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Relative Quot schemes over families of smooth and nodal curves

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Abstract

The present work studies relative Quot schemes parameterizing locally free sheaf quotients with zero–dimensional support on the fibers of a morphism. We focus on families of smooth curves with at most nodal singularities, as well as families of smooth, higher dimensional varieties.

The thesis is organized around three projects, which explore both geometric and combinatorial aspects of these relative Quot schemes.

The first project, carried out in collaboration with Barbara Fantechi and Ajay Gautam [18], provides an explicit geometric description of Quot schemes of families of smooth projective curves. We show that the Quot scheme of a locally free sheaf filtered by line bundles admits a finite locally closed stratification, each stratum being isomorphic to an affine vector bundle over products of symmetric powers of the family, with rank given by an explicit formula. Our result generalizes an existing description by Bifet via an independent approach, which does not rely on Białynicki–Birula decompositions. In addition, it determines the class of the relative Quot scheme in the Grothendieck ring of varieties. Further applications concern nested Quot schemes and Quot schemes of positive rank quotients on a smooth projective curve.

The second project addresses Quot schemes of smooth morphisms of arbitrary relative dimension. By analyzing the Quot–to–Sym morphism and its behavior under the natural stratification by integer partitions, we show that its restriction to each stratum is étale–locally trivial. Combined with the language of power structures on the Grothendieck ring, this result yields two formulas for Quot schemes of smooth morphisms.

The third project concerns Quot schemes over families of nodal curves, with emphasis on Losev–Manin spaces. We first study the geometry of Hilbert schemes over Losev–Manin spaces, showing that they are smooth, irreducible of known dimension. Moreover, by combining existing formulas for smooth curves and nodal singularities, we derive an explicit identity for generating functions of the corresponding Quot scheme classes in the Grothendieck ring. The chapter concludes by addressing some open questions and possible further research directions.

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I studied in Trieste for ten years. This city has given me many opportunities, while offering much less to others. In a world driven by consumerism and capitalism, the local administration aspires to make Trieste into a polished showcase for cruise tourists, and its port into a major commercial bridge between Europe and the Middle East, including Israel. At the same time, the city's residents are growing increasingly poorer. Trieste is also a "safe" harbor for thousands of migrants coming from the Balkan route in search for a better life, many of whom experience extreme living conditions nearby the train station. Just two weeks ago, on December 3, the lifeless body of Hichem Billal, a young Algerian man, was found in one of the several dilapidated buildings of the old port during the latest eviction operation conducted by the police. This is the side of Trieste that many don't (want to) see and that local institutions want to erase in the name of "decorum".

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Introduction

Algebraic geometry is the study of the geometric properties of mathematical objects using techniques arising from commutative algebra. An elementary example is that of conic curves (circles, ellipses, parabolas and hyperbolas), which can be described by degree two polynomial equations in the plane. This algebraic description is reflected, for instance, in the fact that the intersection of a conic with a line consists of (at most) two points.

Some geometric properties describe structural aspects such as smoothness, irreducibility, compactness or connectedness; others are numerical, such as the genus or the Euler characteristic. What makes a property mathematically meaningful is whether it remains the same when the object is transformed in a reasonable way. Besides describing what an object "looks like", the broader goal is to distinguish objects from one another – and ultimately to classify them.

Objects sharing the same geometric properties are collected in a parameter space, called a *moduli* space. For instance, the set of all conic curves forms a five–dimensional projective space – another way of saying that five points in general position determine a unique conic curve. A moduli space is itself an interesting mathematical object, and its geometric properties reveal how the objects it parametrizes vary and relate to one another.

Moduli spaces are one of the central themes of modern algebraic geometry. A key conceptual shift from the classical to the modern viewpoint is the realization that objects (varieties and schemes) are best understood in relation to other objects. Much like, in physics, a particle is studied by observing how it interacts with other particles projected to it, a scheme in algebraic geometry is studied by examining how other schemes map to it. Moduli spaces perfectly fit this perspective, as they are defined by the requirement that families of objects parametrized by a certain scheme correspond to morphisms from that scheme to the moduli space. A moduli *problem* is posed by defining the families of objects that a moduli space should parametrize. The existence of a moduli space solving a given moduli problem is, in general, nontrivial [1].

A prototypical example of moduli space is the *Hilbert scheme*, of which the moduli space of plane conics is a particular instance. In its full generality, the Hilbert scheme parametrizes flat families of subschemes with prescribed properties inside a fixed ambient space. Here, flatness is an algebraic condition which ensures that objects in a family vary in a "nice" and continuous way.

Grothendieck's *Quot scheme* generalizes the notion of Hilbert scheme by parametrizing coherent sheaf quotients (for instance, quotients of a vector bundle) satisfying given conditions. Despite being regarded as "classical" objects in algebraic geometry, Hilbert and Quot schemes constitute a very active research area to this day [41].

The interest in Quot schemes stems from the wide range of its applications. To name a few, Quot schemes arise naturally in the construction of moduli spaces of semistable sheaves via geometric invariant theory (or GIT) [39] [34]. Quot schemes also play a central role in enumerative geometry, a branch of algebraic geometry concerned with counting problems – such as determining how many degree d plane curves pass through $3d-1$ points in general position [44, Theorem 3.3.1]. For instance, Hilbert and Quot schemes find applications in the theory of Donaldson–Thomas invariants [65], a topic of great interest to physicists working in string theory.

The present work focuses on relative Quot schemes parametrizing flat families of locally free sheaf quotients on the fibers of a morphism $f\colon X \to Y$. These schemes provide a natural way to study how the geometry of Quot schemes behaves in families. In the case where Y is a single point, Quot schemes have been studied in depth, while their relative counterparts have received far less attention. This thesis aims to contribute to filling this gap by examining geometric and combinatorial aspects of Quot schemes over families of smooth projective curves, with at most nodal singularities, as well as families of smooth higher dimensional objects.

Overview

Quot schemes were introduced by Grothendieck in [26]; for a more modern exposition of this material, the reader may refer to [53]. A gentle introduction to Hilbert schemes is found in [64], see also [20], [24].

Given a quasiprojective scheme X, a locally free sheaf E on X and a nonnegative integer d, we denote by $\mathrm{Quot}^d_X(E)$ the Quot scheme parametrizing quotients of E with zero dimensional support and degree d. When E is a line bundle, this definition gives us the Hilbert scheme $\mathrm{Hilb}^d(X)$. These definitions extend naturally to the relative setting, when the scheme X is replaced by a quasiprojective morphism $f\colon X \to Y$. In this case, we write $\mathrm{Quot}^d_{X/Y}(E)$ for the relative Quot scheme and $\mathrm{Hilb}^d_Y(X)$ for the relative Hilbert scheme.

One of the recurring tools appearing in this work is the *Grothendieck ring of varieties*, which provides an algebraic framework to operate with varieties up to *cut–and–paste* relations. Through these relations, the Grothendieck ring serves as a universal additive invariant by refining, for instance, Hodge polynomials, Poincaré polynomials and Euler characteristic. A survey on the topic of Grothendieck rings is found in [13, Chapter 2].

Quot schemes on smooth projective curves

Quot schemes over smooth projective curves exhibit a particularly nice behavior. More precisely, for a smooth projective curve X and a locally free sheaf E of rank r, the Quot scheme $\mathrm{Quot}^d_X(E)$ is a smooth projective variety of dimension rd. These spaces have been studied extensively over the years.

A fundamental contribution to the topic is due to Bifet [8], who gave an explicit description of the Quot scheme on a smooth projective curve when the sheaf E is free. Using the Białynicki–Birula decomposition associated with a natural torus action on E, Bifet constructs an explicit stratification of the Quot scheme: each stratum is a vector bundle over products of symmetric powers of the curve, with rank determined by an explicit formula.

This geometric description has interesting arithmetic implications: it provides a closed formula for the class of $\mathrm{Quot}_X^d(\mathcal{O}^{\oplus r})$ in the Grothendieck ring of varieties, and hence recovers several of its invariants. A later result by Bagnarol, Fantechi and Perroni [3] shows that the class of $\mathrm{Quot}_X^d(E)$ depends on E only up to the rank, and not on the bundle itself.

Recently, these results were refined by Marian and Neguț's study of the cohomology of the Quot scheme on a smooth curve [48]; furthermore, a new study by Gautam and Marian [23] provides an explicit presentation of the cohomology ring when $X = \mathbb{P}^1$.

A natural first step towards understanding the geometry of relative Quot schemes would be to address the following question.

Question A. *Does Bifet's geometric description extend to relative Quot schemes over families of smooth projective curves?*

In collaboration with Barbara Fantechi and Ajay Gautam, we find that the answer is affirmative, by constructing a relative version of Bifet's stratification for the Quot scheme $\mathrm{Quot}_{X/Y}^d(E)$ associated to a family $f\colon X \to Y$ of smooth projective curves. Our approach provides an independent proof which does not rely on the Białynicki–Birula decomposition, and yields a formula of the classes of relative Quot schemes in the Grothendieck ring of varieties over Y. In particular, specializing this formula to the case of smooth projective k–curves recovers Bifet's formula.

Quot schemes on smooth varieties of higher dimension

When the underlying space X is smooth of higher dimension, the geometry of the Quot scheme becomes more complex.

For smooth projective surfaces, the Hilbert scheme $\mathrm{Hilb}^d(X)$ is a smooth projective scheme of dimension $2d$ and, in fact, the classical Hilbert–Chow morphism becomes a resolution of singularities of the symmetric power $\mathrm{Sym}^d(X)$ [22]. Quot schemes of degree d quotients on a smooth projective surface are expected to have rational singularities [62], although they remain irreducible [17]. For smooth threefolds, the Hilbert scheme $\mathrm{Hilb}^d(X)$ is known to be smooth for degree $d \leq 3$ [19]; it is normal, Gorenstein for $d \leq 7$ and has only rational singularities for degrees $4 \leq d \leq 6$ [43] [35]. In dimension 16, the geometry of Hilbert scheme of points exhibits a pathological behavior [40]. In general, the geometry of Quot schemes over varieties of dimension three or higher is expected to be singular, due to the presence of higher obstructions.

In these cases, it is natural to focus on invariants in the Grothendieck ring and ways to compute them. While on smooth curves the class of the Quot scheme is determined by an explicit geometric description, a similar approach seems infeasible for higher dimensional varieties, given the complexity of the situation. Still, there are other ways to approach the problem, as we will now explain.

The key aspect of Quot schemes of zero dimensional quotients is the existence of local models – namely punctual Quot and Hilbert schemes – which can be combined together to reconstruct the global object. This feature is captured by a *power structure* on the Grothendieck ring of varieties [27], which provide a natural framework for treating generating functions of the invariants. In particular, one obtains explicit formulas relating the class of the Quot scheme $\mathrm{Quot}_X^d(E)$ to its punctual counterpart. The case of Hilbert schemes was solved by Gusein–Zade, Luengo and Melle–Hernandez [28], building on computations by Göttsche [25]; Ricolfi later generalized the formula to Quot schemes on smooth projective varieties of abitrary dimension [58].

Another instance of the local–to–global nature of Hilbert schemes is the formula proved by Gusein–Zade, Luengo and Melle–Hernandez on the Grothendieck ring of *maps*, which expresses the generating function of Hilbert–Chow morphisms over a smooth quasiprojective variety in terms of local data [31, Theorem 3].

This motivates the following question.

Question B. *Do these power structure formulas for Quot and Hilbert schemes admit an analogue in the case of smooth projective morphisms?*

As will be explained later in this thesis, the answer is affirmative. The relative Quot scheme associated to a smooth morphism $f\colon X \to Y$ satisfies two power structure formulas, respectively, over the Grothendieck ring of Y–varieties and the Grothendieck ring of Y–maps. The proofs adapt some existing results to the relative setting, combining étale–local arguments with a detailed analysis of the Quot–to–Sym morphism.

Quot schemes on singular curves

A further direction of study concerns mildly singular projective curves. In this context, the Quot scheme $\mathrm{Quot}_X^d(E)$ admits a natural stratification depending on how the support of each quotient is distributed along the smooth locus and the singular points. The techniques developed for smooth curves still apply to the contributions coming from the smooth locus, while the punctual Quot schemes supported at singular points require a finer analysis, as they depend on the type of singularity.

Punctual Quot schemes have been studied for several types of curve singularities. One of the main contributions in this direction is due to Ziv Ran, who gave an explicit geometric description of the punctual Hilbert scheme for nodal singularities as a chain of projective lines meeting transversely [55]. This description also determines its class in the Grothendieck ring. The class of the punctual Hilbert scheme is known for other types of curve singularities including cusps [46] and the origin of the coordinate axes in affine spaces of dimension three and higher [5].

More recently, significant progress has been made in higher rank cases by Huang and Jiang [36], [37], who obtained closed formulas for (the generating function of) the class of punctual Quot schemes over reduced plane curve singularities of type $y^2 = x^m$ for all $m \geq 1$ [38, Theorem 1.4]. For even m, a beautiful simplification of their formula appears in Chern's work [15, Theorem 1.7].

These developments naturally raise a question.

Question C. *What can be said about Quot schemes over families of nodal curves?*

In the last part of the present thesis we try to answer this question by analyzing the case of Quot schemes over Losev–Manin spaces, which parametrize stable chains of projective lines with marked points [47]. These spaces have a rich combinatorial structure which allows explicit computations of the classes of the corresponding relative Quot schemes, leading to closed formulas and functional equations for their generating functions.

Main contributions and methods

The thesis is built around three projects concerning the theory of relative Quot schemes. These results aim to provide a combinatorial and, where possible, a geometric understanding of relative Quot schemes, also extending several classical results to the relative setting.

Relative Quot schemes over families of smooth curves

The first project, based on a joint collaboration with Barbara Fantechi and Ajay Gautam [18], establishes a geometric description of Quot schemes over families of smooth projective curves. Its main contribution, stated in full detail in Theorem 2.1.6, can be summarized as follows.

Theorem A.1. *Fix a nonnegative integer d and let $f\colon X \to Y$ be a family of smooth projective curves. Consider a locally free sheaf E of rank r over X, filtered by line bundles. The Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ admits a locally closed stratification indexed by weak compositions of d of length r. Each stratum is isomorphic to an affine vector bundle over products of symmetric powers of the family, with rank given by an explicit formula.*

The assumption that E is filtered by line bundles allows us to carry out the proof by induction on the rank of E. In broad terms, the recursive mechanism can be summarized as follows. At each step, we use flattening stratifications to refine the Quot schemes into smaller strata; each stratum admits a morphism into a product of Quot schemes of locally free sheaves of lower rank (this is the content of Theorem 2.2.1, part (i)). An analysis of the morphism and its fibers, encoded by the Lifting Lemma 2.2.3, allows us to verify that the morphism is an affine vector bundle, as stated in Theorem 2.2.1, part (ii). The iteration of this procedure gives the desired stratification into affine vector bundles and proves the statement.

The case of non–filtered sheaves is treated separately in Section 2.2.5.

Theorem A.1 determines the class of the Quot scheme in the Grothendieck ring of Y–varieties. Here, given a Y–scheme W, we denote its class by $[W \to Y]$.

Corollary A.2. *The following formula holds in the Grothendieck ring of Y–varieties.*

$$[\mathrm{Quot}^d_{X/Y}(E) \to Y] = \sum_{\substack{\alpha_1,\dots,\alpha_r \geq 0 \\ \alpha_1+\cdots+\alpha_r=d}} \prod_{i=1}^{r} [\mathrm{Sym}^{\alpha_i}_Y(X) \to Y][\mathbb{A}^1_Y \to Y]^{\alpha_i(i-1)}.$$

Further applications of this formula, discussed later in the thesis, concern nested Quot schemes (Theorem 2.3.5) and Quot schemes of positive rank quotients on a smooth projective curve (Proposition 2.3.7).

Relative Quot schemes of smooth morphisms

The second project addresses Quot schemes of smooth morphisms of arbitrary relative dimension. In this context, we establish two formulas using the power structure formalism on the Grothendieck ring. The first formula, stated in detail in Theorem 3.1.5, generalizes Ricolfi's exponential formula for generating functions of Quot schemes (Theorem 3.1.1) to the case of smooth morphisms.

Theorem B.1. *Let $f\colon X \to Y$ be a smooth morphism of relative dimension m, and let E be a locally free sheaf of rank r on X. The following formula holds in the ring of formal power series over the Grothendieck ring of Y–varieties.*

$$\sum_{d\geq 0}[\mathrm{Quot}^d_{X/Y}(E) \to Y]t^d = \left(\sum_{d\geq 0}[\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})_0 \times_k Y \to Y]t^d\right)^{[X\to Y]}$$

Here, $\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})_0$ *is the punctual Quot scheme of* $\mathbb{A}^m$ *supported at the origin.*

The expression on the right–hand side of the formula, where the class $[X \to Y]$ appears as an "exponent", is to be interpreted via the notion of a power structure on the Grothendieck ring. Definition 1.2.20 introduces this notation, while Remark 1.2.22 offers a geometric intuition behind it.

As a corollary, Theorem B.1 determines that the class $[\mathrm{Quot}^d_{X/Y}(E) \to Y]$ depends on the morphism $f\colon X \to Y$, on the length d, while the dependence on the choice of sheaf E is limited to the rank. This is the content of Corollary 3.3.2.

The second formula, stated in detail in Theorem 3.1.6, generalizes the formula proved by Gusein–Zade, Luengo and Melle–Hernandez for Hilbert–Chow morphisms on the Grothendieck ring of maps.

Theorem B.2. *For any smooth quasiprojective morphism $f\colon X \to Y$ of relative dimension m and locally free sheaf E on X, the following formula holds in the formal power series ring over the Grothendieck ring of Y–maps.*

$$\sum_{d\geq 0}[\mathrm{Quot}^d_{X/Y}(E) \xrightarrow{\sigma^d_{X/Y}} \mathrm{Sym}^d_Y(X)]t^d = \left(\sum_{d\geq 0}[\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\otimes\,\mathrm{rk}(E)})_0 \times Y \to Y]t^d\right)^{[X\xrightarrow{\mathrm{id}}X]}$$

Again, the expression on the right–hand side, and particularly the use of the "exponent" $[X \xrightarrow{\mathrm{id}} X]$, is to be interpreted via the notion of a power structure on the Grothendieck ring of maps as reported in Definition 1.2.30.

The proofs of Theorem B.1 and Theorem B.2 combine geometric arguments with the power structure formalism on the Grothendieck ring. The key mechanism behind these formulas lies in a detailed analysis of the Quot–to–Sym morphism and its behavior under the natural stratification by integer partitions, which shows that its restriction stratum by stratum is étale–locally trivial with explicit fiber (Theorem 3.1.4).

Relative Quot schemes over families of nodal curves

The third project concerns Quot schemes over families of nodal curves, with focus on the case of Losev–Manin spaces. These spaces, denoted by $\overline{L}_n$, parametrize stable chains of projetive lines with two poles and n marked points. Let $\pi_n\colon \mathcal{X}_n \to \overline{L}_n$ denote the universal family. Fixing a locally free sheaf E on $\mathcal{X}_n$, we consider the relative Quot scheme $\mathrm{Quot}^d_{\mathcal{X}_n/\overline{L}_n}(E)$, which parametrizes torsion quotients supported on the fibers of π_n.

The first contribution, established later in Theorem 4.3.4, concerns the geometry of Hilbert schemes over Losev Manin spaces.

Theorem C.1. *The Hilbert scheme* $\mathrm{Hilb}^d_{\overline{L}_n}(\mathcal{X}_n)$ *is a smooth irreducible projective scheme of dimension* $d+n-1$.

This result follows from a deformation theoretic argument, combined with a smoothness criterion for relative Hilbert schemes over families of nodal curves [57, Corollary 3.4].

Next, we combine existing formulas for smooth curves [58, Theorem 3.2] and nodal singularities [38, Theorem 1.4], [15, Theorem 1.7] to derive an explicit formula for the generating function of Quot schemes in the Grothendieck ring.

Theorem C.2. *The generating function of the classes of Quot schemes over Losev–Manin spaces satisfies the following formula in the Grothendieck ring of k–varieties.*

$$\sum_{d\geq 0}[\mathrm{Quot}^d_{\mathcal{X}_n/\overline{L}_n}(\mathcal{O}^{\oplus r})]t^d = \sum_{\ell=1}^{n} \ell! S(n,\ell)([\mathbb{A}^1]-1)^{n-\ell}\left(\frac{1-t}{(t;[\mathbb{A}^1])_r^2(1-[\mathbb{A}^1]^r t)}\right)^{\ell} \mathrm{N}(\mathcal{O}^{\oplus r},t)^{\ell-1}.$$

Here, $S(n,\ell)$ is a Stirling number of the second kind, $(t;q)_r$ is the q–Pochhammer symbol

$$(t;q)_r := (1-t)(1-qt)\cdots(1-q^{r-1}t),$$

and $\mathrm{N}(\mathcal{O}^{\oplus r},t)$ *is the contribution coming from the nodal singularity.*

A more precise statement of the result is found in Theorem 4.3.10, while an explicit formula for the nodal contribution is recorded in this thesis under Theorem 4.3.5. Stirling numbers are introduced respectively by formula (4.1).

The proof of Theorem C.2 relies on the stratification of Losev–Manin spaces by combinatorial curve type (Theorem 4.2.10). This stratification induces a corresponding decomposition of the Quot scheme, allowing the computation to be carried out strata by strata. After a detailed analysis of the combinatorial strata of $\overline{L}_n$ (Section 4.2.2), and an explicit formula for the class of the Quot scheme over a chain of projective lines (Theorem 4.3.8), the desired global expression follows. The resulting generating series is shown to be rational and to satisfy a functional equation (Corollary 4.3.12).

Outline of the chapters

The present thesis is structured as follows.

Chapter 1: Preliminary notions.

This chapter gathers the background material used throughout the thesis. Section 1.1 recalls the notions of Quot and Hilbert schemes, symmetric products and the Quot–to–Sym morphism; we also recall some generalized constructions such as nested Quot schemes and Quot schemes of positive rank quotients. We then review the Grothendieck ring of varieties and its power structure in Section 1.2. In particular, Section 1.2.3 reviews the definition of Grothendieck ring of maps and the power structure on it. Finally, Section 1.3 introduces Quot zeta functions.

Chapter 2: Relative Quot schemes over families of smooth curves.

After a brief overview of the project and its main results in Section 2.1, Section 2.2 sets up the inductive framework for the proof of the main theorem. We construct a stratification of the relative Quot scheme, and on each stratum an affine vector bundle morphism whose fibers are controlled by a Lifting Lemma; the Lifting Lemma is stated in full generality for abelian categories in Appendix A. The case of non–filtered sheaves is treated in Section 2.2.5. Applications of the main results are discussed in Section 2.3, and concern the class of relative Quot schemes in the Grothendieck ring, the Quot zeta function, and formulas for nested Quot schemes and Quot schemes of positive rank quotients on a smooth curve.

Chapter 3: Power structure formulas for Quot schemes of smooth morphisms.

Section 3.1 provides a brief review of the project and its main contributions. The main results and constructions appear in Section 3.2: we construct étale maps between relative Quot schemes and use them to study the Quot–to–Sym morphism in relation to the stratification by partitions, establishing an étale local triviality statement; these ingredients are then combined in Section 3.3, where two power structure formulas are obtained.

Chapter 4: Relative Quot schemes over moduli spaces of nodal curves.

A brief overview of the project and the main results is given in Section 4.1. In Section 4.2 gathers the necessary background material on Losev–Manin spaces, examining its stratification by combinatorial curve type; we also recall the notion of q–analog. Section 4.3 constitutes the core of the chapter. First, Section 4.3.4 addresses the geometry of the relative Hilbert scheme over Losev–Manin spaces. Next, the study of the class of relative Quot schemes is reduced to a computation on strata, which leads to explicit expressions for the Quot zeta function. To conclude, we prove the rationality of the Quot zeta function and present a functional equation. Lastly, Section 4.4 concludes the chapter with a brief discussion addressing open questions and directions for further studies.

Conventions

Throughout, let k denote an algebraically closed field of characteristic zero. We work with schemes of finite type over k.

Contents

CHAPTER 1

Preliminary notions

We begin by gathering the background material required throughout the thesis. We recall the definitions and main properties of Quot and Hilbert schemes, symmetric powers, the Grothendieck ring of varieties and the power structure on it. Some familiarity with the basics of scheme theory [32] and the language of moduli spaces [52] is required.

1.1 From symmetric powers to Quot schemes

Moduli spaces parameterize objects sharing certain prescribed properties. When the objects are algebro–geometric in nature – such as varieties, schemes, or sheaves – their moduli spaces carry algebro–geometric structures, often taking the form of schemes or stacks. Among the recurring moduli spaces in this thesis are *symmetric powers*, *Hilbert schemes*, and *Quot schemes*. All are defined with respect to a (quasi)projective ambient scheme X, and organize objects in increasing order of complexity.

We begin with a brief intuitive overview of these spaces; precise definitions will be provided in later sections.

Symmetric powers. For a nonnegative integer d, the d–th symmetric power $\mathrm{Sym}^d(X)$ of X parametrizes finite unordered collections of points, counted with multiplicity. A typical point of $\mathrm{Sym}^d(X)$ can be written as

$$p_1 + \cdots + p_d, \qquad p_i \in X,$$

with the p_i not necessarily distinct.

Hilbert schemes. The Hilbert scheme $\mathrm{Hilb}^P(X)$ refines this picture by parameterizing subschemes of X with a prescribed Hilbert polynomial P

$$Z \subset X, \qquad P_Z = P.$$

For example, conic curves in $\mathbb{P}^2$ have Hilbert polynomial $2t+1$ and thus are parameterized by the Hilbert scheme $\mathrm{Hilb}^{2t+1}(\mathbb{P}^2)$, which is isomorphic to $\mathbb{P}^5$. When the polynomial P is constant, the Hilbert scheme parameterizes zero–dimensional subschemes of X. These can be viewed as finite collections of points equipped with additional scheme theoretic data: unlike symmetric powers, the Hilbert scheme not only records the multiplicity, but also infinitesimal information such as the tangent direction or "thickness" of a given point.

Quot schemes. Quot schemes take one step further and can be viewed as a sheaf–theoretic analogues of the Grassmannian. Recall that, given a k–vector space V, the Grassmannian parametrizes short exact sequences of k–vector spaces

$$0 \to W \to V \to V/W \to 0,$$

where W has prescribed dimension. Similarly, for a fixed coherent sheaf E on X and any polynomial P, the Quot scheme $\mathrm{Quot}_X^P(E)$ parametrizes short exact sequences of coherent sheaves on X

$$0 \to S \to E \to F \to 0 \qquad \text{in } \mathrm{Coh}(X),$$

where F has Hilbert polynomial P.

Each of these moduli spaces arises as a solution to a moduli problem: given a scheme T, one defines families of objects over T, which are then organized functorially. The representability of these functors is well established in the literature. In the present work, we will not revisit these foundational aspects, as they are not strictly essential for the understanding of the arguments presented in the thesis.

1.1.1 Quot and Hilbert schemes

Consider a projective morphism $f\colon X \to Y$, a coherent sheaf E on X and an integer valued polynomial $P(t) \in \mathbb{Q}[t]$. Given a Y–scheme T, let $p_X : X \times_Y T \to X$ denote the projection to X.

Definition 1.1.1. The *Quot scheme* $\mathrm{Quot}_{X/Y}^P(E)$ is a quasiprojective Y–scheme which parameterizes flat families of coherent quotients with Hilbert polynomial P. More precisely, a morphism $\tau\colon T \to \mathrm{Quot}_{X/Y}^P(E)$ corresponds to a short exact sequence of coherent sheaves on $X \times_Y T$

$$0 \to S \longrightarrow p_X^* E \xrightarrow{q} F \to 0$$

such that F is flat over T and, for every $t \in T$, the pullback of F to $X \times_Y \{t\}$ has Hilbert polynomial P.

Notation 1.1.2. We shall denote by $0 \to \mathcal{S}_P \to p_X^* E \to \mathcal{F}_P \to 0$ the *universal* sequence of the Quot scheme $\mathrm{Quot}_{X/Y}^P(E)$, corresponding to the identity morphism.

Remark 1.1.3.

(i) The existence of Quot schemes is a classical result due to Grothendieck [26, §3].

(ii) When $f\colon X \to Y$ is a quasiprojective morphism, one additionally requires that $\mathrm{Supp}(F)$ be proper over T. This ensures that the Hilbert polynomial of F on $X \times_Y \{t\}$ is well defined. In this case, the Quot scheme $\mathrm{Quot}_{X/Y}^P(E)$ is only a quasiprojective Y–scheme.

(iii) Two quotients $q\colon p_X^* E \to F$ and $q'\colon p_X^* E \to F'$ define the same point in $\mathrm{Quot}_{X/Y}^P(E)$ if and only if there is an isomorphism $\varphi\colon F \to F'$ such that $q' = \varphi \circ q$.

(iv) When $Y = \mathrm{Spec}(k)$ we use the notation

$$\mathrm{Quot}_X^P(E) := \mathrm{Quot}_{X/\,\mathrm{Spec}(k)}^P(E).$$

For a general Y, we shall refer to $\mathrm{Quot}_{X/Y}^P(E)$ as a *relative* Quot scheme.

Example 1.1.4. When E is the a line bundle, the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ is canonically isomorphic to the Hilbert scheme $\mathrm{Hilb}^P_Y(X)$, which parameterizes flat families of closed subschemes of X with Hilbert polynomial P. A point in the Hilbert scheme corresponds to a closed subscheme Z of X, defined by the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Example 1.1.5. The Grassmannian $\mathrm{Gr}(d, V)$ of d–dimensional quotients of a k–vector space V is isomorphic to the Quot scheme $\mathrm{Quot}^d_{\mathrm{pt}}(V)$. When $V = k^r$, the Grassmannian is denoted $\mathrm{Gr}(d, r)$.

Example 1.1.6. Let $Y = \mathrm{Spec}(k)$ and consider a nonnegative integer d. A point $[E \to F]$ of the Quot scheme $\mathrm{Quot}^d_X(E)$ corresponds to a quotient sheaf F of E, whose stalks F_p are zero everywhere except at finitely many points $p_1, \ldots, p_m$. The lengths $d_1, \ldots, d_m$ of the nonzero stalks F_{p_i} of F sum up to the total length d. Figure 1.1 illustrates this picture.

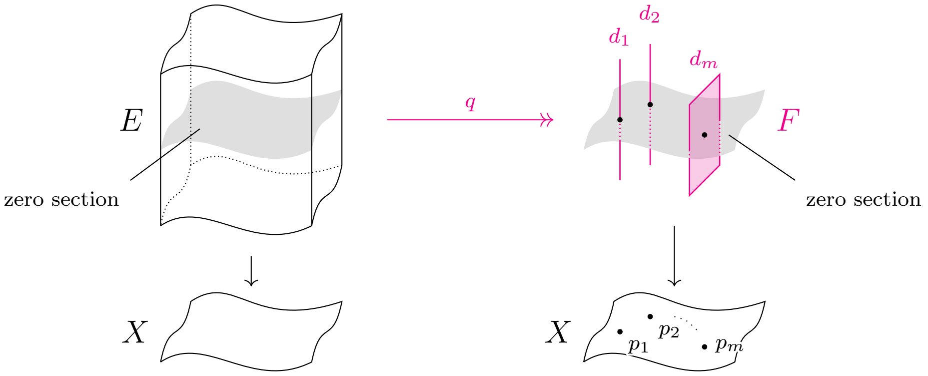

FIGURE 1.1: *Schematic representation of a typical point of* $\mathrm{Quot}^d_X(E)$.

Proposition 1.1.7. *Consider a quasiprojective morphism $f\colon X \to Y$, a coherent sheaf E on X, and a polynomial P. Given any Y–scheme Y', there is an isomorphism*

$$\mathrm{Quot}^P_{X\times_Y Y'/Y'}(p_X^* E) \cong \mathrm{Quot}^P_{X/Y}(E) \underset{Y}{\times} Y'$$

of schemes over Y'.

Remark 1.1.8. Proposition 1.1.7 implies that the construction of $\mathrm{Quot}^P_{X/Y}(E)$ commutes, up to an isomorphism, with base change in Y. Consequently, the fiber of the structure morphism $\mathrm{Quot}^P_{X/Y}(E) \to Y$ over a point $y \in Y$ is precisely the Quot scheme on the fiber $f^{-1}(y)$. In this sense, the relative Quot scheme $\mathrm{Quot}^P_{X/Y}(E)$ organizes the Quot schemes of all fibers of f into a single geometric object.

In the subsequent chapters, we study relative Quot schemes $\mathrm{Quot}^d_{X/Y}(E)$ associated to a projective morphism $f\colon X \to Y$ of relative dimension $m \geq 1$, a locally free sheaf E on X and a nonnegative integer d.

1.1.2 Symmetric powers and the Quot–to–Sym morphism

Symmetric powers of schemes provide a natural parameter space for finite, unordered collections of points counted with multiplicity inside a given ambient scheme. This construction is particularly useful in the study of Quot schemes of finite length quotients. As discussed below, the two notions are related by a natural morphism which assigns to a finite length quotient $[E \to F]$ the cycle corresponding to its zero–dimensional support.

Definition 1.1.9. Consider a quasiprojective morphism $f\colon X \to Y$ and an integer $d \geq 1$. The *(relative) d–th symmetric power* $\mathrm{Sym}^d_Y(X)$ is the quotient of the d-fold product X^d by the symmetric group $\mathfrak{S}_d$, which permutes the factors. For $d = 0$, set $\mathrm{Sym}^0_Y(X) := Y$.

Remark 1.1.10.

(i) The d-th symmetric power $\mathrm{Sym}^d_Y(X)$ exists, for example, when each orbit of the symmetric group action is contained in an affine open subset.

(ii) The symmetric product $\mathrm{Sym}^d_Y(X)$ is a quasiprojective Y–scheme.

(iii) One has $\mathrm{Sym}^d_Y(X \times Y) \cong \mathrm{Sym}^d(X) \times Y$.

Theorem 1.1.11 ([26, §6]) **.** *Consider a quasiprojective morphism $f\colon X \to Y$, a coherent sheaf E on X and a nonnegative integer d. There is a morphism of Y–schemes*

$$\sigma^d_{X/Y}(E)\colon \mathrm{Quot}^d_{X/Y}(E) \to \mathrm{Sym}^d_Y(X)$$

which assigns to each quotient $[E \to F]$ its support cycle

$$\sum_{p \in \mathrm{Supp}(F)} \mathrm{length}_{\mathcal{O}_{X,p}}(F_p)p.$$

Definition 1.1.12. The morphism $\sigma^d_{X/Y}(E)$ is called *Quot–to–Sym* morphism. When E is the structure sheaf $\mathcal{O}_X$, we shall call $\sigma^d_{X/Y}(\mathcal{O}_X)$ the *Hilbert–Chow* morphism.

Notation 1.1.13. When the sheaf E is understood, we shall simply write $\sigma^d_{X/Y} = \sigma^d_{X/Y}(E)$.

Proposition 1.1.14 ([26, §6]) **.** *If $f\colon X \to Y$ if a family of smooth curves, the Hilbert–Chow morphism $\sigma^d_{X/Y}$ defines an isomorphism of $\mathrm{Hilb}^d_Y(X)$ with $\mathrm{Sym}^d_Y(X)$.*

The Quot–to–Sym morphism allows to select a subset of quotients whose support lies within a prescribed locus of X. For instance, when the locus is an open subscheme U of X, we have the following situation.

Theorem 1.1.15. *Consider a projective morphism $f\colon X \to Y$, a coherent sheaf E on X and a nonnegative integer d. Let U be an open subscheme of X. Then the following diagram is cartesian, and the horizontal arrows are open embeddings.*

$$\begin{tikzcd}
\mathrm{Quot}^d_{U/Y}(E|_U) \arrow[r, hook] \arrow[d, "\sigma^d_{U/Y}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{X/Y}(E) \arrow[d, "\sigma^d_{X/Y}"] \\
\mathrm{Sym}^d_Y(U) \arrow[r, hook] & \mathrm{Sym}^d_Y(X)
\end{tikzcd}$$

The study of quotients supported entirely at a point is captured by the notion of punctual Quot scheme.

Definition 1.1.16. Consider a quasiprojective morphism $f\colon X \to Y$, a coherent sheaf E on X and a nonnegative integer d. The *punctual Quot scheme* at $p \in X$ is the inverse image of $d \cdot p$ under the Quot–to–Sym morphism, equipped with the reduced scheme structure.

$$\mathrm{Quot}^d_{X/Y}(E)_p := (\sigma^d_{X/Y})^{-1}(d \cdot p)$$

The geometry of punctual Quot schemes depends on the local structure of X around the point p. In particular, they depend on the sheaf E only via its rank. Around smooth points, we may reduce to the case of affine spaces.

Theorem 1.1.17. *Consider a quasiprojective morphism $f\colon X \to Y$ of relative dimension $m \geq 1$ and let E be a locally free sheaf on X. Let p be a smooth point of X.*

1.1.3 Integer partitions and compositions

Fix a nonnegative integer d.

Definition 1.1.18. A *partition* of d is the denoted by $\omega := (1^{\omega_1} \dots i^{\omega_i} \dots d^{\omega_d})$, where each ω_i is a nonnegative integer counting the parts of size i appearing in the partition. In other words, one has $\sum_{i=1}^d \omega_i i = d$. The *length* of the partition ω is $|\omega| := \sum_{i=1}^d \omega_i$.

Definition 1.1.19. A *composition* of d is an ordered tuple of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_\ell)$ such that $\alpha_1 + \cdots + \alpha_\ell = d$. The length of a composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ is $\ell(\alpha) := \ell$.

Definition 1.1.20. A composition α of an integer d is called *strong* if all parts α_i are nonzero; respectively, it is called *weak* if some parts α_i are zero.

Example 1.1.21.

- The partitions of 3 are: $\{1,1,1\}, \{1,2\}, \{3\}$. Equivalently, we may also write $(1^3), (1^1 2^1), (3^1)$.
- The (strong) compositions of 3 are: $(1,1,1), (1,2), (2,1), (3)$.

The following table establishes the notation for sets of partitions, weak and strong combinations of a given integer.

	any length	length ℓ
Partitions of d	$P(d)$	$P_\ell(d)$
Strong compositions of d	$C(d)$	$C_\ell(d)$
Weak compositions of d	$C^*(d)$	$C^*_\ell(d)$

1.1.4 A stratification by integer partitions

Symmetric products may be stratified by integer partitions, with each stratum parametrizing 0–cycles whose support is distributed according to a given partition. This stratification is naturally inherited by the Quot scheme via the Quot–to–Sym morphism.

Proposition 1.1.22. *The symmetric product $\mathrm{Sym}^d_Y(X)$ admits a canonical stratification*

$$\mathrm{Sym}^d_Y(X) = \coprod_{\omega \in P(d)} \mathrm{Sym}^\omega_Y(X).$$

Each stratum $\mathrm{Sym}^\omega_Y(X)$ *parametrizes* 0*–cycles (supported on the fibers of* $f\colon X\to Y$*) with coefficients prescribed by* ω*: for each* i*, there are exactly* ω_i *points appearing with multiplicity* i*.*

Example 1.1.23. The stratum $\mathrm{Sym}^{(1^3 2^1)}_Y(X)$ of $\mathrm{Sym}^5_Y(X)$ corresponding to the partition $(1^3 2^1)$ parametrizes 0–cycles with three points of multiplicity 1 and one point with multiplicity 2. The symmetric group acts by permuting points with the same multiplicity.

Example 1.1.24. The stratum $\mathrm{Sym}^{(d)}_Y(X)$, corresponding to the trivial partition $(d)=(d^1)$, parameterizes 0-cycles with a single point of multiplicity d, and thus is isomorphic to X.

Proposition 1.1.25. *The relative Quot scheme* $\mathrm{Quot}^d_{X/Y}(E)$ *inherits the stratification of Proposition 1.1.22 via the Quot–to–Sym morphism. For each* $\omega\in P(d)$*, set*

$$\mathrm{Quot}^\omega_{X/Y}(E):=(\sigma^d_{X/Y})^{-1}(\mathrm{Sym}^\omega_Y(X)).$$

1.1.5 Generalized notions of Quot schemes

In this section we recall two generalizations of the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ of length d quotients. Although our applications will concern the case $Y=\mathrm{Spec}(k)$, we introduce these constructions in a relative setting. We fix a quasiprojective morphism $f\colon X\to Y$, a coherent sheaf E on X; given a Y–scheme T, we let $p_X : X\times_Y T\to X$ denote the projection to X.

Nested Quot schemes

The Hilbert scheme $\mathrm{Hilb}^d(X)$ parametrizes zero–dimensional subschemes Z of X of length d. A natural generalization is to consider successive inclusions, or *nestings*, of subschemes

$$Z_1\subseteq\cdots\subseteq Z_s\subset X$$

where each Z_i has prescribed length d_i. This idea extends naturally to higher rank cases, where nested Quot schemes parametrize successive coherent sheaf quotients supported in dimension zero, with prescribed non–increasing lengths.

Definition 1.1.26. Let $\underline{d}=(d_1\le\cdots\le d_s)$ be a vector of positive integers. The *nested Quot scheme* $\mathrm{Quot}^{\underline{d}}_{X/Y}(E)$ is defined by establishing that a morphism $\tau\colon T\to\mathrm{Quot}^{\underline{d}}_{X/Y}(E)$ corresponds to a succession of coherent quotients on $X\times_Y T$

$$p_X^*E\twoheadrightarrow F_s\twoheadrightarrow\cdots\twoheadrightarrow F_1$$

such that each F_i is flat over T and, for every $t\in T$, the pullback of F_i to $X\times_Y\{t\}$ has length d_i.

Remark 1.1.27. The Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ is recovered by setting $s=1$ and $d_1=d$.

Remark 1.1.28. The nested Quot scheme $\mathrm{Quot}^{\underline{d}}_{X/Y}(E)$ can be realized as a closed subscheme of $\prod_i\mathrm{Quot}^{d_i}_{X/Y}(E)$, with fiber product taken over Y, cut out by natural nesting relations between successive quotients. More precisely, the nesting occurs at the level of the kernels $K_i:=\ker(p_X^*E\twoheadrightarrow F_i)$, which must satisfy the relation $K_i\subseteq K_{i+1}$. In particular, it is a projective Y–scheme when f is projective.

Remark 1.1.29. For rank r locally free sheaves E on a smooth projective curve X, the nested Quot scheme $\mathrm{Quot}^{\underline{d}} X(E)$ is smooth of dimension rd_s [49, Proposition 2.1] [50, Proposition 2.1]. In higher dimension, if X irreducible, the nested Quot scheme $\mathrm{Quot}^{\underline{d}}_{X/Y}(E)$ is known to be connected [50, Theorem 1.4]. Smoothness of nested Quot schemes on higher dimensional varieties is discussed in [51, Theorem A].

Quot schemes of positive rank quotients

Another generalization of Quot schemes of finite length quotients is obtained by imposing that the quotient sheaves have prescribed positive rank and degree.

Definition 1.1.30. Suppose E is locally free of rank r, and fix a nonnegative integer $r' \leq r$. The *Quot scheme of rank r' quotients* $\mathrm{Quot}^{d}_{X/Y}(E, r')$ is defined by establishing that a morphism $\tau \colon T \to \mathrm{Quot}^{d}_{X/Y}(E, r')$ corresponds to a short exact sequence of coherent sheaves on $X \times_Y T$

$$0 \to S \longrightarrow p_X^* E \xrightarrow{q} F \to 0$$

such that F is flat over T and, for every $t \in T$, the pullback of F to $X \times_Y \{t\}$ has degree d and rank r'.

Remark 1.1.31. The Quot scheme $\mathrm{Quot}^{d}_{X/Y}(E)$ is recovered by imposing $r' = 0$.

Remark 1.1.32. The existence of Quot schemes of positive rank quotients follows directly from the general representability of Quot functors [26] [53].

Remark 1.1.33. Over the projective line, the Quot scheme $\mathrm{Quot}^{d}_{\mathbb{P}^1}(\mathcal{O}^{\oplus r}, r')$ is an irreducible rational nonsingular projective variety of dimension $rd + r'(r - r')$ [63, Theorem 2.1]. In general, for rank r sheaves E on a smooth projective genus g curve C, the Quot scheme $\mathrm{Quot}^{d}_{C}(E, r')$ is irreducible and generically smooth of dimension

$$rd - r\deg(E) - r'(r - r')(g - 1)$$

for $d \gg 0$ [7, Theorem 4.28] [54].

1.2 The Grothendieck ring of varieties

The Grothendieck ring of varieties provides an algebraic framework to operate with varieties up to *cut–and–paste* relations. Through these relations, it serves as a universal additive invariant by refining Hodge polynomials, Poincaré polynomials and Euler characteristic.

A survey on the topic is found in [13, Chapter 2]. Despite its elementary definition, the Grothendieck ring of varieties provides the foundation for deep mathematical theories such as *motivic integration*, developed by Kontsevich, Denef, Loeser and Looijenga to investigate birational invariants of complex Calabi–Yau varieties.

Fix a variety Y and let (Var_Y) denote the category of Y–varieties. In this section, we recall the definition and main properties of Grothendieck rings of Y–varieties.

Definition 1.2.1. The *Grothendieck ring of varieties (over Y)*, denoted by $\mathrm{K}_0(\mathrm{Var}_Y)$, is the free abelian group generated by isomorphism classes of Y–varieties, together with their structure morphism to Y, subject to the *cut–and–paste* relations

$$[X \to Y] = [Z \to Y] + [X \setminus Z \to Y]$$

for all closed subvarieties Z of X; multiplication is defined via fiber product over Y:

$$[X \to Y] \cdot [X' \to Y] := [X \times_Y X' \to Y].$$

$\mathrm{K}_0(\mathrm{Var}_Y)$ is a commutative ring with zero element $0 = [\emptyset \to Y]$ and unit $1 = [Y \xrightarrow{\mathrm{id}_Y} Y]$. In the literature, classes in the Grothendieck ring are sometimes referred to as *motives*. An element in $\mathrm{K}_0(\mathrm{Var}_Y)$ is said to be *effective* if it is represented by a variety. The subset of effective classes forms a semiring, often denoted by $\mathrm{S}_0(\mathrm{Var}_Y)$.

Notation 1.2.2. The Grothendieck ring of k–varieties is denoted $\mathrm{K}_0(\mathrm{Var}_k)$.

Example 1.2.3.

(i) The class of the affine line is $\mathbb{L}_Y := [\mathbb{A}^1_Y \to Y]$. When $Y = \mathrm{Spec}(k)$, we simply write $\mathbb{L} := \mathbb{L}_{\mathrm{Spec}(k)}$. This element plays a distinguished role in the Grothendieck ring, since many classes can be expressed as polynomials in $\mathbb{L}_Y$. In the literature, it is sometimes called *Lefschetz motive*.

(ii) The class of the affine space $\mathbb{A}^m_Y$ is $[\mathbb{A}^m_Y \to Y] = \mathbb{L}^m_Y$.

(iii) The class of the projective space $\mathbb{P}^m_Y$ is $[\mathbb{P}^m_Y \to Y] = \mathbb{L}^m_Y + \mathbb{L}^{m-1}_Y + \cdots + \mathbb{L}_Y + 1$.

(iv) For any nonempty Y–variety X, the class $-[X \to Y]$ is not effective. Any element in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_Y)$ can be written uniquely as a difference of effective classes.
$$\mathrm{K}_0(\mathrm{Var}_Y) \xrightarrow{\sim} \mathbb{Z}_{\geq 0} \cdot \mathrm{S}_0(\mathrm{Var}_Y) \oplus \mathbb{Z}_{<0} \cdot \mathrm{S}_0(\mathrm{Var}_Y)$$

Remark 1.2.4. The cut–and–paste relations hold for locally closed subvarieties. That is, if X admits a finite partition into locally closed subsvarieties $X = X_1 \sqcup \cdots \sqcup X_s$, then $[X \to Y] = [X_1 \to Y] + \cdots + [X_s \to Y]$.

Remark 1.2.5. Given a morphism $\upsilon\colon Y' \to Y$, there is a natural pullback $\upsilon^*\colon \mathrm{K}_0(\mathrm{Var}_Y) \to \mathrm{K}_0(\mathrm{Var}_{Y'})$ which assigns to a class $[X \to Y]$ the corresponding class $[X \times_Y Y' \to Y']$ obtained via fiber product. The pullback υ^* is a ring homomorphism.
In the other direction, the pushforward map $\upsilon_*\colon \mathrm{K}_0(\mathrm{Var}_{Y'}) \to \mathrm{K}_0(\mathrm{Var}_Y)$ assigns to a class $[X \to Y']$ the class $[X \to Y' \overset{\upsilon}{\to} Y]$ obtained via composition with υ. In general, the pushforward map is not a ring homomorphism because it does not preserve the unit 1 and multiplication.
The operations of pullback and pushforward are related by a *push-pull* formula

$$\upsilon_*(\upsilon^*([X \to Y])[X' \to Y']) = [X \to Y]\upsilon_*[X' \to Y']$$

which holds for any $[X \to Y]$ in $\mathrm{K}_0(\mathrm{Var}_Y)$ and $[X' \to Y']$ in $\mathrm{K}_0(\mathrm{Var}_{Y'})$

For the remainder of the section, we shall review some key properties of $\mathrm{K}_0(\mathrm{Var}_k)$.

Proposition 1.2.6. *Let X, Y, F be k–varieties and consider a Zariski locally trivial fibration $f\colon X \to Y$ with fiber F. Then $[X] = [Y] \cdot [F]$.*

Proposition 1.2.7. *Let X be a k–variety and consider a smooth closed subvariety Z of X. Denote by $\mathrm{Bl}_Z X$ the blowup of X at Z, with blowup morphism $\pi\colon \mathrm{Bl}_Z X \to X$ and exceptional divisor $E = \pi^{-1}(Z)$. Then $[\mathrm{Bl}_Z X] = [X] - [Z] + [E]$.*

What makes the Grothendieck ring particularly useful is the fact that it generalizes the notion of additive invariant.

Definition 1.2.8. Given a ring R, an R-valued *additive invariant* is a map $\lambda\colon (\mathrm{Var}_k) \to R$ satisfying the following properties:

(i) $\lambda(\emptyset) = 0$ and $\lambda(\mathrm{Spec}(k)) = 1$;
(ii) λ is invariant under isomorphism: if X, Y are isomorphic, then $\lambda(X) = \lambda(Y)$;
(iii) λ is additive: if Z is a closed subvariety of X, then $\lambda(X) = \lambda(Z) + \lambda(X \setminus Z)$;
(iv) λ is multiplicative: for any varieties X, Y we have $\lambda(X \times Y) = \lambda(X) \cdot \lambda(Y)$.

Example 1.2.9. Classical examples of additive invariants include Hodge and Poincaré polynomials, and the topological Euler characteristic of a variety. For varieties defined over a finite field $k = \mathbb{F}_q$, another example of additive invariant is the point counting functions, which yields the number of k–points of the variety.

Following is the *universal property* of the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_k)$.

Theorem 1.2.10. *For any R-valued additive invariant $\lambda\colon (\mathrm{Var}_k) \to R$ there exists a unique ring homomorphism $\bar{\lambda} : \mathrm{K}_0(\mathrm{Var}_k) \to R$ such that $\bar{\lambda}([X]) = \lambda(X)$.*

Remark 1.2.11.

(i) In characteristic zero, Bittner [9] showed that $\mathrm{K}_0(\mathrm{Var}_k)$ is generated by classes of smooth irreducible varieties, subject to the blowup relations as stated in Proposition 1.2.7. The proof relies on the *Weak Factorization Theorem*: a birational morphism between smooth projective varieties can be realized as a finite composition of blowups and blowdowns of smooth projective varieties along smooth irreducible centers. A presentation of the relative Grothendieck group is discussed in [9, §5].
(ii) In Definition 1.2.1, replacing varieties by schemes yields the Grothendieck ring of Y–schemes, denoted by $K_0(\mathrm{Sch}_Y)$. The natural map $K_0(\mathrm{Sch}_Y) \to \mathrm{K}_0(\mathrm{Var}_Y)$ is an isomorphism, since $[X \to Y] = [X_{\mathrm{red}} \to Y]$, see for instance [13, Example 1.2.4].
(iii) Definition 1.2.1 may be adapted to broader contexts. For instance, there is a notion of Grothendieck ring of stacks $K_0(\mathrm{St}_k)$, for there is a natural ring isomorphism

$$\mathrm{K}_0(\mathrm{Var}_k)[\mathbb{L}^{-1}][(\mathbb{L}^n - 1)^{-1}, n \geq 1] \xrightarrow{\sim} K_0(\mathrm{St}_k)$$

See for instance [16, Theorem 1.2], [11], [66]. One may also study the Grothendieck ring of k–morphisms, denoted $K_0(\mathrm{Map}_k)$, as well its extension to a relative setting. This generalization is discussed in Section 1.2.3.

1.2.1 Power structures on a (semi)ring

A power structure on a (semi)ring allows one to define expressions of the form $A(t)^x$, where $A(t)$ is a formal power series, in a way that mirrors the usual rules of exponentiation.

Recall that for a ring R, the set $1 + t \cdot R[\![t]\!]$ of power series with constant coefficient 1 is a multiplicative group.

Notation 1.2.12. Denote by $(1-t)^{-1}$ the geometric power series $1 + t + t^2 + \dots$.

Definition 1.2.13. A *power structure* on a (semi)ring R is a map

$$\begin{aligned}(1 + t \cdot R[\![t]\!]) \times R &\longrightarrow 1 + t \cdot R[\![t]\!] \\ (A(t), x) &\longmapsto A(t)^x\end{aligned}$$

satisfying the following properties:

(i) $A(t)^0 = 1$
(ii) $A(t)^1 = A(t)$
(iii) $A(t)^{x+y} = A(t)^x \cdot A(t)^y$
(iv) $A(t)^{xy} = (A(t)^x)^y$
(v) $(A(t) \cdot B(t))^x = A(t)^x \cdot B(t)^x$
(vi) $(1+t)^x = 1 + xt \mod t^2$
(vii) $(A(t^m))^x = (A(t))^x|_{t \mapsto t^m}$

Power structures on a ring may be defined via pre–λ–structures, as reported here.

Definition 1.2.14. A *pre–λ–structure* on a ring R is an additive–to–multiplicative group homomorphism $\lambda_t \colon R \to 1 + t \cdot R[\![t]\!]$ such that for any $x, y \in R$ one has:
(i) $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$;
(ii) $\lambda_t(x) = 1 + xt \mod t^2$.

Definition 1.2.15. A power structure on a ring R is *finitely determined* if for every $n > 0$ there exists an integer $m > 0$ such that the coefficients of $A(t)^x$ modulo t^n are determined by x and the coefficients of $A(t)$ modulo t^m.

Proposition 1.2.16. *A pre–λ–structure on a ring R defines a (finitely determined) power structure on R.*

Proof. Any power series $A(t) \in (1 + tR[\![t]\!])$ can be written as a product $A(t) = \prod_{i \geq 1} \lambda_{t^i}(b_i)$, with $b_i \in R$. This decomposition is unique. Then for any $x \in R$ define

$$A(t)^x := \prod_{i \geq 1} \lambda_{t^i}(xb_i). \qquad \square$$

Remark 1.2.17. Two or more pre–λ–structures λ, λ' on a ring R induce the same power structure if and only if $\lambda_{t^k}(x) \equiv \lambda'_{t^k}(x)$ agree modulo t^{2k} for all $k \geq 1$. See also Example 1.2.24 below.

1.2.2 A power structure on the Grothendieck ring of varieties

This section reviews the power structure on the Grothendieck ring introduced by Gusein-Zade, Luengo and Melle-Hernández in [27], [28], [29]. While the original construction concerns the ring $\mathrm{K}_0(\mathrm{Var}_k)$, its extension to the relative setting is straightforward, as will be explained here.

We start by defining a pre–λ–structure on the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_Y)$.

Definition 1.2.18. Consider a quasiprojective morphism $f \colon X \to Y$. The generating function of the classes $[\mathrm{Sym}^d_Y(X) \to Y]$ in $\mathrm{K}_0(\mathrm{Var}_Y)$ is called *(relative) motivic zeta function*. We denote it by $\zeta_{X/Y}(t)$.

$$\zeta_{X/Y}(t) := \sum_{d \geq 0} [\mathrm{Sym}^d_Y(X) \to Y] t^d$$

Proposition 1.2.19. *The motivic zeta $\zeta_{X/Y}(t)$ defines a pre–λ–structure on the Grothendieck ring* $\mathrm{K}_0(\mathrm{Var}_Y)$.

Proof. Given any two Y–varieties X, X', there is a canonical isomorphism

$$\coprod_{i=0}^{d} \mathrm{Sym}^i_Y(X) \times \mathrm{Sym}^{d-i}_Y(X') \xrightarrow{\sim} \mathrm{Sym}^d_Y(X \amalg X')$$

given by taking the union of points. Therefore one has $\zeta_{X/Y}(t)\zeta_{X'/Y}(t) = \zeta_{X \amalg X'/Y}(t)$. This defines a pre–$\lambda$–structure on the Grothendieck semiring $\mathrm{S}_0(\mathrm{Var}_Y)$. The pre–$\lambda$–structure extends uniquely to the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_Y)$ by setting

$$\lambda_t([X \to Y] - [X' \to Y]) := \zeta_{X/Y}(t)\zeta_{X'/Y}(t)^{-1}. \qquad \square$$

By Proposition 1.2.16, the pre–λ–structure $\lambda_t([X \to Y]) := \zeta_{X/Y}(t)$ defines a finitely determined power structure on $\mathrm{K}_0(\mathrm{Var}_Y)$. Gusein-Zade, Luengo and Melle-Hernández provide a geometric interpretation of this power structure on the Grothendieck semiring $\mathrm{S}_0(\mathrm{Var}_Y)$.

Definition 1.2.20. Let X and $\{A_i\}_{i \geq 1}$ be Y–varieties, and consider their classes $[X \to Y]$ and $[A_i \to Y]$ in $\mathrm{S}_0(\mathrm{Var}_Y)$. The power structure on $\mathrm{S}_0(\mathrm{Var}_Y)$ is defined by the rule

$$\left(1 + \sum_{i \geq 1} [A_i \to Y] t^i \right)^{[X \to Y]} := 1 + \sum_{d \geq 1} [B_d \to Y] t^d \tag{1.1}$$

where B_d is the Y–variety defined as follows

$$B_d := \coprod_{\omega \in P(d)} \left((X^{|\omega|} \setminus \Delta) \times_Y \prod_{i=1}^{d} A_i^{\omega_i} \right) \Big/ \prod_{i=1}^{d} \mathfrak{S}_{\omega_i}. \tag{1.2}$$

In formula (1.2):

- the coproduct runs over partitions ω of d, see Definition 1.1.18;
- all fiber products are taken over Y;
- the large diagonal Δ of $X^{|\omega|} = X^{\sum_i \omega_i}$ is the locus where at least two points coincide;
- for each $i \geq 1$ and $y \in Y$, the symmetric group $\mathfrak{S}_{\omega_i}$ acts simultaneously by permutation on (the fiber over y of) the factors X^{ω_i} and $A_i^{\omega_i}$.

Example 1.2.21. Assume $Y = \mathrm{Spec}(k)$. For small values of d, formula (1.2) yields:

- $B_1 = X \times A_1$;
- $B_2 = (X \times A_2) \sqcup ((X^2 \setminus \Delta) \times A_1^2)/\mathfrak{S}_2$;
- $B_3 = (X \times A_3) \sqcup ((X^2 \setminus \Delta) \times A_1 \times A_2) \sqcup ((X^3 \setminus \Delta) \times A_1^3)/\mathfrak{S}_3$.

Remark 1.2.22. Formulas (1.1) and (1.2) can be interpreted in terms of "configurations of charged particles". In the words of Bryan and Morrison [12], when $Y = \mathrm{Spec}(k)$ one can *think of B_d as parameterizing a finite set of "particles" on X of total "charge" d, where the "internal state space" of a charge i particle is A_i*. For instance:

- B_1 parametrizes configurations of one point of X carrying a particle of charge 1;
- B_2 parametrizes two possible configurations of total charge 2: one point carrying a charge 2 particle, or two distinct points with charge 1 particles;
- B_3 parametrizes three possible configurations with total charge 3: one point carrying a charge 3 particle, one point carrying a charge 1 particle and another with a charge 2 particle, or 3 points carrying a charge 1 particle each.

Moreover, the symmetric group action identifies configurations that differ by permuting points with particles of the same charge.

For general Y, the same interpretation holds fiberwise over the points $y \in Y$: the variety $B_d \times_Y \{y\}$ parametrizes finite sets of "particles" on $X \times_Y \{y\}$ of total "charge" d, where the "internal state space" of a charge i particle is $A_i \times_Y \{y\}$.

Proposition 1.2.23.

(i) *The power structure (1.2) on the Grothendieck semiring* $\mathrm{S}_0(\mathrm{Var}_Y)$ *extends uniquely to a power structure on the Grothendieck ring* $\mathrm{K}_0(\mathrm{Var}_Y)$*.*

(ii) *The power structure on* $\mathrm{K}_0(\mathrm{Var}_Y)$ *defined by the pre–λ–structure* $\zeta_{X/Y}(t)$ *via Proposition 1.2.16 coincides with (i).*

Proof. (i) Consider a formal power series $A(t)$ in $1+t\cdot\mathrm{K}_0(\mathrm{Var}_Y)[\![t]\!]$. By [27, Theorem 2] there exists a series $B(t)$ in $1+t\cdot\mathrm{S}_0(\mathrm{Var}_Y)[\![t]\!]$ such that the coefficients of $C(t):=A(t)B(t)$ are also effective. Define

$$\begin{aligned} A(t)^{[X\to Y]} &:= C(t)^{[X\to Y]}(B(t)^{[X\to Y]})^{-1}\\ A(t)^{[X\to Y]-[X'\to Y]} &:= A(t)^{[X\to Y]}(A(t)^{[X'\to Y]})^{-1}. \end{aligned}$$

(ii) It suffices to check that $\zeta_{X/Y}(t)=(1-t)^{-[X\to Y]}$. By formula (1.2), the coefficients of the series $(1-t)^{-[X\to Y]}=(1+t+t^2+\dots)^{[X\to Y]}$ are given by

$$\begin{aligned} B_d &= \coprod_{\omega\in P(d)}\left((X^{|\omega|}\setminus\Delta)\times_Y\prod_{i=1}^d Y^{\omega_i}\right)\Big/\prod_{i=1}^d\mathfrak{S}_{\omega_i}\\ &= \coprod_{\omega\in P(d)}(X^{|\omega|}\setminus\Delta)/\prod_{i=1}^d\mathfrak{S}_{\omega_i}\\ &= \coprod_{\omega\in P(d)}\mathrm{Sym}^\omega_Y(X)\\ &= \mathrm{Sym}^d_Y(X) \end{aligned}$$

where $\mathrm{Sym}^\omega_Y(X)$ denotes the stratum of $\mathrm{Sym}^d_Y(X)$ parameterizing cycles distributed according to the partition ω, see Proposition 1.1.22. □

Example 1.2.24. Another example of pre–λ–structure on $\mathrm{K}_0(\mathrm{Var}_Y)$ is given by

$$\lambda'_t([X\to Y]):=1+\sum_{d\geq 1}[(X^d\setminus\Delta)/\mathfrak{S}_d\to Y]t^d.$$

For each d, the coefficient $(X^d\setminus\Delta)/\mathfrak{S}_d$ parametrizes configurations of d unordered points supported on the fibers of $X\to Y$. It follows immediately from (1.2) that $\lambda'_t([X\to Y])=(1+t)^{[X\to Y]}$. Since

$$\zeta_{X/Y}(t^k)=1+[X\to Y]t^k+O(t^{2k})$$

and

$$\lambda'_{t^k}([X\to Y])=(1+t^k)^{[X\to Y]}=1+[X\to Y]t^k+O(t^{2k}),$$

the two pre–λ–structures agree modulo t^{2k} for every $k\geq 1$. Hence, by Remark 1.2.17, they induce the same power structure on $\mathrm{K}_0(\mathrm{Var}_Y)$.

Remark 1.2.25. The power structure on the Grothendieck semiring $\mathrm{S}_0(\mathrm{Var}_Y)$ is strictly finer than the power structure defined on $\mathrm{K}_0(\mathrm{Var}_Y)$ via $\zeta_{X/Y}(t)$. For instance, the geometric properties of the coefficients (1.2), such as compactness, could not be deduced from the pre–λ–structure. This can be rephrased by saying that the power structure on $\mathrm{K}_0(\mathrm{Var}_Y)$

is *effective*: if the coefficients of the series $A(t)$ and $[X \to Y]$ are effective in the sense of Definition 1.2.1, so are the coefficients of the series $A(t)^{[X\to Y]}$.
The effectiveness of a power structure is not at all obvious, for instance:

- the power structure on $\mathrm{K}_0(\mathrm{Var}_k)$ defined by the pre–λ–structure $(\zeta_{[X]}(-t))^{-1}$ is not effective [30, Statement 1];
- the power structure on $\mathrm{K}_0(\mathrm{Var}_k)$ defined by $\zeta_{X/Y}(t)$ extends uniquely to a power structure on the Grothendieck ring of stacks $K_0(\mathrm{St}_k)$; however, the resulting power structure is not effective [30, Statement 2].

As a final remark, we note the following rule, which will prove useful later on.

Example 1.2.26. The computation $[\mathrm{Sym}^d(\mathbb{A}^m)] = \mathbb{L}^{md}$ is due to Totaro [25, Lemma 4.4]. It follows immediately that $(1-t)^{-\mathbb{L}^m} = (1-\mathbb{L}^m t)^{-1}$.

1.2.3 A power structure on the Grothendieck ring of maps

Besides the Grothendieck ring of varieties, there is another closely related construction in which the basic objects are *morphisms* rather than varieties. This is the Grothendieck ring of maps, introduced in [31], whose elements are isomorphism classes of morphisms of varieties, subject to natural cut–and–paste relations on both the source and the target. Like the classical Grothendieck ring of varieties, the Grothendieck ring of maps carries a natural power structure which allows one to study generating functions of morphisms. In this section we briefly review the definition and power structure on this ring.

Fix a variety Y and let (Map_Y) denote the category of morphisms of Y–varieties.

Definition 1.2.27. The *Grothendieck ring of maps (over Y)*, denoted by $\mathrm{K}_0(\mathrm{Map}_Y)$, is the free abelian group generated by equivalence classes of morphisms of Y–varieties subject to the following relations:

(i) If two Y–morphisms $g\colon X \to Z$ and $g'\colon X' \to Z'$ fit into a diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\wr"] & Z \arrow[d, "\wr"] \\ X' \arrow[r, "g'"] & Z' \end{tikzcd}$$

where the vertical arrows are given by isomorphisms, then $[X \xrightarrow{g} Z] = [X' \xrightarrow{g} Z']$.

(ii) (*Cut–and–paste on the source*) For any closed $V \subseteq X$:

$$[X \xrightarrow{g} Z] = [V \xrightarrow{g|_V} Z] + [X \setminus V \xrightarrow{g|_{X\setminus V}} Z]$$

(iii) (*Cut–and–paste on the target*) For any closed $V \subseteq Z$:

$$[X \xrightarrow{g} Z] = [g^{-1}(V) \xrightarrow{g|_{g^{-1}(V)}} V] + [X \setminus g^{-1}(V) \xrightarrow{g|_{X\setminus g^{-1}(V)}} Z \setminus V]$$

It follows from these relations that summation is given by

$$[X \xrightarrow{g} Z] + [X' \xrightarrow{g'} Z'] = [X \sqcup X' \xrightarrow{g\sqcup g'} Z \sqcup Z'].$$

Multiplication is defined via fiber product over Y on the source and target, that is

$$[X \to Z] \cdot [X' \to Z] := [X \times_Y X' \to Z \times_Y Z'].$$

With these operations, $\mathrm{K}_0(\mathrm{Map}_Y)$ is a commutative ring with zero element $0 = [\emptyset \to Z]$ and unit $1 = [Y \xrightarrow{\mathrm{id}_Y} Y]$.

Remark 1.2.28. Condition (iii) in Definition 1.2.27 implies that a morphism $[X \to Z]$ can be replaced by $[X \to Z']$ whenever Z' is a subset of Z containing the image of X.

Remark 1.2.29. As in the case of $\mathrm{K}_0(\mathrm{Var}_Y)$, one defines the Grothendieck *semiring* of maps $\mathrm{S}_0(\mathrm{Map}_Y)$ to be the semiring of effective classes representing "genuine" maps.

Definition 1.2.30. Let $g\colon X \to Z$ and $\{\alpha_i\colon A_i \to B_i\}_{i\geq 1}$ be morphisms of Y–varieties, and consider the corresponding classes $[g\colon X \to Z]$ and $[\alpha_i\colon A_i \to B_i]$ in the semiring $\mathrm{S}_0(\mathrm{Map}_Y)$. The power structure on $\mathrm{S}_0(\mathrm{Map}_Y)$ is defined by the rule

$$\left(1+\sum_{i\geq 1}[A_i \xrightarrow{\alpha_i} B_i]t^i\right)^{[X\xrightarrow{g}Z]} := 1+\sum_{d\geq 1}[S_d \xrightarrow{s_d} T_d]t^d$$

where $s_d\colon S_d \to T_d$ is the morphism of Y–varieties defined as follows. For each partition $\omega = (1^{\omega_1}\dots i^{\omega_i}\dots d^{\omega_d})$ of d, define a morphism

$$s_\omega := \left(\left((X^{|\omega|}\setminus\Delta) \xrightarrow{g^{|\omega|}} Z^{|\omega|}\right)\times\prod_{i=1}^{d}(A_i \xrightarrow{\alpha_i} B_i)^{\omega_i}\right)\Big/\prod_{i=1}^{d}\mathfrak{S}_{\omega_i}.$$

Here, Δ is the large diagonal of $X^{|\omega|}$, with $|\omega| = \sum_i \omega_i$. The morphism $s_\omega\colon S_\omega \to T_\omega$ has source and target given by, respectively,

$$S_\omega := \left((X^{|\omega|}\setminus\Delta)\times_Y\prod_{i=1}^{d}A_i^{\omega_i}\right)\Big/\prod_{i=1}^{d}\mathfrak{S}_{\omega_i} \tag{1.3}$$

$$T_\omega := \left(Z^{|\omega|}\times_Y\prod_{i=1}^{d}B_i^{\omega_i}\right)\Big/\prod_{i=1}^{d}\mathfrak{S}_{\omega_i}. \tag{1.4}$$

Each symmetric group α_i acts by diagonal permutation of the factors of X^{ω_i} and $A_i^{\omega_i}$ in the source, and of Z^{ω_i} and $B_i^{\omega_i}$ in the target. Finally, set

$$s_d := \coprod_{\omega\in P(d)} s_\omega, \qquad S_d := \coprod_{\omega\in P(d)} S_\omega \qquad T_d := \coprod_{\omega\in P(d)} T_\omega.$$

Remark 1.2.31. Compare equation (1.3) with (1.2).

The proof of the following statement is analogous to the contents of Proposition 1.2.19 and Proposition 1.2.23, and thus is omitted.

Proposition 1.2.32.

(i) *The power structure on* $\mathrm{S}_0(\mathrm{Map}_Y)$ *given by Definition 1.2.30 extends uniquely to a power structure on the Grothendieck ring of maps* $\mathrm{K}_0(\mathrm{Map}_Y)$.

(ii) *The formal power series*

$$\zeta_{X\to Z}(t) := \sum_{d\geq 0}[\mathrm{Sym}^d_Y(X) \to \mathrm{Sym}^d_Y(Z)]t^d$$

defines a pre–λ–structure on $\mathrm{S}_0(\mathrm{Map}_Y)$.

(iii) *The power structure on* $\mathrm{K}_0(\mathrm{Map}_Y)$ *defined by the pre–λ–structure* $\zeta_{X\to Z}(t)$ *via Proposition 1.2.16 coincides with (i).*

1.3 Quot zeta functions

The power structure on the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_Y)$ provides a natural framework for studying generating functions of Quot scheme classes, which we refer to as *Quot zeta functions*. These generating functions encode, in a single formal expression, the classes of Quot schemes $\mathrm{Quot}^d_{X/Y}(E)$ as the parameter d varies. In particular, they serve as a convenient tool for extracting other geometric or numerical invariants of Quot schemes; see for instance Corollary 4.3.14.

The study of generating functions often leads to questions about their rationality and the existence of symmetries, encoded for instance by functional equations. For Quot zeta functions, the existence of such relations could indicate that the geometry of Quot schemes $\mathrm{Quot}^d_{X/Y}(E)$ is intertwined to that of Quot schemes of lower degree.

Definition 1.3.1. Consider a projective morphism $f\colon X \to Y$ and a coherent sheaf E on X. The *(relative) Quot zeta function* is the generating function of the classes $[\mathrm{Quot}^d_{X/Y}(E) \to Y]$ in $\mathrm{K}_0(\mathrm{Var}_Y)$.

$$Z_{X/Y}(E,t) := \sum_{d\geq 0} [\mathrm{Quot}^d_{X/Y}(E) \to Y] t^d \qquad \in 1 + t \cdot \mathrm{K}_0(\mathrm{Var}_Y)[\![t]\!]$$

When $Y = \mathrm{Spec}(k)$, we shall write $Z_X(E,t) = \sum_{d\geq 0}[\mathrm{Quot}^d_X(E)]t^d$.

Definition 1.3.2. Consider a projective k–scheme X, a coherent sheaf E on X and fix a point p of X. The *punctual Quot zeta function* is the generating function of the classes $[\mathrm{Quot}^d_X(E)_p]$ in $\mathrm{K}_0(\mathrm{Var}_k)$.

$$Z_X(E,t)_p := \sum_{d\geq 0} [\mathrm{Quot}^d_X(E)_p] t^d \qquad \in 1 + t \cdot \mathrm{K}_0(\mathrm{Var}_k)[\![t]\!]$$

In order to capture the class of a Y–scheme as a k–scheme in $\mathrm{K}_0(\mathrm{Var}_k)$, we use the notion of *absolute* Quot zeta functions. These are obtained using the natural pushforward between Grothendieck rings induced by the structure morphism $\nu\colon Y \to k$.

Definition 1.3.3. Consider a projective morphism $f\colon X \to Y$ and a coherent sheaf E on X. The *relative Quot zeta function* is the generating function of the classes $[\mathrm{Quot}^d_{X/Y}(E)]$ in $\mathrm{K}_0(\mathrm{Var}_k)$.

$$Z^{\mathrm{abs}}_{X/Y}(E,t) := \sum_{d\geq 0} \nu_*[\mathrm{Quot}^d_{X/Y}(E) \to Y] t^d \qquad \in 1 + t \cdot \mathrm{K}_0(\mathrm{Var}_k)[\![t]\!]$$

Definition 1.3.4. The generating function of the classes $[\mathrm{Sym}^d_Y(X)]$ is called *relative motivic zeta function*. We denote it by $\zeta^{\mathrm{abs}}_{X/Y}(t)$.

$$\zeta^{\mathrm{abs}}_{X/Y}(t) := \sum_{d\geq 0} \nu_*[\mathrm{Sym}^d_Y(X) \to Y] t^d \qquad \in 1 + t \cdot \mathrm{K}_0(\mathrm{Var}_k)[\![t]\!]$$

CHAPTER 2

Relative Quot schemes over families of smooth curves

The material presented in this chapter is based on the joint preprint [18] with Barbara Fantechi and Ajay Gautam. The project investigates the geometry of the relative Quot scheme associated to families of smooth curves and locally free sheaves on them.

2.1 Overview and main results

Our interest in relative Quot schemes over families of curves originates from the case of smooth projective k–curves, where Quot schemes have been the subject of extensive study. For a smooth projective curve X and a locally free sheaf E of rank r on X, the Quot scheme $\mathrm{Quot}_X^d(E)$ is a smooth projective variety of dimension rd. A complete geometric description of $\mathrm{Quot}_X^d(\mathcal{O}^{\oplus r})$ is due to Emili Bifet, who showed that it admits a smooth stratification, each stratum being isomorphic to a vector bundle over products of symmetric powers of the curve.

Theorem 2.1.1 ([8]) . *Fix two integers $d \geq 0$, $r \geq 1$ and let X be a smooth projective curve. There exists a smooth, locally closed stratification $Q_{d,\alpha}$ of $\mathrm{Quot}_X^d(\mathcal{O}^{\oplus r})$ indexed by weak compositions*[1] *$\alpha = (\alpha_1, \dots, \alpha_r)$ of d of length r. Each stratum is isomorphic to a vector bundle over $\mathrm{Sym}^{\alpha_1}(X) \times \cdots \times \mathrm{Sym}^{\alpha_r}(X)$ of rank $\alpha_2 + 2\alpha_3 + \cdots + (r-1)\alpha_r$.*

The stratification is obtained by applying the Białynicki–Birula decomposition to certain one parameter subgroups of the torus $\mathbb{G}_m^r$, which acts canonically on $\mathrm{Quot}_X^d(\mathcal{O}^{\oplus r})$ by rescaling the fibers of $\mathcal{O}^{\oplus r}$.

Theorem 2.1.1 also has arithmetic implications for Quot schemes on smooth curves, as it determines a closed formula for the class of $\mathrm{Quot}_X^d(\mathcal{O}^{\oplus r})$ in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_k)$. Bagnarol, Fantechi and Perroni later confirmed that this formula continues to hold for any locally free sheaf E on X.

Theorem 2.1.2 ([8], [3]) . *The following formula holds- in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_k)$ for any locally free sheaf E of rank r on a smooth projective curve X.*

$$[\mathrm{Quot}_X^d(E)] = \sum_{\alpha \in C_r(d)} \prod_{i=1}^{r} [\mathrm{Sym}^{\alpha_i}(X)] \mathbb{L}^{(i-1)\alpha_i}$$

---

[1] See Definition 1.1.20.

Corollary 2.1.3. *The Quot zeta function over a smooth curve X factors into a product of motivic zeta functions. In particular, $Z_X(E,t)$ is a rational function in t.*

$$Z_X(E,t) = \zeta_X(t)\zeta_X(\mathbb{L}t)\dots\zeta_X(\mathbb{L}^{\mathrm{rk}(E)-1}t)$$

We were naturally led to the following question.

Question 2.1.4. *Given a family of smooth projective curves and a locally free sheaf on it, is there a way to generalize Bifet's geometric description to relative Quot schemes?*

The answer to this question is positive: not only the geometric picture extends to the relative case, but we also found that it does not necessarily require the use of the Białynicki–Birula decomposition.

The remainder of the section is an overview of our main findings. We shall fix a smooth projective morphism $f\colon X \to Y$ of relative dimension 1, a locally free sheaf E on X and consider the relative Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$.

Our geometric description relies on the assumption that E is filtered by line bundles.

Definition 2.1.5. A locally free sheaf E of rank n on a scheme X is *filtered by line bundles* if there exists a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ where each E_i is locally free of rank i and $L_i := E_i/E_{i-1}$ is a line bundle.

Under this assumption we construct, by recursion on the rank of E, a stratification of the relative Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$. Each stratum is obtained as an iterated composition of affine vector bundles of known ranks over products of symmetric powers of the curve.

Theorem 2.1.6. *Fix an integer $d \geq 0$ and let $f\colon X \to Y$ be a family of smooth projective curves. Consider a locally free sheaf E of rank r over X, filtered by line bundles. There exists a smooth, locally closed stratification $Q_{d,\alpha}$ of $\mathrm{Quot}^d_{X/Y}(E)$ indexed by weak compositions $\alpha = (\alpha_1,\dots,\alpha_r)$ of d of length r. Each stratum is isomorphic to an affine vector bundle over*

$$\mathrm{Sym}^{\alpha_1}_Y(X) \underset{Y}{\times} \dots \underset{Y}{\times} \mathrm{Sym}^{\alpha_r}_Y(X)$$

of rank $\alpha_2 + 2\alpha_3 + \dots + (r-1)\alpha_r$.

In general, not all locally free sheaves admit a filtration by line bundles. Nevertheless, one may always reduce to the filtered case, up to stratifying the base space Y. This is discussed in detail in section 2.2.5.

As an immediate application, Theorem 2.1.6 determines the formula of the class of $\mathrm{Quot}^d_{X/Y}(E)$ in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_Y)$.

Corollary 2.1.7. *Let $f\colon X \to Y$ be a family of smooth projective curves. Consider a locally free sheaf E of rank r over X, filtered by line bundles. For every $d \geq 0$, the following formula*

$$[\mathrm{Quot}^d_{X/Y}(E) \to Y] = \sum_{\alpha \in C_r(d)} \prod_{i=1}^{r} [\mathrm{Sym}^{\alpha_i}_Y(X) \to Y]\mathbb{L}_Y^{\alpha_i(i-1)}$$

holds in $\mathrm{K}_0(\mathrm{Var}_Y)$, where $\mathbb{L}_Y$ denotes the class $[\mathbb{A}^1_Y \to Y]$.

Specializing this formula to $Y = \mathrm{Spec}(k)$ recovers Bifet's formula. More generally, equation (2.7) implies that the Quot zeta function $Z_{X/Y}(E,t)$ factors into a product of motivic zeta functions, thus providing a direct generalization of Corollary 2.1.3.

Further applications, discussed in Section 2.3, include the computation of the classes in $\mathrm{K}_0(\mathrm{Var}_k)$ of nested Quot schemes and Quot schemes of positive rank quotients on a smooth curve.

Notation 2.1.8. Given a Y–scheme X and morphism $g\colon A \to B$ of Y–schemes, denote by $\tilde{g}$ the product morphism $(\mathrm{id}_X, g)\colon X \times_Y A \to X \times_Y B$.

2.2 Main constructions and proofs

2.2.1 Setup for an inductive proof

Theorem 2.1.6 applies to locally free sheaves filtered by line bundles. The choice of a filtration provides a natural framework for an inductive proof on the rank, and the following statement summarizes the key mechanism.

Theorem 2.2.1. *Let $f\colon X \to Y$ be a family of smooth curves. Consider a locally free sheaf E of rank r on X, and choose a short exact sequence of locally free sheaves $0 \to E' \to E \to L \to 0$ where E' has rank $r-1$ and L is a line bundle. Then:*

(i) There exists a locally closed stratification $\{Q_{d,m} \mid m = 0, \ldots, d\}$ of the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ such that each stratum carries a natural morphism

$$\Phi_{d,m}\colon Q_{d,m} \longrightarrow \mathrm{Quot}^{d-m}_{X/Y}(L) \times_Y \mathrm{Quot}^m_{X/Y}(E')$$

(ii) The morphism $\Phi_{d,m}$ is a geometric vector bundle of rank m.

The proof of Theorem 2.2.1 is postponed to the following sections, each addressing a different aspect of the result:

- Section 2.2.2 establishes part (i), where flattening stratifications are used to construct the stratification $Q_{d,m}$ and the morphism $\Phi_{d,m}$;
- Section 2.2.3 introduces a Lifting Lemma, which provides insight into the structure of the fibers of $\Phi_{d,m}$ and sets the stage for part (ii);
- Finally, Section 2.2.4 completes the proof of part (ii) by showing that $\Phi_{d,m}$ is an affine vector bundle.

Assuming Theorem 2.2.1, the proof of Theorem 2.1.6 is carried out as follows.

Proof of Theorem 2.1.6. Proceed by induction on the rank of E. When E is a line bundle, the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ is canonically isomorphic to the Hilbert scheme $\mathrm{Hilb}^d_Y(X)$. Since $f\colon X \to Y$ is a family of smooth curves, the Hilbert–Chow morphism defines an isomorphism of $\mathrm{Hilb}^d_Y(X)$ with $\mathrm{Sym}^d_Y(X)$ [26, §6].

In general, suppose that E has rank $r \geq 2$ and assume the statement holds for filtered sheaves of rank $r-1$. Choose a filtration of E by line bundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

and write $L_{r-1} := E_r/E_{r-1}$. By Theorem 2.2.1 there is a stratification $Q_{d,m}$ of $\mathrm{Quot}^d_{X/Y}(E)$ and, for each m, an affine vector bundle of rank m

$$\Phi_{d,m}\colon Q_{d,m} \longrightarrow \mathrm{Quot}^{d-m}_{X/Y}(L_{r-1}) \times_Y \mathrm{Quot}^m_{X/Y}(E_{r-1}).$$

The Quot scheme $\mathrm{Quot}^{d-m}_{X/Y}(L_{r-1})$ is isomorphic to the symmetric product $\mathrm{Sym}^{d-m}_Y(X)$. By induction on E_{r-1}, we obtain a stratification $Q_{m,\alpha}$ of the Quot scheme $\mathrm{Quot}^m_{X/Y}(E_{r-1})$, indexed by compositions $\alpha = (\alpha_1, \dots, \alpha_{r-1})$ of m. For each α, the morphism

$$\Phi_{m,\alpha} \colon Q_{m,\alpha} \longrightarrow \mathrm{Sym}^{\alpha_1}_Y(X) \times_Y \cdots \times_Y \mathrm{Sym}^{\alpha_{n-1}}_Y(X)$$

is a Zariski affine bundle of rank $\alpha_2 + 2\alpha_2 + \cdots + (r-2)\alpha_{r-1}$. To conclude the proof, write $\beta := (d-m, \alpha_1, \dots, \alpha_{r-1})$ and define $Q_{d,\beta}$ to be the pullback of $\mathrm{Sym}^{d-m}_Y(X) \times_Y Q_{m,\alpha}$ via $\Phi_{d,m}$, as shown in the cartesian diagram below. The arrows in the first column form the morphism $\Phi_{d,\beta}$, which is a Zariski affine bundle of rank $\beta_2 + 2\beta_3 + \cdots + (r-1)\beta_r$, as required.

$$\begin{tikzcd}
Q_{d,\beta} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & Q_{d,m} \arrow[d, "\Phi_{d,m}"] \\
\mathrm{Sym}^{d-m}_Y(X) \times_Y Q_{m,\alpha} \arrow[rr] \arrow[d, "{(\Phi_{m,\alpha},\mathrm{id})}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Sym}^{d-m}_Y(X) \times_Y \mathrm{Quot}^m_{X/Y}(E_{n-1}) \arrow[d, "{(\sigma^m_{X/Y},\mathrm{id})}"] \\
\prod_{i=1}^{r}{}_{Y} \mathrm{Sym}^{\beta_i}_Y(X) \arrow[rr] & & \mathrm{Sym}^{d-m}_Y(X) \times_Y \mathrm{Sym}^m_Y(X)
\end{tikzcd}$$

□

Remark 2.2.2.

(i) By construction, the morphism $\Phi_{d,\beta}$ is compatible with the Quot–to–Sym morphism. The diagram below shows how each stratum maps to symmetric products.

$$\begin{tikzcd}
Q_{d,\beta} \arrow[r, hook] \arrow[d, "\Phi_{d,\beta}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{X/Y}(E) \arrow[d, "\sigma^d_{X/Y}"] \\
\prod_{i=1}^{r}{}_{Y} \mathrm{Sym}^{\beta_i}_Y(X) \arrow[r, hook] & \mathrm{Sym}^d_Y(X)
\end{tikzcd}$$

(ii) The construction carried out in the proof of Theorem 2.1.6 relies on the choice of a filtration, but the final vector bundle structure does not. This ensures that the result is intrinsic to the filtered sheaf E, and is independent of the choice of filtration.

2.2.2 A stratification of the Quot scheme

The goal of this section is to establish part (i) of Theorem 2.2.1. Starting from the short exact sequence

$$0 \to E' \to E \to L \to 0$$

the idea is to stratify the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ by how much of the length of a given quotient $[E \to F]$ is "distributed" among E' and L. More precisely, each stratum $Q_{d,m}$ should parametrize quotients $[E \to F]$ of length d which induce a quotient on E' of length m; the remaining $d-m$ units of length come from the induced quotient on L. The morphism $\Phi_{d,m}$ records this decomposition by mapping each quotient $[E \to F]$ to the induced quotients on E' and L.

Proof of Theorem 2.2.1, part (i). Consider the universal sequence of $\mathrm{Quot}^d_{X/Y}(E)$

$$0 \longrightarrow \mathcal{S}_d \longrightarrow p_X^* E \longrightarrow \mathcal{F}_d \longrightarrow 0$$

where $p_X \colon X \times_Y \mathrm{Quot}^d_{X/Y}(E) \to X$ denotes the projection morphism. The intersection of $\mathcal{S}_d$ with the pullback $p_X^* E'$ produces a subsheaf $\mathcal{S}'_d$ of $p_X^* E'$. This data fits into a short exact sequence on $X \times_Y \mathrm{Quot}^d_{X/Y}(E)$:

$$0 \longrightarrow \mathcal{S}'_d \longrightarrow p_X^* E' \longrightarrow \mathcal{G}_d \longrightarrow 0.$$

A priori, the cokernel $\mathcal{G}_d$ is not flat over $\mathrm{Quot}^d_{X/Y}(E)$, so we consider the flattening stratification of $\mathcal{G}_d$: for each $m = 0, \ldots, d$ there is a locally closed stratum $Q_{d,m}$ of $\mathrm{Quot}^d_{X/Y}(E)$ such that the restriction of $\mathcal{G}_d$ to $X \times_Y Q_{d,m}$ is flat over $Q_{d,m}$ of length m. This construction produces a morphism of Y–schemes

$$\varphi \colon Q_{d,m} \longrightarrow \mathrm{Quot}^m_{X/Y}(E'). \tag{2.1}$$

In particular, the restriction of $\mathcal{G}_d$ to $X \times_Y Q_{d,m}$ is isomorphic to the pullback $\tilde{\varphi}^* \mathcal{F}'_m$, where $\mathcal{F}'_m$ denotes the universal quotient sheaf of $\mathrm{Quot}^m_{X/Y}(E')$.

To construct a morphism from $Q_{d,m}$ to $\mathrm{Quot}^{d-m}_{X/Y}(L)$, consider the following diagram on $X \times_Y Q_{d,m}$ with exact rows and columns, which sums up the data described so far. The sheaves $\mathcal{A}_{d-m}$ and $\mathcal{B}_{d-m}$ denote the respective cokernels of $\mathcal{S}'_d \to \mathcal{S}_d$ and of $\tilde{\varphi}^* \mathcal{F}'_m \to \mathcal{F}_d$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{S}'_d \arrow[r] \arrow[d] & p_X^* E' \arrow[r] \arrow[d] & \tilde{\varphi}^* \mathcal{F}'_m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{S}_d \arrow[r] \arrow[d] & p_X^* E \arrow[r] \arrow[d] & \mathcal{F}_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}_{d-m} \arrow[r] \arrow[d] & p_X^* L \arrow[r] \arrow[d] & \mathcal{B}_{d-m} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.2}$$

The sheaf $\mathcal{B}_{d-m}$ is flat over $Q_{d,m}$ of length $d-m$, as we now explain. If $\pi \colon X \times_Y Q_{d,m} \to Q_{d,m}$ denotes the projection to $Q_{d,m}$, the higher pushforward $R^1 \pi_* \tilde{\varphi}^* \mathcal{F}'_m$ vanishes and this ensures that $\pi_* \mathcal{B}_{d-m}$ is locally free of rank $d-m$; see Lemma 5.5 in [53, §5.3.2].

Thus, the sequence in the bottom row of diagram (2.2) defines a morphism

$$\psi \colon Q_{d,m} \longrightarrow \mathrm{Quot}^{d-m}_{X/Y}(L) \tag{2.3}$$

The product of (2.3) and (2.1) gives the desired morphism $\Phi_{d,m} := (\psi, \varphi)$. □

2.2.3 Lifting Lemma

Before addressing part (ii) of Theorem 2.2.1, let us take a closer look at the morphism $\Phi_{d,m}$. By construction, $\Phi_{d,m}$ takes a length d quotient $[E \to F]$ to the induced quotients

$[L \to B]$ and $[E' \to F']$, of respective lengths $d-m$ and m. Intuitively, the fibers of $\Phi_{d,m}$ should record all possible ways of "gluing" these two pieces back together into a quotient of E of length d.

Formally, this situation is captured by the following diagram with two exact rows and one exact column over $X \times_Y \mathrm{Quot}^{d-m}_{X/Y}(L) \times_Y \mathrm{Quot}^{m}_{X/Y}(E')$. Here, we denote by p_1, p_2 the projections from $\mathrm{Quot}^{d-m}_{X/Y}(L) \times_Y \mathrm{Quot}^{m}_{X/Y}(E')$ to each factor.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \tilde{p}_2^*\mathcal{S}'_m \arrow[r] & p_X^*E' \arrow[r] \arrow[d] & \tilde{p}_2^*\mathcal{F}'_m \arrow[r] & 0 \\
 & & p_X^*E \arrow[d] & & \\
0 \arrow[r] & \tilde{p}_1^*\mathcal{A}_{d-m} \arrow[r] & p_X^*L \arrow[r] \arrow[d] & \tilde{p}_1^*\mathcal{B}_{d-m} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd} \tag{2.4}$$

The top and bottom rows encode the universal families of the Quot schemes $\mathrm{Quot}^{m}_{X/Y}(E')$ and $\mathrm{Quot}^{d-m}_{X/Y}(L)$. The data necessary to fill in the middle row corresponds precisely to a family of quotients of E. From this perspective, the fibers of the morphism $\Phi_{d,m}$ are controlled by all possible completions this diagram to a 3–by–3 diagram with exact rows and columns.

The following Lifting Lemma characterizes how the completions of diagram (2.4) arise. The result holds in full generality for any abelian category; a statement and proof in this general context are provided in Appendix A.

Lemma 2.2.3 (Lifting Lemma) **.** *Let $a\colon \tilde{p}_2^*\mathcal{S}'_m \to p_X^*E$ and $b\colon p_X^*E \to \tilde{p}_1^*\mathcal{B}_{d-m}$ denote the maps appearing in diagram (2.4) obtained via composition.*

(i) Diagram (2.4) induces a canonical short exact sequence

$$0 \longrightarrow \tilde{p}_2^*\mathcal{F}'_m \longrightarrow K_{d,m} \longrightarrow \tilde{p}_1^*\mathcal{A}_{d-m} \longrightarrow 0 \tag{2.5}$$

where $K_{d,m} := \ker(b)/im(a)$.

(ii) The splittings of (2.5) are in bijection with sub-objects $S \subseteq p_X^*E$ (up to equivalence) such that the inclusion of S into the diagram (2.4) extends it to a 3–by–3 diagram with exact rows and columns.

Remark 2.2.4. If a short exact sequence $0 \to A \to B \to C \to 0$ splits, the set of all possible splittings is a torsor under $\mathrm{Hom}(C, A)$.

Corollary 2.2.5. *A morphism $\tau\colon T \to \mathrm{Quot}^{d-m}_{X/Y}(L) \times_Y \mathrm{Quot}^{m}_{X/Y}(E')$ lifts to a morphism $\tau'\colon T \to Q_{d,m}$ such that $\tau = \Phi_{d,m} \circ \tau'$ if and only if the pullback of sequence (2.5) to $X \times_Y T$ splits. The set of all possible splittings is a torsor under $\tilde{\tau}^*(\tilde{p}_1^*\mathcal{A}^\vee_{d-m} \otimes \tilde{p}_2^*\mathcal{F}'_m)$.*

2.2.4 Proof of the inductive step

In this section we prove part (ii) of Theorem 2.2.1, establishing that $\Phi_{d,m}$ is a rank m affine vector bundle. The Lifting Lemma 2.2.3 indicates that the sheaf $\tilde{p}_1^*\mathcal{A}_{d-m}^\vee \otimes \tilde{p}_2^*\mathcal{F}'_m$ plays a key role in the structure of the morphism $\Phi_{d,m}$ by controlling its fibers.

For the purpose of the proof, the following notation is introduced.

Notation 2.2.6. Let us write

$$Z := \mathrm{Quot}^{d-m}_{X/Y}(L) \times_Y \mathrm{Quot}^m_{X/Y}(E'), \qquad \pi\colon X \times_Y Z \to Z.$$

On $X \times_Y Z$, denote by $\mathcal{A}$ and $\mathcal{F}$ the sheaves $\tilde{p}_1^*\mathcal{A}_{d-m}^\vee$ and $\tilde{p}_2^*\mathcal{F}'_m$ on Z defined by diagram (2.4). Note that $\mathcal{A}$ is a line bundle and $\mathcal{F}$ has fiberwise zero–dimensional support. In particular, $\pi_*(\mathcal{A}^\vee \otimes \mathcal{F})$ is locally free of rank m.
Let V be the geometric vector bundle associated to $\pi_*(\mathcal{A}^\vee \otimes \mathcal{F})$. By [53, Theorem 5.8], the vector bundle V carries a projection morphism $\nu\colon V \to Z$ and a universal homomorphism on $X \times_Y V$

$$\mu\colon \tilde{\nu}^*\mathcal{A} \to \tilde{\nu}^*\mathcal{F} \tag{2.6}$$

Given an open subscheme U of Z, we denote by V_U the preimage $\nu^{-1}(U)$, and by $\nu_U\colon V_U \to U$ the restriction of ν to V_U. We have the following cartesian diagram.

$$\begin{tikzcd}
V_U \arrow[r, hook, "\varepsilon"] \arrow[d, "\nu_U"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\nu"] \\
U \arrow[r, hook, "\iota"'] & Z
\end{tikzcd}$$

Finally, we denote by π_U the projection $X \times_Y U \to U$.

The statement of Theorem 2.2.1, part (ii) is local over Z. The first step towards its proof is to show that, affine locally over Z, there are no obstructions to completing the diagram (2.4). This is made precise by the following statement.

Lemma 2.2.7. *Assume notation 2.2.6 and the hypotheses of Theorem 2.2.1. Fix an affine open subscheme U of Z. Then the pullback of sequence (2.5) to $X \times_Y V_U$ splits.*

Proof. The splittings of the short exact sequence

$$0 \to \tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{F} \to \tilde{\varepsilon}^*\tilde{\nu}^*K_{d,m} \to \tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{A} \to 0$$

are controlled by the Ext group

$$\mathrm{Ext}^1_{X\times_Y V_U}(\tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{A}, \tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{F}) = \mathrm{Ext}^1_{X\times_Y V_U}(\tilde{\nu}_U^*\tilde{\iota}^*\mathcal{A}, \tilde{\nu}_U^*\tilde{\iota}^*\mathcal{F}).$$

Since $\mathcal{A}$ is a line bundle on $X \times_Y Z$, this reduces to a computation of the cohomology group $H^1(X \times_Y V_U, \tilde{\nu}_U^*\tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F}))$.
The higher projection formula [32, III, Exercise 8.3] gives

$$R^1(\tilde{\nu}_U)_*\tilde{\nu}_U^*\tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F}) = \tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F}) \otimes R^1(\tilde{\nu}_U)_*\mathcal{O}_{X\times_Y V_U} = 0$$

where the vanishing of $R^1(\tilde{\nu}_U)_*\mathcal{O}_{X\times_Y V_U}$ is ensured by the fact that $\tilde{\nu}_U$ has affine fibers. Apply [32, III, exercise 8.1] to $\tilde{\nu}_U$ to obtain an isomorphism

$$H^1(X \times_Y V_U, \tilde{\nu}_U^*\tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F})) \cong H^1(X \times_Y U, \tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F})).$$

and then again to π_U to deduce that

$$H^1(X \times_Y U, \tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F})) \cong H^1(U, (\pi_U)_*\tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F})) = 0$$

This time, the vanishing of the higher pushforward $R^1(\pi_U)_*\tilde{\iota}^*(\mathcal{A}^\vee \otimes \mathcal{F}) = \iota^* R^1\pi_*(\mathcal{A}^\vee \otimes \mathcal{F})$ is due to the fact that $\mathcal{F}$ has fiberwise zero–dimensional support. ☐

Following is the proof of Theorem 2.2.1, part (ii). In terms of notation 2.2.6, we show that $Q_{d,m}$ is isomorphic to the vector bundle V.

Proof of Theorem 2.2.1, part (ii). The statement is local over Z, so fix any affine open subscheme U of Z. By Lemma 2.2.7, the pullback of sequence (2.5) to $X \times_Y V_U$ splits. The set of all such splittings is a torsor under $\mathrm{Hom}(\tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{A}, \tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{F})$. Choose the splitting induced by the universal homomorphism (2.6), restricted to $X \times_Y V_U$:

$$\mu|_{X \times_Y V_U} \colon \tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{A} \to \tilde{\varepsilon}^*\tilde{\nu}^*\mathcal{F}.$$

By the Lifting Lemma 2.2.3, and in particular by Corollary 2.2.5, the choice of $\mu|_{X \times_Y V_U}$ determines a morphism $V_U \to Q_{d,m}$, which is an isomorphism onto its image and commutes with the natural projection to U. ☐

Remark 2.2.8. Bifet's original proof of Theorem 2.1.1 relies on the use of the Białynicki–Birula decomposition. In contrast, the proofs of Theorems 2.1.6 and 2.2.1 are entirely independent of this method and thus provide an alternative approach.

2.2.5 The case of non-filtered sheaves

In general, a locally free sheaf E on a curve $f \colon X \to Y$ does not admit a filtration by line bundles. However, it is possible to stratify Y so that E is filtered by line bundles on each stratum.

Lemma 2.2.9. *Let $f \colon X \to Y$ be a family of smooth projective curves over Y irreducible, and let E be a locally free sheaf of rank $r > 1$ on X. There is an open subscheme $U \subset Y$ such that E admits a filtration by line bundles on $f^{-1}(U)$.*

Proof. We may twist E by an ample line bundle $\mathcal{O}(1)$ on X so that $E(m) := E \otimes \mathcal{O}(m)$ is generated by global sections. Let $\eta \in Y$ be the generic point of Y and denote by X_η the corresponding curve over η. Then $E(m)|_{X_\eta}$ is a globally generated sheaf of rank $r \geq 2$. Since X_η is a curve, the top Chern class of $E(m)|_{X_\eta}$ vanishes, and thus a general section of $E(m)|_{X_\eta}$ on X_η is nowhere vanishing. Choose a section $s \in \Gamma(E(m))$ whose restriction to X_η is nowhere vanishing and write

$$U := f(X \setminus Z(s)) \subset Y, \qquad X_U := f^{-1}(U).$$

By definition, the generic point η lies in U and X_η is contained in X_U. On X_U, we then have an injective homomorphism

$$\mathcal{O}_{X_U}(-m) \xrightarrow{\cdot s} E|_{X_U}.$$

By applying induction on the rank of the cokernel, we are done. ☐

2.3 Applications

2.3.1 The Quot zeta function of a family of smooth curves

As an immediate application of Theorem 2.1.6, we obtain a closed formula for the class of the Quot scheme $\operatorname{Quot}^d_{X/Y}(E)$ in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_Y)$, which we restate here.

Corollary 2.1.7. *Let $f\colon X \to Y$ be a family of smooth projective curves and consider a locally free sheaf E on X, filtered by line bundles. For every $d \geq 0$, the following formula holds in $\mathrm{K}_0(\mathrm{Var}_Y)$.*

$$[\operatorname{Quot}^d_{X/Y}(E) \to Y] = \sum_{\alpha \in C_r(d)} \prod_{i=1}^{r} [\operatorname{Sym}^{\alpha_i}_Y(X) \to Y] \mathbb{L}_Y^{\alpha_i(i-1)} \tag{2.7}$$

Proof. By Theorem 2.1.6 the Quot scheme $\operatorname{Quot}^d_{X/Y}(E)$ admits a stratification

$$\operatorname{Quot}^d_{X/Y}(E) = \coprod_{\alpha \in C_r(d)} Q_{d,\alpha}$$

with each stratum $Q_{d,\alpha}$ isomorphic to an affine vector bundle of rank $\sum_i \alpha_i(i-1)$ over the product $\operatorname{Sym}^{\alpha_1}_Y(X) \times_Y \cdots \times_Y \operatorname{Sym}^{\alpha_r}_Y(X)$. Consequently, we have the following equality in $\mathrm{K}_0(\mathrm{Var}_Y)$

$$[Q_{d,\alpha} \to Y] = [\operatorname{Sym}^{\alpha_1}_Y(X) \times_Y \cdots \times_Y \operatorname{Sym}^{\alpha_r}_Y(X) \times_k \mathbb{A}^{\sum_i \alpha_i(i-1)} \to Y].$$

In order to express the right–hand side as a product of classes of Y–varieties, we must rewrite the affine factor as $\mathbb{A}_Y^{\sum_i \alpha_i(i-1)} = Y \times_k \mathbb{A}^{\sum_i \alpha_i(i-1)}$. By multiplicativity of $\mathrm{K}_0(\mathrm{Var}_Y)$ with respect to fiber product over Y, this yields the right–hand side of formula (2.7). □

Corollary 2.3.1. *Let $f\colon X \to Y$ be a family of smooth projective curves and consider a locally free sheaf E on X, filtered by line bundles. The Quot zeta function $Z_{X/Y}(E,t)$ factors into a product of relative motivic zeta functions.*

$$Z_{X/Y}(E,t) = \zeta_{X/Y}(t)\zeta_{X/Y}(\mathbb{L}_Y t) \ldots \zeta_{X/Y}(\mathbb{L}_Y^{\mathrm{rk}(E)-1} t).$$

Remark 2.3.2.

(i) The class $[\operatorname{Quot}^d_{X/Y}(E) \to Y]$ depends on the geometry of the family $f\colon X \to Y$, on the length d and on the rank of E. It is independent of the specific choice of sheaf E, as well as of the choice of filtration.

(ii) Theorem 2.1.6, Corollaries 2.1.7 and 2.3.1 apply to the case of smooth projective k–curves, as any locally free sheaf on a smooth curve can be filtered by line bundles. Thus, by setting $Y = \operatorname{Spec}(k)$, we recover Bifet's formula (Theorem 2.1.2), as well as the factorization of the Quot zeta function over a smooth curve into a product of motivic zeta functions (Corollary 2.1.3). The advantage of our approach is that it does not rely on the use of Białynicki–Birula decompositions.

(iii) If $\upsilon\colon Y \to \operatorname{Spec}(k)$ denotes the structure morphism, the pushforward of formula (2.7) along υ (see Remark 1.2.5) yields the class of $\operatorname{Quot}^d_{X/Y}(E)$ as a k–scheme in $\mathrm{K}_0(\mathrm{Var}_k)$.

Since the pushforward between Grothendieck rings is not a homomorphism and generally does not preserve products, the pushforward yields

$$\upsilon_*[\mathrm{Quot}^d_{X/Y}(E) \to Y] = [\mathrm{Quot}^d_{X/Y}(E)] = \sum_{\alpha \in C_r(d)} \left[\prod_{i=1}^r \mathrm{Sym}^{\alpha_i}_Y(X)\right] \mathbb{L}^{\alpha_i(i-1)} \tag{2.8}$$

Here, $\mathbb{L}$ denotes the usual Lefschetz class $[\mathbb{A}^1]$.

Theorem 2.3.3. *Let $f\colon Y \times C \to Y$ be a trivial family of smooth projective curves, and consider a locally free sheaf E on $Y \times C$. The following equality holds in* $\mathrm{K}_0(\mathrm{Var}_k)$*.*

$$[\mathrm{Quot}^d_{Y\times C/Y}(E)] = [Y][\mathrm{Quot}^d_C(\mathcal{O}^{\oplus \mathrm{rk}(E)})]$$

Proof. Suppose that E has a filtration, so that we can apply equation (2.8). The relative symmetric product $\mathrm{Sym}^d_Y(Y \times C)$ is canonically isomorphic to $\mathrm{Sym}^d(C) \times Y$. It follows immediately that

$$\begin{aligned}
\left[\prod_{i=1}^r \mathrm{Sym}^{\alpha_i}_Y(Y \times C)\right] &= [\mathrm{Sym}^{\alpha_1}_Y(Y \times C) \underset{Y}{\times} \cdots \times_Y \mathrm{Sym}^{\alpha_n}_Y(Y \times C)] \\
&= [(\mathrm{Sym}^{\alpha_1}(C) \times Y) \underset{Y}{\times} \dots \underset{Y}{\times} (\mathrm{Sym}^{\alpha_r}(C) \times Y)] \\
&= [\mathrm{Sym}^{\alpha_1}(C) \times \cdots \times \mathrm{Sym}^{\alpha_r}(C) \times Y] \\
&= [\mathrm{Sym}^{\alpha_1}(C)] \dots [\mathrm{Sym}^{\alpha_r}(C)][Y].
\end{aligned}$$

If E is not filtered, proceed by induction on $\dim(Y)$. Suppose that Y has dimension $\dim(Y) = m$, and let $Y_1, \dots, Y_s$ be the m–dimensional irreducible components of Y. Define Y_i° to be the open subset of Y_i that does not meet the other irreducible components.

$$Y_i^\circ := Y_i \setminus \bigcup_{j \neq i} Y_j.$$

Apply Lemma 2.2.9 to each Y_i° to get an open subset $U_i \subset Y_i^\circ$ where E is filtered, and thus the statement holds. Define U to be the union of all such U_i. Then the statement holds on U, as well as on its complement $Z := Y \setminus U$, where we may apply induction since $\dim(Z) < \dim(Y)$. To conclude:

$$\begin{aligned}
[\mathrm{Quot}^d_{Y\times C/Y}(E)] &= [\mathrm{Quot}^d_{Y\times C/Y}(E) \times_Y U] + [\mathrm{Quot}^d_{Y\times C/Y}(E) \underset{Y}{\times} Z] \\
&= [\mathrm{Quot}^d_{U\times C/U}(E|_{U\times C})] + [\mathrm{Quot}^d_{Z\times C/Z}(E|_Z)] \\
&= [\mathrm{Quot}^d_C(E)][Y].
\end{aligned}$$

□

2.3.2 Class of the nested Quot scheme

Let $\underline{d} = [d_1, \dots, d_s]$ be a tuple of integers such that $0 \leq d_1 \leq \cdots \leq d_s$ and consider the nested Quot scheme $\mathrm{Quot}^{\underline{d}}_{X/Y}(E)$, introduced in Section 1.1.5. We use Theorem 2.3.3 to compute its class in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_k)$.

Proposition 2.3.4. *Let $0 \leq d_1 \leq d_2$ be two non-negative integers and consider a trivial family $f\colon Y \times C \to Y$ of smooth projective curves. Write $Q := \mathrm{Quot}^{d_1}_{Y\times C/Y}(E)$. Then there is a natural isomorphism*

$$\mathrm{Quot}^{[d_1,d_2]}_{Y\times C/Y}(E) \cong \mathrm{Quot}^{d_2-d_1}_{Q\times C/Q}(\mathcal{S}_1)$$

where $0 \to \mathcal{S}_1 \to p^*_{Y\times C}E \to \mathcal{F}_1 \to 0$ denotes the universal sequence of $\mathrm{Quot}^{d_1}_{Y\times C/Y}(E)$.

Theorem 2.3.5. *Let $\underline{d} = [d_1, \ldots, d_s]$ be a tuple of integers with $d_0 := 0 \leq d_1 \leq \cdots \leq d_s$. Consider a trivial family $f\colon Y \times C \to Y$ of smooth projective curves and a locally free sheaf E on $Y \times C$. Then the following formula holds in $\mathrm{K}_0(\mathrm{Var}_k)$.*

$$[\mathrm{Quot}^{\underline{d}}_{Y\times C/Y}(E)] = [Y]\prod_{k=1}^{s}[\mathrm{Quot}^{d_k-d_{k-1}}_{C}(\mathcal{O}^{\oplus \operatorname{rk}(E)})]$$

Proof. Proceed by induction over the number s of quotients in the nesting, the case $s = 1$ being solved by Corollary 2.3.3. In general, for any $s \geq 2$, Proposition 2.3.4 implies that

$$\mathrm{Quot}^{[d_1,\ldots,d_s]}_{Y\times C/Y}(E) \cong \mathrm{Quot}^{d_s-d_{s-1}}_{X'/Y'}(\mathcal{S}_{s-1})$$

where $Y' = \mathrm{Quot}^{[d_1,\ldots,d_{s-1}]}_{Y\times C/Y}(E)$ and $X' = Y' \times C$. We apply Corollary 2.3.3 and induction to conclude:

$$\begin{aligned}[\mathrm{Quot}^{[d_1,\ldots,d_s]}_{Y\times C/Y}(E)] &= [\mathrm{Quot}^{d_s-d_{s-1}}_{Y'\times C/Y'}(\mathcal{S}_{s-1})] = \\ &= [Y'][\mathrm{Quot}^{d_s-d_{s-1}}_{C}(\mathcal{O}^{\oplus \operatorname{rk}(E)})] \\ &= [Y]\prod_{k=1}^{s}[\mathrm{Quot}^{d_k-d_{k-1}}_{C}(\mathcal{O}^{\oplus \operatorname{rk}(E)})].\end{aligned}$$

□

Remark 2.3.6. When $Y = \mathrm{Spec}(k)$, Theorem 2.3.5 and Corollary 2.1.7 together recover the formula in Monavari–Ricolfi [50] without the use of Białynicki–Birula.

2.3.3 Class of the Quot scheme of positive rank quotients

Fix integers $1 \leq r' \leq r$ and consider the Quot scheme $\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r')$ of rank r', degree d quotients of $\mathcal{O}^{\oplus r}$ on a smooth projective curve C, introduced in Section 1.1.5. In this section, we apply Theorem 2.3.3 to prove the following statement.

Proposition 2.3.7. *For any smooth projective curve C, the class $[\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r')]$ in $\mathrm{K}_0(\mathrm{Var}_k)$ is divisible by the class $[\mathrm{Gr}(r', r)]$ of the Grassmannian.*

Remark 2.3.8.

(i) For $C = \mathbb{P}^1$, the divisibility of the Poincaré polynomial of $\mathrm{Quot}^d_{\mathbb{P}^1}(\mathcal{O}^{\oplus r}, r')$ by that of $\mathrm{Gr}(r', r)$ is due to Linda Chen [14, Theorem 1]. Her proof relies on the Białynicki–Birula decomposition.

(ii) If there were a Zariski locally trivial fibration $\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r') \to \mathrm{Gr}(r', r)$, then the divisibility stated in Proposition 2.3.7 would follow immediately. However, such a morphism does not exist in general, as shown in the example below.

Example 2.3.9. Set $d = 1$, $r' = 1$, $r = 2$. Over $C = \mathbb{P}^1$ one has $\mathrm{Quot}^1_{\mathbb{P}^1}(\mathcal{O}^{\oplus 2}, 1) \cong \mathbb{P}^3$ and $\mathrm{Gr}(1, 2) \cong \mathbb{P}^1$. Any morphism $\mathbb{P}^3 \to \mathbb{P}^1$ is constant.

Nevertheless, the divisibility of Proposition 2.3.7 can be realized via a simple geometric description: a stratification of the Quot scheme $\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r')$, where each stratum carries an evaluation morphism to $\mathrm{Gr}(r', r)$.

Definition 2.3.10. Let e be a nonnegative integer. We denote by $M_e := \mathrm{Mor}_e(C, \mathrm{Gr}(r', r))$ the *space of degree e morphisms* $\varphi\colon C \to \mathrm{Gr}(r', r)$, as in [6]. For any point $p \in C$, the space $\mathrm{Mor}_e(C, \mathrm{Gr}(r', r))$ is equipped with a canonical *evaluation* morphism

$$\mathrm{ev}_p\colon \mathrm{Mor}_e(C, \mathrm{Gr}(r', r)) \to \mathrm{Gr}(r', r), \quad \varphi \mapsto \varphi(p).$$

Remark 2.3.11. The space of degree e morphisms $\varphi\colon C \to \mathrm{Gr}(r', r)$ admits a canonical action by automorphisms of the Grassmannians via postcomposition.

Lemma 2.3.12. *The evaluation morphism ev_p is a Zariski locally trivial fibration.*

Proof. Acting by the automorphisms of $\mathrm{Gr}(r', r)$, one sees all fibers of ev_p are isomorphic. Fix a point $w_0 \in \mathrm{Gr}(r', r)$ and denote by Y_e the fiber $\mathrm{ev}_p^{-1}(w_0)$.

The Grassmannian $\mathrm{Gr}(r', r)$ can be realized as a quotient of $\mathrm{GL}(r)$ by a parabolic subgroup P of $\mathrm{GL}(r)$ stabilizing w_0, as in [10, §10.3, §11.1]. The quotient map

$$\pi\colon \mathrm{GL}(r) \to \mathrm{Gr}(r', r), \qquad \pi(g) := g \cdot w_0$$

is a principal P–bundle. Since P is special, in the sense of Serre [59, §4.1], π is Zariski locally trivial. Hence, over any open subset $U \subset \mathrm{Gr}(r', r)$ where π trivializes, there exists a local section $s_U : U \to \pi^{-1}(U)$. Note that s_U being a section implies that

$$\pi \circ s_U(u) = s_U(u) \cdot w_0 = u \qquad \forall u \in U.$$

Next, let $\mathrm{GL}(r)$ act on M_e by $(g \cdot \varphi)(t) := g \cdot \varphi(t)$ for all $t \in C$. To conclude, we define a trivialization of ev_p over U by setting

$$\begin{aligned} &a\colon U \times Y_e \to \mathrm{ev}_p^{-1}(U), && a(u, \varphi) := s_U(u)\varphi, \\ &b\colon \mathrm{ev}_p^{-1}(U) \to U \times Y_e, && b(\varphi) := (\mathrm{ev}_p(\varphi), s_U(\mathrm{ev}_p(\varphi))^{-1}\varphi). \end{aligned}$$

One easily verifies that these two maps are inverses to each other. □

Proof of Proposition 2.3.7. Denote by $0 \to \mathcal{S} \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0$ the universal sequence on $\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r') \times C$. Consider the natural stratification of the Quot scheme $\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r')$

$$\mathrm{Quot}^d_C(\mathcal{O}^{\oplus r}, r') = \coprod_{e=0}^{d} Q_e$$

where each stratum Q_e parametrizes quotients $[\mathcal{O}^{\oplus r} \twoheadrightarrow F]$ such that F fits into a short exact sequence

$$0 \to T \to F \to B \to 0$$

with B a locally free of rank $r - r'$ and degree $\deg(B) = e$, and T a torsion sheaf of degree $\deg(T) = d - e$. We denote by $\mathcal{S}_e$ the restriction of the universal subsheaf $\mathcal{S}$ to $Q_e \times C$

and by $\mathcal{F}_e$, respectively, the restriction of the universal quotient sheaf $\mathcal{F}$ to $Q_e \times C$. The sheaf

$$\mathcal{T} := \mathcal{E}xt^1(\mathcal{E}xt^1(\mathcal{F}_e, \mathcal{O}_{Q_e \times C}), \mathcal{O}_{Q_e \times C})$$

is the torsion subsheaf of $\mathcal{F}_e$; that is to say, it parametrizes the torsion sheaves of degree $d-e$ on C; the corresponding cokernel $\mathcal{B} := \operatorname{coker}(\mathcal{T} \to \mathcal{F}_e)$ parametrizes locally free sheaves of degree e on C. Then, on $Q_e \times C$, we have the following diagram with exact rows and columns

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{S}_e \arrow[r] \arrow[d, equal] & \widetilde{\mathcal{S}} \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{S}_e \arrow[r] & \mathcal{O}^{\oplus r} \arrow[r] \arrow[d] & \mathcal{F}_e \arrow[r] \arrow[d] & 0 \\
& & \mathcal{B} \arrow[r, equal] \arrow[d] & \mathcal{B} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

This diagram implies the existence of an isomorphism

$$Q_e \cong \operatorname{Quot}^{d-e}_{M_e \times C/M_e}(\widetilde{\mathcal{S}})$$

where M_e is the space of degree e morphisms $\varphi \colon C \to \operatorname{Gr}(r', r)$ from Definition 2.3.10. Applying Theorem 2.3.3 to Q_e gives

$$[Q_e] = [\operatorname{Mor}_e(C, \operatorname{Gr}(r', r))][\operatorname{Quot}^{d-e}_{\mathcal{O}^{\oplus r}}(C)].$$

Thus, by Lemma 2.3.12, we obtain the following formula in $K_0(\mathrm{Var}_k)$

$$[\operatorname{Quot}^d_C(\mathcal{O}^{\oplus r}, r')] = [\operatorname{Gr}(r', r)] \sum_{e=0}^{d} [Y_e] \cdot [\operatorname{Quot}^{d-e}_{\mathcal{O}^{\oplus r}}(C)]. \qquad \square$$

CHAPTER 3

Power structure formulas for Quot schemes of smooth morphisms

While Chapter 2 explored the geometry of Quot schemes over a family of smooth curves, this chapter extends the discussion to smooth morphisms of arbitrary relative dimension. In this broader setting, we investigate the local structure of the Quot–to–Sym morphism and establish two power structure formulas for Quot schemes of smooth morphisms.

3.1 Overview and main results

As the dimension of the underlying space increases, the geometry of Hilbert and Quot schemes becomes progressively more intricate and difficult to study. Even for line bundles, the Hilbert scheme $\mathrm{Hilb}^d(X)$ of a smooth quasiprojective variety X is known to be smooth for low degrees $d \leq 3$ [19]; in degree $d = 4$, the Hilbert scheme of $\mathbb{P}^3$ is already singular. For this reason, it is natural to focus on Quot scheme invariants and ways to compute them.

We therefore turn our attention to the Quot zeta function, and in particular to the use of power structures as a tool for its study. In [58], Ricolfi studied the Quot zeta function for locally free sheaves on a smooth quasiprojective scheme, establishing a formula that recovers the Quot zeta function from its punctual counterpart via the power structure on the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_k)$. Specializing to rank one, this formula recovers the one proved by Gusein–Zade, Luengo and Melle–Hernandez [28] building on computations by Göttsche [25].

Theorem 3.1.1 ([28], [58]) **.** *Consider a smooth quasiprojective variety X of dimension $m \geq 1$ and let E be a locally free sheaf on X. Fix a point $p \in X$. Then*

$$Z_X(E,t) = Z_X(E,t)_p^{[X]}.$$

This identity reflects the fact that punctual Quot schemes form the basic building blocks of Quot schemes: a length d quotient of a locally free sheaf E on X is the datum of finitely many punctual quotients whose lengths sum up to the total length d. The power structure on $\mathrm{K}_0(\mathrm{Var}_k)$ provides the right framework to express this *local–to–global* behavior; see Example 1.1.6 and Remark 1.2.22.

A direct consequence of Theorem 3.1.1 is that for any smooth variety X the class $[\mathrm{Quot}^d_X(E)]$ depends on E only up to the rank, which provides an immediate generalization of Bagnarol, Fantechi and Perroni's independence result [3] from smooth curves to smooth varieties of arbitrary dimension.

Another example of power structure formula is related to the Grothendieck ring of maps $\mathrm{K}_0(\mathrm{Map}_k)$, introduced in [31] (see also Section 1.2.3). This ring, whose elements are equivalence classes of morphisms of k–varieties, carries a natural power structure; Gusein–Zade, Luengo, and Melle–Hernández used it to formulate a power structure identity for the generating function of Hilbert–Chow morphisms, again expressing global geometry in terms of local data.

Theorem 3.1.2 ([31, Theorem 3]) . *Let X be a smooth quasiprojective variety of dimension m. Then the following formula holds in the formal power series ring* $\mathrm{K}_0(\mathrm{Map}_k)[\![t]\!]$.

$$\sum_{d\geq 0}[\mathrm{Hilb}^d(X)\to \mathrm{Sym}^d(X)]t^d = \left(\sum_{d\geq 0}[\mathrm{Hilb}^d(\mathbb{A}^m)_0 \to \mathrm{Spec}(k)]t^d\right)^{[X\xrightarrow{\mathrm{id}}X]}$$

Here, the notation is to be interpreted according to Definition 1.2.27 and 1.2.30.

The power structures on the Grothendieck rings $\mathrm{K}_0(\mathrm{Var}_k)$ and $\mathrm{K}_0(\mathrm{Map}_k)$ extend seamlessly to their relative counterparts, $\mathrm{K}_0(\mathrm{Var}_Y)$ and $\mathrm{K}_0(\mathrm{Map}_Y)$. This raises the natural question of whether the power structure formulas stated above continue to hold in the relative setting. In more precise terms, we ask:

Question 3.1.3. *Given a smooth morphism $f\colon X\to Y$ and a locally free sheaf E on X, is it possible to extend Theorem 3.1.1 and Theorem 3.1.2 to Quot schemes of smooth morphisms $f\colon X\to Y$?*

The first step in approaching this question is to identify which Y–object takes on the role of the punctual Quot scheme. When $Y=\mathrm{Spec}(k)$, the structure of the punctual Quot scheme is determined étale locally by the pair $(\mathbb{A}^m,\mathcal{O}^{\oplus r})$, see Theorem 1.1.17. Building on this result, we show that the stratification of the Quot–to–Sym morphism by integer partitions (Section 1.1.4) is étale locally trivial with explicit fiber.

Theorem 3.1.4. *Let $f\colon X\to Y$ be a smooth morphism and E a locally free sheaf of rank r on X. Let $m=\dim(X)-\dim(Y)$. Then for all partitions ω of d the restriction of the Quot–to–Sym morphism*

$$\sigma^{\omega}_{X/Y}\colon \mathrm{Quot}^{\omega}_{X/Y}(E)\to \mathrm{Sym}^{\omega}_Y(X)$$

is an étale locally trivial fibration with fiber $\prod_i \mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})_0^{\omega_i}$.

As an application of this result, we establish two power structure formulas generalizing Theorem 3.1.1 and Theorem 3.1.2.

Theorem 3.1.5. *Consider a smooth morphism $f\colon X\to Y$ of relative dimension m and let E be a locally free sheaf on X. The following identity holds in* $\mathrm{K}_0(\mathrm{Var}_Y)[\![t]\!]$.

$$Z_{X/Y}(E,t)=\left(\sum_{d\geq 0}[\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\otimes\,\mathrm{rk}(E)})_0\times Y\to Y]t^d\right)^{[X\to Y]}.$$

The right–hand side of the formula is to be interpreted via the power structure on $\mathrm{K}_0(\mathrm{Var}_Y)$ of Definition 1.2.20.

As a corollary, Theorem 3.1.5 implies that the class $[\mathrm{Quot}^d_{X/Y}(E)\to Y]$ is dependent on the morphism f, on the length d, while it depends on the sheaf E only via its rank. This is the content of Corollary 3.3.2; see Remark 2.3.2 for the same statement when the relative dimension of the morphism f is $m = 1$.

Theorem 3.1.6. *For any smooth quasiprojective morphism $f\colon X \to Y$ of relative dimension m and locally free sheaf E on X, the following formula holds in the formal power series ring $\mathrm{K}_0(\mathrm{Map}_Y)[\![t]\!]$.*

$$\sum_{d\geq 0}[\mathrm{Quot}^d_{X/Y}(E) \xrightarrow{\sigma^d_{X/Y}} \mathrm{Sym}^d_Y(X)]t^d = \left(\sum_{d\geq 0}[\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\otimes \mathrm{rk}(E)})_0 \times Y \to Y]t^d\right)^{[X \xrightarrow{\mathrm{id}} X]}$$

The notation in this formula is to be interpreted using the Grothendieck ring of maps and its power structure, introduced in Section 1.2.3.

3.2 Main constructions and proofs

The power structure on the Grothendieck ring provides a natural framework for expressing the local–to–global nature of Quot schemes. This connection rests on two geometric ingredients that explain why power structures interact so well with the geometry of Quot schemes.

The first is the existence of rational étale maps between Quot schemes induced by étale maps between the base ambient spaces (Lemma 3.2.3). These maps serve a dual purpose: besides allowing comparisons between Quot schemes in terms of the étale local topology of the base, they also provide a gluing mechanism for quotients supported on distinct loci.

The second ingredient is the Quot–to–Sym morphism, which makes it possible to extract quotients whose support is distributed in a prescribed way. In particular, we use the stratification of symmetric powers by integer partitions, introduced in Section 1.1.4, and the corresponding induced stratification on the Quot scheme.

These two elements come together to prove Theorem 3.1.4, which is the key geometric mechanism behind the power structure formulas stated in Theorems 3.1.5 and 3.1.6.

The remainder of the chapter is organized as follows.

- Section 3.2.1 introduces the notion of étale maps between Quot schemes, extending the construction carried out in [4] to the relative setting.
- Section 3.2.2 studies the restriction of the Quot–to–Sym morphism to the strata determined by integer partitions, and proves that these restrictions are étale locally trivial fibrations.
- Finally, Section 3.3 combines these geometric results to establish the two power structure formulas. In particular, Section 3.3.1 proves the power structure formula for Quot schemes, while Section 3.3.2 establishes the formula for the Quot–to–Sym morphism.

3.2.1 Étale morphisms and induced maps on Quot schemes

Let $g\colon X \to X'$ be an étale morphism of Y–schemes, and assume that X' is proper. Given a locally free sheaf E' on X', set $E := g^*E'$. The aim of this section is to construct an étale map

$$g_+\colon \mathrm{Quot}^d_{X/Y}(E) \dashrightarrow \mathrm{Quot}^d_{X'/Y}(E')$$

which is defined on an open subscheme of $\mathrm{Quot}^d_{X/Y}(E)$. In the absolute case $Y = \mathrm{Spec}(k)$, the construction of g_+ is carried out in [4, Appendix A]; here we adapt it to an arbitrary base scheme Y.

Notation 3.2.1. Given a point $y \in Y$, we denote by $g_y \colon X_y \to X'_y$ the restriction of g to the fibers over y. For a sheaf E on X, we denote by E_y its restriction to X_y.

Following [21, §5.1], we introduce the notion of injective locus.

Definition 3.2.2. Let $g \colon X \to X'$ be an étale morphism of Y–schemes. The *injective locus of X^d (with respect to g)* is the set of tuples of points mapped injectively under g. In other words,

$$X^d_{inj} := \{(x_1, \dots, x_d) \in X^d \mid g(x_i) = g(x_j) \Rightarrow x_i = x_j\}.$$

The injective locus X^d_{inj} is open in X^d and $\mathfrak{S}_d$–invariant. We shall denote by $\mathrm{Sym}^d_Y(X)_{inj}$ its image in $\mathrm{Sym}^d_Y(X)$, which is again an open subscheme. Finally, the *injective locus of* $\mathrm{Quot}^d_{X/Y}(E)$ is defined as the open subscheme

$$\mathrm{Quot}^d_{X/Y}(E)_{inj} := (\sigma^d_{X/Y})^{-1}(\mathrm{Sym}^d_Y(X)_{inj})$$

Its points correspond to quotients $[E \twoheadrightarrow F]$ such that g is injective on $\mathrm{Supp}_{set}(F)$.

Lemma 3.2.3. *Consider an étale morphism $g \colon X \to X'$ of quasiprojective Y–schemes. Fix a coherent sheaf E' on X' and let $E := g^*E'$. Then there exists an étale morphism*

$$g_+ \colon\ \mathrm{Quot}^d_{X/Y}(E)_{inj} \to \mathrm{Quot}^d_{X'/Y}(E')$$

which maps a point $[E_y \to F]$ to $[E'_y \to (g_y)_* g_y^* E'_y \to (g_y)_* F]$.

Proof. For $Y = \mathrm{Spec}(k)$, the construction of g_+ is carried out in [4, Appendix A]. The same argument extends to the relative case. First of all, given a point $[q \colon E_y \to F]$ in the injective locus $\mathrm{Quot}^d_{X/Y}(E)_{inj}$ observe that the surjectivity of

$$g_+(q) := [E'_y \to (g_y)_* g_y^* E'_y \to (g_y)_* F] \tag{3.1}$$

can be checked by applying [4, Lemma A.1] to g_y, which is étale by base change.

Next, we recall the main points of [4, Proposition A.2], where one assumes X' proper. The aim is to extend the assignment $q \mapsto g_+(q)$ to an open neighborhood of q. Consider the following diagram with cartesian squares.

$$\begin{tikzcd}
X_y = X \times_Y \{q\} \arrow[r, hook, "i"] \arrow[d, "g_y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_Y \mathrm{Quot}^d_{X/Y}(E) \arrow[r, "p_X"] \arrow[d, "g_Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
X'_y = X' \times_Y \{q\} \arrow[r, hook, "j"'] & X' \times_Y \mathrm{Quot}^d_{X/Y}(E) \arrow[r, "p_{X'}"'] & X'
\end{tikzcd}$$

Let $p_X^* E \twoheadrightarrow \mathcal{F}$ be the universal quotient of $\mathrm{Quot}^d_{X/Y}(E)$; pushing forward along g_Q yields

$$(g_Q)_* p_X^* E \to (g_Q)_* \mathcal{F}.$$

Flat base change ensures that we have an isomorphism $(g_Q)_* p_X^* \cong p_{X'}^* g_*$, so we may compose $(g_Q)_* p_X^* E \to (g_Q)_* \mathcal{F}$ with the adjoint homomorphism $E' \to g_* g^* E'$. This produces a (canonical) homomorphism on $X' \times_Y \mathrm{Quot}^d_{X/Y}(E)$

$$\vartheta \colon p_{X'}^* E' \to p_{X'}^* g_* g^* E' \to (g_Q)_* \mathcal{F}.$$

The pullback of ϑ along j yields the point (3.1). Here, up to further restricting X to an open neighborhood of $\mathrm{Supp}(F)$, one may assume that g is affine, so that $j^*(g_Q)_* \cong (g_y)_* i^*$.

The locus where ϑ is surjective is an open subscheme U of $\mathrm{Quot}^d_{X/Y}(E)$; in fact, U contains the injective locus $\mathrm{Quot}^d_{X/Y}(E)_{inj}$. The restriction of ϑ to $X \times_Y U$ yields a morphism $g_+\colon U \to \mathrm{Quot}^d_{X'/Y}(E')$. The fact that g_+ is étale, and that the construction extends to the case where X' is quasiprojective follows exactly as in [4], up to replacing fiber products over k with fiber products over Y. □

Lemma 3.2.4. *Consider an étale morphism $g\colon X \to X'$ of quasiprojective Y-schemes. Fix a coherent sheaf E' on X' and let $E = g^*E'$. Then the following diagram is cartesian.*

$$\begin{tikzcd}
\mathrm{Quot}^d_{X/Y}(E)_{inj} \arrow[r, "g_+"] \arrow[d, "\sigma^d_{X/Y}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{X'/Y}(E') \arrow[d, "\sigma^d_{X'/Y}"] \\
\mathrm{Sym}^d_Y(X)_{inj} \arrow[r, "S^d(g)"] & \mathrm{Sym}^d_Y(X')
\end{tikzcd} \tag{3.2}$$

Proof. Consider the diagram below.

$$\begin{tikzcd}
\mathrm{Quot}^d_{X/Y}(E)_{inj} \arrow[dr, "w"] \arrow[drr, bend left, "g_+"] \arrow[ddr, bend right, "\sigma^d_{X/Y}"'] & & \\
& W \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{X'/Y}(E') \arrow[d, "\sigma^d_{X'/Y}"] \\
& \mathrm{Sym}^d_Y(X)_{inj} \arrow[r, "S^d(g)"] & \mathrm{Sym}^d_Y(X')
\end{tikzcd}$$

The morphism $S^d(g)$ is étale; by base change, so is the morphism π_2. Together with the étaleness of g_+, we deduce that w is étale as well. To show that w is an isomorphism, it remains to verify that it is bijective on closed points.

Consider two points $[q\colon E_y \twoheadrightarrow F]$ and $[q'\colon E_y \to F']$ such that $w(q) = w(q')$. The projection of $w(q) = w(q')$ via π_1 gives an equality of support cycles

$$\sum_{p \in \mathrm{Supp}(F)} \mathrm{length}_{\mathcal{O}_{X,p}}(F_p)p = \sum_{p \in \mathrm{Supp}(F')} \mathrm{length}_{\mathcal{O}_{X,p}}(F'_p)p.$$

In particular, the underlying sets of $\mathrm{Supp}(F)$ and $\mathrm{Supp}(F')$ must coincide. On the other hand, the projection via π_2 gives an equality of equivalence classes

$$[E'_y \to (g_y)_* g_y^* E'_y \to (g_y)_* F] = [E'_y \to (g_y)_* g_y^* E'_y \to (g_y)_* F'].$$

This corresponds to the existence of an isomorphism $\varphi\colon (g_y)_*F \xrightarrow{\sim} (g_y)_*F'$ compatible with the maps from E'_y. The domain of g_y may be restricted in such a way that g_y is an open embedding around $\mathrm{Supp}(F)$; this can be achieved by removing the points of $g_y^{-1}(g_y(\mathrm{Supp}(F)))$ which do not lie in $\mathrm{Supp}(F)$. Then g_y is an immersion around the support, which ensures that the canonical maps

$$g_y^*(g_y)_*F \xrightarrow{\sim} F \qquad g_y^*(g_y)_*F' \xrightarrow{\sim} F'$$

are isomorphisms. The pullback of φ along g_y gives an isomorphism $F \xrightarrow{\sim} F'$ compatible with the maps from E_y, thus proving injectivity.

To prove surjectivity, pick a point $([q\colon E'_y \to F], Z)$ in W, consisting of a quotient $[q\colon E'_y \to F]$ supported on X'_y and a 0–cycle Z in the injective locus $\mathrm{Sym}^d_Y(X)_{inj}$; the two objects are related by the condition

$$S^d(g)(Z) = \sum_{p \in \mathrm{Supp}(F)} \mathrm{length}_{\mathcal{O}_{X',p}}(F_p)p.$$

Write $Z = \sum_{p\in\Gamma} \lambda_p p$ for a finite set Γ, and note that $g_y(\Gamma)$ coincides with the underlying set of $\mathrm{Supp}(F)$. Consider the quotient $[g_y^* q\colon E_y \to g_y^* F]$ supported on X_y. Because g_y is flat, we have an isomorphism of schemes $\mathrm{Supp}(g_y^* F) \cong g_y^{-1}(\mathrm{Supp}(F))$. Up to restricting X_y to an open neighborhood of Γ, we may assume that $g_y^{-1}(g_y(\Gamma)) = \Gamma$ so that g_y is an open embedding around Γ. Moreover, the lengths of nonzero stalks are preserved under étale pullback

$$\mathrm{length}_{\mathcal{O}_{X,p}}(g^* F)_p = \mathrm{length}_{\mathcal{O}_{X',g(p)}}(F_{g(p)}).$$

This shows that the support cycle of $g_y^* F$ coincides with Z. Therefore, $[E_y \to g_y^* F]$ is a well defined point of $\mathrm{Quot}^d_{X/Y}(E)_{inj}$ which maps to $([E'_y \to F], Z)$ under w. □

3.2.2 Local structure of the Quot–to–Sym morphism

The aim of this section is to analyze the Quot–to–Sym morphism $\sigma^d_{X/Y}\colon \mathrm{Quot}^d_{X/Y}(E) \to \mathrm{Sym}^d_Y(X)$ in relation to the stratification by partitions introduced in Section 1.1.4. We shall denote by

$$\sigma^\omega_{X/Y}\colon \mathrm{Quot}^\omega_{X/Y}(E) \to \mathrm{Sym}^\omega_Y(X) \tag{3.3}$$

the restriction of $\sigma^d_{X/Y}$ to the strata indexed by a partition ω of d.

We start by analyzing the case of affine spaces. When $X = \mathbb{A}^m_Y$, the restriction (3.3) to the deepest stratum, corresponding to the trivial partition $(d) = (d^1)$ of d, admits a global product decomposition.

Lemma 3.2.5. *Let Y be a k–scheme. Then the restriction of the Quot–to–Sym morphism*

$$\sigma^{(d)}_{\mathbb{A}^m_Y/Y}\colon \mathrm{Quot}^{(d)}_{\mathbb{A}^m_Y/Y}(\mathcal{O}^{\oplus r}) \to \mathbb{A}^m_Y$$

is a trivial fibration with fiber $\mathrm{Quot}^d_{A^m}(\mathcal{O}^{\oplus r})_0$.

Proof. Since $\mathbb{A}^m_Y = \mathbb{A}^m_k \times Y$, the Quot–to–Sym morphism $\sigma^d_{\mathbb{A}^m_Y/Y}$ factors as a product map $\sigma^d_{\mathbb{A}^m/k} \times \mathrm{id}_Y$; see Proposition 1.1.7 and Remark 1.1.10. This factorization is compatible with the stratification by partitions ω of d, meaning that for each ω we have a map

$$\sigma^\omega_{\mathbb{A}^m_Y/Y} = \sigma^\omega_{\mathbb{A}^m} \times \mathrm{id}_Y\colon \mathrm{Quot}^\omega_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}) \times Y \to \mathrm{Sym}^\omega(\mathbb{A}^m) \times Y.$$

Thus, the statement reduces to the absolute case: using translations in $\mathbb{A}^m$, one sees that the closed stratum $\mathrm{Quot}^{(d)}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})$ admits a global decomposition

$$\mathrm{Quot}^{(d)}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}) \cong \mathrm{Quot}^d_{A^m}(\mathcal{O}^{\oplus r})_0 \times \mathbb{A}^m$$

and that the morphism $\sigma^{(d)}_{\mathbb{A}^m}$ coincides with the projection to the second factor. □

For a general morphism $f\colon X \to Y$, the restriction of the Quot–to–Sym morphism to the deepest stratum is only Zariski locally trivial.

Lemma 3.2.6. *Consider a smooth morphism $f\colon X \to Y$ and let E be a locally free sheaf on X. Let m denote the relative dimension of f. Then for all $d \geq 0$ the restriction of the Quot–to–Sym morphism*

$$\sigma^d_{X/Y}\colon\ \mathrm{Quot}^{(d)}_{X/Y}(E) \to X$$

is a Zariski locally trivial fibration with fiber $\mathrm{Quot}^d_{A^m}(\mathcal{O}^{\oplus r})_0$.

Proof. Cover X with Zariski open subschemes U equipped with étale morphism $g\colon U \to \mathbb{A}^m_Y$ of Y–schemes. Up to further restricting U, we may assume that $E|_U$ is free of rank n. By Lemma 3.2.3, there is an étale morphism

$$g_+\colon\ \mathrm{Quot}^d_{U/Y}(\mathcal{O}_U^{\oplus r})_{inj} \to \mathrm{Quot}^d_{\mathbb{A}^m_Y/Y}(\mathcal{O}^{\oplus r}_{\mathbb{A}^m_Y})$$

The stratum $\mathrm{Quot}^d_{U/Y}(\mathcal{O}_U^{\oplus r})_{(d)}$ sits inside $\mathrm{Quot}^d_{U/Y}(\mathcal{O}_U^{\oplus r})_{inj}$ as a closed subscheme, and g_+ restricts to a morphism

$$g_+\colon\ \mathrm{Quot}^{(d)}_{U/Y}(\mathcal{O}_U^{\oplus r}) \to \mathrm{Quot}^{(d)}_{\mathbb{A}^m_Y/Y}(\mathcal{O}^{\oplus r}_{\mathbb{A}^m_Y}).$$

By Lemma 3.2.5, it suffices to prove that the diagram below is cartesian.

$$\begin{tikzcd}
\mathrm{Quot}^{(d)}_{U/Y}(\mathcal{O}_U^{\oplus r}) \arrow[r, "g_+"] \arrow[d, "\sigma^d_{U/Y}"'] & \mathrm{Quot}^{(d)}_{\mathbb{A}^m_Y/Y}(\mathcal{O}^{\oplus r}_{\mathbb{A}^m_Y}) \arrow[d, "\sigma^d_{\mathbb{A}^m_Y/Y}"] \\
U \arrow[r, "g"] & \mathbb{A}^m_Y
\end{tikzcd}$$

This is a direct consequence of Lemma 3.2.4, as the diagram above is obtained by base change from diagram (3.2). □

In general, the restriction of the Quot–to–Sym morphism to a stratum ω is shown to be only étale locally trivial.

Theorem 3.2.7. *Let $f\colon X \to Y$ be a smooth morphism and E a locally free sheaf of rank r on X. Let $m = \dim(X) - \dim(Y)$. Then for all partitions ω of d the restriction of the Quot–to–Sym morphism*

$$\sigma^\omega_{X/Y}\colon\ \mathrm{Quot}^\omega_{X/Y}(E) \to \mathrm{Sym}^\omega_Y(X)$$

is an étale locally trivial fibration with fiber $\prod_i \mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})_0^{\omega_i}$.

Proof. Let Q_ω be the locus defined by the following cartesian diagram

$$\begin{tikzcd}
Q_\omega \arrow[r, hookrightarrow] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_i \mathrm{Quot}^{(i)}_{X/Y}(E)^{\omega_i} \arrow[d, "\prod_i (\sigma^{(i)}_{X/Y})^{\omega_i}"] \\
X^\omega_* \arrow[r, hookrightarrow] & \prod_i X^{\omega_i}
\end{tikzcd} \tag{3.4}$$

Here $X^\omega_* := \prod_i X^{\omega_i} \setminus \Delta$ is the complement of the large diagonal Δ, where at least two points coincide. The projection $Q_\omega \to X^\omega_*$ is equivariant with respect to the natural group action of $\mathfrak{S}_\omega := \prod_i \mathfrak{S}_{\omega_i}$ and, by Lemma 3.2.6, Zariski locally trivial with fiber

$$\prod_i \mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})_0^{\omega_i}.$$

Moreover, the action of $\mathfrak{S}_\omega$ on X^ω_* is free and yields a finite étale cover $X^\omega_* \to \mathrm{Sym}^\omega_Y(X)$. The goal is to construct a morphism $\psi_\omega \colon Q_\omega \to \mathrm{Quot}^\omega_{X/Y}(E)$ which completes the following diagram to a cartesian square.

$$\begin{tikzcd}
Q_\omega \arrow[r, dotted, "\psi_\omega"] \arrow[d] & \mathrm{Quot}^\omega_{X/Y}(E) \arrow[d] \\
X^\omega_* \arrow[r] & \mathrm{Sym}^\omega_Y(X)
\end{tikzcd} \tag{3.5}$$

The scheme Q_ω parametrizes tuples of quotients supported on disjoint loci of X. In order to glue these quotients, consider the étale morphism

$$\iota\colon \coprod_{|\omega|} X \to X, \qquad \iota|_X := \mathrm{id}_X, \qquad |\omega| = \sum_i \omega_i,$$

defined on the disjoint union of $|\omega|$ copies of X. The corresponding Quot scheme, where each copy of X carries the sheaf E, admits a canonical decomposition indexed by weak compositions of d of length $|\omega|$.

$$\mathrm{Quot}^d_{\amalg_{|\omega|}X/Y}(E) \xrightarrow{\sim} \coprod_{m \in C^*_{|\omega|}(d)} \prod_{j=1}^{|\omega|} \mathrm{Quot}^{m_j}_{X/Y}(E)$$

Lemma 3.2.3 gives an étale morphism

$$\iota_+ \colon \mathrm{Quot}^d_{\amalg_{|\omega|}X/Y}(E)_{inj} \to \mathrm{Quot}^d_{X/Y}(E).$$

By construction, the locus Q_ω lies in the Quot scheme $\mathrm{Quot}^d_{\amalg_{|\omega|}X/Y}(E)$ as a subscheme of the component $\prod_i \mathrm{Quot}^i_{X/Y}(E)^{\omega_i}$. In fact, Q_ω is entirely contained in the injective locus $\mathrm{Quot}^d_{\amalg_{|\omega|}X/Y}(E)_{inj}$. By a similar argument, one sees that X^ω_* lies in $\mathrm{Sym}^d_Y(\coprod_{|\omega|} X)_{inj}$. These objects fit into a diagram with cartesian squares, the right square being cartesian by Lemma 3.2.4.

$$\begin{tikzcd}
Q_\omega \arrow[r, "j_\omega"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{\amalg_{|\omega|}X/Y}(E)_{inj} \arrow[r, "\iota_+"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{X/Y}(E) \arrow[d] \\
X^\omega_* \arrow[r] & \mathrm{Sym}^d_Y(\coprod_d X)_{inj} \arrow[r] & \mathrm{Sym}^d_Y(X)
\end{tikzcd}$$

Let $g_\omega := \iota_+ \circ j_\omega$ denote the composite map and consider the following diagram.

$$\begin{tikzcd}
Q_\omega \arrow[r, dotted, "\psi_\omega"'] \arrow[d] \arrow[rr, bend left, "g_\omega"] & \mathrm{Quot}^\omega_{X/Y}(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^d_{X/Y}(E) \arrow[d] \\
X^\omega_* \arrow[r] & \mathrm{Sym}^\omega_Y(X) \arrow[r] & \mathrm{Sym}^d_Y(X)
\end{tikzcd}$$

The map g_ω makes the outer square commute and, by the universal property of fiber products, yields the desired morphism ψ_ω. Moreover, since both the right and outer square are cartesian, so is the left one, as required. $\square$

3.3 Applications

In this section, the constructions illustrated so far come together to derive two power structure formulas for Quot schemes of smooth morphisms. The étale local triviality of the Quot–to–Sym morphism, established strata–by–strata in Theorem 3.1.4, combined with the power structure on $\mathrm{K}_0(\mathrm{Var}_Y)$, is the key mechanism which allows the reconstruction of the global Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ from the punctual contributions $\mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\otimes \mathrm{rk}(E)})_0$, leading to the explicit formulas stated below.

3.3.1 A power structure formula for relative Quot schemes

In this section we prove Theorem 3.1.5, whose statement is recalled below for convenience.

Theorem 3.1.5. *Consider a smooth morphism $f\colon X \to Y$ of relative dimension m and let E be a locally free sheaf on X. The following identity holds in $\mathrm{K}_0(\mathrm{Var}_Y)[\![t]\!]$.*

$$Z_{X/Y}(E,t) = \left(\sum_{d\geq 0}[\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\otimes \mathrm{rk}(E)})_0 \times Y \to Y]t^d\right)^{[X\to Y]}.$$

Proof. The stratification of the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$ by partitions ω of d gives

$$[\mathrm{Quot}^d_{X/Y}(E) \to Y] = \sum_{\omega\in P(d)} [\mathrm{Quot}^d_{X/Y}(E)_\omega \to Y].$$

By Theorem 3.1.4, each restricted morphism

$$\mathrm{Quot}^\omega_{X/Y}(E) \to \mathrm{Sym}^\omega_Y(X)$$

is étale locally trivial with fiber $\prod_i \mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})^{\omega_i}_0$. This is expressed by the following diagram with cartesian squares, obtained by combining diagrams (3.4) and (3.5)

$$\begin{tikzcd}
\prod_i \mathrm{Quot}^{(i)}_{X/Y}(E)^{\omega_i} \arrow[d] & Q_\omega \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Quot}^\omega_{X/Y}(E) \arrow[d] \\
\prod_i \mathrm{Sym}^{(i)}_Y(X)^{\omega_i} & X^\omega_* \arrow[l] \arrow[r] & \mathrm{Sym}^\omega_Y(X)
\end{tikzcd}$$

where the vertical maps are the natural support morphisms. The locus Q_ω, defined by the left square, parametrizes tuples with ω_i punctual quotients of size i and pairwise disjoint support. The natural action of $\mathfrak{S}_\omega$ makes $Q_\omega \to X^\omega_*$ an $\mathfrak{S}_\omega$–equivariant morphism which, by Lemma 3.2.6, is Zariski locally trivial with fiber

$$\prod_i \mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})^{\omega_i}_0.$$

Hence, in $\mathrm{K}_0(\mathrm{Var}_Y)$, we have the equality

$$[Q_\omega/S_\omega \to Y] = [(X^\omega_* \times_k \prod_i \mathrm{Quot}^i_{\mathbb{A}^m}(\mathcal{O}^{\oplus r})^{\omega_i}_0)/\mathfrak{S}_\omega \to Y].$$

The right cartesian square shows that $\mathrm{Quot}^\omega_{X/Y}(E) \cong Q_\omega/\mathfrak{S}_\omega$, since $Q_\omega \to \mathrm{Quot}^\omega_{X/Y}(E)$ is a finite étale covers with Galois group $\mathfrak{S}_\omega$. Therefore,

$$[Q_\omega/S_\omega \to Y] = [\mathrm{Quot}^d_{X/Y}(E)_\omega \to Y].$$

Summing over all partitions and degrees gives the stated formula. □

Remark 3.3.1. For $Y = \mathrm{Spec}(k)$, Theorem 3.1.5 recovers the exponential formula in Theorem 3.1.1.

Corollary 3.3.2. *Consider a smooth morphism $f\colon X \to Y$ of relative dimension m and let E be a locally free sheaf on X. Then the dependence of the class $[\mathrm{Quot}^d_{X/Y}(E) \to Y]$ on the choice of locally free sheaf E is limited to the rank.*

3.3.2 A power structure formula for Quot–to–Sym morphisms

The second application of Theorem 3.1.4 concerns the Grothendieck ring of maps $\mathrm{K}_0(\mathrm{Map}_Y)$, whose elements are equivalence classes of morphisms of Y–varieties. The definition of $\mathrm{K}_0(\mathrm{Map}_Y)$ and power structure on it are recalled in Section 1.2.3. Here, we generalize the power structure formula for Hilbert–Chow morphisms computed by Gusein–Zade, Luengo, and Melle–Hernández in [31, Theorem 3].

Definition 3.3.3. Consider a smooth morphism $f\colon X \to Y$ and let E be a locally free sheaf on X. We denote by $\Sigma_{X/Y}(E,t)$ the generating series of the Quot–to–Sym morphisms classes in $\mathrm{K}_0(\mathrm{Map}_Y)$.

$$\Sigma_{X/Y}(E,t) := \sum_{d\geq 0} [\mathrm{Quot}^d_{X/Y}(E) \xrightarrow{\sigma^d_{X/Y}} \mathrm{Sym}^d_Y(X)]t^d.$$

Theorem 3.3.4. *For any smooth quasiprojective morphism $f\colon X \to Y$ of relative dimension m and locally free sheaf E on X, the following formula holds in the formal power series ring $\mathrm{K}_0(\mathrm{Map}_Y)[\![t]\!]$.*

$$\Sigma_{X/Y}(E,t) = \left(\sum_{d\geq 0} [\mathrm{Quot}^d_{\mathbb{A}^m}(\mathcal{O}^{\otimes \mathrm{rk}(E)})_0 \times Y \to Y]t^d\right)^{[X \xrightarrow{\mathrm{id}} X]}$$

Proof. In order to prove this result it suffices to reinterpret the right–hand side of the formula in terms of the power structure on $\mathrm{K}_0(\mathrm{Map}_Y)$, see Definition 1.2.30. In our case, formula (1.3) gives

$$\left(X^\omega_* \times_Y \prod_i (\mathrm{Quot}^i_{\mathbb{A}^m}(O^r)_0 \times_Y Y)^{\omega_i}\right)/\mathfrak{S}_\omega$$

which coincides with the stratum $\mathrm{Quot}^\omega_{X/Y}(E)$ of the Quot scheme $\mathrm{Quot}^d_{X/Y}(E)$. This is exactly the content of Lemma 3.1.4. On the other hand, formula (1.4) gives

$$\left(X^{|\omega|} \times_Y \prod_i Y^{\omega_i}\right)/\mathfrak{S}_\omega = \left(X^{|\omega|}\right)/\mathfrak{S}_\omega$$

Thus, for each partition ω, we obtain a morphism

$$\mathrm{Quot}^{\omega}_{X/Y}(E) \to X^{|\omega|}/\mathfrak{S}_\omega$$

whose image lies in the open subset

$$\mathrm{Sym}^{\omega}_{Y}(X) = X^{\omega}_{*}/\mathfrak{S}_\omega \subset X^{|\omega|}/\mathfrak{S}_\omega.$$

By Remark 1.2.28, we may write

$$[\mathrm{Quot}^{\omega}_{X/Y}(E) \to X^{|\omega|}/\mathfrak{S}_\omega] = [\mathrm{Quot}^{\omega}_{X/Y}(E) \to \mathrm{Sym}^{\omega}_{Y}(X)].$$

Summing over partitions ω of d concludes the proof. □

Chapter 4

Relative Quot schemes over moduli spaces of nodal curves

The focus of this chapter is the study of relative Quot schemes over moduli spaces of curves with nodal singularities, with particular emphasis on the case of Losev–Manin spaces. These spaces provide a controlled and combinatorial setting in which explicit computations are possible.

4.1 Overview and main results

The results established in Chapters 2 and 3 for Quot schemes of *smooth* morphisms naturally raise the question of what happens when singularities enter the picture. A natural starting point is to consider families of curves with at most *nodal* singularities. Despite being the simplest type of curve singularity, nodal curves give rise to a deep and fascinating moduli theory.

Several moduli spaces of nodal curves arise as compactifications of the moduli space $M_{0,n}$, which parametrizes possible distributions of n distinct marked points on a projective line. The choice of such a compactification dictates what happens to the geometry of the projective line whenever two or more marked points collide with one another. Among all these spaces, Losev–Manin spaces exhibit a particularly nice and rich geometry. Denoted by $\overline{L}_n$, they provide a compactification of $M_{0,n+2}$, in which all marked points but two are allowed to collide; the two special points, called *poles*, must remain distinct. This degeneration pattern determines that the boundary curves of $\overline{L}_n$ are connected chains of projective lines with two poles and n marked points; see Figure 4.1. From a structural point of view, Losev–Manin spaces are smooth, irreducible, projective toric varieties of dimension $n-1$.

Losev–Manin spaces were first introduced by Andrey Losev and Yuri Manin in [47], where they also compute the Poincaré polynomial and discuss in depth the toric structure. The interest in Losev–Manin spaces originates from mathematical physics, as their (co)homolgy is directly linked to the solutions to commutativity equations [60].

In the present chapter, we address the following question.

Question 4.1.1. *What can be said about relative Quot schemes over Losev–Manin spaces, both geometrically and at the level of its Grothendieck ring class?*

The existing literature counts several contributions to the study of Hilbert and Quot schemes on nodal curves. In the early 2000's, Ran gave an explicit geometric description of the punctual Hilbert scheme of the nodal singularity, thus determining its class in the Grothendieck ring; he also showed that the relative Hilbert scheme of the germ of the 1–parameter family $xy = t$ is formally smooth.

Theorem 4.1.2 ([55, Theorems 1, 3]) .

(i) Let X be a curve with a node p. The punctual Hilbert scheme $\operatorname{Hilb}^d(X)_p$ is a connected chain of $(d-1)$ projective lines, each meeting the next one transversally.

(ii) Consider the germ $\operatorname{Spec}(k[\![x,y,t]\!]/xy-t) \to \operatorname{Spec}(k[\![t]\!])$ of the 1–parameter family $xy = t$ at $t = 0$. The associated relative Hilbert scheme is formally smooth of dimension $m+1$.

More recently, higher rank cases were addressed by Huang and Jiang, who computed the punctual Quot zeta functions for singularities of type $y^2 = x^m$, including the nodal case [37] [38]. A refinement of their formula was recently provided by Chern [15]; see Theorem 4.3.5. Overall, these results constitute a solid base for the study of Quot schemes over families of nodal curves.

Building on this foundation, we now turn our attention to relative Quot schemes over Losev–Manin spaces. To be precise, let $\mathcal{X}_n \to \overline{L}_n$ denote the universal family of $\overline{L}_n$. For a locally free sheaf E on $\mathcal{X}_n$, we study the relative Quot scheme

$$\operatorname{Quot}^d_{\mathcal{X}_n/\overline{L}_n}(E).$$

Theorem 4.1.2 indicates that Hilbert schemes over families of nodal curves might be smooth in general. Our first result confirms this expectation by establishing the smoothness of the relative Hilbert scheme in the Losev–Manin case.

Theorem 4.1.3. *The Hilbert scheme $\operatorname{Hilb}^d_{\overline{L}_n}(\mathcal{X}_n)$ is a smooth irreducible projective scheme of dimension $d+n-1$.*

The proof builds on a smoothness criterion of Ran [57, Corollary 3.4] for families which are locally versal at the nodes. In the case of Losev–Manin spaces, local versality is granted by the smoothness of the toric structure.

Next, we turn to consider the absolute Quot zeta function with coefficients in $K_0(\mathrm{Var}_k)$,

$$Z^{abs}_{\mathcal{X}_n/\overline{L}_n}(E,t) := \sum_{d\geq 0}[\operatorname{Quot}^d_{\mathcal{X}_n/\overline{L}_n}(E)]t^d.$$

Specializing to $E = \mathcal{O}^{\oplus r}$, we derive an explicit formula.

Theorem 4.1.4. *The absolute Quot zeta function associated to the free sheaf $\mathcal{O}^{\oplus r}$ over the Losev–Manin family $\pi_n\colon \mathcal{X}_n \to \overline{L}_n$ satisfies the formula*

$$Z^{abs}_{\mathcal{X}_n/\overline{L}_n}(\mathcal{O}^{\oplus r},t) = \sum_{\ell=1}^{n} \ell! S(n,\ell)(\mathbb{L}-1)^{n-\ell}\left(\frac{1-t}{(t;\mathbb{L})_r^2(1-\mathbb{L}^r t)}\right)^{\ell} N(\mathcal{O}^{\oplus r},t)^{\ell-1}.$$

Here $S(n,\ell)$ is a Stirling number of the second kind and $(t;\mathbb{L})_r$ is q–Pochhammer symbol, both of which are introduced in Section 4.2; the term $\mathrm{N}(\mathcal{O}^{\oplus r},t)$ is the contribution coming from the nodal singularity, reported in Theorem 4.3.5.

The proof combines existing formulas for Quot schemes over smooth and nodal curves using the power structure formalism. As a result, the Quot zeta function $Z^{abs}_{\mathcal{X}_n/\overline{L}_n}(\mathcal{O}^{\oplus r}, t)$ is shown to be rational in Corollary 4.3.11 and to satisfy a functional equation in Corollary 4.3.12. A specialization of Theorem 4.1.4 is Corollary 4.3.13, which yields the Hilbert zeta function; Corollary 4.3.14 further specializes Corollary 4.3.13 to the topological Euler characteristic. In particular, setting $n = 2$ in Corollary 4.3.14 recovers a computation by Ran [56, Corollary 2.2]; this is shown in Theorem 4.3.16.

Remark 4.1.5. The computation in Theorem 4.1.4 is carried out for coefficients in the Grothendieck ring of k–varieties, rather than in the ring $K_0(\mathrm{Var}_{\overline{L}_n})$. This choice is motivated by the stratification of $\overline{L}_n$ by combinatorial curve type (Theorem 4.2.10) used in the proof.

4.2 Background

In this section, we collect the necessary background material required for the rest of the chapter. We begin by recalling the definition and geometry of Losev–Manin spaces, as well as their stratification by curve type, which will play a central role in the proof of Theorem 4.1.4. We also introduce the notion of q–Pochhammer symbols and, more generally of q–analogs.

4.2.1 Losev–Manin spaces

Definition 4.2.1. A *chain* of projective lines is a connected rational curve with nodal singularities whose dual graph[1] has vertices $\{v_1, \dots, v_\ell\}$ and edges (v_i, v_{i+1}).

Definition 4.2.2. Let $n \geq 1$. The *Losev–Manin space* $\overline{L}_n$ parametrizes stable chains of projective lines equipped with $n + 2$ sections $\{y_0, y_\infty; x_1, \dots, x_n\}$. The sections y_i, called *poles*, and x_j, called *markings*, are subject to the following conditions:

(i) the sections $y_0, y_\infty, x_1, \dots, x_n$ do not contain singular points;
(ii) if the chain has at least two irreducible components, the poles y_0, y_∞ lie in components corresponding to the opposite endpoints of the linear graph;
(iii) each pole y_0, y_∞ is disjoint from all other sections;
(iv) (*Stability*) every irreducible component contains at least three special points (nodes, markings or poles).

Theorem 4.2.3 ([47]) **.** *The Losev–Manin space $\overline{L}_n$ is a smooth separated irreducible proper variety of dimension $n - 1$. Its universal family $\pi_n\colon \mathcal{X}_n \longrightarrow \overline{L}_n$ coincides with the Losev–Manin space $\overline{L}_{n+1}$, while the universal morphism π_n acts by forgetting the point $n + 1$, followed by stabilization if necessary.*

Example 4.2.4. For small values of n, the Losev–Manin spaces are

$$\overline{L}_1 = \mathrm{Spec}(k) \qquad \overline{L}_2 = \mathbb{P}^1 \qquad \overline{L}_3 = \mathrm{Bl}_{(0,0),(\infty,\infty)}\, \mathbb{P}^1 \times \mathbb{P}^1.$$

Notation 4.2.5. We denote by L_n the open locus of $\overline{L}_n$ parametrizing smooth curves.

---

[1] The dual graph of a curve is obtained by assigning a vertex to each irreducible component. An edge between two vertices corresponds to an intersection of the corresponding irreducible components.

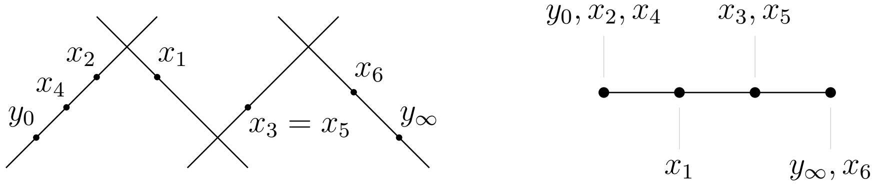

FIGURE 4.1: *On the left, a curve in $\overline{L}_6$ with two overlapping marked points. On the right, its dual graph: each vertex represents an irreducible component, edges represents an intersection between two components. The labels attached to each vertex represent the poles and marked points.*

Remark 4.2.6.

(i) The Losev–Manin space $\overline{L}_n$ is birational to the moduli space $\overline{M}_{0,n+2}$ parametrizing genus zero stable trees of projective lines with $n+2$ marked points. A birational morphism $\overline{M}_{0,n+2} \dashrightarrow \overline{L}_n$ is given as follows. Given a stable curve in $\overline{M}_{0,n+2}$, choose two marked points i, j to play the role of $0, \infty$. These two points are connected by a unique chain of projective lines. Next, contract any component which does not lie in the chain.

(ii) The Losev–Manin space $\overline{L}_n$ is the toric variety associated to the permutohedron of order n. The torus $\mathbb{G}_m^{n-1} = \mathbb{G}_m^n / \mathbb{G}_m$ acts on $\overline{L}_n$ by rescaling all marked points, modulo diagonal rescaling.

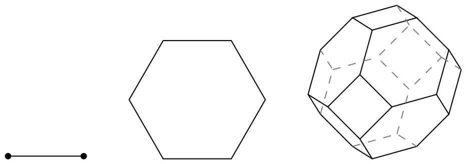

FIGURE 4.2: *Permutohedra of order 2, 3 and 4. The permutohedron of order n is an $(n-1)$–dimensional polytope whose vertices correspond to the permutations of the first n natural numbers. Part of the code used in this figure was adapted from [45].*

Remark 4.2.7. The symmetric group $\mathfrak{S}_n$ acts on $\overline{L}_n$ by permuting the labels of the n marked points. Since no specific ordering of the marked points is part of the structure of $\overline{L}_n$, the notation can be made independent of any chosen order. In other words, for a set B of cardinality n, we write $\overline{L}_B$ for the corresponding Losev–Manin space, and $L_B \subset \overline{L}_B$ for the open locus parametrizing smooth curves. With this notation, the universal family $\mathcal{X}_B$ coincides with the Losev–Manin space $\overline{L}_{B \cup \{*\}}$ [47, Proposition 2.6.2]. From now on we shall adopt this more flexible notation.

4.2.2 Stratification by ordered set partitions

Losev–Manin spaces, similarly to other moduli spaces of curves, admit a stratification by combinatorial curve type. The stratification keeps track of how many irreducible components (equivalently, nodes) each curve contains and how the marked points are distributed among these irreducible components.

Definition 4.2.8. An *ordered partition* σ of a finite set B is a totally ordered set of nonempty subsets σ_i of B whose union is B and whose pairwise intersections are empty. The parts of σ are denoted $\sigma_1, \dots, \sigma_\ell$ in their structure order. The *length* of an ordered partition σ is $\ell(\sigma) := \ell$. We denote by $\mathfrak{P}(B)$ the set of all ordered partitions of B, and by $\mathfrak{P}_\ell(B)$ the set of length ℓ ordered partitions of B.

Remark 4.2.9. The cardinality of $\mathfrak{P}(B)$ is

$$|\mathfrak{P}(B)| = \sum_{\ell=1}^{|B|} |\mathfrak{P}_\ell(B)| = \sum_{\ell=1}^{|B|} \ell! S(|B|, \ell).$$

The quantity $S(n, \ell)$, called *Stirling number of the second kind*, counts the number of unordered partitions of a set of cardinality n into ℓ nonempty subsets. It may be computed using the formula [61, Formula (1.94a)]

$$S(n, \ell) := \frac{1}{\ell!} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^n. \tag{4.1}$$

Theorem 4.2.10 ([47]) **.** *The Losev-Manin space $\overline{L}_B$ admits a locally closed stratification indexed by ordered partitions of B.*

$$\overline{L}_B = \coprod_{\sigma \in \mathfrak{P}(B)} L_B(\sigma)$$

Each stratum $L_B(\sigma)$ parametrizes stable chains with $\ell(\sigma)$ irreducible components, where the i–th irreducible component carries the marked points labeled by σ_i. In particular, there is an isomorphism

$$L_{\sigma_1} \times \cdots \times L_{\sigma_\ell} \xrightarrow{\sim} L_B(\sigma).$$

which, on points, glues the pole y_∞ of the i–th curve to the pole y_0 of the $(i+1)$–th curve.

Example 4.2.11. At the two extremes of the stratification, we find:

- the trivial partition (B), of length 1, corresponding to the open stratum L_B;
- the maximal partitions of length $|B|$, corresponding to closed strata consisting of a single point. The associated curve is a chain of $|B|$ projective lines. The number of maximal partitions is $|B|!$, corresponding to the possible permutations of the marked points on the irreducible components.

The intermediate partitions give rise to positive dimensional strata which are strictly locally closed. For example, the curve depicted in Figure 4.1 lies in the locally closed stratum corresponding to the partition $\sigma = (\{x_2, x_4\}, \{x_1\}, \{x_3, x_5\}, \{x_6\})$ of $B = \{x_1, \dots, x_6\}$.

Definition 4.2.12. Curves parametrized by a stratum $L_B(\sigma)$ are said to have *combinatorial type* σ. We denote by $\mathcal{X}_B(\sigma) := \mathcal{X}_B \times_{\overline{L}_B} L_B(\sigma)$ the restriction of the universal family to the stratum $L_\sigma(B)$.

Theorem 4.2.13. *The projection $\mathcal{X}_B(\sigma) \to L_B(\sigma)$ is a Zariski trivial fibration with fiber a curve C_σ of combinatorial type σ.*

Theorem 4.2.13 requires an intermediate lemma involving truncated partitions.

Definition 4.2.14. Let $\sigma = (\sigma_1, \dots, \sigma_\ell)$ be an ordered partition of B and consider a proper nonempty subset $B' \subsetneq B$. The *truncated partition* $\sigma \cap B'$ is the ordered partition of B' obtained by intersecting each σ_i with B' and discarding any empty subset resulting from this operation.

Lemma 4.2.15. *Let σ be an ordered partition of B. Choose a proper nonempty subset B' of B and suppose that $\sigma_i \cap B'$ is nonempty for all $i = 1, \dots, \ell(\sigma)$. Then the following diagram is cartesian.*

$$\begin{tikzcd}
\mathcal{X}_B(\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}_{B'}(\sigma \cap B') \arrow[d] \\
L_B(\sigma) \arrow[r] & L_{B'}(\sigma \cap B')
\end{tikzcd}$$

Proof. It suffices to prove the case $B = B' \cup \{\xi\}$. Let U_B denote the open locus of curves which do not require stabilization after forgetting the extra marked point ξ. Over U_B, forgetting the extra section ξ induces an isomorphism of the underlying curves, so there is a canonical isomorphism

$$\mathcal{X}_B|_{U_B} \xrightarrow{\sim} \mathcal{X}_{B'} \times_{\overline{L}_{B'}} U_B.$$

Denote by τ the truncated partition $\sigma \cap B'$. By assumption, the partition σ can be recovered from the truncated partition τ by adding ξ to one of the parts τ_i. Geometrically, this corresponds to adding the extra marked point to one of the existing irreducible components of the curve, away from any nodes or poles. Therefore the stratum $L_B(\sigma)$ is contained in U_B. Thus we conclude as follows.

$$\begin{aligned}
\mathcal{X}_B(\sigma) &\cong \mathcal{X}_B \times_{\overline{L}_B} L_B(\sigma) \\
&\cong \mathcal{X}_B|_{U_B} \times_{U_B} L_B(\sigma) \\
&\cong \mathcal{X}_{B'} \times_{\overline{L}_{B'}} U_B \times_{U_B} L_B(\sigma) \\
&\cong \mathcal{X}_{B'} \times_{\overline{L}_{B'}} L_{B'}(\tau) \times_{L_{B'}(\tau)} L_B(\sigma) \\
&\cong \mathcal{X}_{B'}(\tau) \times_{L_{B'}(\tau)} L_B(\sigma)
\end{aligned}$$

□

Proof of Theorem 4.2.13. For each $i = 1, \dots, \ell(\sigma)$, choose an element $b_i \in \sigma_i$ and let $B' := \{b_1, \dots, b_{\ell(\sigma)}\}$. Then the truncated partition $\tau = \sigma \cap B'$ coincides with the maximal partition $(\{b_1\}, \dots, \{b_{l(\sigma)}\})$ of B'. Then the stratum $L_{B'}(\tau)$ is a single point, and the associated curve $\mathcal{X}_{B'}(\tau)$ is isomorphic to C_σ. By Lemma 4.2.15, we conclude that

$$\mathcal{X}_B(\sigma) \cong L_B(\sigma) \underset{L_{B'}(\tau)}{\times} \mathcal{X}_{B'}(\tau) \cong L_B(\sigma) \times C_\sigma.$$

□

Remark 4.2.16. The proof of Theorem 4.2.13 crucially relies on the fact that, after forgetting a suitable subset of marked points, one reaches a stratum consisting of a single point – that is, by taking the maximal partition $(\{b_1\}, \dots, \{b_{\ell(\sigma)}\})$ of B'. This behavior is very specific of Losev–Manin spaces. For more general notions of moduli spaces of stable curves with marked points, such as $\overline{M_{0,n}}$, the Zariski triviality predicted by Theorem 4.2.13 fails. The reason is that, even after forgetting a suitable amount of marked points, the boundary strata may have residual parameters due to the presence of four or more nodes.

Example 4.2.17. The situation described in Remark 4.2.16 occurs in $\overline{M_{0,8}}$. Consider the stratum in parametrizing curves with five irreducible components: a central component meeting four "tail" components at four distinct nodes, with each tail carrying two marked points to ensure stability; see Figure 4.3. While the tails have no moduli, the central component has residual parameters: the stratum is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, with parameter t marking the position of the fourth node. For any given t, corresponding to a curve C_t, there is only a finite number of values t' for which there exists an isomorphism $C_t \cong C_{t'}$. To be precise, these values are the cross–ratios of the four nodes, which must be preserved by the isomorphism. The values are

$$\{t, 1/t, 1-t, 1/(1-t), (t-1)/t, t/(t-1)\}.$$

Thus, the universal family above this stratum is neither Zariski nor étale locally trivial.

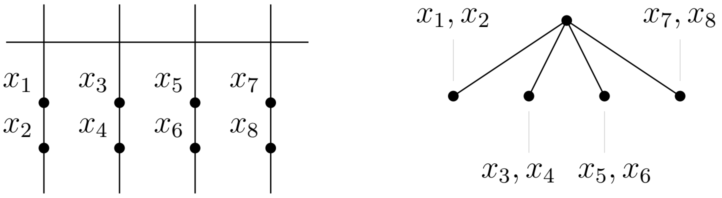

FIGURE 4.3: *On the left, a curve in* $\overline{M_{0,8}}$. *On the right, its dual graph.*

4.2.3 q–analogs

A q–analog is a variation of a combinatorial identity, expression or theorem, which involves a formal parameter q. The classical combinatorial objects are recovered by setting the parameter q to 1, either via direct substitution or by taking a limit. The notion of q–analog is discussed in depth in [2], [61]. Here we recall the notion of q–Pochhammer symbol and of q–binomial coefficients, to be used later on.

Definition 4.2.18. Let t and q be variables and let $n \geq 0$. The *q–Pochhammer symbol* is the finite product

$$(t; q)_n := (1-t)(1-tq) \cdots (1-tq^{n-1})$$

with $(t; q)_0 := 1$.

Remark 4.2.19. The q–Pochhammer symbol is the q–analog of the *Pochhammer polynomial* (or *falling factorial*) $(x)_n := x(x-1)\ldots(x-n+1)$. The two are related by the limit

$$\lim_{q \to 1} \frac{(q^x; q)}{(1-q)^n} = (x)_n.$$

Remark 4.2.20. The Pochhammer polynomial $(x)_k$ and the Stirling number of the second kind $S(n, k)$ both appear in the following identity [61, Formula (1.94d)].

$$\sum_{k=0}^{n} S(n, k)(x)_k = x^n.$$

Definition 4.2.21. The *q-binomial coefficient* is the quantity

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_{m-n}(q;q)_n}.$$

Remark 4.2.22. The q–binomial coefficient is the q–analog of the binomial coefficient.

Remark 4.2.23. The reciprocal of the q-Pochhammer symbol expands into a formal power series with q-binomial coefficients:

$$\frac{1}{(t;q)_{m+1}} = \sum_{d \geq 0} \begin{bmatrix} d+m \\ d \end{bmatrix}_q t^d.$$

4.3 Main constructions and proofs

4.3.1 Geometry of Hilbert schemes over families of nodal curves

In this section we address the geometry of the Hilbert scheme $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$, showing in particular that it is smooth. The argument relies on deformation theoretic techniques.

The following lemma, due to Ran, provides a smoothness criterion for relative Hilbert schemes associated to a families of nodal curves.

Lemma 4.3.1 ([57, Corollary 3.4]) **.** *Let $f\colon X \to Y$ be a family of smooth or nodal curves.*

(i) The structure morphism $\mathrm{Hilb}^d_Y(X) \to Y$ *is a normal crossings morphism;*

(ii) If f is locally versal at the nodes, then the Hilbert scheme $\mathrm{Hilb}^d_Y(X)$ *and the universal family over it are both smooth.*

(iii) If X is irreducible, so is $\mathrm{Hilb}^d_Y(X)$.

We briefly recall the notion of local versality.

Definition 4.3.2. A flat family $f\colon X \to Y$ of nodal curves is *locally versal at the nodes* if for every $y \in Y$ the following natural morphism of deformation functors is smooth.

$$\varphi_{f,y}\colon h_{Y,y} \to \prod_{p \in \mathrm{Sing}(X_y)} \mathrm{Def}_{X_y,p}$$

Remark 4.3.3. Consider a flat family $f\colon X \to Y$ of nodal curves, with central fiber X_y over $y \in Y$, with nodal singularity p. The deformation functor $\mathrm{Def}_{X_y,p}$ is unobstructed and has 1–dimensional tangent space $\mathscr{E}\mathrm{xt}^1(\Omega_{X_y}, \mathcal{O}_{X_y})_p$. Therefore, f is locally versal at the nodes if

(i) X and Y are smooth

(ii) The locus ∂Y of Y of non smooth fibers is a divisor with normal crossings.

Theorem 4.3.4. *The Hilbert scheme* $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$ *is a smooth irreducible projective scheme of dimension* $d + |B| - 1$.

Proof. The universal family $\mathcal{X}_B$ of the Losev–Manin space $\overline{L}_B$ coincides with the space $\overline{L}_{B\cup\{*\}}$ by [48, Proposition 2.6.2], and hence is irreducible by Theorem 4.2.3; by Lemma 4.2.1 so is the corresponding Hilbert scheme.

To prove smoothness, it suffices to show that the universal family $\pi_B\colon \mathcal{X}_B \to \overline{L}_B$ is locally versal at the nodes. Since $\overline{L}_B$ is a smooth toric variety, its boundary $D = \overline{L}_B \setminus L_B$

is a divisor with simple normal crossings. Then the smoothness of $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$ and of its universal family follows directly from Lemma 4.3.1 and Remark 4.3.3.

Since $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$ is smooth and irreducible, its dimension is determined by the dimension of an open subscheme. The universal family over the open stratum L_B is Zariski trivial with fiber $\mathbb{P}^1$, that is: $\mathcal{X}_B \times_{\overline{L}_B} L_B = L_B \times \mathbb{P}^1$. The corresponding open subscheme

$$\mathrm{Hilb}^d_{L_B}(\mathcal{X}_B \times_{\overline{L}_B} L_B) \cong \mathrm{Hilb}^d(\mathbb{P}^1) \times L_B \cong \mathbb{P}^d \times L_B$$

has dimension $d + |B| - 1$. ☐

4.3.2 Quot zeta function of a chain of projective lines

In this section we compute the Quot zeta function of a chain of projective lines. The Quot zeta function of a nodal curve factors into a contribution coming from the smooth locus and another coming from the punctual Quot zeta function of the nodal singularity. We begin by recalling these two ingredients, starting from the nodal contribution. For the numerator of the punctual Quot zeta function, we refer to Chern's formula, obtained by setting $N = 1$, $k = 1$ in formula (1.24) of [15, Theorem 1.7].

Theorem 4.3.5 ([15], [38]) **.** *The punctual Quot zeta function of the nodal singularity* $\{xy = 0\}$ *for a rank r locally free sheaf E is a rational function in t*

$$Z_{node}(E,t) = \frac{N(E,t)}{(t;\mathbb{L})_r^2}$$

with numerator

$$N(E,t) := (t\mathbb{L}^{r-1};\mathbb{L}^{-1})_r \sum_{i=0}^{r} \frac{(t\mathbb{L}^r)^{2i}\mathbb{L}^{-i^2}}{(t\mathbb{L}^{r-1};\mathbb{L}^{-1})_i} \begin{bmatrix} r \\ i \end{bmatrix}_{\mathbb{L}^{-1}}.$$

Theorem 4.3.6 ([38]) **.** *The numerator $N(E,t)$ of the punctual Quot zeta function of the nodal singularity satisfies the functional equation*

$$N(E,\mathbb{L}^{-r}t^{-1}) = \frac{1}{\mathbb{L}^{r^2}t^{2r}} N(E,t).$$

Recall Theorem 3.1.1, which expresses the Quot zeta function in terms of the punctual Quot zeta function, via the power structure on $K_0(\mathrm{Var}_k)$. When specialized to the case of smooth curves, Theorem 3.1.1 is equivalent to Bifet's formula (Therem 2.1.2). In particular, we obtain the following.

Lemma 4.3.7 ([8], [58]) **.** *For a smooth quasiprojective curve C and a rank r locally free sheaf E on C, the Quot zeta function is given by*

$$Z_C(E,t) = (1-t)^{-[C\times\mathbb{P}^{r-1}]}.$$

We can now compute the Quot zeta function of a chain of projective lines.

Theorem 4.3.8. *Let C_ℓ be a chain of ℓ projective lines. For a rank r locally free sheaf E on C_ℓ, the Quot zeta function of C_ℓ is a rational function in t given by*

$$Z_{C_\ell}(E,t) = \left(\frac{1-t}{(t;\mathbb{L})_r^2(1-\mathbb{L}^r t)}\right)^\ell N(\mathcal{O}^{\oplus r},t)^{\ell-1}$$

Proof. Let C_ℓ° denote the smooth locus of C_ℓ, with inclusion morphism $\iota\colon C_\ell^\circ \to C_\ell$. The Quot zeta function over C_ℓ factors into the product of the Quot zeta function over the smooth locus and a product of punctual Quot zeta functions supported at the nodes (there are $\ell - 1$ nodes):

$$Z_{C_\ell}(E,t) = Z_{C_\ell^\circ}(\iota^* E, t) \cdot Z_{\mathrm{node}}(\mathcal{O}^{\oplus r}, t)^{\ell-1}.$$

The punctual Quot zeta function of a node is given by Theorem 4.3.5. On the other hand, the Quot zeta function over the smooth locus C_ℓ° is given by

$$Z_{C_\ell^\circ}(\iota^* E, t) = \frac{1}{(t;\mathbb{L})_r^2}\left(\frac{1-t}{1-\mathbb{L}^r t}\right)^\ell, \tag{4.2}$$

as we now explain. First, note that $[C_\ell^\circ] = \ell(\mathbb{L}-1)+2$, and therefore

$$[C_\ell^\circ \times \mathbb{P}^{r-1}] = \sum_{i=0}^{r} \ell \mathbb{L}^i + (2-\ell)\sum_{i=0}^{r-1}\mathbb{L}^i. \tag{4.3}$$

By Lemma 4.3.7 we have

$$Z_{C_\ell^\circ}(\iota^* E, t) = (1-t)^{-[C_\ell^\circ \times \mathbb{P}^{r-1}]}.$$

By applying formula (4.3) and Example 1.2.26 we obtain

$$\begin{aligned} Z_{C_\ell^\circ}(\iota^* E, t) &= \prod_{i=1}^{r}(1-t)^{-\ell\mathbb{L}^i}\prod_{i=0}^{r-1}(1-t)^{-(2-\ell)\mathbb{L}^i} \\ &= \prod_{i=1}^{r}(1-\mathbb{L}^i t)^{-\ell}\prod_{i=0}^{r-1}(1-\mathbb{L}^i t)^{\ell-2} \\ &= \frac{(1-t)^\ell (t;\mathbb{L})_r^{\ell-2}}{(t;\mathbb{L})_{r+1}^\ell} \end{aligned}$$

Finally, by using the identity $(t;q)_n/(t;q)_{n+1} = (1-tq^n)^{-1}$, it is possible to rewrite the last expression to obtain (4.2), as required. □

Combining Theorem 4.3.6 and Theorem 4.3.8 gives the following result.

Corollary 4.3.9. *The Quot zeta function $Z_{C_\ell}(E,t)$ satisfies the functional equation*

$$Z_{C_\ell}(E, \mathbb{L}^{-r}t^{-1}) = \mathbb{L}^{r^2} t^{2r} Z_{C_\ell}(E,t). \tag{4.4}$$

Proof. By Theorem 4.3.6, we only need to verify that the following equality holds

$$\left(\frac{1-\mathbb{L}^{-r}t^{-1}}{(\mathbb{L}^{-r}t^{-1};\mathbb{L})_r^2(1-t^{-1})}\right)^\ell \frac{1}{(\mathbb{L}^{r^2}t^{2r})^{\ell-1}} = \mathbb{L}^{r^2}t^{2r}\left(\frac{1-t}{(t;\mathbb{L})_r^2(1-\mathbb{L}^r t)}\right)^\ell$$

This follows from a direct computation. In particular, the term $(\mathbb{L}^{-r}t^{-1};\mathbb{L})_r^2$ appearing at the denominator on the left-hand side can be rewritten as follows:

$$(\mathbb{L}^{-r}t^{-1};\mathbb{L})_r^2 = \frac{(t;\mathbb{L})_{r+1}^2}{(1-t)^2\mathbb{L}^{r^2+r}t^{2r}}.$$

□

4.3.3 Main formula, rationality and functional equation

We now compute the Quot–zeta function of the universal family $\pi_B \colon \mathcal{X}_B \to \overline{L}_B$ in terms of the nodal and smooth contributions described in the previous sections.

Theorem 4.3.10. *The absolute Quot zeta function associated to the free sheaf $\mathcal{O}^{\oplus r}$ over the Losev–Manin family $\pi_B \colon \mathcal{X}_B \to \overline{L}_B$ can be written as follows:*

$$Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}, t) = \sum_{\ell=1}^{|B|} \ell! S(|B|, \ell)(\mathbb{L}-1)^{|B|-\ell} \left(\frac{1-t}{(t;\mathbb{L})_r^2(1-\mathbb{L}^r t)} \right)^{\ell} N(\mathcal{O}^{\oplus r}, t)^{\ell-1}.$$

Proof. The stratification of the Losev-Manin space $\overline{L}_B$ (Theorem 4.2.10) induces a locally closed stratification on the relative Quot scheme $\mathrm{Quot}^d_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r})$:

$$\mathrm{Quot}^d_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}) = \coprod_{\sigma \in \mathfrak{P}(B)} \mathrm{Quot}^d_{\mathcal{X}_{B(\sigma)}/L_B(\sigma)}(\mathcal{O}^{\oplus r}).$$

Thus the Quot zeta function decomposes into a sum indexed by ordered partitions:

$$Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}, t) = \sum_{\sigma \in \mathfrak{P}(B)} Z^{abs}_{\mathcal{X}_{B(\sigma)}/L_B(\sigma)}(\mathcal{O}^{\oplus r}, t).$$

By the base change property of Quot schemes, on each stratum we obtain

$$[\mathrm{Quot}^d_{\mathcal{X}_{B(\sigma)}/L_B(\sigma)}(\mathcal{O}^{\oplus r})] = [L_B(\sigma)][\mathrm{Quot}^d_{C_\sigma}(\mathcal{O}^{\oplus r})]$$

where C_σ denotes a fixed curve of combinatorial type σ. The class $[\mathrm{Quot}^d_{C_\sigma}(E)]$ depends on the curve C_σ only up to the length of the partition $\ell(\sigma) = \ell$. Similarly, the class of the stratum $[L_B(\sigma)]$ depends only on $|B|$ and ℓ, and equals $(\mathbb{L}-1)^{|B|-\ell(\sigma)}$.

Counting the number of ordered partitions of length ℓ (Remark 4.2.9), we obtain

$$\sum_{\ell(\sigma)=\ell} [\mathrm{Quot}^d_{\mathcal{X}_{B(\sigma)}/L_B(\sigma)}(\mathcal{O}^{\oplus r})] = \ell! S(|B|, \ell)(\mathbb{L}-1)^{|B|-\ell}[\mathrm{Quot}^d_{C_\ell}(\mathcal{O}^{\oplus r})]$$

where C_ℓ denotes a chain of ℓ projective lines. Summing over ℓ and d gives

$$Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}, t) = \sum_{\ell=1}^{|B|} \ell! S(|B|, \ell)(\mathbb{L}-1)^{|B|-\ell} Z_{C_\ell}(\mathcal{O}^{\oplus r}, t). \tag{4.5}$$

Substituting $Z_{C_\ell}(\mathcal{O}^{\oplus r}, t)$ with the formula from Theorem 4.3.8 proves the statement. ☐

Corollary 4.3.11. $Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(E, t)$ *is a rational function in* t.

Corollary 4.3.12. *The Quot zeta function* $Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}, t)$ *satisfies the functional equation*

$$Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}, \mathbb{L}^{-r}t^{-1}) = (\mathbb{L}^{r^2}t^{2r}) Z_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O}^{\oplus r}, t).$$

Proof. This follows directly from formula (4.5) and Corollary 4.3.9. ☐

We conclude the section with a few remarks about some subcases of Theorem 4.3.10. Specializing to rank $r = 1$ recovers the Hilbert zeta function, which admits a simple, clean formula.

Corollary 4.3.13. *The Hilbert zeta function* $Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O},t)$ *is given by*

$$Z^{abs}_{\mathcal{X}_B/\overline{L}_B}(\mathcal{O},t) = \sum_{\ell=1}^{|B|} \ell! S(|B|,\ell)(\mathbb{L}-1)^{|B|-\ell}\frac{(1-t+\mathbb{L}t^2)^{\ell-1}}{(1-t)^{\ell}(1-\mathbb{L}t)^{\ell}}.$$

Applying the Euler characteristic homomorphism $\chi\colon K_0(\mathrm{Var}_k)\to\mathbb{Z}$ specializes the Hilbert zeta function to a generating series of Euler characteristic of the schemes $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$. In practice, this is achieved by setting $\chi(\mathbb{L})=1$. Under this specialization, the only contribution comes from the summand indexed by $\ell=|B|$, corresponding to single point strata.

Corollary 4.3.14. *The generating function of the Euler characteristic of the Hilbert schemes* $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$ *is given by*

$$\chi_{\mathcal{X}_B/\overline{L}_B}(t) = |B|!\frac{(1-t+t^2)^{|B|-1}}{(1-t)^{2|B|}}$$

In [56], Ziv Ran predicts the Euler characteristic of the relative Hilbert scheme associated to a 1–parameter family of genus g curves with a prescribed number of nodal fibers.

Lemma 4.3.15 ([56, Corollary 2.2]) . *Consider a family of genus* g *curves* $f\colon X\to Y$ *where* $\dim(Y)=1$*; assume the family has* s *singular fibers with exactly one node. Let* $g(Y)$ *denote the genus of* Y*. Then the Euler characteristic of* $\mathrm{Hilb}^d_Y(X)$ *is given by*

$$\chi(\mathrm{Hilb}^d_Y(X)) = (-1)^d\binom{2g-2}{d}(2-2g(Y)) + s\binom{d-2g+2}{d-1}$$

Below, we verify that Ran's formula matches with Corollary 4.3.14 when the Losev–Manin space $\overline{L}_B$ is one dimensional, that is when $|B|=2$.

Theorem 4.3.16. *Assuming* $|B|=2$*, the Euler characteristic of* $\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)$ *predicted by Corollary 4.3.14 and Lemma 4.3.15 coincide.*

Proof. For $|B|=2$, the Losev–Manin family $\mathcal{X}_B\to\overline{L}_B$ has $s=2$ nodal curves. The base of the family $\overline{L}_B$ coincides with the projective line, so $g(\overline{L}_B)=0$. Then Lemma 4.3.15 reads

$$\chi(\mathrm{Hilb}^d_{\overline{L}_B}(\mathcal{X}_B)) = 2(d+1)+2\binom{d+2}{3}.$$

We have used the fact that $\binom{-2}{d}=(-1)^d(d+1)$. The generating series with respect to d is then given by

$$\begin{aligned}\chi_{\mathcal{X}_B/\overline{L}_B}(t) &= 2\sum_{d\geq 0}(d+1)t^d + 2\sum_{d\geq 0}\binom{d+2}{3}t^d\\ &= 2\frac{1}{(1-t)^2}+2\frac{t}{(1-t)^4}\\ &= 2\frac{(1-t+t^2)}{(1-t)^4}.\end{aligned}$$

□

4.3.4 A functional equation for Quot schemes over smooth curves

As a final remark, we note that the functional equation predicted by Huang and Jiang [36, Conjecture 1.5] holds for locally free sheaves on smooth projective genus g curves. The key argument is the existence of a functional equation for the motivic zeta function, proved by Kapranov.

Lemma 4.3.17 ([42, Theorem 1.1.9 (b)]) . *Let C be a smooth projective irreducible curve of genus g. Then the motivic zeta function $\zeta_X(t)$ satisfies the following functional equation:*

$$\zeta_X((\mathbb{L}u)^{-1}) = (\mathbb{L}u^2)^{1-g}\zeta_X(u) \tag{4.6}$$

Theorem 4.3.18. *Let C be a smooth projective irreducible curve over k of genus g. Fix a locally free sheaf E of rank r on C. Then the Quot zeta function $Z_X(E,t)$ satisfies the following functional equation:*

$$Z_X(E,(\mathbb{L}^r t)^{-1}) = (\mathbb{L}^{r^2}t^{2r})^{1-g}Z_X(E,t). \tag{4.7}$$

Proof. Recall that the Quot zeta function over a smooth curve $Z_X(E,t)$ factors into a product of motivic zeta functions (Corollary 2.1.3) as follows

$$Z_X(E,t) = \zeta_X(t)\zeta_X(\mathbb{L}t)\cdot\dots\cdot\zeta_X(\mathbb{L}^{r-1}t).$$

Substituting $t \mapsto (\mathbb{L}^r t)^{-1}$ and re–indexing with $j := r-i-1$ gives

$$Z_X(E,(\mathbb{L}^r t)^{-1}) = \prod_{i=0}^{r-1}\zeta_X((\mathbb{L}^{r-i}t)^{-1}) = \prod_{j=0}^{r-1}\zeta_X((\mathbb{L}\cdot\mathbb{L}^j t)^{-1}).$$

Applying Lemma 4.3.17 to each factor with $u = \mathbb{L}^j t$ yields

$$Z_X(E,(\mathbb{L}^r t)^{-1}) = \prod_{j=0}^{r-1}\left(\mathbb{L}(\mathbb{L}^j t)^2\right)^{1-g}\zeta_X(\mathbb{L}^j t) = \prod_{j=0}^{r-1}\left(\mathbb{L}(\mathbb{L}^j t)^2\right)^{1-g}Z_X(E,t).$$

To conclude, we collect the contributions of the pre–factors as follows

$$\prod_{j=0}^{r-1}\mathbb{L}(\mathbb{L}^j t)^2 = \mathbb{L}^{\sum_{j=0}^{r-1}(1+2j)}t^{2r} = \mathbb{L}^{r+2\frac{r(r-1)}{2}}t^{2r} = \mathbb{L}^{r^2}t^{2r}. \qquad \square$$

Remark 4.3.19. More generally, an analogue of the functional equation (4.7) is known to hold at the level of point counts for Gorenstein curve singularities over $\mathbb{F}_q$; see also [38, Theorem 1.5].

4.4 Further questions and research directions

The results obtained in this chapter establish explicit formulas for the classes and generating functions of relative Quot and Hilbert schemes over Losev–Manin spaces. Several questions remain open, both within this framework and in the broader context of moduli spaces of curves. Here, we outline two natural directions for further investigations.

4.4.1 Independence of the Quot scheme class

As discussed in Corollary 3.3.2, the class of the Quot scheme associated to a smooth morphism depends on the sheaf E only via its rank. A natural question is whether this property persists without the smoothness assumption, that is, for Quot schemes over families of possibly singular curves or higher dimensional objects. More precisely, we ask:

Question 4.4.1. *Consider a morphism $f\colon X \to Y$ with possibly singular fibers, and let E be a rank r locally free sheaf on X. Does the following equality hold in $K_0(\mathrm{Var}_Y)$?*

$$[\mathrm{Quot}^d_{X/Y}(E) \to Y] = [\mathrm{Quot}^d_{X/Y}(\mathcal{O}_X^{\oplus r}) \to Y]$$

As a first approach to this problem, it may be reasonable to impose further assumptions on the morphism, for instance by controlling the relative dimension or the type of singularity.

4.4.2 Extension to other moduli spaces of curves

A second line of inquiry concerns the computation of Quot scheme invariants over other moduli spaces of genus zero nodal curves.

A promising direction for further investigation is provided by Hassett's moduli spaces of weighted stable curves [33]. These moduli spaces generalize both $\overline{M}_{0,n}$ and Losev–Manin spaces by equipping each marked point with a rational number, called *weight*; the role of weights is to determine, within a moduli space, whether or not the curve structure changes as two or more marked points collide with one another.

We briefly recall the relevant definitions for clarity. Fix integers $n \geq 1$, $g \geq 0$.

Definition 4.4.2. A vector $a = (a_1, \dots, a_n) \in \mathbb{Q}^n$ is a *weight vector* if $0 < a_i \leq 1$ for every i and moreover $a_1 + \cdots + a_n > 2 - 2g$.

Definition 4.4.3. A family of nodal curves $\pi\colon (\mathcal{C}, x_1, \dots, x_n) \to B$ is a (g, a)–stable if:

(i) the geometric fibers of π are connected genus g curves with nodal singularities;
(ii) the sections $x_1, \dots, x_n$ do not contain singular points;
(iii) for any subset $\{x_{i_1}, \dots, x_{i_k}\}$ with nonempty intersection, one has $a_{i_1} + \cdots + a_{i_k} \leq 1$;
(iv) (*Stability*) The sheaf $\omega_\pi + \sum_i a_i x_i$ is π–relatively ample, where ω_π is the relative dualizing sheaf.

Theorem 4.4.4 ([33]) **.** *There exists a connected Deligne–Mumford stack $\overline{\mathcal{M}_{g,a}}$ of dimension $3g-3+n$, smooth and proper over $\mathbb{Z}$ parameterizing flat families of (g, a)–stable curves. The corresponding coarse moduli space $\overline{M}_{g,a}$ is projective and has finite quotient singularities.*

The notion of weighted section adds a lot of flexibility by allowing to treat moduli spaces of stable marked curves in great generality.

Example 4.4.5. Important examples of Hassett moduli spaces in genus zero include:

- the Losev–Manin space $\overline{L}_B$, with two marked points of weight 1 (the "poles"), and $|B|$ marked points of weight $\varepsilon < 1/|B|$;
- the moduli space $\overline{M}_{0,n}$ of rational stable curves with n marked points, where all marked points have weight 1;
- the spaces appearing in Kapranov's construction of $\overline{M}_{0,n}$ via a sequence of blowups of $\mathbb{P}^{n-3}$ along certain linear subspaces of increasing dimension. More precisely, the

space obtained at the r-th step, after blowing up linear subspaces of dimension at most $s+r-2$, where $s=1,\dots,n-r-2$, has weight vector

$$a_{r,s}[n] := \Bigg(\underbrace{\frac{1}{n-r-1},\dots,\frac{1}{n-r-1}}_{n-r-1},\frac{s}{n-r-1},\underbrace{1,\dots,1}_{r}\Bigg). \tag{4.8}$$

Remark 4.4.6. In this framework, forgetful morphisms such as the one described in Theorem 4.2.3, are generalized by *reduction morphisms*, which either lower the weight of a section or forget the section entirely.

Question 4.4.7. *Is it possible to carry out computations for Quot zeta functions over Hassett moduli spaces? What can be said about their geometry?*

While Losev–Manin spaces $\overline{L}_B$ provide a particularly tractable model, thanks to the local triviality of the universal family on boundary strata (Theorem 4.2.13), the arguments developed here do not immediately extend to $\overline{M}_{0,n}$, and more generally to Hassett moduli spaces. Indeed, as shown by Example 4.3, the universal family over $\overline{M}_{0,n}$ fails to be locally trivial along boundary strata due to the presence of residual parameters.

Consequently, computations for Quot scheme invariants over Hassett moduli spaces, analogous to the ones carried out in this chapter, may be more accessible at the level of other additive invariants, such as the Euler characteristic or Poincaré polynomials, rather than in the Grothendieck ring.

As the weight vector varies, Hassett moduli spaces exhibit a rich wall–crossing behavior [33, §5]. More precisely, the space of weights admits two wall and chamber decompositions:

- in the *coarse* decomposition, moduli spaces within the same chamber are isomorphic;
- in the *fine* decomposition, spaces in adjacent fine chambers may share the same moduli space but have non isomorphic universal families.

These wall–crossing phenomena raise other natural questions.

Question 4.4.8.

(i) How does the wall–crossing behavior of Hassett moduli spaces affect the geometry of the associated relative Quot schemes?

(ii) How do Quot scheme invariants change as the weights cross a wall? Is it possible to observe discontinuities or perhaps functional relations between the associated generating functions?

A systematic study of these questions could unify the treatment of relative Quot schemes across the various compactifications of $M_{0,n}$.

Appendix A

Lifting Lemma for Abelian categories

In this appendix we formulate a general version of the Lifting Lemma 2.2.3, which plays a key role in the arguments of Chapter 2. More specifically, it serves as a tool to analyze the fibers of the vector bundle morphism described in Theorem 2.2.1. A similar type of result appears in the proof of [20, Theorem 6.4.5], where it is used to construct the tangent–obstruction theory of Quot schemes.

Altough our main application concerns the category of coherent sheaves, the Lifting Lemma holds in full generality for any Abelian category. For completeness, we present the Lemma in its categorical form, accompanied by a detailed proof.

Let $\mathcal{A}$ be an abelian category with enough injective objects, and consider the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & A_1 \arrow[r, "i_A"] \arrow[dr, "\alpha"] & A_2 \arrow[r, "\pi_A"] \arrow[d, "i_2"] & A_3 \arrow[r] & 0 \\
 & & B_2 \arrow[d, "\pi_2"] \arrow[dr, "\beta"] & & \\
0 \arrow[r] & C_1 \arrow[r, "i_C"] & C_2 \arrow[r, "\pi_C"] \arrow[d] & C_3 \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd} \tag{A.1}$$

Lemma A.1.

(i) *The diagram (A.1) induces a short exact sequence*

$$0 \to A_3 \xrightarrow{f} K \xrightarrow{g} C_1 \to 0$$

where $K := \ker(\beta)/im(\alpha)$*.*

(ii) *The splittings of* (i) *are in bijection with sub-objects* $B_1 \subseteq B_2$ *(up to equivalence) such that the inclusion of* B_1 *into the diagram (A.1) extends it to a 3 by 3 diagram with exact rows and columns.*

Proof. (i) Since $\beta \circ i_2 = 0$, the map i_2 factors uniquely via $\ker(\beta)$ as

$$A_2 \xrightarrow{\varphi} \ker(\beta) \overset{i}{\hookrightarrow} B_2.$$

Define f to be map induced by φ after passing to the quotient by $A_1 \cong \mathrm{im}(\alpha)$:

$$f\colon A_2/A_1 \to \ker(\beta)/\mathrm{im}(\alpha).$$

To define the map g, note that the composite map $\pi_C \circ \pi_2 \circ i = \beta \circ i$ is zero, and thus $\pi_2 \circ i$ must factor via C_1 as

$$\ker(\beta) \xrightarrow{\gamma} C_1 \xrightarrow{i_C} C_2.$$

Since $\mathrm{im}(\alpha)$ maps to zero in C_2, the map γ factors through the quotient $\ker(\beta)/\mathrm{im}(\alpha)$ and thus induces the desired map g.

$$g\colon \frac{\ker(\beta)}{\mathrm{im}(\alpha)} \longrightarrow C_1$$

The injectivity of f follows directly from that of i_2, while the surjectivity of g follows directly from that of π_2. In order to check exactness at $\ker(\beta)/\mathrm{im}(\alpha)$, observe that

$$\ker(g) = \frac{\ker(\gamma)}{\mathrm{im}(\alpha)} = \frac{A_2}{A_1} = A_3.$$

(ii) Consider a splitting of the short exact sequence, which is given by a sub-object $D \subseteq K$ mapping isomorphically onto C_1. Let B_1 be the object obtained from D via pullback along the natural projection $\ker(\beta) \to \ker(\beta)/\mathrm{im}(\alpha)$.

$$\begin{tikzcd}
B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ker(\beta) \arrow[d] \\
D \arrow[r] & \frac{\ker(\beta)}{\mathrm{im}(\alpha)}
\end{tikzcd}$$

By construction, B_1 is a sub-object of B_2. Setting $B_3 := B_2/B_1$ completes the middle row of the diagram with an exact sequence

$$0 \to B_1 \xrightarrow{i_B} B_2 \xrightarrow{\pi_B} B_3 \to 0.$$

The inclusions $\mathrm{im}(\alpha) \subseteq B_1 \subseteq \ker(\beta)$ induce maps $i_1\colon A_1 \to B_1$ and $\pi_1\colon B_1 \to C_1$ which make the first column exact. By commutativity, we have induced arrows $i_3\colon A_3 \to B_3$ and $\pi_3\colon B_3 \to C_3$, completing the third column. The surjectivity of π_3 is trivial, while the kernel is

$$\ker(\pi_3) = \ker(\beta)/B_1 \cong A_3$$

where the last isomorphism follows directly from the assumption that $D \cong C_1$. $\square$

Remark A.2. If a short exact sequence $0 \to A \to B \to C \to 0$ splits, the set of all possible splittings is a torsor under $\mathrm{Hom}(C, A)$.

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