to}(g,G)$ in O_2, $v_n\underset{SD_\star}{\to}(g,G)$ in $O_3\setminus\overline{O_1}$, and

$$I^p_\star(g,G,O_2)=\lim_{n\to+\infty}\mathcal{E}(u_n,O_2)\quad\text{and}\quad I^p_\star(g,G,O_3\setminus\overline{O_1})=\lim_{n\to+\infty}\mathcal{E}(v_n,O_3\setminus\overline{O_1}). \tag{3.4.2}$$

Note also that

$$u_n-v_n\to 0\text{ in }\ L^0(O_2\setminus\overline{O_1};\mathbb{R}^k)\text{ as }n\to+\infty. \tag{3.4.3}$$

Let us fix $\delta > 0$ and consider $O_\delta := \{x \in O_2 \colon \operatorname{dist}(x, O_1) < \delta\}$. We let $\mathrm{d}(x) := \operatorname{dist}(x, O_1)$ be the distance function from O_1, which we recall is Lipschitz continuous. Therefore, by the Area Formula [18, (2.47)] we have

$$\int_{O_\delta \setminus \overline{O_1}} |u_n(x) - v_n(x)| \wedge 1 \, J\mathrm{d}(x) \, \mathrm{d}x = \int_0^\delta \Big(\int_{\mathrm{d}^{-1}(y)} |u_n(x) - v_n(x)| \wedge 1 \, \mathrm{d}\mathcal{H}^{d-1}(x) \Big) \, \mathrm{d}y,$$

where we let $J\mathrm{d}$ denote the Jacobian of d. In light of (3.4.3) and of the fact that $J\mathrm{d}$ is bounded, we may apply Fatou's Lemma to obtain, for $\mathcal{L}^1$-a.e. $\rho \in [0, \delta]$,

$$\liminf_{n \to +\infty} \int_{\mathrm{d}^{-1}(\rho)} |u_n - v_n| \wedge 1 \, \mathrm{d}\mathcal{H}^{d-1}(x) = \liminf_{n \to +\infty} \int_{\partial O_\rho} |u_n - v_n| \wedge 1 \, \mathrm{d}\mathcal{H}^{d-1}(x) = 0. \quad (3.4.4)$$

Let us fix $\delta_0 \in [0, \delta]$ such that (3.4.4) holds and $\mathcal{H}^{d-1}(\partial O_{\delta_0}) < +\infty$. The latter can be assumed by the Coarea Formula. By passing to a non-relabelled subsequence of $\{u_n - v_n\}_n$, we may also assume that the liminf in (3.4.4) is actually a limit. We may also assume that O_{δ_0} is a set with locally Lipschitz boundary and then consider u_n, v_n on ∂O_{δ_0} in the sense of traces of $\mathrm{SBV}(\Omega; \mathbb{R}^k)$-functions, and define

$$w_n := \begin{cases} u_n & \text{in } \overline{O_{\delta_0}}, \\ v_n & \text{in } O_3 \setminus \overline{O_{\delta_0}}. \end{cases}$$

It follows that $\{w_n\} \subset \mathrm{SBV}(O_3; \mathbb{R}^k)$, $|[w_n]| = |u_n - v_n| \, \mathcal{H}^{d-1}$-a.e. on ∂O_{δ_0}, and $w_n \underset{SD_\star}{\to} (g, G)$ in O_3. Hence, by ($\gamma 2$), (3.4.4), and (3.4.2) we get

$$\begin{aligned} I^p_\star(g, G, O_3) &\le \liminf_{n \to +\infty} \Big(\int_{O_3} \Psi(x, \nabla w_n) \, \mathrm{d}x + \int_{J_{w_n} \cap O_3} \gamma(x, [w_n], \nu_{w_n}) \, \mathrm{d}\mathcal{H}^{d-1} \Big) \\ &\le \liminf_{n \to +\infty} \Big(\int_{O_2} \Psi(x, \nabla u_n) \, \mathrm{d}x + \int_{J_{u_n} \cap O_2} \gamma(x, [u_n], \nu_{u_n}) \, \mathrm{d}\mathcal{H}^{d-1} \\ &\qquad + \int_{O_3 \setminus \overline{O_1}} \Psi(x, \nabla v_n) \, \mathrm{d}x + \int_{J_{v_n} \cap (O_3 \setminus \overline{O_1})} \gamma(x, [v_n], \nu_{v_n}) \, \mathrm{d}\mathcal{H}^{d-1} \\ &\qquad + \int_{J_{w_n} \cap \partial O_{\delta_0}} C_\gamma |u_n - v_n| \wedge 1 \, \mathrm{d}\mathcal{H}^{d-1} \Big) \\ &= I^p_\star(g, G, O_2) + I^p_\star(g, G, O_3 \setminus \overline{O_1}). \end{aligned}$$

This concludes the proof of (3.4.1).

Next, one can check that $O \mapsto I^p_\star(g, G, O)$ coincides with the restriction to $\mathcal{O}(\Omega)$ of a bounded Radon measure on Ω. We omit the exact argument which can be found in [76, Proposition 2.22, Step 1], using Lemma 3.4.1 and (3.4.1) in place of [76, Lemma 2.18, Lemma 2.21]. This yields (H1) and proves that definition (3.1.13) is well-posed (see the comment after (3.1.13)).

Moreover, by inner regularity on open sets, we have that $I^p_\star$ satisfies the upper bound (3.1.3) with B replaced by O, for every $O \in \mathcal{O}(\Omega)$. Combining this observation with (3.1.13) and Lemma 3.4.1 we get that $I^p_\star$ satisfies (H4) as well (up to a shift).

To prove that $I^p_\star$ satisfies the lower semicontinuity property (H3), it is enough to repeat exactly the arguments used in [76, Proposition 5.1]. Finally, locality property (H2) is a simple consequence of lower semicontinuity property (H3) as observed in [41, Remark 2.1]. This concludes the proof of $I^p_\star \in \mathfrak{F}^p_\star$ and thus of the theorem. □

Proof of Theorem 3.1.14. Throughout the rest of the section, we let $\Phi \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ be a function satisfying (1.3.11), for a fixed $\tau \geq \max\{3, \beta_\gamma + 1\}$, where β_γ is the constant appearing in (γ5). For $R > 0$, we let Φ_R be the function associated to Φ via (1.3.12). The next truncation result will be useful for proving Theorem 3.1.14 as it allows to truncate competitors appearing in suitable minimisation problems. It will also be crucial in the proof of Lemma 3.4.5 below, a key ingredient in the proof of Theorem 3.1.16.

Lemma 3.4.2. *Assume that Ψ and γ satisfy conditions (Ψ1)–(Ψ4), (γ1), (γ2), and (γ5). Let $O \in \mathcal{O}(\Omega)$. Then there exists a constant $C > 0$, depending only on the structural constants $c_\Psi, \bar{C}_\Psi, c_\gamma, C_\gamma$, and β_γ, such that for every $N \in \mathbb{N}$, $R > 0$, $u \in \mathrm{SBV}(O;\mathbb{R}^k)$, and $B \in \mathcal{B}(O)$ it holds*

$$\frac{1}{N}\sum_{i=1}^{N} \mathcal{E}(\Phi_{R_i} \circ u, B) \leq \Big(1 + \frac{C}{N}\Big)\mathcal{E}(u, B) + \mathcal{L}^d(\{|u| \geq R\}) + \frac{C}{N}\mathcal{L}^d(B), \tag{3.4.5}$$

where we have set $R_i := \tau^{i-1}R$. If, in addition, γ satisfies (γ6) as well, then for every $x \in \Omega$ and $\rho > 0$ the same property holds with the same constant C, for the functional $\mathcal{E}_\rho^x$ defined by

$$\mathcal{E}_\rho^x(u, B) := \int_B \Psi(x + \rho y, \nabla u(y))\,\mathrm{d}y + \frac{1}{\rho}\int_{J_u \cap B} \gamma(x + \rho y, \rho[u](y), \nu_u(y))\,\mathrm{d}\mathcal{H}^{d-1}(y) \tag{3.4.6}$$

for every $u \in \mathrm{SBV}(O;\mathbb{R}^k)$ and $B \in \mathcal{B}(O)$.

Proof. The proof of inequality (3.4.5) for $\mathcal{E}$ and $\mathcal{E}_\rho^x$ can be obtained by arguing exactly as in the part of the proof of Proposition 2.1.14 devoted to property ((H8)). □

Later, given a sequence $\{u_n\}_n \subset \mathrm{SBV}(\Omega;\mathbb{R}^k)$ converging in $\mathrm{SD}^p_\star$, it will be essential to determine the limit of the truncated sequence $\Phi_R \circ u_n$ in a related topology. This is taken care of in the next lemma. We remark that the proof below cannot be reproduced in the case $p = 1$, unless we replace in Definition 3.1.2 weak*-convergence in $\mathcal{M}_b(\Omega;\mathbb{R}^{k\times d})$ with weak L^1-convergence.

Lemma 3.4.3. *Let $p > 1$, let $O \in \mathcal{O}(\Omega)$, let $(g, G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, let $\{u_n\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k)$ with $u_n \to_{SD_\star} (g, G)$ in O, and let $R > 0$. Consider the functions $g^R := \Phi_R \circ g$, $u_n^R := \Phi_R \circ u_n$, and $G^R := \nabla\Phi_R(g)G$. Then $\{u_n^R\}_n$ converges to g^R strongly in $L^1(O;\mathbb{R}^k)$ and $\{\nabla u_n^R\}_n$ converges weakly in $L^p(O;\mathbb{R}^{k\times d})$ to G^R. In particular, if $g \in L^\infty(O;\mathbb{R}^k)$ and $R > 2\|g\|_{L^\infty(O;\mathbb{R}^k)}$, then $g^R = g$ and $G^R = G$.*

Proof. By the Chain Rule for BV-functions (see Proposition 1.3.3) we have

$$\nabla u_n^R = \nabla\Phi_R(u_n)\nabla u_n \tag{3.4.7}$$

for every $n \in \mathbb{N}$. We pass to a subsequence $\{u_{n_\ell}\}_\ell$ that converges pointwise a.e. to g as $\ell \to +\infty$. As Φ_R is a C^1-function, from the pointwise convergence of $\{u_{n_\ell}\}_\ell$ and the Dominated Convergence Theorem it follows that $\{u_{n_\ell}^R\}_\ell$ converges to g^R strongly in $L^1(O;\mathbb{R}^k)$ and $\{\nabla\Phi_R(u_{n_\ell})\}_\ell$ converges to $\nabla\Phi_R(g)$ strongly in $L^q(O;\mathbb{R}^{k\times k})$ for every $1 \leq q < +\infty$ and pointwise a.e. in O.

We now prove that $\{\nabla\Phi_R(u_{n_\ell})\nabla u_{n_\ell}\}_\ell$ converges to G^R weakly in $L^p(O;\mathbb{R}^{k\times d})$. To this end, let us fix $\varphi \in L^q(O;\mathbb{R}^{k\times d})$, where $q := p/(p-1)$ denotes the conjugate exponent of p. We have

$$\int_O \varphi : (\nabla\Phi_R(u_{n_\ell})\nabla u_{n_\ell} - \nabla\Phi_R(g)G)\,\mathrm{d}x = \int_O \varphi : (\nabla\Phi_R(g)\nabla u_{n_\ell} - \nabla\Phi_R(g)G)\,\mathrm{d}x + \int_O \varphi : (\nabla\Phi_R(u_{n_\ell})\nabla u_{n_\ell} - \nabla\Phi_R(g)\nabla u_{n_\ell})\,\mathrm{d}x,$$

where for $A, L \in \mathbb{R}^{k\times d}$ the symbol $A : L$ denotes the Frobenius scalar product between A and L. By weak convergence in $L^p(O;\mathbb{R}^{k\times d})$ of $\{\nabla u_{n_\ell}\}_\ell$ to G, the first term converges to 0 as $\ell \to +\infty$. For the second term, we note instead that

$$\Big|\int_O \varphi : (\nabla\Phi_R(u_{n_\ell})\nabla u_{n_\ell} - \nabla\Phi_R(g)\nabla u_{n_\ell})\,\mathrm{d}x\Big| \le \|\nabla u_{n_\ell}\|_{L^p(O)}\Big(\int_O |\varphi|^q|\nabla\Phi_R(u_{n_\ell}) - \nabla\Phi_R(g)|^q\,\mathrm{d}x\Big)^{1/q}.$$

Since $\sup_\ell \|\nabla u_{n_\ell}\|_{L^p(O)} < +\infty$ by weak convergence, $|\nabla\Phi_R| \le 1$ by (1.3.13), and the sequence $\{\nabla\Phi_R(u_{n_\ell})\}_\ell$ converges to $\nabla\Phi_R(g)$ pointwise a.e. in O, an application of the Dominated Convergence Theorem shows that $\{\nabla\Phi_R(u_{n_\ell})\nabla u_{n_\ell}\}_\ell$ converges to G^R weakly in $L^p(O;\mathbb{R}^{k\times d})$ as $\ell \to +\infty$. As the limit does not depend on the chosen subsequence $\{n_\ell\}_\ell$, by (3.4.7) the full sequences $\{u_n^R\}_n$ and $\{\nabla u_n^R\}_n$ converge to g^R and G^R, respectively. This concludes the proof. □

We are now ready to prove Theorem 3.1.14.

Proof of Theorem 3.1.14. Throughout the proof, given $x \in \Omega$, we let $\mathcal{E}_0^x$ be the functional defined by

$$\mathcal{E}_0^x(u,B) := \int_B \Psi(x,\nabla u(y))\,\mathrm{d}y + \int_{J_u\cap B} \gamma^0(x,[u](y),\nu_u(y))\,\mathrm{d}\mathcal{H}^{d-1}(y)$$

for every $u \in \mathrm{SBV}(Q;\mathbb{R}^k)$ and $B \in \mathcal{B}(Q)$. We split the proof into two parts, addressing first the statement concerning the bulk energy density. We divide its proof into several steps.

Step 1. (Bulk: Auxiliary cell formula) For every $x \in \Omega$ and $A, L \in \mathbb{R}^{k\times d}$ we set

$$\widetilde{H}_p(x,A,L) := \inf\Big\{\liminf_{n\to+\infty} \mathcal{E}_0^x(u_n,Q)\colon\ \{u_n\}_n \subset \mathrm{SBV}(Q;\mathbb{R}^k),\ u_n \to \ell_{0,0,A} \text{ strongly in } L^1(Q;\mathbb{R}^k),\ \nabla u_n \rightharpoonup L \text{ weakly in } L^p(Q;\mathbb{R}^{k\times d})\Big\}. \tag{3.4.8}$$

We will prove that for $\mathcal{L}^d$-a.e. $x \in \Omega$ and for all $A, L \in \mathbb{R}^{k\times d}$ it holds

$$\widetilde{H}_p(x,A,L) \le f_{\mathrm{bulk}}(x,A,L) \le H_p(x,A,L). \tag{3.4.9}$$

Since by (Ψ2), (3.1.17), and Remark 3.1.12 the densities $\Psi(x,\cdot)$ and $\gamma^0(x,\cdot)$ satisfy assumptions $(\mathscr{H}1)_p$, $(\mathscr{H}2)$, and $(\mathscr{H}4)$ of [76], it follows from [76, Proposition 3.1] that $\widetilde{H}_p(x,A,L) = H_p(x,A,L)$ for every $x \in \Omega$ and every $A, L \in \mathbb{R}^{k\times d}$. Therefore, proving (3.4.9) is sufficient to show that in (3.1.19) we can write H_p in place of f_{bulk}.

By definition of $f_{\rm bulk}$ (see also Remark 3.1.8) and by Theorem 3.3.5 applied with $\mathcal{F} = I^p_\star$ we have

$$f_{\rm bulk}(x,A,L) = \limsup_{\rho\to 0^+} \frac{\mathfrak{m}(\ell_{x,0,A}, L, Q(x,\rho))}{\rho^d} = \limsup_{\rho\to 0^+} \frac{I^p_\star(\ell_{x,0,A}, L, Q(x,\rho))}{\rho^d} < +\infty \tag{3.4.10}$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$ and for all $A, L \in \mathbb{R}^{k\times d}$. Therefore, to prove (3.4.9) it suffices to show that

$$\widetilde{H}_p(x,A,L) \le f_{\rm bulk}(x,A,L) \le H_p(x,A,L) \tag{3.4.11}$$

for all $x \in \Omega$ such that (3.4.10) holds and for all $A, L \in \mathbb{R}^{k\times d}$.

Step 2. (Bulk: $\widetilde{H}_p \le f_{\rm bulk}$) We let $x \in \Omega$ be such that (3.4.10) holds, and let $A, L \in \mathbb{R}^{k\times d}$. For any $\rho > 0$ we consider a sequence $\{u^\rho_n\}_n \subset {\rm SBV}(\Omega;\mathbb{R}^k)$ with $u^\rho_n \to_{SD_\star} (\ell_{x,0,A}, L)$ in $Q(x,\rho)$, $\{u^\rho_n\}_n$ converging to $\ell_{x,0,A}$ pointwise a.e. in $Q(x,\rho)$, and

$$\lim_{n\to+\infty} \mathcal{E}(u^\rho_n, Q(x,\rho)) = I^p_\star(\ell_{x,0,A}, L, Q(x,\rho)). \tag{3.4.12}$$

We introduce the functions $v^\rho_n \in {\rm SBV}(Q;\mathbb{R}^k)$ defined by $v^\rho_n(y) := \frac{1}{\rho} u^\rho_n(x+\rho y)$ for $y \in Q$, which satisfy

$$v^\rho_n \underset{SD_\star}{\to} (\ell_{0,0,A}, L) \text{ in } Q \ \text{ and } v^\rho_n \to \ell_{0,0,A} \text{ pointwise a.e. as } n \to +\infty, \tag{3.4.13}$$

$$\nabla v^\rho_n(y) = \nabla u^\rho_n(x+\rho y) \quad \text{for } \mathcal{L}^d\text{-a.e. } y \in Q, \tag{3.4.14}$$

$$[v^\rho_n](y) = \frac{1}{\rho}[u^\rho_n](x+\rho y) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in J_{v^\rho_n} = (J_{u^\rho_n} - x)/\rho. \tag{3.4.15}$$

Using (3.4.14) and (3.4.15), a change of variables then shows that for all $n \in \mathbb{N}$ and $\rho > 0$ we have

$$\mathcal{E}(u^\rho_n, Q(x,\rho)) = \rho^d \mathcal{E}^x_\rho(v^\rho_n, Q), \tag{3.4.16}$$

where $\mathcal{E}^x_\rho$ is the functional given by (3.4.6).

We set $R := 2\|\ell_{0,0,A}\|_{L^\infty(Q;\mathbb{R}^k)} \le 2\sqrt{d}|A|$ and $R_i := \tau^{i-1}R$ for all $i \in \mathbb{N}$. Given $N \in \mathbb{N}$, we may apply Lemma 3.4.2 to the functional $\mathcal{E}^\rho_x$ and to v^ρ_n. We find $i_n = i(n,N,\rho) \in \{1,\dots,N\}$ such that

$$\mathcal{E}^x_\rho(\Phi_{R_{i_n}} \circ v^\rho_n, Q) \le \Big(1 + \frac{C}{N}\Big)\mathcal{E}^x_\rho(v^\rho_n, Q) + \mathcal{L}^d(\{|v^\rho_n| \ge R\}) + \frac{C}{N}\mathcal{L}^d(Q) \tag{3.4.17}$$

for $C > 0$ independent of n, ρ, N, A, and L. We remark that by (3.4.13) we have $\limsup_{n\to+\infty} \chi_{\{|v^\rho_n|\ge R\}}(y) \le \chi_{\{|\ell_{0,0,A}|\ge R\}}(y)$ for $\mathcal{L}^d$-a.e. $y \in Q$, so that by Fatou's lemma

$$\limsup_{n\to+\infty} \mathcal{L}^d\big(\{|v^\rho_n| \ge R\}\big) \le \mathcal{L}^d\big(\{|\ell_{0,0,A}| \ge R\} \cap Q\big) = 0 \quad \text{for all } \rho > 0. \tag{3.4.18}$$

We set $w^\rho_n := \Phi_{R_{i_n}} \circ v^\rho_n \in {\rm SBV}(Q;\mathbb{R}^k)$. Then, in view of (3.4.13), Lemma 3.4.3 gives

$$\{w^\rho_n\}_n \text{ converges to } \ell_{0,0,A} \text{ strongly in } L^1(Q;\mathbb{R}^k) \text{ as } n \to +\infty,$$
$$\{\nabla w^\rho_n\}_n \text{ converges to } L \text{ weakly in } L^p(Q;\mathbb{R}^{k\times d}) \text{ as } n \to +\infty,$$

so that, for each $\rho > 0$, the sequence $\{w_n^\rho\}_n$ is a competitor for the minimisation problem in (3.4.8). Therefore,

$$\widetilde{H}_p(x, A, L) \le \liminf_{n\to+\infty} \mathcal{E}_0^x(w_n^\rho, Q) \quad \text{for all } \rho > 0. \tag{3.4.19}$$

In order to use (3.4.16) we still need to compare $\mathcal{E}_0^x(w_n^\rho, Q)$ and $\mathcal{E}_\rho^x(w_n^\rho, Q)$. To this end, we begin by observing that, by the fact that ϑ is nondecreasing, (γ6), (3.4.6), (3.4.17), and the fact that $\|w_n^\rho\|_{L^\infty(Q)} \le \tau^N R$, we obtain

$$\begin{aligned}
&\Big| \int_{J_{w_n^\rho}\cap Q} \gamma^0\big(x+\rho y, [w_n^\rho](y), \nu_{w_n^\rho}(y)\big)\mathrm{d}\mathcal{H}^{d-1}(y) - \int_{J_{w_n^\rho}\cap Q} \frac{1}{\rho}\gamma\big(x+\rho y, \rho[w_n^\rho](y), \nu_{w_n^\rho}(y)\big)\mathrm{d}\mathcal{H}^{d-1}(y)\Big| \\
&\qquad \le \vartheta\big(\tau^N 2R\rho\big) \int_{J_{w_n^\rho}\cap Q} \frac{1}{\rho}\gamma\big(x+\rho y, \rho[w_n^\rho](y), \nu_{w_n^\rho}(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\qquad \le \vartheta\big(\tau^N 2R\rho\big)\Big(\Big(1+\frac{C}{N}\Big)\mathcal{E}_\rho^x(v_n^\rho, Q) + \mathcal{L}^d(\{|v_n^\rho| \ge R\}) + \frac{C}{N}\mathcal{L}^d(Q)\Big).
\end{aligned} \tag{3.4.20}$$

In light of (3.4.12), (3.4.16), and (3.4.18), we have

$$\begin{aligned}
&\limsup_{\rho\to0^+}\limsup_{n\to+\infty}\ \vartheta\big(\tau^N 2R\rho\big)\Big(\Big(1+\frac{C}{N}\Big)\mathcal{E}_\rho^x(v_n^\rho, Q) + \mathcal{L}^d(\{|v_n^\rho| \ge R\}) + \frac{C}{N}\mathcal{L}^d(Q)\Big) \\
&\qquad \le \limsup_{\rho\to0^+} \vartheta\big(\tau^N 2R\rho\big)\Big(\Big(1+\frac{C}{N}\Big)\frac{I_\star^p(\ell_{x,0,A}, L, Q(x,\rho))}{\rho^d} + \frac{C}{N}\mathcal{L}^d(Q)\Big) = 0,
\end{aligned} \tag{3.4.21}$$

where in the last equality we have used that ϑ is a continuous function with $\vartheta(0) = 0$, and that x is such that (3.4.10) holds. Exploiting (3.1.18), (3.4.10), the fact that $\|w_n^\rho\|_{L^\infty(Q)} \le \tau^N R$, and ($\gamma$2) with a similar argument we can prove that

$$\begin{aligned}
\limsup_{\rho\to0^+}\limsup_{n\to+\infty}\Big|\int_{J_{w_n^\rho}\cap Q} &\gamma^0\big(x+\rho y, [w_n^\rho](y), \nu_{w_n^\rho}(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&- \int_{J_{w_n^\rho}\cap Q} \gamma^0\big(x, [w_n^\rho](y), \nu_{w_n^\rho}(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big| = 0.
\end{aligned} \tag{3.4.22}$$

In a similar fashion, using (Ψ5) we can also show that

$$\limsup_{\rho\to0^+}\limsup_{n\to+\infty}\Big|\int_Q \Psi\big(x+\rho y, \nabla w_n^\rho\big)\,\mathrm{d}y - \int_Q \Psi\big(x, \nabla w_n^\rho\big)\,\mathrm{d}y\Big| = 0. \tag{3.4.23}$$

We are now ready to conclude the proof of Step 2. Combining (3.4.10), (3.4.12), and (3.4.16)–(3.4.23), we obtain

$$\begin{aligned}
\widetilde{H}_p(x, A, L) &\le \limsup_{\rho\to0^+}\limsup_{n\to+\infty} \mathcal{E}_0^x(w_n^\rho, Q) = \limsup_{\rho\to0^+}\limsup_{n\to+\infty} \mathcal{E}_\rho^x(w_n^\rho, Q) \\
&\le \limsup_{\rho\to0^+}\limsup_{n\to+\infty} \Big(1+\frac{C}{N}\Big)\mathcal{E}_\rho^x(v_n^\rho, Q) + \frac{C}{N}\mathcal{L}^d(Q) \\
&\le \limsup_{\rho\to0^+} \Big(1+\frac{C}{N}\Big)\frac{I_\star^p(\ell_{x,0,A}, L, Q(x,\rho))}{\rho^d} + \frac{C}{N}\mathcal{L}^d(Q) \\
&= \Big(1+\frac{C}{N}\Big) f_{\mathrm{bulk}}(x, A, L) + \frac{C}{N}\mathcal{L}^d(Q).
\end{aligned}$$

Letting $N \to +\infty$, we obtain the first inequality in (3.4.11).

Step 3. (Bulk: $f_{\rm bulk} \le H_p$) The second inequality in (3.4.11) follows by an argument similar to the one presented in *Step 2*. Let $\eta > 0$ and let $u \in \mathcal{C}^{\rm bulk}_{p,\star}(A,L)$ (see (3.1.21)) be such that

$$\int_Q \Psi\big(x, \nabla u(y)\big)\,\mathrm{d}y + \int_{J_u\cap Q} \gamma^0\big(x,[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le H_p(x,A,L) + \eta. \tag{3.4.24}$$

For every $\rho > 0$, we consider the function $v^\rho := \rho u(\frac{\cdot - x}{\rho}) \in \mathrm{SBV}(Q(x,\rho);\mathbb{R}^k)$ and observe that

$$\nabla v^\rho(z) = \nabla u(\tfrac{z-x}{\rho}) \quad \text{for } \mathcal{L}^d\text{-a.e. } z \in Q(x,\rho), \tag{3.4.25}$$

$$[v^\rho](z) = \rho[u](\tfrac{z-x}{\rho}) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } z \in J_{v^\rho}. \tag{3.4.26}$$

In particular, from (3.4.25) and the fact that $u \in C^{\rm bulk}_{p,\star}(A,L)$ we deduce that $v^\rho = \ell_{x,0,A}$ close to $\partial Q(x,\rho)$ and, by a change of variables,

$$\int_{Q(x,\rho)} \nabla v^\rho\,\mathrm{d}y = \int_{Q(x,\rho)} L\,\mathrm{d}y,$$

so that v^ρ is a competitor for the minimisation problem $\mathfrak{m}(\ell_{x,0,A},L,Q(x,\rho))$. As such,

$$\mathfrak{m}(\ell_{x,0,A},L,Q(x,\rho)) \le \mathcal{E}(v^\rho,Q(x,\rho)). \tag{3.4.27}$$

Using (3.4.25) and (3.4.26), a change of variables gives

$$\mathcal{E}(v^\rho,Q(x,\rho)) = \rho^d \int_Q \Psi\big(x+\rho y,\nabla u(y)\big)\,\mathrm{d}y + \rho^{d-1}\int_{J_u\cap Q} \gamma\big(x+\rho y,\rho[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

From (Ψ5), (γ4), and (3.1.14) it follows that

$$\begin{aligned}\Big|\int_Q \Psi\big(x+\rho y,\nabla u(y)\big)\,\mathrm{d}y - \frac{1}{\rho}\int_{J_u\cap Q}\gamma\big(x+\rho y,\rho[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ - \int_Q \Psi\big(x,\nabla u(y)\big)\,\mathrm{d}y + \frac{1}{\rho}\int_{J_u\cap Q}\gamma\big(x,\rho[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\Big|\\ \le \omega_\Psi(\rho)\int_Q 2\overline{C}_\Psi(|\nabla u|^p+1)\,\mathrm{d}y + \omega_\gamma(\rho)\int_{J_u\cap Q}|[u](y)|\,\mathrm{d}\mathcal{H}^{d-1}(y).\end{aligned} \tag{3.4.28}$$

By inequality (γ2) we also get

$$\frac{1}{\rho}\gamma\big(x,\rho[u](y),\nu_u(y)\big) \le C_\gamma|[u](y)|$$

for $\mathcal{H}^{d-1}$-a.e. $y \in J_u \cap Q$. Hence, combining (3.4.27) with (3.4.28), dividing by ρ^d, letting $\rho \to 0^+$, and using (3.1.15), the Dominated Convergence Theorem, and the fact that ω_Ψ and ω_γ are continuous functions with $\omega_\Psi(0) = 0$ and $\omega_\gamma(0) = 0$, we get

$$f_{\rm bulk}(x,A,L) \le \int_Q \Psi\big(x,\nabla u(y)\big)\,\mathrm{d}y + \int_{J_u\cap Q}\gamma^0\big(x,[u](y),\nu_u(y)\big)\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

Finally, recalling (3.4.24), we conclude $f_{\rm bulk}(x,A,L) \le H_p(x,A,L)+\eta$. Letting $\eta \to 0^+$ we conclude the proof of the second inequality in (3.4.11) and thus of the part of the statement concerning the bulk energy density.

Step 4. (Surface: Auxiliary cell formula) We are left with proving the part of the statement concerning the surface energy density. The proof can be obtained by arguing similarly to [76, Theorem 4.5], taking into account some modifications due to the different growth conditions.

The first step is to define an auxiliary cell formula. For every $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we set

$$\widetilde{h}_p(x,\zeta,\nu) \coloneqq \inf\Big\{ \liminf_{n\to+\infty} \int_{J_{u_n}\cap Q_\nu} \gamma(x,[u_n](y),\nu_{u_n}(y))\,\mathrm{d}\mathcal{H}^{d-1}(y)\colon \{u_n\}_n \subset \mathrm{SBV}(Q_\nu;\mathbb{R}^k) \\ u_n \to u_{\zeta,\nu} \text{ in } L^1(Q_\nu;\mathbb{R}^k),\ \nabla u_n \to 0 \text{ strongly in } L^p(Q_\nu;\mathbb{R}^{k\times d})\Big\},$$

where we have set $u_{\zeta,\nu} \coloneqq u_{0,\zeta,0,\nu}$ (see (3.1.5)). Using arguments very similar to [76, Proposition 4.2], properly taking into account also the ν-dependence (see [149, Remark 3.5, Point 5]), one can show that

$$\widetilde{h}_p(x,\zeta,\nu) = h_p(x,\zeta,\nu) \quad \text{for all } x\in\Omega,\ \zeta\in\mathbb{R}^k, \text{ and } \nu\in\mathbb{S}^{d-1}.$$

Hence, in view of Theorem 3.3.5 and the definition of f_{surf} (see also Remark 3.1.8), to conclude, it is sufficient to show that

$$\widetilde{h}_p(x,\zeta,\nu) \le f_{\mathrm{surf}}(x,\zeta,\nu) \le h_p(x,\zeta,\nu) \tag{3.4.29}$$

for every $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ such that

$$\begin{aligned} f_{\mathrm{surf}}(x,\zeta,\nu) &= \limsup_{\rho\to0^+} \frac{\mathfrak{m}(u_{x,\zeta,0,\nu},0,Q_\nu(x,\rho))}{\rho^{d-1}} \\ &= \limsup_{\rho\to0^+} \frac{I^p_\star(u_{x,\zeta,0,\nu},0,Q_\nu(x,\rho))}{\rho^{d-1}} < +\infty. \end{aligned} \tag{3.4.30}$$

Step 5. (Surface: $\widetilde{h}_p \le f_{\mathrm{surf}} \le h_p$) Let us fix $x \in \Omega$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ as above. Since for any $u \in \mathcal{C}^{\mathrm{surf}}_{p,\star}(\zeta,\nu)$ (see (3.1.22)) the function defined by $v(y) \coloneqq u((y-x)/\rho)$ for $y \in Q_\nu(x,\rho)$ is a competitor for $\mathfrak{m}(u_{x,\zeta,0,\nu},0,Q_\nu(x,\rho))$, the first equality in (3.4.30) and a change of variables prove the second inequality in (3.4.29).

To prove $\widetilde{h}_p \le f_{\mathrm{surf}}$, we argue as follows. For every $\rho > 0$ we let $\{u^\rho_n\}_n$ be a sequence in $\mathrm{SBV}(Q_\nu(x,\rho);\mathbb{R}^k)$ such that $u^\rho_n \underset{SD_\star}{\to} (u_{x,\zeta,0,\nu},0)$ in $Q_\nu(x,\rho)$ and

$$\lim_{n\to+\infty} \mathcal{E}(u^\rho_n, Q_\nu(x,\rho)) = I^p_\star(u_{x,\zeta,0,\nu},0,Q_\nu(x,\rho)). \tag{3.4.31}$$

We let $R \coloneqq 2|\zeta|$, $N \in \mathbb{N}$, and apply Lemma 3.4.2 to obtain $i_n = i(n,N,\rho)$ such that, setting $v^\rho_n \coloneqq \Phi_{R_{i_n}} \circ u_n \in \mathrm{SBV}(Q_\nu(x,\rho);\mathbb{R}^k)$, we have

$$\mathcal{E}(v^\rho_n, Q_\nu(x,\rho)) \le \Big(1+\frac{C}{N}\Big)\mathcal{E}(u^\rho_n, Q_\nu(x,\rho)) + C\rho^d \tag{3.4.32}$$

and $\|v^\rho_n\|_{L^\infty(Q_\nu(x,\rho))} \le \tau^N R$. Thanks to Lemma 3.4.3, the sequence $\{v^\rho_n\}_n$ converges to $u_{x,\zeta,0,\nu}$ strongly in $L^1(Q_\nu(x,\rho);\mathbb{R}^k)$ and $\{\nabla v^\rho_n\}_n$ converges to 0 weakly in $L^p(Q_\nu(x,\rho))$. Moreover, from (3.4.30)–(3.4.32) we deduce that

$$\limsup_{\rho\to0^+}\limsup_{n\to+\infty} \frac{1}{\rho^{d-1}} \int_{J_{v^\rho_n}\cap Q_\nu(x,\rho)} \gamma(y,[v^\rho_n](y),\nu_{v^\rho_n}(y))\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \Big(1+\frac{C}{N}\Big) f_{\mathrm{surf}}(x,\zeta,\nu),$$

which by ($\gamma 2$), ($\gamma 4$), and (3.4.30) implies

$$\limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{1}{\rho^{d-1}}\int_{J_{v_n^\rho}\cap Q_\nu(x,\rho)}\gamma(x,[v_n^\rho](y),\nu_{v_n^\rho}(y))\,\mathrm{d}\mathcal{H}^{d-1}(y)\le\Big(1+\frac{C}{N}\Big)f_{\mathrm{surf}}(x,\zeta,\nu). \tag{3.4.33}$$

We now consider the functions $\{w_n^\rho\}_n\subset \mathrm{SBV}(Q_\nu;\mathbb{R}^k)$ defined by $w_n^\rho(y):=v_n^\rho(x+\rho y)$ for $y\in Q_\nu$, such that

$$\begin{aligned}\nabla w_n^\rho(y)&=\rho\nabla v_n^\rho(x+\rho y)\quad\text{ for }\mathcal{L}^d\text{-a.e. } y\in Q_\nu,\\ [w_n^\rho](y)&=[v_n^\rho](x+\rho y)\quad\text{ for }\mathcal{H}^{d-1}\text{-a.e. } y\in Q_\nu\cap J_{w_n^\rho}=(J_{v_n^\rho}-x)/\rho.\end{aligned}$$

As $\{v_n^\rho\}_n$ converges to $u_{x,\zeta,0,\nu}$, for every $\rho>0$ the sequence $\{w_n^\rho\}_n$ converges to $u_{\zeta,\nu}$ strongly in $L^1(Q_\nu;\mathbb{R}^k)$. A change of variables, ($\Psi 4$), and (3.4.30)–(3.4.32) then show that

$$\begin{aligned}\limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\int_{Q_\nu}|\nabla w_n^\rho|^p\,\mathrm{d}y&=\limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\rho^p}{\rho^d}\int_{Q_\nu(x,\rho)}|\nabla v_n^\rho|^p\,\mathrm{d}y\\ &\le\limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\rho^{p-1}}{c_W}\Big(\frac{\mathcal{E}(v_n^\rho,Q_\nu(x,\rho))}{\rho^{d-1}}+\frac{\rho}{c_W}\Big)\\ &\le\Big(1+\frac{C}{N}\Big)\limsup_{\rho\to 0^+}\frac{\rho^{p-1}}{c_W}f_{\mathrm{surf}}(x,\zeta,\nu)=0.\end{aligned}$$

Finally, with a diagonal argument we can find sequences $\{\rho_\ell\}_\ell$ and $\{n_\ell\}_\ell$, with $\rho_\ell\to 0^+$ and $n_\ell\to+\infty$ as $\ell\to+\infty$, such that the functions defined by $z_\ell:=w_{n_\ell}^{\rho_\ell}$ are competitors for the minimisation problem defining $\widetilde{h}_p(x,\zeta,\nu)$. Therefore, (3.4.33) and a change of variables give

$$\widetilde{h}_p(x,\zeta,\nu)\le\Big(1+\frac{C}{N}\Big)f_{\mathrm{surf}}(x,\zeta,\nu).$$

Letting $N\to+\infty$, we obtain the first inequality in (3.4.29). This concludes the proof. □

We now prove Proposition 3.1.15.

Proof of Proposition 3.1.15. The properties concerning the bulk energy density H_p are a consequence of [31, Theorem 2.10]. It is immediate to check that h_p satisfies ($\gamma 1$)–($\gamma 3$), while ($\gamma 4$) can be obtained by arguing exactly as in [31, Theorem 2.10].

We now check that h_p satisfies ($\gamma 5$). To this end, we let $x\in\Omega$, $\zeta_1,\zeta_2\in\mathbb{R}^k$, with $\beta_\gamma|\zeta_1|\le|\zeta_2|$ and $\zeta_2\neq 0$, and $\nu\in\mathbb{S}^{d-1}$. We consider $u_2\in\mathcal{C}^{\mathrm{surf}}_{p,\star}(\zeta_2,\nu)$ (recall (3.1.22)), set $\lambda:=|\zeta_1|/|\zeta_2|$, and let $R\in\mathrm{SO}(k)$ be such that $R(\lambda\zeta_2)=\zeta_1$. Then, $u_1:=\lambda R(u_2)\in\mathcal{C}^{\mathrm{surf}}_{p,\star}(\zeta_1,\nu)$, and clearly $\beta_\gamma|[u_1]|\le\beta_\gamma\lambda|R[u_2]|\le|[u_2]|$, where we have used that $\beta_\gamma\lambda\le 1$. Since γ satisfies ($\gamma 5$), this implies

$$h_p(x,\zeta_1,\nu)\le\int_{J_{u_1}\cap Q_\nu}\gamma(x,[u_1](y),\nu_{u_1}(y))\,\mathrm{d}\mathcal{H}^{d-1}(y)\le\int_{J_{u_2}\cap Q_\nu}\gamma(x,[u_2](y),\nu_{u_2}(y))\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

From the arbitrariness of $u_2\in\mathcal{C}^{\mathrm{surf}}_{p,\star}(\zeta_2,\nu)$, we conclude that h_p satisfies ($\gamma 5$). □

Proof of Theorem 3.1.16 The rest of the section is devoted to the proof of Theorem 3.1.16. We have already shown in Theorems 3.1.13 and 3.1.14 that the bulk and surface parts $(I_\star^p)^a$ and $(I_\star^p)^j$ admit an integral representation, so that we are left with proving that $(I_\star^p)^c$ can be represented as an integral functional, as well (see Definition 3.1.5). This

is done in three steps, following an argument similar to the one employed in the proof of Theorem 2.3.16. We first use Lemma 3.4.4 below to deduce that $(I^p_\star)^c(g,G,\cdot) = (I^p_\star)^c(g,0,\cdot)$ for all $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. Then, for $\delta > 0$ we will consider a perturbed functional $I^{p,\delta}_\star(g,B)$ defined for $g \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$ which satisfies all the hypotheses of [41, Theorem 3.12], and thus admits a complete integral representation, including its Cantor part. Letting $\delta \to 0^+$ and using Lemma 3.D.1, we recover an integral representation for $(I^p_\star)^c(g,0,B)$ for all $g \in \mathrm{BV}(\Omega;\mathbb{R}^k)$, which can then be extended to all $g \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ by a truncation argument based on Lemma 3.4.5 below.

We now present Lemma 3.4.4, which states that $I^p_\star$ is locally Lipschitz. Note that both in this lemma and in Lemma 3.4.5 below we do not assume that Ψ and γ are independent of the x-variable, a property that will only be used to apply [41, Theorem 3.12] in the proof of Theorem 3.1.16.

Lemma 3.4.4. *Assume that Ψ and γ satisfy* (Ψ1),(Ψ2), (Ψ4) *and* (γ1)–(γ3). *Let* (g_1,G_1), $(g_2,G_2) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$. *Then for every $B \in \mathcal{B}(\Omega)$ it holds*

$$\begin{aligned}\big|I^p_\star(g_1,G_1,B) - I^p_\star(g_2,G_2,B)\big| &\le C_\gamma \mathcal{V}(g_1-g_2,B)\\ &\quad + C\|G_1-G_2\|^p_{L^p(B)} + \widehat{C}C\|G_1-G_2\|_{L^p(B)}, \qquad (3.4.34)\end{aligned}$$

where $\widehat{C} \ge 0$ is given by

$$\widehat{C} \coloneqq I^p_\star(g_1,G_1,B)^{(p-1)/p} + I^p_\star(g_2,G_2,B)^{(p-1)/p} + (\mathcal{L}^d(B))^{(p-1)/p} \qquad (3.4.35)$$

and $C > 0$ is a constant depending only on c_Ψ, C_Ψ, C_γ, p, and d.

Proof. As $I^p_\star \in \mathfrak{F}^p_\star$, in view of (3.1.13), it is enough to prove (3.4.34) for a fixed $O \in \mathcal{O}(\Omega)$. Arguing by symmetry, it is enough to show that

$$\begin{aligned}I^p_\star(g_1,G_1,O) &\le I^p_\star(g_2,G_2,O) + C_\gamma\mathcal{V}(g_1-g_2,O) + C\|G_1-G_2\|^p_{L^p(O)}\\ &\quad + C\Big(I^p_\star(g_2,G_2,O)^{(p-1)/p} + (\mathcal{L}^d(O))^{(p-1)/p}\Big)\|G_1-G_2\|_{L^p(O)} \qquad (3.4.36)\end{aligned}$$

for $C > 0$ depending only on c_Ψ, C_Ψ, C_γ, p, and d.

We consider a sequence of functions $\{u_n\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k)$ with $u_n \underset{SD_\star}{\to} (g_2,G_2)$ in O and

$$\lim_{n\to+\infty} \mathcal{E}(u_n,O) = I^p_\star(g_2,G_2,O). \qquad (3.4.37)$$

We use Alberti's Theorem 3.2.5 to find $h \in \mathrm{SBV}(O;\mathbb{R}^k)$ such that $\nabla h = G_1 - G_2$ $\mathcal{L}^d$-a.e. in O. We then apply Lemma 3.2.3 (see Remark 3.2.4) to $g_1 - g_2 - h$ and obtain a sequence of piecewise constant functions $\{h_n\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k)$ converging to $g_1 - g_2 - h$ in $L^0(O;\mathbb{R}^k)$ and such that

$$\lim_{n\to+\infty} \mathcal{V}(h_n,O) = \mathcal{V}(g_1-g_2-h,O) \le \mathcal{V}(g_1-g_2,O) + (C(d)+1)\|G_1-G_2\|_{L^1(O)}, \qquad (3.4.38)$$

where $C(d)$ is the constant of Theorem 3.2.5. We set $v_n \coloneqq u_n + h + h_n$ and observe that $v_n \to g_1$ in $L^0(O;\mathbb{R}^k)$ and $\nabla v_n \rightharpoonup G_1$ weakly in $L^p(O;\mathbb{R}^{k\times d})$ as $n \to +\infty$, so that $v_n \underset{SD_\star}{\to} (g_1,G_1)$ in O. In particular,

$$I^p_\star(g_1,G_1,O) \le \liminf_{n\to+\infty} \mathcal{E}(v_n,O). \qquad (3.4.39)$$

As $J_{v_n} \subset J_{u_n} \cup J_{h_n} \cup J_h$ for every $n \in \mathbb{N}$, we can choose an orientation for the normals such that, for every $n \in \mathbb{N}$,

$$[v_n] = [u_n] + [h_n] + [h] \quad \mathcal{H}^{d-1}\text{-a.e. on } (J_{u_n} \cup J_{h_n} \cup J_h) \cap O.$$

By the previous observations, using also ($\Psi 2$), ($\gamma 2$), ($\gamma 3$), the fact that $\nabla h = G_1 - G_2$ $\mathcal{L}^d$-a.e. in O, and Hölder's inequality we get

$$\begin{aligned}
\mathcal{E}(v_n, O) &= \int_O \Psi(x, \nabla u_n + \nabla h)\,\mathrm{d}x + \int_{J_{v_n}\cap O} \gamma(x, [v_n], \nu_{v_n})\,\mathrm{d}\mathcal{H}^{d-1} \\
&\le \int_O \Psi(x, \nabla u_n)\,\mathrm{d}x + C_\Psi \int_O |\nabla h|(1 + |\nabla u_n + \nabla h|^{p-1} + |\nabla u_n|^{p-1})\,\mathrm{d}x \\
&\quad + \int_{J_{u_n}\cap O} \gamma(x, [u_n], \nu_{v_n})\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_{h_n}\cap O} \gamma(x, [h_n], \nu_{h_n})\,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + \int_{J_h\cap O} \gamma(x, [h], \nu_h)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\le \int_O \Psi(x, \nabla u_n)\,\mathrm{d}x + \int_{J_{u_n}\cap O} \gamma(x, [u_n], \nu_{u_n})\,\mathrm{d}\mathcal{H}^{d-1} + C_\gamma \mathcal{V}(h_n, O) \\
&\quad + C\big(\|G_1 - G_2\|^p_{L^p(O)} + \|G_1 - G_2\|_{L^1(O)}\big) + C\|\nabla u_n\|^{p-1}_{L^p(O)}\|G_1 - G_2\|_{L^p(O)}
\end{aligned} \tag{3.4.40}$$

for $C > 0$ depending only on c_Ψ, C_Ψ, C_γ, and p. In light of ($\Psi 4$) and (3.4.37), we have

$$\begin{aligned}
\|\nabla u_n\|^{p-1}_{L^p(O)} &\le \Big(\frac{1}{c_\Psi}\int_O \Big(\Psi(x, \nabla u_n) + \frac{1}{c_\Psi}\Big)\,\mathrm{d}x\Big)^{(p-1)/p} \\
&\le 2\Big(\frac{1}{c_\Psi}\Big)^{(p-1)/p} I^p_\star(g_2, G_2, O)^{(p-1)/p} + \Big(\frac{1}{c^2_\Psi}\Big)^{(p-1)/p} (\mathcal{L}^d(O))^{(p-1)/p}
\end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Hence, in view of (3.4.37), (3.4.38), (3.4.40), and Hölder's inequality we deduce

$$\begin{aligned}
\liminf_{n\to+\infty} \mathcal{E}(v_n, O) &\le I^p_\star(g_2, G_2, O) + C_\gamma \mathcal{V}(g_1 - g_2, O) + C\|G_1 - G_2\|^p_{L^p(O)} \\
&\quad + C\Big(I^p_\star(g_2, G_2, O)^{(p-1)/p} + (\mathcal{L}^d(O))^{(p-1)/p}\Big)\|G_1 - G_2\|_{L^p(O)}
\end{aligned}$$

for a different constant $C > 0$, depending only on c_Ψ, C_Ψ, C_γ, p, and d. By (3.4.39), this leads to (3.4.36), which concludes the proof. □

We now study truncation properties of $I^p_\star(\cdot, 0, \cdot)$.

Lemma 3.4.5. *Let $p > 1$ and assume that Ψ and γ satisfy ($\Psi 1$)–($\Psi 4$) and ($\gamma 1$),($\gamma 2$), and ($\gamma 5$). Then for every $N \in \mathbb{N}$, $g \in \mathrm{GBV}_\star(\Omega; \mathbb{R}^k)$, $B \in \mathcal{B}(\Omega)$, and $R > 0$ it holds*

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N I^p_\star\big(\Phi_{R_i} \circ g, 0, B\big) &\le \Big(1 + \frac{C}{N}\Big) I^p_\star(g, 0, B) \\
&\quad + C\mathcal{L}^d\big(\{x \in B\colon\ |g(x)| \ge R\}\big) + \frac{C}{N}\mathcal{L}^d(B)
\end{aligned} \tag{3.4.41}$$

for $C > 0$ depending only on $c_\Psi, \overline{C}_\Psi, \beta_\gamma, c_\gamma, C_\gamma$, and p, where we have set $R_i := \tau^{i-1}R$.

Moreover, for every $g \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, $B \in \mathcal{B}(\Omega)$, and every sequence $\{R_N\}_N \subset (0,+\infty)$, with $R_N \to +\infty$ as $N \to +\infty$, it holds

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} (I^p_\star)^c(\Phi_{\tau^{i-1}R_N} \circ g, 0, B) = (I^p_\star)^c(g,0,B). \tag{3.4.42}$$

Proof. As $I^p_\star \in \mathfrak{F}^p_\star$, in view of (3.1.13), it is enough to prove (3.4.41) and (3.4.42) for all $O \in \mathcal{O}(\Omega)$. Let us fix $N \in \mathbb{N}$, $g \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, $R > 0$, and $O \in \mathcal{O}(\Omega)$. We let $\{u_n\}_n \subset \mathrm{SBV}(O;\mathbb{R}^k)$ be a sequence with $u_n \underset{SD_\star}{\to} (g,0)$ in O such that

$$\lim_{n\to+\infty} \mathcal{E}(u_n, O) = I^p_\star(g,0,O). \tag{3.4.43}$$

We set $v^i_n := \Phi_{R_i} \circ u_n$ and $v^i := \Phi_{R_i} \circ g$. An application of Lemma 3.4.3 yields

$$v^i_n \underset{SD_\star}{\to} (v^i, 0) \quad \text{in } O \text{ as } n \to +\infty \tag{3.4.44}$$

for all $i \in \{1,\dots,N\}$. Thanks to Lemma 3.4.2, there exists $C > 0$, depending only on $c_\Psi, \overline{C}_\Psi, \beta_\gamma, c_\gamma, C_\gamma$, such that

$$\frac{1}{N}\sum_{i=1}^{N} \mathcal{E}(v^i_n, O) \le \Big(1 + \frac{C}{N}\Big)\mathcal{E}(u_n, O) + C\mathcal{L}^d(\{x \in O\colon\ |u_n(x)| \ge R\}) + \frac{C}{N}\mathcal{L}^d(O)$$

for every $n \in \mathbb{N}$. In light of (3.4.43) and of the convergence u_n to g in $L^0(O;\mathbb{R}^k)$, this implies

$$\frac{1}{N}\sum_{i=1}^{N} \liminf_{n\to+\infty} \mathcal{E}(v^i_n, O) \le \Big(1 + \frac{C}{N}\Big) I^p_\star(g,0,O) + C\mathcal{L}^d(\{x \in O\colon\ |g(x)| \ge R\}) + \frac{C}{N}\mathcal{L}^d(O).$$

Combining this inequality with (3.4.44), by definition (3.1.12) of $I^p_\star$ we infer (3.4.41). The proof of (3.4.42) can be obtained by arguing exactly as in Lemma 2.3.2. □

We are finally ready to prove Theorem 3.1.16.

Proof of Theorem 3.1.16. As we have already characterised the bulk and surface parts of $I^p_\star$ in Theorem 3.1.14, we are left with proving that (recall Definition 3.1.5)

$$(I^p_\star)^c(g,G,B) = \int_B H^\infty_p\Big(\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}, 0\Big)\,\mathrm{d}|D^c g| \tag{3.4.45}$$

for all $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$.

Step 1. (Replacement of G by 0) We first show that

$$(I^p_\star)^c(g,G,B) = (I^p_\star)^c(g,0,B) \tag{3.4.46}$$

for all $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$. Equivalently, we prove that for all $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$ it holds

$$\frac{\mathrm{d}I^p_\star(g,G,\cdot)}{\mathrm{d}|D^c g|}(x) = \frac{\mathrm{d}I^p_\star(g,0,\cdot)}{\mathrm{d}|D^c g|}(x) \quad \text{for } |D^c g|\text{-a.e. } x \in \Omega. \tag{3.4.47}$$

To prove this, for every $x \in \Omega$ and $\rho > 0$ we use Lemma 3.4.4 with $g_1 = g_2 = g$ and $B = B(x,\rho)$ to obtain

$$\frac{I^p_\star(g,0,B(x,\rho))}{|D^c g|(B(x,\rho))} \le \frac{I^p_\star(g,G,B(x,\rho))}{|D^c g|(B(x,\rho))} + \frac{C}{|D^c g|(B(x,\rho))}\Big(\|G\|^p_{L^p(B(x,\rho))} + \widehat{C}\|G\|_{L^p(B(x,\rho))}\Big),$$

where $\widehat{C}$ is given by (3.4.35). By the Besicovitch Derivation Theorem this proves that

$$\frac{\mathrm{d}I^p_\star(g,0,\cdot)}{\mathrm{d}|D^c g|}(x) \le \frac{\mathrm{d}I^p_\star(g,G,\cdot)}{\mathrm{d}|D^c g|}(x) \quad \text{for } |D^c g|\text{-a.e. } x \in \Omega.$$

The reverse inequality is obtained in the exact same way. This concludes the proof of (3.4.47) and thus of (3.4.46).

Step 2. (Perturbative argument) For $\delta > 0$, consider the functional $I^{p,\delta}_\star \colon \mathrm{BV}(\Omega;\mathbb{R}^k) \times \mathcal{B}(\Omega) \to [0,+\infty)$ defined by

$$I^{p,\delta}_\star(g,B) \coloneqq I^p_\star(g,0,B) + \delta \int_{J_g \cap B} |[g]|\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for all } g \in \mathrm{BV}(\Omega;\mathbb{R}^k) \text{ and } B \in \mathcal{B}(\Omega).$$

Since $I^p_\star \in \mathfrak{F}^p_\star$ and $I^p_\star$ satisfies the invariance property (3.1.25), the functional $I^{p,\delta}_\star$ satisfies all hypotheses of Theorem 1.5.1. Hence, there exist functions $f^\delta_{\mathrm{bulk}} \colon \mathbb{R}^{k\times d} \to [0,+\infty)$ and $f^\delta_{\mathrm{surf}} \colon \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that

$$I^{p,\delta}_\star(g,B) = \int_B f^\delta_{\mathrm{bulk}}(\nabla g)\,\mathrm{d}x + \int_B f^{\delta,\infty}_{\mathrm{bulk}}\Big(\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\Big)\,\mathrm{d}|D^c g| + \int_{J_g\cap B} f^\delta_{\mathrm{surf}}([g],\nu_g)\,\mathrm{d}\mathcal{H}^{d-1} \tag{3.4.48}$$

for all $g \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$. It is elementary to see that $f^\delta_{\mathrm{bulk}} = H_p$ by Theorem 3.1.14, and thus, in view of Proposition 3.1.15, we have $f^\delta_{\mathrm{bulk}}(A) \le C_H(|A|+1)$ for all $A \in \mathbb{R}^{k\times d}$. Similarly, as h_p satisfies (γ2), it holds that $f^\delta_{\mathrm{surf}}(\zeta,\nu) \le (C_\gamma + \delta)|\zeta|$ for all $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$. Moreover, one can check that f^δ_{surf} and f^δ_{bulk} are nondecreasing in δ, and that the same property holds for $A \mapsto f^{\delta,\infty}_{\mathrm{bulk}}(A)$ when A is a rank-one matrix with unit norm. Thus, we may define

$$\begin{aligned}
\widehat{f}_{\mathrm{bulk}}(A) &= \lim_{\delta\to 0^+} f^\delta_{\mathrm{bulk}}(A) = \inf_{\delta>0} f^\delta_{\mathrm{bulk}}(A) \quad \text{for all } A \in \mathbb{R}^{k\times d},\\
\widehat{f}_{\mathrm{Cantor}}(A) &\coloneqq \lim_{\delta\to 0^+} f^{\delta,\infty}_{\mathrm{bulk}}(A) = \inf_{\delta>0} f^{\delta,\infty}_{\mathrm{bulk}}(A) \quad \text{for all } A \in \mathbb{R}^{k\times d} \text{ with rank 1 and } |A| = 1,\\
\widehat{f}_{\mathrm{surf}}(\zeta,\nu) &\coloneqq \lim_{\delta\to 0^+} f^\delta_{\mathrm{surf}}(\zeta,\nu) = \inf_{\delta>0} f^\delta_{\mathrm{surf}}(\zeta,\nu) \quad \text{for all } \zeta \in \mathbb{R}^k \text{ and } \nu \in \mathbb{S}^{d-1}.
\end{aligned}$$

The function $\widehat{f}_{\mathrm{Cantor}}$ is extended by 1-homogeneity to all rank-one matrices. Together with (3.4.48), the Monotone Convergence Theorem then implies that

$$I^p_\star(g,0,B) = \int_B \widehat{f}_{\mathrm{bulk}}(\nabla g)\,\mathrm{d}x + \int_B \widehat{f}_{\mathrm{Cantor}}\Big(\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|}\Big)\,\mathrm{d}|D^c g| + \int_{J_g\cap B} \widehat{f}_{\mathrm{surf}}([g],\nu_g)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $g \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$. Thus, by Theorem 3.1.14, we obtain

$$\widehat{f}_{\mathrm{bulk}}(A) = H_p(A,0) \quad \text{and} \quad \widehat{f}_{\mathrm{surf}}(\zeta,\nu) = h_p(\zeta,\nu)$$

for all $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

Then, we observe that the corresponding bounds for $f^\delta_{\rm bulk}$ and $f^\delta_{\rm surf}$ also give that $\widehat{f}_{\rm bulk}(A) \le C_H(|A|+1)$ for all $A \in \mathbb{R}^{k\times d}$ and that $\widehat{f}_{\rm surf}(\zeta,\nu) \le C_\gamma|\zeta|$ for all $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$. Since the functional $g \mapsto I^p_\star(g,0,O)$ is $L^1(\Omega;\mathbb{R}^k)$-lower semicontinuous on $\mathrm{BV}(\Omega;\mathbb{R}^k)$ for all $O \in \mathcal{O}(\Omega)$, by Lemma 3.D.1 we then deduce that

$$H^\infty_p(A,0) = \widehat{f}_{\rm Cantor}(A)$$

for all rank-one matrices $A \in \mathbb{R}^{k\times d}$ and

$$I^p_\star(g,0,B) = \int_B H_p(\nabla g,0)\,\mathrm{d}x + \int_B H^\infty_p\Big(\frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|},0\Big)\,\mathrm{d}|D^c g| + \int_{J_g\cap B} h_p([g],\nu_g)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $g \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and $B \in \mathcal{B}(\Omega)$. In light of (3.4.46), this shows that (3.4.45) holds whenever $(g,G) \in \mathrm{SD}^p_\star(\Omega;\mathbb{R}^k)$, with $g \in \mathrm{BV}(\Omega;\mathbb{R}^k)$.

Step 3. (Conclusion) We now deal with the general case $g \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Although the arguments used here follow closely the lines of the last part of the proof of Theorem 2.3.16 (see also [100, Theorem 6.7]), for the sake of completeness, we give here a full proof. For every $R>0$ we consider the set $\widetilde{\Omega}^R_g := \{x \in \Omega\colon \widetilde{g}(x) \text{ exists and } |\widetilde{g}(x)| \le R\}$. We claim that it is enough to prove (3.4.45) for every $B \in \mathcal{B}(\Omega)$ for which there exists $R>0$ such that $B \subset \widetilde{\Omega}^R_g$. Indeed, by Proposition 1.3.21 we have that $\widetilde{\Omega}^R_g \nearrow \Omega_{\rm reg} := \{x \in \Omega\colon \widetilde{g}(x) \text{ exists}\}$ and $|D^c g|(\Omega\setminus\Omega_{\rm reg}) = 0$. Hence, every $B \in \mathcal{B}(\Omega)$ can be written, up to a $|D^c g|$-negligible set, as the increasing union of Borel sets each contained in $\widetilde{\Omega}^{R_n}_g$, for some sequence $R_n \to +\infty$. This proves the claim.

Let us fix $R>0$, a Borel set $B \subset \widetilde{\Omega}^R_g$, and a sequence $\{R_N\}_N$ with $R_N \to +\infty$ and $R_N > R$. Thanks to Lemma 3.4.5, we have

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^N (I^p_\star)^c(v^i_N,0,B) = (I^p_\star)^c(g,0,B), \tag{3.4.49}$$

where we have set $v^i_N := \Phi_{\tau^{i-1}R_N} \circ g$. Since $v^i_N \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ for all $i \in \{1,\dots,N\}$ and $N \in \mathbb{N}$, the integral representation (3.4.45) in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ can be applied, and thus

$$(I^p_\star)^c(v^i_N,0,B) = \int_B H^\infty_p\Big(\frac{\mathrm{d}D^c v^i_N}{\mathrm{d}|D^c v^i_N|},0\Big)\,\mathrm{d}|D^c v^i_N| \tag{3.4.50}$$

for every $i \in \{1,\dots,N\}$ and $N \in \mathbb{N}$. As we assumed that $B \subset \widetilde{\Omega}^R_g$ and $R_N > R$, we have $B \subset \widetilde{\Omega}^{R_N}_g$. Thus, from (ii) of Proposition 1.3.21 we get

$$\begin{aligned} \frac{\mathrm{d}D^c v^i_N}{\mathrm{d}|D^c v^i_N|} &= \frac{\mathrm{d}D^c g}{\mathrm{d}|D^c g|} \quad |D^c g|\text{-a.e. in } B,\\ |D^c v^i_N| &= |D^c g| \quad \text{as Borel measures on } B, \end{aligned}$$

for every $i \in \{1,\dots,N\}$ and $N \in \mathbb{N}$. Together with (3.4.46), (3.4.49), and (3.4.50), these equalities give (3.4.45). This concludes the proof. □

Appendix to Chapter 3

In this appendix, we collect some auxiliary results that have been used throughout the Chapter. In Section 3.A, relying on the slicing properties of $\mathrm{GBV}_\star$-functions developed in Section 1.3, we prove a lemma which is crucial for the Fundamental Estimate of Lemma 3.3.1. We then provide in Section 3.B a suitable extension of the Poincaré inequality of De Giorgi, Carriero, & Leaci to the setting of generalised functions of bounded variation. This inequality is then employed in Section 3.C to analyse the blow-ups of functions of generalised bounded variation. We conclude with Section 3.D, where a well-known fact about lower semicontinuous integral functionals is presented.

3.A Slicing of generalised functions of bounded variation

The following result, which is used in the proof of the Fundamental Estimate above (see Lemma 3.3.1) shows that the product of a smooth function φ with a function $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ still belongs to $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, provided that u satisfies some additional integrability conditions.

Lemma 3.A.1. *Suppose that Ω has Lipschitz boundary. Let $\varphi \in C^1_c(\overline{\Omega};[0,1])$ and let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Assume that $u \in L^1(K;\mathbb{R}^k)$, where we set $K \coloneqq \{y \in \Omega\colon\ 0 < \varphi < 1\}$. Then, $v \coloneqq \varphi u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ and*

$$\nabla v = u \otimes \nabla\varphi + \varphi \nabla u \quad \mathcal{L}^d\text{-a.e. in } \Omega, \tag{3.A.1}$$

$$D^c v = \varphi D^c u \quad \text{as Borel measures on } \Omega, \tag{3.A.2}$$

$$J_v \subset J_u \quad \text{up to an } \mathcal{H}^{d-1}\text{-negligible set and } [v] = \varphi[u] \quad \mathcal{H}^{d-1}\text{-a.e. in } J_v. \tag{3.A.3}$$

Proof. Let us fix $\xi \in \mathbb{S}^{d-1}$. In view of Remark 1.3.29, we have $u^\xi_y \in \mathrm{BV}(\Omega^\xi_y)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$. By the Leibniz rule for one-dimensional BV-functions we then have that $v^\xi_y \in \mathrm{BV}(\Omega^\xi_y)$ and

$$\nabla v^\xi_y = \varphi^\xi_y \nabla u^\xi_y + (\nabla \varphi^\xi_y) u^\xi_y \quad \mathcal{L}^1\text{-a.e. in } \Omega^\xi_y, \tag{3.A.4}$$

$$D^c v^\xi_y = \varphi^\xi_y D^c u^\xi_y \quad \text{as Borel measures on } \Omega^\xi_y, \tag{3.A.5}$$

$$J_{v^\xi_y} = J_{u^\xi_y} \cap \{t \in \Omega^\xi_y\colon\ \varphi^\xi_y(t) \neq 0\} \quad \text{and} \quad (v^\xi_y)^\pm = \varphi^\xi_y (u^\xi_y)^\pm \quad \text{on } J_{u^\xi_y} \tag{3.A.6}$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$. In particular, from the previous equality and the fact that $0 \le \varphi \le 1$, it follows that

$$|[v^\xi_y]| \le |[u^\xi_y]| \quad \text{on } J_{u^\xi_y} \quad \text{and } J^1_{v^\xi_y} \subset J^1_{u^\xi_y}$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$. This inclusion then implies

$$\int_{\pi^\xi(\Omega)} |Dv^\xi_y|(J_{v^\xi_y} \setminus J^1_{v^\xi_y})\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \int_{\pi^\xi(\Omega)} \big(|Du^\xi_y|(J_{u^\xi_y} \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(J^1_{u^\xi_y} \setminus J^1_{v^\xi_y})\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty, \tag{3.A.7}$$

$$\int_{\pi^\xi(\Omega)} \mathcal{H}^0(J^1_{v^\xi_y})\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \int_{\pi^\xi(\Omega)} \mathcal{H}^0(J^1_{u^\xi_y})\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty. \tag{3.A.8}$$

In view of Proposition 1.3.28, we get that v satisfies (1.3.44). Then, by the fact that $u \in L^1(K;\mathbb{R}^k)$ and Fubini's Theorem we find

$$\int_{\pi^\xi(\Omega)} \Big(\int_{\Omega_y^\xi} |\varphi_y^\xi \nabla u_y^\xi|\,\mathrm{d}t \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \int_{\pi^\xi(\Omega)} \Big(\int_{\Omega_y^\xi} |\nabla u_y^\xi|\,\mathrm{d}t \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty, \tag{3.A.9}$$

$$\int_{\pi^\xi(\Omega)} \Big(\int_{\Omega_y^\xi} |(\nabla \varphi_y^\xi) u_y^\xi|\,\mathrm{d}t \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \|\nabla\varphi\|_{L^\infty(\Omega)} \int_{\pi^\xi(\Omega)} \Big(\int_{K_y^\xi} |u_y^\xi|\,\mathrm{d}t \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty, \tag{3.A.10}$$

$$\int_{\pi^\xi(\Omega)} \Big(\int_{\Omega_y^\xi} |\varphi_y^\xi|\,\mathrm{d}|D^c u_y^\xi| \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \int_{\pi^\xi(\Omega)} |D^c u_y^\xi|(\Omega_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty. \tag{3.A.11}$$

Recalling (3.A.4) and (3.A.5), combining (3.A.7)–(3.A.11), we conclude that

$$\int_{\pi^\xi(\Omega)} \big(|Dv_y^\xi|(\Omega_y^\xi \setminus J^1_{v_y^\xi}) + \mathcal{H}^0(\Omega_y^\xi \cap J^1_{v_y^\xi}) \big)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty.$$

As $\xi \in \mathbb{S}^{d-1}$ was arbitrary, Proposition 1.3.28 ensures that $v \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k) \cap L^1(K;\mathbb{R}^k)$.

Equation (3.A.3) follows immediately from (1.3.15) and (3.A.6). To prove (3.A.1), it is enough to combine (b) of Proposition 1.3.12 and (3.A.4). As for (3.A.2), we note that by (1.3.34) applied for v in place of u, (3.A.5), and Lemma 1.3.30 we have

$$D^c v(B)\xi = \int_{\pi^\xi(\Omega)} D^c v_y^\xi(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_{\pi^\xi(\Omega)} \Big(\int_{B_y^\xi} \varphi_y^\xi\,\mathrm{d}D^c u_y^\xi \Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \Big(\int_B \varphi\,\mathrm{d}D^c u \Big)\xi$$

for every $\xi \in \mathbb{S}^{d-1}$ and every Borel set $B \subset \Omega$. Since the previous equality implies (3.A.2), the proof is complete. □

3.B Poincaré inequality for functions with small jump set

This section is devoted to a Poincaré-type inequality for functions in $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, which will be a key tool for a series of results that we will present in Section 3.C of this appendix. It is also employed in the proof of Lemma 3.3.4. The inequality is visibly reminiscent of the one proved by De Giorgi, Carriero, & Leaci [104, Theorem 3.1] for functions in $\mathrm{SBV}(\Omega)$ with small jump set. Indeed, our proof follows closely the argument they devised.

We begin by setting some notation. Throughout this section, we make the further assumption that $d \ge 2$ and $\Omega \subset \mathbb{R}^d$ is connected with Lipschitz boundary. Given $u \in \mathrm{GBV}_\star(\Omega)$, we define

$$u_\star(s,\Omega) \coloneqq \inf\big\{ t \in [-\infty,+\infty] \colon\ \mathcal{L}^d(\{u < t\}) \ge s \big\} \tag{3.B.1}$$

for all $s \in (0, \mathcal{L}^d(\Omega))$. We say that $m \in \mathbb{R}$ is a *median* of u on Ω if

$$\begin{aligned} &\mathcal{L}^d(\{u < t\}) \le \mathcal{L}^d(\Omega)/2 \quad \text{for all } t < m, \\ &\mathcal{L}^d(\{u > t\}) \le \mathcal{L}^d(\Omega)/2 \ \text{ for all } t > m. \end{aligned} \tag{3.B.2}$$

It follows immediately from (3.B.1) that $u_\star(\mathcal{L}^d(\Omega)/2,\Omega)$ is a median of u on Ω. In the vector-valued case $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ with $k > 1$, we say that a vector $m \in \mathbb{R}^k$ is a median of u on Ω if m_i is a median of u_i on Ω for all $i \in \{1,\dots,k\}$. For future use, we report here the statement of the classical relative isoperimetric inequality (see, for instance, [18, (3.43)]). Recall (3.3.1).

Proposition 3.B.1. *There exists a constant $\gamma = \gamma(\Omega) > 0$ such that for every $\rho > 0$, every $x \in \mathbb{R}^d$, and every set $E \subset \mathbb{R}^d$ with $E_{\rho,x} \subset \Omega_{\rho,x}$ having finite perimeter in $\Omega_{\rho,x}$ it holds*

$$\min\Big\{\mathcal{L}^d(E_{\rho,x}\cap\Omega_{\rho,x}), \mathcal{L}^d(\Omega_{\rho,x}\setminus E_{\rho,x})\Big\} \leq \big(\gamma\mathcal{H}^{d-1}\big(\partial^* E_{\rho,x}\cap\Omega_{\rho,x}\big)\big)^{d/(d-1)}.$$

Remark 3.B.2. The isoperimetric constant $\gamma = \gamma(\Omega)$ can be taken to be the same as the one in the Sobolev Embedding Theorem for functions of bounded variation. In particular, the following property holds: let $\mathcal{I} \subset \mathbb{R}$ be a family of indices, and let $\{\Omega_i\}_{i\in\mathcal{I}}$ be a collection of domains with the property that for each $i_1, i_2 \in \mathcal{I}$ there exists a bi-Lipschitz homeomorphism φ_{i_1,i_2} from Ω_{i_1} onto Ω_{i_2} such that, denoting with L_{i_1,i_2} the Lipschitz constant of φ_{i_1,i_2}, it holds $\sup_{i_1,i_2} L_{i_1,i_2} < +\infty$. Then it is possible to choose a positive constant $\gamma = \gamma(\{\Omega_i\}_{i\in\mathcal{I}})$, independent of $i \in \mathcal{I}$, for which Proposition 3.B.1 holds true for every $i \in \mathcal{I}$.

In the following, $\gamma = \gamma(\Omega) > 0$ always denotes the constant of Proposition 3.B.1. We will often assume that the function $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ under consideration satisfies

$$\big(2\gamma\mathcal{H}^{d-1}(J_u^r\cap\Omega)\big)^{d/(d-1)} \leq \frac{\mathcal{L}^d(\Omega)}{2} \tag{3.B.3}$$

for some $r \geq 0$. When (3.B.3) is satisfied, we define the vectors $\tau'(u,\Omega,r), \tau''(u,\Omega,r) \in \mathbb{R}^k$ by setting

$$\begin{aligned}\tau_i'(u,\Omega,r) &\coloneqq (u_i)_\star\big((2\gamma\mathcal{H}^{d-1}(J_{u_i}^r\cap\Omega))^{d/(d-1)},\Omega\big),\\ \tau_i''(u,\Omega,r) &\coloneqq (u_i)_\star\big(\mathcal{L}^d(\Omega) - (2\gamma\mathcal{H}^{d-1}(J_{u_i}^r\cap\Omega))^{d/(d-1)},\Omega\big)\end{aligned} \tag{3.B.4}$$

for all $i \in \{1,\dots,k\}$. For brevity of notation, if $r = 0$, we write $\tau'(u,\Omega)$ and $\tau''(u,\Omega)$ in place of $\tau'(u,\Omega,r)$ and $\tau''(u,\Omega,r)$. Observe that, if m_i is a median of u_i, then it holds

$$\tau_i'(u,\Omega,r) \leq m_i \leq \tau_i''(u,\Omega,r). \tag{3.B.5}$$

We now present a Poincaré inequality for functions in $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ with a small set of large jumps, whose proof follows closely that of [104, Theorem 3.1] (see also [18, Theorem 4.14]).

Theorem 3.B.3. *There exists a constant $C_{\mathrm{Poin}} = C_{\mathrm{Poin}}(\Omega,k) > 0$ such that, for every $r \geq 0$ and $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ satisfying* (3.B.3)*, denoting by $\overline{u}_r \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ the function $\overline{u}_r \coloneqq \big(u \vee \tau'(u,\Omega,r)\big) \wedge \tau''(u,\Omega,r)$, it holds*

$$|D\overline{u}_r|(\Omega) \leq 2(r\vee 1)\mathcal{V}(u,\Omega\setminus J_u^r), \tag{3.B.6}$$

$$\|\overline{u}_r - m\|_{L^{d/(d-1)}(\Omega)} \leq C_{\mathrm{Poin}}(r\vee 1)\mathcal{V}(u,\Omega\setminus J_u^r), \tag{3.B.7}$$

for any median $m \in \mathbb{R}^k$ of u on Ω. Moreover, we have

$$\mathcal{L}^d(\{\overline{u}_r \neq u\}) \leq C_{\mathrm{Poin}}\mathcal{H}^{d-1}(J_u^r\cap\Omega)^{d/(d-1)}. \tag{3.B.8}$$

Proof. Since the proof of the result in the general case $k > 1$ follows immediately from the case $k = 1$, we may assume u to be scalar-valued. Let $r \geq 0$ and $u \in \mathrm{GBV}_\star(\Omega)$ be as in the statement. For simplicity of notation, throughout the proof we write τ' and τ'' in place of $\tau'(u,\Omega,r)$ and $\tau''(u,\Omega,r)$. Without loss of generality, we also assume that $m = 0$, which by (3.B.5) implies $\tau' \leq 0$ and $\tau'' \geq 0$.

Inequality (3.B.8) is a simple consequence of the fact that

$$\{\overline{u}_r \neq u\} = \{u < \tau'\} \cup \{u > \tau''\}$$

and the definition (3.B.4) of τ' and τ''. We now come to the proof of (3.B.6) and (3.B.7). We begin by observing that, thanks to Remark 1.3.17, $\overline{u}_r$ belongs to $\mathrm{BV}(\Omega)$, and by the same remark we have

$$|D\overline{u}_r|(\Omega) \le (r \vee 1)\mathcal{V}(u, \Omega \setminus J_u^r) + (\tau'' - \tau')\mathcal{H}^{d-1}(J_u^r \cap \Omega). \tag{3.B.9}$$

As $\tau' \le \overline{u}_r \le \tau''$ and $\overline{u}_r = u$ on $\{\tau' \le u \le \tau''\}$, by the Coarea Formula we get

$$|D\overline{u}_r|(\Omega) = \int_{\tau'}^{\tau''} \mathcal{H}^{d-1}(\partial^*\{u \ge t\} \cap \Omega)\,\mathrm{d}t. \tag{3.B.10}$$

By Proposition 3.B.1 applied with $\rho = 1$ and $x = 0$, we can estimate from below the right-hand side of the previous equality by

$$\int_{\tau'}^{\tau''} \mathcal{H}^{d-1}(\partial^*\{u \ge t\} \cap \Omega)\,\mathrm{d}t \ge \frac{1}{\gamma}\Big(\int_{\tau'}^{0} \mathcal{L}^d(\{u < t\})^{(d-1)/d}\,\mathrm{d}t + \int_{0}^{\tau''} \mathcal{L}^d(\{u \ge t\})^{(d-1)/d}\,\mathrm{d}t\Big). \tag{3.B.11}$$

From the definitions of τ' and τ'' in (3.B.4) it follows that

$$\begin{aligned} &\mathcal{L}^d(\{u < t\})^{(d-1)/d} \ge 2\gamma\mathcal{H}^{d-1}(J_u^r \cap \Omega) \quad \text{for all } t \in (\tau', 0),\\ &\mathcal{L}^d(\{u \ge t\})^{(d-1)/d} \ge 2\gamma\mathcal{H}^{d-1}(J_u^r \cap \Omega) \quad \text{for all } t \in (0, \tau''), \end{aligned}$$

which, combined with (3.B.10) and (3.B.11), yield

$$2(\tau'' - \tau')\mathcal{H}^{d-1}(J_u^r \cap \Omega) \le |D\overline{u}_r|(\Omega).$$

In light of (3.B.9), this inequality also implies

$$(\tau'' - \tau')\mathcal{H}^{d-1}(J_u^r \cap \Omega) \le (r \vee 1)\mathcal{V}(u, \Omega \setminus J_u^r),$$

which in turn gives (3.B.6). As we assumed Ω to be connected with Lipschitz boundary, the Poincaré-type inequality (3.B.7) then follows from (3.B.6) by applying to $\overline{u}_r$ the classical Poincaré-Sobolev inequality for functions of bounded variation, see, for instance, [18, Theorem 3.51]. □

Remark 3.B.4. When $r = 0$ and $u \in \mathrm{SBV}(\Omega; \mathbb{R}^k)$, the previous result reduces exactly to the Poincaré inequality of De Giorgi, Carriero, & Leaci [104, Theorem 3.1]. If $u \in \mathrm{GBV}_\star(\Omega; \mathbb{R}^k)$ and $\mathcal{H}^{d-1}(J_u^r) = 0$ for some $r \ge 0$, then $u \in \mathrm{BV}(\Omega; \mathbb{R}^k)$ and (by convention) $\tau_i'(u, \Omega, r) = -\infty$ and $\tau_i''(u, \Omega, r) = +\infty$, so that (3.B.7) reduces to the usual Poincaré inequality for functions of bounded variation.

Remark 3.B.5. Inspecting the proof of Theorem 3.B.3, it is clear that the constant C_{Poin} depends on Ω only via the isoperimetric constant $\gamma = \gamma(\Omega)$. In particular, letting $\mathcal{I}$ be a family of indices and $\{\Omega_i\}_{i\in\mathcal{I}}$ be a collection of domains as in Remark 3.B.2, it is possible to choose a uniform constant C_{Poin} for which the conclusions of Theorem 3.B.3 hold for every $i \in \mathcal{I}$.

Remark 3.B.6. Let $\rho > 0$ and $x \in \mathbb{R}^d$, and consider the set $\Omega_{\rho,x}$ given by (3.3.1). By Hölder's inequality and a scaling argument it can be shown that for each $u \in \mathrm{GBV}_\star(\Omega_{\rho,x};\mathbb{R}^k)$ satisfying the hypotheses of Theorem 3.B.3 we have

$$\|\overline{u}_r - m\|_{L^1(\Omega_{\rho,x})} \le \rho C_{\mathrm{Poin}}(r \vee 1)\mathcal{V}(u, \Omega_{\rho,x} \setminus J_u^r),$$

where C_{Poin} is the constant corresponding to $\rho = 1$.

As a simple consequence of the previous result, we show that, when u satisfies zero boundary conditions, the Poincaré-type inequality (3.B.7) holds with $m = 0$ for the original function, at the expense of removing a set ω of small Lebesgue measure.

Corollary 3.B.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a constant $\bar{C}_{\mathrm{Poin}} = \bar{C}_{\mathrm{Poin}}(\Omega, k) > 0$ such that for all $r \ge 0$ and $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, with $u = 0$ on $\partial\Omega$, there is a Borel set $\omega \subset \Omega$, with*

$$\mathcal{L}^d(\omega) \le \bar{C}_{\mathrm{Poin}}\mathcal{H}^{d-1}(J_u^r)^{d/(d-1)},$$

such that

$$\|u\|_{L^{d/(d-1)}(\Omega\setminus\omega)} \le \bar{C}_{\mathrm{Poin}}(r \vee 1)\mathcal{V}(u, \Omega \setminus J_u^r). \tag{3.B.12}$$

Proof. Let $r \ge 0$, let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, and let Ω' be a bounded Lipschitz domain with $\Omega \subset\subset \Omega'$. Since $u = 0$ on $\partial\Omega$, we can extend (without relabelling) u to a function $u \in \mathrm{GBV}_\star(\Omega';\mathbb{R}^k)$ by setting $u = 0$ on $\Omega' \setminus \Omega$. Moreover, it is not restrictive to assume that

$$\mathcal{H}^{d-1}(J_u^r)^{d/(d-1)} \le \frac{\mathcal{L}^d(\Omega' \setminus \Omega)}{2C_{\mathrm{Poin}}}, \tag{3.B.13}$$

where $C_{\mathrm{Poin}} = C_{\mathrm{Poin}}(\Omega', k)$ is the positive constant of Theorem 3.B.3. Indeed, if (3.B.13) is not satisfied, it is enough to set $\omega = \Omega$ and choose $\bar{C}_{\mathrm{Poin}}$ large enough. By possibly choosing a smaller Ω', we may also assume that

$$\mathcal{H}^{d-1}(J_u^r)^{d/(d-1)} \le \frac{\mathcal{L}^d(\Omega' \setminus \Omega)}{2C_{\mathrm{Poin}}} \le \frac{1}{(2\gamma)^{d/(d-1)}}\frac{\mathcal{L}^d(\Omega')}{2}.$$

In view of this inequality, u satisfies (3.B.3), so that we can apply Theorem 3.B.3 to u on Ω'. We obtain a vector $m \in \mathbb{R}^k$ and a Borel set $\omega := \{u \ne \overline{u}_r\} \subset \Omega'$ such that

$$\|u - m\|_{L^{d/(d-1)}(\Omega'\setminus\omega)} \le C_{\mathrm{Poin}}(r \vee 1)\mathcal{V}(u, \Omega \setminus J_u^r) \tag{3.B.14}$$

and

$$\mathcal{L}^d(\omega) \le C_{\mathrm{Poin}}\mathcal{H}^{d-1}(J_u^r)^{d/(d-1)} \le \frac{\mathcal{L}^d(\Omega' \setminus \Omega)}{2}, \tag{3.B.15}$$

where in the second inequality we have used (3.B.13). Next, we note that, since $u = 0$ on $\Omega' \setminus \Omega$, from (3.B.14) it follows that

$$|m| \le \frac{C_{\mathrm{Poin}}}{\mathcal{L}^d(\Omega' \setminus (\Omega \cup \omega))}^{(d-1)/d}(r \vee 1)\mathcal{V}(u, \Omega \setminus J_u^r).$$

Clearly, $\mathcal{L}^d(\Omega' \setminus (\Omega \cup \omega)) \ge \mathcal{L}^d(\Omega' \setminus \Omega) - \mathcal{L}^d(\omega)$, so that by (3.B.15) this implies that $\mathcal{L}^d(\Omega' \setminus (\Omega \cup \omega)) \ge \mathcal{L}^d(\Omega' \setminus \Omega)/2$. Hence,

$$|m| \le \frac{2^{(d-1)/d}C_{\mathrm{Poin}}}{\mathcal{L}^d(\Omega' \setminus \Omega)^{(d-1)/d}}(r \vee 1)\mathcal{V}(u, \Omega \setminus J_u^r).$$

This shows that (3.B.12) holds for a suitable choice of $\bar{C}_{\mathrm{Poin}}$. □

Remark 3.B.8. Let $\rho > 0$ and $x \in \mathbb{R}^d$, and consider the set $\Omega_{\rho,x}$ given by (3.3.1). By Hölder's inequality and a scaling argument, it can be shown that for each $u \in \mathrm{GBV}_\star(\Omega_{\rho,x};\mathbb{R}^k)$, with $u = 0$ on $\partial\Omega_{\rho,x}$, we can find $\omega \subset \Omega_{\rho,x}$ such that

$$\mathcal{L}^d(\omega) \le \bar{C}_{\mathrm{Poin}}\mathcal{H}^{d-1}(J_u^r \cap \Omega_{\rho,x})^{d/(d-1)} \quad \text{and} \quad \|u\|_{L^1(\Omega_{\rho,x}\setminus\omega)} \le \rho\bar{C}_{\mathrm{Poin}}(r \vee 1)\mathcal{V}(u, \Omega_{\rho,x} \setminus J_u^r),$$

where the constant $\bar{C}_{\mathrm{Poin}}$ is the one obtained for $\rho = 1$.

We conclude the section by presenting a last consequence of the Poincaré-type inequality in Theorem 3.B.3, namely a compactness result for sequences $\{u_n\}_n$ with vanishing set of large jumps. Here and in the rest of the work, given a function $u\colon \Omega \to \mathbb{R}^k$, we let $\mathrm{med}(u,\Omega) \in \mathbb{R}^k$ be the vector whose i-th component is given by $(u_i)_\star(\mathcal{L}^d(\Omega)/2, \Omega)$, which we recall is a median of u on Ω.

Corollary 3.B.9. *Let $r \ge 0$ and let $\{u_n\}_n \subset \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ be a sequence such that*

$$\sup_{n\in\mathbb{N}} \mathcal{V}(u_n, \Omega \setminus J_{u_n}^r) < +\infty \quad \text{and} \quad \lim_{n\to+\infty} \mathcal{H}^{d-1}(J_{u_n}^r \cap \Omega) = 0. \tag{3.B.16}$$

Let $\overline{u}_n$ be the function given by $\big((u_n) \vee \tau'(u_n,\Omega,r)\big) \wedge \tau''(u_n,\Omega,r)$. Then the sequence $\{\overline{u}_n - \mathrm{med}(u_n,\Omega)\}_n$ is uniformly bounded in $\mathrm{BV}(\Omega;\mathbb{R}^k)$ and there exist a subsequence, not relabelled, and a function $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ such that

$$\{\overline{u}_n - \mathrm{med}(u_n,\Omega)\}_n \text{ converges to } u \text{ strongly in } L^1(\Omega;\mathbb{R}^k) \text{ as } n \to +\infty, \tag{3.B.17}$$

$$\{u_n - \mathrm{med}(u_n,\Omega)\}_n \text{ converges to } u \text{ in } L^0(\Omega;\mathbb{R}^k) \text{ as } n \to +\infty. \tag{3.B.18}$$

Proof. Thanks to the equality in (3.B.16), we may invoke Theorem 3.B.3 to find that, for n large enough, the function $\overline{u}_n$ belongs to $\mathrm{BV}(\Omega;\mathbb{R}^k)$ and satisfies

$$|D\overline{u}_n|(\Omega) \le 2(r \vee 1)\mathcal{V}(u_n, \Omega \setminus J_{u_n}^r),$$
$$\|\overline{u}_n - \mathrm{med}(u_n,\Omega)\|_{L^{d/(d-1)}(\Omega)} \le C_{\mathrm{Poin}}(r \vee 1)\mathcal{V}(u_n, \Omega \setminus J_{u_n}^r).$$

In light of the inequality in (3.B.16), we deduce that $\{\overline{u}_n - \mathrm{med}(u_n,\Omega)\}_n$ is uniformly bounded in $\mathrm{BV}(\Omega;\mathbb{R}^k)$. Hence, there exist a subsequence, not relabelled, and a function $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ such that (3.B.17) holds. By (3.B.8) applied to u_n and (3.B.16)–(3.B.17) we finally get (3.B.18). □

3.C Blow-ups of $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$-functions

In this section, we analyse properties of blow-ups of functions in $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ around a point $x \in \Omega$. The key tool for this analysis will be the Poincaré-type inequality introduced in the previous section. We carry out this study in two distinct cases: first when x is a point of approximate differentiability of u, and then when x is a jump point of u. In both cases, we will show that it is always possible to find modifications of the function u which are close in measure to u, but that behave better in the blow-up limit. The tools we develop here are crucial in the proof of Propositions 3.3.6 and 3.3.7.

We begin by treating the case of points of approximate differentiability. The proof of the result borrows ideas from [40, Lemma 2].

Lemma 3.C.1 (Blow-up at points of approximate differentiability)**.** *Let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$, and for $x \in \Omega$ let u_ρ and $\overline{u}_\rho$ be the functions defined (recall the notation in Theorem 3.B.3 for $r = 0$) for $y \in Q(x,\rho)$ by*

$$u_\rho(y) \coloneqq (u(y) - \widetilde{u}(x))/\rho \quad \text{and} \quad \overline{u}_\rho(y) \coloneqq \big(u_\rho(y) \vee \tau'(u_\rho, Q(x,\rho))\big) \wedge \tau''(u_\rho, Q(x,\rho)).$$

Then for $\mathcal{L}^d$-a.e. $x \in \Omega$ we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d+1}} \mathcal{L}^d(\{\overline{u}_\rho \neq u_\rho\}) = 0, \tag{3.C.1}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} \Big|\overline{u}_\rho(y) - \nabla u(x)\frac{(y-x)}{\rho}\Big| \,\mathrm{d}y = 0, \tag{3.C.2}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} |\rho\nabla\overline{u}_\rho(y) - \nabla u(x)| \,\mathrm{d}y = 0, \tag{3.C.3}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(J_{\overline{u}_\rho} \cap Q(x,\rho)) = 0 \quad \text{and} \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c\overline{u}_\rho|(Q(x,\rho)) = 0. \tag{3.C.4}$$

Proof. We give the proof only in the scalar valued case $k = 1$, as the general case $k > 1$ is an immediate consequence of the former. We note that for $\mathcal{L}^d$-a.e. $x \in \Omega$ we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} |\nabla u(y) - \nabla u(x)| \,\mathrm{d}y = 0, \tag{3.C.5}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(J_u \cap Q(x,\rho)) = 0 \quad \text{and} \lim_{\rho\to 0^+} \frac{1}{\rho^d} |D^c u|(Q(x,\rho)) = 0, \tag{3.C.6}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{L}^d\Big(\Big\{y \in Q(x,\rho)\colon \frac{|u(y) - \widetilde{u}(x) - \nabla u(x)(y-x)|}{|y-x|} > \lambda\Big\}\Big) = 0 \text{ for all } \lambda > 0. \tag{3.C.7}$$

It suffices to prove the statement for such $x \in \Omega$ satisfying (3.C.5)–(3.C.7). We begin by observing that (3.C.7) implies

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{L}^d\Big(\Big\{y \in Q(x,\rho)\colon \frac{|u(y) - \widetilde{u}(x) - \nabla u(x)\cdot(y-x)|}{\rho} > \lambda\Big\}\Big) = 0 \tag{3.C.8}$$

for all $\lambda > 0$, that $u_\rho \in \mathrm{GBV}_\star(Q(x,\rho))$ and, by Remark 1.3.17, that $\overline{u}_\rho$ belongs to $\mathrm{BV}(Q(x,\rho)) \cap L^\infty(Q(x,\rho))$. By the same remark it also follows that $J_{\overline{u}_\rho} \subset J_u$ up to an $\mathcal{H}^{d-1}$-negligible set and that $|D^c\overline{u}_\rho|(Q(x,\rho)) \le \rho^{-1}|D^c u|(Q(x,\rho))$. Thus, by (3.C.6) we get

$$\limsup_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(J_{\overline{u}_\rho} \cap Q(x,\rho)) \le \limsup_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{H}^{d-1}(J_u \cap Q(x,\rho)) = 0,$$
$$\limsup_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c\overline{u}_\rho|(Q(x,\rho)) \le \limsup_{\rho\to 0^+} \frac{1}{\rho^d} |D^c u|(Q(x,\rho)) = 0.$$

This proves (3.C.4), so that we are left with proving (3.C.1)–(3.C.3).

To this end, we consider suitable rescalings of u_ρ. More precisely, for $z \in Q$ we define $v_\rho(z) \coloneqq u_\rho(x + \rho z)$. We have $\{v_\rho\}_\rho \subset \mathrm{GBV}_\star(Q)$ and by a change of variables it holds

$$\mathcal{V}(v_\rho, Q \setminus J_{v_\rho}) \le \frac{1}{\rho^d}\Big(\int_{Q(x,\rho)} |\nabla u| \,\mathrm{d}x + |D^c u|\big(Q(x,\rho)\big)\Big).$$

Since $x+\rho J_{v_\rho} = J_{u_\rho} = J_u \cap Q(x,\rho)$, up to $\mathcal{H}^{d-1}$-negligible sets, by a change of variables we get

$$\mathcal{H}^{d-1}(Q \cap J_{v_\rho}) = \frac{1}{\rho^{d-1}} \mathcal{H}^{d-1}(J_u \cap Q(x,\rho)).$$

Combining the two previous displayed formulas with (3.C.5) and (3.C.6), we get

$$\limsup_{\rho\to 0^+} \mathcal{V}(v_\rho, Q \setminus J_{v_\rho}) < +\infty \quad \text{and} \quad \lim_{\rho\to 0^+} \mathcal{H}^{d-1}(J_{v_\rho} \cap Q) = 0. \tag{3.C.9}$$

Therefore, for $\rho > 0$ small enough we may apply Theorem 3.B.3 with $r = 0$, and by (3.B.8) we get that the function defined by $\overline{v}_\rho(z) \coloneqq (v_\rho(z) \vee \tau'(v_\rho, Q)) \wedge \tau''(v_\rho, Q)$ for $z \in Q$ satisfies

$$\mathcal{L}^d(\{\overline{v}_\rho \neq v_\rho\}) \leq C_{\mathrm{Poin}} \mathcal{H}^{d-1}(J_{v_\rho} \cap Q)^{d/(d-1)}. \tag{3.C.10}$$

For every $0 < s \leq \rho^d$ we have the equality $(u_\rho)_\star(s, Q(x,\rho)) = (v_\rho)_\star(s/\rho^d, Q)$, so that (recall (3.B.2))

$$\tau'(v_\rho, Q) = \tau'(u_\rho, Q(x,\rho)) \quad \text{and} \quad \tau''(v_\rho, Q) = \tau''(u_\rho, Q(x,\rho)),$$
$$\mathrm{med}(v_\rho, Q) = \mathrm{med}(u_\rho, Q(x,\rho)),$$

and, as consequence, $\overline{v}_\rho(z) = \overline{u}_\rho(x+\rho z)$. A change of variables and (3.C.10) then show that

$$\frac{1}{\rho^d}\mathcal{L}^d(\{\overline{u}_\rho \neq u_\rho\}) \leq C_{\mathrm{Poin}} \Big(\frac{1}{\rho^{d-1}} \mathcal{H}^{d-1}(J_{u_\rho} \cap Q(x,\rho)) \Big)^{d/(d-1)}.$$

Therefore,

$$\begin{aligned} \frac{1}{\rho^{d^2/(d-1)}}\mathcal{L}^d(\{\overline{u}_\rho \neq u_\rho\}) &\leq C_{\mathrm{Poin}} \Big(\frac{1}{\rho^d} \mathcal{H}^{d-1}\big(J_{u_\rho} \cap Q(x,\rho)\big) \Big)^{\frac{d}{d-1}} \\ &= C_{\mathrm{Poin}} \Big(\frac{1}{\rho^d} \mathcal{H}^{d-1}\big(J_u \cap Q(x,\rho)\big) \Big)^{\frac{d}{d-1}}. \end{aligned}$$

Recalling the first equality in (3.C.6), and observing that $d^2/(d-1) \geq (d+1)$, we have just proved

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d+1}} \mathcal{L}^d(\{\overline{u}_\rho \neq u_\rho\}) = 0, \tag{3.C.11}$$

which is exactly (3.C.1). To get (3.C.3), we simply observe that

$$\nabla \overline{u}_\rho(y) = \begin{cases} \rho^{-1} \nabla u(y) & \text{if } y \in \{\overline{u}_\rho = u_\rho\}, \\ 0 & \text{if } y \in Q(x,\rho) \cap \{\overline{u}_\rho \neq u_\rho\}, \end{cases}$$

so that, in view of (3.C.5) and (3.C.11), we infer

$$\begin{aligned} \limsup_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} |\rho \nabla \overline{u}_\rho(y) - \nabla u(x)| \,\mathrm{d}y \\ \leq \lim_{\rho\to 0^+} \Big(\frac{1}{\rho^d} \int_{Q(x,\rho)} |\nabla u(y) - \nabla u(x)| \,\mathrm{d}y + \frac{1}{\rho^d} \mathcal{L}^d(\{\overline{u}_\rho \neq u_\rho\}) |\nabla u(x)| \Big) = 0. \end{aligned}$$

We conclude by proving (3.C.2). To this end, we observe that, thanks to (3.C.9), we may apply Corollary 3.B.9 with $r = 0$ to obtain a sequence $\{\rho_n\}_n$, with $\rho_n \to 0$ as $n \to +\infty$, and a function $v_0 \in \mathrm{BV}(Q)$ such that

$$v_{\rho_n} - \mathrm{med}(v_{\rho_n}, Q) \text{ converges to } v_0 \text{ in } L^0(Q;\mathbb{R}^k) \text{ as } n \to +\infty, \tag{3.C.12}$$

$$\overline{v}_{\rho_n} - \mathrm{med}(v_{\rho_n}, Q) \text{ converges to } v_0 \text{ strongly in } L^1(Q;\mathbb{R}^k) \text{ as } n \to +\infty. \tag{3.C.13}$$

The same arguments used in [40, Lemma 2, Page 196] show that, by using (3.C.8),

$$\lim_{n\to+\infty} \operatorname{med}(v_{\rho_n}, Q) = 0.$$

In view of (3.C.8), (3.C.12), and (3.C.13), this implies that $v_0(z) = \nabla u(x)z$ for all $z \in Q$, so that v_0 does not depend on the chosen subsequence $\{\rho_n\}_n$. Thus,

$$\{\overline{v}_\rho\}_\rho \text{ converges to } \nabla u(x)z \text{ strongly in } L^1(Q;\mathbb{R}^k) \text{ as } \rho \to 0^+.$$

It is now enough to observe that, by another change of variables, this can be rewritten as

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q(x,\rho)} \Big| \overline{u}_\rho(y) - \nabla u(x) \frac{(y-x)}{\rho} \Big| \, \mathrm{d}y = 0,$$

which is exactly (3.C.2). This concludes the proof of the lemma. □

In the following, we analyse the behaviour of blow-ups of functions in $\mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$ around jump points, drawing ideas from [84, Lemma 6.1] and [81, Lemma 3.4]. The strategy is similar to the one employed in Lemma 3.C.1, based on the following observation: since the jump set J_u is $(d-1)$-rectifiable, around a jump point x and for $\rho > 0$ small enough, $J_u \cap Q(x,\rho)$ is mostly contained in a smooth hypersurface which subdivides the cube into two connected halves. As the restrictions of the function u to each of the two halves have small jump sets, it is then possible to perform the modifications proposed in Lemma 3.C.1 in each half. Recall (3.3.1) and (3.1.5).

Lemma 3.C.2 (Blow-up at jump points)**.** *Let $u \in \mathrm{GBV}_\star(\Omega;\mathbb{R}^k)$. Then, for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$, letting $\nu := \nu_u(x)$ and $u_x := v_{x,u^+(x),u^-(x),\nu}$, there exists $u_\rho \in \mathrm{BV}(Q_\nu(x,\rho);\mathbb{R}^k)$ such that*

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{L}^d(\{u_\rho \neq u\} \cap Q_\nu(x,\rho)) = 0, \tag{3.C.14}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q_\nu(x,\rho)} |u_\rho - u_x| \, \mathrm{d}y = 0, \tag{3.C.15}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \mathcal{H}^{d-1}(J_{u_\rho} \cap E_{\rho,x}) \le \mathcal{H}^{d-1}(\Pi^\nu_x \cap E) \quad \textit{for all Borel sets } E \subset Q_\nu(x,1), \tag{3.C.16}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \int_{Q_\nu(x,\rho)} |\nabla u_\rho| \, \mathrm{d}y = 0 \quad \textit{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c u_\rho|(Q_\nu(x,\rho)) = 0. \tag{3.C.17}$$

Note that the function u_ρ is different from the one constructed in Lemma 3.C.1 although we use the same notation.

Proof. Clearly, it is enough to prove the result in the scalar case $k = 1$.

We observe that for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ the following holds: first, we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \mathcal{H}^{d-1}(J_u \cap Q_\nu(x,\rho)) = 1, \tag{3.C.18}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} \int_{Q_\nu(x,\rho)} |\nabla u| \, \mathrm{d}y = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}} |D^c u|(Q_\nu(x,\rho)) = 0, \tag{3.C.19}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{L}^d(\{y \in Q_\nu(x,\rho) \colon |u(y) - u_x(y)| > \lambda\}) = 0 \quad \text{for all } \lambda > 0. \tag{3.C.20}$$

Indeed, almost every jump point is a point of density 1 for J_u, i.e., (3.C.18) holds. Condition (3.C.19) is satisfied for $\mathcal{H}^{d-1}$-a.e. point of Ω since the measures $\nabla u\mathcal{L}^d$ and $D^c u$ are diffuse (see, for instance, [18, Theorem 2.56]). Condition (3.C.20) is a rephrasing of the definition of jump points given in Definition 1.3.9.

Moreover, there exist $\overline{\rho} > 0$ and a $(d-1)$-dimensional surface $\Gamma \subset Q_\nu(x,\overline{\rho})$ of class C^1 with $x \in \Gamma$ such that Γ is the graph of a function h defined on the hyperplane $\Pi_x^\nu = \{(y-x)\cdot\nu = 0\}$, the hyperplane Π_x^ν is tangent to Γ in x, for all $0 < \rho < \overline{\rho}$ the surface $\Gamma \cap Q_\nu(x,\rho)$ separates $Q_\nu(x,\rho)$ into two open connected components $Q_\nu^{\Gamma,\pm}(x,\rho)$, and

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\mathcal{H}^{d-1}\big((J_u \triangle \Gamma) \cap Q_\nu(x,\rho)\big) = 0, \tag{3.C.21}$$

$$\lim_{\rho\to 0^+} \frac{1}{\rho^{d-1}}\mathcal{H}^{d-1}(\Gamma \cap E_{\rho,x}) = \mathcal{H}^{d-1}(\Pi_x^\nu \cap E) \quad \text{for all Borel sets } E \subset Q_\nu(x,1). \tag{3.C.22}$$

This property can be justified by arguing exactly as in the beginning of the proof of [66, Theorem 2] (see also the comments in [67, Section 3]), noting that the arguments rely only on the rectifiability of the set J_u.

We start with a first observation: the Lipschitz constant of $h \llcorner_{\Pi_x^\nu \cap Q_\nu(x,\rho)}$ converges to zero as $\rho \to 0^+$. This implies that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\mathcal{L}^d\big(Q_\nu^{\Gamma,\pm}(x,\rho) \triangle Q_\nu^{\pm}(x,\rho)\big) = 0, \tag{3.C.23}$$

where we have set $Q_\nu^{\pm}(x,\rho) := \{y \in Q_\nu(x,\rho)\colon \pm(y-x)\cdot\nu > 0\}$.

In view of (3.C.21), the restriction of u to $Q_\nu^{\Gamma,\pm}(x,\rho)$ satisfies the hypothesis of Theorem 3.B.3 with Ω replaced by $Q_\nu^{\Gamma,\pm}(x,\rho)$. As $\{\frac{1}{\rho}(\Gamma \cap Q_\nu(x,\rho))\}_\rho$ are graphs of functions whose Lipschitz constant converges to zero as $\rho \to 0^+$, the constant in the Poincaré-type inequality can be chosen independently of ρ, see Remarks 3.B.5–3.B.6. Hence, letting $\overline{u}_\rho^\pm \in \mathrm{BV}(Q_\nu^{\Gamma,\pm}(x,\rho))$ be the functions defined by

$$\overline{u}_\rho^\pm := \big(u \vee \tau'(u, Q_\nu^{\Gamma,\pm}(x,\rho))\big) \wedge \tau''(u, Q_\nu^{\Gamma,\pm}(x,\rho))$$

and letting $m_\rho^\pm := \mathrm{med}(u, Q_\nu^{\Gamma,\pm}(x,\rho))$, recalling also Remark 3.B.6, we have that

$$|D\overline{u}_\rho^\pm|(Q_\nu^{\Gamma,\pm}(x,\rho)) \le 2\mathcal{V}\big(u, Q_\nu^{\Gamma,\pm}(x,\rho)\setminus J_u\big), \tag{3.C.24}$$

$$\int_{Q_\nu^{\Gamma,\pm}(x,\rho)} |\overline{u}_\rho^\pm - m_\rho^\pm|\,\mathrm{d}y \le \rho C_{\mathrm{Poin}}\mathcal{V}\big(u, Q_\nu^{\Gamma,\pm}(x,\rho)\setminus J_u\big), \tag{3.C.25}$$

$$\mathcal{L}^d(\{\overline{u}_\rho^\pm \neq u\}) \le C_{\mathrm{Poin}}\mathcal{H}^{d-1}\big(J_u \cap Q_\nu^{\Gamma,\pm}(x,\rho)\big)^{d/(d-1)}, \tag{3.C.26}$$

$$J_{\overline{u}_\rho^\pm} \subset J_u \cap Q_\nu^{\Gamma,\pm}(x,\rho) \quad \text{up to an } \mathcal{H}^{d-1}\text{-negligible set} \tag{3.C.27}$$

for a constant $C_{\mathrm{Poin}} > 0$ independent of ρ. Note that from (3.C.21) and (3.C.26) it follows that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d}\mathcal{L}^d\big(\{\overline{u}_\rho^\pm \neq u\} \cap Q_\nu^{\Gamma,\pm}(x,\rho)\big) = 0. \tag{3.C.28}$$

We now introduce the modifications $u_\rho \in \mathrm{BV}(Q_\nu(x,\rho))$ by setting

$$u_\rho(y) := \begin{cases} \overline{u}_\rho^+(y) & \text{if } y \in Q_\nu^{\Gamma,+}(x,\rho), \\ \overline{u}_\rho^-(y) & \text{if } y \in Q_\nu^{\Gamma,-}(x,\rho), \end{cases}$$

and show that $\{u_\rho\}_\rho$ satisfy properties (3.C.14)–(3.C.17). Property (3.C.14) has already been addressed in (3.C.28). To prove (3.C.16), we first observe that by (3.C.27) we have

$$J_{u_\rho} \subset \Gamma \cup J_u \quad \text{ up to an } \mathcal{H}^{d-1}\text{-negligible set,}$$

which, together with (3.C.21) and (3.C.22), shows (3.C.16). The equalities in (3.C.17) are a consequence of (3.C.19) and of (3.C.24).

It remains to prove (3.C.15). We note that this would follow from

$$\lim_{\rho\to 0^+} m_\rho^\pm = u^\pm(x). \tag{3.C.29}$$

Indeed, (3.C.19) and (3.C.25) give that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q_\nu^{\Gamma,\pm}(x,\rho)} |u_\rho - m_\rho^\pm| \,\mathrm{d}y = 0.$$

Recalling (3.C.23), by the triangle inequality this implies

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{Q_\nu^{\pm}(x,\rho)} |u_\rho - m_\rho^\pm| \,\mathrm{d}y = 0.$$

Then, (3.C.29) and the triangle inequality yield, in turn, (3.C.15).

We now show (3.C.29). In view of (3.C.20), by a diagonal argument we can find a sequence $\{\lambda_\rho\}_\rho \subset (0,+\infty)$ with $\lambda_\rho \to 0$ as $\rho \to 0^+$ such that the sets

$$\omega_\rho^\pm \coloneqq \{y \in Q_\nu^\pm(x,\rho)\colon\ |u(y) - u^\pm(x)| > \lambda_\rho\}$$

satisfy

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \mathcal{L}^d(\omega_\rho^\pm) = 0. \tag{3.C.30}$$

Recalling (3.C.25), we have that

$$\int_{Q_\nu^{\Gamma,\pm}(x,\rho)\setminus\{\overline{u}_\rho^\pm \neq u\}} |u(y) - m_\rho^\pm| \,\mathrm{d}y \le \rho C_{\mathrm{Poin}} \mathcal{V}\big(u, Q_\nu^{\Gamma,\pm}(x,\rho) \setminus J_u\big). \tag{3.C.31}$$

By the definition of $\omega_\rho^\pm$, using (3.C.31) and the triangle inequality, we then get

$$\begin{aligned} |u^\pm(x) - m_\rho^\pm| \mathcal{L}^d\big((Q_\nu^\pm(x,\rho) \cap Q_\nu^{\Gamma,\pm}(x,\rho)) \setminus (\{\overline{u}_\rho^\pm \neq u\} \cup \omega_\rho^\pm)\big) \\ \le \rho C_{\mathrm{Poin}} \mathcal{V}\big(u, Q_\nu^{\Gamma,\pm}(x,\rho) \setminus J_u\big) + \rho^d \lambda_\rho. \end{aligned} \tag{3.C.32}$$

From (3.C.23), (3.C.28), and (3.C.30) we deduce the existence of $\rho_0 > 0$ such that

$$\frac{1}{\rho^d} \mathcal{L}^d\big((Q_\nu^\pm(x,\rho) \cap Q_\nu^{\Gamma,\pm}(x,\rho)) \setminus (\{\overline{u}_\rho^\pm \neq u\} \cup \omega_\rho^\pm)\big) \ge \frac{1}{4}$$

for every $\rho \in (0,\rho_0)$. Then from inequality (3.C.32) we infer

$$|u^\pm(x) - m_\rho^\pm| \le \frac{4C_{\mathrm{Poin}}}{\rho^{d-1}} \mathcal{V}\big(u, Q_\nu^{\Gamma,\pm}(x,\rho) \setminus J_u\big) + 4\lambda_\rho$$

for all $\rho \in (0,\rho_0)$. Letting $\rho \to 0^+$ and resorting to (3.C.19) and to the fact that $\lambda_\rho \to 0$ as $\rho \to 0^+$, we obtain (3.C.29). This concludes the proof of the lemma. □

3.D A lower semicontinuity lemma

In this section, we present a well-known property for lower semicontinuous integral functionals on $\mathrm{BV}(\Omega;\mathbb{R}^k)$ with linear growth, employed in the proof of Theorem 3.1.16. In the following, we denote by $\mathbb{R}^{k\times d}_1 \subset \mathbb{R}^{k\times d}$ the subspace of rank-one matrices.

Lemma 3.D.1. *Let $f\colon \mathbb{R}^{k\times d} \to [0,+\infty)$, $h\colon \mathbb{R}^{k\times d}_1 \to [0,+\infty)$, and $g\colon \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ be Borel integrands such that*

$$0 \le f(A) \le C(|A|+1) \quad \text{for all } A \in \mathbb{R}^{k\times d},$$
$$h \text{ is positively 1-homogeneous},$$
$$g(\zeta,\nu) = g(-\zeta,-\nu) \quad \text{and} \quad 0 \le g(\zeta,\nu) \le C|\zeta| \quad \text{for all } \zeta \in \mathbb{R}^k \text{ and } \nu \in \mathbb{S}^{d-1}$$

for some $C>0$. Consider the functional $F\colon \mathrm{BV}(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)\to[0,+\infty)$ given by

$$F(u,O) := \int_O f(\nabla u)\,\mathrm{d}x + \int_O h\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big)\,\mathrm{d}|D^c u| + \int_{J_u\cap O} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $u \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and all $O \in \mathcal{O}(\Omega)$, and assume that $F(\cdot,O)$ is $L^1(\Omega;\mathbb{R}^k)$-lower semicontinuous for all $O\in\mathcal{O}(\Omega)$. Then

$$h(A) = f^\infty(A) \quad \text{for all } A \in \mathbb{R}^{k\times d}_1,$$

where $f^\infty(A) := \limsup_{t\to+\infty} \frac{f(tA)}{t}$.

Proof. Let us fix $a\in\mathbb{R}^k$, $\nu\in\mathbb{S}^{d-1}$, and set $A := a\otimes\nu$. Without loss of generality, we may assume that $Q_\nu \subset \Omega$, where we recall that Q_ν is a cube centred at 0 with two faces orthogonal to ν and sides of unit length.

We first prove that $h(A) \le f^\infty(A)$. To this end, we let $\psi\colon[-1/2,1/2]\to[-1/2,1/2]$ (be a translation of) Cantor's staircase function and we consider the function defined for $x\in Q_\nu$ by $u(x) := a\psi(x\cdot\nu)$. Clearly, $u\in\mathrm{BV}(Q_\nu;\mathbb{R}^k)$, $|Du|(Q_\nu) = |D^c u|(Q_\nu) = |A|$, and

$$\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|} = \frac{A}{|A|} \quad |D^c u|\text{-a.e. on } Q_\nu.$$

In particular, $F(u,Q_\nu) = f(0) + h(A)$. Then, we let $\{\psi_n\}_n$ be the sequence of piecewise affine functions considered in the construction of ψ, which converges to ψ uniformly and satisfies

$$\psi_n' = t_n \chi_{E_n} \quad \text{and} \quad \mathcal{L}^1(E_n) = t_n^{-1}$$

for all $n\in\mathbb{N}$, where $t_n := (3/2)^n$ and E_n is the union of the 2^n intervals remaining in the n-th step of the construction of the Cantor set. We set $u_n(x) := a\psi_n(x\cdot\nu)$. We note that $\{u_n\}_n$ converges to u strongly in $L^1(Q_\nu;\mathbb{R}^k)$ and that $\nabla u_n(x) = t_n\chi_{E_n}(x\cdot\nu)A$. Thus,

$$F(u_n,Q_\nu) = \int_{Q_\nu} f(\nabla u_n)\,\mathrm{d}x = (1-t_n^{-1})f(0) + \frac{f(t_nA)}{t_n}.$$

By the L^1-lower semicontinuity of F, this implies

$$f(0)+h(A) = F(u,Q_\nu) \le \liminf_{n\to+\infty} F(u_n,Q_\nu) \le f(0) + f^\infty(A),$$

which proves $h(A) \le f^\infty(A)$.

To prove the converse inequality, we let $t > 0$ and consider the affine map defined by $v(x) = ta(x \cdot \nu)$ for $x \in Q_\nu$. Then, we set

$$w_n(s) \coloneqq \frac{1}{n}\big(\lfloor ns \rfloor + \frac{1}{2} + \psi(ns - \lfloor ns \rfloor - \frac{1}{2})\big) \quad \text{for } s \in [-1/2, 1/2],$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Clearly, $\{w_n\}_n$ converges uniformly to the identity on $[-1/2, 1/2]$, and $|Dw_n|(-1/2, 1/2) = |D^c w_n|(-1/2, 1/2) = 1$ for all $n \in \mathbb{N}$, so that the functions $v_n(x) \coloneqq taw_n(x \cdot \nu)$ converge to v strongly in $L^1(Q_\nu; \mathbb{R}^k)$, $|Dv_n|(Q_\nu) = |D^c v_n|(Q_\nu) = t|a| = t|A|$, and

$$\frac{\mathrm{d}D^c v_n}{\mathrm{d}|D^c v_n|} = \frac{A}{|A|} \quad |D^c v_n|\text{-a.e. in } Q_\nu.$$

Thus, $F(v_n, Q_\nu) = f(0) + th(A)$, which by lower semicontinuity gives

$$f(tA) = F(v, Q_\nu) \le \liminf_{n \to +\infty} F(v_n, Q_\nu) = f(0) + th(A).$$

Dividing by t and letting $t \to +\infty$ we obtain $f^\infty(A) \le h(A)$, concluding the proof. □

Γ-CONVERGENCE AND HOMOGENISATION OF FREE DISCONTINUITY PROBLEMS: THE SYMMETRIC GRADIENT CASE

In this chapter, we study a problem closely connected to the one treated in Chapter 2, although in the different setting of functions of bounded deformation $\mathrm{BD}(\Omega)$. More precisely, we study the Γ-convergence as $n \to +\infty$ of sequences of functionals of the form

$$F_n(u) \coloneqq \int_\Omega f_n(x, \mathcal{E}u)\,\mathrm{d}x + \int_\Omega f_n^\infty\Big(x, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap\Omega} g_n(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \quad (4.0.1)$$

where Ω is a bounded open subset of $\mathbb{R}^d$, $u \in \mathrm{BD}(\Omega)$, the Borel measurable integrands $f_n\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, +\infty)$ and $g_n\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$ satisfy

$$c\big(|A| - 1\big) \le f_n(x, A) \le C\big(|A| + 1\big) \quad \text{for all } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \quad (4.0.2)$$

$$c|\zeta \odot \nu| \le g_n(x, \zeta, \nu) \le C|\zeta \odot \nu| \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{S}^{d-1}, \quad (4.0.3)$$

for suitable positive constants c, C, and the recession function f_n^∞ is given by (1.5.6), with f_{bulk} replaced by f_n. We observe that, with respect to what was assumed in Chapter 2, the growth conditions assumed in (4.0.3) are not of the *cohesive*-type of (2.0.3), but are instead purely linear. Hence, the right functional domain for the study of (4.0.1) is $\mathrm{BD}(\Omega)$ and not the generalised space $\mathrm{GBD}(\Omega)$, which will be the main object of study of Chapters 5 and 6.

With the aim of proving a general Γ-convergence theorem for sequences of the form (4.0.1), we adopt a strategy similar to Chapter 2, which relies on the *localisation method* of Γ-convergence (see [45, Chapter 3]) and on a recent integral representation result of Caroccia, Focardi, & Van Goethem [63] (see Lemma 1.5.5). We observe, however, that due to the applications we have in mind, it is not possible to apply directly the integral representation theorem 1.5.4. Indeed, it relies crucially on the equicontinuity with respect to translations (see (H4) of [63]), which is a property that, in the case of stochastic homogenisation dealt with in Section 4.8, we are not able to directly verify.

A crucial difficulty will be that of identifying suitable assumptions on $\{f_n\}_n$ and on $\{g_n\}_n$ (see Section 4.4) that allow us to proceed with the method developed in Section 2.5. We will see that, in contrast with the results of that chapter, the crucial assumption to make on the surface integrands $\{g_n\}_n$ is not on their behaviour near the origin, but on their distance from a fixed line at infinity. This is due to the fact that, as truncation techniques are not expedient in $\mathrm{BD}(\Omega)$, it is not possible to enforce uniform boundedness of the jump openings of competitors to minimisation problems, which was one of the key ingredients of the analysis carried out in $\mathrm{GBV}_\star$.

After having established general results for the Γ-convergence of sequences of the form (4.0.1), as an application, we study the Γ-convergence of certain sequences $\{F_\varepsilon\}_\varepsilon$ defined by rescaling, from both a deterministic and probabilistic point of view. More precisely, we study functionals that for $\varepsilon > 0$ are defined by

$$F_\varepsilon(u) \coloneqq \int_\Omega f\Big(\frac{x}{\varepsilon}, \mathcal{E}u\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap\Omega} \varepsilon g\Big(\frac{x}{\varepsilon}, \frac{[u]}{\varepsilon}, \nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \quad (4.0.4)$$

with f and g satisfying (4.0.2) and (4.0.3). Although the sequence $\{F_\varepsilon\}_\varepsilon$ is not of the form usually considered in homogenisation problems (compare with (2.0.1)), in the case where $\zeta \mapsto g(x,\zeta,\nu)$ is positively homogeneous of degree 1, we have $\varepsilon g(\frac{x}{\varepsilon}, \frac{[u]}{\varepsilon}, \nu_u) = g(\frac{x}{\varepsilon}, [u], \nu_u)$. Hence, under this additional assumption $\{F_\varepsilon\}_\varepsilon$ belongs to the usual class of functionals considered in the homogenisation theory of free discontinuity problems. The particular choice of the scaling for the surface term in (4.0.4) allows us to circumvent the use of vertical truncations and obtain suitable change of variable formulas, which are a crucial ingredient of the proof (see Lemmas 4.7.4 and 4.7.6).

We briefly describe the organisation of the chapter. We begin by introducing in Section 4.1 a general class of functionals defined on $\mathrm{BD}(\Omega)$ and denoted by $\mathfrak{F}$, which includes, in particular, many functionals $\{F_n\}$ of the form (4.0.1). We then investigate in Section 2.2 the compactness properties with respect to Γ-convergence of $\mathfrak{F}$ (see Theorem 2.2.1) and show that every sequence $\{F_n\}_n \subset \mathfrak{F}$ admits a subsequence Γ-converging to a lower semicontinuous functional in F.

We then begin analysing the structure of possible Γ-limits $F \in \mathfrak{F}$. To this end, we apply a result of Ebobisse & Toader [115] (see also Theorem 1.5.4) to prove in Section 4.3 that the bulk and surface parts of every such F can be represented as integral functionals associated to a suitable choice of f and g satisfying (4.0.2) and (4.0.3) (see Lemma 4.3.3).

To obtain a complete integral representation of functionals in $\mathfrak{F}$, including also their Cantor part, we then introduce in Section 4.4 a subclass of $\mathfrak{F}$, which we call $\mathfrak{F}^{\alpha,\infty}$, and prove its closedness under Γ-convergence (see Proposition 4.4.6). This additional property (see Definition 4.4.4) implies that the integrands f and g obtained in Lemma 4.3.3 satisfy quantitative estimates relating f to f^∞ and g to the function g^∞ defined by $g^\infty(x,\zeta,\nu) \coloneqq \lim_{t\to+\infty} g(x,t\zeta,\nu)/t$. These estimates are then employed in Section 4.5 to prove that, if F is the Γ-limit of a sequence $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\infty}$ and the function f given by Lemma 4.3.3 does not depend on x, then F admits a complete integral representation in terms of the integrands f, f^∞, and g obtained in Lemma 4.3.3.

In Section 4.6, the relation between Γ-convergence of a sequence $\{F_n\}_n \subset \mathfrak{F}$ and the asymptotic behaviour of associated boundary value problems is investigated. Together with the theory developed in Section 4.4, these results allow us to derive in Section 4.6 the desired necessary and sufficient conditions for the Γ-convergence of the functionals $\{F_\varepsilon\}_\varepsilon$ given by (4.0.4) to an integral functional F independent of the spatial variable x (see Theorem 4.6.3). Finally, in Section 4.8 we resort to the techniques developed by Cagnetti, Dal Maso, Scardia, & Zeppieri in [58, 59] to prove the desired Γ-convergence of $\{F_\varepsilon\}_\varepsilon$ in the probabilistic setting (see Theorem 2.7.9).

The content of this chapter is part of the joint paper with Dal Maso [92].

4.1 Volume and surface integrands

In this section we introduce the collections of integrands and of functionals that are going to be our main objects of study. We refer the reader to Section 1.1 for the notation concerning families of open/Borel sets and to Section 1.4 for the definition and main properties of functions of bounded deformation.

We fix non-negative constants $c_1, \dots, c_5 \in [0, +\infty)$, with $0 < c_1 \le c_3$, and a constant $\alpha \in (0,1)$. We also fix a continuous function $\tau\colon [0,+\infty) \to [0,+\infty)$ with the property that

$\tau(0) = 0$ and

$$\tau^\infty := \limsup_{t\to+\infty} \frac{\tau(t)}{t} < +\infty. \tag{4.1.1}$$

Definition 4.1.1. Let $\mathscr{F}$ be the collection of all integrands $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ satisfying the following conditions:

(f1) (*Measurability*) f is Borel measurable on $\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$;

(f2) (*Upper and lower bound*) for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we have

$$c_1|A| - c_2 \le f(x,A) \le c_3|A| + c_4;$$

(f3) (*Lipschitz property*) for every $x \in \mathbb{R}^d$ and $A_1, A_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we have

$$|f(x,A_1) - f(x,A_2)| \le c_5|A_1 - A_2|.$$

Let $\mathscr{G}$ be the collection of all integrands $g\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfying the following conditions:

(g1) (*Measurability*) g is Borel measurable on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$;

(g2) (*Symmetry*) for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$g(x,\zeta,\nu) = g(x,-\zeta,-\nu);$$

(g3) (*Upper and lower bounds*) for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$c_1|\zeta \odot \nu| \le g(x,\zeta,\nu) \le c_3|\zeta \odot \nu|;$$

(g4) (*Continuity in* ζ)for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$|g(x,\zeta_1,\nu) - g(x,\zeta_2,\nu)| \le \tau(|\zeta_1 - \zeta_2|).$$

To every function $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ we associate its *recession function* f^∞ (with respect to the variable A), defined by

$$f^\infty(x,A) := \limsup_{t\to+\infty} \frac{f(x,tA)}{t} \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{4.1.2}$$

Remark 4.1.2. Given $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfying (f2) and (f3), one can check that for every $x \in \mathbb{R}^d$, $A, A_1, A_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the function f^∞ satisfies

$$\begin{aligned} c_1|A| \le f^\infty(x,A) \le c_3|A|, \\ |f^\infty(x,A_1) - f^\infty(x,A_2)| \le c_5|A_1 - A_2|. \end{aligned} \tag{4.1.3}$$

We now introduce a class of integral functionals associated with $\mathscr{F}$ and $\mathscr{G}$.

Definition 4.1.3. Given $f \in \mathscr{F}$ and $g \in \mathscr{G}$, for every open set $O \subset \mathbb{R}^d$ and $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ we set

$$F^{f,g}(u,O) := \int_O f(x,\mathcal{E}u)\,\mathrm{d}x + \int_O f^\infty\Big(x,\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap O} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \quad (4.1.4)$$

if $u \in \mathrm{BD}_{\rm loc}(O)$, while $F^{f,g}(u,O) = +\infty$ otherwise. The definition is then extended to every Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ by setting

$$F^{f,g}(u,B) := \inf\{F^{f,g}(u,O) : O \subset \mathbb{R}^d \text{ open, with } O \supset B\}. \quad (4.1.5)$$

Note that, if $O \subset \mathbb{R}^d$ is a bounded open set, $B \in \mathcal{B}(O)$, $u \in \mathrm{BD}_{\rm loc}(O)$, and $|Eu|(O) < +\infty$, then

$$F^{f,g}(u,B) = \int_B f(x,\mathcal{E}u)\,\mathrm{d}x + \int_B f^\infty\Big(x,\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}. \quad (4.1.6)$$

Indeed, in this case the upper bounds in (f2), (4.1.3), and (g3) imply

$$\int_O f(x,\mathcal{E}u)\,\mathrm{d}x + \int_O f^\infty\Big(x,\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap O} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} < +\infty,$$

so that (4.1.6) follows from (4.1.4) and (4.1.5).

We now introduce a collection of abstract functionals related to the integral functionals considered above. In the rest of the chapter, we let $\mathcal{R}$ denote the collections of all rigid motions, i.e., all functions $u\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $u = Ax + b$ for some $A \in \mathbb{R}^{d\times d}_{\rm skew}$ and $b \in \mathbb{R}^d$.

Definition 4.1.4. Let $\mathfrak{F}$ be the collection of functionals $F\colon L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ with the following properties:

(H1) (*Measure property*) for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ the set function $F(u,\cdot)$ is a Borel measure that for every $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies

$$F(u,B) := \inf\{F(u,O) : O \subset \mathbb{R}^d \text{ open}, O \supset B\}; \quad (4.1.7)$$

(H2) (*Locality on open sets*) for every bounded open set $O \subset \mathbb{R}^d$ and $u,v \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ with $u = v$ $\mathcal{L}^d$-a.e on O we have $F(u,O) = F(v,O)$;

(H3) (*Upper and lower bounds*) if $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $O \subset \mathbb{R}^d$ is a bounded open set, then

$$c_1|Eu|(O) - c_2\mathcal{L}^d(O) \le F(u,O) \le c_3|Eu|(O) + c_4\mathcal{L}^d(O) \quad \text{if } u \in \mathrm{BD}(O),$$

while $F(u,O) = +\infty$ if $u \notin \mathrm{BD}(O)$;

(H4) (*Invariance under addition of rigid motions*) for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $w \in \mathcal{R}$, and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u+w,B) = F(u,B).$$

(H5) (*Bulk estimate*) for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u+\ell_A,B) \le F(u,B) + c_5|A|\mathcal{L}^d(B);$$

(H6) (*Surface estimate*) for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u + u_{x,\zeta,\nu}, B) \le F(u, B) + \tau(|\zeta|)\mathcal{H}^{d-1}(B \cap \Pi^\nu_x),$$

where $\Pi^\nu_x := \{y \in \mathbb{R}^d : (y-x)\cdot\nu = 0\}$ is the hyperplane orthogonal to ν containing x.

We denote by $\mathfrak{F}_{\rm sc}$ the collection of functionals $F \in \mathfrak{F}$ such that for every open $O \subset \mathbb{R}^d$ the functional $F(\cdot, O)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous.

Remark 4.1.5. Combining properties (H1) and (H2) we obtain that the functionals in $\mathfrak{F}$ satisfy the following locality property: if $B \in \mathcal{B}(\mathbb{R}^d)$, $u, v \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, and $u = v$ $\mathcal{L}^d$-a.e. on a neighbourhood of B, then $F(u, B) = F(v, B)$. Easy examples show that, in general, this equality does not hold if we have only $u = v$ $\mathcal{L}^d$-a.e. on B.

For this reason, if $O \subset \mathbb{R}^d$ is an open set and $u \in L^1(O;\mathbb{R}^d)$, for every $B \in \mathcal{B}(O)$ we can define $F(u, B) := F(v, B)$, where $v \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ is any function such that $v = u$ $\mathcal{L}^d$-a.e. on O. The locality property described above implies that the value of $F(u, B)$ does not depend on the chosen extension v of u.

Remark 4.1.6. From (4.1.7) and (H3) we deduce the following inequalities for Borel sets. If $O \subset \mathbb{R}^d$ is a bounded open set and $u \in {\rm BD}(O)$, then for every $B \in \mathcal{B}(O)$ we have

$$c_1|Eu|(B) - c_2\mathcal{L}^d(B) \le F(u, B) \le c_3|Eu|(B) + c_4\mathcal{L}^d(B).$$

Remark 4.1.7. Thanks to property (H1) a functional $F \in \mathfrak{F}$ belongs to $\mathfrak{F}_{\rm sc}$ if and only if for every open bounded set $O \subset \mathbb{R}^d$ the functional $F(\cdot, O)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous.

We now show that the functional introduced in Definition 4.1.3 belongs to $\mathfrak{F}$.

Proposition 4.1.8. *Let $f \in \mathscr{F}$ and $g \in \mathscr{G}$. Then the functional $F^{f,g}$ introduced in Definition 4.1.3 belongs to $\mathfrak{F}$.*

Proof. By (4.1.5) $F^{f,g}$ satisfies (4.1.7), while the rest of property (H1) is obvious when $u \in {\rm BD}(\mathbb{R}^d)$. The general case $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ can be obtained by a straightforward argument; it can also be treated using the De Giorgi-Letta criterion for measures (see Lemma 4.2.4 below). Property (H2) is obvious. Property (H3) follows from (f2), (4.1.3), and (g3). To see that $F^{f,g}$ satisfies property (H4), it is enough to observe that for every $w \in \mathcal{R}$ we have $Ew = 0$.

To see that $F^{f,g}$ satisfies property (H5) we observe that by (f3) for every $B \in \mathcal{B}(\mathbb{R}^d)$ and $A \in \mathbb{R}^{d\times d}_{\rm sym}$ we have

$$|F^{f,g}(u + \ell_A, B) - F^{f,g}(u, B)| \le c_5|A|\mathcal{L}^d(B).$$

A similar argument shows that $F^{f,g}$ satisfies property (H6). □

4.2 A compactness result for $\mathfrak{F}$

In this section we state and prove a compactness result for sequences of functionals in $\mathfrak{F}$. The proof is based on the classical localisation method for Γ-convergence (see [45, Section 3.3] or [88, Chapter 18]) and follows a similar structure of the proof of Theorem 2.2.1.

Theorem 4.2.1. *Let $\{F_n\}_n \subset \mathfrak{F}$. Then there exist a functional $F \in \mathfrak{F}_{\rm sc}$ and a subsequence, not relabelled, such that for every bounded open set $O \subset \mathbb{R}^d$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ in the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$.*

Proof. For every open set $O \subset \mathbb{R}^d$ we consider the functionals defined on $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ by

$$F'(\cdot,O) = \Gamma\text{-}\liminf_{n\to+\infty} F_n(\cdot,O) \quad \text{and} \quad F''(\cdot,O) = \Gamma\text{-}\limsup_{n\to+\infty} F_n(\cdot,O), \tag{4.2.1}$$

$$F'_-(\cdot,O) = \sup_{O'\in\mathcal{O}_c(O)} F'(\cdot,O') \quad \text{and} \quad F''_-(\cdot,O) = \sup_{O'\in\mathcal{O}_c(O)} F''(\cdot,O'), \tag{4.2.2}$$

where the Γ-liminf and Γ-limsup are computed with respect to the $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-convergence and we recall that $\mathcal{O}_c(O)$ is the collection of open sets compactly contained in O.

It is immediate to check that for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ the set functions $F_n(u,\cdot)$, $F'(u,\cdot)$, $F''(u,\cdot)$, $F'_-(u,\cdot)$, and $F''_-(u,\cdot)$ are increasing. Thus, we can apply the compactness theorem for sequences of increasing set functionals [88, Theorem 16.9] to obtain that there exist a subsequence of $\{F_n\}_n$, not relabelled, and a functional $F\colon L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\times\mathcal{O}(\mathbb{R}^d)$ such that

$$F(u,O) = F'_-(u,O) = F''_-(u,O) \quad \text{for every } u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d) \text{ and } O \in \mathcal{O}(\mathbb{R}^d). \tag{4.2.3}$$

For every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we set

$$F(u,B) \coloneqq \inf\{F(u,O)\colon\ O \in \mathcal{O}(\mathbb{R}^d) \text{ with } O \supset B\}. \tag{4.2.4}$$

We want to prove that $F \in \mathfrak{F}$. The approximation property (4.1.7) is obvious, while the measure property (H1) will be proved in Lemma 4.2.4 below. The locality property (H2) and the lower semicontinuity of F follow from general results about Γ-limits of local functionals (see [88, Proposition 16.15 and Remark 16.3]).

We note that it is enough to prove properties (H4)–(H6) for $O \in \mathcal{O}_c(\mathbb{R}^d)$. Indeed, by the measure property (H1) they can be extended to $O \in \mathcal{O}(\mathbb{R}^d)$ using the inner regularity of Borel measures. The further extension to Borel sets follows immediately from (4.2.4).

Let us fix $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $O \in \mathcal{O}_c(\mathbb{R}^d)$. We begin by proving the upper and lower bounds in (H3). We consider first the case $u \in {\rm BD}(O)$. Under this hypothesis, the inequality

$$F''(u,O) \le c_3|Eu|(O) + c_4\mathcal{L}^d(O) \tag{4.2.5}$$

is a consequence of the corresponding inequality for F_n and of the fact that the Γ-limsup is less than or equal to the pointwise limsup. The upper bound in (H3) for F follows immediately from (4.2.2), (4.2.3), and (4.2.5). To prove the lower bound for F'', we observe that our assumption $u \in {\rm BD}(O)$ implies that $F''(u,O) < +\infty$. Given $O' \in \mathcal{O}_c(O)$, we fix a sequence $\{u_n\}_n \subset L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ converging to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\limsup_n F_n(u_n,O') = F''(u,O') < +\infty$. Since for every $n \in \mathbb{N}$ the functional F_n satisfies (H3) we have $u_n \in {\rm BD}(O')$ for $n \in \mathbb{N}$ large enough and

$$c_1|Eu_n|(O') - c_2\mathcal{L}^d(O') \le F_n(u_n,O').$$

From the lower semicontinuity of $v \mapsto |Ev|(O')$ with respect to the L^1-convergence and from the previous inequality we obtain

$$c_1|Eu|(O') - c_2\mathcal{L}^d(O') \le F''(u,O').$$

Using again (4.2.2) and (4.2.3), we deduce the corresponding lower bound for $F(u,O)$.

Assume now that $u \notin \mathrm{BD}(O)$. We want to prove that $F(u,O) = +\infty$. Assume by contradiction that $F(u,O) < +\infty$. Given $O' \in \mathcal{O}_c(O)$, we consider a sequence $\{u_n\}_n \subset L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ converging to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\limsup_n F_n(u_n,O') = F''(u,O') \leq F(u,O) < +\infty$. Arguing as before, we obtain that $u_n \in \mathrm{BD}(O')$ for $n \in \mathbb{N}$ large enough and

$$\limsup_{n\to+\infty} |Eu_n|(O') \leq \frac{1}{c_1} \limsup_{n\to+\infty} F_n(u_n,O') + \frac{c_2}{c_1}\mathcal{L}^d(O') \leq \frac{F(u,O)}{c_1} + \frac{c_2}{c_1}\mathcal{L}^d(O),$$

which implies $u \in \mathrm{BD}(O')$ and $|Eu|(O') \leq \frac{1}{c_1}F(u,O) + \frac{c_2}{c_1}\mathcal{L}^d(O)$. As this holds for every $O' \in \mathcal{O}_c(O)$, we obtain $u \in \mathrm{BD}(O)$, contradicting our hypothesis. This concludes the proof of the implication $u \notin \mathrm{BD}(O) \implies F(u,O) = +\infty$.

Properties (H4)–(H6) for F follow immediately from the same properties for F_n.

In Lemma 4.2.5 we will prove that for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and $O \in \mathcal{O}_c(\mathbb{R}^d)$ we have

$$F(u,O) = F'(u,O) = F''(u,O), \tag{4.2.6}$$

hence for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ in the topology of $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. □

In the rest of this section, we assume that F' and F'' are defined by (4.2.1) and that F satisfies (4.2.3). Our aim is to prove equality (4.2.6) and the measure property (H1) of Definition 4.1.4 for this functional. The proof of the measure property (H1) is based on the De Giorgi-Letta criterion for measures, which is based on the subadditivity of the function $O \mapsto F(u,O)$. For this reason, we begin by proving that F'' satisfies the *nested* subadditivity condition.

Lemma 4.2.2. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and $O', O, U \in \mathcal{O}_c(\mathbb{R}^d)$, with $O' \subset\subset O$. Then*

$$F''(u,O' \cup U) \leq F''(u,O) + F''(u,U). \tag{4.2.7}$$

Proof. Without loss of generality, we may assume that $F''(u,O) + F''(u,U) < +\infty$. Since $F'' \geq F$ and F satisfies property (H3), this implies that $u \in \mathrm{BD}(O \cup U)$. Recalling the definition of F'' (see (4.2.1)), we can find two sequences $\{u_n\}_n$ and $\{v_n\}_n$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, converging to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and such that

$$F''(u,O) = \limsup_{n\to+\infty} F_n(u_n,O) \quad \text{and} \quad F''(u,U) = \limsup_{n\to+\infty} F_n(v_n,U). \tag{4.2.8}$$

Thanks to property (H3) for F_n and the finiteness of $F''(u,O)$ and $F''(u,U)$ we can assume that $\{u_n\}_n \subset \mathrm{BD}(O)$, $\{v_n\}_n \subset \mathrm{BD}(U)$, and that

$$M \coloneqq c_3 \sup_{n\in\mathbb{N}} \Big(|Eu_n|(O) + |Ev_n|(U) \Big) + c_4\mathcal{L}^d(O \cup U) < +\infty. \tag{4.2.9}$$

Let us fix $m \in \mathbb{N}$ and set $\eta \coloneqq \mathrm{dist}(O', \partial O) > 0$. For $j \in \{0,\dots,m\}$ we also consider the sets $O_j \coloneqq \{x \in O\colon \mathrm{dist}(x,\partial O) > \eta - \frac{j\eta}{m}\}$. Clearly, we have $O' \subset O_0 \subset\subset \cdots \subset\subset O_{m-1} \subset O_m = O$.

For every $j \in \{1,\dots,m\}$ we consider a function $\varphi_j \in C^\infty_c(\mathbb{R}^d;[0,1])$ satisfying $\varphi_j = 1$ on a neighbourhood of $\overline{O}_{j-1}$, $\varphi_j = 0$ on a neighbourhood of $\mathbb{R}^d \setminus O_j$, and

$$|\nabla\varphi_j| \leq \frac{2m}{\eta} \quad \text{on } \mathbb{R}^d. \tag{4.2.10}$$

We set

$$w_n^j \coloneqq \varphi_j u_n + (1-\varphi_j)v_n \quad \text{and} \quad S_j \coloneqq (O_j \setminus \overline{O}_{j-1}) \cap U, \tag{4.2.11}$$

and observe that $w_n^j \in \mathrm{BD}(O' \cup U)$ and

$$Ew_n^j = \varphi_j Eu_n + (1-\varphi_j)Ev_n + (u_n - v_n) \odot \nabla\varphi_j \, \mathcal{L}^d \tag{4.2.12}$$

as Radon measures on $O' \cup U$. Moreover, one checks immediately that for every choice of $j_n \in \{1, \dots, m\}$ the sequence $\{w_n^{j_n}\}_{n\in\mathbb{N}}$ converges to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ as $n \to +\infty$.

Let us fix $j \in \{1, \dots, m-1\}$ and $n \in \mathbb{N}$. We observe that by the measure property (H1), by the properties of φ_j, and by the locality property of Remark 4.1.5 we have

$$\begin{aligned} F_n(w_n^j, O' \cup U) &\le F_n(w_n^j, \overline{O}_{j-1}) + F_n(w_n^j, S_j) + F_n(w_n^j, U \setminus O_j) \\ &= F_n(u_n, \overline{O}_{j-1}) + F_n(w_n^j, S_j) + F_n(v_n, U \setminus O_j) \\ &\le F_n(u_n, O) + F_n(w_n^j, S_j) + F_n(v_n, U). \end{aligned} \tag{4.2.13}$$

We now estimate the term involving the strip S_j in (4.2.13). Thanks to (H3) we have

$$F_n(w_n^j, S_j) \le c_3 |Ew_n^j|(S_j) + c_4 \mathcal{L}^d(S_j),$$

so that by (4.2.10) and (4.2.12) we get

$$|Ew_n^j|(S_j) \le |Eu_n|(S_j) + |Ev_n|(S_j) + \frac{2m}{\eta} \int_{S_j} |u_n - v_n| \, \mathrm{d}x.$$

From these two inequalities, we deduce that for every $n \in \mathbb{N}$ we can find $j_n \in \{1, \dots, m\}$ such that, setting $w_n \coloneqq w_n^{j_n}$, we have

$$\begin{aligned} F_n(w_n, S_{j_n}) &\le \frac{1}{m} \sum_{j=1}^m \Big(c_3|Eu_n|(S_j) + c_3|Ev_n|(S_j) + \frac{2c_3 m}{\eta} \int_{S_j} |u_n - v_n| \, \mathrm{d}x + c_4 \mathcal{L}^d(S_j) \Big) \\ &\le \frac{1}{m} \big(c_3|Eu_n|(O) + c_3|Ev_n|(U) + c_4 \mathcal{L}^d(O \cup U) \big) + \frac{2c_3}{\eta} \int_{O\cap U} |u_n - v_n| \, \mathrm{d}x \\ &\le \frac{M}{m} + \frac{2c_3}{\eta} \int_{O\cap U} |u_n - v_n| \, \mathrm{d}x, \end{aligned} \tag{4.2.14}$$

where in the third inequality we used (4.2.9).

Finally, combining (4.2.13) and (4.2.14), we deduce that

$$F_n(w_n, O' \cup U) \le F_n(u_n, O) + F_n(v_n, U) + \frac{M}{m} + \frac{2c_3}{\eta} \int_{O\cap U} |u_n - v_n| \, \mathrm{d}x \tag{4.2.15}$$

for every $n \in \mathbb{N}$. Recalling that $\{u_n - v_n\}_n$ converges to 0 strongly in $L^1(O \cap U; \mathbb{R}^d)$, that $\{w_n\}_n$ converges to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, we may let $n \to +\infty$ in (4.2.15) and from (4.2.8) we deduce that

$$F''(u, O' \cup U) \le \limsup_{n\to+\infty} F_n(w_n, O' \cup U) \le F''(u, O) + F''(u, U) + \frac{M}{m}.$$

Letting $m \to +\infty$ in this inequality, we obtain (4.2.7). □

We are now ready to prove the subadditivity of $F(u, \cdot)$ on $\mathcal{O}(\mathbb{R}^d)$.

Lemma 4.2.3. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and $O, U \in \mathcal{O}(\mathbb{R}^d)$. Then*

$$F(u, O \cup U) \leq F(u, O) + F(u, U).$$

Proof. The result is a consequence of Lemma 4.2.2 and of classical arguments concerning increasing set functions (see, for instance, [88, Lemma 18.4]). □

We now prove that F satisfies the measure property (H1).

Lemma 4.2.4. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Then $F(u,\cdot)$ is a Borel measure.*

Proof. It is enough to use the De Giorgi-Letta criterion for measures (see [106, Théorème 5.1] or [88, Theorem 14.21]), which ensures that the claim is proved, provided that we show that the set function $O \mapsto F(u, O)$ is subadditive, superadditive, and inner regular on $\mathcal{O}(\mathbb{R}^d)$. Subadditivity follows from Lemma 4.2.3, superadditivity is a consequence of classic results of Γ-convergence (see [88, Proposition 16.12]), and inner regularity follows from (4.2.3), recalling (4.2.2). □

Finally, we conclude the section by showing that Γ-convergence takes place for every $O \in \mathcal{O}_c(\mathbb{R}^d)$.

Lemma 4.2.5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and $O \in \mathcal{O}_c(\mathbb{R}^d)$. Then*

$$F(u, O) = F'(u, O) = F''(u, O).$$

Proof. Since the inequalities $F(u, O) \leq F'(u, O) \leq F''(u, O)$ are obvious, we only have to show that $F''(u, O) \leq F(u, O)$, assuming that $F(u, O) < +\infty$. By property (H3) for the functional F, this implies that $u \in \mathrm{BD}(O)$.

Let $\varepsilon > 0$ and consider a compact set $K \subset O$ such that

$$c_3|Eu|(O \setminus K) + c_4\mathcal{L}^d(O \setminus K) < \varepsilon.$$

Consider now $O' \in \mathcal{O}_c(\mathbb{R}^d)$ with $K \subset O' \subset\subset O$. Using Lemma 4.2.2 with $U = O \setminus K$, from the previous inequality and (4.2.5) we obtain

$$F''(u, O) \leq F''(u, O') + F''(u, O \setminus K) < F(u, O) + \varepsilon.$$

The conclusion then follows from the arbitrariness of ε. □

4.3 A partial integral representation for abstract functionals

In this section we use Theorem 1.5.4 to provide a partial integral representation of every functional $F \in \mathfrak{F}_{\mathrm{sc}}$. More precisely, we show that the "absolutely continuous" part and the "jump" part of every such F can be written as integrals associated to suitable integrands $f \in \mathscr{F}$ and $g \in \mathscr{G}$.

We begin by introducing a decomposition of functionals in $\mathfrak{F}$ that reflects the usual decomposition of the measure Eu described in (1.4.1).

Definition 4.3.1. Let $F \in \mathfrak{F}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $u \in \mathrm{BD}(O)$. By (H1) and (H3) of Definition 4.1.4 $F(u,\cdot)$ is a bounded Radon measure on O. Its absolutely continuous and the singular part with respect to $\mathcal{L}^d$, defined on every $B \in \mathcal{B}(O)$, are denoted by $F^a(u,B)$ and $F^s(u,B)$, respectively. We also introduce the bounded Radon measures $F^c(u,\cdot)$ and $F^j(u,\cdot)$ defined by

$$F^c(u,B) \coloneqq F^s(u, B \setminus J_u) \text{ and } F^j(u,B) \coloneqq F(u, B \cap J_u)$$

for every $B \in \mathcal{B}(O)$. Note that we have

$$F(u,B) = F^a(u,B) + F^c(u,B) + F^j(u,B) \tag{4.3.1}$$

for every $B \in \mathcal{B}(O)$.

Remark 4.3.2. Let $F \in \mathfrak{F}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $u \in \mathrm{BD}(O)$. It follows directly from the bounds in (H3) of Definition 4.1.4 that $F^c(u,\cdot)$ and $F^j(u,\cdot)$, considered as measures defined on $\mathcal{B}(O)$, are absolutely continuous with respect to $|E^c u|$ and $\mathcal{H}^{d-1} \llcorner J_u$, respectively.

Similarly to the previous two chapters, we will often work with functions defined by using auxiliary minimisation problems. For this reason, given a functional $F \in \mathfrak{F}$, a set $O \in \mathcal{O}_c(\mathbb{R}^d)$ with Lipschitz boundary ∂O, and a function $v \in \mathrm{BD}(O)$, we set

$$\mathfrak{m}^F(v,O) \coloneqq \inf\{F(u,O)\colon\ u \in \mathrm{BD}(O) \text{ and } u = v \text{ on } \partial O\},$$

where the equality $u = v$ is understood in the sense of traces. As v is a competitor for the minimisation problem $\mathfrak{m}^F(v,O)$, by the upper bounds in Definition 4.1.4 it follows that

$$\mathfrak{m}^F(v,O) \le F(v,O) \le c_3|Ev|(O) + c_4\mathcal{L}^d(O) < +\infty.$$

Given $F \in \mathfrak{F}$, for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, and $\nu \in \mathbb{S}^{d-1}$ we set

$$f(x,A) \coloneqq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d}, \tag{4.3.2}$$

$$g(x,\zeta,\nu) \coloneqq \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}, \tag{4.3.3}$$

where we recall that $Q(x,\rho)$ and $Q_\nu(x,\rho)$ are the cubes introduced at the beginning of Section 1.1 of Chapter 1 , while the functions ℓ_A and $u_{x,\zeta,\nu}$ are those introduced in (1.5.2) and (1.5.3), respectively.

We now state the partial integral representation result.

Lemma 4.3.3. *Let $F \in \mathfrak{F}_{\mathrm{sc}}$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $u \in \mathrm{BD}(O)$. Let f and g be the two functions defined by* (4.3.2) *and* (4.3.3)*. Then $f \in \mathscr{F}$, $g \in \mathscr{G}$, and*

$$F^a(u,B) = \int_B f(x, \mathcal{E}u)\,\mathrm{d}x, \tag{4.3.4}$$

$$F^j(u,B) = \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{4.3.5}$$

for every set $B \in \mathcal{B}(O)$.

Proof. Equalities (4.3.4) and (4.3.5) follow from Theorem 1.5.4, while the inclusions $f \in \mathscr{F}$ and $g \in \mathscr{G}$ are proved in Lemmas 4.3.6 and 4.3.8 below. □

Remark 4.3.4. We remark that properties (H5) and (H6) of Definition 4.1.4 are not used to obtain (4.3.4) and (4.3.5). Properties (H5) and (H6) are used only to prove that f and g satisfy (f3) and (g4) .

To prove that f is Borel measurable we will need the following lemma, which will be used also in Section 4.7 for different purposes.

Lemma 4.3.5. *Let $F \in \mathfrak{F}$, $O', O \in \mathcal{O}_c(\mathbb{R}^d)$, with Lipschitz boundary and $O' \subset\subset O$, and let $w \in \mathrm{BD}(O)$. Then*

$$\mathfrak{m}^F(w, O) \le \mathfrak{m}^F(w, O') + c_3|Ew|(O \setminus O') + c_4\mathcal{L}^d(O \setminus O'). \tag{4.3.6}$$

Proof. Let $\eta > 0$ and $u \in \mathrm{BD}(O')$ with $u = w$ on $\partial O'$ be such that

$$F(u, O') \le \mathfrak{m}^F(w, O') + \eta. \tag{4.3.7}$$

We can then extend u to O by setting $u = w$ on $O \setminus O'$, so that $u = w$ on ∂O, which implies

$$\mathfrak{m}^F(w, O) \le F(u, O).$$

Using properties (H1) and (H3) of Definition 4.1.4 we get

$$\begin{aligned} F(u, O) &= F(u, O') + F(u, O \setminus O') \\ &\le F(u, O') + c_3|Eu|(O \setminus \overline{O'}) + c_3|Eu|(\partial O') + c_4\mathcal{L}^d(O \setminus O'). \end{aligned} \tag{4.3.8}$$

Since $u = w$ on the open set $O\setminus\overline{O'}$, we have $Eu = Ew$ as measures defined on $\mathcal{B}(O\setminus\overline{O'})$, hence $|Eu|(O\setminus\overline{O'}) = |Ew|(O\setminus\overline{O'})$. Moreover, since $|Eu|\llcorner\partial O' = |(u^+ - u^-)\odot\nu_{\partial O'}|\mathcal{H}^{d-1}\llcorner(J_u \cap \partial O')$ and $|Ew|\llcorner\partial O' = |(w^+ - w^-)\odot\nu_{\partial O'}|\mathcal{H}^{d-1}\llcorner(J_w \cap \partial O')$, the equalities $u^+ = w^+$ and $u^- = w^-$ on $\partial O'$ imply that $|Eu|(\partial O') = |Ew|(\partial O')$. Combining this equality with (4.3.7) and (4.3.8), we get

$$\mathfrak{m}^F(w, O) \le \mathfrak{m}^F(w, O') + c_3|Ew|(O \setminus \overline{O'}) + c_3|Ew|(\partial O') + c_4\mathcal{L}^d(O \setminus O') + \eta,$$

which, by arbitrariness of η, implies (4.3.6). □

We are ready to prove that $f \in \mathscr{F}$.

Lemma 4.3.6. *Let $F \in \mathfrak{F}$. Then the function f defined by* (4.3.2) *belongs to $\mathscr{F}$.*

Proof. The proof of property (f1) can be obtained following the lines of the proof of [100, Lemma 5.3], replacing the Lemma 4.11 of that paper by our Lemma 4.3.5.

Let us fix $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. We want to show that

$$f(x, A) \le c_3|A| + c_4. \tag{4.3.9}$$

To this aim, we note that for every $\rho > 0$ from the upper bound in (H3) of Definition 4.1.4 and the fact that ℓ_A is a competitor for the minimisation problem $\mathfrak{m}^F(\ell_A, Q(x, \rho))$ it follows that

$$\mathfrak{m}^F(\ell_A, Q(x, \rho)) \le F(\ell_A, Q(x, \rho)) \le (c_3|A| + c_4)\rho^d.$$

We now divide by ρ^d, take the limsup as $\rho \to 0^+$, and use (4.3.2) to obtain (4.3.9).

To conclude the proof of (f2) we show that

$$c_1|A| - c_2 \le f(x, A). \tag{4.3.10}$$

To prove this, let us fix $\rho > 0$ and observe that, by the lower bound in (H3) of Definition 4.1.4, we have

$$c_1|Eu|(Q(x,\rho)) - c_2\rho^d \le F(u, Q(x,\rho))$$

for every $u \in \mathrm{BD}(Q(x,\rho))$, so that

$$c_1 \min\{|Eu|(Q(x,\rho))\colon\ u \in \mathrm{BD}(Q(x,\rho)) \text{ with } u = \ell_A \text{ on } \partial Q(x,\rho)\} - c_2\rho^d \le \mathfrak{m}^F(\ell_A, Q(x,\rho)).$$

By Jensen's inequality the function ℓ_A minimises the problem on the left-hand side of the previous inequality. Hence,

$$c_1|A|\rho^d - c_2\rho^d \le \mathfrak{m}^F(\ell_A, Q(x,\rho)).$$

Dividing by ρ^d, taking the limsup as $\rho \to 0^+$, and recalling (4.3.2), we obtain (4.3.10).

We now prove (f3) . To this aim, let us fix $x \in \mathbb{R}^d$, $A_1, A_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\rho > 0$, and $\eta > 0$. Consider a function $u_2 \in \mathrm{BD}(Q(x,\rho))$ with $u_2 = \ell_{A_2}$ on $\partial Q(x,\rho)$ and such that

$$F(u_2, Q(x,\rho)) \le \mathfrak{m}^F(\ell_{A_2}, Q(x,\rho)) + \eta\rho^d.$$

Then, we set $u_1 \coloneqq u_2 - \ell_{A_2} + \ell_{A_1}$ and observe that $u_1 = \ell_{A_1}$ on $\partial Q(x,\rho)$. Using property (H5) of Definition 4.1.4 and the previous inequality, it follows that

$$\begin{aligned}\mathfrak{m}^F(\ell_{A_1}, Q(x,\rho)) &\le F(u_1, Q(x,\rho)) \le F(u_2, Q(x,\rho)) + c_5|A_1 - A_2|\rho^d \\ &\le \mathfrak{m}^F(\ell_{A_2}, Q(x,\rho)) + c_5|A_1 - A_2|\rho^d + \eta\rho^d. \end{aligned} \tag{4.3.11}$$

Dividing by ρ^d, taking first the limsup as $\rho \to 0^+$ and then as $\eta \to 0^+$, by (4.3.2) we obtain

$$f(x, A_1) \le f(x, A_2) + c_5|A_1 - A_2|.$$

Exchanging the roles of A_1 and A_2, we obtain (f3) . □

To prove that g is Borel measurable we will need the following lemma, which will be used also in Section 4.7 for different purposes.

Lemma 4.3.7. *Let $F \in \mathfrak{F}$. Then there exists a continuous function $\omega\colon [0,+\infty)\times[0,+\infty) \to [0,+\infty)$, increasing with respect to each variable and with $\omega(0,0) = 0$, such that for every $x_1, x_2 \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$, and $0 < \rho_1 < \rho_2$, the inclusion $Q_{\nu_1}(x_1,\rho_1) \subset\subset Q_{\nu_2}(x_2,\rho_2)$ implies*

$$\begin{aligned}\mathfrak{m}^F(u_{x_2,\zeta,\nu_2}, Q_{\nu_2}(x_2,\rho_2)) \le{}& \mathfrak{m}^F(u_{x_1,\zeta,\nu_1}, Q_{\nu_1}(x_1,\rho_1)) + c_3|\zeta|(\rho_2^{d-1} - \rho_1^{d-1}) \\ &+ c_4(\rho_2^d - \rho_1^d) + c_3|\zeta|\omega(\tfrac{|x_2-x_1|}{\rho_1}, |\nu_1 - \nu_2|)\rho_1^{d-1}.\end{aligned}$$

Proof. The proof is obtained by the same argument used in the proof of [100, Lemma 5.9], replacing the use of the distributional gradient D by that of the symmetric distributional gradient E. □

We are now in position to prove that $g \in \mathscr{G}$.

Lemma 4.3.8. *Let $F \in \mathfrak{F}$. Then the function g defined by (4.3.3) belongs to $\mathscr{G}$.*

Proof. The proof of property (g1) can be obtained arguing as in [100, Lemma 5.12], replacing Lemma 5.9 of that paper by our Lemma 4.3.7.

Property (g2) follows immediately from (H4) of Definition 4.1.4 and (4.3.3) once we observe that $u_{x,-\zeta,-\nu} = u_{x,\zeta,\nu} - \zeta$ and that $Q_\nu(x,\rho) = Q_{-\nu}(x,\rho)$ (see property 4 of the paragraph about cubes of Section 1.1).

We prove property (g3) . Let us fix $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. As for every $\rho > 0$ the function $u_{x,\zeta,\nu}$ is a competitor for the minimisation problem $\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho))$, by the upper bound in (H3) of Definition 4.1.4 we obtain that

$$\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le c_3|\zeta \odot \nu|\rho^{d-1} + c_4\rho^d.$$

Dividing by ρ^{d-1}, letting $\rho \to 0^+$, and recalling (4.3.3) we obtain the upper bound in (g3) .

To obtain the lower bound in (g3) we argue as follows. For every $\rho > 0$ let $u \in \mathrm{BD}(Q_\nu(x,\rho))$ be a function with $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and observe that by the lower bound in (H3) of Definition 4.1.4 we have

$$c_1|Eu|(Q_\nu(x,\rho)) - c_2\rho^d \le F(u, Q_\nu(x,\rho)),$$

so that

$$c_1 \inf\{|Eu|(Q_\nu(x,\rho))\colon\ u = u_{x,\zeta,\nu} \text{ on } \partial Q_\nu(x,\rho)\} - c_2\rho^d \le \mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)). \tag{4.3.12}$$

Consider the function $\tilde{g}\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ defined for every $\zeta \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ by

$$\widetilde{g}(\zeta,\nu) \coloneqq \limsup_{\rho\to 0^+} \frac{\inf\{|Eu|(Q_\nu(x,\rho))\colon\ u = u_{x,\zeta,\nu} \text{ on } \partial Q_\nu(x,\rho)\}}{\rho^{d-1}}, \tag{4.3.13}$$

where x is an arbitrary point of $\mathbb{R}^d$. By Lemma 4.3.3 applied to the functional $\widetilde{F}(u,O) \coloneqq |Eu|(O)$, we have

$$\widetilde{F}^j(u,O) = \int_{J_u\cap O} \widetilde{g}([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \tag{4.3.14}$$

for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ and $u \in \mathrm{BD}(O)$. In particular, for $O = Q_\nu(x,1)$ and $u = u_{x,\zeta,\nu}$ equality (4.3.14) leads to $|\zeta \odot \nu| = \widetilde{g}(\zeta,\nu)$. Dividing (4.3.12) by ρ^{d-1}, letting $\rho \to 0^+$, and using (4.3.3) and (4.3.13), we obtain the lower bound in (g3) .

We conclude by showing that g satisfies property (g4) . Let $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $\rho > 0$, and $\eta > 0$. Consider a function $u_2 \in \mathrm{BD}(Q_\nu(x,\rho))$ with $u_2 = u_{x,\zeta_2,\nu}$ on $\partial Q_\nu(x,\rho)$ and such that

$$F(u_2, Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \eta\rho^{d-1}.$$

Then, we set $u_1 \coloneqq u_2 - u_{x,\zeta_2,\nu} + u_{x,\zeta_1,\nu}$ and observe that $u_1 = u_{x,\zeta_1,\nu}$ on $\partial Q_\nu(x,\rho)$. Using property (H6) of Definition 4.1.4 and the previous inequality, it follows that

$$\begin{aligned}\mathfrak{m}^F(u_{x,\zeta_1,\nu}, Q_\nu(x,\rho)) &\le F(u_1, Q_\nu(x,\rho)) \le F(u_2, Q_\nu(x,\rho)) + \tau(|\zeta_1-\zeta_2|)\rho^{d-1} \\ &\le \mathfrak{m}^F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \tau(|\zeta_1-\zeta_2|)\rho^{d-1} + \eta\rho^{d-1}.\end{aligned} \tag{4.3.15}$$

Dividing this inequality by ρ^{d-1}, taking first the limsup for $\rho \to 0^+$ and then the limit for $\eta \to 0^+$, by (4.3.3) we obtain

$$g(x,\zeta_1,\nu) \le g(x,\zeta_2,\nu) + \tau(|\zeta_1-\zeta_2|).$$

Exchanging the roles of ζ_1 and ζ_2, we obtain (g4) . □

4.4 A smaller collection of functionals

In this section we introduce two subcollections of $\mathscr{F}$ and $\mathscr{G}$, denoted by $\mathscr{F}^\alpha$ and $\mathscr{G}^\infty$, by prescribing suitable conditions on the integrands, related to their behaviour when $|A| \to +\infty$ and $|\zeta| \to +\infty$ (see Remarks 4.4.2 and 4.4.3 below). We then define a subcollection of $\mathfrak{F}$, denoted by $\mathfrak{F}^{\alpha,\infty}$, which is stable under Γ-convergence (see Proposition 4.4.6) and includes the functionals $F^{f,g}$ with $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\infty$. We shall see in Section 4.5 that all lower semicontinuous functionals $F \in \mathfrak{F}^{\alpha,\infty}$ admit a complete integral representation, including their Cantor part F^c, provided they satisfy an additional condition (see (4.5.1) below).

In the rest of the chapter, we fix three constants $0 < \alpha < 1$ and $c_6, c_7 \geq 0$.

Definition 4.4.1. Let $\mathscr{F}^\alpha$ be the collection of all functions $f \in \mathscr{F}$ such that, for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $s, t > 0$, we have

$$\Big|\frac{f(x,sA)}{s} - \frac{f(x,tA)}{t}\Big| \leq \frac{c_6}{s} f(x,sA)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t} f(x,tA)^{1-\alpha} + \frac{c_6}{t}. \tag{f4}$$

Let $\mathscr{G}^\infty$ be the collection of all functions $g \in \mathscr{G}$ such that, for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $s, t > 0$, we have

$$\Big|\frac{g(x,s\zeta,\nu)}{s} - \frac{g(x,t\zeta,\nu)}{t}\Big| \leq c_7\Big(\frac{g(x,s\zeta,\nu)}{s} + \frac{g(x,t\zeta,\nu)}{t}\Big)\Big(\frac{1}{s} + \frac{1}{t}\Big). \tag{g5}$$

Remark 4.4.2. A property closely related to (f4) was already introduced in Definition 2.4.1. We recall (see Remark 2.4.2) that the upper bound in (f2) and (f4), are equivalent to the two conditions

$$f^\infty(x,A) = \lim_{t\to+\infty} \frac{f(x,tA)}{t} \quad \text{for all } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}, \tag{4.4.1}$$

$$\Big|\frac{f(x,tA)}{t} - f^\infty(x,A)\Big| \leq \frac{c_6}{t} + \frac{c_6}{t} f(x,tA)^{1-\alpha} \quad \text{for all } x \in \mathbb{R}^d,\ t > 0, \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{4.4.2}$$

Observe that by the growth conditions in (f2) , inequality (4.4.2) implies that

$$\Big|\frac{f(x,tA)}{t} - f^\infty(x,A)\Big| \leq \frac{C_A}{t^\alpha} \quad \text{for every } x \in \mathbb{R}^d,\ t \geq 1, \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}, \tag{4.4.3}$$

where $C_A \coloneqq c_6(1 + c_4^{1-\alpha} + c_3^{1-\alpha}|A|^{1-\alpha})$.

Remark 4.4.3. If $g \in \mathscr{G}^\infty$, then for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ there exists the limit

$$g^\infty(x,\zeta,\nu) \coloneqq \lim_{t\to+\infty} \frac{g(x,t\zeta,\nu)}{t}. \tag{4.4.4}$$

Indeed, (g3) and (g5) imply the Cauchy condition for the function $t \mapsto g(x,t\zeta,\nu)/t$. Passing to the limit in (g3) we obtain

$$c_1|\zeta \odot \nu| \leq g^\infty(x,\zeta,\nu) \leq c_3|\zeta \odot \nu| \tag{4.4.5}$$

for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, while passing to the limit in (g4) and using (4.1.1) we get

$$|g^\infty(x,\zeta_1,\nu) - g^\infty(x,\zeta_2,\nu)| \leq \tau^\infty|\zeta_1 - \zeta_2| \tag{g4$'$}$$

for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$, where τ^∞ is the non-negative constant introduced in (4.1.1).

Letting $s \to +\infty$ in (g5) , we also obtain

$$\big|\frac{g(x,t\zeta,\nu)}{t} - g^\infty(x,\zeta,\nu)\big| \le c_7\Big(\frac{g(x,t\zeta,\nu)}{t} + g^\infty(x,\zeta,\nu)\Big)\frac{1}{t} \tag{4.4.6}$$

for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $t > 0$. Thanks to the bounds in (g3) and (4.4.5), this inequality implies that

$$\big|\frac{g(x,t\zeta,\nu)}{t} - g^\infty(x,\zeta,\nu)\big| \le \frac{2c_3c_7|\zeta \odot \nu|}{t}, \tag{4.4.7}$$

while the inequality

$$\big|\frac{g(x,t\zeta,\nu)}{t} - g^\infty(x,\zeta,\nu)\big| \le \frac{2c_1c_7|\zeta \odot \nu|}{t}$$

implies (4.4.6).

Conversely, assume that g is a function in $\mathscr{G}$ such that the limit in (4.4.4) exists and the inequality

$$g^\infty(x,\zeta,\nu) \le \frac{g(x,t\zeta,\nu)}{t} \tag{4.4.8}$$

is satisfied for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $t > 0$. If g satisfies also (4.4.6), then $g \in \mathscr{G}^\infty$. Indeed, by (4.4.6) and (4.4.8) we have

$$\big|\frac{g(x,t\zeta,\nu)}{t} - g^\infty(x,\zeta,\nu)\big| \le c_7\Big(\frac{g(x,t\zeta,\nu)}{t} + \frac{g(x,s\zeta,\nu)}{s}\Big)\frac{1}{t} \tag{4.4.9}$$

for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $s,t > 0$. Exchanging the roles of s and t, we obtain

$$\big|\frac{g(x,s\zeta,\nu)}{s} - g^\infty(x,\zeta,\nu)\big| \le c_7\Big(\frac{g(x,s\zeta,\nu)}{s} + \frac{g(x,t\zeta,\nu)}{t}\Big)\frac{1}{s}. \tag{4.4.10}$$

Using the triangle inequality, from (4.4.9) and (4.4.10) we obtain (g5) .

Note that the existence of the limit in (4.4.4) and inequality (4.4.8) is always satisfied when $t \mapsto g(x,t\zeta,\nu)$ is concave.

We now introduce a collection of functionals closely related to conditions (f4) and (g5).

Definition 4.4.4. Let $\mathfrak{F}^{\alpha,\infty}$ be the collection of all functionals $F \in \mathfrak{F}$ such that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$, $u \in \mathrm{BD}(O)$, and $s,t > 0$ we have

$$\begin{aligned}\big|\frac{F(su,O)}{s} - \frac{F(tu,O)}{t}\big| &\le \frac{c_6}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha} + \frac{c_6}{s}\mathcal{L}^d(O)\\ +\frac{c_6}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(O) &+ c_7\Big(\frac{F(su,O)}{s} + \frac{F(tu,O)}{t}\Big)\Big(\frac{1}{s} + \frac{1}{t}\Big).\end{aligned} \tag{g}$$

The collection of functionals $F \in \mathfrak{F}^{\alpha,\infty}$ such that for every $O \in \mathcal{O}(\mathbb{R}^d)$ the functional $F(\cdot,O)$ is $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous is denoted by $\mathfrak{F}^{\alpha,\infty}_{\mathrm{sc}}$.

We now show that integral functionals of the form $F^{f,g}$ belong to $\mathfrak{F}^{\alpha,\infty}$ whenever $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\infty$.

Proposition 4.4.5. *Let $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\infty$. Then the functional $F^{f,g}$ introduced in Definition 4.1.3 belongs to $\mathfrak{F}^{\alpha,\infty}$.*

Proof. Thanks to Proposition 4.1.8, we only need to prove that $F^{f,g}$ satisfies property (g). To this aim, let us fix $O \in \mathcal{O}_c(\mathbb{R}^d)$, $u \in \mathrm{BD}(O)$, and $s,t > 0$. Using (f4) and Hölder's inequality we see that

$$\begin{aligned}\Big|\int_O \frac{f(x,s\mathcal{E}u)}{s}\,\mathrm{d}x - \int_O \frac{f(x,t\mathcal{E}u)}{t}\,\mathrm{d}x\Big| &\le \int_O \Big(\frac{c_6}{s}f(x,s\mathcal{E}u)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t}f(x,t\mathcal{E}u)^{1-\alpha} + \frac{c_6}{t}\Big)\mathrm{d}x\\ &\le \frac{c_6}{s}\mathcal{L}^d(O)^\alpha F^{f,g}(su,O)^{1-\alpha} + \frac{c_6}{s}\mathcal{L}^d(O) + \frac{c_6}{t}\mathcal{L}^d(O)^\alpha F^{f,g}(tu,O)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(O). \quad (4.4.11)\end{aligned}$$

Since f^∞ is positively 1-homogeneous, we immediately see that

$$\frac{1}{s}\int_O f^\infty\Big(x, s\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| = \frac{1}{t}\int_O f^\infty\Big(x, t\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u|. \quad (4.4.12)$$

Condition (g5) implies that

$$\begin{aligned}&\Big|\int_{J_u\cap O}\Big(\frac{g(x,s[u],\nu_u)}{s} - \frac{g(x,t[u],\nu_u)}{t}\Big)\mathrm{d}\mathcal{H}^{d-1}\Big|\\ &\qquad\le c_7\Big(\int_{J_u\cap O}\frac{g(x,s[u],\nu_u)}{s}\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_u\cap O}\frac{g(x,t[u],\nu_u)}{t}\,\mathrm{d}\mathcal{H}^{d-1}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\qquad\le c_7\Big(\frac{F^{f,g}(su,O)}{s} + \frac{F^{f,g}(tu,O)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big). \quad (4.4.13)\end{aligned}$$

Finally, combining (4.4.11)–(4.4.13) we obtain that $F^{f,g}$ satisfies property (g). □

We now prove that the closure of the class $\mathfrak{F}^{\alpha,\infty}$ with respect to Γ-convergence is $\mathfrak{F}_{\mathrm{sc}}^{\alpha,\infty}$.

Proposition 4.4.6. *Let $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\infty}$. Assume that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Then the functional F belongs to $\mathfrak{F}_{\mathrm{sc}}^{\alpha,\infty}$.*

Proof. Thanks to Theorem 4.2.1 we have $F \in \mathfrak{F}_{\mathrm{sc}}$. Hence, to conclude we only have to show that F satisfies property (g). The argument we use is a variant of the one presented in the proof of Proposition 2.4.12.

Let us fix $O \in \mathcal{O}_c(\mathbb{R}^d)$, $u \in \mathrm{BD}(O)$, and $s,t > 0$. To prove that (g) holds it is enough to show that

$$\begin{aligned}&\frac{F(su,O)}{s} - \frac{c_6}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(O) - c_7\frac{F(su,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\quad\le \frac{F(tu,O)}{t} + \frac{c_6}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(O) + c_7\frac{F(tu,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big). \quad (4.4.14)\end{aligned}$$

Indeed, exchanging the role of s and t we obtain (g). We note that, if the left-hand side of (4.4.14) is less than or equal to zero, then the inequality is trivial because the right-hand side is non-negative. Hence, we may assume that the left-hand side is positive.

Let $\{u_n\}_n \subset \mathrm{BD}(O)$ be a sequence converging to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and such that $F_n(tu_n,O)$ converges to $F(tu,O)$ as $n \to +\infty$. It follows from property (g) applied to F_n that for every $n \in \mathbb{N}$ we have

$$\begin{aligned}&\frac{F_n(su_n,O)}{s} - \frac{c_6}{s}\mathcal{L}^d(O)^\alpha F_n(su_n,O)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(O) - c_7\frac{F_n(su_n,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\quad\le \frac{F_n(tu_n,O)}{t} + \frac{c_6}{t}\mathcal{L}^d(O)^\alpha F_n(tu_n,O)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(O) + c_7\frac{F_n(tu_n,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big). \quad (4.4.15)\end{aligned}$$

Thanks to our choice of $\{u_n\}_n$, we have

$$\lim_{n\to+\infty}\Big(\frac{F_n(tu_n,O)}{t}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha F_n(tu_n,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\frac{F_n(tu_n,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)\Big)$$
$$=\frac{F(tu,O)}{t}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\frac{F(tu,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Hence, by (4.4.15) to conclude we only need to show that

$$\frac{F(su,O)}{s}-\frac{c_6}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(O)-c_7\frac{F(su,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big) \tag{4.4.16}$$
$$\le\liminf_{n\to+\infty}\Big(\frac{F_n(su_n,O)}{s}-\frac{c_6}{s}\mathcal{L}^d(O)^\alpha F_n(su_n,O)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(O)-c_7\frac{F_n(su_n,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\Big).$$

We consider the function $\Phi\colon[0,+\infty)\to\mathbb{R}$ defined for every $z\in[0,+\infty)$ by

$$\Phi(z):=\frac{z}{s}-\frac{c_6}{s}\mathcal{L}^d(O)^\alpha z^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(O)-c_7\frac{z}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

We observe that $\Phi(F(su,O))$ coincides with the left-hand side of (4.4.14) and (4.4.16), while the right-hand side of (4.4.16) coincides with $\liminf_n\Phi(F_n(su_n,O))$. If $1-c_7(1/s+1/t)\le 0$ then $\Phi(z)\le 0$ for every $z\in[0,+\infty)$, so that (4.4.14) is satisfied. If $1-c_7(1/s+1/t)>0$, then we set $z_0:=\big(c_6(1-\alpha)\mathcal{L}^d(O)^\alpha\big)^{1/\alpha}\big(1-c_7\big(1/s+1/t\big)\big)^{-1/\alpha}$. By direct computation of Φ' we see that Φ is strictly decreasing in $[0,z_0]$ and strictly increasing in $[z_0,+\infty)$. Since $\Phi(0)\le 0$ we deduce that $\Phi(z)\le 0$ for $z\in[0,z_0]$. This implies that if $\Phi(z)>0$ then $z>z_0$. Since $\Phi(F(su,O))$ coincides with the left-hand side of (4.4.16), which we assumed to be positive, we have that $F(su,O)>z_0$. Moreover, by the Γ-liminf inequality we have $F(su,O)\le\liminf_n F_n(su_n,O)$, so that, using that Φ is increasing on $[z_0,+\infty)$, we deduce that

$$\Phi(F(su,O))\le\liminf_{n\to+\infty}\Phi(F_n(su_n,O)).$$

This proves (4.4.16) and concludes the proof of the proposition. □

We now present an inequality that allows to estimate the difference between the minimum values of some minimisation problems with Dirichlet boundary conditions involving functionals in $\mathfrak{F}^{\alpha,\infty}$. The aim of this lemma is twofold. On the one hand, it is useful to establish that the functions f and g defined by (4.3.2) and (4.3.3) satisfy (f4) and (g5). On the other hand, this lemma will be crucial in the proof of the integral representation of the Cantor part of functionals in $\mathfrak{F}^{\alpha,\infty}_{\rm sc}$. The proof of this lemma closely resembles that of Proposition 4.4.6. It follows the lines of the proofs of Lemmas 2.4.13 and 2.4.14, removing any truncation arguments, that are not available in our context.

Lemma 4.4.7. *Let $F\in\mathfrak{F}^{\alpha,\infty}$, let $O\in\mathcal{O}_c(\mathbb{R}^d)$ with Lipschitz boundary, and let $w\in\mathrm{BD}(O)$. Then*

$$\Big|\frac{\mathfrak{m}^F(sw,O)}{s}-\frac{\mathfrak{m}^F(tw,O)}{t}\Big|\le\frac{c_6}{s}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(sw,O)^{1-\alpha}+\frac{c_6}{s}\mathcal{L}^d(O)$$
$$+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(tw,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\Big(\frac{\mathfrak{m}^F(sw,O)}{s}+\frac{\mathfrak{m}^F(tw,O)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Proof. It is enough to prove that

$$\begin{aligned}&\frac{\mathfrak{m}^F(sw,O)}{s}-\frac{c_6}{s}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(sw,O)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(O)-c_7\frac{\mathfrak{m}^F(sw,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\&\le\frac{\mathfrak{m}^F(tw,O)}{t}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(tw,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\frac{\mathfrak{m}^F(tw,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)\end{aligned}\tag{4.4.17}$$

whenever the left-hand side is positive.

Let $\eta>0$ and consider a function $u\in\mathrm{BD}(O)$ with $u=w$ on ∂O and such that

$$F(tu,O)\le\mathfrak{m}^F(tw,O)+\eta.$$

Since F satisfies property (g) of Definition 4.4.4, from this inequality we deduce that

$$\begin{aligned}&\frac{F(su,O)}{s}-\frac{c_6}{s}\mathcal{L}^d(O)^\alpha F(su,O)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(O)-c_7\frac{F(su,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\&\quad\le\frac{F(tu,O)}{t}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha F(tu,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\frac{F(tu,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\&\quad\le\frac{\mathfrak{m}^F(tw,O)}{t}+\frac{\eta}{t}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(tw,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha\eta^{1-\alpha}\\&\qquad+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\frac{\mathfrak{m}^F(tw,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)+c_7\frac{\eta}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

Exploiting the fact that $\mathfrak{m}^F(sw,O)\le F(su,O)$ and using the same argument employed at the end of the proof of Proposition 4.4.6 we conclude that

$$\begin{aligned}&\frac{\mathfrak{m}^F(sw,O)}{s}-\frac{c_6}{s}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(sw,O)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(O)-c_7\frac{\mathfrak{m}^F(sw,O)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\&\quad\le\frac{\mathfrak{m}^F(tw,O)}{t}+\frac{\eta}{t}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha\mathfrak{m}^F(tw,O)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(O)^\alpha\eta^{1-\alpha}\\&\qquad+\frac{c_6}{t}\mathcal{L}^d(O)+c_7\frac{\mathfrak{m}^F(tw,O)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)+c_7\frac{\eta}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

As η is arbitrary, we obtain (4.4.17). □

We conclude the section by proving a further property of the functionals $F\in\mathfrak{F}^{\alpha,\infty}$.

Lemma 4.4.8. *Let $F\in\mathfrak{F}^{\alpha,\infty}$. Then the functions f and g defined by* (4.3.2) *and* (4.3.3) *belong to $\mathscr{F}^\alpha$ and $\mathscr{G}^\infty$, respectively.*

Proof. Thanks to Lemmas 4.3.6 and 4.3.8, to conclude it is enough to show that f and g satisfy properties (f4) and (g5).

We first prove that f satisfies (f4). Let us fix $x\in\mathbb{R}^d$, $A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$, and $s,t>0$. Thanks to Lemma 4.4.7 applied with $O=Q(x,\rho)$ and $w=\ell_A$, we have that

$$\begin{aligned}&\frac{\mathfrak{m}^F(s\ell_A,Q(x,\rho))}{s}\le\frac{\mathfrak{m}^F(t\ell_A,Q(x,\rho))}{t}+\frac{c_6}{s}\rho^{\alpha d}\mathfrak{m}^F(s\ell_A,Q(x,\rho))^{1-\alpha}+\frac{c_6}{s}\rho^d\\&+\frac{c_6}{t}\rho^{\alpha d}\mathfrak{m}^F(t\ell_A,Q(x,\rho))^{1-\alpha}+\frac{c_6}{t}\rho^d+c_7\Big(\frac{\mathfrak{m}^F(s\ell_A,Q(x,\rho))}{s}+\frac{\mathfrak{m}^F(t\ell_A,Q(x,\rho))}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

Dividing this inequality by ρ^d, letting $\rho\to0^+$, and recalling (4.3.2), we obtain

$$\begin{aligned}\frac{f(x,sA)}{s}&\le\frac{f(x,tA)}{t}+\frac{c_6}{s}f(x,sA)^{1-\alpha}+\frac{c_6}{s}+\frac{c_6}{t}f(x,tA)^{1-\alpha}\\&\quad+\frac{c_6}{t}+c_7\Big(\frac{f(x,sA)}{s}+\frac{f(x,tA)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

Exchanging the roles of s and t we obtain

$$\frac{f(x,tA)}{t} \le \frac{f(x,sA)}{s} + \frac{c_6}{s} f(x,sA)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t} f(x,tA)^{1-\alpha} + \frac{c_6}{t} + c_7\Big(\frac{f(x,sA)}{s} + \frac{f(x,tA)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

As these inequalities hold for every s and t, we may let $s \to +\infty$ and obtain for every $t > 0$ that

$$\big|f^\infty(x,A) - \frac{f(x,tA)}{t}\big| \le \frac{c_6}{t} f(x,tA)^{1-\alpha} + \frac{c_6}{t} + \frac{c_7}{t}\Big(f^\infty(x,A) + \frac{f(x,tA)}{t}\Big).$$

Let us fix $\lambda > 0$ and apply this inequality with A replaced by λA to get

$$\big|\lambda f^\infty(x,A) - \frac{f(x,t\lambda A)}{t}\big| \le \frac{c_6}{t} f(x,t\lambda A)^{1-\alpha} + \frac{c_6}{t} + \frac{c_7}{t}\Big(\lambda f^\infty(x,A) + \frac{f(x,t\lambda A)}{t}\Big)$$

for every $t > 0$, hence

$$\big|f^\infty(x,A) - \frac{f(x,t\lambda A)}{\lambda t}\big| \le \frac{c_6}{\lambda t} f(x,t\lambda A)^{1-\alpha} + \frac{c_6}{\lambda t} + \frac{c_7}{t}\Big(f^\infty(x,A) + \frac{f(x,t\lambda A)}{\lambda t}\Big)$$

for every $t > 0$ and $\lambda > 0$. Setting $\tau = t\lambda$

$$\big|f^\infty(x,A) - \frac{f(x,\tau A)}{\tau}\big| \le \frac{c_6}{\tau} f(x,\tau A)^{1-\alpha} + \frac{c_6}{\tau} + \frac{c_7}{t}\Big(f^\infty(x,A) + \frac{f(x,\tau A)}{\tau}\Big)$$

for every $t > 0$ and $\tau > 0$, so that, letting $t \to +\infty$ we obtain

$$\big|f^\infty(x,A) - \frac{f(x,\tau A)}{\tau}\big| \le \frac{c_6}{\tau} f(x,\tau A)^{1-\alpha} + \frac{c_6}{\tau} \tag{4.4.18}$$

for every $\tau > 0$. Recalling that f satisfies (f2) , we may use Remark 4.4.2 to deduce from (4.4.18) that f satisfies (f4), concluding the proof of the inclusion $f \in \mathscr{F}^\alpha$.

We now prove that g satisfies property (g5) . Let $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. Lemma 4.4.7 applied with $O = Q_\nu(x,\rho)$ and $w = u_{x,\zeta,\nu}$ implies that for every $s,t > 0$ we have

$$\begin{aligned}\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} \le{}& \frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t} + \frac{c_6}{s}\rho^{\alpha d}\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \frac{c_6}{s}\rho^d \\ &+ \frac{c_6}{t}\rho^{\alpha d}\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \frac{c_6}{t}\rho^d \\ &+ c_7\Big(\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} + \frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

Dividing this inequality by ρ^{d-1}, letting $\rho \to 0^+$, and recalling (4.3.3) we deduce

$$\frac{g(x,s\zeta,\nu)}{s} \le \frac{g(x,t\zeta,\nu)}{t} + c_7\Big(\frac{g(x,s\zeta,\nu)}{s} + \frac{g(x,t\zeta,\nu)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Exchanging the roles of s and t we obtain (g5) . □

4.5 Full integral representation

In this section we prove a complete integral representation result for functionals F in $\mathfrak{F}_{\rm sc}^{\alpha,\infty}$, including the representation of the Cantor part F^c, assuming that the function f, defined in (4.3.2), does not depend on x.

A full integral representation result for functionals $F \in \mathfrak{F}_{\rm sc}$ has recently been proved in [63], under the additional assumption of uniform continuity of F with respect to translations of the independent variables or, as stated in Theorem 1.5.4, under a complete independence with respect to such translations.

In the case of periodic homogenisation one can easily check that the Γ-limit functional $F_{\rm hom}$ (see Theorem 4.7.7 below) is invariant under translations (see, for instance, [63, Theorem 6.14]). However, in general, this invariance (nor the continuity with respect to translations) cannot be checked directly in the case of non-periodic homogenisation. For this reason, we will prove Theorem 4.5.1 below under the sole additional assumption (4.5.1), which is much weaker than invariance under translations (see Example 4.5.3). We shall see in Section 4.7 that condition (4.5.1) is satisfied almost surely under the standard hypotheses of stochastic homogenisation.

The following theorem is the main result of this section.

Theorem 4.5.1. *Let $F \in \mathfrak{F}_{\rm sc}^{\alpha,\infty}$ and let g be the function defined by* (4.3.3)*. Assume that there exist a function $f\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and a Borel set $N \subset \mathbb{R}^d$, σ-finite with respect to $\mathcal{H}^{d-1}$, such that*

$$f(A) = \lim_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} \quad \text{for every } x \in \mathbb{R}^d \setminus N \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{4.5.1}$$

Then for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ and $u \in {\rm BD}(O)$ we have

$$F(u,B) = \int_B f(\mathcal{E}u)\,{\rm d}x + \int_B f^\infty\Big(\frac{{\rm d}E^c u}{{\rm d}|E^c u|}\Big)\,{\rm d}|E^c u| + \int_{J_u\cap B} g(x,[u],\nu_u)\,{\rm d}\mathcal{H}^{d-1}$$

for every $B \in \mathcal{B}(O)$.

We recall the notation for the parallelograms $P_\lambda^{a,b}(x,\rho)$ and $P_\lambda^x(x,\rho)$, which has been set in Section 1.5 of Chapter 1, before the statement of Lemma 1.5.5

The following result shows that, under suitable assumptions on F, in definition (4.3.2) we may replace the cube $Q(x,\rho)$ by parallelograms of the form $P_\lambda^{a,b}(x,\rho)$.

Lemma 4.5.2. *Let $F \in \mathfrak{F}$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, $a,b \in \mathbb{R}^d$, and $\lambda > 0$. Assume that there exists $\mu \ge 0$ such that*

$$\mathfrak{m}^F(\ell_A, Q(y,\rho)) \le \mu\rho^d$$

for every $y \in \mathbb{R}^d$ and $\rho > 0$. Then

$$\mathfrak{m}^F(\ell_A, P_\lambda^{a,b}(y,\rho)) \le \mu\mathcal{L}^d(P_\lambda^{a,b}(y,\rho))$$

for every $y \in \mathbb{R}^d$ and $\rho > 0$. If, in addition, for some $x \in \mathbb{R}^d$ we have

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} = \mu,$$

then

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, P_\lambda^{a,b}(x,\rho))}{\mathcal{L}^d(P_\lambda^{a,b}(x,\rho))} = \mu.$$

Proof. The proof can be obtained arguing as in [101, Lemma 5.3], observing that in the last part of that lemma limsup can be replaced by lim. We also remark that, in contrast with [101, Lemma 5.3], we deal with minimisation problems without constraints on the oscillation of the competitors, which simplifies the proof. □

We are now ready to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. By (4.5.1) we have

$$f(A) = \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and every } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{4.5.2}$$

Let us fix $O \in \mathcal{O}_c(\mathbb{R}^d)$ and $u \in \mathrm{BD}(O)$. Thanks to (4.3.1), (4.5.2), and Proposition 4.3.3, it is enough to show that

$$F^c(u,B) = \int_B f^\infty\Big(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\mathrm{d}|E^c u|$$

for every $B \in \mathcal{B}(O)$. To prove this we will show that

$$\frac{\mathrm{d}F^c(u,\cdot)}{\mathrm{d}|E^c u|}(x) = f^\infty\Big(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\Big) \tag{4.5.3}$$

for $|E^c u|$-a.e. $x \in O$. Let $C(u) \subset O$ be the set of Lemma 1.5.5 and let us fix $x \in C(u) \setminus N$, where N is as in the statement.

Thanks to Lemma 1.5.5 there exist a positive sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ converging to 0 as $j \to +\infty$, and for every $j \in \mathbb{N}$ a positive sequence $\{\rho_{i,j}\}_{i\in\mathbb{N}}$, converging to 0 as $i \to +\infty$, such that (1.5.13) and (1.5.14) hold. For simplicity of notation, we set $P^x_j(x,\rho_{i,j}) \coloneqq P^x_{\lambda_j}(x,\rho_{i,j})$. To obtain (4.5.3) is enough to show that

$$f^\infty\Big(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\Big) = \lim_{j\to+\infty}\limsup_{i\to+\infty} \frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}, \tag{4.5.4}$$

where $A = A(x)$ and $s_{i,j}$ are defined by (1.5.12).

By (4.3.2), (4.3.4), and (4.5.1) we have

$$\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j})) \le F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j})) = f(s_{i,j}A)\mathcal{L}^d(P^x_j(x,\rho_{i,j})),$$

so that by (1.5.13) we have

$$\limsup_{i\to+\infty} \frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} \le \limsup_{i\to+\infty} \frac{f(s_{i,j}A)}{s_{i,j}} = f^\infty(A), \tag{4.5.5}$$

for every $j \in \mathbb{N}$.

Then, we observe that for every $y \in \mathbb{R}^d$, $\rho > 0$, and $t > 0$ by (4.3.2), (4.3.4), and (4.5.1) we have

$$\mathfrak{m}^F(t\ell_A, Q(y,\rho)) \le F(t\ell_A, Q(y,\rho)) = f(tA)\rho^d \quad \text{and} \quad \lim_{\rho\to0^+} \frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{\rho^d} = f(tA).$$

Hence, we can use Lemma 4.5.2 to deduce that

$$\lim_{\rho\to0^+} \frac{\mathfrak{m}^F(t\ell_A, P^x_\lambda(x,\rho))}{\mathcal{L}^d(P^x_\lambda(x,\rho))} = f(tA) \tag{4.5.6}$$

for every $\lambda, t > 0$. Recalling (4.4.1), from the previous equality it follows that

$$\lim_{t\to+\infty}\lim_{\rho\to 0^+}\frac{\mathfrak{m}^F(t\ell_A, P^x_\lambda(x,\rho))}{t\mathcal{L}^d(P^x_\lambda(x,\rho))} = f^\infty(A). \tag{4.5.7}$$

Then, we apply Lemma 4.4.7 with $w = \ell_A$ and $O = P^x_j(x,\rho_{i,j})$ to get for every $t > 0$

$$\begin{aligned}\Big|\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} - \frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big| &\le \frac{c_6}{s_{i,j}}\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha}\\ &+\frac{c_6}{s_{i,j}} + \frac{c_6}{t}\Big(\frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha} + \frac{c_6}{t}\\ &+c_7\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} + \frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)\Big(\frac{1}{s_{i,j}} + \frac{1}{t}\Big).\end{aligned} \tag{4.5.8}$$

We observe that for every $t > 0$ it follows from the upper bound in (H3) of Definition 4.1.4 that

$$\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j})) \le F(t\ell_A, P^x_j(x,\rho_{i,j})) \le (c_3 t|A| + c_4)\mathcal{L}^d(P^x_j(x,\rho_{i,j})). \tag{4.5.9}$$

Let us fix $\varepsilon > 0$. Recalling (4.4.1), we may find $t > 0$ such that

$$\Big|\frac{f(tA)}{t} - f^\infty(A)\Big| < \varepsilon \quad\text{and}\quad \frac{c_6}{t} + \frac{c_6}{t^\alpha}(c_3|A| + \frac{c_4}{t})^{1-\alpha} + c_7\big(2c_3|A| + \frac{c_4}{t}\big)\frac{1}{t} < \varepsilon. \tag{4.5.10}$$

Thanks to (4.5.6), from the first inequality in (4.5.10) we deduce that for every $j \in \mathbb{N}$ we have

$$f^\infty(A) - \varepsilon < \frac{f(tA)}{t} = \lim_{i\to+\infty}\frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}.$$

Combining this inequality with (4.5.8), we get

$$\begin{aligned}f^\infty(A) - \varepsilon \le \lim_{j\to+\infty}\lim_{i\to+\infty}\frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} &\le \limsup_{j\to+\infty}\limsup_{i\to+\infty}\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\\ &+\frac{c_6}{s_{i,j}}\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha} + \frac{c_6}{s_{i,j}} + \frac{c_6}{t}\Big(\frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha} + \frac{c_6}{t}\\ &+c_7\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} + \frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)\Big(\frac{1}{s_{i,j}} + \frac{1}{t}\Big)\Big).\end{aligned} \tag{4.5.11}$$

Using (4.5.9), we see that the right-hand side of the previous chain of inequalities can be bounded from above by

$$\begin{aligned}limsup_{j\to+\infty}\limsup_{i\to+\infty}\Big(&\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} + \frac{c_6}{s^\alpha_{i,j}}\Big(c_3|A| + \frac{c_4}{s_{i,j}}\Big)^{1-\alpha} + \frac{c_6}{s_{i,j}}\\ &+\frac{c_6}{t^\alpha}(c_3|A| + \frac{c_4}{t})^{1-\alpha} + \frac{c_6}{t} + c_7\Big(2c_3|A| + \frac{c_4}{s_{i,j}} + \frac{c_4}{t}\Big)\Big(\frac{1}{s_{i,j}} + \frac{1}{t}\Big)\Big).\end{aligned} \tag{4.5.12}$$

Recalling (4.5.5) and (1.5.13), from (4.5.10)-(4.5.12) we infer

$$f^\infty(A) - \varepsilon < \lim_{j\to+\infty}\limsup_{i\to+\infty}\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} + \varepsilon \le f^\infty(A) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we obtain (4.5.4). Since $x \in C(u) \setminus N$, $|E^c u|(O \setminus C(u)) = 0$, and $|E^c u|(N) = 0$, this concludes the proof. □

It is easy to produce examples of functionals $F \in \mathfrak{F}_{\mathrm{sc}}^{\alpha,\infty}$ which are not invariant under translation of the independent variable, but satisfy (4.5.1).

Example 4.5.3. Assume that $0 < c_1 < c_3$ and let f and g be the functions defined for every $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ by

$$f(A) \coloneqq c_3|A| \quad \text{and} \quad g(x,\zeta,\nu) \coloneqq \begin{cases} c_3|\zeta \odot \nu| & \text{if } x \in \mathbb{R}^d \setminus \Pi^{e_d}, \\ c_1|\zeta \odot \nu| & \text{if } x \in \Pi^{e_d}, \end{cases}$$

where we recall that $e_d = (0,\dots,1)$ and $\Pi^{e_d} \coloneqq \{y \in \mathbb{R}^d : y \cdot e_d = 0\}$. Clearly, $f \in \mathscr{F}$ and $g \in \mathscr{G}$, and since they are positively homogeneous of degree one we also have $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\infty$. Hence, the functional $F \coloneqq F^{f,g}$ belongs to $\mathfrak{F}^{\alpha,\infty}$ by Proposition 4.4.5. Let $\psi\colon \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$\psi(x) \coloneqq \begin{cases} c_3 & \text{if } x \in \mathbb{R}^d \setminus \Pi^{e_d}, \\ c_1 & \text{if } x \in \Pi^{e_d}. \end{cases}$$

We observe that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ and $u \in \mathrm{BD}(O)$ we have

$$F(u,O) = \int_O \psi \,\mathrm{d}|Eu|.$$

Since ψ is lower semicontinuous in $\mathbb{R}^d$, this equality shows that $F(\cdot,O)$ is $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous, hence $F \in \mathfrak{F}_{\mathrm{sc}}^{\alpha,\infty}$.

4.6 Characterisation of Γ-convergence using minima on small cubes

In this section we determine the integrands of the Γ-limit of a sequence $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\infty}$ by means of limits of the minimum values of suitable minimisation problems for F_n on small cubes. The next lemma shows that the Γ-convergence of a sequence $\{F_n\}_n \subset \mathfrak{F}$ to a limit functional F allows us to compare $\mathfrak{m}^F$ with $\{\mathfrak{m}^{F_n}\}_n$.

Lemma 4.6.1. *Let $\{F_n\}_n \subset \mathfrak{F}$. Assume that there exists $F \in \mathfrak{F}_{\mathrm{sc}}$ such that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Let $O,U \in \mathcal{O}_c(\mathbb{R}^d)$ be with Lipschitz boundaries and with $U \subset\subset O$, and let $w \in \mathrm{BD}(O)$. Then*

$$\mathfrak{m}^F(w,O) \le \liminf_{n\to+\infty} \mathfrak{m}^{F_n}(w,U) + c_3|Ew|(O \setminus U) + c_4\mathcal{L}^d(O \setminus U), \tag{4.6.1}$$

$$\limsup_{n\to+\infty} \mathfrak{m}^{F_n}(w,O) \le \mathfrak{m}^F(w,O). \tag{4.6.2}$$

Proof. We consider a subsequence, not relabelled, such that the liminf in the right-hand side of (4.6.1) is actually a limit. To prove (4.6.1) we fix $\delta > 0$ and for every $n \in \mathbb{N}$ we select $v_n \in \mathrm{BD}(U)$ such that $v_n = w$ on ∂U and

$$F_n(v_n,U) < \mathfrak{m}^{F_n}(w,U) + \delta. \tag{4.6.3}$$

We observe that by the upper bound in (H3) of Definition 4.1.4, we have

$$F_n(v_n, U) < \mathfrak{m}^{F_n}(w, U) + \delta \leq F_n(w, U) + \delta \leq c_3|Ew|(U) + c_4\mathcal{L}^d(U) + \delta,$$

so that by the lower bound in (H3) we also get that $|Ev_n|(U)$ is bounded uniformly with respect to n. We extend v_n to O by setting $v_n = w$ on $O \setminus U$ and observe that $|Ev_n|(O)$ is uniformly bounded in n as well. Hence, there exist a subsequence, not relabelled, and a function $v \in \mathrm{BD}(O)$ such that $v_n \to v$ in $L^1(O; \mathbb{R}^d)$ and $v = w$ on $O \setminus U$, so that

$$\mathfrak{m}^F(w, O) \leq F(v, O) \leq \liminf_{n\to+\infty} F_n(v_n, O), \tag{4.6.4}$$

where the last inequality is due to Γ-convergence. Since the inner trace of v_n on ∂U equals the inner trace of w, we have $|Ev_n|(\partial U) = |Ew|(\partial U)$. Hence, the upper bound in (H3) of Definition 4.1.4 gives

$$F_n(v_n, O \setminus U) \leq c_3|Ev_n|(O \setminus U) + c_4\mathcal{L}^d(O \setminus U) = c_3|Ew|(O \setminus U) + c_4\mathcal{L}^d(O \setminus U),$$

where the last equality is due to the fact that $v_n = w$ in the open set $O \setminus \overline{U}$ and $|Ev_n|(\partial U) = |Ew|(\partial U)$. Combining this inequality with (4.6.3) and (4.6.4), and letting $\delta \to 0^+$ we obtain (4.6.1).

To prove (4.6.2), let us fix $\delta > 0$ and let $u \in \mathrm{BD}(O)$ be such that $u = w$ on ∂O and $F(u, O) < \mathfrak{m}^F(w, O) + \delta$. By Γ-convergence there exists a sequence $\{u_n\}_n \subset \mathrm{BD}(O)$ converging to u strongly in $L^1(O; \mathbb{R}^d)$ and such that

$$\lim_{n\to+\infty} F_n(u_n, O) = F(u, O) < \mathfrak{m}^F(w, O) + \delta. \tag{4.6.5}$$

We now fix a compact set $K \subset O$ such that

$$c_3|Eu|(O \setminus K) + c_4\mathcal{L}^d(O \setminus K) < \delta. \tag{4.6.6}$$

We also consider two additional open sets O'' and O' with the property that $K \subset O'' \subset\subset O' \subset\subset O$. We now argue as in the part of the proof of Lemma 4.2.2 that starts from (4.2.11), with $U = O \setminus \overline{O}''$, v_n replaced by u, and for every $n \in \mathbb{N}$ we construct a function $v_n \in \mathrm{BD}(O)$ such that $v_n = u$ in a neighbourhood of ∂O and

$$F_n(v_n, O) \leq \; F_n(u_n, O) + F_n(u, O \setminus O'') + \frac{M}{m} + \frac{2c_3}{\eta}\|u_n - u\|_{L^1(O)},$$

where $M \geq 0$ is the constant introduced in (4.2.9) and $\eta := \mathrm{dist}(O'', \partial O')$. Since $v_n = u = w$ on ∂O, we have $\mathfrak{m}^{F_n}(w, O) \leq F_n(v_n, O)$. Hence, using the upper bounds in property (H3) of Definition 4.1.4 for F_n, (4.6.6), and the inclusion $K \subset O''$, from the previous displayed formula we obtain

$$\mathfrak{m}^{F_n}(w, O) \leq F_n(u_n, O) + \delta + \frac{M}{m} + \frac{2c_3}{\eta}\|u_n - u\|_{L^1(O)}.$$

We now pass to the limsup as $n \to +\infty$ and, recalling that $u_n \to u$ in $L^1(O; \mathbb{R}^d)$ as $n \to +\infty$, from (4.6.5) we get

$$\limsup_{n\to+\infty} \mathfrak{m}^{F_n}(w, O) \leq \mathfrak{m}^F(w, O) + 2\delta + \frac{M}{m}.$$

Taking the limit as $m \to +\infty$ and $\delta \to 0^+$ we obtain (4.6.2). □

The next result shows that for a functional $F \in \mathfrak{F}_{\rm sc}$ arising as Γ-limit of a sequence of functionals in $\mathfrak{F}$ the integrands f and g of the bulk and surface parts can be obtained using limits of the minimum values of the minimisation problems $\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))$ and $\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))$ for the functionals F_n.

Lemma 4.6.2. *Let $\{F_n\}_n \subset \mathfrak{F}$. Assume that there exists $F \in \mathfrak{F}_{\rm sc}$ such that for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, and let f and g be the functions associated to F by (4.3.2) and (4.3.3). Then for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have*

$$f(x,A) = \limsup_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} = \limsup_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d},$$
$$g(x,\zeta,\nu) = \limsup_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}.$$

Proof. The proof can be obtained arguing exactly as in [101, Lemma 3.3], replacing D with E, ∇ with $\mathcal{E}$, and Propositions 3.1 and 3.2 of that paper with our Lemma 4.6.1. □

We conclude this section by proving a sufficient condition for the Γ-convergence of a sequence F_n, based on limits of minimum values of minimisation problems for F_n on small cubes. Note that we require that the limits corresponding to the volume integrand do not depend on x.

Theorem 4.6.3. *Let $\{F_n\}_n \subset \mathfrak{F}^{\alpha,\infty}$. Assume that there exist $\hat f\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and $\hat g\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that*

$$\hat f(A) = \lim_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} = \lim_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d}, \tag{4.6.7}$$
$$\hat g(x,\zeta,\nu) = \limsup_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}, \tag{4.6.8}$$

for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. Then $\hat f \in \mathscr{F}^\alpha$, $\hat g \in \mathscr{G}^\infty$, and for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ we have that $\{F_n(\cdot,O)\}_n$ Γ-converges to $F^{\hat f,\hat g}(\cdot,O)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, where $F^{\hat f,\hat g}$ is the functional introduced in Definition 4.1.3.

Proof. By Theorem 4.2.1 there exists a subsequence, not relabelled, and a functional $F \in \mathfrak{F}_{\rm sc}$ such that $\{F_n(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ for every $O \in \mathcal{O}_c(\mathbb{R}^d)$. Let f and g be the functions defined by (4.3.2) and (4.3.3) corresponding to F. Using Lemmas 4.6.1 and 4.6.2, together with (4.6.7) and (4.6.8), we see that $f(x,A) = \hat f(A)$ and $g(x,\zeta,\nu) = \hat g(x,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $\nu \in \mathbb{S}^{d-1}$. Moreover, thanks to Lemma 4.6.1, we obtain that

$$\hat f(A) = \lim_{\rho\to 0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} \le \liminf_{\rho\to 0^+} \frac{\mathfrak{m}^{F}(\ell_A, Q(x,\rho))}{\rho^d},$$
$$\hat f(A) = \lim_{\rho\to 0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho-\rho^2))}{(\rho-\rho^2)^d} \ge \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^{F}(\ell_A, Q(x,\rho))}{\rho^d},$$

hence

$$\hat{f}(A) = \lim_{\rho \to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

Thus, the functional F satisfies the hypotheses of Theorem 4.5.1, so that $F = F^{\hat{f},\hat{g}}$.

Moreover, by (4.6.7) and (4.6.8) the functions $\hat{f}$ and $\hat{g}$, and hence the functional $F^{\hat{f},\hat{g}}$, do not depend on the subsequence chosen at the beginning of the proof, so that by the Urysohn property of Γ-convergence (see [88, Proposition 8.3]) for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the whole sequence $\{F_n(\cdot, O)\}_n$ Γ-converges to $F^{\hat{f},\hat{g}}(\cdot, O)$. □

4.7 Functionals defined by rescaling

In this section we use the theory developed in the previous sections to deal with the problem of Γ-convergence of oscillating functionals obtained by rescaling a single functional. In particular, we prove a general theorem which provides sufficient conditions for the Γ-convergence of these functionals (see Theorem 4.7.7).

We now introduce the notation we will use in the rest of the work. Throughout this section we keep fixed $f \in \mathscr{F}^\alpha$ and $g \in \mathscr{G}^\infty$ and set

$$F \coloneqq F^{f,g} \in \mathfrak{F}^{\alpha,\infty},$$

where $F^{f,g}$ is the functional introduced in Definition 4.1.3. We also assume that the modulus of continuity introduced in (4.1.1) satisfies

$$\tau(t) = \tau_1 t \quad \text{for every } t \geq 0, \tag{4.7.1}$$

for some constant $\tau_1 > 0$. With this hypothesis, condition (g4) of Definition 4.1.1 reads

$$|g(x,\zeta_1,\nu) - g(x,\zeta_2,\nu)| \leq \tau_1 |\zeta_1 - \zeta_2|,$$

while the surface continuity estimate (H6) of Definition 4.1.4 becomes

$$F(u + u_{x,\zeta,\nu}, B) \leq F(u,B) + \tau_1 |\zeta| \mathcal{H}^{d-1}(\Pi^\nu_x \cap B).$$

Definition 4.7.1. For $0 < \varepsilon \leq 1$, we consider the integrands $f_\varepsilon \in \mathscr{F}^\alpha$ and $g_\varepsilon \in \mathscr{G}^\infty$ defined by

$$f_\varepsilon(x,A) \coloneqq f\big(\tfrac{x}{\varepsilon}, A\big) \quad \text{for } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{4.7.2}$$

$$g_\varepsilon(x,\zeta,\nu) \coloneqq \varepsilon g\big(\tfrac{x}{\varepsilon}, \tfrac{\zeta}{\varepsilon}, \nu\big) \quad \text{for } x \in \mathbb{R}^d, \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{S}^{d-1}, \tag{4.7.3}$$

and we set $F_\varepsilon \coloneqq F^{f_\varepsilon, g_\varepsilon} \in \mathfrak{F}^{\alpha,\infty}$.

Remark 4.7.2. If for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ the function $t \mapsto g(x, t\zeta, \nu)$ is positively homogeneous of degree one, that is to say

$$g(x,t\zeta,\nu) = t g(x,\zeta,\nu) \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d,\ \nu \in \mathbb{S}^{d-1}, \text{ and } t \geq 0, \tag{4.7.4}$$

then

$$g_\varepsilon(x,\zeta,\nu) = g(\tfrac{x}{\varepsilon},\zeta,\nu) \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d,\ \nu \in \mathbb{S}^{d-1}, \text{ and } \varepsilon > 0.$$

In this case, for $O \in \mathcal{O}_c(\mathbb{R}^d)$ and $u \in \mathrm{BD}(O)$ the functionals $F_\varepsilon(u,O)$ of Definition 4.7.1 become

$$\int_O f\Big(\frac{x}{\varepsilon}, \mathcal{E}u\Big)\,\mathrm{d}x + \int_O f^\infty\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap O} g\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{4.7.5}$$

which are the functionals commonly considered in homogenisation of free discontinuity problems.

Our choice in the definition of g_ε given by (4.7.3) is justified by the fact that the corresponding functional F_ε defined by $F^{f_\varepsilon,g_\varepsilon}$ satisfies good change of variables formulas (see Lemmas 4.7.4 and 4.7.6 below) even when g does not satisfy (4.7.4). This will allow us to prove a very general Γ-convergence result for $F^{f_\varepsilon,g_\varepsilon}$ (see Theorem 4.7.7), which implies the Γ-convergence of the functionals in (4.7.5) when g satisfies the additional condition (4.7.4). Unfortunately, in the BD case we are not able to extend the truncation arguments that were crucial to study the analogue of (4.7.5) in BV (see Chapter 2).

In the rest of the section, we need the following technical result about a change of variables formula involving the Cantor part of a BD function.

Lemma 4.7.3. *Let $\varepsilon,\rho \in (0,1)$, $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $u \in \mathrm{BD}(Q_\nu(x,\rho))$, and consider the function $v \in \mathrm{BD}(Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))$ given by $v(z) = \frac{1}{\varepsilon}u(\varepsilon z)$ for every $z \in Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$. Then*

$$\int_{Q_\nu(x,\rho)} f^\infty_\varepsilon\Big(y, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| = \varepsilon^d \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty\Big(z, \frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\Big)\,\mathrm{d}|E^c v|. \tag{4.7.6}$$

Proof. Let $\psi \in C_c^\infty(Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon});\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and let $\psi_\varepsilon := \psi\big(\frac{\cdot}{\varepsilon}\big) \in C_c^\infty(Q_\nu(x,\rho);\mathbb{R}^{d\times d}_{\mathrm{sym}})$. By change of variables and integration by parts we have

$$\begin{aligned}
-\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \psi\,\mathrm{d}Ev &= \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} v\,\mathrm{div}\,\psi\,\mathrm{d}z = \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \tfrac{1}{\varepsilon}u(\varepsilon z)\,\mathrm{div}\,\psi\,\mathrm{d}z \\
&= \frac{1}{\varepsilon^d}\int_{Q_\nu(x,\rho)} \frac{u(y)}{\varepsilon}\mathrm{div}\,\psi\big(\tfrac{y}{\varepsilon}\big)\,\mathrm{d}y = \frac{1}{\varepsilon^d}\int_{Q_\nu(x,\rho)} u\,\mathrm{div}\,\psi_\varepsilon\,\mathrm{d}y \\
&= -\frac{1}{\varepsilon^d}\int_{Q_\nu(x,\rho)} \psi_\varepsilon\,\mathrm{d}Eu = -\frac{1}{\varepsilon^d}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \psi\,\mathrm{d}\Big(\big(\tfrac{\cdot}{\varepsilon}\big)_\# Eu\Big),
\end{aligned}$$

where $\big(\frac{\cdot}{\varepsilon}\big)_\# Eu$ denotes the push-forward of Eu via the function $y \mapsto \frac{y}{\varepsilon}$ (see, for instance, [35, Theorem 3.6.1]). This implies that

$$Ev = \frac{1}{\varepsilon^d}\Big(\frac{\cdot}{\varepsilon}\Big)_\# Eu \quad \text{as Borel measures on } Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})$$

and, passing to their singular parts,

$$E^s v = \frac{1}{\varepsilon^d}\Big(\frac{\cdot}{\varepsilon}\Big)_\# E^s u \quad \text{as Borel measures on } Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}),$$

where $E^s v$ and $E^s u$ denote the singular part of Ev and Eu with respect to the Lebesgue measure. Restricting the previous equality to $Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}) \setminus J_v = Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}) \setminus (\frac{1}{\varepsilon}J_u)$, we deduce that

$$E^c v = \frac{1}{\varepsilon^d}\Big(\frac{\cdot}{\varepsilon}\Big)_\# E^c u \quad \text{as Borel measures on } Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}).$$

This implies that

$$\frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\big(\frac{y}{\varepsilon}\big) = \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(y) \quad \text{for } |E^c u|\text{-a.e. } y \in Q_\nu(x,\rho).$$

Hence, by the integration formula for the push-forward of measures (see, for instance, [35, Theorem 3.6.1]) we get (4.7.6). □

We state two preliminary results that allow us to rewrite the minimisation problems on small cubes for the functionals F_ε by means of minimisation problems on large cubes for F and F^{f^∞,g^∞}, where f^∞ and g^∞ are the functions defined by (4.4.1) and (4.4.4), respectively.

Lemma 4.7.4. *Let $\varepsilon, \rho \in (0,1)$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Then*

$$\mathfrak{m}^{F_\varepsilon}(\ell_A, Q(x,\rho)) = \varepsilon^d \mathfrak{m}^{F^{f,g}}(\ell_A, Q(\tfrac{x}{\varepsilon}, \tfrac{\rho}{\varepsilon})).$$

Proof. Let $u \in \mathrm{BD}(Q(x,\rho))$, with $u = \ell_A$ on $\partial Q(x,\rho)$, and let $v(z) := \frac{1}{\varepsilon}u(\varepsilon z)$ for $z \in Q(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$. It is easy to see that $v = \ell_A$ on $\partial Q(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$. Thanks to (4.7.2) and (4.7.3), a change of variables and Lemma 4.7.3 show that

$$\begin{aligned}
\int_{Q(x,\rho)} f_\varepsilon(y, \mathcal{E}u)\,\mathrm{d}y &= \varepsilon^d \int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f(z, \mathcal{E}v)\,\mathrm{d}z,\\
\int_{Q(x,\rho)} f_\varepsilon^\infty\Big(y, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| &= \varepsilon^d \int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty\Big(z, \frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\Big)\,\mathrm{d}|E^c v|,\\
\int_{J_u\cap Q(x,\rho)} g_\varepsilon(y,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} &= \varepsilon^d \int_{J_v\cap Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Thus, we infer

$$\mathfrak{m}^{F_\varepsilon}(\ell_A, Q(x,\rho)) = \varepsilon^d \mathfrak{m}^{F}(\ell_A, Q(\tfrac{x}{\varepsilon}, \tfrac{\rho}{\varepsilon})),$$

concluding the proof. □

Definition 4.7.5. We set

$$F^\infty := F^{f^\infty,g^\infty} \in \mathfrak{F}^{\alpha,\infty},$$

where f^∞ and g^∞ are the functions defined by (4.4.1) and (4.4.4).

Lemma 4.7.6. *Let $\varepsilon \in (0, 1/(2c_6))$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $\rho \in (0,1)$. Then*

$$\big|\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) - \varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu}, Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))\big| \le C\rho^{d-1+\alpha} + C\varepsilon\rho^{d-1}, \tag{4.7.7}$$

where $C = C_\zeta > 0$ is a constant depending only on $|\zeta|$ and on the structural constants α, c_1, c_3, c_6, and c_7, but is independent of ε, x, ν, and ρ.

Proof. Let $u \in \mathrm{BD}(Q_\nu(x,\rho))$ with $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and let $v(z) := u(\varepsilon z)$ for $z \in Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$, so that $v = u_{\frac{x}{\varepsilon},\zeta,\nu}$ on $\partial Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$. Recalling (4.7.2) and (4.7.3), a change of variables and Lemma 4.7.3 imply that

$$\int_{Q_\nu(x,\rho)} f_\varepsilon(y, \mathcal{E}u)\,\mathrm{d}y = \varepsilon^{d-1} \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \varepsilon f\Big(z, \frac{1}{\varepsilon}\mathcal{E}v\Big)\,\mathrm{d}z, \tag{4.7.8}$$

$$\int_{Q_\nu(x,\rho)} f_\varepsilon^\infty\Big(y, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| = \varepsilon^{d-1} \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty\Big(z, \frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\Big)\,\mathrm{d}|E^c v|, \tag{4.7.9}$$

$$\int_{J_u\cap Q_\nu(x,\rho)} g_\varepsilon(y,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} = \varepsilon^{d-1} \int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \varepsilon g\Big(z, \frac{1}{\varepsilon}[v],\nu_v\Big)\,\mathrm{d}\mathcal{H}^{d-1}. \tag{4.7.10}$$

We can now exploit (4.4.2) of Remark 4.4.2 to obtain for $\mathcal{L}^d$-a.e. $z \in Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$

$$|\varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v) - f^\infty(z, \mathcal{E}v)| \le c_6\varepsilon + c_6\varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v)^{1-\alpha}, \tag{4.7.11}$$

while using (4.4.7) of Remark 4.4.3 and inequalities (g3) and (4.4.5), for $\mathcal{H}^{d-1}$-a.e. $z \in J_v \cap Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$ we get

$$|\varepsilon g(z, \tfrac{1}{\varepsilon}[v], \nu_v) - g^\infty(z, [v], \nu_v)| \le 2c_3c_7\varepsilon|[v] \odot \nu_v| \le C_1\varepsilon^2 g(z, \tfrac{1}{\varepsilon}[v], \nu_v), \tag{4.7.12}$$

$$|\varepsilon g(z, \tfrac{1}{\varepsilon}[v], \nu_v) - g^\infty(z, [v], \nu_v)| \le 2c_3c_7\varepsilon|[v] \odot \nu_v| \le C_1\varepsilon g^\infty(z, [v], \nu_v), \tag{4.7.13}$$

where we have set $C_1 \coloneqq (2c_3c_7)/c_1$. From (4.7.11) we then deduce for $\mathcal{L}^d$-a.e. $z \in Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$ that

$$f^\infty(z, \mathcal{E}v) \le \varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v) + c_6\varepsilon + c_6\varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v)^{1-\alpha}, \tag{4.7.14}$$

$$\begin{aligned} f^\infty(z, \mathcal{E}v) &\ge \varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v) - c_6\varepsilon - c_6\varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v)^{1-\alpha} \\ &\ge \tfrac{\varepsilon}{2} f(z, \tfrac{1}{\varepsilon}\mathcal{E}v) - c_6\varepsilon - c_6\varepsilon(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}, \end{aligned} \tag{4.7.15}$$

where we have used the inequality $t^{1-\alpha} \le \frac{1}{2c_6}t + (2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}$ for every $t \ge 0$ and $\varepsilon \in (0, 1/(2c_6))$, while from (4.7.12) and (4.7.13) we obtain for $\mathcal{H}^{d-1}$-a.e. $z \in J_v \cap Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$

$$g^\infty(z, [v], \nu_v) \le \varepsilon g(z, \tfrac{1}{\varepsilon}[v], \nu_v) + C_1\varepsilon^2 g(z, \tfrac{1}{\varepsilon}[v], \nu_v), \tag{4.7.16}$$

$$g^\infty(z, [v], \nu_v) \ge \varepsilon g(z, \tfrac{1}{\varepsilon}[v], \nu_v) - C_1\varepsilon g^\infty(z, [v], \nu_v). \tag{4.7.17}$$

Note that (4.7.15) implies that

$$\varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v) \le 2f^\infty(z, \mathcal{E}v) + C_2\varepsilon,$$

where $C_2 \coloneqq 2c_6 + 2c_6(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}$, so that from (4.7.11) we infer

$$\varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v) \le f^\infty(z, \mathcal{E}v) + c_6\varepsilon + c_6\varepsilon^\alpha(2f^\infty(z, \mathcal{E}v) + C_2\varepsilon)^{1-\alpha}. \tag{4.7.18}$$

Integrating (4.7.14) on $Q_\nu\big(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon}\big)$, by Hölder's inequality we get

$$\begin{aligned} \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} f^\infty(z, \mathcal{E}v)\,\mathrm{d}z \le{}& \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} \varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v)\,\mathrm{d}z + c_6\rho^d \\ &+ c_6\Big(\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} \varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v)\,\mathrm{d}z\Big)^{1-\alpha}\rho^{\alpha d}, \end{aligned} \tag{4.7.19}$$

while from (4.7.18) we get

$$\begin{aligned} \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} \varepsilon f(z, \tfrac{1}{\varepsilon}\mathcal{E}v)\,\mathrm{d}z \le{}& \varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} f^\infty(z, \mathcal{E}v)\,\mathrm{d}z + c_6\rho^d \\ &+ c_6\Big(2\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} f^\infty(z, \mathcal{E}v)\,\mathrm{d}z + C_2\rho^d\Big)^{1-\alpha}\rho^{\alpha d}. \end{aligned} \tag{4.7.20}$$

Integrating (4.7.16) and (4.7.17) on $J_v \cap Q_\nu\big(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon}\big)$ we obtain

$$\begin{aligned} \varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} g^\infty(z, [v], \nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \le{}& \varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} \varepsilon g(z, \tfrac{1}{\varepsilon}[v], \nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \\ &+ \varepsilon^d C_1\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} \varepsilon g(z, \tfrac{1}{\varepsilon}[v], \nu_v)\,\mathrm{d}\mathcal{H}^{d-1}, \end{aligned} \tag{4.7.21}$$

and

$$\varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g^\infty(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \geq \varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} - \varepsilon^d C_1\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g^\infty(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1}. \quad (4.7.22)$$

Recalling (4.7.8)–(4.7.10) and combining (4.7.19) and (4.7.21), we obtain

$$\varepsilon^{d-1}F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \leq F_\varepsilon(u,Q_\nu(x,\rho)) + c_6\rho^d + c_6F_\varepsilon(u,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + C_1\varepsilon F_\varepsilon(u,Q_\nu(x,\rho)),$$

while from (4.7.20) and (4.7.22) we get

$$F_\varepsilon(u,Q_\nu(x,\rho)) \leq \varepsilon^{d-1}F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d + c_6\Big(2\varepsilon^{d-1}F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + C_2\rho^d\Big)^{1-\alpha}\rho^{\alpha d} + C_1\varepsilon^d F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})).$$

Since $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ if and only if $v = u_{\frac{x}{\varepsilon},\zeta,\nu}$ on $\partial Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$, the last two inequalities imply

$$\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \leq \mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + c_6\rho^d + c_6\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + C_1\varepsilon\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)). \quad (4.7.23)$$

and

$$\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) \leq \varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d + c_6\Big(2\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + C_2\rho^d\Big)^{1-\alpha}\rho^{\alpha d} + C_1\varepsilon^d\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})). \quad (4.7.24)$$

To conclude, as $u_{x,\zeta,\nu}$ and $u_{\frac{x}{\varepsilon},\zeta,\nu}$ are competitors for the problems $\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))$ and $\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))$, respectively, using the upper bounds in (g3) and (4.4.5) we obtain

$$\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) \leq c_3|\zeta\odot\nu|\rho^{d-1},$$
$$\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \leq c_3|\zeta\odot\nu|\frac{\rho^{d-1}}{\varepsilon^{d-1}}.$$

Finally, combining these two inequalities with (4.7.23) and (4.7.24) we get

$$\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) \leq \mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + c_6\rho^d + c_6(c_3|\zeta\odot\nu|)^{1-\alpha}\rho^{d-1+\alpha} + C_1c_3|\zeta\odot\nu|\varepsilon\rho^{d-1},$$

and

$$\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) \leq \varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d + c_6\big(2c_3|\zeta\odot\nu| + C_2\big)^{1-\alpha}\rho^{d-1+\alpha} + C_1c_3|\zeta\odot\nu|\varepsilon\rho^{d-1},$$

which imply (4.7.7) for $C \coloneqq \max\{c_6, c_6(c_3|\zeta|)^{1-\alpha}, C_1c_3|\zeta|, c_6\big(2c_3|\zeta| + C_2\big)^{1-\alpha}\}$. □

The following theorem constitutes the main result of this section. We shall see in the next section that its hypotheses are satisfied under the standard hypotheses of periodic or stochastic homogenisation.

Theorem 4.7.7. *Let $f \in \mathscr{F}^\alpha$, $g \in \mathscr{G}^\infty$, and let $F_\varepsilon \coloneqq F^{f_\varepsilon, g_\varepsilon}$ be the functionals introduced in Definition 4.7.1. Assume that there exist functions $f_{\rm lim}\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and $g_{\rm lim}\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that*

$$f_{\rm lim}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g}}(\ell_A, Q(rx,r))}{r^d} \quad \text{for all } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}, \tag{4.7.25}$$

$$g_{\rm lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{4.7.26}$$

Then $f_{\rm lim} \in \mathscr{F}^\alpha$, $g_{\rm lim} \in \mathscr{G}^\infty$, $g_{\rm lim}$ satisfies (4.7.4), and the following property holds: for every positive sequence $\{\varepsilon_n\}_n$ converging to 0 as $n \to +\infty$ and for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_{\varepsilon_n}(\cdot, O)\}_n$ Γ-converges to $F^{f_{\rm lim}, g_{\rm lim}}(\cdot, O)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, we have the equality

$$g_{\rm lim}(\zeta,\nu) = f^\infty_{\rm lim}(\zeta \odot \nu) \quad \text{for all } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}, \tag{4.7.27}$$

which implies that for every $O \in \mathcal{O}(\mathbb{R}^d)$, $u \in \mathrm{BD}(O)$, and $B \in \mathcal{B}(O)$ we have

$$F^{f_{\rm lim}, g_{\rm lim}}(u,B) = \int_B f_{\rm lim}(\mathcal{E}u)\,\mathrm{d}x + \int_B f^\infty_{\rm lim}\Big(\frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}\Big)\,\mathrm{d}|E^s u|, \tag{4.7.28}$$

where $E^s u$ denotes the singular part of Eu with respect to the Lebesgue measure.

Proof. Let us fix a sequence $\{\varepsilon_n\}_n \subset (0,1)$ converging to 0 as $n \to +\infty$. Since $\{F_{\varepsilon_n}\}_n \subset \mathfrak{F}^{\alpha,\infty}$, to prove the Γ-convergence part of the statement it is enough to check that the hypotheses of Theorem 4.6.3 are satisfied. Using Lemma 4.7.4, we see that for every $\rho \in (0,1)$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{d\times d}_{\rm sym}$ we have

$$\frac{1}{\rho^d}\mathfrak{m}^{F_{\varepsilon_n}}(\ell_A, Q(x,\rho)) = \frac{\varepsilon_n^d}{\rho^d}\mathfrak{m}^F(\ell_A, Q(\tfrac{x}{\varepsilon_n}, \tfrac{\rho}{\varepsilon_n})),$$

so that, setting $r_n \coloneqq \rho/\varepsilon_n$, we may rewrite the previous equality as

$$\frac{1}{\rho^d}\mathfrak{m}^{F_{\varepsilon_n}}(\ell_A, Q(x,\rho)) = \frac{1}{r_n^d}\mathfrak{m}^F(\ell_A, Q(r_n\tfrac{x}{\rho}, r_n)).$$

By (4.7.25) applied with x replaced by x/ρ, we then obtain for every $A \in \mathbb{R}^{d\times d}_{\rm sym}$ and $\rho \in (0,1)$ that

$$\lim_{n\to+\infty} \frac{1}{\rho^d}\mathfrak{m}^{F_{\varepsilon_n}}(\ell_A, Q(x,\rho)) = f_{\rm lim}(A),$$

which proves (4.6.7) for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}_{\rm sym}$.

Let us fix $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. Using Lemma 4.7.6, we see that for every $\rho \in (0,1)$ and n large enough we have

$$\begin{aligned}\frac{\varepsilon_n^{d-1}}{\rho^{d-1}}\mathfrak{m}^{F^{f^\infty, g^\infty}}(u_{\frac{x}{\varepsilon_n},\zeta,\nu}, Q_\nu(\tfrac{x}{\varepsilon_n}, \tfrac{\rho}{\varepsilon_n})) - C(\rho^\alpha + \varepsilon_n) &\le \frac{1}{\rho^{d-1}}\mathfrak{m}^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \\ &\le \frac{\varepsilon_n^{d-1}}{\rho^{d-1}}\mathfrak{m}^{F^{f^\infty, g^\infty}}(u_{\frac{x}{\varepsilon_n},\zeta,\nu}, Q_\nu(\tfrac{x}{\varepsilon_n}, \tfrac{\rho}{\varepsilon_n})) + C(\rho^\alpha + \varepsilon_n),\end{aligned}$$

so that, setting again $r_n := \rho/\varepsilon_n$, by (4.7.26) we may pass to the limit first as $n \to +\infty$ and then as $\rho \to 0^+$ to obtain

$$\lim_{\rho\to 0^+} \lim_{n\to+\infty} \frac{1}{\rho^{d-1}} \mathfrak{m}^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) = g_{\lim}(\zeta,\nu),$$

proving (4.6.8) for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$.

The fact that $f_{\lim} \in \mathscr{F}^\alpha$ and $g_{\lim} \in \mathscr{G}^\infty$ follows again by Theorem 4.6.3, while property (4.7.4) for $g_{\lim}$ follows from the observation that, being f^∞ and g^∞ both positively homogeneous of degree one (in the variables A and ζ, respectively), so is the function $t \mapsto \mathfrak{m}^{F^\infty}(u_{x,t\zeta,\nu}, Q_\nu(x,\rho))$. By (4.7.26) this leads to the positive homogeneity of $g_{\lim}$.

Finally, since for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the functional $F^{f_{\lim},g_{\lim}}(\cdot, O)$ is $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ lower semicontinuous and the functions $f_{\lim}$ and $g_{\lim}$ are independent of the variable x, equality (4.7.27) follows from Lemma 4.7.10 below. Equality (4.7.28) follows immediately from Definition 4.1.3. □

For the application to stochastic homogenisation it is useful to obtain the conclusion of the previous theorem assuming only that the limits in (4.7.25) and (4.7.26) hold on countable dense collections of A, ζ, and ν. This is made possible by the following two lemmas.

Lemma 4.7.8. *Let $f \in \mathscr{F}$, $g \in \mathscr{G}$, and $\mathbb{D} \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be a dense subset. Assume that there exists a function $f_{\lim}\colon \mathbb{D} \to [0,+\infty)$ such that*

$$f_{\lim}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g}}(\ell_A, Q(rx,r))}{r^d} \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{D}. \tag{4.7.29}$$

Then there exists a unique continuous extension $f_{\lim}\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ and this extension is Lipschitz continuous with Lipschitz constant c_5 and satisfies

$$f_{\lim}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g}}(\ell_A, Q(rx,r))}{r^d} \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{4.7.30}$$

Proof. Arguing exactly as in the proof of (4.3.11), we obtain

$$|\mathfrak{m}^{F^{f,g}}(\ell_{A_1}, Q(rx,r)) - \mathfrak{m}^{F^{f,g}}(\ell_{A_2}, Q(rx,r))| \le c_5|A_2 - A_1|r^d \quad \text{for every } A_1, A_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

Combining the previous inequality with (4.7.29), we obtain that there exists a unique continuous extension of $f_{\lim}$ and that this extension is c_5-Lipschitz continuous and satisfies (4.7.30). □

Lemma 4.7.9. *Let $f \in \mathscr{F}^\alpha$, $g \in \mathscr{G}^\infty$, and $\mathbb{D}_1 \subset \mathbb{R}^d$ and $\mathbb{D}_2 \subset \mathbb{S}^{d-1}$ be dense subsets. Assume that there exists a function $g_{\lim}\colon \mathbb{D}_1 \times \mathbb{D}_2 \to [0,+\infty)$ such that*

$$g_{\lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{D}_1, \text{ and } \nu \in \mathbb{D}_2. \tag{4.7.31}$$

Then there exists a unique extension $g_{\lim}\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that

$$g_{\lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{4.7.32}$$

Proof. We begin by proving that there exists an extension $g_{\rm lim}\colon \mathbb{R}^d \times \mathbb{D}_2 \to [0,+\infty)$ such that

$$g_{\rm lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for all } x\in\mathbb{R}^d,\ \zeta\in\mathbb{R}^d, \text{ and } \nu\in\mathbb{D}_2. \tag{4.7.33}$$

To this aim, we observe that by (g4′) and (4.7.1) we have

$$|\mathfrak{m}^{F^\infty}(u_{rx,\zeta_1,\nu}, Q_\nu(rx,r)) - \mathfrak{m}^{F^\infty}(u_{rx,\zeta_2,\nu}, Q_\nu(rx,r))| \le \tau_1|\zeta_1-\zeta_2|r^{d-1}$$

for every $x\in\mathbb{R}^d$, $\zeta_1,\zeta_2\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$. This can be proved arguing as in the proof of inequality (4.3.15) in Lemma 4.3.8. Combining this inequality with (4.7.31), we obtain that the limit in (4.7.33) exists for every $\zeta\in\mathbb{R}^d$ and $\nu\in\mathbb{D}_2$.

We now prove that there exists an extension of $g_{\rm lim}$ satisfying (4.7.32). We begin by observing that, given $0<\eta<1$, by the continuity condition on the map $\nu\mapsto R_\nu$ introduced in the paragraph about cubes of Section 1.1, there exists $0<\delta_\eta<\eta$ such that

$$Q_{\nu_1}(0,1) \subset\subset Q_{\nu_2}(0,1+\eta) \quad \text{for every } \nu_1,\nu_2\in\mathbb{S}^{d-1}_\pm \text{ with } |\nu_1-\nu_2|<\delta_\eta. \tag{4.7.34}$$

Let us fix $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, $r>0$, and $\eta>0$. We observe that (4.7.34) gives

$$Q_{\nu_1}(rx,r) \subset\subset Q_{\nu_2}(rx,(1+\eta)r) \quad \text{for every } \nu_1,\nu_2\in\mathbb{S}^{d-1}_\pm \text{ with } |\nu_1-\nu_2|<\delta_\eta.$$

Hence, we can apply Lemma 4.3.7 with $x_1=x_2=rx$, $\rho_1=r$, $\rho_2=(1+\eta)r$, and $\nu_1,\nu_2\in\mathbb{S}^{d-1}_\pm$ with $|\nu_1-\nu_2|<\delta_\eta$. We observe that F^∞ satisfies the upper estimate in (H3) of Definition 4.1.4 with $c_4=0$ because of (4.1.3) and Proposition 4.1.8, so that we can omit the term containing c_4 in Lemma 4.3.7. Using the inequality $(1+\eta)^{d-1}-1\le 2^{d-1}\eta$, for every $\nu_1,\nu_2\in\mathbb{S}^{d-1}_\pm$ with $|\nu_1-\nu_2|<\delta_\eta<\eta$ we then obtain

$$\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_2}, Q_{\nu_2}(rx,(1+\eta)r)) \le \mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_1}, Q_{\nu_1}(rx,r)) + c_3|\zeta|(2^{d-1}\eta+\omega(0,\eta))r^{d-1}.$$

We set $x_\eta := \frac{x}{1+\eta}$ and $r_\eta := (1+\eta)r$. Dividing the previous inequality by r^{d-1}, we obtain

$$\frac{\mathfrak{m}^{F^\infty}(u_{r_\eta x_\eta,\zeta,\nu_2}, Q_{\nu_2}(r_\eta x_\eta, r_\eta))}{r_\eta^{d-1}} \le \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_1}, Q_{\nu_1}(rx,r))}{r^{d-1}} + c_3|\zeta|(2^{d-1}\eta+\omega(0,\eta)). \tag{4.7.35}$$

Exchanging the roles of ν_1 and ν_2 we also get

$$\frac{\mathfrak{m}^{F^\infty}(u_{r_\eta x_\eta,\zeta,\nu_1}, Q_{\nu_1}(r_\eta x_\eta, r_\eta))}{r_\eta^{d-1}} \le \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_2}, Q_{\nu_2}(rx,r))}{r^{d-1}} + c_3|\zeta|(2^{d-1}\eta+\omega(0,\eta)). \tag{4.7.36}$$

Moreover, applying (4.7.35) with x replaced by $x^\eta := (1+\eta)x$ we obtain

$$\frac{\mathfrak{m}^{F^\infty}(u_{r_\eta x,\zeta,\nu_2}, Q_{\nu_2}(r_\eta x, r_\eta))}{r_\eta^{d-1}} \le \frac{\mathfrak{m}^{F^\infty}(u_{rx^\eta,\zeta,\nu_1}, Q_{\nu_1}(rx^\eta,r))}{r^{d-1}} + c_3|\zeta|(2^{d-1}\eta+\omega(0,\eta)). \tag{4.7.37}$$

Let us fix $\nu\in\mathbb{S}^{d-1}_\pm$. For every $\eta>0$ we can find $\nu_\eta\in\mathbb{S}^{d-1}_\pm\cap\mathbb{D}_2$ with $|\nu_\eta-\nu|<\delta_\eta$, where $0<\delta_\eta<\eta$ is the constant given by (4.7.34). Using (4.7.36) with $\nu_1=\nu_\eta$ and $\nu_2=\nu$, by (4.7.33), which clearly holds with r replaced by r_η, we obtain

$$g_{\rm lim}(\zeta,\nu_\eta) - c_3|\zeta|(2^{d-1}\eta+\omega(0,\eta)) \le \liminf_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}}, \tag{4.7.38}$$

while using (4.7.37) with $\nu_1 = \nu_\eta$ and $\nu_2 = \nu$ by (4.7.33) we get

$$\limsup_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \le g_{\lim}(\zeta,\nu_\eta) + c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)), \tag{4.7.39}$$

where we have also used that by (4.7.33) the limit appearing the definition of $g_{\lim}(\zeta,\nu_\eta)$ is the same for x and x^η. Combining (4.7.38) and (4.7.39), we obtain

$$\limsup_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} - \liminf_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \le 2c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)).$$

Letting $\eta \to 0^+$ we conclude the proof of the lemma. □

We conclude this section by showing that, if $F^{f,g}(\cdot, O)$ is $L^1(O;\mathbb{R}^d)$ lower semicontinuous for some bounded open set $O \subset \mathbb{R}^d$, then $f^\infty(\zeta \odot \nu) = g(\zeta,\nu)$, provided that f and g are independent of x and g is positively homogeneous of degree one in ζ. Although variants of this result are well-known (see, for instance, [63, Step 3 of Theorem 6.14]), we present here a complete statement and a full proof.

Lemma 4.7.10. *Let $f\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ and $g\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ be Borel functions satisfying* (f2) , (g2) , (g3) *, and* (4.7.4)*. Assume that there exists a non-empty bounded open set $O \subset \mathbb{R}^d$ such that the functional $F^{f,g}(\cdot, O)$ is $L^1(O;\mathbb{R}^d)$-lower semicontinuous. Then we have*

$$g(\zeta,\nu) = f^\infty(\zeta \odot \nu)$$

for every $\zeta \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$.

Proof. Let us fix $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $x \in \mathbb{R}^d$ such that

$$0 < \mathcal{H}^{d-1}(\Pi^\nu_x \cap O), \tag{4.7.40}$$

$$0 \text{ is a Lebesgue point of the function } \mathbb{R} \ni t \mapsto \mathcal{H}^{d-1}(\Pi^\nu_{x+t\nu} \cap O), \tag{4.7.41}$$

where we recall that for $\nu_0 \in \mathbb{S}^{d-1}$ and $x_0 \in \mathbb{R}^d$ the symbol $\Pi^{\nu_0}_{x_0}$ denotes the hyperplane passing through x_0 and orthogonal to ν_0, given by $\{y \in \mathbb{R}^d : (y-x_0)\cdot\nu_0 = 0\}$. Thanks to the Fubini's Theorem and the Lebesgue Differentiation Theorem, the set of points $x \in \mathbb{R}^d$ for which the two previous conditions are satisfied has positive $\mathcal{L}^d$ measure.

We now show that

$$g(\zeta,\nu) \le f^\infty(\zeta \odot \nu). \tag{4.7.42}$$

To prove this, we approximate the function $u_{x,\zeta,\nu}$ by the sequence of piecewise affine functions $\{u_n\}_n \subset \mathrm{BD}(O)$ defined for every $y \in O$ by

$$u_n(y) \coloneqq \begin{cases} 0 & \text{if } (y-x)\cdot\nu \le -\frac{1}{2n}, \\ n((y-x)\cdot\nu + \frac{1}{2n})\zeta & \text{if } -\frac{1}{2n} \le (y-x)\cdot\nu \le \frac{1}{2n}, \\ \zeta & \text{if } (y-x)\cdot\nu \ge \frac{1}{2n}. \end{cases}$$

Clearly $\{u_n\}_n$ converges to $u_{x,\zeta,\nu}$ strongly in $L^1(O;\mathbb{R}^d)$ as $n \to +\infty$.

Setting $H_n \coloneqq \{y \in O\colon\ -\frac{1}{2n} \le (y-x)\cdot\nu \le \frac{1}{2n}\}$, one checks immediately that

$$\mathcal{E}u_n = n\zeta \odot \nu\chi_{H_n} \quad \mathcal{L}^d\text{-a.e. in } O, \tag{4.7.43}$$

where χ_{H_n} is the characteristic function of H_n. Moreover, the Fubini Theorem implies that

$$\mathcal{L}^d(H_n) = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \mathcal{H}^{d-1}(\Pi^\nu_{x+t\nu} \cap O)\,\mathrm{d}t,$$

which together with (4.7.41) gives

$$\lim_{n\to+\infty} n\mathcal{L}^d(H_n) = \mathcal{H}^{d-1}(\Pi^\nu_x \cap O). \tag{4.7.44}$$

In light of the lower semicontinuity of $F^{f,g}(\cdot, O)$, a direct computation shows that

$$\begin{aligned} f(0)\mathcal{L}^d(O) + g(\zeta,\nu)\mathcal{H}^{d-1}(\Pi^\nu_x \cap O) &= F^{f,g}(u_{x,\zeta,\nu}, O) \le \liminf_{n\to+\infty} F^{f,g}(u_n, O) \\ &= \liminf_{n\to+\infty} \Big(\int_{H_n} f(\mathcal{E}u_n)\,\mathrm{d}x + f(0)\mathcal{L}^d(O \setminus H_n)\Big). \end{aligned}$$

Since $\mathcal{L}^d(H_n)$ converges to 0 as $n \to +\infty$, the previous inequality gives

$$g(\zeta,\nu)\mathcal{H}^{d-1}(\Pi^\nu_x \cap O) \le \liminf_{n\to+\infty} \int_{H_n} f(\mathcal{E}u_n)\,\mathrm{d}x. \tag{4.7.45}$$

Using (4.7.43), we see that

$$\int_{H_n} f(\mathcal{E}u_n)\,\mathrm{d}x = f(n\zeta \odot \nu)\mathcal{L}^d(H_n). \tag{4.7.46}$$

Combining (4.7.44)–(4.7.46), recalling the definition of f^∞ given by (4.1.2), we have

$$g(\zeta,\nu)\mathcal{H}^{d-1}(\Pi^\nu_x \cap O) \le \liminf_{n\to+\infty} \frac{f(n\zeta \odot \nu)}{n}(n\mathcal{L}^d(H_n)) \le f^\infty(\zeta \odot \nu)\mathcal{H}^{d-1}(\Pi^\nu_x \cap O),$$

which by (4.7.40) implies (4.7.42).

We now prove

$$f^\infty(\zeta \odot \nu) \le g(\zeta,\nu). \tag{4.7.47}$$

To this aim, we set $T > \operatorname{diam}(O)$ and let $x \in O$. By the Fubini Theorem we have

$$\mathcal{L}^d(O) = \int_{-T}^{T} \mathcal{H}^{d-1}(\Pi^\nu_{x+t\nu} \cap O)\,\mathrm{d}t.$$

By the well-known approximation properties of Lebesgue integrals by means of Riemann sums (see, for instance, Lemma 5.A.1 below) we can select $x \in O$ in such a way that

$$\mathcal{L}^d(O) = \lim_{n\to+\infty} \frac{T}{n} \sum_{i=-n+1}^{n} \mathcal{H}^{d-1}\big(\Pi^\nu_{x+\frac{i-1}{n}T\nu} \cap O\big). \tag{4.7.48}$$

We set $A \coloneqq \zeta \otimes \nu$ and fix $t > 0$. To prove (4.7.47), we construct a sequence of pure jump functions approximating ℓ_{tA} in $L^1(O;\mathbb{R}^d)$. Given $n \in \mathbb{N}$, let $\sigma_n \colon (-T,T) \to \mathbb{R}$ be defined by

$$\sigma_n(s) \coloneqq \sum_{i=-n+1}^{n} \tfrac{iT}{n} \chi_{(\frac{i-1}{n}T, \frac{i}{n}T)}(s) \quad \text{for every } s \in (-T,T).$$

For every $n \in \mathbb{N}$, let $u_n \in \mathrm{BD}(O)$ be the function defined by

$$u_n(y) \coloneqq t\zeta\sigma_n((y-x)\cdot\nu) \quad \text{for every } y \in O.$$

It is easy to see that $\{u_n\}_n$ converges to $\ell_{tA} - tAx$ strongly in $L^1(O;\mathbb{R}^d)$ as $n \to +\infty$. Thus, by the L^1-lower semicontinuity of $F^{f,g}(\cdot, O)$ we obtain

$$f(tA^{\mathrm{sym}})\mathcal{L}^d(O) = \int_O f(tA^{\mathrm{sym}})\,\mathrm{d}x = F^{f,g}(\ell_{tA}, O) \le \liminf_{n\to+\infty} F^{f,g}(u_n, O). \tag{4.7.49}$$

On the other hand, using the homogeneity of g given by (4.7.4) we immediately check that

$$\begin{aligned} F^{f,g}(u_n, O) &= f(0)\mathcal{L}^d(O) + \sum_{i=-n+1}^{n} g\Big(\frac{t\zeta}{n}T, \nu\Big)\mathcal{H}^{d-1}\big(\Pi^{\nu}_{x+\frac{i-1}{n}T\nu} \cap O\big) \\ &= f(0)\mathcal{L}^d(O) + tg(\zeta,\nu)\frac{T}{n}\sum_{i=-n+1}^{n} \mathcal{H}^{d-1}\big(\Pi^{\nu}_{x+\frac{i-1}{n}T\nu} \cap O\big). \end{aligned}$$

Taking the liminf as $n \to +\infty$ and using (4.7.48) and (4.7.49) we get

$$f(tA^{\mathrm{sym}})\mathcal{L}^d(O) \le f(0)\mathcal{L}^d(O) + tg(\zeta,\nu)\mathcal{L}^d(O).$$

Hence, we get $\frac{1}{t}f(tA^{\mathrm{sym}}) \le \frac{1}{t}f(0) + g(\zeta,\nu)$, and letting $t \to +\infty$ we obtain (4.7.47). □

4.8 Stochastic homogenisation

In this section, we use the results of the previous section to study the problem of stochastic homogenisation of free discontinuity functionals defined on the space BD. In the following we still assume that the modulus of continuity introduced in (4.1.1) satisfies (4.7.1).

We refer the reader to Section 2.7 of Chapter 2 for the notation and main definitions concerning the probabilistic setting of stochastic homogenisation. Throughout this section, $(\Theta, \mathcal{T}, P)$ is a probability space endowed with a group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ of P-preserving transformations on $(\Theta, \mathcal{T}, P)$.

In analogy with Definition 2.7.3 of Chapter 2, we introduce the following two classes of stochastic integrands.

Definition 4.8.1. $\mathscr{F}^{\alpha}_{\mathrm{stoc}}$ denotes the set of all $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}})$-measurable functions $f\colon \Theta\times\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ such that for every $\theta\in\Theta$ the function $f(\theta) \coloneqq f(\theta,\cdot,\cdot)$ belongs to $\mathscr{F}^{\alpha}$ and the following stochastic periodicity condition holds:

$$f(\theta, x+z, A) = f(\tau_z(\theta), x, A)$$

for every $\theta\in\Theta$, $z\in\mathbb{Z}^d$, $x\in\mathbb{R}^d$, and $A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

$\mathscr{G}^{\infty}_{\mathrm{stoc}}$ denotes the set of all $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1})$-measurable functions $g\colon \Theta\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\to[0,+\infty)$ such that for every $\theta\in\Theta$ the function $g(\theta)\coloneqq g(\theta,\cdot,\cdot,\cdot)$ belongs to $\mathscr{G}^{\infty}$, and the following stochastic periodicity condition holds:

$$g(\theta, x+z, \zeta, \nu) = g(\tau_z(\theta), x, \zeta, \nu)$$

for every $\theta\in\Theta$, $z\in\mathbb{Z}^d$, $x,\zeta\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$.

The following is the main result concerning the bulk part. We recall the collection of half-closed rectangles $\mathscr{R}$ introduced in (2.7.5) and Definition 2.7.4 of subadditive (ergodic) process.

Proposition 4.8.2. *Let $f \in \mathscr{F}^{\alpha}_{\mathrm{stoc}}$ and $g \in \mathscr{G}^{\infty}_{\mathrm{stoc}}$. Then there exist $\Theta' \in \mathcal{T}$, with $P(\Theta') = 1$, and a function $f_{\mathrm{lim}} \colon \Theta \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$, with $f_{\mathrm{lim}}(\theta,\cdot)$ continuous for every $\theta \in \Theta'$ and $f_{\mathrm{lim}}(\cdot, A)$ $\mathcal{T}$-measurable for every $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, such that*

$$f_{\mathrm{lim}}(\theta, A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f(\theta),g(\theta)}}(\ell_A, Q(rx,r))}{r^d} \tag{4.8.1}$$

for every $\theta \in \Theta'$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. If, in addition, $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Θ' appropriately we have that f_{lim} is independent of θ.

Proof. Let us fix $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and consider the function $\Phi_A \colon \Theta \times \mathscr{R} \to [0,+\infty)$ defined by

$$\Phi_A(\theta, R) \coloneqq \mathfrak{m}^{F^{f(\theta),g(\theta)}}(\ell_A, R^\circ)$$

for every $\theta \in \Theta$ and $R \in \mathscr{R}$, where R° denotes the interior of R. This function defines a covariant subadditive process. Indeed, it is easy to check that properties (b), (c), and (d) of Definition 2.7.4 hold (see [59, Theorem 9.1] for the details), while the proof of the measurability property (a) can be obtained by slightly modifying the Appendix of [59], replacing Du and ∇u by Eu and $\mathcal{E}u$.

Let $\mathbb{D}$ be a countable dense subset of $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. We may then apply Theorem 2.7.5 with $\mu = \Phi_A$ for every $A \in \mathbb{D}$ to obtain a set $\Theta' \in \mathcal{T}$, with $P(\Theta') = 1$, and a function $\phi_A \colon \Theta \to [0,+\infty)$ such that

$$\phi_A(\theta) = \lim_{r\to+\infty} \frac{\Phi_A(\theta, \widetilde{Q}(rx,r))}{r^d} = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f(\theta),g(\theta)}}(\ell_A, Q(rx,r))}{r^d}$$

for every $\theta \in \Theta'$, $x \in \mathbb{R}^d$, and $A \in \mathbb{D}$. By Lemma 4.7.8 for every $\theta \in \Theta'$ there exists a unique continuous function $f_{\mathrm{lim}}(\theta,\cdot) \colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ such that $f_{\mathrm{lim}}(\theta, A) = \phi_A(\theta)$ for every $A \in \mathbb{D}$. Moreover, $A \mapsto f_{\mathrm{lim}}(\theta, A)$ is c_5-Lipschitz continuous and satisfies (4.8.1).

Finally, the statement concerning the ergodic hypothesis follows from the last part of Theorem 2.7.5. □

The following proposition collects the main results concerning the surface term.

Proposition 4.8.3. *Let $f \in \mathscr{F}^{\alpha}_{\mathrm{stoc}}$ and $g \in \mathscr{G}^{\infty}_{\mathrm{stoc}}$. Then there exist $\Theta' \in \mathcal{T}$, with $P(\Theta') = 1$, and a function $g_{\mathrm{lim}} \colon \Theta \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that*

$$\begin{gathered}
\zeta \mapsto g_{\mathrm{lim}}(\theta,\zeta,\nu) \text{ is Lipschitz continuous for every } \theta \in \Theta' \text{ and } \nu \in \mathbb{S}^{d-1},\\
\mathbb{S}^{d-1}_{\pm} \ni \nu \mapsto g_{\mathrm{lim}}(\theta,\zeta,\nu) \text{ is continuous for every } \theta \in \Theta' \text{ and } \zeta \in \mathbb{R}^d,\\
\theta \mapsto g_{\mathrm{lim}}(\theta,\zeta,\nu) \text{ is } \mathcal{T}\text{-measurable for every } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1},
\end{gathered}$$

and

$$g_{\mathrm{lim}}(\theta,\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{\infty}(\theta)}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \tag{4.8.2}$$

for every $\theta \in \Theta'$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$, where F^∞ is the functional introduced in Definition 4.7.5. If, in addition, $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Θ' appropriately we have that g_{lim} is independent of θ.

Proof. It is enough to repeat the arguments of [59, Proposition 9.4 and 9.5], replacing Step 2 of their Propositions 9.4 and 9.5 by our Lemma 4.7.9. □

Collecting the results of Propositions 4.8.2 and 4.8.3 we obtain the following almost sure convergence result.

Theorem 4.8.4. *Let $f \in \mathscr{F}^\alpha_{\mathrm{stoc}}$, $g \in \mathscr{G}^\infty_{\mathrm{stoc}}$ and let $F_\varepsilon(\theta) := F^{f_\varepsilon(\theta),g_\varepsilon(\theta)}$ be the functionals introduced in Definition 4.7.1. Then there exist $\Theta' \in \mathcal{T}$, with $P(\Theta') = 1$, and functions $f_{\lim} \in \mathscr{F}^\alpha_{\mathrm{stoc}}$ and $g_{\lim} \in \mathscr{G}^\infty_{\mathrm{stoc}}$, with $g_{\lim}(\theta)$ satisfying* (4.7.4) *for every $\theta \in \Theta'$, and with the following property: for every sequence $\{\varepsilon_n\}_n \subset (0,1)$ converging to 0 as $n \to +\infty$, $O \in \mathcal{O}_c(\mathbb{R}^d)$, and $\theta \in \Theta'$ we have that $\{F_{\varepsilon_n}(\theta)(\cdot,O)\}_n$ Γ-converges, with respect to the topology of $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, to $F^{f_{\lim}(\theta),g_{\lim}(\theta)}(\cdot,O)$ as $n \to +\infty$. Moreover, we have the equality*

$$g_{\lim}(\theta,\zeta,\nu) = f^\infty_{\lim}(\theta,\zeta\odot\nu) \quad \text{for every } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{4.8.3}$$

If, in addition, $\{\tau_z\}_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Θ' appropriately we have that $f_{\lim}$ and $g_{\lim}$ are independent of θ.

Proof. Thanks to Propositions 4.8.2 and 4.8.3, there exist $\Theta' \in \mathcal{T}$, with $P(\Theta') = 1$, $f_{\lim}\colon \Theta \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$, and $g_{\lim}\colon \Theta \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that (4.8.1) and (4.8.2) hold. Thus, by Theorem 4.7.7 we obtain the two inclusions $f_{\lim} \in \mathscr{F}^\alpha_{\mathrm{stoc}}$ and $g_{\lim} \in \mathscr{G}^\infty_{\mathrm{stoc}}$, together with the fact that $g_{\lim}(\theta)$ satisfies (4.7.4) for every $\theta \in \Theta'$, as well as the Γ-convergence part of the statement and equality (4.8.3).

The last part of the statement follows from the final parts of the statements of Propositions 4.8.2 and 4.8.3. □

As a particular case of the previous theorem we deduce the following result in the periodic deterministic setting.

Corollary 4.8.5. *Let $f \in \mathscr{F}^\alpha$, $g \in \mathscr{G}^\infty$, and let $F_\varepsilon := F^{f_\varepsilon,g_\varepsilon}$ be the functionals introduced in Definition 4.7.1. Assume that both f and g are 1-periodic in the x variable. Then there exist functions $f_{\lim} \in \mathscr{F}^\alpha$ and $g_{\lim} \in \mathscr{G}^\infty$, with $f_{\lim}$ satisfying* (4.7.25)*, and with $g_{\lim}$ satisfying* (4.7.4) *and* (4.7.26)*, and consequently the following property holds: for every sequence $\{\varepsilon_n\}_n \subset (0,1)$ converging to 0 as $n \to +\infty$ and for every $O \in \mathcal{O}_c(\mathbb{R}^d)$ the sequence $\{F_{\varepsilon_n}(\cdot,O)\}_n$ Γ-converges to $F^{f_{\lim},g_{\lim}}(\cdot,O)$ with respect to the topology of $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, we have the equality*

$$g_{\lim}(\zeta,\nu) = f^\infty_{\lim}(\zeta\odot\nu) \quad \text{for every } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}.$$

Proof. It is enough to apply Theorem 4.8.4 in the case where Θ consists of a single point. □

A MATRIX-VALUED MEASURE ASSOCIATED WITH GENERALISED FUNCTIONS OF BOUNDED DEFORMATION

This chapter is devoted to the study of the fine properties of generalised functions of bounded deformation. In his seminal work [89], Dal Maso showed that, remarkably, although the definition of such functions relies solely on one-dimensional slicing, this suffices to recover many of the fine properties that, in the setting of functions of bounded deformation, are usually obtained exploiting the structure of the symmetric distributional derivative Eu. In particular (see Theorem 1.4.13), generalised functions of bounded deformation admit an approximate symmetric gradient $\mathcal{E}u$, their jump set J_u is $(d-1)$-rectifiable, and the unilateral traces $u^{\pm}$ on J_u are compatible, in the sense of slicing, with the corresponding traces of the one-dimensional slices u_y^ξ, where, for simplicity of notation, throughout this chapter we write u_y^ξ in place of $\hat{u}_y^\xi$ for the function defined by (1.1.4).

The aim of this chapter is to complete this picture by providing, for functions in GBD, a suitable analogue of the Cantor part $E^c u$ of the symmetric distributional gradient Eu. In particular, this allows to introduce a suitable analogue of Eu for such functions. This is a crucial step in the study of cohesive fracture models in the fully vectorial case, as the relevant functionals to be minimised should be naturally defined on GBD and, as the lower semicontinuity results of [99, Theorem 6.1] in the antiplane case suggest, should also include a dependence on this Cantor-type contribution (see the discussion in the Introduction).

This generalisation is achieved in Theorem 5.5.1, which shows that, for every $u \in \mathrm{GBD}(\Omega)$, it is possible to associate a measure $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that

$$\mu_u(B)\xi \cdot \xi = \int_{\Pi^\xi} Du_y^\xi(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y), \tag{5.0.1}$$

for every Borel $B \subset \Omega$ with $\mathcal{H}^{d-1}(J_u \cap B) = 0$ and $\xi \in \mathbb{S}^{d-1}$. The Cantor part is then defined by setting $\mu_u^c := \mu_u^s \llcorner (\Omega \setminus J_u)$, where μ_u^s is the singular part of μ_u with respect to $\mathcal{L}^d$. Moreover, Proposition 5.5.8 shows that μ_u^c satisfies the analogue of property (ii) displayed under (1.4.1), i.e. it vanishes on all Borel sets which are σ-finite with respect to $\mathcal{H}^{d-1}$. Finally, we show in Proposition 5.5.5 that

$$\mu_u(B) = \int_B \mathcal{E}u\, \mathrm{d}x + \mu_u^c(B) + \int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \tag{5.0.2}$$

for all Borel sets $B \subset \Omega$, where we recall that $J_u^1 = \{x \in J_u \colon |[u](x)| \geq 1\}$. Comparing this decomposition with (1.4.1), we see that μ_u plays for functions in GBD the role of Eu.

To prove (5.0.1), we will show that for every Borel set $B \subset \Omega$, the function

$$\mathbb{R}^d \setminus \{0\} \ni \xi \mapsto \sigma_u^\xi(B) := |\xi| \int_{\Pi^\xi} Du_y^\xi(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y), \tag{5.0.3}$$

is quadratic, in the sense of Definition 5.1.1 below, a fact that can be checked by verifying three conditions: a lower bound, 2-homogeneity, and the parallelogram identity. While the first two properties follow rather directly, proving the parallelogram identity requires

substantially more work and constitutes the central part of the chapter. It will be obtained by dealing first with the planar case $d = 2$, which is then extended, through a Fubini-type argument, to the general case $d > 2$. The planar case $d = 2$ is treated by means of a technical discretisation argument based on classical approximation properties of Lebesgue integrals by Riemann sums.

We briefly discuss the organisation of the chapter. In the following, we let u be a fixed function in $\mathrm{GBD}(\Omega)$. In Section 5.1, we study some properties of the measures σ_u^ξ given by (5.0.3). More precisely, we investigate their relation with $D_\xi(\tau(u \cdot \xi))$, for a suitable class of truncation functions $\tau\colon \mathbb{R} \to \mathbb{R}$ (see (a) of Definition 1.4.3 and Proposition 5.1.5) and prove the 2-homogeneity of $\xi \mapsto \sigma_u^\xi(B)$. We also prove two useful preliminary results in Proposition 5.1.8 and Lemma 5.1.9, which show that, in order to prove that $\xi \mapsto \sigma_u^\xi(B)$ is quadratic, it is enough to establish this property under an additional constraint on the jump set J_u.

Sections 5.2 and 5.3 are fully devoted to the proof of the parallelogram identity for the function $\xi \mapsto \sigma_u^\xi(B)$ in the planar case $d = 2$. In the first section, we introduce the discretisation argument we employ and show that the parallelogram identity follows from a technical result, namely, Theorem 5.3.1. The proof of this latter result is the most delicate part of the argument and will be the main object of Section 5.3.

In Section 5.4, we present some new results concerning the restriction of GBD-functions to affine spaces of arbitrary dimension. Combining these results with those of Section 5.2, we deduce that $\xi \mapsto \sigma_u^\xi(B)$ is quadratic in dimension $d > 2$. Having established this latter property, in Section 5.5 we construct the measure μ_u that satisfies (5.0.1). We then prove (5.0.2) and further investigate the properties of μ_u. Finally, we characterise the space $\mathrm{GSBD}(\Omega)$ (see Definition 1.4.3) as the space of functions $u \in \mathrm{GBD}(\Omega)$ with $\mu_u^c = 0$. We conclude by observing that, combining this characterisation with recent results of Chambolle & Crismale [74], it is possible to deduce another characterisation of $\mathrm{GSBD}(\Omega)$ based on slicing in a finite number of directions.

The content of this chapter is based on the joint work with Dal Maso [91].

5.1 An auxiliary family of measures

We refer the reader to Section 1.1 for the notation concerning slicing. We stress that, throughout the chapter, we write u_y^ξ in place of $\widehat{u}_y^\xi$ for the function defined by (1.1.4). We also refer the reader to Section 1.4 for the general properties of functions in BD and GBD.

This section is devoted to the study of an auxiliary family of signed measures. More precisely, given $u \in \mathrm{GBD}(\Omega)$, we associate to it a family of measures, closely related to the measures λ_u^ξ introduced in (1.4.14). For every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$ we set

$$\sigma_u^\xi(B) \coloneqq |\xi| \int_{\Pi^\xi} Du_y^\xi((B \setminus J_u^1)_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.1.1}$$

For $\xi = 0$, we set $\sigma_u^\xi(B) = 0$ for every Borel set $B \subset \Omega$.

We recall the definition of a quadratic function.

Definition 5.1.1. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is quadratic if there exists a matrix $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that $f(\xi) = A\xi \cdot \xi$ for every $\xi \in \mathbb{R}^d$.

We recall the following characterisation of quadratic functions.

Proposition 5.1.2. *A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is quadratic if and only if the following conditions are satisfied:*

(a) 2-homogeneity: $f(t\xi) = t^2 f(\xi)$ for every $\xi \in \mathbb{R}^d$ and every $t \in \mathbb{R}$;

(b) parallelogram identity: for every $\xi, \eta \in \mathbb{R}^d$ we have

$$f(\xi+\eta) + f(\xi-\eta) = 2f(\xi) + 2f(\eta);$$

(c) lower bound: there exists a constant $c > 0$ such that

$$f(\xi) \geq -c|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^d.$$

Proof. Assume that (a)–(c) hold. By applying [88, Proposition 11.9] to the function $g(\xi) = f(\xi) + c|\xi|^2$ we obtain a matrix $B \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that $g(\xi) = B\xi\cdot\xi$ for every $\xi \in \mathbb{R}^d$. Setting $A \coloneqq B - cI$, where I is the identity matrix, we obtain $f(\xi) = A\xi\cdot\xi$ for every $\xi \in \mathbb{R}^d$.

The converse implication is trivial. □

Remark 5.1.3. Suppose that $u \in \mathrm{BD}(\Omega)$ and let Eu be the distributional symmetric gradient of u, which by definition belongs to the space $\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. From the Structure Theorem 1.3.4 it follows that

$$\sigma_u^\xi(B) = Eu(B \setminus J_u^1)\xi\cdot\xi$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{R}^d$. Since σ_u^ξ is a measure, this equality can be extended to all functions $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ such that $Eu \in \mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. This implies that, in this case, the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic in the sense of Definition 5.1.1. In particular, if $d = 1$, this happens for every $u \in \mathrm{GBD}(\Omega)$ thanks to Remark 1.4.5.

Remark 5.1.4. Let λ_u^ξ be the measure introduced in (1.4.14). One can see that for every $\xi \in \mathbb{S}^{d-1}$, we have the equality

$$|\sigma_u^\xi| = \lambda_u^\xi \llcorner (\Omega \setminus J_u^1) \tag{5.1.2}$$

as Borel measures on Ω. Indeed, in view of Proposition 1.3.2, this is an easy consequence of (5.1.1) and of the fact that, by Theorem 1.4.13, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have the inclusion $J^1_{u_y^\xi} \subset (J_u^1)_y^\xi$. This fact, together with (1.4.15) and the 2-homogeneity of the function $\xi \mapsto \sigma_u^\xi(\Omega)$ proved in Proposition 5.1.6 below, implies that

$$|\sigma_u^\xi|(\Omega) \leq |\xi|^2 \lambda_u(\Omega \setminus J_u^1). \tag{5.1.3}$$

For $R > 0$ let $\tau_R\colon \mathbb{R} \to \mathbb{R}$ be the 1-Lipschitz functions defined by

$$\tau_R(s) \coloneqq \begin{cases} -\frac{R}{2} & \text{if } s \leq -\frac{R}{2}, \\ s & \text{if } \frac{-R}{2} \leq s \leq \frac{R}{2}, \\ \frac{R}{2} & \text{if } s \geq \frac{R}{2}. \end{cases} \tag{5.1.4}$$

Thanks to Remark 1.4.6, we have that $D_\xi(\tau_R(u\cdot\xi)) \in \mathcal{M}_b(\Omega;\mathbb{R})$ for every $u \in \mathrm{GBD}(\Omega)$, $\xi \in \mathbb{S}^{d-1}$, and $R > 0$. The following result shows that for Borel sets that do not intersect J_u^1 we can obtain the value of $\sigma_u^\xi(B)$ by considering the limit of $D_\xi(\tau_R(u\cdot\xi))(B)$ as $R \to +\infty$.

Proposition 5.1.5. *Let $u \in \mathrm{GBD}(\Omega)$. Then for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\sigma_u^\xi(B) = \lim_{R\to+\infty} D_\xi(\tau_R(u\cdot\xi))(B), \tag{5.1.5}$$

$$\lambda_u^\xi(B) = \lim_{R\to+\infty} |D_\xi(\tau_R(u\cdot\xi))|(B), \tag{5.1.6}$$

for every Borel set $B \subset \Omega$ with $\mathcal{H}^{d-1}(B \cap J_u^1) = 0$.

Proof. Let us fix $\xi \in \mathbb{S}^{d-1}$ and a Borel set $B \subset \Omega$ as in the statement. Since $\mathcal{H}^{d-1}(B\cap J_u^1) = 0$, it follows that $\mathcal{H}^{d-1}(\pi^\xi(B \cap J_u^1)) = 0$ and hence $B_y^\xi \cap (J_u^1)_y^\xi = \emptyset$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Recalling that by Theorem 1.4.13 for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the inclusion in (1.4.18) holds, we also have that

$$B_y^\xi \cap J^1_{u_y^\xi} = \emptyset \text{ for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi. \tag{5.1.7}$$

Thus, by (1.4.14) and (5.1.1) we have

$$\sigma_u^\xi(B) = \int_{\Pi^\xi} Du_y^\xi((B\setminus J_u^1)_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_{\Pi^\xi} Du_y^\xi(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{5.1.8}$$

$$\lambda_u^\xi(B) = \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi \setminus J^1_{u_y^\xi})\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.1.9}$$

In particular, since $\lambda_u^\xi \in \mathcal{M}_b^+(\Omega)$, we have

$$\int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty. \tag{5.1.10}$$

We then remark that from (b) of Definition 1.4.3 and the chain rule for BV-functions (see [18, Theorem 3.99]) it follows that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $\tau_R(u_y^\xi) \in BV_{\mathrm{loc}}(\Omega_y^\xi)$ and

$$D(\tau_R(u_y^\xi)) = \tau_R'(v_y^\xi)\nabla u_y^\xi \mathcal{L}^1 + \tau_R'(v_y^\xi)D^c u_y^\xi + [\tau_R(u_y^\xi)]\mathcal{H}^0 \llcorner J_{\tau_R(u_y^\xi)}, \tag{5.1.11}$$

where for every $t \in \Omega_y^\xi \setminus J_{u_y^\xi}$ the function v_y^ξ is defined by

$$v_y^\xi(t) \coloneqq \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_y^\xi(t+s)\,\mathrm{d}s,$$

and we set $\tau_R'(\pm\frac{R}{2}) = 0$. Since the measures on the right-hand side of (5.1.11) are mutually singular, we have also

$$|D(\tau_R(u_y^\xi))| = |\tau_R'(v_y^\xi)\nabla u_y^\xi|\mathcal{L}^1 + |\tau_R'(v_y^\xi)||D^c u_y^\xi| + |[\tau_R(u_y^\xi)]|\mathcal{H}^0 \llcorner J_{\tau_R(u_y^\xi)}. \tag{5.1.12}$$

Moreover, we observe that for every $y \in \Pi^\xi$ such that $u_y^\xi \in BV_{\mathrm{loc}}(\Omega_y^\xi)$ we have

$$|D^c u_y^\xi|(\Omega_y^\xi \setminus J_{u_y^\xi}) = |D^c u_y^\xi|(\Omega_y^\xi),$$

$$\lim_{R\to+\infty} \tau_R'(v_y^\xi)(t) = 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega_y^\xi, \tag{5.1.13}$$

$$\lim_{R\to+\infty} \tau_R'(v_y^\xi)(t) = 1 \quad \text{for } |D^c u_y^\xi|\text{-a.e. } t \in \Omega_y^\xi, \tag{5.1.14}$$

$$J_{u_y^\xi} = \bigcup_{R>0} J_{\tau_R(u_y^\xi)} \quad \text{and} \quad \lim_{R\to+\infty}[\tau_R(u_y^\xi)](t) = [u_y^\xi](t) \quad \text{for every } t \in J_{u_y^\xi}. \tag{5.1.15}$$

Additionally, from (5.1.4) and (5.1.12) it follows that

$$|D(\tau_R(u_y^\xi))|(B_y^\xi) \le \int_{B_y^\xi} |\nabla u_y^\xi| \,\mathrm{d}t + |D^c u_y^\xi|(B_y^\xi) + \int_{B_y^\xi \cap J_{u_y^\xi}} |[u_y^\xi]| \,\mathrm{d}\mathcal{H}^0 = |Du_y^\xi|(B_y^\xi). \quad (5.1.16)$$

Recalling that by Remark 1.4.4 for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $|Du_y^\xi|(B_y^\xi) < +\infty$, from (5.1.11)–(5.1.15) and the Dominated Convergence Theorem we deduce for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ that

$$\lim_{R\to+\infty} D(\tau_R(u_y^\xi))(B_y^\xi) = Du_y^\xi(B_y^\xi), \quad (5.1.17)$$

$$\lim_{R\to+\infty} |D(\tau_R(u_y^\xi))|(B_y^\xi) = |Du_y^\xi|(B_y^\xi). \quad (5.1.18)$$

Thanks to Proposition 1.3.2 and [89, Proposition 3.1], for every $R > 0$ we have that

$$D_\xi(\tau_R(u\cdot\xi))(B) = \int_{\Pi^\xi} D(\tau_R(u_y^\xi))(B_y^\xi) \,\mathrm{d}\mathcal{H}^{d-1}(y), \quad (5.1.19)$$

$$|D_\xi(\tau_R(u\cdot\xi))|(B) = \int_{\Pi^\xi} |D(\tau_R(u_y^\xi))|(B_y^\xi) \,\mathrm{d}\mathcal{H}^{d-1}(y), \quad (5.1.20)$$

so that by (5.1.11) and (5.1.12) we have

$$\begin{aligned} D_\xi(\tau_R(u\cdot\xi))(B) = \int_{\Pi^\xi} \Big(& \int_{B_y^\xi} \tau_R'(v_y^\xi)\nabla u_y^\xi \,\mathrm{d}t \\ & + \int_{B_y^\xi} \tau_R'(v_y^\xi) \frac{\mathrm{d}D^c u_y^\xi}{\mathrm{d}|D^c u_y^\xi|} \,\mathrm{d}|D^c u_y^\xi| + \int_{B_y^\xi \cap J_{\tau_R(u_y^\xi)}} [\tau_R(u_y^\xi)] \,\mathrm{d}\mathcal{H}^0 \Big) \mathrm{d}\mathcal{H}^{d-1}(y), \\ |D_\xi(\tau_R(u\cdot\xi))|(B) = \int_{\Pi^\xi} \Big(& \int_{B_y^\xi} |\tau_R'(v_y^\xi)\nabla u_y^\xi| \,\mathrm{d}t \\ & + \int_{B_y^\xi} \tau_R'(v_y^\xi) \,\mathrm{d}|D^c u_y^\xi| + \int_{B_y^\xi \cap J_{\tau_R(u_y^\xi)}} |[\tau_R(u_y^\xi)]| \,\mathrm{d}\mathcal{H}^0 \Big) \mathrm{d}\mathcal{H}^{d-1}(y), \end{aligned}$$

Finally, recalling (1.4.5), (5.1.7), and (5.1.16), we can apply the Dominated Convergence Theorem and from (5.1.8), (5.1.9), and (5.1.17)–(5.1.20) we obtain (5.1.5) and (5.1.6). □

The main goal of Sections 5.2–5.4 will be proving that for every $u \in \mathrm{GBD}(\Omega)$ and every Borel set $B \subset \Omega$ the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic. Toward this end, in the rest of this section we investigate some of the properties of the measure defined by (5.1.1). We first show that the function $\xi \mapsto \sigma_u^\xi(B)$ is 2-homogeneous.

Proposition 5.1.6. *Let $u \in \mathrm{GBD}(\Omega)$ and let $B \subset \Omega$ be a Borel set. Then the function $\xi \mapsto \sigma_u^\xi(B)$ is 2-homogeneous.*

Proof. Let us fix $t \neq 0$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. By (1.4.9), with B replaced by $B \setminus J_u^1$, we have

$$Du_y^{t\xi}((B \setminus J_u^1)_y^{t\xi}) = |t| Du_y^\xi((B \setminus J_u^1)_y^\xi).$$

Hence,

$$\sigma_u^{t\xi}(B) = |t\xi| \int_{\Pi^\xi} Du_y^{t\xi}((B\setminus J_u^1)_y^{t\xi}) \,\mathrm{d}\mathcal{H}^{d-1}(y) = t^2|\xi| \int_{\Pi^\xi} Du_y^\xi\big((B\setminus J_u^1)_y^\xi\big) \,\mathrm{d}\mathcal{H}^{d-1}(y) = t^2 \sigma_u^\xi(B).$$

This shows that $\xi \mapsto \sigma_u^\xi(B)$ is 2-homogeneous, concluding the proof. □

In the next proposition we give an explicit formula for $\sigma_u^\xi(B)$ when the Borel set B is contained in J_u. This shows in particular that in this case the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic in the sense of Definition 5.1.1.

Proposition 5.1.7. *Let $u \in \mathrm{GBD}(\Omega)$ and let $B \subset J_u$ be a Borel set. Then for every $\xi \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\sigma_u^\xi(B) = \int_{B \setminus J_u^1} (([u] \odot \nu_u)\xi \cdot \xi)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.1.21}$$

Proof. Let us fix $\xi \in \mathbb{R}^d \setminus \{0\}$. Since a change of sign of ν_u implies a change of sign of $[u]$, we may assume without loss of generality that $\nu_u \cdot \xi \geq 0$. Thanks to Proposition 1.4.14 the integral in the right-hand side of (5.1.21) is well-defined. Hence, by (1.4.17) and the Area Formula [164, 12.4] we infer

$$\begin{aligned}\sigma_u^\xi(B) &= |\xi| \int_{\Pi^\xi} Du_y^\xi((B \setminus J_u^1)_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y) = |\xi| \int_{\Pi^\xi} \Big(\int_{(B \setminus J_u^1)_y^\xi} [u_y^\xi](t)\, \mathrm{d}\mathcal{H}^0(t) \Big) \mathrm{d}\mathcal{H}^{d-1}(y) \\ &= \int_{B \setminus J_u^1} ([u] \cdot \xi)(\nu_u \cdot \xi)\, \mathrm{d}\mathcal{H}^{d-1}(y) = \int_{B \setminus J_u^1} (([u] \odot \nu_u)\xi \cdot \xi)\, \mathrm{d}\mathcal{H}^{d-1}(y),\end{aligned}$$

which concludes the proof of (5.1.21). □

Given two $\mathcal{H}^{d-1}$-measurable sets $A, B \subset \mathbb{R}^d$, we write $A \simeq B$ when $\mathcal{H}^{d-1}(A \triangle B) = 0$, where $\triangle$ denotes the symmetric difference. We now present a decomposition result for $\mathrm{GBD}(\Omega)$ functions, which states that any function $u \in \mathrm{GBD}(\Omega)$ can be written as the sum of two functions v and w, with $v \in \mathrm{SBV}(\Omega; \mathbb{R}^d)$, $w \in \mathrm{GBD}(\Omega)$, $J_v \simeq J_u \setminus J_u^1$, and $J_w \simeq J_u^1$. We refer to [18] for the definition and properties of the space SBV of *special functions of bounded variation*.

Proposition 5.1.8. *Let $u \in \mathrm{GBD}(\Omega)$. Then there exists $v \in \mathrm{SBV}(\Omega; \mathbb{R}^d)$ such that*

$$J_v \simeq J_u \setminus J_u^1, \tag{5.1.22}$$

$$[v] = [u] \quad \text{and} \quad \nu_v = \nu_u \quad \mathcal{H}^{d-1}\text{-a.e. on } J_u \setminus J_u^1. \tag{5.1.23}$$

In particular, setting $w := u - v$, we have $w \in \mathrm{GBD}(\Omega)$, $u = w + v$,

$$J_w \simeq J_w^1 \simeq J_u^1, \tag{5.1.24}$$

$$[w] = [u] \quad \mathcal{H}^{d-1}\text{-a.e. } J_u^1. \tag{5.1.25}$$

Proof. Since $J_u \setminus J_u^1$ is $(\mathcal{H}^{d-1}, d-1)$-rectifiable and $[u]$ is integrable on $J_u \setminus J_u^1$ by Proposition 1.4.14, the proof of the statements concerning v can be obtained arguing as in [98, Theorems 3.1 and 4.1].

The inclusion $w \in \mathrm{GBD}(\Omega)$ is due to the vector space properties of $\mathrm{GBD}(\Omega)$. Equalities (5.1.24) and (5.1.25) follow from (5.1.22) and (5.1.23). □

From this result, we derive the following useful consequence.

Lemma 5.1.9. *Let $u \in \mathrm{GBD}(\Omega)$ and $v \in \mathrm{SBV}(\Omega; \mathbb{R}^d)$ be as in Proposition 5.1.8 and let $w := u - v$. Assume that for a Borel set $B \subset \Omega$ the function $\xi \mapsto \sigma_w^\xi(B)$ is quadratic. Then the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic as well.*

Proof. As $v \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$, from Remark 5.1.3 it follows immediately that $\xi \mapsto \sigma_v^\xi(B)$ is quadratic, so that the function $\xi \mapsto \sigma_w^\xi(B) + \sigma_v^\xi(B)$ is also quadratic. We claim that for every $\xi \in \mathbb{R}^d \setminus \{0\}$ we have

$$\sigma_u^\xi(B) = \sigma_w^\xi(B) + \sigma_v^\xi(B). \tag{5.1.26}$$

To prove this, let us fix $\xi \in \mathbb{R}^d \setminus \{0\}$. By definition we have $u = w + v$. Thus, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ it holds

$$Du_y^\xi = Dw_y^\xi + Dv_y^\xi \tag{5.1.27}$$

as Borel measures on Ω_y^ξ. Moreover, by construction we have $J_v^1 \simeq \emptyset$ and $J_w \simeq J_w^1 \simeq J_u^1$. This implies that

$$Dw_y^\xi((B \setminus J_w^1)_y^\xi) = Dw_y^\xi((B \setminus J_u^1)_y^\xi) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi.$$

Since $v \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$, from (5.1.22) we deduce that $|Dv|(J_u^1) = 0$. By slicing we obtain that

$$|Dv_y^\xi|((J_u^1)_y^\xi) = 0 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi.$$

Recalling that $J_v^1 \simeq \emptyset$ by (5.1.22) and (5.1.23), this implies that

$$Dv_y^\xi((B \setminus J_v^1)_y^\xi) = Dv_y^\xi(B_y^\xi) = Dv_y^\xi((B \setminus J_u^1)_y^\xi).$$

These remarks, together with (5.1.27), imply that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ it holds

$$Dw_y^\xi((B \setminus J_w^1)_y^\xi) + Dv_y^\xi((B \setminus J_v^1)_y^\xi) = D(w_y^\xi + v_y^\xi)((B \setminus J_u^1)_y^\xi) = Du_y^\xi((B \setminus J_u^1)_y^\xi).$$

Integrating this equality, by (5.1.1) we obtain (5.1.26), concluding the proof. □

5.2 The case of dimension d = 2

In this and the next section we assume that $\Omega \subset \mathbb{R}^2$. Our aim is to prove the following result.

Theorem 5.2.1. *Let $u \in \mathrm{GBD}(\Omega)$ and let $B \subset \Omega$ be a Borel set. Then the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic.*

Proof. Thanks to Proposition 5.1.8 and Lemma 5.1.9, it is not restrictive to assume that $J_u \simeq J_u^1$. Moreover, it is enough to prove the result when B is an open set, which will be denoted by U.

To prove that $\xi \mapsto \sigma_u^\xi(U)$ is quadratic we use Proposition 5.1.2. Since by Proposition 5.1.6 the function $\xi \mapsto \sigma_u^\xi(U)$ is 2-homogeneous and by Remark 5.1.4 it satisfies the lower bound (c) of Proposition 5.1.2, we are left with proving the *parallelogram identity*

$$\sigma_u^{\xi+\eta}(U) + \sigma_u^{\xi-\eta}(U) = 2\sigma_u^\xi(U) + 2\sigma_u^\eta(U), \tag{5.2.1}$$

for every $\xi, \eta \in \mathbb{R}^2 \setminus \{0\}$.

To this aim, we fix $\xi, \eta \in \mathbb{R}^2 \setminus \{0\}$. Note that, if ξ and η are not linearly independent, then the parallelogram identity follows from 2-homogeneity, so we may assume ξ and η to be linearly independent. We also note that it is not restrictive to assume that U is a parallelogram of the form

$$U = \{s\xi + t\eta\colon\ s \in (0,\alpha) \text{ and } t \in (0,\beta)\}, \tag{5.2.2}$$

for suitable constants $\alpha, \beta > 0$ and with $U \subset\subset \Omega$. Indeed, every open set U contained in Ω can be approximated by a sequence $\{U_k\}_k$ of disjoint unions of such parallelograms for which $\sigma_u^\zeta(U) = \lim_k \sigma_u^\zeta(U_k)$ for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$.

The vectors ξ and η, as well as the parallelogram U, are kept fixed throughout the rest of the proof.

To prove the parallelogram identity (5.2.1), we will use Lemma 5.A.1 to approximate, by means of Riemann sums, each integral appearing in the definition of the terms occurring in (5.2.1). This will allow us to prove that the obtained approximations satisfy, up to an arbitrarily small error, the parallelogram identity.

In order to construct these approximations, we need to introduce some notation first. Given a point $\omega \in \mathbb{R}^2$, for every $k \in \mathbb{N}$ and for every $i, j \in \mathbb{Z}$ we set (see Figure 5.1)

$$x_{i,j}^k := \omega + \tfrac{i}{k}\xi + \tfrac{j}{k}\eta. \tag{5.2.3}$$

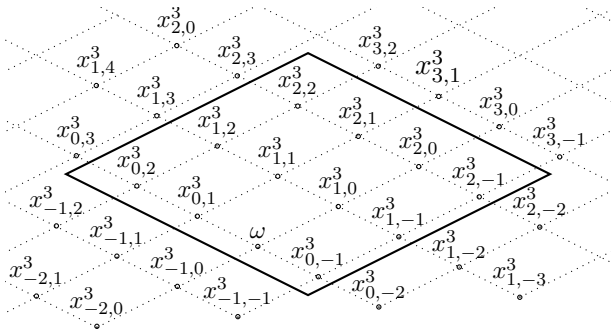


Figure 5.1: The parallelogram U and the grid of points $x_{i,j}^k$ associated to $\omega \in U$ and $k = 3$

Since the points $x_{i,j}^k$ will be instrumental to the discretisation of the summands in (5.2.1), which are integrals over the straight lines Π^ζ for $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, we consider also the projections of the points $x_{i,j}^k$ onto these lines. For every $k \in \mathbb{N}$, $i, j \in \mathbb{Z}$, and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we set

$$y_{i,j}^{k,\zeta} := \pi^\zeta(x_{i,j}^k) = \pi^\zeta(\omega) + \tfrac{i}{k}\pi^\zeta(\xi) + \tfrac{j}{k}\pi^\zeta(\eta) \in \Pi^\zeta. \tag{5.2.4}$$

We observe that $y_{i,j}^{k,\xi}$ depends only on j and $y_{i,j}^{k,\eta}$ depends only on i, while $y_{i,j}^{k,\xi+\eta}$ depends only on $i-j$ and $y_{i,j}^{k,\xi-\eta}$ depends only on $i+j$. When we want to underline the dependence of these families on a single index, we set

$$\begin{aligned} y_j^{k,\xi} = y_{0,j}^{k,\xi}, \quad y_i^{k,\eta} = y_{i,0}^{k,\eta}, \\ y_j^{k,\xi+\eta} = y_{j,0}^{k,\xi+\eta} = y_{0,-j}^{k,\xi+\eta}, \quad y_j^{k,\xi-\eta} = y_{j,0}^{k,\xi-\eta} = y_{0,j}^{k,\xi-\eta}, \end{aligned} \tag{5.2.5}$$

see Figure 5.2.

It is clear from these definitions that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $i, j \in \mathbb{Z}$ there exists a unique real number $t_{i,j}^{k,\zeta}$ such that

$$x_{i,j}^k = y_{i,j}^{k,\zeta} + t_{i,j}^{k,\zeta}\zeta. \tag{5.2.6}$$

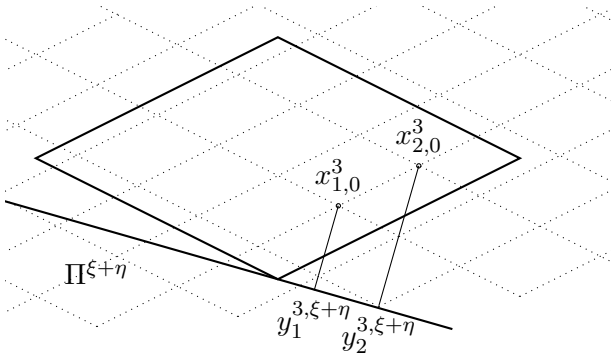


Figure 5.2: Projections of the points $x^k_{i,j}$ onto the straight line $\Pi^{\xi+\eta}$

Let $C_{\xi,\eta} \coloneqq \big(|\xi|^2|\eta|^2 - (\xi\cdot\eta)^2\big)^{1/2} > 0$. We observe that for $i,j\in\mathbb{Z}$ we have

$$\begin{aligned} k|y^{k,\xi}_{i,j+1} - y^{k,\xi}_{i,j}| &= |\pi^{\xi}(\eta)| = \frac{1}{|\xi|}C_{\xi,\eta},\\ k|y^{k,\eta}_{i+1,j} - y^{k,\eta}_{i,j}| &= |\pi^{\eta}(\xi)| = \frac{1}{|\eta|}C_{\xi,\eta}\\ k|y^{k,\xi+\eta}_{i,j} - y^{k,\xi+\eta}_{i,j+1}| = k|y^{k,\xi+\eta}_{i,j} - y^{k,\xi+\eta}_{i+1,j}| &= |\pi^{\xi+\eta}(\xi)| = |\pi^{\xi+\eta}(\eta)| = \frac{1}{|\xi+\eta|}C_{\xi,\eta},\\ k|y^{k,\xi-\eta}_{i,j} - y^{k,\xi-\eta}_{i,j+1}| = k|y^{k,\xi-\eta}_{i,j} - y^{k,\xi-\eta}_{i-1,j}| &= |\pi^{\xi-\eta}(\xi)| = |\pi^{\xi-\eta}(\eta)| = \frac{1}{|\xi-\eta|}C_{\xi,\eta}. \end{aligned} \tag{5.2.7}$$

For technical reasons, which will appear in Lemmas 5.3.2 and 5.D.1, it is convenient to replace the set $J_u = J^1_u$ by a set $J\subset U$ that can be written as countable union of compact sets. Since $\mathcal{H}^1(J^1_u) < +\infty$ by Remark 1.4.8, there exist a countable family of compact sets $K_n \subset J^1_u\cap U$ and two Borel sets $N_1\subset N\subset U$ with $\mathcal{H}^1(N_1) = \mathcal{H}^1(N) = 0$ such that

$$J^1_u\cap U = \Big(\bigcup_{n\in\mathbb{N}} K_n\Big)\cup N_1 \quad\text{and}\quad J_u\cap U = \Big(\bigcup_{n\in\mathbb{N}} K_n\Big)\cup N. \tag{5.2.8}$$

We set

$$J \coloneqq \bigcup_{n\in\mathbb{N}} K_n \tag{5.2.9}$$

and observe that for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ and for $\mathcal{H}^1$-a.e. $y\in\Pi^\zeta$ we have the equality $(J^1_u\cap U)^\zeta_y = (J_u\cap U)^\zeta_y = J^\zeta_y$. In particular, by (5.1.1) we have that

$$\sigma^\zeta_u(B) \coloneqq |\zeta|\int_{\Pi^\zeta} Du^\zeta_y((B\setminus J)^\zeta_y)\,\mathrm{d}\mathcal{H}^1(y) \tag{5.2.10}$$

for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ and every Borel set $B\subset U$.

The following two lemmas will be used in the choice of ω to obtain an approximation of $\sigma^\zeta_u(U)$ by means of suitable Riemann sums.

Lemma 5.2.2. *Let $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$. Then for $\mathcal{L}^2$-a.e. $\omega\in U$ the following conditions are simultaneously satisfied:*

(a) (*Properties of the slices*) *for every $k\in\mathbb{N}$ and $i,j\in\mathbb{Z}$ we have*

$$u^\zeta_{y^{k,\zeta}_{i,j}} \in \mathrm{BV}(U^\zeta_{y^{k,\zeta}_{i,j}}) \quad \text{and} \quad (J^1_u\cap U)^\zeta_{y^{k,\zeta}_{i,j}} = J^\zeta_{y^{k,\zeta}_{i,j}};$$

(b) (*The points $x^k_{i,j}$ are directional Lebesgue points*) *for every $k \in \mathbb{N}$ and $i,j \in \mathbb{Z}$, with $x^k_{i,j} \in U$, we have*

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |u(x^k_{i,j}+s\zeta)\cdot\zeta - u(x^k_{i,j})\cdot\zeta|\,\mathrm{d}s = 0. \tag{5.2.11}$$

Proof. Taking into account that $U \subset\subset \Omega$ and recalling the definition of $\mathrm{GBD}(\Omega)$ and Remark 1.4.4, from (5.2.8) and (5.2.9) it follows that there exists a Borel set $N_\zeta \subset \Pi^\zeta$, with $\mathcal{H}^1(N_\zeta) = 0$, such that for every $y \in \Pi^\zeta \setminus N_\zeta$ we have $u^\zeta_y \in \mathrm{BV}(U^\zeta_y)$ and $(J^1_u \cap U)^\zeta_y = J^\zeta_y$. Let $N^\infty_\zeta := \bigcup_{k\in\mathbb{N}}\bigcup_{(i,j)\in\mathbb{Z}^2}\big(N_\zeta - \frac{i}{k}\pi^\zeta(\xi) - \frac{j}{k}\pi^\zeta(\eta)\big)$. It is immediate to check that $\mathcal{H}^1(N^\infty_\zeta) = 0$. By (5.2.4) we have

$$y^{k,\zeta}_{i,j} = \pi^\zeta(\omega) + \tfrac{i}{k}\pi^\zeta(\xi) + \tfrac{j}{k}\pi^\zeta(\eta),$$

so that if $\pi^\zeta(\omega) \notin N^\infty_\zeta$, we have $u^\zeta_{y^{k,\zeta}_{i,j}} \in \mathrm{BV}(U^\zeta_{y^{k,\zeta}_{i,j}})$ and that $(J^1_u\cap U)^\zeta_{y^{k,\zeta}_{i,j}} = J^\zeta_{y^{k,\zeta}_{i,j}}$. This proves that for $\mathcal{L}^2$-a.e. $\omega \in U$ condition (a) holds.

Let us prove (b). Let B be the $\mathcal{L}^2$-measurable set defined by

$$B := \Big\{x \in U\colon\ \limsup_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|u(x+s\zeta)\cdot\zeta - u(x)\cdot\zeta|\,\mathrm{d}s > 0\Big\}.$$

For every $y \in \Pi^\zeta \setminus N_\zeta$ we have $u^\zeta_y \in \mathrm{BV}(U^\zeta_y)$ and the slices B^ζ_y satisfy

$$B^\zeta_y = \Big\{t \in U^\zeta_y\colon\ \limsup_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|u^\zeta_y(s+t) - u^\zeta_y(t)|\,\mathrm{d}s > 0\Big\}.$$

Therefore, by the Lebesgue Differentiation Theorem $\mathcal{L}^1(B^\zeta_y) = 0$ for every $y \in \Pi^\zeta \setminus N_\zeta$ and by the Fubini Theorem this implies that $\mathcal{L}^2(B) = 0$. We observe that

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|u^\zeta_y(t+s) - u^\zeta_y(t)|\,\mathrm{d}s = 0 \tag{5.2.12}$$

for every $t \in U^\zeta_y \setminus B^\zeta_y$.

Recalling that by (5.2.6) we have that

$$x^k_{i,j} = y^{k,\zeta}_{i,j} + t^{k,\zeta}_{i,j}\zeta, \tag{5.2.13}$$

and that $y^{k,\zeta}_{i,j} \notin N_\zeta$ by the first step, from (5.2.12) we deduce that (5.2.11) holds whenever

$$t^{k,\zeta}_{i,j} \notin B^\zeta_{y^{k,\zeta}_{i,j}}. \tag{5.2.14}$$

Thus, to prove (b) it is enough to show that, for given i,j, and ζ, condition (5.2.14) holds for $\mathcal{L}^2$-a.e. $\omega \in U$. Observing that $y^{k,\zeta}_{i,j}\cdot\zeta = 0$ and recalling (5.2.3), if we multiply (5.2.13) by $\zeta/|\zeta|^2$ we obtain that

$$t^{k,\zeta}_{i,j} = \tfrac{x^k_{i,j}\cdot\zeta}{|\zeta|^2} = \tfrac{\omega\cdot\zeta}{|\zeta|^2} + \tfrac{i}{k}\tfrac{\xi\cdot\zeta}{|\zeta|^2} + \tfrac{j}{k}\tfrac{\eta\cdot\zeta}{|\zeta|^2}.$$

Hence, (5.2.14) holds whenever

$$\tfrac{\omega\cdot\zeta}{|\zeta|^2} \notin B^\zeta_{y^{k,\zeta}_{i,j}} - \tfrac{i}{k}\tfrac{\xi\cdot\zeta}{|\zeta|^2} - \tfrac{j}{k}\tfrac{\eta\cdot\zeta}{|\zeta|^2}. \tag{5.2.15}$$

Recalling that by (5.2.4) $y^{k,\zeta}_{i,j}$ has the form $\pi^\zeta(\omega) + z$ for some $z \in \Pi^\zeta$, depending on k, ζ, i, and j, we deduce that (5.2.15) holds for $\mathcal{L}^2$-a.e. $\omega \in U$. This proves that (5.2.14) holds for $\mathcal{L}^2$-a.e. $\omega \in U$, concluding the proof. □

Proof of Theorem 5.2.1 (continuation). Given $i,j\in\mathbb{Z}$ and $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ we set

$$I^{k,\zeta}_{i,j}:=[t^{k,\zeta}_{i,j},t^{k,\zeta}_{i,j}+\tfrac{1}{k}), \tag{5.2.16}$$

where $t^{k,\zeta}_{i,j}$ are defined in (5.2.6). We note that

$$[x^k_{i,j},x^k_{i,j}+\tfrac{1}{k}\zeta)=\{y^{k,\zeta}_{i,j}+t\zeta\colon\ t\in I^{k,\zeta}_{i,j}\}. \tag{5.2.17}$$

For $h\in\mathbb{N}$ we set

$$\mathcal{J}^k_h:=\{(i,j)\in\mathbb{Z}^2\colon\ x^k_{i,j}\pm\tfrac{h}{k}\zeta\in U\text{ for every }\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}\}. \tag{5.2.18}$$

Since every $\omega\in\mathbb{R}^2$ can be written in a unique way as $\omega=s\xi+t\eta$ with $s,t\in\mathbb{R}$, by (5.2.2) and (5.2.3) we have

$$\mathcal{J}^k_h=\{(i,j)\in\mathbb{Z}^2\colon\ 0<s+\tfrac{i\pm h}{k}<\alpha\text{ and }0<t+\tfrac{j\pm h}{k}<\beta\}. \tag{5.2.19}$$

In the following lemma, given a sequence $\{\omega_k\}_k$ of elements of U, we consider the points $x^k_{i,j}$ and $y^{k,\zeta}_{i,j}$ defined by (5.2.3) and (5.2.4) with $\omega=\omega_k$. We recall that $C_{\xi,\eta}>0$ is the constant which appears in (5.2.7).

Lemma 5.2.3. *There exists an infinite set $K\subset\mathbb{N}$ and, for every $\varepsilon>0$, a Borel set $U_\varepsilon\subset U$, with $\mathcal{L}^2(U\setminus U_\varepsilon)\le\varepsilon$, such that for every sequence $\{\omega_k\}_{k\in\mathbb{N}}$ in U_ε and for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ conditions (a) and (b) of Lemma 5.2.2 are satisfied and*

$$\lim_{\substack{k\to+\infty\\ k\in K}}\frac{C_{\xi,\eta}}{k}\sum_{(i,j)\in\mathcal{J}^k}Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}\setminus J^\zeta_{y^{k,\zeta}_{i,j}})=\sigma^\zeta_u(U) \tag{5.2.20}$$

for every sequence $\{\mathcal{J}^k\}_k$ of subsets of $\mathbb{Z}^2$ for which there exists $h\in\mathbb{N}$ such that $\mathcal{J}^k_h\subset\mathcal{J}^k\subset\mathcal{J}^k_1$ for every $k\in\mathbb{N}$.

Proof. Thanks to Lemma 5.2.2, there exists a Borel set $U_0\subset U$, with $\mathcal{L}^2(U_0)=\mathcal{L}^2(U)$, such that (a) and (b) hold for every $\omega\in U_0$, so that we only need to show that there exist an infinite set $K\subset\mathbb{N}$ and for every $\varepsilon>0$, a Borel set $U_\varepsilon\subset U_0$, with $\mathcal{L}^2(U\setminus U_\varepsilon)\le\varepsilon$, such that (5.2.20) holds. The proof for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ will be carried out in four steps, first for $\zeta=\xi$, then for $\zeta=\eta$, next for $\zeta=\xi+\eta$, and finally for $\zeta=\xi-\eta$. Starting from the second step, $\mathbb{N}$ is replaced by the set K of the previous step and we may assume that U_ε is contained in the corresponding set of the previous step, so that the sets K and U_ε obtained at the end satisfy (5.2.20) for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$.

We begin by proving the result for $\zeta=\xi$. We observe that every $\omega\in\mathbb{R}^2$ can be written in a unique way as

$$\omega=z_1\eta+z_2\xi,$$

with $z_1,z_2\in\mathbb{R}$. For every $k\in\mathbb{N}$ and $i,j\in\mathbb{Z}$ by (5.2.4) we have

$$y^{k,\xi}_{i,j}=\big(z_1+\tfrac{j}{k}\big)\pi^\xi(\eta). \tag{5.2.21}$$

We set $I:=(0,\beta)=\{t\in\mathbb{R}:t\pi^\xi(\eta)\in\pi^\xi(U)\}$. For every $h,k\in\mathbb{N}$ and $z_1\in I$ we define

$$\mathcal{I}^k_h(z_1):=\big\{j\in\mathbb{Z}\colon\ z_1+\tfrac{j\pm h}{k}\in I\big\}=\big\{j\in\mathbb{Z}\colon\ y^{k,\xi}_{0,j}\pm\tfrac{h}{k}\pi^\xi(\eta)\in\pi^\xi(U)\big\}, \tag{5.2.22}$$

$$\mathcal{F}^k_h(z_1):=\big\{j\in\mathbb{Z}\colon\ z_1+\tfrac{j}{k}\in I\big\}\setminus\mathcal{I}^k_h(z_1). \tag{5.2.23}$$

Let $N_\xi \subset \Pi^\xi$ be the $\mathcal{H}^1$-negligible Borel set introduced at the beginning of the proof of Lemma 5.2.2 for $\zeta = \xi$ and consider the Borel set $M_\xi := \{t \in \mathbb{R} : t\pi^\xi(\eta) \in N_\xi\}$. Applying Lemma 5.A.1 to the function defined for $t \in \mathbb{R}$ by

$$f(t) := \begin{cases} Du^\xi_{t\pi^\xi(\eta)}((U \setminus J)^\xi_{t\pi^\xi(\eta)}) & \text{if } t \in \mathbb{R} \setminus M_\xi, \\ 0 & \text{if } t \in M_\xi, \end{cases}$$

which vanishes out of I, and recalling (5.2.7), (5.2.10), and (5.2.21), we obtain an infinite set $H \subset \mathbb{N}$ and, for every $\varepsilon > 0$, a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \le \varepsilon$ and $I_\varepsilon \cap M_\xi = \emptyset$, such that for every $h \in \mathbb{N}$ we have

$$\lim_{\substack{k\to+\infty \\ k\in H}} \frac{C_{\xi,\eta}}{k} \sum_{j\in\mathcal{I}^k_h(z_1)} Du^\xi_{y^{k,\xi}_{0,j}}((U \setminus J)^\xi_{y^{k,\xi}_{0,j}}) = |\xi||\pi^\xi(\eta)| \int_I Du^\xi_{s\pi^\xi(\eta)}((U \setminus J)^\xi_{s\pi^\xi(\eta)})\,\mathrm{d}s$$

$$= |\xi| \int_{\Pi^\xi} Du^\xi_y((U \setminus J)^\xi_y)\,\mathrm{d}\mathcal{H}^1(y) = \sigma^\xi_u(U) \qquad \text{uniformly for } z_1 \in I_\varepsilon, \tag{5.2.24}$$

$$\lim_{\substack{k\to+\infty \\ k\in H}} \frac{1}{k} \sum_{j\in\mathcal{F}^k_h(z_1)} Du^\xi_{y^{k,\xi}_{0,j}}((U \setminus J)^\xi_{y^{k,\xi}_{0,j}}) = 0 \qquad \text{uniformly for } z_1 \in I_\varepsilon. \tag{5.2.25}$$

We set

$$V_\varepsilon := \{\omega \in U\colon\ \omega = z_1\eta + z_2\xi,\ z_1 \in I_\varepsilon,\ z_2 \in \mathbb{R}\} \tag{5.2.26}$$

and observe that $\mathcal{L}^2(U \setminus V_\varepsilon) \le c\varepsilon$ for a constant $c > 0$ depending only on ξ, η, α, and β. For every $h, k \in \mathbb{N}$ we set

$$W^k_h := \{x \in U\colon\ \text{either } x + \tfrac{h}{k}\xi \text{ or } x - \tfrac{h}{k}\xi \text{ does not belong to } U\}, \tag{5.2.27}$$

$$f^k_h(t, z_2) := \begin{cases} |Du^\xi_{t\pi^\xi(\eta)}|((W^k_h \setminus J)^\xi_{t\pi^\xi(\eta)}) & \text{if } t \in I \setminus M_\xi \text{ and } z_2 \in (0, \alpha+\beta), \\ 0 & \text{otherwise,} \end{cases} \tag{5.2.28}$$

$$g(t) := \begin{cases} |Du^\xi_{t\pi^\xi(\eta)}|((U \setminus J)^\xi_{t\pi^\xi(\eta)}) & \text{for } t \in I \setminus M_\xi, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.29}$$

We observe that $0 \le f^k_h(t, z_2) \le g(t)$ for $\mathcal{L}^1$-a.e. $t \in I$ and $\mathcal{L}^1$-a.e. $z_2 \in \mathbb{R}$. Let λ be a measure as in Definition 1.4.3. Since $\lambda(W^k_h)$ converges to 0 as $k \to +\infty$, from (1.4.10) and (1.4.18) we deduce that the sequence $\{f^k_h\}_k$ converges to 0 in $L^1(I \times \mathbb{R})$ as $k \to +\infty$. Thanks to Lemma 5.A.2 and Remark 5.A.3, applied with $\mathbb{N}$ replaced by H, we can find an infinite set $K \subset H \subset \mathbb{N}$ and a Borel set $U_\varepsilon \subset V_\varepsilon \subset U$, with $\mathcal{L}^2(U \setminus U_\varepsilon) \le c\varepsilon$ for a constant $c > 0$ depending only on ξ, η, α, and β, such that for every $h \in \mathbb{N}$

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k} \sum_{j\in\mathcal{I}^k_h(z_1)} |Du^\xi_{y^{k,\xi}_{0,j}}|((W^k_{h+1} \setminus J)^\xi_{y^{k,\xi}_{0,j}}) = \lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k} \sum_{j\in\mathcal{I}^k_h(z_1)} f^k_{h+1}(z_1 + \tfrac{j}{k}, z_2) = 0 \tag{5.2.30}$$

uniformly for $\omega = z_1\eta + z_2\xi \in U_\varepsilon$, where the first equality follows from (5.2.21).

For every $j \in \mathbb{Z}$ let $\mathcal{J}^k(j) := \{i \in \mathbb{Z}\colon\ (i, j) \in \mathcal{J}^k\}$. Since $\mathcal{J}^k_h \subset \mathcal{J}^k \subset \mathcal{J}^k_1$, by (5.2.16) and (5.2.27) for every $j \in \mathcal{I}^k_h(z_1)$ it holds

$$(U \setminus J)^\xi_{y^{k,\xi}_{0,j}} = (W^k_{h+1} \setminus J)^\xi_{y^{k,\xi}_{0,j}} \cup \bigcup_{i\in\mathcal{J}^k(j)} \big(I^{k,\xi}_{i,j} \setminus J^\xi_{y^{k,\xi}_{0,j}}\big).$$

Hence,

$$\Big|Du^{\xi}_{y^{k,\xi}_{0,j}}\big((U\setminus J)^{\xi}_{y^{k,\xi}_{0,j}}\big)-\sum_{i\in\mathcal{J}^k(j)}Du^{\xi}_{y^{k,\xi}_{0,j}}\big(I^{k,\xi}_{i,j}\setminus J^{\xi}_{y^{k,\xi}_{0,j}}\big)\Big|\le\big|Du^{\xi}_{y^{k,\xi}_{0,j}}\big|\big((W^k_{h+1}\setminus J)^{\xi}_{y^{k,\xi}_{0,j}}\big).$$

Recalling that $y^{k,\xi}_{i,j}=y^{k,\xi}_{0,j}$, the previous inequality gives

$$\begin{aligned}&\Big|\sum_{j\in\mathcal{I}^k_h(z_1)}Du^{\xi}_{y^{k,\xi}_{0,j}}\big((U\setminus J)^{\xi}_{y^{k,\xi}_{0,j}}\big)-\sum_{(i,j)\in\mathcal{J}^k}Du^{\xi}_{y^{k,\xi}_{0,j}}\big(I^{k,\xi}_{i,j}\setminus J^{\xi}_{y^{k,\xi}_{0,j}}\big)\Big|\\&\le\sum_{j\in\mathcal{I}^k_h(z_1)}\big|Du^{\xi}_{y^{k,\xi}_{0,j}}\big|\big((W^k_{h+1}\setminus J)^{\xi}_{y^{k,\xi}_{0,j}}\big)+\sum_{j\in\mathcal{F}^k_h(z_1)}\big|Du^{\xi}_{y^{k,\xi}_{0,j}}\big|\big((U\setminus J)^{\xi}_{y^{k,\xi}_{0,j}}\big).\end{aligned}$$

Combining (5.2.24), (5.2.25), and (5.2.30), we obtain (5.2.20) for $\zeta=\xi$. The proof for the case $\zeta=\eta$ can be obtained by arguing as above, exchanging the roles of ξ and η.

In the case $\zeta=\xi+\eta$ we argue as follows. First, we write every $\omega\in\mathbb{R}^2$ as

$$\omega=z_1\eta+z_2(\xi+\eta),$$

with $z_1,z_2\in\mathbb{R}$, so that by (5.2.4)

$$y^{k,\xi+\eta}_{i,j}=(z_1+\tfrac{j}{k})\pi^{\xi+\eta}(\eta)+\tfrac{i}{k}\pi^{\xi+\eta}(\xi)=(z_1+\tfrac{j-i}{k})\pi^{\xi+\eta}(\eta).$$

Setting $m:=j-i$, we have

$$y^{k,\xi+\eta}_{i,j}=(z_1+\tfrac{m}{k})\pi^{\xi+\eta}(\eta).\tag{5.2.31}$$

We now set $I:=\{t\in\mathbb{R}\colon\ t\pi^{\xi+\eta}(\eta)\in\pi^{\xi+\eta}(U)\}$ and for every $k\in\mathbb{N}$ and $z_1\in I$ we define

$$\mathcal{I}^k(z_1):=\{m\in\mathbb{Z}\colon\ z_1+\tfrac{m}{k}\in I\}.\tag{5.2.32}$$

Let $N_{\xi+\eta}$ be the $\mathcal{H}^1$-negligible Borel set introduced at the beginning of the proof of Lemma 5.2.2 for $\zeta=\xi+\eta$ and consider the Borel set $M_{\xi+\eta}:=\{t\in\mathbb{R}\colon\ t\pi^{\xi+\eta}(\eta)\in N_{\xi+\eta}\}$. We can apply Lemma 5.A.1 to the function defined for $t\in\mathbb{R}$ by

$$h(t):=\begin{cases}Du^{\xi+\eta}_{t\pi^{\xi+\eta}(\eta)}\big((U\setminus J)^{\xi+\eta}_{t\pi^{\xi+\eta}(\eta)}\big) & \text{if } t\in\mathbb{R}\setminus M_{\xi+\eta},\\ 0 & \text{if } t\in M_{\xi+\eta}.\end{cases}$$

and arguing as in the previous part of the proof we obtain an infinite set H contained in the set K obtained in the previous steps and, for every $\varepsilon>0$, a Borel set $I_\varepsilon\subset I$, with $\mathcal{L}^1(I\setminus I_\varepsilon)\le\varepsilon$ and $I_\varepsilon\cap M_{\xi+\eta}=\emptyset$, such that

$$\lim_{\substack{k\to+\infty\\ k\in H}}\frac{C_{\xi,\eta}}{k}\sum_{m\in\mathcal{I}^k(z_1)}Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}\big((U\setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}\big)=\sigma^{\xi+\eta}_u(U)\qquad\text{uniformly for } z_1\in I_\varepsilon.\tag{5.2.33}$$

For every $h,k\in\mathbb{N}$ now define

$$W^k_h:=\{x\in U\colon\ \text{there exists } \zeta\in\{\pm\xi,\pm\eta,\pm(\xi+\eta),\pm(\xi-\eta)\}\text{ such that } x+\tfrac{h}{k}\zeta\notin U\}$$

and we observe that (5.2.19) and the inclusions $\mathcal{J}^k_h\subset\mathcal{J}^k\subset\mathcal{J}^k_1$ imply that

$$\{(i,j)\in\mathbb{Z}^2\colon\ x^k_{i,j}\in U\setminus W^k_h\}\subset\mathcal{J}^k,$$
$$\mathcal{J}^k\subset\{(i,j)\in\mathbb{Z}^2\colon\ [x^k_{i,j},x^k_{i,j}\pm\tfrac{1}{k}\zeta]\subset U\text{ for every }\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}\}.$$

It follows from (5.2.16) and (5.2.17) that

$$(U \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}} = (W^k_{h+1} \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}} \cup \bigcup_{\substack{(i,j)\in\mathcal{J}^k \\ j-i=m}} (I^{k,\xi+\eta}_{i,j} \setminus J^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}). \tag{5.2.34}$$

For every $k \in \mathbb{N}$ we now define V_ε, f^k_h, and g as in (5.2.26), (5.2.28), and (5.2.29), with ξ replaced by $\xi+\eta$. Arguing as in the first part of the proof, we obtain that $\{f^k_h\}_k$ converges to 0 in $L^1(I\times\mathbb{R})$ as $k\to+\infty$. Hence, recalling (5.2.19), we may apply Lemma 5.A.2, with $\mathbb{N}$ replaced by H, to obtain an infinite set $K \subset H \subset \mathbb{N}$ and a Borel set $U_\varepsilon \subset V_\varepsilon \subset U$, with $\mathcal{L}^2(U \setminus U_\varepsilon) \le c\varepsilon$ for a constant $c>0$ depending only on ξ, η, α, and β, such that

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k} \sum_{m\in\mathcal{I}^k(z_1)} |Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}|((W^k_{h+1} \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}) = \lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k} \sum_{m\in\mathcal{I}^k(z_1)} f^k_{h+1}(z_1 + \tfrac{m}{k}, z_2) = 0 \tag{5.2.35}$$

uniformly for $\omega = z_1\eta + z_2(\xi+\eta) \in U_\varepsilon$. Recalling the equality $y^{k,\xi+\eta}_{i,j} = y^{k,\xi+\eta}_{0,m}$ for $j-i=m$, from (5.2.34) it follows that

$$|Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}((U \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}) - \sum_{\substack{(i,j)\in\mathcal{J}^k \\ j-i=m}} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}(I^{k,\xi+\eta}_{i,j} \setminus J^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}})| \le |Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}|((W^k_{h+1} \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}).$$

Since by (5.2.31) and (5.2.32) we have

$$\sum_{m\in\mathcal{I}^k(z_1)} \sum_{\substack{(i,j)\in\mathcal{J}^k \\ j-i=m}} = \sum_{(i,j)\in\mathcal{J}^k},$$

combining (5.2.33) and (5.2.35) with the previous inequality, we obtain (5.2.20) for $\zeta = \xi+\eta$. The proof for $\zeta = \xi-\eta$ is similar. □

Proof of Theorem 5.2.1 (continuation). By Lemma 5.2.3 we may choose a sequence $\{\omega_k\}_k \subset U$ and an infinite set $K \subset \mathbb{N}$ such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ conditions (a) and (b) of Lemma 5.2.2 hold and

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{J}^k} Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j} \setminus J^\zeta_{y^{k,\zeta}_{i,j}}) = \sigma^\zeta_u(U),$$

for every $\mathcal{J}^k$ such that $\mathcal{J}^k_3 \subset \mathcal{J}^k \subset \mathcal{J}^k_1$ (see (5.2.18)), where the projections $y^{k,\zeta}_{i,j}$ are defined taking $\omega = \omega_k$ in (5.2.4).

To present the technique we will employ in the sequel, let us assume for a moment that for every $(i,j) \in \mathcal{J}^k_1$ the segments $[x^k_{i,j}, x^k_{i+1,j}]$, $[x^k_{i,j}, x^k_{i,j+1}]$, $[x^k_{i,j}, x^k_{i+1,j+1}]$, $[x^k_{i+1,j}, x^k_{i+1,j+1}]$, $[x^k_{i,j+1}, x^k_{i+1,j+1}]$, and $[x^k_{i+1,j}, x^k_{i,j+1}]$ do not intersect the set J, which implies $I^{k,\zeta}_{i,j} \setminus J^\zeta_{y^{k,\zeta}_{i,j}} = I^{k,\zeta}_{i,j}$, hence

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{J}^k} Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = \sigma^\zeta_u(U), \tag{5.2.36}$$

whenever $\mathcal{J}^k_3 \subset \mathcal{J}^k \subset \mathcal{J}^k_1$. By (a) and (b) of Lemma 5.2.2 and (5.2.17), for every $k \in \mathbb{N}$ and $(i,j) \in \mathcal{J}^k_1$ we have that $u^\zeta_{y^{k,\zeta}_{i,j}} \in \mathrm{BV}(U^\zeta_{y^{k,\zeta}_{i,j}})$ and

$$Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = (u(x^k_{i,j} + \tfrac{1}{k}\zeta) - u(x^k_{i,j})) \cdot \zeta \quad \text{for every } \zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}. \tag{5.2.37}$$

Let $e_1 := (1,0)$ and $e_2 := (0,1)$ and set

$$\mathcal{J}_2^k + e_1 := \{(i+1,j)\colon\ (i,j) \in \mathcal{J}_2^k\} \quad \text{and} \quad \mathcal{J}_2^k + e_2 := \{(i,j+1)\colon\ (i,j) \in \mathcal{J}_2^k\},$$

and observe that by (5.2.19) we have $\mathcal{J}_3^k \subset \mathcal{J}_2^k + e_1 \subset \mathcal{J}_1^k$ and $\mathcal{J}_3^k \subset \mathcal{J}_2^k + e_2 \subset \mathcal{J}_1^k$. From (5.2.3) and (5.2.37) it follows that

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{J}_2^k} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}_2^k} \big(u(x^k_{i+1,j}) - u(x^k_{i,j})\big)\cdot\xi,\\
\sum_{(i,j)\in\mathcal{J}_2^k+e_2} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}_2^k+e_2} \big(u(x^k_{i+1,j}) - u(x^k_{i,j})\big)\cdot\xi,\\
\sum_{(i,j)\in\mathcal{J}_2^k} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}_2^k} \big(u(x^k_{i,j+1}) - u(x^k_{i,j})\big)\cdot\eta,\\
\sum_{(i,j)\in\mathcal{J}_2^k+e_1} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}_2^k+e_1} \big(u(x^k_{i,j+1}) - u(x^k_{i,j})\big)\cdot\eta,\\
\sum_{(i,j)\in\mathcal{J}_2^k} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{i,j}}(I^{k,\xi+\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}_2^k} \big(u(x^k_{i+1,j+1}) - u(x^k_{i,j})\big)\cdot(\xi+\eta),\\
\sum_{(i,j)\in\mathcal{J}_2^k+e_2} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}_2^k} \big(u(x^k_{i+1,j}) - u(x^k_{i,j+1})\big)\cdot(\xi-\eta).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{J}_2^k} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{i,j}}(I^{k,\xi+\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{J}_2^k+e_2} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j})\\
&= \sum_{(i,j)\in\mathcal{J}_2^k} \big(u(x^k_{i+1,j+1}) - u(x^k_{i,j})\big)\cdot(\xi+\eta) + \sum_{(i,j)\in\mathcal{J}_2^k} \big(u(x^k_{i+1,j}) - u(x^k_{i,j+1})\big)\cdot(\xi-\eta)\\
&= \sum_{(i,j)\in\mathcal{J}_2^k} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) + \sum_{(i,j)\in\mathcal{J}_2^k+e_2} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j})\\
&\quad + \sum_{(i,j)\in\mathcal{J}_2^k} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{J}_2^k+e_1} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}).
\end{aligned} \tag{5.2.38}$$

Thanks to (5.2.36) we obtain

$$\sigma_u^{\xi+\eta}(U) + \sigma_u^{\xi-\eta}(U) = 2\sigma_u^{\xi}(U) + 2\sigma_u^{\eta}(U),$$

which implies that $\xi \mapsto \sigma_u^{\xi}(U)$ is quadratic.

Unfortunately, the hypothesis that for every $k \in \mathbb{N}$ and $(i,j) \in \mathcal{J}_1^k$ each of the six segments $[x^k_{i,j}, x^k_{i+1,j}]$, $[x^k_{i,j}, x^k_{i,j+1}]$, $[x^k_{i,j}, x^k_{i+1,j+1}]$, $[x^k_{i+1,j}, x^k_{i+1,j+1}]$, $[x^k_{i,j+1}, x^k_{i+1,j+1}]$, and $[x^k_{i+1,j}, x^k_{i,j+1}]$ does not intersect the set J is almost never satisfied. Therefore, for every $k \in \mathbb{N}$ we introduce the set $\mathcal{G}^k \subset \mathbb{Z}^2$ of good indices, defined as

$$\begin{aligned}
\mathcal{G}^k := \big\{(i,j) \in \mathcal{J}_2^k\colon\ &\text{none of the segments } [x^k_{i,j}, x^k_{i+1,j}],\ [x^k_{i,j}, x^k_{i,j+1}],\ [x^k_{i,j}, x^k_{i+1,j+1}],\\
&[x^k_{i+1,j}, x^k_{i+1,j+1}],\ [x^k_{i,j+1}, x^k_{i+1,j+1}],\ [x^k_{i+1,j}, x^k_{i,j+1}] \text{ intersects } J\big\},
\end{aligned} \tag{5.2.39}$$

Note that by (5.2.17) we have

$$\begin{aligned} I^{k,\zeta}_{i,j} \cap J^{\zeta}_{y^{k,\zeta}_{i,j}} = \emptyset \quad &\text{for every } (i,j) \in \mathcal{G}^k, \quad \zeta \in \{\xi, \eta, \xi+\eta\}, \\ I^{k,\xi}_{i,j} \cap J^{\xi}_{y^{k,\xi}_{i,j}} = \emptyset \quad &\text{for every } (i,j) \in \mathcal{G}^k + e_2, \\ I^{k,\eta}_{i,j} \cap J^{\eta}_{y^{k,\eta}_{i,j}} = \emptyset \quad &\text{for every } (i,j) \in \mathcal{G}^k + e_1, \\ I^{k,\xi-\eta}_{i,j} \cap J^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}} = \emptyset \quad &\text{for every } (i,j) \in \mathcal{G}^k + e_2. \end{aligned} \tag{5.2.40}$$

where

$$\mathcal{G}^k + e_1 := \{(i+1, j) \in \mathbb{Z}^2 \colon\ (i,j) \in \mathcal{G}^k\} \quad \text{and} \quad \mathcal{G}^k + e_2 := \{(i, j+1) \colon\ (i,j) \in \mathcal{G}^k\}.$$

To prove the result in the general case, in the next section (see Theorem 5.3.1) we shall show that the sequence $\{\omega_k\}_k \subset U$ and the infinite set $K \subset \mathbb{N}$ can be chosen in such a way that conditions (a) and (b) of Lemma 5.2.2 hold and, in addition, for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$,

$$\begin{aligned} \lim_{\substack{k\to+\infty \\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k} Du^{\zeta}_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = \sigma^{\zeta}_u(U), \quad \zeta \in \{\xi, \eta, \xi+\eta\}, \\ \lim_{\substack{k\to+\infty \\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k+e_2} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) = \sigma^{\xi}_u(U), \\ \lim_{\substack{k\to+\infty \\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k+e_1} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) = \sigma^{\eta}_u(U), \\ \lim_{\substack{k\to+\infty \\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k+e_2} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j}) = \sigma^{\xi-\eta}_u(U). \end{aligned} \tag{5.2.41}$$

Assuming that these equalities hold, we now conclude the proof in the general case. Observing that (5.2.37) still holds for $(i,j) \in \mathcal{G}^k$, and also for $(i,j) \in \mathcal{G}^k + e_1$ when $\zeta = \eta$ and for $(i,j) \in \mathcal{G}^k + e_2$ when $\zeta \in \{\xi, \xi-\eta\}$, repeating the arguments that led to (5.2.38) we obtain

$$\begin{aligned} &\sum_{(i,j)\in\mathcal{G}^k} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{i,j}}(I^{k,\xi+\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{G}^k+e_2} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j}) \\ &\quad = \sum_{(i,j)\in\mathcal{G}^k} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) + \sum_{(i,j)\in\mathcal{G}^k+e_2} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) \\ &\quad + \sum_{(i,j)\in\mathcal{G}^k+e_1} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{G}^k} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}). \end{aligned}$$

Multiplying the previous equality by $C_{\xi,\eta}/k$ and using (5.2.41) we obtain (5.2.1). This concludes the proof. □

5.3 Conclusion of the proof in dimension d = 2

In this section we prove a technical result, which concludes the proof of Theorem 5.2.1. Throughout this section u, ξ, η, and U are as in Section 5.2 and we use the notation introduced in the proof of Theorem 5.2.1. In particular, we recall that $\mathcal{G}^k$ is defined by

(5.2.39). Before stating the main result of this section we introduce the set of bad indices $\mathcal{B}^k \subset \mathbb{Z}^2$, defined as

$$\mathcal{B}^k := \big\{(i,j) \in \mathcal{J}_2^k\colon \text{ one of the segments } [x_{i,j}^k, x_{i+1,j}^k],\, [x_{i,j}^k, x_{i,j+1}^k]\,, [x_{i,j}^k, x_{i+1,j+1}^k],\\ [x_{i+1,j}^k, x_{i+1,j+1}^k],\, [x_{i,j+1}^k, x_{i+1,j+1}^k],\, [x_{i+1,j}^k, x_{i,j+1}^k] \text{ intersects } J\big\}, \tag{5.3.1}$$

and the set

$$\mathcal{P} := \{(\xi,0), (\eta,0), (\xi+\eta,0), (\xi,e_2), (\eta,e_1), (\xi-\eta,e_2)\}.$$

Theorem 5.3.1. *There exist an infinite set $K \subset \mathbb{N}$ and a sequence $\{\omega_k\}_k \subset U$ such that conditions (a), (b) of Lemma 5.2.2 and* (5.2.20) *hold and, for every $(\zeta, a) \in \mathcal{P}$, we have*

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k+a} Du^{\zeta}_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = \sigma^{\zeta}_u(U), \tag{5.3.2}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{(i,j)\in\mathcal{B}^k+a} |Du^{\zeta}_{y^{k,\zeta}_{i,j}}|(I^{k,\zeta}_{i,j} \setminus J^{\zeta}_{y^{k,\zeta}_{i,j}}) = 0. \tag{5.3.3}$$

where the points $y^{k,\zeta}_{i,j}$ introduced in (5.2.4) *are defined by taking $\omega = \omega_k$.*

The crucial part in the proof of this result is proving (5.3.3), as (5.3.2) can then be obtained from (5.2.20) by difference, using (5.2.40). This proof is rather technical. The arguments we are going to use require some additional notation.

Given $\zeta \in \{\xi, \eta, \xi-\eta, \xi+\eta\}$, we set

$$\bar\zeta := \xi \ \text{ if } \ \zeta \in \{\eta, \xi+\eta\} \qquad \text{and} \qquad \bar\zeta := \eta \ \text{ if } \ \zeta \in \{\xi, \xi-\eta\}. \tag{5.3.4}$$

We observe that ζ and $\bar\zeta$ are linearly independent. For $y \in \mathbb{R}^2$, $k \in \mathbb{N}$, $j \in \mathbb{Z}$, and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ let $t_j^{k,\zeta}(y)$ be the real number characterised by

$$y + t_j^{k,\zeta}(y)\zeta \in \{\omega + \tfrac{j}{k}\zeta + s\bar\zeta\colon\ s \in \mathbb{R}\}. \tag{5.3.5}$$

We also set

$$x_j^{k,\zeta}(y) := y + t_j^{k,\zeta}(y)\zeta. \tag{5.3.6}$$

In other words, $x_j^{k,\zeta}(y)$ is the intersection of the straight lines $\{y + t\zeta\colon\ t \in \mathbb{R}\}$ and $\{\omega + \frac{j}{k}\zeta + s\bar\zeta\colon\ s \in \mathbb{R}\}$. Note that the family of straight lines $\{\{\omega + \frac{j}{k}\zeta + s\bar\zeta\colon\ s \in \mathbb{R}\}\}_{j\in\mathbb{Z}}$ coincides with the family of the straight lines parallel to $\bar\zeta$ passing through one of the points $x_{i,j}^k$ for $i, j \in \mathbb{Z}$.

Note that for every $j \in \mathbb{Z}$ and $t \in \mathbb{R}$ we have

$$t_{j+1}^{k,\zeta}(y) = t_j^{k,\zeta}(y) + \tfrac{1}{k} \quad \text{and} \quad t_j^{k,\zeta}(y + t\zeta) = t_j^{k,\zeta}(y) - t,$$

which give

$$x_{j+1}^{k,\zeta}(y) = x_j^{k,\zeta}(y) + \tfrac{1}{k}\zeta \quad \text{and} \quad x_j^{k,\zeta}(y + t\zeta) = x_j^{k,\zeta}(y). \tag{5.3.7}$$

Moreover,

$$[x_j^{k,\zeta}(y), x_{j+1}^{k,\zeta}(y)) = \{y + t\zeta\colon\ t \in [t_j^{k,\zeta}(y), t_{j+1}^{k,\zeta}(y))\}. \tag{5.3.8}$$

Therefore for every $y \in \mathbb{R}^2$ each straight line $\{y + t\zeta\colon\ t \in \mathbb{R}\}$ can be written as disjoint union of segments in the following way

$$\{y + t\zeta : t \in \mathbb{R}\} = \bigcup_{j\in\mathbb{Z}} [x_j^{k,\zeta}(y), x_{j+1}^{k,\zeta}(y)). \tag{5.3.9}$$

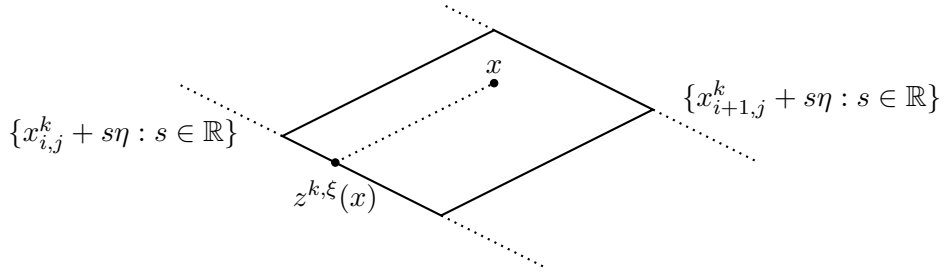


Figure 5.3: The map $x \mapsto z^{k,\xi}(x)$.

We need to introduce some sets which are useful to establish (5.3.3) and whose definition requires some additional notation. Let us fix $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $k \in \mathbb{N}$. In view of (5.3.9), for every $x \in \mathbb{R}^2$ there exists a unique $j \in \mathbb{Z}$ such that

$$x \in [x^{k,\zeta}_j(x), x^{k,\zeta}_{j+1}(x)). \tag{5.3.10}$$

We define the map $z^{k,\zeta}\colon \mathbb{R}^2 \to \mathbb{R}^2$ (see Figure 5.3) as

$$z^{k,\zeta}(x) = x^{k,\zeta}_j(x), \tag{5.3.11}$$

where $j \in \mathbb{Z}$ is the unique index such that (5.3.10) holds.

By (5.3.7) and (5.3.8) for every $y \in \mathbb{R}^2$

$$z^{k,\zeta}(y + t\zeta) = x^{k,\zeta}_j(y) \quad \text{for every } t \in [t^{k,\zeta}_j(y), t^{k,\zeta}_{j+1}(y)). \tag{5.3.12}$$

Geometrically (see Figure 5.3), $z^{k,\zeta}(x)$ is given by $x - t\zeta$ where $t \geq 0$ is the smallest number such that $x - t\zeta$ belongs to one of the straight lines parallel to $\bar{\zeta}$ passing through one of the points $x^k_{i,j}$. By this geometric characterisation, $z^{k,\zeta}$ is a Borel function.

We consider the union S of the sides and the diagonals of the parallelogram of vertices at 0, $\xi, \xi+\eta, \eta$, i.e.,

$$S := [0,\xi] \cup [0,\eta] \cup [0,\xi+\eta] \cup [\eta,\xi] \cup [\eta,\xi+\eta] \cup [\xi,\xi+\eta]. \tag{5.3.13}$$

For $k \in \mathbb{N}$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we introduce the set

$$E^{k,\zeta} := \{x \in \mathbb{R}^2 \colon (z^{k,\zeta}(x) + \tfrac{1}{k}S) \cap J \neq \emptyset\}. \tag{5.3.14}$$

For $y \in \mathbb{R}^2$ we define

$$\begin{gathered} \mathcal{I}^{k,\zeta}(y) := \{i \in \mathbb{Z} \colon (x^{k,\zeta}_i(y) + \tfrac{1}{k}S) \cap J \neq \emptyset\}, \quad \mathcal{N}^{k,\zeta}(y) := \mathcal{H}^0(\mathcal{I}^{k,\zeta}(y)), \\ E^{k,\zeta}(y) := \bigcup_{i \in \mathcal{I}^{k,\zeta}(y)} [t^{k,\zeta}_i(y), t^{k,\zeta}_{i+1}(y)). \end{gathered} \tag{5.3.15}$$

By (5.3.12) we have the equality

$$E^{k,\zeta}(y) = (E^{k,\zeta})^\zeta_y \quad \text{for every } y \in \mathbb{R}^2 \tag{5.3.16}$$

and by (5.3.7) we have

$$\mathcal{I}^{k,\zeta}(y + t\zeta) = \mathcal{I}^{k,\zeta}(y) \quad \text{and} \quad \mathcal{N}^{k,\zeta}(y + t\zeta) = \mathcal{N}^{k,\zeta}(y) \quad \text{for every } t \in \mathbb{R}. \tag{5.3.17}$$

Let $x \in \mathbb{R}^2$ and let $j \in \mathbb{Z}$ be the unique index such that (5.3.10) holds. By (5.3.7) for every $i \in \mathbb{Z}$ we have

$$x_i^{k,\zeta}(x) = x_i^{k,\zeta}(x + \tfrac{i-j}{k}\zeta) = z^{k,\zeta}(x + \tfrac{i-j}{k}\zeta),$$

where the last equality follows from (5.3.11), since $x + \frac{i-j}{k}\zeta \in [x_i^{k,\zeta}(x + \frac{i-j}{k}\zeta), x_{i+1}^{k,\zeta}(x + \frac{i-j}{k}\zeta))$ by (5.3.7). Recalling the definition of $E^{k,\zeta}$ in (5.3.14), the equalities above imply that

$$\mathcal{I}^{k,\zeta}(x) = \{i \in \mathbb{Z}\colon\ (z^{k,\zeta}(x + \tfrac{i-j}{k}\zeta) + \tfrac{1}{k}S) \cap J \neq \emptyset\} = \{i \in \mathbb{Z}\colon\ x + \tfrac{i-j}{k}\zeta \in E^{k,\zeta}\},$$

which gives

$$\mathcal{N}^{k,\zeta}(x) = \mathcal{H}^0(\{i \in \mathbb{Z} : x + \tfrac{i}{k}\zeta \in E^{k,\zeta}\}). \tag{5.3.18}$$

For $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, $k, m \in \mathbb{N}$, and $y \in \mathbb{R}^2$ we set

$$\widehat{E}_m^{k,\zeta} \coloneqq \{x \in E^{k,\zeta}\colon\ \mathcal{N}^{k,\zeta}(x) \le m\}, \quad \check{E}_m^{k,\zeta} \coloneqq \{x \in E^{k,\zeta}\colon\ \mathcal{N}^{k,\zeta}(x) > m\}, \tag{5.3.19}$$

$$\widehat{E}_m^{k,\zeta}(y) \coloneqq \begin{cases} E^{k,\zeta}(y) & \text{if } \mathcal{N}^{k,\zeta}(y) \le m, \\ \emptyset & \text{otherwise,} \end{cases} \quad \check{E}_m^{k,\zeta}(y) \coloneqq \begin{cases} E^{k,\zeta}(y) & \text{if } \mathcal{N}^{k,\zeta}(y) > m, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.3.20}$$

By (5.3.16) and (5.3.17) we have

$$\widehat{E}_m^{k,\zeta}(y) = (\widehat{E}_m^{k,\zeta})_y^\zeta \quad \text{and} \quad \check{E}_m^{k,\zeta}(y) = (\check{E}_m^{k,\zeta})_y^\zeta. \tag{5.3.21}$$

All these sets are Borel measurable as the following lemma shows.

Lemma 5.3.2. *The sets $E^{k,\zeta}$, $\widehat{E}_m^{k,\zeta}$, and $\check{E}_m^{k,\zeta}$ are Borel measurable. Moreover, the function $\mathcal{N}^{k,\zeta}$ is Borel measurable on $\mathbb{R}^2$.*

Proof. For every set $B \subset \mathbb{R}^2$ we define

$$E_B^{k,\zeta} \coloneqq \{x \in \mathbb{R}^2\colon\ (z^{k,\zeta}(x) + \tfrac{1}{k}S) \cap B \neq \emptyset\},$$
$$F_B \coloneqq \{z \in \mathbb{R}^2\colon\ (z + \tfrac{1}{k}S) \cap B \neq \emptyset\}.$$

We begin by proving that for a compact set $K \subset \mathbb{R}^2$ the set $E_K^{k,\zeta}$ is Borel measurable. To this aim we note that the set F_K is closed and that $E_K^{k,\zeta} = \{x \in \mathbb{R}^2\colon\ z^{k,\zeta}(x) \in F_K\}$. Recalling that $z^{k,\zeta}$ is Borel measurable, we conclude that $E_K^{k,\zeta}$ is Borel measurable. By (5.2.9) we have $J = \bigcup_{n\in\mathbb{N}} K_n$, where K_n are compact sets. This gives that $E^{k,\zeta} = \bigcup_{n\in\mathbb{N}} E_{K_n}^{k,\zeta}$. Since the sets $E_{K_n}^{k,\zeta}$ are Borel measurable, so is $E^{k,\zeta}$.

To prove that $\widehat{E}_m^{k,\zeta}$ Borel measurable, we observe that by (5.3.18) and (5.3.19) a point x belongs to $\widehat{E}_m^{k,\zeta}$ if and only if the number of indices $i \in \mathbb{Z}$ such that $x \in E^{k,\zeta} - \frac{i}{k}\zeta$ is less than or equal to m. This implies that, setting $E_i^{k,\zeta} \coloneqq E^{k,\zeta} - \frac{i}{k}\zeta$, we have

$$\widehat{E}_m^{k,\zeta} = E^{k,\zeta} \cap \big\{x \in \mathbb{R}^2\colon\ \textstyle\sum_i \chi_{E_i^{k,\zeta}}(x) \le m\big\},$$

where $\chi_{E_i^{k,\zeta}}$ is the characteristic function of $E_i^{k,\zeta}$. Since the sets $E_i^{k,\zeta}$ are Borel measurable, we deduce that $\widehat{E}_m^{k,\zeta}$ is Borel measurable. The Borel measurability of $\check{E}_m^{k,\zeta}$ follows from the equality $\check{E}_m^{k,\zeta} = E^{k,\zeta} \setminus \widehat{E}_m^{k,\zeta}$.

To prove that the function $\mathcal{N}^{k,\zeta}$ is Borel measurable, it is enough to observe that by (5.3.18) we have $\mathcal{N}^{k,\zeta} = \sum_{i\in\mathbb{Z}} \chi_{E_i^{k,\zeta}}$. □

Remark 5.3.3. All the sets defined in (5.3.14) and (5.3.19) depend non-trivially on $\omega \in \mathbb{R}^2$, since by (5.2.3) every choice of ω determines different points $x^k_{i,j}$ and thus, by (5.3.6), different sets in (5.3.14) and (5.3.19) as well. Not to overburden the notation, we do not indicate the dependence of such objects on ω. The measurability issues with respect to ω will be dealt with in the Appendix.

To use the sets $\widehat{E}^{k,\zeta}_m$ and $\check{E}^{k,\zeta}_m$ in our estimates we need the properties proved in the following two lemmas, whose proofs are postponed. We observe that, since ξ and η are linearly independent, given $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, every point $\omega \in \mathbb{R}^2$ can be written in a unique way as

$$\omega = z_1\bar{\zeta} + z_2\zeta, \tag{5.3.22}$$

for suitable $z_1, z_2 \in \mathbb{R}$. We set

$$I^\zeta := \begin{cases} (0,\alpha) & \text{if } \zeta = \xi \text{ or } \zeta = \xi - \eta, \\ (0,\beta) & \text{if } \zeta = \eta \text{ or } \zeta = \xi + \eta, \end{cases} \tag{5.3.23}$$

and we observe that by (5.2.2)

$$\pi^{\bar{\zeta}}(U) = \{z_2\pi^{\bar{\zeta}}(\zeta)\colon\ z_2 \in I^\zeta\}. \tag{5.3.24}$$

Lemma 5.3.4. *There exists a constant $C > 0$ such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and for every $\varepsilon > 0$ there exist an infinite set $K^\zeta_\varepsilon \subset \mathbb{N}$ and a Borel set $I^\zeta_\varepsilon \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I^\zeta_\varepsilon) \le \varepsilon$, such that*

$$\mathcal{H}^1\big(\pi^\zeta(\check{E}^{k,\zeta}_m)\big) \le C/m \tag{5.3.25}$$

for every $m \in \mathbb{N}$, $\omega \in U^\zeta_\varepsilon := U \cap \{z_1\bar{\zeta} + z_2\zeta\colon\ z_1 \in \mathbb{R},\ z_2 \in I^\zeta_\varepsilon\}$, and $k \in K^\zeta_\varepsilon$.

Lemma 5.3.5. *Let $m \in \mathbb{N}$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$. Then we have*

$$\lim_{k\to+\infty} |Du^\zeta_y|(\widehat{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y) = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } y \in \Pi^\zeta, \tag{5.3.26}$$

$$\lim_{k\to+\infty} \int_{\Pi^\zeta} |Du^\zeta_y|(\widehat{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y)\,\mathrm{d}\mathcal{H}^1(y) = 0. \tag{5.3.27}$$

To prove Lemma 5.3.4 we need the following elementary result.

Lemma 5.3.6. *Let $F \subset \mathbb{R}$ be a finite set and let $a > 0$ and $b < c$. Then*

$$\int_{\mathbb{R}} \mathcal{H}^0([at+b, at+c] \cap F)\,\mathrm{d}t = \frac{c-b}{a}\mathcal{H}^0(F).$$

Proof. By the Fubini Theorem we have that

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{H}^0([at+b, at+c] \cap F)\,\mathrm{d}t &= \int_{\mathbb{R}} \Big(\int_F \chi_{[at+b,at+c]}(s)\,\mathrm{d}\mathcal{H}^0(s)\Big)\mathrm{d}t \\ &= \int_F \Big(\int_{\mathbb{R}} \chi_{[\frac{s-c}{a}, \frac{s-b}{a}]}(t)\,\mathrm{d}t\Big)\mathrm{d}\mathcal{H}^0(s) = \frac{c-b}{a}\mathcal{H}^0(F). \end{aligned}$$

This concludes the proof. □

Proof of Lemma 5.3.4. We observe that $\pi^\zeta(\check{E}^{k,\zeta}_m) = \{y \in \Pi^\zeta \colon \mathcal{N}^{k,\zeta}(y) > m\}$ by (5.3.17) and (5.3.19). Since $\mathcal{N}^{k,\zeta}$ is Borel measurable by Lemma 5.3.2, $\pi^\zeta(\check{E}^{k,\zeta}_m)$ is a Borel set. By Čebyšëv's inequality we obtain

$$\mathcal{H}^1\big(\pi^\zeta(\check{E}^{k,\zeta}_m)\big) = \mathcal{H}^1\big(\{y \in \Pi^\zeta \colon \mathcal{N}^{k,\zeta}(y) > m\}\big) \le \frac{1}{m}\int_{\Pi^\zeta} \mathcal{N}^{k,\zeta}(y)\,\mathrm{d}\mathcal{H}^1(y) \tag{5.3.28}$$

for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, $k, m \in \mathbb{N}$, $\omega \in U$. To conclude the proof it is enough to show that there exist a constant $C > 0$ and, for every $\varepsilon > 0$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, an infinite set $K^\zeta_\varepsilon \subset \mathbb{N}$ and a Borel set $I^\zeta_\varepsilon \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I^\zeta_\varepsilon) \le \varepsilon$, such that

$$\int_{\Pi^\zeta} \mathcal{N}^{k,\zeta}(y)\,\mathrm{d}\mathcal{H}^1(y) \le C \tag{5.3.29}$$

for every $\omega \in U^\zeta_\varepsilon := U \cap \{z_1\bar{\zeta} + z_2\zeta \colon z_1 \in \mathbb{R},\, z_2 \in I^\zeta_\varepsilon\}$ and $k \in K^\zeta_\varepsilon$.

We observe that by (5.3.13) we can write $S = S_1 \cup \cdots \cup S_6$, where

$$S_1 := [0, \bar{\zeta}], \qquad S_2 := \begin{cases} S_1 + \xi & \text{if} \quad \zeta \in \{\xi, \xi-\eta\}, \\ S_1 + \eta & \text{if} \quad \zeta \in \{\eta, \xi+\eta\}, \end{cases} \tag{5.3.30}$$

while $S_3, \dots, S_6$ are the other four segments of S, which are transversal to $\bar{\zeta}$. By (5.3.15) this implies that

$$\mathcal{I}^{k,\zeta}(y) = \bigcup_{h=1}^{6} \{j \in \mathbb{Z} \colon (x^{k,\zeta}_j(y) + \tfrac{1}{k}S_h) \cap J \ne \emptyset\}$$

for every $y \in \Pi^\zeta$, hence

$$\mathcal{N}^{k,\zeta}(y) \le \sum_{h=1}^{6} \mathcal{H}^0(\{j \in \mathbb{Z} \colon (x^{k,\zeta}_j(y) + \tfrac{1}{k}S_h) \cap J \ne \emptyset\}).$$

Thus,

$$\int_{\Pi^\zeta} \mathcal{N}^{k,\zeta}(y)\,\mathrm{d}\mathcal{H}^1(y) \le \sum_{h=1}^{6} \int_{\Pi^\zeta} \mathcal{H}^0(\{j \in \mathbb{Z} \colon (x^{k,\zeta}_j(y) + \tfrac{1}{k}S_h) \cap J \ne \emptyset\})\,\mathrm{d}\mathcal{H}^1(y). \tag{5.3.31}$$

Let us fix $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $\varepsilon > 0$. We claim that there exist a constant $c_1 > 0$, independent of ω, k, and ε, an infinite set $K^\zeta_\varepsilon \subset \mathbb{N}$, independent of ω, and a Borel set $I^\zeta_\varepsilon \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I^\zeta_\varepsilon) \le \varepsilon$, such that

$$\int_{\Pi^\zeta} \mathcal{H}^0(\{j \in \mathbb{Z} \colon (x^{k,\zeta}_j(y) + \tfrac{1}{k}S_h) \cap J \ne \emptyset\})\,\mathrm{d}\mathcal{H}^1(y) \le c_1, \tag{5.3.32}$$

for every $\omega \in U^\zeta_\varepsilon$, $k \in K^\zeta_\varepsilon$, and $h \in \{1, \dots, 6\}$. To prove this claim, we first observe that for every $y \in \Pi^\zeta$ we have

$$\mathcal{H}^0(\{j \in \mathbb{Z} \colon (x^{k,\zeta}_j(y) + \tfrac{1}{k}S_h) \cap J \ne \emptyset\}) \le \sum_{j \in \mathbb{Z}} \mathcal{H}^0((x^{k,\zeta}_j(y) + \tfrac{1}{k}S_h) \cap J). \tag{5.3.33}$$

We consider first the case $h = 1$. We prove that there exists a constant $c > 0$, independent of ω, k, ε, and j, such that

$$\int_{\Pi^\zeta} \mathcal{H}^0((x^{k,\zeta}_j(y) + \tfrac{1}{k}S_1) \cap J)\,\mathrm{d}\mathcal{H}^1(y) \le \frac{c}{k}\mathcal{H}^0(J^{\bar{\zeta}}_{y^{k,\bar{\zeta}}_j}) \tag{5.3.34}$$

for every $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. If $\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) = +\infty$ there is nothing to prove. Let us fix k and j such that $\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) < +\infty$.

We parametrise Π^ζ by $y = s\pi^\zeta(\bar\zeta)$ with $s \in \mathbb{R}$ and observe that (5.3.34) is equivalent to

$$\int_{\mathbb{R}} \mathcal{H}^0((x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta)) + \tfrac{1}{k}S_1) \cap J)\,\mathrm{d}s \le \frac{c}{k}\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) \tag{5.3.35}$$

for a possibly different constant c, independent of ω, k, ε, and j. By (5.2.4), (5.2.5), and (5.3.4) we have that $\pi^{\bar\zeta}(\omega + \frac{j}{k}\zeta) = y_j^{k,\bar\zeta}$. Therefore by the comments after (5.3.6)

$$\{x_j^{k,\zeta}(y)\} = \{x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta))\} = \{s\pi^\zeta(\bar\zeta) + t\zeta\colon\ t \in \mathbb{R}\} \cap \{y_j^{k,\bar\zeta} + t\bar\zeta\colon\ t \in \mathbb{R}\} \tag{5.3.36}$$

for all $s \in \mathbb{R}$. This implies that for every $s \in \mathbb{R}$ there exists a unique $\tau_j^{k,\bar\zeta}(s) \in \mathbb{R}$ such that $y_j^{k,\bar\zeta} + \tau_j^{k,\bar\zeta}(s)\bar\zeta = x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta))$, so that by (5.3.30) we have

$$\mathcal{H}^0((x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta)) + \tfrac{1}{k}S_1) \cap J) = \mathcal{H}^0([\tau_j^{k,\bar\zeta}(s), \tau_j^{k,\bar\zeta}(s) + \tfrac{1}{k}] \cap J^{\bar\zeta}_{y_j^{k,\bar\zeta}}).$$

Elementary geometric arguments show that there exist a constant $c > 0$, independent of ω, k, ε, and j, and a constant $d \in \mathbb{R}$, depending on ω, k, and j, such that

$$\tau_j^{k,\bar\zeta}(s) = \frac{s}{c} + d \quad \text{for every } s \in \mathbb{R}.$$

Since by assumption $\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) < +\infty$, we may apply Lemma 5.3.6 to obtain that

$$\int_{\mathbb{R}} \mathcal{H}^0([\tau_j^{k,\bar\zeta}(s), \tau_j^{k,\bar\zeta}(s) + \tfrac{1}{k}] \cap J^{\bar\zeta}_{y_j^{k,\bar\zeta}})\,\mathrm{d}s \le \frac{c}{k}\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}). \tag{5.3.37}$$

As the constant c depends only on ξ and η, this proves (5.3.35), which gives (5.3.34). Arguing in a similar way, we prove that

$$\int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k}S_2) \cap J)\,\mathrm{d}\mathcal{H}^1(y) \le \frac{c}{k}\mathcal{H}^0(J^{\bar\zeta}_{y_{j\pm1}^{k,\bar\zeta}}), \tag{5.3.38}$$

where the sign in ±1 depends on the specific value of ζ, according to (5.3.30).

By (5.2.4), (5.2.5), (5.3.4), and (5.3.22) we have the equality $y_j^{k,\bar\zeta} = (z_2 + \frac{j}{k})\pi^{\bar\zeta}(\zeta)$ for every $j \in \mathbb{Z}$ and for a suitable $z_2 \in \mathbb{R}$. Let $f\colon \mathbb{R} \to [0, +\infty]$ be defined by

$$f(s) \coloneqq \mathcal{H}^0(J^{\bar\zeta}_{s\pi^{\bar\zeta}(\zeta)}) \quad \text{for every } s \in \mathbb{R}.$$

Observe that, since $J \subset U$, by (5.3.24) the function f vanishes outside of I^ζ. Since by (5.2.8) and (5.2.9) we have $\mathcal{H}^1(J) = \mathcal{H}^1(J^1_u \cap U)$ and $\mathcal{H}^1(J^1_u \cap U) < +\infty$ by Remark 1.4.8, the function f is integrable by Lemma 5.B.1. Thus, we may apply Lemma 5.A.1 and we obtain an infinite set $K_\varepsilon^\zeta \subset \mathbb{N}$ and a Borel set $I_\varepsilon^\zeta \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I_\varepsilon^\zeta) \le \varepsilon$, such that

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon^\zeta}} \frac{1}{k}\sum_{j\in\mathbb{Z}} \mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) = \int_{I^\zeta} \mathcal{H}^0(J^{\bar\zeta}_{s\pi^{\bar\zeta}(\zeta)})\,\mathrm{d}s = \frac{1}{|\pi^{\bar\zeta}(\zeta)|}\int_{\Pi^{\bar\zeta}} \mathcal{H}^0(J^{\bar\zeta}_y)\,\mathrm{d}\mathcal{H}^1(y) \le \frac{1}{|\pi^{\bar\zeta}(\zeta)|}\mathcal{H}^1(J)$$

uniformly for $z_2 \in I_\varepsilon^\zeta$. Hence, up to removing a finite number of elements from K_ε^ζ, we may assume that

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} \mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) \le \frac{1}{|\pi^{\bar\zeta}(\zeta)|}\mathcal{H}^1(J) + 1 \tag{5.3.39}$$

for every $k \in K_\varepsilon^\zeta$ and $z_2 \in I_\varepsilon^\zeta$. Together with (5.3.33), (5.3.34), and (5.3.38), this implies (5.3.32) for $h = 1$ and $h = 2$.

Let us now fix $h \in \{3, \dots, 6\}$. For every $j \in \mathbb{Z}$, let L_j be the strip defined by $L_j := \{x \in \mathbb{R}^2 \colon \pi^{\bar\zeta}(x) \in [y_{j-1}^{k,\bar\zeta}, y_{j+1}^{k,\bar\zeta}]\}$, that is to say, the region of the plane between the straight lines $\{y_{j-1}^{k,\bar\zeta} + s\bar\zeta \colon s \in \mathbb{R}\}$ and $\{y_{j+1}^{k,\bar\zeta} + s\bar\zeta \colon s \in \mathbb{R}\}$. Since $\pi^{\bar\zeta}(S_h)$ is equal to $[0, \pi^\xi(\eta)]$ or $[\pi^\xi(\eta), 0]$ if $\bar\zeta = \xi$, while $\pi^{\bar\zeta}(S_h) = [0, \pi^\eta(\xi)]$ if $\bar\zeta = \eta$, from (5.2.4) and (5.2.5) we see that in every case we have the inclusion

$$\bigcup_{s\in\mathbb{R}} \{y_j^{k,\bar\zeta} + s\bar\zeta + \tfrac{1}{k}S_h\} \subset L_j \quad \text{for every } j \in \mathbb{Z}.$$

Let $\tilde\zeta \in \mathbb{R}^2 \setminus \{0\}$ be a vector in the direction of the segment S_h. Recalling that $x_j^{k,\zeta}(y) \in \{y_j^{k,\bar\zeta} + s\bar\zeta \colon s \in \mathbb{R}\}$ for every $y \in \Pi^\zeta$ by (5.3.36), we deduce from the previous inclusion that $(x_j^{k,\zeta}(y) + \frac{1}{k}S_h) \cap J \subset J \cap L_j$, hence

$$\mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k}S_h) \cap J) \le \mathcal{H}^0((J \cap L_j)^{\tilde\zeta}_{\pi^{\tilde\zeta}(x_j^{k,\zeta}(y))}) \tag{5.3.40}$$

for every $j \in \mathbb{Z}$. As ζ and $\bar\zeta$ are linearly independent and the same holds for $\tilde\zeta$ and $\bar\zeta$, the map $y \mapsto \pi^{\tilde\zeta}(x_j^{k,\zeta}(y))$ from Π^ζ into $\Pi^{\tilde\zeta}$ is affine and invertible. Moreover, its linear part is independent of ω, k, and j, since it depends only on ζ, $\bar\zeta$, and $\tilde\zeta$. By Lemma 5.B.1 we then have

$$\int_{\Pi^{\tilde\zeta}} \mathcal{H}^0((J \cap L_j)^{\tilde\zeta}_z)\,\mathrm{d}\mathcal{H}^1(z) \le \mathcal{H}^1(J \cap L_j).$$

Using the map $y \mapsto \pi^{\tilde\zeta}(x_j^{k,\zeta}(y))$ as change of variables, from the previous inequality and (5.3.40) it follows that

$$\int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k}S_h) \cap J)\,\mathrm{d}\mathcal{H}^1(y) \le \gamma\mathcal{H}^1(J \cap L_j)$$

for every $j \in \mathbb{Z}$, $k \in \mathbb{N}$, and $\omega \in U$, with $\gamma > 0$, a constant depending only on ζ, $\bar\zeta$, and $\tilde\zeta$. Observing that every point $x \in \mathbb{R}^2$ belongs at most to three strips of the form L_j, from the previous inequality it follows that

$$\sum_{j\in\mathbb{Z}} \int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k}S_h) \cap J)\,\mathrm{d}\mathcal{H}^1(y) \le 3\gamma\mathcal{H}^1(J).$$

This inequality, together with (5.3.33), yields

$$\int_{\Pi^\zeta} \mathcal{H}^0(\{j \in \mathbb{Z} \colon (x_j^{k,\zeta}(y) + \tfrac{1}{k}S_h) \cap J \neq \emptyset\})\,\mathrm{d}\mathcal{H}^1(y) \le 3\gamma\mathcal{H}^1(J).$$

Since $\mathcal{H}^1(J) < +\infty$, this proves (5.3.32) for $h \in \{3, \dots, 6\}$. Therefore, (5.3.32) holds for every $h \in \{1, \dots, 6\}$. Thanks to (5.3.31), from (5.3.32) we obtain (5.3.29), which by (5.3.28) concludes the proof. □

Before proving Lemma 5.3.5, we state a result about one-dimensional measures, which shows that non-atomic measures satisfy a suitable uniform absolute continuity property.

Lemma 5.3.7. *Let $I = [a, b] \subset \mathbb{R}$ be a bounded closed interval and let $\mu \in \mathcal{M}_b^+(I)$ be a measure such that $\mu(\{t\}) = 0$ for every $t \in I$. Then for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, m) > 0$ such that for every $\delta \in (0, \delta(\varepsilon, m))$ we have*

$$\mu\big(\bigcup_{\ell=1}^{m}(t_\ell - \delta, t_\ell + \delta) \cap I\big) \leq \varepsilon \quad \text{for every } (t_1, \dots, t_m) \in I^m.$$

Proof. We argue by contradiction. Suppose that there exist $m \in \mathbb{N}$ and $\varepsilon > 0$ such that for a sequence $\delta_k > 0$ converging to 0 we have

$$\mu\big(\bigcup_{\ell=1}^{m}(t_\ell^k - \delta_k, t_\ell^k + \delta_k) \cap I\big) > \varepsilon \quad \text{for some } (t_1^k, \dots, t_m^k) \in I^m.$$

Given $\delta > 0$, this implies that

$$\mu\big(\bigcup_{\ell=1}^{m}(t_\ell^k - \delta, t_\ell^k + \delta) \cap I\big) > \varepsilon \quad \text{for all } k \in \mathbb{N} \text{ sufficiently large.}$$

Since I is compact, there exists a subsequence, not relabelled, and a point $(t_1, \dots, t_m) \in I^m$ such that $(t_1^k, \dots, t_m^k)$ converges to $(t_1, \dots, t_m)$ as $k \to +\infty$. Since $(t_\ell^k - \delta, t_\ell^k + \delta) \subset (t_\ell - 2\delta, t_\ell + 2\delta)$ for k large we deduce that

$$\mu\big(\bigcup_{\ell=1}^{m}(t_\ell - 2\delta, t_\ell + 2\delta) \cap I\big) > \varepsilon.$$

Since $\delta > 0$ is arbitrary, we obtain

$$\mu\big(\bigcup_{\ell=1}^{m}\{t_\ell\}\big) \geq \varepsilon,$$

in contradiction with our hypotheses. This concludes the proof. □

Proof of Lemma 5.3.5. We begin by noting that for $\mathcal{H}^1$-a.e. $y \in \Pi^\zeta$ we have $u_y^\zeta \in \mathrm{BV}(U_y^\zeta)$ by Remark 1.4.4 and, recalling (5.2.8) and (5.2.9), by Theorem 1.4.13 we have also $J_{u_y^\zeta} \cap U_y^\zeta \subset J_y^\zeta = (J_u^1)_y^\zeta$. Let us fix $y \in \Pi^\zeta$ such that these two conditions hold. By standard properties of BV-functions in dimension one we have $Du_y^\zeta(\{t\}) = 0$ for every $t \in (U \setminus J)_y^\zeta$. We recall that by definition $\widehat{E}_m^{k,\zeta}(y)$ is the union of at most m intervals of length $\frac{1}{k}$ (see (5.3.15)). Hence, Lemma 5.3.7, applied to the measure $\mu := |Du_y^\zeta| \llcorner (U_y^\zeta \setminus J_y^\zeta)$ defined on the closure of the interval U_y^ζ, implies that (5.3.26) holds.

We observe that the integral in (5.3.27) is well-defined, since by Lemma 5.3.2 the set $\widehat{E}_m^{k,\zeta}$ is Borel measurable and $\widehat{E}_m^{k,\zeta}(y)$ is its slice by (5.3.21). Since $|Du_y^\zeta|(\widehat{E}_m^{k,\zeta}(y) \cap U_y^\zeta \setminus J_y^\zeta) \leq |Du_y^\zeta|(U_y^\zeta \setminus J_y^\zeta)$ for $\mathcal{H}^1$-a.e. $y \in \Pi^\zeta$ and the function $y \mapsto |Du_y^\zeta|(U_y^\zeta \setminus J_y^\zeta)$ is integrable on Π^ζ with respect to $\mathcal{H}^1$ by (1.4.12), the Dominated Convergence Theorem implies (5.3.27), concluding the proof. □

For technical reasons, instead of a single $\omega \in U$ we have to consider a sequence $\{\omega_k\}_k \subset U$; accordingly, the points $y_j^{k,\zeta}$, introduced in (5.2.5), and the sets $\widehat{E}_m^{k,\zeta}(y)$ and $\check{E}_m^{k,\zeta}(y)$ are defined by taking $\omega = \omega_k$.

Lemma 5.3.8. *For every $\varepsilon > 0$ there exist $m_\varepsilon \in \mathbb{N}$ and an infinite set $K_\varepsilon \subset \mathbb{N}$ with the following property: for every $k \in K_\varepsilon$ there exists a Borel set $U_\varepsilon^k \subset U$, with $\mathcal{L}^2(U \setminus U_\varepsilon^k) \le \varepsilon$, such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and for every sequence $\{\omega_k\}_{k\in K_\varepsilon}$, with $\omega_k \in U_\varepsilon^k$ for every $k \in K_\varepsilon$, the following conditions are simultaneously satisfied:*

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\check{E}^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) \le \varepsilon \quad \text{for every } k \in K_\varepsilon, \tag{5.3.41}$$

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon}} \frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\widehat{E}^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) = 0. \tag{5.3.42}$$

Proof. Let us fix $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $\varepsilon > 0$. Arguing as in Lemma 5.2.3 we see that it is enough to prove that there exist $m_\varepsilon \in \mathbb{N}$ and an infinite set $K_\varepsilon \subset \mathbb{N}$ with the following property: for every $k \in K_\varepsilon$ there exists a Borel set $U_\varepsilon^k \subset U$, with $\mathcal{L}^2(U \setminus U_\varepsilon^k) \le \varepsilon$, such that for every sequence $\{\omega_k\}_{k\in K_\varepsilon}$, with $\omega_k \in U_\varepsilon^k$ for every $k \in K_\varepsilon$, conditions (5.3.41) and (5.3.42) hold for this particular ζ.

We observe that every $\omega \in \mathbb{R}^2$ can be written in a unique way as $\omega = z_1\bar{\zeta} + z_2\zeta$, where $z_1, z_2 \in \mathbb{R}$ and $\bar{\zeta}$ is defined by (5.3.4). By (5.3.5) the numbers $t_i^{k,\zeta}(y)$ depend on ω only through z_2, hence the same holds for $x_j^{k,\zeta}(y)$, $\widehat{E}^{k,\zeta}_m(y)$, and $\check{E}^{k,\zeta}_m(y)$ (see (5.3.6) and (5.3.20)). We also remark that there exists a constant $c_1 > 0$ such that if $\omega \in U$ then $|z_1| \le c_1/2$. Moreover, there exists a constant $c_2 > 0$ such that, if $B \subset \mathbb{R}^2$ is a Borel set and $A = \{\omega \in U\colon\ \omega = z_1\bar{\zeta} + z_2\zeta \text{ with } (z_1, z_2) \in B\}$, then

$$\mathcal{L}^2(A) \le c_2\mathcal{L}^2(B). \tag{5.3.43}$$

Let $C > 0$ be the constant of Lemma 5.3.4. Thanks to Lemma 5.3.4, we can find an infinite set $H_\varepsilon^\zeta \subset \mathbb{N}$ and a Borel set $I_\varepsilon^\zeta \subset I^\zeta$, with

$$\mathcal{L}^1(I^\zeta \setminus I_\varepsilon^\zeta) \le \varepsilon/(4c_1c_2), \tag{5.3.44}$$

such that (5.3.25) holds for every $m \in \mathbb{N}$, $k \in H_\varepsilon^\zeta$, and $\omega \in U_\varepsilon^\zeta := U \cap \{z_1\bar{\zeta} + z_2\zeta\colon\ z_1 \in \mathbb{R},\ z_2 \in I_\varepsilon^\zeta\}$. Moreover, let $N_\zeta \subset \Pi^\zeta$ be the $\mathcal{H}^1$-negligible Borel set introduced at the beginning of the proof of Lemma 5.2.2.

For every $m \in \mathbb{N}$, $k \in H_\varepsilon^\zeta$, $y \in \Pi^\zeta$, and $z_2 \in \mathbb{R}$ we define

$$h^{k,\zeta}_{m,\varepsilon}(y, z_2) := \begin{cases} |Du^\zeta_y|(\check{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y) & \text{if } y \in \Pi^\zeta \setminus N_\zeta \text{ and } z_2 \in I_\varepsilon^\zeta,\\ 0 & \text{if } y \in N_\zeta \text{ or } z_2 \notin I_\varepsilon^\zeta, \end{cases} \tag{5.3.45}$$

where $\check{E}^{k,\zeta}_m(y)$ is defined using ω of the form $z_1\bar{\zeta} + z_2\zeta$ with an arbitrary $z_1 \in \mathbb{R}$. We observe that the function $h^{k,\zeta}_{m,\varepsilon}$ is Borel measurable on $\Pi^\zeta \times \mathbb{R}$ thanks to Lemma 5.D.2. We also define

$$g^\zeta(y) := \begin{cases} |Du^\zeta_y|((U \setminus J)^\zeta_y) & \text{if } y \in \Pi^\zeta \setminus N_\zeta,\\ 0 & \text{if } y \in N_\zeta. \end{cases} \tag{5.3.46}$$

Note that for every $y \in \Pi^\zeta$ we have $\check{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y \subset (U \setminus J)^\zeta_y$, hence $h^{k,\zeta}_{m,\varepsilon}(y, z_2) \le g^\zeta(y)$ for every $y \in \Pi^\zeta$ and $z_2 \in \mathbb{R}$.

We set

$$M^\zeta := \frac{\mathcal{H}^1(\pi^\zeta(U))}{|\pi^\zeta(\bar{\zeta})|}.$$

By (1.4.12) we have $g^\zeta \in L^1_{\mathcal{H}^1}(\Pi^\zeta)$, so that by the absolute continuity of the integral there exists $\delta_\varepsilon > 0$ such that

$$\int_B g^\zeta(y)\,\mathrm{d}\mathcal{H}^1(y) < \frac{\varepsilon^2|\pi^\zeta(\bar\zeta)|}{4(M^\zeta+1)\max\{\alpha,\beta\}c_2} \tag{5.3.47}$$

for every $\mathcal{H}^1$-measurable set $B \subset \Pi^\zeta$ with $\mathcal{H}^1(B) < \delta_\varepsilon$.

Since by (5.3.21) and (5.3.45) the inequality $h^{k,\zeta}_{m,\varepsilon}(y,z_2) > 0$ implies $y \in \pi^\zeta(\breve E^{k,\zeta}_m)$ and $z_2 \in I^\zeta_\varepsilon$, we have

$$\begin{aligned}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} h^{k,\zeta}_{m,\varepsilon}(z_1\pi^\zeta(\bar\zeta),z_2)\,\mathrm{d}z_1\Big)\mathrm{d}z_2 &= |\pi^\zeta(\bar\zeta)|^{-1}\int_{I^\zeta_\varepsilon}\Big(\int_{\pi^\zeta(\breve E^{k,\zeta}_m)} h^{k,\zeta}_{m,\varepsilon}(y,z_2)\,\mathrm{d}\mathcal{H}^1(y)\Big)\,\mathrm{d}z_2\\ &\le |\pi^\zeta(\bar\zeta)|^{-1}\int_{I^\zeta_\varepsilon}\Big(\int_{\pi^\zeta(\breve E^{k,\zeta}_m)} g^\zeta(y)\,\mathrm{d}\mathcal{H}^1(y)\Big)\mathrm{d}z_2.\end{aligned} \tag{5.3.48}$$

Let us fix $m_\varepsilon \in \mathbb{N}$ such that $m_\varepsilon > C\delta_\varepsilon^{-1}$, where δ_ε is a constant such that (5.3.47) holds. By (5.3.25) we have $\mathcal{H}^1(\pi^\zeta(\breve E^{k,\zeta}_{m_\varepsilon})) < \delta_\varepsilon$, for every $k \in H^\zeta_\varepsilon$ and $\omega = z_1\bar\zeta + z_2\zeta$ with $z_2 \in I^\zeta_\varepsilon$. Since $\mathcal{L}^1(I^\zeta_\varepsilon) \le \mathcal{L}^1(I^\zeta) \le \max\{\alpha,\beta\}$ by (5.3.23), from (5.3.47) and (5.3.48) we infer that

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(z_1\pi^\zeta(\bar\zeta),z_2)\mathrm{d}z_1\Big)\mathrm{d}z_2 < \frac{\varepsilon^2}{4(M^\zeta+1)c_2} \tag{5.3.49}$$

for every $k \in H^\zeta_\varepsilon$.

By (5.2.4), (5.2.5), and (5.3.4), if $\omega = z_1\bar\zeta + z_2\zeta$, then

$$y^{k,\zeta}_j = \Big(z_1 + \frac{j}{k}\Big)\pi^\zeta(\bar\zeta) \quad \text{for every } j \in \mathbb{Z}. \tag{5.3.50}$$

We set $A^\zeta := \{(z_1,z_2) \in \mathbb{R}^2\colon\ \omega = z_1\bar\zeta + z_2\zeta \in U\}$. By Tonelli's Theorem and the change of variables $s = z_1 + \frac{j}{k}$, we get

$$\begin{aligned}&\int_{A^\zeta}\frac{1}{k}\sum_{j\in\mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}\Big(y^{k,\zeta}_j,z_2\Big)\,\mathrm{d}z_1\,\mathrm{d}z_2\\ &\quad= \int_{\mathbb{R}^2} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(s\pi^\zeta(\bar\zeta),z_2)\Big(\frac{1}{k}\sum_{j\in\mathbb{Z}}\chi_{A^\zeta}\big(s-\tfrac{j}{k},z_2\big)\Big)\mathrm{d}s\,\mathrm{d}z_2.\end{aligned}$$

For every fixed (s,z_2), the number of indices j for which $(s-j/k,z_2) \in A^\zeta$ is at most $kM^\zeta+1$. Therefore, $\frac{1}{k}\sum_{j\in\mathbb{Z}}\chi_{A^\zeta}\big(s-\frac{j}{k},z_2\big) \le (M^\zeta+1)$, so that

$$\begin{aligned}&\int_{A^\zeta}\frac{1}{k}\sum_{j\in\mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}\Big(y^{k,\zeta}_j,z_2\Big)\,\mathrm{d}z_1\,\mathrm{d}z_2\\ &\quad\le (M^\zeta+1)\int_{\mathbb{R}^2} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(s\pi^\zeta(\bar\zeta),z_2)\,\mathrm{d}s\,\mathrm{d}z_2 < \frac{\varepsilon^2}{4c_2}\end{aligned} \tag{5.3.51}$$

for every $k \in H^\zeta_\varepsilon$.

We now set

$$V^{k,\zeta}_\varepsilon := \Big\{z \in A^\zeta\colon\ \frac{1}{k}\sum_{j\in\mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(y^{k,\zeta}_j,z_2) > \varepsilon\Big\} \cup F^{k,\zeta}, \tag{5.3.52}$$

where

$$F^{k,\zeta} := \Big\{z \in A^\zeta\colon\ y^{k,\zeta}_j \in N_\zeta \text{ for some } j \in \mathbb{Z}\Big\}.$$

Since $\mathcal{H}^1(N_\zeta)=0$ by (5.3.50) we obtain that $\mathcal{L}^2(F^{k,\zeta})=0$. In light of Cebyšëv's inequality and (5.3.51), we infer that

$$\mathcal{L}^2(V_\varepsilon^{k,\zeta})\le\frac{\varepsilon}{4c_2}.$$

We also introduce the set

$$W_\varepsilon^{k,\zeta}:=V_\varepsilon^{k,\zeta}\cup\Big\{z\in A^\zeta\colon\ z_2\in I^\zeta\setminus I_\varepsilon^\zeta\Big\}\tag{5.3.53}$$

and observe that, since $\mathcal{L}^2(\{z\in A^\zeta\colon\ z_2\in I^\zeta\setminus I_\varepsilon^\zeta\})\le c_1\mathcal{L}^1(I^\zeta\setminus I_\varepsilon^\zeta)\le\varepsilon/(4c_2)$ by (5.3.44), we have

$$\mathcal{L}^2(W_\varepsilon^{k,\zeta})\le\frac{\varepsilon}{2c_2}.\tag{5.3.54}$$

It follows immediately from (5.3.45), (5.3.52), and (5.3.53) that

$$\frac1k\sum_{j\in\mathbb{Z}}|Du^\zeta_{y_j^{k,\zeta}}|(\check E^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta})\cap U^\zeta_{y_j^{k,\zeta}}\setminus J^\zeta_{y_j^{k,\zeta}})=\frac1k\sum_{j\in\mathbb{Z}}h^{k,\zeta}_{m_\varepsilon,\varepsilon}(y_j^{k,\zeta},z_2)\le\varepsilon\tag{5.3.55}$$

for every $k\in H_\varepsilon^\zeta$ and for every $z\in A^\zeta\setminus W_\varepsilon^{k,\zeta}$.

For every $k,m\in\mathbb{N}$, $y\in\Pi^\zeta$, and $z_2\in\mathbb{R}$ we set

$$g_m^{k,\zeta}(y,z_2):=\begin{cases}|Du_y^\zeta|(\widehat E^{k,\zeta}_m(y)\cap U_y^\zeta\setminus J_y^\zeta) & \text{if } y\in\Pi^\zeta\setminus N_\zeta \text{ and } z_2\in I^\zeta,\\ 0 & \text{if } y\in N_\zeta \text{ or } z_2\notin I^\zeta.\end{cases}\tag{5.3.56}$$

By Lemma 5.D.2 the function $g_m^{k,\zeta}$ is Borel measurable on $\Pi^\zeta\times\mathbb{R}$. We also observe that by Lemma 5.3.5 for every $z_2\in\mathbb{R}$ and for $\mathcal{H}^1$-a.e. $y\in\pi^\zeta(U)$ the sequence $g_m^{k,\zeta}(y,z_2)$ converges to 0 as $k\to+\infty$. Arguing as before, we obtain that $g_m^{k,\zeta}(y,z_2)\le g^\zeta(y)$ for every $y\in\Pi^\zeta$ and $z_2\in\mathbb{R}$, where g^ζ is the function defined by (5.3.46).

We set

$$f_\varepsilon^{k,\zeta}(z_1,z_2):=g^{k,\zeta}_{m_\varepsilon}(z_1\pi^\zeta(\bar\zeta),z_2)\tag{5.3.57}$$

for every $(z_1,z_2)\in\mathbb{R}^2$ and observe that $f_\varepsilon^{k,\zeta}=0$ out of a suitable bounded set. Using Lemma 5.A.2 and Remark 5.A.3 with $\mathbb{N}$ replaced by H_ε^ζ, we obtain an infinite set $K_\varepsilon^\zeta\subset H_\varepsilon^\zeta$ and a Borel set $B_\varepsilon^\zeta\subset A^\zeta$, with $\mathcal{L}^2(B_\varepsilon^\zeta)\le\varepsilon/(2c_2)$, such that

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon^\zeta}}\frac1k\sum_{j\in\mathbb{Z}}f_\varepsilon^{k,\zeta}(z_1+\tfrac jk,z_2)=0\ \text{ uniformly for } (z_1,z_2)\in A^\zeta\setminus B_\varepsilon^\zeta.\tag{5.3.58}$$

We set $C_\varepsilon^\zeta:=\{\omega\in U\colon\ \omega=z_1\bar\zeta+z_2\zeta \text{ with } (z_1,z_2)\in B_\varepsilon^\zeta\}$ and observe that from (5.3.43) it follows that

$$\mathcal{L}^2(C_\varepsilon^\zeta)\le\frac{\varepsilon}{2}.\tag{5.3.59}$$

Moreover, from (5.3.56), (5.3.57), and (5.3.58) we deduce that

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon^\zeta}}\frac1k\sum_{j\in\mathbb{Z}}|Du^\zeta_{y_j^{k,\zeta}}|(\widehat E^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta})\cap U^\zeta_{y_j^{k,\zeta}}\setminus J^\zeta_{y_j^{k,\zeta}})=0\ \text{ uniformly for } \omega\in U\setminus C_\varepsilon^\zeta.\tag{5.3.60}$$

We set $D_\varepsilon^{k,\zeta}=\{\omega\in U\colon\ \omega=z_1\bar\zeta+z_2\zeta \text{ with } (z_1,z_2)\in W_\varepsilon^{k,\zeta}\}$ and observe that by (5.3.43), (5.3.54), and (5.3.55) we have

$$\mathcal{L}^2(D_\varepsilon^{k,\zeta})\le\frac{\varepsilon}{2}\quad\text{for every } k\in K_\varepsilon^\zeta,\tag{5.3.61}$$

$$\frac1k\sum_{j\in\mathbb{Z}}|Du^\zeta_{y_j^{k,\zeta}}|(\check E^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta})\cap U^\zeta_{y_j^{k,\zeta}}\setminus J^\zeta_{y_j^{k,\zeta}})\le\varepsilon\quad\text{for every } k\in K_\varepsilon^\zeta \text{ and } \omega\in U\setminus D_\varepsilon^{k,\zeta}.\tag{5.3.62}$$

Let $U_\varepsilon^{k,\zeta} := U \setminus (C_\varepsilon^\zeta \cup D_\varepsilon^{k,\zeta})$. Combining (5.3.59) and (5.3.61) we obtain that $\mathcal{L}^2(U \setminus U_\varepsilon^{k,\zeta}) \le \varepsilon$. Finally, from (5.3.60) and (5.3.62) it follows that (5.3.41) and (5.3.42) hold for every sequence $\{\omega_k\}_{k\in K_\varepsilon^\zeta}$ such that $\omega_k \in U_\varepsilon^{k,\zeta}$ for every $k \in K_\varepsilon^\zeta$. This concludes the proof. □

Remark 5.3.9. We observe that Lemma 5.3.8 continues to hold, with the same proof, for the six pairs $(\zeta, a) \in \mathcal{P}$ in place of the four directions ζ. To see this, let $(\zeta, a) \in \mathcal{P}$ and set

$$w^k := \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{k}\bar\zeta & \text{if } a \neq 0, \end{cases} \tag{5.3.63}$$

so that $x_{i,j}^k = x_{(i,j)-a}^k + w^k$. Indeed, in $\mathcal{P}$ the index $a = e_2$ occurs only for $\zeta \in \{\xi, \xi - \eta\}$, for which $\bar\zeta = \eta$ by (5.3.4), while $a = e_1$ occurs only for $\zeta = \eta$, for which $\bar\zeta = \xi$. We then set

$$E^{k,\zeta,a} := E^{k,\zeta} + w^k, \quad \widehat{E}_m^{k,\zeta,a} := \widehat{E}_m^{k,\zeta} + w^k, \text{ and } \check{E}_m^{k,\zeta,a} := \check{E}_m^{k,\zeta} + w^k \tag{5.3.64}$$

and note that for $a = 0$ these sets coincide with those introduced in (5.3.14) and (5.3.19).

Note that $\pi^\zeta(\check{E}_m^{k,\zeta,a})$ is a translate of $\pi^\zeta(\check{E}_m^{k,\zeta})$ inside Π^ζ, so that Lemma 5.3.4 holds for it with the same constant C and the same sets K_ε^ζ, I_ε^ζ. In addition, for every $y \in \Pi^\zeta$ the slice $(\widehat{E}_m^{k,\zeta} + w^k)_y^\zeta$ is a translate of $\widehat{E}_m^{k,\zeta}(y - \pi^\zeta(w^k))$, hence it is still the union of at most m intervals of length $1/k$, so that Lemma 5.3.5 holds for it as well. Finally, these sets (and the associated sets considered in Lemma 5.D.1) are Borel, since they are translates of Borel sets and w^k does not depend on ω. Thus, the joint measurability in (x, ω) of the associated functions follows from Lemma 5.D.2.

Combining the previous observations, the proof of Lemma 5.3.8 can be carried out over the six pairs of $\mathcal{P}$ instead of the four directions ζ.

Remark 5.3.10. For future use, we also observe that the sets introduced in (5.3.64) can be rewritten in a form analogous to (5.3.14) and (5.3.19). To see this, we first observe that, since the straight lines $\{\omega + \frac{j}{k}\zeta + s\bar\zeta\colon\ s \in \mathbb{R}\}$ considered in (5.3.5) are invariant under translation by w^k, we have $z^{k,\zeta}(x + w^k) = z^{k,\zeta}(x) + w^k$ for every $x \in \mathbb{R}^2$. Hence, setting $\mathcal{N}^{k,\zeta,a}(x) := \mathcal{N}^{k,\zeta}(x - w^k)$, which by (5.3.17) satisfies $\mathcal{N}^{k,\zeta,a}(x + t\zeta) = \mathcal{N}^{k,\zeta,a}(x)$ for every $t \in \mathbb{R}$, we deduce from (5.3.14), (5.3.19), and (5.3.64) that, if $a \neq 0$, then

$$\begin{aligned} E^{k,\zeta,a} &= \{x \in \mathbb{R}^2\colon\ (z^{k,\zeta}(x) + \tfrac{1}{k}(S - \bar\zeta)) \cap J \neq \emptyset\}, \\ \widehat{E}_m^{k,\zeta,a} &= \{x \in E^{k,\zeta,a}\colon\ \mathcal{N}^{k,\zeta,a}(x) \le m\}, \\ \check{E}_m^{k,\zeta,a} &= \{x \in E^{k,\zeta,a}\colon\ \mathcal{N}^{k,\zeta,a}(x) > m\}. \end{aligned} \tag{5.3.65}$$

We are finally ready to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. Recalling (5.3.1) and (5.3.13) for every $k \in \mathbb{N}$ we have

$$\mathcal{B}^k = \big\{(i,j) \in \mathcal{J}_2^k\colon\ (x_{i,j}^k + \tfrac{1}{k}S) \cap J \neq \emptyset\big\}. \tag{5.3.66}$$

Since by (5.3.11) we have

$$z^{k,\zeta}(x_{i,j}^k) = x_{i,j}^k \tag{5.3.67}$$

(see also the comments after (5.3.12)), from (5.3.14) and (5.3.66) we deduce that for every $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\}$ we have

$$\mathcal{B}^k = \{(i,j) \in \mathcal{J}_2^k\colon\ x_{i,j}^k \in E^{k,\zeta}\}. \tag{5.3.68}$$

For every $k, m \in \mathbb{N}$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we set

$$\begin{aligned} \widehat{\mathcal{B}}^{k,\zeta}_m &:= \{(i,j) \in \mathcal{J}^k_2 \colon\ x^k_{i,j} \in \widehat{E}^{k,\zeta}_m\}, \\ \check{\mathcal{B}}^{k,\zeta}_m &:= \{(i,j) \in \mathcal{J}^k_2 \colon\ x^k_{i,j} \in \check{E}^{k,\zeta}_m\}, \end{aligned} \tag{5.3.69}$$

and observe that $\mathcal{B}^k = \widehat{\mathcal{B}}^{k,\zeta}_m \cup \check{\mathcal{B}}^{k,\zeta}_m$.

Let us fix $0 < \varepsilon < \mathcal{L}^2(U)/2$. For every $n \in \mathbb{N}$ we apply Lemma 5.2.3 with ε replaced by $\varepsilon/2^n$ and obtain an infinite set H_n and a Borel set U_n, with $\mathcal{L}^2(U \setminus U_n) \le \varepsilon/2^n$, such that all conditions of Lemma 5.2.3 hold with U_ε and K_ε replaced by U_n and H_n. In the step $n+1$ we can replace $\mathbb{N}$ in Lemma 5.2.3 by H_n, obtaining $H_{n+1} \subset H_n$ for every $n \in \mathbb{N}$. Then, we apply Lemma 5.3.8 and Remark 5.3.9, with ε replaced by $\varepsilon/2^n$ and $\mathbb{N}$ replaced by H_n, and we obtain $m_n \in \mathbb{N}$, with $m_{n+1} \ge m_n$, an infinite set $K_n \subset H_n$, and for every $k \in K_n$ a Borel set $U^k_n \subset U$, with $\mathcal{L}^2(U \setminus U^k_n) \le \varepsilon/2^n$, with the following property: for every sequence $\{\omega_k\}_{k\in K_n}$ such that $\omega_k \in U^k_n$ for every $k \in K_n$ and for every $(\zeta, a) \in \mathcal{P}$ we have

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y^{k,\zeta}_j}|(\check{E}^{k,\zeta,a}_{m_n}(y^{k,\zeta}_j) \cap U^\zeta_{y^{k,\zeta}_j} \setminus J^\zeta_{y^{k,\zeta}_j}) \le \frac{\varepsilon}{2^n} \quad \text{for every } k \in K_n, \tag{5.3.70}$$

$$\lim_{\substack{k\to+\infty \\ k\in K_n}} \frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y^{k,\zeta}_j}|(\widehat{E}^{k,\zeta,a}_{m_n}(y^{k,\zeta}_j) \cap U^\zeta_{y^{k,\zeta}_j} \setminus J^\zeta_{y^{k,\zeta}_j}) = 0 \tag{5.3.71}$$

(see (5.3.64) for the definition of $\check{E}^{k,\zeta,a}_m$ and $\widehat{E}^{k,\zeta,a}_m$). Replacing U^k_n by $U^k_n \cap U_n$ we obtain that $U^k_n \subset U_n$ and $\mathcal{L}^2(U_n \setminus U^k_n) \le \varepsilon/2^n$. Moreover, repeating the same argument used when we pass from n to $n+1$, it is not restrictive to assume that $K_{n+1} \subset K_n$ for every $n \in \mathbb{N}$.

By a diagonal argument, we can find an infinite set $K \subset \mathbb{N}$ and a strictly increasing sequence $\{k_n\}_n \subset \mathbb{N}$ such that $K \cap [k_n, +\infty) \subset K_n$ for every $n \in \mathbb{N}$. For every $k \in K$ we also define $n_k \in \mathbb{N}$ as the largest integer such that $k_n \le k$ and observe that $n_k \to +\infty$ as $k \to +\infty$. For $k \in K$ consider the set $U^k := \bigcap_{n=1}^{n_k} U^k_n$ and observe that $\mathcal{L}^2(U^k) \ge \mathcal{L}^2(U) - 2\varepsilon$ for every $k \in K$, hence $U^k \ne \emptyset$. We fix $\{\omega_k\}_{k\in K} \subset U$ such that $\omega_k \in U^k$ for every $k \in K$. By (5.3.71) we have

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y^{k,\zeta}_j}|(\widehat{E}^{k,\zeta,a}_{m_n}(y^{k,\zeta}_j) \cap U^\zeta_{y^{k,\zeta}_j} \setminus J^\zeta_{y^{k,\zeta}_j}) = 0 \tag{5.3.72}$$

for every $(\zeta, a) \in \mathcal{P}$ and for every $n \in \mathbb{N}$.

We now prove (5.3.3) for $(\zeta, a) = (\xi, 0)$. Let us fix $(i,j) \in \mathcal{B}^k$ and $t \in I^{k,\xi}_{i,j}$. By (5.3.66) we have

$$(x^k_{i,j} + \tfrac{1}{k}S) \cap J \ne \emptyset. \tag{5.3.73}$$

Recalling (5.2.16), we also have $t^{k,\xi}_{i,j} \le t < t^{k,\xi}_{i,j} + \frac{1}{k}$, where $t^{k,\xi}_{i,j}$ is defined in (5.2.6). Since $x^k_{i,j} \in \{\omega_k + \frac{i}{k}\xi + s\eta \colon\ s \in \mathbb{R}\}$, by (5.2.6) and (5.3.5) we have $t^{k,\xi}_i(y^{k,\xi}_j) = t^{k,\xi}_{i,j}$, so that $t^{k,\xi}_i(y^{k,\xi}_j) \le t < t^{k,\xi}_i(y^{k,\xi}_j) + \frac{1}{k}$. From (5.3.12) and (5.3.67) we deduce that

$$z^{k,\xi}(y^{k,\xi}_j + t\xi) = z^{k,\xi}(y^{k,\xi}_j + t^{k,\xi}_i(y^{k,\xi}_j)\xi) = z^{k,\xi}(y^{k,\xi}_j + t^{k,\xi}_{i,j}\xi) = z^{k,\xi}(x^k_{i,j}) = x^k_{i,j}.$$

In light of (5.3.73), this implies that

$$(z^{k,\xi}(y^{k,\xi}_j + t\xi) + \tfrac{1}{k}S) \cap J \ne \emptyset,$$

which, by (5.3.14), implies that $y_j^{k,\xi}+t\xi\in E^{k,\xi}$. Recalling (5.3.16), this is equivalent to $t\in E^{k,\xi}(y_j^{k,\xi})$. Since $(i,j)\in\mathcal{J}_2^k$, this shows that

$$I_{i,j}^{k,\xi}\subset E^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}. \tag{5.3.74}$$

If $(i,j)\in\widehat{\mathcal{B}}_m^{k,\xi}$ for some $m\in\mathbb{N}$, it follows from (5.3.20) and (5.3.69) that $\mathcal{N}^{k,\xi}(x_{i,j}^k)\le m$, where $\mathcal{N}^{k,\xi}$ is defined by (5.3.15). By (5.3.17) we have $\mathcal{N}^{k,\xi}(x_{i,j}^k)=\mathcal{N}^{k,\xi}(y_j^{k,\xi}+t\xi)$ for every $t\in\mathbb{R}$, so that from (5.3.74) we also deduce that

$$I_{i,j}^{k,\xi}\subset \widehat{E}_m^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}. \tag{5.3.75}$$

In a similar way we prove that if $(i,j)\in\check{\mathcal{B}}_m^{k,\xi}$ for some $m\in\mathbb{N}$ then

$$I_{i,j}^{k,\xi}\subset \check{E}_m^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}. \tag{5.3.76}$$

For every $k,m\in\mathbb{N}$ and $j\in\mathbb{Z}$ we set $\mathcal{B}^k(j):=\{i\in\mathbb{Z}\colon (i,j)\in\mathcal{B}^k\}$, $\widehat{\mathcal{B}}_m^{k,\xi}(j):=\{i\in\mathbb{Z}\colon (i,j)\in\widehat{\mathcal{B}}_m^{k,\xi}\}$, and $\check{\mathcal{B}}_m^{k,\xi}(j):=\{i\in\mathbb{Z}\colon (i,j)\in\check{\mathcal{B}}_m^{k,\xi}\}$. From (5.3.75) and (5.3.76) we deduce that for every $k,m\in\mathbb{N}$ and $j\in\mathbb{Z}$ we have

$$\bigcup_{i\in\widehat{\mathcal{B}}_m^{k,\xi}(j)} I_{i,j}^{k,\xi}\subset \widehat{E}_m^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}\quad\text{and}\quad \bigcup_{i\in\check{\mathcal{B}}_m^{k,\xi}(j)} I_{i,j}^{k,\xi}\subset \check{E}_m^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}.$$

Therefore, for every $k,m\in\mathbb{N}$ we have

$$\sum_{i\in\widehat{\mathcal{B}}_m^{k,\xi}(j)}|Du^{\xi}_{y_j^{k,\xi}}|(I_{i,j}^{k,\xi}\setminus J^{\xi}_{y_j^{k,\xi}})\le |Du^{\xi}_{y_j^{k,\xi}}|(\widehat{E}_m^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}\setminus J^{\xi}_{y_j^{k,\xi}}), \tag{5.3.77}$$

$$\sum_{i\in\check{\mathcal{B}}_m^{k,\xi}(j)}|Du^{\xi}_{y_j^{k,\xi}}|(I_{i,j}^{k,\xi}\setminus J^{\xi}_{y_j^{k,\xi}})\le |Du^{\xi}_{y_j^{k,\xi}}|(\check{E}_m^{k,\xi}(y_j^{k,\xi})\cap U^{\xi}_{y_j^{k,\xi}}\setminus J^{\xi}_{y_j^{k,\xi}}). \tag{5.3.78}$$

From (5.3.70) and (5.3.78) for every $n\in\mathbb{N}$ we obtain that

$$\frac{1}{k}\sum_{j\in\mathbb{Z}}\sum_{i\in\check{\mathcal{B}}_{mn}^{k,\xi}(j)}|Du^{\xi}_{y_{i,j}^{k,\xi}}|(I_{i,j}^{k,\xi}\setminus J^{\xi}_{y_{i,j}^{k,\xi}})\le\frac{\varepsilon}{2^n}\quad\text{for every } k\in K \text{ with } k\ge k_n, \tag{5.3.79}$$

while (5.3.72) and (5.3.77) give

$$\lim_{\substack{k\to+\infty\\ k\in K}}\frac{1}{k}\sum_{j\in\mathbb{Z}}\sum_{i\in\widehat{\mathcal{B}}_{mn}^{k,\xi}(j)}|Du^{\xi}_{y_{i,j}^{k,\xi}}|(I_{i,j}^{k,\xi}\setminus J^{\xi}_{y_{i,j}^{k,\xi}})=0\quad\text{for every } n\in\mathbb{N}. \tag{5.3.80}$$

Combining (5.3.79) and (5.3.80), we deduce that

$$\limsup_{\substack{k\to+\infty\\ k\in K}}\ \frac{1}{k}\sum_{(i,j)\in\mathcal{B}^k}|Du^{\xi}_{y_{i,j}^{k,\xi}}|(I_{i,j}^{k,\xi}\setminus J^{\xi}_{y_{i,j}^{k,\xi}})\le\frac{\varepsilon}{2^n}\quad\text{for every } n\in\mathbb{N},$$

which, after letting $n\to+\infty$, concludes the proof of (5.3.3) for $(\zeta,a)=(\xi,0)$. The proof of (5.3.3) for $(\zeta,a)\in\mathcal{P}$ with $a=0$ is obtained with the same exact arguments.

Let now $(\zeta, a) \in \mathcal{P}$ with $a \neq 0$. Since $\mathcal{B}^k \subset \mathcal{J}_2^k$, from (5.3.68), (5.3.63), and the fact that $x_{i,j}^k = x_{(i,j)-a}^k + w^k$ (see Remark 5.3.9) we obtain

$$\mathcal{B}^k + a \subset \{(i,j) \in \mathcal{J}_2^k + a \colon\ x_{i,j}^k \in E^{k,\zeta,a}\}. \tag{5.3.81}$$

Note that $\mathcal{J}_2^k + a \subset \mathcal{J}_1^k$, so that $[x_{i,j}^k, x_{i,j}^k + \frac{1}{k}\zeta) \subset U$ for all $(i,j) \in \mathcal{J}_2^k + a$. Thus, the proof given above for $(\xi, 0)$ can be repeated as it is, replacing the set $\mathcal{J}_2^k$ by $\mathcal{J}_2^k + a$ in (5.3.69), $\mathcal{B}^k$ by $\mathcal{B}^k + a$, the set S by $S - \bar{\zeta}$ in (5.3.73) and (5.3.14), and $\mathcal{N}^{k,\zeta}$ by $\mathcal{N}^{k,\zeta,a}$ (see (5.3.65) and Remark 5.3.10). This concludes the proof of (5.3.3) for all $(\zeta, a) \in \mathcal{P}$.

Equalities (5.3.2) can then be obtained by difference from (5.2.20). This concludes the proof of Theorem 5.3.1. □

5.4 The case of dimension d > 2

We now show that Theorem 5.2.1 can be extended to the general case $d > 2$. This is done by means of a Fubini-type argument. To this aim, we present a few preliminary results.

We first prove a short lemma that shows that the measure λ_u introduced in (1.4.15) does not charge Borel sets that are σ-finite with respect to $\mathcal{H}^{d-1}$ and do not intersect the jump set.

Lemma 5.4.1. *Let $d \geq 1$, let $u \in \mathrm{GBD}(\Omega)$, and let $B \subset \Omega$ be a Borel set that is σ-finite with respect to $\mathcal{H}^{d-1}$. Then $\lambda_u(B \setminus J_u) = 0$.*

Proof. It is not restrictive to assume that $\mathcal{H}^{d-1}(B) < +\infty$. By (1.4.15) to prove the claim it is enough to show that for every $\xi \in \mathbb{S}^{d-1}$ the measure λ_u^ξ defined by (1.4.14) satisfies $\lambda_u^\xi(B \setminus J_u) = 0$. Let us fix $\xi \in \mathbb{S}^{d-1}$. Since $J_{u_y^\xi} \subset (J_u)_y^\xi$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ by Theorem 1.4.13, it follows from (1.4.14) that

$$\lambda_u^\xi(B \setminus J_u) \leq \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi \setminus J_{u_y^\xi})\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.4.1}$$

Recalling that by assumption $\mathcal{H}^{d-1}(B) < +\infty$, Lemma 5.B.1 implies that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the slice B_y^ξ is a finite set. By well-known properties of BV-functions of one variable (see [18, Corollary 3.33]), this implies that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $|Du_y^\xi|(B_y^\xi \setminus J_{u_y^\xi}) = 0$. By (5.4.1) this equality gives $\lambda_u^\xi(B \setminus J_u) = 0$, concluding the proof. □

The second tool we need is the following result, which shows that the space $\mathrm{GBD}(\Omega)$ behaves nicely with respect to restriction to affine subspaces of the domain Ω.

Theorem 5.4.2. *Assume $d \geq 2$. Let $u \in \mathrm{GBD}(\Omega)$, let V be a vector subspace of $\mathbb{R}^d$ of dimension k, with $1 \leq k \leq d-1$, let $V^\perp$ be its orthogonal subspace, and let $\pi_V \colon \mathbb{R}^d \to V$ be the orthogonal projection onto V. For every $z \in V^\perp$ and $E \subset \Omega$ we set $E_z^V := \{x \in V \colon\ z + x \in E\} = V \cap (E - z)$ and consider the function $u_z^V \colon \Omega_z^V \to V$ defined by $u_z^V(x) := \pi_V(u(z + x))$. Then the following properties hold:*

(a) for $\mathcal{H}^{d-k}$-a.e. $z \in V^\perp$ we have $u_z^V \in \mathrm{GBD}(\Omega_z^V)$;

(b) for $\mathcal{H}^{d-k}$-a.e. $z \in V^\perp$ we have $J_{u_z^V} \subset (J_u)_z^V \cup N_z$ for a Borel set $N_z \subset V$ with $\mathcal{H}^{k-1}(N_z) = 0$.

To prove property (b) we need the following result, due to Dal Maso [89, Theorem 7.1], concerning the relation between the jump points of a function $u \in \mathrm{GBD}(\Omega)$ and the jump points of its restriction to a hyperplane that does not intersect the set S_u of approximate discontinuity points.

Proposition 5.4.3. *Assume $d \geq 2$. Let $u \in \mathrm{GBD}(\Omega)$, $x_0 \in \Omega$, and $\xi \in \mathbb{S}^{d-1}$. Assume that*

$$\mathcal{H}^{d-1}(S_u \cap (x_0 + \Pi^\xi)) = 0. \tag{5.4.2}$$

Let $v\colon (\Omega - x_0) \cap \Pi^\xi \to \Pi^\xi$ be the function defined by $v(y) := \pi^\xi(\widetilde{u}(x_0 + y))$ for $\mathcal{H}^{d-1}$-a.e. $y \in (\Omega - x_0) \cap \Pi^\xi$. Assume that there exist a direction $\nu \in \mathbb{S}^{d-1} \cap \Pi^\xi$ and two vectors $b^\pm \in \Pi^\xi$, with $b^+ \neq b^-$, such that for every $\varepsilon > 0$ we have

$$\limsup_{\rho \to 0^+} \frac{\mathcal{H}^{d-1}(\{y \in B_\rho(0) \cap \Pi^\xi : \pm y \cdot \nu > 0,\ |v(y) - b^\pm| > \varepsilon\})}{\rho^{d-1}} = 0. \tag{5.4.3}$$

Then $x_0 \in \Theta_u$.

We are now ready to prove Theorem 5.4.2.

Proof of Theorem 5.4.2. The proof of (a) can be found in [89, Theorem 4.19].

We divide the proof of (b) into two steps.

Step 1. Assume that $k = d - 1$ and let $\xi \in \mathbb{S}^{d-1}$ be such that $V = \Pi^\xi$. We claim that for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ there exists a Borel set $N_s \subset V$, with $\mathcal{H}^{d-2}(N_s) = 0$, such that

$$J_{u^V_{s\xi}} \subset (J_u)^V_{s\xi} \cup N_s.$$

To prove this property, we observe that by the Fubini Theorem the equality $\mathcal{L}^d(S_u) = 0$ implies that for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ and for every $y_0 \in \Pi^\xi = V$ condition (5.4.2) holds with $x_0 = s\xi + y_0$, while the equality $\widetilde{u} = u$ $\mathcal{L}^d$-a.e. in Ω implies that for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ we have

$$\widetilde{u}(s\xi + y) = u(s\xi + y) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi. \tag{5.4.4}$$

Let us fix $s \in \mathbb{R}$ with these properties. Given $y_0 \in J_{u^V_{s\xi}}$, we consider the function $v(y) := \pi^\xi(\widetilde{u}(s\xi + y_0 + y))$ for $y \in \Pi^\xi$ and observe that by (5.4.4) we have

$$v(y) = u^V_{s\xi}(y_0 + y) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi.$$

Since $y_0 \in J_{u^V_{s\xi}}$, v satisfies (5.4.3). We can then apply Proposition 5.4.3 to obtain that $s\xi + y_0 \in \Theta_u$, which gives $y_0 \in (\Theta_u)^V_{s\xi}$. Setting $N_s := (\Theta_u \setminus J_u)^V_{s\xi}$, we have $y_0 \in (J_u)^V_{s\xi} \cup N_s$, hence $J_{u^V_{s\xi}} \subset (J_u)^V_{s\xi} \cup N_s$. Since by Theorem 1.4.13 we have $\mathcal{H}^{d-1}\Theta_u \setminus J_u = 0$, it follows from [118, Theorem 2.10.25] that $\mathcal{H}^{d-2}(N_s) = 0$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. This concludes the proof of the claim. Of course, if $d = 2$ this Step also completes the proof of (b).

Step 2. Assume $d > 2$. We now prove (b) by induction on the codimension of V. Step 1 gives (b) when the dimension of V is $d - 1$. Given $1 \leq k \leq d - 2$, we assume now that (b) holds for every subspace of dimension $k+1$ and we want to prove that it holds in dimension k. Let us fix a subspace V of dimension k and $\xi \in V^\perp \cap \mathbb{S}^{d-1}$. We consider the vector space $\widetilde{V}$ generated by V and ξ. By (a) for $\mathcal{H}^{d-k-1}$-a.e. $z \in \widetilde{V}^\perp$ we have that $u^{\widetilde{V}}_z \in \mathrm{GBD}(\Omega^{\widetilde{V}}_z)$. Using the inductive hypothesis, we deduce that for $\mathcal{H}^{d-k-1}$-a.e. $z \in \widetilde{V}^\perp$ there exists a Borel set $\widetilde{N}_z \subset \widetilde{V}$, with $\mathcal{H}^k(\widetilde{N}_z)=0$, such that

$$J_{u^{\widetilde{V}}_z} \subset (J_u)^{\widetilde{V}}_z \cup \widetilde{N}_z. \tag{5.4.5}$$

Let us fix $z \in \widetilde{V}^\perp$ satisfying both these properties. We observe that for every $s \in \mathbb{R}$ and $E \subset \Omega$ we have

$$E^V_{z+s\xi} = (E^{\widetilde{V}}_z)^V_{s\xi} \quad \text{and} \quad u^V_{z+s\xi} = (u^{\widetilde{V}}_z)^V_{s\xi} \ \text{ on } \Omega^V_{z+s\xi}. \tag{5.4.6}$$

Applying Step 1 with $\mathbb{R}^d$ replaced by $\widetilde{V}$ and with u replaced by $u^{\widetilde{V}}_z$, we have that for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ there exists a Borel set $N_s \subset V$, with $\mathcal{H}^{k-1}(N_s) = 0$, such that

$$J_{(u^{\widetilde{V}}_z)^V_{s\xi}} \subset (J_{u^{\widetilde{V}}_z})^V_{s\xi} \cup N_s.$$

By (5.4.5) this implies that

$$J_{(u^{\widetilde{V}}_z)^V_{s\xi}} \subset ((J_u)^{\widetilde{V}}_z)^V_{s\xi} \cup (\widetilde{N}_z)^V_{s\xi} \cup N_s.$$

By (5.4.6) this gives

$$J_{u^V_{z+s\xi}} \subset (J_u)^V_{z+s\xi} \cup (\widetilde{N}_z)^V_{s\xi} \cup N_s. \tag{5.4.7}$$

By [118, Theorem 2.10.25] we deduce that for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ we have $\mathcal{H}^{k-1}((\widetilde{N}_z)^V_{s\xi}) = 0$. Setting $N_{z+s\xi} \coloneqq N_s \cup (\widetilde{N}_z)^V_{s\xi}$, it follows that $\mathcal{H}^{k-1}(N_{z+s\xi}) = 0$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. Finally, from (5.4.7) we deduce that

$$J_{u^V_{z+s\xi}} \subset (J_u)^V_{z+s\xi} \cup N_{z+s\xi}. \tag{5.4.8}$$

Since $V^\perp$ is the space generated by $\widetilde{V}^\perp$ and ξ and (5.4.8) holds for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ and for $\mathcal{H}^{d-k-1}$-a.e. $z \in \widetilde{V}^\perp$, property (b) for V follows from the Fubini Theorem. This concludes the proof of the inductive step and hence of the theorem. □

Finally, we prove the desired extension of Theorem 5.2.1 to the general case $d > 2$.

Theorem 5.4.4. *Let $d > 2$, let $u \in \mathrm{GBD}(\Omega)$, and let $B \subset \Omega$ be a Borel set. Then the function $\xi \mapsto \sigma^\xi_u(B)$ is quadratic.*

Proof. By Proposition 5.1.2 it is enough to show that $\xi \mapsto \sigma^\xi_u(B)$ is 2-homogeneous, satisfies the parallelogram identity, and is lower bounded in the sense of (c) of Proposition 5.1.2. Since by Proposition 5.1.6 the function $\xi \mapsto \sigma^\xi_u(B)$ is 2-homogeneous and by Remark 5.1.4 it satisfies the correct lower bound, we only need to prove the parallelogram identity.

We decompose $\sigma^\xi_u(B)$ as

$$\sigma^\xi_u(B) = \sigma^\xi_u(B \setminus J_u) + \sigma^\xi_u(B \cap J_u)$$

and observe that by Propositions 5.1.7 the function $\xi \mapsto \sigma^\xi_u(B \cap J_u)$ is quadratic. Thus, to conclude we only need to prove that $\xi \mapsto \sigma^\xi_u(B \setminus J_u)$ satisfies the parallelogram identity.

Let $\xi, \eta \in \mathbb{R}^d$ be two linearly independent vectors and consider the 2-dimensional vector space V generated by ξ and η. Let $\pi_V \colon \mathbb{R}^d \to V$ be the orthogonal projection onto V. For $z \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$ let $E^V_z \coloneqq \{y \in V : z + y \in E\} = V \cap (E - z)$ and let $u^V_z \colon \Omega^V_z \to V$ be the function defined for every $y \in \Omega^V_z$ by $u^V_z(y) \coloneqq \pi_V(u(z + y))$. By (a) of Theorem 5.4.2 for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ we have $u^V_z \in \mathrm{GBD}(\Omega^V_z)$.

We observe that for every $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\} \subset V$ and for every $E \subset \mathbb{R}^d$ we have

$$(E^V_z)^\zeta_y = E^\zeta_{z+y} \quad \text{for every } z \in V^\perp \text{ and } y \in V \cap \Pi^\zeta. \tag{5.4.9}$$

Moreover, for every $z \in V^\perp$, $y \in V \cap \Pi^\zeta$, and $t \in (\Omega_z^V)_y^\zeta = \Omega_{z+y}^\zeta$ we have

$$(u_z^V)_y^\zeta(t) = u_{z+y}^\zeta(t). \tag{5.4.10}$$

This implies that for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for $\mathcal{H}^1$-a.e. $y \in V \cap \Pi^\zeta$ we have

$$D(u_z^V)_y^\zeta = Du_{z+y}^\zeta \tag{5.4.11}$$

as Borel measures on $(\Omega_z^V)_y^\zeta = \Omega_{y+z}^\zeta$.

We then apply Theorem 5.4.2 to deduce that for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ there exists a Borel set $N_z \subset V$ such that

$$\mathcal{H}^1(N_z) = 0 \quad \text{and} \quad J_{u_z^V} \subset (J_u)_z^V \cup N_z. \tag{5.4.12}$$

By Remark 1.4.8 the set J_u is σ-finite with respect to $\mathcal{H}^{d-1}$. Therefore, by [118, Theorem 2.10.25] we have that $(J_u)_z^V$ is σ-finite with respect to $\mathcal{H}^1$ for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$. We can then apply Lemma 5.4.1, with $d = 2$, Ω replaced by Ω_z^V, u replaced by u_z^V, and $B = (J_u)_z^V$, and we obtain that $\lambda_{u_z^V}((J_u)_z^V \setminus J_{u_z^V}) = 0$ for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$. By Definition 1.4.3, applied to $u_z^V \in \mathrm{GBD}(\Omega_z^V)$, and from this equality it follows that

$$\int_{V\cap\Pi^\zeta} |D(u_z^V)_y^\zeta|((J_u)_z^V \setminus J_{u_z^V})_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y) \le \lambda_{u_z^V}((J_u)_z^V \setminus J_{u_z^V}) = 0 \tag{5.4.13}$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$.

Since for every triple of sets A_1, A_2, A_3 we have $(A_1 \setminus A_2) \setminus (A_1 \setminus A_3) \subset A_3 \setminus A_2$, we obtain that

$$(B_z^V \setminus J_{u_z^V}) \setminus (B_z^V \setminus (J_u)_z^V) \subset (J_u)_z^V \setminus J_{u_z^V},$$

hence by (5.4.13)

$$|D(u_z^V)_y^\zeta|((B_z^V \setminus J_{u_z^V})_y^\zeta \setminus (B_z^V \setminus (J_u)_z^V)_y^\zeta) = 0 \tag{5.4.14}$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for $\mathcal{H}^1$-a.e. $y \in V \cap \Pi^\zeta$.

The inclusion in (5.4.12) implies that for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for every $y \in V \cap \Pi^\zeta \setminus \pi^\zeta(N_z)$ we have $(J_{u_z^V})_y^\zeta \subset ((J_u)_z^V)_y^\zeta$, and hence $(B_z^V \setminus (J_u)_z^V)_y^\zeta \subset (B_z^V \setminus J_{u_z^V})_y^\zeta$. Observing that $\mathcal{H}^1(\pi^\zeta(N_z)) = 0$ by the equality in (5.4.12), we deduce from this inclusion and from (5.4.14) that

$$D(u_z^V)_y^\zeta((B_z^V \setminus J_{u_z^V})_y^\zeta) = D(u_z^V)_y^\zeta((B_z^V \setminus (J_u)_z^V)_y^\zeta)$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for $\mathcal{H}^1$-a.e. $y \in V \cap \Pi^\zeta$.

Integrating this equality with respect to y we obtain that

$$\int_{V\cap\Pi^\zeta} D(u_z^V)_y^\zeta((B \setminus J_u)_z^V)_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y) = \int_{V\cap\Pi^\zeta} D(u_z^V)_y^\zeta((B_z^V \setminus J_{u_z^V})_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y)$$

for $\mathcal{H}^{d-2}$- a.e. $z \in V^\perp$, so that, setting $y' = z + y$, by (5.4.9)–(5.4.11) and the Fubini Theorem we have

$$\begin{aligned}\int_{\Pi^\zeta} Du_{y'}^\zeta((B \setminus J_u)_{y'}^\zeta)\,\mathrm{d}\mathcal{H}^{d-1}(y') \\ = \int_{V^\perp} \Big(\int_{V\cap\Pi^\zeta} D(u_z^V)_y^\zeta((B_z^V \setminus J_{u_z^V})_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y)\Big)\mathrm{d}\mathcal{H}^{d-2}(z).\end{aligned}$$

Taking into account the definition of σ_u^ζ (see (5.1.1)), this last equality can be written as

$$\sigma_u^\zeta(B \setminus J_u) = \int_{V^\perp} \sigma_{u_z^V}^\zeta(B_z^V \setminus J_{u_z^V})\,\mathrm{d}\mathcal{H}^{d-2}(z). \tag{5.4.15}$$

We may now apply Theorem 5.2.1 to the function $u_z^V \in \mathrm{GBD}(\Omega_z^V)$ to obtain

$$\sigma_{u_z^V}^{\xi+\eta}(B_z^V \setminus J_{u_z^V}) + \sigma_{u_z^V}^{\xi-\eta}(B_z^V \setminus J_{u_z^V}) = 2\sigma_{u_z^V}^{\xi}(B_z^V \setminus J_{u_z^V}) + 2\sigma_{u_z^V}^{\eta}(B_z^V \setminus J_{u_z^V})$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$. Integrating this equality with respect to z and exploiting (5.4.15) we deduce that

$$\sigma_u^{\xi+\eta}(B \setminus J_u) + \sigma_u^{\xi-\eta}(B \setminus J_u) = 2\sigma_u^{\xi}(B \setminus J_u) + 2\sigma_u^{\eta}(B \setminus J_u).$$

This shows that the function $\xi \mapsto \sigma_u^\xi(B \setminus J_u)$ satisfies the parallelogram identity, concluding the proof. □

5.5 A matrix-valued measure associated to a GBD function

In this section, for every $u \in \mathrm{GBD}(\Omega)$ we introduce a matrix-valued measure μ_u that generalises the distributional symmetric gradient Eu of $\mathrm{BD}(\Omega)$ functions. We then analyse some of its properties and deduce some useful consequences. We recall that all matrices $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfy the polarisation identity, i.e.,

$$\begin{aligned} A\xi\cdot\eta &= \frac{1}{4}\big(A(\xi+\eta)\cdot(\xi+\eta) - A(\xi-\eta)\cdot(\xi-\eta)\big) \\ &= \tfrac{1}{2}\big(A(\xi+\eta)\cdot(\xi+\eta) - A\xi\cdot\xi - A\eta\cdot\eta\big) \end{aligned} \tag{5.5.1}$$

for every $\xi, \eta \in \mathbb{R}^d$.

The following constitutes the main result of the chapter.

Theorem 5.5.1. *Let $d \geq 1$ and $u \in \mathrm{GBD}(\Omega)$. Then there exists a matrix-valued measure $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\mu_u(B)\xi\cdot\xi = \sigma_u^\xi(B) = \lim_{R\to+\infty} D_\xi(\tau_R(u\cdot\xi))(B \setminus J_u^1) \quad \text{for every Borel set } B \subset \Omega, \tag{5.5.2}$$

where σ_u^ξ is the measure defined in (5.1.1) *and τ_R are the truncation functions defined in* (5.1.4). *Moreover, the variation $|\mu_u|$ with respect to the operator norm in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfies*

$$|\mu_u|(J_u^1) = 0, \tag{5.5.3}$$

$$|\mu_u| = \lambda_u \llcorner (\Omega \setminus J_u^1) \quad \text{as Borel measures on } \Omega, \tag{5.5.4}$$

where λ_u is the positive measure defined by (1.4.15).

Proof. If $d = 1$ these results follow from Remarks 5.1.3, 5.1.4, and Proposition 5.1.5. We may thus assume that $d \geq 2$. From Theorems 5.2.1 and 5.4.4 it follows that the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic for every Borel set $B \subset \Omega$. Thus, there exists a set function μ_u defined on the σ-algebra of all Borel subsets of Ω and with values in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

$$\sigma_u^\xi(B) = \mu_u(B)\xi\cdot\xi \tag{5.5.5}$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. Observing that $\sigma_u^\xi(B \cap J_u^1) = 0$ by (5.1.1), we may apply Proposition 5.1.5 to $B \setminus J_u^1$ and we obtain

$$\sigma_u^\xi(B) = \lim_{R \to +\infty} D_\xi(\tau_R(u \cdot \xi))(B \setminus J_u^1)$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{S}^{d-1}$. As (5.5.3) is an obvious consequence of (5.5.4), we are left with proving that $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$ and that equality (5.5.4) holds.

Since $B \mapsto \sigma_u^\xi(B)$ is a bounded scalar-valued Radon measure for every $\xi \in \mathbb{R}^d \setminus \{0\}$, it follows from (5.5.5) that the same property holds for $B \mapsto \mu_u(B)\xi \cdot \xi$. The polarisation identity (5.5.1) then implies that $B \mapsto \mu_u(B)\xi \cdot \eta$ belongs to $\mathcal{M}_b(\Omega)$ for every $\xi, \eta \in \mathbb{R}^d$, hence $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$.

To prove (5.5.4) let us first show that

$$|\mu_u| \le \lambda_u \llcorner (\Omega \setminus J_u^1) \quad \text{as Borel measures on } \Omega. \tag{5.5.6}$$

To this aim, we observe that, since μ_u takes values in $\mathbb{R}^{d\times d}_{\rm sym}$, for every Borel set $B \subset \Omega$ the operator norm $|\mu_u(B)|$ satisfies

$$|\mu_u(B)| = \sup_{\xi \in \mathbb{S}^{d-1}} |(\mu_u(B)\xi \cdot \xi)| = \sup_{\xi \in \mathbb{S}^{d-1}} |\sigma_u^\xi(B)|,$$

so that by (1.4.15) and (5.1.2) we have

$$|\mu_u|(B) = \sup \sum |\sigma_u^{\xi_i}(B_i)| \le \sup \sum \lambda_u^{\xi_i}(B_i \setminus J_u^1) = \lambda_u(B \setminus J_u^1),$$

where the *supremum* is taken over all finite Borel partitions $(B_i)_i$ of B and all finite collections of vectors $(\xi_i)_i \subset \mathbb{S}^{d-1}$. This shows (5.5.6).

To prove the inequality

$$|\mu_u| \ge \lambda_u \llcorner (\Omega \setminus J_u^1) \quad \text{as Borel measures on } \Omega, \tag{5.5.7}$$

we argue as follows. Consider the measure $\lambda \in \mathcal{M}_b^+(\Omega)$ defined for Borel set $B \subset \Omega$ by

$$\lambda(B) \coloneqq |\mu_u|(B) + \mathcal{H}^{d-1}(B \cap J_u^1).$$

Thanks to (5.5.6), Lemma 5.B.1, Remark 1.4.8, and Theorem 1.4.13 it follows that λ satisfies (1.4.5). Since λ_u is the minimal measure that satisfies (1.4.5) it follows that $\lambda_u(B) \le \lambda(B)$ for every Borel set $B \subset \Omega$, which implies (5.5.7). □

Remark 5.5.2. By Remark 5.1.3 it follows immediately from (5.5.5) that if $u \in \mathrm{BD}(\Omega)$, then $\mu_u = (Eu) \llcorner (\Omega \setminus J_u^1)$ as Borel measures on Ω.

Given $u \in \mathrm{GBD}(\Omega)$, the Lebesgue Decomposition Theorem allows us to decompose the measure μ_u as the sum of a measure μ_u^a, which is absolutely continuous with respect to $\mathcal{L}^d$, and a measure μ_u^s, which is singular with respect to $\mathcal{L}^d$. In the following definition, we introduce a further decomposition of μ_u, which closely resembles the classical decomposition (1.4.1) of Eu for a function $u \in \mathrm{BD}(\Omega)$.

Definition 5.5.3. For $u \in \mathrm{GBD}(\Omega)$ we introduce the measures $\mu_u^c, \mu_u^j \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$, called the *Cantor* part and the *jump* part of μ_u, defined for every Borel set $B \subset \Omega$ by

$$\begin{aligned} \mu_u^c(B) &\coloneqq \mu_u^s(B \setminus J_u), \\ \mu_u^j(B) &\coloneqq \mu_u^s(B \cap J_u) = \mu_u(B \cap J_u). \end{aligned}$$

Since $\mu_u = \mu_u^a + \mu_u^s$, we have $\mu_u = \mu_u^a + \mu_u^c + \mu_u^j$ as Borel measures on Ω.

Remark 5.5.4. It follows from Remark 5.5.2 that if $u \in \mathrm{BD}(\Omega)$ then the measure μ_u^c of Definition 5.5.3 coincides with the Cantor part $E^c u$ (see [17, Definition 4.1]) of the symmetrised gradient Eu.

We recall that for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every $y \in \Pi^\xi$ such that $u_y^\xi \in BV_{\mathrm{loc}}(\Omega_y^\xi)$ and $Du_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$ we can consider the measures $D^a u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$ and $D^s u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$, defined as the absolutely continuous and the singular part of Du_y^ξ with respect to the one-dimensional Lebesgue measure, and the measures $D^c u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$ and $D^j u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$, defined for every Borel set $B \subset \Omega_y^\xi$ by

$$D^c u_y^\xi(B) := D^s u_y^\xi(B \setminus J_{u_y^\xi}),$$
$$D^j u_y^\xi(B) := D^s u_y^\xi(B \cap J_{u_y^\xi}) = Du_y^\xi(B \cap J_{u_y^\xi}).$$

Since $Du_y^\xi = D^a u_y^\xi + D^s u_y^\xi$, we have $Du_y^\xi = D^a u_y^\xi + D^c u_y^\xi + D^j u_y^\xi$ as Borel measures on Ω_y^ξ.

We now show that the measures μ_u^a and μ_u^j can be expressed as suitable integrals depending on the approximate symmetric gradient $\mathcal{E}u$ (see Theorem 1.4.13) and $[u]$, respectively, and that μ_u^c can be expressed by means of $D^c u_y^\xi$.

Proposition 5.5.5. *Let $u \in \mathrm{GBD}(\Omega)$. Then*

$$\mu_u^a(B) = \int_B \mathcal{E}u \,\mathrm{d}x, \tag{5.5.8}$$
$$\mu_u^j(B) = \int_{(J_u \setminus J_u^1) \cap B} ([u] \odot \nu_u)\, \mathrm{d}\mathcal{H}^{d-1}, \tag{5.5.9}$$

for every Borel set $B \subset \Omega$. Moreover, for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$ we have

$$\mu_u^c(B)\xi \cdot \xi = |\xi| \int_{\Pi^\xi} D^c u_y^\xi(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.5.10}$$

Proof. Let us fix a Borel set $B \subset \Omega$. By definition of μ_u^a and of μ_u^j, it follows from the polarisation identity (5.5.1) and from (5.5.5) that

$$\mu_u^a(B)\xi \cdot \eta = \frac{1}{4}((\sigma_u^{\xi+\eta})^a(B) - (\sigma_u^{\xi-\eta})^a(B)), \tag{5.5.11}$$
$$\mu_u^j(B)\xi \cdot \eta = \frac{1}{4}(\sigma_u^{\xi+\eta}(B \cap J_u) - \sigma_u^{\xi-\eta}(B \cap J_u)), \tag{5.5.12}$$

for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$, where for a vector $\zeta \in \mathbb{R}^d \setminus \{0\}$ the measure $(\sigma_u^\zeta)^a$ is the absolutely continuous part of σ_u^ζ with respect to $\mathcal{L}^d$. By Lemma 5.C.2 for every $\zeta \in \mathbb{R}^d \setminus \{0\}$ we have

$$(\sigma_u^\zeta)^a(B) = |\zeta| \int_{\Pi^\zeta} D^a u_y^\zeta(B_y^\zeta)\, \mathrm{d}\mathcal{H}^{d-1}(y), \tag{5.5.13}$$
$$(\sigma_u^\zeta)^s(B) = |\zeta| \int_{\Pi^\zeta} D^s u_y^\zeta((B \setminus J_u^1)_y^\zeta)\, \mathrm{d}\mathcal{H}^{d-1}(y), \tag{5.5.14}$$

where $(\sigma_u^\zeta)^s$ is the singular part of σ_u^ζ with respect to $\mathcal{L}^d$. In light of Theorem 1.4.13, using (5.5.13) and the Fubini Theorem, we deduce that

$$(\sigma_u^\zeta)^a(B) = \int_B \mathcal{E}u\, \zeta \cdot \zeta \,\mathrm{d}x$$

for every $\zeta \in \mathbb{R}^d \setminus \{0\}$. Combining this equality with (5.5.11), we obtain that

$$\mu_u^a(B)\xi \cdot \eta = \int_B \mathcal{E}u\,\xi \cdot \eta \,\mathrm{d}x = \big(\int_B \mathcal{E}u \,\mathrm{d}x\big)\xi \cdot \eta$$

for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$, which proves (5.5.8).

To prove (5.5.9), we observe that by Proposition 5.1.7 for every $\zeta \in \mathbb{R}^d \setminus \{0\}$ we have

$$\sigma_u^\zeta(B \cap J_u) = \int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u)\zeta \cdot \zeta \,\mathrm{d}\mathcal{H}^{d-1}.$$

Together with (5.5.12), this equality shows that

$$\mu_u^j(B)\xi \cdot \eta = \int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u)\xi \cdot \eta \,\mathrm{d}\mathcal{H}^{d-1} = \big(\int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\big)\xi \cdot \eta$$

for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$. This proves (5.5.9).

To conclude, we fix $\xi \in \mathbb{R}^d \setminus \{0\}$ and a Borel set $B \subset \Omega$ and show that (5.5.10) holds. We observe that for every $y \in \Pi^\xi$ such that (1.4.16) holds we have $|D^j u_y^\xi|((B \setminus J_u)_y^\xi) = 0$, hence

$$D^s u_y^\xi((B \setminus J_u)_y^\xi) = D^c u_y^\xi((B \setminus J_u)_y^\xi).$$

Since (1.4.16) holds for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, from this equality, (5.5.5), (5.5.14), and Definition 5.5.3 it follows that

$$\mu_u^c(B)\xi \cdot \xi = (\sigma_u^\xi)^s(B \setminus J_u) = |\xi| \int_{\Pi^\xi} D^c u_y^\xi((B \setminus J_u)_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.5.15}$$

Recalling Lemma 5.B.1, by Remark 1.4.8 for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the set $(J_u)_y^\xi$ is finite or countable. Recalling the properties of the derivatives of BV functions in dimension one, this implies that $D^c u_y^\xi((J_u)_y^\xi) = 0$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Therefore (5.5.15) implies (5.5.10). □

The following corollary shows that in the previous results we can replace J_u^1 by J_u^r (see (1.3.10)) for an arbitrary $r > 0$ and that the absolutely continuous and the Cantor part of the corresponding measure $\mu_{u,r}$ do not depend on r.

Corollary 5.5.6. *Let $d \geq 1$, $u \in \mathrm{GBD}(\Omega)$, and $r > 0$. Then there exists a measure $\mu_{u,r} \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\mu_{u,r}(B)\xi \cdot \xi = \lim_{R\to+\infty} D_\xi(\tau_R(u \cdot \xi))(B \setminus J_u^r) \quad \text{for every Borel set } B \subset \Omega, \tag{5.5.16}$$

where τ_R are the truncation functions defined in (5.1.4)*. Moreover, setting $\mu_{u,r}^j := \mu_{u,r} \llcorner J_u$, we have*

$$\mu_{u,r}^j(B) = \int_{(J_u \setminus J_u^r)\cap B} ([u] \odot \nu_u)\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for every Borel set } B \subset \Omega. \tag{5.5.17}$$

Finally, we have $\mu_{u,r} = \mu_u^a + \mu_u^c + \mu_{u,r}^j$ as Borel measures on Ω.

Proof. Let $v := u/r$ and $\mu_{u,r} := r\mu_v$. Using the equalities

$$\tau_R(v \cdot \xi) = \frac{1}{r}\tau_{(rR)}(u \cdot \xi) \quad \text{and} \quad J_v^1 = J_u^r, \tag{5.5.18}$$

from (5.5.2) we obtain (5.5.16). Using the equalities $J_v^1 = J_u^r$ and $J_v = J_u$, from (5.5.9) we deduce (5.5.17).

Let $\mu_{u,r}^a$ and $\mu_{u,r}^s$ be the absolutely continuous and the singular part of $\mu_{u,r}$ with respect to $\mathcal{L}^d$. By (5.5.2) and (5.5.16), with B replaced by $B\setminus J_u$, we obtain $\mu_{u,r}(B\setminus J_u) = \mu_u(B\setminus J_u)$. This implies that $\mu_{u,r}^a(B) = \mu_u^a(B)$ and $\mu_{u,r}^s(B\setminus J_u) = \mu_u^s(B\setminus J_u) = \mu_u^c(B)$ for every Borel set $B \subset \Omega$. Hence, $\mu_{u,r}(B) = \mu_{u,r}^a(B) + \mu_{u,r}^s(B \setminus J_u) + \mu_{u,r}^s(B \cap J_u) = \mu_u^a(B) + \mu_u^c(B) + \mu_{u,r}^j(B)$ for every Borel set $B \subset \Omega$. □

Remark 5.5.7. Using the function $v := u/r$, it follows from Remark 5.5.2 that, if $u \in \mathrm{BD}(\Omega)$, then $\mu_{u,r} = (Eu) \llcorner (\Omega \setminus J_u^r)$ as Borel measures on Ω.

The following result shows that, in analogy with $E^c u$, the measure μ_u^c does not charge Borel sets which are σ-finite with respect to $\mathcal{H}^{d-1}$.

Proposition 5.5.8. *Let $u \in \mathrm{GBD}(\Omega)$ and let B be a Borel set that is σ-finite with respect to $\mathcal{H}^{d-1}$. Then $|\mu_u^c|(B) = 0$.*

Proof. It is not restrictive to assume that $\mathcal{H}^{d-1}(B) < +\infty$. Let us fix $\xi \in \mathbb{S}^{d-1}$. Thanks to Lemma 5.B.1, we have $\mathcal{H}^0(B_y^\xi) < +\infty$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. By the properties of one-dimensional BV-functions this implies that $D^c u_y^\xi(B_y^\xi) = 0$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Hence, by (5.5.10) we have $\mu_u^c(B)\xi \cdot \xi = 0$. Since this is true for every $\xi \in \mathbb{S}^{d-1}$, we obtain $\mu_u^c(B) = 0$. As this property holds also for every Borel subset of B, we deduce that $|\mu_u^c|(B) = 0$. □

The definition and properties of μ_u allow us to give a new characterisation of the space $\mathrm{GSBD}(\Omega)$, originally defined by slicing. We recall that $\mathrm{GSBD}(\Omega)$ (see [89, Definition 4.2]) is the space of all $u \in \mathrm{GBD}(\Omega)$ such that for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have

$$u_y^\xi \in \mathrm{SBV}_{\mathrm{loc}}(\Omega_y^\xi). \tag{5.5.19}$$

Theorem 5.5.9. *Let $u \in \mathrm{GBD}(\Omega)$. Then $u \in \mathrm{GSBD}(\Omega)$ if and only if $\mu_u^c = 0$.*

Proof. Assume that $\mu_u^c = 0$ as a Borel measure on Ω. Recalling the uniqueness of the disintegration of measures (see [18, Theorem 2.28]), it follows from (5.5.10) that for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $D^c u_y^\xi = 0$ as a Borel measure on Ω_y^ξ, i.e., (5.5.19) holds. By definition this implies that $u \in \mathrm{GSBD}(\Omega)$.

Conversely, if $u \in \mathrm{GSBD}(\Omega)$ it follows from (5.5.10) that $\mu_u^c(B)\xi \cdot \xi = 0$ for every $\xi \in \mathbb{S}^{d-1}$ and every Borel set $B \subset \Omega$. This implies that $\mu_u^c = 0$. □

Combining this result with recent results of [74], where new characterisations of the spaces $\mathrm{GBD}(\Omega)$ and $\mathrm{GSBD}^p(\Omega)$, for $p > 1$, are obtained, we can give an analogous characterisation for $\mathrm{GSBD}(\Omega)$. More precisely, we show that an $\mathcal{L}^d$-measurable function $u\colon \Omega \to \mathbb{R}^d$ belongs to $\mathrm{GSBD}(\Omega)$ if and only if (1.4.5) and (5.5.19) hold for a suitable finite number of directions $\xi \in \mathbb{S}^{d-1}$.

Theorem 5.5.10. *Let $u\colon \Omega \to \mathbb{R}^d$ be an $\mathcal{L}^d$-measurable function. Assume that there exists an orthonormal basis $\{\xi_i\colon\ i = 1, \dots, d\}$ such that for every $\xi \in \Xi := \{\xi_i\colon\ i = 1, \dots, d\} \cup \{\xi_i + \xi_j\colon\ 1 \le i \le j \le d\}$ the two following conditions hold:*

$$u_y^\xi \in \mathrm{SBV}_{\mathrm{loc}}(\Omega_y^\xi) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi, \tag{5.5.20}$$

$$\Lambda_u^\xi := \int_{\Pi^\xi} |Du_y^\xi|(\Omega_y^\xi \setminus J^1_{u_y^\xi}) + \mathcal{H}^0(J^1_{u_y^\xi})\, \mathrm{d}\mathcal{H}^{d-1}(y) < +\infty. \tag{5.5.21}$$

Then $u \in \mathrm{GSBD}(\Omega)$ and, setting $\Lambda := \sum_{\xi\in\Xi} \Lambda_u^\xi$, there exists a constant $C_d > 0$, depending only on d, such that

$$\lambda_u(\Omega) \le C_d \Lambda, \tag{5.5.22}$$

where λ_u is the measure defined by (1.4.15).

Proof. Since the inclusion $u \in \mathrm{GBD}(\Omega)$ and inequality (5.5.22) follow directly from [74, Theorem 1, Corollary 1], to conclude we only need to show that $u \in \mathrm{GSBD}(\Omega)$.

To prove this, we observe that from (5.5.10) it follows that given a Borel set $B \subset \Omega$ we have that

$$\mu_u^c(B)\xi \cdot \xi = |\xi| \int_{\Pi^\xi} D^c u_y^\xi(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y)$$

for every $\xi \in \Xi$. By (5.5.20) this equality implies that $\mu_u^c(B)\xi \cdot \xi = 0$ for every $\xi \in \Xi$. From the polarisation identity (5.5.1) we obtain $\mu_u^c(B)\xi_i \cdot \xi_j = 0$ for every $i, j = 1, \dots, d$. Recalling that $\{\xi_i\}_i$ is a basis of $\mathbb{R}^d$, we deduce that $\mu_u^c(B) = 0$. Since this property holds for every Borel set $B \subset \Omega$, from Theorem 5.5.9 we obtain that $u \in \mathrm{GSBD}(\Omega)$, concluding the proof. □

Remark 5.5.11. An alternative proof of the Theorem above, which does not rely on the existence of the matrix-valued measure μ_u, can be deduced by combining Proposition 5.1.8 with the arguments of [74, Remark 2]. In that remark, Chambolle & Crismale show that the conclusion of Theorem 5.5.10 holds for all $u \in L^0(\Omega; \mathbb{R}^d)$ satisfying (5.5.20)–(5.5.21) and the additional constraint $\mathcal{H}^{d-1}(J_u) < +\infty$.

Given a function $u \in L^0(\Omega; \mathbb{R}^d)$ that satisfies (5.5.20)–(5.5.21) but not $\mathcal{H}^{d-1}(J_u) < +\infty$, we may use [74, Corollary 1] to obtain that $u \in \mathrm{GBD}(\Omega)$ and then Proposition 5.1.8 to find $v \in \mathrm{SBV}(\Omega; \mathbb{R}^d)$ such that $w := u - v$ satisfies $J_w \simeq J_u^1$, which, by Remark 1.4.8, implies $\mathcal{H}^{d-1}(J_w) < +\infty$. Since $v \in \mathrm{SBV}(\Omega; \mathbb{R}^d)$, it follows immediately from the analogous properties for u that w satisfies also (5.5.20) and (5.5.21); the remark of Chambolle & Crismale then implies $w \in \mathrm{GSBD}(\Omega)$. As u differs from w by a $\mathrm{SBV}(\Omega; \mathbb{R}^d)$-function, it belongs to $\mathrm{GSBD}(\Omega)$ as well, concluding the proof.

Appendix to Chapter 5

The purpose of this appendix is to collect some auxiliary results needed in the previous sections. In Section 5.A, we prove some variants of classical results concerning the approximation of Lebesgue integrals by means of Riemann sums. These results constitute one of the main ingredients of the discretisation argument developed in Sections 5.2 and 5.3.

Section 5.B is devoted to an elementary lemma in geometric measure theory. The aim of Sections 5.C and 5.D is to prove the Borel measurability of the functions $g_m^{k,\zeta}$ and $h_m^{k,\zeta}$, defined in (5.3.56) and (5.3.45), respectively, and to establish some general properties of measures defined through integration.

Results related to those proved in these two last sections are well-established in the literature. Nevertheless, because of the particular form of the functions considered here, the available results do not seem to apply directly in our setting. We therefore state and prove precisely the auxiliary results required in the sequel.

5.A Approximations by Riemann sums

In the proof of the main result of this chapter, we need to approximate various integrals with well-chosen Riemann sums, using a suitable version of a result that goes back to Hahn (see [138]). A similar result is also proved in [61, Lemma A.1]. For technical reasons, we use a construction described in [114, Page 63] and further developed in [95, Lemma 4.12]. Since this result is crucial for our arguments, we give here the precise statement and a detailed proof.

Lemma 5.A.1. *Let $I = [a,b] \subset \mathbb{R}$ be a bounded closed interval. For every $z \in I$, $k \in \mathbb{N}$, and $i \in \mathbb{Z}$ let $t_i^k := z + i/k$. Let $(X, \|\cdot\|)$ be a Banach space and let $f\colon \mathbb{R} \to X$ be a Bochner integrable function such that $f = 0$ on $\mathbb{R} \setminus I$. Then there exist an infinite subset $K \subset \mathbb{N}$ and an $\mathcal{L}^1$-negligible set $N \subset I$ such that*

$$\lim_{\substack{k\to+\infty \\ k\in K}} \sum_{i\in\mathbb{Z}} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\| \,\mathrm{d}t = 0, \tag{5.A.1}$$

for every $z \in I \setminus N$. Moreover, if for $h \in \mathbb{N}$ we set $\mathcal{I}_h^k := \{i \in \mathbb{Z}\colon\ [t_{i-h}^k, t_{i+h}^k] \subset I\}$ and $\mathcal{F}_h^k := \{i \in \mathbb{Z}\colon\ t_i^k \in I\} \setminus \mathcal{I}_h^k$, for every $\varepsilon > 0$ there exists a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \le \varepsilon$, such that for every $h \in \mathbb{N}$ one has

$$\lim_{\substack{k\to+\infty \\ k\in K}} \sum_{i\in\mathbb{Z}} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\| \,\mathrm{d}t = 0 \quad \textit{uniformly for } z \in I_\varepsilon, \tag{5.A.2}$$

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k} \sum_{i\in\mathcal{I}_h^k} f(t_i^k) = \int_I f(t) \,\mathrm{d}t \quad \textit{uniformly for } z \in I_\varepsilon, \tag{5.A.3}$$

$$\lim_{\substack{k\to+\infty \\ k\in K}} \frac{1}{k} \sum_{i\in\mathcal{F}_h^k} \|f(t_i^k)\| = 0 \quad \textit{uniformly for } z \in I_\varepsilon. \tag{5.A.4}$$

Proof. We follow closely the proof of [95, Lemma 4.12]. Given $k \in \mathbb{N}$, we consider the set

$$\mathcal{J}^k := \{i \in \mathbb{Z}\colon\ k(a-b) - 1 \le i \le k(b-a)\}. \tag{5.A.5}$$

Note that for every $z \in I$ we have

$$[t_i^k, t_{i+1}^k] \cap I \neq \emptyset \quad \Longrightarrow \quad i \in \mathcal{J}^k.$$

Integrating with respect to z the sum on the left-hand side of (5.A.1) and using Fubini's theorem we obtain

$$\begin{aligned}
\int_I \Big(\sum_{i\in\mathbb{Z}} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\| \,\mathrm{d}t \Big) \mathrm{d}z &= \int_I \Big(\sum_{i\in\mathcal{J}^k} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\| \,\mathrm{d}t \Big) \mathrm{d}z \\
&= \int_I \Big(\sum_{i\in\mathcal{J}^k} \int_0^{\frac{1}{k}} \|f(z+\tfrac{i}{k}) - f(z+\tfrac{i}{k}+s)\| \,\mathrm{d}s \Big) \mathrm{d}z \\
&\leq \sum_{i\in\mathcal{J}^k} \int_0^{\frac{1}{k}} \Big(\int_{-\infty}^{+\infty} \|f(z+\tfrac{i}{k}) - f(z+\tfrac{i}{k}+s)\| \,\mathrm{d}z \Big) \mathrm{d}s \\
&= \sum_{i\in\mathcal{J}^k} \int_0^{\frac{1}{k}} \Big(\int_{-\infty}^{+\infty} \|f(z) - f(z+s)\| \,\mathrm{d}z \Big) \mathrm{d}s. \qquad (5.A.6)
\end{aligned}$$

By the L^1-continuity of translations for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-\infty}^{+\infty} \|f(z) - f(z+s)\| \,\mathrm{d}z < \varepsilon$$

whenever $0 \leq s < \delta$. Hence,

$$\int_0^{\frac{1}{k}} \Big(\int_{-\infty}^{+\infty} \|f(z) - f(z+s)\| \,\mathrm{d}z \Big) \mathrm{d}s < \frac{\varepsilon}{k}$$

for $k > 1/\delta$. Observing that the number of elements of $\mathcal{J}^k$ satisfies $\mathcal{H}^0(\mathcal{J}^k) \leq 2k(b-a)+2$, from the previous inequality and (5.A.6) we deduce that

$$\lim_{k\to+\infty} \int_I \Big(\sum_{i\in\mathbb{Z}} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\| \,\mathrm{d}t \Big) \mathrm{d}z = 0.$$

Hence, there exists an infinite set $K \subset \mathbb{N}$ and an $\mathcal{L}^1$-negligible set $N \subset I$ such that (5.A.1) holds.

To prove the second part of the statement, we first observe that by Egorov's theorem there exists a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \leq \varepsilon$, such that (5.A.2) holds.

To prove (5.A.3) and (5.A.4), we first observe that, since the number of elements of

$$\widehat{\mathcal{F}}_h^k := \{i \in \mathbb{Z}\colon\ [t_i^k, t_{i+1}^k] \cap I \neq \emptyset\} \setminus \mathcal{I}_h^k$$

is less than $2h+2$, by the absolute continuity of the integral we have

$$\lim_{k\to+\infty} \sum_{i\in\widehat{\mathcal{F}}_h^k} \int_{t_i^k}^{t_{i+1}^k} \|f(t)\| \,\mathrm{d}t = 0 \quad \text{ uniformly for } z \in I_\varepsilon. \qquad (5.A.7)$$

Since

$$\frac{1}{k} \sum_{i\in\mathcal{F}_h^k} \|f(t_i^k)\| \leq \sum_{i\in\mathbb{Z}} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\| \,\mathrm{d}t + \sum_{i\in\mathcal{F}_h^k} \int_{t_i^k}^{t_{i+1}^k} \|f(t)\| \,\mathrm{d}t,$$

(5.A.4) follows from (5.A.2), (5.A.7), and from the inclusion $\mathcal{F}_h^k \subset \widehat{\mathcal{F}}_h^k$. Similarly, the inequality

$$\Big\| \frac{1}{k} \sum_{i \in \mathcal{I}_h^k} f(t_i^k) - \int_I f(t)\,\mathrm{d}t \Big\| \le \sum_{i \in \mathcal{I}_h^k} \int_{t_i^k}^{t_{i+1}^k} \|f(t_i^k) - f(t)\|\,\mathrm{d}t + \sum_{i \in \widehat{\mathcal{F}}_h^k} \int_{t_i^k}^{t_{i+1}^k} \|f(t)\|\,\mathrm{d}t,$$

together with (5.A.2) and (5.A.7), gives (5.A.3). □

In the next technical result we will consider the Riemann sums associated with functions converging to 0 in L^1 and depending on an additional parameter. This result is crucial in Section 5.2.

Lemma 5.A.2. *Let $I := [a,b]$ and $J := [c,d]$ be bounded closed intervals. For every $z_1 \in I$, $k \in \mathbb{N}$, and $i \in \mathbb{Z}$ let $t_i^k := z_1 + \frac{i}{k}$. Let $(X, \|\cdot\|)$ be a Banach space and let $f_k \colon \mathbb{R} \times \mathbb{R} \to X$ be a sequence of Bochner integrable functions, with $f_k = 0$ on $(\mathbb{R} \times \mathbb{R}) \setminus (I \times J)$, such that*

$$\lim_{k \to +\infty} f_k(t, z_2) = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in I \text{ and } \mathcal{L}^1\text{-a.e. } z_2 \in J. \tag{5.A.8}$$

Assume also that there exists an integrable function $g \colon \mathbb{R} \to [0, +\infty)$ such that

$$\|f_k(t, z_2)\| \le g(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in I,\ \mathcal{L}^1\text{-a.e. } z_2 \in J, \text{ and every } k \in \mathbb{N}. \tag{5.A.9}$$

Then there exists an $\mathcal{L}^2$-negligible set $N \subset I \times J$ and an infinite set $K \subset \mathbb{N}$ such that for every $(z_1, z_2) \in (I \times J) \setminus N$ we have

$$\lim_{\substack{k \to +\infty \\ k \in K}} \frac{1}{k} \sum_{i \in \mathbb{Z}} \|f_k(t_i^k, z_2)\| = 0. \tag{5.A.10}$$

Moreover, for every $\varepsilon > 0$ there exists a Borel set $A_\varepsilon \subset I \times J$, with $\mathcal{L}^2((I \times J) \setminus A_\varepsilon) \le \varepsilon$, such that

$$\lim_{\substack{k \to +\infty \\ k \in K}} \frac{1}{k} \sum_{i \in \mathbb{Z}} \|f_k(t_i^k, z_2)\| = 0 \text{ uniformly for } (z_1, z_2) \in A_\varepsilon. \tag{5.A.11}$$

Proof. For $k \in \mathbb{N}$ let $\mathcal{J}^k$ be given by (5.A.5). Integrating the sum on the left-hand side of (5.A.10) with respect to $z = (z_1, z_2) \in I \times J$

$$\begin{aligned} &\int_J \Big(\int_I \Big(\frac{1}{k} \sum_{i \in \mathbb{Z}} \|f_k(t_i^k, z_2)\| \Big) \mathrm{d}z_1 \Big) \mathrm{d}z_2 = \int_J \Big(\int_I \Big(\frac{1}{k} \sum_{i \in \mathbb{Z}} \|f_k(z_1 + \tfrac{i}{k}, z_2)\| \Big) \mathrm{d}z_1 \Big) \mathrm{d}z_2 \\ &\le \frac{1}{k} \sum_{i \in \mathcal{J}^k} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \|f_k(z_1 + \tfrac{i}{k}, z_2)\|\,\mathrm{d}z_1 \Big) \mathrm{d}z_2 = \frac{\mathcal{H}^0(\mathcal{J}^k)}{k} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \|f_k(z_1, z_2)\|\,\mathrm{d}z_1 \Big) \mathrm{d}z_2, \end{aligned} \tag{5.A.12}$$

where in the inequality we have used the hypothesis on the supports of f_k. By (5.A.8) for $\mathcal{L}^2$-a.e. $z \in I \times J$ the sequence $\|f_k(z_1, z_2)\|$ converges to 0 as $k \to +\infty$. Thanks to (5.A.9), by the Dominated Convergence Theorem we have $\lim_k \|f_k\|_{L^1(\mathbb{R}^2; X)} = 0$. This fact, together with (5.A.12) and the boundedness of $\mathcal{H}^0(\mathcal{J}^k)/k$, implies that

$$\lim_{k \to +\infty} \int_{I \times J} \Big(\frac{1}{k} \sum_{i \in \mathbb{Z}} \|f_k(t_i^k, z_2)\| \Big) \mathrm{d}z = 0,$$

which gives (5.A.10). By Egorov's Theorem we obtain also (5.A.11), concluding the proof. □

Remark 5.A.3. Suppose that for every $n \in \mathbb{N}$ we have a function f^n that satisfies the hypotheses of Lemma 5.A.1. Arguing as in [95, Remark 4.13], one can find sets K and I_ε as in the statement of Lemma 5.A.1 for which (5.A.2)–(5.A.4) hold with $f = f^n$ for every $n \in \mathbb{N}$.

Similarly, if for every $n \in \mathbb{N}$ we have a sequence of functions $\{f_k^n\}_k$ that satisfies the hypotheses of Lemma 5.A.2, the same argument shows that one can find sets K and A_ε as in the statement of Lemma 5.A.2 for which (5.A.10) and (5.A.11) hold with $f_k = f_k^n$ for every $n \in \mathbb{N}$.

5.B An elementary lemma in geometric measure theory

Lemma 5.B.1. *Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and let $B \subset \mathbb{R}^d$ be a Borel set. Then the function $y \mapsto \mathcal{H}^0(B_y^\xi)$ is $\mathcal{H}^{d-1}$-measurable on Π^ξ and*

$$\int_{\Pi^\xi} \mathcal{H}^0(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \mathcal{H}^{d-1}(B). \tag{5.B.1}$$

Proof. For every $k \in \mathbb{N}$ and $i \in \mathbb{Z}$ we set $B_{k,i} \coloneqq \{x \in B\colon\ i/2^k \le x \cdot \xi < (i+1)/2^k\}$ and we consider the function $f_k\colon \Pi^\xi \to [0, +\infty]$ defined by

$$f_k(y) \coloneqq \sum_{i \in \mathbb{Z}} \chi_{\pi^\xi(B_{k,i})}(y), \tag{5.B.2}$$

where $\chi_{\pi^\xi(B_{k,i})}$ is the characteristic function of the projection of $B_{k,i}$ onto Π^ξ. We observe that $f_k \le f_{k+1}$ for every $k \in \mathbb{N}$.

It follows from the definition of B_y^ξ that

$$\mathcal{H}^0(B_y^\xi) = \lim_{k \to +\infty} f_k(y) \quad \text{for every } y \in \Pi^\xi. \tag{5.B.3}$$

By the Projection Theorem (see, e.g., [79, Proposition 8.4.4]) for every $k \in \mathbb{N}$ and $i \in \mathbb{Z}$ the set $\pi^\xi(B_{k,i})$ is $\mathcal{H}^{d-1}$-measurable on Π^ξ. By (5.B.2) and (5.B.3) this implies that the same is true for the function $y \mapsto \mathcal{H}^0(B_y^\xi)$. By the Monotone Convergence Theorem we obtain from (5.B.3) that

$$\int_{\Pi^\xi} \mathcal{H}^0(B_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \lim_{k \to +\infty} \int_{\Pi^\xi} f_k(y)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.B.4}$$

By (5.B.2) we have

$$\int_{\Pi^\xi} f_k(y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \sum_{i \in \mathbb{Z}} \mathcal{H}^{d-1}(\pi^\xi(B_{k,i})) \le \sum_{i \in \mathbb{Z}} \mathcal{H}^{d-1}(B_{k,i}) = \mathcal{H}^{d-1}(B),$$

which, together with (5.B.4), gives (5.B.1). □

5.C Lebesgue decomposition of measures defined by integration

In this subsection we consider measures defined on the slices of a set, depending on a parameter $\omega \in \mathbb{R}^k$, and the measures that can be obtained by integrating with respect to the parameters corresponding to the slices. We are interested in a formula for the Lebesgue decomposition of these measures.

We begin by a lemma concerning measurability conditions with respect to these parameters. Given $h, k \in \mathbb{N}$, a Borel set $B \subset \mathbb{R}^h \times \mathbb{R}^k$, and $\omega \in \mathbb{R}^k$ we set

$$B(\omega) \coloneqq \{x \in \mathbb{R}^h \colon (x, \omega) \in B\}. \tag{5.C.1}$$

Lemma 5.C.1. *Let $k \in \mathbb{N}$ and let $\zeta \in \mathbb{R}^d \setminus \{0\}$. For every $y \in \Pi^\zeta$ and $\omega \in \mathbb{R}^k$ let $\mu_y^\omega \in \mathcal{M}_b(\mathbb{R})$ be a signed measure. The following three measurability conditions are equivalent:*

(a) for every $\psi \in C_c^0(\mathbb{R} \times \mathbb{R}^k)$

$$\text{the function } (y, \omega) \mapsto \int_{-\infty}^{+\infty} \psi(t, \omega) \, \mathrm{d}\mu_y^\omega(t) \text{ is Borel measurable on } \Pi^\zeta \times \mathbb{R}^k;$$

(b) for every $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$

$$\text{the function } (y, \omega) \mapsto \int_{-\infty}^{+\infty} \varphi(y + t\zeta, \omega) \, \mathrm{d}\mu_y^\omega(t) \text{ is Borel measurable on } \Pi^\zeta \times \mathbb{R}^k;$$

(c) for every Borel set $B \subset \mathbb{R}^d \times \mathbb{R}^k$ the function $(y, \omega) \mapsto \mu_y^\omega(B(\omega)_y^\zeta)$ is Borel measurable on $\Pi^\zeta \times \mathbb{R}^k$.

Moreover, if the previous conditions are satisfied, then for every Borel set $B \subset \mathbb{R}^d \times \mathbb{R}^k$ the function $(y, \omega) \mapsto |\mu_y^\omega|(B(\omega)_y^\zeta)$ is Borel measurable on $\Pi^\zeta \times \mathbb{R}^k$.

Proof. For simplicity of notation we consider only the case $\zeta = e_d$, the d-th vector of the canonical basis, the proof in the other cases being analogous. For $x \in \mathbb{R}^d$ we set $x = (x', x_d)$, where $x' \coloneqq (x_1, \dots, x_{d-1})$, and we identify $\Pi^{e_d} = \mathbb{R}^{d-1} \times \{0\}$ with $\mathbb{R}^{d-1}$. Therefore the measure $\mu_{(x',0)}^\omega$ is denoted simply by $\mu_{x'}^\omega$ and for the slicing of sets we use the notation $B_{x'}$ instead of $B_{(x',0)}^{e_d}$.

Assume (a). For every pair of functions $w \in C_c^0(\mathbb{R}^{d-1} \times \mathbb{R}^k)$ and $\psi \in C_c^0(\mathbb{R} \times \mathbb{R}^k)$ it follows from (a) that the function

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} w(x', \omega) \psi(x_d, \omega) \, \mathrm{d}\mu_{x'}^\omega(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$.

By an argument based on partition of unities corresponding to coverings by open sets with small diameter we can prove that the class of linear combinations of functions $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$ of the form $\varphi(x, \omega) = w(x', \omega)\psi(x_d, \omega)$ is dense in $C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$ with respect to the uniform convergence. This implies that the function

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} \varphi((x', x_d), \omega) \, \mathrm{d}\mu_{x'}^\omega(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$ for every $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$, which is condition (b) in the case $\zeta = e_d$.

Assume (b) and consider the class $\mathcal{F}$ of bounded Borel functions $f \colon \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ such that

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} f((x', x_d), \omega) \, \mathrm{d}\mu_{x'}^\omega(x_d) \text{ is Borel measurable on } \mathbb{R}^{d-1} \times \mathbb{R}^k.$$

It is easy to check that $\mathcal{F}$ is a monotone class, that is,

(i) if $\{f_n\}_n \subset \mathcal{F}$, with $f_n \le g$, for some $g \in \mathcal{F}$, and $f_n \nearrow f$, then $f \in \mathcal{F}$;

(ii) if $\{f_n\}_n \subset \mathcal{F}$, with $f_n \ge g$, for some $g \in \mathcal{F}$, and $f_n \searrow f$, then $f \in \mathcal{F}$.

Moreover, thanks to (b), we have that $C_c^0(\mathbb{R}^d \times \mathbb{R}^k) \subset \mathcal{F}$. Hence, from the Monotone Class Theorem (see [170, Section 3.14]) we deduce that for every bounded Borel function $f\colon \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ the function

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} f((x', x_d), \omega)\, \mathrm{d}\mu_{x'}^{\omega}(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$. By taking as f the characteristic function of a Borel set $B \subset \mathbb{R}^d \times \mathbb{R}^k$ we obtain that the function

$$(x', \omega) \mapsto \mu_{x'}^{\omega}(B(\omega)_{x'})$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$, which is condition (c) in the case $\zeta = e_d$.

Assume now (c). Let $E \subset \mathbb{R} \times \mathbb{R}^k$ be a Borel set and let $B := \mathbb{R}^{d-1} \times E \subset \mathbb{R}^d \times \mathbb{R}^k$. Clearly, $B(\omega)_{x'} = E(\omega)$ for every $x' \in \mathbb{R}^{d-1}$, so that by (c) the function $(x', \omega) \mapsto \mu_{x'}^{\omega}(E(\omega))$ is $\mathbb{R}^{d-1} \times \mathbb{R}^k$ measurable. Hence,

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} \chi_E(x_d, \omega)\, \mathrm{d}\mu_{x'}^{\omega}(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$. By linearity we obtain that

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} g(x_d, \omega)\, \mathrm{d}\mu_{x'}^{\omega}(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$ for every simple function $g\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. Since every function $\psi \in C_c^0(\mathbb{R} \times \mathbb{R}^k)$ can be approximated by a uniformly bounded sequence of simple functions, an application of the Dominated Convergence Theorem yields (a).

We now show that, if (a)-(c) hold, then the last part of the statement holds. By the equivalence of (a)-(c) for $|\mu_{x'}^{\omega}|$, to conclude the proof it is enough to show that for every function $\psi \in C_c^0(\mathbb{R} \times \mathbb{R}^k)$ the function

$$(x', \omega) \mapsto \int_{-\infty}^{+\infty} \psi(x_d, \omega)\, \mathrm{d}|\mu_{x'}^{\omega}|(x_d) \tag{5.C.2}$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$. Assume for a moment that $\psi \ge 0$. By definition of total variation of a measure we have

$$\int_{-\infty}^{+\infty} \psi(x_d, \omega)\, \mathrm{d}|\mu_{x'}^{\omega}|(x_d) = \sup_{\substack{\varphi \in C_c^0(\mathbb{R} \times \mathbb{R}^k) \\ |\varphi| \le \psi}} \int_{-\infty}^{+\infty} \varphi(x_d, \omega)\, \mathrm{d}\mu_{x'}^{\omega}(x_d).$$

Since the supremum above can be reduced to a countable dense subset of $C_c^0(\mathbb{R} \times \mathbb{R}^k)$, this equality, together with (a) for $\mu_{x'}^{\omega}$, implies that the function in (5.C.2) is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$ when $\psi \ge 0$. In the general case, one can split ψ into its positive and negative part. □

Let $\zeta \in \mathbb{R}^d \setminus \{0\}$ and for every $y \in \Pi^\zeta$ let $\mu_y \in \mathcal{M}_b(\Omega_y^\zeta)$. For every $y \in \Pi^\zeta$ let μ_y^a and μ_y^s be the absolutely continuous part and the singular part of μ_y with respect to the one dimensional Lebesgue measure. In the following lemma we consider some general conditions on $\{\mu_y\}_{y\in\Pi^\zeta}$ which allow us to define a measure μ on Ω by integrating μ_y with respect to y. We then show that the absolutely continuous part μ^a and its singular part μ^s of μ with respect to $\mathcal{L}^d$ can be obtained by integrating μ_y^a and μ_y^s with respect to y.

Lemma 5.C.2. *Let $\zeta \in \mathbb{R}^d \setminus \{0\}$ and for every $y \in \Pi^\zeta$ let $\mu_y \in \mathcal{M}_b(\Omega_y^\zeta)$. Assume that for every Borel set $B \subset \Omega$*

$$\text{the function } y \mapsto \mu_y(B_y^\zeta) \text{ is Borel measurable on } \Pi^\zeta \tag{5.C.3}$$

and that there exists $g \in L^1(\Pi^\zeta, \mathcal{H}^{d-1})$ such that

$$|\mu_y|(\Omega_y^\zeta) \le g(y) \text{ for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\zeta. \tag{5.C.4}$$

Consider the measure defined for every Borel set $B \subset \Omega$ by

$$\mu(B) \coloneqq \int_{\Pi^\zeta} \mu_y(B_y^\zeta)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{5.C.5}$$

Let μ^a and μ^s be the absolutely continuous part and the singular part of μ with respect to the Lebesgue measure $\mathcal{L}^d$. Then for every Borel set $B \subset \Omega$ the functions

$$y \mapsto \mu_y^a(B_y^\zeta) \text{ and } y \mapsto \mu_y^s(B_y^\zeta) \text{ are Borel measurable and } \mathcal{H}^{d-1}\text{-integrable on } \Pi^\zeta \tag{5.C.6}$$

and we have

$$\mu^a(B) = \int_{\Pi^\zeta} \mu_y^a(B_y^\zeta)\, \mathrm{d}\mathcal{H}^{d-1}(y) \quad \text{and} \quad \mu^s(B) = \int_{\Pi^\zeta} \mu_y^s(B_y^\zeta)\, \mathrm{d}\mathcal{H}^{d-1}(y) \tag{5.C.7}$$

for every Borel set $B \subset \Omega$.

Proof. As in the previous lemma we consider only the case $\zeta = e_d$ and use the notation of the proof of the previous lemma. We also drop the hypothesis that Ω is bounded and assume that $\Omega = \mathbb{R}^d$, as the result for a general Ω then easily follows.

We now prove (5.C.6). To this aim let

$$S \coloneqq \left\{ x \in \mathbb{R}^d \colon\ x = (x', x_d) \text{ and } \limsup_{\rho\to 0^+} \frac{|\mu_{x'}|((x_d-\rho, x_d+\rho))}{2\rho} = +\infty \right\} \tag{5.C.8}$$

and, for every $x' \in \mathbb{R}^{d-1}$, let $S_{x'}$ be the corresponding slice. By the Besicovitch Derivation Theorem (see [18, Theorem 2.22]) for every $x' \in \mathbb{R}^{d-1}$ we have

$$\mu_{x'}^s(B_{x'}) = \mu_{x'}(B_{x'} \cap S_{x'}) \quad \text{and} \quad \mu_{x'}^a(B_{x'}) = \mu_{x'}(B_{x'} \setminus S_{x'}) \tag{5.C.9}$$

for every Borel set $B \subset \mathbb{R}^d$.

Therefore, by (5.C.3) to prove (5.C.6) it is enough to show that the set S is Borel measurable. To this end, we note that, as the function $\rho \mapsto |\mu_{x'}|((x_d-\rho, x_d+\rho))$ is left-continuous, in the lim sup in (5.C.8) we can reduce to considering ρ varying in a countable dense set. Hence, to conclude that S is Borel measurable we only need to prove that for every $\rho > 0$ the function

$$(x', x_d) \mapsto |\mu_{x'}|((x_d-\rho, x_d+\rho)) = \int_{\mathbb{R}} \chi_{(-\rho,\rho)}(x_d - t)\, \mathrm{d}|\mu_{x'}|(t) \tag{5.C.10}$$

is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}$, where $\chi_{(-\rho,\rho)}$ denotes the characteristic function of $(-\rho,\rho)$. Let $\{\psi_n\}_n \subset C_c^0(\mathbb{R})$ be a sequence of functions with $\psi_n \le \psi_{n+1}$ for every $n\in\mathbb{N}$ and converging pointwise to $\chi_{(-\rho,\rho)}$ as $n\to+\infty$. By the Monotone Convergence Theorem for every $x'\in\mathbb{R}^{d-1}$ we have

$$\int_{\mathbb{R}} \chi_{(-\rho,\rho)}(x_d-t)\,\mathrm{d}|\mu_{x'}|(t) = \lim_{n\to+\infty}\int_{\mathbb{R}} \psi_n(x_d-t)\,\mathrm{d}|\mu_{x'}|(t). \tag{5.C.11}$$

For every $n\in\mathbb{N}$ the function

$$(x',x_d)\mapsto \int_{\mathbb{R}} \psi_n(x_d-t)\,\mathrm{d}|\mu_{x'}|(t)$$

is Borel measurable in x' for x_d fixed, thanks to (5.C.3) and Lemma 5.C.1, and continuous in x_d for x' fixed. Thus, it is Borel measurable in the product space $\mathbb{R}^{d-1}\times\mathbb{R}$. Thanks to (5.C.11), this implies that the function (5.C.10) is Borel measurable, which proves that the set S is Borel measurable and concludes the proof of the measurability property in (5.C.6). The integrability follows from (5.C.4).

Thanks to (5.C.6) we can define two bounded Radon measures on $\mathbb{R}^d$ by setting

$$\nu_1(B) \coloneqq \int_{\mathbb{R}^{d-1}} \mu^a_{x'}(B_{x'})\,\mathrm{d}\mathcal{H}^{d-1}(x') \quad \text{and} \quad \nu_2(B) \coloneqq \int_{\mathbb{R}^{d-1}} \mu^s_{x'}(B_{x'})\,\mathrm{d}\mathcal{H}^{d-1}(x')$$

for every Borel set $B\subset\mathbb{R}^d$. It follows from the Fubini Theorem that ν_1 is absolutely continuous with respect to $\mathcal{L}^d$ and that $\mathcal{L}^d(S)=0$. By (5.C.9) we also deduce that $\mu^s_{x'}(B_{x'}) = \mu^s_{x'}(B_{x'}\cap S_{x'})$, hence $\nu_2(B)=\nu_2(B\cap S)$ for every Borel set $B\subset\mathbb{R}^d$. This shows that ν_2 is singular with respect to the Lebesgue measure. Since $\mu=\nu_1+\nu_2$, the equalities in (5.C.7) follow from the uniqueness of the Lebesgue decomposition. □

5.D Measurability of the auxiliary functions used in Section 5.3

In this subsection we prove the measurability of the functions $g_m^{k,\zeta}$ and $h_{m,\varepsilon}^{k,\zeta}$ defined in (5.3.56) and (5.3.45).

As in the proof of Theorem 5.2.1, Ω is a bounded open set of $\mathbb{R}^2$, u is a function in $\mathrm{GBD}(\Omega)$ with $J_u^1 = J_u$, ξ and η are two linearly independent vectors in $\mathbb{R}^2$ and U is the parallelogram defined by (5.2.2). We keep u, ξ,η, and U fixed throughout the rest of the subsection.

We also recall that $J\subset U$ is the set defined in (5.2.9), and that the sets $E^{k,\zeta}$, $\widehat{E}_m^{k,\zeta}$, and $\check{E}_m^{k,\zeta}$ are defined in (5.3.14) and (5.3.19). Since it will be important to keep track of the dependence of such sets on ω, in the following we underline their dependence on ω by writing $E^{k,\zeta,\omega}$, $\widehat{E}_m^{k,\zeta,\omega}$, and $\check{E}_m^{k,\zeta,\omega}$.

We introduce some sets which will play a crucial role in our arguments. For every $k\in\mathbb{N}$, $m\in\mathbb{N}$, and $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ we set

$$\begin{aligned}
\mathfrak{E}^{k,\zeta} &\coloneqq \{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ x\in E^{k,\zeta,\omega}\}\subset\mathbb{R}^4,\\
\widehat{\mathfrak{E}}_m^{k,\zeta} &\coloneqq \{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ x\in \widehat{E}_m^{k,\zeta,\omega}\}\subset\mathbb{R}^4,\\
\check{\mathfrak{E}}_m^{k,\zeta} &\coloneqq \{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ x\in \check{E}_m^{k,\zeta,\omega}\}\subset\mathbb{R}^4,\\
\mathfrak{J} &\coloneqq \{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ x\in J\}\subset\mathbb{R}^4.
\end{aligned} \tag{5.D.1}$$

The following lemma addresses the Borel measurability of these sets. The proof is very similar to that of Lemma 5.3.2, but for the sake of completeness we give here all details.

Lemma 5.D.1. *The sets* $\mathfrak{E}^{k,\zeta}$, $\widehat{\mathfrak{E}}^{k,\zeta}_m$, $\check{\mathfrak{E}}^{k,\zeta}_m$, *and* $\mathfrak{J}$ *are Borel measurable subsets of* $\mathbb{R}^4$.

Proof. The property for $\mathfrak{J}$ is trivial. To prove the result for the other sets we consider the map $z^{k,\zeta}\colon \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ defined by (5.3.11) (see Figure 5.3), where the dependence on the variable ω is made clear by (5.2.3). By elementary geometrical arguments, it follows that $(x,\omega)\mapsto z^{k,\zeta}(x,\omega)$ is Borel measurable.

For every set $B\subset\mathbb{R}^2$ we define

$$\mathfrak{E}^{k,\zeta}_B := \{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ (z^{k,\zeta}(x,\omega)+\tfrac{1}{k}S)\cap B\neq\emptyset\},$$
$$F_B := \{z\in\mathbb{R}^2\colon\ (z+\tfrac{1}{k}S)\cap B\neq\emptyset\},$$

where S is defined by (5.3.13). For a compact set $K\subset\mathbb{R}^2$ the set F_K is closed, so that, observing that $\mathfrak{E}^{k,\zeta}_K=\{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ z^{k,\zeta}(x,\omega)\in F_K\}$ and recalling that $z^{k,\zeta}$ is Borel measurable, we conclude that $\mathfrak{E}^{k,\zeta}_K$ is Borel measurable. By definition, $\mathfrak{E}^{k,\zeta}=\mathfrak{E}^{k,\zeta}_J$. Since $J=\bigcup_{n\in\mathbb{N}}K_n$ with K_n compact, this gives that $\mathfrak{E}^{k,\zeta}=\bigcup_{n\in\mathbb{N}}\mathfrak{E}^{k,\zeta}_{K_n}$. Since the sets $\mathfrak{E}^{k,\zeta}_{K_n}$ are Borel measurable, so is $\mathfrak{E}^{k,\zeta}$.

To prove that $\widehat{\mathfrak{E}}^{k,\zeta}_m$ is Borel measurable, we observe that by (5.3.18) a pair (x,ω) belongs to $\widehat{\mathfrak{E}}^{k,\zeta}_m$ if and only if the number of indices $i\in\mathbb{Z}$ such that $x\in E^{k,\zeta,\omega}-\frac{i}{k}\zeta$ is less than or equal to m. Hence, setting $\mathfrak{E}^{k,\zeta}_i := \{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ x\in E^{k,\zeta,\omega}-\frac{i}{k}\zeta\}$, we have that

$$\widehat{\mathfrak{E}}^{k,\zeta}_m=\mathfrak{E}^{k,\zeta}\cap\big\{(x,\omega)\in\mathbb{R}^2\times\mathbb{R}^2\colon\ \textstyle\sum_i\chi_{\mathfrak{E}^{k,\zeta}_i}(x,\omega)\le m\big\},$$

where $\chi_{\mathfrak{E}^{k,\zeta}_i}$ is the characteristic function of $\mathfrak{E}^{k,\zeta}_i$. Since the sets $\mathfrak{E}^{k,\zeta}_i$ are Borel measurable, we deduce that $\widehat{\mathfrak{E}}^{k,\zeta}_m$ is a Borel set as well. From the equality $\check{\mathfrak{E}}^{k,\zeta}_m=\mathfrak{E}^{k,\zeta}\setminus\widehat{\mathfrak{E}}^{k,\zeta}_m$, we deduce that $\check{\mathfrak{E}}^{k,\zeta}_m$ is Borel measurable too, concluding the proof. □

We are now ready to state and prove the final result of this subsection.

Lemma 5.D.2. *Let* $k,m\in\mathbb{N}$, *let* $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$, *and let* $N_\zeta\subset\Pi^\zeta$ *be a Borel set such that* $\mathcal{H}^{d-1}(N_\zeta)=0$ *and* $u^\zeta_y\in\mathrm{BV}(U^\zeta_y)$ *for every* $y\in\Pi^\zeta\setminus N_\zeta$. *Then the functions* $g^{k,\zeta}_m\colon\Pi^\zeta\times\mathbb{R}^2\to[0,+\infty)$ *and* $h^{k,\zeta}_m\colon\Pi^\zeta\times\mathbb{R}^2\to[0,+\infty)$ *defined by*

$$g^{k,\zeta}_m(y,\omega) := \begin{cases}|Du^\zeta_y|(\widehat{E}^{k,\zeta,\omega}_m(y)\cap U^\zeta_y\setminus J^\zeta_y) & \text{if } y\in\Pi^\zeta\setminus N_\zeta,\\ 0 & \text{if } y\in N_\zeta,\end{cases}\tag{5.D.2}$$

$$h^{k,\zeta}_m(y,\omega) := \begin{cases}|Du^\zeta_y|(\check{E}^{k,\zeta,\omega}_m(y)\cap U^\zeta_y\setminus J^\zeta_y) & \text{if } y\in\Pi^\zeta\setminus N_\zeta,\\ 0 & \text{if } y\in N_\zeta,\end{cases}\tag{5.D.3}$$

are Borel measurable on $\Pi^\zeta\times\mathbb{R}^2$.

Proof. We begin by observing that by (5.C.1) and (5.D.1) for every $y\in\Pi^\zeta$ and $\omega\in\mathbb{R}^2$ we have

$$\widehat{E}^{k,\zeta,\omega}_m(y)\cap U^\zeta_y\setminus J^\zeta_y=\widehat{\mathfrak{E}}(\omega)^\zeta_y\cap U^\zeta_y\setminus\mathfrak{J}(\omega)^\zeta_y,\tag{5.D.4}$$

$$\check{E}^{k,\zeta,\omega}_m(y)\cap U^\zeta_y\setminus J^\zeta_y=\check{\mathfrak{E}}(\omega)^\zeta_y\cap U^\zeta_y\setminus\mathfrak{J}(\omega)^\zeta_y.\tag{5.D.5}$$

By Lemma 5.D.1, the sets $\widehat{\mathfrak{E}}_m^{k,\zeta}$, $\check{\mathfrak{E}}_m^{k,\zeta}$, and $\mathfrak{J}$ are Borel measurable subsets of $\mathbb{R}^2 \times \mathbb{R}^2$, so that the same property holds for $\widehat{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$ and $\check{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$.

We want to apply Lemma 5.C.1 with the measure $\mu_y^\omega \in \mathcal{M}_b(\mathbb{R})$ defined for every Borel set $B \subset \mathbb{R}$ by

$$\mu_y^\omega(B) \coloneqq \begin{cases} Du_y^\zeta(B \cap U_y^\zeta) & \text{if } y \in \Pi^\zeta \setminus N_\zeta, \\ 0 & y \in N_\zeta. \end{cases} \tag{5.D.6}$$

To show that condition (b) of Lemma 5.C.1 holds we fix a function $\varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and observe that

$$\int_{-\infty}^{+\infty} \varphi(y + t\zeta, \omega) \, \mathrm{d}\mu_y^\omega(t) = \int_{U_y^\zeta} \varphi(y + t\zeta, \omega) \, \mathrm{d}Du_y^\zeta(t)$$

for every $y \in \Pi^\zeta \setminus N_\zeta$. Let us consider a bounded sequence $\{\rho_n\}_n \subset C_c^\infty(U)$ converging to 1 on U as $n \to +\infty$ and set $\varphi_n \coloneqq \rho_n \varphi \in C_c^\infty(U \times \mathbb{R}^2)$. For every $y \in \Pi^\zeta \setminus N_\zeta$ by the Dominated Convergence Theorem we have

$$\int_{-\infty}^{+\infty} \varphi(y + t\zeta, \omega) \, \mathrm{d}\mu_y^\omega(t) = \lim_{n \to +\infty} \int_{U_y^\zeta} \varphi_n(y + t\zeta, \omega) \, \mathrm{d}Du_y^\zeta(t). \tag{5.D.7}$$

Moreover, integrating by parts, we have

$$\int_{U_y^\zeta} \varphi_n(y + t\zeta, \omega) \, \mathrm{d}Du_y^\zeta(t) = -\int_{U_y^\zeta} u_y^\zeta(t) \nabla_x \varphi_n(y + t\zeta, \omega) \cdot \zeta \, \mathrm{d}t, \tag{5.D.8}$$

where for a given $\omega \in \mathbb{R}^2$, we denote by $\nabla_x \varphi_n(y + t\zeta, \omega) \in \mathbb{R}^2$ the vector whose two components are the partial derivatives of φ_n with respect to x at the point $(y + t\zeta, \omega)$. By the Fubini Theorem, it follows from (5.D.8) that

$$(y, \omega) \mapsto \int_{U_y^\zeta} \varphi_n(y + t\zeta, \omega) \, \mathrm{d}Du_y^\zeta(t)$$

is Borel measurable on $(\Pi^\zeta \setminus N_\zeta) \times \mathbb{R}^2$. It follows from (5.D.7) that the same property holds for

$$(y, \omega) \mapsto \int_{-\infty}^{+\infty} \varphi(y + t\zeta, \omega) \, \mathrm{d}\mu_y^\omega(t). \tag{5.D.9}$$

Since by definition $\mu_y^\omega(B) = 0$ for $y \in N_\zeta$, it follows that the function (5.D.9) is Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$. As every function in $C_c^0(\mathbb{R}^2 \times \mathbb{R}^2)$ can be approximated uniformly by a sequence of functions in $C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, the function (5.D.9) is Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$ for every $\varphi \in C_c^0(\mathbb{R}^2 \times \mathbb{R}^2)$. Hence, μ_y^ω satisfies condition (b) of Lemma 5.C.1.

By Lemma 5.C.1 applied to the measure μ_y^ω defined by (5.D.6) and with $B = \widehat{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$ and $B = \check{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$ we obtain that the functions $(y, \omega) \mapsto |Du_y^\zeta|(\widehat{\mathfrak{E}}(\omega)_y^\zeta \cap U_y^\zeta \setminus \mathfrak{J}(\omega)_y^\zeta)$ and $(y, \omega) \mapsto |Du_y^\zeta|(\check{\mathfrak{E}}(\omega)_y^\zeta \cap U_y^\zeta \setminus \mathfrak{J}(\omega)_y^\zeta)$ are Borel measurable on $(\Pi^\zeta \setminus N_\zeta) \times \mathbb{R}^2$. By (5.D.2)–(5.D.5) this implies that the functions $g_m^{k,\zeta}$ and $h_m^{k,\zeta}$ are Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$. □

A Korn-Poincaré-Sobolev Inequality for generalised functions of bounded deformation

The aim of this chapter is to present a variant of the classical Korn-Poincaré-Sobolev inequality for generalised functions of bounded deformation proved by Kohn [141] for $\mathrm{BD}(\Omega)$. This inequality is close in spirit to the one obtained by Chambolle, Conti, & Francfort [68] for functions in SBD^p and, indeed, its proof closely follows the same line of argument.

We recall the definition of λ_u given in Proposition 1.4.10 and of the set $J_u^\sigma := \{x \in J_u \colon |[u](x)| \geq \sigma\}$. The following is the main result of the chapter, containing a Korn-Poincaré-Sobolev inequality, together with a related estimate for the trace on $\partial\Omega$.

Theorem 6.0.1 (Korn-Poincaré-Sobolev inequality)**.** *Let $d \geq 2$ and let Ω be a bounded connected open set with Lipschitz boundary. Then there exist positive constants κ_Ω and K_Ω, depending only on Ω, such that for all $\sigma \geq 0$ and $u \in \mathrm{GBD}(\Omega)$ there exist Borel sets $\omega \subset \Omega$ and $\omega_\partial \subset \partial\Omega$, with*

$$\mathcal{L}^d(\omega) \leq \kappa_\Omega \mathcal{H}^{d-1}(J_u^\sigma)^{d/(d-1)} \quad \text{and} \quad \mathcal{H}^{d-1}(\omega_\partial) \leq \kappa_\Omega \mathcal{H}^{d-1}(J_u^\sigma), \tag{6.0.1}$$

and an affine map $a(x) := Ax + b$, with $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$, such that

$$\Big(\int_{\Omega\setminus\omega} |u-a|^{d/(d-1)} \,\mathrm{d}x \Big)^{(d-1)/d} \leq K_\Omega \Big(\lambda_u(\Omega \setminus J_u) + \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} \Big), \tag{6.0.2}$$

$$\int_{\partial\Omega\setminus\omega_\partial} |u_{\partial\Omega} - a| \,\mathrm{d}\mathcal{H}^{d-1} \leq K_\Omega \Big(\lambda_u(\Omega \setminus J_u) + \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} \Big), \tag{6.0.3}$$

where $u_{\partial\Omega}$ denotes the trace of u on $\partial\Omega$ (see [89, Theorem 5.3]).

We refer the reader to the introduction for a discussion on the statement of this inequality.

The chapter is divided into two sections. In Section 6.1, we present some auxiliary results needed in the proof of Theorem 6.0.1, i.e., a change of variables lemma, an extension result for functions in GBD, and an improvement of Proposition 5.1.8. We then show in Section 6.2 how, using the results of Section 6.1, the arguments of [68] adapt to this generalised setting.

The results of this chapter are based on the joint work with Dal Maso [93].

6.1 Some auxiliary results on functions in GBD

In this section, we prove some results concerning generalised functions of bounded deformation. As in the previous chapter, in the following we write u_y^ξ in place of $\hat{u}_y^\xi$ for the function defined by (1.1.4). We refer the reader to Section 1.1 for the notation concerning slicing and to 1.4 for properties of GBD functions.

We begin by proving that GBD is stable under a simultaneous linear change of coordinates of the domain and of the range, a property we will exploit in the proof of Lemma 6.2.2.

Lemma 6.1.1. *Let $u \in \mathrm{GBD}(\Omega)$ and let $A \in \mathbb{R}^{d\times d}$ be invertible. Then the function v defined for $x \in \Omega_A := A^{-1}(\Omega)$ by $v(x) := A^T u(Ax)$ belongs to $\mathrm{GBD}(\Omega_A)$. Moreover, there exist constants c_A and $\widetilde{c}_A$, depending only on A, such that for every $\sigma \geq 0$ and $\xi \in \mathbb{S}^{d-1}$, setting $\eta := A\xi/|A\xi|$, the following properties hold:*

$$|Dv_y^\xi|((\Omega_A \setminus J_v)_y^\xi) \leq c_A |Du_{Ay}^\eta|((\Omega\setminus J_u)_{Ay}^\eta) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \pi^\xi(\Omega_A), \tag{6.1.1}$$

$$\lambda_v(\Omega_A \setminus J_v) \leq c_A \lambda_u(\Omega \setminus J_u), \tag{6.1.2}$$

$$\int_{J_v\setminus J_v^\sigma} |[v]|\,\mathrm{d}\mathcal{H}^{d-1} \leq c_A \int_{J_u\setminus J_u^{\widetilde{\sigma}}} |[u]|\,\mathrm{d}\mathcal{H}^{d-1}, \tag{6.1.3}$$

where $\widetilde{\sigma} := \sigma\widetilde{c}_A$.

Proof. Let us fix $\xi \in \mathbb{S}^{d-1}$ and let $\eta \in \mathbb{S}^{d-1}$ be as in the statement. For every $y \in \pi^\xi(\Omega_A)$ and $t \in (\Omega_A)_y^\xi$ we have

$$v_y^\xi(t) = A^T u(A(y+t\xi))\cdot\xi = u(Ay + |A\xi|t\eta)\cdot\eta|A\xi| = |A\xi|u_{Ay}^\eta(|A\xi|t). \tag{6.1.4}$$

We then observe that for every Borel set $B \subset \Omega_A$ and $y \in \pi^\xi(\Omega_A)$ we have

$$t \in B_y^\xi \Leftrightarrow y + t\xi \in B \Leftrightarrow Ay + tA\xi \in A(B) \Leftrightarrow Ay + |A\xi|t\eta \in A(B) \Leftrightarrow |A\xi|t \in (A(B))_{Ay}^\eta,$$

which implies

$$|A\xi|B_y^\xi = (A(B))_{Ay}^\eta. \tag{6.1.5}$$

Equalities (6.1.4) and (6.1.5) imply the following useful property: given $y \in \pi^\xi(\Omega_A)$, $v_y^\xi \in \mathrm{BV}_{\mathrm{loc}}((\Omega_A)_y^\xi)$ if and only if $u_{Ay}^\eta \in \mathrm{BV}_{\mathrm{loc}}(\Omega_{Ay}^\eta)$. Moreover, for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega_A)$ and for every Borel set $B \subset \Omega_A$ we have

$$Dv_y^\xi(B_y^\xi) = |A\xi|Du_{Ay}^\eta\big((A(B))_{Ay}^\eta\big), \tag{6.1.6}$$

$$A(J_v) = J_u \quad \text{and} \quad |A\xi|(J_v)_y^\xi = (J_u)_{Ay}^\eta, \tag{6.1.7}$$

$$|[v](x)| = |A^T[u](Ax)|, \tag{6.1.8}$$

$$|[v](y+t\xi)| = |A^T[u](Ay + |A\xi|t\eta)| \quad \text{for all } t \in (J_v)_y^\xi. \tag{6.1.9}$$

By the definition of v and the first equality in (6.1.7), for every $\sigma > 0$ we have

$$J_u^\sigma \subset A(J_v^{\widehat{\sigma}}) \quad \text{and} \quad A(J_v^\sigma) \subset J_u^{\breve{\sigma}}, \tag{6.1.10}$$

where $\widehat{\sigma} = \widehat{c}_A\sigma$, $\breve{\sigma} = \breve{c}_A\sigma$, with $\widehat{c}_A := \min_{\nu\in\mathbb{S}^{d-1}} |A^T\nu|$ and $\breve{c}_A := 1/(\max_{\nu\in\mathbb{S}^{d-1}} |A^T\nu|)$. In light of (6.1.5), the inclusions in (6.1.10) imply that

$$(J_u^\sigma)_{Ay}^\eta \subset |A\xi|(J_v^{\widehat{\sigma}})_y^\xi \quad \text{and} \quad |A\xi|(J_v^\sigma)_y^\xi \subset (J_u^{\breve{\sigma}})_{Ay}^\eta \tag{6.1.11}$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega_A)$.

In view of [89, Remark 4.6] and (1.4.19), we have

$$\begin{aligned}
&|Dv_y^\xi|(B_y^\xi \setminus J^1_{v_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{v_y^\xi}) \\
&\quad = |Dv_y^\xi|(B_y^\xi \setminus J_{v_y^\xi}) + \int_{B_y^\xi \cap J_{v_y^\xi}} |[v_y^\xi](t)| \wedge 1 \, \mathrm{d}\mathcal{H}^0(t) \\
&\quad = |Dv_y^\xi|\big((B \setminus J_v)_y^\xi\big) + \int_{(B\cap J_v)_y^\xi} |[v_y^\xi](t)| \wedge 1 \, \mathrm{d}\mathcal{H}^0(t) \\
&\quad \le |Dv_y^\xi|\big((B \setminus J_v)_y^\xi\big) + \int_{(B\cap (J_v \setminus J_v^1))_y^\xi} |[v_y^\xi](t)| \, \mathrm{d}\mathcal{H}^0(t) + \mathcal{H}^0\big((B \cap J_v^1)_y^\xi\big) \\
&\quad = |Dv_y^\xi|\big((B \setminus J_v^1)_y^\xi\big) + \mathcal{H}^0\big((B \cap J_v^1)_y^\xi\big).
\end{aligned}$$

Recalling (6.1.5)–(6.1.7), and (6.1.9), we may use (6.1.11), with $\sigma = 1$, to obtain

$$\begin{aligned}
&|Dv_y^\xi|(B_y^\xi \setminus J^1_{v_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{v_y^\xi}) \\
&\qquad \le |A\xi| |Du^\eta_{Ay}|\big((A(B) \setminus J_u^{1/\widehat{c}_A})^\eta_{Ay}\big) + \mathcal{H}^0\big((A(B) \cap J_u^{\check{c}_A})^\eta_{Ay}\big).
\end{aligned} \tag{6.1.12}$$

In particular, in view of (6.1.7), evaluating this last inequality at $B = \Omega_A \setminus J_v$, we obtain (6.1.1).

To prove (6.1.2) and (6.1.3), we first make some preliminary observations. For every $y \in \Pi^\xi$ let $\zeta(y) \coloneqq \pi^\eta(Ay) \in \Pi^\eta$ and let $\tau(y) \in \mathbb{R}$ be such that $Ay = \zeta(y) + \tau(y)\eta$. We first remark that $\zeta\colon \Pi^\xi \to \Pi^\eta$ and $\tau\colon \Pi^\xi \to \mathbb{R}$ are linear. Let us prove that

$$y \mapsto \zeta(y) \text{ is invertible,} \tag{6.1.13}$$

$$\zeta(\pi^\xi(\Omega_A)) = \pi^\eta(\Omega). \tag{6.1.14}$$

By linearity, to prove (6.1.13) it is enough to show that $\zeta(y) = 0$ implies $y = 0$. To see this, let us fix $y \in \Pi^\xi$ such that $\zeta(y) = 0$. By definition $Ay - \tau(y)\eta = \zeta(y) = 0$, so that $Ay = \tau(y)\eta = \tau(y)|A\xi|^{-1}A\xi$. Since A is invertible, this implies $y = \tau(y)|A\xi|^{-1}\xi$ and, as y is orthogonal to ξ, the previous equality shows that $y = 0$. This proves (6.1.13).

To prove (6.1.14), let $y \in \pi^\xi(\Omega_A)$ and let $x \in \Omega$ be such that $y = \pi^\xi(A^{-1}x)$, so that $A^{-1}x = y + s\xi$ for some $s \in \mathbb{R}$. As A is invertible, this implies that $x = Ay + sA\xi = Ay + s|A\xi|\eta$, which, in turn, yields $\pi^\eta(x) = \pi^\eta(Ay) = \zeta(y)$. This implies that $\zeta(y) \in \pi^\eta(\Omega)$ and shows that $\zeta(\pi^\xi(\Omega_A)) \subset \pi^\eta(\Omega)$. The same argument, in reverse order, proves the other inclusion and concludes the proof of (6.1.14).

It follows from the definition of ζ that for every $y \in \pi^\xi(\Omega_A)$ we have

$$u^\eta_{Ay}(s) = u^\eta_{\zeta(y)}(s + \tau(y)) \quad \text{for every } s \in \Omega^\eta_{Ay}$$

and

$$(A(B))^\eta_{Ay} = (A(B))^\eta_{\zeta(y)} - \tau(y) \quad \text{for every Borel set } B \subset \Omega_A.$$

Thus, for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega_A)$ we may rewrite (6.1.12) as

$$\begin{aligned}
&|Dv_y^\xi|(B_y^\xi \setminus J^1_{v_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{v_y^\xi}) \\
&\qquad \le |A\xi| |Du^\eta_{\zeta(y)}|\big((A(B) \setminus J_u^{1/\widehat{c}_A})^\eta_{\zeta(y)}\big) + \mathcal{H}^0\big((A(B) \cap J_u^{\check{c}_A})^\eta_{\zeta(y)}\big).
\end{aligned}$$

Recalling (1.4.14), we may integrate the previous inequality over $y \in \pi^\xi(\Omega_A)$ and use [89, Remark 4.9] to obtain

$$\begin{aligned}\lambda_v^\xi(B) &= \int_{\pi^\xi(\Omega_A)} \big(|Dv_y^\xi|(B_y^\xi \setminus J^1_{v_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{v_y^\xi})\big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ &\le c\int_{\pi^\eta(\Omega)} \big(|A\xi||Du_z^\eta|\big((A(B)\setminus J_u^{1/\widehat{c}_A})_z^\eta\big) + \mathcal{H}^0\big((A(B)\cap J_u^{\check{c}_A})_z^\eta\big)\big)\,\mathrm{d}\mathcal{H}^{d-1}(z) \quad (6.1.15)\end{aligned}$$

for some constant c, depending only on A, where we have used that $y \mapsto \zeta(y)$ is linear and invertible from $\pi^\xi(\Omega_A)$ to $\pi^\eta(\Omega)$ thanks to (6.1.13) and (6.1.14). In light of Remark 1.4.8 and Lemma 5.B.1, inequality (6.1.15) also gives

$$\lambda_v^\xi(B) \le c_A \lambda_u(\Omega \cap A(B)) \quad \text{for every Borel set } B \subset \Omega_A$$

for a suitable constant c_A, so that $v \in \mathrm{GBD}(\Omega_A)$. From the arbitrariness of ξ, recalling (1.4.15) and (6.1.7), this implies (6.1.2).

To prove (6.1.3), it suffices to note that by (6.1.8), (6.1.10), and the Area Formula (see [164, 12.4]) it holds

$$\int_{J_v \setminus J_v^\sigma} |[v]|\,\mathrm{d}\mathcal{H}^{d-1} \le c_A \int_{J_u \setminus J_u^{\widetilde{\sigma}}} |[u]|\,\mathrm{d}\mathcal{H}^{d-1}$$

for a possibly larger constant c_A and with $\widetilde{c}_A := 1/\widehat{c}_A$ (recall that $\widetilde{\sigma} = \sigma\widetilde{c}_A$). □

The following lemma shows that, under certain assumptions on Ω, the extension by zero of a GBD function is still GBD.

Lemma 6.1.2. *Let Ω, $\Omega' \subset \mathbb{R}^d$ be bounded open sets, with $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$ and $\Omega \subset \Omega'$, and let $u \in \mathrm{GBD}(\Omega)$. Consider the function $v\colon \Omega' \to \mathbb{R}^d$ defined by*

$$v := \begin{cases} u & \text{on } \Omega,\\ 0 & \text{on } \Omega' \setminus \Omega. \end{cases}$$

Then $v \in \mathrm{GBD}(\Omega')$ and

$$J_v \subset J_u \cup \partial\Omega, \quad (6.1.16)$$

$$\lambda_v(\Omega' \setminus J_v) \le \lambda_u(\Omega \setminus J_u), \quad (6.1.17)$$

$$|Dv_y^\xi|((\Omega' \setminus J_v)_y^\xi) = |Du_y^\xi|((\Omega \setminus J_u)_y^\xi), \quad (6.1.18)$$

for all $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$.

Proof. The fact that $v \in \mathrm{GBD}(\Omega')$ and (6.1.16) follow from [168, Proposition 2.20], while (6.1.17) follows from [168, formula (2.18)] with $B = \Omega' \setminus (J_u \cup \partial\Omega)$.

To prove (6.1.18), we begin by noting that for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$ we have $\mathcal{H}^0((\partial\Omega)_y^\xi) < +\infty$ (see Lemma 5.B.1) and $u_y^\xi \in \mathrm{BV}_{\mathrm{loc}}(\Omega_y^\xi)$. In particular, for every such $y \in \pi^\xi(\Omega)$ the slice Ω_y^ξ is a finite union of disjoint open intervals so that by Remark 1.4.4 we have $u_y^\xi \in \mathrm{BV}(\Omega_y^\xi)$.

We then observe that by (1.4.19) it holds

$$|Dv_y^\xi|((\Omega' \setminus J_v)_y^\xi) = |Dv_y^\xi|((\Omega')_y^\xi \setminus J_{v_y^\xi}) \quad \text{and} \quad |Du_y^\xi|((\Omega \setminus J_u)_y^\xi) = |Du_y^\xi|(\Omega_y^\xi \setminus J_{u_y^\xi})$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$. Moreover, as $v = 0$ in $\Omega' \setminus \overline{\Omega}$, we have

$$|Dv_y^\xi|((\Omega')_y^\xi \setminus J_{v_y^\xi}) = |Dv_y^\xi|((\overline{\Omega})_y^\xi \setminus J_{v_y^\xi}) = |Dv_y^\xi|((\partial\Omega)_y^\xi \setminus J_{v_y^\xi}) + |Du_y^\xi|(\Omega_y^\xi \setminus J_{u_y^\xi})$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$. Since $\mathcal{H}^0((\partial\Omega)_y^\xi) < +\infty$ for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$, we have that $|Dv_y^\xi|((\partial\Omega)_y^\xi \setminus J_{v_y^\xi}) = 0$ for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^\xi(\Omega)$, as the atomic part of $|Dv_y^\xi|$ is concentrated on $J_{v_y^\xi}$. Combining all these equalities, we obtain (6.1.18), concluding the proof. □

Throughout the rest of the chapter, given two $\mathcal{H}^{d-1}$-measurable sets B_1, B_2, we write $B_1 \simeq B_2$ when $\mathcal{H}^{d-1}(B_1 \triangle B_2) = 0$, where $\triangle$ denotes the symmetric difference.

We conclude the section with an improved version of Proposition 5.1.8 showing that for every function $u \in \mathrm{GBD}(\Omega)$ and every $\sigma > 0$ it is always possible to find a function $w \in \mathrm{GBD}(\Omega)$, with $J_w \simeq J_u^\sigma$, that satisfies also some estimates which will allow us to obtain the Korn-Poincaré-Sobolev inequality for u from the analogous property for w (see the proof of Lemma 6.2.3). We observe that the right-hand sides of the inequalities in (6.1.21) are finite by Proposition 1.4.14.

Proposition 6.1.3. *Let $u \in \mathrm{GBD}(\Omega)$, let $\varepsilon \in (0,1)$, and let $\sigma > 0$. Then there exists $v \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$ such that*

$$J_v \simeq J_u \setminus J_u^\sigma, \tag{6.1.19}$$

$$[v] = [u] \quad \text{and} \quad \nu_v = \nu_u \quad \mathcal{H}^{d-1}\text{-a.e. on } J_u \setminus J_u^\sigma, \tag{6.1.20}$$

$$\int_\Omega |v| \,\mathrm{d}x \le \varepsilon \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} \quad \text{and} \quad \int_\Omega |\nabla v| \,\mathrm{d}x \le (1+\varepsilon) \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}. \tag{6.1.21}$$

In particular, setting $w := u - v$, we have $w \in \mathrm{GBD}(\Omega)$, $u = w + v$,

$$J_w \simeq J_w^\sigma \simeq J_u^\sigma, \tag{6.1.22}$$

$$[w] = [u] \quad \mathcal{H}^{d-1}\text{-a.e. on } J_u^\sigma, \tag{6.1.23}$$

$$\lambda_w(\Omega \setminus J_w) \le \lambda_u(\Omega \setminus J_u) + 4 \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}. \tag{6.1.24}$$

If, in addition, Ω has Lipschitz boundary, then

$$\Big(\int_\Omega |v|^{d/(d-1)} \,\mathrm{d}x \Big)^{(d-1)/d} \le 4C_{\mathrm{BV}}(\Omega) \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}, \tag{6.1.25}$$

$$\int_{\partial\Omega} |v_{\partial\Omega}| \,\mathrm{d}\mathcal{H}^{d-1} \le 4C_{\mathrm{Tr}}(\Omega) \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}, \tag{6.1.26}$$

where $v_{\partial\Omega}$ denotes the trace of v on $\partial\Omega$, the constant $C_{\mathrm{BV}}(\Omega) > 0$ is the norm of the embedding $\mathrm{BV}(\Omega;\mathbb{R}^d) \to L^{d/(d-1)}(\Omega;\mathbb{R}^d)$, while the constant $C_{\mathrm{Tr}}(\Omega)$ is the norm of the trace operator from $\mathrm{BV}(\Omega;\mathbb{R}^d)$ to $L^1_{\mathcal{H}^{d-1}}(\partial\Omega;\mathbb{R}^d)$.

Proof. The existence of a v such that (6.1.19)–(6.1.21) hold can be obtained by arguing exactly as in [98, Theorem 3.1] (see also Proposition 5.1.8 for the adaptation to GBD). Inequality (6.1.24) follows from the fact that $\lambda_w \le \lambda_u + \lambda_v$, (6.1.19), (6.1.22), (6.1.23), and the inequalities

$$\lambda_u(J_u \setminus J_w) \le \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} \quad \text{and} \quad \lambda_v(\Omega \setminus J_w) \le 3 \int_{J_u \setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}.$$

Finally, (6.1.25) and (6.1.26) are obtained by combining (6.1.19)–(6.1.21) and the classical embedding and trace theorems for $\mathrm{BV}(\Omega;\mathbb{R}^d)$. □

6.2 Proof of the Korn-Poincaré-Sobolev inequality

In this section, we prove Theorem 6.0.1. To this end, we recall the following result, due to Chambolle & Crismale [73, Theorem 2.3], which is obtained by adapting the arguments of [68, Proposition 2.1]. Given $\rho > 0$, we let $Q_\rho := (-\frac{\rho}{2}, \frac{\rho}{2})^d$.

Theorem 6.2.1. *There exist positive constants c and C, depending only on d, such that for all $\rho > 0$, $\sigma \geq 0$, and $u \in \mathrm{GBD}(Q_\rho)$ there exist a Borel set $\omega \subset Q_\rho$, with $\mathcal{L}^d(\omega) \leq c\rho\mathcal{H}^{d-1}(J_u^\sigma)$, and an affine map $a(x) := Ax + b$, with $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$, such that*

$$\int_{Q_\rho\setminus\omega} |u-a|\,\mathrm{d}x \leq C\rho\Big(\lambda_u(Q_\rho\setminus J_u) + \int_{J_u\setminus J_u^\sigma} |[u]|\,\mathrm{d}\mathcal{H}^{d-1}\Big).$$

We now prove a technical result. Its proof is obtained by using a standard tool in the proof of the classical Sobolev-Gagliardo-Nirenberg inequality (see, for instance, [52, Lemma 9.4]) and properties of generalised functions of bounded deformation. It will be used in the proof of Lemma 6.2.3 below.

Lemma 6.2.2. *Let $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded open set with $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$, and let $\Xi := \{\xi_1, \dots, \xi_d\} \subset \mathbb{S}^{d-1}$ be a collection of linearly independent vectors. Then there exists a constant $M_\Xi > 0$, depending only on Ξ, such that for every $u \in \mathrm{GBD}(\Omega)$ and $v \in L^1(\Omega)$ we have*

$$\int_\Omega \prod_{i=1}^d \Big(\int_{\Omega_x^{\xi_i}} |v(x+t\xi_i)|\,\mathrm{d}t + |Du_x^{\xi_i}|((\Omega\setminus J_u)_x^{\xi_i})\Big)^{1/(d-1)} \mathrm{d}x \\ \leq M_\Xi\big(\|v\|_{L^1} + \lambda_u(\Omega\setminus J_u)\big)^{d/(d-1)}. \tag{6.2.1}$$

Proof. By a linear change of coordinates (see (6.1.1) and (6.1.2) of Lemma 6.1.1) we may assume that $\Xi = \{e_i\}_{i=1}^d$, the canonical basis of $\mathbb{R}^d$. Thanks to Lemma 6.1.2, it is not restrictive to assume that Ω is a cube with edges parallel to the axes, and, by a rescaling argument, we may assume that $\Omega = Q := Q_2 = (-1,1)^d$.

Let us fix $u \in \mathrm{GBD}(Q)$ and $v \in L^1(Q)$. We make some preliminary observations. By Remark 1.4.4 for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{e_i}(Q)$ and for all $t \in (-1,1)$ we have $u^{e_i}_{y+te_i} \in \mathrm{BV}(Q^{e_i}_{y+te_i})$ and

$$|Du^{e_i}_{y+te_i}|((Q\setminus J_u)^{e_i}_{y+te_i}) = |Du^{e_i}_{y+te_i+se_i}|((Q\setminus J_u)^{e_i}_{y+te_i+se_i}) \quad \text{for all } s \in Q^{e_i}_{y+te_i}. \tag{6.2.2}$$

Moreover, by Fubini's Theorem for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{e_i}(Q)$ and for all $t \in (-1,1)$ we have

$$\int_{Q^{e_i}_{y+te_i}} |v(y+(t+\tau)e_i)|\,\mathrm{d}\tau = \int_{Q^{e_i}_{y+(t+s)e_i}} |v(y+(s+t+\tau)e_i)|\,\mathrm{d}\tau \quad \text{for all } s \in Q^{e_i}_{y+te_i}. \tag{6.2.3}$$

For every $i \in \{1,\dots,d\}$ let us define for $\mathcal{L}^d$-a.e. $x \in Q$ the function

$$f_i(x) := \Big(\int_{Q_x^{e_i}} |v(x+\tau e_i)|\,\mathrm{d}\tau + |Du_x^{e_i}|((Q\setminus J_u)_x^{e_i})\Big)^{1/(d-1)}.$$

Thanks to (6.2.2) and (6.2.3), we have that for all $i \in \{1,\dots,d\}$ the function f_i is independent of the x_i-component. Moreover, inequality (1.4.5) in the definition of GBD and Fubini's Theorem imply that f_i, seen as a function of $(d-1)$-variables, belongs to $L^{d-1}(Q_{d-1})$, where $Q_{d-1} := (-1,1)^{d-1}$.

Let $f(x) \coloneqq f_1(x) \dots f_d(x)$. By [52, Lemma 9.4] we have $f \in L^1(Q)$ and

$$\|f\|_{L^1(Q)} \le \prod_{i=1}^{d} \|f_i\|_{L^{d-1}(Q_{d-1})}.$$

By definition,

$$\|f\|_{L^1(Q)} = \int_Q \prod_{i=1}^{d} \Big(\int_{Q_x^{e_i}} |v(x+te_i)|\,\mathrm{d}t + |Du_x^{e_i}|((Q\setminus J_u)_x^{e_i})\Big)^{1/(d-1)} \mathrm{d}x,$$

and

$$\|f_i\|^{d-1}_{L^{d-1}(Q_{d-1})} = \int_{\pi^{e_i}(Q)} \Big(\int_{Q_y^{e_i}} |v(y+\tau e_i)|\,\mathrm{d}\tau\Big)\mathrm{d}\mathcal{H}^{d-1}(y)$$
$$+ \int_{\pi^{e_i}(Q)} |Du_y^{e_i}|((Q\setminus J_u)_y^{e_i})\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \|v\|_{L^1(Q)} + \lambda_u(Q\setminus J_u).$$

Combining the last three formulas we get (6.2.1). □

Using the two results above and following the arguments of [68, Proposition 4.2], we now prove Theorem 6.0.1 for Lipschitz domains Ω of a special form. The case of a general Lipschitz domain will then follow from a covering argument.

Lemma 6.2.3. *Let $d \ge 2$, let $\rho > 0$, and let $\varphi\colon (-\rho,\rho)^{d-1} \to \mathbb{R}$ be a Lipschitz function, with Lipschitz constant L, and whose* inf *is equal to ρ. We set*

$$\Omega \coloneqq \{(x', x_d) \in (-\rho,\rho)^{d-1}\times\mathbb{R}\colon\ -\rho < x_d < \varphi(x')\},$$
$$\widetilde{\Omega} \coloneqq \{(x', x_d) \in (-\rho/2,\rho/2)^{d-1}\times\mathbb{R}\colon\ 0 < x_d < \varphi(x')\},$$
$$\widetilde{\Gamma} \coloneqq \{(x', \varphi(x'))\colon\ x' \in (-\rho/2,\rho/2)^{d-1}\}.$$

Then there exist constants c_L and C_L, depending only on d and on L, such that for every $u \in \mathrm{GBD}(\Omega)$ and $\sigma \ge 0$ there exist Borel sets $\omega \subset \widetilde{\Omega}$ and $\omega_{\widetilde{\Gamma}} \subset \widetilde{\Gamma}$, with

$$\mathcal{L}^d(\omega) \le c_L \mathcal{H}^{d-1}(J_u^\sigma)^{d/(d-1)} \quad \text{and} \quad \mathcal{H}^{d-1}(\omega_{\widetilde{\Gamma}}) \le c_L \mathcal{H}^{d-1}(J_u^\sigma), \tag{6.2.4}$$

and an affine map $a(x) \coloneqq Ax + b$, with $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$, such that

$$\Big(\int_{\widetilde{\Omega}\setminus\omega} |u-a|^{d/(d-1)}\,\mathrm{d}x\Big)^{(d-1)/d} \le C_L\Big(\lambda_u(\Omega\setminus J_u) + \int_{J_u\setminus J_u^\sigma} |[u]|\,\mathrm{d}\mathcal{H}^{d-1}\Big), \tag{6.2.5}$$

$$\int_{\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}}} |u_{\widetilde{\Gamma}} - a|\,\mathrm{d}\mathcal{H}^{d-1} \le C_L\Big(\lambda_u(\Omega\setminus J_u) + \int_{J_u\setminus J_u^\sigma} |[u]|\,\mathrm{d}\mathcal{H}^{d-1}\Big), \tag{6.2.6}$$

where $u_{\widetilde{\Gamma}}$ is the trace of u on $\widetilde{\Gamma}$.

Proof. By the scaling properties of GBD functions (see Remark 1.4.11) it is enough to consider the case $\rho = 1$.

Let us fix $u \in \mathrm{GBD}(\Omega)$ and $\sigma \ge 0$. We claim that, when $\sigma > 0$, it is enough to prove the result assuming also that $J_u \simeq J_u^\sigma$. Indeed, suppose that we have proved the result for every $w \in \mathrm{GBD}(\Omega)$ with $J_w \simeq J_w^\sigma$. Then we can employ Proposition 6.1.3 to obtain a function $v \in \mathrm{SBV}(\Omega;\mathbb{R}^d)$ satisfying (6.1.19)–(6.1.21), (6.1.25), and (6.1.26), so

that the function $w := u - v \in \mathrm{GBD}(\Omega)$ satisfies (6.1.22)–(6.1.24). Hence, there exist Borel sets $\omega \subset \widetilde{\Omega}$ and $\omega_{\widetilde{\Gamma}} \subset \widetilde{\Gamma}$, with $\mathcal{L}^d(\omega) \le c_L \mathcal{H}^{d-1}(J_w)^{d/(d-1)} = c_L \mathcal{H}^{d-1}(J_u^\sigma)^{d/(d-1)}$ and $\mathcal{H}^{d-1}(\omega_{\widetilde{\Gamma}}) \le c_L \mathcal{H}^{d-1}(J_w) = c_L \mathcal{H}^{d-1}(J_u^\sigma)$, and an affine map $a(x) := Ax + b$, with $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$, such that

$$\Big(\int_{\widetilde{\Omega}\setminus\omega} |w-a|^{d/(d-1)} \,\mathrm{d}x \Big)^{(d-1)/d} \le C_L \lambda_w(\Omega \setminus J_w),$$

$$\int_{\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}}} |w_{\widetilde{\Gamma}} - a| \,\mathrm{d}\mathcal{H}^{d-1} \le C_L \lambda_w(\Omega \setminus J_w).$$

By (6.1.24), (6.1.25), and the triangle inequality, the first estimate implies

$$\Big(\int_{\widetilde{\Omega}\setminus\omega} |u-a|^{d/(d-1)} \,\mathrm{d}x \Big)^{(d-1)/d} \le 4(C_{\mathrm{BV}}(\Omega)+C_L)\Big(\lambda_u(\Omega\setminus J_u) + \int_{J_u\setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}\Big). \tag{6.2.7}$$

By the linearity of the trace, the triangle inequality, and (6.1.26), we obtain

$$\int_{\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}}} |u_{\widetilde{\Gamma}} - a| \,\mathrm{d}\mathcal{H}^{d-1} \le 4(C_{\mathrm{Tr}}(\Omega) + C_L)\Big(\lambda_u(\Omega \setminus J_u) + \int_{J_u\setminus J_u^\sigma} |[u]| \,\mathrm{d}\mathcal{H}^{d-1}\Big). \tag{6.2.8}$$

Observe that the particular shape of the domain Ω implies that the constants $C_{\mathrm{BV}}(\Omega)$ and $C_{\mathrm{Tr}}(\Omega)$ depend only on d and L. Therefore, (6.2.5) and (6.2.6) follow from (6.2.7) and (6.2.8), after updating the value of the constant C_L.

Throughout the rest of the proof, we always write J in place of $J_u^\sigma \simeq J_u$. Recalling that our choice $\rho = 1$ implies $Q := Q_2 \subset \Omega$, we apply Theorem 6.2.1 and obtain a Borel set $\widehat{\omega} \subset Q$, with $\mathcal{L}^d(\widehat{\omega}) \le 2c_d \mathcal{H}^{d-1}(J)$, and an affine map $a(x) := Ax + b$, with $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$, such that

$$\int_{Q\setminus\widehat{\omega}} |u - a| \,\mathrm{d}x \le 2C_d \lambda_u(\Omega \setminus J). \tag{6.2.9}$$

In the rest of the proof we shall find two constants c_L and C_L, depending only on d and L, and we shall construct two Borel sets $\omega \subset \widetilde{\Omega}$ and $\omega_{\widetilde{\Gamma}} \subset \widetilde{\Gamma}$ such that (6.2.4)–(6.2.6) hold with this choice of the affine map a.

Without loss of generality, we may assume that

$$a = 0 \quad \text{in } \mathbb{R}^d. \tag{6.2.10}$$

Indeed, if $a \neq 0$, we may simply replace u by $u - a$, thanks to Remark 1.4.12.

Given a small constant $r > 0$, it is convenient to introduce the set

$$S_r := \Big\{ \xi \in \mathbb{S}^{d-1} \colon\ |\xi - e_d| \le r \Big\}.$$

Elementary geometric considerations show that we can choose r small enough, depending on L, so that, for every x in the closure of $\widetilde{\Omega}$ and $\xi \in S_r$, the line $x + \mathbb{R}\xi$ meets $\partial\Omega$ at exactly two points, one lying on $\Gamma := \{(x', x_d) \colon\ x' \in (-1,1)^{d-1}$ and $x_d = \varphi(x')\}$ and one on $(-1,1)^{d-1} \times \{-1\}$. Hence, Ω_x^ξ is an interval and

$$\mathcal{L}^1(Q_x^\xi) \ge 2 \quad \text{for every } x \in \widetilde{\Omega} \text{ and } \xi \in S_r. \tag{6.2.11}$$

Moreover, we may assume that for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma$ and every $\xi \in S_r$ we have

$$\nu_\Gamma(x) \cdot \xi \ge \frac{1}{2\sqrt{1+L^2}}, \tag{6.2.12}$$

where $\nu_\Gamma(x)$ is the outward unit normal to Γ at x.

Thanks to Remark 1.4.4 and Theorem 1.4.13, for every $\xi \in S_r$ there exists a Borel set $N(\xi) \subset \Pi^\xi$, with $\mathcal{H}^{d-1}(N(\xi)) = 0$, such that for every $y \in \pi^\xi(\Omega) \setminus N(\xi)$ we have $u_y^\xi \in \mathrm{BV}(\Omega_y^\xi)$ and $J_{u_y^\xi} \subset J_y^\xi$. From the Lebesgue Differentiation Theorem in 1-dimension we deduce that

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t+\tau)\xi)\cdot\xi\,\mathrm{d}\tau = u(y+t\xi)\cdot\xi$$

for every $y \in \pi^\xi(\Omega)\setminus N(\xi)$ and for $\mathcal{L}^1$-a.e. $t \in \Omega_y^\xi$. Setting $x = y + t\xi$, by Fubini's Theorem this implies that

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(x+\tau\xi)\cdot\xi\,\mathrm{d}\tau = u(x)\cdot\xi \tag{6.2.13}$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$.

By the Fundamental Theorem of Calculus for 1-dimensional BV-functions, for every $y \in \pi^\xi(\Omega)\setminus N(\xi)$ we have

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t_2+\tau)\xi)\cdot\xi\,\mathrm{d}\tau - \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t_1+\tau)\xi)\cdot\xi\,\mathrm{d}\tau = Du_y^\xi((t_1,t_2))$$

for every $t_1, t_2 \in \Omega_y^\xi \setminus J_{u_y^\xi}$ with $t_1 < t_2$.

Moreover, if $J_y^\xi = \emptyset$, the inclusion $J_{u_y^\xi} \subset J_y^\xi$ and the previous equality imply that

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t_2+\tau)\xi)\cdot\xi\,\mathrm{d}\tau - \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t_1+\tau)\xi)\cdot\xi\,\mathrm{d}\tau = Du_y^\xi((t_1,t_2)\setminus J_y^\xi)$$

for every $t_1, t_2 \in \Omega_y^\xi$ with $t_1 < t_2$, which gives

$$\Big|\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t_1+\tau)\xi)\cdot\xi\,\mathrm{d}\tau\Big| \le \Big|\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+(t_2+\tau)\xi)\cdot\xi\,\mathrm{d}\tau\Big| + |Du_y^\xi|((\Omega\setminus J)_y^\xi)$$

for every $t_1, t_2 \in \Omega_y^\xi$. Setting $x = y + t_1\xi$ and $s = t_2 - t_1$, by the previous displayed formula for every $x \in \widetilde{\Omega}$ with $\pi^\xi(x) \notin N(\xi)$ and $J_x^\xi = \emptyset$ we have

$$\Big|\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(x+\tau\xi)\cdot\xi\,\mathrm{d}\tau\Big| \le \Big|\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} u(x+(s+\tau)\xi)\cdot\xi\,\mathrm{d}\tau\Big| + |Du_x^\xi|((\Omega\setminus J)_x^\xi)$$

for every $s \in \Omega_x^\xi$. Thanks to (6.2.13), this implies that

$$|u(x)\cdot\xi| \le |u(x+s\xi)\cdot\xi| + |Du_x^\xi|((\Omega\setminus J)_x^\xi) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \Omega_x^\xi \tag{6.2.14}$$

for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega}$ with $\pi^\xi(x) \notin N(\xi)$ and $J_x^\xi = \emptyset$.

For every $\xi \in S_r$ we set

$$\widetilde{\omega}(\xi) := \{x \in \widetilde{\Omega}\colon \ \text{either } \pi^\xi(x) \in N(\xi) \text{ or } \pi^\xi(x) \notin N(\xi) \text{ and } J_x^\xi \neq \emptyset\}.$$

In particular, this implies $\pi^\xi(\widetilde{\omega}(\xi)) \subset \pi^\xi(J) \cup N(\xi)$, hence

$$\mathcal{H}^{d-1}\big(\pi^\xi(\widetilde{\omega}(\xi))\big) \le \mathcal{H}^{d-1}(\pi^\xi(J)) \le \mathcal{H}^{d-1}(J). \tag{6.2.15}$$

Given $\xi \in S_r$, we define the set

$$\omega(\xi) := \widetilde{\omega}(\xi) \cup \{x \in \widetilde{\Omega}\colon \ \mathcal{L}^1(\widehat{\omega}_x^\xi) \ge 1\} \subset \widetilde{\Omega}, \tag{6.2.16}$$

where $\widehat{\omega}$ is the Borel set appearing in (6.2.9), which satisfies $\mathcal{L}^d(\widehat{\omega}) \le 2c_d\mathcal{H}^{d-1}(J)$. Observing that $\pi^\xi(\{x \in \widetilde{\Omega}\colon\ \mathcal{L}^1(\widehat{\omega}_x^\xi) \ge 1\}) = \{y \in \pi^\xi(\widetilde{\Omega})\colon\ \mathcal{L}^1(\widehat{\omega}_y^\xi) \ge 1\}$, by Fubini's Theorem (in the orthogonal decomposition relative to ξ and Π^ξ) we obtain

$$\mathcal{H}^{d-1}(\pi^\xi(\{x \in \widetilde{\Omega}\colon\ \mathcal{L}^1(\widehat{\omega}_x^\xi) \ge 1\})) \le 2c_d\mathcal{H}^{d-1}(J),$$

which together with (6.2.15) gives

$$\mathcal{H}^{d-1}\big(\pi^\xi(\omega(\xi))\big) \le (2c_d + 1)\mathcal{H}^{d-1}(J). \tag{6.2.17}$$

Since $\widetilde{\omega}(\xi) \subset \omega(\xi)$, by definition of $\widetilde{\omega}(\xi)$ and the comment after (6.2.14) for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega} \setminus \omega(\xi)$ we have that $\pi^\xi(x) \notin N(\xi)$ and (6.2.14) holds. Recalling (6.2.11), by definition of $\omega(\xi)$ we have that $\mathcal{L}^1((Q \setminus \widehat{\omega})_x^\xi) \ge 1$ for every $x \in \widetilde{\Omega} \setminus \omega(\xi)$. We can then integrate inequality (6.2.14) over $s \in (Q \setminus \widehat{\omega})_x^\xi$ and divide by $\mathcal{L}^1((Q \setminus \widehat{\omega})_x^\xi)$ to get

$$\begin{aligned} |u(x) \cdot \xi| &\le \frac{1}{\mathcal{L}^1((Q \setminus \widehat{\omega})_x^\xi)} \int_{(Q\setminus\widehat{\omega})_x^\xi} |u(x + s\xi)|\,\mathrm{d}s + |Du_x^\xi|((\Omega \setminus J)_x^\xi) \\ &\le \int_{(Q\setminus\widehat{\omega})_x^\xi} |u(x + s\xi)|\,\mathrm{d}s + |Du_x^\xi|((\Omega \setminus J)_x^\xi) \end{aligned} \tag{6.2.18}$$

for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega} \setminus \omega(\xi)$.

We choose a collection $S \subset S_r \subset \mathbb{S}^{d-1}$ of $d^2 + d - 1$ vectors, with the following property: every $\Xi \subset S$ consisting of d vectors is a basis of $\mathbb{R}^d$. Given any such $\Xi \subset S$, being the vectors in Ξ linearly independent, there exists a positive constant C_Ξ, depending only on Ξ, such that

$$|\eta| \le C_\Xi \sum_{\xi\in\Xi} |\eta \cdot \xi| \quad \text{ for every } \eta \in \mathbb{R}^d,$$

so that from (6.2.18) it follows that

$$|u(x)| \le C_\Xi \sum_{\xi\in\Xi} \Big(\int_{(Q\setminus\widehat{\omega})_x^\xi} |u(x + s\xi)|\,\mathrm{d}s + |Du_x^\xi|((\Omega \setminus J)_x^\xi)\Big) \tag{6.2.19}$$

for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega} \setminus \bigcup_{\xi\in\Xi} \omega(\xi)$.

We now fix $Z \subset S$ consisting of d^2 vectors and set $\omega_Z := \bigcup_{\xi\in Z} \omega(\xi)$. We consider $\Xi_1, \dots, \Xi_d \subset Z$ disjoint collections of d vectors in Z (whose union obviously coincides with Z) and multiply the associated inequalities (6.2.19) to get

$$|u(x)|^d \le \prod_{i=1}^d \Big(C_{\Xi_i} \sum_{\xi\in\Xi_i} \Big(\int_{(Q\setminus\widehat{\omega})_x^\xi} |u(x + s\xi)|\,\mathrm{d}s + |Du_x^\xi|((\Omega \setminus J)_x^\xi)\Big)\Big)$$

for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega} \setminus \omega_Z$. Using the distributive property, we obtain that there exists a positive constant C_Z, depending only on Z, such that

$$|u(x)|^d \le C_Z \sum_{\substack{\Xi\subset Z \\ \#\Xi=d}} \prod_{\xi\in\Xi} \Big(\int_{(Q\setminus\widehat{\omega})_x^\xi} |u(x + s\xi)|\,\mathrm{d}s + |Du_x^\xi|((\Omega \setminus J)_x^\xi)\Big)$$

for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega} \setminus \omega_Z$. By the subadditivity of the function $s \mapsto s^{1/(d-1)}$, we get

$$|u(x)|^{d/(d-1)} \le C_Z^{1/(d-1)} \sum_{\substack{\Xi \subset Z \\ \#\Xi = d}} \prod_{\xi \in \Xi} \Big(\int_{(Q\setminus\widehat{\omega})_x^\xi} |u(x+s\xi)|\,\mathrm{d}s + |Du_x^\xi|((\Omega \setminus J)_x^\xi) \Big)^{1/(d-1)}$$

for $\mathcal{L}^d$-a.e. $x \in \widetilde{\Omega} \setminus \omega_Z$. Therefore, integrating over $x \in \widetilde{\Omega} \setminus \omega_Z$ and applying Lemma 6.2.2, with $v = |u|\chi_{Q\setminus\widehat{\omega}}$, we obtain

$$\int_{\widetilde{\Omega}\setminus\omega_Z} |u(x)|^{d/(d-1)}\,\mathrm{d}x \le K_{d,Z} \Big(\int_{Q\setminus\widehat{\omega}} |u(x)|\,\mathrm{d}x + \lambda_u(\Omega \setminus J_u) \Big)^{d/(d-1)},$$

where

$$K_{d,Z} \coloneqq C_Z^{1/(d-1)} \sum_{\substack{\Xi \subset Z \\ \#\Xi = d}} M_\Xi,$$

and M_Ξ are the constants of Lemma 6.2.2. Recalling that we have assumed $a = 0$ (see (6.2.10)), by (6.2.9) this implies

$$\Big(\int_{\widetilde{\Omega}\setminus\omega_Z} |u(x)|^{d/(d-1)}\,\mathrm{d}x \Big)^{(d-1)/d} \le K_{d,Z}^{(d-1)/d} (2C_d + 1) \lambda_u(\Omega \setminus J_u). \tag{6.2.20}$$

We now introduce the exceptional set $\omega \subset \widetilde{\Omega}$ by setting

$$\omega \coloneqq \bigcap_{\substack{Z \subset S \\ \#Z = d^2}} \omega_Z, \quad \text{which implies} \quad \widetilde{\Omega} \setminus \omega = \bigcup_{\substack{Z \subset S \\ \#Z = d^2}} (\widetilde{\Omega} \setminus \omega_Z).$$

As there is only a finite number of such Z, from (6.2.20) we deduce that there exists a constant $\widetilde{C}_{d,S}$, depending only on d and on S, such that

$$\Big(\int_{\widetilde{\Omega}\setminus\omega} |u(x)|^{d/(d-1)}\,\mathrm{d}x \Big)^{(d-1)/d} \le \widetilde{C}_{d,S} \lambda_u(\Omega \setminus J_u). \tag{6.2.21}$$

Since the choice of S depends only on S_r and, in turn, r depends only on the Lipschitz constant L (see the arguments leading to (6.2.11)), we have that $\widetilde{C}_{d,S}$ depends only on d and L. Recalling that we are assuming $\mathcal{H}^{d-1}(J_u \setminus J_u^\sigma) = 0$ (see the discussion at the beginning of the proof) and $a = 0$ (see (6.2.10)), we obtain that (6.2.21) implies (6.2.5) with $C_L = \widetilde{C}_{d,S}$.

To prove that $\mathcal{L}^d(\omega) \le c_L \mathcal{H}^{d-1}(J_u)^{d/(d-1)}$ for a suitable constant c_L, it is enough to repeat the arguments of [68, Proposition 4.2, pp. 1394–1395], exploiting (6.2.17). This concludes the proof of the first inequality in (6.2.4). The specific choice of the collection $S \subset S_r$ is used only in this part of the proof.

We now deal with the trace $u_{\widetilde{\Gamma}}$ of u on $\widetilde{\Gamma}$. To this end, for every $\xi \in S_r$ we set

$$\omega_{\widetilde{\Gamma}}(\xi) \coloneqq \{x \in \widetilde{\Gamma} \colon\ \mathcal{L}^1(\omega(\xi)_x^\xi) > 0\},$$

where $\omega(\xi)$ is the set introduced in (6.2.16). By (6.2.12) and the Area Formula [164, 12.4] we have

$$\begin{aligned} \mathcal{H}^{d-1}(\omega_{\widetilde{\Gamma}}(\xi)) &\le 2\sqrt{1+L^2} \int_{\omega_{\widetilde{\Gamma}}(\xi)} \xi \cdot \nu_\Gamma(x)\,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &= 2\sqrt{1+L^2} \int_{\pi^\xi(\omega_{\widetilde{\Gamma}}(\xi))} \mathcal{H}^0(\omega_{\widetilde{\Gamma}}(\xi)_y^\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y). \end{aligned}$$

Observing that the arguments that lead to (6.2.11) imply that $\mathcal{H}^0(\omega_{\widetilde{\Gamma}}(\xi)^\xi_y) = 1$ for all $y \in \pi^\xi(\omega_{\widetilde{\Gamma}}(\xi))$, from the inclusion $\pi^\xi(\omega_{\widetilde{\Gamma}}(\xi)) \subset \pi^\xi(\omega(\xi))$ we infer

$$\int_{\pi^\xi(\omega_{\widetilde{\Gamma}}(\xi))} \mathcal{H}^0(\omega_{\widetilde{\Gamma}}(\xi)^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \mathcal{H}^{d-1}\big(\pi^\xi(\omega_{\widetilde{\Gamma}}(\xi))\big) \le \mathcal{H}^{d-1}(\pi^\xi\big(\omega(\xi)\big)) \le (2c_d+1)\mathcal{H}^{d-1}(J),$$

where in the last inequality we have used (6.2.17).

Combining the previous inequalities, we obtain

$$\mathcal{H}^{d-1}(\omega_{\widetilde{\Gamma}}(\xi)) \le 2\sqrt{1+L^2}\,(2c_d+1)\mathcal{H}^{d-1}(J) \tag{6.2.22}$$

for all $\xi \in S_r$.

We now let $\{\xi_1,\dots,\xi_d\}$ be a collection of linearly independent vectors belonging to S_r and set

$$\omega_{\widetilde{\Gamma}} \coloneqq \omega_{\widetilde{\Gamma}}(\xi_1) \cup \dots \cup \omega_{\widetilde{\Gamma}}(\xi_d). \tag{6.2.23}$$

By (6.2.22) we have

$$\mathcal{H}^{d-1}(\omega_{\widetilde{\Gamma}}) \le 2d\sqrt{1+L^2}\,(2c_d+1)\mathcal{H}^{d-1}(J),$$

which proves the second inequality in (6.2.4).

To prove (6.2.6) we observe that, if $r>0$ is small enough, for all $i \in \{1,\dots,d\}$ there exists a Lipschitz function $\varphi_i\colon \pi^{\xi_i}(\widetilde{\Gamma}) \to \mathbb{R}$ such that $y + t\xi_i \in \widetilde{\Gamma}$ if and only if $t = \varphi_i(y)$.

Moreover, by the definition (6.2.23) of $\omega_{\widetilde{\Gamma}}$, for all $y \in \pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})$ we have $\mathcal{L}^1(\omega(\xi_i)^{\xi_i}_y) = 0$. This implies that for any such y we have $y + t\xi_i \in \widetilde{\Omega}\setminus\omega(\xi_i)$ for $\mathcal{L}^1$-a.e. $t \in \widetilde{\Omega}^{\xi_i}_y$. Hence, for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})$ and $\mathcal{L}^1$-a.e. $t \in \widetilde{\Omega}^{\xi_i}_y$ it follows from (6.2.18) that

$$\begin{aligned}|u(y+t\xi_i)\cdot\xi_i| &\le \int_{(Q\setminus\widehat{\omega})^{\xi_i}_{y+t\xi_i}} |u(y+t\xi_i+s\xi_i)|\,\mathrm{d}s + |Du^{\xi_i}_{y+t\xi_i}|((\Omega\setminus J)^{\xi_i}_{y+t\xi_i})\\ &= \int_{(Q\setminus\widehat{\omega})^{\xi_i}_{y}} |u(y+s\xi_i)|\,\mathrm{d}s + |Du^{\xi_i}_{y}|((\Omega\setminus J)^{\xi_i}_{y}).\end{aligned}$$

Using [89, Theorem 5.5], we may let $t \to \varphi_i(y)^-$ and obtain

$$|u_{\widetilde{\Gamma}}(y+\varphi_i(y)\xi_i)\cdot\xi_i| \le \int_{(Q\setminus\widehat{\omega})^{\xi_i}_{y}} |u(y+s\xi_i)|\,\mathrm{d}s + |Du^{\xi_i}_{y}|((\Omega\setminus J)^{\xi_i}_{y}) \tag{6.2.24}$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})$. By (1.4.5), Proposition 1.4.10, (6.2.9), and (6.2.10) we have

$$\begin{aligned}&\int_{\pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})} \Big(\int_{(Q\setminus\widehat{\omega})^{\xi_i}_{y}} |u(y+s\xi_i)|\,\mathrm{d}s + |Du^{\xi_i}_{y}|((\Omega\setminus J)^{\xi_i}_{y})\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ &\qquad\le \int_{Q\setminus\widehat{\omega}} |u|\,\mathrm{d}x + \lambda_u(\Omega\setminus J) \le (2C_d+1)\lambda_u(\Omega\setminus J).\end{aligned}$$

Therefore, integrating (6.2.24) over $y \in \pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})$, we obtain

$$\int_{\pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})} |u_{\widetilde{\Gamma}}(y+\varphi_i(y)\xi_i)\cdot\xi_i|\,\mathrm{d}\mathcal{H}^{d-1}(y) \le (2C_d+1)\lambda_u(\Omega\setminus J).$$

By the Area Formula, for all $i \in \{1, \dots, d\}$ there exists a constant $C_i > 0$ such that

$$\int_{\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}}} |u_{\widetilde{\Gamma}}(x)\cdot\xi_i|\,\mathrm{d}\mathcal{H}^{d-1}(x) \le C_i \int_{\pi^{\xi_i}(\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}})} |u_{\widetilde{\Gamma}}(y+\varphi_i(y)\xi_i)\cdot\xi_i|\,\mathrm{d}\mathcal{H}^{d-1}(y).$$

Hence,

$$\int_{\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}}} |u_{\widetilde{\Gamma}}(x)\cdot\xi_i|\,\mathrm{d}\mathcal{H}^{d-1}(x) \le C_i(2C_d+1)\lambda_u(\Omega\setminus J).$$

Since $\{\xi_1, \dots, \xi_d\}$ are linearly independent, summing over $i \in \{1, \dots, d\}$ we get

$$\int_{\widetilde{\Gamma}\setminus\omega_{\widetilde{\Gamma}}} |u_{\widetilde{\Gamma}}(x)|\,\mathrm{d}\mathcal{H}^{d-1}(x) \le C_L\lambda_u(\Omega\setminus J)$$

for a constant C_L depending only on d and on L. Recalling that we are assuming $\mathcal{H}^{d-1}(J_u \setminus J_u^\sigma) = 0$ (see the discussion at the beginning of the proof) and $a = 0$ by (6.2.10), this proves (6.2.6) and concludes the proof of the lemma. □

We conclude the chapter by proving Theorem 6.0.1.

Proof of Theorem 6.0.1. It is enough to repeat the arguments of [68, Theorem 1.1], replacing Proposition 4.2 of that paper by our Lemma 6.2.3. □

Part II

Singular perturbation models for phase transitions

CHAPTER 7

Higher-order singular perturbation models for phase transitions

This chapter concerns higher-order variants of the classical Modica-Mortola approximation of the perimeter functional. The main difference with respect to the standard first-order theory (and with the second-order theory developed by Fonseca & Mantegazza [123]) is the presence of singular perturbations depending on derivatives of arbitrary order k, which makes the analysis of transitions between the two wells more delicate. Despite this additional difficulty, the limit picture remains a sharp-interface model; sequences with equi-bounded energy converge to functions taking only two values, and the cost of each transition is described by an optimal-profile problem.

More precisely, we investigate compactness properties and Γ-convergence as $\varepsilon \to 0^+$ of the sequence of functionals $\{F_\varepsilon\}_\varepsilon \colon \mathrm{H}^k(a,b) \to [0,+\infty)$ defined by

$$F_\varepsilon(u) \coloneqq \int_{(a,b)} \Big(\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} (u^{(k)})^2 \Big) \, \mathrm{d}t, \tag{7.0.1}$$

where $W \colon \mathbb{R} \to [0,+\infty)$ is a double-well potential, with wells at $\{-1,1\}$, and $k \in \mathbb{N}$ is an arbitrarily large fixed natural number.

We here prove three main results:

(i) a compactness result for $\{F_\varepsilon\}_\varepsilon$ (see Theorem 7.1.1): every sequence $\{u_\varepsilon\}_\varepsilon$ with equi-bounded energy converges in measure, up to subsequences, to a piecewise constant function;

(ii) a Γ-convergence result (see Theorem 7.1.1): the sequence $\{F_\varepsilon\}_\varepsilon$ Γ-converges to a sharp-interface energy, which for $u \in \mathrm{BV}((a,b);\{-1,+1\})$ takes the form $m_k \# J_u$ for a positive constant $m_k > 0$ determined by an optimal profile problem (see (7.1.2));

(iii) the Γ-convergence to a sharp interface model of a suitable d-dimensional generalisation of (7.0.1), in which the singular perturbation involves the operator norm of the k-th order symmetric tensor $\nabla^{(k)}$ (see Theorem 7.1.2).

The proofs of (i) and (ii) require much more care with respect to the case $k = 1$, since, in the higher-order case, some transitions may have infinitesimally small cost and therefore do not contribute to the limit energy. To overcome this difficulty, drawing inspiration from Fonseca & Mantegazza [123], we will perform a careful analysis of points of (a,b) where sequences $\{u_\varepsilon\}_\varepsilon$ are close to the wells and their derivatives up to order $k-1$ are small. This analysis is based on the use of local interpolation inequalities for higher-order Sobolev spaces and allows us to identify the transitions with fixed cost, namely those which, after rescaling, become competitors for the minimisation problem defining m_k (see (7.1.2)). Once (i) and (ii) are established, (iii) will follow from a blow-up argument *á la* Fonseca & Müller, together with slicing techniques, for the lower bound, and a standard construction for the upper bound. In Chapter 8, we will generalise this last result by studying more general energies in which the operator norm is replaced by an arbitrary norm on the space of k-th order symmetric tensors.

We briefly discuss the organisation of the chapter. In Section 7.1, we introduce the hypotheses we assume on W and give the precise statements of the results mentioned in (i)–(iii). Section 7.2 is devoted to the analysis of the points of (a,b) where sequences $\{u_\varepsilon\}_\varepsilon$ are close to the wells, which represents the cornerstone of the analysis we perform. We then study in Section 7.3 properties of the optimal profile problem m_k. Most notably, we prove that it is strictly positive, a property which is crucial for the arguments we employ. Collecting the results obtained in these last two sections, we conclude in Section 7.4 the proof of the 1-dimensional Γ-convergence and compactness result (i)–(ii). Finally, in Section 7.5 we prove the Γ-convergence in the d-dimensional case (iii).

The content of this chapter is based on the joint work with Brusca & Solci [54].

7.1 Setting and main results

We refer the reader to Sections 1.2 and 1.3 for the definitions and main properties of Γ-convergence and of sets of finite perimeter.

Let $W\colon \mathbb{R} \to [0,+\infty)$ be a continuous function such that $W(-1) = W(1) = 0$ and

$$W(z) \geq \alpha_W \min\{(z+1)^2, (z-1)^2, \beta_W\}$$

for some $\alpha_W, \beta_W > 0$.

Let $k \geq 1$ be an integer and let (a,b) be an open bounded interval. For any $\varepsilon > 0$ we define on $\mathrm{H}^k(a,b)$ the functional F_ε given by

$$F_\varepsilon(u) = \int_{(a,b)} \Big(\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} (u^{(k)})^2\Big)\,\mathrm{d}t \tag{7.1.1}$$

with $u \in \mathrm{H}^k(a,b)$.

We will prove the following compactness and Γ-convergence theorem. The result for $k = 1$ and $k = 2$ has been shown in [154] and in [123], respectively. In the latter a condition of linear growth from below for W is required; our proofs allow us to remove that assumption also in the case $k = 2$. Note that the domain of the Γ-limit in this case is the space $\mathrm{BV}((a,b);\{-1,1\})$ of functions with bounded variation u on the interval (a,b) with $u(t) \in \{-1,1\}$ almost everywhere, which are just the piecewise-constant functions on (a,b) with values equal to -1 or 1.

Theorem 7.1.1 (Coerciveness and Γ-convergence)**.** *Let F_ε be defined by* (7.1.1) *on $\mathrm{H}^k(a,b)$. Then, the family $\{F_\varepsilon\}_\varepsilon$ is equicoercive with respect to the convergence in measure; that is, for all $\varepsilon_j \to 0$ and $\{u_j\}_j$ with $\sup_j F_{\varepsilon_j}(u_j) < +\infty$, there exists $u \in \mathrm{BV}((a,b);\{-1,1\})$ such that, up to subsequences, $u_j \to u$ in measure. Moreover, the family $\{F_\varepsilon\}_\varepsilon$ Γ-converges with respect to the convergence in measure and in $L^1(a,b)$ to the functional F with domain $\mathrm{BV}((a,b);\{-1,1\})$ given by*

$$F(u) = m_k \# J_u,$$

where

$$m_k = \inf\Big\{\int_{-\infty}^{+\infty} \big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t\colon\ u \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}),\ \lim_{t\to\pm\infty} u(t) = \pm 1\Big\}. \tag{7.1.2}$$

More precisely, for all $u \in \mathrm{BV}((a,b);\{-1,1\})$ we have that

(i) *for all sequences* $u_\varepsilon \to u$ *in measure,*

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq m_k \# J_u;$$

(ii) *there exists a recovery sequence* $u_\varepsilon \to u$ *in* $L^1(a,b)$ *such that*

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) = m_k \# J_u.$$

From the one-dimensional result, we will recover an analog in dimension $d > 1$ by applying blow-up and localisation techniques. We consider $\Omega \subset \mathbb{R}^d$ an open and bounded set with Lipschitz boundary, and, for $\varepsilon > 0$, the functional defined for $u \in \mathrm{H}^k(\Omega)$ as

$$F_\varepsilon(u) := \int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} \|\nabla^{(k)} u\|^2\Big) \mathrm{d}x, \tag{7.1.3}$$

where $W\colon \mathbb{R} \to [0,+\infty)$ is a function as above and $\|\nabla^{(k)} u\|$ denotes the operator norm of the tensor $\nabla^{(k)} u$. In accordance with Theorem 7.1.1, the domain of the Γ-limit will be the space $\mathrm{BV}(\Omega;\{-1,1\})$ of functions u with bounded variation on the set Ω with $u(x) \in \{-1,1\}$ almost everywhere. We recall that in this case the set of essential discontinuity points J_u is a set with finite $(d-1)$-dimensional Hausdorff measure, and that $\mathcal{H}^{d-1}(J_u)$ is equal to the perimeter of the set $\{x\colon\ u(x) = 1\}$ in Ω in the sense of the theory of sets of finite perimeter (see Section 1.3).

Theorem 7.1.2 (Higher dimensional case)**.** *The family* $\{F_\varepsilon\}_\varepsilon$ *defined by* (7.1.3) *on the Sobolev space* $\mathrm{H}^k(\Omega)$ Γ*-converges with respect to the* $L^1(\Omega)$*-convergence to the functional* F *with domain* $\mathrm{BV}(\Omega;\{-1,1\})$ *given by*

$$F(u) = m_k \mathcal{H}^{d-1}(J_u) = m_k \mathrm{Per}(\{u = 1\}, \Omega),$$

where m_k *is as defined in* (7.1.2) *and* $\mathrm{Per}(A,\Omega)$ *is the perimeter of* A *in* Ω *(see Section 1.3).*

Remark 7.1.3. Note that, if we strengthen the growth assumptions on the double-well potential W or alternatively we assume the equi-integrability of $\{u_\varepsilon\}_\varepsilon$, we may deduce that a sequence $\{u_\varepsilon\}_\varepsilon$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ is strongly precompact in $L^1(\Omega)$. This follows by a result due to Alberti, Bouchitté, Seppecher [7, Theorem 6.6]. A related result will be proved in Chapter 8 (see Theorem 8.4.1 and also [51, Theorem 2.3]).

To prove the results, it is useful to introduce the localised functionals

$$F_\varepsilon(u, I) = \int_I \Big(\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} (u^{(k)})^2\Big)\, \mathrm{d}t$$

for any interval $I \subset (a,b)$. Finally, to simplify the notation, in the following we only consider $(a,b) = (0,1)$, the changes in the proofs for a general interval being obvious.

7.2 Analysis of oscillations close to 1 and –1

From now on, we shall assume that $k \geq 2$. In the proofs of the compactness and of the lower bound, we will make use of some optimal-profile problems in order to estimate the

number of transitions between wells of sequence of functions with equibounded energy. In particular, we will make use of the family of minimum problems given by

$$m_k(\eta,N,T) \coloneqq \inf\Big\{\int_{-T}^{T}\big(W(u)+(u^{(k)})^2\big)\,\mathrm{d}t\colon\ u\in \mathrm{H}^k(-T,T),\\ |u(-T)+1|\le\eta, |u(T)-1|\le\eta,\\ |u^{(\ell)}(\pm T)|\le \tfrac{1}{N} \text{ for all } \ell\in\{1,\dots,k-1\}\Big\}. \tag{7.2.1}$$

To that end, in this section, we will study the properties of intervals of points where functions u_ε with equibounded energy are close to either -1 or 1, and their derivatives up to order $k-1$ are small. We will show that every sufficiently large interval (with respect to ε) intersects such sets. The technical tool used to prove this result is the simultaneous application of interpolation inequalities for all derivatives in the intervals where u_ε is close to either -1 or 1. This is an adaptation and simplification of the technique devised in [167] to treat an approximation of free-discontinuity problems by singular perturbation with higher-order derivatives.

Let $\{u_\varepsilon\}_\varepsilon$ be a family of $\mathrm{H}^k(0,1)$ functions with equibounded energy, and let

$$S=\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon).$$

For any fixed $\eta\in(0,1)$ and $\varepsilon>0$, we define the set

$$A_\varepsilon^\eta=\{t\in(0,1)\colon\ ||u_\varepsilon|-1|<\eta\}. \tag{7.2.2}$$

Recalling that $W(z)\ge\alpha_W\min\{(z-1)^2,(z+1)^2,\beta_W\}$ for some $\alpha_W,\beta_W>0$, we will consider $\eta\in(0,\sqrt{\beta_W})$, so that $W(z)\ge\alpha_W(z-1)^2$ if $|z-1|\le\eta$ and $W(z)\ge\alpha_W(z+1)^2$ if $|z+1|\le\eta$.

We now fix an integer $N\ge1$, and prove an upper estimate for the length of an interval I such that at all points of $I\cap A_\varepsilon^\eta$ there exists at least one $\ell\in\{1,\dots,k-1\}$ such that $|u_\varepsilon^{(\ell)}|\ge\frac{1}{N\varepsilon^\ell}$.

Lemma 7.2.1. *Let $\eta\in(0,\sqrt{\beta_W})$ and $N\ge1$ be fixed, and let $\{u_\varepsilon\}_\varepsilon$ be a sequence with the property that $\sup_\varepsilon F_\varepsilon(u_\varepsilon)=S<+\infty$. Then, there exists a constant $R_k(\eta,N,S)>0$ such that, for all $\varepsilon>0$ and for any interval I satisfying the hypotheses:*

(i) $I\subset A_\varepsilon^\eta$;

(ii) $\mathcal{L}^1\Big(\Big\{t\in I\colon\ |u_\varepsilon^{(\ell)}(t)|<\dfrac{1}{N\varepsilon^\ell}\text{ for all }\ell\in\{1,\dots,k-1\}\Big\}\Big)=0$;

it follows that

$$\mathcal{L}^1(I)\le R_k(\eta,N,S)\varepsilon.$$

Proof. The proof is based on a classical interpolation estimate. There exists a constant $r_k>0$ such that for any interval I and for any $v\in\mathrm{H}^k(I)$ the following estimate holds

$$\|v^{(\ell)}\|_{L^2(I)}\le r_k\Big(\|v\|_{L^2(I)}^{\theta}\|v^{(k)}\|_{L^2(I)}^{1-\theta}+\mathcal{L}^1(I)^{-\ell}\|v\|_{L^2(I)}\Big) \tag{7.2.3}$$

for all $\ell\in\{1,\dots,k-1\}$, with $\theta=\frac{k-\ell}{k}$ (see e. g. [143, Theorem 7.41]). Setting

$$a=\varepsilon^{-\theta}\|v\|_{L^2(I)}^{2\theta}\quad\text{and}\quad b=\varepsilon^{(2k-1)(1-\theta)}\|v^{(k)}\|_{L^2(I)}^{2(1-\theta)},$$

by using the convexity inequality $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$ with $p = \frac{1}{\theta}$ and $q = \frac{1}{1-\theta}$, we then deduce that there exists a constant $R_k > 0$ such that

$$\varepsilon^{2\ell-1}\|v^{(\ell)}\|^2_{L^2(I)} \le R_k\Big(\frac{1}{\varepsilon}\|v\|^2_{L^2(I)} + \varepsilon^{2k-1}\|v^{(k)}\|^2_{L^2(I)} + \frac{\varepsilon^{2\ell-1}}{\mathcal{L}^1(I)^{2\ell}}\|v\|^2_{L^2(I)}\Big). \tag{7.2.4}$$

Let now I be as in the hypothesis. It is not restrictive to suppose that $|u_\varepsilon - 1| \le \eta$ in I, the case $|u_\varepsilon + 1| \le \eta$ being completely analogous. Choosing $v = u_\varepsilon - 1$ in (7.2.4), and recalling that $W(u_\varepsilon) \ge \alpha_W(u_\varepsilon - 1)^2$ in the interval I, we get

$$\varepsilon^{2\ell-1}\|u_\varepsilon^{(\ell)}\|^2_{L^2(I)} \le \widetilde{R}_k\Big(F_\varepsilon(u_\varepsilon, I) + \varepsilon^{2\ell-1}\mathcal{L}^1(I)^{-2\ell}\|u_\varepsilon - 1\|^2_{L^2(I)}\Big), \tag{7.2.5}$$

for any $\varepsilon > 0$, where $\widetilde{R}_k = R_k \max\{\frac{1}{\alpha_W}, 1\}$.

Note that hypothesis (ii) implies that for at least one $\ell \in \{1, \dots, k-1\}$

$$\mathcal{L}^1\Big(\Big\{t \in I : |u_\varepsilon^{(\ell)}(t)| \ge \frac{1}{N\varepsilon^\ell}\Big\}\Big) \ge \frac{\mathcal{L}^1(I)}{k+1}.$$

By using this lower estimate in (7.2.5), recalling that $(u_\varepsilon - 1)^2 \le \eta^2$ and that $F_\varepsilon(u_\varepsilon, I) \le S$ we obtain

$$\varepsilon^{-1}\frac{1}{(k+1)N^2}\mathcal{L}^1(I) \le \widetilde{R}_k\Big(S + \eta^2\varepsilon^{2\ell-1}\mathcal{L}^1(I)^{1-2\ell}\Big)$$

for any $\varepsilon > 0$. We then conclude that

$$\mathcal{L}^1(I) \le \widetilde{R}_k(k+1)N^2S\varepsilon + \widetilde{R}_k(k+1)N^2\eta^2\varepsilon^{2\ell}\mathcal{L}^1(I)^{1-2\ell}.$$

Hence, if $\mathcal{L}^1(I) > 2\widetilde{R}_k(k+1)N^2S\varepsilon$, it follows that

$$\mathcal{L}^1(I) \le 2\widetilde{R}_k(k+1)N^2\eta^2\varepsilon^{2\ell}\mathcal{L}^1(I)^{1-2\ell},$$

so that $\mathcal{L}^1(I)^{2\ell} \le 2\widetilde{R}_k(k+1)N^2\eta^2\varepsilon^{2\ell}$. Hence,

$$\mathcal{L}^1(I) \le \max\Big\{2\widetilde{R}_k(k+1)N^2S, \max_{1\le\ell\le k-1}\big\{(2\widetilde{R}_k(k+1)N^2\eta^2)^{\frac{1}{2\ell}}\big\}\Big\}\varepsilon.$$

With a little abuse of notation, by letting $R_k(\eta, N, S)$ denote the constant in this inequality, we obtain the claim. □

We set

$$A^\eta_\varepsilon(N) = \Big\{t \in A^\eta_\varepsilon\colon\ |u_\varepsilon^{(\ell)}(t)| < \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\}\Big\}, \tag{7.2.6}$$

where we recall that A^η_ε is defined in (7.2.2) as the set of t such that $||u_\varepsilon(t)| - 1| < \eta$. The sets $A^\eta_\varepsilon(N)$ describe the regions close to $\{\pm 1\}$ where all derivatives of u_ε up to $k-1$ are "small". We now show that points in such sets cannot have mutual distance much larger than ε.

Lemma 7.2.2. *Let $\eta \in (0, \sqrt{\beta_W})$, $N \ge 1$ and let $\{u_\varepsilon\}_\varepsilon$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) = S < +\infty$. Let $\varepsilon_r \to 0$ as $r \to +\infty$, and let $\{I_{\varepsilon_r}\}$ be a family of intervals such that*

$$\lim_{r\to+\infty}\frac{\mathcal{L}^1(I_{\varepsilon_r})}{\varepsilon_r} = +\infty.$$

Then, there exists $r_0 \in \mathbb{N}$ such that

$$\mathcal{L}(I_{\varepsilon_r} \cap A^\eta_{\varepsilon_r}(N)) > 0$$

for all $r \ge r_0$.

Proof. Let $\{I_{\varepsilon_r}\}$ be the family of intervals in the hypothesis. By contradiction, we suppose that $\mathcal{L}^1(I_{\varepsilon_r} \cap A^\eta_{\varepsilon_r}(N)) = 0$ for a subsequence (not relabeled) $\varepsilon_r \to 0$ as $r \to +\infty$. The proof of the contradiction is divided in two steps.

Step 1. Let r be fixed, and let $J \subset I_{\varepsilon_r}$ be an interval such that

$$\mathcal{L}^1(J) \geq \varepsilon_r \Big(\frac{2S}{\alpha_W \eta^2} + (2k+3) R_k(\eta, N, S) \Big),$$

where $R_k(\eta, N, S) > 0$ is the constant given in Lemma 7.2.1. Then for all $\ell \in \{1, \dots, k-1\}$ there exists a point in J where $u^{(\ell)}_{\varepsilon_r}$ vanishes. This can be proved as follows.

We first note that $\mathcal{L}^1(\{||u_{\varepsilon_r}| - 1| \geq \eta\}) \leq \varepsilon_r \frac{S}{\alpha_W \eta^2}$. Since $\mathcal{L}^1(I_{\varepsilon_r} \cap A^\eta_{\varepsilon_r}(N)) = 0$, we can apply Lemma 7.2.1 deducing that J contains at least $2k+3$ intervals such that in each one of them $|u_{\varepsilon_r} - 1| < \eta$, with the equality holding at the endpoints, or $|u_{\varepsilon_r} + 1| < \eta$, again with the equality holding at the endpoints. Let $\{I_n = (a_n, b_n)\}_{n=1}^{2k+3}$ denote the set of these intervals, with a_n and b_n increasing.

Now, we show that in each interval (a_n, a_{n+2}) with $n \leq 2k+1$ there exists at least a point where u'_{ε_r} vanishes. Indeed, let $n \leq 2k+1$ and suppose that $|u_{\varepsilon_r} - 1| < \eta$ in I_n. If $u'_{\varepsilon_r} \neq 0$ in the whole I_n, then u_{ε_r} is strictly monotone. We first consider the case u_{ε_r} strictly increasing in I_n, with $u_{\varepsilon_r}(b_n) = 1 + \eta$. If $u'_{\varepsilon_r}(b_n) = 0$ the claim is proven; otherwise, it follows that $a_{n+1} > b_n$, and u_{ε_r} cannot be strictly increasing in the interval (b_n, a_{n+1}) since the value $u_{\varepsilon_r}(a_{n+1})$ is at most $1+\eta$. Then, there exists a local maximizer in (b_n, a_{n+1}), and the claim follows. Otherwise, if u_{ε_r} is strictly decreasing in I_n, we note that u_{ε_r} cannot be decreasing in the interval (b_n, a_{n+2}); indeed, in this case $u_{\varepsilon_r}(b_n) = 1-\eta$, and if u_{ε_r} is strictly decreasing in (b_n, b_{n+1}), then $u_{\varepsilon_r}(a_{n+1}) = -1 + \eta$ and $u_{\varepsilon_r}(b_{n+1}) = -1 - \eta$. Since $u_{\varepsilon_r}(a_{n+2}) \geq -1 - \eta$, this shows that u_{ε_r} can not be strictly decreasing in both (b_n, b_{n+1}) and (b_{n+1}, a_{n+2}). Then, $[b_n, a_{n+2})$ contains a local minimiser, proving the claim. The corresponding argument can be applied if $|u_{\varepsilon_r} + 1| < \eta$ in I_n.

Hence, there exist at least k internal points in J where $u'_{\varepsilon_r} = 0$. By an iteration of the application of Lagrange Theorem, this implies that for each $\ell \in \{1, \dots, k-1\}$ there exists at least a point where $u^{(\ell)}_{\varepsilon_r} = 0$.

Step 2. Let N_r be the integer part of $\frac{\mathcal{L}^1(I_{\varepsilon r})}{\varepsilon_r L}$, where

$$L = L_k(\eta, N, S) = \frac{2S}{\alpha_W \eta^2} + (2k+3) R_k(\eta, N, S),$$

so that each interval I_{ε_r} contains N_r disjoint intervals with length $\varepsilon_r L$ and by hypothesis $N_r \to +\infty$. Then, for any r there exists one of these intervals, denoted by $J_r = (a_r, a_r + \varepsilon_r L)$, such that

$$F_{\varepsilon_r}(u_{\varepsilon_r}, J_r) \leq \frac{S}{N_r}.$$

We can then write

$$\begin{aligned} \frac{S}{N_r} &\geq \int_{a_r}^{a_r + \varepsilon_r L} \Big(\frac{1}{\varepsilon_r} W(u_{\varepsilon_r}) + \varepsilon_r^{2k-1} (u^{(k)}_{\varepsilon_r})^2 \Big) \, \mathrm{d}t \\ &= \int_0^L \Big(W(v_r) + (v_r^{(k)})^2 \Big) \, \mathrm{d}s, \end{aligned} \tag{7.2.7}$$

where $v_r(t) = u_{\varepsilon_r}(\varepsilon_r t + a_r)$. By Step 1, for any $\ell \in \{1, \dots, k-1\}$ there exists a point t^ℓ_r in J_r where $u^{(\ell)}_{\varepsilon_r}$ vanishes, hence $v^{(\ell)}_r = 0$ in $s^\ell_r = \frac{t^\ell_r - a_r}{\varepsilon_r} \in (0, L)$. An iterated application of the

Fundamental Theorem of Calculus then gives

$$\|v_r\|^2_{H^k(0,L)} \le C(L)\big(\min_{[0,L]}|v_r|^2 + \|v_r^{(k)}\|^2_{L^2(0,L)}\big),$$

where $C(L) > 0$ depends on L (and then on η, N, S and k). Moreover, note that there exists at least a point in J_r such that $||u_{\varepsilon_r}| - 1| = \eta$, and the same holds for v_r in a point in $(0, L)$. Then, since $N_r \to +\infty$, by (7.2.7) we deduce that v_r is equibounded in $\mathrm{H}^k(0, L)$, thus $v_r \rightharpoonup v$ in $\mathrm{H}^k(0, L)$ as $r \to +\infty$. Since $v_r \to v$ uniformly, by (7.2.7) we deduce that

$$0 = \liminf_{r\to+\infty} \int_0^L W(v_r)\,\mathrm{d}s \ge \int_0^L W(v)\,\mathrm{d}s,$$

so that either $v = 1$ in $(0, L)$ or $v = -1$ in $(0, L)$. This gives a contradiction since for any r there exists a point $s_r \in (0, L)$ such that $||v_r(s_r)| - 1| = \eta$. □

7.3 Optimal-profile problems

In the proof of the equicoerciveness of the family $\{F_\varepsilon\}_\varepsilon$ and of the lower estimate for the Γ-limit, we will make use of the optimal-profile problem

$$m_k = \inf\Big\{\int_{-\infty}^{+\infty}\big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t\colon\ u \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}),\ \lim_{t\to\pm\infty} u(t) = \pm 1\Big\}.$$

It is convenient to interpret this optimal-profile problem as an optimisation on intervals and functions; to that end, we define

$$\begin{aligned}\widetilde{m}_k &\coloneqq \inf_{T>0}\inf\Big\{\int_{-\infty}^{+\infty}\big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t\colon\ u \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}),\\ &\qquad\qquad u(t) = -1 \text{ for } t \le -T, u(t) = 1 \text{ for } t \ge T\Big\} \qquad (7.3.1)\\ &= \inf_{T>0} m_k(T) \;=\; \lim_{T\to+\infty} m_k(T),\end{aligned}$$

where

$$\begin{aligned}m_k(T) \coloneqq \inf\Big\{\int_{-T}^{T}\big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t\colon\ u \in \mathrm{H}^k(-T, T), u(\pm T) = \pm 1,\\ u^{(\ell)}(\pm T) = 0 \text{ for all } \ell \in \{1, \dots, k-1\}\Big\}.\end{aligned}$$

As a consequence of Lemma 7.2.2, we can prove that these two definitions coincide.

Proposition 7.3.1. *The equality $m_k = \widetilde{m}_k$ holds.*

Proof. Since all test functions for $\widetilde{m}_k$ are test functions for m_k, one inequality is trivial.

We then consider a function $u \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$ such that $\lim_{t\to\pm\infty} u(t) = \pm 1$ and $\int_{-\infty}^{+\infty}\big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t < +\infty$. With fixed η and N, we can apply Lemma 7.2.2 twice, to the sequences $u_\varepsilon(t) = u(\frac{t-1}{\varepsilon})$ and $u_\varepsilon(t) = u(\frac{t}{\varepsilon})$, respectively, and with $I_\varepsilon = (0, \frac12)$ and $I_\varepsilon = (\frac12, 1)$, respectively. In terms of u, we deduce that there are $t_\varepsilon^- \in (-\frac{1}{\varepsilon}, 0)$ and $t_\varepsilon^+ \in (0, \frac{1}{\varepsilon})$ such that $|u(t_\varepsilon^-) + 1| < \eta$, $|u(t_\varepsilon^+) - 1| < \eta$, and $|u^{(\ell)}(t_\varepsilon^\pm)| \le \frac{1}{N}$ for all $\ell \in \{1, \dots, k-1\}$. We now choose $\eta_j \to 0$ and $N_j \to +\infty$, and consider the correspondingly defined t_j^- and t_j^+.

We want to extend the restriction of the test function u to (t_j^-, t_j^+) in order to obtain an admissible test function u_j for $\widetilde{m}_k$ with an energy not greater than that of u up to an infinitesimal term. To this end, we consider a family of auxiliary minimum problems defined as follows. Given z_j^ℓ for $\ell \in \{0, \dots, k-1\}$, let v_j be a minimiser of the problem

$$\theta_j = \min\Big\{ \int_0^1 (v^{(k)})^2 \,\mathrm{d}t\colon\ v^{(\ell)}(0) = z_j^\ell, v^{(\ell)}(1) = 0 \text{ for all } \ell \in \{0, \dots, k-1\}\Big\},$$

where we used the notation $v^{(0)} = v$. By choosing $z_j^0 = u(t_j^+) - 1$, $z_j^\ell = u^{(\ell)}(t_j^+)$ for $\ell \in \{1, \dots, k-1\}$, we have $\theta_j \to 0$, and moreover, by the uniform convergence of the corresponding minimisers v_j to 0, we also have

$$\lim_{j\to+\infty} \int_0^1 W(v_j(t) + 1) \,\mathrm{d}t = 0.$$

We then define u_j^+ on $(t_j^+, +\infty)$ as

$$u_j^+(t) = \begin{cases} 1 + v_j(t - t_j^+) & \text{if } t \in (t_j^+, t_j^+ + 1), \\ 1 & \text{if } t \geq t_j^+ + 1. \end{cases}$$

Note that

$$\int_{t_j^+}^{+\infty} \big(W(u_j^+) + ((u_j^+)^{(k)})^2\big) \,\mathrm{d}t = \theta_j + \int_0^1 W(v_j(t) + 1) \,\mathrm{d}t = o(1)$$

as $j \to +\infty$. We define symmetrically u_j^- on $(-\infty, t_j^-)$, so that the functions

$$u_j(t) = \begin{cases} u_j^-(t) & \text{if } t \leq t_j^-, \\ u(t) & \text{if } t_j^- < t < t_j^+, \\ u_j^+(t) & \text{if } t \geq t_j^+ \end{cases}$$

are test functions for $\widetilde{m}_k$ and

$$\widetilde{m}_k \leq \int_{-\infty}^{+\infty} \big(W(u_j) + (u_j^{(k)})^2\big) \,\mathrm{d}t \leq \int_{-\infty}^{+\infty} \big(W(u) + (u^{(k)})^2\big) \,\mathrm{d}t + o(1)$$

as $j \to +\infty$. Hence,

$$\widetilde{m}_k \leq \int_{-\infty}^{+\infty} \big(W(u) + (u^{(k)})^2\big) \,\mathrm{d}t.$$

By the arbitrariness of u this proves the claim. □

Proposition 7.3.2. *We have $m_k > 0$.*

Proof. Since $m_k = \widetilde{m}_k$, we equivalently prove $\widetilde{m}_k > 0$.

Let $\{v_j\}_j$ be a sequence of admissible functions for $\widetilde{m}_k$ such that

$$\lim_{j\to+\infty} \int_{-\infty}^{+\infty} \big(W(v_j) + (v_j^{(k)})^2\big) \,\mathrm{d}t = \widetilde{m}_k.$$

For each $j \geq 1$, let T_j be a positive number such that $v_j(t) = -1$ for $t < -T_j$ and $v_j(t) = 1$ for $t > T_j$.

We fix $\eta < \min\{1, \sqrt{\beta_W}\}$, in view of the application of Lemma 7.2.2, and consider the family of open subintervals (a, b) of $(-T_j, T_j)$ such that $|v_j(a) + 1| \le \eta$, $|v_j(b) - 1| \le \eta$, and the derivatives of v_j at the endpoints satisfy $|v_j^{(\ell)}| \le 1$ for all $\ell \in \{1, \dots, k-1\}$. Let (a_j, b_j) be a minimal such interval; that is, any subinterval of (a_j, b_j) does not belong to this family. Hence, (a_j, b_j) does not intersect the set of the points t such that $||v_j(t)| - 1| \le \eta$ and $|v_j^{(\ell)}(t)| \le 1$ for all $\ell \in \{1, \dots, k-1\}$.

In order to apply Lemma 7.2.2 to prove that $b_j - a_j$ is equibounded, we use the scaling factor $\varepsilon_j = \frac{1}{2T_j}$ to define $u_j(t) = v_j(\frac{t}{\varepsilon_j} - T_j)$. We then have

$$\int_0^1 \Big(\frac{1}{\varepsilon_j} W(u_j) + \varepsilon_j^{2k-1} (u_j^{(k)})^2 \Big) \, \mathrm{d}t = \int_{-T_j}^{T_j} \big(W(v_j) + (v_j^{(k)})^2 \big) \, \mathrm{d}t,$$

which is equibounded. Assume by contradiction that there exists a (not relabeled) subsequence $b_j - a_j \to +\infty$. Then, we can apply Lemma 7.2.2 to the sequence of scaled intervals $((a_j + T_j)\varepsilon_j, (b_j + T_j)\varepsilon_j)$ and deduce that there exists $j_0 \in \mathbb{N}$ such that

$$\mathcal{L}^1\big(((a_j + T_j)\varepsilon_j, (b_j + T_j)\varepsilon_j) \cap A^\eta_{\varepsilon_j}(1))\big) > 0$$

for all $j \ge j_0$, where we recall that $A^\eta_\varepsilon(N)$ is defined as in (7.2.6). By scaling again to $(-T_j, T_j)$, this gives a contradiction.

Up to subsequences and up to translations, we can then assume that there exists $T^* > 0$ such that $a_j \to -T^*$ and $b_j \to T^*$ as $j \to +\infty$. Up to an asymptotically negligible change of variable, it is not restrictive to assume that $a_j = -T^*$ and $b_j = T^*$. By the equiboundedness of the boundary condition and by the Fundamental Theorem of Calculus we deduce that $\{v_j\}_j$ is equibounded in $\mathrm{H}^k(-T^*, T^*)$. Hence, we extract a (not relabeled) subsequence $\{v_j\}_j$ weakly converging to a function v in $\mathrm{H}^k(-T^*, T^*)$.

Assume now by contradiction that $\widetilde{m}_k = 0$. Since $\{v_j\}_j$ is a minimising sequence, by the lower semicontinuity of the L^2-norm, we have

$$0 = \lim_{j \to +\infty} \int_{-T^*}^{T^*} \big(W(v_j) + (v_j^{(k)})^2 \big) \, \mathrm{d}t \ge \liminf_{j \to +\infty} \int_{-T^*}^{T^*} (v_j^{(k)})^2 \, \mathrm{d}t \ge \int_{-T^*}^{T^*} (v^{(k)})^2 \, \mathrm{d}t.$$

Hence, we obtain that $v^{(k)} = 0$ on $(-T^*, T^*)$, so that v is a polynomial of degree at most $k - 1$. Moreover, by the uniform convergence of $v_j \to v$, we have

$$\int_{-T^*}^{T^*} W(v) \, \mathrm{d}t = \lim_{j \to +\infty} \int_{-T^*}^{T^*} W(v_j) \, \mathrm{d}t = 0,$$

so that $W(v) = 0$ almost everywhere, and then, since $\{W = 0\} = \{-1, 1\}$, we deduce that v is either identically -1 or 1 on $(-T^*, T^*)$. Since $|v(-T^*) + 1| \le \eta$ and $|v(T^*) - 1| \le \eta$, and $\eta < 1$, this gives a contradiction, and the proof is complete. □

As it is usual in phase-transition problems, we will need some auxiliary results involving optimal profile problems approximating m_k, defined via the minimisation of $\int_{I_T} (W(v) + (v^{(k)})^2) \, \mathrm{d}t$, where I_T is an interval with length $2T$ and v in $\mathrm{H}^k(I_T)$ satisfies suitable boundary conditions; in particular, we will minimise over sets of functions with "small" values of the derivatives at the endpoints of the interval. To that end, for all $\eta \in (0, \sqrt{\beta_W})$ and for all integers $N \ge 1$, we define an approximating constant given by

$$m_k(\eta, N) \coloneqq \inf_{T > 0} m_k(\eta, N, T), \tag{7.3.2}$$

where $m_k(\eta, N, T)$ is the minimum defined in (7.2.1).

Lemma 7.3.3. *Let $\widetilde{m}_k$ and $m_k(\eta, N)$ be defined by* (7.3.1) *and* (7.3.2)*, respectively. Then*

$$\lim_{\eta\to 0}\lim_{N\to+\infty} m_k(\eta, N) = \widetilde{m}_k.$$

Proof. Since $m_k(\eta, N)$ is increasing as $\eta \to 0$ and as $N \to +\infty$, the limits exist. For all $T > 0$, η and N, we have $m_k(T) \geq m_k(\eta, N, T)$, so that $\widetilde{m}_k \geq m_k(\eta, N)$ for all η and N. Hence,

$$\widetilde{m}_k \geq \lim_{\eta\to 0}\lim_{N\to+\infty} m_k(\eta, N).$$

We now prove the converse inequality. Let $\eta_j \to 0$ and $N_j \to +\infty$ be such that

$$\lim_{j\to+\infty} m_k(\eta_j, N_j) = \lim_{\eta\to 0}\lim_{N\to+\infty} m_k(\eta, N).$$

For any j, let $T_j > 0$ and $v_j \in \mathrm{H}^k(-T_j, T_j)$ be an admissible test function for $m_k(\eta_j, N_j, T_j)$ such that

$$\lim_{j\to+\infty} m_k(\eta_j, N_j) = \lim_{j\to+\infty} \int_{-T_j}^{T_j} \big(W(v_j) + (v_j^{(k)})^2\big)\,\mathrm{d}t.$$

To define an admissible test function u_j for $\widetilde{m}_k$, we can proceed as in the proof of Proposition 7.3.1 with $-T_j$ in the place of t_j^-, T_j in the place of t_j^+, and v_j in the place of the restriction of u to (t_j^-, t_j^+).

In this way, the resulting u_j is a test function for $\widetilde{m}_k$ and

$$\lim_{j\to+\infty} m_k(\eta_j, N_j) = \lim_{j\to+\infty} \int_{-\infty}^{+\infty} \big(W(u_j) + (u_j^{(k)})^2\big)\,\mathrm{d}t \geq \widetilde{m}_k,$$

proving the equality

$$\lim_{\eta\to 0}\lim_{N\to+\infty} m_k(\eta, N) = \widetilde{m}_k,$$

and the claim. □

We introduce some quantities that will allow us to estimate oscillations of order η. For all $0 < \eta < \min\{1, \sqrt{\beta_W}\}$, integers $N \geq 1$ and $\overline{T} > 0$, we define

$$\begin{aligned} m_k^+(\eta, N, \overline{T}) = \inf_{0<T\leq\overline{T}} \min\Big\{ &\int_0^T \big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t\colon\ u \in \mathrm{H}^k(0,T),\\ &|u(0) - 1| \leq \eta, |u(T) - 1| = 2\eta,\\ &\text{and } |u^{(\ell)}(0)| \leq \frac{1}{N} \text{ for all } \ell \in \{1, \dots, k-1\}\Big\}, \end{aligned} \tag{7.3.3}$$

and

$$\begin{aligned} m_k^-(\eta, N, \overline{T}) = \inf_{0<T\leq\overline{T}} \min\Big\{ &\int_0^T \big(W(u) + (u^{(k)})^2\big)\,\mathrm{d}t\colon\ u \in \mathrm{H}^k(0,T),\\ &|u(0) + 1| \leq \eta, |u(T) + 1| = 2\eta,\\ &\text{and } |u^{(\ell)}(0)| \leq \frac{1}{N} \text{ for all } \ell \in \{1, \dots, k-1\}\Big\}. \end{aligned}$$

Lemma 7.3.4. *For every $\overline{T} > 0$, it holds*

$$m_k^*(\eta, N, \overline{T}) \coloneqq \min\{m_k^+(\eta, N, \overline{T}), m_k^-(\eta, N, \overline{T})\} > 0. \tag{7.3.4}$$

Proof. We only prove that $m_k^+(\eta, N, \overline{T}) > 0$, the other case being analogous. We argue by contradiction.

Suppose there exists a sequence $\{T_j\}_j \subset (0, \overline{T}]$ and functions $v_j \in \mathrm{H}^k(0, T_j)$ which are admissible for (7.3.3) and such that

$$\lim_{j\to+\infty} \int_0^{T_j} \big(W(v_j) + (v_j^{(k)})^2\big)\, \mathrm{d}t = 0. \tag{7.3.5}$$

Without loss of generality, we may suppose that T_j converges to $T \in [0, \overline{T}]$ as $j \to +\infty$. We distinguish the two cases $T = 0$ and $T > 0$.

Case 1: $(T = 0)$. By (7.3.5) it holds $\|v_j^{(k)}\|_{L^2(0,T_j)} \to 0$ as $j \to +\infty$; hence, recalling that v_j satisfies the boundary conditions appearing in (7.3.3), we apply the Fundamental Theorem of Calculus to deduce that $\|v_j\|_{\mathrm{H}^k(0,T_j)} \le C$ for every j. We then extend each function v_j to the whole interval $(0, 1)$ by setting it equal to p_j, the polynomial of degree at most $k-1$ obtained by imposing $p_j^{(\ell)}(T_j) = v_j^{(\ell)}(T_j)$ for $\ell \in \{0, \dots, k-1\}$. The sequence $\{v_j\}_j$ is then equibounded in $\mathrm{H}^k(0, 1)$ so that there exists a subsequence, not relabeled, which converges uniformly to some $v \in \mathrm{H}^k(0, 1)$. This produces a contradiction since $T_j \to 0$ and $|v_j(T_j) - v_j(0)| \ge \eta$.

Case 2: $(T > 0)$. Up to an asymptotically negligible change of variable we may suppose $T_j = T$ for every j. Then, by (7.3.5), we have that $\{v_j\}_j$ is equibounded in $\mathrm{H}^k(0, T)$ so that a certain subsequence converges uniformly on $[0, T]$ to a function v in $\mathrm{H}^k(0, T)$. Again by (7.3.5) and by the lower semicontinuity of the L^2-norm, we have that either $v = -1$ or $v = 1$ on $(0, T)$. But $|v_j(T_j) - 1| = 2\eta$ for every j, hence, $|v(T) - 1| = 2\eta$, which is a contradiction since $\eta < 1$. □

7.4 Proof of the main result

We divide the proof of Theorem 7.1.1 in proving separately compactness, lower and upper bounds. We start by proving the equicoerciveness of the family $\{F_\varepsilon\}_\varepsilon$.

Proposition 7.4.1 (Compactness)**.** *Let F_ε be defined by* (7.1.1) *on $\mathrm{H}^k(0, 1)$, for all $\varepsilon_j \to 0$ and $\{u_j\}$ with $\sup_j F_{\varepsilon_j}(u_j) < +\infty$, there exists $u \in \mathrm{BV}((0, 1); \{-1, 1\})$ such that, up to subsequences, $u_j \to u$ in measure.*

Proof. Up to subsequences, we may assume that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) = S < +\infty$. We consider for any fixed $0 < \eta < \min\{1, \sqrt{\beta_W}\}$ and $\varepsilon > 0$ the sets of indices $\mathcal{I}_\varepsilon^{1,\eta}$ and $\mathcal{I}_\varepsilon^{-1,\eta}$ such that

$$\{t \in (0, 1)\colon\ |u_\varepsilon(t) - 1| < \eta\} = \bigcup_{j\in\mathcal{I}_\varepsilon^{1,\eta}} I_j,$$
$$\{t \in (0, 1)\colon\ |u_\varepsilon(t) + 1| < \eta\} = \bigcup_{j\in\mathcal{I}_\varepsilon^{-1,\eta}} I_j,$$

where each $I_j = (a_j, b_j)$ is a maximal open interval (that is, if $a_j \neq 0$ then $||u_\varepsilon(a_j)| - 1| = \eta$, and if $b_j \neq 1$ then $||u_\varepsilon(b_j)| - 1| = \eta$, respectively) and they are pairwise disjoint.

Let $N \in \mathbb{N}$ be fixed. A first remark is that for ε small enough the set $(0, 1) \cap A_\varepsilon^\eta(N)$ of the points where $||u_\varepsilon| - 1| < \eta$ and $|u_\varepsilon^{(\ell)}| < \frac{1}{N\varepsilon^\ell}$ is not empty. This follows by the application

of Lemma 7.2.2 to the constant sequence of intervals $I_{\varepsilon_r} = (0,1)$ and ε_r any infinitesimal sequence. We can then define

$$a_\varepsilon = \inf\big((0,1) \cap A_\varepsilon^\eta(N)\big) \quad \text{and} \quad b_\varepsilon = \sup\big((0,1) \cap A_\varepsilon^\eta(N)\big),$$

noting that

$$\mathcal{L}^1((0,1) \setminus (a_\varepsilon, b_\varepsilon)) = o(1)_{\varepsilon\to 0^+}.$$

This can again be proved by applying Lemma 7.2.2 to the family of intervals $I_\varepsilon = (0, a_\varepsilon)$, which actually implies that $a_\varepsilon = O(\varepsilon)$, since otherwise the claim of Lemma 7.2.2 would contradict the definition of a_ε. Similarly one proves that $1 - b_\varepsilon = O(\varepsilon)$.

Let $\mathcal{I}_\varepsilon^{1,\eta}(N)$ denote the subset of indices of $\mathcal{I}_\varepsilon^{1,\eta}$ for which there exists an open interval $I = (a_j', b_j') \subseteq I_j = (a_j, b_j)$ such that

$$|u_\varepsilon^{(\ell)}(a_j')| \le \frac{1}{N\varepsilon^\ell} \quad \text{and} \quad |u_\varepsilon^{(\ell)}(b_j')| \le \frac{1}{N\varepsilon^\ell} \quad \text{for } \ell \in \{1, \dots, k-1\}. \tag{7.4.1}$$

Analogously, we define $\mathcal{I}_\varepsilon^{-1,\eta}(N)$.

For each $j \in \mathcal{I}_\varepsilon^{\pm 1,\eta}(N)$, we consider the maximal open interval (a_j^N, b_j^N) for which (7.4.1) holds true; that is,

$$a_j^N \coloneqq \inf\Big\{t \in [a_j, b_j)\colon\ |u_\varepsilon^{(\ell)}(t)| \le \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\}\Big\},$$
$$b_j^N \coloneqq \sup\Big\{t \in (a_j, b_j]\colon\ |u_\varepsilon^{(\ell)}(t)| \le \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\}\Big\}.$$

For any $t \in (a_\varepsilon, b_\varepsilon) \setminus \bigcup_{j \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_j^N, b_j^N)$, we set

$$\tau(t) = \sup\{b_j^N\colon\ b_j^N \le t,\ j \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)\},$$
$$\sigma(t) = \inf\{a_j^N\colon\ a_j^N \ge t,\ j \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)\}.$$

Note that $\tau(t)$ and $\sigma(t)$ may coincide. For any t the following properties hold:

(i) $[\tau(t), \sigma(t)] \cap \bigcup_{j \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_j^N, b_j^N) = \emptyset$, so that, recalling that $A_\varepsilon^\eta(N)$ denotes the set of points t such that $||u_\varepsilon(t)| - 1| < \eta$ and $|u_\varepsilon^{(\ell)}(t)| < \frac{1}{N\varepsilon^\ell}$ for every $\ell \in \{1, \dots, k-1\}$, it follows

$$\mathcal{L}^1((\tau(t), \sigma(t)) \cap A_\varepsilon^\eta(N)) = 0;$$

(ii) if $I_j \cap [\tau(t), \sigma(t)] \neq \emptyset$ for $j \in \big(\mathcal{I}_\varepsilon^{1,\eta} \cup \mathcal{I}_\varepsilon^{-1,\eta}\big) \setminus \big(\mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)\big)$, then $I_j \subset [\tau(t), \sigma(t)]$;

(iii) if $(a_j, a_j^N] \cap [\tau(t), \sigma(t)] \neq \emptyset$, then $(a_j, a_j^N] \subset [\tau(t), \sigma(t)]$; the same property holds for $[b_j^N, b_j)$;

(iv) by the continuity of u_ε and of its derivatives,

$$||u_\varepsilon(\tau(t))| - 1| \le \eta \ \text{ and } \ |u_\varepsilon^{(\ell)}(\tau(t))| \le \frac{1}{N\varepsilon^\ell} \ \text{ for all } \ \ell \in \{1, \dots, k-1\},$$

and the same properties hold for $\sigma(t)$.

Moreover, for different t the intervals $(\tau(t),\sigma(t))$ either coincide or are disjoint; indeed, if $s\in(\tau(t),\sigma(t))$ for some t, then $(\tau(t),\sigma(t))=(\tau(s),\sigma(s))$.

Then, we can write the set $(a_\varepsilon,b_\varepsilon)\setminus\bigcup_{j\in\mathcal{I}_\varepsilon^{1,\eta}(N)\cup\mathcal{I}_\varepsilon^{-1,\eta}(N)}(a_j^N,b_j^N)$ as a disjoint union of intervals of the form $[\tau(t),\sigma(t)]$; that is, there exists a set of indices $\Lambda_\varepsilon(\eta,N)$ such that

$$(a_\varepsilon,b_\varepsilon)\setminus\bigcup_{j\in\mathcal{I}_\varepsilon^{1,\eta}(N)\cup\mathcal{I}_\varepsilon^{-1,\eta}(N)}(a_j^N,b_j^N)=\bigcup_{\lambda\in\Lambda_\varepsilon(\eta,N)}[\tau_\lambda,\sigma_\lambda],$$

where the interior of intervals are pairwise disjoint and the endpoints τ_λ and σ_λ belong to the sets

$$\{\tau(t)\colon\ t\in(a_\varepsilon,b_\varepsilon)\setminus\bigcup_{j\in\mathcal{I}_\varepsilon^{1,\eta}(N)\cup\mathcal{I}_\varepsilon^{-1,\eta}(N)}(a_j^N,b_j^N)\},$$
$$\{\sigma(t)\colon\ t\in(a_\varepsilon,b_\varepsilon)\setminus\bigcup_{j\in\mathcal{I}_\varepsilon^{1,\eta}(N)\cup\mathcal{I}_\varepsilon^{-1,\eta}(N)}(a_j^N,b_j^N)\},$$

respectively.

Since the measure of the set of points t such that $||u_\varepsilon(t)|-1|\geq\eta$ is bounded by $\varepsilon\frac{S}{\alpha_W\eta^2}$, we deduce that there exists a countable subfamily $\Lambda_\varepsilon^*(\eta,N)\subset\Lambda_\varepsilon(\eta,N)$ such that

$$(a_\varepsilon,b_\varepsilon)\setminus\bigcup_{j\in\mathcal{I}_\varepsilon^{1,\eta}(N)\cup\mathcal{I}_\varepsilon^{-1,\eta}(N)}(a_j^N,b_j^N)=J_\varepsilon\cup\bigcup_{\lambda\in\Lambda_\varepsilon^*(\eta,N)}(\tau_\lambda,\sigma_\lambda),$$

where $\mathcal{L}^1(J_\varepsilon)\to0$ as $\varepsilon\to0^+$.

We now prove that there exists a constant $T_k(\eta,N,S)>0$ such that

$$\mathcal{L}^1((\tau_\lambda,\sigma_\lambda))\leq T_k(\eta,N,S)\varepsilon \tag{7.4.2}$$

for any $\varepsilon>0$ and $\lambda\in\Lambda_\varepsilon^*(\eta,N)$.

We reason by contradiction. If the claim does not hold, then for any $r\in\mathbb{N}$ there exist $\varepsilon_r>0$ and $\lambda_r\in\Lambda_{\varepsilon_r}^*(\eta,N)$ such that

$$\mathcal{L}^1((\tau_{\lambda_r},\sigma_{\lambda_r}))>r\varepsilon_r.$$

Since $(\tau_{\lambda_r},\sigma_{\lambda_r})\subset(0,1)$, this implies that $\varepsilon_r\to0$ as $r\to+\infty$. We can then apply Lemma 7.2.2 to the family of intervals $I_{\varepsilon_r}=(\tau_{\lambda_r},\sigma_{\lambda_r})$, deducing that there exists $r_0\in\mathbb{N}$ such that

$$\mathcal{L}^1(A_{\varepsilon_r}^\eta(N)\cap(\tau_{\lambda_r},\sigma_{\lambda_r}))>0$$

for any $r\geq r_0$. This gives a contradiction, since property (i) holds for any $(\tau_{\lambda_r},\sigma_{\lambda_r})$.

We say that an interval $(\tau_\lambda,\sigma_\lambda)$ with $\lambda\in\Lambda_\varepsilon^*(\eta,N)$ is an *effective transition interval* if either $|u_\varepsilon(\tau_\lambda)-1|\leq\eta$ and $|u_\varepsilon(\sigma_\lambda)+1|\leq\eta$, or $|u_\varepsilon(\tau_\lambda)+1|\leq\eta$ and $|u_\varepsilon(\sigma_\lambda)-1|\leq\eta$. We now prove that the number of such intervals is equibounded. Indeed, setting $v_\varepsilon(s)\coloneqq u_\varepsilon(\varepsilon s+\frac{\sigma_\lambda+\tau_\lambda}{2})$, by a change of variable we obtain

$$F_\varepsilon(u_\varepsilon,(\tau_\lambda,\sigma_\lambda))=\int_{-\frac{\sigma_\lambda-\tau_\lambda}{2\varepsilon}}^{\frac{\sigma_\lambda-\tau_\lambda}{2\varepsilon}}\left(W(v_\varepsilon)+(v_\varepsilon^{(k)})^2\right)\mathrm{d}s\geq m_k(\eta,N)\ \geq\ \frac{m_k}{2}, \tag{7.4.3}$$

where the last inequality follows by Lemma 7.3.3. Since $m_k>0$, the number of intervals of effective transitions is equibounded by a constant which is independent of ε,η and N; more precisely, by $\frac{2S}{m_k}$.

We now consider intervals $(\tau_\lambda, \sigma_\lambda)$ which are not intervals of effective transitions; that is, either $|u_\varepsilon(\tau_\lambda) - 1| \le \eta$ and $|u_\varepsilon(\sigma_\lambda) - 1| \le \eta$ or $|u_\varepsilon(\tau_\lambda) + 1| \le \eta$ and $|u_\varepsilon(\sigma_\lambda) + 1| \le \eta$. Let $(\tau_\lambda, \sigma_\lambda)$ be such that $|u_\varepsilon(\tau_\lambda) - 1| \le \eta$ and $|u_\varepsilon(\sigma_\lambda) - 1| \le \eta$, and suppose that there exists a point t in $(\tau_\lambda, \sigma_\lambda)$ such that $|u_\varepsilon(t) - 1| \ge 2\eta$. Then, setting $t' = \inf\{t\colon\ |u_\varepsilon(t) - 1| \ge 2\eta\}$, by a change of variable we have that

$$\begin{aligned} F_\varepsilon(u_\varepsilon, (\tau_\lambda, \sigma_\lambda)) \ge F_\varepsilon(u_\varepsilon, (\tau_\lambda, t')) &= \int_{\tau_\lambda}^{t'} \Big(\frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon^{2k-1}(u_\varepsilon^{(k)})^2\Big)\,\mathrm{d}t \\ &= \int_0^{\frac{t'-\tau_\lambda}{\varepsilon}} \Big(W(v_\varepsilon) + (v_\varepsilon^{(k)})^2\Big)\,\mathrm{d}s \\ &\ge \inf_{T \le T_k(\eta,N,S)} \min_v \Big\{ \int_0^T \Big(W(v_\varepsilon) + (v_\varepsilon^{(k)})^2\Big)\,\mathrm{d}s \Big\}, \end{aligned}$$

where the minimum is taken over $v \in H^k(0,T)$ such that $|v(T) - 1| = 2\eta$, $|v(0) - 1| \le \eta$, $|v^{(\ell)}(0)| \le \frac{1}{N}$ for all $\ell \in \{1, \dots, k-1\}$, and we used the fact that $|(\tau_\lambda, \sigma_\lambda)| \le T_k(\eta, N, S)\varepsilon$ by (7.4.2). Then, by applying Lemma 7.3.4 with $\eta > 0$ and $\overline{T} = T_k(\eta, N, S)$, we obtain

$$F_\varepsilon(u_\varepsilon, (\tau_\lambda, \sigma_\lambda)) \ge m_k^*(\eta, N, T_k(\eta, N, S)) > 0.$$

Since a corresponding argument holds if $|u_\varepsilon(\tau_\lambda) + 1| \le \eta$ and $|u_\varepsilon(\sigma_\lambda) + 1| \le \eta$, we can conclude that

$$||u_\varepsilon(t)| - 1| \le 2\eta \ \text{ in } \ (\tau_\lambda, \sigma_\lambda)$$

except for a finite number of indices λ (which of course depends on η and N). We let $c_k(\eta, N)$ denote the number of such λ.

Let now $\{(\tau_i, \sigma_i)\}_{i=1}^{M_\varepsilon}$ with $\tau_i < \tau_{i+1}$ denote the set of the effective transition intervals. Since $M_\varepsilon \le \frac{2S}{m_k}$, we may suppose M_ε is independent of ε, up to passing to a subsequence, and we set $M_\varepsilon = M$. We construct a sequence of functions $u_\varepsilon^{\eta,N}$ in $(0,1)$ as follows. To give a definition in the intervals (σ_i, τ_{i+1}), we note that, by the definitions of $\tau(t)$ and $\sigma(t)$, there exists an index $j \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)$ such that $(a_j^N, b_j^N) \subseteq (\sigma_i, \tau_{i+1})$. Moreover, if there exists an index $j \in \mathcal{I}_\varepsilon^{-1,\eta}(N)$ such that $(a_j^N, b_j^N) \subseteq (\sigma_i, \tau_{i+1})$, then $(\sigma_i, \tau_{i+1}) \cap \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N)} (a_h^N, b_h^N) = \emptyset$. Indeed, otherwise (σ_i, τ_{i+1}) would contain an effective transition interval. The analogous property holds if there exists an index $j \in \mathcal{I}_\varepsilon^{1,\eta}(N)$ such that $(a_j^N, b_j^N) \subseteq (\sigma_i, \tau_{i+1})$. We then set

$$u_\varepsilon^{\eta,N} = -1 \text{ in } (\sigma_i, \tau_{i+1})$$

if there exists $j \in \mathcal{I}_\varepsilon^{-1,\eta}(N)$ such that $(a_j^N, b_j^N) \subseteq (\sigma_i, \tau_{i+1})$, and

$$u_\varepsilon^{\eta,N} = 1 \text{ in } (\sigma_i, \tau_{i+1})$$

if there exists $j \in \mathcal{I}_\varepsilon^{1,\eta}(N)$ such that $(a_j^N, b_j^N) \subseteq (\sigma_i, \tau_{i+1})$. Finally, we extend the function $u_\varepsilon^{\eta,N}$ on the whole interval $(0,1)$ with values $-1, 1$ in such a way that

$$\#J_{u_\varepsilon^{\eta,N}} = M, \qquad \text{for every } \varepsilon, \eta, N. \tag{7.4.4}$$

Moreover, by (7.4.3) we have that

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \ge M m_k(\eta, N). \tag{7.4.5}$$

Up to subsequences, by (7.4.4) we can suppose that $u_\varepsilon^{\eta_\varepsilon,N_\varepsilon} \to u$ in L^1, the limit u takes values in $\{-1,1\}$, and its jump set J_u is composed of no more than M points.

By construction, we also have that

$$\mathcal{L}^1(\{t \in (0,1)\colon |u_\varepsilon(t) - u_\varepsilon^{\eta,N}(t)| > 2\eta\}) \le C_k(\eta,N)\,\varepsilon, \tag{7.4.6}$$

where $C_k(\eta,N) \coloneqq (\frac{2S}{m_k} + c_k(\eta,N))T_k(\eta,N,S)$. By this estimate, we show that, up to subsequences, $\{u_\varepsilon\}_\varepsilon$ converges in measure to the same limit u. Indeed, with fixed $\delta > 0$, by the convergence of $\{u_\varepsilon^{\eta_\varepsilon,N_\varepsilon}\}$ we have that

$$\mathcal{L}^1(\{|u_\varepsilon^{\eta_\varepsilon,N_\varepsilon} - u| > \delta\}) = o(1)$$

as $\varepsilon \to 0^+$. By a triangular inequality and (7.4.6), for ε small enough we have

$$\begin{aligned}\mathcal{L}^1(\{|u_\varepsilon - u| > 2\delta\}) &\le \mathcal{L}^1(\{|u_\varepsilon - u_\varepsilon^{\eta_\varepsilon,N_\varepsilon}| > 2\eta_\varepsilon\}) + \mathcal{L}^1(\{|u_\varepsilon^{\eta_\varepsilon,N_\varepsilon} - u| > \delta\}) \\ &\le C_k(\eta_\varepsilon,N_\varepsilon)\varepsilon + o(1) = o(1)\end{aligned}$$

as $\varepsilon \to 0^+$, concluding the proof. □

Finally, we prove the Γ-convergence result.

Proposition 7.4.2 (Lower bound)**.** *Let $u \in \mathrm{BV}((0,1);\{-1,1\})$ and $u_\varepsilon \to u$ in measure. Then*

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \ge m_k \# J_u.$$

Proof. From the last part of the proof of the previous proposition, and by (7.4.5), we deduce that, up to the extraction of subsequences,

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \ge m_k(\eta,N) \# J_u.$$

We can then let $N \to +\infty$ and $\eta \to 0$ and apply Lemma 7.3.3 and Proposition 7.3.1. □

Proposition 7.4.3 (Upper bound)**.** *Let $u \in \mathrm{BV}((0,1);\{-1,1\})$. Then there exists a recovery sequence $u_\varepsilon \to u$ in $L^1(0,1)$ such that*

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) = m_k \# J_u.$$

Proof. Let $u \in \mathrm{BV}((0,1);\{-1,1\})$ and denote by $x_1,\dots,x_r$ its jump points. We set $J_1 \coloneqq \{n \in \{1,\dots,r\}\colon u(x_n^+) - u(x_n^-) = 2\}$, $J_2 \coloneqq \{n \in \{1,\dots,r\}\colon u(x_n^+) - u(x_n^-) = -2\}$, and for convenience, we also set $x_0 = 0$ and $x_{r+1} = 1$. For a fixed real positive $\delta < \min\{x_n - x_{n-1}\colon n = 1,\dots,r+1\}$, consider a function v admissible for $\widetilde{m}_k$ such that

$$\int_{-\infty}^{+\infty} \big(W(v) + (v^{(k)})^2\big)\mathrm{d}t \le m_k + \frac{\delta}{r}. \tag{7.4.7}$$

Recall that there exists $T > 0$ such that $v = -1$ in $(-\infty,-T)$ and $v = 1$ in $(T,+\infty)$.

Let $\varepsilon_j \to 0^+$ and suppose that $\frac{\delta}{2\varepsilon_j} > T$ for every j. For $n \in \{1,\dots,r\}$, define intervals $I_n \coloneqq [(x_{n-1}+x_n)/2, (x_n+x_{n+1})/2]$ and consider the sequence defined as

$$u_j(t) \coloneqq \begin{cases} v\Big(\dfrac{t-x_n}{\varepsilon_j}\Big) & \text{if } t \in I_n \text{ and } n \in J_1, \\ v\Big(-\dfrac{t-x_n}{\varepsilon_j}\Big) & \text{if } t \in I_n \text{ and } n \in J_2, \\ u(t) & \text{otherwise.} \end{cases}$$

We note that the condition $\frac{\delta}{2\varepsilon_j} > T$ implies that $\{u_j\} \subset \mathrm{H}^k(0,1)$; and moreover, $u_j \to u$ in $L^1(0,1)$ as $j \to +\infty$. To conclude, we estimate

$$\begin{aligned}
\lim_{j\to+\infty} F_{\varepsilon_j}(u_j) &= \lim_{j\to+\infty} \sum_{n=1}^{r} \int_{I_n} \Big(\frac{1}{\varepsilon_j} W(u_j(t)) + \varepsilon_j^{2k-1} (u_j^{(k)}(t))^2 \Big) \mathrm{d}t \\
&= \lim_{j\to+\infty} \sum_{n\in J_1} \int_{\frac{x_{n-1}-x_n}{2\varepsilon_j}}^{\frac{x_{n+1}-x_n}{2\varepsilon_j}} \big(W(v(s)) + (v^{(k)}(s))^2 \big) \mathrm{d}s \\
&\quad + \lim_{j\to+\infty} \sum_{n\in J_2} \int_{\frac{x_{n-1}-x_n}{2\varepsilon_j}}^{\frac{x_{n+1}-x_n}{2\varepsilon_j}} \big(W(v(-s)) + (v^{(k)}(-s))^2 \big) \mathrm{d}s \\
&= \sum_{n\in J_1} \int_{-\infty}^{+\infty} \big(W(v(s)) + (v^{(k)}(s))^2 \big) \mathrm{d}s \\
&\quad + \sum_{n\in J_2} \int_{-\infty}^{+\infty} \big(W(v(-s)) + (v^{(k)}(-s))^2 \big) \mathrm{d}s \\
&\leq m_k \# J_u + \delta,
\end{aligned}$$

where in the last inequality we have used (7.4.7). The arbitrariness of δ and a diagonal argument allow us to conclude. □

7.5 Γ-limit in dimension d > 1

In this section we prove Theorem 7.1.2 in the general d-dimensional case. We refer the reader to Section 1.1 for the notation concerning one-dimensional slicing adopted in the proof of the lower bound.

To that end, we use some standard notions in the theory of sets of finite perimeter, for which we refer to Section 1.3. Moreover, we use some notation for d-dimensional sets. We recall that $Q_1 = (-\frac{1}{2}, \frac{1}{2})^d$, and that $Q_\nu(x,\rho)$ denotes a cube with centre x, side length ρ and one face orthogonal to ν. For brevity of notation, we also set $Q_\rho \coloneqq Q_{e_j}(0,\rho)$ if $x = 0$ and $\nu = e_j$, the j-th element of the canonical basis of $\mathbb{R}^d$.

We first prove the lower bound using a variant of the blow-up technique by Fonseca and Müller [124] adapted to Γ-converging families of functionals in the spirit of [50], combined with a slicing argument that allows us to reduce to the one-dimensional case in Theorem 7.1.1.

Proposition 7.5.1 (Lower bound). *Let $u \in \mathrm{BV}(\Omega;\{-1,1\})$ and let $\{u_\varepsilon\}_\varepsilon$ be a sequence in $\mathrm{H}^k(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$. Then*

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq m_k \mathcal{H}^{d-1}(J_u).$$

Proof. Without loss of generality, we assume $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Let $\varepsilon_j \to 0^+$ be such that

$$\lim_{j\to+\infty} F_{\varepsilon_j}(u_{\varepsilon_j}) = \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)$$

and let $\{u_{\varepsilon_j}\}$ be a sequence converging to u in $L^1(\Omega)$. Since $u \in \mathrm{BV}(\Omega;\{-1,1\})$, we can write $u = 2\chi_E - 1$, where $E = \{x \in \Omega\colon\ u(x) = 1\}$ is a set of finite perimeter in Ω. The set

J_u of discontinuity points of u coincides in Ω with $\partial^* E$, the reduced boundary of E, up to an $\mathcal{H}^{d-1}$-negligible set. The set $\partial^* E$ is a $(d-1)$-rectifiable set, and for every $x \in \partial^* E$ there exists $\nu_E(x)$, the inner unit normal to E in the measure-theoretical sense.

For every positive integer j, we introduce the measure μ_{ε_j} defined for $B \subset \Omega$ Borel as

$$\mu_{\varepsilon_j}(B) \coloneqq \int_B \Big(\frac{1}{\varepsilon_j} W(u_{\varepsilon_j}) + \varepsilon_j^{2k-1} \|\nabla^{(k)} u_{\varepsilon_j}\|^2\Big) \mathrm{d}x.$$

Since $\sup_j \mu_{\varepsilon_j}(\Omega) < +\infty$, up to a not relabeled subsequence, we may assume that $\{\mu_{\varepsilon_j}\}$ weakly* converges to a finite Borel measure μ. By the Besicovitch Derivation Theorem, for $\mathcal{H}^{d-1}$-almost every x in $\partial^* E$, it holds

$$\frac{d\mu}{d(\mathcal{H}^{d-1} \llcorner \partial^* E)}(x) = \lim_{\rho \to 0} \frac{\mu(Q_\nu(x,\rho))}{\mathcal{H}^{d-1}(Q_\nu(x,\rho) \cap \partial^* E)} = \lim_{\rho \to 0} \frac{\mu(Q_\nu(x,\rho))}{\rho^{d-1}}, \tag{7.5.1}$$

where $\nu = \nu_E(x)$ is the inner unit normal to E at x and in the last equality we have used that the $(d-1)$-dimensional density of $\partial^* E$ is equal to 1 for $\mathcal{H}^{d-1}$-almost every point in the reduced boundary of E. Since μ is a finite measure, we also have

$$\lim_{j \to +\infty} \mu_{\varepsilon_j}(Q_\nu(x,\rho)) = \mu(Q_\nu(x,\rho)) \tag{7.5.2}$$

for all $\rho > 0$ but a countable set. Hence, if we fix a point $x \in \partial^* E$ such that (7.5.1) holds, a diagonal argument and (7.5.2) imply that there exists $\rho_j = \rho(\varepsilon_j) \to 0^+$ such that

$$\frac{d\mu}{d(\mathcal{H}^{d-1} \llcorner \partial^* E)}(x) = \lim_{j \to +\infty} \frac{\mu_{\varepsilon_j}(Q_\nu(x,\rho_j))}{\rho_j^{d-1}} \tag{7.5.3}$$

and $\frac{\varepsilon_j}{\rho_j} \to 0$.

Up to a translation and rotation argument, we may suppose that $x = 0 \in \partial^* E$ is such that (7.5.3) holds and that $\nu_E(0) = e_d$. In this case, the following blow-up condition holds:

$$\chi_{\frac{1}{\rho_j} E} \xrightarrow{L^1_{\mathrm{loc}}(\mathbb{R}^d)} \chi_{\{x_d \geq 0\}} =: H \ \text{ as } j \to +\infty, \tag{7.5.4}$$

where H is the characteristic function of the half-space given by $x_d \geq 0$.

Applying the change of variable $y \coloneqq x/\rho_j$ and setting $v_{\varepsilon_j}(y) \coloneqq u_{\varepsilon_j}(\rho_j y)$, we get

$$\begin{aligned} \frac{\mu_{\varepsilon_j}(Q_{\rho_j})}{\rho_j^{d-1}} &= \frac{1}{\rho_j^{d-1}} \int_{Q_{\rho_j}} \Big(\frac{1}{\varepsilon_j} W(u_{\varepsilon_j}(x)) + \varepsilon_j^{2k-1} \|\nabla^{(k)} u_{\varepsilon_j}(x)\|^2\Big) \mathrm{d}x \\ &= \int_{Q_1} \Big(\frac{1}{\delta_j} W(v_{\varepsilon_j}(y)) + \delta_j^{2k-1} \|\nabla^{(k)} v_{\varepsilon_j}(y)\|^2\Big) \mathrm{d}y, \end{aligned}$$

where $\delta_j \coloneqq \frac{\varepsilon_j}{\rho_j}$. We set $y = (\xi, t)$, with $\xi \in [-1/2, 1/2]^{d-1}$, $t \in [-\frac{1}{2}, \frac{1}{2}]$ and $v^\xi_{\varepsilon_j}(t) \coloneqq v_{\varepsilon_j}(\xi, t)$, and we note that

$$\|\nabla^{(k)} v_{\varepsilon_j}\| \geq \Big|\frac{d^k}{\mathrm{d}t^k} v^\xi_{\varepsilon_j}(t)\Big|.$$

Therefore, we have

$$\frac{\mu_{\varepsilon_j}(Q_{\rho_j})}{\rho_j^{d-1}} \geq \int_{[-\frac{1}{2},\frac{1}{2}]^{d-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\delta_j} W(v^\xi_{\varepsilon_j}(t)) + \delta_j^{2k-1} \Big(\frac{d^k}{\mathrm{d}t^k} v^\xi_{\varepsilon_j}(t)\Big)^2\Big) \mathrm{d}t \, \mathrm{d}\xi$$

and by Fatou's Lemma

$$\liminf_{j\to+\infty}\frac{\mu_{\varepsilon_j}(Q_{\rho_j})}{\rho_j^{d-1}}\geq\int_{[-\frac12,\frac12]^{d-1}}\liminf_{j\to+\infty}\int_{-\frac12}^{\frac12}\Big(\frac{1}{\delta_j}W(v^\xi_{\varepsilon_j}(t))+\delta_j^{2k-1}\Big(\frac{d^k}{\mathrm{d}t^k}v^\xi_{\varepsilon_j}(t)\Big)^2\Big)\mathrm{d}t\,\mathrm{d}\xi.$$

Since $\{u_{\varepsilon_j}\}$ converges to the function $2\chi_E-1$ in $L^1(\Omega)$, by (7.5.4) we have that $v_{\varepsilon_j}\to 2H-1$ in $L^1(Q_1)$, and, in turn, up to subsequences, $v^\xi_{\varepsilon_j}\to 2\chi_{[0,\frac12)}-1=:v^\xi$ for Lebesgue-almost every $\xi\in[-\frac12,\frac12]^{d-1}$. We apply the lower estimate in Theorem 7.1.1 to infer that

$$\liminf_{j\to+\infty}\frac{\mu_{\varepsilon_j}(Q_{\rho_j})}{\rho_j^{d-1}}\geq\int_{[-\frac12,\frac12]^{d-1}}m_k\#J_{v^\xi}\,\mathrm{d}\xi=m_k.$$

Since this argument is valid for all $x\in\partial^*E$, recalling (7.5.3), we obtain

$$\begin{aligned}\mu(\Omega)&\geq\int_{\partial^*E}\frac{d\mu}{d(\mathcal{H}^{d-1}\llcorner\partial^*E)}(x)\mathrm{d}\mathcal{H}^{d-1}(x)\\&\geq m_k\mathcal{H}^{d-1}(\partial^*E)=m_k\mathcal{H}^{d-1}(J_u),\end{aligned}$$

and since the weak* convergence of μ_{ε_j} to μ implies that $\liminf_{j\to+\infty}\mu_{\varepsilon_j}(\Omega)\geq\mu(\Omega)$, finally we get

$$\liminf_{\varepsilon\to0^+}\mu_\varepsilon(\Omega)=\lim_{j\to+\infty}\mu_{\varepsilon_j}(\Omega)\geq\mu(\Omega)\geq m_k\mathcal{H}^{d-1}(J_u),$$

and the proof is complete. □

We conclude the proof of the Γ-convergence by showing the existence of a recovery sequence. In what follows C denotes a generic positive constant which may vary from line to line.

Proposition 7.5.2 (Upper bound)**.** *Let $u\in\mathrm{BV}(\Omega;\{-1,1\})$. Then there exists a recovery sequence $u_\varepsilon\in\mathrm{H}^k(\Omega)$ converging to u in $L^1(\Omega)$ such that*

$$\limsup_{\varepsilon\to0^+}F_\varepsilon(u_\varepsilon)\leq m_k\mathcal{H}^{d-1}(J_u).\tag{7.5.5}$$

Proof. Let E be the set of finite perimeter in Ω such that $u=2\chi_E-1$. Then, there exists a sequence $\{E_j\}$ of bounded open sets in $\mathbb{R}^d$ with C^∞ boundary such that $\chi_{E_j\cap\Omega}\to\chi_E$ in $L^1(\Omega)$ and $\mathrm{Per}(E_j,\Omega)\to\mathrm{Per}(E,\Omega)$. Therefore, since $\mathcal{H}^{d-1}(J_u)=\mathrm{Per}(E,\Omega)$, in view of Proposition 1.2.5, it is enough to prove the Γ-limsup inequality (7.5.5) only for functions $u=2\chi_E-1$ when $E=\widetilde{E}\cap\Omega$ for some smooth bounded set $\widetilde{E}$ in $\mathbb{R}^d$. We set $M:=\partial\widetilde{E}$. Moreover, arguing again by density, it is not restrictive to assume that $\widetilde{E}$ meets $\partial\Omega$ transversally.

Let $\varepsilon_j\to0^+$, and consider $\{v_j\}_j$ a sequence of functions in $\mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$ such that

$$\lim_{j\to+\infty}\int_{-\infty}^{+\infty}\Big(W(v_j)+(v_j^{(k)})^2\Big)\mathrm{d}t=m_k\tag{7.5.6}$$

and

$$v_j(t)=\begin{cases}-1 & \text{if } t\leq-\frac{1}{\sqrt{\varepsilon_j}},\\ 1 & \text{if } t\geq\frac{1}{\sqrt{\varepsilon_j}}.\end{cases}$$

Since M is a smooth manifold in $\mathbb{R}^d$, there exists $\delta_0 > 0$ such that for every $\delta < \delta_0$ there exists a unique smooth projection from U_δ on M, where $U_\delta := \{x \in \mathbb{R}^d\colon\ d(x, M) < \delta\}$ is a tubular neighbourhood of M. Therefore, denoting by $\widetilde{d}_M$ the signed distance function from M which, for convenience, we assume to be positive inside of $\widetilde{E}$, we have that $\widetilde{d}_M \in C^\infty(U_\delta)$ for every $\delta < \delta_0$.

We suppose that $\sqrt{\varepsilon_j} < \delta_0/2$ for every j and define

$$u_j(x) = \begin{cases} -1 & \text{if } x \in \Omega \setminus (E \cup U_j), \\ v_j\big(\frac{\widetilde{d}_M(x)}{\varepsilon_j}\big) & \text{if } x \in \Omega \cap U_j, \\ 1 & \text{if } x \in E \setminus U_j, \end{cases} \tag{7.5.7}$$

where $U_j := U_{\sqrt{\varepsilon_j}}$.

In order to estimate $F_{\varepsilon_j}(u_{\varepsilon_j})$, we compute the derivatives of order k of the function u_j. Let α be a multi-index of order k. We observe that

$$\partial^\alpha u_j = \frac{1}{\varepsilon_j^k} v_j^{(k)}\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big)(\partial_{\alpha_1}\widetilde{d}_M)\cdots(\partial_{\alpha_k}\widetilde{d}_M) + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big) P_\alpha^\ell(A(\partial^\alpha \widetilde{d}_M)),$$

where $A(\partial^\alpha \widetilde{d}_M)$ represents the ordered vector of all the derivatives $\partial^\beta \widetilde{d}_M$ with $\beta \subset \alpha, |\beta| \geq 1$, and $P_\alpha^\ell : \mathbb{R}^{2^{|\alpha|}-1} \to \mathbb{R}$ are polynomials of degree at most k. Therefore, the k-tensor $\nabla^{(k)} u_j$ can be written in the form

$$\nabla^{(k)} u_j = \frac{1}{\varepsilon_j^k} v_j^{(k)}\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big)(\nabla\widetilde{d}_M \otimes \cdots \otimes \nabla\widetilde{d}_M) + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big) T^\ell, \tag{7.5.8}$$

where T^ℓ, $\ell \in \{1, \dots, k-1\}$ are k-tensors having equibounded operator norm. Indeed, the coefficients of T^ℓ only depend on the derivatives of $\widetilde{d}_M$ up to order k, and these are uniformly bounded on U_j since we assumed $\sqrt{\varepsilon_j} < \delta_0/2$, which implies the existence of a compact set K such that $U_j \subset\subset K \subset U_{\delta_0}$.

By (7.5.7) and (7.5.8) we obtain

$$\begin{aligned} \limsup_{j\to+\infty} F_{\varepsilon_j}(u_j) = \limsup_{j\to+\infty} \bigg(& \int_{U_j\cap\Omega} \frac{1}{\varepsilon_j} W\Big(v_j\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big)\Big)\,\mathrm{d}x \\ & + \varepsilon_j^{2k-1} \int_{U_j\cap\Omega} \bigg\| \frac{1}{\varepsilon_j^k} v_j^{(k)}\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big)(\nabla\widetilde{d}_M \otimes \cdots \otimes \nabla\widetilde{d}_M) + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\Big(\frac{\widetilde{d}_M}{\varepsilon_j}\Big) T^\ell \bigg\|^2 \mathrm{d}x \bigg). \end{aligned}$$

Let $\Phi : M \times (-\delta_0/2, \delta_0/2) \to U_{\delta_0/2}$ be the diffeomorphism defined by $\Phi(y,t) := y - t\nu(y)$, where $\nu(y)$ is the outer unit normal to M at y, and denote by $J(y,t)$ its Jacobian. Applying this change of variable, we get

$$\begin{aligned} \limsup_{j\to+\infty} F_{\varepsilon_j}(u_j) = \limsup_{j\to+\infty} \int_{M\cap\Omega} \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \bigg(& \frac{1}{\varepsilon_j} W\Big(v_j\Big(\frac{t}{\varepsilon_j}\Big)\Big) \\ & + \varepsilon_j^{2k-1} \bigg\| \frac{1}{\varepsilon_j^k} v_j^{(k)}\Big(\frac{t}{\varepsilon_j}\Big)(\nabla\widetilde{d}_M \otimes \cdots \otimes \nabla\widetilde{d}_M)(\Phi(y,t)) \\ & + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\Big(\frac{t}{\varepsilon_j}\Big) T^\ell(\Phi(y,t)) \bigg\|^2 \bigg) J(y,t)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y). \end{aligned} \tag{7.5.9}$$

We set

$$I_j^k := \int_{M\cap\Omega}\int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \Big(\frac{1}{\varepsilon_j}W\Big(v_j\Big(\frac{t}{\varepsilon_j}\Big)\Big) + \frac{1}{\varepsilon_j}\Big(v_j^{(k)}\Big(\frac{t}{\varepsilon_j}\Big)\Big)^2\Big)J(y,t)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y),$$
$$I_j^\ell := \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \varepsilon_j^{2k-1-2\ell}\Big(v_j^{(\ell)}\Big(\frac{t}{\varepsilon_j}\Big)\Big)^2\,\mathrm{d}t \qquad \text{if } \ell < k,$$

and

$$I_j^{r,\ell} := \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \varepsilon_j^{2k-1-r-\ell}\Big|v_j^{(r)}\Big(\frac{t}{\varepsilon_j}\Big)v_j^{(\ell)}\Big(\frac{t}{\varepsilon_j}\Big)\Big|\mathrm{d}t \qquad \text{if } r < \ell \le k.$$

Taking into account that the Jacobian and the operator norm of all T^ℓ are bounded on $M\times(-\delta_0/2,\delta_0/2)$, expanding the square in the right-hand side of (7.5.9) we have

$$\begin{aligned}\limsup_{j\to+\infty} F_{\varepsilon_j}(u_j) &\le \limsup_{j\to+\infty}\Big(I_j^k + C\Big(\sum_{\ell=1}^{k-1} I_j^\ell + \sum_{\ell=1}^{k-1} I_j^{\ell,k} + \sum_{\ell=1}^{k-1}\sum_{r=1}^{\ell-1} I_j^{r,\ell}\Big)\Big)\\ &= \limsup_{j\to+\infty}\Big(I_j^k + C\Big(\sum_{\ell=1}^{k-1} I_j^\ell + \sum_{\ell=1}^{k}\sum_{r=1}^{\ell-1} I_j^{r,\ell}\Big)\Big).\end{aligned}$$

We have also used that $\|\nabla\widetilde{d}_M\otimes\cdots\otimes\nabla\widetilde{d}_M\| \le |\nabla\widetilde{d}_M|^k = 1$ since $\widetilde{d}_M$ is a 1-Lipschitz function.

We first prove that

$$\limsup_{j\to+\infty} I_j^k \le m_k\mathcal{H}^{d-1}(J_u).$$

By a change of variable, we have

$$\begin{aligned}I_j^k &= \int_{M\cap\Omega}\int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \Big(\frac{1}{\varepsilon_j}W\Big(v_j\Big(\frac{t}{\varepsilon_j}\Big)\Big) + \frac{1}{\varepsilon_j}\Big(v_j^{(k)}\Big(\frac{t}{\varepsilon_j}\Big)\Big)^2\Big)J(y,t)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ &= \int_{M\cap\Omega}\int_{-\frac{1}{\sqrt{\varepsilon_j}}}^{\frac{1}{\sqrt{\varepsilon_j}}} \big(W(v_j(s)) + (v_j^{(k)}(s))^2\big)J(y,s\varepsilon_j)\,\mathrm{d}s\,\mathrm{d}\mathcal{H}^{d-1}(y)\\ &\le \Big(\sup_{\substack{y\in M\\ t\in(-\sqrt{\varepsilon_j},\sqrt{\varepsilon_j})}} J(y,t)\Big)\int_{M\cap\Omega}\int_{-\infty}^{+\infty}(W(v_j(s)) + (v_j^{(k)}(s))^2)ds\,\mathrm{d}\mathcal{H}^{d-1}(y).\end{aligned}$$

Since M is compact, $J(y,t)$ converges uniformly to 1 as $t\to 0$; hence, by (7.5.6) we obtain

$$\limsup_{j\to+\infty} I_j^k \le m_k\mathcal{H}^{d-1}(M\cap\Omega) = m_k\mathrm{Per}(E,\Omega) = m_k\mathcal{H}^{d-1}(J_u).$$

We now prove that for fixed $\ell\in\{1,\dots,k-1\}$ we have $\limsup_{j\to+\infty} I_j^\ell = 0$. To this end, we set $w_j(t) := v_j(t/\varepsilon_j)$ and $A_j := (-\sqrt{\varepsilon_j},\sqrt{\varepsilon_j})$, and write

$$I_j^\ell = \varepsilon_j^{2k-1-2\ell}\int_{A_j}\Big(v_j^{(\ell)}\Big(\frac{t}{\varepsilon_j}\Big)\Big)^2\,\mathrm{d}t = \varepsilon_j^{2k-1}\int_{A_j}(w_j^{(\ell)}(t))^2\,\mathrm{d}t. \tag{7.5.10}$$

By a change of variable, we have

$$\int_{A_j}\Big(\frac{1}{\varepsilon_j}W(w_j) + \varepsilon_j^{2k-1}(w_j^{(k)})^2\Big)\mathrm{d}t = \int_{-\infty}^{+\infty}\big(W(v_j) + (v_j^{(k)})^2\big)\,\mathrm{d}t;$$

hence, by (7.5.6) and by the fact that $W \geq 0$, we infer that

$$\|w_j^{(k)}\|_{L^2(A_j)} \leq C\varepsilon_j^{\frac{1-2k}{2}}. \tag{7.5.11}$$

Note that for every positive integer j and for any $h \in \{0, \dots, k-1\}$ there exists $z_j^h \in [-\sqrt{\varepsilon_j}, \sqrt{\varepsilon_j}]$ such that $w_j^{(h)}(z_j^h) = 0$. Indeed, this holds with $z_j^h = \sqrt{\varepsilon_j}$ for $h \in \{1, \dots, k-1\}$, while the existence of z_j^0 follows by the continuity of w_j. Hence, applying the Fundamental Theorem of Calculus and using (7.5.11), we get

$$|w_j(t)| \leq \varepsilon_j^{\frac{2k-1}{4}} \|w_j^{(k)}\|_{L^2(A_j)} \leq C\varepsilon_j^{\frac{1-2k}{4}} \quad \text{for every } t \in A_j,$$

so that

$$\|w_j\|_{L^2(A_j)} \leq C\varepsilon_j^{\frac{1-k}{2}}. \tag{7.5.12}$$

By the interpolation inequality (7.2.3), it holds

$$\|w_j^{(\ell)}\|_{L^2(A_j)} \leq C\Big(\|w_j^{(k)}\|_{L^2(A_j)}^{\frac{\ell}{k}} \|w_j\|_{L^2(A_j)}^{\frac{k-\ell}{k}} + \mathcal{L}^1(A_j)^{-\ell}\|w_j\|_{L^2(A_j)}\Big).$$

Hence, combining (7.5.11) and (7.5.12) we get

$$\|w_j^{(\ell)}\|_{L^2(A_j)} \leq C\Big(\varepsilon_j^{(\frac{1-2k}{2})(\frac{\ell}{k})}\varepsilon_j^{(\frac{1-k}{2})(\frac{k-\ell}{k})} + \varepsilon_j^{-\frac{\ell}{2}+\frac{1-k}{2}}\Big) = C\varepsilon_j^{\frac{-k-\ell+1}{2}}. \tag{7.5.13}$$

Note that the above formula, up to a constant factor that is independent of j, reduces to (7.5.11) when $\ell = k$. Recalling (7.5.10) we conclude

$$I_j^\ell = \varepsilon_j^{2k-1}\|w_j^{(\ell)}\|_{L^2(A_j)}^2 \leq C\varepsilon_j^{k-\ell},$$

which tends to 0 since $\ell < k$.

Finally, we prove that for fixed $r < \ell$, we have $\limsup_{j\to+\infty} I_j^{r,\ell} = 0$.

Recalling that $w_j(t) = v_j(t/\varepsilon_j)$, by Hölder's inequality we have

$$\begin{aligned} I_j^{r,\ell} &= \varepsilon_j^{2k-1-r-\ell} \int_{A_j} \Big|v_j^{(r)}\Big(\frac{t}{\varepsilon_j}\Big)\Big| \, \Big|v_j^{(\ell)}\Big(\frac{t}{\varepsilon_j}\Big)\Big| \, \mathrm{d}t \\ &= \varepsilon_j^{2k-1} \int_{A_j} |w_j^{(r)}(t)| \, |w_j^{(\ell)}(t)| \, \mathrm{d}t \\ &\leq \varepsilon_j^{2k-1}\|w_j^{(r)}\|_{L^2(A_j)} \, \|w_j^{(\ell)}\|_{L^2(A_j)}. \end{aligned}$$

Since (7.5.13) holds for every $\ell \in \{1, \dots, k\}$, we get

$$I_j^{r,\ell} \leq C\varepsilon_j^{2k-1}\varepsilon_j^{\frac{-k-r+1}{2}}\varepsilon_j^{\frac{-k-\ell+1}{2}} = C\varepsilon_j^{k-\frac{r+\ell}{2}},$$

which tends to 0 since $r + \ell < 2k$, concluding the proof. □

Anisotropic higher-order singular perturbation models for phase transitions

This chapter is devoted to an extension of the results presented in the previous chapter to a broader class of singularly perturbed energies. In particular, we consider energies of the form

$$F_\varepsilon(u,\Omega) := \begin{cases} \displaystyle\int_\Omega \Big(\frac{1}{\varepsilon}W(u) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\Big)\,\mathrm{d}x & \text{if } u \in \mathrm{H}^k(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus \mathrm{H}^k(\Omega), \end{cases} \tag{8.0.1}$$

where Ω is a bounded open subset of $\mathbb{R}^d$ with C^1 boundary, k is a fixed positive integer larger than 1, the coefficients $q_1,\dots,q_{k-1}$ are real constants, possibly negative, $q_k = 1$, and $|\cdot|_\ell$ is a norm on the space of symmetric ℓ-tensors for every $\ell \in \{1,\dots,k\}$.

The novelty with respect to the functionals considered in (7.0.1) and (7.1.3) is that we consider:

- singular perturbations of intermediate order, possibly negative;
- arbitrary norms $|\cdot|_\ell$ that allow for anisotropies in the limit process.

As in the previous chapter, our aim is to establish compactness and Γ-convergence results for the family $\{F_\varepsilon\}_\varepsilon$ as $\varepsilon \to 0^+$. More precisely, we prove that cluster points of sequences with equibounded energy are sharp interfaces, and that the corresponding Γ-limit is an anisotropic perimeter functional.

The strategy we adopt combines ideas from [75] and [78], where analogous problems have been studied in the case $k = 2$. Following the approach of [78], we first prove suitable nonlinear interpolation estimates, which allow us to control the negative intermediate-order terms (see Theorem 8.1.1). These estimates allow one to compare $\{F_\varepsilon\}_\varepsilon$ with the functionals introduced in (7.1.3); therefore, by using the results obtained in the previous chapter, we obtain the compactness part of the result (see Theorem 8.1.2). We then use arguments in the spirit of [75], to identify the Γ-limit (see Theorem 8.1.3). The nonlinear interpolation estimates are crucial also in this step, as they allow us to prove a lemma which enables us to impose boundary conditions in an energetically convenient way. We stress that, because of the general tensor norms $|\cdot|_\ell$ considered here, the lower bound cannot be reduced to a one-dimensional analysis by slicing, as it was previously done in Section 7.5. A genuine blow-up argument is therefore needed.

Finally, in the isotropic case, we carry out a careful analysis of the properties of the optimal profile problems that determine the expression of the limit energy density, which is, in this case, a multiple of the usual perimeter.

The chapter is structured as follows. In Section 8.1, we introduce the setting and present the three main results of the chapter: the nonlinear interpolation estimate Theorem 8.1.1, compactness Theorem 8.1.2, and the Γ-convergence result Theorem 8.1.3. In Section 8.3, we derive the nonlinear interpolation inequalities which are crucial in the rest of the chapter; this

is done by first analysing the one-dimensional case and then extended to the d-dimensional case relying on arguments devised in [75]. These inequalities are then exploited in Section 8.4 to deduce compactness Theorem 8.1.2. The proof of the Γ-limit result is split in two different sections; in Section 8.5 we prove the lower bound, while the upper bound is dealt with in Section 8.6. Finally, in Section 8.7, we analyse the optimal profile problems defining the limit energy density in dimension one. In particular, we show in Proposition 8.7.3 that, under suitable assumptions on W, these problems admit a minimiser.

The content of this chapter is based on the joint paper with Brusca & Trifone [55].

8.1 Setting and main results

In this section, we introduce the functional we study in this chapter and give the statements of its main results. We refer the reader to Section 1.1 for the basic notation for sets and slicing we adopt and to Section 1.3 for the properties of sets of finite perimeter.

We recall that $\Omega \subset \mathbb{R}^d$ is a bounded open set with C^1 boundary, k is a fixed positive integer larger than 1, F_ε is given by (8.0.1). We also recall that the coefficients $q_1, \dots, q_{k-1}$ are real constants, possibly negative, $q_k = 1$, and $|\cdot|_\ell$ is a norm on the space of symmetric ℓ-tensors for every $\ell \in \{1, \dots, k\}$. In what follows, for a symmetric k-tensor T, its operator norm is denoted by $\|T\|_k$. It is also convenient to set a different notation for the functionals introduced in (7.1.3), and we set

$$E_\varepsilon(u, \Omega) := \begin{cases} \displaystyle\int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} \|\nabla^{(k)} u\|_k^2 \Big) \, \mathrm{d}x & \text{if } u \in \mathrm{H}^k(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \mathrm{H}^k(\Omega), \end{cases} \tag{8.1.1}$$

We assume that $W \colon \mathbb{R} \to [0, +\infty)$ is a continuous function satisfying the following assumptions:

(H1) $W(s) = 0$ if and only if $s \in \{-1, 1\}$;

(H2) $W(s) \geq \min\{(s+1)^2, (s-1)^2\}$ for every $s \in \mathbb{R}$;

(H3) $W(s) \leq W(t) + 1$ for every $t \in \mathbb{R}$ and for every $s \in \mathbb{R}$ with $|s| \leq |t|$.

We remark that, with some minor adaptations in the computations, we could also assume, more in general, that, for some positive constants α, β, the following conditions hold:

(H2') $W(s) \geq \alpha \min\{(s+1)^2, (s-1)^2\}$ for every $s \in \mathbb{R}$;

(H3') $W(s) \leq \beta W(t) + \beta$ for every $t \in \mathbb{R}$ and for every $s \in \mathbb{R}$ with $|s| \leq |t|$,

with our main results still holding unchanged.

Our first main result is a global interpolation inequality at scale ε, in the spirit of [75] and [78], that establishes bounds on the square of the L^2 norm of $\nabla^{(\ell)} u$ in terms of $\int_\Omega W(u) \, \mathrm{d}x$ and the square of the L^2 norm of $\nabla^{(k)} u$. This inequality is fundamental in the asymptotic analysis of F_ε, and determines lower bounds for the constants $q_1, \dots, q_{k-1}$ for which our asymptotic analysis can be carried out.

Theorem 8.1.1. *Let $k > 1$ be an integer and assume that* (H2) *is satisfied. Then, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant q_ℓ^* independent of Ω such that for every $q < q_\ell^*$ there exists $\varepsilon_0 = \varepsilon_0(q, \Omega)$ such that it holds*

$$q\varepsilon^{2\ell} \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 \, \mathrm{d}x \le \int_\Omega \Big(W(u) + \varepsilon^{2k} \|\nabla^{(k)} u\|_k^2 \Big) \, \mathrm{d}x$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathrm{H}^k(\Omega)$. Moreover, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant $\widetilde{q}_\ell^*$ independent of Ω such that for every $q < \widetilde{q}_\ell^*$ there exists $\varepsilon_0 = \varepsilon_0(q, \Omega)$ such that it holds

$$q\varepsilon^{2\ell} \int_\Omega |\nabla^{(\ell)} u|_\ell^2 \, \mathrm{d}x \le \int_\Omega \Big(W(u) + \varepsilon^{2k} |\nabla^{(k)} u|_k^2 \Big) \, \mathrm{d}x$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathrm{H}^k(\Omega)$. In particular, $\widetilde{q}_\ell^*$ depends on $q_\ell^*, |\cdot|_\ell$, and $|\cdot|_k$.

The above interpolation inequality is proved by, first, adapting the arguments in [78], which yield the result in the case $d = 1$, and then by applying the slicing method as in [75] to recover the general case. We remark that assumption (H2), despite not being fully satisfactory in describing the model as it prescribes the growth rate at infinity of the double-well potential, is needed to apply classical, global L^2-interpolation estimates. Indeed, the presence of negative terms in the energy seems to prevent the use of the local arguments applied in Chapter 7, as it is not evident to us how to estimate the energy when far from the bottom of the wells.

Having at disposal such an interpolation estimate allows us to find a priori bounds on the functionals (8.0.1), provided the coefficients $q_1, \dots, q_{k-1}$ are not too negative, and this is the key ingredient to get the main compactness result.

Theorem 8.1.2. *Let $\{F_\varepsilon\}_\varepsilon$ be defined as in* (8.0.1) *with $k > 1$ an integer, let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} \colon q_\ell \le 0\}$, and let $\{\varepsilon_n\}_n$ be a positive sequence converging to* 0*. Assume that* (H1) *and* (H2) *are satisfied and that one of the following conditions holds:*

(i) $\mathcal{N} = \emptyset$*;*

(ii) $\mathcal{N} \neq \emptyset$ *and $q_\ell > -\overline{q}_\ell$ for every $\ell \in \mathcal{N}$ for suitable positive constants $\{\overline{q}_\ell \colon \ell \in \mathcal{N}\}$.*

If $\{u_n\}_n \subset \mathrm{H}^k(\Omega)$ is a sequence such that

$$\sup_n F_{\varepsilon_n}(u_n, \Omega) < +\infty,$$

then there exist a subsequence $\{u_{n_j}\}_j$ and a function $u \in \mathrm{BV}(\Omega; \{-1, 1\})$ such that $u_{n_j} \to u$ in $L^2(\Omega)$ as $j \to +\infty$. In particular, if $\mathcal{N} \neq \emptyset$, letting $\widetilde{q}_1^*, \dots, \widetilde{q}_{k-1}^*$ denote the same positive constants appearing in Theorem 8.1.1 and given $\{\alpha_\ell \colon \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$, it holds that $\overline{q}_\ell = \alpha_\ell \widetilde{q}_\ell^*$ for all $\ell \in \mathcal{N}$; hence, $\{\overline{q}_\ell \colon \ell \in \mathcal{N}\}$ are independent of Ω.

Clearly, if the coefficients $q_1, \dots, q_{k-1}$ are non-negative, the compactness result of Chapter 8 suggests there is no need to resort to Theorem 8.1.1 since, by the equivalence of the tensor norms, compactness of the functionals $\{F_\varepsilon\}_\varepsilon$ can be inferred by compactness of the functionals $\{E_\varepsilon\}_\varepsilon$ defined in (8.1.1) through a direct comparison. For this reason, we also prove a compactness result for functionals $\{E_\varepsilon\}_\varepsilon$ (Proposition 8.4.3). The first step is enhancing the compactness result for the 1-dimensional functionals treated in Chapter 7

taking advantage of assumption (H2). Indeed, this assumption implies that W has quadratic growth at infinity, and this condition, in turn, allows to improve precompactness in measure into precompactness in L^2. Then, we recover the general d-dimensional case by resorting to a precompactness criterion in L^1 by slicing that has been introduced in [7] (see Proposition 8.4.2 below), which in our case is sufficient to obtain precompactness in L^2.

Afterwards, we compute the Γ-limit of the family $\{F_\varepsilon\}_\varepsilon$ with respect to the strong convergence in $L^2(\Omega)$. In order to state our result, we introduce some notation. We set $\mathbb{S}^{d-1} \coloneqq \{\xi \in \mathbb{R}^d \colon |\xi| = 1\}$. For brevity of notation, in this chapter we let the symbol Q_T^ν denote an open cube of side length $T > 0$, centred at the origin, with two faces orthogonal to $\nu \in \mathbb{S}^{d-1}$,

$$Q_T^\nu \coloneqq \Big\{x \in \mathbb{R}^d \colon |x \cdot \nu| < \frac{T}{2},\, |x \cdot \nu_i| < \frac{T}{2} \text{ for every } i \in \{1, \dots, d-1\}\Big\}, \tag{8.1.2}$$

where $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^d$ that we fix arbitrarily.

We fix a function $\overline{u} \in H^k_{\mathrm{loc}}(\mathbb{R})$ such that $\overline{u}(t) = -1$ if $t \le -1/8$, $\overline{u}(t) = 1$ if $t \ge 1/8$, $|\overline{u}| \le 1$, and

$$0 < \int_{-\infty}^{+\infty} \Big(W(\overline{u}) + \sum_{\ell=1}^{k} |q_\ell| |\overline{u}^{(\ell)}|^2\Big)\, \mathrm{d}t < +\infty,$$

then, for fixed $\nu \in \mathbb{S}^{d-1}$ we define the function

$$\overline{u}^\nu(x) \coloneqq \overline{u}(x \cdot \nu) \quad \text{ for all } x \in \mathbb{R}^d,$$

and for every $\varepsilon > 0$ we further define

$$\overline{u}^\nu_\varepsilon(x) \coloneqq \overline{u}^\nu\big(\frac{x}{\varepsilon}\big) = \overline{u}\big(\frac{x \cdot \nu}{\varepsilon}\big) \quad \text{ for all } x \in \mathbb{R}^d.$$

We introduce a class of admissible functions on unit cubes as

$$\mathcal{A}^\nu_\varepsilon \coloneqq \big\{v \in \mathrm{H}^k(Q_1^\nu) \colon v = \overline{u}^\nu_\varepsilon \text{ near } \partial Q_1^\nu\big\}, \tag{8.1.3}$$

where by 'near ∂Q_1^ν', we mean 'in $Q_1^\nu \setminus Q_r^\nu$ for some $r \in (0,1)$'.

Finally, for every $\nu \in \mathbb{S}^{d-1}$ we introduce the function

$$g(\nu) \coloneqq \inf\Big\{ \int_{Q_1^\nu} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|^2_\ell\Big)\, \mathrm{d}x \colon v \in \mathcal{A}^\nu_\varepsilon, \varepsilon \in (0,1)\Big\}, \tag{8.1.4}$$

and we state our last main result.

Theorem 8.1.3. *Let $\{F_\varepsilon\}_\varepsilon$ be defined as in* (8.0.1) *with $k > 1$ an integer and let $\mathcal{N} \coloneqq \{\ell \in \{1, \dots, k-1\} \colon q_\ell \le 0\}$. Assume that* (H1)–(H3) *are satisfied and that one of the following conditions holds:*

(i) *$\mathcal{N} = \emptyset$;*

(ii) *$\mathcal{N} \neq \emptyset$ and $q_\ell > -\overline{q}_\ell$ for every $\ell \in \mathcal{N}$ with the same positive constants $\{\overline{q}_\ell \colon \ell \in \mathcal{N}\}$ independent of Ω appearing in Theorem 8.1.2.*

Then the family of functionals $\{F_\varepsilon\}_\varepsilon$ Γ*-converges, with respect to the strong convergence in* $L^2(\Omega)$*, to the functional* F_0 *given by*

$$F_0(u,\Omega) \coloneqq \begin{cases} \displaystyle\int_{\partial^* E\cap\Omega} g(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} & \text{if } u\in \mathrm{BV}(\Omega;\{-1,1\}),\ \text{ with } u=2\chi_E-1,\\ +\infty & \text{if } u\in L^2(\Omega)\setminus \mathrm{BV}(\Omega;\{-1,1\}),\end{cases}$$

where E *is a set of finite perimeter in* Ω*,* $\partial^* E$ *is its reduced boundary,* ν_E *is its inner unit normal, and* g *is defined as in* (8.1.4).

In the special case $q_1=\cdots=q_{k-1}=0$, this result improves Theorem 7.1.2, as it deals with arbitrary norms on the spaces of tensors, upon assuming stronger hypotheses on the double-well potential.

In order to compute the Γ-limit, we follow the strategy presented in [75]. The lower bound is proved by using the blow-up argument and relying on a method due to De Giorgi that allows to vary boundary conditions on converging sequences (Lemma 8.5.2). The construction of a recovery sequence is confined to the case of polyhedra, which are seen to be dense in energy for the limit F_0, being the surface energy density g upper semicontinuous (Proposition 8.2.1).

As stated below in Remark 8.2.2, the density (8.1.4) can be described by the following asymptotic formula

$$\begin{aligned} g(\nu) &= \lim_{\varepsilon\to 0}\ \inf\Big\{\int_{Q_1^\nu}\Big(\frac{1}{\varepsilon}W(v)+\sum_{\ell=1}^k q_\ell\varepsilon^{2\ell-1}|\nabla^{(\ell)}v|_\ell^2\Big)\,\mathrm{d}x\colon\ v\in\mathcal{A}_\varepsilon^\nu\Big\}\\ &= \lim_{T\to+\infty}\frac{1}{T^{d-1}}\ \inf\Big\{\int_{Q_T^\nu}\Big(W(v)+\sum_{\ell=1}^k q_\ell|\nabla^{(\ell)}v|_\ell^2\Big)\,\mathrm{d}x\colon\ v\in\widetilde{\mathcal{A}}_T^\nu\Big\},\end{aligned}$$

where

$$\widetilde{\mathcal{A}}_T^\nu \coloneqq \big\{v\in \mathrm{H}^k(Q_T^\nu)\colon\ v=\overline{u}^\nu \text{ near } \partial Q_T^\nu\big\},$$

and the last equality follows by setting $T\coloneqq\frac{1}{\varepsilon}$ and a change of variables.

The formula appearing in (8.1.4) should be compared with the one in [75]. In that work such density is a constant because of the absence of anisotropies, and it is given by

$$\inf\Big\{\int_Q\Big(\frac{1}{\varepsilon}W(v)-q\varepsilon\|\nabla v\|_1^2+\varepsilon^3\|\nabla^{(2)}v\|_2^2\Big)\,\mathrm{d}x\colon\ v\in\mathcal{A},\varepsilon\in(0,1)\Big\},$$

with $Q=(-\frac12,\frac12)^d$, and

$$\begin{aligned}\mathcal{A}\coloneqq\Big\{v\in\mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}^d)\colon\ & v=-1 \text{ near } x\cdot e_d=-\frac12, v=1 \text{ near } x\cdot e_d=\frac12,\\ & v(x+e_i)=v(x) \text{ for all } x\in\mathbb{R}^d, i\in\{1,\dots,d-1\}\Big\},\end{aligned}$$

where by 'near $x\cdot e_d=\pm\frac12$' it is meant 'in a neighbourhood of the set $\{x\in\mathbb{R}^d\colon\ x\cdot e_d=\pm\frac12\}$'.

In light of this observation, we expect that our density g could be characterised as

$$\inf\Big\{\int_{Q_1^\nu}\Big(\frac{1}{\varepsilon}W(v)+\sum_{\ell=1}^k q_\ell\varepsilon^{2\ell-1}|\nabla^{(\ell)}v|_\ell^2\Big)\,\mathrm{d}x\colon\ v\in\widetilde{\mathcal{A}}^\nu,\varepsilon\in(0,1)\Big\},$$

with

$$\widetilde{\mathcal{A}}^\nu := \big\{ v \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}^d) \colon \ v = -1 \text{ near } x\cdot\nu = -\frac{1}{2}, v = 1 \text{ near } x\cdot\nu = \frac{1}{2}, \\ v(x+\nu_i) = v(x) \text{ for all } x \in \mathbb{R}^d, i \in \{1,\dots,d-1\}\big\},$$

where $\{\nu_1,\dots,\nu_{d-1},\nu\}$ is the orthonormal basis that we fixed in the definitions (8.1.2) and (8.1.3). To prove this fact, we would need to impose boundary conditions resorting to Lemma 8.5.2. In turn, to do this we would need to improve Theorem 8.1.1, even just in the case of the operator norms, to the case where Ω is an open cube. In [75], this result is obtained by slicing, which is well suited because, roughly speaking, the operator norm of a 1-tensor, the gradient, coincides with its Euclidean norm as a vector of $\mathbb{R}^d$. Because of the higher order of derivatives that our interpolation estimates involve, such argument seems to be difficult to adapt. As an additional consequence of the lack of interpolation estimates on cubes, the proofs of Lemma 8.5.2 (and its statement) and of the lower bound (Proposition 8.5.1) present several differences with respect to their counterparts in [75]. In particular, we will need to apply our interpolation inequalities on a C^1 subset of the cube Q_1^ν that coincides with this cube inside the strip $\{x \in \mathbb{R}^d \colon |x\cdot\nu| \le \frac{1}{4}\}$ which is close to the interface on which most of the energy is stored.

8.2 Limit energy density

In this section we prove a characterisation of the density introduced in (8.1.4). In view of the asymptotic analysis, the following proposition makes irrelevant the choice of the orthonormal basis $\{\nu_1,\dots,\nu_{d-1},\nu\}$ fixed in the definitions (8.1.2) and (8.1.3). In the following we let $\mathrm{SO}(d)$ denote the group of $d\times d$ orthogonal matrices with determinant equal to 1.

Proposition 8.2.1. *For every $\nu \in \mathbb{S}^{d-1}$ and for every $R \in \mathrm{SO}(d)$ such that $R\nu = \nu$ it holds*

$$g(\nu) = \inf\Big\{ \int_{R(Q_1^\nu)} \Big(\frac{1}{\varepsilon}W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\,\mathrm{d}x \colon \ v\circ R \in \mathcal{A}^\nu_\varepsilon,\ \varepsilon\in(0,1)\Big\} \tag{8.2.1}$$

$$= \inf\Big\{ \int_{R'(Q_1^\nu)} \Big(\frac{1}{\varepsilon}W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\,\mathrm{d}x \colon \ v\circ R' \in \mathcal{A}^\nu_\varepsilon,\ \varepsilon\in(0,1), \\ R' \in \mathrm{SO}(d) \text{ with } R'\nu = \nu\Big\}. \tag{8.2.2}$$

Moreover, g is upper semicontinuous.

Proof. We first prove (8.2.2). Fix $\nu \in \mathbb{S}^{d-1}$ and let $\varepsilon \in (0,1)$, $R \in \mathrm{SO}(d)$ satisfying $R\nu = \nu$, and $v \circ R \in \mathcal{A}^\nu_\varepsilon$. We claim that for every $\eta \in (0,\varepsilon)$, there exists a function $w_\eta \in \mathcal{A}^\nu_\eta$ such that

$$\limsup_{\eta\to 0} \int_{Q_1^\nu} \Big(\frac{1}{\eta}W(w_\eta) + \sum_{\ell=1}^k q_\ell \eta^{2\ell-1} |\nabla^{(\ell)} w_\eta|_\ell^2\Big)\,\mathrm{d}x \tag{8.2.3}$$

$$\le \int_{R(Q_1^\nu)} \Big(\frac{1}{\varepsilon}W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\,\mathrm{d}x. \tag{8.2.4}$$

For the sake of exposition, we set $T \coloneqq 1/\varepsilon$ and $\widetilde{v}(x) \coloneqq v(\frac{1}{T}x)$ for all $x \in R(Q_T^\nu)$. For $S > T$, we introduce

$$\mathcal{Z}(T,S) \coloneqq \{z = T(z_1 R\nu_1 + \cdots + z_{d-1}R\nu_{d-1})\colon\ z + R(Q_T^\nu) \subset Q_S^\nu,\ (z_1,\dots,z_{d-1}) \in \mathbb{Z}^{d-1}\}.$$

where $\{\nu_1,\dots,\nu_{d-1},\nu\}$ is the orthonormal basis that we arbitrarily fixed in the definitions (8.1.2) and (8.1.3), and

$$E(T,S) \coloneqq \bigcup_{z\in\mathcal{Z}(T,S)} \big(z + R(Q_T^\nu)\big).$$

Then, we define the function

$$\widetilde{w}_S(x) \coloneqq \begin{cases} \widetilde{v}(x-z) & \text{if } x \in z + R(Q_T^\nu) \text{ for some } z \in \mathcal{Z}(T,S),\\ \overline{u}^\nu(x) & \text{otherwise in } Q_S^\nu, \end{cases}$$

and we note that $\widetilde{w}_S \in \mathrm{H}^k(Q_S^\nu)$ with $\widetilde{w}_S = \overline{u}^\nu$ close to ∂Q_S^ν. Recalling the notation (8.0.1), we have

$$\begin{aligned}\frac{F_1(\widetilde{w}_S, Q_S^\nu)}{S^{d-1}} &= \frac{F_1(\widetilde{w}_S, E(T,S)) + F_1(\widetilde{w}_S, Q_S^\nu \setminus E(T,S))}{S^{d-1}}\\ &= \Big(\sum_{z\in\mathcal{Z}(T,S)} \frac{F_1(\widetilde{w}_S, z + R(Q_T^\nu))}{S^{d-1}}\Big) + \frac{F_1(\widetilde{w}_S, Q_S^\nu \setminus E(T,S))}{S^{d-1}}\\ &= \#\mathcal{Z}(T,S)\frac{F_1(\widetilde{v}, R(Q_T^\nu))}{S^{d-1}} + \frac{\mathcal{H}^{d-1}\big(\pi^\nu(Q_S^\nu) \setminus \pi^\nu(E(T,S))\big)}{S^{d-1}} c_0,\end{aligned}$$

having set $c_0 \coloneqq \int_{-\infty}^{+\infty}\Big(W(\overline{u}) + \sum_{\ell=1}^k q_\ell |\overline{u}^{(\ell)}|^2\Big)\,\mathrm{d}t$ and letting π^ν denote the orthogonal projection on the hyperplane $\nu^\perp$. We observe that

$$\lim_{S\to+\infty} \frac{\#\mathcal{Z}(T,S)}{(S/T)^{d-1}} = 1,$$

and

$$\mathcal{H}^{d-1}\big(\pi^\nu(Q_S^\nu) \setminus \pi^\nu(E(T,S))\big) = S^{d-1} - \#\mathcal{Z}(T,S)T^{d-1}$$

therefore,

$$\limsup_{S\to+\infty} \frac{F_1(\widetilde{w}_S, Q_S^\nu)}{S^{d-1}} \le \frac{F_1(\widetilde{v}, R(Q_T^\nu))}{T^{d-1}};$$

that is

$$\begin{aligned}\limsup_{S\to+\infty} \frac{1}{S^{d-1}} \int_{Q_S^\nu} \Big(W(\widetilde{w}_S) + \sum_{\ell=1}^k q_\ell |\nabla^{(\ell)}\widetilde{w}_S|_\ell^2\Big)\,\mathrm{d}x&\\ \le \frac{1}{T^{d-1}} \int_{R(Q_T^\nu)} \Big(W(\widetilde{v}) + \sum_{\ell=1}^k q_\ell |\nabla^{(\ell)}\widetilde{v}|_\ell^2\Big)\,\mathrm{d}x.&\end{aligned} \tag{8.2.5}$$

Finally, we set

$$\eta \coloneqq \frac{1}{S}, \qquad w_\eta(x) \coloneqq \widetilde{w}_{\frac{1}{\eta}}\Big(\frac{1}{\eta}x\Big) \text{ for all } x \in Q_1^\nu,$$

so that $w_\eta \in \mathcal{A}_\eta^\nu$. By applying the change of variables $y \coloneqq x/\eta$ on the left-hand side and $y \coloneqq x/\varepsilon$ on the right-hand side, (8.2.5) can be written as

$$\limsup_{\eta\to0} \int_{Q_1^\nu} \Big(\frac{1}{\eta}W(w_\eta) + \sum_{\ell=1}^k q_\ell \eta^{2\ell-1} |\nabla^{(\ell)} w_\eta|_\ell^2\Big)\mathrm{d}y \le \int_{R(Q_1^\nu)} \Big(\frac{1}{\varepsilon}W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\mathrm{d}y,$$

which is (8.2.3). By the arbitrariness of ε, R, and v, inequality (8.2.3) and a diagonal argument yield (8.2.2).

As for the proof of (8.2.1), consider $R_1, R_2 \in \mathrm{SO}(d)$ satisfying $R_1\nu = R_2\nu = \nu$ and $v \circ R_1 \in \mathcal{A}_\varepsilon^\nu$. With a similar argument to the one employed above, we may prove that for every $\eta \in (0, \varepsilon)$, there exists a function w_η such that $w_\eta \circ R_2 \in \mathcal{A}_\eta^\nu$ and

$$\limsup_{\eta\to 0} \int_{R_2(Q_1^\nu)} \Big(\frac{1}{\eta} W(w_\eta) + \sum_{\ell=1}^k q_\ell \eta^{2\ell-1} |\nabla^{(\ell)} w_\eta|_\ell^2\Big)\,\mathrm{d}x \\ \leq \int_{R_1(Q_1^\nu)} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\,\mathrm{d}x,$$

which, by the arbitrariness of the rotations R_1, R_2 and of the function v, proves (8.2.1).

Finally, we prove that g is upper semicontinuous. Let $\nu \in \mathbb{S}^{d-1}$ and $\{\nu_n\}_n \subset \mathbb{S}^{d-1}$ such that $\nu_n \to \nu$. We aim to show that

$$\limsup_{n\to+\infty} g(\nu_n) \leq g(\nu).$$

Fix $\eta > 0$ and let $\varepsilon \in (0,1)$ and $v \in \mathcal{A}_\varepsilon^\nu$ such that

$$g(\nu) + \eta \geq \int_{Q_1^\nu} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\,\mathrm{d}x. \tag{8.2.6}$$

For each $n \in \mathbb{N}$ we can find $R_n = (R_n^{i,j})_{i,j=1}^d \in \mathrm{SO}(d)$ such that $R_n\nu = \nu_n$ and $R_n \to \mathbb{I}_d$. Consider the cube $R_n(Q_1^\nu)$ and, for each n, a rotation $S_n \in \mathrm{SO}(d)$ such that

$$S_n\nu_n = \nu_n \text{ and } R_n(Q_1^\nu) = S_n(Q_1^{\nu_n}).$$

Define $v_n(x) \coloneqq v(R_n^T x)$ for every $x \in R_n(Q_1^\nu) = S_n(Q_1^{\nu_n})$ and note that $v_n \circ S_n \in \mathcal{A}_\varepsilon^{\nu_n}$. By (8.2.2) and a change of variables

$$\begin{aligned} g(\nu_n) &\leq \int_{S_n(Q_1^{\nu_n})} \Big(\frac{1}{\varepsilon} W(v_n(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(x)|_\ell^2\Big)\,\mathrm{d}x \\ &= \int_{R_n(Q_1^\nu)} \Big(\frac{1}{\varepsilon} W(v_n(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(x)|_\ell^2\Big)\,\mathrm{d}x \\ &= \int_{Q_1^\nu} \Big(\frac{1}{\varepsilon} W(v(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(R_n x)|_\ell^2\Big)\,\mathrm{d}x. \end{aligned} \tag{8.2.7}$$

We show that, for each $\ell \in \{1, \dots, k\}$ and $\mathcal{L}^d$-a.e. $x \in Q_1^\nu$, we have $\nabla^{(\ell)} v_n(R_n x) \to \nabla^{(\ell)} v(x)$ in the space of ℓ-tensors as $n \to +\infty$. By the equivalence of the norms, to prove this it is sufficient to show that for $\mathcal{L}^d$-a.e. $x \in Q_1^\nu$ it holds

$$\lim_{n\to+\infty} \nabla^{(\ell)} v_n(R_n x)(e_{i_1}, \dots, e_{i_\ell}) = \nabla^{(\ell)} v(x)(e_{i_1}, \dots, e_{i_\ell}) \text{ for } (i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell. \tag{8.2.8}$$

Let us fix $(i_1,\dots,i_\ell)\in\{1,\dots,d\}^\ell$. For $\mathcal{L}^d$-a.e. $x\in Q_1^\nu$ we have

$$\begin{aligned}\lim_{n\to+\infty}\nabla^{(\ell)}v_n(R_nx)(e_{i_1},\dots,e_{i_\ell}) &= \lim_{n\to+\infty}\frac{\partial^\ell v_n}{\partial x_{i_1}\dots\partial x_{i_\ell}}(R_nx)\\ &= \lim_{n\to+\infty}\sum_{j_1=1}^d\cdots\sum_{j_\ell=1}^d R_n^{i_1,j_1}\dots\ R_n^{i_\ell,j_\ell}\frac{\partial^\ell v}{\partial x_{j_1}\dots\partial x_{j_\ell}}(x)\\ &= \sum_{j_1=1}^d\cdots\sum_{j_\ell=1}^d\delta_{i_1j_1}\dots\delta_{i_\ell j_\ell}\frac{\partial^\ell v}{\partial x_{j_1}\dots\partial x_{j_\ell}}(x)\\ &= \frac{\partial^\ell v}{\partial x_{i_1}\dots\partial x_{i_\ell}}(x)=\nabla^{(\ell)}v(x)(e_{i_1},\dots,e_{i_\ell}),\end{aligned}$$

and this equality proves (8.2.8).

To conclude, we first observe that, since $R_n\to\mathrm{Id}$, we may assume that $|R_n^{i,j}|\le 2$ for all $i,j\in\{1,\dots,d\}$ and $n\in\mathbb{N}$ so that

$$\begin{aligned}|\nabla^{(\ell)}v_n(R_nx)|_\ell &\le C\max\Big\{\Big|\frac{\partial^\ell v}{\partial x_{i_1}\dots\partial x_{i_\ell}}(R_nx)\Big|\colon\ (i_1,\dots,i_\ell)\in\{1,\dots,d\}^\ell\Big\}\\ &= C\max\Big\{\Big|\sum_{j_1=1}^d\cdots\sum_{j_\ell=1}^d R_n^{i_1,j_1}\dots\ R_n^{i_\ell,j_\ell}\frac{\partial^\ell v}{\partial x_{j_1}\dots\partial x_{j_\ell}}(x)\Big|\colon\ (i_1,\dots,i_\ell)\in\{1,\dots,d\}^\ell\Big\}\\ &\le C\max\Big\{\Big|\frac{\partial^\ell v}{\partial x_{i_1}\dots\partial x_{i_\ell}}(x)\Big|\colon\ (i_1,\dots,i_\ell)\in\{1,\dots,d\}^\ell\Big\}\le C|\nabla^{(\ell)}v(x)|_\ell,\end{aligned}\tag{8.2.9}$$

where C is a positive constant (that may vary from line to line) which only depends on d and ℓ as it arises by the equivalence of the tensor norms. Finally, combining (8.2.8) with (8.2.9), we apply the Dominated Convergence Theorem in (8.2.7), so that by (8.2.6) we obtain

$$\begin{aligned}\limsup_{n\to+\infty}g(\nu_n) &\le \lim_{n\to+\infty}\int_{Q_1^\nu}\Big(\frac{1}{\varepsilon}W(v(x))+\sum_{\ell=1}^k q_\ell\varepsilon^{2\ell-1}|\nabla^{(\ell)}v_n(R_nx)|_\ell^2\Big)\,\mathrm{d}x\\ &= \int_{Q_1^\nu}\Big(\frac{1}{\varepsilon}W(v(x))+\sum_{\ell=1}^k q_\ell\varepsilon^{2\ell-1}|\nabla^{(\ell)}v(x)|_\ell^2\Big)\,\mathrm{d}x\le g(\nu)+\eta.\end{aligned}$$

The arbitrariness of η concludes the proof. □

Remark 8.2.2. With the previous proof, we have also shown that

$$g(\nu)=\lim_{\varepsilon\to0}\inf\Big\{\int_{Q_1^\nu}\Big(\frac{1}{\varepsilon}W(v)+\sum_{\ell=1}^k q_\ell\varepsilon^{2\ell-1}|\nabla^{(\ell)}v|_\ell^2\Big)\,\mathrm{d}x\colon\ v\in\mathcal{A}_\varepsilon^\nu\Big\}.$$

This observation will be useful in the last section in order to study the density g in the one-dimensional case.

8.3 Interpolation

This section is devoted to the proof of Theorem 8.1.1. Furthermore, we obtain a useful result (Corollary 8.3.7) that we will use in the proof of Theorem 8.1.2. Throughout this

section, we keep track of the dependence of the constants on the involved parameters with the only exceptions of d, the dimension of the ambient Euclidean space, and k, the highest order of derivation.

We start by mentioning a classical interpolation estimate on intervals (see Theorem 7.41 in [143]). For simplicity of notation, for every interval I we let $|I|$ denote its length.

Theorem 8.3.1. *Let $I \subset \mathbb{R}$ be a bounded, open interval and let k be a positive integer. Then there exists a positive constant c independent of I such that*

$$\int_I |u^{(k-1)}|^2 \,\mathrm{d}t \le c\mathcal{L}^1(I)^{-2(k-1)} \int_I u^2 \,\mathrm{d}t + c \int_I |u^{(k)}|^2 \,\mathrm{d}t$$

for every $u \in \mathrm{H}^k(I)$.

We adapt the argument of Lemma 3.1 in [78] to bound the square of the L^2-norm of the derivative of order $k-1$ by means of the integral of the double-well potential and the square of the L^2-norm of the derivative of order k. Then, we deduce the same kind of estimate for any derivative of intermediate order $\ell \in \{1, \dots, k-1\}$.

Lemma 8.3.2. *Let $I \subset \mathbb{R}$ be a bounded, open interval, let $k > 1$ be an integer, and assume that* (H2) *is satisfied. Then there exists a positive constant q'_{k-1} independent of I such that*

$$q \int_I |u^{(k-1)}(t)|^2 \,\mathrm{d}t \le \mathcal{L}^1(I)^{-2(k-1)} \int_I W(u) \,\mathrm{d}t + \mathcal{L}^1(I)^2 \int_I |u^{(k)}|^2 \,\mathrm{d}t$$

for every $q < q'_{k-1}$ and $u \in \mathrm{H}^k(I)$.

Proof. It is sufficient to prove the statement for $I = (0,1)$, the general case being obtained by translating and rescaling. We set

$$z \coloneqq \int_0^1 u^{(k-1)} \,\mathrm{d}t,$$

and by the Fundamental Theorem of Calculus we infer that

$$|u^{(k-1)}(t) - z| \le \int_0^1 |u^{(k)}| \,\mathrm{d}t \qquad \text{for all } t \in (0,1). \tag{8.3.1}$$

As a consequence

$$\int_0^1 |u^{(k-1)}|^2 \,\mathrm{d}t \le 2 \int_0^1 |u^{(k)}|^2 \,\mathrm{d}t + 2z^2;$$

therefore, the proof is complete if we prove that

$$z^2 \le C \int_0^1 \Big(W(u) + |u^{(k)}|^2 \Big) \,\mathrm{d}t \tag{8.3.2}$$

for some $C > 0$. If $z^2 \le 4 \int_0^1 |u^{(k)}|^2 \,\mathrm{d}t$, then (8.3.2) is immediately satisfied. Otherwise, by Jensen's inequality, we have that $|z| > 2 \int_0^1 |u^{(k)}| \,\mathrm{d}t$, which together with (8.3.1), yields

$$|u^{(k-1)}(t)| \ge \frac{|z|}{2} > 0 \qquad \text{for all } t \in (0,1). \tag{8.3.3}$$

Consider the intervals $(\frac{j}{k}, \frac{j+1}{k})$, $j \in \{0, \dots, k-1\}$. We observe that there exists $j^* \in \{0, \dots, k-1\}$ such that $u \neq 0$ on $(\frac{j^*}{k}, \frac{j^*+1}{k})$. To see this, assume by contradiction that u

vanishes on each of these intervals; then, u has at least k zeros on I. By Lagrange's Theorem, this implies that u' vanishes at $k-1$ points of I, and then, iterating this argument, we obtain that $u^{(k-1)}$ vanishes at a point of I, contradicting (8.3.3). We suppose $u > 0$ in $(\frac{j^*}{k}, \frac{j^*+1}{k})$, the case $u < 0$ being analogous. Applying Theorem 8.3.1 to the function $u - 1$ on the interval $(\frac{j^*}{k}, \frac{j^*+1}{k})$ and resorting to (H2), we obtain

$$\begin{aligned}\int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} |u^{(k-1)}|^2 \,\mathrm{d}t &\le ck^{2(k-1)} \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} (u-1)^2 \,\mathrm{d}t + c \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} |u^{(k)}|^2 \,\mathrm{d}t \\ &\le ck^{2(k-1)} \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} W(u) \,\mathrm{d}t + c \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} |u^{(k)}|^2 \,\mathrm{d}t,\end{aligned}$$

therefore, taking into account (8.3.3),

$$\begin{aligned}z^2 &\le 4k \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} |u^{(k-1)}|^2 \,\mathrm{d}t \\ &\le 4ck^{2k-1} \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} W(u) \,\mathrm{d}t + 4ck \int_{\frac{j^*}{k}}^{\frac{j^*+1}{k}} |u^{(k)}|^2 \,\mathrm{d}t,\end{aligned}$$

which, by the positivity of W, proves (8.3.2) and concludes the proof. □

Corollary 8.3.3. *Let $I \subset \mathbb{R}$ be a bounded, open interval, let $k > 1$ be an integer, and assume that* (H2) *is satisfied. Then, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant q'_ℓ independent of I such that*

$$q \int_I |u^{(\ell)}|^2 \,\mathrm{d}t \le \mathcal{L}^1(I)^{-2\ell} \int_I W(u) \,\mathrm{d}t + \mathcal{L}^1(I)^{2(k-\ell)} \int_I |u^{(k)}|^2 \,\mathrm{d}t$$

for every $q < q'_\ell$ and $u \in \mathrm{H}^k(I)$.

Proof. The thesis follows by applying iteratively Lemma 8.3.2. □

Now we adapt the interpolation inequality at scale ε, in view of its application to functionals (8.0.1).

Proposition 8.3.4. *Let $I \subset \mathbb{R}$ be a bounded, open interval, let $k > 1$ be an integer, and assume that* (H2) *is satisfied. Then, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant q''_ℓ independent of I such that, for every $\varepsilon \in (0, \mathcal{L}^1(I)/2)$, it holds*

$$q\varepsilon^{2\ell} \int_I |u^{(\ell)}|^2 \,\mathrm{d}t \le \int_I \Big(W(u) + \varepsilon^{2k} |u^{(k)}|^2 \Big) \,\mathrm{d}t$$

for every $q < q''_\ell$ and $u \in \mathrm{H}^k(I)$.

Proof. We set $v(t) \coloneqq u(\varepsilon t)$ for $t \in \frac{1}{\varepsilon} I$ so that

$$\varepsilon^{2\ell-1} \int_I |u^{(\ell)}|^2 \,\mathrm{d}t = \int_{\frac{I}{\varepsilon}} |v^{(\ell)}|^2 \,\mathrm{d}t.$$

Then, we set $n_\varepsilon := \lfloor \frac{\mathcal{L}^1(I)}{\varepsilon} \rfloor$ and we subdivide the rescaled interval $\frac{1}{\varepsilon} I$ in n_ε pairwise disjoint, open intervals I_ε^j, $j \in \{1, \dots, n_\varepsilon\}$ each of length $\frac{\mathcal{L}^1(I)}{\varepsilon n_\varepsilon}$. Then we apply Corollary 8.3.3 on each subinterval I_ε^j with $q < q'_\ell$ and we obtain

$$\begin{aligned}
\varepsilon^{2\ell-1} \int_I |u^{(\ell)}|^2 \,\mathrm{d}t &= \sum_{j=1}^{n_\varepsilon} \int_{I_\varepsilon^j} |v^{(\ell)}|^2 \,\mathrm{d}t \\
&\le \sum_{j=1}^{n_\varepsilon} \frac{1}{q} \Big\{ \mathcal{L}^1(I_\varepsilon^j)^{-2\ell} \int_{I_\varepsilon^j} W(v) \,\mathrm{d}t + \mathcal{L}^1(I_\varepsilon^j)^{2(k-\ell)} \int_{I_\varepsilon^j} |v^{(k)}|^2 \,\mathrm{d}t \Big\} \\
&\le \frac{1}{q} \Big\{ 2^{2\ell} \int_{\frac{1}{\varepsilon} I} W(v) \,\mathrm{d}t + \Big(\frac{3}{2}\Big)^{2(k-\ell)} \int_{\frac{1}{\varepsilon} I} |v^{(k)}|^2 \,\mathrm{d}t \Big\} \\
&\le \frac{\max\{2^{2\ell}, (3/2)^{2(k-\ell)}\}}{q} \int_I \Big(\frac{W(u)}{\varepsilon} + \varepsilon^{2k-1} |u^{(k)}|^2 \Big) \,\mathrm{d}t,
\end{aligned}$$

where the second inequality follows by the fact $\mathcal{L}^1(I_\varepsilon^j) \in (\frac{1}{2}, \frac{3}{2})$ if $\varepsilon \in (0, |I|/2)$.
We multiply the above inequality by ε to obtain that

$$\widetilde{q} \varepsilon^{2\ell} \int_I |u^{(\ell)}|^2 \,\mathrm{d}t \le \int_I \Big(W(u) + \varepsilon^{2k} |u^{(k)}|^2 \Big) \,\mathrm{d}t$$

for every $\widetilde{q} < q''_\ell$ and $\varepsilon \in (0, |I|/2)$, where we set

$$q''_\ell := \frac{q'_\ell}{\max\{2^{2\ell}, (3/2)^{2(k-\ell)}\}};$$

therefore, the proof is complete. □

The following immediate corollary proves Theorem 8.1.1 in dimension 1.

Corollary 8.3.5. *Let $\Omega \subset \mathbb{R}$ be the union of finitely many bounded, open, disjoint intervals, let $k > 1$ be an integer, and assume that* (H2) *is satisfied. Then, denoting by $\mathcal{L}^1(I_o)$ the length of the shortest connected component of Ω, for every $\ell \in \{1, \dots, k-1\}$ and for every $\varepsilon \in (0, |I_o|/2)$ it holds*

$$q \varepsilon^{2\ell} \int_\Omega |u^{(\ell)}|^2 \,\mathrm{d}t \le \int_\Omega \Big(W(u) + \varepsilon^{2k} |u^{(k)}|^2 \Big) \,\mathrm{d}t$$

for every $q < q''_\ell$ and $u \in \mathrm{H}^k(\Omega)$.

Proof. The proof follows by an immediate application of Proposition 8.3.4. □

Finally, we prove the main result of this section by using the argument presented in the proof of Theorem 1.2 in [75]. Below, we will make use of notation about slicing that has been introduced in Section 2.

Proof of Theorem 8.1.1. It suffices to prove the part of the statement concerning the operator norms, since the other case follows by the equivalence of the norms. We follow the proof of Theorem 1.2 in [75]. For $r > 0$, $\xi \in \mathbb{S}^{d-1}$, and $y \in \pi^\xi(\Omega)$, we define $\Omega_r^{\xi,y}$ as the finite union of the connected components of the slice Ω_y^ξ having length larger than r. Then we set $\Omega(\xi, r) := \{y + t\xi \in \mathbb{R}^d \colon\ y \in \pi^\xi(\Omega),\ t \in \Omega_r^{\xi,y}\}$ and we observe that $(\Omega(\xi, r))_y^\xi = \Omega_r^{\xi,y}$.

Applying Corollary 8.3.5 we obtain

$$\widetilde{q}\varepsilon^{2\ell}\int_{\Omega_r^{\xi,y}}|(u_y^\xi)^{(\ell)}|^2\,\mathrm{d}t\le\int_{\Omega_r^{\xi,y}}\Big(W(u_y^\xi)+\varepsilon^{2k}|(u_y^\xi)^{(k)}|^2\Big)\,\mathrm{d}t$$

for all $\widetilde{q}<q_\ell''$ and $\varepsilon<r/2$. We integrate the above inequality on $\xi^\perp$ to get

$$\widetilde{q}\varepsilon^{2\ell}\int_{\Omega(\xi,r)}|(\nabla^{(\ell)}u)(\xi,\dots,\xi)|^2\,\mathrm{d}x\le\int_\Omega\Big(W(u)+\varepsilon^{2k}\|\nabla^{(k)}u\|_k^2\Big)\,\mathrm{d}x,$$

and setting $A(\xi,r):=\Omega\setminus\Omega(\xi,r)$ we obtain

$$\widetilde{q}\varepsilon^{2\ell}\int_\Omega|(\nabla^{(\ell)}u)(\xi,\dots,\xi)|^2\,\mathrm{d}x-\widetilde{q}\varepsilon^{2\ell}\int_{A(\xi,r)}\|\nabla^{(\ell)}u\|_\ell^2\,\mathrm{d}x\le\int_\Omega\Big(W(u)+\varepsilon^{2k}\|\nabla^{(k)}u\|_k^2\Big)\,\mathrm{d}x.$$

Averaging this inequality on $\mathbb{S}^{d-1}$ yields

$$\begin{aligned}\varepsilon^{2\ell}\frac{\widetilde{q}}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}}\int_\Omega|(\nabla^{(\ell)}u)(\xi,\dots,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{d-1}(\xi)&\\-\varepsilon^{2\ell}\frac{\widetilde{q}}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}}\int_{A(\xi,r)}\|\nabla^{(\ell)}u\|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{d-1}(\xi)&\\\le\int_\Omega\Big(W(u)+\varepsilon^{2k}\|\nabla^{(k)}u\|_k^2\Big)\,\mathrm{d}x,&\end{aligned}$$

where we set $\sigma_{d-1}:=\mathcal{H}^{d-1}(\mathbb{S}^{d-1})$.

By a simple compactness argument, there exists a positive constant $\tau(\ell)$ such that

$$\frac{1}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}}|T(\xi,\dots,\xi)|^2\,\mathrm{d}\mathcal{H}^{d-1}(\xi)\ge\tau(\ell)\|T\|_\ell^2$$

for every symmetric ℓ-tensor T. Therefore, by Fubini's Theorem,

$$\frac{1}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}}\int_\Omega|(\nabla^{(\ell)}u)(\xi,\dots,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{d-1}(\xi)\ge\tau(\ell)\int_\Omega\|\nabla^{(\ell)}u\|_\ell^2\,\mathrm{d}x.\tag{8.3.4}$$

Again by Fubini's Theorem, we have

$$\frac{1}{\sigma_{d-1}}\int_{\mathbb{S}^{d-1}}\int_{A(\xi,r)}\|\nabla^{(\ell)}u\|_\ell^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{d-1}(\xi)=\frac{1}{\sigma_{d-1}}\int_\Omega\|\nabla^{(\ell)}u\|_\ell^2\mathcal{H}^{d-1}(D(x,r))\,\mathrm{d}x,$$

where $D(x,r):=\{\xi\in\mathbb{S}^{d-1}\colon\ x\in A(\xi,r)\}$; hence, to conclude the proof, we have to select r small enough so that $\mathcal{H}^{d-1}(D(x,r))$ is small uniformly in x. Given $x\in\partial\Omega$, we let $C_R(x)$ denote the cylinder having height $2R$ and radius R with axis parallel to the normal $\nu(x)$ to $\partial\Omega$ at x.

Let $q<\tau(\ell)q_\ell''$, and fix $\widetilde{q}$ such that $\frac{q}{\tau(\ell)}<\widetilde{q}<q_\ell''$. Fix $\eta=\eta(q)>0$ such that

$$q<\widetilde{q}(\tau(\ell)-\eta);\tag{8.3.5}$$

since Ω has C^1 boundary, there exists R such that, for every $x\in\partial\Omega$, it holds $C_R(x)\cap\partial\Omega$ is the graph of a C^1 function with respect to the base of the cylinder, and moreover,

$$\mathcal{H}^{d-1}(\{\xi\in\mathbb{S}^{d-1}\colon\ \xi\in T_{\partial\Omega}(y),\ y\in C_R(x)\cap\partial\Omega\})\le\eta\sigma_{d-1}.$$

Employing the same argument presented in the proof of Theorem 1.2 in [75], it holds that,

$$D(x,r) \subset \{\xi \in \mathbb{S}^{d-1} \colon \xi \in T_{\partial\Omega}(y),\, y \in C_R(x) \cap \partial\Omega\}$$

up to choosing $r < R/2$; as a consequence, $\mathcal{H}^{d-1}(D(x,r)) \le \eta$ for all $x \in \partial\Omega$ if $r < R/2$, and then

$$\frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{A(\xi,r)} \|\nabla^{(\ell)} u\|_\ell^2 \,\mathrm{d}x \,\mathrm{d}\mathcal{H}^{d-1}(\xi) \le \eta \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 \,\mathrm{d}x, \tag{8.3.6}$$

for all $r < R/2$.

Gathering (8.3.4) and (8.3.6), we obtain that

$$\widetilde{q}(\tau(\ell) - \eta)\varepsilon^{2\ell} \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 \,\mathrm{d}x \le \int_\Omega \Big(W(u) + \varepsilon^{2k} \|\nabla^{(k)} u\|_k^2 \Big) \,\mathrm{d}x \tag{8.3.7}$$

for all $\widetilde{q} < q''_\ell$ and $\varepsilon < R/2$, where $R = R(\eta, \Omega)$. Recalling (8.3.5), by (8.3.7) we get

$$q\varepsilon^{2\ell} \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 \,\mathrm{d}x \le \int_\Omega \Big(W(u) + \varepsilon^{2k} \|\nabla^{(k)} u\|_k^2 \Big) \,\mathrm{d}x.$$

The above inequality is valid for all $q < \tau(\ell) q''_\ell$ and $\varepsilon < R(\eta, \Omega)/2$. Then, the thesis holds with $q^*_\ell \coloneqq \tau(\ell) q''_\ell$ and $\varepsilon_0 \coloneqq R(\eta, \Omega)/2$, which in turn, since η depends on q, only depends on q and Ω. □

Remark 8.3.6. We are not able to prove the interpolation inequality on an open cube. The same slicing arguments employed in [75] cannot be applied in our case because of the high order of the derivatives that we want to estimate. This complicates our arguments in view of the proof of Theorem 8.1.3 since, in what follows, it will be important to apply Theorem 8.1.1 on a class of C^1 sets $\{\Omega_\rho \colon \rho \in (\rho_1, \rho_2)\}$ for some $\rho_1, \rho_2 > 0$ for which the threshold $\varepsilon_0 = \varepsilon_0(q, \Omega_\rho)$ can be chosen independently of $\rho \in (\rho_1, \rho_2)$. Inspecting the proof of Theorem 8.1.1, this amounts to prove that for each properly fixed q, given $\eta > 0$ such that

$$q < \widetilde{q}(\tau(\ell) - \eta),$$

there exists $R > 0$ such that, for every $\rho \in (\rho_1, \rho_2)$ and for every $x \in \partial\Omega_\rho$, it holds that

$$C_R(x) \cap \partial\Omega_\rho \ \text{ is the graph of a } C^1 \text{ function} \tag{8.3.8}$$

and

$$\mathcal{H}^{d-1}(\{\xi \in \mathbb{S}^{d-1} \colon \xi \in T_{\partial\Omega_\rho}(y),\, y \in C_R(x) \cap \partial\Omega_\rho\}) \le \eta \mathcal{H}^{d-1}(\mathbb{S}^{d-1}). \tag{8.3.9}$$

We claim that the above conditions are satisfied if we consider Ω a bounded, open set with C^2 boundary, and

$$\Omega_\rho \coloneqq \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) < \rho\}$$

for $\rho \in (\rho^*, \rho^* + R^*)$, where $\rho^* > 0$ is fixed and $R^* = R^*(\rho^*) > 0$. In particular, we fix ρ^* small enough so that $\{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) = \rho\}$ is a $(d-1)$-manifold of class C^1 and there exists a C^1 projection from Ω_ρ onto $\partial\Omega$ for every $\rho \le 2\rho^*$.

Let $\eta' = \eta'(\eta, d)$ be such that

$$\mathcal{H}^{d-1}\Big(\Big\{\xi \in \mathbb{S}^{d-1} \colon |\xi_d| \le \eta'\Big\}\Big) \le \eta \mathcal{H}^{d-1}(\mathbb{S}^{d-1}). \tag{8.3.10}$$

Since $\partial\Omega$ is of class C^1, for every $x \in \partial\Omega$ there exist $R > 0$ and a C^1 function φ defined on the base of $C_R(x)$ such that

$$C_R(x) \cap \partial\Omega = C_R(x) \cap \operatorname{graph} \varphi.$$

By a rotation argument, we can further assume that $\nabla\varphi(x) = 0$, and upon taking a smaller R, that

$$\|\nabla\varphi\|_{L^\infty(C_R(x))} \le \eta'.$$

Moreover, since $\partial\Omega$ is compact, we can select R as above independent of $x \in \partial\Omega$.

Since $\partial\Omega_{\rho^*}$ is of class C^1 and compact, using the same argument, we find a possibly smaller R with the property that for all $x \in \partial\Omega_{\rho^*} \cap \Omega$ there exists a C^1 function φ_{ρ^*} defined on the base of $C_R(x)$ such that

$$C_R(x) \cap \partial\Omega_{\rho^*} = C_R(x) \cap \text{graph } \varphi_{\rho^*} \quad \text{with} \quad \|\nabla\varphi_{\rho^*}\|_{L^\infty(C_R(x))} \le \eta'.$$

Finally, we assume without loss of generality that $R < \rho^*$.

Now, let $x \in \partial\Omega_{\rho^*} \cap \Omega$ and $0 < r < R/4$ and note that

$$\{y \in C_{R-r}(x)\colon \text{dist}(y,\ \text{graph } \varphi_{\rho^*}) = r\} = C_{R-r}(x) \cap \partial\Omega_{\rho^*+r}.$$

Then, letting $x_r \in \partial\Omega_{\rho^*+r}$ denote the unique point such that $|x - x_r| = r$, we have that $C_{R-2r}(x_r) \subset C_{R-r}(x)$, which implies

$$\{y \in C_{R-2r}(x_r)\colon \text{dist}(y,\ \text{graph } \varphi_{\rho^*}) = r\} = C_{R-2r}(x_r) \cap \partial\Omega_{\rho^*+r}.$$

We also note that there exists a C^1 function φ_{ρ^*+r} such that

$$\{y \in C_{R-2r}(x_r)\colon \text{dist}(y,\ \text{graph } \varphi_{\rho^*}) = r\} = C_{R-2r}(x_r) \cap \text{ graph } \varphi_{\rho^*+r},$$

and

$$\|\nabla\varphi_{\rho^*+r}\|_{L^\infty(C_{R-2r}(x_r))} \le \|\nabla\varphi_{\rho^*}\|_{L^\infty(C_{R-2r}(x_r))} \le \|\nabla\varphi_{\rho^*}\|_{L^\infty(C_R(x))}.$$

Recalling that there exists a C^1 projection from Ω_ρ onto $\partial\Omega$ for every $\rho \in (\rho^*, \rho^* + R/4)$, the above observations prove that for every $\rho \in (\rho^*, \rho^* + R/4)$ and for every $x \in \partial\Omega_\rho$, it holds that

$$C_{R-2r}(x) \cap \partial\Omega_\rho = C_{R-2r}(x) \cap \text{graph } \varphi_\rho,$$

where φ_ρ is a C^1 function such that

$$\|\nabla\varphi_\rho\|_{L^\infty(C_{R-2r}(x))} \le \|\nabla\varphi_{\rho^*}\|_{L^\infty(C_{R-2r}(x))} \le \eta'.$$

As r ranges in $(0, R/4)$, in particular we obtain

$$C_{R/2}(x) \cap \partial\Omega_\rho = C_{R/2}(x) \cap \text{graph } \varphi_\rho \tag{8.3.11}$$

and

$$\|\nabla\varphi_\rho\|_{L^\infty(C_{R/2}(x))} \le \|\nabla\varphi_{\rho^*}\|_{L^\infty(C_{R/2}(x))} \le \eta'. \tag{8.3.12}$$

At this point, we conclude that our claim is proved with $R^* = R/4$. Indeed, (8.3.8) follows by (8.3.11), and we are left with proving that (8.3.12) implies (8.3.9). To see this, note that for fixed $\rho \in (\rho^*, \rho^* + R/4)$ and $x \in \partial\Omega_\rho$, (8.3.12) implies that

$$\begin{aligned}
&\{\xi \in \mathbb{S}^{d-1}\colon \xi \in T_{\partial\Omega_\rho}(y),\ y \in C_{R/2}(x) \cap \partial\Omega_\rho\} \\
&\quad \subset \Big\{\xi \in \mathbb{S}^{d-1}\colon |\xi_d| = |\nabla\varphi_\rho(y_1, \dots, y_{d-1}) \cdot (\xi_1, \dots, \xi_{d-1})|,\ y \in C_{R/2}(x) \cap \partial\Omega_\rho\Big\} \\
&\quad \subset \Big\{\xi \in \mathbb{S}^{d-1}\colon |\xi_d| \le \eta'\Big\};
\end{aligned}$$

therefore, by (8.3.10) we conclude that

$$\mathcal{H}^{d-1}(\{\xi \in \mathbb{S}^{d-1}\colon \xi \in T_{\partial\Omega_\rho}(y),\ y \in C_{R/2}(x) \cap \partial\Omega_\rho\}) \le \eta\mathcal{H}^{d-1}(\mathbb{S}^{d-1}).$$

Being the above estimates uniform in ρ and x, (8.3.9) is satisfied.

The following corollary will come into play in the forthcoming sections.

Corollary 8.3.7. *Let $k > 1$ be an integer, and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} \colon q_\ell \le 0\}$. Assume that $\mathcal{N} \neq \emptyset$ and that* (H2) *is satisfied. There exist positive constants $\{\overline{q}_\ell \colon \ell \in \mathcal{N}\}$ independent of Ω such that if $q_\ell > -\overline{q}_\ell$ for every $\ell \in \mathcal{N}$ there exist positive constants $\varepsilon_0 = \varepsilon_0(\{q_\ell \colon \ell \in \mathcal{N}\}, \Omega)$ and $\delta = \delta(\{q_\ell \colon \ell \in \mathcal{N}\})$ such that it holds*

$$\int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\Big)\,\mathrm{d}x \ge \delta \int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^{k} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\Big)\,\mathrm{d}x$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathrm{H}^k(\Omega)$. In particular, letting $\widetilde{q}_1^*, \dots, \widetilde{q}_{k-1}^*$ denote the same positive constants appearing in Theorem 8.1.1 and given $\{\alpha_\ell \colon \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$, it holds that $\overline{q}_\ell = \alpha_\ell \widetilde{q}_\ell^*$ for all $\ell \in \mathcal{N}$.

Proof. By Theorem 8.1.1, for every $\ell \in \{1, \dots, k-1\}$ there exists a positive constant $\widetilde{q}_\ell^*$ independent of Ω such that for every $q > -\widetilde{q}_\ell^*$ there exists $\varepsilon_0 = \varepsilon_0(q, \Omega)$ such that it holds

$$\int_\Omega \Big(\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2\Big)\,\mathrm{d}x + q \int_\Omega \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\,\mathrm{d}x \ge 0 \tag{8.3.13}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathrm{H}^k(\Omega)$.

Let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} \colon q_\ell \le 0\}$ and consider $\{\alpha_\ell \colon \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$. Suppose that $q_\ell > -\alpha_\ell \widetilde{q}_\ell^*$ for every $\ell \in \mathcal{N}$ and let δ be a positive real number smaller than 1 such that

$$\frac{1}{\alpha_\ell} \frac{q_\ell - \delta}{1 - \delta} > -\widetilde{q}_\ell^* \tag{8.3.14}$$

for all $\ell \in \mathcal{N}$. Note that

$$\begin{aligned}
&\alpha_\ell \frac{W(u)}{\varepsilon} + q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2 \\
&= (1-\delta)\Big(\alpha_\ell \frac{W(u)}{\varepsilon} + \frac{q_\ell - \delta}{1-\delta} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2\Big) \\
&\quad + \delta\Big(\alpha_\ell \frac{W(u)}{\varepsilon} + \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2\Big).
\end{aligned}$$

We integrate the above equality and, taking into account (8.3.14), we apply (8.3.13) with $q = \frac{1}{\alpha_\ell} \frac{q_\ell - \delta}{1-\delta}$ to obtain

$$\begin{aligned}
&\int_\Omega \Big(\alpha_\ell \frac{W(u)}{\varepsilon} + q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2\Big)\,\mathrm{d}x \\
&\ge \delta \int_\Omega \Big(\alpha_\ell \frac{W(u)}{\varepsilon} + \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2\Big)\,\mathrm{d}x
\end{aligned} \tag{8.3.15}$$

for every $\ell \in \mathcal{N}$, $\varepsilon < \varepsilon_0(q_\ell, \Omega)$.

Clearly, for $\ell \notin \mathcal{N}$ the coefficient q_ℓ is strictly positive; hence it holds

$$\int_\Omega q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\,\mathrm{d}x \ge \delta \int_\Omega q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\,\mathrm{d}x \tag{8.3.16}$$

since $\delta < 1$. Therefore, summing over $\ell \in \{1, \dots, k-1\}$ and gathering (8.3.15) and (8.3.16), upon considering a smaller ε_0 depending on $\{q_\ell \colon \ell \in \mathcal{N}\}$ and Ω, we get

$$\int_\Omega \Big(\frac{W(u)}{\varepsilon} + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\Big)\,\mathrm{d}x \ge \delta \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \sum_{\ell=1}^{k} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2\Big)\,\mathrm{d}x,$$

for every $\varepsilon < \varepsilon_0$, completing the proof. □

Remark 8.3.8. The above statement clearly holds (for every ε) if $\mathcal{N} = \emptyset$ with $\delta := \min\{q_\ell \colon \ell \in \{1, \dots, k\}\}$, which is strictly positive.

8.4 Compactness

In this section we prove Theorem 8.1.2. In what follows, we let $E^1_\varepsilon \colon$ be the one-dimensional functional we studied in the previous chapter, which for any bounded open set $I \subset \mathbb{R}$ is defined by

$$E^1_\varepsilon(u, I) := \begin{cases} \displaystyle\int_I \Big[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |u^{(k)}|^2\Big]\, dt & \text{if } u \in H^k(I), \\ +\infty & \text{if } u \in L^2(I) \setminus H^k(I), \end{cases} \tag{8.4.1}$$

and we also consider its Γ-limit

$$E^1_0(u, I) := \begin{cases} m_k \#(J_u \cap I) & \text{if } u \in BV(I; \{-1, 1\}), \\ +\infty & \text{if } u \in L^2(I) \setminus BV(I; \{-1, 1\}), \end{cases} \tag{8.4.2}$$

where m_k is given by (7.1.2).

We now show that, under our strengthened growth condition on W, a stronger version of the compactness part of Theorem 7.1.1 holds true.

Proposition 8.4.1. *Let $\{E^1_\varepsilon\}_\varepsilon$ be defined as in* (8.4.1) *on a bounded, open set $I \subset \mathbb{R}$, let $\{\varepsilon_n\}_n$ be a positive sequence converging to* 0, *and assume that* (H1) *and* (H2) *are satisfied. If $\{u_n\}_n \subset \mathrm{H}^k(I)$ is a sequence such that $\sup_n E^1_{\varepsilon_n}(u_n, I) < +\infty$, then there exist a subsequence $\{u_{n_j}\}_j$ and a function $u \in \mathrm{BV}(I; \{-1, 1\})$ such that $u_{n_j} \to u$ in $L^2(I)$ as $j \to +\infty$. Moreover, the family of functionals $\{E^1_\varepsilon\}_\varepsilon$ Γ-converges with respect to the convergence in $L^2(I)$ to the functional E^1_0 defined in* (8.4.2).

Proof. It suffices to prove the statement in the case that I be an interval. Let $\{u_n\}_n \subset \mathrm{H}^k(I)$ be a sequence such that $\sup_n E^1_{\varepsilon_n}(u_n, I) < +\infty$. Since (H2) implies ($\widetilde{\text{H2}}$), we apply Theorem 7.1.1 to deduce that, up to subsequences, $u_n \to u$ in measure for some $u \in \mathrm{BV}(I; \{-1, 1\})$. Let $A \subset I$ be measurable, by (H2) we have

$$\int_A |u_n|^2 \, \mathrm{d}t = \int_{A \cap \{|u_n| \le 1\}} |u_n|^2 \, \mathrm{d}t + \int_{A \cap \{|u_n| > 1\}} |u_n|^2 \, \mathrm{d}t \le \mathcal{L}^1(A) + 2 \int_A \Big(W(u_n) + 1\Big) \, \mathrm{d}t,$$

and since $\sup_n E^1_{\varepsilon_n}(u_n, I) < +\infty$, it holds that $\int_I W(u_n) \, \mathrm{d}t \to 0$ as $n \to +\infty$. Hence, the sequence $\{|u_n|^2\}_n$ is equi-integrable, which, together with its convergence in measure, implies $u_n \to u$ in $L^2(I)$.

As for the Γ-limit, the lower bound follows again by Theorem 7.1.1, while a recovery sequence for the $L^2(I)$-convergence is obtained exactly as in Proposition 7.4.3. □

At this point, we prove an analogous compactness result for functionals (8.1.1), replacing I with $\Omega \subset \mathbb{R}^d$, $d > 1$. This is possible by applying a compactness criterion by slicing that appears in [7]. We introduce some notation specific for its statement: here, Ω denotes a 2-dimensional rectangle which is the product of the bounded, open intervals I, J, and a point $x \in \Omega$ is denoted by (y, z), with $y \in I, z \in J$. Given a function u defined on Ω, for every $y \in I$ we let u^y denote the function $u^y(z) := u(y, z)$ for every $z \in J$, and for every $z \in J$ we let u^z denote the function $u^z(y) := u(y, z)$ for every $y \in I$. We say that two sequences $\{u_n\}_n$ and $\{v_n\}_n$ are δ-close if $\|u_n - v_n\|_{L^1(\Omega)} \le \delta$.

Proposition 8.4.2. *Assume that a sequence $\{u_n\}_n \subset L^1(\Omega)$ is equi-integrable, and that for every $\delta > 0$ there exist sequences $\{v_n\}_n$ and $\{w_n\}_n$ δ-close to $\{u_n\}_n$, and such that $\{v_n^y\}_n$ is precompact in $L^1(J)$ for a.e. $y \in I$, and $\{w_n^z\}_n$ is precompact in $L^1(I)$ for a.e. $z \in J$. Then $\{u_n\}_n$ is precompact in $L^1(\Omega)$.*

We employ this result exactly as in [123]. For the convenience of the reader, we sketch the proof, and for simplicity, we consider only the case $d = 2$ when needed.

Proposition 8.4.3. *Let $\{E_\varepsilon\}_\varepsilon$ be defined as in* (8.1.1)*, let $\{\varepsilon_n\}_n$ be a positive sequence converging to* 0*, and assume that* (H1) *and* (H2) *are satisfied. If $\{u_n\}_n \subset \mathrm{H}^k(\Omega)$ is a sequence such that $\sup_n E_{\varepsilon_n}(u_n, \Omega) < +\infty$, then there exist a subsequence $\{u_{n_j}\}_j$ and a function $u \in \mathrm{BV}(\Omega; \{-1, 1\})$ such that, $u_{n_j} \to u$ in $L^2(\Omega)$ as $j \to +\infty$.*

Proof. For simplicity, in the first part of the proof we suppose $d = 2$ and that Ω be the rectangle $I \times J$, otherwise, it suffices to cover a general Ω by essentially disjoint rectangles in order to reduce the proof to our simpler case. Using the notation previously introduced, we recall that $\{u_n^y\}_n \subset \mathrm{H}^k(J)$ for a.e. $y \in I$, and, analogously, that $\{u_n^z\}_n \subset \mathrm{H}^k(I)$ for a.e. $z \in J$. In particular, it holds

$$\frac{\mathrm{d}^k u_n^y}{\mathrm{d}z^k}(z) = \frac{\partial^k u_n}{\partial z^k}(x), \qquad \frac{\mathrm{d}^k u_n^z}{\mathrm{d}y^k}(y) = \frac{\partial^k u_n}{\partial y^k}(x)$$

for a.e. $x = (y, z) \in \Omega$ and for all n. As a consequence, by the definition of the operator norm, we have $\|\nabla^{(k)} u_n\|_k \geq \max\Big\{\Big|\frac{\mathrm{d}^k u_n^y}{\mathrm{d}z^k}\Big|, \Big|\frac{\mathrm{d}^k u_n^z}{\mathrm{d}y^k}\Big|\Big\}$, and then

$$E_\varepsilon(u_n, \Omega) \geq \int_I E_\varepsilon^1(u_n^y, J)\mathrm{d}y, \qquad E_\varepsilon(u_n, \Omega) \geq \int_J E_\varepsilon^1(u_n^z, I)\,\mathrm{d}z. \tag{8.4.3}$$

Arguing as in the proof of Proposition 8.4.1, we find that $\{u_n\}_n$ and $\{|u_n|^2\}_n$ are equi-integrable sequences; therefore, we fix $\delta > 0$ and we find $\delta' \in (0, \delta)$ such that

$$\mathcal{L}^2(A) \leq \delta' \mathcal{L}^1(J) \implies \sup_{n \in \mathbb{N}} \int_A (|u_n| + 1)\,\mathrm{d}x \leq \delta. \tag{8.4.4}$$

Letting S denote $\sup_n E_{\varepsilon_n}(u_n, \Omega)$, for every $n \in \mathbb{N}$ we set

$$v_n^y(z) \coloneqq \begin{cases} u_n^y(z) & \text{if } E^1_{\varepsilon_n}(u_n^y, J) \leq S/\delta', \\ 1 & \text{otherwise.} \end{cases}$$

We also put $Y_n \coloneqq \{y \in I\colon\ u_n^y - v_n^y \text{ is not identically } 0\}$ and we observe that

$$\mathcal{L}^1(Y_n) \leq \mathcal{L}^1(\{y \in I\colon\ E^1_{\varepsilon_n}(u_n^y, J) > S/\delta'\}) \leq \frac{\delta'}{S} \int_I E^1_{\varepsilon_n}(u_n^y, J)\mathrm{d}y \leq \delta',$$

where the last inequality follows by (8.4.3). This inequality, combined with (8.4.4), implies that

$$\|u_n - v_n\|_{L^1(\Omega)} \leq \int_{Y_n \times J} (|u_n| + 1)\,\mathrm{d}x \leq \delta;$$

that is, $\{u_n\}_n$ and $\{v_n\}_n$ are δ-close. Moreover, by construction and by (H1), we have that for every $y \in I$ the sequence $\{v_n^y\}_n$ is such that $\sup_n E^1_{\varepsilon_n}(v_n^y, J) \leq S/\delta'$. Hence, by Proposition 8.4.1, this sequence is precompact in $L^2(J)$, and thus in $L^1(J)$.

A similar argument allows to find a sequence $\{w_n\}_n$ that is δ-close to $\{u_n\}_n$ and such that, for a.e. $z \in J$, the sequence $\{w_n^z\}_n$ is precompact in $L^1(I)$. Therefore, by applying Proposition 8.4.2 we deduce that $\{u_n\}_n$ is precompact in $L^1(\Omega)$. Finally, since $\{|u_n|^2\}_n$ is equi-integrable, we deduce the precompactness in $L^2(\Omega)$.

To conclude, we prove, without further assumptions on d and Ω, that if the subsequence $\{u_{n_j}\}_j$ converges to u in $L^2(\Omega)$ as $j \to +\infty$, then $u \in \mathrm{BV}(\Omega; \{-1, 1\})$. To see this, using the notation introduced in Section 1.3, we prove that for every $m \in \{1, \dots, d\}$, the one-dimensional slice $u_y^{e_m}$ has bounded variation on the open set $\Omega_y^{e_m}$ for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{e_m}(\Omega)$, and

$$\int_{\pi^{e_m}(\Omega)} \#(J_{u_y^{e_m}} \cap \Omega_y^{e_m}) \, \mathrm{d}\mathcal{H}^{d-1}(y) < +\infty.$$

Arguing as previously done for (8.4.3), we have that

$$E_{\varepsilon_{n_j}}(u_{n_j}, \Omega) \geq \int_{\pi^{e_m}(\Omega)} E^1_{\varepsilon_{n_j}}((u_{n_j})_y^{e_m}, \Omega_y^{e_m}) \, \mathrm{d}\mathcal{H}^{d-1}(y).$$

Moreover, upon extracting a further subsequence that we do not relabel, it holds that $(u_{n_j})_y^{e_m} \to u_y^{e_m}$ in $L^2(\Omega_y^{e_m})$ for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{e_m}(\Omega)$. Hence, by Fatou's lemma and Proposition 8.4.1 applied with $I = \Omega_y^{e_m}$ and $u = u_y^{e_m}$, we get

$$\begin{aligned} +\infty > S \geq \liminf_{j \to +\infty} E_{\varepsilon_{n_j}}(u_{n_j}, \Omega) &\geq \int_{\pi^{e_m}(\Omega)} \liminf_{j \to +\infty} E^1_{\varepsilon_{n_j}}((u_{n_j})_y^{e_m}, \Omega_y^{e_m}) \, d\mathcal{H}^{d-1}(y) \\ &\geq \int_{\pi^{e_m}(\Omega)} E^1_0(u_y^{e_m}, \Omega_y^{e_m}) \, \mathrm{d}\mathcal{H}^{d-1}(y). \end{aligned}$$

As a consequence, $E^1_0(u_y^{e_m}, \Omega_y^{e_m})$ is finite for $\mathcal{H}^{d-1}$-a.e. $y \in \pi^{e_m}(\Omega)$, which implies that

$$\begin{aligned} +\infty > S &\geq \int_{\pi^{e_m}(\Omega)} E^1_0(u_y^{e_m}, \Omega_y^{e_m}) \, \mathrm{d}\mathcal{H}^{d-1}(y) \\ &\geq m_k \int_{\pi^{e_m}(\Omega)} \#(J_{u_y^{e_m}} \cap \Omega_y^{e_m}) \, \mathrm{d}\mathcal{H}^{d-1}(y). \end{aligned}$$

Then, since $m_k > 0$, for such y it holds that $u_y^{e_m} \in \mathrm{BV}(\Omega_y^{e_m}; \{-1, 1\})$, hence, the conclusion follows applying Proposition 1.3.6. □

Finally, the proof of our main result is an immediate consequence.

Proof of Theorem 8.1.2. Let S denote $\sup_n F_{\varepsilon_n}(u_n, \Omega)$. If $\mathcal{N} = \emptyset$, by the equivalence of the norms there exists $c > 0$ such that $F_{\varepsilon_n}(u_n, \Omega) \geq c E_{\varepsilon_n}(u_n, \Omega)$ for every n so that

$$S/c \geq \sup_n E_{\varepsilon_n}(u_n, \Omega)$$

and the conclusion follows by Proposition 8.4.3.

If $\mathcal{N} \neq \emptyset$, again by the equivalence of the norms and by Corollary 8.3.7, upon restricting to large n, it holds

$$S/\delta' \geq \sup_n E_{\varepsilon_n}(u_n, \Omega)$$

for some $\delta' > 0$; therefore, the conclusion follows by Proposition 8.4.3. □

8.5 Lower bound

In this section we prove the liminf inequality of Theorem 8.1.3. In particular, we prove the following result.

Proposition 8.5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set with C^1 boundary and let $\mathcal{N} := \{\ell \in \{1,\dots,k-1\}\colon q_\ell \le 0\}$. Assume that* (H1)–(H3) *are satisfied and that one of the following conditions holds:*

(i) $\mathcal{N} = \emptyset$;

(ii) $\mathcal{N} \neq \emptyset$ *and* $q_\ell > -\overline{q}_\ell$ *for every* $\ell \in \mathcal{N}$ *with the same positive constants* $\{\overline{q}_\ell\colon \ell \in \mathcal{N}\}$ *independent of* Ω *appearing in Theorem 8.1.2.*

Then for every positive sequence $\{\varepsilon_n\}_n$ converging to 0 and $\{u_n\}_n \subset \mathrm{H}^k(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ it holds

$$F_0(u) \le \liminf_{n\to+\infty} F_{\varepsilon_n}(u_n).$$

In the following, C denotes a generic positive constant that may change from line to line, but which is independent of the relevant parameters. We recall that $\overline{u}$ is a function in $\mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$ such that $\overline{u}(t) = -1$ if $t \le -1/8$ and $\overline{u}(t) = 1$ if $t \ge 1/8$, $|\overline{u}| \le 1$, and

$$0 < \int_{-\infty}^{+\infty} \Big(W(\overline{u}) + \sum_{\ell=1}^{k} |q_\ell||\overline{u}^{(\ell)}|^2\Big)\,\mathrm{d}t < +\infty. \tag{8.5.1}$$

Moreover, for every $\varepsilon > 0$ we have set $\overline{u}^\nu_\varepsilon(x) := \overline{u}(\frac{x\cdot\nu}{\varepsilon})$ for all $x \in \mathbb{R}^d$. We observe that, thanks to (8.5.1), the definition of $\overline{u}^\nu_\varepsilon$, and Fubini's Theorem, for every open set $V \subset \Omega$ there exists a constant $C > 0$ such that for every $\nu \in \mathbb{S}^{d-1}$ and $\varepsilon > 0$ we have that

$$\begin{aligned} F_\varepsilon(\overline{u}_\varepsilon, V) &\le C \int_{\pi^\nu(V)} \int_{-\infty}^{+\infty} \Big(W(\overline{u}) + \sum_{\ell=1}^{k} |q_\ell||\overline{u}^{(\ell)}|^2\Big)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\ &\le C \int_{-\infty}^{+\infty} \Big(W(\overline{u}) + \sum_{\ell=1}^{k} |q_\ell||\overline{u}^{(\ell)}|^2\Big)\,\mathrm{d}t \le C, \end{aligned} \tag{8.5.2}$$

where π^ν denotes the orthogonal projection onto $\nu^\perp$ and where we have also used that $(\overline{u}^\nu_\varepsilon)^{\nu,y}(t) = \overline{u}(\frac{t}{\varepsilon})$ for every $\varepsilon > 0$, $y \in \nu^\perp$, and $t \in \mathbb{R}$.

We further set

$$u^\nu_0(x) := \mathrm{sgn}(x\cdot\nu), \tag{8.5.3}$$

where sgn denotes the sign function and remark that, for every $K \subset \mathbb{R}^d$ compact, $\{\overline{u}^\nu_\varepsilon\}_\varepsilon$ converges to u^ν_0 in $L^2(K)$ as $\varepsilon \to 0$.

We start by proving a lemma which allows to modify boundary conditions on a converging sequence at the cost of an asymptotically negligible amount of energy.

For every $x \in \mathbb{R}^d, \nu \in \mathbb{S}^{d-1}$, we introduce an open subset $\widetilde{Q}_\nu(x)$ of $Q_\nu(x)$ having C^2 boundary and such that

$$\widetilde{Q}_\nu(x) \cap \{y \in \mathbb{R}^d\colon |y\cdot\nu| \le \tfrac{1}{4}\} = Q_\nu(x) \cap \{y \in \mathbb{R}^d\colon |y\cdot\nu| \le \tfrac{1}{4}\}.$$

We suppose that all the above sets are congruent, namely, that they are obtained starting by $\widetilde{Q}_{e_d}(0)$ by a translation and a rotation. Also for the *rounded* cube $\widetilde{Q}_\nu(x)$ we use the convention that when $x = 0$ we simply write $\widetilde{Q}_\nu$.

Lemma 8.5.2. *Let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0, $\nu \in \mathbb{S}^{d-1}$, and let $\{v_n\}_n \subset \mathrm{H}^k(\widetilde{Q}_\nu)$ be a sequence converging to u_0^ν in $L^2(\widetilde{Q}_\nu)$. Assume that (H1)–(H3) are satisfied and that one of the two conditions (i) and (ii) of Proposition 8.5.1 holds. Then there exists a sequence $\{w_n\}_n \subset \mathrm{H}^k(\widetilde{Q}_\nu)$ such that*

(i) $w_n \to u_0^\nu$ as $n \to +\infty$ $L^2(\widetilde{Q}_\nu)$,

(ii) the support of $w_n - \overline{u}_{\varepsilon_n}$ is compactly contained in $\widetilde{Q}_\nu$ for every $n \in \mathbb{N}$,

(iii) it holds

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(w_n, \widetilde{Q}_\nu) \le \liminf_{n\to+\infty} F_{\varepsilon_n}(v_n, \widetilde{Q}_\nu).$$

Proof. For the sake of exposition, we only consider the case $\nu = e_d$, the others being analogous. For simplicity of notation, throughout the rest of the proof we set $\overline{u}_n := \overline{u}_{\varepsilon_n}^{e_d}$, and we set $\widetilde{Q} := \widetilde{Q}_{e_d}(0)$. It is not restrictive to assume that

$$\|v_n - \overline{u}_n\|_{L^2(\widetilde{Q})} > 0 \text{ for every } n \in \mathbb{N}$$

as, if $v_n = \overline{u}_n$ for infinitely many $n \in \mathbb{N}$, it is enough to set $w_n := \overline{u}_n$ to conclude. An analogous argument shows that we may also assume that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(v_n, \widetilde{Q}) = \lim_{n\to+\infty} F_{\varepsilon_n}(v_n, \widetilde{Q}) \le C < +\infty. \tag{8.5.4}$$

By Corollary 8.3.7, (8.5.2), and (8.5.4) there exists a constant $\delta > 0$, independent of n and $\widetilde{Q}$, such that

$$\int_{\widetilde{Q}} \Big(\frac{W(v_n)}{\varepsilon_n} + \sum_{\ell=1}^{k} \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2 \Big) \,\mathrm{d}x \le \frac{1}{\delta} F_{\varepsilon_n}(v_n, \widetilde{Q}) \le C, \tag{8.5.5}$$

$$\int_{\widetilde{Q}} \Big(\frac{W(\overline{u}_n)}{\varepsilon_n} + \sum_{\ell=1}^{k} \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} \overline{u}_n|_\ell^2 \Big) \,\mathrm{d}x \le \frac{1}{\delta} F_{\varepsilon_n}(\overline{u}_n, \widetilde{Q}) \le C, \tag{8.5.6}$$

for $n \in \mathbb{N}$ large enough depending on $\{q_\ell \colon q_\ell \le 0\}$ and $\widetilde{Q}$. Additionally, as both v_n and $\overline{u}_n$ converge to $u_0^{e_d}$ in $L^2(\widetilde{Q})$ as $n \to +\infty$, we have that

$$\int_{\widetilde{Q}} |v_n - \overline{u}_n|^2 \,\mathrm{d}x \le C \tag{8.5.7}$$

for every $n \in \mathbb{N}$.

Since the set $\widetilde{Q}$ is C^2, there exists $\rho_0 > 0$ such that for every $0 < \rho < \rho_0$ there is a unique C^1 projection from the neighbourhood $\{x \in \widetilde{Q} \colon \operatorname{dist}(x, \partial\widetilde{Q}) < \rho\}$ onto $\partial\widetilde{Q}$. In particular, for every $0 < \rho < \rho_0$ the sets $\{x \in \widetilde{Q} \colon \operatorname{dist}(x, \partial\widetilde{Q}) = \rho\}$ are $(d-1)$-dimensional manifolds in $\mathbb{R}^d$ of class C^1.

Let $0 < \rho < \rho_0/2$ and $0 < \sigma < \rho_0/2$, and let $\lceil \varepsilon_n^{-1} \rceil$ be the smallest integer larger than or equal to ε_n^{-1}, so that

$$\tfrac{\varepsilon_n}{2} \le \lceil \varepsilon_n^{-1} \rceil^{-1} \le \varepsilon_n \tag{8.5.8}$$

for $\varepsilon_n < 1$. In light of the observations above, the set of class C^1 given by

$$\{x \in \widetilde{Q} \colon \operatorname{dist}(x, \partial\widetilde{Q}) < \rho + \sigma\}$$

can be subdivided in $\lceil \varepsilon_n^{-1} \rceil$ open subsets with C^1 boundary defined as

$$L^i_{n,\sigma,\rho} \coloneqq \Big\{ x \in \widetilde{Q} \colon\ (i-1)\frac{\sigma}{\lceil \varepsilon_n^{-1} \rceil} + \rho < \operatorname{dist}(x, \partial \widetilde{Q}) < i \frac{\sigma}{\lceil \varepsilon_n^{-1} \rceil} + \rho \Big\}. \tag{8.5.9}$$

for $i \in \{1, \dots, \lceil \varepsilon_n^{-1} \rceil\}$. We observe that

$$\mathcal{L}^d(L^i_{n,\sigma,\rho}) \le C \frac{\sigma}{\lceil \varepsilon_n^{-1} \rceil}. \tag{8.5.10}$$

From (8.5.5)-(8.5.7) we deduce that

$$\begin{aligned} \sum_{i=1}^{\lceil \varepsilon_n^{-1} \rceil} \Big(\int_{L^i_{n,\sigma,\rho}} \Big(\frac{W(v_n)}{\varepsilon_n} + \sum_{\ell=1}^k \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2 \Big) \,\mathrm{d}x \\ + \int_{L^i_{n,\sigma,\rho}} \Big(\frac{W(\overline{u}_n)}{\varepsilon_n} + \sum_{\ell=1}^k \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} \overline{u}_n|_\ell^2 \Big) \,\mathrm{d}x + \int_{L^i_{n,\sigma,\rho}} \frac{|v_n - \overline{u}_n|^2}{\|v_n - \overline{u}_n\|^2_{L^2(\widetilde{Q})}} \,\mathrm{d}x \Big) \le C, \end{aligned}$$

for every n large enough. Hence, by (8.5.8) we find $i^* = i^*(n, \sigma, \rho) \in \{1, \dots, \lceil \varepsilon_n^{-1} \rceil\}$ such that

$$\begin{aligned} \Big(\int_{L^{i^*}_{n,\sigma,\rho}} \Big(\frac{W(v_n)}{\varepsilon_n} + \sum_{\ell=1}^k \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2 \Big) \,\mathrm{d}x \\ + \int_{L^{i^*}_{n,\sigma,\rho}} \Big(\frac{W(\overline{u}_n)}{\varepsilon_n} + \sum_{\ell=1}^k \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} \overline{u}_n|_\ell^2 \Big) \,\mathrm{d}x + \int_{L^{i^*}_{n,\sigma,\rho}} \frac{|v_n - \overline{u}_n|^2}{\|v_n - \overline{u}_n\|^2_{L^2(\widetilde{Q})}} \,\mathrm{d}x \Big) \le C\varepsilon_n. \end{aligned}$$

In particular, for every $\ell \in \{1, \dots, k\}$ we have

$$\varepsilon_n^{2\ell-1} \| |\nabla^{(\ell)} v_n|_\ell \|^2_{L^2(L^{i^*}_{n,\sigma,\rho})} \le C\varepsilon_n, \tag{8.5.11}$$

$$\varepsilon_n^{2\ell-1} \| |\nabla^{(\ell)} \overline{u}_n|_\ell \|^2_{L^2(L^{i^*}_{n,\sigma,\rho})} \le C\varepsilon_n, \tag{8.5.12}$$

$$\frac{1}{\varepsilon_n} \int_{L^{i^*}_{n,\sigma,\rho}} W(v_n) \,\mathrm{d}x + \frac{1}{\varepsilon_n} \int_{L^{i^*}_{n,\sigma,\rho}} W(\overline{u}_n) \,\mathrm{d}x \le C\varepsilon_n, \tag{8.5.13}$$

$$\int_{L^{i^*}_{n,\sigma,\rho}} |v_n - \overline{u}_n|^2 \,\mathrm{d}x \le C\varepsilon_n \|v_n - \overline{u}_n\|^2_{L^2(\widetilde{Q})}. \tag{8.5.14}$$

We now introduce the sets

$$\begin{aligned} A_{n,\sigma,\rho} &\coloneqq \Big\{ x \in \widetilde{Q} \colon\ \operatorname{dist}(x, \partial \widetilde{Q}) \le \rho + (i^* - 1) \tfrac{\sigma}{\lceil \varepsilon_n^{-1} \rceil} \Big\}, \\ B_{n,\sigma,\rho} &\coloneqq \Big\{ x \in \widetilde{Q} \colon\ \operatorname{dist}\big(x, \partial \widetilde{Q}\big) > \rho + i^* \tfrac{\sigma}{\lceil \varepsilon_n^{-1} \rceil} \Big\}, \end{aligned}$$

and cut-off functions $\phi_{n,\sigma,\rho} \in C_c^\infty(\widetilde{Q}; [0,1])$ such that

$$\begin{cases} \phi_{n,\sigma,\rho}(x) = 0 \quad \text{if } x \in A_{n,\sigma,\rho}, \\ \phi_{n,\sigma,\rho}(x) = 1 \quad \text{if } x \in B_{n,\sigma,\rho}, \\ \| |\nabla^{(\ell)} \phi_{n,\sigma,\rho}| \ell \|_{L^\infty(\widetilde{Q})} \le C(\sigma\varepsilon_n)^{-\ell} \qquad \text{for every } \ell \in \{1, \dots, k\}. \end{cases} \tag{8.5.15}$$

For $x \in \widetilde{Q}$ we define

$$w_n(x) = w_n^{\sigma,\rho}(x) \coloneqq \phi_{n,\sigma,\rho}(x) v_n(x) + (1 - \phi_{n,\sigma,\rho}(x)) \overline{u}_n(x), \tag{8.5.16}$$

and we observe that $w_n \in \mathrm{H}^k(\widetilde{Q})$ and that $w_n - \overline{u}_n$ is compactly supported in $\widetilde{Q}$. Also note that, since both $\{v_n\}_n$ and $\{\overline{u}_n\}_n$ converge to $u_0^{e_d}$ in $L^2(\widetilde{Q})$, for every choice of σ and ρ the sequence $\{w_n\}_n$ converges in $L^2(\widetilde{Q})$ to $u_0^{e_d}$ as well.

By (8.5.15) and (8.5.16), we have

$$\begin{aligned} F_{\varepsilon_n}(w_n, \widetilde{Q}) &= F_{\varepsilon_n}(v_n, B_{n,\sigma,\rho}) + F_{\varepsilon_n}(w_n, L^{i^*}_{n,\sigma,\rho}) + F_{\varepsilon_n}(\overline{u}_n, A_{n,\sigma,\rho}) \\ &\le F_{\varepsilon_n}(v_n, B_{n,\sigma,\rho}) + \int_{L^{i^*}_{n,\sigma,\rho}} \Big(\frac{W(w_n)}{\varepsilon_n} + \sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} w_n|_\ell^2 \Big) \,\mathrm{d}x \\ &\quad + \int_{A_{n,\sigma,\rho}} \Big(\frac{W(\overline{u}_n)}{\varepsilon_n} + \sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} \overline{u}_n|_\ell^2 \Big) \,\mathrm{d}x =: I_n^1 + I_n^2 + I_n^3. \end{aligned} \tag{8.5.17}$$

We now study the asymptotic behaviour of these three quantities.
We claim that

$$\limsup_{n\to+\infty} I_n^1 = \limsup_{n\to+\infty} F_{\varepsilon_n}(v_n, B_{n,\sigma,\rho}) \le \lim_{n\to+\infty} F_{\varepsilon_n}(v_n, \widetilde{Q}) \tag{8.5.18}$$

for every fixed σ, ρ small enough. This inequality follows from an application of Remark 8.3.6 with $\Omega = \widetilde{Q}$, $\rho^* = \rho$, and $\sigma < R^*(\rho)$. The choice of these parameters is allowed by our construction since we will conclude our proof by letting $n \to +\infty$, $\sigma \to 0$, and $\rho \to 0$, in this order.

To prove (8.5.18), we set

$$\widetilde{Q}(\tau) \coloneqq \{x \in \widetilde{Q} \colon \operatorname{dist}(x, \partial\widetilde{Q}) < \tau\},$$

for $\tau \in (\rho, \rho+\sigma)$. Then, by Remark 8.3.6, for every $\ell \in \{1, \dots, k-1\}$ we find $\varepsilon_0 = \varepsilon_0(q_\ell, \widetilde{Q}, \rho, \sigma) > 0$ such that

$$\int_{\widetilde{Q}_\tau} \Big(\frac{1}{\varepsilon_n} W(v_n) + q_\ell \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2 + \varepsilon_n^{2k-1} |\nabla^{(k)} v_n|_k^2 \Big) \,\mathrm{d}x \ge 0$$

holds for every n sufficiently large so that $\varepsilon_n \le \varepsilon_0$ and for every $\tau \in (\rho, \rho+\sigma)$. Arguing as in the proof of Corollary 8.3.7, we deduce that there exists a possibly smaller $\varepsilon_0 = \varepsilon_0(\{q_\ell \colon q_\ell \le 0\}, \widetilde{Q}, \rho, \sigma)$ such that

$$\int_{\widetilde{Q}_\tau} \Big(\frac{1}{\varepsilon_n} W(v_n) + \sum_{\ell=1}^{k} q_\ell \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2 \Big) \,\mathrm{d}x \ge 0 \tag{8.5.19}$$

for every n sufficiently large so that $\varepsilon_n \le \varepsilon_0$ and for every $\tau \in (\rho, \rho+\sigma)$. We now apply (8.5.19) with $\tau = \rho + i^* \frac{\sigma}{\lceil \varepsilon_n^{-1} \rceil}$, so that $\widetilde{Q}_\tau = \widetilde{Q} \setminus B_{n,\sigma,\rho}$, and obtain that

$$F_{\varepsilon_n}(v_n, \widetilde{Q}) = F_{\varepsilon_n}(v_n, \widetilde{Q} \setminus B_{n,\sigma,\rho}) + F_{\varepsilon_n}(v_n, B_{n,\sigma,\rho}) \ge F_{\varepsilon_n}(v_n, B_{n,\sigma,\rho}),$$

which implies (8.5.18).

We now study the term I_n^3. From Fubini's Theorem, the definition of $\overline{u}_n$, a change of variable, and (8.5.1) it follows that

$$I_n^3 \le C\mathcal{H}^{d-1}(A_{n,\sigma,\rho} \cap \Pi^\nu) \Big(\int_{-\infty}^{+\infty} [W(\overline{u}(t)) + \sum_{\ell=1}^{k} |q_\ell| |\overline{u}^{(\ell)}|^2] \,\mathrm{d}t \Big) \le C(\rho+\sigma),$$

so that

$$\limsup_{n\to+\infty} I_n^3 \le C(\rho+\sigma). \tag{8.5.20}$$

We now show that I_n^2 is asymptotically negligible. We begin by studying the term involving the double well potential. We remark that the continuity of W and (H3) imply there exists $C>0$ such that

$$W(s) \le W(t) + C \tag{8.5.21}$$

for every $s=(1-\theta)t+\theta t_0$ with $\theta\in[0,1)$ and $|t_0|\le 1$. Indeed, if $|t|\le 1$, then $|s|\le 1$, so that $W(s)\le \sup_{\tau\in[-1,1]} W(\tau)$, while if instead $|t|>1$ then $s\le (1-\theta)|t|+\theta|t_0|\le |t|$, so that (8.5.21) follows from (H3). Thus, recalling that $|\overline{u}_n|\le 1$, from (8.5.21), (8.5.13), (8.5.10), and (8.5.8) we deduce that

$$\begin{aligned}\frac{1}{\varepsilon_n}\int_{L^{i^*}_{n,\sigma,\rho}} W(w_n)\,\mathrm{d}x &\le \frac{1}{\varepsilon_n}\Big(\int_{L^{i^*}_{n,\sigma,\rho}} W(v_n)\,\mathrm{d}x + C\mathcal{L}^d(L^{i^*}_{n,\sigma,\rho})\Big)\\ &\le C\varepsilon_n + \frac{C\sigma}{\varepsilon_n\lceil\varepsilon_n^{-1}\rceil} \le C(\varepsilon_n+\sigma).\end{aligned} \tag{8.5.22}$$

We now investigate the remaining terms in I_n^2. We note that for every $\ell\in\{1,\dots,k\}$ and multi-index α of order $|\alpha|=\ell$ we have

$$\begin{aligned}\partial^\alpha w_n &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\Big(\partial^\beta\phi_{n,\sigma,\rho}\partial^{\alpha-\beta}v_n + \partial^\beta(1-\phi_{n,\sigma,\rho})\partial^{\alpha-\beta}\overline{u}_n\Big)\\ &= \phi_{n,\sigma,\rho}\partial^\alpha v_n + (1-\phi_{n,\sigma,\rho})\partial^\alpha\overline{u}_n + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\big(\partial^\beta\phi_{n,\sigma,\rho}\,\partial^{\alpha-\beta}(v_n-\overline{u}_n)\big)\end{aligned}$$

so that, by (8.5.15), we deduce that for every $\ell\in\{1,\dots,k\}$ the following inequality holds

$$\begin{aligned}|\nabla^{(\ell)}w_n|_\ell^2 \le C\Big(&|\nabla^{(\ell)}v_n|_\ell^2 + |\nabla^{(\ell)}\overline{u}_n|_\ell^2\\ &+(\sigma\varepsilon_n)^{-2\ell}|v_n-\overline{u}_n|^2 + \sum_{j=1}^{\ell-1}(\sigma\varepsilon_n)^{-2(\ell-j)}|\nabla^{(j)}(v_n-\overline{u}_n)|_j^2\Big).\end{aligned}$$

Combining this inequality with (8.5.11), (8.5.12), (8.5.14), and (8.5.8) we obtain

$$\begin{aligned}\varepsilon_n^{2\ell-1}\||\nabla^{(\ell)}w_n|_\ell\|^2_{L^2(L^{i^*}_{n,\sigma,\rho})} &\le C\varepsilon_n + C\sigma^{-2\ell}\varepsilon_n^{-1}\int_{L^{i^*}_{n,\sigma,\rho}}|v_n-\overline{u}_n|^2\,\mathrm{d}x\\ &\quad + C\sum_{j=1}^{\ell-1}\frac{\varepsilon_n^{2j-1}}{\sigma^{2(\ell-j)}}\||\nabla^{(j)}(v_n-\overline{u}_n)|_j\|^2_{L^2(L^{i^*}_{n,\sigma,\rho})}\\ &\le C\varepsilon_n + C\sigma^{-2\ell}\|v_n-\overline{u}_n\|^2_{L^2(\widetilde{Q})} + C\Big(\sum_{j=1}^{\ell-1}\frac{1}{\sigma^{2(\ell-j)}}\Big)\varepsilon_n.\end{aligned}$$

Finally, this last inequality, together with (8.5.22), yields

$$I_n^2 \le C\varepsilon_n + C\sigma + C\Big(\sum_{\ell=1}^{k}\sigma^{-2\ell}\Big)\|v_n-\overline{u}_n\|^2_{L^2(\widetilde{Q})} + C\Big(\sum_{\ell=1}^{k}\sum_{j=1}^{\ell-1}\frac{1}{\sigma^{2(\ell-j)}}\Big)\varepsilon_n. \tag{8.5.23}$$

Since $\{v_n - \overline{u}_n\}_n$ converges to 0 in $L^2(\widetilde{Q})$ as $n \to +\infty$, by (8.5.17), (8.5.18), (8.5.20), and (8.5.23) we infer that

$$\limsup_{\rho\to 0}\limsup_{\sigma\to 0}\limsup_{n\to+\infty} F_{\varepsilon_n}(w_n, \widetilde{Q}) \le \lim_{n\to+\infty} F_{\varepsilon_n}(v_n, \widetilde{Q}).$$

Recalling that for every fixed σ and ρ the sequence $\{w_n\}_n = \{w_n^{\rho,\sigma}\}_n$ converges to u_0^ν in $L^2(\widetilde{Q})$ and is such that the support of $w_n - \overline{u}_n$ is compactly contained in $\widetilde{Q}$ for all $n \in \mathbb{N}$, we let σ and ρ vary in two positive vanishing sequences and use a diagonalisation argument to conclude the proof. □

We are now ready to prove the liminf inequality. The proof is based on the blow-up argument of Fonseca and Müller [124].

Proof of Proposition 8.5.1. Let us fix a positive sequence $\{\varepsilon_n\}_n$ converging to 0 as $n \to +\infty$. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) = \lim_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) \le C < +\infty. \tag{8.5.24}$$

In light of Theorem 8.1.2, this implies that $u \in \mathrm{BV}(\Omega; \{-1, 1\})$ so that $u = 2\chi_E - 1$ for E a set of finite perimeter in Ω and $J_u \cap \Omega = \partial^* E \cap \Omega$, where we recall that $\partial^* E$ denotes the reduced boundary of E.

For every $n \in \mathbb{N}$, we consider the measure μ_n defined by

$$\mu_n(B) \coloneqq F_{\varepsilon_n}(u_n, B) \quad \text{for every } B \text{ Borel subset of } \Omega.$$

Note that by Corollary 8.3.7 and (8.5.24), upon assuming n large enough, the sequence $\{f_n \coloneqq \frac{W(u_n)}{\varepsilon_n} + \sum_{\ell=1}^k q_\ell \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} u_n|_\ell^2\}_n$ is equibounded in $L^1(\Omega)$. Therefore, letting $|M|$ denote the total variation of a measure M, we have $\sup_n |\mu_n|(\Omega) \le C$, so that, up to passing to a not relabelled subsequence, we may assume that $\{\mu_n\}_n$ and $\{|\mu_n|\}_n$ converge weakly* to two finite Radon measures μ and λ, respectively. We now show that $\mu \ge 0$. To this aim, we use the Besicovitch Derivation Theorem (see [121, Theorem 1.153]) to write for $|\mu|$-a.e. $x \in \Omega$

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) = \lim_{r\to 0^+} \frac{\mu(B_r(x))}{|\mu|(B_r(x))},$$

where $\mathrm{d}\mu/\mathrm{d}|\mu|$ is the Radon-Nikodým derivative of μ with respect to its total variation $|\mu|$. Let us fix a point x such that the previous equality holds. Since the set $\{r \in (0, +\infty)\colon \lambda(\partial B_r(x)) > 0\}$ is at most countable, we may choose a sequence of positive real numbers $\{r_m\}_m$, vanishing as $m \to +\infty$, such that

$$\frac{d\mu}{d|\mu|}(x) = \lim_{m\to+\infty}\lim_{n\to+\infty} \frac{1}{|\mu|(B_{r_m}(x))} \int_{B_{r_m}(x)} f_n(y)\mathrm{d}y.$$

For every $m \in \mathbb{N}$ fixed, by Corollary 8.3.7 the integral in the right-hand side of the previous equality is non-negative for $n \in \mathbb{N}$ large enough, hence, we conclude that $\mathrm{d}\mu/\mathrm{d}|\mu|$ is non-negative $|\mu|$-a.e. and then $\mu \ge 0$.

By the Radon-Nikodým Theorem we write $\mu = f\mathcal{H}^{d-1} \llcorner (\partial^* E \cap \Omega) + \sigma$, for f a non-negative function and for σ a non-negative measure which is singular with respect to $\mathcal{H}^{d-1} \llcorner (\partial^* E \cap \Omega)$.

We now show that the function f can be estimated from below in terms of the function g defined by (8.1.4). To see this, for every $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$ we consider $\widetilde{Q}_\nu(x)$ an open subset of $Q_\nu(x)$ having C^2 boundary and such that

$$\widetilde{Q}_\nu(x) \cap \{y \in \mathbb{R}^d \colon |y \cdot \nu| \le \tfrac{1}{4}\} = Q_\nu(x) \cap \{y \in \mathbb{R}^d \colon |y \cdot \nu| \le \tfrac{1}{4}\}. \tag{8.5.25}$$

For every $x \in \partial^* E \cap \Omega$ we consider $\nu = \nu_E(x)$ the inner unit normal to $\partial^* E \cap \Omega$ at x. By (8.5.25), using Theorem 1.3.4 one can show that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E \cap \Omega$ we have

$$\lim_{r \to 0} \frac{\mathcal{H}^{d-1}(\widetilde{Q}^\nu_r(x) \cap \partial^* E)}{r^{d-1}} = 1, \tag{8.5.26}$$

where we used the notation $\widetilde{Q}^\nu_r(x) = x + r\widetilde{Q}_\nu$. By the Besicovitch Derivation Theorem, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E \cap \Omega$ we have

$$f(x) = \lim_{r \to 0^+} \frac{\mu(\widetilde{Q}^\nu_r(x))}{\mathcal{H}^{d-1}(\widetilde{Q}^\nu_r(x) \cap \partial^* E)}, \tag{8.5.27}$$

Additionally, given $x \in \partial^* E \cap \Omega$ such that (8.5.27) holds, since $\{\mu_n\}_n$ weakly*-converges to μ, it follows from (8.5.26) and a diagonal argument that there exists a vanishing sequence $\{r_n\}_n$, with $\varepsilon_n \ll r_n$, such that

$$f(x) = \lim_{n \to +\infty} \frac{\mu_n(\widetilde{Q}^\nu_{r_n}(x))}{r_n^{d-1}}. \tag{8.5.28}$$

Upon translating the point x, it is not restrictive to assume that $x = 0 \in \partial^* E \cap \Omega$; moreover, for the sake of notation and simplicity of exposition, we only discuss the case $\nu_E(x) = e_d$, the other cases being analogous.

We write $\widetilde{Q}$ in place of $\widetilde{Q}_{e_d}(0)$ and $\widetilde{Q}_n$ in place of $\widetilde{Q}^{e_d}_{r_n}(0)$. Then, we set $v_n(y) \coloneqq u_n(r_n y)$ for every $y \in \widetilde{Q}$ and, by Theorem 1.3.4, we have that $\{v_n\}_n$ converges in $L^2(\widetilde{Q})$ to the function $u_0^{e_d}$ defined by (8.5.3). We further set $\tau_n \coloneqq \varepsilon_n / r_n$ and use the change of variables $y = x/r_n$ to obtain

$$\frac{\mu_n(\widetilde{Q}_n)}{r_n^{d-1}} = \frac{F_{\varepsilon_n}(u_n, \widetilde{Q}_n)}{r_n^{d-1}} = \int_{\widetilde{Q}} \Big(\frac{1}{\tau_n} W(v_n) + \sum_{\ell=1}^{k} q_\ell \tau_n^{2\ell-1} |\nabla^{(\ell)} v_n|^2_\ell \Big) \, \mathrm{d}y. \tag{8.5.29}$$

Since $\{v_n\}_n$ satisfies all the hypotheses of Lemma 8.5.2, we can apply such Lemma to obtain a sequence $\{w_n\}_n \subset \mathrm{H}^k(\widetilde{Q})$ which satisfies *(i)–(iii)* of said Lemma.

Noting that $\tau_n \to 0$ as $n \to +\infty$, by (8.5.28), (8.5.29) and *(iii)* of Lemma 8.5.2, it follows

$$f(0) \ge \liminf_{n \to +\infty} \int_{\widetilde{Q}} \Big(\frac{1}{\tau_n} W(w_n) + \sum_{\ell=1}^{k} q_\ell \tau_n^{2\ell-1} |\nabla^{(\ell)} w_n|^2_\ell \Big) \mathrm{d}y. \tag{8.5.30}$$

Now we observe that, by *(ii)* of Lemma 8.5.2, putting

$$w_n \coloneqq \overline{u}^{e_d}_{\tau_n} \quad \text{ on } \ Q_1^{e_d}(0) \setminus \widetilde{Q}$$

for all $n \in \mathbb{N}$, we have $\{w_n\}_n \subset \mathcal{A}^{e_d}_{\tau_n}$. Moreover, since $|\overline{u}^{e_d}(y)| = 1$ if $y \in \{z \colon |z \cdot e_d| > \frac{1}{8}\}$, it follows that $|\overline{u}^{e_d}_{\tau_n}(y)| = 1$ if $y \in \{z \colon |z \cdot e_d| > \frac{\tau_n}{8}\}$.

Therefore, by (8.5.25), we have

$$f(0) \geq \liminf_{n\to+\infty} \int_{Q_1^{e_d}(0)} \Big(\frac{1}{\tau_n} W(w_n) + \sum_{\ell=1}^{k} q_\ell \tau_n^{2\ell-1} |\nabla^{(\ell)} w_n|_\ell^2 \Big) \mathrm{d}y,$$

and by the definition of g in (8.1.4), we conclude

$$f(0) \geq g(e_d).$$

Since the above argument holds for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E \cap \Omega$ and every $\nu = \nu_E(x)$, we obtain

$$f(x) \geq g(\nu_E(x)). \tag{8.5.31}$$

Finally, by the inequality $\liminf_n \mu_n(\Omega) \geq \mu(\Omega)$ we infer that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) = \lim_{n\to+\infty} \mu_n(\Omega) \geq \mu(\Omega) = \int_{\partial^* E \cap \Omega} f(x)\, \mathrm{d}\mathcal{H}^{d-1}(x) + \sigma(\Omega)$$

and, recalling that $\sigma(\Omega) \geq 0$ and exploiting (8.5.31), this last inequality leads to

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) \geq \int_{\partial^* E \cap \Omega} f(x)\, \mathrm{d}\mathcal{H}^{d-1}(x) \geq \int_{\partial^* E \cap \Omega} g(\nu_E(x)) \mathrm{d}\mathcal{H}^{d-1}(x),$$

concluding the proof. □

Remark 8.5.3. We note that, thanks to (8.5.30), as a by-product of the above proof we have also shown that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u, \Omega) \geq \begin{cases} \displaystyle\int_{\partial^* A \cap \Omega} \overline{g}(\nu_A)\, \mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in \mathrm{BV}(\Omega; \{-1, 1\}), \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \mathrm{BV}(\Omega; \{-1, 1\}), \end{cases}$$

where

$$\overline{g}(\nu) := \inf \Big\{ \int_{\widetilde{Q}_\nu} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \Big)\, \mathrm{d}x \colon\ v \in \overline{\mathcal{A}}_\varepsilon^\nu,\ \varepsilon \in (0,1) \Big\}$$

for every $\nu \in \mathbb{S}^{d-1}$ and we have set

$$\overline{\mathcal{A}}_\varepsilon^\nu := \big\{ v \in \mathrm{H}^k(\widetilde{Q}_\nu) \colon\ v = \overline{u}_\varepsilon^\nu \text{ near } \partial \widetilde{Q}_\nu \big\}.$$

Such observation will be useful in order to prove that the Γ-limit is strictly positive.

8.6 Upper bound

In this section we conclude the proof of Theorem 8.1.3 by proving the limsup inequality. As in Section 8.5, C denotes a generic positive constant that may change from line to line, but which is independent of the relevant parameters. We now state the main result of the section.

Proposition 8.6.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set with C^1 boundary. Let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0 and let $u \in \mathrm{BV}(\Omega; \{-1, +1\})$. Assume that* (H1)–(H3) *are satisfied. Then there exists a sequence $\{u_n\}_n \subset \mathrm{H}^k(\Omega)$ converging to u in $L^2(\Omega)$ as $n \to +\infty$ and such that*

$$\limsup_{n\to+\infty} F_{\varepsilon_n}(u_n) \leq F_0(u).$$

Proof. We subdivide the proof in three parts.

Step 1. (Γ-limsup inequality for hyperplanes) First, we consider the case where Ω has Lipschitz boundary (and not necessarily C^1) and $u = 2\chi_E - 1$, where

$$E := \{x \in \Omega\colon\ (x - x_0) \cdot \nu > 0\}$$

for some $x_0 \in \Omega$ and $\nu \in \mathbb{S}^{d-1}$ such that

$$\nu_\Omega(x) \perp \nu \ \text{ for every } x \in \partial\Omega \text{ with } |(x - x_0) \cdot \nu| \text{ small enough,} \tag{8.6.1}$$

recalling that $\nu_\Omega(x)$ is the inner unit normal to Ω at x. For the sake of exposition, we assume $x_0 = 0$ and $\nu = e_d$, the other cases being analogous, so that for every $x \in \Omega$ we have $u(x) = u_0^{e_d}(x)$, where by definition $u_0^{e_d}(x) = \operatorname{sgn}(x \cdot e_d)$, and

$$F_0(u) = g(e_d)\mathcal{H}^{d-1}\left(\Omega \cap \{x_d = 0\}\right).$$

For every $\eta > 0$, there exist $0 < \varepsilon_0 < 1$ and $v \in \mathcal{A}_{\varepsilon_0}^{e_d}$ such that $v = \overline{u}_{\varepsilon_0}^{e_d}$ in a neighbourhood of $\partial Q_1^{e_d}$, and

$$\int_{Q_1^{e_d}} \Big(\frac{1}{\varepsilon_0} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon_0^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big) \mathrm{d}x \le g(e_d) + \eta. \tag{8.6.2}$$

By Proposition 8.2.1, upon rotating $Q_1^{e_d}$ leaving e_d fixed, we may assume that $Q_1^{e_d} = Q := (-\frac{1}{2}, \frac{1}{2})^d$. We then extend v to a function in $\mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}^d)$ which is periodic in the first $d-1$ variables; that is, such that

$$v(x) = v(x + e_i), \tag{8.6.3}$$

for all $i \in \{1, \dots, d-1\}$. We also set $\Omega_n := \{x \in \Omega\colon\ |x_d| \le \frac{\varepsilon_n}{2\varepsilon_0}\}$ for every $n \in \mathbb{N}$.

We claim that a recovery sequence for u is given by the sequence of functions $\{u_n\} \subset \mathrm{H}^k(\Omega)$ defined by

$$u_n(x) := \begin{cases} v\big(\frac{\varepsilon_0 x}{\varepsilon_n}\big) & \text{if } x \in \Omega_n, \\ u(x) & \text{if } x \in \Omega \setminus \Omega_n. \end{cases} \tag{8.6.4}$$

To prove the claim, we first show that

$$\|u_n - u\|_{L^2(\Omega)} \to 0 \text{ as } n \to +\infty. \tag{8.6.5}$$

To this aim, we set $\Omega' := \big\{x' \in \mathbb{R}^{d-1}\colon\ (x', 0) \in \Omega\big\}$ and observe that by (8.6.1), it holds

$$\Omega_n = \Omega' \times \Big(-\frac{\varepsilon_n}{2\varepsilon_0}, \frac{\varepsilon_n}{2\varepsilon_0}\Big)$$

for n sufficiently large.
Using the change of variables $(x', x_d) \to (x', \frac{\varepsilon_n}{\varepsilon_0} x_d)$, by Fubini's Theorem and (8.6.4) we have

$$\begin{aligned} \int_{\Omega_n} |u_n|^2 \,\mathrm{d}x &= \int_{\Omega'} \int_{-\frac{\varepsilon_n}{2\varepsilon_0}}^{\frac{\varepsilon_n}{2\varepsilon_0}} \Big|v\Big(\frac{\varepsilon_0 x'}{\varepsilon_n}, \frac{\varepsilon_0 x_d}{\varepsilon_n}\Big)\Big|^2 \mathrm{d}x_d \mathrm{d}x' \\ &= \frac{\varepsilon_n}{\varepsilon_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Omega'} \Big|v\Big(\frac{\varepsilon_0 x'}{\varepsilon_n}, x_d\Big)\Big|^2 \mathrm{d}x' \mathrm{d}x_d. \end{aligned}$$

We observe that, since $v \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}^d)$, the function $f\colon \mathbb{R}^{d-1} \to \mathbb{R}$ defined by

$$f(y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |v(y, x_d)|^2 \,\mathrm{d}x_d \quad \text{for every } y \in \mathbb{R}^{d-1}$$

is integrable on Ω' and that, by definition of v, it is periodic. We may then apply the Riemann-Lebesgue Lemma to the function f to obtain

$$\lim_{n\to+\infty}\int_{\Omega'}\int_{-\frac12}^{\frac12}\Big|v\Big(\frac{\varepsilon_0 x'}{\varepsilon_n},x_d\Big)\Big|^2\mathrm{d}x_d\,\mathrm{d}x'=\int_{\Omega'}\int_{(-\frac12,\frac12)^{d-1}}\int_{-\frac12}^{\frac12}\big|v(y',x_d)\big|^2\,\mathrm{d}x_d\mathrm{d}y'\,\mathrm{d}x'$$
$$=\mathcal{L}^{d-1}(\Omega')\|v\|^2_{L^2(Q)},\tag{8.6.6}$$

which implies

$$\int_{\Omega_n}|u_n|^2\,\mathrm{d}x\le C\frac{\varepsilon_n}{\varepsilon_0}\|v\|^2_{L^2(Q)}.\tag{8.6.7}$$

From this inequality, (8.6.4), and the fact that u takes values only in $\{\pm1\}$, we deduce that

$$\|u_n-u\|^2_{L^2(\Omega)}\le 2\int_{\Omega_n}|u_n|^2\,\mathrm{d}x+2\mathcal{L}^d(\Omega_n)\le C\frac{\varepsilon_n}{\varepsilon_0}+2\mathcal{L}^d(\Omega_n),$$

which tends to 0 as $n\to+\infty$. This inequality proves (8.6.5).

We now show that $\{u_n\}_n$ is a recovery sequence for u. To this aim, we first note that for every $x\in\Omega\setminus\Omega_n$ we have

$$u_n(x)=u(x)\in\{-1,1\}\ \text{ and }\ \nabla^{(\ell)}u_n(x)=0\ \text{ for every }\ell\in\{1,\dots,k\},$$

while, if $x\in\Omega_n$, we have

$$\nabla^{(\ell)}u_n(x)=\Big(\frac{\varepsilon_0}{\varepsilon_n}\Big)^\ell\nabla^{(\ell)}v\Big(\frac{\varepsilon_0 x}{\varepsilon_n}\Big)\quad\text{for every }\ell\in\{1,\dots,k\}.$$

Then, we use (H1), (8.6.4), Fubini's Theorem, and a change of variable to obtain

$$\begin{aligned}F_{\varepsilon_n}(u_n)&=\int_{\Omega_n}\Big(\frac{1}{\varepsilon_n}W(u_n)+\sum_{\ell=1}^k q_\ell\varepsilon_n^{2\ell-1}|\nabla^{(\ell)}u_n|^2_\ell\Big)\mathrm{d}x\\&=\int_{\Omega'}\int_{-\frac{\varepsilon_n}{2\varepsilon_0}}^{\frac{\varepsilon_n}{2\varepsilon_0}}\Big(\frac{1}{\varepsilon_n}W\Big(v\Big(\frac{\varepsilon_0x'}{\varepsilon_n},\frac{\varepsilon_0x_d}{\varepsilon_n}\Big)\Big)+\sum_{\ell=1}^k q_\ell\varepsilon_n^{2\ell}\Big|\nabla^{(\ell)}v\Big(\frac{\varepsilon_0x'}{\varepsilon_n},\frac{\varepsilon_0x_d}{\varepsilon_n}\Big)\Big|^2_\ell\Big)\mathrm{d}x_d\,\mathrm{d}x'\\&=\int_{\Omega'}\int_{-\frac12}^{\frac12}\Big(\frac{1}{\varepsilon_0}W\Big(v\Big(\frac{\varepsilon_0x'}{\varepsilon_n},x_d\Big)\Big)+\sum_{\ell=1}^k q_\ell\varepsilon_0^{2\ell-1}\Big|\nabla^{(\ell)}v\Big(\frac{\varepsilon_0x'}{\varepsilon_n},x_d\Big)\Big|^2_\ell\Big)\mathrm{d}x_d\mathrm{d}x'\end{aligned}\tag{8.6.8}$$

for every $n\in\mathbb{N}$. Since v satisfies (8.6.2), the function $h\colon\mathbb{R}^{d-1}\to\mathbb{R}^d$ defined for $y\in\mathbb{R}^{d-1}$ by

$$h(y):=\int_{-\frac12}^{\frac12}\Big(\frac{1}{\varepsilon_0}W(v(y',x_d))+\sum_{\ell=1}^k q_\ell\varepsilon_0^{2\ell-1}|\nabla^{(\ell)}v(y',x_d)|^2_\ell\Big)\mathrm{d}x_d$$

is integrable and, by (8.6.3), is also periodic. Therefore, we may apply once again the Riemann-Lebesgue Lemma to obtain

$$\lim_{n\to+\infty}\int_{\Omega'}\int_{-\frac12}^{\frac12}\Big(\frac{1}{\varepsilon_0}W\Big(v\Big(\frac{\varepsilon_0x'}{\varepsilon_n},x_d\Big)\Big)+\sum_{\ell=1}^k q_\ell\varepsilon_0^{2\ell-1}\Big|\nabla^{(\ell)}v\Big(\frac{\varepsilon_0x'}{\varepsilon_n},x_d\Big)\Big|^2_\ell\Big)\mathrm{d}x_d\,\mathrm{d}x'$$
$$=\mathcal{L}^{d-1}(\Omega')\int_Q\Big(\frac{1}{\varepsilon_0}W(v)+\sum_{\ell=1}^k q_\ell\varepsilon_0^{2\ell-1}|\nabla^{(\ell)}v|^2_\ell\Big)\mathrm{d}x.$$

Combining this equality with (8.6.8) and (8.6.2), we conclude that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = \mathcal{L}^{d-1}(\Omega') \int_Q \Big(\frac{1}{\varepsilon_0} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon_0^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big) \mathrm{d}x \le F_0(u) + \mathcal{L}^{d-1}(\Omega')\eta.$$

Since η was arbitrary, a diagonal argument allows to prove the thesis in this special case.

Step 2. (Γ-limsup inequality for polygonal sets) We now assume that $u = 2\chi_E - 1$, with $E = P \cap \Omega$ and P polyhedral. This means that $\partial P = \bigcup_{i=1}^M H_i \cup F$, where the sets $\{H_i\}_{i=1}^M$ are pairwise disjoint and relatively open convex polyhedra of dimension $d-1$, while F is the union of a finite number of convex polyhedra of dimension $d-2$. In particular, there exists $\nu_1, \dots, \nu_M \in \mathbb{S}^{d-1}$ and $x_1, \dots, x_M \in \mathbb{R}^d$ such that

$$H_i \subset \{x \in \mathbb{R}^d \colon\ (x - x_i) \cdot \nu_i = 0\},$$

and each ν_i is the inner unit normal to ∂P on H_i. For simplicity, we may also assume that $\partial\Omega \cap \partial P$ is the union of a finite number of C^1 manifolds of dimension $d-2$.

Let us fix $0 < \sigma < 1$ and let $H_1', \dots, H_M'$ relatively open subsets of $H_1, \dots, H_M$ with $(d-2)$-dimensional boundary of class C^1, such that

$$\big\{x \in H_i \cap \Omega \colon\ \mathrm{dist}(x, \partial\Omega \cup F) \ge \tfrac{\sigma}{2}\big\} \subset H_i' \subset \overline{H_i'} \subset H_i.$$

Fix $0 < \eta < \frac{\sigma}{2}$ and define for each $i \in \{1, \dots, M\}$

$$\Omega_i \coloneqq \{x + t\nu_i \colon\ x \in H_i',\, t \in (-\eta, \eta)\}.$$

We can choose η so small that the sets $\Omega_1, \dots, \Omega_M$ are pairwise disjoint. Since each Ω_i, satisfies (8.6.1), by the previous step, we obtain sequences $\{u_n^i\}_n \subset \mathrm{H}^k(\Omega_i)$ such that

$$u_n^i \to u \text{ in } L^2(\Omega_i), \tag{8.6.9}$$

$$u_n^i(x) = u(x) \ \text{ for every } x \in \big\{x \in \overline{\Omega}_i \colon\ \mathrm{dist}(x, H_i) \ge \tfrac{\varepsilon_n}{2\varepsilon_0^i}\big\},$$

and

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n^i, \Omega_i) \le (g(\nu_i) + \eta)\, \mathcal{H}^{d-1}(\Omega_i \cap H_i), \tag{8.6.10}$$

where for every $x \in \Omega_i$ it holds

$$u_n^i(x) \coloneqq \begin{cases} v^i\big(\frac{\varepsilon_0^i (x - x_i)}{\varepsilon_n}\big) & \text{if } |(x - x_i)\cdot\nu_i| \le \frac{\varepsilon_n}{2\varepsilon_0^i}, \\ u(x) & \text{if } |(x - x_i)\cdot\nu_i| > \frac{\varepsilon_n}{2\varepsilon_0^i}, \end{cases}$$

and, for each $i \in \{1, \dots, M\}$, v^i and ε_0^i are such that

$$\int_{Q_1^{\nu_i}} \Big(\frac{1}{\varepsilon_0^i} W(v) + \sum_{\ell=1}^{k} q_\ell (\varepsilon_0^i)^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big) \mathrm{d}x \le g(\nu_i) + \eta.$$

In order to present the candidate recovery sequence, we let $\{\Psi_\varepsilon\}_{\varepsilon>0}$ be a radial smooth convolution kernel, with $\mathrm{supp}\Psi_\varepsilon \subset B_\varepsilon(0)$ and $\int \Psi_\varepsilon \,\mathrm{d}x = 1$ for every $\varepsilon > 0$. We extend u to the whole $\mathbb{R}^d$ by setting $u \coloneqq 2\chi_P - 1$ and introduce smooth functions $\widetilde{u}_n$ defined for every $n \in \mathbb{N}$ by

$$\widetilde{u}_n \coloneqq u * \Psi_{\varepsilon_n},$$

where $*$ denotes the operation of convolution. Observe that these functions satisfy the following conditions

$$\widetilde{u}_n(x) = u(x) \text{ for every } x \in \{x \in \Omega\colon \operatorname{dist}(x, \partial P) \geq \varepsilon_n\}, \tag{8.6.11}$$

$$\|\widetilde{u}_n\|_{L^\infty(\mathbb{R}^d)} \leq 1 \quad \text{and} \quad \||\nabla^{(\ell)}\widetilde{u}_n|_\ell\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\varepsilon^\ell} \text{ for every } \ell \in \{1, \dots, k\}. \tag{8.6.12}$$

For $\tau \in (0,1)$ we set

$$U(\tau) \coloneqq \{x \in \Omega\colon \operatorname{dist}(x, F \cup (\partial\Omega \cap \partial P)) \leq \tau\}$$

and choose cut-off functions $\phi_\sigma \in C_c^\infty(\mathbb{R}^d; [0,1])$ such that

$$\begin{cases} \phi_\sigma(x) = 1 & \text{if } x \in U(\sigma), \\ \phi_\sigma(x) = 0 & \text{if } x \in \mathbb{R}^d \setminus U(2\sigma), \\ \||\nabla^{(\ell)}\phi_\sigma|_\ell\|_{L^\infty(\mathbb{R}^d)} \leq C\sigma^{-\ell} & \text{for every } \ell \in \{1, \dots, k\}. \end{cases} \tag{8.6.13}$$

We claim that the sequence of functions $\{u_n\}_n \subset \mathrm{H}^k(\Omega)$ defined by

$$u_n \coloneqq \begin{cases} (1-\phi_\sigma)u_n^i + \phi_\sigma\widetilde{u}_n & \text{in } \overline{\Omega}_i \text{ for } \ i \in \{1, \dots, M\}, \\ \widetilde{u}_n & \text{in } \Omega' \coloneqq \Omega \setminus \cup_{i=1}^M \overline{\Omega}_i \end{cases}$$

is a recovery sequence for u.

The fact that $\{u_n\}_n \subset \mathrm{H}^k(\Omega)$ follows from the same argument used in [75, Theorem 1.3, Substep 2B]. Also, (8.6.9) and the fact that $\widetilde{u}_n \to u$ in $L^2(\Omega)$ imply that $u_n \to u$ in $L^2(\Omega)$ as $n \to +\infty$.

We now study the asymptotic behaviour of the energy of $\{u_n\}_n$. First, we show that $F_{\varepsilon_n}(u_n, \Omega')$ is negligible. To this aim, for every $n \in \mathbb{N}$ we set

$$\mathcal{R}_n \coloneqq \Big\{x \in \Omega\colon \operatorname{dist}(x, \partial P) \leq \max\big\{\varepsilon_n, \frac{\varepsilon_n}{2\varepsilon_0^1}, \dots, \frac{\varepsilon_n}{2\varepsilon_0^M}\big\}\Big\}. \tag{8.6.14}$$

We observe that $\Omega' \cap \mathcal{R}_n \subset U(\sigma)$ for n large enough and that $\mathcal{H}^{d-1}(\partial P \cap U(\sigma)) \leq C\sigma$. Hence, by the definition of Ω', (8.6.11), and (8.6.12), we have

$$F_{\varepsilon_n}(u_n, \Omega') = \int_{\Omega' \cap \mathcal{R}_n} \Big(\frac{1}{\varepsilon_n} W(\widetilde{u}_n) + \sum_{\ell=1}^k |q_\ell| \varepsilon_n^{2\ell-1} |\nabla^{(\ell)}\widetilde{u}_n|_\ell^2\Big) \mathrm{d}x \leq C\sigma. \tag{8.6.15}$$

We now study the energetic contribution of the remaining terms. We fix $i \in \{1, \dots, M\}$ and compute

$$\begin{aligned} F_{\varepsilon_n}(u_n,\Omega_i) &= F_{\varepsilon_n}(u_n^i, \Omega_i \setminus U(2\sigma)) + F_{\varepsilon_n}(u_n, \Omega_i \cap (U(2\sigma) \setminus U(\sigma))) + F_{\varepsilon_n}(\widetilde{u}_n, \Omega_i \cap U(\sigma)) \\ &\leq F_{\varepsilon_n}(u_n^i, \Omega_i \setminus U(2\sigma)) + \int_{\Omega_i \cap (U(2\sigma) \setminus U(\sigma))} \Big(\frac{1}{\varepsilon_n} W(u_n) + \sum_{\ell=1}^k |q_\ell| \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} u_n|_\ell^2\Big) \,\mathrm{d}x \\ &\quad + \int_{\Omega_i \cap U(\sigma)} \Big(\frac{1}{\varepsilon_n} W(\widetilde{u}_n) + \sum_{\ell=1}^k |q_\ell| \varepsilon_n^{2\ell-1} |\nabla^{(\ell)}\widetilde{u}_n|_\ell^2\Big) \,\mathrm{d}x \\ &\eqqcolon \mathcal{K}_n^1 + \mathcal{K}_n^2 + \mathcal{K}_n^3. \end{aligned}$$

We separately investigate the three terms $\mathcal{K}_n^1$, $\mathcal{K}_n^2$, and $\mathcal{K}_n^3$, starting from $\mathcal{K}_n^1$. Recalling that $q_k = 1$, we observe that

$$\begin{aligned}\mathcal{K}_n^1 &= \int_{\Omega_i\setminus U(2\sigma)} \Big(\frac{1}{\varepsilon_n} W(u_n^i) + \varepsilon_n^{2k-1}|\nabla^{(k)} u_n^i|_k^2\Big)\,\mathrm{d}x + \int_{\Omega_i\setminus U(2\sigma)} \Big(\sum_{\ell=1}^{k-1} q_\ell \varepsilon_n^{2\ell-1}|\nabla^{(\ell)} u_n^i|_\ell^2\Big)\,\mathrm{d}x\\ &\le \int_{\Omega_i} \Big(\frac{1}{\varepsilon_n} W(u_n^i) + \varepsilon_n^{2k-1}|\nabla^{(k)} u_n^i|_k^2\Big)\,\mathrm{d}x + \int_{\Omega_i} \Big(\sum_{\ell=1}^{k-1} q_\ell \varepsilon_n^{2\ell-1}|\nabla^{(\ell)} u_n^i|_\ell^2\Big)\,\mathrm{d}x\\ &\quad - \int_{\Omega_i\cap U(2\sigma)} \Big(\sum_{\ell=1}^{k-1} q_\ell \varepsilon_n^{2\ell-1}|\nabla^{(\ell)} u_n^i|_\ell^2\Big)\,\mathrm{d}x\\ &\le F_{\varepsilon_n}(u_n^i, \Omega_i) + \int_{\Omega_i\cap U(2\sigma)} \Big(\sum_{\ell=1}^{k-1} |q_\ell| \varepsilon_n^{2\ell-1}|\nabla^{(\ell)} u_n^i|_\ell^2\Big)\,\mathrm{d}x. \end{aligned} \tag{8.6.16}$$

By the periodicity of u_n^i with respect to the variables tangential to H_i, we may argue as in *Step 1*, using Fubini's Theorem and the Riemann-Lebesgue Lemma to deduce that

$$\lim_{n\to+\infty} \int_{\Omega_i\cap U(2\sigma)} \Big(\sum_{\ell=1}^{k-1} |q_\ell| \varepsilon_n^{2\ell-1}|\nabla^{(\ell)} u_n^i|_\ell^2\Big)\,\mathrm{d}x \le C\sigma \int_{Q_1^{\nu_i}} \Big(\sum_{\ell=1}^{k-1} |\nabla^{(\ell)} v^i|_\ell^2\Big)\,\mathrm{d}x \le C\sigma.$$

Hence, it follows from (8.6.16) and (8.6.10) that

$$\limsup_{n\to+\infty} \mathcal{K}_n^1 \le (g(\nu_i) + \eta)\mathcal{H}^{d-1}(H_i \cap \Omega_i) + C\sigma. \tag{8.6.17}$$

We now investigate the term $\mathcal{K}_n^2$. We begin observing that for every $\ell \in \{1, \dots, k\}$ and multi-index α of order $|\alpha| = \ell$, we have

$$\begin{aligned}\partial^\alpha u_n &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \Big(\partial^\beta(1-\phi_\sigma)\partial^{\alpha-\beta} u_n^i + \partial^\beta \phi_\sigma \partial^{\alpha-\beta}\widetilde{u}_n\Big)\\ &= (1-\phi_\sigma)\partial^\alpha u_n^i + \phi_\sigma \partial^\alpha \widetilde{u}_n + \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \big(\partial^\beta \phi_\sigma\, \partial^{\alpha-\beta}(\widetilde{u}_n - u_n^i)\big),\end{aligned}$$

which by (8.6.13) implies

$$\begin{aligned}|\nabla^{(\ell)} u_n|_\ell^2 \le C\Big(&|\nabla^{(\ell)} u_n^i|_\ell^2 + |\nabla^{(\ell)}\widetilde{u}_n|_\ell^2\\ &+\sigma^{-2\ell}|\widetilde{u}_n - u_n^i|^2 + \sum_{j=1}^{\ell-1} \sigma^{-2(\ell-j)}|\nabla^{(j)}(\widetilde{u}_n - u_n^i)|_j^2\Big).\end{aligned} \tag{8.6.18}$$

Since by (H3) we have that

$$W(u_n) \le W(u_n^i) + C, \tag{8.6.19}$$

it follows from (8.6.14), (8.6.18), and (8.6.19) that

$$\begin{aligned}\mathcal{K}_n^2 \le C \int_{\Omega_i\cap(U(2\sigma)\setminus U(\sigma))\cap\mathcal{R}_n} \Big(&\frac{1}{\varepsilon_n} W(u_n^i) + \frac{1}{\varepsilon_n}\\ &+ \sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1}\Big(|\nabla^{(\ell)} u_n^i|_\ell^2 + \sum_{j=1}^{\ell-1} \sigma^{-2(\ell-j)}|\nabla^{(j)} u_n^i|_j^2\Big)\Big)\,\mathrm{d}x\end{aligned}$$

$$+ C \int_{\Omega_i \cap (U(2\sigma)\setminus U(\sigma)) \cap \mathcal{R}_n} \Big(\sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1} \Big(|\nabla^{(\ell)} \widetilde{u}_n|_\ell^2 + \sigma^{-2\ell} |u_n^i - \widetilde{u}_n|^2 + \sum_{j=1}^{\ell-1} \sigma^{-2(\ell-j)} |\nabla^{(j)} \widetilde{u}_n|_j^2 \Big) \Big) \, \mathrm{d}x.$$

Arguing as in (8.6.6) and (8.6.8), we obtain that

$$\begin{aligned} \lim_{n\to+\infty} \int_{\Omega_i \cap (U(2\sigma)\setminus U(\sigma)) \cap \mathcal{R}_n} \Big(& \frac{1}{\varepsilon_n} W(u_n^i) + \frac{1}{\varepsilon_n} \\ & + \sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1} \Big(|\nabla^{(\ell)} u_n^i|_\ell^2 + \sum_{j=1}^{\ell-1} \sigma^{-2(\ell-j)} |\nabla^{(j)} u_n^i|_j^2 \Big) \Big) \, \mathrm{d}x \\ & \le C \mathcal{H}^{d-1}(H_i \cap U(2\sigma)) \le C\sigma. \end{aligned} \tag{8.6.20}$$

Then, we observe that using (8.6.7), with u_n replaced by u_n^i, v replaced by v^i, and Ω replaced by Ω_i, together with (8.6.12), we have

$$\begin{aligned} & \int_{\Omega_i \cap (U(2\sigma)\setminus U(\sigma)) \cap \mathcal{R}_n} \Big(\sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1} \Big(|\nabla^{(\ell)} \widetilde{u}_n|_\ell^2 + \sigma^{-2\ell} |\widetilde{u}_n - u_n^i|^2 + \sum_{j=1}^{\ell-1} \sigma^{-2(\ell-j)} |\nabla^{(j)} \widetilde{u}_n|_j^2 \Big) \Big) \, \mathrm{d}x \\ & \quad \le \frac{C}{\varepsilon_n} \Big(\sum_{\ell=1}^{k} \Big(1 + \sum_{j=1}^{\ell-1} \frac{\varepsilon_n^{2(\ell-j)}}{\sigma^{2(\ell-j)}} \Big) \Big) \mathcal{L}^d \big(\Omega_i \cap (U(2\sigma) \setminus U(\sigma)) \cap \mathcal{R}_n \big) \\ & \qquad + C \sum_{\ell=1}^{k} \Big(\frac{\varepsilon_n^{2\ell}}{\sigma^{2\ell}} \|v^i\|_{L^2(\Omega_i)}^2 + \frac{\varepsilon_n^{2\ell-1}}{\sigma^{2\ell}} \mathcal{L}^d \big(\Omega_i \cap (U(2\sigma) \setminus U(\sigma)) \cap \mathcal{R}_n \big) \Big) \\ & \quad \le C \Big(\sum_{\ell=1}^{k} \Big(1 + \sum_{j=1}^{\ell-1} \frac{\varepsilon_n^{2(\ell-j)}}{\sigma^{2(\ell-j)}} \Big) \Big) \sigma + C \sum_{\ell=1}^{k} \Big(\frac{\varepsilon_n^{2\ell}}{\sigma^{2\ell}} + \frac{\varepsilon_n^{2\ell}}{\sigma^{2\ell-1}} \Big). \end{aligned} \tag{8.6.21}$$

As

$$\sum_{\ell=1}^{k} \sum_{j=1}^{\ell-1} \frac{\varepsilon_n^{2(\ell-j)}}{\sigma^{2(\ell-j)}} + \sum_{\ell=1}^{k} \Big(\frac{\varepsilon_n^{2\ell}}{\sigma^{2\ell}} + \frac{\varepsilon_n^{2\ell}}{\sigma^{2\ell-1}} \Big) \longrightarrow 0 \quad \text{for } n \to +\infty,$$

combining (8.6.20) with (8.6.21), we obtain that

$$\limsup_{n\to+\infty} \mathcal{K}_n^2 \le C\sigma. \tag{8.6.22}$$

To conclude, we use (8.6.12) to estimate the remaining term $\mathcal{K}_n^3$ as

$$\begin{aligned} \mathcal{K}_n^3 & \le \int_{U(\sigma) \cap \{x:\ |(x-x_i)\cdot\nu_i| \le \varepsilon_n\}} \Big(\frac{1}{\varepsilon_n} W(\widetilde{u}_n) + \sum_{\ell=1}^{k} |q_\ell| \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} \widetilde{u}_n|_\ell^2 \Big) \, \mathrm{d}x \\ & \le \frac{C}{\varepsilon_n} \mathcal{L}^d \left(\{ x \in U(\sigma) :\ |(x-x_i)\cdot\nu_i| \le \varepsilon_n \} \right) \le C \mathcal{H}^{d-1}(U(\sigma) \cap H_i) \le C\sigma. \end{aligned} \tag{8.6.23}$$

Finally, gathering (8.6.15), (8.6.17),(8.6.22), (8.6.23), and summing over $i \in \{1, \dots, M\}$ we

obtain that

$$\begin{aligned}\limsup_{n\to+\infty} F_{\varepsilon_n}(u_n,\Omega) &\le \sum_{i=1}^{M} \limsup_{n\to+\infty} F_{\varepsilon_n}(u_n,\Omega_i) + \limsup_{n\to+\infty} F_{\varepsilon_n}(u_n,\Omega') \\ &= \sum_{i=1}^{M} (g(\nu_i)+\eta)\mathcal{H}^{d-1}(\Omega_i\cap H_i) + C\sigma \le F_0(u) + C(\eta+\sigma).\end{aligned}$$

Since $\eta>0$ and $\sigma>0$ were arbitrary, a diagonal argument allows to conclude.

Step 3. (Conclusion of the proof) Finally, we deal with the general case $u=2\chi_E-1$, for $E\subset\Omega$ a set of relatively finite perimeter in Ω. By Theorem 1.3.5, we can find a sequence of polyhedral sets $\{E_n\}_n$ such that $\chi_{E_n}\to\chi_E$ in $L^1(\Omega)$. By the previous step, setting $u_n := 2\chi_{E_n}-1$,

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u_n) \le F_0(u_n) \text{ for every } n\in\mathbb{N},$$

where we recall that the Γ-limsup is the functional defined in Section 1.2. By the lower semicontinuity of the Γ-limsup we then infer

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u) \le \liminf_{n\to\infty} \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u_n) \le \liminf_{n\to\infty} F_0(u_n).$$

Since g is upper semicontinous on $\mathbb{S}^{d-1}$ (see Proposition 8.2.1 above), by [88, Remark 9.11, Theorem 9.15] there exists a sequence $\{g_m\}_m$ of Lipschitz continuous functions such that $g\le g_{m+1}\le g_m\le \max_{\mathbb{S}^{d-1}} g$ and

$$g(\nu) = \inf_{m\in\mathbb{N}} g_m(\nu) \quad \text{for every } \nu\in\mathbb{S}^{d-1}.$$

Using again Theorem 1.3.5 and (1.3.5), we obtain for every $m\in\mathbb{N}$ that

$$\lim_{n\to+\infty}\int_{\partial^* E_n\cap\Omega} g_m(\nu_{E_n}(x))d\mathcal{H}^{d-1}(x) = \int_{\partial^* E\cap\Omega} g_m(\nu_E(x))d\mathcal{H}^{d-1}(x)$$

so that by monotonicity

$$\begin{aligned}\liminf_{n\to\infty} F_0(u_n) &= \liminf_{n\to\infty}\int_{\partial^* E_n\cap\Omega} g(\nu_{E_n}(x))d\mathcal{H}^{d-1}(x) \\ &\le \lim_{n\to\infty}\int_{\partial^* E_n\cap\Omega} g_m(\nu_{E_n}(x))d\mathcal{H}^{d-1}(x) \\ &= \int_{\partial^* E\cap\Omega} g_m(\nu_E(x))d\mathcal{H}^{d-1}(x).\end{aligned}$$

Finally, by Monotone Convergence we obtain that

$$\begin{aligned}\liminf_{n\to\infty} F_0(u_n) &\le \lim_{m\to\infty}\int_{\partial^* E\cap\Omega} g_m(\nu_E(x))d\mathcal{H}^{d-1}(x) \\ &\le \int_{\partial^* E\cap\Omega} g(\nu_E(x))d\mathcal{H}^{d-1}(x) = F_0(u),\end{aligned}$$

which concludes the proof. □

We conclude this section proving that the surface energy density g is strictly positive. This observation makes the Γ-limit non-degenerate.

Proposition 8.6.2. *Let g be the function defined in* (8.1.4) *with $k > 1$ an integer and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\}\colon\ q_\ell \le 0\}$. Assume that* (H1)–(H3) *are satisfied, and, if $\mathcal{N} \neq \emptyset$, further assume that $q_\ell > -\overline{q}_\ell$ for all $\ell \in \mathcal{N}$, where $\{\overline{q}_\ell\colon\ \ell \in \mathcal{N}\}$ denote the same positive constants appearing in Corollary 8.3.7. Then* $\inf_{\mathbb{S}^{d-1}} g > 0$.

Proof. Similarly to Section 5, we introduce $\widetilde{Q}_\nu$ an open set with C^2 boundary contained in Q^ν such that

$$\widetilde{Q}^\nu \cap \{y \in \mathbb{R}^d\colon\ |y \cdot \nu| \le \tfrac{1}{4}\} = Q^\nu \cap \{y \in \mathbb{R}^d\colon\ |y \cdot \nu| \le \tfrac{1}{4}\}$$

and, letting $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ denote an orthonormal basis describing Q^ν, with the additional property that

$$\widetilde{Q}^\nu \cap \bigcap_{i=1}^{d-1} \{y \in \mathbb{R}^d\colon\ |y \cdot \nu_i| \le \tfrac{1}{4}\} = Q^\nu \cap \bigcap_{i=1}^{d-1} \{y \in \mathbb{R}^d\colon\ |y \cdot \nu_i| \le \tfrac{1}{4}\} =: R^\nu. \tag{8.6.24}$$

Recalling the notation in Remark 8.5.3, we note that the function $\overline{g}\colon \mathbb{S}^{d-1} \to \mathbb{R}$ given by

$$\overline{g}(\nu) := \inf\Big\{ \int_{\widetilde{Q}_\nu} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2\Big)\, \mathrm{d}x\colon\ v \in \overline{\mathcal{A}}_\varepsilon^\nu, \varepsilon \in (0,1)\Big\}$$

satisfies $\overline{g}(\nu) \ge g(\nu)$ for every $\nu \in \mathbb{S}^{d-1}$. We combine this observation with Remark 8.5.3 and Proposition 8.6.1 to infer that

$$\begin{aligned}
\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,\Omega) &\ge \begin{cases} \displaystyle\int_{\partial^* A \cap \Omega} \overline{g}(\nu_A)\, \mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in \mathrm{BV}(\Omega;\{-1,1\}),\ \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \mathrm{BV}(\Omega;\{-1,1\}), \end{cases} \\
&\ge \begin{cases} \displaystyle\int_{\partial^* A \cap \Omega} g(\nu_A)\, \mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in \mathrm{BV}(\Omega;\{-1,1\}),\ \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \mathrm{BV}(\Omega;\{-1,1\}) \end{cases} \\
&\ge \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u,\Omega),
\end{aligned}$$

which, in turn, implies that $\overline{g}(\nu) = g(\nu)$ for every $\nu \in \mathbb{S}^{d-1}$. For this reason, we equivalently prove that $\inf_{\mathbb{S}^{d-1}} \overline{g} > 0$.

Fix $\nu \in \mathbb{S}^{d-1}$, and let $\{\varepsilon_n\}_n$, $v_n \in \overline{\mathcal{A}}_{\varepsilon_n}^\nu$, $n \in \mathbb{N}$ such that

$$\overline{g}(\nu) = \lim_{n\to+\infty} \int_{\widetilde{Q}_\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \sum_{\ell=1}^{k} q_\ell \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2\Big)\, \mathrm{d}x.$$

Arguing as in Remark 8.2.2, it is not restrictive to suppose $\varepsilon_n \to 0$ and moreover, we may further assume that for all $n \in \mathbb{N}$ it holds $\varepsilon_n < \varepsilon_0(\{q_\ell\colon\ \ell \in \mathcal{N}\}, \widetilde{Q}_\nu)$, the constant appearing in Corollary 8.3.7.

Clearly, the set R^ν defined in (8.6.24) is contained in $\widetilde{Q}_\nu$; then, we apply Corollary 8.3.7 with $\Omega = \widetilde{Q}_\nu$ and, by the equivalence of the norms, we obtain that

$$\begin{aligned}
\int_{\widetilde{Q}_\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \sum_{\ell=1}^{k} q_\ell \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2\Big)\, \mathrm{d}x &\ge \delta \int_{\widetilde{Q}_\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \sum_{\ell=1}^{k} \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2\Big)\, \mathrm{d}x \\
&\ge \delta \int_{R^\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \sum_{\ell=1}^{k} \varepsilon_n^{2\ell-1} |\nabla^{(\ell)} v_n|_\ell^2\Big)\, \mathrm{d}x \\
&\ge \delta' \int_{R^\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \varepsilon_n^{2k-1} \|\nabla^{(k)} v_n\|_k^2\Big)\, \mathrm{d}x
\end{aligned}$$

for every $n \in \mathbb{N}$ and some $\delta' > 0$ which is independent of $\widetilde{Q}_\nu$. Applying Fubini's Theorem, inequality

$$\|\nabla^{(k)} v_n(x)\|_k \geq \Big|\frac{d^k (v_n)_y^\nu}{dt^k}(t)\Big|, \qquad \text{for } \mathcal{L}^d\text{-a.e. } x \in \widetilde{Q}_\nu,$$

and a change of variable, we obtain

$$\begin{aligned}
&\int_{R^\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \varepsilon_n^{2k-1} \|\nabla^{(k)} v_n\|_k^2\Big)\,\mathrm{d}x \\
&\qquad \geq \int_{R^\nu \cap \Pi^\nu} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\varepsilon_n} W((v_n)_y^\nu) + \varepsilon_n^{2k-1} |((v_n)_y^\nu)^{(k)}|^2\Big)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\qquad = \int_{R^\nu \cap \Pi^\nu} \int_{-\frac{1}{2\varepsilon_n}}^{\frac{1}{2\varepsilon_n}} \Big(W\big((w_n)_y^\nu(t)\big) + |((w_n)_y^\nu)^{(k)}(t)|^2\Big)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\qquad \geq \frac{1}{2^{d-1}} \inf\Big\{ \int_{-\frac{1}{2\varepsilon_n}}^{\frac{1}{2\varepsilon_n}} \Big(W(w) + |w^{(k)}|^2\Big)\mathrm{d}t, w \in \mathrm{H}^k\big((-\frac{1}{2\varepsilon_n}, \frac{1}{2\varepsilon_n})\big), \\
&\qquad\qquad w(t) = \mathrm{sgn}(t) \text{ if } |t| > M \text{ for some } M \in \Big(0, \frac{1}{2\varepsilon_n}\Big)\Big\},
\end{aligned}$$

where we set $(w_n)_y^\nu(t) \coloneqq (v_n)_y^\nu(\varepsilon_n t)$ and in the last inequality we used that $v_n \in \overline{\mathcal{A}}_{\varepsilon_n}^\nu$ for all $n \in \mathbb{N}$. As (H1) and (H2) are satisfied, we are in position to apply Propositions 4.1 and 4.2 in [39], deducing that

$$\overline{g}(\nu) \geq \liminf_{n\to+\infty} \delta' \int_{R^\nu} \Big(\frac{1}{\varepsilon_n} W(v_n) + \varepsilon_n^{2k-1} \|\nabla^{(k)} v_n\|_k^2\Big)\,\mathrm{d}x \geq \frac{\delta'}{2^{d-1}} m_k > 0,$$

m_k being as in (7.1.2), which concludes the proof. □

8.7 Analysis of the cell problem in dimension one

In this final section we analyze the surface energy density g defined in (8.1.4) when $d = 1$. In this instance, this density is constant, and we show that it is given by an optimal-profile problem on the real line. It is not restrictive to assume that $|\cdot|_\ell$ coincides with the absolute value for every $\ell \in \{1, \dots, k\}$ as any norm on $\mathbb{R}$ is a multiple of the Euclidean one.

By Remark 8.2.2 and by a change of variable, we find that

$$\begin{aligned}
g &= \inf_{T>\frac{1}{2}} \inf\Big\{ \int_{-T}^{T} \Big(W(v) + \sum_{\ell=1}^{k} q_\ell |v^{(\ell)}|^2\Big)\,\mathrm{d}t\colon \ \ v \in \mathrm{H}^k((-T,T)), \\
&\qquad\qquad v(t) = \mathrm{sgn}(t) \text{ if } |t| > M \text{ for some } M \in (0,T)\Big\} \\
&= \lim_{T\to+\infty} \inf\Big\{ \int_{-T}^{T} \Big(W(v) + \sum_{\ell=1}^{k} q_\ell |v^{(\ell)}|^2\Big)\,\mathrm{d}t\colon \ \ v \in \mathrm{H}^k((-T,T)), \\
&\qquad\qquad v(t) = \mathrm{sgn}(t) \text{ if } |t| > M \text{ for some } M \in (0,T)\Big\},
\end{aligned}$$

hence, for $T > 1/2$, we set

$$\begin{aligned}
m(T) \coloneqq \inf\Big\{ &\int_{-T}^{T} \Big(W(v) + \sum_{\ell=1}^{k} q_\ell |v^{(\ell)}|^2\Big)\,\mathrm{d}x,\ v \in \mathrm{H}^k((-T,T)), \\
&v(t) = \mathrm{sgn}(t) \text{ if } |t| > M \text{ for some } M \in (0,T)\Big\},
\end{aligned}$$

in such a way that

$$g = \lim_{T\to+\infty} m(T),$$

and we put

$$m := \inf\Big\{\int_{-\infty}^{+\infty}\Big(W(u)+\sum_{\ell=1}^{k} q_\ell|u^{(\ell)}|^2\Big)\,\mathrm{d}t\colon\ u\in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R}),\ \lim_{t\to\pm\infty} u(t)=\pm1\Big\}. \tag{8.7.1}$$

The main result of this section is that, under the by now usual, suitable assumptions on the coefficients $q_1,\dots,q_{k-1}$, the equality $g=m$ holds. In order to prove this, we state the one-dimensional version of Corollary 8.3.7 at scale $\varepsilon=1$.

Corollary 8.7.1. *Let $I\subset\mathbb{R}$ be an open interval of length 1, let $k>1$ be an integer, and let $\mathcal{N} := \{\ell\in\{1,\dots,k-1\}\colon q_\ell\le 0\}$. Assume that* (H2) *is satisfied, and, if $\mathcal{N}\neq\emptyset$, further assume that $q_\ell > -\overline{q}'_\ell$ for all $\ell\in\mathcal{N}$, where, letting $\{q'_\ell\colon \ell\in\mathcal{N}\}$ denote the same positive constants appearing in Corollary 8.3.3 and given $\{\alpha_\ell\colon \ell\in\mathcal{N}\}$ such that $\alpha_\ell>0$ for all $\ell\in\mathcal{N}$ and $\sum_{\ell\in\mathcal{N}}\alpha_\ell=1$, it holds that $\overline{q}'_\ell=\alpha_\ell q'_\ell$ for all $\ell\in\mathcal{N}$. Then, there exists a positive constant $\delta=\delta(\{q_\ell\colon \ell\in\mathcal{N}\})$ such that*

$$\int_I\Big(W(u)+\sum_{\ell=1}^{k} q_\ell|u^{(\ell)}|^2\Big)\,\mathrm{d}t \ge \delta\int_I\Big(W(u)+\sum_{\ell=1}^{k}|u^{(\ell)}|^2\Big)\,\mathrm{d}t$$

for every $u\in\mathrm{H}^k(I)$.

Proof. The proof is obtained by following the same line of the one of Corollary 8.3.7, with Corollary 8.3.3 used for every $\ell\in\{1,\dots,k-1\}$ in place of Theorem 8.1.1. □

Proposition 8.7.2. *Let $k>1$ be an integer, assume that $q_\ell>-\overline{q}'_\ell$ for all $\ell\in\mathcal{N}$, with $\mathcal{N}$ and $\{\overline{q}'_\ell\colon \ell\in\mathcal{N}\}$ as in Corollary 8.7.1, and assume that* (H1) *and* (H2) *are satisfied. Then $m=\lim_{T\to+\infty} m(T)$.*

Proof. Clearly, $m(T)\ge m$ for every T; hence, $\lim_{T\to+\infty} m(T)\ge m$.

To prove the converse, let $u\in\mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$ be such that $\lim_{t\to\pm\infty}u(t)=\pm1$ and $\int_{-\infty}^{+\infty}[W(u)+\sum_{\ell=1}^{k} q_\ell|u^{(\ell)}|^2]\,\mathrm{d}t<+\infty$, and let $T_o>0$ be such that $||u(t)|-1|<1/2$ if $|t|>T_o$.

Consider a function $\phi\in C^\infty(\mathbb{R};[0,1])$ that is supported in $(-\infty,1/2)$ and such that $\phi(t)=1$ for every $t<0$. For every $n\in\mathbb{N}$ we set

$$v_n(t) := \phi(t-n)u(t)+1-\phi(t-n),$$

and we observe that v_n coincides with u on $(-\infty,n)$ and it is constantly equal to 1 on $(n+1/2,+\infty)$. As a consequence

$$\begin{aligned}\int_{-\infty}^{+\infty}\Big(W(v_n)+\sum_{\ell=1}^{k} q_\ell|v_n^{(\ell)}|^2\Big)\,\mathrm{d}t &= \int_{-\infty}^{n-\frac12}\Big(W(u)+\sum_{\ell=1}^{k} q_\ell|u^{(\ell)}|^2\Big)\,\mathrm{d}t\\ &\quad+\int_{n-\frac12}^{n+\frac12}\Big(W(v_n)+\sum_{\ell=1}^{k} q_\ell|v_n^{(\ell)}|^2\Big)\,\mathrm{d}t.\end{aligned}$$

We estimate the energy of the function v_n on the interval $(n-1/2,n+1/2)$; that is, we estimate

$$\int_{n-\frac12}^{n+\frac12}\Big(W(v_n)+\sum_{\ell=1}^{k} q_\ell|v_n^{(\ell)}|^2\Big)\,\mathrm{d}t. \tag{8.7.2}$$

First, we observe that $\min\{u(t),1\} \le v_n(t) \le \max\{u(t),1\}$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$; therefore, since $u(t) \to 1$ as $t \to +\infty$, for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $|v_n(t) - 1| < \varepsilon$ for all $t > T_\varepsilon$ and $n \in \mathbb{N}$. Hence, the continuity of W and (H1) imply that

$$\lim_{n\to+\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} W(v_n)\,\mathrm{d}t = 0. \tag{8.7.3}$$

As for the derivatives of v_n, we observe that for all $\ell \in \{1,\dots,k\}$, there exists a positive constant C depending on k, ℓ, and the function ϕ such that

$$|v_n^{(\ell)}(t)|^2 \le C\Big\{|u(t)-1|^2 + \sum_{j=1}^{\ell} |u^{(j)}(t)|^2\Big\} \qquad \text{for all } t \in \mathbb{R},$$

therefore,

$$\begin{aligned}\Big|\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(\sum_{\ell=1}^{k} q_\ell |v_n^{(\ell)}|^2\Big)\,\mathrm{d}t\Big| &\le q_0 \sum_{\ell=1}^{k} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(C\Big\{|u-1|^2 + \sum_{j=1}^{\ell} |u^{(j)}|^2\Big\}\Big)\,\mathrm{d}t \\ &\le q_0 C k \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(W(u) + \sum_{\ell=1}^{k} |u^{(\ell)}|^2\Big)\,\mathrm{d}t,\end{aligned}$$

where $q_0 \coloneqq \max\{|q_1|,\dots,|q_{k-1}|,1\}$, and the last inequality follows by (H2) assuming that $n-1 > T_o$.
By Corollary 8.7.1, there exists a positive δ such that

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(W(u) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}|^2\Big)\,\mathrm{d}t \ge \delta \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(W(u) + \sum_{\ell=1}^{k} |u^{(\ell)}|^2\Big)\,\mathrm{d}t$$

for every n; therefore

$$\Big|\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(\sum_{\ell=1}^{k} q_\ell |v_n^{(\ell)}|^2\Big)\,\mathrm{d}t\Big| \le \frac{q_0 C k}{\delta} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(W(u) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}|^2\Big)\,\mathrm{d}t, \tag{8.7.4}$$

which tends to 0 as $n \to +\infty$.

Gathering (8.7.3) and (8.7.4), we obtain that (8.7.2) vanishes as $n \to +\infty$. With a similar reasoning, we obtain functions $\{w_n\}_n \subset \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$ which coincide with u on $(-n,n)$, which are equal to the sign function outside $(-n-\frac{1}{2}, n+\frac{1}{2})$, and such that

$$\int_{-\infty}^{+\infty} \Big(W(w_n) + \sum_{\ell=1}^{k} q_\ell |w_n^{(\ell)}|^2\Big)\,\mathrm{d}t = \int_{-n-\frac{1}{2}}^{n+\frac{1}{2}} \Big(W(u) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}|^2\Big)\,\mathrm{d}t + o_n(1).$$

Since each function w_n is admissible for the minimum problem $m(n+1)$, letting $n \to +\infty$ the conclusion follows by the arbitrariness of u. □

Finally, we prove the existence of an optimal-profile. We omit some minor details in the proof as the arguments involved are similar to those that appear in the proofs of Propositions 7.3.1 and 7.3.2 and Lemma 2.5 in [123].

Proposition 8.7.3. *Let $k > 1$ be an integer, assume that $q_\ell > -\overline{q}'_\ell$ for all $\ell \in \mathcal{N}$, with $\mathcal{N}$ and $\{\overline{q}'_\ell\colon\ \ell \in \mathcal{N}\}$ as in Corollary 8.7.1, and assume that* (H1) *and* (H2) *are satisfied. Then, the infimum* (8.7.1) *is a minimum.*

Proof. Consider $\{u_n\}_n$ a minimising sequence and, by translation invariance, assume that $u_n(0) = 0$ for every $n \in \mathbb{N}$. Applying Corollary 8.7.1 on each interval $(j, j+1)$, $j \in \mathbb{Z}$ and summing over j, we have

$$\int_{-\infty}^{+\infty} \Big(W(u_n) + \sum_{\ell=1}^{k} q_\ell |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t \geq \delta \int_{-\infty}^{+\infty} \Big(W(u_n) + \sum_{\ell=1}^{k} |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t \tag{8.7.5}$$

for all $n \in \mathbb{N}$, and then

$$\sup_n \int_{-\infty}^{+\infty} \Big(W(u_n) + \sum_{\ell=1}^{k} |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t < +\infty. \tag{8.7.6}$$

We apply the Fundamental Theorem of Calculus and Hölder's inequality to obtain

$$|u_n(t)| \leq \int_0^t |u_n'(s)| \, ds \leq \sqrt{t} \|u_n'\|_{L^2(\mathbb{R})} \qquad \text{for all } t \in \mathbb{R},$$

which, combined with (8.7.6), implies that for every $T > 0$ there exists a positive constant $C(T)$ such that

$$\sup_n \|u_n\|_{\mathrm{H}^k((-T,T))} \leq C(T).$$

Therefore, there exists $u \in \mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$ such that, up to subsequences, $\{u_n^{(\ell)}\}_n$ converges locally uniformly on compact subsets of $\mathbb{R}$ to $u^{(\ell)}$ for all $\ell \in \{0, \dots, k-1\}$, with the convention that $v^{(0)} = v$, and $u_n^{(k)} \rightharpoonup u^{(k)}$ weakly in $L^2(\mathbb{R})$. Note that, in particular, $u(0) = 0$.
As for the lower semicontinuity of the energy, we first note that, as a consequence of Corollary 8.7.1, we have

$$\int_{-\infty}^{-T} \Big(W(u_n) + \sum_{\ell=1}^{k} q_\ell |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t + \int_{T}^{+\infty} \Big(W(u_n) + \sum_{\ell=1}^{k} q_\ell |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t \geq 0 \tag{8.7.7}$$

for all $n \in \mathbb{N}$ and $T \in \mathbb{N}$. Then, we observe that Rellich's Theorem implies that $u_n^{(\ell)} \to u^{(\ell)}$ in $L^2((-T,T))$ as $n \to +\infty$ for every $\ell \in \{1, \dots, k-1\}$ and $T > 0$, which in turn yields

$$\lim_{n \to +\infty} \int_{-T}^{T} \Big(\sum_{\ell=1}^{k-1} q_\ell |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t = \int_{-T}^{T} \Big(\sum_{\ell=1}^{k-1} q_\ell |u^{(\ell)}|^2 \Big) \, \mathrm{d}t \tag{8.7.8}$$

for every $T > 0$; moreover, by Fatou's lemma and the lower semicontinuity of the L^2-norm, we have

$$\liminf_{n \to +\infty} \int_{-\infty}^{+\infty} \Big(W(u_n) + |u_n^{(k)}|^2 \Big) \, \mathrm{d}t \geq \int_{-\infty}^{+\infty} \Big(W(u) + |u^{(k)}|^2 \Big) \, \mathrm{d}t. \tag{8.7.9}$$

Combining (8.7.7), (8.7.8), and (8.7.9), and recalling that $q_k = 1$, we obtain

$$\begin{aligned} m = \liminf_{n \to +\infty} \int_{-\infty}^{+\infty} \Big(W(u_n) + \sum_{\ell=1}^{k} q_\ell |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t &\geq \liminf_{n \to +\infty} \int_{-T}^{T} \Big(W(u_n) + \sum_{\ell=1}^{k} q_\ell |u_n^{(\ell)}|^2 \Big) \, \mathrm{d}t \\ &\geq \int_{-T}^{T} \Big(W(u) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}|^2 \Big) \, \mathrm{d}t \end{aligned} \tag{8.7.10}$$

for all $T \in \mathbb{N}$.

Recall that $\{u_n\}_n$ is a minimising sequence, hence, by (8.7.5) and Fatou's Lemma, we infer

$$m = \liminf_{n\to+\infty} \int_{-\infty}^{+\infty} \Big(W(u_n) + \sum_{\ell=1}^{k} q_\ell |u_n^{(\ell)}|^2\Big)\,\mathrm{d}t \geq \delta \int_{-\infty}^{+\infty} \Big(W(u) + \sum_{\ell=1}^{k} |u^{(\ell)}|^2\Big)\,\mathrm{d}t;$$

therefore, we can pass to the limit as $T \to +\infty$ in (8.7.10) by Dominated Convergence Theorem to obtain

$$m \geq \int_{-\infty}^{+\infty} \Big(W(u) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}|^2\Big)\,\mathrm{d}t.$$

If $\lim_{t\to\pm\infty} u(t) = \pm 1$, the proof is completed; while, if $\lim_{t\to\pm\infty} u(t) = \mp 1$, a minimiser is $u(-t)$. For this reason, in order to conclude, we prove that both $\lim_{t\to-\infty} u(t)$ and $\lim_{t\to+\infty} u(t)$ exist, and that $\{\lim_{t\to-\infty} u(t), \lim_{t\to+\infty} u(t)\} = \{-1, 1\}$.

As a first step we prove the existence of the limit at $+\infty$, the other case being analogous. To see this, it is useful to observe that $u' \in H^1(\mathbb{R})$, which implies

$$\lim_{t\to\pm\infty} u'(t) = 0. \tag{8.7.11}$$

If $\lim_{t\to+\infty} u(t)$ exists, it equals -1 or 1, otherwise $\int_{-\infty}^{+\infty} W(u)\,\mathrm{d}t = +\infty$ which is a contradiction. We are left with considering the case there exist real numbers a, b, with $a < b$, and two increasing sequences $\{x_n\}_n, \{y_n\}_n$ such that

$$\lim_{n\to+\infty} x_n = +\infty, \quad \lim_{n\to+\infty} y_n = +\infty, \quad \lim_{n\to+\infty} u(x_n) = a, \quad \lim_{n\to+\infty} u(y_n) = b.$$

Without loss of generality, we can suppose

$$x_n < y_n < x_{n+1}, \quad u(x_n) \leq a + \frac{b-a}{8}, \quad u(y_n) \geq b - \frac{b-a}{8} \qquad \text{for all } n \in \mathbb{N};$$

moreover, by the continuity of u, we can also assume that both -1 and 1 are not contained in the interval (a, b).

If there exists $\{n_j\}_j$ such that $|x_{n_j} - y_{n_j}| \leq 2$ for every $j \in \mathbb{N}$, then, by Lagrange's Theorem, there exists $z_{n_j} \in (x_{n_j}, y_{n_j})$ such that

$$|u'(z_{n_j})| \geq \frac{|u(y_{n_j}) - u(x_{n_j})|}{2} \geq \frac{3}{8}(b-a),$$

hence, $\limsup_{t\to+\infty} |u'(t)| \geq 3(b-a)/8$, which contradicts (8.7.11). Otherwise, we may assume that $|x_n - y_n| \geq 2$ for every $n \in \mathbb{N}$. We set

$$\widetilde{x}_n := \max\{t \in (x_n, y_n)\colon\ u(t) \leq a + \frac{b-a}{4}\}$$

and

$$\widetilde{y}_n := \min\{t \in (x_n, y_n)\colon\ u(t) \geq b - \frac{b-a}{4}\}.$$

If $|\widetilde{x}_n - \widetilde{y}_n| \leq 1$ for infinitely many $n \in \mathbb{N}$, the conclusion follows by Lagrange's Theorem as before, otherwise, it holds that $|\widetilde{x}_n - \widetilde{y}_n| \geq 1$ for all $n \geq N_o$ with some $N_o \in \mathbb{N}$ large enough. Recalling that the interval (a, b) does not contain the points $-1, 1$, the above observation and (H1) imply

$$\int_{-\infty}^{+\infty} W(u)\,\mathrm{d}t \geq \sum_{n\geq N_o} \int_{\widetilde{x}_n}^{\widetilde{y}_n} W(u)\,\mathrm{d}t \geq \sum_{n\geq N_o} \min\big\{W(t)\colon\ t \in \big(a + \frac{b-a}{4}, b - \frac{b-a}{4}\big)\big\} = +\infty,$$

which is a contradiction and concludes the proof of the existence of the limits at infinity. We finally prove that they are different.
By contradiction suppose $\lim_{t\to\pm\infty} u(t) = -1$. By Lemma 3.2 in [39] and by the convergence $u_n \to u$ in $\mathrm{H}^k_{\mathrm{loc}}(\mathbb{R})$, we find points $\{t_j\}_j$ and a subsequence of indices $\{n_j\}_j$ such that

$$t_j \to +\infty, \quad u_{n_j}(t_j) \to -1, \qquad u^{(\ell)}_{n_j}(t_j) \to 0 \quad \text{as } j \to +\infty, \ \text{ for all } \ell \in \{1, \dots, k-1\}. \tag{8.7.12}$$

Then we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\Big(W(u_{n_j})+\sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t &= \int_{-\infty}^{t_j}\Big(W(u_{n_j}) \\ &\quad + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t + \int_{t_j}^{+\infty}\Big(W(u_{n_j})+\sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)dt \\ &\geq \int_{-\infty}^{t_j}\Big(W(u_{n_j}) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t \\ &\quad + m - \widetilde{m}(u_{n_j}(t_j), \dots, u^{(k-1)}_{n_j}(t_j)),\end{aligned}$$

where we set

$$\begin{aligned}\widetilde{m}(u_0, \dots, u_{k-1}) \coloneqq \inf\Big\{\int_0^1 \Big(W(v) + \sum_{\ell=1}^{k} q_\ell |v^{(\ell)}|^2\Big)\,\mathrm{d}t\colon\ & v(0) = -1, \\ & v^{(\ell)}(0) = 0 \text{ for all } \ell \in \{1, \dots, k-1\}, \\ & v^{(\ell)}(1) = u_\ell \text{ for all } \ell \in \{0, \dots, k-1\}\Big\},\end{aligned}$$

with the convention that $v^{(0)} = v$. Since $\widetilde{m}(u_0, \dots, u_{k-1}) \to 0$ as $(u_0, \dots, u_{k-1}) \to (-1, 0, \dots, 0)$, by (8.7.12) we infer that

$$\begin{aligned} m &= \liminf_{j\to+\infty} \int_{-\infty}^{+\infty} \Big(W(u_{n_j}) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t \\ &\geq \liminf_{j\to+\infty} \int_{-\infty}^{t_j} \Big(W(u_{n_j}) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t + m - \widetilde{m}(u_{n_j}(t_j), \dots, u^{(k-1)}_{n_j}(t_j)) \\ &\geq \liminf_{j\to+\infty} \int_{-\infty}^{t_j} \Big(W(u_{n_j}) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t + m, \end{aligned}$$

which implies

$$\liminf_{j\to+\infty} \int_{-\infty}^{t_j} \Big(W(u_{n_j}) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t \leq 0.$$

But, by Corollary 8.7.1 and Fatou's Lemma, it holds that

$$\begin{aligned}\liminf_{j\to+\infty} \int_{-\infty}^{t_j} \Big(W(u_{n_j}) + \sum_{\ell=1}^{k} q_\ell |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t &\geq \delta \liminf_{j\to+\infty} \int_{-\infty}^{t_j} \Big(W(u_{n_j}) + \sum_{\ell=1}^{k} |u^{(\ell)}_{n_j}|^2\Big)\,\mathrm{d}t \\ &\geq \delta \int_{-\infty}^{+\infty} \Big(W(u) + \sum_{\ell=1}^{k} |u^{(\ell)}|^2\Big)\,\mathrm{d}t \geq 0;\end{aligned}$$

therefore, we deduce that u is identically equal to -1. This contradicts $u(0) = 0$, which concludes the proof. □

Remark 8.7.4. Restoring the general dimension d, we may deal with the case of special norms on tensors that are 'compatible with slicing' upon supposing the non-negativity of the coefficients $q_1, \dots, q_{k-1}$. More precisely, we suppose that for every $\ell \in \{1, \dots, k\}$ the norm $|\cdot|_\ell$ satisfies

$$|T|_\ell \geq |T(\xi, \dots, \xi)| \qquad \text{for every } \xi \in \mathbb{S}^{d-1}, \tag{8.7.13}$$

and we assume that $q_\ell \geq 0$ for every $\ell \in \{1, \dots, k-1\}$. Also in this instance, the energy density g defined in (8.1.4) is the constant m given by (8.7.1).

The argument is similar to the one in the proof of Proposition 8.6.2. For fixed $\nu \in \mathbb{S}^{d-1}$ and $\eta > 0$, let v and ε be such that

$$g(\nu) + \eta \geq \int_{Q_1^\nu} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \Big) \, \mathrm{d}x.$$

Applying Fubini's Theorem, (8.7.13), and the slicing properties of Sobolev functions, we obtain

$$\begin{aligned}
g(\nu) + \eta &\geq \int_{Q_1^\nu} \Big(\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \Big) \, \mathrm{d}x \\
&\geq \int_{Q_1^\nu \cap \Pi^\nu} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{\varepsilon} W(v_y^\nu(t)) + \sum_{\ell=1}^{k} q_\ell \varepsilon^{2\ell-1} |(v_y^\nu)^{(\ell)}(t)|^2 \Big) \, \mathrm{d}t \, \mathrm{d}\mathcal{H}^{d-1}(y) \\
&\geq \inf \Big\{ \int_{-\frac{1}{2\varepsilon}}^{\frac{1}{2\varepsilon}} \Big(W(v) + \sum_{\ell=1}^{k} q_\ell |v^{(\ell)}|^2 \Big) \, \mathrm{d}x, \ v \in \mathrm{H}^k\big((-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon})\big), \\
&\qquad\qquad v(t) = \operatorname{sgn}(t) \text{ if } |t| > M \text{ for some } M \in \big(0, \frac{1}{2\varepsilon}\big) \Big\} \\
&= m\Big(\frac{1}{2\varepsilon}\Big),
\end{aligned}$$

where the second inequality also follows by the assumption $q_\ell \geq 0$ for every $\ell \in \{1, \dots, k-1\}$. By the arbitrariness of η and ε, the previous chain of inequalities implies $g \geq m$. The converse inequality is obtained using as a test function for $g(\nu)$ the function $v(x) := u(\frac{x \cdot \nu}{\varepsilon})$, with u an (almost) optimal one-dimensional profile for $m(1/(2\varepsilon))$.

Part III

Nonlocal-to-local variational problems

Another look at elliptic homogenisation

In this chapter, we consider nonlocal functionals of the form

$$F_{\varepsilon,s}(u) \coloneqq \begin{cases} (1-s) \iint_{\Omega\times\Omega} a\left(\frac{x}{\varepsilon}\right) \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} \mathrm{d}x\,\mathrm{d}y & \text{if } u \in \mathrm{H}^s(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus \mathrm{H}^s(\Omega), \end{cases} \tag{9.0.1}$$

where $\varepsilon, s \in (0,1)$, $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $u \in \mathrm{H}^s(\Omega)$, having set

$$\mathrm{H}^s(\Omega) \coloneqq \Big\{ u \in L^2(\Omega) \colon\ [u]_{\mathrm{H}^s(\Omega)} = \Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{2}} < +\infty \Big\}.$$

The coefficient $a\colon \mathbb{R}^d \to (0,+\infty)$ in (9.0.1) is assumed to be a 1-periodic continuous function satisfying

$$0 < \alpha \le a(x) \le \beta < +\infty \quad \text{for every } x \in \mathbb{R}^d \tag{9.0.2}$$

for two suitable constants α, β.

When ε is fixed and $s \to 1^-$, the celebrated Bourgain-Brezis-Mironescu formula [42] shows that, for fixed $u \in \mathrm{H}^1(\Omega)$, $F_{\varepsilon,s}(u)$ converges to (a multiple of)

$$\int_\Omega a\Big(\frac{x}{\varepsilon}\Big) |\nabla u|^2\,\mathrm{d}x.$$

Letting then $\varepsilon \to 0^+$, one obtains the usual homogenised functional (see [47, Theorem 14.7]) given by

$$\frac{\sigma_{d-1}}{2d} \int_\Omega A_{\mathrm{hom}} \nabla u \cdot \nabla u\,\mathrm{d}x,$$

where A_{hom} is a symmetric $d \times d$ matrix characterised by the *homogenisation formula*

$$A_{\mathrm{hom}} z \cdot z = \min\Big\{ \int_{(0,1)^d} a(y)|z + \nabla\varphi(y)|^2 \mathrm{d}y \colon\ \varphi \text{ 1-periodic} \Big\}. \tag{9.0.3}$$

The aim of this chapter is to identify a regime in which this separation of scales remains valid even though the two limits are taken simultaneously. In what follows, we assume that ε is a positive parameter and that $s \in (0,1)$ is a function of ε that approaches 1 as $\varepsilon \to 0^+$. Since $s = s(\varepsilon)$, for brevity, $F_{\varepsilon,s}$ will be denoted by F_ε.

We prove that if $1 - s \ll |\log(\varepsilon)|^{-1}$, then the separation of scales described above holds, as if we were passing to the limit first letting $s \to 1$ and then letting $\varepsilon \to 0^+$. The following constitutes the main result of the chapter.

Theorem 9.0.1. *Let Ω be a bounded open set with Lipschitz boundary, $\varepsilon \in (0,1)$, $s = s_\varepsilon$, and let F_ε be the functional defined in* (9.0.1)*. If*

$$\lim_{\varepsilon \to 0^+} (1-s)|\log(\varepsilon)| = 0, \tag{9.0.4}$$

then $\{\mathfrak{F}_\varepsilon\}_\varepsilon$ Γ*-converges with respect to the strong* $L^2(\Omega)$*-topology to the functional* F_{hom} *defined by*

$$F_{\mathrm{hom}}(u) \coloneqq \begin{cases} \dfrac{\sigma_{d-1}}{2d} \displaystyle\int_\Omega A_{\mathrm{hom}} \nabla u \cdot \nabla u \,\mathrm{d}x & \text{if } u \in H^1(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H^1(\Omega), \end{cases} \tag{9.0.5}$$

for every $u \in L^2(\Omega)$*, where the* Γ*-limit is computed with respect to the* $L^2(\Omega)$*-convergence,* A_{hom} *is given by* (9.0.3)*, and* $\sigma_{d-1} \coloneqq \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$.

We note that the hypothesis that Ω has a Lipschitz boundary is exploited only in proving, by means of a density argument, the Γ-lim sup inequality.

Remark 9.0.2. Theorem 9.0.1 can be generalized to functionals modelled on (s,p)-Gagliardo seminorms with $p > 1$, of the form

$$F_{\varepsilon,s}(u) = (1-s) \iint_{\Omega\times\Omega} a\left(\frac{x}{\varepsilon}\right) \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}} \mathrm{d}x \,\mathrm{d}y \tag{9.0.6}$$

for $u \in \mathrm{W}^{s,p}(\Omega)$. More generally, we may consider integrals

$$F_{\varepsilon,s}(u) = (1-s) \iint_{\Omega\times\Omega} f\left(\frac{x}{\varepsilon}, \frac{|u(x)-u(y)|}{|x-y|}\right) \frac{1}{|x-y|^{d-p(1-s)}} \mathrm{d}x \,\mathrm{d}y, \tag{9.0.7}$$

with f a standard periodic integrand with p growth (see e.g. [47]), possibly with u vector-valued. An argument analogous to the one for $p = 2$ suggests that the subcritical regime is still $1 - s \ll |\log(\varepsilon)|^{-1}$, and the resulting Γ-limit is

$$C_{d,p} \int_\Omega f_{\mathrm{hom}}(\nabla u) \,\mathrm{d}x,$$

defined on $\mathrm{W}^{1,p}(\Omega)$, with $C_{d,p}$ the constant appearing in the corresponding result in [42], and f_{hom} the homogenised energy density for f (see [8, 47]). Since we focus on determining subcritical scales, here we concentrate on quadratic functionals, for which a proof can be achieved by a discretisation argument.

Remark 9.0.3. In the original paper [46] on which this chapter is based on, we proved Theorem 9.0.1 under the more restrictive assumption $(1-s) \ll \varepsilon^2$. We later realised that a slight variant of the same arguments (compare Lemma 9.1.1 below with Lemma 3 of [46]) shows that the separation of scales occurs also under the assumption (9.0.4).

9.1 Preliminary results

We here gather a few results that will be used in the following. We begin by stating a lemma that will allow us in the proof of the upper bound to take into account only those interactions due to pairs of sufficiently close points in Ω.

Lemma 9.1.1. *Let* $\{u_\varepsilon\}_\varepsilon \subseteq \mathrm{W}^{1,\infty}(\Omega)$ *be a sequence of uniformly Lipschitz functions and let* $\{r_\varepsilon\}_\varepsilon \subseteq (0,1)$ *be a sequence such that*

$$\lim_{\varepsilon\to 0^+} r_\varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} r_\varepsilon^{2(1-s_\varepsilon)} = 1. \tag{9.1.1}$$

Then

$$\lim_{\varepsilon\to 0^+} \left(F_\varepsilon(u_\varepsilon) - (1-s_\varepsilon) \iint_{\Omega\times\Omega\cap\{(x,y)\colon |x-y|\le r_\varepsilon\}} a\left(\frac{x}{\varepsilon}\right) \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s_\varepsilon}} \,\mathrm{d}x \,\mathrm{d}y \right) = 0.$$

Proof. Let $L \coloneqq \sup_{\varepsilon>0} \mathrm{Lip}(u_\varepsilon, \Omega) < +\infty$, where $\mathrm{Lip}(u_\varepsilon, \Omega)$ denotes the Lipschitz constant of u_ε on Ω, and let $D \coloneqq \mathrm{diam}(\Omega)$. By (9.0.2), we have

$$\begin{aligned}
F_\varepsilon(u_\varepsilon) - (1-s_\varepsilon) &\iint_{\Omega\times\Omega\cap\{(x,y)\colon |x-y|\le r_\varepsilon\}} a\left(\frac{x}{\varepsilon}\right) \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s_\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y \\
&= (1-s_\varepsilon) \iint_{\Omega\times\Omega\cap\{(x,y)\colon |x-y|> r_\varepsilon\}} a\left(\frac{x}{\varepsilon}\right) \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s_\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le (1-s_\varepsilon)\,\beta L^2 \iint_{\Omega\times\Omega\cap\{(x,y)\colon |x-y|> r_\varepsilon\}} \frac{|x-y|^2}{|x-y|^{d+2s_\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le (1-s_\varepsilon)\,\beta L^2 \mathcal{L}^d(\Omega) \int_{B(0,D)\setminus B(0,r_\varepsilon)} \frac{1}{|\xi|^{d+2s_\varepsilon-2}}\mathrm{d}\xi, \\
&= (1-s_\varepsilon)\beta L^2 \mathcal{L}^d(\Omega)\sigma_{d-1} \int_{r_\varepsilon}^{D} \rho^{1-2s_\varepsilon}\mathrm{d}\rho \\
&= \frac{\beta L^2 \mathcal{L}^d(\Omega)\sigma_{d-1}}{2}(D^{2(1-s_\varepsilon)} - r_\varepsilon^{2(1-s_\varepsilon)}),
\end{aligned}$$

which tends to 0 in view of (9.1.1). This concludes the proof. □

Remark 9.1.2. We will apply the previous lemma setting $r_\varepsilon \coloneqq \varepsilon^2$. Under (9.0.4), it is easy to see that

$$\lim_{\varepsilon\to 0^+} \frac{r_\varepsilon}{\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} r_\varepsilon^{2(1-s_\varepsilon)} = 1, \tag{9.1.2}$$

so that condition (9.1.1) is satisfied.

In the proof of the liminf inequality we take advantage of a discretisation argument based on the construction of proper lattices, which can be parameterized on orthogonal bases.

Remark 9.1.3 (Notation for the set of orthogonal bases)**.** Following the notation of [165] we define the set of orthonormal bases of $\mathbb{R}^d$

$$V \coloneqq \{\overline{\nu} = (\nu_1, \dots, \nu_d)\colon\ \nu_j \in \mathbb{S}^{d-1} \text{ such that } \nu_i \cdot \nu_j = 0 \text{ for } i \neq j\}$$

and observe that V has Hausdorff dimension equal to $k_d \coloneqq d(d-1)/2$. For every $n \in \{1, \dots, d\}$ and fixed $\nu \in \mathbb{S}^{d-1}$ we define

$$V_n^\nu \coloneqq \{\overline{\nu} \in V \text{ such that } \nu_n = \nu\},$$

whose Hausdorff dimension is $k_d - (d-1)$. Note that we have

$$\mathcal{H}^{k_d-(d-1)}(V_n^\nu) = \frac{\mathcal{H}^{k_d}(V)}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})}, \tag{9.1.3}$$

and that the formula

$$\int_V f(\overline{\nu})\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) = \int_{\mathbb{S}^{d-1}} \int_{V_n^\nu} f^\nu(\overline{\nu}_n)\,\mathrm{d}\mathcal{H}^{k_d-(d-1)}(\overline{\nu}_n)\,\mathrm{d}\mathcal{H}^{d-1}(\nu) \tag{9.1.4}$$

holds with $\overline{\nu}_n \coloneqq (\nu_1, \dots, \nu_{n-1}, \nu_{n+1}, \dots, \nu_d)$ and $f^\nu(\overline{\nu}_n) \coloneqq f(\overline{\nu})$ for every $n \in \{1, \dots, d\}$ and f non-negative measurable function.

Given $\rho > 0$ and $\overline{\nu} \in V$, we define $\mathbb{Z}^d_{\rho\overline{\nu}} \coloneqq \{z_1\rho\nu_1 + z_2\rho\nu_2 + \cdots + z_{\mathrm{d}}\rho\nu_d\colon\ (z_1, \dots, z_d) \in \mathbb{Z}^d\}$ and $Q_{\rho\overline{\nu}}$ as the cube described by the orthogonal basis $\{\rho\nu_1, \dots, \rho\nu_d\}$.

Remark 9.1.4 (Kuhn's decomposition)**.** A cube $Q_{\rho\overline{\nu}}$ defined as in the previous remark can be further decomposed into $d!$ d-simplices of size $\rho^d/d!$ through *Kuhn's decomposition* in the following way: for every τ permutation of d indices, we define $\Delta^{\tau}_{\rho\overline{\nu}}$ as the simplex described by the vertices $\rho\nu_{\tau(1)}$, $\rho\nu_{\tau(1)}+\rho\nu_{\tau(2)},\dots,\rho\nu_{\tau(1)}+\rho\nu_{\tau(2)}+\cdots+\rho\nu_{\tau(d)}$. As shown in [142, Lemma 1], as τ varies among all the permutations, we get a family of $d!$ simplices which constitutes the desired partition.

We denote the vertices of each simplex by

$$\Delta^{\tau,0}_{\rho\overline{\nu}}=0,\quad \Delta^{\tau,j}_{\rho\overline{\nu}}\coloneqq\rho\nu_{\tau(1)}+\rho\nu_{\tau(2)}+\cdots+\rho\nu_{\tau(j)}\qquad\text{for } j=1,\dots,d$$

and observe that, with this choice, it holds

$$\Delta^{\tau,j}_{\rho\overline{\nu}}-\Delta^{\tau,j-1}_{\rho\overline{\nu}}=\rho\nu_{\tau(j)}\qquad\text{for } j=1,\dots,d. \tag{9.1.5}$$

9.2 Proof of Theorem 9.0.1

We divide the proof of the theorem into two parts, addressing the lower and upper bound separately. We begin with the lower bound.

Liminf inequality. In what follows, we write $s\coloneqq s_\varepsilon$, $r\coloneqq r_\varepsilon$ and $\sigma_{d-1}\coloneqq\mathcal{H}^{d-1}(\mathbb{S}^{d-1})$ in order to simplify the notation.

Let $u\in L^2(\Omega)$ and consider a sequence $\{u_\varepsilon\}_\varepsilon$ converging to u in $L^2(\Omega)$. Without loss of generality, we may assume that $\sup_\varepsilon F_\varepsilon(u_\varepsilon)<\infty$ which implies that $u_\varepsilon\in \mathrm{H}^s(\Omega)$ for every $\varepsilon>0$.

Since we aim at proving that $u\in \mathrm{H}^1(\Omega')$ for every Ω' open subset well contained in Ω and that

$$\begin{aligned}\liminf_{\varepsilon\to0^+}F_\varepsilon(u_\varepsilon)&\geq\liminf_{\varepsilon\to0^+}(1-s)\iint_{\Omega\times\Omega\cap\{(x,y):|x-y|\leq r\}}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\\&\geq\frac{\sigma_{d-1}}{2d}\int_{\Omega'}A_{\mathrm{hom}}\nabla u\cdot\nabla u\,\mathrm{d}x,\end{aligned}\tag{9.2.1}$$

where $r\coloneqq\varepsilon^2$. By the independence of the left-hand side from Ω', this would imply that $u\in \mathrm{H}^1(\Omega)$. Moreover, as the first inequality is a simple consequence of the positivity of the integrand in F_ε, we only need to prove the second. We divide the proof of this inequality into several steps.

Step 1. (Integration over the orthonormal bases) By the change of variables $\eta\coloneqq y-x$, $\xi\coloneqq\eta/r$, and the coarea formula, we get

$$\begin{aligned}&(1-s)\iint_{\Omega\times\Omega\cap\{(x,y):\,|x-y|\leq r\}}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad=(1-s)\int_{B(0,r)}\frac{1}{|\eta|^{d-2(1-s)}}\int_{\{x\in\Omega:\,x+\eta\in\Omega\}}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x+\eta)-u_\varepsilon(x)|^2}{|\eta|^2}\,\mathrm{d}x\,\mathrm{d}\eta\\&\quad=(1-s)\int_{B(0,1)}\frac{r^{2(1-s)}}{|\xi|^{d-2(1-s)}}\int_{\{x\in\Omega:\,x+r\xi\in\Omega\}}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x+r\xi)-u_\varepsilon(x)|^2}{|r\xi|^2}\,\mathrm{d}x\,\mathrm{d}\xi\\&\quad=(1-s)\int_0^1\rho^{d-1}\int_{\mathbb{S}^{d-1}}\frac{r^{2(1-s)}}{\rho^{d-2(1-s)}}\\&\qquad\qquad\int_{\{x\in\Omega:\,x+r\rho\nu\in\Omega\}}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x+r\rho\nu)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{d-1}(\nu)\,\mathrm{d}\rho\end{aligned}$$

Taking into account (9.1.3), we rewrite the above term as

$$(1-s)\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_{\mathbb{S}^{d-1}} \frac{1}{d}\sum_{n=1}^{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_{V_n^\nu}\int_{\{x\in\Omega:\ x+r\rho\nu\in\Omega\}} a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x+r\rho\nu)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d-(d-1)}(\overline{\nu}_n)\,\mathrm{d}\mathcal{H}^{d-1}(\nu)\,\mathrm{d}\rho,$$

then we apply formula (9.1.4) to get

$$\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{n=1}^{d}\int_{\{x\in\Omega:\ x+r\rho\nu_n\in\Omega\}} a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x+r\rho\nu_n)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho, \tag{9.2.2}$$

where ν_n is the n-th element of the orthonormal basis $\overline{\nu}$.

For $\overline{\nu}\in V$ (recall the notation of 9.1.3) we set

$$\mathcal{I}^r_{\rho\overline{\nu}} \coloneqq \{k\in\mathbb{Z}^d_{\rho\overline{\nu}}\colon rk+rQ_{\rho\overline{\nu}}\subset\subset\Omega^{r\rho}\},$$

where, given $\eta>0$, we set

$$\Omega^\eta \coloneqq \{x\in\Omega\colon\ \mathrm{dist}(x,\partial\Omega)>\eta\}.$$

Note that for every $\overline{\nu}\in V$ and for every $n\in\{1,\dots,d\}$ it holds that

$$\bigcup_{k\in\mathcal{I}^r_{\rho\overline{\nu}}} rk+rQ_{\rho\overline{\nu}} \subseteq \Omega^{r\rho} \subseteq \{x\in\Omega\colon x+r\rho\nu_n\in\Omega\};$$

hence, (9.2.2) is bounded below by

$$\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{n=1}^{d}\int_{\bigcup_{k\in\mathcal{I}^r_{\rho\overline{\nu}}} rk+rQ_{\rho\overline{\nu}}} a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x+r\rho\nu_n)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho. \tag{9.2.3}$$

We shall exploit the uniform continuity of the function a to factor it out from the inner integral; for this reason, we introduce a modulus of continuity $\omega:[0,+\infty)\to[0,+\infty)$; that is, an increasing continuous function such that $\omega(0)=0$ and $|a(x_1)-a(x_2)|\le\omega(|x_1-x_2|)$ for every $x_1,x_2\in\mathbb{R}^d$.

We rewrite the right-hand side of (9.2.3) as

$$\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{n=1}^{d}\sum_{k\in\mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right)\int_{rk+rQ_{\rho\overline{\nu}}}\frac{|u_\varepsilon(x+r\rho\nu_n)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho$$

$$+\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{n=1}^{d}\sum_{k\in\mathcal{I}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}}\left(a\left(\frac{x}{\varepsilon}\right)-a\left(\frac{rk}{\varepsilon}\right)\right)\frac{|u_\varepsilon(x+r\rho\nu_n)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho$$

and we note that the last term is negligible as $\varepsilon \to 0^+$ as its absolute value is estimated from above by

$$\omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big)\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{n=1}^d \int_{\{x\in\Omega:\ x+r\rho\nu_n\in\Omega\}} \frac{|u_\varepsilon(x+r\rho\nu_n)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho \le \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big)\frac{1}{\alpha}F_\varepsilon(u_\varepsilon)+o_\varepsilon(1)$$

which tends to 0 since we assumed $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ and since $r/\varepsilon \to 0^+$ by the first equality in (9.1.2).

Combining the previous estimates, we then obtain the inequality

$$F_\varepsilon(u_\varepsilon)+o_\varepsilon(1) \ge \frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{n=1}^d \sum_{k\in\mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right)\int_{rk+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(x+r\rho\nu_n)-u_\varepsilon(x)|^2}{|r\rho|^2}\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho. \quad (9.2.4)$$

Step 2. (Piecewise-affine auxiliary functions) To bound from below (9.2.4), we take advantage of a discretisation argument based on piecewise-affine functions.

Given a sufficiently small $\varepsilon > 0$, $\rho \in (0,1)$ and $\overline{\nu} \in V$ we define the function $u_\varepsilon^{\rho\overline{\nu}}$ in two steps:

- first, we assign values on the lattice $\mathcal{I}^r_{\rho\overline{\nu}}$ setting

$$u_\varepsilon^{\rho\overline{\nu}}(rk) = \frac{1}{|r\rho|^d}\int_{rk+rQ_{\rho\overline{\nu}}} u_\varepsilon\,\mathrm{d}x \qquad \text{for every } k \in \mathcal{I}^r_{\rho\overline{\nu}};$$

- then, given k in the lattice

$$\mathcal{J}^r_{\rho\overline{\nu}} \coloneqq \{k \in \mathbb{Z}^d_{\rho\overline{\nu}}:\ rk + 2rQ_{\rho\overline{\nu}} \subset\subset \Omega^{r\rho}\} \subset \mathcal{I}^r_{\rho\overline{\nu}},$$

we consider the cube $rk + rQ_{\rho\overline{\nu}}$, τ a permutation of the indices $\{1,\dots,d\}$, and $rk + r\Delta^\tau_{\rho\overline{\nu}}$ the corresponding simplex in Kuhn's decomposition; on this simplex we define $u_\varepsilon^{\rho\overline{\nu}}$ being the affine interpolation of the previously defined values $u_\varepsilon^{\rho\overline{\nu}}(rk) = u_\varepsilon^{\rho\overline{\nu}}(rk + r\Delta^{\tau,0}_{\rho\overline{\nu}})$, $u_\varepsilon^{\rho\overline{\nu}}(rk + r\Delta^{\tau,1}_{\rho\overline{\nu}}), \dots, u_\varepsilon^{\rho\overline{\nu}}(rk + r\Delta^{\tau,d}_{\rho\overline{\nu}})$.

We remark that, in the second step, our definition is well posed since $k \in \mathcal{J}^r_{\rho\overline{\nu}}$ implies $k + \Delta^{\tau,j}_{\rho\overline{\nu}} \in \mathcal{I}^r_{\rho\overline{\nu}}$ for every τ permutation of d indices and $j \in \{0,\dots,d\}$; that is, on the simplex denoted by τ, to each vertex has already been assigned a value in the first step of the construction. For the same reason, $u_\varepsilon^{\rho\overline{\nu}}$ cannot be defined this way on the cubes $rk + rQ_{\rho\overline{\nu}}$ with $k \in \mathcal{I}^r_{\rho\overline{\nu}} \setminus \mathcal{J}^r_{\rho\overline{\nu}}$. This means that $u_\varepsilon^{\rho\overline{\nu}}$ is not defined on each cell of the above form contained in Ω, but these exceptions only occur near the boundary of Ω and for our purpose this issue is overcome simply setting to 0 the function $u_\varepsilon^{\rho\overline{\nu}}$ outside $\bigcup_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} rk + rQ_{\rho\overline{\nu}}$.

Note that $u_\varepsilon^{\rho\overline{\nu}}$ is well defined as a continuous piecewise-affine function on $\bigcup_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} rk +$

$rQ_{\rho\overline{\nu}}$; hence, on each simplex $rk + r\Delta^{\tau}_{\rho\overline{\nu}}$, its gradient is constant and by (9.1.5) it holds

$$\begin{aligned}
\int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} |\nabla u^{\rho\overline{\nu}}_{\varepsilon}|^2 \,\mathrm{d}x &= \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} \frac{|u^{\rho\overline{\nu}}_{\varepsilon}(rk + r\Delta^{\tau,j}_{\rho\overline{\nu}}) - u^{\rho\overline{\nu}}_{\varepsilon}(rk + r\Delta^{\tau,j-1}_{\rho\overline{\nu}})|^2}{|r\rho|^2} \,\mathrm{d}x \\
&= \frac{1}{|r\rho|^2} \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} \Big| \frac{1}{|r\rho|^d} \Big(\int_{rk+r\Delta^{\tau,j}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} u_\varepsilon \,\mathrm{d}y - \int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} u_\varepsilon \,\mathrm{d}y \Big) \Big|^2 \,\mathrm{d}x \\
&= \frac{1}{|r\rho|^2} \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} \Big| \frac{1}{|r\rho|^d} \Big(\int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} u_\varepsilon(y + r\rho\nu_{\tau(j)}) - u_\varepsilon(y) \,\mathrm{d}y \Big) \Big|^2 \,\mathrm{d}x \\
&\le \frac{1}{|r\rho|^2} \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} \frac{1}{|r\rho|^d} \int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} |u_\varepsilon(y + r\rho\nu_{\tau(j)}) - u_\varepsilon(y)|^2 \mathrm{d}y \,\mathrm{d}x \\
&= \frac{1}{|r\rho|^d} \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} \int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_{\tau(j)}) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y \,\mathrm{d}x \\
&= \frac{1}{d!} \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_{\tau(j)}) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y,
\end{aligned}$$

where we used Jensen's inequality and the fact that $|rk + r\Delta^{\tau}_{\rho\overline{\nu}}| = |r\rho|^d/d!$.

Summing over all permutations τ and $k \in \mathcal{J}^r_{\rho\overline{\nu}}$ and reinstating the coefficient a, we get

$$\begin{aligned}
&\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} a\Big(\frac{rk}{\varepsilon}\Big) \int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u^{\rho\overline{\nu}}_{\varepsilon}|^2 \,\mathrm{d}x \\
&= \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \sum_{\tau} \int_{rk+r\Delta^{\tau}_{\rho\overline{\nu}}} |\nabla u^{\rho\overline{\nu}}_{\varepsilon}|^2 \,\mathrm{d}x \\
&\le \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \sum_{\tau} \frac{1}{d!} \sum_{j=1}^{d} \int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_{\tau(j)}) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y \\
&= \frac{1}{d!} \sum_{n=1}^{d} \sum_{\{\tau,\, j\colon\ \tau(j)=n\}} \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \int_{rk+r\Delta^{\tau,j-1}_{\rho\overline{\nu}}+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y. \qquad (9.2.5)
\end{aligned}$$

Keeping n, j, τ fixed, we put $h := k + \Delta^{\tau,j-1}_{\rho\overline{\nu}}$ and note, once again, that $k \in \mathcal{J}^r_{\rho\overline{\nu}}$ implies $h \in \mathcal{I}^r_{\rho\overline{\nu}}$, so that (9.2.5) is less than or equal to

$$\begin{aligned}
&\frac{1}{d!} \sum_{n=1}^{d} \sum_{\{\tau,\, j\colon\ \tau(j)=n\}} \sum_{h\in\mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rh}{\varepsilon}\right) \int_{rh+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y + E_\varepsilon(\rho,\overline{\nu}) \\
&\qquad = \sum_{n=1}^{d} \sum_{h\in\mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rh}{\varepsilon}\right) \int_{rh+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y + E_\varepsilon(\rho,\overline{\nu}),
\end{aligned}$$

where we used that the cardinality of the set $\{\tau, j\colon\ \tau(j) = n\}$ is $d!$ for every $n = 1, \dots, d$, and we set

$$E_\varepsilon(\rho,\overline{\nu}) := \frac{1}{d!} \sum_{n=1}^{d} \sum_{\{\tau,\, j\colon\ \tau(j)=n\}} \sum_{h\in\mathcal{I}^r_{\rho\overline{\nu}}} \Big| a\Big(\frac{rh - r\Delta^{\tau,j-1}_{\rho\overline{\nu}}}{\varepsilon}\Big) - a\Big(\frac{rh}{\varepsilon}\Big) \Big| \int_{rh+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y.$$

Summarising, we have proved that for every $\varepsilon, \rho, \overline{\nu}$, it holds

$$\sum_{n=1}^{d} \sum_{k \in \mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \int_{rk+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y$$
$$\geq \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u_\varepsilon^{\rho\overline{\nu}}|^2 \,\mathrm{d}x - \ E_\varepsilon(\rho, \overline{\nu}), \tag{9.2.6}$$

and therefore, having achieved an estimate in terms of the inner integral in (9.2.4), we get

$$\begin{aligned}
&F_\varepsilon(u_\varepsilon) + o_\varepsilon(1) \\
&\quad \geq \frac{1-s}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \\
&\qquad \int_V \sum_{n=1}^{d} \sum_{k \in \mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \int_{rk+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(x + r\rho\nu_n) - u_\varepsilon(x)|^2}{|r\rho|^2} \,\mathrm{d}x \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho \qquad (9.2.7) \\
&\quad \geq \frac{1-s}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u_\varepsilon^{\rho\overline{\nu}}|^2 \,\mathrm{d}x \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho \qquad (9.2.8) \\
&\qquad - \ \frac{1-s}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V E_\varepsilon(\rho, \overline{\nu}) \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho. \qquad (9.2.9)
\end{aligned}$$

Note that the term in (9.2.9) vanishes as $\varepsilon \to 0^+$; indeed, considering the modulus of continuity, the bound from above (9.0.2) and the bound from above involving (9.2.7), we have

$$\begin{aligned}
&\frac{1-s}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \int_V E_\varepsilon(\rho, \overline{\nu}) \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho \\
&\quad \leq \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big) \frac{1-s}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \\
&\qquad \int_V \sum_{n=1}^{d} \sum_{h \in \mathcal{I}^r_{\rho\overline{\nu}}} \int_{rh+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho \\
&\quad \leq \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big) \frac{1-s}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \frac{1}{\alpha} \int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \\
&\qquad \int_V \sum_{n=1}^{d} \sum_{h \in \mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rh}{\varepsilon}\right) \int_{rh+rQ_{\rho\overline{\nu}}} \frac{|u_\varepsilon(y + r\rho\nu_n) - u_\varepsilon(y)|^2}{|r\rho|^2} \,\mathrm{d}y \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho \\
&\quad \leq \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big) \frac{1}{\alpha} F_\varepsilon(u_\varepsilon) + o_\varepsilon(1),
\end{aligned}$$

which tends to 0 since $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ and since $r/\varepsilon \to 0^+$ by the first equality in (9.1.2).

As for (9.2.8), we can reinsert the coefficient a in the integral; in particular, taking into account the modulus of continuity and applying (9.0.2), (9.2.6), and once more the

inequality leading to (9.2.7), we have

$$\begin{aligned}
&\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\Big|\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}}\int_V \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}}\Big(a\Big(\frac{rk}{\varepsilon}\Big)-a\Big(\frac{x}{\varepsilon}\Big)\Big)|\nabla u_\varepsilon^{\rho\overline{\nu}}|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho\Big| \\
&\quad\le \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big)\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}}\int_V \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}}|\nabla u_\varepsilon^{\rho\overline{\nu}}|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho \\
&\quad\le \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big)\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\frac{1}{\alpha}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}} \\
&\qquad\int_V\Big(\sum_{n=1}^d\sum_{k\in\mathcal{I}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right)\int_{rk+rQ_{\rho\overline{\nu}}}\frac{|u_\varepsilon(y+r\rho\nu_n)-u_\varepsilon(y)|^2}{|r\rho|^2}\,\mathrm{d}y + E_\varepsilon(\rho,\overline{\nu})\Big)\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho \\
&\quad\le \omega\Big(\frac{r\sqrt{d}}{\varepsilon}\Big)\frac{1}{\alpha}F_\varepsilon(u_\varepsilon)+o_\varepsilon(1),
\end{aligned}$$

and we conclude as above.

We finally obtain

$$\begin{aligned}
F_\varepsilon(u_\varepsilon)+o_\varepsilon(1) &\ge \frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}}\int_V \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}} a\left(\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon^{\rho\overline{\nu}}|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho \\
&\ge \frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)}\int_0^1 \frac{r^{2(1-s)}}{\rho^{-1+2s}}\int_V\int_{\Omega'} a\left(\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon^{\rho\overline{\nu}}|^2\,\mathrm{d}x\,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})\,\mathrm{d}\rho, \qquad (9.2.10)
\end{aligned}$$

where Ω' is any open subset well contained in Ω that we may assume to be well contained in $\bigcup_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} rk+rQ_{\rho\overline{\nu}}$ for ε sufficiently small.

Step 3. (Passage to the classical Homogenisation Theorem) In the final part of the proof of the lower bound, we average the auxiliary functions obtained by discretisation weighting them with the appropriate kernel.

For every $\varepsilon>0$ sufficiently small we define the probability measure μ_ε on $[0,1]\times V$ by

$$\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \coloneqq 2(1-s)\rho^{1-2s}\mathrm{d}\rho\frac{\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu})}{\mathcal{H}^{k_d}(V)};$$

then we rewrite (9.2.10) as

$$\begin{aligned}
&\frac{\sigma_{d-1}}{2d}\int_{[0,1]\times V} r^{2(1-s)}\int_{\Omega'} a\left(\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon^{\rho\overline{\nu}}|^2\,\mathrm{d}x\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \\
&\qquad = \frac{\sigma_{d-1}}{2d}\int_{\Omega'} r^{2(1-s)}a\left(\frac{x}{\varepsilon}\right)\int_{[0,1]\times V}|\nabla u_\varepsilon^{\rho\overline{\nu}}|^2\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\,\mathrm{d}x \\
&\qquad \ge \frac{\sigma_{d-1}}{2d}\int_{\Omega'} r^{2(1-s)}a\left(\frac{x}{\varepsilon}\right)\Big|\int_{[0,1]\times V}\nabla u_\varepsilon^{\rho\overline{\nu}}\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\Big|^2\,\mathrm{d}x \qquad (9.2.11)
\end{aligned}$$

by Jensen's inequality.

We set

$$\overline{u}_\varepsilon(x) \coloneqq \int_{[0,1]\times V} u_\varepsilon^{\rho\overline{\nu}}(x)\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}),$$

and note that, since $u_\varepsilon^{\rho\overline{\nu}}$ is continuous on Ω', this function belongs to $\mathrm{H}^1(\Omega')$ with

$$\nabla\overline{u}_\varepsilon(x) = \int_{[0,1]\times V}\nabla u_\varepsilon^{\rho\overline{\nu}}(x)\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})$$

for $\mathcal{L}^d$-a.e. $x \in \Omega'$. As a consequence, (9.2.11) reduces to

$$\frac{\sigma_{d-1}}{2d} \int_{\Omega'} r^{2(1-s)} a\Big(\frac{x}{\varepsilon}\Big) |\nabla \overline{u}_\varepsilon(x)|^2 \,\mathrm{d}x.$$

Our final goal is to show that

$$\liminf_{\varepsilon \to 0^+} \frac{\sigma_{d-1}}{2d} \int_{\Omega'} a\Big(\frac{x}{\varepsilon}\Big) |\nabla \overline{u}_\varepsilon(x)|^2 \,\mathrm{d}x \;\geq\; \frac{\sigma_{d-1}}{2d} \int_{\Omega'} A_{\mathrm{hom}} \nabla u \cdot \nabla u \,\mathrm{d}x.$$

To prove this inequality it is sufficient to prove that $\overline{u}_\varepsilon \to u$ in $L^2(\Omega')$. Indeed, this implies that u is the $L^2(\Omega')$-limit of a bounded sequence in $\mathrm{H}^1(\Omega')$, so that $u \in \mathrm{H}^1(\Omega')$ and the conclusion follows from the liminf inequality provided by the Sobolev Homogenisation Theorem (see [47, Theorem 14.7]) and the fact that $r^{2(1-s_\varepsilon)} \to 1$ by the second equality in (9.1.2).

First, we prove the convergence under the additional assumption that $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(\Omega)$. If this is the case, it suffices to prove that $\overline{u}_\varepsilon - u_\varepsilon \to 0^+$ in $L^1(\Omega')$. Indeed, since $\{\overline{u}_\varepsilon\}_\varepsilon$ is bounded by construction in $L^\infty(\Omega')$, we conclude by interpolation.

As $\Omega' \subseteq \bigcup_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} rk + rQ_{\rho\overline{\nu}}$, we have

$$\begin{aligned}
\|\overline{u}_\varepsilon - u_\varepsilon\|_{L^1(\Omega')} &= \int_{\Omega'} \Big| u_\varepsilon(x) - \int_{[0,1]\times V} u_\varepsilon^{\rho\overline{\nu}}(x) \mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \Big| \,\mathrm{d}x \\
&\leq \int_{[0,1]\times V} \int_{\Omega'} |u_\varepsilon(x) - u_\varepsilon^{\rho\overline{\nu}}(x)| \,\mathrm{d}x \,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \\
&\leq \int_{[0,1]\times V} \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}} |u_\varepsilon(x) - u_\varepsilon^{\rho\overline{\nu}}(x)| \,\mathrm{d}x \,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \\
&\leq \int_{[0,1]\times V} \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}} |u_\varepsilon(x) - u_\varepsilon^{\rho\overline{\nu}}(rk)| \,\mathrm{d}x \,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \qquad (9.2.12) \\
&\quad + \int_{[0,1]\times V} \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}} |u_\varepsilon^{\rho\overline{\nu}}(x) - u_\varepsilon^{\rho\overline{\nu}}(rk)| \,\mathrm{d}x \,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}). \qquad (9.2.13)
\end{aligned}$$

We first prove that (9.2.12) tends to 0 as $\varepsilon \to 0^+$. Applying Hölder's inequality and a rescaled version of Poincaré-Wirtinger's inequality (see [144, Theorem 6.33]), we get

$$\begin{aligned}
\sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}} |u_\varepsilon(x) - u_\varepsilon^{\rho\overline{\nu}}(rk)| \,\mathrm{d}x \;&\leq |r\rho|^{\frac{d}{2}} \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} \Big(\int_{rk+rQ_{\rho\overline{\nu}}} |u_\varepsilon(x) - u_\varepsilon^{\rho\overline{\nu}}(rk)|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \\
&\leq P|r\rho|^{\frac{d}{2}+s} \sum_{k \in \mathcal{J}^r_{\rho\overline{\nu}}} \Big(\iint_{(rk+rQ_{\rho\overline{\nu}})\times(rk+rQ_{\rho\overline{\nu}})} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y \Big)^{\frac{1}{2}}, \qquad (9.2.14)
\end{aligned}$$

where P is the Poincaré-Wirtinger constant for the d-dimensional unit cube. Then we make

use of the concavity of $x \mapsto x^{\frac{1}{2}}$ so that

$$\begin{aligned}
&P|r\rho|^{\frac{d}{2}+s}\#\mathcal{J}^r_{\rho\overline{\nu}} \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} \frac{1}{\#\mathcal{J}^r_{\rho\overline{\nu}}}\Big(\iint_{(rk+rQ_{\rho\overline{\nu}})\times(rk+rQ_{\rho\overline{\nu}})} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}\\
&\quad\le P|r\rho|^{\frac{d}{2}+s}\#\mathcal{J}^r_{\rho\overline{\nu}}\Big(\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} \frac{1}{\#\mathcal{J}^r_{\rho\overline{\nu}}}\iint_{(rk+rQ_{\rho\overline{\nu}})\times(rk+rQ_{\rho\overline{\nu}})} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}\\
&\quad\le P|r\rho|^{\frac{d}{2}+s}(\#\mathcal{J}^r_{\rho\overline{\nu}})^{\frac{1}{2}}\Big(\iint_{\Omega\times\Omega} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}\\
&\quad\le P|r\rho|^{\frac{d}{2}+s}|r\rho|^{-\frac{d}{2}}\mathcal{L}^d(\Omega)^{\frac{1}{2}}[u_\varepsilon]_{\mathrm{H}^s(\Omega)}\\
&\quad= P|r\rho|^{s}\,\mathcal{L}^d(\Omega)^{\frac{1}{2}}[u_\varepsilon]_{\mathrm{H}^s(\Omega)}.
\end{aligned}$$

After integration we obtain that (9.2.12) is bounded above by

$$2Pr^s\,\mathcal{L}^d(\Omega)^{\frac{1}{2}}(1-s)[u_\varepsilon]_{\mathrm{H}^s(\Omega)}\int_0^1 \rho^{1-s}\mathrm{d}\rho = 2Pr^s\mathcal{L}^d(\Omega)^{\frac{1}{2}}(1-s)[u_\varepsilon]_{\mathrm{H}^s(\Omega)}\frac{1}{2-s}.$$

Then we use that $(1-s)[u_\varepsilon]^2_{\mathrm{H}^s(\Omega)} \le F_\varepsilon(u_\varepsilon)/\alpha$, which is uniformly bounded in ε by assumption, and that $r^s = \varepsilon^{2s} \to 0$ as $\varepsilon \to 0^+$ to deduce that the above term vanishes.

We treat (9.2.13) with a similar argument. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_{[0,1]\times V}\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}} |u^{\rho\overline{\nu}}_\varepsilon(x)-u^{\rho\overline{\nu}}_\varepsilon(rk)|\,\mathrm{d}x\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\\
&\qquad\le \int_{[0,1]\times V}\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u^{\rho\overline{\nu}}_\varepsilon(x)||x-rk|\,\mathrm{d}x\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\\
&\qquad\le \sqrt{d}\int_{[0,1]\times V}|r\rho|\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u^{\rho\overline{\nu}}_\varepsilon(x)|\,\mathrm{d}x\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})
\end{aligned}$$

and by Hölder's inequality and the concavity of $x \mapsto x^{\frac{1}{2}}$, this is bounded above by

$$\begin{aligned}
&\sqrt{d}\int_{[0,1]\times V}|r\rho|^{\frac{d}{2}+1}\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\Big(\int_{rk+rQ_{\rho\overline{\nu}}}|\nabla u^{\rho\overline{\nu}}_\varepsilon(x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\\
&\qquad= \sqrt{d}\int_{[0,1]\times V}|r\rho|^{\frac{d}{2}+1}\#\mathcal{J}^r_{\rho\overline{\nu}}\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\frac{1}{\#\mathcal{J}^r_{\rho\overline{\nu}}}\Big(\int_{rk+rQ_{\rho\overline{\nu}}}|\nabla u^{\rho\overline{\nu}}_\varepsilon(x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\\
&\qquad\le \sqrt{d}\int_{[0,1]\times V}|r\rho|^{\frac{d}{2}+1}(\#\mathcal{J}^r_{\rho\overline{\nu}})^{\frac{1}{2}}\Big(\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}}|\nabla u^{\rho\overline{\nu}}_\varepsilon(x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\\
&\qquad\le r\sqrt{d\,\mathcal{L}^d(\Omega)}\int_{[0,1]\times V}\Big(\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}}|\nabla u^{\rho\overline{\nu}}_\varepsilon(x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}),
\end{aligned}$$

which by Jensen's inequality is less than or equal to

$$r\sqrt{d\,\mathcal{L}^d(\Omega)}\Big(\int_{[0,1]\times V}\sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}}\int_{rk+rQ_{\rho\overline{\nu}}}|\nabla u^{\rho\overline{\nu}}_\varepsilon(x)|^2\,\mathrm{d}x\,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu})\Big)^{\frac{1}{2}}. \tag{9.2.15}$$

Now we write

$$\begin{aligned}
&\int_{[0,1]\times V} \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u_\varepsilon^{\rho\overline{\nu}}(x)|^2 \,\mathrm{d}x \,\mathrm{d}\mu_\varepsilon(\rho,\overline{\nu}) \\
&= \frac{2(1-s)}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{1}{\rho^{-1+2s}} \int_V \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} \int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u_\varepsilon^{\rho\overline{\nu}}|^2 \,\mathrm{d}x \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho \\
&\leq \Big(\frac{2d}{\alpha\sigma_{d-1}}\Big)\frac{1-s}{d}\frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{1}{\rho^{-1+2s}} \int_V \sum_{k\in\mathcal{J}^r_{\rho\overline{\nu}}} a\left(\frac{rk}{\varepsilon}\right) \int_{rk+rQ_{\rho\overline{\nu}}} |\nabla u_\varepsilon^{\rho\overline{\nu}}|^2 \,\mathrm{d}x \,\mathrm{d}\mathcal{H}^{k_d}(\overline{\nu}) \,\mathrm{d}\rho
\end{aligned}$$

and we note that this last term equals (9.2.8) (up to a constant factor), therefore, as a byproduct of the previous computations (see (9.2.10) and (9.2.11)), it is bounded above by

$$\frac{2d}{\alpha\sigma_{d-1}} r^{-2(1-s)} F_\varepsilon(u_\varepsilon) + o_\varepsilon(1),$$

which is uniformly bounded for small ε (recall the second equality in (9.1.2)). We conclude that (9.2.15) vanishes since $r \to 0^+$.

Step 4. (The unbounded case) Finally, we remove the L^∞-boundedness condition by means of a truncation argument. We set $u^M \coloneqq (u \wedge M) \vee -M$ and note that $u_\varepsilon \to u$ in $L^2(\Omega)$ implies $u_\varepsilon^M \to u^M$ in $L^2(\Omega)$. Since $F_\varepsilon(u_\varepsilon) \geq F_\varepsilon(u_\varepsilon^M)$, we have

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon^M) \geq \frac{\sigma_{d-1}}{2d} \int_\Omega A_{\mathrm{hom}} \nabla u^M \cdot \nabla u^M \,\mathrm{d}x,$$

and then

$$\sup_{M>0} \|\nabla u^M\|^2_{L^2(\Omega)} \leq \frac{2d}{\alpha\sigma_{d-1}} \sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty,$$

so that $u \in \mathrm{H}^1(\Omega)$ with $\nabla u^M \to \nabla u$. The proof is concluded once we let $M \to +\infty$.

Proof of the limsup inequality. To prove the upper bound, we first study the case where u is piecewise-affine. We then recover the inequality on the whole space $\mathrm{H}^1(\Omega)$ by means of a density argument. For the sake of clarity, we first deal with the case where u is affine.

Step 1 (u is affine) We first assume that $u(x) = z \cdot x$ for some $z \in \mathbb{R}^d$. We recall that by the classical homogenisation formula for A_{hom} (see [47, Theorem 14.7]) and by density, it holds

$$A_{\mathrm{hom}} z \cdot z = \inf\Big\{ \int_{(0,1)^d} a(y)|\nabla\varphi(y) + z|^2 \mathrm{d}y \colon\ \varphi \in C^\infty(\mathbb{R}^d),\ \text{1-periodic} \Big\}. \tag{9.2.16}$$

Fix $\delta > 0$ and let $\varphi \in C^\infty(\mathbb{R}^d)$ be a 1-periodic function such that

$$\int_{(0,1)^d} a(y)|\nabla\varphi(y) + z|^2 \mathrm{d}y < A_{\mathrm{hom}} z \cdot z + \delta; \tag{9.2.17}$$

we claim that $u_\varepsilon(x) = \langle z, x\rangle + \varepsilon\varphi\left(\frac{x}{\varepsilon}\right)$ is an approximate recovery sequence for u, i.e., it is such that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \leq \frac{\sigma_{d-1}}{2d} \int_\Omega A_{\mathrm{hom}} z \cdot z \,\mathrm{d}x + \delta. \tag{9.2.18}$$

This, combined with a diagonal argument and the arbitrariness of $\delta > 0$, will lead to the limsup inequality.

In order to prove (9.2.18), we first observe that $\{u_\varepsilon\}_\varepsilon$ are uniformly Lipschitz and equibounded in $L^2(\Omega)$, so that Lemma 9.1.1 applies to $\{u_\varepsilon\}_\varepsilon$ with $r_\varepsilon = \varepsilon^2$ (see Remark 9.1.2). Using this lemma and the first-order Taylor expansion of φ, we get

$$\begin{aligned}
&F_\varepsilon(u_\varepsilon) + o_\varepsilon(1)\\
&\quad = (1-s)\iint_{\Omega\times\Omega} a\left(\frac{x}{\varepsilon}\right)\frac{|z\cdot(x-y)+\varepsilon\varphi\left(\frac{x}{\varepsilon}\right)-\varepsilon\varphi\left(\frac{y}{\varepsilon}\right)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y\\
&\quad = (1-s)\iint_{\Omega\times\Omega\cap\{|x-y|\le r_\varepsilon\}} a\left(\frac{x}{\varepsilon}\right)\frac{|(z+\nabla\varphi\left(\frac{x}{\varepsilon}\right))\cdot(x-y)+\frac{1}{\varepsilon}R(|x-y|)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y + o_\varepsilon(1),
\end{aligned}$$

where the function $R(|x-y|)$ is the remainder in Lagrange form of φ and it is such that

$$R(|x-y|) \le d^2\|\nabla^{(2)}\varphi\|_{L^\infty(\mathbb{R}^d)}|x-y|^2. \tag{9.2.19}$$

For brevity, in what follows we denote by R both the remainder and its absolute value.

For every $\eta > 0$, using the inequality $|a+b|^2 \le (1+\eta)a^2 + (1+1/\eta)b^2$ and the change of variables $\xi := y - x$, we deduce

$$F_\varepsilon(u_\varepsilon) + o_\varepsilon(1) \le (1+\eta)(1-s)\int_\Omega\int_{B(0,r)} a\left(\frac{x}{\varepsilon}\right)\frac{|\langle z+\nabla\varphi\left(\frac{x}{\varepsilon}\right),\xi\rangle|^2}{|\xi|^{d+2s}}\,\mathrm{d}\xi\,\mathrm{d}x \tag{9.2.20}$$

$$+\left(1+\frac{1}{\eta}\right)\frac{1-s}{\varepsilon^2}\int_\Omega\int_{B(0,r)} a\left(\frac{x}{\varepsilon}\right)\frac{R(|\xi|)^2}{|\xi|^{d+2s}}\,\mathrm{d}\xi\,\mathrm{d}x. \tag{9.2.21}$$

For a fixed $\eta > 0$, the term in (9.2.21) vanishes as $\varepsilon \to 0^+$. Indeed using (9.0.2), we have

$$\begin{aligned}
\int_\Omega\int_{B(0,r)} a\left(\frac{x}{\varepsilon}\right)\frac{R(|\xi|)^2}{|\xi|^{d+2s}}\,\mathrm{d}\xi\,\mathrm{d}x &\le d^4\beta\mathcal{L}^d(\Omega)\|\nabla^{(2)}\varphi\|^2_{L^\infty(\mathbb{R}^d)}\int_{B(0,r)}\frac{1}{|\xi|^{d+2s-4}}\mathrm{d}\xi\\
&= d^4\sigma_{d-1}\beta\mathcal{L}^d(\Omega)\|\nabla^{(2)}\varphi\|^2_{L^\infty(\mathbb{R}^d)}\int_0^r \rho^{3-2s}\mathrm{d}\rho\\
&= d^4\frac{\sigma_{d-1}}{4-2s}\beta\mathcal{L}^d(\Omega)\|\nabla^{(2)}\varphi\|^2_{L^\infty(\mathbb{R}^d)}r^{2(1-s)}r^2,
\end{aligned} \tag{9.2.22}$$

so that the term in (9.2.21) is bounded from above by

$$C_{\eta,\varphi}(1-s)\Big(\frac{r}{\varepsilon}\Big)^2 r^{2(1-s)},$$

for a constant $C_{\eta,\varphi} > 0$ depending only on η and φ. Using the two equalities in (9.1.2), we deduce that (9.2.21) converges to 0 as $\varepsilon \to 0^+$.

As for the term in (9.2.20), note that by the symmetry of the integrand, it holds

$$\begin{aligned}
\int_{B(0,r)}\frac{|(z+\nabla\varphi\left(\frac{x}{\varepsilon}\right))\cdot\xi|^2}{|\xi|^{d+2s}}\,\mathrm{d}\xi &= \int_0^r\int_{\partial B(0,\rho)}\frac{\left|(z+\nabla\varphi\left(\frac{x}{\varepsilon}\right))\cdot\nu\right|^2}{\rho^{d+2s}}\mathrm{d}\mathcal{H}^{d-1}(\nu)\,\mathrm{d}\rho\\
&= \int_0^r\frac{1}{\rho^{d+2(s-1)}}\frac{1}{d}\int_{\partial B(0,\rho)}\left|z+\nabla\varphi\left(\frac{x}{\varepsilon}\right)\right|^2\mathrm{d}\mathcal{H}^{d-1}(\nu)\,\mathrm{d}\rho\\
&= \frac{\sigma_{d-1}}{d}\left|z+\nabla\varphi\left(\frac{x}{\varepsilon}\right)\right|^2\int_0^r\rho^{1-2s}\mathrm{d}\rho\\
&= \frac{\sigma_{d-1}}{2d}\left|z+\nabla\varphi\left(\frac{x}{\varepsilon}\right)\right|^2\frac{r^{2(1-s)}}{(1-s)}.
\end{aligned} \tag{9.2.23}$$

Recalling that both φ and a are 1-periodic, we may substitute (9.2.23) in (9.2.20) to get

$$\begin{aligned}(1+\eta)(1-s)&\int_\Omega\int_{B(0,r)} a\left(\frac{x}{\varepsilon}\right)\frac{|(z+\nabla\varphi\left(\frac{x}{\varepsilon}\right))\cdot\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\&=(1+\eta)\frac{\sigma_{d-1}}{2d}r^{2(1-s)}\int_\Omega a\left(\frac{x}{\varepsilon}\right)\left|\nabla\varphi\left(\frac{x}{\varepsilon}\right)+z\right|^2\,\mathrm{d}x\\&=(1+\eta)\frac{\sigma_{d-1}}{2d}r^{2(1-s)}\frac{\mathcal{L}^d(\Omega)}{\varepsilon^d}\int_{(0,\varepsilon)^d} a\left(\frac{x}{\varepsilon}\right)\left|\nabla\varphi\left(\frac{x}{\varepsilon}\right)+z\right|^2\,\mathrm{d}x+o_\varepsilon(1)\\&=(1+\eta)\frac{\sigma_{d-1}}{2d}r^{2(1-s)}\mathcal{L}^d(\Omega)\int_{(0,1)^d} a(y)\left|\nabla\varphi\left(y\right)+z\right|^2\,\mathrm{d}y+o_\varepsilon(1),\end{aligned}\tag{9.2.24}$$

which, in view of the second equality in (9.1.2) and (9.2.17), implies

$$\limsup_{\varepsilon\to0^+}F_\varepsilon(u_\varepsilon)\le(1+\eta)\Big(\frac{\sigma_{d-1}}{2d}\int_\Omega A_{\mathrm{hom}}z\cdot z\,\mathrm{d}x+\delta\Big),$$

and we conclude by letting η tend to 0.

Step 2. (u is piecewise-affine) We now assume that the function u is piecewise-affine in Ω. More precisely, we suppose that there exists a finite family of d-simplices $\{\Delta_i\}_{i\in\mathcal{I}}$ covering Ω, vectors $\{z_i\}_{i\in\mathcal{I}}\subseteq\mathbb{R}^d$, and constants $\{m_i\}_{i\in\mathcal{I}}\subseteq\mathbb{R}$ such that $u(x)=\langle x,z_i\rangle+m_i$ for every $x\in\Delta_i\cap\Omega$ and for every $i\in\mathcal{I}$. To simplify the notation, from now on we will always write Δ_i in place of $\Delta_i\cap\Omega$.

In analogy with the case treated in *Step* 1, for a fixed $\delta>0$ and for every $i\in\mathcal{I}$, we consider a 1-periodic function $\varphi_i\in C^\infty(\mathbb{R}^d)$ satisfying

$$\int_{(0,1)^d}a(y)|\nabla\varphi_i(y)+z_i|^2\mathrm{d}y<A_{\mathrm{hom}}z_i\cdot z_i+\frac{\delta}{\#\mathcal{I}}.\tag{9.2.25}$$

Having in mind the construction of the recovery sequence performed in the previous case, we want to construct an approximate recovery sequence for u by perturbing, in a suitable way, u with the functions φ_i. To this end, for every Δ_i, we consider the subset defined by

$$\Delta_i^\varepsilon:=\{x\in\Omega\colon\ \mathrm{dist}(x,\Delta_i^c)>\varepsilon\}.$$

Given $\varepsilon>0$ and $i\in\mathcal{I}$, we consider a positive function $\psi_\varepsilon^i\in C_c^\infty(\mathbb{R}^d)$, satisfying

$$\begin{cases}\psi_\varepsilon^i(x)=0 & \text{for } x\notin\Delta_i^\varepsilon\\ \psi_\varepsilon^i(x)=1 & \text{for } x\in\Delta_i^{2\varepsilon},\\ 0\le\psi_\varepsilon^i(x)\le1 & \text{for } x\in\mathbb{R}^d,\\ \|\nabla\psi_\varepsilon^i\|_{L^\infty(\mathbb{R}^d)}\le\frac{2}{\varepsilon}.\end{cases}\tag{9.2.26}$$

We claim that an approximate recovery sequence for u is given by

$$u_\varepsilon(x)=\sum_{i\in\mathcal{I}}\Big(\langle z_i,x\rangle+\varepsilon\varphi_i\left(\frac{x}{\varepsilon}\right)\psi_\varepsilon^i(x)+m_i\Big)\chi_{\Delta_i}(x),$$

where χ_{Δ_i} denotes the characteristic function of Δ_i. Note that u_ε is a continuous function, since it coincides with u on the boundary of each simplex Δ_i.

Let $A \coloneqq \Omega \times \Omega \cap \{|x-y| \le r\}$, where $r = \varepsilon^2$. Since the functions $\{u_\varepsilon\}_\varepsilon$ are uniformly Lipschitz and equibounded in $L^2(\Omega)$, in view of Lemma 9.1.1 and Remark 9.1.2, to conclude the proof in this case it is enough to show that

$$\limsup_{\varepsilon\to 0^+}(1-s)\iint_A a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y \le \frac{\sigma_{d-1}}{2d}\int_\Omega A_{\hom}\nabla u\cdot\nabla u\,\mathrm{d}x+\delta.$$

To make the computations more manageable, we consider a suitable partition of A. For $i\in\mathcal{I}$, we set

$$B_i \coloneqq \Delta_i^{2\varepsilon}\times\Delta_i^{2\varepsilon}\cap\{|x-y|\le r\},$$
$$C_i \coloneqq \Delta_i\times\Delta_i\cap\{|x-y|\le r \text{ and } x \text{ or } y\in\Delta_i\setminus\Delta_i^{2\varepsilon}\},$$

and for $j\in\mathcal{I}$ with $j\neq i$, we set

$$D_{ij}\coloneqq\Delta_i\times\Delta_j\cap\{|x-y|\le r\},$$

so that $A=\bigcup_{i\in\mathcal{I}}(B_i\cup C_i)\cup\bigcup_{i\neq j}D_{ij}$.

Taking advantage of this decomposition and of the subadditivity of the lim sup, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+}(1-s)\iint_A a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y\\
&\qquad\le\limsup_{\varepsilon\to0^+}\sum_{i\in\mathcal{I}}(1-s)\iint_{B_i}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y &(9.2.27)\\
&\qquad\quad+\sum_{i\in\mathcal{I}}\limsup_{\varepsilon\to0^+}(1-s)\iint_{C_i}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y &(9.2.28)\\
&\qquad\quad+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I},\,j\neq i}\limsup_{\varepsilon\to0^+}(1-s)\iint_{D_{ij}}a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y. &(9.2.29)
\end{aligned}$$

We begin by estimating (9.2.27). Since for fixed $i\in\mathcal{I}$ we have $\psi_\varepsilon^i=1$ on B_i, by the first-order Taylor expansion of φ_i, for a fixed $\eta>0$ we get

$$\begin{aligned}
&\sum_{i\in\mathcal{I}}(1-s)\iint_{B_i}a\left(\frac{x}{\varepsilon}\right)\frac{|(z_i+\nabla\varphi_i\left(\frac{x}{\varepsilon}\right))\cdot(x-y)+\frac{1}{\varepsilon}R_i(|x-y|)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y\\
&\qquad\le\sum_{i\in\mathcal{I}}(1+\eta)(1-s)\int_{\Delta_i^{2\varepsilon}}\int_{B(0,r)}a\left(\frac{x}{\varepsilon}\right)\frac{|(z_i+\nabla\varphi_i\left(\frac{x}{\varepsilon}\right))\cdot\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x &(9.2.30)\\
&\qquad\quad+\sum_{i\in\mathcal{I}}\left(1+\frac{1}{\eta}\right)\frac{1-s}{\varepsilon^2}\int_{\Delta_i^{2\varepsilon}}\int_{B(0,r)}a\left(\frac{x}{\varepsilon}\right)\frac{R_i(|\xi|)^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x, &(9.2.31)
\end{aligned}$$

where R_i is the remainder in Lagrange form of the first-order Taylor expansion of φ_i, which satisfies

$$R_i(|x-y|)\le d^2\max_{i\in\mathcal{I}}\|\nabla^{(2)}\varphi_i\|_{L^\infty(\mathbb{R}^d)}|x-y|^2=:M|x-y|^2. \tag{9.2.32}$$

Thus, we may estimate (9.2.31) with

$$\beta M^2\sum_{i\in\mathcal{I}}\left(1+\frac{1}{\eta}\right)\frac{1-s}{\varepsilon^2}\int_{\Delta_i}\int_{B(0,r)}\frac{|\xi|^4}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x,$$

which is seen to converge to 0 for $\varepsilon \to 0^+$ by the same computations that have been used in the previous step to prove (9.2.22) (with Ω replaced by Δ_i).

As for (9.2.30), passing to the slightly larger domains $\Delta_i^{2\varepsilon}$ and taking advantage of the computations leading to (9.2.24), we see that

$$\sum_{i\in\mathcal{I}}(1+\eta)(1-s)\int_{\Delta_i^{2\varepsilon}}\int_{B(0,r)} a\left(\frac{x}{\varepsilon}\right)\frac{|(z_i+\nabla\varphi_i\left(\frac{x}{\varepsilon}\right))\cdot\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x$$
$$\leq (1+\eta)\frac{\sigma_{d-1}}{2d}r^{2(1-s)}\sum_{i\in\mathcal{I}}\mathcal{L}^d(\Delta_i)\int_{(0,1)^d} a(y)\left|\nabla\varphi_i\left(y\right)+z_i\right|^2\mathrm{d}y+o_\varepsilon(1).$$

Recalling (9.2.25) and the second equality in (9.1.2), we finally conclude

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}&\sum_{i\in\mathcal{I}}(1-s)\iint_{B_i} a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq \limsup_{\varepsilon\to 0^+}(1+\eta)\frac{\sigma_{d-1}}{2d}r^{2(1-s)}\sum_{i\in\mathcal{I}}\Big(\int_{\Delta_i}A_{\mathrm{hom}}z_i\cdot z_i\,\mathrm{d}x+\frac{\delta}{\#\mathcal{I}}\Big)\\ &\leq (1+\eta)\frac{\sigma_{d-1}}{2d}\int_\Omega A_{\mathrm{hom}}\nabla u\cdot\nabla u\,\mathrm{d}x+\delta,\end{aligned}\tag{9.2.33}$$

so that letting $\eta\to 0^+$, we obtain the desired estimate for (9.2.27).

We now show that (9.2.28) converges to 0 as $\varepsilon\to 0^+$. Since $\#\mathcal{I}$ is finite and independent of ε, it suffices to consider separately each single term of the sum. By the convexity of $x\mapsto x^2$, we have

$$(1-s)\iint_{C_i} a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y\leq 3\beta(1-s)\iint_{C_i}\frac{|z_i|^2|x-y|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y \tag{9.2.34}$$
$$+3\beta(1-s)\iint_{C_i}\frac{\left|\varepsilon\psi_\varepsilon^i(y)\left(\varphi_i\left(\frac{x}{\varepsilon}\right)-\varphi_i\left(\frac{y}{\varepsilon}\right)\right)\right|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y \tag{9.2.35}$$
$$+3\beta(1-s)\iint_{C_i}\frac{\left|\varepsilon\varphi_i\left(\frac{x}{\varepsilon}\right)\left(\psi_\varepsilon^i(x)-\psi_\varepsilon^i(y)\right)\right|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y. \tag{9.2.36}$$

We start with (9.2.34). By a change of variables, we see that

$$\begin{aligned}(1-s)\iint_{C_i}\frac{|z_i|^2|x-y|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y&\leq 2(1-s)|z_i|^2\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{|\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\ &\leq 2(1-s)|z_i|^2\mathcal{L}^d(\Delta_i\setminus\Delta_i^{2\varepsilon})\int_{B(0,r)}\frac{|\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\\ &=\sigma_{d-1}|z_i|^2\mathcal{L}^d(\Delta_i\setminus\Delta_i^{2\varepsilon})r^{2(1-s)},\end{aligned}$$

which converges to zero as $\varepsilon\to 0^+$, in view of the second equality in (9.1.2) and the fact that $\mathcal{L}^d(\Delta_i\setminus\Delta_i^{2\varepsilon})\to 0$.

As for (9.2.35), using the first-order Taylor expansions of φ_i, (9.2.26), and recalling

(9.2.32), one gets

$$
\begin{aligned}
&(1-s)\iint_{C_i}\frac{\left|\varepsilon\psi^i_\varepsilon(y)\left(\varphi_i\left(\frac{x}{\varepsilon}\right)-\varphi_i\left(\frac{y}{\varepsilon}\right)\right)\right|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y\\
&\le 2(1-s)\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{\left|(\nabla\varphi_i\left(\frac{x}{\varepsilon}\right))\cdot\xi+\frac{1}{\varepsilon}R_i(|\xi|)\right|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\
&\le 4(1-s)\Big(\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{\left|(\nabla\varphi_i\left(\frac{x}{\varepsilon}\right))\cdot\xi\right|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x+\frac{M^2}{\varepsilon^2}\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{|\xi|^4}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\Big).
\end{aligned}
\tag{9.2.37}
$$

The first term can be estimated with

$$
\begin{aligned}
&(1-s)\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{|(\nabla\varphi_i\left(\frac{x}{\varepsilon}\right))\cdot\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\
&\qquad\le(1-s)\|\nabla\varphi_i\|^2_{L^\infty(\mathbb{R}^d)}\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{|\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\
&\qquad=(1-s)\|\nabla\varphi_i\|^2_{L^\infty(\mathbb{R}^d)}\mathcal{L}^d(\Delta_i\setminus\Delta_i^{2\varepsilon})\int_{B(0,r)}\frac{|\xi|^2}{|\xi|^{d+2s}}\,\mathrm{d}\xi\\
&\qquad\le\frac{\sigma_{d-1}}{2}\|\nabla\varphi_i\|^2_{L^\infty(\mathbb{R}^d)}\mathcal{L}^d(\Delta_i\setminus\Delta_i^{2\varepsilon})r^{2(1-s)},
\end{aligned}
\tag{9.2.38}
$$

while for the second term we have

$$
\frac{1-s}{\varepsilon^2}\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{|\xi|^4}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\le(1-s)\frac{\sigma_{d-1}}{4-2s}\mathcal{L}^d(\Delta_i\setminus\Delta_i^{2\varepsilon})\Big(\frac{r}{\varepsilon}\Big)^2r^{2(1-s)},
\tag{9.2.39}
$$

and therefore both terms vanish as ε tends to 0.

We note that from (9.2.26) we deduce that

$$
\varepsilon^2|\psi^i_\varepsilon(x)-\psi^i_\varepsilon(y)|^2\le 4|x-y|^2\quad\text{for all } x,y\in\mathbb{R}^d,
$$

so that we may estimate (9.2.36) as

$$
\begin{aligned}
&(1-s)\iint_{C_i}\frac{\left|\varepsilon\varphi_i\left(\frac{x}{\varepsilon}\right)(\psi^i_\varepsilon(x)-\psi^i_\varepsilon(y))\right|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y\\
&\qquad\le 4(1-s)\|\varphi_i\|^2_{L^\infty(\mathbb{R}^d)}\int_{\Delta_i\setminus\Delta_i^{2\varepsilon}}\int_{B(0,r)}\frac{|\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x
\end{aligned}
$$

which goes to zero by the same computations that led to (9.2.38). Combining the previous estimates, we conclude that (9.2.28) tends to 0 for $\varepsilon\to 0^+$.

Finally, we prove that (9.2.29) converges to 0 as $\varepsilon\to 0^+$. Also in this case, since $\#\mathcal{I}$ is finite and independent of ε, it suffices to take into account just one summand.

Note that, if $(x,y)\in D_{ij}$, then $x\in\Delta_i\setminus\Delta_i^\varepsilon$ and $y\in\Delta_j\setminus\Delta_j^\varepsilon$, so that $\psi^i_\varepsilon(x)=\psi^j_\varepsilon(y)=0$, which, in turn, gives $u_\varepsilon(x)=u(x)$ and $u_\varepsilon(y)=u(y)$. Thus, taking advantage of the Lipschitz

continuity of u, we get

$$\begin{aligned}(1-s)\iint_{D_{ij}} a\left(\frac{x}{\varepsilon}\right)\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\,\mathrm{d}y &\leq \beta(1-s)\int_{\Delta_i\setminus\Delta_i^\varepsilon}\int_{B(0,r)}\frac{|u(x+\xi)-u(x)|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\ &\leq \beta\mathrm{Lip}(u,\Omega)(1-s)\int_{\Delta_i\setminus\Delta_i^\varepsilon}\int_{B(0,r)}\frac{|\xi|^2}{|\xi|^{d+2s}}\mathrm{d}\xi\,\mathrm{d}x\\ &= \beta\mathrm{Lip}(u,\Omega)(1-s)\mathcal{L}^d(\Delta_i\setminus\Delta_i^\varepsilon)\sigma_{d-1}\frac{1}{2-2s}r^{2(1-s)}\\ &\leq \beta\mathrm{Lip}(u,\Omega)\frac{\sigma_{d-1}}{2}\mathcal{L}^d(\Delta_i\setminus\Delta_i^\varepsilon)\end{aligned}$$

and, since $\mathcal{L}^d(\Delta_i\setminus\Delta_i^\varepsilon)\to 0$, this last term converges to zero and so does (9.2.29). As we have shown that in the limit (9.2.27) can be bounded from above by (9.2.33) and that (9.2.28) and (9.2.29) go to zero, in this case the proof of the upper bound is concluded.

Step 3. (Conclusion) The functional F_{hom} defined in (9.0.5), which has been proven to be the desired Γ-limit on piecewise-affine functions, is continuous in the strong topology of $\mathrm{H}^1(\Omega)$. Since Ω is an open bounded set with Lipschitz boundary, every function $u\in\mathrm{H}^1(\Omega)$ may be extended to $\overline{u}\in\mathrm{H}^1_0(B(0,R))$ via the standard extension operator and a cut-off function. It is a known fact that a dense class in $\mathrm{H}^1_0(B(0,R))$ is the class of piecewise-affine functions, namely, functions which are piecewise-affine on a finite family of simplices covering $\overline{B}(0,R)$ (see [116, Proposition 2.1]). Therefore, the class of piecewise-affine functions discussed in Step 2 is dense in $\mathrm{H}^1_0(B(0,R))$ and *a fortiori* in $\mathrm{H}^1(\Omega)$. Having proved that the Γ-lim sup inequality holds on a dense subset of $\mathrm{H}^1(\Omega)$, we conclude that it holds for any $u\in\mathrm{H}^1(\Omega)$ (see Proposition 1.2.5). □

CHAPTER 10

Γ-CONVERGENCE OF NONLOCAL FUNCTIONALS TO FREE DISCONTINUITY PROBLEMS

In this final chapter, we study compactness properties with respect to Γ-convergence of a family of nonlocal functionals. More precisely, we study energies fashioned upon Gobbino's finite-difference approximation of the Mumford-Shah functional [132], which take the form

$$F_\varepsilon(u) \coloneqq \frac{1}{\varepsilon^d}\int_\Omega\int_\Omega f_\varepsilon\Big(x, \frac{y-x}{\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\Big)\mathrm{d}y\,\mathrm{d}x \quad \text{for } u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k), \tag{10.0.1}$$

and investigate their behaviour as the parameter $\varepsilon > 0$ goes to 0, forcing a concentration phenomenon to occur. The integrands f_ε are rather general non-negative functions, satisfying suitable upper and lower bounds making F_ε comparable with Gobbino's functional (see (10.1.6) below). Under these hypotheses, we prove that every sequence of the form (10.0.1) Γ-converges, up to subsequences, to a local integral functional made of the sum of a bulk and surface term (see Theorem 10.2.3). Relying on this result we then deduce necessary and sufficient conditions for the Γ-convergence of sequences as in (10.0.1) (see Theorem 10.2.4).

The proof of the compactness result requires several steps. It starts in Section 10.3, after having fixed the notation in Section 10.1 and presented the main results in Section 10.2. The first step consists in showing that, under our assumptions, the functionals F_ε "localise", so that as $\varepsilon \to 0^+$ we may neglect interactions between points x, y that are too far away from each other. This is done relying on techniques devised in [8]. This localisation lemma is crucial in the application of the localisation method of Γ-convergence we use in Section 10.4 to prove compactness properties of $\{F_\varepsilon\}_\varepsilon$. The proof of the compactness theorem is then concluded with the use of an integral representation result in $\mathrm{GSBV}^2(\Omega;\mathbb{R}^k)$, proved by means of a perturbation/truncation argument based on the results of [40]. The chapter ends with Section 10.6, where, after having proved certain asymptotic properties of minimisation problems for F_ε , we deduce necessary and sufficient conditions for Γ-convergence of such sequences.

This chapter is based on the joint paper with Brusca, Scalabrino, Trifone, & Voglino [53].

10.1 Notation and preliminary results

In this section, we fix the notation and present some preliminary results concerning the functionals that constitute the upper and lower bounds for our energies (10.0.1).

Notation. We let $\Omega \subset \mathbb{R}^d$ denote a bounded open set with Lipschitz boundary. We recall the notation for collections of open/Borel sets introduced in Section 1.1. In addition to the notation presented therein, in what follows we let $\mathcal{O}_{\mathrm{reg}}(\Omega)$ denote the collection of all open subsets of Ω with Lipschitz boundary. Given $E \subset \mathbb{R}^d, \varepsilon > 0$, and $\xi \in \mathbb{R}^d$, we also introduce the notation

$$E_\varepsilon(\xi) \coloneqq \{x \in E\colon\ x + \varepsilon\xi \in E\}.$$

For brevity of notation, in this chapter we let $B_r \coloneqq B(0,r)$ and $B_r(x) \coloneqq B(x,r)$.

For the definition and the main properties of the various spaces of functions of bounded variation used here, we refer the reader to Section 1.3. In particular, we recall the spaces

$\mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ and $\mathrm{GSBV}^2(\Omega;\mathbb{R}^k)$ introduced in Definition 1.3.13. We also recall that the latter is a vector space and that it is closed by components and vertical truncations, thanks to Lemma 1.3.14. Justified by this lemma, for every $u\colon \mathbb{R}^d \to \mathbb{R}^k$ and $M > 0$, u^M will always denote the truncated function whose i-th component is given by $u_i^M := (u_i \vee -M) \wedge M$.

Remark 10.1.1. Let $u \in \mathrm{GSBV}^2(\Omega;\mathbb{R}^k)$. One can check that the following conditions hold:

(i) $\nabla u^M = \nabla u$ for $\mathcal{L}^d$-a.e. $x \in \Omega$ such that $|u(x)| \le M$ so that ∇u^M converges $\mathcal{L}^d$-a.e. in Ω (and in $L^1(\Omega;\mathbb{R}^k)$) to ∇u as $M \to +\infty$, and $|\nabla u^M|$ converges monotonically to $|\nabla u|$;

(ii) for every $M > 0$ the jump set J_{u^M} is contained in J_u, up to a $\mathcal{H}^{d-1}$-negligible set, and $\mathcal{H}^{d-1}(J_u \setminus \bigcup_{M>0} J_{u^M}) = 0$. Moreover, for $\mathcal{H}^{d-1}$-a.e. $x \in J_{u^M}$, we have $(u^M(x))^\pm = (u^\pm(x))^M$;

(iii) the map $\zeta \mapsto \zeta^M$ is 1-Lipschitz, $(\zeta^{M+1})^M = \zeta^M$ for all $\zeta \in \mathbb{R}^k$ and $M > 0$; hence, $|[u^M]| = |[(u^{M+1})^M]| \le |[u^{M+1}]|$ for all $M > 0$ and $\mathcal{H}^{d-1}$-a.e. in J_{u^M}.

Reference functionals. We now introduce some families of nonlocal functionals that constitute upper and lower bounds for the general family of energies that we mean to study.

For $i \in \{1,2\}$ we let $\rho_i : \mathbb{R}^d \to [0,+\infty]$ be a radially symmetric kernel; in particular, we suppose that

$$\int_{\mathbb{R}^d} (1 + |\xi| + |\xi|^2)\rho_i(\xi)\,\mathrm{d}\xi < +\infty, \tag{10.1.1}$$

$$\rho_1(\xi) \ge c_0 \qquad \text{for } \mathcal{L}^d\text{-a.e. } \xi \in B_{r_0} \tag{10.1.2}$$

for some positive constants c_0, r_0. We let $\{\psi_\varepsilon\}_\varepsilon$ be a family of non-negative radially symmetric kernels such that

$$\int_{\mathbb{R}^d} |\xi|\psi_\varepsilon(\xi)\,\mathrm{d}\xi = 1 \quad \text{for every } \varepsilon > 0, \tag{10.1.3}$$

for every $\delta > 0$ there exists $r_\delta > 0$ such that

$$\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^d \setminus B_{r_\delta}} |\xi|\psi_\varepsilon(\xi)\,\mathrm{d}\xi < \delta, \tag{10.1.4}$$

and

$$\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^d} \psi_\varepsilon(\xi)\,\mathrm{d}\xi < +\infty. \tag{10.1.5}$$

For $\varepsilon > 0$, $O \in \mathcal{O}(\Omega)$, and $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, we define the functionals

$$G_{i,\varepsilon}(u,O) := \int_{O\times O} \frac{1}{\varepsilon^d}\rho_i\Big(\frac{y-x}{\varepsilon}\Big)\min\Big\{\frac{|u(y)-u(x)|^2}{\varepsilon^2}, \frac{1}{\varepsilon}\Big\}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for } i \in \{1,2\},$$

and

$$P_\varepsilon(u,O) := \int_{O\times O} \frac{1}{\varepsilon^d}\psi_\varepsilon\Big(\frac{y-x}{\varepsilon}\Big)\frac{|u(y)-u(x)|}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}y.$$

Recalling that $O_\varepsilon(\xi) := \{x \in O\colon x + \varepsilon\xi \in O\}$, we perform the change of variables $\xi := (y-x)/\varepsilon$ to obtain

$$G_{i,\varepsilon}(u,O) = \int_{\mathbb{R}^d} \rho_i(\xi) \int_{O_\varepsilon(\xi)} g_\varepsilon\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi \qquad \text{for } i \in \{1,2\}, \tag{10.1.6}$$

where we set

$$g_\varepsilon(z) \coloneqq \min\Big\{|z|^2, \frac{1}{\varepsilon}\Big\} \quad \text{for all } z \in \mathbb{R}^k, \tag{10.1.7}$$

and

$$P_\varepsilon(u, O) = \int_{\mathbb{R}^d} \psi_\varepsilon(\xi) \int_{O_\varepsilon(\xi)} \frac{|u(x+\varepsilon\xi) - u(x)|}{\varepsilon}\, \mathrm{d}x\, \mathrm{d}\xi.$$

Conditions (10.1.1)–(10.1.4) play distinct roles in our analysis. On the one hand, inequalities (10.1.1) and (10.1.3) guarantee that the Γ-limits of $\{G_{i,\varepsilon}\}_\varepsilon, i \in \{1,2\}$, and $\{P_\varepsilon\}_\varepsilon$ are finite. On the other hand, (10.1.2) is useful in order to obtain suitable coercivity conditions in GSBV^2. Assumption (10.1.4) corresponds to the assumption (H2) introduced in [8] and ensures that short-range interactions are favoured when $\varepsilon \to 0^+$; this condition is in fact stronger than the analogous hypothesis appearing in [102, 162], which reads as

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_{\frac{r}{\varepsilon}}} |\xi| \psi_\varepsilon(\xi)\, \mathrm{d}\xi = 0 \quad \text{for every } r > 0.$$

We mention some useful results concerning the above functionals. For $i \in \{1,2\}$ and $O \in \mathcal{O}(\Omega)$, we introduce the Mumford-Shah functional

$$\mathrm{MS}_i(u, O) \coloneqq \begin{cases} \lambda_i \displaystyle\int_O |\nabla u|^2\, \mathrm{d}x + \mu_i \mathcal{H}^{d-1}(J_u \cap O) & \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \cap \mathrm{GSBV}^2(O; \mathbb{R}^k), \\ +\infty & \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \setminus \mathrm{GSBV}^2(O; \mathbb{R}^k), \end{cases}$$

where λ_i and μ_i are two positive constants given by

$$\lambda_i \coloneqq \frac{1}{d} \int_{\mathbb{R}^d} |\xi|^2 \rho_i(\xi)\, \mathrm{d}\xi, \qquad \mu_i \coloneqq 2\mathcal{H}^{d-1}(\mathbb{S}^{d-1}) \int_{\mathbb{R}^d} |\xi| \rho_i(\xi)\, \mathrm{d}\xi, \tag{10.1.8}$$

and we introduce (a multiple of) the total variation

$$\mathrm{TV}_\kappa(u, O) \coloneqq \kappa |Du|(O) \qquad \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \cap \mathrm{BV}(O; \mathbb{R}^k),$$

where the constant κ is given by

$$\kappa \coloneqq \Big(\frac{1}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} |e_1 \cdot \sigma|\, \mathrm{d}\mathcal{H}^{d-1}(\sigma) \Big). \tag{10.1.9}$$

Finally, we introduce the reference functionals

$$\begin{aligned} H_\varepsilon(u, O) &\coloneqq G_{2,\varepsilon}(u, O) + P_\varepsilon(u, O) \qquad \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k), \\ H(u, O) &\coloneqq \mathrm{MS}_2(u, O) + \mathrm{TV}_\kappa(u, O) \qquad \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \cap \mathrm{BV}(O; \mathbb{R}^k). \end{aligned}$$

In the next theorem, we summarise the needed auxiliary results concerning the scalar case $k = 1$, which is the setting of the works of Ponce [162], Dávila [102], and Gobbino [132].

Theorem 10.1.2. *Let $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ and assume $k = 1$. Then the following two conditions hold:*

(i) for all $u \in L^1_{\mathrm{loc}}(\Omega) \cap \mathrm{BV}(O)$ and for all $\{u_\varepsilon\}_\varepsilon$ converging to u in $L^1_{\mathrm{loc}}(\Omega)$, we have

$$\liminf_{\varepsilon \to 0^+} P_\varepsilon(u_\varepsilon, O) \geq \mathrm{TV}_\kappa(u, O);$$

(ii) for all $u \in L^1_{\mathrm{loc}}(\Omega) \cap \mathrm{BV}(O)$*, we have*

$$\lim_{\varepsilon \to 0^+} P_\varepsilon(u, O) = \mathrm{TV}_\kappa(u, O).$$

If we assume, in addition, that for $i \in \{1, 2\}$ *we have*

$$\int_{\mathbb{R}^d} |\xi|^m \rho_i(\xi)\, \mathrm{d}\xi < +\infty \tag{10.1.10}$$

for every $m \in \mathbb{N}$*, then:*

(iii) for all $u \in L^1_{\mathrm{loc}}(\Omega)$ *and for all* $\{u_\varepsilon\}_\varepsilon$ *converging to* u *in* $L^1_{\mathrm{loc}}(\Omega)$*, we have*

$$\liminf_{\varepsilon \to 0^+} G_{i,\varepsilon}(u_\varepsilon, O) \geq \mathrm{MS}_i(u, O);$$

(iv) for all $u \in L^1_{\mathrm{loc}}(\Omega)$*, we have*

$$\lim_{\varepsilon \to 0^+} G_{i,\varepsilon}(u, O) = \mathrm{MS}_i(u, O).$$

In particular, assuming (10.1.10)*, for all* $u \in L^1_{\mathrm{loc}}(\Omega) \cap \mathrm{BV}(O)$ *it holds that*

$$\Gamma\text{-}\lim_{\varepsilon \to 0^+} H_\varepsilon(u, O) = H(u, O).$$

Proof. Part (i) follows from [162, Theorem 8], and part (ii) from [102, Theorem 1].

The proofs of (iii) and (iv) are the content of [132, Theorem 4.4 and Section 7]; finally, the computation of the Γ-limit of $\{H_\varepsilon(\cdot, O)\}_\varepsilon$ is obtained combining (i)–(iv). □

Remark 10.1.3. If $k > 1$, we immediately obtain from Theorem 10.1.2 that there exist positive constants c_1 and c_2, depending only on d and k, such that for all $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ we have

$$c_1 \mathrm{MS}_1(u, O) \leq \Gamma\text{-}\liminf_{\varepsilon \to 0^+} G_{1,\varepsilon}(u, O) \text{ for all } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k), \tag{10.1.11}$$

$$\Gamma\text{-}\limsup_{\varepsilon \to 0^+} H_\varepsilon(u, O) \leq c_2 H(u, O) \text{ for all } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \cap \mathrm{BV}(O; \mathbb{R}^k). \tag{10.1.12}$$

Remark 10.1.4. We note that (iii) and (iv) of Theorem 10.1.2, and therefore also inequalities (10.1.11), (10.1.12), hold by simply assuming (10.1.1) instead of (10.1.10), which is an assumption introduced in [132, Section 7]. To see this, we resort to the main result of Section 10.3, which allows us to compute the Γ-limit of a general family of functionals, including $\{G_{i,\varepsilon}\}_\varepsilon, i \in \{1, 2\}$, by first limiting the analysis to kernels with compact support.

For $i \in \{1, 2\}$, let $T > 0$, and consider functionals

$$G^T_{i,\varepsilon}(u, O) := \int_{B_T} \rho_i(\xi) \int_{O_\varepsilon(\xi)} g_\varepsilon\Big(\frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\Big)\, \mathrm{d}x\, \mathrm{d}\xi \tag{10.1.13}$$

for all $(u, O) \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \times \mathcal{O}_{\mathrm{reg}}(\Omega)$. Since ρ_i satisfies (10.1.1), letting χ_{B_T} denote the characteristic function of B_T, we infer that $\rho_i \chi_{B_T}$ fulfils (10.1.10), and then, using Theorem 10.1.2, we obtain

$$\Gamma\text{-}\lim_{\varepsilon \to 0^+} G^T_{i,\varepsilon}(u, O) = \begin{cases} \lambda_i^T \displaystyle\int_O |\nabla u|^2\, \mathrm{d}x + \mu_i^T \mathcal{H}^{d-1}(J_u \cap O) & \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \cap \mathrm{GSBV}^2(O; \mathbb{R}^k), \\ +\infty & \text{if } u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k) \setminus \mathrm{GSBV}^2(O; \mathbb{R}^k), \end{cases}$$

with

$$\lambda_i^T \coloneqq \frac{1}{d}\int_{B_T} |\xi|^2 \rho_i(\xi)\,\mathrm{d}\xi, \qquad \mu_i^T \coloneqq 2\mathcal{H}^{d-1}(\mathbb{S}^{d-1})\int_{B_T} |\xi|\rho_i(\xi)\,\mathrm{d}\xi.$$

Applying Proposition 10.3.2, we conclude

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} G_{i,\varepsilon}(u,O) = \lim_{T\to+\infty} \Gamma\text{-}\lim_{\varepsilon\to 0^+} G_{i,\varepsilon}^T(u,O) = \mathrm{MS}_i(u,O)$$

for every $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, which is the claim.

10.2 Setting of the problem and main results

In this section, we introduce the functionals that are going to be our main object of investigation and state the main results of the Chapter. We recall that ρ_1, ρ_2 are radially symmetric non-negative kernels satisfying (10.1.1) and (10.1.2), that the family of radially symmetric non-negative kernels $\{\psi_\varepsilon\}_\varepsilon$ satisfies (10.1.3)–(10.1.5) and that, for all $\varepsilon > 0$, the functions g_ε are those introduced in (10.1.7).

Setting of the problem. We fix a family of non-negative functions $\{\eta_\varepsilon\}_\varepsilon$ such that for every $\delta > 0$ there exists $r_\delta > 0$ such that

$$\limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^d\setminus B_{r_\delta}} \eta_\varepsilon(\xi)\,\mathrm{d}\xi < \delta, \tag{10.2.1}$$

and

$$\Lambda \coloneqq \limsup_{\varepsilon\to 0^+} \|\eta_\varepsilon\|_{L^1(\mathbb{R}^d)} < +\infty. \tag{10.2.2}$$

For every $\varepsilon > 0$ we consider an integrand $f_\varepsilon\colon \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^k \to [0,+\infty)$ satisfying the following properties:

(i) (*Measurability*) f_ε is Borel measurable;

(ii) (*Growth conditions*) for almost every $x,\xi \in \mathbb{R}^d$ and $z \in \mathbb{R}^k$ it holds

$$\rho_1(\xi)g_\varepsilon(z) \le f_\varepsilon(x,\xi,z) \le \rho_2(\xi)g_\varepsilon(z) + \psi_\varepsilon(\xi)|z| + \eta_\varepsilon(\xi); \tag{10.2.3}$$

(iii) (*Monotonicity*) for almost every $x,\xi \in \mathbb{R}^d$ and for every $z_1, z_2 \in \mathbb{R}^k$ with $|z_1| \le |z_2|$ it holds

$$f_\varepsilon(x,\xi,z_1) \le f_\varepsilon(x,\xi,z_2). \tag{10.2.4}$$

We study functionals

$$F_\varepsilon(u,O) \coloneqq \int_{\mathbb{R}^d}\int_{O_\varepsilon(\xi)} f_\varepsilon\Big(x,\xi,\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi, \tag{10.2.5}$$

where $\varepsilon > 0$ and $(u,O) \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)$.

Remark 10.2.1. Thanks to the upper and lower bounds in (10.2.3), it follows immediately that

$$G_{1,\varepsilon}(u,O) \le F_\varepsilon(u,O) \le H_\varepsilon(u,O) + \|\eta_\varepsilon\|_{L^1(\mathbb{R}^d)}\mathcal{L}^d(O). \tag{10.2.6}$$

We also note that the presence of the term P_ε in the upper bound allows us to treat limit energies that, contrary to the Mumford-Shah functional, admit linear growth with respect to the size of the jump $|[u]|$.

Remark 10.2.2. Hypothesis (iii) above was already considered in the discrete setting in [24]. It will be used repeatedly throughout the work to allow us to restrict our attention to functions in L^∞. More in general, given $(u,O)\in L^1_{\rm loc}(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)$ and a 1-Lipschitz function $\Phi\colon\mathbb{R}^k\to\mathbb{R}^k$, it follows immediately from (10.2.4) that

$$F_\varepsilon(\Phi\circ u,O)\le F_\varepsilon(u,O). \tag{10.2.7}$$

As a consequence, we obtain that

$$\begin{aligned} F_\varepsilon(u+a,O)&=F_\varepsilon(u,O) &&\text{for every } a\in\mathbb{R}^k,\\ F_\varepsilon(Ru,O)&=F_\varepsilon(u,O) &&\text{for every } R\in \mathrm{SO}(k),\\ F_\varepsilon(u^M,O)&\le F_\varepsilon(u,O) &&\text{for every } M>0. \end{aligned}$$

Before presenting the main results of the chapter, we recall some notation concerning minimisation problems introduced in Section 1.5. Given a functional $\mathcal{F}\colon \mathrm{SBV}^2(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)\to[0,+\infty)$, a function $w\in\mathrm{SBV}^2(\Omega;\mathbb{R}^k)$, and $O\in\mathcal{O}_{\rm reg}(\Omega)$ we set

$$\mathfrak{m}^{\mathcal{F}}(w,O):=\inf\{\mathcal{F}(u,O)\colon\ u\in\mathrm{SBV}^2(O;\mathbb{R}^k)\text{ with } u=w \text{ near } \partial O\}. \tag{10.2.8}$$

Given $A\in\mathbb{R}^{k\times d}$ we let ℓ_A denote the corresponding linear map $\ell_A:\mathbb{R}^d\to\mathbb{R}^k$. Given $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$, we let $u_{x,\zeta,\nu}\colon\mathbb{R}^d\to\mathbb{R}^k$ be the function defined for every $y\in\mathbb{R}^d$ by

$$u_{x,\zeta,\nu}(y):=\begin{cases}\zeta & \text{if } (y-x)\cdot\nu\ge 0,\\ 0 & \text{otherwise.}\end{cases}$$

We recall that $Q_\nu(x,r)$ denotes an open cube of side length r and centre x with two faces orthogonal to ν and that $Q_\nu(x,r)=Q_{-\nu}(x,r)$. We also recall that $Q(x,r):=x+(-\frac{r}{2},\frac{r}{2})^d$.

The following constitutes the main contribution of the chapter.

Theorem 10.2.3. *Let $\{F_\varepsilon\}_\varepsilon$ be as in* (10.2.5)*, and let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0 as $n\to+\infty$. Then, there exists a subsequence $\{\varepsilon_{n_m}\}_m$ and a functional $F\colon L^1_{\rm loc}(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)\to[0,+\infty]$ such that*

$$\Gamma\text{-}\lim_{m\to+\infty}F_{\varepsilon_{n_m}}(u,O)=F(u,O)$$

for every $(u,O)\in L^1_{\rm loc}(\Omega;\mathbb{R}^k)\times\mathcal{O}_{\rm reg}(\Omega)$. In particular, for $O\in\mathcal{O}(\Omega)$ we have

$$F(u,O)=\int_O f_{\rm bulk}(x,\nabla u)\mathrm{d}x+\int_{J_u\cap O}f_{\rm surf}(x,[u],\nu_u)\mathrm{d}\mathcal{H}^{d-1}$$

if $u\in L^1_{\rm loc}(\Omega;\mathbb{R}^k)\cap\mathrm{GSBV}^2(O;\mathbb{R}^k)$ and $F(u,O)=+\infty$ otherwise, where

$$f_{\rm bulk}(x,A):=\limsup_{r\to0^+}\frac{\mathfrak{m}^F(\ell_A,Q(x,r))}{r^d}\quad\text{for all } x\in\Omega \text{ and } A\in\mathbb{R}^{k\times d}, \tag{10.2.9}$$

$$f_{\rm surf}(x,\zeta,\nu):=\limsup_{r\to0^+}\frac{\mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,r))}{r^{d-1}}\quad\text{for all } x\in\Omega,\ \zeta\in\mathbb{R}^k, \text{ and } \nu\in\mathbb{S}^{d-1}. \tag{10.2.10}$$

Letting $\lambda_1,\lambda_2,\mu_1,\mu_2$ be as in (10.1.8)*, κ as in* (10.1.9)*, c_1,c_2 as in Remark 10.1.3, and Λ as in* (10.2.2)*, we have*

$$\begin{aligned} c_1\lambda_1|A|^2\le f_{\rm bulk}(x,A)&\le c_2(\lambda_2|A|^2+\kappa|A|)+\Lambda\quad\text{for all } x\in\Omega \text{ and } A\in\mathbb{R}^{k\times d},\\ c_1\mu_1\le f_{\rm surf}(x,\zeta,\nu)&\le c_2(\mu_2+\kappa|\zeta|)\quad\text{for all } x\in\Omega,\ \zeta\in\mathbb{R}^k\setminus\{0\}, \text{ and } \nu\in\mathbb{S}^{d-1}. \end{aligned}$$

Moreover, for all $x \in \Omega$ and $\nu \in \mathbb{S}^{d-1}$ we have

$$f_{\rm surf}(x,\zeta_1,\nu) \le f_{\rm surf}(x,\zeta_2,\nu) \quad \text{if } |\zeta_1| \le |\zeta_2|,$$

so that

$$\begin{aligned} f_{\rm surf}(x,\zeta_1,\nu) &= f_{\rm surf}(x,\zeta_2,\nu) \quad \text{if } |\zeta_1| = |\zeta_2|,\\ f_{\rm surf}(x,\zeta,\nu) &= f_{\rm surf}(x,\zeta,-\nu) \quad \text{for all } \zeta \in \mathbb{R}^k. \end{aligned}$$

Proof. It is sufficient to combine Propositions 10.4.5 and 10.5.1. □

We now introduce some relevant minimisation problems for the nonlocal functionals we are interested in. Given $O \in \mathcal{O}(\Omega)$, $w \in L^1_{\rm loc}(\Omega;\mathbb{R}^k)$, and $s > 0$, we set

$$\mathcal{D}_{s,w}(O) := \big\{u \in L^1_{\rm loc}(\Omega;\mathbb{R}^k)\colon\ u = w \text{ for } \mathcal{L}^d\text{-a.e. } x \in O \text{ with } \operatorname{dist}(x,\mathbb{R}^d \setminus O) < s\big\}, \quad (10.2.11)$$

and consider the minimisation problems

$$\mathfrak{m}_s^{F_{\varepsilon_n}}(w,O) := \inf\big\{F_{\varepsilon_n}(u,O)\colon u \in \mathcal{D}_{\varepsilon_n s,w}(O)\big\}. \quad (10.2.12)$$

We observe that for every $0 < s_1 < s_2$ we clearly have the inclusion $\mathcal{D}_{s_2,w}(O) \subset \mathcal{D}_{s_1,w}(O)$, which implies

$$\mathfrak{m}_{s_1}^{F_{\varepsilon_n}}(w,O) \le \mathfrak{m}_{s_2}^{F_{\varepsilon_n}}(w,O).$$

For every $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we define

$$\begin{aligned} f'_{\rm bulk}(x,A) &:= \limsup_{r\to 0^+} \sup_{s>0} \liminf_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A, Q(x,r))}{r^d},\\ f''_{\rm bulk}(x,A) &:= \limsup_{r\to 0^+} \sup_{s>0} \limsup_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A, Q(x,r))}{r^d},\\ f'_{\rm surf}(x,\zeta,\nu) &:= \limsup_{r\to 0^+} \sup_{s>0} \liminf_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}},\\ f''_{\rm surf}(x,\zeta,\nu) &:= \limsup_{r\to 0^+} \sup_{s>0} \limsup_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}}. \end{aligned}$$

Theorem 10.2.4. *Let $\{F_\varepsilon\}_\varepsilon$ be as in (10.2.5), and let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0 as $n \to +\infty$. If there exists a functional $F\colon L^1_{\rm loc}(\Omega;\mathbb{R}^k) \times \mathcal{O}(\Omega) \to [0,+\infty]$ such that for every $O \in \mathcal{O}_{\rm reg}(\Omega)$ the sequence $\{F_{\varepsilon_n}(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ as $n \to +\infty$, then the functions $f_{\rm bulk}$ and $f_{\rm surf}$ defined in (10.2.9) and (10.2.10), respectively, satisfy*

$$\begin{aligned} f_{\rm bulk}(x,A) &= f'_{\rm bulk}(x,A) = f''_{\rm bulk}(x,A),\\ f_{\rm surf}(x,\zeta,\nu) &= f'_{\rm surf}(x,\zeta,\nu) = f''_{\rm surf}(x,\zeta,\nu), \end{aligned}$$

for all $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

Conversely, if there exist functions $\hat f_{\rm bulk}\colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0,+\infty)$ and $\hat f_{\rm surf}\colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that

$$\hat f_{\rm bulk}(x,A) = f'_{\rm bulk}(x,A) = f''_{\rm bulk}(x,A),$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$ and for all $A \in \mathbb{R}^{k\times d}$ and

$$\hat f_{\rm surf}(x,\zeta,\nu) = f'_{\rm surf}(x,\zeta,\nu) = f''_{\rm surf}(x,\zeta,\nu)$$

for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$, for all $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$, then for every $O \in \mathcal{O}_{\rm reg}(\Omega)$ the sequence $\{F_{\varepsilon_n}(\cdot, O)\}_n$ Γ-converges as $n \to +\infty$ to the functional $F(\cdot, O)$ defined by

$$F(u, O) := \int_O \hat{f}_{\rm bulk}(x, \nabla u)\mathrm{d}x + \int_{J_u \cap O} \hat{f}_{\rm surf}(x, [u], \nu_u)\mathrm{d}\mathcal{H}^{d-1}$$

if $u \in L^1_{\rm loc}(\Omega; \mathbb{R}^k) \cap \mathrm{GSBV}^2(O; \mathbb{R}^k)$ and $F(u, O) = +\infty$ otherwise.

Proof. It is enough to combine Proposition 10.6.5 and 10.6.6. □

The proofs of the two main theorems are subdivided as follows: Section 10.4 is devoted to the proof of the compactness property, Section 10.5 is devoted to the proof of the integral representation, and Section 10.6 to the representation of the energy densities as limits of minimisation problems.

10.3 Reduction to finite-range interactions

In this section we show that the computation of the Γ-limit of functionals $\{F_\varepsilon\}_\varepsilon$ given by (10.2.5) can be achieved by first computing the Γ-limits of functionals $\{F^T_\varepsilon\}_\varepsilon$, that are obtained considering interactions between pairs of points whose distance is at most εT, and then letting T to $+\infty$.

For any $T > 0$, we define the functionals $F^T_\varepsilon : L^1_{\rm loc}(\Omega; \mathbb{R}^k) \times \mathcal{O}(\Omega) \to [0, +\infty]$ as

$$F^T_\varepsilon(u, O) := \int_{B_T} \int_{O_\varepsilon(\xi)} f_\varepsilon\Big(x, \xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi, \tag{10.3.1}$$

and, given a positive sequence $\{\varepsilon_n\}_n$ converging to 0, we let

$$F'^{,T}(u, O) := \Gamma\text{-}\liminf_{n\to+\infty} F^T_{\varepsilon_n}(u, O), \quad F''^{,T}(u, O) := \Gamma\text{-}\limsup_{n\to+\infty} F^T_{\varepsilon_n}(u, O) \tag{10.3.2}$$

and

$$F'(u, O) := \Gamma\text{-}\liminf_{n\to+\infty} F_{\varepsilon_n}(u, O), \quad F''(u, O) := \Gamma\text{-}\limsup_{n\to+\infty} F_{\varepsilon_n}(u, O) \tag{10.3.3}$$

for all $(u, O) \in L^1_{\rm loc}(\Omega; \mathbb{R}^k) \times \mathcal{O}(\Omega)$, where Γ-liminf and Γ-limsup are defined as in Section 1.2.

It is simple to observe that, due to (10.2.7), all the above functionals are continuous with respect to vertical truncations.

Lemma 10.3.1. *Let $T > 0$, $\Phi\colon \mathbb{R}^k \to \mathbb{R}^k$ be a 1-Lipschitz function, and let $F'^{,T}$ and $F''^{,T}$ be the functionals defined by* (10.3.2)*. Then*

$$F'^{,T}(\Phi \circ u, O) \le F'^{,T}(u, O) \quad \text{and} \quad F''^{,T}(\Phi \circ u, O) \le F''^{,T}(u, O) \tag{10.3.4}$$

for all $(u, O) \in L^1_{\rm loc}(\Omega; \mathbb{R}^k) \times \mathcal{O}(\Omega)$. Moreover,

$$F'^{,T}(u, O) = \lim_{M\to+\infty} F'^{,T}(u^M, O) \quad \text{and} \quad F''^{,T}(u, O) = \lim_{M\to+\infty} F''^{,T}(u^M, O). \tag{10.3.5}$$

In addition, the same conclusions hold for the functionals F', F'' defined by (10.3.3)*.*

Proof. We only prove (10.3.4) and (10.3.5) for $F''^{,T}$, the other cases being analogous. We consider a sequence of functions $\{u_n\}_n \subset L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ as $n \to +\infty$ and such that

$$F''^{,T}(u,O) = \lim_{n\to+\infty} F^T_{\varepsilon_n}(u_n,O).$$

Thanks to (10.2.7), it follows immediately that $F^T_{\varepsilon_n}(\Phi\circ u_n,O) \le F^T_{\varepsilon_n}(u_n,O)$ for every $n\in\mathbb{N}$, and therefore, we get

$$\limsup_{n\to+\infty} F^T_{\varepsilon_n}(\Phi\circ u_n,O) \le \limsup_{n\to+\infty} F^T_{\varepsilon_n}(u_n,O) = F''^{,T}(u,O).$$

Since $\Phi\circ u_n \to \Phi\circ u$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ as $n\to+\infty$, (10.3.4) follows by the definition (10.3.2) of $F''^{,T}$. To conclude, we note that, as $F''^{,T}(\cdot,O)$ is lower semicontinuous with respect to the $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ convergence, we also have

$$F''^{,T}(u,O) \le \liminf_{M\to+\infty} F''^{,T}(u^M,O),$$

which, together with (10.3.4), implies (10.3.5). □

The following proposition constitutes the main result of this section.

Proposition 10.3.2. *Let $(u,O)\in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)\times\mathcal{O}_{\mathrm{reg}}(\Omega)$. It holds*

$$F'(u,O) = \lim_{T\to+\infty} F'^{,T}(u,O), \quad \text{and} \quad F''(u,O) = \lim_{T\to+\infty} F''^{,T}(u,O).$$

In particular, if $\{T_j\}_j$ is a positive sequence such that $T_j\to+\infty$ as $j\to+\infty$ and

$$\Gamma\text{-}\lim_{n\to+\infty} F^{T_j}_{\varepsilon_n}(u,O) = F^{T_j}(u,O)$$

for all $(u,O)\in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)\times\mathcal{O}_{\mathrm{reg}}(\Omega)$ and $j\in\mathbb{N}$, then we have

$$\Gamma\text{-}\lim_{n\to+\infty} F_{\varepsilon_n}(u,O) = \lim_{j\to+\infty} F^{T_j}(u,O)$$

for all $(u,O)\in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)\times\mathcal{O}_{\mathrm{reg}}(\Omega)$.

The proof of Proposition 10.3.2 requires several preliminary results, which are obtained by adapting some of the arguments presented in [8, Chapter 4]. We pursue the same approach in a slightly more general and flexible framework. In the rest of the section, we let C denote a positive constant whose value may change from line to line, for which the dependence on the relevant quantities shall be emphasized.

For every $\varepsilon>0$, we let $a_\varepsilon:\mathbb{R}^k\to[0,+\infty)$ be a radially increasing function and assume there exist $c>0$ and $p\in[1,+\infty)$ such that

$$|a_\varepsilon(z)| \le c|z|^p \quad \text{for every } z\in\mathbb{R}^k, \tag{10.3.6}$$

and that, given any positive integer m and $\{z_1,z_2,\dots,z_m\}\subset\mathbb{R}^k$, it holds

$$a_\varepsilon\Big(\sum_{i=1}^m z_i\Big) \le m^{p-1}\sum_{i=1}^m a_\varepsilon(z_i). \tag{10.3.7}$$

Then we set

$$A^r_\varepsilon(u,O) := \int_{B_r}\int_{O_\varepsilon(\xi)} a_\varepsilon\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi, \tag{10.3.8}$$

for $r>0$ and $(u,O)\in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)\times\mathcal{O}(\mathbb{R}^d)$.

Theorem 10.3.3. *Let $O \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $r > 0$. There exist a bounded open set $\widetilde{O} \supset\supset O$, a continuous linear map*

$$\mathrm{Ext}\colon L^p(O;\mathbb{R}^k) \to L^p(\widetilde{O};\mathbb{R}^k),$$

and three positive constants $C_1 = C_1(r,O)$, $r_1 = r_1(r,O)$, and $\varepsilon' = \varepsilon'(r,O)$ such that

$$\mathrm{Ext}u = u \quad \mathcal{L}^d\text{-a.e. in } O$$

and

$$\|\mathrm{Ext}(u)\|^p_{L^p(\widetilde{O})} + A^{r_1}_\varepsilon(\mathrm{Ext}(u),\widetilde{O}) \le C_1(\|u\|^p_{L^p(O)} + A^r_\varepsilon(u,O))$$

for all $u \in L^p(O;\mathbb{R}^k)$ and $\varepsilon < \varepsilon'$.

Proof. Thanks to the Lipschitz regularity of ∂O, there exists an open cover $\{O_i\}_{i=1}^n$ of ∂O with the property that for all $i \in \{1,\dots,n\}$ there exists a bi-Lipschitz continuous map $\Psi_i\colon Q \to O_i$ such that

$$\Psi_i(Q^+) = O_i \cap O, \quad \Psi_i(Q^-) = O_i \setminus O, \quad \Psi_i(Q^0) = O_i \cap \partial O,$$

where $Q \coloneqq (-1,1)^d$, $Q^\pm \coloneqq \{x \in Q\colon \pm x_1 > 0\}$, and $Q^0 \coloneqq \{x \in Q\colon x_1 = 0\}$. Let then O_0 be an open set such that $O \setminus \bigcup_{i=1}^n O_i \subset O_0 \subset O$ and set $\widetilde{O} \coloneqq O_0 \cup \bigcup_{i=1}^n O_i$.

For $i \in \{1,\dots,n\}$ we define the map $R_i\colon O_i \setminus O \to O_i \cap O$ as

$$R_i(x) \coloneqq \Psi_i(-y_1, y_2, \dots, y_d),$$

where $(y_1,\dots,y_d) = \Psi_i^{-1}(x) \in Q^-$, and given $u \in L^p(O;\mathbb{R}^k)$, we let

$$u_i(x) \coloneqq \begin{cases} u(x) & \text{if } x \in O_i \cap O, \\ u(R_i(x)) & \text{if } x \in O_i \setminus O. \end{cases}$$

Consider now $\{\varphi_i\}_{i=0}^n \subset C_c^\infty(\mathbb{R}^d)$ a partition of unity subordinated to the open cover $\{O_i\}_{i=0}^n$, and let

$$\widetilde{u}_0(x) \coloneqq \varphi_0(x)u(x), \quad \widetilde{u}_i(x) \coloneqq \varphi_i(x)u_i(x) \text{ for all } i \in \{1,\dots,n\},$$

and finally set $\mathrm{Ext}(u) \coloneqq \sum_{i=0}^n \widetilde{u}_i$.

By a change of variables, we have

$$\int_{O_i} |u_i|^p \,\mathrm{d}x \le C \int_{O\cap O_i} |u|^p \,\mathrm{d}x \tag{10.3.9}$$

for all $i \in \{1,\dots,n\}$, where C is a positive constant depending only on O. Therefore, we have

$$\|\mathrm{Ext}(u)\|_{L^p(\widetilde{O})} \le C\|u\|_{L^p(O)} \tag{10.3.10}$$

for all $u \in L^p(O;\mathbb{R}^k)$, and then

$$\mathrm{Ext}\colon L^p(O;\mathbb{R}^k) \to L^p(\widetilde{O};\mathbb{R}^k)$$

is a linear continuous map such that $\mathrm{Ext}(u) = u$ $\mathcal{L}^d$-a.e. on O.

We claim that there exists a positive constant $C = C(O)$ such that

$$A^{r_1}_\varepsilon(u_i, O_i) \le C A^r_\varepsilon(u, O_i \cap O) \tag{10.3.11}$$

for all $i \in \{1,\dots,n\}$, $\varepsilon > 0$, and $u \in L^p(O;\mathbb{R}^k)$, where

$$r_1 := \frac{r}{1+S^2} \quad \text{and} \quad S := \max_{i\in\{1,\dots,n\}} \|D\Psi_i^{-1}\|_{L^\infty(O_i)}.$$

Recalling (10.3.8), by a change of variables we get

$$\begin{aligned} A_\varepsilon^{r_1}(u_i,O_i) &= \frac{1}{\varepsilon^d}\int_{O_i}\int_{O_i\cap B_{\varepsilon r_1}(x)} a_\varepsilon\Big(\frac{u_i(y)-u_i(x)}{\varepsilon}\Big)\,\mathrm{d}y\,\mathrm{d}x \\ &\le A_\varepsilon^r(u,O_i\cap O) \\ &\quad + \frac{1}{\varepsilon^d}\int_{O_i\cap O}\int_{(O_i\setminus O)\cap B_{\varepsilon r_1}(x)} a_\varepsilon\Big(\frac{u_i(y)-u_i(x)}{\varepsilon}\Big)\,\mathrm{d}y\,\mathrm{d}x \qquad (10.3.12) \\ &\quad + \frac{1}{\varepsilon^d}\int_{O_i\setminus O}\int_{O_i\cap O\cap B_{\varepsilon r_1}(x)} a_\varepsilon\Big(\frac{u_i(y)-u_i(x)}{\varepsilon}\Big)\,\mathrm{d}y\,\mathrm{d}x \qquad (10.3.13) \\ &\quad + \frac{1}{\varepsilon^d}\int_{O_i\setminus O}\int_{(O_i\setminus O)\cap B_{\varepsilon r_1}(x)} a_\varepsilon\Big(\frac{u_i(y)-u_i(x)}{\varepsilon}\Big)\,\mathrm{d}y\,\mathrm{d}x. \qquad (10.3.14) \end{aligned}$$

We have that the Lipschitz constant of R_i is less than S^2, and then, with our choice of r_1, it holds that $R_i((O_i\setminus O)\cap B_{\varepsilon r_1}(x)) \subset O_i\cap O\cap B_{\varepsilon r}(x)$ for all $i\in\{1,\dots,n\}$ and $x\in O_i$. Therefore, using the change of variables $y' = R_i(y)$, we obtain

$$\begin{aligned} &\int_{O_i\cap O}\int_{(O_i\setminus O)\cap B_{\varepsilon r_1}(x)} a_\varepsilon\Big(\frac{u_i(y)-u_i(x)}{\varepsilon}\Big)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad\le S^{2d}\int_{O_i\cap O}\int_{O_i\cap O\cap B_{\varepsilon r}(x)} a_\varepsilon\Big(\frac{u(y')-u(x)}{\varepsilon}\Big)\,\mathrm{d}y'\,\mathrm{d}x \\ &\qquad\le S^{2d}A_\varepsilon^r(u,O_i\cap O), \end{aligned}$$

which provides an upper bound for (10.3.12). With the same argument, one proves that the term in (10.3.13) is bounded above by $S^{2d}A_\varepsilon^r(u,O_i\cap O)$ as well, and the term in (10.3.14) is bounded above by $S^{4d}A_\varepsilon^r(u,O_i\cap O)$. The combination of these estimates yields (10.3.11).

Finally, for every $i\in\{0,\dots,n\}$ we set $U_i := \operatorname{supp}\varphi_i$ and

$$\varepsilon' = \varepsilon'(r,O) := \frac{1}{r_1}\min_{i\in\{0,\dots,n\}} \operatorname{dist}(U_i,\mathbb{R}^d\setminus O_i) > 0.$$

If $\varepsilon<\varepsilon'$ and $i\in\{0,\dots,n\}$, summing and subtracting $\varphi_i(x+\varepsilon\xi)u_i(x)$ and using (10.3.7), we get

$$\begin{aligned} A_\varepsilon^{r_1}(\widetilde{u}_i,\widetilde{O}) &= \int_{B_{r_1}}\int_{(O_i)_\varepsilon(\xi)} a_\varepsilon\Big(\frac{\widetilde{u}_i(x+\varepsilon\xi)-\varphi_i(x+\varepsilon\xi)u_i(x)+\varphi_i(x+\varepsilon\xi)u_i(x)-\widetilde{u}_i(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\le 2^{p-1}\int_{B_{r_1}}\int_{(O_i)_\varepsilon(\xi)} a_\varepsilon\Big(\frac{(u_i(x+\varepsilon\xi)-u_i(x))\varphi_i(x+\varepsilon\xi)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\quad + 2^{p-1}\int_{B_{r_1}}\int_{(O_i)_\varepsilon(\xi)} a_\varepsilon\Big(\frac{(\varphi_i(x+\varepsilon\xi)-\varphi_i(x))u_i(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\le 2^{p-1}\int_{B_{r_1}}\int_{(O_i)_\varepsilon(\xi)} a_\varepsilon\Big(\frac{u_i(x+\varepsilon\xi)-u_i(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\quad + 2^{p-1}\int_{B_{r_1}}\int_{(O_i)_\varepsilon(\xi)} \Big|\frac{\varphi_i(x+\varepsilon\xi)-\varphi_i(x)}{\varepsilon}\Big|^p |u_i(x)|^p\mathrm{d}\xi, \end{aligned}$$

where in the last inequality we have used that a_ε is radially increasing and (10.3.6). We then infer

$$A_\varepsilon^{r_1}(\widetilde{u}_i,\widetilde{O}) \le C(A_\varepsilon^{r_1}(u_i,O_i)+\|u_i\|^p_{L^p(O_i)}),$$

for all $i\in\{0,\dots,n\}$, where $C=C(r_1,\nabla\varphi_0,\dots,\nabla\varphi_n)=C(r,O)$.

If $i=0$, we have

$$A_\varepsilon^{r_1}(\widetilde{u}_0,\widetilde{O}) \le C(A_\varepsilon^{r}(u,O_0)+\|u\|^p_{L^p(O_0)}),$$

while, if $i\in\{1,\dots,n\}$, we apply (10.3.11) and (10.3.9) to conclude

$$A_\varepsilon^{r_1}(\widetilde{u}_i,\widetilde{O}) \le C(A_\varepsilon^{r}(u,O_i\cap O)+\|u\|^p_{L^p(O_i\cap O)}).$$

Finally, summing over $i\in\{0,\dots,n\}$ we obtain

$$A_\varepsilon^{r_1}(\mathrm{Ext}(u),\widetilde{O}) \le \sum_{i=0}^n A_\varepsilon^{r_1}(\widetilde{u}_i,\widetilde{O}) \le C(A_\varepsilon^{r}(u,O)+\|u\|^p_{L^p(O)})$$

for every $\varepsilon<\varepsilon'$ and for a possibly larger constant C. This estimate, together with (10.3.10), concludes the proof. □

Remark 10.3.4. It is easily seen that

$$\mathrm{dist}(\mathrm{supp}\ \mathrm{Ext}(u),\mathbb{R}^d\setminus\widetilde{O})>\varepsilon' r_1,$$

and therefore,

$$A_\varepsilon^{r_1}(\mathrm{Ext}(u),\widetilde{O}) = A_\varepsilon^{r_1}(\mathrm{Ext}(u),\mathbb{R}^d)$$

for every $\varepsilon<\varepsilon'$. As a consequence, we observe that Theorem 10.3.3 remains valid (with the same constants) replacing $\widetilde{O}$ with any larger bounded set.

The following result shows that, for ε small enough, it is possible to control the inner integral in (10.3.8) by means of the total energy A_ε^r.

Lemma 10.3.5. *Let V be a bounded open set. For every $r>0$ there exists a positive constant $C_2=C_2(r)$ such that, for every open set $U\subset V$, $\xi\in\mathbb{R}^d$, and $\varepsilon>0$, with*

$$\varepsilon r<\mathrm{dist}(U+(0,\varepsilon)\xi,\mathbb{R}^d\setminus V), \tag{10.3.15}$$

it holds

$$\int_U a_\varepsilon\Big(\frac{v(x+\varepsilon\xi)-v(x)}{\varepsilon}\Big)\mathrm{d}x \le C_2(|\xi|^p+1)A_\varepsilon^r(v,U_{\varepsilon,\xi}) \tag{10.3.16}$$

for every $v\in L^p(V;\mathbb{R}^k)$, where $U_{\varepsilon,\xi}:=U+(0,\varepsilon)\xi+B_{\varepsilon r}$ and $(0,\varepsilon)\xi:=\{s\xi\colon\ s\in(0,\varepsilon)\}$.

Proof. By condition (10.3.15) it holds that $U_{\varepsilon,\xi}\subset V$, so that all the terms in inequality (10.3.16) are well defined.

Set $\widetilde{\varepsilon}:=\frac{\varepsilon r}{\sqrt{d+3}}$ and let R_ξ be a rotation in $\mathbb{R}^d$ such that $R_\xi e_1=\frac{\xi}{|\xi|}$. We introduce the lattice $\mathcal{L}_\varepsilon:=\{R_\xi i\colon\ i\in\widetilde{\varepsilon}\mathbb{Z}^d\}$ and, for any $j\in\mathcal{L}_\varepsilon$, we define $Q_\varepsilon^j:=j+R_\xi(-\widetilde{\varepsilon}/2,\widetilde{\varepsilon}/2)^d$ and introduce the indices

$$\mathcal{I}_\varepsilon:=\{j\in\mathcal{L}_\varepsilon\colon\ Q_\varepsilon^j\cap U\neq\emptyset\}.$$

Let $m:=\lceil\varepsilon|\xi|/\widetilde{\varepsilon}\rceil$, where $\lceil a\rceil$ denotes the smaller integer greater than or equal to a, and, for fixed $j_0\in\mathcal{I}_\varepsilon$ and $0\le\ell\le m-1$, define $j_\ell:=j_0+\ell\widetilde{\varepsilon}\frac{\xi}{|\xi|}$. We let x_ℓ denote any point in $Q_\varepsilon^{j_\ell}$ for all $0\le\ell\le m-1$, and set $x_m:=x_0+\varepsilon\xi$ and $Q_\varepsilon^{j_m}:=Q_\varepsilon^{j_0}+\varepsilon\xi$.

Given points $x_0, x_1, \dots, x_m = x_0 + \varepsilon\xi$ in accordance with the above notation, we use (10.3.7) with $z_\ell = (v(x_\ell) - v(x_{\ell-1}))/\varepsilon$ for $\ell \in \{1, \dots, m\}$, and we get

$$a_\varepsilon\Big(\frac{v(x_0+\varepsilon\xi)-v(x_0)}{\varepsilon}\Big) \le m^{p-1}\sum_{\ell=1}^{m} a_\varepsilon\Big(\frac{v(x_\ell)-v(x_{\ell-1})}{\varepsilon}\Big).$$

Integrating this inequality with respect to the variables $(x_0, \dots, x_{m-1})$, we obtain

$$\begin{aligned}
&\int_{Q_\varepsilon^{j_0}} a_\varepsilon\Big(\frac{v(x_0+\varepsilon\xi)-v(x_0)}{\varepsilon}\Big)\,\mathrm{d}x_0 \\
&\quad\le \frac{m^{p-1}}{(\widetilde{\varepsilon})^d}\sum_{\ell=1}^{m}\int_{Q_\varepsilon^{j_{\ell-1}}}\int_{Q_\varepsilon^{j_\ell}} a_\varepsilon\Big(\frac{v(x_\ell)-v(x_{\ell-1})}{\varepsilon}\Big)\,\mathrm{d}x_\ell\,\mathrm{d}x_{\ell-1} \\
&\quad\le \frac{m^{p-1}}{(\widetilde{\varepsilon})^d}\sum_{\ell=1}^{m}\int_{Q_\varepsilon^{j_{\ell-1}}}\int_{B_{\varepsilon r}(x_{\ell-1})} a_\varepsilon\Big(\frac{v(y)-v(x_{\ell-1})}{\varepsilon}\Big)\,\mathrm{d}y\,\mathrm{d}x_{\ell-1},
\end{aligned}$$

where in the last line of the above inequality we used that for every $\ell \in \{1, \dots, m\}$ it holds $Q_\varepsilon^{j_\ell} \subset B_{\varepsilon r}(x_{\ell-1})$, regardless of the choice of the point $x_{\ell-1} \in Q_\varepsilon^{j_{\ell-1}}$. Then, by the change of variables $\xi' := (y - x_{\ell-1})/\varepsilon$ and Fubini's Theorem, it follows

$$\begin{aligned}
&\int_{Q_\varepsilon^{j_0}} a_\varepsilon\Big(\frac{v(x_0+\varepsilon\xi)-v(x_0)}{\varepsilon}\Big)\,\mathrm{d}x_0 \\
&\quad\le \Big(\frac{\varepsilon}{\widetilde{\varepsilon}}\Big)^d m^{p-1}\sum_{\ell=1}^{m}\int_{B_r}\int_{Q_\varepsilon^{j_{\ell-1}}} a_\varepsilon\Big(\frac{v(x_{\ell-1}+\varepsilon\xi')-v(x_{\ell-1})}{\varepsilon}\Big)\,\mathrm{d}x_{\ell-1}\,\mathrm{d}\xi' \\
&\quad= \Big(\frac{\sqrt{d+3}}{r}\Big)^d m^{p-1}\int_{B_r}\int_{S_\varepsilon(j_0)} a_\varepsilon\Big(\frac{v(x+\varepsilon\xi')-v(x)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\xi',
\end{aligned}$$

where $S_\varepsilon(j_0) := \cup_{\ell=0}^{m-1} Q_\varepsilon^{j_\ell}$ and we note that $S_\varepsilon(j_0) \subset U_{\varepsilon,\xi}$ for every $j_0 \in \mathcal{I}_\varepsilon$.

Since the sets $\{S_\varepsilon(j_0)\colon\ j_0 \in \mathcal{I}_\varepsilon\}$ overlap at most $2m-1$ times, summing over the indices $j_0 \in \mathcal{I}_\varepsilon$ and recalling that $m = \lceil \varepsilon|\xi|/\widetilde{\varepsilon}\,\rceil$, we get

$$\int_U a_\varepsilon\Big(\frac{v(x+\varepsilon\xi)-v(x)}{\varepsilon}\Big)\,\mathrm{d}x \le C(|\xi|^p+1)A_\varepsilon^r(v, U_{\varepsilon,\xi})$$

and the proof is concluded. □

Combining Theorem 10.3.3 and Lemma 10.3.5 we obtain the following result.

Corollary 10.3.6. *Let $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ and let $r > 0$. There exist two positive constants $C_3 = C_3(r, O)$ and $\varepsilon'' = \varepsilon''(r, O)$ such that*

$$\int_{O_\varepsilon(\xi)} a_\varepsilon\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)\mathrm{d}x \le C_3(|\xi|^p+1)(\|u\|_{L^p(O)}^p + A_\varepsilon^r(u, O))$$

for every $\xi \in \mathbb{R}^d$, $u \in L^p(O;\mathbb{R}^k)$, and $\varepsilon < \varepsilon''$.

Proof. Let $\widetilde{O}$, r_1, and ε' be as in the statement of Theorem 10.3.3 and, in light of Remark 10.3.4, suppose that $\mathrm{co}(O) \subset\subset \widetilde{O}$, where we let $\mathrm{co}(O)$ denote the closed convex hull of O, and let

$$\varepsilon'' = \varepsilon''(r, O) := \min\Big\{\varepsilon', \frac{\mathrm{dist}(\mathrm{co}(O), \mathbb{R}^d \setminus \widetilde{O})}{r_1}\Big\} > 0.$$

Since we assumed that

$$O_\varepsilon(\xi) + (0,\varepsilon)\xi \subset \mathrm{co}(O)$$

for every $\xi \in \mathbb{R}^d$ and $\varepsilon > 0$, it holds that

$$\varepsilon r_1 < \mathrm{dist}(O_\varepsilon(\xi) + (0,\varepsilon)\xi, \mathbb{R}^d \setminus \widetilde{O})$$

for every $\xi \in \mathbb{R}^d$ and $\varepsilon < \varepsilon''$. Hence, the set $(O_\varepsilon(\xi))_{\varepsilon,\xi} = O_\varepsilon(\xi) + (0,\varepsilon)\xi + B_{\varepsilon r_1}$ is contained in $\widetilde{O}$. Applying Lemma 10.3.5 with

$$r = r_1, \quad V = \widetilde{O}, \quad U = O_\varepsilon(\xi), \quad v = \mathrm{Ext}(u),$$

and Theorem 10.3.3, we then get

$$\begin{aligned}\int_{O_\varepsilon(\xi)} a_\varepsilon\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)\,\mathrm{d}x &\le C_2(|\xi|^p+1)A_\varepsilon^{r_1}(\mathrm{Ext}(u),(O_\varepsilon(\xi))_{\varepsilon,\xi})\\ &\le C_2(|\xi|^p+1)A_\varepsilon^{r_1}(\mathrm{Ext}(u),\widetilde{O})\\ &\le C_1C_2(|\xi|^p+1)(\|u\|^p_{L^p(O)} + A_\varepsilon^r(u,O)),\end{aligned}$$

which concludes the proof. □

We introduce further reference functionals. For every $\varepsilon, r > 0$ and $(u,O) \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)\times \mathcal{O}(\mathbb{R}^d)$ we set

$$\begin{aligned}G_\varepsilon^r(u,O) &:= \int_{B_r}\int_{O_\varepsilon(\xi)} \min\Big\{\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^2, \frac{1}{\varepsilon}\Big\}\,\mathrm{d}x\,\mathrm{d}\xi\\ &= \int_{B_r}\int_{O_\varepsilon(\xi)} g_\varepsilon\Big(\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big)\mathrm{d}x\,\mathrm{d}\xi, \qquad (10.3.17)\end{aligned}$$

and

$$P_\varepsilon^r(u,O) := \int_{B_r}\int_{O_\varepsilon(\xi)}\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|\,\mathrm{d}x\,\mathrm{d}\xi,$$

and finally

$$H_\varepsilon^r(u,O) := G_\varepsilon^r(u,O) + P_\varepsilon^r(u,O).$$

According to the following elementary lemma, g_ε fulfils (10.3.6) and (10.3.7) with $p = 2$ and, therefore, all the previous results apply to functionals $\{G_\varepsilon^r\}_\varepsilon$. The same conclusion clearly holds for $\{P_\varepsilon^r\}_\varepsilon$ with $a_\varepsilon(z) = |z|$.

Lemma 10.3.7. *Let m be a positive integer and let $\{z_1, z_2, \dots, z_m\} \subset \mathbb{R}^k$. Then for every $\varepsilon > 0$ we have*

$$g_\varepsilon\Big(\sum_{i=1}^m z_i\Big) \le m\sum_{i=1}^m g_\varepsilon(z_i).$$

Proof. If there exists an index $i^* \in \{1,\dots,m\}$ such that $g_\varepsilon(z_{i^*}) = \frac{1}{\varepsilon}$, then

$$m\sum_{i=1}^m g_\varepsilon(z_i) \ge \frac{1}{\varepsilon} \ge \min\Big\{\Big|\sum_{i=1}^m z_i\Big|^2, \frac{1}{\varepsilon}\Big\} = g_\varepsilon\Big(\sum_{i=1}^m z_i\Big),$$

otherwise,

$$m\sum_{i=1}^m g_\varepsilon(z_i) = m\sum_{i=1}^m |z_i|^2 \geq \Big|\sum_{i=1}^m z_i\Big|^2 \geq \min\Big\{\Big|\sum_{i=1}^m z_i\Big|^2, \frac{1}{\varepsilon}\Big\} = g_\varepsilon\Big(\sum_{i=1}^m z_i\Big),$$

which concludes the proof. □

The next proposition shows that for sequences of functions $\{u_\varepsilon\}_\varepsilon$ that are essentially bounded, a uniform bound on $G^r_\varepsilon(u_\varepsilon, O)$ implies a uniform bound on $H^r_\varepsilon(u_\varepsilon, O)$.

Proposition 10.3.8. *Let $u \in L^\infty(\Omega;\mathbb{R}^k)$, $O \in \mathcal{O}(\Omega)$, and $\varepsilon, r > 0$. We have*

$$H^r_\varepsilon(u,O) \leq (1 + 2\|u\|_{L^\infty(\Omega)})G^r_\varepsilon(u,O) + (\mathcal{L}^d(O)\mathcal{L}^d(B_r)G^r_\varepsilon(u,O))^{1/2}.$$

Proof. We claim that

$$\begin{aligned}
&\int_{B_r}\int_{O_\varepsilon(\xi)} \frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq \mathcal{L}^d(O)^{1/2}\mathcal{L}^d(B_r)^{1/2}\Big(\int_{B_r}\int_{O_\varepsilon(\xi)} \min\Big\{\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^2, \frac{1}{\varepsilon}\Big\}\,\mathrm{d}x\,\mathrm{d}\xi\Big)^{1/2} \\
&\quad + 2\|u\|_{L^\infty(\Omega)}\int_{B_r}\int_{O_\varepsilon(\xi)} \min\Big\{\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^2, \frac{1}{\varepsilon}\Big\}\,\mathrm{d}x\,\mathrm{d}\xi. \qquad (10.3.18)
\end{aligned}$$

To prove this, we let

$$O(\varepsilon,\xi) \coloneqq \{x \in O_\varepsilon(\xi)\colon\ |u(x+\varepsilon\xi)-u(x)| \leq \varepsilon^{1/2}\}$$

for all $\varepsilon > 0, \xi \in \mathbb{R}^d$. By Hölder's inequality, we infer that

$$\int_{O(\varepsilon,\xi)} \frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon}\,\mathrm{d}x \leq \mathcal{L}^d(O)^{1/2}\Big(\int_{O(\varepsilon,\xi)}\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^2\,\mathrm{d}x\Big)^{1/2} \qquad (10.3.19)$$

and, by the boundedness of u,

$$\int_{O_\varepsilon(\xi)\setminus O(\varepsilon,\xi)} \frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon}\,\mathrm{d}x \leq 2\|u\|_{L^\infty(\Omega)}\int_{O_\varepsilon(\xi)\setminus O(\varepsilon,\xi)} \frac{1}{\varepsilon}\,\mathrm{d}x. \qquad (10.3.20)$$

Summing (10.3.19) and (10.3.20) and integrating with respect to ξ, we apply Jensen's inequality to get

$$\begin{aligned}
\int_{B_r}\int_{O_\varepsilon(\xi)} \frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}\xi &\leq \mathcal{L}^d(O)^{1/2}\int_{B_r}\Big(\int_{O(\varepsilon,\xi)}\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^2\mathrm{d}x\Big)^{1/2}\mathrm{d}\xi \\
&\quad + 2\|u\|_{L^\infty(\Omega)}\int_{B_r}\int_{O_\varepsilon(\xi)\setminus O(\varepsilon,\xi)} \frac{1}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq \mathcal{L}^d(O)^{1/2}\mathcal{L}^d(B_r)^{1/2}\Big(\int_{B_r}\int_{O(\varepsilon,\xi)}\Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^2\,\mathrm{d}x\,\mathrm{d}\xi\Big)^{1/2} \\
&\quad + 2\|u\|_{L^\infty(\Omega)}\int_{B_r}\int_{O_\varepsilon(\xi)\setminus O(\varepsilon,\xi)} \frac{1}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}\xi.
\end{aligned}$$

This proves the claim.

By (10.3.18), we have

$$H^r_\varepsilon(u,O) = G^r_\varepsilon(u,O) + \int_{B_r}\int_{O_\varepsilon(\xi)} \frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}\xi$$
$$\le G^r_\varepsilon(u,O) + \mathcal{L}^d(O)^{1/2}\mathcal{L}^d(B_r)^{1/2}(G^r_\varepsilon(u,O))^{1/2} + 2\|u\|_{L^\infty(\Omega)}G^r_\varepsilon(u,O),$$

which concludes the proof. □

The last preliminary result will also be useful in Section 10.6.

Lemma 10.3.9. *Let $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$, let $r_0 > 0$ be as in (10.1.2), and let $\varepsilon'' = \varepsilon''(r_0, O)$ be as in Corollary 10.3.6. Consider a sequence $\{u_n\}_n \subset L^\infty(\Omega;\mathbb{R}^k)$ such that*

$$\sup_n \big(\|u_n\|_{L^\infty(\Omega)} + F^T_{\varepsilon_n}(u_n,O)\big) < +\infty, \tag{10.3.21}$$

for some $T > r_0$. Then, $S := \sup_n H^{r_0}_{\varepsilon_n}(u_n,O) < +\infty$. Moreover, there exists a positive constant $C^* = C^*(S, \sup_n \|u_n\|_{L^\infty(\Omega)})$ such that for every $\delta > 0$ there exist $T^* = T^*(\delta) > r_0$ and $\varepsilon^* = \varepsilon^*(\delta)$ such that

$$F_{\varepsilon_n}(u_n,O) \le F^T_{\varepsilon_n}(u_n,O) + C^*\delta \tag{10.3.22}$$

for every $T > T^*$ and for every $\varepsilon_n < \varepsilon^*$.

Proof. In order to prove the first part of the statement, we observe that, by (10.1.2), (10.2.3), and the fact that $T > r_0$, we have

$$c_0 G^{r_0}_{\varepsilon_n}(u_n,O) \le F^T_{\varepsilon_n}(u_n,O);$$

therefore, taking into account (10.3.21), we deduce $\sup_n G^{r_0}_{\varepsilon_n}(u_n,O) < +\infty$, and consequently we infer $S := \sup_n H^{r_0}_{\varepsilon_n}(u_n,O) < +\infty$ in light of Proposition 10.3.8.

To prove (10.3.22) we write

$$F_{\varepsilon_n}(u_n,O) = F^T_{\varepsilon_n}(u_n,O) + \int_{\mathbb{R}^d\setminus B_T}\int_{O_{\varepsilon_n}(\xi)} f_{\varepsilon_n}\Big(x,\xi,\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)\,\mathrm{d}x\,\mathrm{d}\xi$$

and, by (10.2.3), we obtain

$$F_{\varepsilon_n}(u_n,O) \le F^T_{\varepsilon_n}(u_n,O) + \int_{\mathbb{R}^d\setminus B_T}\rho_2(\xi)\int_{O_{\varepsilon_n}(\xi)} g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)\mathrm{d}x\,\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^d\setminus B_T}\psi_{\varepsilon_n}(\xi)\int_{O_{\varepsilon_n}(\xi)}\Big|\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big|\,\mathrm{d}x\,\mathrm{d}\xi + \mathcal{L}^d(\Omega)\int_{\mathbb{R}^d\setminus B_T}\eta_{\varepsilon_n}(\xi)\,\mathrm{d}\xi.$$

Applying Corollary 10.3.6 with $r = r_0, a_\varepsilon(z) = g_\varepsilon(z)$, and with $r = r_0, a_\varepsilon(z) = |z|$, we get that for every $\varepsilon_n < \varepsilon''$ it holds

$$F_{\varepsilon_n}(u_n,O) \le F^T_{\varepsilon_n}(u_n,O) + C_3(G^{r_0}_{\varepsilon_n}(u_n,O) + \|u_n\|^2_{L^2(O)})\int_{\mathbb{R}^d\setminus B_T}\rho_2(\xi)(|\xi|^2+1)\,\mathrm{d}\xi$$
$$+ C_3(P^{r_0}_{\varepsilon_n}(u_n,O) + \|u_n\|_{L^1(O)})\int_{\mathbb{R}^d\setminus B_T}\psi_{\varepsilon_n}(\xi)(|\xi|+1)\,\mathrm{d}\xi + \mathcal{L}^d(\Omega)\int_{\mathbb{R}^d\setminus B_T}\eta_{\varepsilon_n}(\xi)\,\mathrm{d}\xi$$
$$\le F^T_{\varepsilon_n}(u_n,O) + C(G^{r_0}_{\varepsilon_n}(u_n,O) + \|u_n\|^2_{L^\infty(\Omega)})\int_{\mathbb{R}^d\setminus B_T}\rho_2(\xi)|\xi|^2\,\mathrm{d}\xi$$
$$+ C(P^{r_0}_{\varepsilon_n}(u_n,O) + \|u_n\|_{L^\infty(\Omega)})\int_{\mathbb{R}^d\setminus B_T}\psi_{\varepsilon_n}(\xi)|\xi|\,\mathrm{d}\xi + \mathcal{L}^d(\Omega)\int_{\mathbb{R}^d\setminus B_T}\eta_{\varepsilon_n}(\xi)\,\mathrm{d}\xi$$

$$\leq F^T_{\varepsilon_n}(u_n,O) + C(H^{r_0}_{\varepsilon_n}(u_n,O) + \|u_n\|^2_{L^\infty(\Omega)} + \|u_n\|_{L^\infty(\Omega)} + \mathcal{L}^d(\Omega))$$
$$\times \int_{\mathbb{R}^d\setminus B_T} (\rho_2(\xi)|\xi|^2 + \psi_{\varepsilon_n}(\xi)|\xi| + \eta_{\varepsilon_n}(\xi))\,\mathrm{d}\xi.$$

By (10.1.1), (10.1.4), and (10.2.1) there exists $T^* = T^*(\delta)$ such that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^d\setminus B_{T^*}} \Big(\rho_2(\xi)|\xi|^2 + \psi_{\varepsilon_n}(\xi)|\xi| + \eta_{\varepsilon_n}(\xi)\Big)\,\mathrm{d}\xi < \delta$$

and therefore, there exists $\varepsilon^*(T^*) = \varepsilon^*(\delta)$ such that

$$\int_{\mathbb{R}^d\setminus B_{T^*}} \Big(\rho_2(\xi)|\xi|^2 + \psi_{\varepsilon_n}(\xi)|\xi| + \eta_{\varepsilon_n}(\xi)\Big)\,\mathrm{d}\xi < 2\delta$$

for every $\varepsilon_n < \varepsilon^*$; hence, we obtain

$$F_{\varepsilon_n}(u_n,O) \leq F^T_{\varepsilon_n}(u_n,O) + C\big(S + \mathcal{L}^d(O) + \sup_n(\|u_n\|^2_{L^\infty(\Omega)} + \|u_n\|_{L^\infty(\Omega)})\big)\delta$$

for every $T > T^*$ and $\varepsilon_n < \varepsilon^*$, concluding the proof. □

Before proceeding with the proof of Proposition 10.3.2, we make an observation that will be used several times.

Remark 10.3.10. Consider $u \in L^\infty(\Omega;\mathbb{R}^k)$ and let $\{u_n\}_n$ be such that

$$u_n \to u \text{ in } L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \quad \text{and} \quad F_{\varepsilon_n}(u_n,O) \to F'(u,O) \quad \text{as } n \to +\infty.$$

In light of (10.2.7), by a truncation argument we can always assume that for every $n \in \mathbb{N}$ we have $\|u_n\|_{L^\infty(\Omega)} \leq \sqrt{k}\|u\|_{L^\infty(\Omega)}$. Note that an analogous property holds if we replace the Γ-lim inf with the Γ-lim sup; that is, considering F'' in place of F'. Moreover, the same conclusions hold for $F'^{,T}$ and $F''^{,T}$ if we consider $\{F^T_\varepsilon\}_\varepsilon$ in place of $\{F_\varepsilon\}_\varepsilon$.

We are finally ready to prove Proposition 10.3.2.

Proof of Proposition 10.3.2. We only prove the statement for F', the other case being analogous.

Since $F^T_\varepsilon(u,O) \leq F_\varepsilon(u,O)$ for every $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ and $\varepsilon, T > 0$, one inequality is trivial; therefore, we only prove the opposite inequality assuming first that $u \in L^\infty(\Omega;\mathbb{R}^k)$.

Given $T > r_0$, let $\{u_n\}_n \subset L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ be such that

$$u_n \to u \text{ in } L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \quad \text{and} \quad F^T_{\varepsilon_n}(u_n,O) \to F'^{,T}(u,O) \quad \text{as } n \to +\infty. \tag{10.3.23}$$

It is not restrictive to suppose that $F'^{,T}(u,O) < +\infty$ and, by Remark 10.3.10, we may also assume

$$\sup_n \|u_n\|_{L^\infty(\Omega)} \leq \sqrt{k}\|u\|_{L^\infty(\Omega)}. \tag{10.3.24}$$

Thanks to (10.3.23) and (10.3.24), inequality (10.3.21) holds. Hence, by Lemma 10.3.9, for any $\delta > 0$ it holds

$$F_{\varepsilon_n}(u_n,O) \leq F^T_{\varepsilon_n}(u_n,O) + C^*\delta$$

for every $T > T^*$ and n large enough. We now pass to the liminf as $n \to +\infty$ and obtain

$$F'(u,O) \leq F'^{,T}(u,O) + C^*\delta,$$

for every $\delta > 0$, $T > T^*$; the conclusion follows letting first $T \to +\infty$ and then $\delta \to 0^+$.

If $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, let $\{u^M\}_M$ be the sequence obtained by truncation. By the previous step we have that

$$F'(u^M, O) = \lim_{T\to+\infty} F'^{,T}(u^M, O) = \sup_T F'^{,T}(u^M, O),$$

hence, using Lemma 10.3.1, we obtain

$$\begin{aligned} F'(u, O) &= \sup_M F'(u^M, O) = \sup_M \sup_T F'^{,T}(u^M, O) \\ &= \sup_T \sup_M F'^{,T}(u^M, O) = \sup_T F'^{,T}(u, O) = \lim_{T\to+\infty} F'^{,T}(u, O), \end{aligned}$$

which concludes the proof. □

10.4 Compactness

For this section, let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0 and recall the definitions of $F', F'', F'^{,T}, F''^{,T}$ given in Section 10.3. In the next two sections, we shall prove Theorem 10.2.3. For the reader's convenience, we divide the proof in two different propositions, proving separately the compactness and the integral representation of the obtained Γ-limit. In this section, we address the compactness part, which is the content of Proposition 10.4.5.

In this section, we let C denote a positive constant whose value may change from line to line.

We begin by proving that the functionals F' and F'' introduced in (10.3.3) satisfy suitable upper and lower bounds.

Proposition 10.4.1. *Let $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ and let $\{F_\varepsilon\}_\varepsilon$ be defined as in* (10.2.5)*. There exist positive constants c_1, c_2, Λ independent of O such that*

$$c_1 \mathrm{MS}_1(u, O) \le F'(u, O) \text{ for all } u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k), \tag{10.4.1}$$

$$F''(u, O) \le c_2 H(u, O) + \Lambda \mathcal{L}^d(O) \text{ for all } u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \cap \mathrm{BV}(O;\mathbb{R}^k). \tag{10.4.2}$$

In particular, if $F'(u, O) < +\infty$, then $u \in \mathrm{GSBV}^2(O;\mathbb{R}^k)$.

Proof. The proof immediately follows by (10.2.6), (10.1.11), and (10.1.12). □

In the proof of Proposition 10.4.5 below, we shall use the localisation method for Γ-convergence. As we have seen in Chapters 2 and 4, it relies on the De Giorgi-Letta criterion for measures, which requires one to prove superadditivity, subadditivity, and inner regularity. To this aim, we first prove that F'' satisfies a version of the *nested subadditivity* property.

Lemma 10.4.2. *Let $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, let $O', O, U', U \in \mathcal{O}(\Omega)$ such that $O' \subset\subset O$, $U' \subset\subset U$, and $O' \cup U' \in \mathcal{O}_{\mathrm{reg}}(\Omega)$. Let F'' be the functional defined by* (10.3.3)*: it holds*

$$F''(u, O' \cup U') \le F''(u, O) + F''(u, U).$$

Proof. Assume first that $u \in L^\infty(\Omega;\mathbb{R}^k)$ and, without loss of generality, also assume that $F''(u, O)$ and $F''(u, U)$ are both finite. Consider two sequences $u_n, v_n \to u$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ such that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n, O) = F''(u, O) \qquad \text{and} \qquad \lim_{n\to+\infty} F_{\varepsilon_n}(v_n, U) = F''(u, U),$$

and, in light of Remark 10.3.10, assume that

$$\sup_n \|u_n\|_{L^\infty(\Omega)} \le \sqrt{k}\|u\|_{L^\infty(\Omega)}, \quad \sup_n \|v_n\|_{L^\infty(\Omega)} \le \sqrt{k}\|u\|_{L^\infty(\Omega)}.$$

Fix $0 < R < \min\{\operatorname{dist}(O', \mathbb{R}^d \setminus O), \operatorname{dist}(U', \mathbb{R}^d \setminus U)\}$, let $N > 4$ be a positive integer, and let

$$O_i := \Big\{x \in \mathbb{R}^d \colon \operatorname{dist}(x, O') < \frac{iR}{N}\Big\}, \quad i \in \{1, \dots, N\}.$$

For all $i \in \{1, \dots, N-1\}$ we note that $O' \subset\subset O_i \subset\subset O_{i+1} \subset\subset O$, and we consider a cut-off function $\varphi_i \in C_c^\infty(\mathbb{R}^d; [0,1])$ such that

$$\begin{cases} \varphi_i = 1 & \text{in } \overline{O}_i, \\ \varphi_i = 0 & \text{in } \mathbb{R}^d \setminus O_{i+1}, \\ |\nabla \varphi_i| \le \frac{2N}{R} & \text{in } \mathbb{R}^d. \end{cases} \tag{10.4.3}$$

Set $w_n^i := \varphi_i u_n + (1-\varphi_i)v_n$ and note that, for any $\xi \in \mathbb{R}^d, n \in \mathbb{N}$, and $x \in \Omega_{\varepsilon_n}(\xi)$, we have

$$\begin{aligned} \frac{w_n^i(x+\varepsilon_n\xi) - w_n^i(x)}{\varepsilon_n} &= \varphi_i(x)\frac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n} + (1-\varphi_i(x))\frac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n} \\ &\quad + \frac{\varphi_i(x+\varepsilon_n\xi) - \varphi_i(x)}{\varepsilon_n}\left(u_n(x+\varepsilon_n\xi) - v_n(x+\varepsilon_n\xi)\right) \end{aligned} \tag{10.4.4}$$

and, in particular,

$$\frac{w_n^i(x+\varepsilon_n\xi) - w_n^i(x)}{\varepsilon_n} = \begin{cases} \dfrac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n} & \text{if } x \in (O_i)_{\varepsilon_n}(\xi), \\ \dfrac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n} & \text{if } x \in (\Omega \setminus \overline{O}_{i+1})_{\varepsilon_n}(\xi). \end{cases} \tag{10.4.5}$$

We observe that, using (10.4.3) and (10.4.4), we have

$$\begin{aligned} \Big|\frac{w_n^i(x+\varepsilon_n\xi) - w_n^i(x)}{\varepsilon_n}\Big| &\le \Big|\frac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n}\Big| + \Big|\frac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n}\Big| \\ &\quad + \Big(\frac{2N}{R}\Big)|\xi||u_n(x+\varepsilon_n\xi) - v_n(x+\varepsilon_n\xi)|; \end{aligned} \tag{10.4.6}$$

and similarly, recalling the definition of g_ε and exploiting Lemma 10.3.7, we get

$$\begin{aligned} g_{\varepsilon_n}\Big(\frac{w_n^i(x+\varepsilon_n\xi) - w_n^i(x)}{\varepsilon_n}\Big) &\le 3g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n}\Big) + 3g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n}\Big) \\ &\quad + 3\Big(\frac{2N}{R}\Big)^2|\xi|^2|u_n(x+\varepsilon_n\xi) - v_n(x+\varepsilon_n\xi)|^2, \end{aligned} \tag{10.4.7}$$

for every $\xi \in \mathbb{R}^d$, $n \in \mathbb{N}$, and $x \in \Omega_{\varepsilon_n}(\xi)$.

For $i \in \{1, \dots, N-1\}$ we set

$$S_{n,\xi}^i := (O' \cup U')_{\varepsilon_n}(\xi) \setminus \big[(O_i)_{\varepsilon_n}(\xi) \cup (\Omega \setminus \overline{O}_{i+1})_{\varepsilon_n}(\xi)\big]$$

and we observe that if $\xi \in B_T$ for some $T > 0$ and $\varepsilon_n T < \frac{R}{N}$, then

$$S_{n,\xi}^i \subset [(O_{i+1} \setminus \overline{O}_i) + (-\varepsilon_n, \varepsilon_n)\xi] \cap (U')_{\varepsilon_n}(\xi), \tag{10.4.8}$$

and, in particular,

$$\bigcup_{i=1}^{N-4} S^i_{n,\xi} \subset (O_{N-2} \setminus \overline{O'}) \cap U'. \tag{10.4.9}$$

Let $T > 0$ be fixed. By (10.4.5) and (10.2.3), we have

$$\begin{aligned}
F^T_{\varepsilon_n}(w^i_n, O' \cup U') &= \int_{B_T} \int_{(O_i \cap (O' \cup U'))_{\varepsilon_n}(\xi)} f_{\varepsilon_n}\Big(x, \xi, \frac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n}\Big) \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \int_{B_T} \int_{((\Omega \setminus \overline{O}_{i+1}) \cap U')_{\varepsilon_n}(\xi)} f_{\varepsilon_n}\Big(x, \xi, \frac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n}\Big) \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \int_{B_T} \int_{S^i_{n,\xi}} f_{\varepsilon_n}\Big(x, \xi, \frac{w^i_n(x+\varepsilon_n\xi) - w^i_n(x)}{\varepsilon_n}\Big) \mathrm{d}x \, \mathrm{d}\xi \\
&\le F_{\varepsilon_n}(u_n, O) + F_{\varepsilon_n}(v_n, U) \\
&\quad + \int_{B_T} \rho_2(\xi) \int_{S^i_{n,\xi}} g_{\varepsilon_n}\Big(\frac{w^i_n(x+\varepsilon_n\xi) - w^i_n(x)}{\varepsilon_n}\Big) \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \int_{B_T} \psi_{\varepsilon_n}(\xi) \int_{S^i_{n,\xi}} \Big|\frac{w^i_n(x+\varepsilon_n\xi) - w^i_n(x)}{\varepsilon_n}\Big| \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \int_{B_T} \eta_{\varepsilon_n}(\xi) |S^i_{n,\xi}| \mathrm{d}\xi,
\end{aligned}$$

and then, using (10.4.7) and (10.4.6), we get

$$\begin{aligned}
F^T_{\varepsilon_n}(w^i_n, O' \cup U') &\le F_{\varepsilon_n}(u_n, O) + F_{\varepsilon_n}(v_n, U) \\
&\quad + 3 \int_{B_T} \rho_2(\xi) \int_{S^i_{n,\xi}} g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n}\Big) + g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n}\Big) \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \frac{12N^2}{R^2} \int_{B_T} \rho_2(\xi) |\xi|^2 \int_{S^i_{n,\xi}} |u_n(x+\varepsilon_n\xi) - v_n(x+\varepsilon_n\xi)|^2 \, \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \int_{B_T} \psi_{\varepsilon_n}(\xi) \int_{S^i_{n,\xi}} \Big|\frac{u_n(x+\varepsilon_n\xi) - u_n(x)}{\varepsilon_n}\Big| + \Big|\frac{v_n(x+\varepsilon_n\xi) - v_n(x)}{\varepsilon_n}\Big| \, \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \frac{2N}{R} \int_{B_T} \psi_{\varepsilon_n}(\xi) |\xi| \int_{S^i_{n,\xi}} |u_n(x+\varepsilon_n\xi) - v_n(x+\varepsilon_n\xi)| \, \mathrm{d}x \, \mathrm{d}\xi \\
&\quad + \int_{B_T} \eta_{\varepsilon_n}(\xi) |S^i_{n,\xi}| \, \mathrm{d}\xi.
\end{aligned} \tag{10.4.10}$$

We estimate the summands in (10.4.10).

As for the integrals in the third and fifth line of (10.4.10), we use (10.1.1) and (10.4.8) to get that, for n large enough,

$$\begin{aligned}
&\frac{12N^2}{R^2} \int_{B_T} \rho_2(\xi) |\xi|^2 \int_{S^i_{n,\xi}} |u_n(x+\varepsilon_n\xi) - v_n(x+\varepsilon_n\xi)|^2 \, \mathrm{d}x \, \mathrm{d}\xi \\
&\le \frac{12N^2}{R^2} \Big(\int_{\mathbb{R}^d} \rho_2(\xi) |\xi|^2 \, \mathrm{d}\xi \Big) \|u_n - v_n\|^2_{L^2(U')} \\
&\le \frac{24\sqrt{k}N^2}{R^2} \Big(\int_{\mathbb{R}^d} \rho_2(\xi) |\xi|^2 \, \mathrm{d}\xi \Big) \|u\|_{L^\infty(\Omega)} \|u_n - v_n\|_{L^1(V')} \\
&\le CN^2 \|u_n - v_n\|_{L^1(U')},
\end{aligned} \tag{10.4.11}$$

and similarly, also using (10.1.3),

$$\frac{2N}{R}\int_{B_T}\psi_{\varepsilon_n}(\xi)|\xi|\int_{S^i_{n,\xi}}|u_n(x+\varepsilon_n\xi)-v_n(x+\varepsilon_n\xi)|\mathrm{d}x\,\mathrm{d}\xi \\ \le CN\|u_n-v_n\|_{L^1(U')}. \tag{10.4.12}$$

In order to estimate the integrals in the second, fourth, and sixth line of (10.4.10), we note that, by (10.4.8), for $i\in\{1,\dots,N-4\}$ every point belongs at most to two sets of the form $S^i_{n,\xi}$, when n is large enough. As a consequence, setting $E\coloneqq O_{N-2}\cap U'$ and using (10.4.9), we have

$$\begin{aligned}&\sum_{i=1}^{N-4}\Big[\int_{B_T}\rho_2(\xi)\int_{S^i_{n,\xi}}g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)+g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big)\mathrm{d}x\,\mathrm{d}\xi\\ &+\int_{B_T}\psi_{\varepsilon_n}(\xi)\int_{S^i_{n,\xi}}\Big|\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big|+\Big|\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big|\,\mathrm{d}x\,\mathrm{d}\xi+\int_{B_T}\eta_{\varepsilon_n}(\xi)|S^i_{n,\xi}|\mathrm{d}\xi\Big]\\ &\le 2\int_{B_T}\rho_2(\xi)\int_E g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)+g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big)\,\mathrm{d}x\,\mathrm{d}\xi\\ &+2\int_{B_T}\psi_{\varepsilon_n}(\xi)\int_E\Big|\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big|+\Big|\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big|\,\mathrm{d}x\,\mathrm{d}\xi+4\Lambda\mathcal{L}^d(\Omega),\end{aligned} \tag{10.4.13}$$

with Λ as in (10.2.2).

Let now r_0 be as in (10.1.2). Assuming $\varepsilon_n(r_0+T)<\operatorname{dist}(U',\mathbb{R}^d\setminus\Omega)$ and $\xi\in B_T$, we have

$$\operatorname{dist}(E+(0,\varepsilon_n)\xi,\mathbb{R}^d\setminus\Omega)\ge\operatorname{dist}(U'+B_{\varepsilon_nT},\mathbb{R}^d\setminus\Omega)\ge\operatorname{dist}(U',\mathbb{R}^d\setminus\Omega)-\varepsilon_nT>\varepsilon_nr_0;$$

hence, we are in position to apply Lemma 10.3.5 which yields

$$\begin{aligned}&\int_E g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)+g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big)\,\mathrm{d}x\\ &\quad\le C_2(|\xi|^2+1)(G^{r_0}_{\varepsilon_n}(u_n,E_{\varepsilon_n,\xi})+G^{r_0}_{\varepsilon_n}(v_n,E_{\varepsilon_n,\xi}))\\ &\quad\le C_2(|\xi|^2+1)(G^{r_0}_{\varepsilon_n}(u_n,O)+G^{r_0}_{\varepsilon_n}(v_n,U)),\end{aligned} \tag{10.4.14}$$

where we have used that

$$E_{\varepsilon_n,\xi}\coloneqq E+(0,\varepsilon_n)\xi+B_{\varepsilon_nr_0}\subset O\cap U$$

upon further assuming that n is large enough so that $\varepsilon_n(r_0+T)<2\frac{R}{N}$.

By (10.1.2) and (10.2.6), we have that $c_0G^{r_0}_{\varepsilon_n}\le G_{1,\varepsilon_n}\le F_{\varepsilon_n}$; therefore, recalling that

$$\sup_n F_{\varepsilon_n}(u_n,O)<+\infty,\qquad\sup_n F_{\varepsilon_n}(v_n,U)<+\infty,$$

we deduce from (10.4.14) that

$$\int_E g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)+g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big)\,\mathrm{d}x\le C(|\xi|^2+1), \tag{10.4.15}$$

where the constant C is independent of n and ξ.

With a similar argument based on Lemma 10.3.5, we get

$$\begin{aligned}\int_E \Big|\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big| + \Big|\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big|\,\mathrm{d}x \\ &\le C_2(|\xi|+1)(P^{r_0}_{\varepsilon_n}(u_n,O)+P^{r_0}_{\varepsilon_n}(v_n,U)) \\ &\le C_2(|\xi|+1)(H^{r_0}_{\varepsilon_n}(u_n,O)+H^{r_0}_{\varepsilon_n}(v_n,U)) \\ &\le C(|\xi|+1), \end{aligned} \tag{10.4.16}$$

where we have used Proposition 10.3.8 in the last inequality.

Putting together (10.4.13), (10.4.15), and (10.4.16) we deduce that, upon taking ε_n small enough,

$$\begin{aligned}\sum_{i=1}^{N-4}\Big[\int_{B_T}\rho_2(\xi)\int_{S^i_{n,\xi}} g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)+g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big)\mathrm{d}x\,\mathrm{d}\xi \\ + \int_{B_T}\psi_{\varepsilon_n}(\xi)\int_{S^i_{n,\xi}}\Big|\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big|+\Big|\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big|\,\mathrm{d}x\,\mathrm{d}\xi \\ + \int_{B_T}\eta_{\varepsilon_n}(\xi)|S^i_{n,\xi}|\mathrm{d}\xi\Big] \\ \le C\int_{B_T}\rho_2(\xi)(|\xi|^2+1)+\psi_{\varepsilon_n}(\xi)(|\xi|+1)\,\mathrm{d}\xi + 4\Lambda\mathcal{L}^d(\Omega),\end{aligned}$$

where the constant C is independent of n. Recalling (10.1.1), (10.1.3), and (10.1.5), we deduce that there exists $i^*\in\{1,\dots,N-4\}$ such that

$$\begin{aligned}\int_{B_T}\rho_2(\xi)\int_{S^{i^*}_{n,\xi}} g_{\varepsilon_n}\Big(\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big)+g_{\varepsilon_n}\Big(\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big)\mathrm{d}x\,\mathrm{d}\xi \\ + \int_{B_T}\psi_{\varepsilon_n}(\xi)\int_{S^{i^*}_{n,\xi}}\Big|\frac{u_n(x+\varepsilon_n\xi)-u_n(x)}{\varepsilon_n}\Big|+\Big|\frac{v_n(x+\varepsilon_n\xi)-v_n(x)}{\varepsilon_n}\Big|\,\mathrm{d}x\,\mathrm{d}\xi \\ + \int_{B_T}\eta_{\varepsilon_n}(\xi)|S^{i^*}_{n,\xi}|\,\mathrm{d}\xi \le \frac{C}{N-4}\end{aligned} \tag{10.4.17}$$

for all n large enough.

Finally, substituting (10.4.11), (10.4.12), and (10.4.17) in (10.4.10), we get

$$F^T_{\varepsilon_n}(w^{i^*}_n, O'\cup U') \le F_{\varepsilon_n}(u_n,O)+F_{\varepsilon_n}(v_n,U)+\frac{C}{N-4}+C(N^2+N)\|u_n-v_n\|_{L^1(U')},$$

then, letting $n\to+\infty$ and $N\to+\infty$ in this order, recalling that $u_n, v_n, w^{i^*}_n \to u$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, we achieve

$$F''^{,T}(u, O'\cup U') \le F''(u,O)+F''(u,U)$$

and the thesis follows by Proposition 10.3.2 since $O'\cup U'\in\mathcal{O}_{\mathrm{reg}}(\Omega)$.

If $u\in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, by the previous case we have

$$F''(u^M, O'\cup U') \le F''(u^M,O)+F''(u^M,U)$$

for all $M>0$; hence, the conclusion follows by Lemma 10.3.1 letting $M\to+\infty$. □

We now investigate the inner regularity of the functionals F' and F''.

Lemma 10.4.3. *Let $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$. Then $F'(u,\cdot)$ and $F''(u,\cdot)$ are increasing set functions and are inner regular on $\mathcal{O}_{\mathrm{reg}}(\Omega)$; that is, $F'(u,\emptyset) = F''(u,\emptyset) = 0$, given $O, U \in \mathcal{O}(\Omega)$ such that $O \subset U$, it holds*

$$F'(u,O) \le F'(u,U), \qquad F''(u,O) \le F''(u,U),$$

and, for every $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$, it holds

$$F'(u,O) = \sup\{F'(u,O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\}, \tag{10.4.18}$$
$$F''(u,O) = \sup\{F''(u,O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\}. \tag{10.4.19}$$

Proof. The fact that both $F'(u,\cdot)$ and $F''(u,\cdot)$ are increasing set-functions follows once we observe that, for every $n \in \mathbb{N}$, we have $F_{\varepsilon_n}(u,\emptyset) = 0$ and $F_{\varepsilon_n}(u,O) \le F_{\varepsilon_n}(u,U)$ if $O \subset U$.

Let $O, O' \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ with $O' \subset\subset O$, and assume $u \notin \mathrm{GSBV}^2(O;\mathbb{R}^k)$. Using (10.4.1), we have

$$F'(u,O) \ge F'(u,O') \ge c_1 \mathrm{MS}_1(u,O'),$$

and by the arbitrariness of O', we infer

$$\begin{aligned} F'(u,O) &\ge \sup\{F'(u,O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\} \\ &\ge \sup\{F'(u,O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}_{\mathrm{reg}}(\Omega)\} \\ &\ge c_1 \mathrm{MS}_1(u,O) = +\infty, \end{aligned}$$

from which (10.4.18) readily follows. The same proof yields also (10.4.19).

Now we prove (10.4.19) in the case $u \in \mathrm{GSBV}^2(O;\mathbb{R}^k)$, the proof of (10.4.18) being analogous. We first suppose that $u \in L^\infty(\Omega;\mathbb{R}^k)$ and note that, since $F''(u,\cdot)$ is an increasing set-function, it is enough to prove that

$$F''(u,O) \le \sup\{F''(u,O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\}.$$

Let $\delta > 0$. As $u \in \mathrm{SBV}^2(O;\mathbb{R}^k)$, we find an open set $O_\delta \subset\subset O$ such that $O \setminus \overline{O}_\delta \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ and

$$c_2 H(u, O \setminus \overline{O}_\delta) + \Lambda \mathcal{L}^d(O \setminus \overline{O}_\delta) < \delta, \tag{10.4.20}$$

where c_2 and Λ are the constants of Proposition 10.4.1.

Let us consider an open set O' such that $O_\delta \subset\subset O' \subset\subset O$ and let $R \coloneqq \mathrm{dist}(O_\delta, \Omega \setminus O')$. We consider sequences $\{u_n\}_n, \{v_n\}_n \subset L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ and such that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n, O') = F''(u,O'), \qquad \lim_{n\to+\infty} F_{\varepsilon_n}(v_n, O \setminus \overline{O}_\delta) = F''(u, O \setminus \overline{O}_\delta), \tag{10.4.21}$$

and which, thanks to Remark 10.3.10, we may also assume to be satisfying

$$\sup_n \|u_n\|_{L^\infty(\Omega)} \le \sqrt{k}\|u\|_{L^\infty(\Omega)}, \qquad \sup_n \|v_n\|_{L^\infty(\Omega)} \le \sqrt{k}\|u\|_{L^\infty(\Omega)}.$$

Letting N be a positive integer, for $i \in \{1,\dots,N\}$, we set $O_i \coloneqq \{x \in O \colon \mathrm{dist}(x, O_\delta) < \frac{iR}{N}\}$ and we introduce a cut-off function $\varphi_i \in C^\infty_c(\mathbb{R}^d;[0,1])$ such that

$$\begin{cases} \varphi_i = 1 & \text{in } \overline{O}_i, \\ \varphi_i = 0 & \text{in } \mathbb{R}^d \setminus O_{i+1}, \\ |\nabla \varphi_i| \le \frac{2N}{R} & \text{in } \mathbb{R}^d. \end{cases}$$

We set $w_n^i \coloneqq \varphi_i u_n + (1-\varphi_i)v_n$ and

$$S_{n,\xi}^i \coloneqq O_{\varepsilon_n}(\xi) \setminus [(O_i)_{\varepsilon_n}(\xi) \cup (O \setminus \overline{O}_{i+1})_{\varepsilon_n}(\xi)].$$

Let us fix $T > 0$. We have

$$\begin{aligned} F_{\varepsilon_n}^T(w_n^i, O) &= F_{\varepsilon_n}^T(u_n, O_i) + F_{\varepsilon_n}^T(v_n, O \setminus \overline{O}_{i+1}) \\ &\quad + \int_{B_T} \int_{S_{n,\xi}^i} f_{\varepsilon_n}\Big(x, \xi, \frac{w_n^i(x+\varepsilon_n\xi) - w_n^i(x)}{\varepsilon_n}\Big)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\le F_{\varepsilon_n}(u_n, O') + F_{\varepsilon_n}(v_n, O \setminus \overline{O}_\delta) \\ &\quad + \int_{B_T} \int_{S_{n,\xi}^i} f_{\varepsilon_n}\Big(x, \xi, \frac{w_n^i(x+\varepsilon_n\xi) - w_n^i(x)}{\varepsilon_n}\Big)\,\mathrm{d}x\,\mathrm{d}\xi. \end{aligned}$$

Arguing exactly as in the part of the proof of Lemma 10.4.2 that led to (10.4.11), (10.4.12), and (10.4.17), we infer the existence of $i^* \in \{1, \dots, N-4\}$ such that

$$\int_{B_T} \int_{S_{n,\xi}^{i^*}} f_{\varepsilon_n}\Big(x, \xi, \frac{w_n^{i^*}(x+\varepsilon_n\xi) - w_n^{i^*}(x)}{\varepsilon_n}\Big)\mathrm{d}x\mathrm{d}\xi \le \frac{C}{N-4} + C(N^2+N)\|u_n - v_n\|_{L^1(O' \setminus \overline{O}_\delta)};$$

then, letting $n \to +\infty$ and using (10.4.21), we get

$$F''^{,T}(u, O) \le F''(u, O') + F''(u, O \setminus \overline{O}_\delta) + \frac{C}{N-4}.$$

Having supposed $O \setminus \overline{O}_\delta \in \mathcal{O}_{\mathrm{reg}}(\Omega)$, we use (10.4.2) to obtain

$$F''^{,T}(u, O) \le F''(u, O') + c_2 H(u, O \setminus \overline{O}_\delta) + \Lambda \mathcal{L}^d(O \setminus \overline{O}_\delta) + \frac{C}{N-4};$$

hence, by (10.4.20), we get

$$F''^{,T}(u, O) \le F''(u, O') + \delta + \frac{C}{N-4}.$$

Since $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$, we apply Proposition 10.3.2 so that the proof follows letting $T \to +\infty$ and by the arbitrariness of δ and N.

If $u \in \mathrm{GSBV}^2(O; \mathbb{R}^k) \cap L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^k)$, thanks to Lemma 10.3.1 and to (10.4.19) applied with $u = u^M$, we obtain that

$$\begin{aligned} F''(u, O) &= \sup_M F''(u^M, O) \\ &= \sup_M \sup\{F''(u^M, O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\} \\ &= \sup\{\sup_M F''(u^M, O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\} \\ &= \sup\{F''(u, O') \colon O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\}, \end{aligned}$$

which proves (10.4.19) in the general case. □

Remark 10.4.4. Resorting to Lemma 10.4.3, one checks immediately that in Lemma 10.4.2 holds unaltered if we drop the hypothesis that $O' \cup U' \in \mathcal{O}_{\mathrm{reg}}(\Omega)$.

We conclude the section by proving the compactness of the sequence $\{F_{\varepsilon_n}\}_n$ with respect to Γ-convergence, and also provide some relevant properties of the limit functional, which are going to be useful to prove an integral representation.

Proposition 10.4.5. *Let $\{F_\varepsilon\}_\varepsilon$ be as in (10.2.5), and let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0 as $n \to +\infty$. Then there exists a subsequence $\{\varepsilon_{n_k}\}_k$ and a functional $F\colon L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \times \mathcal{O}(\Omega) \to [0,+\infty]$ satisfying the following properties:*

(a) (Γ-convergence) *for every $(u,O) \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \times \mathcal{O}_{\mathrm{reg}}(\Omega)$ it holds*

$$F(u,O) = \Gamma\text{-}\lim_{m\to+\infty} F_{\varepsilon_{n_m}}(u,O);$$

(b) (Lower semicontinuity) *for every $O \in \mathcal{O}(\Omega)$ the functional $F(\cdot,O)$ is $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$-lower semicontinuous;*

(c) (Locality) *for every $O \in \mathcal{O}(\Omega)$ and $u,v \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ with $u = v$ $\mathcal{L}^d$-a.e. on O we have $F(u,O) = F(v,O)$;*

(d) (Growth conditions) *there exist positive constants c_1, c_2, Λ such that for every $O \in \mathcal{O}(\Omega)$ we have*

$$c_1 \mathrm{MS}_1(u,O) \le F(u,O) \text{ for all } u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k), \tag{10.4.22}$$

$$F(u,O) \le c_2 H(u,O) + \Lambda \mathcal{L}^d(O) \text{ for all } u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \cap \mathrm{BV}(O;\mathbb{R}^k); \tag{10.4.23}$$

(e) (Measure property) *for every $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ the set function $F(u,\cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a Borel measure on Ω;*

(f) (Monotonicity) *for every 1-Lipschitz function $\Phi\colon \mathbb{R}^k \to \mathbb{R}^k$ and $(u,O) \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \times \mathcal{O}(\Omega)$, we have*

$$F(\Phi \circ u, O) \le F(u,O).$$

In particular,

$$\begin{aligned} F(u+a,O) &= F(u,O) \quad \text{for every } a \in \mathbb{R}^k,\\ F(Ru,O) &= F(u,O) \quad \text{for every } R \in \mathrm{SO}(k),\\ F(u^M,O) &\le F(u,O) \quad \text{for every } M > 0. \end{aligned}$$

Proof. Consider the functionals $F', F''\colon L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \times \mathcal{O}(\Omega) \to [0,+\infty]$ given by

$$F'(u,O) \coloneqq \Gamma\text{-}\liminf_{n\to+\infty} F_{\varepsilon_n}(u,O), \qquad F''(u,O) \coloneqq \Gamma\text{-}\limsup_{n\to+\infty} F_{\varepsilon_n}(u,O) \tag{10.4.24}$$

and, for fixed $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, define the inner regular envelopes

$$\begin{aligned} F'_-(u,O) &\coloneqq \sup\{F'(u,O')\colon\ O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\},\\ F''_-(u,O) &\coloneqq \sup\{F''(u,O')\colon\ O' \subset\subset O \text{ with } O' \in \mathcal{O}(\Omega)\} \end{aligned}$$

for every $O \in \mathcal{O}(\Omega)$. By [88, Theorem 16.9], there exists a subsequence $\varepsilon_{n_m} \to 0^+$ as $m \to +\infty$ such that

$$F'_-(u,O) = F''_-(u,O) =: F(u,O) \tag{10.4.25}$$

for every $(u,O) \in L^1_{\rm loc}(\Omega;\mathbb{R}^k) \times \mathcal{O}(\Omega)$, where F'_-, F''_- in (10.4.25) are obtained replacing $\{\varepsilon_n\}_n$ in (10.4.24) with the subsequence $\{\varepsilon_{n_k}\}_k$.

Using Lemma 10.4.3, we obtain that

$$F'(u,O) = F'_-(u,O) = F''_-(u,O) = F''(u,O)$$

for every $(u,O) \in L^1_{\rm loc}(\Omega;\mathbb{R}^k) \times \mathcal{O}_{\rm reg}(\Omega)$, which proves property (a). Property (b) is a consequence of the lower semicontinuity of Γ-limits, while (c) follows from the fact that F_ε satisfies property (c), and the latter is stable under Γ-limits (see [88, Chapter 16])

In order to prove the measure property (e) we use De Giorgi-Letta criterion for measures (see, e.g., [47, Theorem 10.2] or [88, Theorem 14.21]). The proof then amounts to show that, for every $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^k)$, the set function $F(u,\cdot)$ is increasing, inner regular on $\mathcal{O}(\Omega)$, superadditive, and subadditive.

Let us fix $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^k)$. By Lemma 10.4.3, (10.4.25), and [88, Remark 14.5] the set function $F(u,\cdot)$ is increasing, while it is inner regular on $\mathcal{O}(\Omega)$ by definition.

We now prove that $F(u,\cdot)$ is subadditive. Let $O, U \in \mathcal{O}(\Omega)$. By inner regularity of F, it suffices to show that $F(u,V) \le F(u,O) + F(u,U)$ for any $V \in \mathcal{O}_{\rm reg}(\Omega)$ such that $V \subset\subset O \cup U$. We choose $O', O'', U', U'' \in \mathcal{O}_{\rm reg}(\Omega)$ with $O' \subset\subset O'' \subset\subset O$ and $U' \subset\subset U'' \subset\subset U$ and such that $V \subset O' \cup U'$. Using Lemma 10.4.2 and Remark 10.4.4, we obtain that

$$F(u,V) \le F(u,O' \cup U') \le F''(u,O' \cup U') \le F''(u,O'') + F''(u,U'') \le F(u,O) + F(u,U).$$

We are left with proving that $F(u,\cdot)$ is superadditive. Let $O, U \in \mathcal{O}(\Omega)$ with $O \cap U = \emptyset$. One can easily check that $O_{\varepsilon_{nm}}(\xi) \cup U_{\varepsilon_{nm}}(\xi) \subset (O \cup U)_{\varepsilon_{nm}}(\xi)$ for every $m \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$. Since the sets $O_{\varepsilon_{nm}}(\xi)$ and $U_{\varepsilon_{nm}}(\xi)$ are disjoint, for every $\{u_m\}_m$ converging to u in $L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ we obtain

$$F_{\varepsilon_{n_m}}(u_m,O) + F_{\varepsilon_{n_m}}(u_m,U) \le F_{\varepsilon_{n_m}}(u_m, O \cup U),$$

which, letting $m \to +\infty$, implies the superadditivity, thus, property (e) is verified.

For $U \in \mathcal{O}_{\rm reg}(\Omega)$, property (d) holds by Proposition 10.4.1 and property (f) is a consequence of Lemma 10.3.1. Both are extended to every element of $\mathcal{O}(\Omega)$ resorting to (e). □

10.5 Integral Representation

In this section we complete the proof of Theorem 10.2.3 by proving its integral representation part. We recall that the notation concerning the minimisation problems below was introduced in Section 10.2; see, in particular, (10.2.8). The following is the main result of the section.

Proposition 10.5.1. *Let $F\colon L^1_{\rm loc}(\Omega;\mathbb{R}^k) \times \mathcal{O}(\Omega) \to [0,+\infty]$ be a functional satisfying properties* (b)–(f) *of Proposition 10.4.5. Then for every $O \in \mathcal{O}(\Omega)$ and $u \in {\rm GSBV}^2(O;\mathbb{R}^k) \cap L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ we have*

$$F(u,O) = \int_O f_{\rm bulk}(x,\nabla u)\,{\rm d}x + \int_{J_u \cap O} f_{\rm surf}(x,[u],\nu_u)\,{\rm d}\mathcal{H}^{d-1},$$

where

$$f_{\rm bulk}(x,A) \coloneqq \limsup_{r \to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,r))}{r^d} \tag{10.5.1}$$

for all $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, *and*

$$f_{\rm surf}(x,\zeta,\nu) := \limsup_{r\to 0^+} \frac{\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}} \tag{10.5.2}$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^k$, *and* $\nu \in \mathbb{S}^{d-1}$.

Moreover, we have

$$\begin{aligned} c_1\lambda_1|A|^2 \le f_{\rm bulk}(x,A) \le c_2(\lambda_2|A|^2 + \kappa|A|) + \Lambda \quad &\text{for all } x\in\Omega \text{ and } A\in\mathbb{R}^{k\times d},\\ c_1\mu_1 \le f_{\rm surf}(x,\zeta,\nu) \le c_2(\mu_2 + \kappa|\zeta|) \quad &\text{for all } x\in\Omega,\ \zeta\in\mathbb{R}^k\setminus\{0\}, \text{ and } \nu\in\mathbb{S}^{d-1}, \end{aligned}$$

and

$$f_{\rm surf}(x,\zeta_1,\nu) \le f_{\rm surf}(x,\zeta_2,\nu) \quad \text{if } |\zeta_1|\le|\zeta_2|.$$

for all $x \in \Omega$ *and* $\nu \in \mathbb{S}^{d-1}$. *In particular, for all* $x \in \Omega$ *and* $\nu \in \mathbb{S}^{d-1}$ *it holds*

$$\begin{aligned} &f_{\rm surf}(x,\zeta_1,\nu) = f_{\rm surf}(x,\zeta_2,\nu) \quad \text{if } |\zeta_1| = |\zeta_2|,\\ &f_{\rm surf}(x,\zeta,\nu) = f_{\rm surf}(x,\zeta,-\nu) \quad \text{for all } \zeta\in\mathbb{R}^k. \end{aligned}$$

Remark 10.5.2. Thanks to (f) of Proposition 10.4.5, the functional F considered in Proposition 10.5.1 is invariant under vertical translations. In particular, for every $x \in \Omega$ and $A \in \mathbb{R}^{k\times d}$ we have

$$f_{\rm bulk}(x,A) := \limsup_{r\to 0^+} \frac{\mathfrak{m}^F(A(\cdot - x), Q(x,r))}{r^d}. \tag{10.5.3}$$

We now prove Proposition 10.5.1.

Proof of Proposition 10.5.1. It follows immediately from (b), (c), (e), and (f) that the restriction of the functional F to $\mathrm{SBV}^2(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)$ satisfies conditions ((X1))–(X4a) of Section 1.5 with $X = \mathrm{SBV}^2(\Omega;\mathbb{R}^k)$. By (10.4.23), F satisfies the upper bound in (1.5.9) of Theorem 1.5.3. However, by hypothesis it only satisfies the weaker lower bound in (10.4.22), and not the desired lower bound of (1.5.9). To overcome this difficulty, we use a perturbative approach similar to the one employed in Chapters 2 and 3. For every $\delta > 0$ we introduce the functional F^δ defined by

$$F^\delta(u,O) := F(u,O) + \delta\int_{J_u\cap O} |[u]|\,\mathrm{d}\mathcal{H}^{d-1},$$

for every $u \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ and $O \in \mathcal{O}(\Omega)$, which satisfies all hypotheses of Theorem 1.5.3. Hence, we deduce the integral representation

$$F^\delta(u,O) = \int_O f^\delta_{\rm bulk}(x,\nabla u)\,\mathrm{d}x + \int_{J_u\cap O} f^\delta_{\rm surf}(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{10.5.4}$$

for all $(u,O) \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)$, where $f^\delta_{\rm bulk}$ and $f^\delta_{\rm surf}$ are defined as in (10.5.1) and (10.5.2) with F replaced by F^δ.

At this point, we aim to pass to the limit as $\delta \to 0^+$ in (10.5.4) in order to obtain an integral representation for F on $\mathrm{SBV}^2(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)$. Clearly, $F^\delta(u,O) \to F(u,O)$ as $\delta\to 0^+$; therefore, the conclusion follows provided we are in position to apply the Dominated Convergence Theorem on the right-hand side of (10.5.4).

We preliminarily observe that, using ℓ_A as a test function for the minimisation problems $\mathfrak{m}^F(\ell_A, Q(x,r))$ and $\mathfrak{m}^{F^\delta}(\ell_A, Q(x,r))$, from the upper bound in (10.4.23), we have

$$f_{\rm bulk}(x,A) \le c_2(\lambda_2|A|^2 + \kappa|A|) + \Lambda \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d}, \tag{10.5.5}$$

$$f^\delta_{\rm bulk}(x,A) \le c_2(\lambda_2|A|^2 + \kappa|A|) + \Lambda \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d} \tag{10.5.6}$$

and, applying Theorem 1.5.3 with $\mathcal{F} = \mathrm{MS}_1$, we also obtain the lower bound

$$c_1\lambda_1|A|^2 \le f_{\rm bulk}(x,A) \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d}.$$

Analogously, we have

$$\begin{aligned} c_1\mu_1 \le f_{\rm surf}(x,\zeta,\nu) \le c_2(\mu_2 + \kappa|\zeta|) \quad &\text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^k \setminus \{0\},\ \text{and } \nu \in \mathbb{S}^{d-1},\\ f^\delta_{\rm surf}(x,\zeta,\nu) \le c_2\mu_2 + (c_2\kappa + \delta)|\zeta| \quad &\text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^k,\ \text{and } \nu \in \mathbb{S}^{d-1}. \end{aligned} \tag{10.5.7}$$

Now we prove that

$$\lim_{\delta\to 0^+} f^\delta_{\rm bulk}(x,A) = f_{\rm bulk}(x,A) \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d}, \tag{10.5.8}$$

$$\lim_{\delta\to 0^+} f^\delta_{\rm surf}(x,\zeta,\nu) = f_{\rm surf}(x,\zeta,\nu) \quad \text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^k,\ \text{and } \nu \in \mathbb{S}^{d-1}. \tag{10.5.9}$$

To this aim, we first observe that, since $F(u,Q(x,r)) \le F^\delta(u,Q(x,r))$ for every $\delta > 0$ and $\{F^\delta\}_\delta$ is increasing in δ, it follows immediately that

$$f_{\rm bulk}(x,A) \le \inf_{\delta>0} f^\delta_{\rm bulk}(x,A) = \lim_{\delta\to 0^+} f^\delta_{\rm bulk}(x,A) \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d},$$

$$f_{\rm surf}(x,\zeta,\nu) \le \inf_{\delta>0} f^\delta_{\rm surf}(x,\zeta,\nu) = \lim_{\delta\to 0^+} f^\delta_{\rm surf}(x,\zeta,\nu) \quad \text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^k, \text{ and } \nu \in \mathbb{S}^{d-1}.$$

To obtain the converse inequalities, we will prove that for every $\delta > 0$ we have

$$f^\delta_{\rm bulk}(x,A) \le f_{\rm bulk}(x,A) \quad \text{for all } x \in \Omega \text{ and } A \in \mathbb{R}^{k\times d}, \tag{10.5.10}$$

and that there exists a constant $C > 0$, independent of δ and of $|\zeta|$, such that

$$f^\delta_{\rm surf}(x,\zeta,\nu) \le (1 + C|\zeta|\delta) f_{\rm surf}(x,\zeta,\nu) \quad \text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^k, \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{10.5.11}$$

Let us fix $x \in \Omega$ and $A \in \mathbb{R}^{k\times d}$. In view of Remark 10.5.2, for $r > 0$ we consider a function $u_r \in \mathrm{SBV}^2(Q(x,r);\mathbb{R}^k)$, with $u_r = A(\cdot - x)$ in a neighbourhood of $\partial Q(x,r)$ such that

$$F(u_r, Q(x,r)) \le \mathfrak{m}^F(A(\cdot - x), Q(x,r)) + r^{d+1}. \tag{10.5.12}$$

By a truncation argument we assume $\|u_r\|_{L^\infty(Q(x,r);\mathbb{R}^k)} \le \sqrt{kd}|A|r$. We compare the two values $F^\delta(u_r,Q(x,r))$ with $F(u_r,Q(x,r))$. By definition of F^δ, the lower bound (10.4.22), and (10.5.12) we get

$$\begin{aligned} F^\delta(u_r,Q(x,r)) &= F(u_r,Q(x,r)) + \delta\int_{J_{u_r}\cap Q(x,r)} |[u_r]|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\le F(u_r,Q(x,r)) + 2\delta\sqrt{kd}|A|r\mathcal{H}^{d-1}(J_{u_r}\cap Q(x,r))\\ &\le \big(1 + \frac{2\delta\sqrt{kd}|A|r}{c_1\mu_1}\big)F(u_r,Q(x,r))\\ &\le \big(1 + \frac{2\delta\sqrt{kd}|A|r}{c_1\mu_1}\big)\big(\mathfrak{m}^F(A(\cdot - x),Q(x,r)) + r^{d+1}\big). \end{aligned}$$

As u_r is a competitor for the minimisation problem $\mathfrak{m}^{F^\delta}(A(\cdot-x),Q(x,r))$, dividing by r^d the previous inequality we get that

$$\frac{\mathfrak{m}^{F^\delta}(A(\cdot-x),Q(x,r))}{r^d}\le \big(1+\frac{2\delta\sqrt{kd}|A|r}{c_1\mu_1}\big)\big(\frac{\mathfrak{m}^{F}(A(\cdot-x),Q(x,r))}{r^d}+r\big),$$

and then, letting $r\to 0^+$ and using (10.5.3), we obtain (10.5.10), which in turn implies (10.5.8).

We now show (10.5.11). Let us fix $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$. Arguing as in the proof of (10.5.10), for every $r>0$ we find a function $u_r\in \mathrm{SBV}^2(Q_\nu(x,r);\mathbb{R}^k)$, with $u_r=u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q_\nu(x,r)$ and $\|u_r\|_{L^\infty(Q_\nu(x,r))}\le\sqrt{k}|\zeta|$, such that

$$F(u_r,Q_\nu(x,r))\le \mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,r))+r^d.$$

Using this inequality and the lower bound (10.4.22), we obtain that

$$\begin{aligned}F^\delta(u_r,Q_\nu(x,r))&=F(u_r,Q_\nu(x,r))+\delta\int_{J_{u_r}\cap Q_\nu(x,r)}|[u_r]|\,\mathrm{d}\mathcal{H}^{d-1}\\&\le F(u_r,Q_\nu(x,r))+2\sqrt{k}\delta|\zeta|\mathcal{H}^{d-1}(J_{u_r}\cap Q_\nu(x,r))\\&\le \big(1+\frac{2\sqrt{k}\delta|\zeta|}{c_1\mu_1}\big)F(u_r,Q_\nu(x,r))\\&\le \big(1+\frac{2\sqrt{k}\delta|\zeta|}{c_1\mu_1}\big)\big(\mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,r))+r^d\big),\end{aligned}$$

then, dividing this last inequality by r^{d-1} and using the fact that $u_r=u_{x,\zeta,\nu}$ in a neighbourhood of $\partial Q_\nu(x,r)$, we get

$$\frac{\mathfrak{m}^{F^\delta}(u_{x,\zeta,\nu},Q_\nu(x,r))}{r^{d-1}}\le \big(1+\frac{2\sqrt{k}\delta|\zeta|}{c_1\mu_1}\big)\big(\frac{\mathfrak{m}^{F}(u_{x,\zeta,\nu},Q_\nu(x,r))}{r^{d-1}}+r\big).$$

Letting $r\to 0^+$, we get (10.5.11), which leads to (10.5.9).

Having proved the pointwise convergence, we finally resort to the uniform bounds given by (10.5.6) and (10.5.7) in order to apply the Dominated Convergence Theorem in (10.5.4) and to obtain

$$F(u,O)=\int_O f_{\mathrm{bulk}}(x,\nabla u)\,\mathrm{d}x+\int_{J_u\cap O} f_{\mathrm{surf}}(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\tag{10.5.13}$$

for every $(u,O)\in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)$. Note that, by (c), this integral representation yields an identical integral representation on every $O\in\mathcal{O}_{\mathrm{reg}}(\Omega)$ and $u\in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)\cap \mathrm{SBV}^2(O;\mathbb{R}^k)$. Indeed, if $O\in\mathcal{O}_{\mathrm{reg}}(\Omega)$, any function $u\in\mathrm{SBV}^2(O;\mathbb{R}^k)$ can be extended to a function $\widetilde{u}\in\mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ and by locality we then have $F(u,O)=F(\widetilde{u},O)$.

In order to extend the integral representation to GSBV^2, we begin showing that

$$|\zeta_1|\le|\zeta_2|\implies f_{\mathrm{surf}}(x,\zeta_1,\nu)\le f_{\mathrm{surf}}(x,\zeta_2,\nu)\tag{10.5.14}$$

for all $x\in\mathbb{R}^d$ and $\nu\in\mathbb{S}^{d-1}$. Let us fix $\zeta_1,\zeta_2\in\mathbb{R}^k$ and $r>0$ and let $v_2\in\mathrm{SBV}^2(Q_\nu(x,r);\mathbb{R}^k)$ with $v_2=u_{x,\zeta_2,\nu}$ in a neighbourhood of $\partial Q_\nu(x,r)$. We consider a rotation $R\in\mathrm{SO}(k)$ that

sends $\zeta_2/|\zeta_2|$ to $\zeta_1/|\zeta_1|$, so that $v_1 := \frac{|\zeta_1|}{|\zeta_2|} R v_2$ equals $u_{x,\zeta_1,\nu}$ in a neighbourhood of $\partial Q_\nu(x,r)$ and therefore, by (f), we obtain

$$\mathfrak{m}^F(u_{x,\zeta_1,\nu}, Q_\nu(x,r)) \le F(v_1, Q_\nu(x,r)) \le F(v_2, Q_\nu(x,r)).$$

From the arbitrariness of v_2 and of r we deduce (10.5.14).

We are now ready to conclude the proof of the integral representation. Let us fix $O \in \mathcal{O}(\Omega)$ and $u \in \mathrm{GSBV}^2(O;\mathbb{R}^k) \cap L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$. Since $u^M \in \mathrm{SBV}^2(O;\mathbb{R}^k) \cap L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ for every $M > 0$, by (10.5.13) we obtain that

$$F(u^M, O) = \int_O f_{\mathrm{bulk}}(x, \nabla u^M)\,\mathrm{d}x + \int_{J_{u^M} \cap O} f_{\mathrm{surf}}(x, [u^M], \nu_{u^M})\,\mathrm{d}\mathcal{H}^{d-1}. \tag{10.5.15}$$

Combining (b) with (f), we deduce that

$$\lim_{M \to +\infty} F(u^M, O) = F(u, O).$$

Then, recalling Remark 10.1.1, (10.5.5), and (10.5.14), and using the dominated convergence theorem for the bulk part and the monotone convergence theorem for the surface part, we let $M \to +\infty$ in (10.5.15) and get

$$F(u, O) = \int_O f_{\mathrm{bulk}}(x, \nabla u)\,\mathrm{d}x + \int_{J_u \cap O} f_{\mathrm{surf}}(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

which is the thesis. □

10.6 Convergence of minima and Γ-convergence

In this section, we fix a positive sequence $\{\varepsilon_n\}_n$ converging to 0 as $n \to +\infty$ and a sequence $\{F_{\varepsilon_n}\}_n$ given by (10.2.5), with $\{f_{\varepsilon_n}\}_n$ satisfying conditions (i)-(iii) of Section 10.2. We show that thanks to general properties of Γ-convergence and to results obtained in previous sections, the asymptotic behaviour of minimisation problems associated to F_{ε_n} on small cubes determines the energy densities in the integral representation of the Γ-limit F.

We first state a compactness result for sequences with uniformly bounded energy, originally proved by Gobbino in [132, Theorem 5.4] (see also [43, Theorem 5.23]).

Lemma 10.6.1. *Let $\{u_n\}_n \subset L^\infty(\Omega;\mathbb{R}^k)$, let $r > 0$, and assume that*

$$\sup_n \left(\|u_n\|_{L^\infty(\Omega;\mathbb{R}^k)} + G^r_{1,\varepsilon_n}(u_n, \Omega) \right) < +\infty,$$

where G^r_{1,ε_n} are the functionals defined by (10.3.17)*. Then there exists a subsequence $\{\varepsilon_{n_m}\}_m$ and a function $u \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)$ such that $\{u_{n_m}\}_m$ converges to u in $L^1(\Omega;\mathbb{R}^k)$ as $m \to +\infty$.*

We recall that the notation concerning the minimisation problems below was introduced in Section 10.2; see, in particular, (10.2.11) and (10.2.12).

The following result shows that, if $\{F_{\varepsilon_n}\}_n$ Γ-converges to F on a regular open set $O \subset \Omega$ and if $U \subset\subset O$, the minimum value of $\mathfrak{m}^F(w, O)$ can be controlled from above by the asymptotic value of the minimisation problems $\mathfrak{m}_s^{F_{\varepsilon_n}}(w, U)$ and $H(w, O \setminus U)$.

Lemma 10.6.2. *Let $w \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)\cap L^\infty(\Omega;\mathbb{R}^k)$, and let $O, U \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ with $U \subset\subset O$. Assume that $\{F_{\varepsilon_n}(\cdot, O)\}_n$ Γ-converges to $F(\cdot, O)$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$. Then*

$$\mathfrak{m}^F(w, O) \le \sup_{s>0} \liminf_{n\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_n}}(w, U) + c_2 H(w, O \setminus U) + \Lambda \mathcal{L}^d(O \setminus U). \tag{10.6.1}$$

Proof. We consider a sequence of functions $\{u_n\}_n \subset \mathcal{D}_{\varepsilon_n s, w}(U)$ such that

$$F_{\varepsilon_n}(u_n, U) \le \mathfrak{m}_s^{F_{\varepsilon_n}}(w, U) + \varepsilon_n, \tag{10.6.2}$$

and which, by truncation, we may assume to be satisfying $\|u_n\|_{L^\infty(\Omega)} \le \sqrt{k}\|w\|_{L^\infty(\Omega)}$.

Let us fix $s > 2r_0$, where r_0 is the constant introduced in (10.1.2). Thanks to (10.1.2) and (10.2.6), from this inequality we obtain

$$c_0 G^{r_0}_{\varepsilon_n}(u_n, U) \le F_{\varepsilon_n}(u_n, U) \le \mathfrak{m}_s^{F_{\varepsilon_n}}(w, U) + \varepsilon_n \le H_{\varepsilon_n}(w, U) + \|\eta_{\varepsilon_n}\|_{L^1(\mathbb{R}^d)} \mathcal{L}^d(U) + \varepsilon_n,$$

where c_0 is the positive constant introduced in (10.1.2) and $G^{r_0}_{\varepsilon_n}$ is as in (10.3.17). Recalling (10.1.12), this chain of inequalities implies

$$\sup_n G^{r_0}_{\varepsilon_n}(u_n, U) \le \sup_n \frac{1}{c_0}\Big(H_{\varepsilon_n}(w, O) + \|\eta_{\varepsilon_n}\|_{L^1(\mathbb{R}^d)} \mathcal{L}^d(O) + 1\Big) < +\infty \tag{10.6.3}$$

upon assuming $\varepsilon_n < 1$ for every $n \in \mathbb{N}$, with the right hand side being independent of $s > 2r_0$.

We introduce the extension $\widetilde{u}_n$ of the function u_n obtained by setting $\widetilde{u}_n := u_n$ on U and $\widetilde{u}_n := w$ on $O \setminus U$. We observe that

$$G^{r_0}_{\varepsilon_n}(\widetilde{u}_n, O) \le G^{r_0}_{\varepsilon_n}(u_n, U) + G^{r_0}_{\varepsilon_n}\big(w, \big((O \setminus U) + B_{\varepsilon_n r_0}\big) \cap O\big),$$

which by (10.6.3) implies

$$\sup_n G^{r_0}_{\varepsilon_n}(\widetilde{u}_n, O) \le \big(1 + \frac{1}{c_0}\big) \sup_n \Big(H_{\varepsilon_n}(w, O) + \|\eta_{\varepsilon_n}\|_{L^1(\mathbb{R}^d)} \mathcal{L}^d(O) + 1\Big) < +\infty. \tag{10.6.4}$$

Observe that by Proposition 10.3.8, inequality (10.6.4) also implies that

$$S := \sup_n H^{r_0}_{\varepsilon_n}(\widetilde{u}_n, O) < +\infty.$$

We now pass to a subsequence $\{\varepsilon_{n_m}\}_m$ such that

$$\liminf_{n\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_n}}(w, U) = \lim_{m\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_{n_m}}}(w, U).$$

Thanks to (10.6.4), we may apply Lemma 10.6.1 to obtain a further subsequence, not relabelled, such that $\{\widetilde{u}_{n_m}\}_m$ converges strongly in $L^1(O;\mathbb{R}^k)$ to a function $u \in \mathrm{SBV}^2(O;\mathbb{R}^k)$, which also satisfies the boundary conditions $u = w$ $\mathcal{L}^d$-a.e. in $O \setminus U$.

For every $T > 0$ let us consider the functional $F^T_{\varepsilon_n}$ defined by (10.3.1). Thanks to Lemma 10.3.9, there is a constant C^*, depending only on S and on $\|w\|_{L^\infty(\Omega)}$, and which is therefore independent of s, such that for every $\delta > 0$ it holds

$$F_{\varepsilon_n}(\widetilde{u}_n, O) \le F^T_{\varepsilon_n}(\widetilde{u}_n, O) + C^*\delta$$

for every $T > T^* = T^*(\delta)$ and n large enough. Hence, given $\delta > 0$ and further assuming that $s > T^*$, recalling that u is a competitor for the minimisation problem $\mathfrak{m}^F(w, O)$, by Γ-convergence we have

$$\begin{aligned}\mathfrak{m}^F(w,O) &\le F(u,O) \le \liminf_{m\to+\infty} F_{\varepsilon_{n_m}}(\widetilde{u}_{n_m}, O) \le \liminf_{m\to+\infty} \Big(F^T_{\varepsilon_{n_m}}(\widetilde{u}_{n_m}, O) + C^*\delta\Big)\\ &\le \liminf_{m\to+\infty} \Big(F^T_{\varepsilon_{n_m}}(u_{n_m}, U) + F^T_{\varepsilon_{n_m}}\big(w, \big((O\setminus U) + B_{\varepsilon_{n_m}T}\big)\cap O\big)\Big) + C^*\delta,\end{aligned}$$

for every $T \in (T^*, s]$. Using (10.2.6) and (10.1.12), from this last chain of inequalities we obtain that for every $\sigma > 0$

$$\begin{aligned}\mathfrak{m}^F(w,O) \le \liminf_{m\to+\infty} F_{\varepsilon_{n_m}}(u_{n_m}, U) &+ c_2 H\big(w, \big((O\setminus U) + B_\sigma\big)\cap O\big)\\ &+ \Lambda\mathcal{L}^d(((O\setminus U) + B_\sigma)\cap O)) + C^*\delta.\end{aligned}$$

By arbitrariness of $\sigma > 0$ and by (10.6.2) we get

$$\begin{aligned}\mathfrak{m}^F(w,O) &\le \liminf_{m\to+\infty} F_{\varepsilon_{n_m}}(u_{n_m}, U) + c_2 H(w, O\setminus U) + \Lambda\mathcal{L}^d(O\setminus U) + C^*\delta\\ &\le \lim_{m\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_{n_m}}}(w, U) + c_2 H(w, O\setminus U) + \Lambda\mathcal{L}^d(O\setminus U) + C^*\delta.\end{aligned}$$

Taking the limit as $s \to +\infty$ and as $\delta \to 0^+$, we obtain (10.6.1), concluding the proof. □

Remark 10.6.3. Following the above proof, it is easily seen that the statement of the previous lemma can be improved under the additional assumption that there exists $T > 0$ such that $f_{\varepsilon_n}(x, \xi, z) = 0$ for $\mathcal{L}^d$-a.e. $x \in \Omega, \xi \in \mathbb{R}^d \setminus B_T, z \in \mathbb{R}^k$, and $n \in \mathbb{N}$. In this case, we obtain that

$$\mathfrak{m}^F(w,O) \le \liminf_{n\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_n}}(w, U) + c_2 H(w, O\setminus U) + \Lambda\mathcal{L}^d(O\setminus U)$$

for all $s \ge T$.

In the following lemma we show that, if $\{F_{\varepsilon_n}\}_n$ Γ-converges to F on a regular open set $U \subset \Omega$, the minimum value of $\mathfrak{m}^F(w, O)$ is always larger than the asymptotic value of the minimisation problems $\mathfrak{m}_s^{F_{\varepsilon_n}}(w, O)$.

Lemma 10.6.4. *Let $w \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k) \cap L^\infty(\Omega;\mathbb{R}^k)$, let $s > 0$, and let $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$. Assume that $\{F_{\varepsilon_n}(\cdot, O)\}_n$ Γ-converges to $F(\cdot, O)$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ as $n \to +\infty$. Then*

$$\limsup_{n\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_n}}(w, O) \le \mathfrak{m}^F(w, O). \tag{10.6.5}$$

Proof. Since $w \in \mathrm{SBV}^2(\Omega;\mathbb{R}^k)$, we have that $H(w, O) < +\infty$, which by (10.2.6) clearly implies $\mathfrak{m}^F(w, O) < +\infty$. Let us fix $\sigma > 0$ and $u \in \mathrm{SBV}^2(O;\mathbb{R}^k) \cap L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ such that $F(u, O) < \mathfrak{m}^F(w, O) + \sigma$ and $u = w$ in a neighbourhood of ∂O. Observe that, since F satisfies property (f) of Proposition 10.4.5 on O, it is not restrictive to assume that $u \in L^\infty(\Omega;\mathbb{R}^k)$.

We pass to a subsequence, not relabelled, such that the limsup in (10.6.5) is actually a limit and consider a sequence $\{u_n\}_n \subset L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$, such that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n, O) = F(u, O) < \mathfrak{m}^F(w, O) + \sigma. \tag{10.6.6}$$

By Lemma 10.3.1 we can also assume that $\|u_n\|_{L^\infty(\Omega)} \le \sqrt{k}\|u\|_{L^\infty(\Omega)}$ for every $n \in \mathbb{N}$, so that, possibly passing to a further subsequence, we can assume that $\{u_n\}_n$ converges to u in $L^1(O;\mathbb{R}^k)$ by Lemma 10.6.1.

Let $U \in \mathcal{O}_{\rm reg}(O)$ be obtained intersecting O with a neighbourhood of ∂O on which $u = w$ $\mathcal{L}^d$-a.e. and let $O' \subset\subset O'' \subset\subset O$ be open sets such that $O \setminus O' \subset U$. Having fixed $T > 0$ and $N \in \mathbb{N}$, we may argue as in the proof of Lemma 10.4.2 to find a sequence of cutoff functions $\{\varphi_n\}_n$ compactly supported in O'' such that $\varphi_n = 1$ in $\overline{O'}$ and, setting $v_{n,N} \coloneqq \varphi_n u_n + (1-\varphi_n)u$, we have

$$F^T_{\varepsilon_n}(v_{n,N}, O) \le F_{\varepsilon_n}(u_n, O) + F_{\varepsilon_n}(w, U) + \frac{C}{N-4} + C(N^2+N)\|u_n - w\|_{L^1(U)}, \quad (10.6.7)$$

for a positive constant $C > 0$, independent of n and N, and for n large enough. Note that $\{v_{n,N}\}_n$ converges to u in $L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ as $n \to +\infty$ and that $\|v_{n,N}\|_{L^\infty(\Omega;\mathbb{R}^k)} \le 2\sqrt{k}\|u\|_{L^\infty(\Omega)}$ for every $n \in \mathbb{N}$. Moreover, we observe that, since the supports of $\{\varphi_n\}_n$ are compactly contained in O'', $O \setminus O' \subset U$, and $u = w$ $\mathcal{L}^d$-a.e. on U, we have $v_{n,N} \in \mathcal{D}_{\varepsilon_n s,w}(O)$ for every n large enough.

Then, by (10.2.6) and (10.1.12) we get

$$\limsup_{n\to+\infty} F_{\varepsilon_n}(w, U) \le \limsup_{n\to+\infty} H_{\varepsilon_n}(w, U) + \Lambda\mathcal{L}^d(U) \le c_2 H(w, U) + \Lambda\mathcal{L}^d(U).$$

Recalling that $\|u_n - w\|_{L^1(U;\mathbb{R}^k)} \to 0$ as $n \to +\infty$ and (10.6.6), we may pass to the limsup as $n \to +\infty$ in (10.6.7) and obtain

$$\limsup_{n\to+\infty} F^T_{\varepsilon_n}(v_{n,N}, O) \le \mathfrak{m}^F(w, O) + \sigma + c_2 H(w, U) + \Lambda\mathcal{L}^d(U) + \frac{C}{N-4}$$

for every $N \in \mathbb{N}$, and by arbitrariness of σ and U, we get

$$\limsup_{n\to+\infty} F^T_{\varepsilon_n}(v_{n,N}, O) \le \mathfrak{m}^F(w, O) + \frac{C}{N-4}. \quad (10.6.8)$$

To conclude, we show that for every $N \in \mathbb{N}$ we have

$$\limsup_{n\to+\infty} \mathfrak{m}_s^{F_{\varepsilon_n}}(w, O) \le \sup_{T>0} \limsup_{n\to+\infty} F^T_{\varepsilon_n}(v_{n,N}, O). \quad (10.6.9)$$

To see this, we begin noting that from (10.6.8) and Lemma 10.3.9 it follows that

$$S \coloneqq \sup_n H^{r_0}_{\varepsilon_n}(v_{n,N}, O) < +\infty$$

and that there is a positive constant C^*, depending only on S, N, and $\|u\|_{L^\infty(\Omega)}$, such that for every $\delta > 0$

$$F_{\varepsilon_n}(v_{n,N}, O) \le F^T_{\varepsilon_n}(v_{n,N}, O) + C^*\delta,$$

for every $T > T^*(\delta)$ and n large enough. Recalling that $v_{n,N} \in \mathcal{D}_{\varepsilon_n s,w}(O)$, this implies that

$$\mathfrak{m}_s^{F_{\varepsilon_n}}(w, O) \le F^T_{\varepsilon_n}(v_{n,N}, O) + C^*\delta$$

for every $T > T^*(\delta)$. Passing to the limsup as $n \to +\infty$ and to the limit as $T \to +\infty$ and $\delta \to 0^+$, we obtain (10.6.9). Finally, combining (10.6.8) and (10.6.9) and passing to the limit as $N \to +\infty$, we get (10.6.5), concluding the proof. □

We recall the definition of the functions $f'_{\rm bulk}$, $f''_{\rm bulk}$, $f'_{\rm surf}$, and $f''_{\rm surf}$ given in Section 10.2. For every $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we let

$$f'_{\rm bulk}(x,A) \coloneqq \limsup_{r\to 0^+} \sup_{s>0} \liminf_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A, Q(x,r))}{r^d}, \tag{10.6.10}$$

$$f''_{\rm bulk}(x,A) \coloneqq \limsup_{r\to 0^+} \sup_{s>0} \limsup_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A, Q(x,r))}{r^d}, \tag{10.6.11}$$

$$f'_{\rm surf}(x,\zeta,\nu) \coloneqq \limsup_{r\to 0^+} \sup_{s>0} \liminf_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}}, \tag{10.6.12}$$

$$f''_{\rm surf}(x,\zeta,\nu) \coloneqq \limsup_{r\to 0^+} \sup_{s>0} \limsup_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}}. \tag{10.6.13}$$

The following two propositions prove Theorem 10.2.4.

Proposition 10.6.5. *Assume that there exists a functional $F\colon L^1_{\rm loc}(\Omega;\mathbb{R}^k)\times\mathcal{O}(\Omega)\to[0,+\infty]$ such that for every $O\in\mathcal{O}_{\rm reg}(\Omega)$ the sequence $\{F_{\varepsilon_n}(\cdot,O)\}_n$ Γ-converges to $F(\cdot,O)$ with respect to the topology of $L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ as $n\to+\infty$. Then the functions $f_{\rm bulk}$ and $f_{\rm surf}$ defined in* (10.2.9) *and* (10.2.10)*, respectively, satisfy*

$$f_{\rm bulk}(x,A) = f'_{\rm bulk}(x,A) = f''_{\rm bulk}(x,A), \tag{10.6.14}$$

$$f_{\rm surf}(x,\zeta,\nu) = f'_{\rm surf}(x,\zeta,\nu) = f''_{\rm surf}(x,\zeta,\nu), \tag{10.6.15}$$

for all $x\in\Omega$, $A\in\mathbb{R}^{k\times d}$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$.

Proof. Let us fix $x\in\Omega$, $A\in\mathbb{R}^{k\times d}$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$. We begin proving (10.6.14). Given $r>0$, we set $\sigma\coloneqq r+r^2$ and apply Lemma 10.6.2 with $w=\ell_A$, $O=Q(x,\sigma)$, and $U=Q(x,r)$ to get

$$\frac{\mathfrak{m}^F(\ell_A,Q(x,\sigma))}{\sigma^d} \le \sup_{s>0}\liminf_{n\to+\infty}\frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A,Q(x,r))}{\sigma^d} + [c_2(\lambda_2|A|^2+\kappa|A|)+\Lambda]\Big(1-\frac{r^d}{\sigma^d}\Big).$$

Since $\sigma/r\to 1$ as $r\to 0^+$, from the previous inequality we get $f_{\rm bulk}(x,A)\le f'_{\rm bulk}(x,A)$. The inequality $f''_{\rm bulk}(x,A)\le f_{\rm bulk}(x,A)$ is obtained applying Lemma 10.6.4.

To prove (10.6.15), we apply again Lemma 10.6.2 with $w=u_{x,\zeta,\nu}$, $O=Q_\nu(x,\sigma)$, and $U=Q_\nu(x,r)$ to get

$$\begin{aligned}\frac{\mathfrak{m}^F(u_{x,\zeta,\nu},Q_\nu(x,\sigma))}{\sigma^{d-1}} \le& \sup_{s>0}\liminf_{n\to+\infty}\frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(u_{x,\zeta,\nu},Q_\nu(x,r))}{\sigma^{d-1}}\\ &+ c_2(\mu_2+\kappa|\zeta|)\Big(1-\frac{r^{d-1}}{\sigma^{d-1}}\Big)+\Lambda\Big(\frac{\sigma^d-r^d}{\sigma^{d-1}}\Big),\end{aligned}$$

Since $\sigma/r\to 1$ as $r\to 0^+$, from the previous inequality we get $f_{\rm surf}(x,\zeta,\nu)\le f'_{\rm surf}(x,\zeta,\nu)$. The inequality $f''_{\rm surf}(x,\zeta,\nu)\le f_{\rm surf}(x,\zeta,\nu)$ follows from an application of Lemma 10.6.4. □

Proposition 10.6.6. *Assume that there exist functions $\hat f_{\rm bulk}\colon\Omega\times\mathbb{R}^{k\times d}\to[0,+\infty)$ and $\hat f_{\rm surf}\colon\Omega\times\mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$ such that*

$$\hat f_{\rm bulk}(x,A) = f'_{\rm bulk}(x,A) = f''_{\rm bulk}(x,A), \tag{10.6.16}$$

for $\mathcal{L}^d$-a.e. $x \in \Omega$ and for all $A \in \mathbb{R}^{k\times d}$ and

$$\hat{f}_{\rm surf}(x,\zeta,\nu) = f'_{\rm surf}(x,\zeta,\nu) = f''_{\rm surf}(x,\zeta,\nu) \tag{10.6.17}$$

for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$, for all $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$. Then for every $O \in \mathcal{O}_{\rm reg}(\Omega)$ the sequence $\{F_{\varepsilon_n}(\cdot,O)\}_n$ Γ-converges with respect to the topology of $L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ to the functional $F(\cdot,O)$ defined by

$$F(u,O) \coloneqq \int_O \hat{f}_{\rm bulk}(x,\nabla u)\mathrm{d}x + \int_{J_u\cap O} \hat{f}_{\rm surf}(x,[u],\nu_u)\mathrm{d}\mathcal{H}^{d-1}$$

if $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^k) \cap \mathrm{GSBV}^2(O;\mathbb{R}^k)$ and $F(u,O) = +\infty$ otherwise.

Proof. Thanks to Theorem 10.2.3, there exists a subsequence $\{\varepsilon_{n_m}\}_m$ such that for every open set $O \in \mathcal{O}_{\rm reg}(\Omega)$ the sequence $\{F_{\varepsilon_{n_m}}(\cdot,O)\}_m$ Γ-converges in the $L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ topology as $m \to +\infty$ to the functional $F(\cdot,O)$ defined for $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^k)$ by

$$F(u,O) \coloneqq \int_O f_{\rm bulk}(x,\nabla u)\mathrm{d}x + \int_{J_u\cap O} f_{\rm surf}(x,[u],\nu_u)\mathrm{d}\mathcal{H}^{d-1}$$

if $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^k) \cap \mathrm{GSBV}^2(O;\mathbb{R}^k)$ and $F(u,O) = +\infty$ otherwise, where $f_{\rm bulk}$ and $f_{\rm surf}$ are given by (10.2.9) and (10.2.10), respectively.

By Proposition 10.6.5, we note that

$$\begin{aligned} f_{\rm bulk}(x,A) &= \limsup_{r\to 0^+} \sup_{s>0} \liminf_{m\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_{n_m}}}(\ell_A, Q(x,r))}{r^d} \\ &\geq \limsup_{r\to 0^+} \sup_{s>0} \liminf_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A, Q(x,r))}{r^d} = \hat{f}_{\rm bulk}(x,A); \end{aligned}$$

and similarly

$$\begin{aligned} f_{\rm bulk}(x,A) &= \limsup_{r\to 0^+} \sup_{s>0} \limsup_{m\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_{n_m}}}(\ell_A, Q(x,r))}{r^d} \\ &\leq \limsup_{r\to 0^+} \sup_{s>0} \limsup_{n\to+\infty} \frac{\mathfrak{m}_s^{F_{\varepsilon_n}}(\ell_A, Q(x,r))}{r^d} = \hat{f}_{\rm bulk}(x,A), \end{aligned}$$

which implies $f_{\rm bulk}(x,A) = \hat{f}_{\rm bulk}(x,A)$ for $\mathcal{L}^d$-a.e. $x \in \Omega$ and for every $A \in \mathbb{R}^{k\times d}$. A similar argument proves that $f_{\rm surf}(x,\zeta,\nu) = \hat{f}_{\rm surf}(x,\zeta,\nu)$ for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$ and for every $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$.

Finally, since the functions $\hat{f}_{\rm bulk}$ and $\hat{f}_{\rm surf}$ are, by (10.6.16) and (10.6.17), independent of the chosen subsequence, by the Urysohn property of Γ-convergence (see [88, Proposition 8.3]) the original sequence $\{F_{\varepsilon_n}(\cdot,O)\}$ Γ-converges to $F(\cdot,O)$ for every $O \in \mathcal{O}_{\rm reg}(\Omega)$, concluding the proof. □

Remark 10.6.7. In view of Remark 10.6.3, if for some $T > 0$ it holds $f_{\varepsilon_n}(x,\xi,z) = 0$ for $\mathcal{L}^d$-a.e. $x \in \Omega$, $\xi \in \mathbb{R}^d \setminus B_T$ and for all $z \in \mathbb{R}^k$ and $n \in \mathbb{N}$, then in definitions (10.6.10)–(10.6.13) we may neglect the supremum over $s > 0$ and simply consider a fixed $s \geq T$, and have Proposition 10.6.5 and Proposition 10.6.6 hold unaltered.

We conclude mentioning a further characterisation of the energy densities in the Γ-limit that involves functionals taking into account finite-range interactions.

Proposition 10.6.8. *Let $\{T_j\}_j$ be a monotonically increasing sequence such that $T_j \to +\infty$ as $j \to +\infty$, and assume that for every $j \in \mathbb{N}$ and for every $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ the sequence $\{F_{\varepsilon_n}^{T_j}(\cdot, O)\}_n$ Γ-converges with respect to the topology of $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ as $n \to +\infty$ to the functional $F^{T_j}(\cdot, O)$ defined by*

$$F^{T_j}(u,O) \coloneqq \int_O f_{\mathrm{bulk}}^{T_j}(x,\nabla u)\mathrm{d}x + \int_{J_u\cap O} f_{\mathrm{surf}}^{T_j}(x,[u],\nu_u)\mathrm{d}\mathcal{H}^{d-1}$$

if $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \cap \mathrm{GSBV}^2(O;\mathbb{R}^k)$ and $F^{T_j}(u,O) = +\infty$ otherwise. Then, for every $O \in \mathcal{O}_{\mathrm{reg}}(\Omega)$ the sequence $\{F_{\varepsilon_n}(\cdot,O)\}_n$ Γ-converges with respect to the topology of $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ as $n \to +\infty$ to the functional $F(\cdot,O)$ defined by

$$F(u,O) \coloneqq \int_O f_{\mathrm{bulk}}(x,\nabla u)\mathrm{d}x + \int_{J_u\cap O} f_{\mathrm{surf}}(x,[u],\nu_u)\mathrm{d}\mathcal{H}^{d-1}$$

if $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^k) \cap \mathrm{GSBV}^2(O;\mathbb{R}^k)$ and $F(u,O) = +\infty$ otherwise.

Moreover, we have

$$\begin{aligned} f_{\mathrm{bulk}}(x,A) &= \lim_{j\to+\infty} \limsup_{r\to0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(\ell_A, Q(x,r))}{r^d} \\ &= \lim_{j\to+\infty} \limsup_{r\to0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(\ell_A, Q(x,r))}{r^d} \end{aligned}$$

and

$$\begin{aligned} f_{\mathrm{surf}}(x,\zeta,\nu) &= \lim_{j\to+\infty} \limsup_{r\to0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}} \\ &= \lim_{j\to+\infty} \limsup_{r\to0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}}. \end{aligned}$$

for all $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

Proof. The first part of the statement follows by Proposition 10.3.2. Since $\{f_{\mathrm{bulk}}^{T_j}\}_j$ and $\{f_{\mathrm{surf}}^{T_j}\}_j$ are sequences of monotonically increasing functions, we have that

$$f_{\mathrm{bulk}} = \lim_{j\to+\infty} f_{\mathrm{bulk}}^{T_j}, \qquad f_{\mathrm{surf}} = \lim_{j\to+\infty} f_{\mathrm{surf}}^{T_j}. \tag{10.6.18}$$

Moreover, applying Proposition 10.6.5 and Remark 10.6.7, we obtain

$$f_{\mathrm{bulk}}^{T_j}(x,A) = \limsup_{r\to0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(\ell_A, Q(x,r))}{r^d} = \limsup_{r\to0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(\ell_A, Q(x,r))}{r^d} \tag{10.6.19}$$

and

$$f_{\mathrm{surf}}^{T_j}(x,\zeta,\nu) = \limsup_{r\to0^+} \liminf_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}} = \limsup_{r\to0^+} \limsup_{n\to+\infty} \frac{\mathfrak{m}_{T_j}^{F_{\varepsilon_n}^{T_j}}(u_{x,\zeta,\nu}, Q_\nu(x,r))}{r^{d-1}} \tag{10.6.20}$$

for all $x \in \Omega$, $A \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$. Therefore, the thesis follows by (10.6.18), (10.6.19), and (10.6.20). □

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