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Abstract

Derived categories are deeply explored invariants of algebraic varieties, often studied on their own merit. In this thesis we focus on two categorical analogues of classical algebro-geometric constructions, in the spirit of *homological projective geometry* ([KP21a, §1.5]). We present two results obtained in this theoretical framework.

In Chapter § 1, based on the preprint [Cat23b], we consider two odd isotropic Grassmannians as examples of horospherical varieties of Picard rank one. We construct a full rectangular Lefschetz collection on $\mathsf{IGr}(3, 9)$, confirming in this case a version of Dubrovin's conjecture.

In § 1.6, we study a generalization of this approach to $\mathsf{IGr}(3, 11)$. We construct a rectangular Lefschetz collection and propose an element to complete it to a full exceptional collection. To do so, we partially rely on code developed with SageMath and available online at [Cat23a]. Despite not being able to prove yet that the final collection is exceptional, we have several partial results (e.g. Proposition 1.6.18, Theorem 1.6.24, Theorem 1.6.25).

We hope that the construction of these exceptional collections will be useful to find a general categorical construction for horospherical varieties of Picard rank one. We prove a statement of this kind for Grothendieck groups of horospherical varieties in Appendix § A.

Chapter § 2 is based on the joint work [Cat+23]. In this work we propose a generalization of the notion of nodal singularity to triangulated categories. With some technical caveats, we prove that the derived category of a variety admitting a simple double point is given by the Verdier localization of its categorical resolution by a 2 or 3-spherical object. We propose this notion as a starting point to define "nodal categories".

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Introduction

In this dissertation we present two contributions to the study of derived categories of algebraic varieties in the framework of *homological projective geometry* and *non-commutative algebraic geometry* in the sense of Kontsevich and Rosenberg. We provide an overview of these topics and a motivation for the study of these objects.

Triangulated categories as *non-commutative algebraic varieties*

Vector bundles on manifolds are very classical objects of study. The category of vector bundles on a variety is an *exact category* in the sense of Quillen [Qui10]. It is an additive category with a natural notion of exact sequences, but it is not abelian, preventing its study with algebraic techniques. Specifically, in the case of algebraic varieties, the category of vector bundles is replaced with the category of coherent sheaves. If X is a smooth algebraic variety, we denote the category of coherent sheaves as $\mathbf{Coh}(X)$. The category of vector bundles is a full subcategory of $\mathbf{Coh}(X)$, but additionally, $\mathbf{Coh}(X)$ is abelian.

A classical result of Gabriel and Rosenberg (see for instance the survey [Ros98]) shows that if X and Y are two smooth projective varieties with equivalent categories of coherent sheaves, then X and Y are isomorphic themselves. This shows that the category of coherent sheaves on a variety is a complete invariant.

The derived category of an abelian category (cf. [Huy06, Chapter 2]) is a general categorical construction. The definition of its morphisms is quite technical, but otherwise, its objects are simply complexes of objects in the original abelian category. As $\mathbf{Coh}(X)$ is abelian, we can construct its derived category. The complexes with a finite number of nonzero cohomologies form the bounded derived category of X, which we denote as $\mathbf{D}^{\mathrm{b}}(X)$.

First of all, $\mathbf{D}^{\mathrm{b}}(X)$ is not a complete invariant, i.e. there are varieties with equivalent derived categories which are not isomorphic. There are many examples, see for instance the pioneering work [Muk81], which proved that an abelian variety and its dual are derived equivalent. Nonetheless, the derived category captures a lot of the starting geometry: Bondal and Orlov proved in [BO01, Theorem 2.5] that if X has ample canonical or anticanonical sheaf and $\mathbf{D}^{\mathrm{b}}(X) \cong \mathbf{D}^{\mathrm{b}}(Y)$, then $X \cong Y$. Notice that this result includes Fano varieties.

The derived category of a smooth projective variety is a triangulated category endowed with an auto-equivalence induced by Serre duality, called a Serre functor (cf. [Huy06, Chapter 1]). The results mentioned before (and many others) encouraged the study of triangulated categories as a generalization of algebraic varieties, the so-called *non-commutative algebraic varieties*, in opposition to the derived categories of algebraic varieties which could be referred to as *commutative algebraic varieties* (see for instance [Kon95], [Ros98], [Gin05]).

Homological projective geometry (see [KP21a, §1.5]) aims to study generalizations of classical algebraic geometry constructions in categorical terms. There are two souls to this topic, often interconnected.

The first and most classical part consists in the study of projective geometry constructions and determining how they reflect on the derived category, see for instance [Orl92], [BO02, §4] and related work.

The second part is the study of generalizations of projective geometry constructions in a non-commutative sense, with the hope that the categorical analogue has nicer properties. To better illustrate this idea, we give two examples, relevant to this dissertation.

The first example is given by *categorical resolutions of singularities* (see for instance [Van04], [BO02, §5], [Kuz08b], [KL15]). It is known that for algebraic varieties in dimension 3 or higher there is no minimal resolution of singularities, as there are birational maps (non-morphisms!) between resolutions. In the Minimal Model Program, this problem is addressed by allowing models with mild singularities to find a minimal representative in a class of birational varieties. In the same way, in the derived category setting, we allow non-commutative varieties as possible resolutions of singularities. It is conjectured that varieties with mild singularities have minimal categorical resolutions.

Another example can be found in [KP21a], which improves on the theory of *projective joins*. The projective join of two varieties living in two different projective spaces is constructed in the following way. We first embed the two ambient projective spaces in a common projective space, then we consider the union of all the lines with a point on the first variety and another on the second. The resulting variety is the classical projective join. The projective join is singular unless the two varieties are linear subspaces. To solve this issue, the *categorical join* is a nonsingular non-commutative variety instead, which is actually a categorical resolution of the projective join.

We mention here some techniques and approaches common to both topics of the dissertation.

Lefschetz collections on Fano varieties and residual categories

The derived category $\mathbf{D}^{\mathrm{b}}(X)$ can often be split in smaller subcategories. A semiorthogonal collection is a sequence of triangulated subcategories $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_p \subseteq \mathbf{D}^{\mathrm{b}}(X)$, such that there are no morphisms (in $\mathbf{D}^{\mathrm{b}}(X)$) from a category to one on their left. A semiorthogonal collection is *full* (or a *decomposition*) if the smallest triangulated category containing it is $\mathbf{D}^{\mathrm{b}}(X)$. Ideally, the subcategories in a collection are generated by a single object with the easiest possible structure: an *exceptional object*. A set of semiorthogonal exceptional objects is called an *exceptional collection*.

Recall that the *index* $w \geq 0$ of a Fano variety X is the maximal positive integer such that the canonical class K_X is divisible by w in $\operatorname{Pic} X$. In this case, we write $\omega_X = \mathcal{O}(-w)$, where ω_X is the canonical bundle and $\mathcal{O}(1)$ is a primitive ample line bundle on X.

Lefschetz collections are the bread and butter of homological projective geometry and the study of derived categories. A *rectangular Lefschetz collection* is an exceptional collection formed by a smaller exceptional collection $E_1, \dots, E_p$, the first block, and its twists by a line bundle $\mathcal{O}(1)$, that is:

$$\mathbf{D}^{\mathrm{b}}(X) \supseteq \langle E_1, \dots, E_p, E_1(1), \dots, E_p(1), \dots, E_1(w-1), \dots, E_p(w-1) \rangle.$$

For a more detailed discussion, we refer to § 1.2. In Chapter § 2, we need Lefschetz collections as a key ingredient to cook up a categorical resolution of a nodal variety.

We define the residual category of a rectangular Lefschetz collection, $\mathscr{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$, from the semiorthogonal decomposition:

$$\mathbf{D}^{\mathrm{b}}(X) = \langle\, \mathscr{A}, E_1, \dots, E_p, E_1(1), \dots, E_p(1), \dots, E_1(w-1), \dots, E_p(w-1) \,\rangle,$$

i.e. $\mathscr{A}$ is the right orthogonal to the Lefschetz collection (cf. § 1.4).

The *qualitative Dubrovin's conjecture* claims that a Fano variety admits a full exceptional collection if and only if its quantum cohomology is semisimple. In [KS21, Conjecture 1.3], it is proposed an enhancement of Dubrovin's conjecture that keeps track of the Lefschetz part of the decomposition and the residual category. That is, if the quantum cohomology is semisimple, the conjecture claims that the residual category to a maximal rectangular Lefschetz collection is generated by a fully orthogonal exceptional sequence. According to this claim, in Chapter § 1, we find a full Lefschetz collection of $\mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(3,9))$, as the residual category vanishes. In the study of $\mathsf{IGr}(3,11)$ (in § 1.6), we find the basis of a maximal rectangular Lefschetz collection and construct an object that we conjecture to belong to the residual category (see Theorem 1.6.24).

Homogeneous varieties of simple algebraic groups

The study of derived categories of homogeneous varieties was started by the results of Beilinson [Bei78] on $\mathbb{P}^n$ and Kapranov [Kap88] on Grassmannians and quadrics. After the work of Kapranov, it was conjectured that every homogeneous space G/P, for semisimple G and a parabolic subgroup P, admits an exceptional collection of $\mathbf{D}^{\mathrm{b}}(\mathsf{G}/\mathsf{P})$. We refer to [KP16, § 1.1-§ 1.2] and [Bel20] for a survey on this conjecture. A strong reason for the study of derived categories of homogeneous varieties can be found in the abundance of G-equivariant bundles.

Simple algebraic groups are classified in four infinite series $\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n$ and some sporadic groups. If G is a simple simply connected algebraic group, then there is an equivalence between the category of P-representations and the category of G-equivariant vector bundles on G/P. Since the subgroup P is not reductive, the structure of $\mathbf{Rep}\,\mathsf{P}$ is not straightforward. Restricting our attention to semisimple representations of P, which naturally restrict to representations of the Levi quotient L, we obtain a full exceptional collection in the derived category of equivariant sheaves (see [KP16, Theorem 3.4]), although it is not exceptional in the category of coherent sheaves.

The main tool to handle the cohomology of homogeneous bundles is the Borel–Bott–Weil Theorem, which relates the weights of the irreducible representations of L with the cohomology of their associated bundles. In Theorem 1.3.1, Theorem 1.3.10 and Theorem 2.2.29 we provide the detailed statements necessary for our applications.

The theory of homogeneous spaces is crucial to both chapters. In Chapter § 1, even though *odd isotropic Grassmannians* are not homogeneous themselves, they are subvarieties of homogeneous spaces of type A_n and C_n (respectively standard and even isotropic Grassmannians). As a consequence, we can use these embedding and the Borel–Bott–Weil Theorem to obtain some vanishing results, e.g. Corollary 1.3.14 and some ad-hoc computations, e.g. § 1.4, § 1.6. In Chapter § 2, several computations are possible because the exceptional locus of the resolution of a nodal point is a quadric, which is a homogeneous space for groups of type B_n or D_n, depending on dimension.

Overview of the dissertation

In this dissertation we present two independent contributions to homological projective geometry.

In Chapter § 1 (based on the preprint [Cat23b]) we were motivated to find a categorical analogue of the construction that associates a *horospherical variety of Picard rank one* to a pair of homogeneous varieties following the spirit of [KP21a]. We obtain a full exceptional collection on $\mathsf{IGr}(3, 9)$. We generalize this approach to $\mathsf{IGr}(3, 11)$ in § 1.6. In the latter section, we construct the basis of a rectangular collection and propose a object to complete it to a full (non rectangular) Lefschetz collection. To do so, we partially rely on code developed with SageMath and available online at [Cat23a]. Despite not being able to prove yet that the final collection is exceptional, we have several partial results (e.g. Proposition 1.6.18, Theorem 1.6.24, Theorem 1.6.25).

We expect that the study of exceptional collections in $\mathsf{IGr}(3, 9)$ and $\mathsf{IGr}(3, 11)$ can clarify how to construct an exceptional collection on general odd isotropic Grassmannians and on other horospherical varieties, starting from exceptional collections on their homogeneous orbits.

In Chapter § 2 (based on the joint work [Cat+23]) we present a result on the categorical resolution of a nodal variety. The existence of a categorical resolution in this case is provided by [Kuz08b]. In this work, we characterize the derived category of a quasi-projective nodal variety as the Verdier localization of its categorical resolution by a 2 or 3-spherical object, depending on the parity of the dimension of the starting variety.

Chapter 1

The derived category of some odd isotropic Grassmannians

1.1 Introduction

The bounded derived category of coherent sheaves of a smooth projective variety X is one of its most remarkable invariants. The derived category (from now on, denoted as $\mathbf{D}^{\mathrm{b}}(X)$) can often be split in smaller triangulated subcategories. In the best case, these subcategories are generated by a single object with the easiest possible structure, an exceptional object. The work of Kapranov [Kap88] sparked the interest in the study of the derived category of homogeneous spaces by finding full exceptional collections on Grassmannians and quadrics. This led to the following conjecture. As in the rest of the work, we fix $\mathbb{C}$ as base field.

Conjecture 1.1.1. *If* G *is a semisimple algebraic group over* $\mathbb{C}$ *and* $\mathsf{P} \subset \mathsf{G}$ *is a parabolic subgroup, then there is a full exceptional collection in* $\mathbf{D}^{\mathrm{b}}(\mathsf{G}/\mathsf{P})$.

We refer to [KP16, § 1.1-§ 1.2] and [Bel20] for a survey on this conjecture. In addition, it was proved in [KP16, Theorem 1.2] that any G/P for G classical admits an exceptional collection of maximal possible length $r = \operatorname{\mathsf{rank}} K_0(\mathsf{G}/\mathsf{P}) = \operatorname{\mathsf{dim}} \mathrm{H}^\bullet(\mathsf{G}/\mathsf{P}, \mathbb{C})$, even though its fullness is not known.

Dubrovin's conjecture [Dub98, Conjecture 4.2.2] claims that the existence of a full exceptional collection is equivalent to the generic semisimplicity of the big quantum cohomology ring. We consider the following version of the conjecture, which additionally involves Lefschetz collections. For a more detailed discussion, we refer to [KS21, § 1]. Recall that a rectangular Lefschetz collection with respect to a line bundle $\mathcal{L}$ is given by an exceptional collection and its twists by powers of $\mathcal{L}$ (cf. Definition 1.2.4). Recall that the *index* $w \geq 0$ of a Fano variety X is the maximal positive integer such that the canonical class K_X is divisible by w in $\operatorname{Pic} X$. In that case, we write $\omega_X = \mathcal{O}(-w)$, where ω_X is the canonical bundle and $\mathcal{O}(1)$ is a primitive ample line bundle on X. We denote the big and small quantum cohomology of X by $\mathrm{BQH}(X)$ and $\mathrm{QH}(X)$. Recall that $\mathrm{QH}(X)$ is an algebra over the quantum parameters $\mathbb{Q}[q_1, \dots, q_s]$ corresponding to the functions on the affine space $\operatorname{Pic} X \otimes \mathbb{Q}$. Consider the algebra $\mathrm{QH}(X)_{can} = \mathrm{QH}(X) \otimes_{\mathbb{Q}[q_1,\dots,q_s]} \mathbb{C}$, where $\mathbb{Q}[q_1, \dots, q_s] \to \mathbb{C}$ is induced by $K_X \in \operatorname{Pic} X \otimes \mathbb{Q}$. Notice that $\mathrm{QH}(X)_{can} \cong \mathrm{H}^\bullet(X, \mathbb{C})$ as a vector space, but it is endowed with a different (quantum) multiplication.

Conjecture 1.1.2 ([KS21, Conjecture 1.3.(i)]). *Let X be a Fano variety with index w and assume that $\mathrm{BQH}(X)$ is generically semisimple. If the class $[K_X] \in \mathrm{H}^2(X, \mathbb{C}) \subset \mathrm{QH}(X)_{can}$ is invertible (with respect to the quantum multiplication), then there is an exceptional collection $E_1, \dots, E_p$ extending to a full rectangular Lefschetz collection of $\mathbf{D}^{\mathrm{b}}(X)$, where $p = \frac{1}{w} \operatorname{\mathsf{dim}} \mathrm{H}^\bullet(X, \mathbb{C})$.*

A good testing ground for this conjecture is the class of *horospherical varieties*, introduced by [Pas08]. These are normal algebraic varieties on which a reductive group acts with an open orbit isomorphic to a torus bundle over a homogeneous variety. Naturally, this includes homogeneous spaces and toric varieties. Therefore, we believe that studying exceptional collections on horospherical varieties can shed light on exceptional collections in homogeneous spaces.

Let G be a semisimple algebraic group over $\mathbb{C}$. Pasquier proved that except for homogeneous varieties, any smooth G-horospherical variety of Picard rank one is a G-variety which has exactly two disjoint closed orbits Y, Z under the action of G. The stabilizers of Y and Z are maximal parabolic subgroups $\mathsf{P}_Y, \mathsf{P}_Z \subset \mathsf{G}$. The following Theorem 1.1.3 provides a classification of horospherical varieties. We write $\mathrm{Type}(\mathsf{G})$ for the Dynkin type of G and P_k for the maximal parabolic subgroup of G associated to the k-th fundamental weight with respect to Bourbaki notation.

Theorem 1.1.3 ([Pas09, Theorem 0.1]). *Let X be a smooth projective G-horospherical variety of Picard rank one. Then either X is homogeneous, or X can be constructed from a triple $(\mathrm{Type}(\mathsf{G}), \mathsf{P}_Y, \mathsf{P}_Z)$ belonging to the following list:*

1. $(\mathsf{B}_n, \mathsf{P}_n, \mathsf{P}_{n-1})$ *with* $n \geq 3$;
2. $(\mathsf{B}_3, \mathsf{P}_3, \mathsf{P}_1)$;
3. $(\mathsf{C}_n, \mathsf{P}_k, \mathsf{P}_{k-1})$ *with* $n \geq 2$ *and* $k \in \{2, \dots, n\}$;
4. $(\mathsf{F}_4, \mathsf{P}_3, \mathsf{P}_2)$;
5. $(\mathsf{G}_2, \mathsf{P}_2, \mathsf{P}_1)$.

An explicit construction of X out of $(\mathrm{Type}(\mathsf{G}), \mathsf{P}_Y, \mathsf{P}_Z)$ can be found in [Pas09] and it is summarized in [Gon+22, Proposition 1.6]. As the homogeneous pieces Y and Z are enough to identify the horospherical variety X, it would be interesting to describe $\mathbf{D}^{\mathrm{b}}(X)$ in terms of the homogeneous varieties Y, Z. As a first step towards such a description, we prove in Proposition A.0.3 that

$$K_0(X) \cong K_0(Y) \oplus K_0(Z).$$

Following the computations of [Gon+22], we expect that if X is a horospherical variety of Picard rank one, then $\mathbf{D}^{\mathrm{b}}(X)$ admits a full exceptional collection and the big quantum cohomology ring $\mathrm{BQH}(X)$ is generically semisimple.

We summarize here the cases of horospherical varieties for which full exceptional collections are already known, grouped as in the classification of Theorem 1.1.3:

2. $(\mathsf{B}_3, \mathsf{P}_3, \mathsf{P}_1)$ by [Kuz06, §6.2];

3. $(\mathsf{C}_n, \mathsf{P}_2, \mathsf{P}_1)$ by [Pec13], [Kuz08a]; $(\mathsf{C}_3, \mathsf{P}_3, \mathsf{P}_2)$ by [Fon22];

5. $(\mathsf{G}_2, \mathsf{P}_2, \mathsf{P}_1)$ by [Gon+22].

The most interesting case in the list of Theorem 1.1.3 is $(\mathsf{C}_n, \mathsf{P}_k, \mathsf{P}_{k-1})$. The corresponding horospherical variety X is the odd isotropic Grassmannian $\mathsf{IGr}(k, 2n+1)$ of k-dimensional subspaces in a $(2n+1)$-dimensional space endowed with a skew-symmetric form ψ of maximal possible rank $2n$. We refer to [Mih07] for a survey on the properties of these varieties. In [Gon+22, Theorem 5.17], the authors provide a presentation of the small quantum cohomology ring and show its semisimplicity for $\mathsf{IGr}(2, 2n+1)$ and $\mathsf{IGr}(3, 7)$. Based on this description, Belmans could verify computationally (cf. [Bel21]) that the small quantum cohomology of $\mathsf{IGr}(k, 2n+1)$ for $1 \leq k \leq n \leq 7$ is generically semisimple. Consequently, the extended Dubrovin conjecture (Conjecture 1.1.2) predicts that in all these cases $\mathsf{IGr}(k, 2n+1)$ has a full exceptional collection. The main result of this paper, Theorem 1.1.4 stated below, proves this on $\mathsf{IGr}(3, 9)$, the first case not covered by previous results. The prediction of Conjecture 1.1.2 in this case claims there should be an exceptional collection of 8 elements that extends to a rectangular Lefschetz collection generating $\mathbf{D}^{\mathrm{b}}(X)$.

To state our result we need to introduce some notation. Let V be a 9-dimensional vector space endowed with a skew-symmetric form ψ of maximal rank 8. Let us fix $X = \mathsf{IGr}(3, 9)$, the isotropic Grassmannian of 3-subspaces in V with respect to ψ. Let $\mathcal{U}$ be the tautological subbundle on X, recall that

$$\mathcal{O}(1) = \wedge^3 \mathcal{U}^*$$

is the ample generator of $\operatorname{Pic} X$. In this work, we construct a full rectangular Lefschetz collection of $\mathbf{D}^{\mathrm{b}}(X)$ with respect to $\mathcal{O}(1)$.

Given a GL_3-dominant weight λ, we denote by $\mathcal{U}^\lambda$ the bundle associated to the irreducible representation of GL_3 of highest weight λ and the frame bundle of $\mathcal{U}^*$, so that

$$\mathcal{U}^{m,0,0} = S^m \mathcal{U}^*, \quad \mathcal{U}^{0,0,-m} = S^m \mathcal{U}, \quad \mathcal{U}^{l,l,l} = \mathcal{O}(l).$$

In other words, $\mathcal{U}^\lambda$ is obtained by an application of the Schur functor associated to λ to the vector bundle $\mathcal{U}^*$

Consider the following collections of vector bundles on $\mathsf{IGr}(3, 9)$:

$$\begin{aligned} \mathbf{B}_1 &= \{\, \mathcal{U}^{0,0,-2},\ \mathcal{U}^{0,0,-1},\ \mathcal{U}^{1,0,-1},\ \mathcal{U}^{2,0,-1},\ \mathcal{U}^{0,0,0},\ \mathcal{U}^{1,0,0},\ \mathcal{U}^{2,0,0} \,\}, \\ \mathbf{B}_2 &= \{\, \mathcal{U}^{0,0,-1},\ \mathcal{U}^{1,0,-1},\ \mathcal{U}^{2,0,-1},\ \mathcal{U}^{0,0,0},\ \mathcal{U}^{1,0,0},\ \mathcal{U}^{2,0,0},\ \mathcal{U}^{3,0,0} \,\}. \end{aligned}$$

Notice that $\mathbf{B}_1$ and $\mathbf{B}_2$ have 6 elements in common, while their union has length 8. We check in Corollary 1.4.10 that both $\mathbf{B}_1$ and $\mathbf{B}_2$ induce non full Lefschetz collections of length 7. One could hope that $\mathbf{B}_1 \cup \mathbf{B}_2$ is an exceptional collection, but this is not the case. In fact, by Lemma 1.4.7, we have

$$\mathrm{Hom}^\bullet(\mathcal{U}^{3,0,0},\mathcal{U}^{0,0,-2}) = \mathbb{C}[-4], \qquad \mathrm{Hom}^\bullet(\mathcal{U}^{0,0,-2},\mathcal{U}^{3,0,0}) \neq 0,$$

so we cannot have both $\mathcal{U}^{3,0,0}$ and $\mathcal{U}^{0,0,-2}$ in the same exceptional collection.

To solve this problem, we will replace the bundle $\mathcal{U}^{3,0,0} \in \mathbf{B}_2 \setminus \mathbf{B}_1$ by another object $\mathcal{H}$, using a procedure analogous to the one used in [Gus20] and [Nov20]. More precisely, we consider the following bicomplex:

$$\begin{tikzcd}
\wedge^3 V^* \otimes \mathcal{O} \arrow[r] & \wedge^2 V^* \otimes \mathcal{U}^{1,0,0} \arrow[r] & V^* \otimes \mathcal{U}^{2,0,0} \arrow[r] & \mathcal{U}^{3,0,0} \\
\wedge^2 V^* \otimes \mathcal{U}^{0,0,-1} \arrow[r] \arrow[u] & V^* \otimes \mathcal{U}^{1,0,-1} \arrow[r] \arrow[u] & \mathcal{U}^{2,0,-1}, \arrow[u] &
\end{tikzcd} \tag{1.1}$$

where the rows are given by the stupid truncations of the staircase complexes associated to $\mathcal{U}^{3,0,0}$ and $\mathcal{U}^{2,0,-1}$ (cf. Theorem 1.3.6) and the vertical arrows are induced by the form ψ. The object $\mathcal{H}$ is the totalization of this bicomplex. See (1.38) and Proposition 1.4.13 for an alternative description of the object $\mathcal{H}$. Denote by $\mathbf{B}$ the following collection of bundles on X:

$$\mathbf{B} = \{\mathcal{H}, \mathcal{U}^{0,0,-2}, \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}\} = \{\mathcal{H}\} \cup \mathbf{B}_1. \tag{1.2}$$

Our main result is the following.

Theorem 1.1.4. *Let $X = \mathsf{IGr}(3,9)$. Then $\mathbf{B}$ is an exceptional collection of Sp_9-equivariant objects and it extends to a full rectangular Lefschetz collection given by:*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{B}, \mathbf{B}(1), \mathbf{B}(2), \mathbf{B}(3), \mathbf{B}(4), \mathbf{B}(5), \mathbf{B}(6) \rangle.$$

The proof proceeds as follows. We prove that $\mathbf{B}_1$ and $\mathbf{B}_2$ defined before both induce Lefschetz bases of length 7. Additionally, they satisfy $\mathrm{Hom}^\bullet(\mathbf{B}_2(l), \mathbf{B}_1) = 0$ for $l = 1, \dots, 6$. Let $\mathscr{B}$ be the triangulated category generated by the objects in $\mathbf{B}_1$ and $\mathbf{B}_2$. Then, the collection of triangulated subcategories $\mathscr{B}, \mathscr{B}(1), \dots, \mathscr{B}(6)$ is semiorthogonal.

Using the staircase complexes associated to $\mathcal{U}^{3,0,0}$ and $\mathcal{U}^{2,0,-1}$, we prove that $\mathcal{H}$ is the left mutation of $\mathcal{U}^{3,0,0}$ through $\mathbf{B}_1$, hence $\mathbf{B} = \{\mathcal{H}\} \cup \mathbf{B}_1$ is a full exceptional collection of $\mathscr{B}$ (actually, they are all vector bundles, cf. Remark 1.4.16), proving that $\mathbf{B}$ is a Lefschetz basis.

To prove semiorthogonality we mainly rely on the embedding of X in $\mathsf{IGr}(3,10)$, using the fact that $\mathsf{IGr}(3,10)$ is homogeneous under the action of Sp_{10}. We prove a vanishing criterion for bundles on odd isotropic Grassmannians (Corollary 1.3.14) as application of Borel–Bott–Weil Theorem. To prove that $\mathbf{B}$ induces a full Lefschetz collection, we follow the algorithmic method developed in [Nov20].

Overview of the work. In §§ 1.2 and 1.3 we cover the preliminaries and prove more general versions (Proposition 1.3.11 and Corollary 1.3.14) of cohomology vanishing lemmas previously established in [Fon22]. In § 1.4.1, we prove that $\mathbf{B}_1$ and $\mathbf{B}_2$ are exceptional collections extending to (non full) rectangular Lefschetz collections. In § 1.4.2, we show that the object $\mathcal{H}$ is the left mutation of $\mathcal{U}^{3,0,0}$ through $\mathbf{B}_1$ and that $\mathbf{B}$ is an exceptional collection. In § 1.5, we show the fullness of the rectangular Lefschetz collection induced by $\mathbf{B}$.

Notation. We fix $\mathbb{C}$ as base field. Unless otherwise noted, $X = \mathsf{IGr}(3,9)$ with $\mathbf{D}^{\mathrm{b}}(X)$ its bounded derived category of coherent sheaves. We define the graded vector space $\mathrm{Hom}^\bullet(-,-)$ as $\oplus_i \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(-,-[i])[-i]$.

1.2 Exceptional collections

In this section X is an arbitrary smooth projective variety. A standard reference for this foundational material, except where otherwise noted, is [BK89]. Let $\mathscr{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be a full triangulated subcategory. We define the right and left orthogonal of $\mathscr{A}$ as:

$$\mathscr{A}^\perp = \{E \in \mathbf{D}^{\mathrm{b}}(X) \,|\, \mathrm{Hom}(\mathscr{A}, E) = 0\}, \qquad {}^\perp\mathscr{A} = \{E \in \mathbf{D}^{\mathrm{b}}(X) \,|\, \mathrm{Hom}(E, \mathscr{A}) = 0\}.$$

Definition 1.2.1. *Let $\mathscr{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be a full triangulated subcategory. Then $\mathscr{A}$ is called admissible if the embedding functor admits left and right adjoints.*

Definition 1.2.2. *A sequence of full triangulated subcategories $\mathscr{A}_1, \dots, \mathscr{A}_p \subseteq \mathbf{D}^{\mathrm{b}}(X)$ is called a semiorthogonal collection if $\mathscr{A}_i \subseteq \mathscr{A}_j^\perp$ for $1 \le i < j \le p$. In that case, we denote the smallest triangulated subcategory containing all the $\mathscr{A}_i$ by $\langle \mathscr{A}_1, \dots, \mathscr{A}_p \rangle \subseteq \mathbf{D}^{\mathrm{b}}(X)$. We say that a semiorthogonal collection $\mathscr{A}_1, \dots, \mathscr{A}_p$ is a decomposition of $\mathbf{D}^{\mathrm{b}}(X)$ if $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathscr{A}_1, \dots, \mathscr{A}_p \rangle$ and every $\mathscr{A}_i$ is admissible.*

If $\mathscr{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ is admissible, then the following:

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathscr{A}^\perp, \mathscr{A} \rangle, \qquad \mathbf{D}^{\mathrm{b}}(X) = \langle \mathscr{A}, {}^\perp\mathscr{A} \rangle,$$

are semiorthogonal decompositions.

Definition 1.2.3. *An object $E \in \mathbf{D}^{\mathrm{b}}(X)$ is exceptional if $\mathrm{Hom}^\bullet(E, E) = \mathbb{C}$.*

An exceptional collection is a sequence of exceptional objects $E_1, \dots, E_p \in \mathbf{D}^{\mathrm{b}}(X)$ with $E_i \in E_j^\perp$ for $1 \le i < j \le p$. An exceptional collection is said to be full if $\mathbf{D}^{\mathrm{b}}(X) = \langle E_1, \dots, E_p \rangle$.

Recall that every subcategory $\mathscr{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ generated by an exceptional collection is admissible, cf. [BK89, Theorem 2.10].

The notion of rectangular Lefschetz decomposition is a natural way of studying decompositions of $\mathbf{D}^{\mathrm{b}}(X)$ knowing a line bundle on X. Recall that the *index* $w \geq 0$ of a Fano variety X is the maximal positive integer such that $\omega_X = \mathcal{O}(-w)$, where ω_X is the canonical bundle and $\mathcal{O}(1)$ is a primitive ample line bundle on X.

Definition 1.2.4 ([Kuz07, Definition 4.1])**.** *A rectangular Lefschetz exceptional collection of* $\mathbf{D}^{\mathrm{b}}(X)$ *with respect to* $\mathcal{O}(1)$ *is an exceptional collection of the form:*

$$\langle\, E_1, \dots, E_p, E_1(1), \dots, E_p(1), \dots, E_1(w-1), \dots, E_p(w-1)\,\rangle \subseteq \mathbf{D}^{\mathrm{b}}(X)\,.$$

We say that $E_1, \dots, E_p$ *is the basis of the rectangular Lefschetz collection and* p *is its length.*

Recall that for any $0 \neq F \in \mathbf{D}^{\mathrm{b}}(X)$, we have by Serre duality

$$\mathrm{Hom}^{\bullet}(F(w), F) = \mathrm{Hom}^{\bullet}(F, F)^{*}[-\mathsf{dim}\, X] \neq 0;$$

therefore, w is the maximal number of twists of $E_1, \dots, E_p$ which can be semiorthogonal.

We state here for completeness a criterion to verify that an exceptional collection extends to a rectangular Lefschetz collection.

Lemma 1.2.5 ([Fon22, Lemma 2.18])**.** *A collection of objects* $E_1, \dots, E_p \in \mathbf{D}^{\mathrm{b}}(X)$ *is a basis of a rectangular Lefschetz collection if and only if*

- $E_1, \dots, E_p$ *is an exceptional collection, and*
- $\mathrm{Hom}^{\bullet}(E_j(t), E_i) = 0$ *for* $1 \leq i \leq j \leq p$ *and all* $1 \leq t \leq w-1$,

where w *is the index of* X*. Let* $\mathscr{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ *be the smallest full triangulated subcategory containing* $E_1, \dots, E_p$*. If only the second condition holds, then* $\mathscr{A}, \mathscr{A}(1), \dots, \mathscr{A}(w-1)$ *is only a semiorthogonal collection.*

Proof. The second hypothesis proves one half of the required semiorthogonality conditions; that is $\mathrm{Hom}^{\bullet}(E_j(t), E_i) = 0$ for $j \geq i$. The other half follows by Serre duality. This proves the second claim.

Taking the first hypothesis into account, we also obtain the first claim. □

We now introduce the mutation functors with respect to an admissible subcategory $\mathscr{A}$. For any object $F \in \mathbf{D}^{\mathrm{b}}(X)$ there are unique and functorial triangles:

$$\mathbb{R}_{\mathscr{A}} F \to F \to F' \quad \text{and} \quad F'' \to F \to \mathbb{L}_{\mathscr{A}} F,$$

where $F', F'' \in \mathscr{A}$ while $\mathbb{R}_{\mathscr{A}} F \in {}^{\perp}\mathscr{A}$ and $\mathbb{L}_{\mathscr{A}} F \in \mathscr{A}^{\perp}$. Both $\mathbb{R}_{\mathscr{A}}$ and $\mathbb{L}_{\mathscr{A}}$ vanish on $\mathscr{A}$ and induce mutually inverse equivalences between ${}^{\perp}\mathscr{A}$ and $\mathscr{A}^{\perp}$.

The mutation functors $\mathbb{R}_{\mathscr{A}}$ and $\mathbb{L}_{\mathscr{A}}$ take a more explicit form when $\mathscr{A} = \langle\, E_1, \dots, E_p\,\rangle$, where $E_1, \dots, E_p$ is an exceptional collection. It is immediate to verify:

$$\mathbb{R}_{\langle\, E_1, \dots, E_p\,\rangle} = \mathbb{R}_{E_p} \circ \dots \circ \mathbb{R}_{E_1} \quad \text{and} \quad \mathbb{L}_{\langle\, E_1, \dots, E_p\,\rangle} = \mathbb{L}_{E_1} \circ \dots \circ \mathbb{L}_{E_p}.$$

Moreover, for any exceptional object E we can write:

$$\mathbb{R}_E F = \mathrm{Cone}(F \to \mathrm{Hom}^\bullet(F,E)^* \otimes E)[-1] \quad \text{and} \quad \mathbb{L}_E F = \mathrm{Cone}(\mathrm{Hom}^\bullet(E,F) \otimes E \to F),$$

where the morphisms are the canonical coevaluation and evaluation maps.

1.3 Cohomology on isotropic Grassmannians

1.3.1 Grassmannians

Let V be an m-dimensional vector space and let $\mathsf{GL}(V)$ be the corresponding linear group. For $1 \leq k \leq m-1$, we denote the Grassmannian of k-planes in V by $\mathsf{Gr}(k,V)$. We recall the tautological exact sequence of bundles on $\mathsf{Gr}(k,V)$:

$$0 \to \mathcal{U} \to V \otimes \mathcal{O} \to \mathcal{Q} \to 0, \tag{1.3}$$

where $\mathcal{U}$ is the tautological subbundle of rank k and $\mathcal{Q}$ is the tautological quotient bundle of rank $m-k$. Note that the canonical sheaf satisfies:

$$\omega_{\mathsf{Gr}(k,V)} = \mathcal{O}(-m),$$

where $\mathcal{O}(1) = \det \mathcal{U}^*$ is the ample generator of the Picard group. We use the following notation:

$$\mathcal{U}^\perp = \mathcal{Q}^*. \tag{1.4}$$

In the coming sections, we discuss the preliminaries that will be needed in the computations on $\mathbf{D}^b(\mathsf{Gr}(k,m))$: Schur functors, Borel–Bott-Weil Theorem, the Littlewood–Richardson rule and Koszul and staircase complexes. The only new result in this section is Proposition 1.3.8, where we describe the restriction of a staircase complex to a smaller Grassmannian.

Schur functors, Borel–Bott-Weil Theorem and Littlewood–Richardson rule

We summarize here some basic facts about the representation theory of GL_k and the associated Schur functors. The weight lattice of GL_k is isomorphic to $\mathbb{Z}^k$, and its subset

$$P_k^+ = \{\lambda \in \mathbb{Z}^k \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k\}$$

is the cone of dominant weights. Given $\lambda \in P_k^+$, we denote by V_{GL}^λ the $\mathsf{GL}(V)$-representation with highest weight λ. In particular, the symmetric and wedge powers are:

$$V_{\mathsf{GL}}^{p,0,\ldots,0} = S^p V^*, \qquad V_{\mathsf{GL}}^{1,\ldots,1,0\ldots,0} = \wedge^p V^*.$$

If $\lambda \in P_k^+$, we denote:

$$-\lambda = (-\lambda_k, -\lambda_{k-1}, \ldots, -\lambda_1) \qquad \text{and} \qquad |\lambda| = \sum \lambda_i.$$

We recall that:

$$V_{\mathsf{GL}}^{-\lambda} = (V_{\mathsf{GL}}^{\lambda})^* = (V_{\mathsf{GL}}^*)^{\lambda}.$$

There is a natural partial ordering on P_k^+ given by

$$\mu \subseteq \lambda \Leftrightarrow \mu_i \leq \lambda_i \quad \text{for all} \quad i = 1, \dots, k. \tag{1.5}$$

The Weyl group of GL_k is isomorphic to the symmetric group $\mathbf{S}_k$, acting on $\mathbb{Z}^k$ by permutation. Let $\ell : \mathbf{S}_k \to \mathbb{Z}$ be the length function. We recall that the length of an element σ of the Weyl group is computed in the following way: the Weyl group is generated by reflections by simple roots, thus $\ell(\sigma)$ is the smallest number of reflection by simple roots necessary to decompose σ. In the case of GL_k, reflections by simple roots act on $\mathbb{Z}^k$ as inversions of the basis vectors. The sum of the fundamental weights of GL_k is

$$\rho_{\mathsf{GL}} = (k, k-1, \dots, 1).$$

Let $\mathcal{E}$ be a vector bundle of rank k on an algebraic variety and consider the principal GL_k-bundle of its frames. Let $\mathcal{E}^\lambda$ be the vector bundle associated to the representation $V_{\mathsf{GL}}^{\lambda}$. Once we fixed $\lambda \in P_k^+$, the functor $\mathbf{VB}_k(X) \to \mathbf{VB}(X)$, $\mathcal{E} \mapsto \mathcal{E}^\lambda$ defined as above is the *Schur functor* associated to λ. In particular, the symmetric and wedge powers of a vector bundle are Schur functors:

$$\mathcal{E}^{p,0,\dots,0} = S^p\mathcal{E}^*, \qquad \mathcal{E}^{1,\dots,1,0\dots,0} = \wedge^p\mathcal{E}^*. \tag{1.6}$$

It also follows that:

$$\mathcal{E}^{-\lambda} = (\mathcal{E}^\lambda)^* = (\mathcal{E}^*)^\lambda. \tag{1.7}$$

We are now ready to state the classical Borel–Bott–Weil theorem.

Theorem 1.3.1 (Borel–Bott–Weil for GL, [Dem76]). *Let $\alpha \in P_k^+$ and $\beta \in P_{n-k}^+$. Consider*

$$(\gamma_1, \dots, \gamma_n) := (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}) \in \mathbf{Z}^n,$$

the concatenation of α and β. Assume that all entries of $\gamma + \rho$ are distinct integers. Let $\sigma \in \mathbf{S}_n$ be the unique element of the Weyl group such that $\gamma' = \sigma(\gamma+\rho)-\rho$ is GL_n-dominant, then we have:

$$\mathrm{H}^\bullet(\mathsf{Gr}(k,V), \mathcal{U}^\alpha \otimes \mathcal{Q}^\beta) = \Sigma^{\gamma'} V^*[-\ell(\sigma)].$$

Otherwise:

$$\mathrm{H}^\bullet(\mathsf{Gr}(k,V), \mathcal{U}^\alpha \otimes \mathcal{Q}^\beta) = 0.$$

The *Littlewood–Richardson rule* allows one to decompose the representation $V_{\mathsf{GL}}^\alpha \otimes V_{\mathsf{GL}}^\beta$ as a direct sum of representations V_{GL}^γ, possibly with multiplicities. Considering their associated Schur functors, we obtain an induced GL-equivariant decomposition of $\mathcal{E}^\alpha \otimes \mathcal{E}^\beta$ in $\mathcal{E}^\gamma$. We will often use the notation

$$\mathcal{E}^\gamma \Subset \mathcal{E}^\alpha \otimes \mathcal{E}^\beta$$

to say that $\mathcal{E}^\gamma$ is a direct summand of the right hand side. A statement of the Littlewood–Richardson rule is out of the scope of this work, for that we refer to [Wey03, Theorem 2.3.4]. Instead, we describe some simple special cases.

Proposition 1.3.2 (Pieri's Formulas, [Wey03, Corollary 2.3.5]). *Let $\lambda \in P_k^+$ and let j be a positive integer. Then, there are direct sum decompositions*

$$\mathcal{E}^\lambda \otimes S^j \mathcal{E}^* = \bigoplus_{\gamma \in \mathrm{HS}_\lambda^j} \mathcal{E}^\gamma \quad \text{and} \quad \mathcal{E}^\lambda \otimes \wedge^j \mathcal{E}^* = \bigoplus_{\gamma \in \mathrm{VS}_\lambda^j} \mathcal{E}^\gamma$$

where

$$\mathrm{HS}_\lambda^j = \{\gamma \in P_k^+ \text{ with } |\gamma| - |\lambda| = j \text{ and } \gamma_1 \geq \lambda_1 \geq \gamma_2 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \gamma_k\},$$
$$\mathrm{VS}_\lambda^j = \{\gamma \in P_k^+ \text{ with } |\gamma| - |\lambda| = j \text{ and } \lambda_i + 1 \geq \gamma_i \geq \lambda_i \text{ for all } i = 1, \dots, k\}.$$

In Proposition 1.3.2 above, the notation HS and VS comes from the representation of weights as Young diagrams. Indeed, if $\gamma \in \mathrm{HS}_\lambda^j$, then $\gamma \setminus \lambda$ is a *horizontal strip* (j boxes, at most one box per column), while if $\gamma \in \mathrm{VS}_\lambda^j$, then $\gamma \setminus \lambda$ is a *vertical strip* (j boxes, at most one box per row). The following corollary is an immediate application of Pieri's formulas.

Corollary 1.3.3. *Let V be a vector space of rank k. Alternatively, let $\mathcal{E}$ be a vector bundle of rank k. Let $\lambda \in P_k^+$ and let l be an integer. Then:*

$$V_{\mathsf{GL}}^\lambda \otimes V_{\mathsf{GL}}^{(l,l,\dots,l)} = V_{\mathsf{GL}}^{\lambda+(l,l,\dots,l)}, \qquad \mathcal{E}^\lambda \otimes \mathcal{E}^{(l,l,\dots,l)} = \mathcal{E}^{\lambda+(l,l,\dots,l)},$$

where $(l, l, \dots, l)$ denotes the vector of P_k^+ with k entries equal to l.

During the course of the work, we will have to study tensor products of representations where neither weights are elementary enough to apply Pieri's formulas. In those cases, we will use the following lemmas.

Lemma 1.3.4 ([Wey03, Proposition 2.3.1], [Gus20, Lemma 3.3]). *Let $\alpha, \beta \in P_k^+$. Suppose*

$$\mathcal{E}^\gamma \Subset \mathcal{E}^{-\alpha} \otimes \mathcal{E}^\beta.$$

Then

$$\beta_k - \alpha_{k+1-i} \leq \gamma_i \leq \beta_i - \alpha_k \quad \text{for all} \quad 1 \leq i \leq k \quad \text{and} \quad |\gamma| = |\beta| - |\alpha|.$$

As a straightforward consequence:

$$\gamma_i - \gamma_{i+1} \leq (\beta_i - \beta_k) + (\alpha_{k-i} - \alpha_k) \quad \text{for all} \quad 1 \leq i < k. \tag{1.8}$$

Lemma 1.3.5. *Let $\alpha, \beta \in P_k^+$, and let $l \in \mathbb{Z}$. Let*

$$\mathcal{E}^{-\alpha} \otimes \mathcal{E}^\beta = \bigoplus_\gamma \mathcal{E}^\gamma$$

be the Littlewood–Richardson decomposition. Then the following statements are equivalent:

1. $\beta - \alpha = (l, \dots, l)$,

2. the weight $(l, \dots, l)$ *appears once and only once among* γ.

3. the weight $(l, \dots, l)$ *appears among* γ.

Proof. By Schur's Lemma, $\mathcal{E}^{(l,\dots,l)}$ is a direct summand of the Littlewood–Richardson decomposition of $\mathcal{E}^{-\alpha} \otimes \mathcal{E}^{\beta}$ if and only if

$$\mathrm{Hom}(V_{\mathsf{GL}}^{(l,\dots,l)}, V_{\mathsf{GL}}^{-\alpha} \otimes V_{\mathsf{GL}}^{\beta}) \neq 0.$$

On the other hand:

$$\mathrm{Hom}(V_{\mathsf{GL}}^{(l,\dots,l)}, V_{\mathsf{GL}}^{-\alpha} \otimes V_{\mathsf{GL}}^{\beta}) = \mathrm{Hom}(V_{\mathsf{GL}}^{(l,\dots,l)} \otimes V_{\mathsf{GL}}^{\alpha}, V_{\mathsf{GL}}^{\beta}) = \mathrm{Hom}(V_{\mathsf{GL}}^{\alpha+(l,\dots,l)}, V_{\mathsf{GL}}^{\beta}),$$

where the first equality holds by (1.7) and the second is an application of Corollary 1.3.3. The last Hom-space being nonzero is equivalent to $\alpha + (l, \dots, l) = \beta$. As we have that $\mathsf{dim}\,\mathrm{Hom}(V_{\mathsf{GL}}^{\alpha+(l,\dots,l)}, V_{\mathsf{GL}}^{\beta})$ equals the multiplicity of V_{GL}^{β} in $V_{\mathsf{GL}}^{\alpha+(l,\dots,l)}$, if it is nonzero, it must be 1. □

Koszul and staircase complexes

We recall here some exact sequences on $\mathsf{Gr}(k, m)$. For each p, the tautological sequence (1.3) induces the following exact sequence, which is called the *Koszul complex*:

$$0 \to \wedge^p \mathcal{U}^{\perp} \to \wedge^p V^* \otimes \mathcal{O} \to \dots \to V^* \otimes S^{p-1}\mathcal{U}^* \to S^p \mathcal{U}^* \to 0. \tag{1.9}$$

Applying Borel–Bott–Weil Theorem (see [Dem76]), it is immediate to see that the differentials in (1.9) are the unique nonzero $\mathsf{GL}(V)$-equivariant maps between the terms in the complex.

We will now introduce the second and most important family of exact sequences for this work, *staircase complexes*, introduced in [Fon13]. First we recall the general theory. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in P_k^+$, with $\lambda_1 = m - k$ and $\lambda_k \geq 0$. We define

$$\lambda' = (\lambda_2, \dots, \lambda_k, 0) \in P_k^+.$$

Theorem 1.3.6 (Staircase complex, [Fon13, Proposition 5.3])**.** *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in P_k^+$ *with* $\lambda_1 = m - k$ *and* $\lambda_k \geq 0$. *Then there exist a sequence of weights* $\{\mu_i\}_{i \in \{1,\dots,m-k\}} \in P_k^+$ *and an exact sequence:*

$$0 \to \mathcal{U}^{\lambda'}(-1) \xrightarrow{\gamma_{m-k+1}} \wedge^{\nu_{m-k}} V^* \otimes \mathcal{U}^{\mu_{m-k}} \xrightarrow{\gamma_{m-k}} \dots \xrightarrow{\gamma_2} \wedge^{\nu_1} V^* \otimes \mathcal{U}^{\mu_1} \xrightarrow{\gamma_1} \mathcal{U}^{\lambda} \to 0, \tag{1.10}$$

where $\nu_i = |\lambda| - |\mu_i|$ *and the morphisms* $\{\gamma_i\}_{i \in \{1,\dots,m-k+1\}}$ *are the unique nonzero* $\mathsf{GL}(V)$*-equivariant maps between the terms of the complex. The sequence* $\{\mu_i\}_{i \in \{1,\dots,m-k\}}$ *is totally ordered by inclusion; moreover, the weights* μ_i *are all positive and are contained in* λ.

We do not report the construction of the weights μ_i, except in the case $k = 3$ (see (1.16) below), which is the only case used in the body of the paper. Note however that if $\lambda_1 > \lambda_2$, then $\mu_1 = (\lambda_1 - 1, \lambda_2, \dots, \lambda_k)$.

Remark 1.3.7. Notice that the conditions $\lambda_1 = m - k$ and $\lambda_k \geq 0$ can be achieved for any $\lambda \in P_k^+$ with $\lambda_1 - \lambda_k \leq m - k$ by twisting appropriately. We will still refer to this complex as the staircase complex associated to $\mathcal{U}^\lambda$. With the notation of Theorem 1.3.6, we denote both the acyclic complex defined above and the one obtained in Remark 1.3.7 as $\mathsf{Stair}(\mathcal{U}^\lambda)$.

We now consider the problem of the restriction of a staircase complex on $\mathsf{Gr}(k, m+1)$ to $\mathsf{Gr}(k, m)$. Let V be a m-dimensional vector space. Fix a $(m+1)$-dimensional vector space $\widetilde{V}$ such that $V \subset \widetilde{V}$ is a hyperplane, then we obtain an induced embedding $j : \mathsf{Gr}(k, V) \to \mathsf{Gr}(k, \widetilde{V})$. Let $\widetilde{\mathcal{U}}$ be the tautological bundle on $\mathsf{Gr}(k, \widetilde{V})$. Recall that:

$$j^*(\widetilde{V} \otimes \mathcal{O}) \cong (V \otimes \mathcal{O}) \oplus \mathcal{O}, \quad j^*\widetilde{\mathcal{U}}^\lambda \cong \mathcal{U}^\lambda. \tag{1.11}$$

for every $\lambda \in P_k^+$. We prove our result.

Proposition 1.3.8. *Let $\lambda \in P_k^+$ with $\lambda_1 = m + 1 - k > \lambda_2$ and $\lambda_k \geq 1$. Let $\mathsf{Stair}(\widetilde{\mathcal{U}}^\lambda)$ be the associated staircase complex on $\mathsf{Gr}(k, m+1)$. Then the restriction of the complex to $\mathsf{Gr}(k, m)$ splits:*

$$j^* \mathsf{Stair}(\widetilde{\mathcal{U}}^\lambda) \cong \mathsf{Stair}(\mathcal{U}^\lambda) \oplus \mathsf{Stair}(\mathcal{U}^{\mu_1})[1].$$

Proof. By [Fon13, Lemma 5.1], $\mathsf{Stair}(\widetilde{\mathcal{U}}^\lambda)$ can be characterized in the following way:

$$\varepsilon \in \mathrm{Hom}^{m+1-k}(\widetilde{\mathcal{U}}^\lambda, \widetilde{\mathcal{U}}^{\lambda'}(-1)) \cong \mathrm{H}^{m+1-k}(\mathsf{Gr}(k, m+1), \widetilde{\mathcal{U}}^{-1,\dots,-1,-(m+1-k)-1}) = \mathbb{C}, \tag{1.12}$$

where ε is the unique nonzero extension. By Theorem 1.3.6, the central part of $\mathsf{Stair}(\widetilde{\mathcal{U}}^\lambda)$ is obtained by decomposing the complex

$$\mathrm{Cone}(\widetilde{\mathcal{U}}^\lambda \xrightarrow{\varepsilon} \widetilde{\mathcal{U}}^{\lambda'}(-1)[m+1-k]). \tag{1.13}$$

with respect to the exceptional collection on $\mathsf{Gr}(k, m+1)$ given by $\langle \widetilde{\mathcal{U}}^\gamma \mid 0 \subseteq \gamma \subseteq \mu_1 \subset \lambda \rangle$ (see [Fon13, Theorem 2.1]). Since $\lambda_1 > \lambda_2$, we have $\mu_1 = (m - k,\ \lambda_2,\ \dots,\ \lambda_k)$; hence, the restriction of this collection to $\mathsf{Gr}(k, m)$:

$$\langle \mathcal{U}^\gamma \mid 0 \subseteq \gamma \subseteq \mu_1 \rangle = \langle j^*\widetilde{\mathcal{U}}^\gamma \mid 0 \subseteq \gamma \subseteq \mu_1 \rangle, \tag{1.14}$$

is an exceptional collection as well, by [Fon13, Theorem 2.1].

We now prove that $j^*\varepsilon = 0$ on $\mathsf{Gr}(k, m)$. Similarly to the proof of [Fon13, Lemma 5.1], we apply Borel–Bott–Weil Theorem on $\mathsf{Gr}(k, m)$ (see [Dem76]) obtaining:

$$\mathrm{H}^p(\mathsf{Gr}(k, m), \mathcal{U}^{-1,\dots,-1,-(m+1-k)-1}) = \begin{cases} V \otimes \wedge^m V & \text{if } p = m - k, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

As $j^*\varepsilon \in \mathrm{H}^{m+1-k}(\mathsf{Gr}(k,m), j^*\widetilde{\mathcal{U}}^{-1,\dots,-1,-(m+1-k)-1}) = 0$ by (1.12) and (1.15), we obtain:

$$j^*\mathrm{Cone}(\varepsilon) = \mathrm{Cone}(j^*\varepsilon) = \mathrm{Cone}(0) = \mathcal{U}^{\lambda}[1] \oplus \mathcal{U}^{\lambda'}(-1)[m-k].$$

Now, consider the left resolution of $j^*\widetilde{\mathcal{U}}^{\lambda} = \mathcal{U}^{\lambda}$ given by the staircase complex on $\mathsf{Gr}(k,m)$ (cf. Remark 1.3.7). We introduce the following weights:

$$\overline{\lambda} = (\lambda_1 - 1, \dots, \lambda_k - 1) \in P_k^+ \qquad \text{and} \qquad \overline{\lambda}' = (\lambda_2 - 1, \dots, \lambda_k - 1, 0) \in P_k^+.$$

Notice that $\overline{\lambda}$ satisfies the conditions in Theorem 1.3.6. As a consequence, we find that

$$\mathcal{U}^{\lambda} = \mathcal{U}^{\overline{\lambda}}(1) \in \langle \mathcal{U}^{\overline{\lambda}'}, \mathcal{U}^{\gamma} \mid (1,\dots,1) \subseteq \gamma \subseteq \mu_1 \rangle \subseteq \langle \mathcal{U}^{\gamma} \mid 0 \subseteq \gamma \subseteq \mu_1 \rangle,$$

where the rightmost term is the exceptional collection in (1.14).

On the other hand, $\mathcal{U}^{\lambda'}(-1)$ belongs to (1.14) because it is the leftmost term of $\mathsf{Stair}(\mathcal{U}^{\mu_1})$ in $\mathsf{Gr}(k,m)$, as $\mu_1 = (m-k, \lambda_2, \dots, \lambda_k)$. As a consequence, we can recover $\mathsf{Stair}(\mathcal{U}^{\mu_1})$ as the decomposition of $\mathcal{U}^{\lambda'}(-1)$ with respect to (1.14). Taking the direct sum of these resolutions finally provides a decomposition of $\mathcal{U}^{\lambda}[1] \oplus \mathcal{U}^{\lambda'}(-1)[m-k]$ in terms of (1.14).

By construction, the central truncation of $j^*\,\mathsf{Stair}(\widetilde{\mathcal{U}}^{\lambda})$ gives a resolution of $j^*\mathrm{Cone}(\varepsilon)$ with respect to (1.14). By uniqueness of the decomposition, it must agree with the decomposition given by the direct sum of the truncated staircase complexes. This proves the claim. □

We restate here the description of a staircase complex on $\mathsf{Gr}(3,m)$. In Theorem 1.3.6, fix any GL_3-dominant weight $\lambda = (a, 0, -b)$ such that $a + b \le m - 3$. Then the following collection of $m-1$ weights $\{\mu_i\}_{i \in \{0,\dots,m-2\}}$ of GL_3

$$\mu_i := \begin{cases} (a-i, 0, -b), & \text{if} \quad 0 \le i \le a; \\ (-1, a-i, -b), & \text{if} \quad a < i \le a+b; \\ (-1, -b-1, a-i), & \text{if} \quad a+b < i \le m-2. \end{cases} \tag{1.16}$$

are the weights appearing in the statement; note that $\mu_0 = \lambda$.

1.3.2 Even isotropic Grassmannian

Let W be a $2n$-dimensional vector space endowed with a symplectic form ψ and let $\mathsf{Sp}(W)$ be the corresponding symplectic group. Note that ψ induces a $\mathsf{Sp}(W)$-equivariant isomorphism $W \cong W^*$, which we will use quite often.

For $1 \le k \le n$, we denote the isotropic Grassmannian of k-planes in W by $\mathsf{IGr}(k,W)$. We often refer to $\mathsf{IGr}(k,W)$ as the *even isotropic Grassmannian* to emphasize the difference with the *odd isotropic Grassmannian* defined in § 1.3.3.

The variety $\mathsf{IGr}(k,W)$ is isomorphic to the zero locus of

$$\psi \in \wedge^2 W^* = \mathrm{H}^0(\mathsf{Gr}(k,W), \wedge^2 \mathcal{U}^*).$$

The tautological sequence on $\mathsf{IGr}(k, W)$ is the restriction of the one in $\mathsf{Gr}(k, W)$, i.e.:

$$0 \to \mathcal{U} \to W \otimes \mathcal{O} \to \mathcal{Q} \to 0. \tag{1.17}$$

The symplectic form induces a canonical embedding $\mathcal{U} \hookrightarrow \mathcal{U}^\perp$ (cf. (1.4)), so that we can define the quotient bundle $\mathcal{S}$:

$$0 \to \mathcal{U} \to \mathcal{U}^\perp \to \mathcal{S} \to 0. \tag{1.18}$$

Additionally, $\mathcal{S}$ is also the cohomology sheaf in the middle term of the complex:

$$0 \to \mathcal{U} \to W \otimes \mathcal{O} \xrightarrow{\psi} \mathcal{U}^* \to 0. \tag{1.19}$$

Notice that $\mathcal{S}$ is a bundle of $\mathsf{rank}\,\mathcal{S} = 2(n-k)$, endowed with a symplectic isomorphism $\mathcal{S} \cong \mathcal{S}^*$ induced by ψ. The following result is well known, cf. [KP16, Lemma 2.19, Proposition 9.7].

Lemma 1.3.9. *The dimension d and the index w of $\mathsf{IGr}(k, 2n)$ are respectively:*

$$d = \frac{k(4n-3k+1)}{2}, \quad w = 2n+1-k.$$

Moreover, the Grothendieck group is a free abelian group and its rank r is:

$$r = \binom{n}{k} \cdot 2^k.$$

In the rest of the section, we state Borel–Bott–Weil Theorem on the even isotropic Grassmannian and we apply it to obtain a cohomology vanishing result.

Borel–Bott–Weil Theorem

The weight lattice of Sp_{2n} is isomorphic to $\mathbb{Z}^n$, and its subset

$$T_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$$

is the cone of dominant weights. Given $\lambda \in T_n^+$, we denote by W_{Sp}^λ the $\mathsf{Sp}(W)$-representation of highest weight λ. Our convention is such that:

$$W_{\mathsf{Sp}}^{m,0,\ldots,0} = S^m W^* \cong S^m W.$$

The Weyl group of Sp_{2n} is the semidirect product $\mathbf{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ acting on $\mathbb{Z}^n$, where $\mathbf{S}_n$ acts by permutation and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by changing signs of the coordinates. Let $\ell : \mathbf{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}$ be the length function. Thus, $\ell(\sigma)$ is the smallest number of simple reflections required to decompose σ. In this case, simple reflections are of two types: transposition of adjacent terms and the change of sign $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n) \mapsto (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$. We recall that the sum of fundamental weights of Sp_{2n} is

$$\rho_{\mathsf{Sp}_{2n}} = (n, n-1, \dots, 1).$$

We recall here an immediate application of the more general Borel–Bott–Weil theorem, which will cover most of the computations in this work.

Theorem 1.3.10 (Borel–Bott–Weil Theorem, [Dem76])**.** *Let $\lambda \in P_k^+$. Let*

$$\gamma = (\gamma_1, \dots, \gamma_n) := (\lambda_1, \dots, \lambda_k, 0, \dots, 0) \in \mathbb{Z}^n,$$

be the extension of λ by $n-k$ zeros. Suppose that $\gamma + \rho_{\mathsf{Sp}_{2n}}$ has a zero entry or two entries with same absolute value, then:

$$\mathrm{H}^\bullet(\mathsf{IGr}(k, W), \mathcal{U}^\lambda) = 0.$$

Otherwise, let $\sigma \in \mathbf{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the unique element of the Weyl group such that

$$\gamma' = \sigma(\gamma + \rho_{\mathsf{Sp}_{2n}}) - \rho_{\mathsf{Sp}_{2n}} \in T_n^+$$

is Sp_{2n}-dominant, then we have:

$$\mathrm{H}^\bullet(\mathsf{IGr}(k, W), \mathcal{U}^\lambda) = W_{\mathsf{Sp}}^{\gamma'}[-\ell(\sigma)].$$

Vanishing

The next result and the following Corollary 1.3.14 are a straightforward generalization of the ideas presented in [Fon22, § 4.1], where they were proved only in the case $k = n$. For further convenience, we state the following result for $\mathsf{IGr}(k, 2n+2)$ instead of $\mathsf{IGr}(k, 2n)$.

Proposition 1.3.11. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \in P_k^+$ be a dominant weight of the group GL_k such that*

1. $\lambda_k < 0$,
2. $\lambda_1 \geq -2(n+1) + k$,
3. $\lambda_i - \lambda_{i+1} \leq 2(n+2-k) - 1$ *for* $i = 1, \dots, k-1$.

Then we have:

$$\mathrm{H}^\bullet(\mathsf{IGr}(k, 2n+2), \mathcal{U}^\lambda) = 0,$$

that is, the bundle $\mathcal{U}^\lambda$ on $\mathsf{IGr}(k, 2n+2)$ is acyclic.

Proof. Suppose that $\mathcal{U}^\lambda$ is not acyclic. Then, according to Theorem 1.3.10, the absolute values of the entries in the sequence

$$\gamma = (n+1+\lambda_1, n+\lambda_2, \dots, n+2-k+\lambda_k, n+1-k, \dots, 1)$$

have distinct nonzero absolute values. In particular, the entries γ_i, $1 \leq i \leq k$, satisfy

$$\gamma_i \geq n+2-k \quad \text{or} \quad \gamma_i \leq -(n-k+2). \tag{1.20}$$

Using conditions 1 and 2 we have:

$$\gamma_1 = n+1+\lambda_1 \geq -(n+1-k) \quad \text{and} \quad \gamma_k = n+2-k+\lambda_k \leq n+1-k,$$

which, together with (1.20) imply

$$\gamma_1 \geq n+2-k \quad \text{and} \quad \gamma_k \leq -(n+2-k).$$

As λ is dominant, the first k entries of γ are strictly decreasing, hence there must be some $j \in \{1, \dots, k-1\}$ such that

$$\gamma_j \geq n+2-k \quad \text{and} \quad \gamma_{j+1} \leq -(n+2-k).$$

On the other hand, from condition 3 we obtain:

$$\gamma_j - \gamma_{j+1} = n+2-j+\lambda_j - (n+2-(j+1)+\lambda_{j+1}) \leq 1+\lambda_j - \lambda_{j+1} \leq 2(n+2-k).$$

Finally, we obtain:

$$\gamma_j = n+2-k \quad \text{and} \quad \gamma_{j+1} = -(n+2-k),$$

so we have $|\gamma_j| = |\gamma_{j+1}|$, contradicting the assumption that the absolute values of the entries of γ are distinct. □

1.3.3 Odd isotropic Grassmannian

Let V be a $(2n+1)$-dimensional vector space endowed with $\psi \in \wedge^2 V^*$, a skew-symmetric form of maximal possible rank $2n$. Fix a $(2n+2)$-dimensional symplectic vector space $(\widetilde{V}, \widetilde{\psi})$ such that $V \subset \widetilde{V}$ is a hyperplane and $\widetilde{\psi}|_V = \psi$.

The odd isotropic Grassmannian $X = \mathsf{IGr}(k, V)$ is the variety parametrising isotropic k-subspaces of V, that is,

$$\mathsf{IGr}(k, V) = \mathsf{Gr}(k, V) \cap \mathsf{IGr}(k, \widetilde{V})$$

where the intersection is considered in $\mathsf{Gr}(k, \widetilde{V})$.

We denote the stabilizer of ψ in $\mathsf{GL}(V)$ by $\mathsf{Sp}(V) = \mathsf{Sp}_{2n+1}$ (cf. [Mih07, §3.3], [Pro88]) in analogy with the classical symplectic group. We refer to Sp_{2n+1} as the *odd symplectic group*, it is connected and nonreductive. The natural action of Sp_{2n+1} on $\mathsf{IGr}(k, V)$ is quasi-homogeneous.

Realization

We can define X as the zero locus of a global section of a vector bundle on $\mathsf{IGr}(k, \widetilde{V})$. We denote by $\widetilde{\mathcal{U}}$ the tautological bundle of $\mathsf{IGr}(k, \widetilde{V})$. Consider any nonzero section:

$$\widetilde{v} \in \widetilde{V}^* = \mathrm{H}^0(\mathsf{IGr}(k, \widetilde{V}), \widetilde{\mathcal{U}}^*),$$

i.e. a nonzero linear form on $\widetilde{V}$, with $V = \operatorname{Ker} \widetilde{v}$. Then $\mathsf{IGr}(k, V)$ is the zero locus of $\widetilde{v}$. As $\mathsf{IGr}(k, V)$ is a smooth variety of expected codimension by [Mih07, Proposition 4.1], $\widetilde{v}$ is a regular section. We denote the embedding as $j : \mathsf{IGr}(k, V) \to \mathsf{IGr}(k, \widetilde{V})$.

We have the following Koszul resolution in $\mathsf{IGr}(k, \widetilde{V})$:

$$0 \to \wedge^k \widetilde{\mathcal{U}} \to \wedge^{k-1} \widetilde{\mathcal{U}} \to \cdots \to \widetilde{\mathcal{U}} \to \mathcal{O}_{\mathsf{IGr}(k,\widetilde{V})} \to j_* \mathcal{O}_{\mathsf{IGr}(k,V)} \to 0. \tag{1.21}$$

The following proposition provides an odd-dimensional analogue to Lemma 1.3.9.

Proposition 1.3.12. *The dimension d and the index w of $\mathsf{IGr}(k, 2n+1)$ are respectively:*

$$d = \frac{k(4n - 3k + 3)}{2}, \quad w = 2n + 2 - k.$$

Moreover, the Grothendieck group is a free abelian group and its rank r is:

$$r = \binom{n}{k-1} \cdot \frac{2^{k-1}(2n + 2 - k)}{k}.$$

Proof. The first two formulas are well known and can be obtained from the the embedding of $\mathsf{IGr}(k, V)$ in $\mathsf{IGr}(k, \widetilde{V})$, cf. [Mih07, Proposition 4.1]. The formula for the rank of the Grothendieck group is an immediate consequence of Lemma 1.3.9 and Proposition A.0.3. We give a detailed proof of Proposition A.0.3 in Appendix § A. ☐

Remark 1.3.13. A full exceptional collection induces a basis of the Grothendieck group. If a variety X of index w admits a full rectangular Lefschetz collection with a basis of p elements, this induces a basis of $K_0(X)$ with $r = wp$ elements. In particular, the index divides the rank of the Grothendieck group. By Proposition 1.3.12, in the case of the odd isotropic Grassmannians $\mathsf{IGr}(3, 2n+1)$ it is possible to have a full rectangular Lefschetz collection only if

$$2n - 1 \quad \text{divides} \quad \binom{n}{2} \cdot \frac{4(2n-1)}{3} \quad \Leftrightarrow \quad 3 \quad \text{divides} \quad 4 \cdot \binom{n}{2} = 2n(n-1),$$

this is equivalent to $n \equiv 0$ or $1 \bmod 3$. In this case, $p = \frac{2}{3} n(n-1)$.

For $\mathsf{IGr}(3, 9)$ we have $d = 15$, $w = 7$, $r = 56$; therefore $p = 8$.

Vanishing

We give here an acyclicity criterion similar to Proposition 1.3.11, which will cover most bundles on the odd Grassmannian $X = \mathsf{IGr}(k, 2n+1)$ studied in this work.

Corollary 1.3.14. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_k^+$ be a dominant weight of the group GL_k such that*

1. $\lambda_k < 0$,
2. $\lambda_1 \geq -2n + k - 1$,
3. $\lambda_i - \lambda_{i+1} \leq 2(n+1-k)$ *for* $i = 1, \ldots, k-1$.

Then the bundle $\mathcal{U}^\lambda$ on $X = \mathsf{IGr}(k, 2n+1)$ is acyclic.

Proof. Fix an embedding $j : X \to \mathsf{IGr}(k, 2n+2) = \widetilde{X}$. Recall the notation fixed in § 1.3.3. By the projection formula, we have an isomorphism:

$$\mathrm{H}^\bullet(X, \mathcal{U}^\lambda) = \mathrm{H}^\bullet(\widetilde{X}, j_* \mathcal{U}^\lambda) = \mathrm{H}^\bullet(\widetilde{X}, \widetilde{\mathcal{U}}^\lambda \otimes j_* \mathcal{O}_X).$$

Replacing $j_* \mathcal{O}_X$ by its Koszul resolution (1.21), we obtain the following spectral sequence:

$$E_1^{-p,q} = \mathrm{H}^q(\widetilde{X}, \widetilde{\mathcal{U}}^\lambda \otimes \wedge^p \widetilde{\mathcal{U}}) \Rightarrow \mathrm{H}^{q-p}(X, \mathcal{U}^\lambda) \tag{1.22}$$

for $q \geq 0$ and $p = 0, \ldots, k$. Hence, it is enough to compute the cohomology of the direct summands

$$\widetilde{\mathcal{U}}^\gamma \mathrel{\in\!\!\!\!+} \widetilde{\mathcal{U}}^\lambda \otimes \wedge^p \widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}^\lambda \otimes \wedge^k \widetilde{\mathcal{U}} \otimes (\wedge^{k-p} \widetilde{\mathcal{U}})^* = \widetilde{\mathcal{U}}^{\lambda - (1, \ldots, 1)} \otimes (\wedge^{k-p} \widetilde{\mathcal{U}})^*.$$

By Proposition 1.3.2 and Corollary 1.3.3, it is enough to verify that the bundles $\widetilde{\mathcal{U}}^\gamma$ are acyclic for any

$$\gamma \in \bigcup_{p \in \{0, \ldots, k\}} \mathrm{VS}^p_{\lambda - (1, \ldots, 1)}.$$

Let us verify the conditions of Proposition 1.3.11 for $\widetilde{\mathcal{U}}^\gamma$. First of all, we have

$$\lambda_i - 1 \leq \gamma_i \leq \lambda_i \quad \text{for} \quad i = 1, \ldots, k. \tag{1.23}$$

As $\gamma_k \leq \lambda_k < 0$, by (1.23) and the hypothesis, the first condition is satisfied. Using the second hypothesis, we obtain:

$$(-2n + k - 1) - 1 \leq \lambda_1 - 1 \leq \gamma_1,$$

so the second condition of Proposition 1.3.11 is satisfied. Finally,

$$\gamma_i - \gamma_{i+1} \leq \lambda_i - \lambda_{i+1} + 1 \leq 2(n+1-k) + 1,$$

so that the last condition is verified and $\widetilde{\mathcal{U}}^\gamma$ is acyclic by Proposition 1.3.11. Therefore, the spectral sequence (1.22) vanishes at the first page, proving that $\mathcal{U}^\lambda$ is acyclic. $\square$

1.4 Exceptionality

Let us fix $X = \mathsf{IGr}(3,9)$, the odd isotropic Grassmannian of 3-dimensional subspaces in V, where V is a 9-dimensional vector space endowed with a skew-symmetric form of maximal rank 8, ψ. By Proposition 1.3.12, X has index $w = 7$. Recall that in our notation we have:

$$\mathcal{U}^{m,0,0} = S^m\mathcal{U}^*, \quad \mathcal{U}^{0,0,-m} = S^m\mathcal{U}, \quad \mathcal{U}^{l,l,l} = \mathcal{O}(l).$$

We will often need to consider the embedding $j : \mathsf{IGr}(3,9) \to \mathsf{IGr}(3,10)$. We will denote the tautological bundle of $\mathsf{IGr}(3,10)$ as $\widetilde{\mathcal{U}}$. We restate here Proposition 1.3.11 and Corollary 1.3.14 for $\mathsf{IGr}(3,10)$ and $\mathsf{IGr}(3,9)$, they will be used as key tools to show semiorthogonality in § 1.4.1.

Corollary 1.4.1. *Let* $(\lambda_1,\lambda_2,\lambda_3) \in P_3^+$ *be a dominant weight of the group* GL_3 *such that*

1. $\lambda_3 < 0$,
2. $\lambda_1 \geq -7$,
3. $\lambda_1 - \lambda_2 \leq 5$ *and* $\lambda_2 - \lambda_3 \leq 5$.

Then, $\mathrm{H}^\bullet(\mathsf{IGr}(3,10), \widetilde{\mathcal{U}}^\lambda) = 0$. *If* $\lambda_1 \geq -1$ *instead,* $\mathrm{H}^\bullet(\mathsf{IGr}(3,10), \widetilde{\mathcal{U}}^\lambda(-l)) = 0$ *for* $l = 0, \dots, 6$.

Corollary 1.4.2. *Let* $(\lambda_1,\lambda_2,\lambda_3) \in P_3^+$ *be a dominant weight of the group* GL_3 *such that*

1. $\lambda_3 < 0$,
2. $\lambda_1 \geq -6$,
3. $\lambda_1 - \lambda_2 \leq 4$ *and* $\lambda_2 - \lambda_3 \leq 4$.

Then, $\mathrm{H}^\bullet(\mathsf{IGr}(3,9), \mathcal{U}^\lambda) = 0$. *If* $\lambda_1 \geq 0$ *instead,* $\mathrm{H}^\bullet(\mathsf{IGr}(3,9), \mathcal{U}^\lambda(-l)) = 0$ *for* $l = 0, \dots, 6$.

1.4.1 Two collections

Recall that in the § 1.1 we defined two collections in $\mathbf{D}^{\mathrm{b}}(X)$ of length 7:

$$\begin{aligned} \mathbf{B}_1 &= \{\, \mathcal{U}^{0,0,-2},\ \mathcal{U}^{0,0,-1},\ \mathcal{U}^{1,0,-1},\ \mathcal{U}^{2,0,-1},\ \mathcal{U}^{0,0,0},\ \mathcal{U}^{1,0,0},\ \mathcal{U}^{2,0,0} \,\}, \\ \mathbf{B}_2 &= \{\, \mathcal{U}^{0,0,-1},\ \mathcal{U}^{1,0,-1},\ \mathcal{U}^{2,0,-1},\ \mathcal{U}^{0,0,0},\ \mathcal{U}^{1,0,0},\ \mathcal{U}^{2,0,0},\ \mathcal{U}^{3,0,0} \,\}. \end{aligned}$$

We remark that all the weights $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ such that $\mathcal{U}^\lambda \in \mathbf{B}_1 \cup \mathbf{B}_2$ satisfy $\lambda_2 = 0$. Notice that

$$\mathbf{B}_1 = \{\mathcal{U}^{0,0,-2}\} \cup (\mathbf{B}_1 \cap \mathbf{B}_2), \qquad \mathbf{B}_2 = (\mathbf{B}_1 \cap \mathbf{B}_2) \cup \{\mathcal{U}^{3,0,0}\}.$$

Moreover, $\mathbf{B}_1$ and $\mathbf{B}_2$ are ordered lexicographically with entries read from right to left, i.e.:

$$(\beta_1, \beta_2, \beta_3) < (\alpha_1, \alpha_2, \alpha_3) \Leftrightarrow \begin{cases} \beta_3 < \alpha_3 & \text{or} \\ \beta_3 = \alpha_3 & \text{and} \quad \beta_2 < \alpha_2 \quad \text{or} \\ \beta_3 = \alpha_3 & \text{and} \quad \beta_2 = \alpha_2 \quad \text{and} \quad \beta_1 < \alpha_1 \end{cases} \tag{1.24}$$

which refines the partial order $\subseteq$ on P_3^+ presented in (1.5). Recall that an additional object $\mathcal{H} \in \mathbf{D}^{\mathrm{b}}(X)$ was introduced as the totalization of the bicomplex (1.1) and that the collection

$$\mathbf{B} = \{\mathcal{H}\} \cup \mathbf{B}_1$$

was defined in (1.2).

In this section, we prove that $\mathbf{B}_1$ and $\mathbf{B}_2$ are bases of two (non full) rectangular Lefschetz exceptional collections (cf. Corollary 1.4.10). In addition, we prove the following intermediate statement.

Proposition 1.4.3. *Let $\mathscr{B} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be the subcategory generated by $\mathbf{B}$. Then the collection of subcategories $\mathscr{B}$, $\mathscr{B}(1)$, $\mathscr{B}(2)$, $\mathscr{B}(3)$, $\mathscr{B}(4)$, $\mathscr{B}(5)$, $\mathscr{B}(6)$ is semiorthogonal.*

Notice that this proposition does not claim that $\mathscr{B}$ is generated by an exceptional collection or that $\mathscr{B}$ is admissible. These properties are indeed true, but they will only be proved in Theorem 1.4.15, after a more detailed study of $\mathcal{H}$.

To prove Proposition 1.4.3, the starting point is to observe that $\mathcal{H}$ lies in the category $\langle \mathbf{B}_2 \rangle \subset \mathscr{B}$, so that $\mathbf{B}_1 \cup \mathbf{B}_2$ is an alternative set of generators of $\mathscr{B}$. Therefore, we need to compute $\mathrm{Hom}^\bullet$-groups between various twists of the generators of $\mathbf{B}_1$ and $\mathbf{B}_2$. We will need some ad hoc computations of cohomology, which we group here at the beginning of the section.

We start with some computations on $\mathsf{IGr}(3, 10)$. Recall the notation introduced in § 1.3.3.

Lemma 1.4.4. *We have the following isomorphisms:*

$$\mathrm{H}^\bullet(\mathsf{IGr}(3,10), \widetilde{\mathscr{U}}^{0,0,-6}(-l)) = \begin{cases} \mathbb{C}[-5] & \text{if } l = 0, \\ 0 & \text{if } l = 1, \dots, 7, \end{cases}$$

$$\mathrm{H}^\bullet(\mathsf{IGr}(3,10), \widetilde{\mathscr{U}}^{0,-1,-7}(-l)) = \begin{cases} \mathbb{C}[-6] & \text{if } l = 0, \\ 0 & \text{if } l = 1, \dots, 7. \end{cases}$$

Proof. To prove both results, we apply Theorem 1.3.10. We focus on the first isomorphism. To do so, take $\lambda = (-l, -l, -l-6)$, and consider $\gamma = (-l, -l, -(l+6), 0, 0)$ its trivial extension, so that:

$$\gamma + \rho_{\mathsf{Sp}_{10}} = (5 - l, 4 - l, -l - 3, 2, 1).$$

If $l = 0$ we have $\gamma+\rho_{\mathsf{Sp}_{10}} = (5, 4, -3, 2, 1)$, so all the entries are distinct in absolute value and nonzero. Let σ be the element of the Weil group such that $\sigma(5, 4, -3, 2, 1) = (5, 4, 3, 2, 1) \in T_5^+$, hence:

$$\gamma' = \sigma(5, 4, -3, 2, 1) - \rho_{\mathsf{Sp}_{10}} = (0, 0, 0, 0, 0).$$

We observe that $\ell(\sigma) = 5$, obtaining the result for $l = 0$. On the other hand, for $l = 1, \dots , 7$, there is always a repetition of absolute values. The vanishing for these l follows.

In the second case, $\gamma = (-l, -1 - l, -7 - l, 0, 0)$, obtaining

$$\gamma + \rho_{\mathsf{Sp}_{10}} = (5 - l, 3 - l, -4 - l, 2, 1),$$

and the same argument applies. □

We continue with simple cohomology computations on $\mathsf{IGr}(3, 9)$.

Lemma 1.4.5. *We have the following isomorphism:*

$$\mathrm{H}^\bullet(\mathsf{IGr}(3, 9), \mathcal{U}^{0,0,-5}(-l)) = \begin{cases} \mathbb{C}[-4] & \text{if } l = 0, \\ 0 & \text{if } l = 1, \dots , 6, \end{cases}$$

$$\mathrm{H}^\bullet(\mathsf{IGr}(3, 9), \mathcal{U}^{0,-1,-6}(-l)) = \begin{cases} \mathbb{C}[-5] & \text{if } l = 0, \\ 0 & \text{if } l = 1, \dots , 6. \end{cases}$$

Proof. Let us fix an embedding $j : \mathsf{IGr}(3, 9) \to \mathsf{IGr}(3, 10)$ and consider the spectral sequence (1.22), which we rewrite here for convenience:

$$E_1^{-p,q} = \mathrm{H}^q(\mathsf{IGr}(3, 10), \widetilde{\mathcal{U}}^\lambda \otimes \wedge^p \widetilde{\mathcal{U}}(-l)) \Rightarrow \mathrm{H}^{q-p}(\mathsf{IGr}(3, 9), \mathcal{U}^\lambda(-l)),$$

where $\widetilde{\mathcal{U}}$ is the tautological bundle on $\mathsf{IGr}(3, 10)$. We prove that the spectral sequence for $\lambda = (0, 0, -5)$ and $\lambda = (0, -1, -6)$ either vanishes or, if $l = 0$, it has only one nonzero term. By Proposition 1.3.2 and Corollary 1.3.3 (compare with the proof of Corollary 1.3.14), the terms in the spectral sequence are the cohomologies on $\mathsf{IGr}(3, 10)$ of the bundles $\widetilde{\mathcal{U}}^\gamma(-l)$ for

$$\gamma \in \bigcup_{p\in\{0,\dots,k\}} \mathrm{VS}^p_{\lambda-(1,\dots,1)} = \{(\gamma_1, \gamma_2, \gamma_3) \mid \gamma_1 \geq \gamma_2 \geq \gamma_3, \quad \lambda_i \geq \gamma_i \geq \lambda_i - 1\}.$$

In particular, for both λ, we have $\gamma_1 \geq \lambda_1 - 1 \geq -1$ and $-5 \geq \lambda_3 \geq \gamma_3$. Moreover, we have:

$$\gamma_1 - \gamma_2 \leq (\lambda_1 - \lambda_2) + 1 \leq 2, \qquad \gamma_2 - \gamma_3 \leq (\lambda_2 - \lambda_3) + 1 \leq 6.$$

If $\gamma_2 - \gamma_3 \leq 5$, $\widetilde{\mathcal{U}}^\gamma(-l)$ is acyclic for $l = 0, \dots , 6$ by Corollary 1.4.1. It is immediate to determine the bundles appearing in both spectral sequences which satisfy $\gamma_2 - \gamma_3 = 6$:

- $\widetilde{\mathcal{U}}^{0,0,-6}(-l) \in \widetilde{\mathcal{U}}^{0,0,-6} \otimes \widetilde{\mathcal{U}}(-l)$ if $\lambda = (0, 0, -5)$;

- $\widetilde{\mathcal{U}}^{0,0,-6}(-l-1) \mathrel{\in\!\!\!+} \widetilde{\mathcal{U}}^{0,-1,-6} \otimes \wedge^2\widetilde{\mathcal{U}}(-l)$ and $\widetilde{\mathcal{U}}^{0,-1,-7}(-l) \mathrel{\in\!\!\!+} \widetilde{\mathcal{U}}^{0,-1,-6} \otimes \widetilde{\mathcal{U}}(-l)$ in the case $\lambda = (0,-1,-6)$.

By Lemma 1.4.4, the spectral sequence vanishes identically for both λ if $l = 1, \ldots, 6$. If $l = 0$, there is one nonzero entry in both spectral sequences by Lemma 1.4.4, proving the result. □

We need some extra computations of cohomology for § 1.4.2.

Lemma 1.4.6. *We have the following isomorphisms:*

$$\begin{aligned} \mathrm{H}^\bullet(\mathsf{IGr}(3,9), \mathcal{U}^{1,0,-2} \otimes \mathcal{U}^{0,0,-3}(-1)) &= \mathbb{C}[-5], \\ \mathrm{H}^\bullet(\mathsf{IGr}(3,9), \mathcal{U}^{1,0,-3} \otimes \mathcal{U}^{\lambda}(-2)) &= 0, \end{aligned}$$

with $\lambda \in \{(1,0,-1),\ (1,0,0),\ (2,0,0)\} \subset P_3^+$.

Proof. We prove the first equality. Applying Pieri's formula (Proposition 1.3.2), we obtain:

$$\begin{aligned} \mathcal{U}^{1,0,-2} \otimes \mathcal{U}^{0,0,-3}(-1) = &\, \mathcal{U}^{0,-1,-6} \oplus \mathcal{U}^{0,-2,-5} \oplus \mathcal{U}^{-1,-1,-5} \oplus \\ & \oplus \mathcal{U}^{0,-3,-4} \oplus \mathcal{U}^{-1,-2,-4} \oplus \mathcal{U}^{-1,-3,-3}. \end{aligned}$$

By Corollary 1.4.2, all the bundles in the decomposition are acyclic except possibly $\mathcal{U}^{0,-1,-6}$. Applying Lemma 1.4.5, we deduce the first equality.

We now prove the last three vanishings. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in P_3^+$ such that:

$$\mathcal{U}^{\gamma} \mathrel{\in\!\!\!+} \mathcal{U}^{1,0,-3} \otimes \mathcal{U}^{\lambda}(-2),$$

for $\lambda \in \{(1,0,-1),\ (1,0,0),\ (2,0,0)\}$. We can deduce the following facts from Lemma 1.3.4, fixing $\alpha = (5,2,1) = (3,0,-1) + (2,2,2)$ and $\beta = \lambda$ as above:

$$\beta_1 - 1 \geq \gamma_1 \geq \beta_3 - 1, \quad \beta_2 - 1 \geq \gamma_2 \geq \beta_3 - 2, \quad \beta_3 - 1 \geq \gamma_3 \geq \beta_3 - 5, \quad |\gamma| = |\beta| - 8.$$

It is immediate to see that any possible $\mathcal{U}^{\gamma}$ is acyclic by Corollary 1.4.2, unless $\gamma = (-1,-1,-6)$ and $\beta = (1,0,-1)$. The bundle $\mathcal{U}^{-1,-1,-6}$ is acyclic by Lemma 1.4.5, proving the claim. □

The next lemma shows that $\mathbf{B}_1 \cup \mathbf{B}_2$ is not an exceptional collection; that is the only reason preventing $\mathbf{B}_1 \cup \mathbf{B}_2$ from extending to a Lefschetz collection.

Lemma 1.4.7. *There is the following isomorphism:*

$$\mathrm{Hom}^\bullet(\mathcal{U}^{3,0,0}(l), \mathcal{U}^{0,0,-2}) = \begin{cases} \mathbb{C}[-4] & \textit{if } l = 0, \\ 0 & \textit{if } l = 1, \ldots, 6. \end{cases}$$

Proof. We have $(\mathcal{U}^{3,0,0})^* \cong \mathcal{U}^{0,0,-3}$ and by Pieri's formula:

$$\mathcal{U}^{0,0,-3} \otimes \mathcal{U}^{0,0,-2} = \mathcal{U}^{0,0,-5} \oplus \mathcal{U}^{0,-1,-4} \oplus \mathcal{U}^{0,-2,-3}.$$

Using Corollary 1.4.2, we verify that for all bundles $\mathcal{U}^\gamma \Subset \mathcal{U}^{0,0,-3} \otimes \mathcal{U}^{0,0,-2}$, the bundle $\mathcal{U}^\gamma(-l)$ is acyclic for $l = 0, \ldots, 6$, except for $\gamma = (0,0,-5)$. For $\gamma = (0,0,-5)$, we apply Lemma 1.4.5 and we deduce the claim. □

In light of the vanishing criterion given by Corollary 1.4.2, we state the main property of the collections $\mathbf{B}_1$ and $\mathbf{B}_2$.

Lemma 1.4.8. *Let $\alpha, \beta \in P_3^+$ such that $\mathcal{U}^\alpha, \mathcal{U}^\beta \in \mathbf{B}_j$ for the same $j = 1, 2$. Suppose that*

$$\mathcal{U}^\gamma \Subset \mathcal{U}^{-\alpha} \otimes \mathcal{U}^\beta.$$

Then, γ satisfies:

$$\gamma_1 - \gamma_2 \leq 4 \quad \textit{and} \quad \gamma_2 - \gamma_3 \leq 4.$$

Proof. Let λ be such that $\mathcal{U}^\lambda \in \mathbf{B}_1 \cup \mathbf{B}_2$, then it satisfies

$$\lambda_1 - \lambda_3 \leq 3 \quad \text{and} \quad \lambda_2 - \lambda_3 \leq 2,$$

where the first equality is attained for $\lambda = (2,0,-1) \in \mathbf{B}_1 \cap \mathbf{B}_2$ and $\lambda = (3,0,0) \in \mathbf{B}_2 \setminus \mathbf{B}_1$, while the second only for $\lambda = (0,0,-2) \in \mathbf{B}_1 \setminus \mathbf{B}_2$. By (1.8), we have:

$$\gamma_1 - \gamma_2 \leq 4 \quad \text{and} \quad \gamma_2 - \gamma_3 \leq 4,$$

except possibly for $\alpha = (0,0,-2)$ and $\beta = (2,0,-1)$ or $(3,0,0)$ or vice versa. This finishes the proof for $\mathcal{U}^\alpha, \mathcal{U}^\beta \in \mathbf{B}_2$. Finally, if $\alpha = (0,0,-2)$ and $\beta = (2,0,-1)$, we have by Pieri's formula:

$$\mathcal{U}^{2,0,0} \otimes \mathcal{U}^{2,0,-1} = \mathcal{U}^{4,0,-1} \oplus \mathcal{U}^{3,1,-1} \oplus \mathcal{U}^{3,0,0} \oplus \mathcal{U}^{2,2,-1} \oplus \mathcal{U}^{2,1,0},$$

so that we can verify the claim. □

Now we finished preparations and we can prove the Lefschetz property for $\mathbf{B}_1$ and $\mathbf{B}_2$.

Proposition 1.4.9. *Let $\alpha, \beta \in P_3^+$, with $\beta \leq \alpha$ with respect to (1.24), such that $\mathcal{U}^\alpha, \mathcal{U}^\beta \in \mathbf{B}_j$ for the same $j = 1, 2$. Then*

$$\operatorname{Hom}^\bullet(\mathcal{U}^\alpha(l), \mathcal{U}^\beta) = 0 \quad \textit{for} \quad l = 0, \ldots 6,$$

unless $l = 0$ and $\alpha = \beta$. In that case, we have:

$$\operatorname{Hom}^\bullet(\mathcal{U}^\alpha, \mathcal{U}^\alpha) = \mathbb{C}.$$

Proof. With the notation presented above, consider the decomposition:

$$\mathcal{U}^{\gamma} \mathrel{\ooalign{$\in$\cr\hfil$+$\hfil}} \mathcal{U}^{-\alpha} \otimes \mathcal{U}^{\beta}.$$

We claim that either:

- $\gamma_1 \geq 0$ and $\gamma_3 < 0$; or
- $\gamma = (0, 0, 0)$ and $\alpha = \beta$, and it appears with multiplicity one.

Applying Lemma 1.3.4 and recalling $\beta \leq \alpha$, we obtain:

$$\gamma_1 \geq \beta_3 - \alpha_3, \quad 0 \geq \beta_3 - \alpha_3 \geq \gamma_3, \quad \gamma_1 + \gamma_2 + \gamma_3 = |\gamma| = |\beta| - |\alpha|. \tag{1.25}$$

We first prove that every γ satisfies $\gamma_1 \geq 0$. If $\alpha_3 = \beta_3$, we obtain $\gamma_1 \geq 0$. Otherwise, at least one between $\alpha_3, \beta_3 \neq -1$, so we obtain that either $\mathcal{U}^{-\alpha} = S^{\alpha_1}\mathcal{U}$ if $\alpha_3 = 0$ or $\mathcal{U}^{\beta} = S^{-\beta_3}\mathcal{U}$ if $\beta_3 = -2$, while the other factor is some $\mathcal{U}^{\lambda} \in \mathbf{B}_1 \cup \mathbf{B}_2$, with $\lambda_2 = 0$. Applying Pieri's formula, we conclude that $\gamma_1 \geq \lambda_2 = 0$. This proves that all γ satisfy $\gamma_1 \geq 0$.

We now focus on γ_3. If $\gamma_3 = 0$, then $\beta_3 = \alpha_3$ by (1.25) and $\beta_1 \leq \alpha_1$ by (1.24), obtaining $|\gamma| \leq 0$. This shows $\gamma = (0, 0, 0)$. By Lemma 1.3.5, we obtain $\alpha = \beta$ and γ has multiplicity one. Alternatively, by (1.25), we have $\gamma_3 < 0$, hence γ falls in the first case of the initial claim.

By Lemma 1.4.8 we already know $\gamma_1 - \gamma_2 \leq 4$ and $\gamma_2 - \gamma_3 \leq 4$, applying Corollary 1.4.2, this is enough to conclude. ☐

The following corollary is an immediate consequence of Proposition 1.4.9 and Lemma 1.2.5.

Corollary 1.4.10. *The collections*

$$\begin{array}{llllllllll}
\mathbf{B}_1 = \{ & \mathcal{U}^{0,0,-2}, & \mathcal{U}^{0,0,-1}, & \mathcal{U}^{1,0,-1}, & \mathcal{U}^{2,0,-1}, & \mathcal{U}^{0,0,0}, & \mathcal{U}^{1,0,0}, & \mathcal{U}^{2,0,0} & & \}, \\
\mathbf{B}_2 = \{ & & \mathcal{U}^{0,0,-1}, & \mathcal{U}^{1,0,-1}, & \mathcal{U}^{2,0,-1}, & \mathcal{U}^{0,0,0}, & \mathcal{U}^{1,0,0}, & \mathcal{U}^{2,0,0}, & \mathcal{U}^{3,0,0} & \}.
\end{array}$$

are bases of rectangular Lefschetz exceptional collections of length 7*.*

We conclude the section proving Proposition 1.4.3.

Proof of Proposition 1.4.3. As $\mathcal{H}$ is the totalization of the bicomplex (1.1), $\mathcal{H} \in \langle \mathbf{B}_2 \rangle$ because it admits a resolution where all terms belong to $\mathbf{B}_2$. On the other hand, the same resolution of $\mathcal{H}$ shows that $\mathcal{U}^{3,0,0} \in \mathscr{B}$, the category generated by $\mathbf{B}$ (which is possibly not an exceptional sequence, cf. Theorem 1.4.15). Hence, the subcategory $\mathscr{B}$ can be generated by $\mathbf{B}_1 \cup \mathbf{B}_2$ (which is not an exceptional sequence).

By Lemma 1.2.5, to prove semiorthogonality it is enough to verify

$$\operatorname{Hom}^{\bullet}(\mathcal{U}^{\alpha}(l), \mathcal{U}^{\beta}) = 0 \quad \text{for} \quad \beta \leq \alpha \quad \text{and} \quad 1 \leq l \leq 6, \tag{1.26}$$

with $\mathcal{U}^\alpha, \mathcal{U}^\beta \in \mathbf{B}_1 \cup \mathbf{B}_2$. By Corollary 1.4.10, the equation (1.26) holds if both $\alpha, \beta \in \mathbf{B}_1$ or both $\alpha, \beta \in \mathbf{B}_2$. On the other hand, if $\alpha = (3,0,0)$ and $\beta = (0,0,-2)$, we have

$$\operatorname{Hom}^\bullet(\mathcal{U}^{3,0,0}(l), \mathcal{U}^{0,0,-2}) = 0 \quad \text{for} \quad 1 \leq l \leq 6$$

by Lemma 1.4.7, proving the initial statement. □

1.4.2 Properties of $\mathcal{H}$

Recall the definition of the object $\mathcal{H}$ as convolution of the bicomplex (1.1). The purpose of this section is to show that $\mathcal{H}$ is an exceptional object (cf. Proposition 1.4.13) completing $\mathbf{B}_1$ to a Lefschetz basis (cf. Theorem 1.4.15), hence we need to show that $\mathcal{H}$ is exceptional and $\mathcal{H} \in \mathbf{B}_1^\perp$.

We first outline the properties of the lines of the bicomplex. Let us consider the (acyclic!) staircase complex on $\mathsf{IGr}(3,9)$ (cf. Theorem 1.3.6) associated to $\mathcal{U}^{3,0,0}$:

$$0 \to \mathcal{U}^{0,0,-3}(-1) \to \wedge^8 V^* \otimes \mathcal{U}^{0,0,-2}(-1) \to \wedge^7 V^* \otimes \mathcal{U}^{0,0,-1}(-1) \to \wedge^6 V^* \otimes \mathcal{O}(-1) \to \\ \to \wedge^3 V^* \otimes \mathcal{O} \to \wedge^2 V^* \otimes \mathcal{U}^{1,0,0} \to V^* \otimes \mathcal{U}^{2,0,0} \to \mathcal{U}^{3,0,0} \to 0. \quad (1.27)$$

The complex (1.27) is self-dual, as it is the unique complex with nonzero GL-equivariant differentials and these terms (see Theorem 1.3.6). We define the object $\mathcal{E}$ as the stupid truncation of the complex between the lines. As the complex is exact, this induces two resolutions of $\mathcal{E}$ by Sp_9-equivariant bundles (cf. § 1.3.3). More explicitly, there are two exact sequences:

$$0 \to \mathcal{U}^{0,0,-3}(-1) \to \wedge^8 V^* \otimes \mathcal{U}^{0,0,-2}(-1) \to \\ \to \wedge^7 V^* \otimes \mathcal{U}^{0,0,-1}(-1) \to \wedge^6 V^* \otimes \mathcal{O}(-1) \to \mathcal{E} \to 0 \quad (1.28)$$

and

$$0 \to \mathcal{E} \to \wedge^3 V^* \otimes \mathcal{O} \to \wedge^2 V^* \otimes \mathcal{U}^{1,0,0} \to V^* \otimes \mathcal{U}^{2,0,0} \to \mathcal{U}^{3,0,0} \to 0. \quad (1.29)$$

As we can deduce from the self-duality of (1.27),

$$\mathcal{E} \cong \mathcal{E}^*(-1). \quad (1.30)$$

We now consider the (acyclic!) staircase complex on $\mathsf{IGr}(3,9)$ induced by $\mathcal{U}^{2,0,-1}$:

$$0 \to \mathcal{U}^{1,0,-3}(-2) \to \wedge^8 V^* \otimes \mathcal{U}^{1,0,-2}(-2) \to \\ \to \wedge^7 V^* \otimes \mathcal{U}^{1,0,-1}(-2) \to \wedge^6 V^* \otimes \mathcal{U}^{1,0,0}(-2) \to \wedge^4 V^* \otimes \mathcal{O}(-1) \\ \to \wedge^2 V^* \otimes \mathcal{U}^{0,0,-1} \to V^* \otimes \mathcal{U}^{1,0,-1} \to \mathcal{U}^{2,0,-1} \to 0. \quad (1.31)$$

The object $\mathcal{F}$ is the stupid truncation of the complex above between the middle and the lowest line. We obtain two resolutions of $\mathcal{F}$. More explicitly, there are two exact sequences:

$$\begin{aligned} 0 \to \mathcal{U}^{1,0,-3}(-2) \to \wedge^8 V^* \otimes \mathcal{U}^{1,0,-2}(-2) \to \wedge^7 V^* \otimes \mathcal{U}^{1,0,-1}(-2) \to \\ \to \wedge^6 V^* \otimes \mathcal{U}^{1,0,0}(-2) \to \wedge^4 V^* \otimes \mathcal{O}(-1) \to \mathcal{F} \to 0 \end{aligned} \quad (1.32)$$

and

$$0 \to \mathcal{F} \to \wedge^2 V^* \otimes \mathcal{U}^{0,0,-1} \to V^* \otimes \mathcal{U}^{1,0,-1} \to \mathcal{U}^{2,0,-1} \to 0. \quad (1.33)$$

We observe that

$$\mathcal{E}, \mathcal{F} \in \mathscr{B} \quad (1.34)$$

because all the terms in their right resolutions (1.29) and (1.33) belong to $\mathscr{B}$. Notice that $\mathcal{E}$ and $\mathcal{F}$ are vector bundles by [EH16, Theorem 19.2].

We first compute the $\mathrm{Hom}^\bullet$-groups between $\mathcal{E}$ and $\mathcal{F}$ and show that the sequences (1.29) and (1.33) induce mutation triangles. To do so, we introduce the following subsets of $\mathbf{B}_1$:

$$\begin{aligned} \mathbf{S}_1 &= \{\mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}\}, \\ \mathbf{S}_2 &= \{\mathcal{U}^{0,0,-2}, \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}\}, \\ \mathbf{S} &= \mathbf{B}_1 \setminus \{\mathcal{U}^{2,0,-1}\} = \mathbf{S}_1 \cup \mathbf{S}_2. \end{aligned} \quad (1.35)$$

An observation that simplifies many computations in this section is that:

$$\mathbf{S}^* = \mathbf{S} \subset \mathscr{B}, \quad (1.36)$$

where by $\mathbf{S}^*$ we denote all the duals of the elements in $\mathbf{S}$.

Proposition 1.4.11. *The bundle $\mathcal{E}$ is right orthogonal to $\mathcal{F}$:*

$$\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{F}) = 0.$$

Moreover, $\mathcal{E}$ and $\mathcal{F}$ are right orthogonal to $\mathbf{S}$, that is:

$$\mathrm{Hom}^\bullet(\mathbf{S}, \mathcal{E}) = \mathrm{Hom}^\bullet(\mathbf{S}, \mathcal{F}) = 0.$$

Finally, the following isomorphisms hold:

$$\mathcal{E}[3] = \mathbb{L}_{\mathbf{S}_1 \cap \mathbf{B}_2} \mathcal{U}^{3,0,0} = \mathbb{L}_{\mathbf{S}} \mathcal{U}^{3,0,0}, \qquad \mathcal{F}[2] = \mathbb{L}_{\mathbf{S}_2 \cap \mathbf{B}_2} \mathcal{U}^{2,0,-1} = \mathbb{L}_{\mathbf{S}} \mathcal{U}^{2,0,-1}.$$

Proof. As $\mathcal{E} \cong \mathcal{E}^*(-1)$ by (1.30), we have:

$$\mathrm{Hom}^\bullet(\mathbf{S}, \mathcal{E}) = \mathrm{Hom}^\bullet(\mathbf{S}, \mathcal{E}^*(-1)) = \mathrm{Hom}^\bullet(\mathcal{E}(1), \mathbf{S}^*) = \mathrm{Hom}^\bullet(\mathcal{E}(1), \mathbf{S}),$$

where the last equality holds by the symmetry of $\mathbf{S}$, cf. (1.36). As $\mathcal{E}(1) \in \mathscr{B}(1)$ by (1.34) and $\mathbf{S} \subset \mathscr{B}$, the last $\mathrm{Hom}^\bullet$-group vanishes by Proposition 1.4.3, proving that $\mathcal{E} \in \mathbf{S}^\perp$.

We now focus on $\mathcal{F}$. We observe that:

$$\mathrm{Hom}^\bullet(\mathbf{S}_1, \mathcal{F}) = 0, \qquad \mathrm{Hom}^\bullet(\mathcal{U}^{3,0,0}, \mathcal{F}) = 0,$$

because $\mathbf{B}_2$ is an exceptional collection by Corollary 1.4.10 and all the terms in the right resolution (1.33) of $\mathcal{F}$ belong to $\langle\, \mathcal{U}^{0,0,-1},\, \mathcal{U}^{1,0,-1},\, \mathcal{U}^{2,0,-1}\,\rangle$, as a consequence, we obtain that

$$\mathcal{F} \in \langle\, \mathcal{U}^{0,0,0},\, \mathcal{U}^{1,0,0},\, \mathcal{U}^{2,0,0},\, \mathcal{U}^{3,0,0}\,\rangle^\perp = \mathbf{S}_1^\perp \cap (\mathcal{U}^{3,0,0})^\perp.$$

Finally, $\mathcal{E} \in \langle\, \mathcal{U}^{0,0,0},\, \mathcal{U}^{1,0,0},\, \mathcal{U}^{2,0,0},\, \mathcal{U}^{3,0,0}\,\rangle$ by (1.29), hence we obtain $\mathcal{F} \in \mathcal{E}^\perp$.

Notice that we have:

$$\mathrm{Hom}^\bullet(\mathbf{S}_2, \mathcal{F}) \cong \mathrm{Hom}^\bullet(\mathcal{F}^*, \mathbf{S}_2^*). \tag{1.37}$$

By (1.32), we have that $\mathcal{F}^*$ belongs to the category generated by $\mathbf{B}_2(1)$, $\mathbf{B}_2(2)$ and $\mathcal{U}^{3,0,-1}(2)$, while $\mathbf{S}_2^* \subset \langle\, \mathbf{B}_2\,\rangle$. As a consequence, if $\mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-1}(2), \mathbf{S}_2^*) = 0$, the right side of (1.37) vanishes by Proposition 1.4.3. But this follows from Lemma 1.4.6, proving that $\mathcal{F} \in \mathbf{S}^\perp$.

Finally, by (1.29) and (1.33), we have:

$$\mathrm{Cone}(\mathcal{U}^{3,0,0} \to \mathcal{E}[3]) \in \langle\, \mathbf{S}_1 \cap \mathbf{B}_2\,\rangle, \qquad \mathrm{Cone}(\mathcal{U}^{2,0,-1} \to \mathcal{F}[2]) \in \langle\, \mathbf{S}_2 \cap \mathbf{B}_2\,\rangle.$$

As we proved that $\mathcal{E}, \mathcal{F} \in \mathbf{S}^\perp$, the sequences (1.29) and (1.33) induce mutation triangles. □

Proposition 1.4.12. *We have the following isomorphisms of* $\mathrm{Hom}^\bullet$*-groups:*

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{E}) = \mathbb{C}.$$

Proof. By (1.28) and (1.33), we have

$$\begin{aligned}\mathrm{Cone}(\mathcal{U}^{2,0,-1}[-2] \to \mathcal{F}) &\in \langle \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}\,\rangle \subset \mathscr{B} \cap \mathscr{B}^*,\\ \mathrm{Cone}(\mathcal{E} \to \mathcal{U}^{0,0,-3}(-1)[3]) &\in \langle \mathcal{U}^{0,0,-2}(-1), \mathcal{U}^{0,0,-1}(-1), \mathcal{O}(-1)\,\rangle \subset \mathscr{B}(-1).\end{aligned}$$

Applying Proposition 1.4.3 and recalling that $\mathrm{Hom}^\bullet(\mathscr{B}^*, \mathcal{U}^{0,0,-3}(-1)) \cong \mathrm{Hom}^\bullet(\mathcal{U}^{3,0,0}(1), \mathscr{B}) = 0$, because $\mathcal{U}^{3,0,0}(1) \in \mathscr{B}(1)$. Hence we have:

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{E}) = \mathrm{Hom}^\bullet(\mathcal{U}^{2,0,-1}[-2], \mathcal{U}^{0,0,-3}(-1)[3]) = \mathrm{Hom}^\bullet(\mathcal{U}^{2,0,-1}, \mathcal{U}^{0,0,-3}(-1))[5].$$

Finally,

$$\mathrm{Hom}^\bullet(\mathcal{U}^{2,0,-1}, \mathcal{U}^{0,0,-3}(-1)) \cong \mathrm{H}^\bullet(X, \mathcal{U}^{1,0,-2} \otimes \mathcal{U}^{0,0,-3}(-1)) \cong \mathbb{C}[-5]$$

by Lemma 1.4.6. □

Let $\phi \in \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{E}) = \mathbb{C}$ be the nonzero morphism, which is unique up to scalar. Then, we can define $\mathcal{H}$ as

$$\mathcal{H} = \mathrm{Cone}(\mathcal{F} \xrightarrow{\phi} \mathcal{E}). \tag{1.38}$$

By [Gus20, Lemma 5.1], the morphism $\phi : \mathcal{F} \to \mathcal{E}$ lifts uniquely to a morphism of the right resolutions of $\mathcal{F}$ and $\mathcal{E}$ ((1.33) and (1.29)), finally defining the maps in the bicomplex (1.1); hence $\mathcal{H}$ defined by (1.38) is isomorphic to the totalization of (1.1).

As an immediate consequence, we obtain the following.

Proposition 1.4.13. *The objects $\mathcal{E}, \mathcal{F}$ and $\mathcal{H}$ are exceptional. Moreover, $\mathcal{H} = \mathbb{L}_{\mathcal{F}}\mathcal{E}$.*

Proof. We recall from § 1.2 that given an admissible subcategory $\mathscr{A}$, the mutation functor $\mathbb{L}_{\mathscr{A}}$ induces an equivalence between ${}^{\perp}\mathscr{A}$ and $\mathscr{A}^{\perp}$. In particular, if $G \in {}^{\perp}\mathscr{A}$ is exceptional, then $\mathbb{L}_{\mathscr{A}}G$ is exceptional as well. As $\mathbf{B}_1$ and $\mathbf{B}_2$ are exceptional collections by Corollary 1.4.10, the objects $\mathcal{U}^{3,0,0}$ and $\mathcal{U}^{2,0,-1}$ are exceptional and we have

$$\mathcal{U}^{3,0,0} \in {}^{\perp}(\mathbf{S}_1 \cap \mathbf{B}_2) \quad \text{and} \quad \mathcal{U}^{2,0,-1} \in {}^{\perp}(\mathbf{S}_2 \cap \mathbf{B}_2),$$

hence the objects $\mathcal{E}$ and $\mathcal{F}$ are exceptional by Proposition 1.4.11. Finally, as $\mathcal{E} \in {}^{\perp}\mathcal{F}$ by Proposition 1.4.11 and $\operatorname{Hom}^\bullet(\mathcal{F}, \mathcal{E}) \cong \mathbb{C}$ by Proposition 1.4.12, we conclude that

$$\mathcal{H} = \operatorname{Cone}(\mathcal{F} \xrightarrow{\phi} \mathcal{E}) = \mathbb{L}_{\mathcal{F}}\mathcal{E} \tag{1.39}$$

is exceptional. □

Proposition 1.4.14. *The object $\mathcal{H}$ is right orthogonal to $\mathbf{B}_1$, that is:*

$$\operatorname{Hom}^\bullet(\mathbf{B}_1, \mathcal{H}) = 0.$$

Hence, $\mathcal{H} = \mathbb{L}_{\mathbf{B}_1}\mathcal{U}^{3,0,0}[3]$.

Proof. We have $\mathcal{H} \in \mathbf{S}^{\perp}$, because $\mathcal{E}, \mathcal{F} \in \mathbf{S}^{\perp}$ by Proposition 1.4.11. By Proposition 1.4.13, we have $\mathcal{H} \in \mathcal{F}^{\perp}$. But $\mathcal{U}^{2,0,-1} \in \langle \mathcal{F}, \mathbf{S} \rangle$ by (1.33), hence $\mathcal{H} \in (\mathcal{U}^{2,0,-1})^{\perp}$. Finally, as we have $\mathbf{B}_1 = \mathbf{S} \cup \{\mathcal{U}^{2,0,-1}\}$, we deduce that $\mathcal{H} \in \mathbf{B}_1^{\perp}$. This also proves $\mathcal{H} = \mathbb{L}_{\mathbf{B}_1}\mathcal{E}$. By Proposition 1.4.11 we obtain the second claim. □

Finally we can state the main result of the section. Recall that $\mathbf{B} = \{\mathcal{H}\} \cup \mathbf{B}_1$.

Theorem 1.4.15. *The bounded derived category of coherent sheaves on $\mathsf{IGr}(3, 9)$ admits a rectangular Lefschetz exceptional collection of length 8 composed by Sp_9-equivariant objects given by:*

$$\langle \mathbf{B}, \mathbf{B}(1), \mathbf{B}(2), \mathbf{B}(3), \mathbf{B}(4), \mathbf{B}(5), \mathbf{B}(6) \rangle.$$

Proof. Since $\{\mathcal{H}\} \cup \mathbf{B}_1 \subset \mathscr{B}$, the blocks are semiorthogonal by Proposition 1.4.3. On the other hand, semiorthogonality within a block follows from Corollary 1.4.10 and Proposition 1.4.14. The object $\mathcal{H}$ itself is exceptional by Proposition 1.4.13. Applying Lemma 1.2.5, we deduce the exceptionality claim.

It is immediate to see that all the elements in $\mathbf{B}_1$ admit an Sp_9-equivariant structure. To show that $\mathcal{H}$ admits one as well, we recall [Pol11, Lemma 2.2.(1)], which proves that any exceptional object in $\mathsf{IGr}(k, 2n+1)$ is Sp_{2n+1}-equivariant.

Recall that $\mathsf{Sp}_{2n+1} = (\mathbb{C}^* \times \mathsf{Sp}_{2n}) \rtimes U$, by [Mih07, §3], where $\mathbb{C}^* \times \mathsf{Sp}_{2n}$ is the Levi subgroup, and $U \cong \mathbb{C}^{2n}$ is the unipotent radical. As the Levi subgroup is reductive, we can apply [Pol11, Lemma 2.2.(2)] to $\mathbb{C}^* \times \mathsf{Sp}_{2n}$. Since $U \cong \mathbb{C}^{2n}$, we see that the hypothesis of [Pol11, Lemma 2.2.(1)] hold for Sp_{2n+1} as well. In particular, $\mathcal{H}$ is Sp_9-equivariant. □

Remark 1.4.16. It is possible to show that $\mathcal{H}$ is a vector bundle. To do so, we need to prove that the morphism ϕ defined by Proposition 1.4.12 is surjective. Approaching the problem in $\mathsf{IGr}(3, 9)$ is hard, as $\mathsf{IGr}(3, 9)$ is not homogeneous and Sp_9 is not semisimple.

A lighter solution is to consider the embedding of X in $\mathsf{IGr}(3, 10)$. The staircase complexes of $\widetilde{\mathcal{U}}^{3,0,0}$ and $\widetilde{\mathcal{U}}^{2,0,-1}$ on $\mathsf{IGr}(3, 10)$, appropriately truncated, provide lifts of $\mathcal{E}$ and $\mathcal{F}$. We can prove that the corresponding map is surjective, similarly to [Gus20, Lemma 5.3], then we can obtain the result on the original morphism applying Proposition 1.3.8.

1.5 Fullness

In [Nov20, § 4-§ 5], a procedure to show the fullness of an exceptional collection on the even isotropic Grassmannian was developed. We adapt it to the case of an odd isotropic Grassmannian and we summarize it in § 1.5.2. This reduces the question of fullness to a combinatorial statement (cf. Proposition 1.5.6).

The procedure consists of consequent applications of steps of two different kinds to produce more and more objects in the category $\mathscr{D}$ generated by the exceptional collection:

1. using staircase complexes, already discussed in Theorem 1.3.6;
2. using so-called "symplectic bundle relations", see § 1.5.1.

When we have produced sufficiently many objects, we can conclude that $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(X)$ by Proposition 1.5.6. We give more details on this strategy in § 1.5.2 below and implement it to prove the fullness in § 1.5.3 in nine steps. From now on, we represent $\mathcal{U}^{i,0,\dots,0,-j}$ with the shorthand notation

$$\mathcal{U}^{i,-j}.$$

1.5.1 Symplectic bundle relations

In this section we work with any odd isotropic Grassmannian $\mathsf{IGr}(k, V)$, with $\mathsf{dim}\, V = 2n+1$.

Consider the embedding $j : \mathsf{IGr}(k, V) \to \mathsf{IGr}(k, \widetilde{V})$ where $\widetilde{V}$ is a symplectic vector space with $\dim \widetilde{V} = 2n + 2$. Recall the notation fixed in § 1.3.3. On the even isotropic Grassmannian $\mathsf{IGr}(k, \widetilde{V})$, the symplectic bundle $\widetilde{\mathcal{S}} = \widetilde{\mathcal{U}}^{\perp}/\widetilde{\mathcal{U}}$ is the bundle of rank $2(n+1-k)$ defined as the cohomology in degree 0 of the complex (1.19). We consider the restriction to $\mathsf{IGr}(k, V)$ of the sequence (1.19), that is:

$$0 \to \mathcal{U} \to \widetilde{V} \otimes \mathcal{O} \to \mathcal{U}^* \to 0, \tag{1.40}$$

which has only cohomology in degree 0, which is isomorphic to $j^*\widetilde{\mathcal{S}}$.

Proposition 1.5.1. *For any $p \geq 0$, the object $\wedge^p j^*\widetilde{\mathcal{S}} \in \mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(k, V))$ is quasi-isomorphic to a complex with entries given by direct sums, possibly with multiplicities, of $\mathcal{U}^{i,-j}$ for $i, j \geq 0$ and $i + j \leq p$. Moreover, if $i + j = p$, the bundle $\mathcal{U}^{i,-j}$ appears exactly once among the direct summands of the terms of this complex.*

Proof. The complex obtained from the sequence (1.40), which we will denote as $\mathcal{C}^\bullet$ is quasi-isomorphic to $j^*\widetilde{\mathcal{S}}$. The cohomology of the complex $\wedge^p\mathcal{C}^\bullet$ is isomorphic to the cohomology of $\wedge^p j^*\widetilde{\mathcal{S}}$ (see for instance [Dol58, § 6.6]). As $\wedge^p j^*\widetilde{\mathcal{S}}$ is concentrated in degree 0, we have $\wedge^p\mathcal{C}^\bullet \cong \wedge^p j^*\widetilde{\mathcal{S}}$ in $\mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(k,V))$. Recall that the monoidal structure of $\mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(k,V))$ is defined in such a way that $\wedge^p(\mathcal{F}[\pm 1]) \cong (S^p\mathcal{F})[\pm p]$ for every vector bundle $\mathcal{F}$.

The terms of the complex $\wedge^p\mathcal{C}^\bullet$ are given by direct sums of:

$$\wedge^{p_1}(\mathcal{U}^*[-1]) \otimes \wedge^{p_2}(\widetilde{V}\otimes\mathcal{O}) \otimes \wedge^{p_3}(\mathcal{U}[1]) = \mathcal{U}^{p_1,0}[-p_1] \otimes \wedge^{p_2}(\widetilde{V}\otimes\mathcal{O}) \otimes \mathcal{U}^{0,-p_3}[p_3] \quad (1.41)$$

with

$$p_1 + p_2 + p_3 = p, \qquad p_1, p_2, p_3 \geq 0$$

By Pieri's formula, we can decompose the factors above as:

$$\mathcal{U}^{p_1,0} \otimes \mathcal{U}^{0,-p_3} = \bigoplus_{0\leq t\leq \min(p_1,p_3)} \mathcal{U}^{p_1-t,t-p_3}.$$

We now determine the multiplicity of $\mathcal{U}^{i,-j}$ with $i+j=p$. If $\mathcal{U}^{i,-j}$ comes from $\mathcal{U}^{p_1,0}\otimes\mathcal{U}^{0,-p_3}$, then $i+j=(p_1-t)+(p_3-t)\leq p_1+p_3$. Then we obtain $p_2=0, i=p_1, j=p_3$ and $t=0$. This shows that $\mathcal{U}^{i,-j}$ appears as a direct summand of a single term of $\wedge^p\mathcal{C}^\bullet$. Moreover, its multiplicity is $\wedge^{p_2}\widetilde{V}\cong\mathbb{C}$, proving the claim. □

Restricting the symplectic isomorphism on $\mathsf{IGr}(k,\widetilde{V})$ we obtain $j^*\widetilde{\mathcal{S}} \cong j^*\widetilde{\mathcal{S}}^*$, which induces:

$$\wedge^p j^*\widetilde{\mathcal{S}} \cong \wedge^{2(n+1-k)-p} j^*\widetilde{\mathcal{S}}^* \cong \wedge^{2(n+1-k)-p} j^*\widetilde{\mathcal{S}}. \quad (1.42)$$

Applying Proposition 1.5.1 to $\wedge^p j^*\widetilde{\mathcal{S}}$ and $\wedge^{2(n+1-k)-p} j^*\widetilde{\mathcal{S}}$, we obtain the following key proposition.

Proposition 1.5.2. *Let $\mathscr{D} \subseteq \mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(k,V))$ be a triangulated subcategory and let $l\in\mathbb{Z}$. Let $i+j=p$ with $p>n+1-k$. If $\mathcal{U}^{i',-j'}(l)\in\mathscr{D}$ for every $i',j'\geq 0$, with $i'+j'\leq p$ and $(i,j)\neq(i',j')$, then $\mathcal{U}^{i,-j}(l)\in\mathscr{D}$.*

Proof. Assume $l=0$. By hypothesis, we have $2(n+1-k)-p < n+1-k$. We first apply Proposition 1.5.1 to show that $\wedge^{2(n+1-k)-p} j^*\widetilde{\mathcal{S}} \in \mathscr{D}$. By (1.42), that is equivalent to $\wedge^p j^*\widetilde{\mathcal{S}}\in\mathscr{D}$. Applying again Proposition 1.5.1, $\wedge^p j^*\widetilde{\mathcal{S}}$ admits a filtration with factors given by direct sums of $\mathcal{U}^{i,-j}$ and $\mathcal{U}^{i',-j'}$ as above.

By hypothesis, all the factors in the filtration belong to $\mathscr{D}$ except possibly $\mathcal{U}^{i,-j}$, which appears exactly once as a factor, hence $\mathcal{U}^{i,-j}\in\mathscr{D}$ as well. This proves the claim.

If $l\neq 0$, we apply the proven result to a twist of $\mathscr{D}$. □

1.5.2 Strategy

We show here how to reduce the proof of the fullness of the Lefschetz collection $\mathbf{B}$ to an algorithmic question. Let us consider the case of $X = \mathsf{IGr}(3, 2n+1)$. Consider the set of bundles

$$\mathbf{T} = \{\mathcal{U}^{i,-j} \mid 0 \leq i, j;\ i+j \leq 2n-2\} \tag{1.43}$$

Proposition 1.5.3. *Let $\mathscr{D} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be an admissible subcategory. If the set $\mathbf{T}(-l)$ is contained in $\mathscr{D}$ for every $l \geq 0$ then $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(X)$.*

Proof. Since $\mathscr{D}$ is admissible, consider the decomposition $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathscr{D}^{\perp}, \mathscr{D} \rangle$. Recall that $\mathcal{O}(-l)$ for $l \geq 0$ is a spanning class (cf. [Huy06, Corollary 3.19]). This implies $\mathscr{D}^{\perp} = 0$ as $\mathcal{O}(-l) \in \mathbf{T}(-l) \subset \mathscr{D}$, proving $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(X)$. □

We rephrase here in a more compact form how to apply Theorem 1.3.6. For each triple of integers (a, b, c), with $a, c \geq -1$, $b \geq 0$ and $a+b+c \leq 2n-2$, we introduce a set of $a+b+c+2$ bundles named $\mathbf{S}_b^{a,c}$ and defined as:

$$\mathbf{S}_b^{a,c} := \begin{cases} \mathcal{U}^{i,-b} & \text{for} \quad 0 \leq i \leq a, \\ \mathcal{U}^{i,i-b+1}(-i-1) & \text{for} \quad 0 \leq i \leq b-1, \\ \mathcal{U}^{b,-i}(-b-1) & \text{for} \quad 0 \leq i \leq c. \end{cases}$$

It is immediate to verify that

$$(\mathbf{S}_b^{a,c})^* = \mathbf{S}_b^{c,a}(b+1). \tag{1.44}$$

We also remark that

$$\mathbf{S}_b^{a+1,c} = \{\mathcal{U}^{a+1,-b}\} \sqcup \mathbf{S}_b^{a,c}, \quad \mathbf{S}_b^{a,c+1} = \mathbf{S}_b^{a,c} \sqcup \{\mathcal{U}^{b,-c-1}(-b-1)\}. \tag{1.45}$$

We now explain the relationship of $\mathbf{S}_b^{a,c}$ with staircase complexes.

If $a, b \geq 0$ and $\mathsf{max}\{a, b, a+b\} \leq 2n-2$, then we can apply Theorem 1.3.6 to compute the staircase complex of $\mathcal{U}^{a,-b}$ on $\mathsf{IGr}(3, 2n+1)$. As $a+b+c+2 \leq 2n$, the collection $\mathbf{S}_b^{a,c}$ is the set of the rightmost $a+b+c+2$ vector bundles appearing in this staircase complex (cf. (1.16)). If $a+b+c = 2n-2$, then $\mathbf{S}_b^{a,c}$ corresponds to the whole staircase complex.

We now consider the case where a or c is -1. Let us fix $a+b+c = 2n-3$, which corresponds to one term fewer than the amount of entries of a staircase complex on $\mathsf{IGr}(3, 2n+1)$. From the previous consideration and the identities stated in (1.45), we obtain that if $a, b \geq 0$ and $c = -1$, then $\mathbf{S}_b^{a,-1} = \mathbf{S}_b^{a,0} \setminus \{\mathcal{U}^{b,0}(-b-1)\}$. That is, $\mathbf{S}_b^{a,-1}$ is the set of entries of the staircase complex associated to $\mathcal{U}^{a,-b}$ except $\mathcal{U}^{b,0}(-b-1)$.

Similarly, if $a = -1$, $2n-2 \geq b \geq 0$ and $c \geq 0$, then $\mathbf{S}_b^{-1,c} = \mathbf{S}_b^{0,c} \setminus \{\mathcal{U}^{0,-b}\}$, which corresponds to all the entries of the staircase complex associated to $\mathcal{U}^{0,-b}$, except $\mathcal{U}^{0,-b}$ itself.

Proposition 1.5.4. *Let* $\mathscr{D} \subseteq \mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(3, 2n+1))$ *be a full triangulated subcategory and* $l \in \mathbb{Z}$. *Let* (a, b, c) *be a triple with* $a + b + c = 2n - 3$ *and* $a, c \geq -1$ *and* $2n - 2 \geq b \geq 0$. *If* $\mathbf{S}_b^{a,c}(l) \subset \mathscr{D}$, *then we have the following cases:*

- *if* $a \geq -1$ *and* $b, c \geq 0$, *then* $\mathcal{U}^{a+1,-b}(l) \in \mathscr{D}$;
- *if* $a, b \geq 0$ *and* $c \geq -1$, *then* $\mathcal{U}^{b,-c-1}(l - b - 1) \in \mathscr{D}$.

Proof. In the first case, we consider the staircase complex of $\mathcal{U}^{a+1,-b}(l)$ as the left resolution of its rightmost term. The terms in the resolution are multiples of the bundles in $\mathbf{S}_b^{a,c}(l) \subset \mathscr{D}$, we conclude that $\mathcal{U}^{a+1,-b}(l) \in \mathscr{D}$.

For the second statement, we apply (1.44). From the hypotheses, we have that $(\mathbf{S}_b^{a,c}(l))^* = \mathbf{S}_b^{c,a}(b+1-l) \subset \mathscr{D}^*$. Applying the first statement, we obtain $\mathcal{U}^{c+1,-b}(b+1-l) \in \mathscr{D}^*$, hence $\mathcal{U}^{b,-c-1}(l - b - 1) \in \mathscr{D}$, proving the second claim. □

The following Proposition 1.5.5 formalizes the idea of applying the staircase complex several times in a row. Consider the following partitions of the set $\mathbf{T}$ defined in (1.43):

$$\mathbf{T} = \bigsqcup_{0 \leq b \leq 2n-2} \mathbf{P}_b = \bigsqcup_{0 \leq a \leq 2n-2} \mathbf{N}_a. \tag{1.46}$$

where

$$\mathbf{P}_b = \{\mathcal{U}^{i,-b} \mid 0 \leq i \leq 2n - 2 - b\}, \quad \mathbf{N}_a = \{\mathcal{U}^{a,-i} \mid 0 \leq i \leq 2n - 2 - a\}.$$

Let $\mathscr{D}$ be a triangulated subcategory. Fixing b, we observe that the staircase complexes associated to $\mathcal{U}^{i,-b}$ contains the left resolution of $\mathcal{U}^{i+1,-b}$. This allows us to generate $\mathcal{U}^{i+1,-b}$ starting from the staircase complex of $\mathcal{U}^{i,-b}$. Proceeding inductively, this allows to generate all the bundles $\mathcal{U}^{j,b}$ with $j \geq i$, which are all contained in $\mathbf{P}_b$. The same works with the right ends of the staircase complexes and $\mathbf{N}_a$.

This fact is shown in the following picture (fig. 1.1), where we represent the weights in $\mathbf{T}$ for $n = 4$. In this picture, the sets $\mathbf{P}_i$ are given by the ascending diagonals with second coordinate equal to i, while the sets $\mathbf{N}_i$ are given by descending diagonals with first coordinate i.

We circle the elements of $\mathbf{S}_1^{5,0}$, $\mathbf{S}_1^{4,1}$ and $\mathbf{S}_1^{3,2}$, which correspond to the bundles belonging to three staircase complexes; notice that they have several elements in common. This shows that once we proved that $\mathbf{S}_1^{2,2} \subset \mathscr{D}$, then $\mathbf{S}_1^{3,2} \subset \mathscr{D}$ by Proposition 1.5.4, but then, the left part of the staircase complex of $\mathcal{U}^{4,-1}$ is contained in $\mathbf{S}_1^{3,2}$, so we can apply repeatedly Proposition 1.5.4.

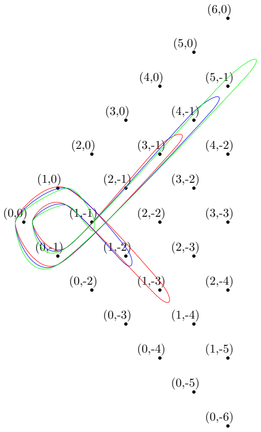

Figure 1.1: The weights in $\mathbf{T}$ for $n = 4$. The sets $\mathbf{S}_1^{5,0}$, $\mathbf{S}_1^{4,2}$, $\mathbf{S}_1^{3,3}$ are in green, blue, red, respectively.

Using this observation, we prove the following.

Proposition 1.5.5. *Let $\mathscr{D} \subseteq \mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(3, 2n+1))$ be a triangulated subcategory and $l \in \mathbb{Z}$. Let (a, b, c) be a triple with $a + b + c = 2n - 3$ and $a, c \geq -1$ and $2n - 2 \geq b \geq 0$. If $\mathbf{S}_b^{a,c}(l) \subset \mathscr{D}$, then:*

$$\mathbf{P}_b(l) \subset \mathscr{D} \quad \textit{and} \quad \mathbf{N}_b(l - b - 1) \subset \mathscr{D}.$$

Proof. Assume $l = 0$. We prove the statement for any triple (a, b, c) with the properties stated above, in particular, $a = 2n - 3 - c - b$. We prove $\mathbf{P}_b \subset \mathscr{D}$, proceeding by induction on c. If $c = -1$, we have $a = 2n - 2 - b$. By definition:

$$\mathbf{S}_b^{a,-1} = \{\mathcal{U}^{i,-b} \mid 0 \leq i \leq a\} \sqcup \{\mathcal{U}^{b,-i}(-i-1) \mid 0 \leq i \leq b-1\} \supseteq \mathbf{P}_b,$$

proving the base of induction. Suppose $c \geq 0$, then by Proposition 1.5.4 we have $\mathcal{U}^{a+1,-b} \in \mathscr{D}$. Applying (1.45), we have $\mathbf{S}_b^{a+1,c} \subset \mathscr{D}$. Finally with (1.45), $\mathbf{S}_b^{a+1,c-1} \subset \mathbf{S}_b^{a+1,c} \subset \mathscr{D}$. By the induction hypothesis, $\mathbf{P}_b \subset \mathscr{D}$, proving the first part of the statement. When $l \neq 0$, we apply the result to a twist of $\mathscr{D}$.

From the statement on $\mathbf{P}_b$ we deduce the result for $\mathbf{N}_b$ by dualizing and applying (1.44) (as in the final part of Proposition 1.5.4). □

Proposition 1.5.6. *Let $\mathscr{D} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be a full admissible subcategory. If $\mathbf{T}(l) \subset \mathscr{D}$ for $l = 0, \ldots, 2n-2$, then $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(X)$.*

Proof. Let us consider the set

$$\begin{aligned}\mathbf{S}_b^{2n-2-b,-1} &= \{\mathcal{U}^{i,-b} \mid 0 \leq i \leq 2n-2-b\} \sqcup \{\mathcal{U}^{b,-i}(-i-1) \mid 0 \leq i \leq b-1\} \subseteq \\ &\subseteq \mathbf{T} \sqcup \bigsqcup_{0 \leq i \leq b-1} \mathbf{T}(-i-1).\end{aligned}$$

In particular, twisting the previous inclusion by $\mathscr{O}(b)$, for $0 \le b \le 2n-2$, we obtain:

$$\mathbf{S}_b^{2n-2-b,-1}(b) \subseteq \mathbf{T}(b) \sqcup \bigsqcup_{0\le i\le b-1} \mathbf{T}(b-i-1) \subset \mathscr{D}.$$

By Proposition 1.5.5, we have $\mathbf{N}_b(-1) \subset \mathscr{D}$ for any $0 \le b \le 2n-2$. As $\mathbf{T}(-1)$ can be partitioned as the disjoint union of $\mathbf{N}_b(-1)$ for $0 \le b \le 2n-2$ by (1.46), we obtain $\mathbf{T}(-1) \subset \mathscr{D}$. Inductively, we obtain $\mathbf{T}(-l) \subset \mathscr{D}$ for every $l \ge 0$, concluding the proof with Proposition 1.5.3. ☐

Finally, we are able to show the fullness of the Lefschetz collection with basis $\mathbf{B}$ on $\mathsf{IGr}(3,9)$.

1.5.3 Proof of fullness

We go back to the case $X = \mathsf{IGr}(3,9)$. Let

$$\mathscr{D} := \langle\, \mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2),\, \mathbf{B}(3),\, \mathbf{B}(4),\, \mathbf{B}(5),\, \mathbf{B}(6)\,\rangle \subseteq \mathbf{D}^{\mathrm{b}}(X)\,.$$

By Theorem 1.4.15, $\mathscr{D}$ is generated by an exceptional collection, hence it is admissible (cf. § 1.2). We want to prove that $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(X)$. Recall that

$$\mathbf{B}_1 \cup \mathbf{B}_2 = \{\mathcal{U}^{0,0,-2}, \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}\} \subset \langle\, \mathbf{B}\,\rangle, \quad (1.47)$$

hence $(\mathbf{B}_1 \cup \mathbf{B}_2)(l) \subset \mathscr{D}$ for $l = 0, \dots, 6$. To prove that $\mathscr{D}$ is full, we go through the following algorithm, which consists in the iteration of these two steps:

- determine all the $\mathbf{S}_b^{a,c}(l) \subset \mathscr{D}$ with $a+b+c=6$ and apply Proposition 1.5.5;
- apply Proposition 1.5.2 where possible.

The procedure terminates if we prove that the following set of bundles

$$\mathbf{T},\, \mathbf{T}(1),\, \dots,\, \mathbf{T}(6),$$

where $\mathbf{T}$ was defined in (1.43), is contained in $\mathscr{D}$. Then we can apply Proposition 1.5.6 and prove fullness. Notice that it is not certain that the presented algorithm terminates, even if the starting collection is full. In our specific case, we are able to show in nine steps that the condition in Proposition 1.5.6 holds.

Theorem 1.5.7. *The condition in Proposition 1.5.6 holds for $X = \mathsf{IGr}(3,9)$ and the following admissible subcategory*

$$\mathscr{D} := \langle\, \mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2),\, \mathbf{B}(3),\, \mathbf{B}(4),\, \mathbf{B}(5),\, \mathbf{B}(6)\,\rangle.$$

Hence, $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(X)$.

For clarity of exposition we present the steps in this procedure next to diagrams to illustrate the progress in the proof. We apply a label $\boxed{l_1 \div l_2}$ next to every colored area to denote for which twists we know already that the bundles in the area belong to $\mathscr{D}$.

Step 1.

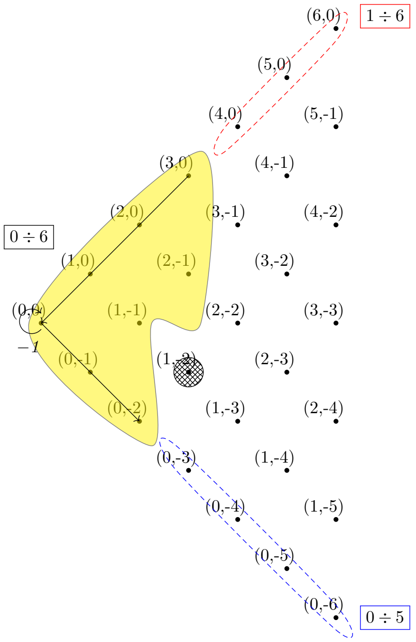

Figure 1.2: The set $\mathbf{S}_0^{3,2}$ and the bundle $\mathcal{U}^{1,-2}$.

As $\mathbf{S}_0^{3,2}(l) \subset \mathscr{D}$ for $l = 1, \dots, 6$, using Proposition 1.5.5 we obtain:

$$\begin{aligned} &\mathbf{P}_0(l) \subset \mathscr{D}, && \text{for} \quad l = 1, \dots, 6, \\ &\mathbf{N}_0(l) \subset \mathscr{D}, && \text{for} \quad l = 0, \dots, 5. \end{aligned}$$

Moreover, we notice that:

$$\begin{aligned} \mathcal{U}^{i,-j}(l) \in \mathscr{D} \quad & \text{for} \quad l = 0, \dots, 5 \\ & \text{and} \quad i + j \leq 3, \end{aligned}$$

except for $\mathcal{U}^{1,-2}(l)$. Applying Proposition 1.5.2, we conclude

$$\mathcal{U}^{1,-2}(l) \in \mathscr{D} \quad \text{for} \quad l = 0, \dots, 5.$$

Step 2.

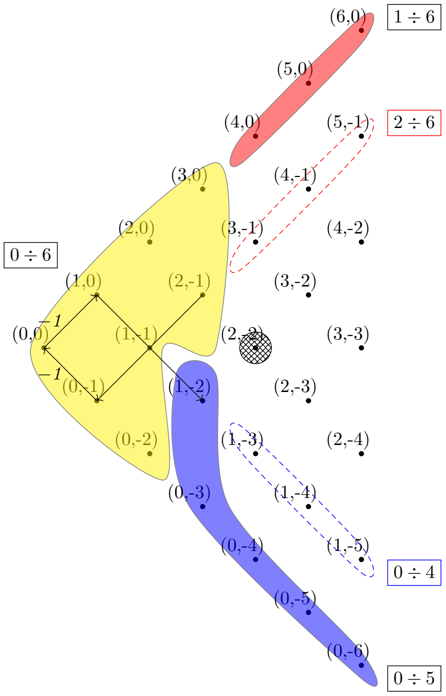

Figure 1.3: The set $\mathbf{S}_1^{2,2}$ and $\mathcal{U}^{2,-2}$.

Given that $\mathbf{S}_1^{2,2}(l) \subset \mathscr{D}$ for $l = 2, \ldots, 6$, we show:

$$\begin{aligned} &\mathbf{P}_1(l) \subset \mathscr{D}, && \text{for} \quad l = 2, \ldots, 6, \\ &\mathbf{N}_1(l) \subset \mathscr{D}, && \text{for} \quad l = 0, \ldots, 4. \end{aligned}$$

Applying Proposition 1.5.2 to $\mathcal{U}^{2,-2}$, we obtain that

$$\mathcal{U}^{2,-2}(l) \in \mathscr{D} \quad \text{for} \quad l = 2, \ldots, 4.$$

Step 3.

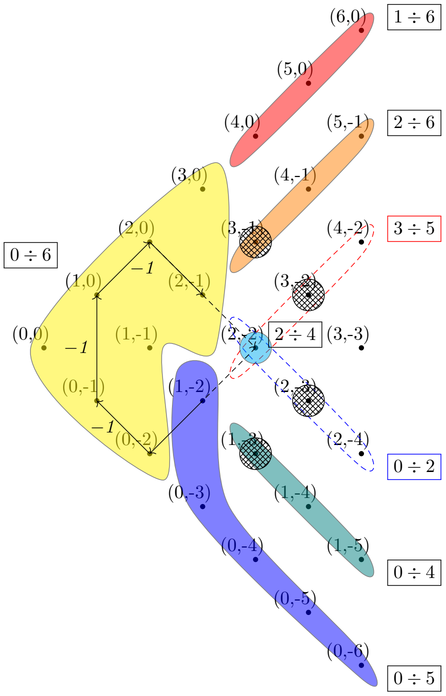

Figure 1.4: In solid black arrows, $\mathbf{S}_2^{1,2} \cap \mathbf{S}_2^{2,1}$. Inside the circles, the bundles obtained applying Proposition 1.5.2.

As $\mathbf{S}_2^{1,2}(l) \subset \mathscr{D}$ for $l = 5$ and $\mathbf{S}_2^{2,1}(l) \subset \mathscr{D}$ for $l = 3, 4$, we have:

$$\begin{aligned} &\mathbf{P}_2(l) \subset \mathscr{D}, && \text{for} \quad l = 3, \dots, 5, \\ &\mathbf{N}_2(l) \subset \mathscr{D}, && \text{for} \quad l = 0, \dots, 2. \end{aligned}$$

Applying Proposition 1.5.2, we obtain:

$$\begin{aligned} \mathcal{U}^{3,-1}(1) &\in \mathscr{D}, \\ \mathcal{U}^{1,-3}(5) &\in \mathscr{D}, \\ \mathcal{U}^{3,-2}(2) &\in \mathscr{D}, \\ \mathcal{U}^{2,-3}(3), \mathcal{U}^{2,-3}(4) &\in \mathscr{D}. \end{aligned}$$

Step 4.

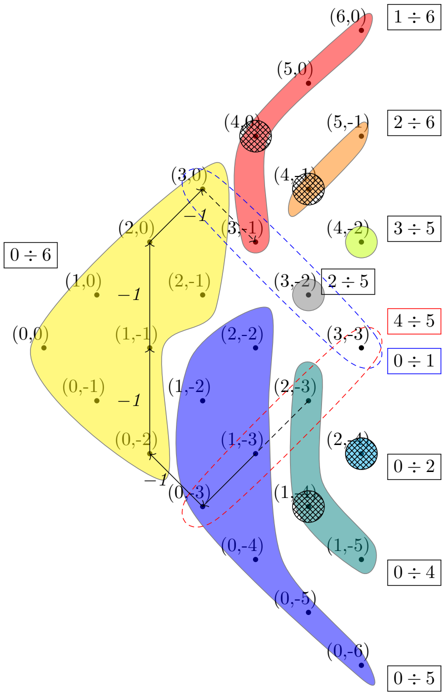

Figure 1.5: In continuous black arrows, $\mathbf{S}_3^{2,0} \cap \mathbf{S}_3^{1,1}$. Inside the circles, the bundles obtained applying Proposition 1.5.2.

As $\mathbf{S}_3^{2,0}(4) \subset \mathscr{D}$ and $\mathbf{S}_3^{1,1}(5) \subset \mathscr{D}$, we have:

$$\begin{aligned} &\mathbf{P}_3(l) \subset \mathscr{D}, && \text{for} \quad l = 4, 5, \\ &\mathbf{N}_3(l) \subset \mathscr{D}, && \text{for} \quad l = 0, 1. \end{aligned}$$

Applying Proposition 1.5.2, we obtain the following bundles:

$$\begin{aligned} \mathcal{U}^{4,0} &\in \mathscr{D}, \\ \mathcal{U}^{4,-1}(1) &\in \mathscr{D}, \\ \mathcal{U}^{1,-4}(5) &\in \mathscr{D}, \\ \mathcal{U}^{2,-4}(4) &\in \mathscr{D}. \end{aligned}$$

Step 5.

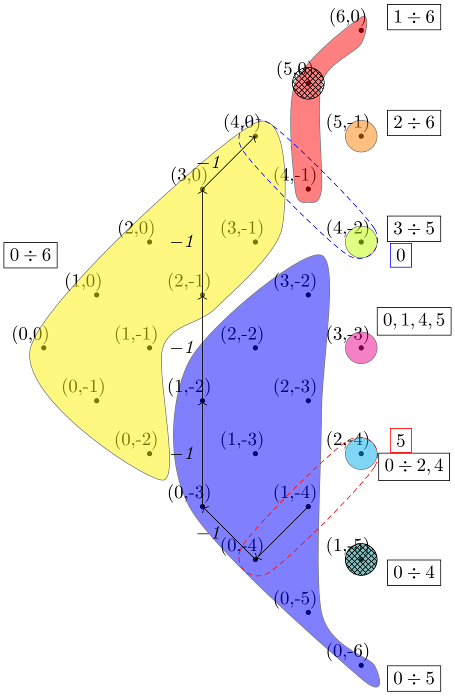

Figure 1.6: The set $\mathbf{S}_4^{1,0}$ and the bundles obtained by applying Proposition 1.5.2.

As $\mathbf{S}_4^{1,0}(5) \subset \mathscr{D}$, we obtain:

$$\mathbf{P}_4(5) \subset \mathscr{D},$$
$$\mathbf{N}_4(0) \subset \mathscr{D}.$$

Applying Proposition 1.5.2, we obtain:

$$\mathcal{U}^{5,0} \in \mathscr{D},$$
$$\mathcal{U}^{1,-5}(5) \in \mathscr{D}.$$

Step 6.

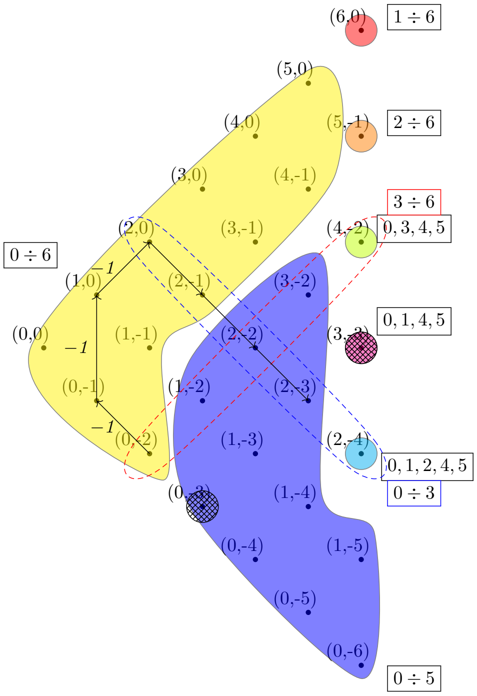

Figure 1.7: The set $\mathbf{S}_2^{0,3}$ and the bundles obtained by applying Proposition 1.5.2.

As $\mathbf{S}_2^{0,3}(6) \subset \mathscr{D}$, we have:

$$\mathbf{P}_2(6) \subset \mathscr{D},$$
$$\mathbf{N}_2(3) \subset \mathscr{D}.$$

Applying Proposition 1.5.2 for $i + j = 3$ and $l = 6$, we obtain

$$\mathcal{U}^{0,-3}(6) \in \mathscr{D};$$

while using $i + j = 6$ and $l = 3$, we obtain

$$\mathcal{U}^{3,-3}(3) \in \mathscr{D}.$$

Step 7.

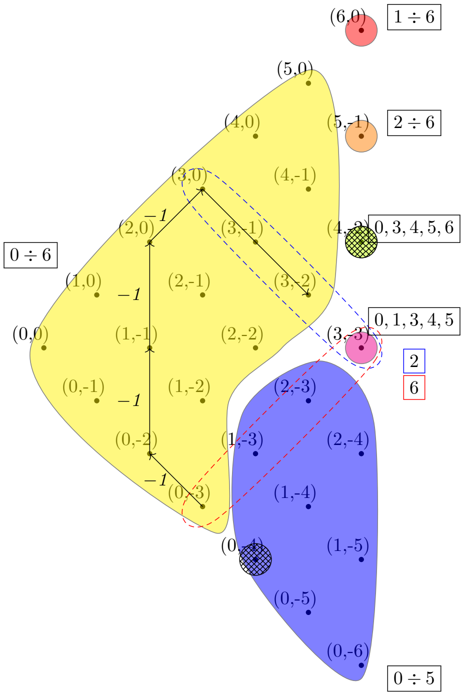

Figure 1.8: The set $\mathbf{S}_3^{0,2}$ and the bundles obtained by applying Proposition 1.5.2.

As $\mathbf{S}_3^{0,2}(6) \subset \mathscr{D}$, we have:

$$\mathbf{P}_3(6) \subset \mathscr{D},$$
$$\mathbf{N}_3(2) \subset \mathscr{D}.$$

Applying Proposition 1.5.2, we obtain:

$$\mathcal{U}^{0,-4}(6) \in \mathscr{D},$$
$$\mathcal{U}^{4,-2}(2) \in \mathscr{D}.$$

Step 8.

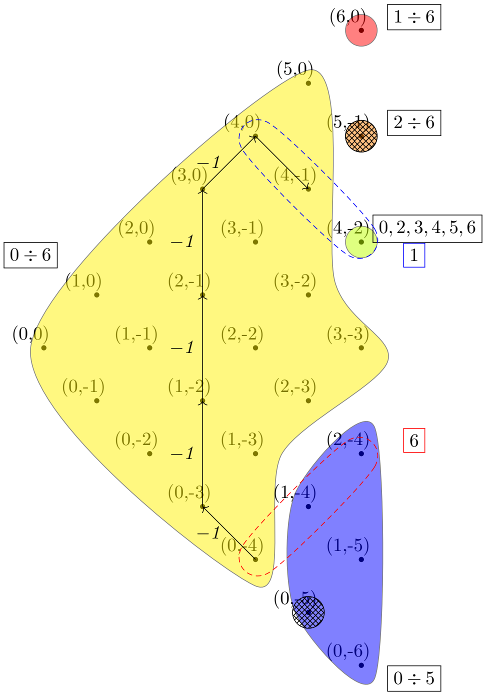

Figure 1.9: The set $\mathbf{S}_4^{0,1}$ and the bundles obtained by applying Proposition 1.5.2.

As $\mathbf{S}_4^{0,1}(6) \subset \mathscr{D}$, we have:

$$\mathbf{P}_4(6) \subset \mathscr{D},$$
$$\mathbf{N}_4(1) \subset \mathscr{D}.$$

As all bundles $\mathcal{U}^{i,-j}(6) \in \mathscr{D}$ for $i+j \leq 5$, applying Proposition 1.5.2, we obtain:

$$\mathcal{U}^{0,-5}(6) \in \mathscr{D};$$

analogously, for $\mathcal{U}^{i,-j}(1) \in \mathscr{D}$ for $i+j \leq 6$, we obtain:

$$\mathcal{U}^{1,-5}(1) \in \mathscr{D}.$$

Step 9.

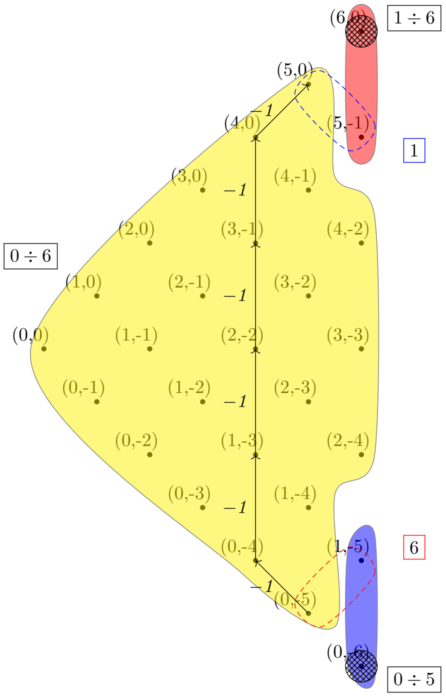

Figure 1.10: The set $\mathbf{S}_5^{0,0}$ and the remaining bundles.

As $\mathbf{S}_5^{0,0}(6) \subset \mathscr{D}$, we have:

$$\mathbf{P}_5(6) \subset \mathscr{D},$$
$$\mathbf{N}_5(0) \subset \mathscr{D}.$$

Applying Proposition 1.5.2, we finally generate the remaining bundles.

Conclusion

After these nine iterations, we have shown that

$$\mathbf{T},\, \mathbf{T}(1),\, \ldots,\, \mathbf{T}(6) \subset \mathscr{D}.$$

Applying Proposition 1.5.6 to $\mathscr{D}$, we show that the Lefschetz collection induced by $\mathbf{B}$ is full, finally proving Theorem 1.5.7.

1.6 On the derived category of $\mathsf{IGr}(3,11)$

We now focus our attention on the odd isotropic Grassmannian $\mathsf{IGr}(3,11)$. Let V be a 11-dimensional vector space endowed with a skew-symmetric form ψ of maximal rank 10. Let us fix $X = \mathsf{IGr}(3,11)$, the isotropic Grassmannian of 3-subspaces in V with respect to ψ. By Proposition 1.3.12, we know that the index of $\mathsf{IGr}(3,11)$ and $\mathsf{rank}\, K_0(\mathsf{IGr}(3,11))$ are, respectively:

$$w = 9, \qquad r = 120.$$

Accordingly to [KS21, Conjecture 1.3.(ii)], we do not expect to find a full rectangular Lefschetz collection. As

$$120 = 13 \cdot 9 + 3,$$

a minimal full (non rectangular) Lefschetz collection of $\mathbf{D}^{\mathrm{b}}(X)$ would have 3 blocks of length 14 and the remaining blocks of length 13. We start focusing our efforts in the construction of the rectangular part of the collection, which is expected to be induced by a Lefschetz basis of length 13.

In this section, we construct $\mathbf{B}$, a basis of a rectangular Lefschetz collection of length 13 and propose an object to complete the rectangular collection induced by $\mathbf{B}$ to a full Lefschetz decomposition, namely $\mathcal{G}$ (cf. § 1.6.4). The construction of $\mathbf{B}$ is very similar to the $\mathsf{IGr}(3,9)$ case, hence we simply outline it in § 1.6.1, giving more details in the construction of the non-rectangular completion of the Lefschetz collection. We summarize some of the properties of $\mathcal{G}$ in Proposition 1.6.16 and Proposition 1.6.18 and conjecture a relation that implies semiorthogonality (cf. Conjecture 1.6.21). We show in Theorem 1.6.23 how assuming the remaining vanishings would also prove the fullness of the collection induced by $\{\mathcal{G}\} \sqcup \mathbf{B}$, following the method we presented in § 1.5.

Remark 1.6.1. In most proofs of this section, we used some computations obtained through computer algebra implemented in Sage [Ste+21].

In particular, to compute $\mathrm{Hom}^\bullet(\mathcal{U}^\alpha, \mathcal{U}^\beta)$, we compute the Littlewood–Richardson coefficients of the tensor product $\mathcal{U}^{-\alpha} \otimes \mathcal{U}^\beta$ with the package [Buc11].

We provide an implementation of this functions in a notebook available at [Cat23a]

Recall that $j : \mathsf{IGr}(3,11) \to \mathsf{Gr}(3,11)$ is the zero locus of the skew symmetric form:

$$\psi \in \wedge^2 V^* = \mathrm{H}^0(\mathsf{Gr}(3,11), \mathcal{U}^{1,1,0}).$$

Using the Koszul resolution of $\mathsf{IGr}(3,11)$ in $\mathsf{Gr}(3,11)$:

$$0 \to \mathcal{O}(-1) \to \mathcal{U}^{0,0,-1}(-1) \to \mathcal{U}^{0,-1,-1} \to \mathcal{O}_{\mathsf{Gr}(3,11)} \to \mathcal{O}_{\mathsf{IGr}(3,11)} \to 0, \tag{1.48}$$

or the Koszul resolution of the structure sheaf in $\mathsf{IGr}(3,12)$ (1.21), we obtain a spectral sequence (cf. Corollary 1.3.14) to compute the cohomology of a bundle $\mathcal{U}^\lambda$.

To compute $\mathrm{H}^\bullet(X, \mathcal{U}^\lambda)$, we verify that one of the spectral sequences induced by (1.21) or possibly (1.48) has at most one non vanishing entry (this fact can be easily verified implementing the combinatorial conditions in Theorem 1.3.1 and Theorem 1.3.10). In that case, we can determine the cohomology (see for instance Lemma 1.4.5 and Lemma 1.4.6).

1.6.1 The rectangular part

Similarly to the case $\mathsf{IGr}(3,9)$, we have two collections of 12 elements each:

$$\begin{aligned}\mathbf{B}_1 = \{&\mathcal{U}^{0,0,-3},\\ &\mathcal{U}^{0,0,-2}, \mathcal{U}^{1,0,-2}, \mathcal{U}^{2,0,-2},\\ &\mathcal{U}^{0,0,-1},\mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1},\\ &\mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}\}\\ \mathbf{B}_2 = \{&\mathcal{U}^{0,0,-2}, \mathcal{U}^{1,0,-2}, \mathcal{U}^{2,0,-2},\\ &\mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1},\\ &\mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}, \mathcal{U}^{4,0,0}\}.\end{aligned}$$

Notice that $\mathbf{B}_1$ and $\mathbf{B}_2$ have 11 elements in common and their union has 13 elements.

Lemma 1.6.2. *The collections $\mathbf{B}_1$ are $\mathbf{B}_2$ are bases of (non full) Lefschetz collections.*

Proof. The proof that $\mathbf{B}_1$ and $\mathbf{B}_2$ are Lefschetz collections is similar to Proposition 1.4.3. Otherwise, it can be verified with Sage. □

Similarly to the case $\mathsf{IGr}(3,9)$, $\mathbf{B}_1 \cup \mathbf{B}_2$ is not an exceptional collection, as we have:

$$\mathrm{Hom}^\bullet(\mathcal{U}^{4,0,0},\mathcal{U}^{0,0,-3}) = \mathbb{C}[-6], \qquad \mathrm{Hom}^\bullet(\mathcal{U}^{0,0,-3},\mathcal{U}^{4,0,0}) \neq 0.$$

Similarly to Lemma 1.4.7, we can easily verify:

$$\mathrm{Hom}^\bullet(\mathcal{U}^{4,0,0}(l),\mathcal{U}^{0,0,-3}) = 0 \quad \text{for} \quad l = 1,\dots,8. \tag{1.49}$$

We define:

$$\begin{aligned}\mathbf{B} = \{\mathcal{H}\} \cup \mathbf{B}_1 = \{&\mathcal{H},\\ &\mathcal{U}^{0,0,-3},\\ &\mathcal{U}^{0,0,-2}, \mathcal{U}^{1,0,-2}, \mathcal{U}^{2,0,-2},\\ &\mathcal{U}^{0,0,-1},\mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1},\\ &\mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}\},\end{aligned}$$

where the object $\mathcal{H}$ is isomorphic to the totalization of this bicomplex:

$$\begin{tikzcd}
\wedge^4 V^* \otimes \mathcal{O} \arrow[r] & \wedge^3 V^* \otimes \mathcal{U}^{1,0,0} \arrow[r] & \wedge^2 V^* \otimes \mathcal{U}^{2,0,0} \arrow[r] & V^* \otimes \mathcal{U}^{3,0,0} \arrow[r] & \mathcal{U}^{4,0,0} \\
\wedge^3 V^* \otimes \mathcal{U}^{0,0,-1} \arrow[r] \arrow[u] & \wedge^2 V^* \otimes \mathcal{U}^{1,0,-1} \arrow[r] \arrow[u] & V^* \otimes \mathcal{U}^{2,0,-1} \arrow[r] \arrow[u] & \mathcal{U}^{3,0,-1} \arrow[u] & \\
\wedge^2 V^* \otimes \mathcal{U}^{0,0,-2} \arrow[r] \arrow[u] & V^* \otimes \mathcal{U}^{1,0,-2} \arrow[r] \arrow[u] & \mathcal{U}^{2,0,-2}, \arrow[u] & &
\end{tikzcd} \tag{1.50}$$

where the vertical maps are induced by the skew symmetric form ψ (cf. [Gus20, Lemma 5.1.(a), Proposition 5.2]), while the rows are truncations of the staircase complexes associated to the bundles $\mathcal{U}^{4,0,0}, \mathcal{U}^{3,0,-1}$ and $\mathcal{U}^{2,0,-2}$. We recall them here:

$$\begin{aligned}0 \to \mathcal{U}^{0,0,-4}(-1) \to \wedge^{10}V^* \otimes \mathcal{U}^{0,0,-3}(-1) \to \wedge^{9}V^* \otimes \mathcal{U}^{0,0,-2}(-1) \to\\ \to \wedge^{8}V^* \otimes \mathcal{U}^{0,0,-1}(-1) \to \wedge^{7}V^* \otimes \mathcal{O}(-1) \to\\ \to \wedge^{4}V^* \otimes \mathcal{O} \to \wedge^{3}V^* \otimes \mathcal{U}^{1,0,0} \to \wedge^{2}V^* \otimes \mathcal{U}^{2,0,0} \to V^* \otimes \mathcal{U}^{3,0,0} \to \mathcal{U}^{4,0,0} \to 0,\end{aligned}$$

$$\begin{aligned}0 \to \mathcal{U}^{1,0,-4}(-2) \to \wedge^{10}V^* \otimes \mathcal{U}^{1,0,-3}(-2) \to \wedge^{9}V^* \otimes \mathcal{U}^{1,0,-2}(-2) \to\\ \to \wedge^{8}V^* \otimes \mathcal{U}^{1,0,-1}(-2) \to \wedge^{7}V^* \otimes \mathcal{U}^{1,0,0}(-2) \to \wedge^{5}V^* \otimes \mathcal{O}(-1) \to\\ \to \wedge^{3}V^* \otimes \mathcal{U}^{0,0,-1} \to \wedge^{2}V^* \otimes \mathcal{U}^{1,0,-1} \to V^* \otimes \mathcal{U}^{2,0,-1} \to \mathcal{U}^{3,0,-1} \to 0,\end{aligned}$$

and finally

$$\begin{aligned}0 \to \mathcal{U}^{2,0,-4}(-3) \to \wedge^{10}V^* \otimes \mathcal{U}^{2,0,-3}(-3) \to \wedge^{9}V^* \otimes \mathcal{U}^{2,0,-2}(-3) \to \wedge^{8}V^* \otimes \mathcal{U}^{2,0,-1}(-3) \to\\ \to \wedge^{7}V^* \otimes \mathcal{U}^{2,0,0}(-3) \to \wedge^{5}V^* \otimes \mathcal{U}^{1,0,0}(-2) \to \wedge^{4}V^* \otimes \mathcal{U}^{0,0,-1}(-1) \to\\ \to \wedge^{2}V^* \otimes \mathcal{U}^{0,0,-2} \to V^* \otimes \mathcal{U}^{1,0,-2} \to \mathcal{U}^{2,0,-2} \to 0.\end{aligned}$$

The stupid truncation between the second-to-lowest and the lowest line of each complex presented above induces the lines in complex (1.50).

As in Proposition 1.4.13, we can characterize $\mathcal{H}$ as a mutation to prove that $\mathbf{B}$ is an exceptional collection.

Proposition 1.6.3. *The object $\mathcal{H}$ is exceptional. Moreover, we have the following isomorphisms:*

$$\mathcal{H}[6] = \mathbb{L}_{\mathbf{B}_1}\mathcal{U}^{4,0,0} = \mathbb{L}_{\mathbf{B}_1 \cap \mathbf{B}_2}\mathcal{U}^{4,0,0}.$$

Proof. From the totalization of the bicomplex in (1.50) we can extract the following:

$$\mathrm{Cone}(\mathcal{U}^{4,0,0} \to \mathcal{H}[6]) \in \langle\, \mathbf{B}_1 \cap \mathbf{B}_2 \,\rangle \subset \langle\, \mathbf{B}_1 \,\rangle.$$

We need to verify $\mathcal{H} \in \mathbf{B}_1^{\perp}$. First of all, by construction, similarly to Proposition 1.4.14, we verify that

$$\mathcal{H} \in \langle \mathcal{U}^{2,0,-2}, \mathcal{U}^{3,0,-1} \rangle^{\perp}.$$

Considering the left resolution of the complex defining $\mathcal{H}$ we can prove the remaining vanishings with Sage, obtaining:

$$\mathcal{H}[6] = \mathbb{L}_{\mathbf{B}_1}\mathcal{U}^{4,0,0}.$$

As $\mathcal{H} \in (\mathbf{B}_1 \cap \mathbf{B}_2)^{\perp}$ as well, we have that $\mathcal{H}[6] = \mathbb{L}_{\mathbf{B}_1 \cap \mathbf{B}_2}\mathcal{U}^{4,0,0}$. Since $\mathcal{U}^{4,0,0} \in {}^{\perp}(\mathbf{B}_1 \cap \mathbf{B}_2)$ because $\mathbf{B}_2$ is an exceptional collection, the mutation is an equivalence on $\mathcal{U}^{4,0,0}$, hence $\mathcal{H}$ is exceptional. ☐

Corollary 1.6.4. *The collection* $\mathbf{B}$ *is the basis of a (non full!) Lefschetz collection.*

Proof. Since $\{\mathcal{H}\} \cup \mathbf{B}_1 \subset \langle \mathbf{B}_1 \cup \{\mathcal{U}^{4,0,0}\} \rangle$, the blocks are semiorthogonal by Lemma 1.6.2 and (1.49). On the other hand, exceptionality within a block follows from Lemma 1.6.2 and Proposition 1.6.3. Applying Lemma 1.2.5, we deduce that $\mathbf{B}$ is a Lefschetz basis. ☐

Remark 1.6.5. We notice that to study the odd isotropic Grassmannian $\mathsf{IGr}(3, 2n+1)$ with $n \geq 6$, it will be necessary to involve more ideas than what was done so far, as there is a behaviour similar to what was observed for homogeneous varieties in [KP16]. It was proved in [Fon22] that the set of exceptional bundles:

$$\{\mathcal{U}^{0,0,-1}, \mathcal{O}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}\}$$

is a set of generators of a full Lefschetz collection of $\mathsf{IGr}(3,7)$. Compare this collection with the Lefschetz basis constructed for $\mathsf{IGr}(3,9)$ in [Cat23b], which is obtained mutating $\mathcal{U}^{3,0,0}$ in the following sequence:

$$\{\mathcal{U}^{0,0,-2}, \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{0,0,0}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}\}$$

and the set of bundles we presented for $\mathsf{IGr}(3,11)$:

$$\begin{aligned} &\{\mathcal{U}^{0,0,-3},\\ &\ \mathcal{U}^{0,0,-2}, \mathcal{U}^{1,0,-2}, \mathcal{U}^{2,0,-2},\\ &\ \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1},\\ &\ \mathcal{O}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}, \mathcal{U}^{4,0,0}\}. \end{aligned}$$

The emerging pattern would suggest that the (non exceptional) set of bundles

$$\{\mathcal{U}^{a,0,-b} \mid a+b \leq n-1\} \setminus \{\mathcal{U}^{0,0,-(n-1)}, \mathcal{U}^{1,0,-(n-2)}\}$$

would be the set of generators of a rectangular Lefschetz basis of expected length. This cannot be, because this set asymptotically has $\frac{1}{2}n^2$ elements, while the expected length of the rectangular part is asymptotically $\frac{2}{3}n^2$ by Remark 1.3.13. This lack of elements is first observed in $\mathsf{IGr}(3, 2n+1)$ if $n = 6$.

Hoping to complete $\mathbf{B}$ to a basis of a full collection, we try to apply the method presented in § 1.5 to prove the fullness of the collection $\mathbf{B}$ (we replace $\mathcal{H}$ by $\mathcal{U}^{4,0,0}$). Clearly, the Lefschetz collection induced by $\mathbf{B}$ is not full as it has the wrong number of elements. We notice that the algorithmic procedure is able to generate the majority of the twists of

$$\mathcal{U}^{a,0,-b} \qquad \text{for} \quad a+b \leq 4.$$

Following this method, we are not able to generate the bundles $\mathcal{U}^{3,0,-2}$, $\mathcal{U}^{2,0,-3}$ and their twists. We now construct a bundle $\mathcal{G}$ starting from the bundle $\mathcal{U}^{3,0,-2}$ to complete the exceptional collection $\mathbf{B}$. The construction of this object is done in 2 steps, along the coming sections.

We denote the first step of the mutation as $\mathcal{G}_2$ (cf. (1.52)) and we study some of its properties.

1.6.2 Properties of $\mathcal{G}_2$

Consider the bundle $\mathcal{U}^{3,0,-2}$. The first obstacle to be part of a non-rectangular Lefschetz block is that:

$$\mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-2}(2), \mathcal{U}^{3,0,-2}) = \mathrm{H}^\bullet(\mathsf{IGr}(3,11), \mathcal{U}^{1,0,-7}) = \mathbb{C}[-6].$$

We now present some properties of $\mathcal{U}^{3,0,-2}$.

Lemma 1.6.6. *The bundle* $\mathcal{U}^{3,0,-2}$ *is exceptional. We have the following vanishings:*

$$\mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-2}(l), \mathcal{U}^{2,0,-3}) = \begin{cases} 0 & \text{for} \quad l = 0,4,5,7,8, \\ \mathbb{C}[-8] & \text{for} \quad l = 3. \end{cases}$$

The representative of $\mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-2}(3), \mathcal{U}^{2,0,-3}) = \mathbb{C}[-8]$, *which is unique up to scalar, is the staircase complex* $\mathsf{Stair}(\mathcal{U}^{3,0,-2})$.

Proof. Both results have been obtained with Sage, following the method presented in Remark 1.6.1. The second result is obtained analogously, using Sage:

$$\mathcal{U}^{-1,-1,-9} \Subset \mathcal{U}^{2,0,-3} \otimes (\mathcal{U}^{3,0,-2}(3))^*$$

is the only acyclic term. Applying the associate spectral sequence as in Remark 1.6.1, we have:

$$\mathrm{H}^\bullet(\mathsf{IGr}(3,11), \mathcal{U}^{-1,-1,-9}) = \mathbb{C}[-8].$$

We verify that the last term in $\mathsf{Stair}(\mathcal{U}^{3,0,-2}(3))$ is $\mathcal{U}^{2,0,-3}$ (see (1.51)) □

Proposition 1.6.7. *We have the following vanishings:*

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathbf{B}(l), \mathcal{U}^{3,0,-2}) &= 0 \quad \text{for} \quad l = 1, \ldots, 6, \\ \mathrm{Hom}^\bullet(\mathbf{B}(l), \mathcal{U}^{2,0,-3}) &= 0 \quad \text{for} \quad l = 3, \ldots, 8. \end{aligned}$$

Proof. We can replace $\mathbf{B}$ with $\mathbf{B}_1 \cup \mathbf{B}_2$, then both results can be verified with Sage. □

We introduce the bundle $\mathcal{G}_2$ obtained as the stupid truncation of the staircase complexes associated to $\mathcal{U}^{3,0,-2}(2)$ between the second and the third line from the bottom:

$$\begin{aligned} 0 \to \mathcal{U}^{2,0,-3}(-1) \to \wedge^{10} V^* \otimes \mathcal{U}^{2,0,-2}(-1) \to \wedge^9 V^* \otimes \mathcal{U}^{2,0,-1}(-1) \to \\ \to \wedge^8 V^* \otimes \mathcal{U}^{2,0,0}(-1) \to \wedge^6 V^* \otimes \mathcal{U}^{1,0,0} \to \wedge^5 V^* \otimes \mathcal{U}^{0,0,-1}(1) \to \\ \to \wedge^3 V^* \otimes \mathcal{U}^{0,0,-2}(2) \to \wedge^2 V^* \otimes \mathcal{U}^{1,0,-2}(2) \to \\ \to V^* \otimes \mathcal{U}^{2,0,-2}(2) \to \mathcal{U}^{3,0,-2}(2) \to 0. \quad (1.51) \end{aligned}$$

More explicitly, there are two acyclic complexes. The right resolution of $\mathcal{G}_2$:

$$\begin{aligned} 0 \to \mathcal{G}_2 \to \wedge^6 V^* \otimes \mathcal{U}^{1,0,0} \to \wedge^5 V^* \otimes \mathcal{U}^{0,0,-1}(1) \to \\ \to \wedge^3 V^* \otimes \mathcal{U}^{0,0,-2}(2) \to \wedge^2 V^* \otimes \mathcal{U}^{1,0,-2}(2) \to \\ \to V^* \otimes \mathcal{U}^{2,0,-2}(2) \to \mathcal{U}^{3,0,-2}(2) \to 0, \end{aligned} \quad (1.52)$$

and the left resolution of $\mathcal{G}_2$:

$$\begin{aligned} 0 \to \mathcal{U}^{2,0,-3}(-1) \to \wedge^{10} V^* \otimes \mathcal{U}^{2,0,-2}(-1) \to \wedge^9 V^* \otimes \mathcal{U}^{2,0,-1}(-1) \to \\ \to \wedge^8 V^* \otimes \mathcal{U}^{2,0,0}(-1) \to \mathcal{G}_2 \to 0. \end{aligned} \quad (1.53)$$

We outline some properties of $\mathcal{G}_2$.

Proposition 1.6.8. *We have the following vanishings:*

$$\mathrm{Hom}(\mathcal{U}^\lambda(l), \mathcal{G}_2) = 0 \quad \textit{for} \quad \mathcal{U}^\lambda \in \mathbf{B}_1 \cup \mathbf{B}_2 \quad \textit{and} \quad l = 0, \dots, 8,$$

unless, $l = 1$ *and* $\mathcal{U}^\lambda \in \{\mathcal{U}^{2,0,-2}, \mathcal{U}^{3,0,-1}\}$ *or* $l = 0$ *and* $\mathcal{U}^\lambda \in \{\mathcal{U}^{3,0,-1}, \mathcal{U}^{4,0,0}\}$.

Proof. This result can be obtained with Sage or possibly combining the vanishings in Lemma 1.6.2, Lemma 1.6.6 and Proposition 1.6.7. If $l = 0, \dots, 7$, we can prove the vanishings using the left resolution of $\mathcal{G}_2$ (1.53), the vanishings for $l = 8$ can be obtained with the right resolution (1.52). □

Proposition 1.6.9. *The collection* $\mathcal{G}_2$, $\mathcal{G}_2(1)$, $\mathcal{G}_2(2)$ *is exceptional.*

Proof. Consider the morphism induced by (1.52):

$$\mathrm{Cone}(\mathcal{U}^{3,0,-2}(2) \to \mathcal{G}_2[5]) \in \langle \mathcal{U}^{1,0,0}, \mathcal{U}^{0,0,-1}(1), \mathcal{U}^{0,0,-2}(2), \mathcal{U}^{1,0,-2}(2), \mathcal{U}^{2,0,-2}(2) \rangle.$$

We now show that $\mathcal{G}_2[5]$ is actually the left mutation of $\mathcal{U}^{3,0,-2}(2)$ through the category on the right. Using the result in Proposition 1.6.8, we easily obtain that:

$$\mathcal{G}_2 \in \langle \mathcal{U}^{1,0,0}, \mathcal{U}^{0,0,-1}(1), \mathcal{U}^{0,0,-2}(2), \mathcal{U}^{1,0,-2}(2), \mathcal{U}^{2,0,-2}(2) \rangle^\perp.$$

Since

$$\mathcal{U}^{3,0,-2}(2) \in {}^\perp\langle \mathcal{U}^{1,0,0}, \mathcal{U}^{0,0,-1}(1), \mathcal{U}^{0,0,-2}(2), \mathcal{U}^{1,0,-2}(2), \mathcal{U}^{2,0,-2}(2) \rangle$$

by Serre duality and Proposition 1.6.7, the mutation is an equivalence on $\mathcal{U}^{3,0,-2}(2)$, which is an exceptional bundle by Lemma 1.6.6, hence $\mathcal{G}_2$ is exceptional as well.

We can now verify that $\mathcal{G}_2$, $\mathcal{G}_2(1)$, $\mathcal{G}_2(2)$ is an exceptional sequence by computing

$$\mathrm{Hom}^\bullet(\mathcal{G}_2(l), \mathcal{G}_2) = 0 \qquad \text{for} \quad l = 1, 2.$$

To do so, we replace the left term with the resolution of $\mathcal{G}_2$ given by (1.52) and on the right term we use the resolution (1.53), then apply Lemma 1.6.2 and Lemma 1.6.6. Alternatively, this can be verified explicitly using Sage. This settles the claim. □

1.6.3 Construction of $\mathcal{G}$

In this section we prepare the ground to define an explicit object $\mathcal{G}$. We expect that:

- $\mathcal{G} = \mathbb{L}_{\mathbf{B},\mathbf{B}(1)}\mathcal{G}_2$,
- $\{\mathcal{G}\} \sqcup \mathbf{B}$ is a basis of a full Lefschetz collection.

We summarize in Proposition 1.6.16 and Proposition 1.6.18 the properties of $\mathcal{G}$ that we are able to prove. Consider the following set of bundles:

$$\mathbf{A} = \{\mathcal{O}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}, \mathcal{U}^{0,0,-1}(1), \mathcal{U}^{1,0,-1}(1), \mathcal{U}^{2,0,-1}(1), \mathcal{U}^{3,0,-1}(1)\} \subset \mathbf{B} \cup \mathbf{B}(1)$$

and

$$\mathbf{A}' = \mathbf{A} \setminus \{\mathcal{U}^{3,0,0}, \mathcal{U}^{3,0,-1}(1)\} \subset \mathbf{B} \cup \mathbf{B}(1).$$

We summarize some properties of $\mathbf{A}$.

Lemma 1.6.10. *The collection* $\mathbf{A}$ *is a basis of a (non full) Lefschetz collection.*

Proof. Consider the following exceptional collection:

$$\{\mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1}, \mathcal{O}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}, \mathcal{U}^{3,0,0}\} \subset \mathbf{B},$$

this is clearly a Lefschetz basis by Corollary 1.6.4. A standard way to construct a second Lefschetz basis is to consider a partial twist by $\mathcal{O}(-1)$:

$$\{\mathcal{O}(-1), \mathcal{U}^{1,0,0}(-1), \mathcal{U}^{2,0,0}(-1), \mathcal{U}^{3,0,0}(-1), \mathcal{U}^{0,0,-1}, \mathcal{U}^{1,0,-1}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1}\}.$$

To verify that this collection is still a Lefschetz basis, use Lemma 1.2.5 and Serre Duality. Since the collection $\mathbf{A}$ is given by twisting the latter collection by $\mathcal{O}(1)$, $\mathbf{A}$ is a Lefschetz basis. □

Lemma 1.6.11. *For* $\mathcal{U}^\lambda \in \mathbf{A}$ *and* $l = 0, \dots, 8$, *we have:*

$$\mathrm{Hom}^\bullet(\mathcal{U}^\lambda(l), \mathcal{G}_2) = \begin{cases} \mathbb{C}[-4] & \text{if } \mathcal{U}^\lambda = \mathcal{U}^{3,0,-1}(1) \text{ and } l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Most vanishings were already computed in Proposition 1.6.8, we can obtain the remaining vanishings for $l = 8$ with Sage using the right resolution of $\mathcal{G}_2$ (1.52).

If $\mathcal{U}^\lambda = \mathcal{U}^{3,0,-1}(1)$ and $l = 0$, we consider the left resolution of $\mathcal{G}_2$, (1.53). Since $\mathbf{B}$ is a Lefschetz basis, we have:

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-1}(1), \mathcal{G}_2) &= \mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-1}(1), \mathcal{U}^{2,0,-3}(-1)[3]) = \\ &= \mathrm{H}^\bullet(\mathsf{IGr}(3,11), \mathcal{U}^{0,-1,-8}))[3] = \mathbb{C}[-4] \neq 0, \end{aligned}$$

proving the last isomorphism in the claim. □

We now provide explicit descriptions of $\mathbb{L}_{\mathbf{A}'}\mathcal{U}^{3,0,-1}(1)$ and $\mathbb{L}_{\mathbf{A}'}\mathcal{U}^{3,0,0}$. The bundle $\mathcal{G}_1$ is obtained as the stupid truncation of the staircase complexes associated to $\mathcal{U}^{3,0,-1}(1)$ between the second and the third line from the bottom, that is:

$$\begin{aligned} 0 \to \mathcal{U}^{1,0,-4}(-1) \to \wedge^{10}V^* \otimes \mathcal{U}^{1,0,-3}(-1) \to \wedge^{9}V^* \otimes \mathcal{U}^{1,0,-2}(-1) \to \\ \to \wedge^{8}V^* \otimes \mathcal{U}^{1,0,-1}(-1) \to \wedge^{7}V^* \otimes \mathcal{U}^{1,0,0}(-1) \to \\ \to \wedge^{5}V^* \otimes \mathcal{O} \to \wedge^{3}V^* \otimes \mathcal{U}^{0,0,-1}(1) \to \wedge^{2}V^* \otimes \mathcal{U}^{1,0,-1}(1) \to \\ \to V^* \otimes \mathcal{U}^{2,0,-1}(1) \to \mathcal{U}^{3,0,-1}(1) \to 0. \end{aligned} \tag{1.54}$$

More explicitly, we have two exact sequences:

$$\begin{aligned} 0 \to \mathcal{G}_1 \to \wedge^{5}V^* \otimes \mathcal{O} \to \wedge^{3}V^* \otimes \mathcal{U}^{0,0,-1}(1) \to \wedge^{2}V^* \otimes \mathcal{U}^{1,0,-1}(1) \to \\ \to V^* \otimes \mathcal{U}^{2,0,-1}(1) \to \mathcal{U}^{3,0,-1}(1) \to 0 \end{aligned} \tag{1.55}$$

and

$$\begin{aligned} 0 \to \mathcal{U}^{1,0,-4}(-1) \to \wedge^{10}V^* \otimes \mathcal{U}^{1,0,-3}(-1) \to \wedge^{9}V^* \otimes \mathcal{U}^{1,0,-2}(-1) \to \\ \to \wedge^{8}V^* \otimes \mathcal{U}^{1,0,-1}(-1) \to \wedge^{7}V^* \otimes \mathcal{U}^{1,0,0}(-1) \to \mathcal{G}_1 \to 0. \end{aligned} \tag{1.56}$$

We introduce the bundle $\mathcal{G}_0$ obtained as the stupid truncation of the staircase complexes associated to $\mathcal{U}^{3,0,0}$ between the second and the third line from the bottom:

$$\begin{aligned} 0 \to \mathcal{U}^{0,0,-5}(-1) \to \wedge^{10}V^* \otimes \mathcal{U}^{0,0,-4}(-1) \to \wedge^{9}V^* \otimes \mathcal{U}^{0,0,-3}(-1) \to \\ \to \wedge^{8}V^* \otimes \mathcal{U}^{0,0,-2}(-1) \to \wedge^{7}V^* \otimes \mathcal{U}^{0,0,-1}(-1) \to \wedge^{6}V^* \otimes \mathcal{O}(-1) \to \\ \to \wedge^{3}V^* \otimes \mathcal{O} \to \wedge^{2}V^* \otimes \mathcal{U}^{1,0,0} \to V^* \otimes \mathcal{U}^{2,0,0} \to \mathcal{U}^{3,0,0} \to 0. \end{aligned}$$

We have the two exact sequences:

$$0 \to \mathcal{G}_0 \to \wedge^{3}V^* \otimes \mathcal{O} \to \wedge^{2}V^* \otimes \mathcal{U}^{1,0,0} \to V^* \otimes \mathcal{U}^{2,0,0} \to \mathcal{U}^{3,0,0} \to 0 \tag{1.57}$$

and

$$\begin{aligned} 0 \to \mathcal{U}^{0,0,-5}(-1) \to \wedge^{10}V^* \otimes \mathcal{U}^{0,0,-4}(-1) \to \wedge^{9}V^* \otimes \mathcal{U}^{0,0,-3}(-1) \to \\ \to \wedge^{8}V^* \otimes \mathcal{U}^{0,0,-2}(-1) \to \wedge^{7}V^* \otimes \mathcal{U}^{0,0,-1}(-1) \to \wedge^{6}V^* \otimes \mathcal{O}(-1) \to \mathcal{G}_0 \to 0. \end{aligned} \tag{1.58}$$

As a starting point, we prove the following immediate properties.

Lemma 1.6.12. *The objects $\mathcal{G}_0$ and $\mathcal{G}_1$ are exceptional. Moreover, $\mathcal{G}_0[3] = \mathbb{L}_{\mathbf{A}'}\mathcal{U}^{3,0,0}$ and $\mathcal{G}_1[4] = \mathbb{L}_{\mathbf{A}'}\mathcal{U}^{3,0,-1}(1)$. Finally, we have:*

$$\operatorname{Hom}^\bullet(\mathcal{G}_0, \mathcal{G}_1) = \mathbb{C}, \qquad \operatorname{Hom}^\bullet(\mathcal{G}_1, \mathcal{G}_0) = 0.$$

Proof. First of all, considering the left resolutions of $\mathcal{G}_0$ and $\mathcal{G}_1$ ((1.58) and (1.56)), we can compute with Sage:

$$\mathrm{Hom}^\bullet(\mathbf{A}', \mathcal{G}_0) = 0, \qquad \mathrm{Hom}^\bullet(\mathbf{A}', \mathcal{G}_1) = 0, \tag{1.59}$$

hence we obtain the following isomorphisms:

$$\mathcal{G}_0[3] = \mathbb{L}_{\mathbf{A}'}\mathcal{U}^{3,0,0} \qquad \mathcal{G}_1[4] = \mathbb{L}_{\mathbf{A}'}\mathcal{U}^{3,0,-1}(1).$$

Moreover, considering the right resolutions ((1.57) and (1.55)), the left mutation $\mathbb{L}_{\mathbf{A}'}$ on $\mathcal{U}^{3,0,0}$ and $\mathcal{U}^{3,0,-1}(1)$ coincides with the mutation by

$$\langle\, \mathcal{O}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}\,\rangle, \qquad \langle\, \mathcal{O}, \mathcal{U}^{0,0,-1}(1), \mathcal{U}^{1,0,-1}(1), \mathcal{U}^{2,0,-1}(1)\,\rangle,$$

respectively. As $\mathbf{A}$ is an exceptional collection, we have:

$$\mathcal{U}^{3,0,0} \in {}^\perp\langle\, \mathcal{O}, \mathcal{U}^{1,0,0}, \mathcal{U}^{2,0,0}\,\rangle, \qquad \mathcal{U}^{3,0,-1}(1) \in {}^\perp\langle\, \mathcal{O}, \mathcal{U}^{0,0,-1}(1), \mathcal{U}^{1,0,-1}(1), \mathcal{U}^{2,0,-1}(1)\,\rangle.$$

As a consequence, the mutation is an equivalence on $\mathcal{U}^{3,0,0}$ and $\mathcal{U}^{3,0,-1}(1)$ (cf. Proposition 1.4.13), hence $\mathcal{G}_0$ and $\mathcal{G}_1$ are exceptional.

We now prove the last two isomorphisms. Considering the resolutions (1.57) and (1.56), recalling that $\mathbf{B}_1$ is a Lefschetz basis by Lemma 1.6.2, we have:

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathcal{G}_0, \mathcal{G}_1) &= \mathrm{Hom}^\bullet(\mathcal{G}_1^*, \mathcal{G}_0^*) = \\ &= \mathrm{Hom}^\bullet(\mathcal{U}^{4,0,-1}(1)[-4], \mathcal{G}_0^*) = \\ &= \mathrm{Hom}^\bullet(\mathcal{U}^{4,0,-1}(1), \mathcal{U}^{0,0,-3})[7] = \mathbb{C}, \end{aligned}$$

where the equality between second and third line is verified with Sage.

Considering the resolution (1.55) for $\mathcal{G}_1$ and (1.57) for $\mathcal{G}_0$, recalling that $\mathbf{B}_1$ is a Lefschetz basis and all entries of $\mathcal{G}_1$ belong to $\mathbf{B}_1(1)$ except for $\wedge^5 V^* \otimes \mathcal{O}$ and $\mathcal{G}_0 \in \langle\, \mathbf{B}_1\,\rangle$, we obtain:

$$\mathrm{Hom}^\bullet(\mathcal{G}_1, \mathcal{G}_0) = \mathrm{Hom}^\bullet(\wedge^5 V^* \otimes \mathcal{O}, \mathcal{G}_0) = 0,$$

where the final vanishing can be verified considering the left resolution of $\mathcal{G}_0$ (1.57). This concludes the proof. □

Proposition 1.6.13. *The object* $\mathcal{G}_{0,1} = \mathbb{L}_{\mathcal{G}_0}\mathcal{G}_1$ *is exceptional. Moreover, we have the following isomorphisms:*

$$\mathcal{G}_{0,1} = \mathbb{L}_{\mathcal{G}_0}\mathcal{G}_1 = \mathrm{Cone}(\mathcal{G}_0 \to \mathcal{G}_1) = \mathbb{L}_{\mathbf{A}\setminus\{\mathcal{U}^{3,0,-1}(1)\}}\,\mathcal{G}_1.$$

Proof. From the last couple of vanishings in Lemma 1.6.12, we have that $\mathcal{G}_{0,1}$ is exceptional and

$$\mathcal{G}_{0,1} = \mathrm{Cone}(\mathcal{G}_0 \to \mathcal{G}_1).$$

We have that:

$$\mathcal{G}_0 \in \mathbf{A}'^{\perp}, \qquad \mathcal{G}_1 \in \mathbf{A}'^{\perp},$$

by Lemma 1.6.12, hence

$$\mathcal{G}_{0,1} \in \mathbf{A}'^{\perp}.$$

Since:

$$\mathcal{U}^{3,0,0} \in \langle\, \mathcal{G}_0,\, \mathbf{A}' \,\rangle,$$

and we have $\mathcal{G}_{0,1} \in \mathcal{G}_0^{\perp}$ by construction, we finally have that $\mathcal{G}_{0,1} \in (\mathbf{A} \setminus \{\mathcal{U}^{3,0,-1}(1)\})^{\perp}$, proving the last isomorphism in the statement. ☐

Consider the nonzero map $\phi : \mathcal{G}_0 \to \mathcal{G}_1$ which is unique up to scalar by Lemma 1.6.12. Verifying the appropriate vanishings, we can apply [Gus20, Lemma 5.1.(a)] and prove that $\phi : \mathcal{G}_0 \to \mathcal{G}_1$ lifts uniquely to a nonzero morphism of complexes:

$$\begin{tikzcd}
\wedge^5 V^* \otimes \mathcal{O} \arrow[r] & \wedge^3 V^* \otimes \mathcal{U}^{0,0,-1}(1) \arrow[r] & \wedge^2 V^* \otimes \mathcal{U}^{1,0,-1}(1) \arrow[r] & V^* \otimes \mathcal{U}^{2,0,-1}(1) \arrow[r] & \mathcal{U}^{3,0,-1}(1) \\
\wedge^3 V^* \otimes \mathcal{O} \arrow[r] \arrow[u] & \wedge^2 V^* \otimes \mathcal{U}^{1,0,0} \arrow[r] \arrow[u] & V^* \otimes \mathcal{U}^{2,0,0} \arrow[r] \arrow[u] & \mathcal{U}^{3,0,0}. \arrow[u] &
\end{tikzcd} \tag{1.60}$$

Thus, $\mathcal{G}_{0,1}$ is isomorphic to the totalization of the bicomplex (1.60). We now aim to mutate the bundle $\mathcal{G}_2$ by $\mathcal{G}_{0,1}$ to construct $\mathcal{G} = \mathbb{L}_{\mathbf{A}}\mathcal{G}_2$. We compute some preliminary results.

Lemma 1.6.14. *We have the following isomorphisms:*

$$\mathrm{Hom}^\bullet(\mathcal{G}_0,\, \mathcal{G}_2) = 0, \qquad \mathrm{Hom}^\bullet(\mathcal{G}_1,\, \mathcal{G}_2) = \mathbb{C}$$

and

$$\mathrm{Hom}^\bullet(\mathcal{G}_2,\, \mathcal{G}_0) = 0, \qquad \mathrm{Hom}^\bullet(\mathcal{G}_2,\, \mathcal{G}_1) = 0.$$

As a consequence we have:

$$\mathrm{Hom}^\bullet(\mathcal{G}_{0,1},\, \mathcal{G}_2) = \mathbb{C}, \qquad \mathrm{Hom}^\bullet(\mathcal{G}_2,\, \mathcal{G}_{0,1}) = 0.$$

Proof. To compute the first pair of isomorphisms, we proceed in the usual way. We can easily see, considering the right resolution of $\mathcal{G}_0$ (1.58) and Proposition 1.6.8 that

$$\mathrm{Hom}^\bullet(\mathcal{G}_0,\, \mathcal{G}_2) = 0.$$

Considering the sequences (1.56) and (1.53), we get:

$$\begin{aligned}\mathrm{Hom}^\bullet(\mathcal{G}_1,\, \mathcal{G}_2) &= \mathrm{Hom}^\bullet(\mathcal{U}^{3,0,-1}(1)[-4], \mathcal{U}^{2,0,-3}(-1)[3]) = \\ &= \mathrm{H}^\bullet(X, \mathcal{U}^{0,-1,-8})[7] = \mathbb{C}.\end{aligned}$$

Finally, we can prove the following vanishings with Sage:

$$\mathrm{Hom}^\bullet(\mathcal{G}_2,\, \mathcal{G}_0) = 0, \qquad \mathrm{Hom}^\bullet(\mathcal{G}_2,\, \mathcal{G}_1) = 0,$$

replacing $\mathcal{G}_2$ by its resolution (1.52) and the terms $\mathcal{G}_0, \mathcal{G}_1$ by (1.57) and (1.55).

The last pair of isomorphisms is obtained applying the computations of $\mathrm{Hom}^\bullet(\mathcal{G}_i, \mathcal{G}_j)$ to the description of $\mathcal{G}_{0,1}$ given by Proposition 1.6.13. □

We define

$$\mathcal{G} = \mathbb{L}_{\mathcal{G}_{0,1}} \mathcal{G}_2, \tag{1.61}$$

where the object $\mathcal{G}_{0,1} = \mathbb{L}_{\mathbf{A} \setminus \{\mathcal{U}^{3,0,-1}(1)\}} \mathcal{G}_1$ was defined in Proposition 1.6.13.

1.6.4 Work in progress: properties of $\mathcal{G}$

We now prove some properties of $\mathcal{G}$ (cf. (1.61)).

Proposition 1.6.15. *The object $\mathcal{G}$ is exceptional. Moreover we have the following isomorphisms:*

$$\mathcal{G} = \mathbb{L}_{\mathcal{G}_{0,1}} \mathcal{G}_2 = \mathrm{Cone}(\mathcal{G}_{0,1} \to \mathcal{G}_2) = \mathrm{Cone}(\mathrm{Cone}(\mathcal{G}_0 \to \mathcal{G}_1) \to \mathcal{G}_2) = \mathbb{L}_{\mathbf{A}} \mathcal{G}_2.$$

Proof. The outline of the proof is very similar to Proposition 1.6.13. From the last couple of vanishings in Lemma 1.6.14, we have that $\mathcal{G}$ is exceptional and

$$\mathcal{G} = \mathrm{Cone}(\mathcal{G}_{0,1} \to \mathcal{G}_2).$$

We have that:

$$\mathcal{G}_{0,1} \in (\mathbf{A} \setminus \{\mathcal{U}^{3,0,-1}(1)\})^\perp, \qquad \mathcal{G}_2 \in (\mathbf{A} \setminus \{\mathcal{U}^{3,0,-1}(1)\})^\perp,$$

by Proposition 1.6.13 and by Lemma 1.6.11, hence

$$\mathcal{G} \in (\mathbf{A} \setminus \{\mathcal{U}^{3,0,-1}(1)\})^\perp.$$

Since:

$$\mathcal{U}^{3,0,-1}(1) \in \langle\, \mathcal{G}_{0,1},\, \mathbf{A} \setminus \{\mathcal{U}^{3,0,-1}(1)\}\,\rangle,$$

and we have $\mathcal{G} \in \mathcal{G}_{0,1}^\perp$ by construction, we finally have that $\mathcal{G} \in \mathbf{A}^\perp$, proving the last isomorphism in the statement. □

First of all, we determine a representation of $\mathrm{Cone}(\mathcal{G}_1 \to \mathcal{G}_2)$ similar to the one we obtained for $\mathrm{Cone}(\mathcal{G}_0 \to \mathcal{G}_1)$. Verifying the appropriate vanishings, we can apply [Gus20, Lemma 5.1.(a)], hence $\tau : \mathcal{G}_1 \to \mathcal{G}_2$ lifts uniquely to a nonzero morphism of complexes:

$$\begin{tikzcd}
\wedge^6 V^* \otimes \mathcal{U}^{1,0,0} \arrow[r] & \cdots \arrow[r] & \wedge^2 V^* \otimes \mathcal{U}^{1,0,-2}(2) \arrow[r] & V^* \otimes \mathcal{U}^{2,0,-2}(2) \arrow[r] & \mathcal{U}^{3,0,-2}(2) \\
\wedge^5 V^* \otimes \mathcal{O} \arrow[r] \arrow[u] & \cdots \arrow[r] & V^* \otimes \mathcal{U}^{2,0,-1}(1) \arrow[r] \arrow[u] & \mathcal{U}^{3,0,-1}(1). \arrow[u] &
\end{tikzcd} \tag{1.62}$$

Finally, the object $\mathcal{G}$ is isomorphic to the totalization of the following bicomplex with lines given from bottom to top by (1.58), (1.56) and (1.52), where the vertical morphisms are ϕ (1.60) and τ (1.62):

$$\begin{tikzcd}
\wedge^6 V^* \otimes \mathcal{U}^{1,0,0} \arrow[r] & \cdots \arrow[r] & \wedge^2 V^* \otimes \mathcal{U}^{1,0,-2}(2) \arrow[r] & V^* \otimes \mathcal{U}^{2,0,-2}(2) \arrow[r] & \mathcal{U}^{3,0,-2}(2) \\
\wedge^5 V^* \otimes \mathcal{O} \arrow[r] \arrow[u] & \cdots \arrow[r] & V^* \otimes \mathcal{U}^{2,0,-1}(1) \arrow[r] \arrow[u] & \mathcal{U}^{3,0,-1}(1) \arrow[u] & \\
\wedge^3 V^* \otimes \mathcal{O} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{U}^{3,0,0}. \arrow[u] & &
\end{tikzcd} \tag{1.63}$$

Proposition 1.6.16. *The collection $\mathcal{G}, \mathcal{G}(1), \mathcal{G}(2)$ is exceptional.*

Proof. In Proposition 1.6.15 we showed that $\mathcal{G}$ is an exceptional object. To prove that the collection in the statement is exceptional, we need to show:

$$\operatorname{Hom}^\bullet(\mathcal{G}(l), \mathcal{G}) = 0 \qquad \text{for} \quad l = 1, 2.$$

To do so, as $\mathcal{G} = \mathbb{L}_{\mathbf{A}} \mathcal{G}_2$, we can verify the following vanishings for $l = 1, 2$:

$$\begin{aligned}
\operatorname{Hom}^\bullet(\mathcal{G}_2(l), \mathcal{G}_2) &= 0 &\qquad& \text{(by Proposition 1.6.9)}, \\
\operatorname{Hom}^\bullet(\mathbf{A}(l), \mathcal{G}_2) &= 0 && \text{(by Lemma 1.6.11)}, \\
\operatorname{Hom}^\bullet(\mathcal{G}_2(l), \mathbf{A}) &= 0 && \text{(by Serre duality and Lemma 1.6.11)}, \\
\operatorname{Hom}^\bullet(\mathbf{A}(l), \mathbf{A}) &= 0 && \text{(by Lemma 1.6.10)},
\end{aligned}$$

proving the claim. ☐

Proposition 1.6.17. *We have the following vanishings:*

$$\operatorname{Hom}^\bullet(\mathbf{B}(l), \mathcal{G}) = 0 \qquad \textit{for} \quad l = 2, \ldots, 8.$$

Proof. By Proposition 1.6.15, it is enough to prove the respective vanishings for $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$. As $\mathcal{G}_0, \mathcal{G}_1 \in \langle \mathbf{B}, \mathbf{B}(1) \rangle$ by (1.58) and (1.56), the vanishings hold since $\mathbf{B}$ is a Lefschetz basis. The vanishings with respect to $\mathcal{G}_2$ were proved in Proposition 1.6.8. This settles the claim. ☐

If now we were able to show the following series of vanishings:

$$\operatorname{Hom}^\bullet(\mathbf{B}(l), \mathcal{G}) = 0 \quad \text{for} \quad l = 0, 1,$$

we would be able to conclude that $\{\mathcal{G}\} \sqcup \mathbf{B}$ is the basis of a non rectangular Lefschetz collection of expected length (cf. Theorem 1.6.24). Actually, we show in Theorem 1.6.23 that if $\{\mathcal{G}\} \sqcup \mathbf{B}$ is a Lefschetz basis, we could apply the machinery of § 1.5 and we would actually obtain that the collection is full.

We are able to prove some additional vanishings to reach the following result.

Proposition 1.6.18. *For $\mathcal{U}^\lambda \in \mathbf{B}_1 \cup \mathbf{B}_2$ and $l = 0, \dots, 8$, we have:*

$$\mathrm{Hom}^\bullet(\mathcal{U}^\lambda(l), \mathcal{G}) = 0,$$

except possibly for $l = 1$ and $\mathcal{U}^\lambda = \mathcal{U}^{2,0,-2}$ and $l = 0$ and $\mathcal{U}^\lambda \in \{\mathcal{U}^{2,0,-2}, \mathcal{U}^{2,0,-1}, \mathcal{U}^{3,0,-1}\}$.

At this point, we start conjecturing on how to prove the remaining vanishings. We provide some partial insights that could be useful for a future proof.

The most promising idea is to show explicitly the following:

$$\mathcal{G}(3) \in \langle\, \{\mathcal{G}\} \sqcup \mathbf{B}, \mathbf{B}(1), \mathbf{B}(2)\,\rangle, \tag{1.64}$$

that is, $\mathcal{G}(3)$ can be generated by the (a priori) non-exceptional sequence of objects of the right. As a matter of fact, if $\{\mathcal{G}\} \sqcup \mathbf{B}$ were a Lefschetz basis, we should be able to find a decomposition as (1.64).

We cannot follow the approach of § 1.5, as multiple terms of the complexes describing $\mathcal{G}(3)$ do not belong to $\langle\, \{\mathcal{G}\} \sqcup \mathbf{B}, \mathbf{B}(1), \mathbf{B}(2)\,\rangle$, but we conjecture that $\mathcal{G}(3)$ should (cf. Conjecture 1.6.21). In the rest of the section, we show that excluding the information related to morphisms and looking at $K_0(X)$, the statement in (1.64) holds (cf. Proposition 1.6.20).

We start by considering the right resolution of $\mathcal{G}(3)$ as the totalization of the following bicomplex (the left resolution of (1.63)):

$$\begin{tikzcd}
 & & \mathcal{U}^{2,0,-3}(2) \arrow[r] & \cdots \arrow[r] & \wedge^3 V \otimes \mathcal{U}^{2,0,0}(2) \\
 & \mathcal{U}^{1,0,-4}(2) \arrow[r] & V \otimes \mathcal{U}^{1,0,-3}(2) \arrow[r] \arrow[u] & \cdots \arrow[r] & \wedge^4 V \otimes \mathcal{U}^{1,0,0}(2) \arrow[u] \\
\mathcal{U}^{0,0,-5}(2) \arrow[r] & V \otimes \mathcal{U}^{0,0,-4}(2) \arrow[r] \arrow[u] & \wedge^2 V \otimes \mathcal{U}^{0,0,-3}(2) \arrow[r] \arrow[u] & \cdots \arrow[r] & \wedge^5 V \otimes \mathcal{O}(2) \arrow[u]
\end{tikzcd} \tag{1.65}$$

Consider the following morphism of (non-exact) complexes, where the lower line is the restriction of (1.19) on $\mathsf{IGr}(3,12)$ to $\mathsf{IGr}(3,11)$:

$$\begin{tikzcd}
\mathcal{U}^{0,0,-1} \arrow[r] \arrow[d, "\cong"] & V \arrow[r] \arrow[d] & \mathcal{U}^{1,0,0} \arrow[d, "\cong"] \\
j^*\widetilde{\mathcal{U}}^{0,0,-1} \arrow[r] & j^*\widetilde{V} \arrow[r] & j^*\widetilde{\mathcal{U}}^{1,0,0},
\end{tikzcd} \tag{1.66}$$

where both the morphisms on the right are induced by the composition of the map induced by the skew symmetric form (ψ and $\widetilde{\psi}$ respectively) and the canonical projection.

We denote the upper line of (1.66) as $\mathcal{C}^\bullet$ and the lower as $\widetilde{\mathcal{C}}^\bullet$, then we have the following short exact sequence of complexes:

$$0 \to \mathcal{C}^\bullet \to \widetilde{\mathcal{C}}^\bullet \to \mathcal{O}[0] \to 0. \tag{1.67}$$

Recall that $\widetilde{\mathcal{C}}^\bullet \cong j^*\widetilde{\mathcal{S}}$, where $\widetilde{\mathcal{S}}$ is the symplectic vector bundle on $\mathsf{IGr}(3,12)$ and $\mathsf{rank}\,\widetilde{\mathcal{S}} = 6$. Thus

$$\mathcal{C}^\bullet = \operatorname{Cone}(j^*\widetilde{\mathcal{S}} \to \mathcal{O})[-1],$$

hence it is a complex concentrated in degree $0, 1$, with cohomology:

$$\mathcal{H}^0(\mathcal{C}^\bullet) = \mathcal{S}, \qquad \mathcal{H}^1(\mathcal{C}^\bullet) = \mathcal{O}_Z,$$

where $Z = \mathsf{IGr}(2,10) \subset \mathsf{IGr}(3,11)$ is the orbit of 3-spaces containing the isotropic vector of V and $\mathcal{S} = \operatorname{Ker}(j^*\widetilde{\mathcal{S}} \to \mathcal{O})$. In a way, we can think of $\mathcal{S}$ as an odd dimensional analogue of the symplectic bundle: the "odd symplectic sheaf". As $\mathcal{C}^\bullet$ is a complex concentrated in two degrees, we have:

$$\mathcal{C}^\bullet \to \mathcal{O}_Z[-1] \to \mathcal{S}[1].$$

We now prove a result similar to Proposition 1.5.2. The idea is that $\wedge^i\mathcal{C}^\bullet$ and $\wedge^i\widetilde{\mathcal{C}}^\bullet$ are very similar, except for the multiplicity of most factors in the complexes.

Lemma 1.6.19. *We have for every i:*

$$\wedge^i\mathcal{C}^\bullet \in \langle\, \mathbf{B} \,\rangle.$$

Proof. Computing $\wedge^i\mathcal{C}^\bullet$ with $i \leq 3$, the claim holds because every term in $\wedge^i\mathcal{C}^\bullet$ belongs to $\langle\, \mathbf{B} \,\rangle$ as

$$\mathcal{U}^{p_1,0,-p_3} \in \langle\, \mathbf{B} \,\rangle \qquad \text{for} \quad p_1 + p_3 \leq 3;$$

we refer to Proposition 1.5.1 for an explicit description of the complex $\wedge^i\mathcal{C}^\bullet$.

We proceed by induction on $i \geq 4$. From (1.67), we obtain an exact triangle of complexes:

$$\wedge^i\mathcal{C}^\bullet \to \wedge^i\widetilde{\mathcal{C}}^\bullet \to \wedge^{i-1}\mathcal{C}^\bullet,$$

and

$$\wedge^i\widetilde{\mathcal{C}}^\bullet \cong \wedge^i\widetilde{\mathcal{S}} \cong \wedge^{6-i}\widetilde{\mathcal{S}}.$$

As $i \geq 4$, we have $\wedge^{6-i}\widetilde{\mathcal{S}} \in \langle\, \mathbf{B} \,\rangle$ (cf. Proposition 1.5.1), $\wedge^{i-1}\mathcal{C}^\bullet \in \langle\, \mathbf{B} \,\rangle$ by the inductive hypothesis, hence $\wedge^i\mathcal{C}^\bullet \in \langle\, \mathbf{B} \,\rangle$. This proves the claim. □

We now prove a weaker version of (1.64) at the level of $K_0(\mathsf{IGr}(3,11))$.

Proposition 1.6.20. *The following result holds in the Grothendieck group:*

$$[\mathcal{G}(3)] - [\mathcal{G}] \in \langle\, [\,\mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2)\,]\,\rangle.$$

That is, $[\mathcal{G}(3)] - [\mathcal{G}]$ is in the span of the classes of the elements of $\mathbf{B}, \mathbf{B}(1), \mathbf{B}(2)$.

Proof. Consider the left resolution of $\mathcal{G}(3)$ given by the totalization of the bicomplex (1.65). We can easily represent its class in $K_0(\mathsf{IGr}(3,11))$ as

$$[\mathcal{G}(3)] = \sum_{\substack{p_1,p_2,p_3\geq 0,\\ p_1+p_2+p_3=5,\\ p_1\leq 2}} (-1)^{p_1-p_3+1}\binom{11}{p_2}\left[\mathcal{U}^{p_1,0,-p_3}(2)\right]; \tag{1.68}$$

notice that all bundles with $p_2 = 0$ appear with multiplicity 1 and all bundles with $p_2 = 1$ appear with multiplicity 11.

Consider the class of $\wedge^5\mathcal{C}^\bullet(2) \in \langle\, \mathbf{B}\,\rangle$, recall from Proposition 1.5.1 the explicit description of its terms (rearranged in terms of their Littlewood-Richardson factors):

$$\left[\wedge^5\mathcal{C}^\bullet(2)\right] = \sum_{\substack{p_1,p_2,p_3\geq 0,\\ p_1+p_2+p_3=5}} (-1)^{p_1-p_3+1} \sum_{p_2\geq 2k\geq 0}\binom{11}{p_2-2k}\left[\mathcal{U}^{p_1,0,-p_3}(2)\right] \in \langle\, [\,\mathbf{B}(2)\,]\,\rangle;$$

notice that all the terms with $p_2 = 0, 1$ appear with multiplicity 1 or 11 respectively, similarly to (1.68). We obtain:

$$\begin{aligned}\left[\wedge^5\mathcal{C}^\bullet(2)\right] = [\mathcal{G}(3)] + &\sum_{\substack{p_1,p_2,p_3\geq 0,\\ p_1+p_2+p_3=5,\\ p_1\geq 3}} (-1)^{p_1-p_3+1} \sum_{p_2\geq 2k\geq 0}\binom{11}{p_2-2k}\left[\mathcal{U}^{p_1,0,-p_3}(2)\right] + \\ + &\sum_{\substack{p_1,p_2,p_3\geq 0,\\ p_1+p_2+p_3=5,\\ p_1\leq 2}} (-1)^{p_1-p_3+1} \sum_{p_2\geq 2k> 0}\binom{11}{p_2-2k}\left[\mathcal{U}^{p_1,0,-p_3}(2)\right] \in \langle\, [\,\mathbf{B}(2)\,]\,\rangle\end{aligned}$$

Notice that all the terms on the right side of the equality belong to $\langle\, [\,\mathbf{B}(2)\,]\,\rangle$, except for $[\mathcal{G}(3)]$ and possibly the following:

$$[\mathcal{U}^{3,0,-2}(2)],\ [\mathcal{U}^{4,0,-1}(2)],\ [\mathcal{U}^{5,0,0}(2)].$$

Using the description of $\mathcal{G}(3)$ as the totalization of the bicomplex (1.63) and the staircase complexes associated to $\mathcal{U}^{5,0,0}(2)$ and $\mathcal{U}^{4,0,-1}(2)$, we obtain:

$$\begin{aligned}[\mathcal{G}] + [\mathcal{U}^{3,0,-2}(2)] &\in \langle\, [\,\mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2)\,]\,\rangle,\\ [\mathcal{U}^{5,0,0}(2)] &\in \langle\, [\,\mathbf{B}(1),\, \mathbf{B}(2)\,]\,\rangle,\\ [\mathcal{U}^{4,0,-1}(2)] - [\mathcal{U}^{1,0,-3}] &\in \langle\, [\,\mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2)\,]\,\rangle.\end{aligned}$$

Considering the right resolution of $\mathcal{U}^{0,0,-4}$ given by the staircase complex of $\mathcal{U}^{4,0,0}(1)$ and symplectic relations, we obtain the following:

$$[\mathcal{U}^{0,0,-4}] \in \langle\, [\,\mathbf{B},\, \mathbf{B}(1)\,]\,\rangle, \qquad [\mathcal{U}^{1,0,-3}] \in \langle\, [\,\mathbf{B},\, \mathbf{B}(1)\,]\,\rangle.$$

Substituting in the previous equations, we obtain:

$$[\mathcal{G}(3)] - [\mathcal{G}] \in \langle\, [\,\mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2)\,]\,\rangle,$$

proving the claim. ☐

Categorifying the result in Proposition 1.6.20, we conjecture the following key fact.

Conjecture 1.6.21. *The following inclusion holds:*

$$\mathrm{Cone}(\mathcal{G}(3) \to \mathcal{G}) \in \langle\, \mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2)\,\rangle.$$

As immediate consequence, we would have the following results.

Proposition 1.6.22. *If Conjecture 1.6.21 holds, then* $\{\mathcal{G}\} \sqcup \mathbf{B}$ *is a (non-rectangular) Lefschetz basis.*

Proof. Recalling Proposition 1.6.16 and Proposition 1.6.18, it is sufficient to verify that

$$\mathrm{Hom}^\bullet(\mathbf{B}(l), \mathcal{G}) = 0,$$

for $l = 0, 1$, knowing that it holds for $l = 2, \ldots, 8$, by Proposition 1.6.17. This is immediate, fix $l = 0, 1$:

$$\mathrm{Hom}^\bullet(\mathbf{B}(l), \mathcal{G}) = \mathrm{Hom}^\bullet(\mathbf{B}(l+3), \mathcal{G}(3)) = \mathrm{Hom}^\bullet(\mathbf{B}(l+3), \mathcal{G}),$$

where the equality is an immediate consequence of Conjecture 1.6.21 and Corollary 1.6.4. Finally, we obtain:

$$\mathrm{Hom}^\bullet(\mathbf{B}(l+3), \mathcal{G}) = 0,$$

by Proposition 1.6.17, proving the claim. ☐

As a further clue that $\mathcal{G}$ is the right candidate to complete the basis $\mathbf{B}$, we have the following result.

Theorem 1.6.23. *If Conjecture 1.6.21 holds, then the induced (non-rectangular) Lefschetz collection is full.*

Proof. Let $\mathscr{D} \subset \mathbf{D}^{\mathrm{b}}(X)$ be the subcategory generated by

$$\mathcal{G},\, \mathbf{B},\, \mathcal{G}(1),\, \mathbf{B}(1),\, \mathcal{G}(2),\, \mathbf{B}(2),\, \ldots,\, \mathbf{B}(3),\, \ldots, \mathbf{B}(8).$$

If the hypothesis holds, $\mathscr{D}$ is an admissible subcategory of $\mathbf{D}^{\mathrm{b}}(X)$. Using (1.63), we notice that

$$\mathcal{U}^{3,0,-2}(2),\, \mathcal{U}^{3,0,-2}(3),\, \mathcal{U}^{3,0,-2}(4),\, (\mathbf{B}_1 \cup \mathbf{B}_2),\, (\mathbf{B}_1 \cup \mathbf{B}_2)(1),\, \ldots,\, (\mathbf{B}_1 \cup \mathbf{B}_2)(8) \subset \mathscr{D}.$$

Following the procedure presented in § 1.5, we can determine that the condition in Proposition 1.5.6 holds in 9 steps. This verification is completely algorithmic. We provide the associated code in [Cat23a]. ☐

We now show that on $\mathsf{IGr}(3, 11)$, Conjecture 1.6.21 implies [KS21, Conjecture 1.3].

Theorem 1.6.24. *If Conjecture 1.6.21 holds, then the bounded derived category of coherent sheaves on* $\mathsf{IGr}(3, 11)$ *admits a full (non-rectangular!) Lefschetz collection with the first* 3 *blocks given by* $\{\mathcal{G}\} \sqcup \mathbf{B}$ *and its twists and the remaining blocks are given by* $\mathbf{B}$*, that is:*

$$\mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(3, 11)) = \langle\, \mathcal{G},\, \mathbf{B},\, \mathcal{G}(1),\, \mathbf{B}(1),\, \mathcal{G}(2),\, \mathbf{B}(2),\, \mathbf{B}(3),\, \dots,\, \mathbf{B}(8)\,\rangle.$$

To state the last conjecture, we recall the notion of induced polarization in the specific case of $\mathsf{IGr}(3, 11)$ (for more context, see [BKS23, § 2], [KS21, § 1]). Given the rectangular Lefschetz collection with basis $\mathbf{B}$ with respect to the bundle $\mathcal{O}(1)$, we denote by $\mathscr{R}$ the residual category of the Lefschetz collection:

$$\mathbf{D}^{\mathrm{b}}(\mathsf{IGr}(3, 11)) = \langle\, \mathscr{R},\, \mathbf{B},\, \mathbf{B}(1),\, \mathbf{B}(2),\, \mathbf{B}(3),\, \dots,\, \mathbf{B}(8)\,\rangle.$$

We define the induced polarization $\tau_{\mathscr{R}}$ (also known as the rotation functor) as:

$$\begin{aligned} \tau_{\mathscr{R}} : \mathscr{R} &\to \mathscr{R} \\ \mathcal{F} &\mapsto \mathbb{L}_{\mathbf{B}}(\mathcal{F}(1)). \end{aligned}$$

Notice that the induced polarization $\tau_{\mathscr{R}}$ is an autoequivalence of $\mathscr{R}$.

According to [KS21, Conjecture 1.3], the residual category $\mathscr{R}$ admits a totally orthogonal decomposition and the action of the induced polarization permutes the components of this decomposition of $\mathscr{R}$.

Theorem 1.6.25. *If Conjecture 1.6.21 holds, then the residual category* $\mathscr{R}$ *to the rectangular Lefschetz collection* $\mathbf{B}$ *admits a totally orthogonal exceptional collection and* $\tau_{\mathscr{R}}$ *acts transitively on any set of completely orthogonal triples of generators.*

Proof. Assuming that Theorem 1.6.24 holds, applying the result [BKS23, Proposition 2.4], we obtain the following exceptional collection:

$$\mathscr{R} = \langle\, \mathcal{G},\, \tau_{\mathscr{R}}(\mathcal{G}),\, \tau^2_{\mathscr{R}}(\mathcal{G})\,\rangle = \langle\, \mathcal{G},\, \mathbb{L}_{\mathbf{B}}\mathcal{G}(1),\, \mathbb{L}_{\mathbf{B},\mathbf{B}(1)}\mathcal{G}(2)\,\rangle.$$

Using Conjecture 1.6.21, we furthermore obtain that $\tau^3_{\mathscr{R}}(\langle\, \mathcal{G}\,\rangle) = \langle\, \mathcal{G}\,\rangle$, verifying that $\tau_{\mathscr{R}}$ permutes the exceptional collection. As $\tau_{\mathscr{R}}$ is an autoequivalence on $\mathscr{R}$, we obtain that

$$\mathscr{R} = \langle\, \tau_{\mathscr{R}}(\mathcal{G}),\, \tau^2_{\mathscr{R}}(\mathcal{G}),\, \mathcal{G}\,\rangle = \langle\, \tau^2_{\mathscr{R}}(\mathcal{G}),\, \mathcal{G},\, \tau_{\mathscr{R}}(\mathcal{G})\,\rangle$$

are exceptional collections as well, proving that they are all completely orthogonal.

Finally, the triples of completely exceptional collections of $\mathscr{R}$ are simply given by shifts and permutations of

$$\mathcal{G},\, \tau_{\mathscr{R}}(\mathcal{G}),\, \tau^2_{\mathscr{R}}(\mathcal{G}),$$

proving the claim. □

Chapter 2

Nodal categorical singularities

2.1 Introduction

The resolution of singularities is a central topic studied in algebraic geometry. Since Hironaka [Hir64] proved that singularities of varieties in characteristic 0 can be resolved, there has been much progress in studying singularities, their resolutions, and their applications in birational geometry. On the other hand, derived categories provide a strong technique for understanding algebraic varieties, for example two smooth Fano (or general type) varieties with equivalent derived categories are isomorphic [BO01]. For other varieties, derived categories can yield information about their birational geometry, for example flops of three dimensional varieties induce derived equivalences of their derived categories [Bri02].

One can often study a singularity by considering the properties of a resolution of it, and for relatively simple varieties and singularities, this might be done concretely. From the categorical viewpoint, let Y be a singular variety and let $\sigma\colon \widetilde{Y} \to Y$ be a resolution of singularities, then we have derived functors between their derived categories

$$\sigma^*\colon\ \mathbf{D}^{\mathrm{perf}}(Y) \to \mathbf{D}^{\mathrm{perf}}(\widetilde{Y}), \quad \sigma_*\colon\ \mathbf{D}^{\mathrm{b}}(\widetilde{Y}) \to \mathbf{D}^{\mathrm{b}}(Y).$$

Since $\widetilde{Y}$ is a smooth variety, we have $\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) = \mathbf{D}^{\mathrm{perf}}(\widetilde{Y})$. The two functors are related by the projection formula

$$\sigma_*\sigma^*(\mathcal{F}) = \mathcal{F} \otimes \sigma_*\mathcal{O}_{\widetilde{Y}}.$$

Inspired by the geometric picture, Kuznetsov introduced in [Kuz08b] the definition of "abstract" categorical resolution of singularities (see Definition 2.2.20). In the case of $\mathbf{D}^{\mathrm{b}}(Y)$, it consists of a triple $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$, where $\widetilde{\mathcal{D}}$ is a geometric triangulated category, $\sigma_*\colon \widetilde{\mathcal{D}} \to \mathbf{D}^{\mathrm{b}}(Y)$ and $\sigma^*\colon\ \mathbf{D}^{\mathrm{perf}}(Y) \to \widetilde{\mathcal{D}}$ are functors such that σ^* is left adjoint to σ_*, and the natural morphism of functors $\mathrm{id}_{\mathcal{D}^{\mathrm{perf}}} \to \sigma_*\sigma^*$ is an isomorphism.

Now an interesting question is whether or not this categorical viewpoint allows one to characterize the singularity geometrically. To shed some light on this, we investigate in this paper one special kind of singularities and their categorical resolutions, namely *nodal singularities.*

Before stating our main result, we briefly recall a few notions. A resolution of singularities is crepant if its relative canonical class is trivial. Crepant resolutions are interesting since they are considered minimal resolutions in the case of Gorenstein varieties, but they are also rare. On the other hand, a categorical resolution of singularities $\sigma_*\colon \widetilde{\mathcal{D}} \to \mathbf{D}^{\mathrm{b}}(Y)$ is *weakly crepant* if the left adjoint σ^* of σ_* is also its right adjoint (see Definition 2.2.20). An object $\mathcal{T} \in \widetilde{\mathcal{D}}$ is called k*-spherical* if $\mathrm{Hom}^\bullet(\mathcal{T}, \mathcal{T}) = \mathbb{C} \oplus \mathbb{C}[-k]$ and there is an isomorphism of functors $\mathrm{Hom}(\mathcal{T}, -) = \mathrm{Hom}(-, \mathcal{T}[k])^\vee$; and $\mathcal{E} \in \widetilde{\mathcal{D}}$ is *exceptional* if $\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{E}) = \mathbb{C}$.

Theorem 2.1.1. *Let* Y *be a quasiprojective variety with an isolated nodal singularity, and assume* $\dim(Y) \geq 2$*. Then there exists a weakly crepant categorical resolution* $\sigma_*\colon \widetilde{\mathcal{D}} \to \mathbf{D}^{\mathrm{b}}(Y)$ *such that:*

1. *The kernel* $\mathrm{Ker}(\sigma_*)$ *of* σ_* *is classically generated by a single object* $\mathcal{T}$ *which is* 2*-spherical if* $\dim(Y)$ *is even, and* 3*-spherical otherwise.*
2. *The resolution* σ_* *is a localization functor up to direct summands, cf. Definition 2.2.23.*

Note that the existence of a weakly crepant categorical resolution is a direct application of [Kuz08b]. We remark that the constructed categorical resolution has the advantage of being weakly crepant in any dimension, while the geometric resolution $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ is not. In [Kuz08b] another notion of crepancy, called strong crepancy, was introduced. The resolution $\widetilde{\mathcal{D}}$ in Theorem 2.1.1 is not strongly crepant, as computed in Proposition 2.3.10, if the dimension of Y is bigger than 3.

Our contribution is the explicit description of the kernel of the categorical resolution. We will define the resolution $\widetilde{\mathcal{D}}$ as an admissible component of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$, where $\widetilde{Y}$ is the blow-up at the isolated nodal singularity. Here, the object $\mathcal{T}$ has a clear geometric meaning: if $\dim(Y)$ is even, the object $\mathcal{T}$ is the pushforward to $\widetilde{Y}$ of the spinor bundle on the quadric exceptional divisor; and if $\dim(Y)$ is odd, the object $\mathcal{T}$ is described as the right mutation of the pushforward of one of the spinor bundles through the other, see Proposition 2.3.6.

Remark 2.1.2. Theorem 2.1.1 has been recently proven independently by Kuznetsov and Shinder in [KS23a, Theorem 5.8] with a similar strategy. Furthermore, [KS23a, Theorem 5.2] explains that one can drop "classically" and "up to direct summands" in Theorem 2.1.1; see also [MS23] for a discussion about this. Finally, note that the case when Y is 1-dimensional has been recently studied in [Sun22].

Based on Theorem 2.1.1, we propose the following definitions of categorical nodal singularities.

Definition 2.1.3 ((Abstract) nodal category)**.** *A triangulated category* $\mathscr{T}$ *is called* (abstract) nodal *if there is a categorical resolution* $\sigma_*\colon \widetilde{\mathcal{D}} \to \mathscr{T}$ *which is weakly crepant and whose kernel is (classically) generated by a single spherical object.*

Definition 2.1.4 (Geometric nodal category)**.** *A triangulated category $\mathscr{T}$ is called* geometric nodal *if it is an admissible subcategory of the derived category $\mathbf{D}^{\mathrm{b}}(Y)$ of a normal quasiprojective variety Y which has only an isolated nodal singularity, such that $\mathcal{T}^{\mathrm{perf}}$ is not smooth*[1]*.*

Using Theorem 2.1.1 we show the following relation between the above definitions.

Theorem 2.1.5 (Theorem 2.3.11)**.** *If $\mathscr{T}$ is a geometric nodal category, then $\mathscr{T}$ is an abstract nodal category. Furthermore, the constructed categorical resolution $\sigma_* \colon \widetilde{\mathcal{D}} \to \mathscr{T}$ as in the definition of an abstract nodal category is a localization up to direct summands.*

However, there are some questions around the definition of abstract nodal category.

Question 2.1.6.

1. The sphericalness property depends on the dimension of the variety. What should be a suitable definition of dimension of an abstract triangulated category?
2. It is not clear to us whether the definition characterizes nodal singularities in the geometric picture. In other words, if Y is a variety such that $\mathbf{D}^{\mathrm{b}}(Y)$ is abstract nodal, is then $\mathbf{D}^{\mathrm{b}}(Y)$ a geometric nodal category?
3. Does the sphericalness property of the kernel generator already imply that the resolution is weakly crepant?
4. Suppose that there is a 2 or 3-spherical object $\mathcal{T}$ in $\mathbf{D}^{\mathrm{b}}(X)$ where X is a smooth projective variety, and let $\mathscr{T} \subset \mathbf{D}^{\mathrm{b}}(X)$ be the triangulated subcategory classically generated by $\mathcal{T}$. Is the quotient $\mathbf{D}^{\mathrm{b}}(X)/\mathscr{T}$ a *geometric* nodal category?

Remark 2.1.7. A positive answer to question 2.1.6.(c) has been recently given in [KS23b, Lemma 5.8].

To address the last problem above, we study a concrete example: Let $Y \subset \mathbb{P}^5$ be a nodal cubic fourfold, with hyperplane section class H. By [Kuz10] there is a semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(Y)$ given by

$$\mathbf{D}^{\mathrm{b}}(Y) = \langle \mathscr{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H) \rangle,$$

where $\mathscr{A}_Y := \langle \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H) \rangle^{\perp}$ and $\mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)$ form an exceptional collection of line bundles. Then a categorical resolution of $\mathscr{A}_Y$ is provided by $\mathbf{D}^{\mathrm{b}}(S)$, where S is a K3 surface of degree 6 obtained as the intersection in $\mathbb{P}^4$ of a smooth quadric hypersurface Q with a cubic hypersurface. In this situation, we have the following application of Theorem 2.1.1, which provides an answer to [Kuz10, Remark 5.9].

---

[1] When $\mathcal{T}$ is a triangulated category, we say in this article that $\mathcal{T}^{\mathrm{perf}}$ is *smooth* if $\mathcal{T}$ can be realized as an admissible subcategory of the bounded derived category $\mathbf{D}^{\mathrm{b}}(X)$ of a smooth variety X. This means in particular that $\mathcal{T}^{\mathrm{perf}} = \mathcal{T}$ by [Orl06, Proposition 1.10] and the fact that $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^{\mathrm{b}}(X)$.

Theorem 2.1.8. *If Y is a nodal cubic fourfold, then the kernel of the categorical resolution $\mathbf{D}^{\mathrm{b}}(S) \to \mathscr{A}_Y$ is classically generated by $t^*\mathcal{S}$, where $t\colon S \to Q$ is the inclusion map of the K3 surface into the defining quadric Q of S, and $\mathcal{S}$ denotes the spinor bundle on Q.*

Remark 2.1.9. Note that the object $t^*\mathcal{S} \in \mathbf{D}^{\mathrm{b}}(S)$ is 2-spherical. This is similar to the situation of a nodal K3 surface, in which the spherical objects $\mathcal{O}_{E_i}(-1)$ appear in the kernel, where E_i are the exceptional curves in the resolution, cf. [Kuz21, Lemma 2.3] and [Bri02, Lemma 3.1].

Overview of the work In § 2.2, we recall the definitions and theorems that we use in the following sections. In particular, we review the definitions and properties of nodal singularities, the construction of their categorical resolution via a Lefschetz decomposition following [Kuz08b], and some results in [Efi20] which we use to compute the kernel of these categorical resolutions.

§ 2.3 is about the proof of Theorem 2.1.1. We first use a Lefschetz decomposition of quadrics to construct a categorical resolution of varieties with an isolated nodal singularity as in [Kuz08b]. Then by results of [Efi20], we find the kernel generator and check the sphericalness property.

In § 2.4, we focus on the case of nodal cubic fourfolds, proving Theorem 2.1.8 as a consequence of Theorem 2.1.1.

Notations and Conventions By variety we mean an integral scheme that is separated and of finite type over $\mathbb{C}$. If not otherwise mentioned, all functors between derived categories are implicitly derived. We use $\mathbb{R}_{\mathscr{A}}$ and $\mathbb{L}_{\mathscr{A}}$ to denote the right and left mutation with respect to an admissible subcategory $\mathscr{A}$, and use $\mathbb{T}_B$ to denote the twist functor $-\otimes B$. We define $\mathrm{Hom}^\bullet(-,-) = \bigoplus_i \mathrm{Hom}(-,-[i])[-i]$. If $\mathscr{T}$ is a triangulated category, a *classical generator* of $\mathscr{T}$ is an object $\mathcal{T} \in \mathscr{T}$ such that the smallest strictly full triangulated subcategory of $\mathscr{T}$ which is closed under direct summands and containing $\mathcal{T}$ is equal to $\mathscr{T}$, in symbols $\mathscr{T} = \langle \mathcal{T} \rangle^{\oplus}$. We take the liberty to write most isomorphisms as equalities.

2.2 Preliminaries

In this section, we briefly recall the notation and tools that we will use in subsequent sections. In particular, we discuss nodal singularities, semiorthogonal decompositions, categorical resolutions arising from Lefschetz decompositions and some results from [Efi20] that allow to compute the kernel of certain categorical resolutions. Finally we review some properties of spinor bundles on quadrics, and perform some cohomology computations we need in later sections.

2.2.1 Nodal singularities

Let X be a variety of dimension n. We recall the definition of a nodal singularity, which is the simplest kind of hypersurface singularity.

Definition 2.2.1. *An isolated singular point $x \in X$ is a* nodal point *(or* ordinary double point*) if the variety X is formally locally around x isomorphic to the singularity defined by the origin of the zero locus of $x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2$ inside $\mathbb{A}_{\mathbb{C}}^{n+1}$, i.e. $\widehat{\mathcal{O}_{X,x}} \simeq \mathbb{C}[\![x_0, \dots, x_n]\!]/(x_0^2 + \cdots + x_n^2)$.*

Remark 2.2.2. Since we are working over $\mathbb{C}$, we can replace "formally locally" with "analytically locally" and obtain an equivalent definition. Indeed, the completions of the algebraic and the analytic local rings coincide, cf. [Ser56, Proposition 3], and two analytic germs are equivalent if and only if the completions of their analytic local rings are isomorphic, cf. [Ish18, Theorem 4.2.3].

Assume that X has only one nodal singularity $x \in X$ and is smooth elsewhere. Since hypersurface singularities are Gorenstein, so is X (recall that being Gorenstein can be checked after completion of local rings, cf. [BH93, Proposition 3.1.19.(c)]). Now let $\sigma\colon \widetilde{X} \to X$ be the blow-up of X at x. Then σ is a resolution of singularities whose exceptional locus $j\colon Q \to \widetilde{X}$ is the smooth projective quadric hypersurface defined by $x_0^2 + x_1^2 + x_2^2 + \dots + x_n^2$. The conormal bundle of $Q \subset \widetilde{X}$ is $\mathcal{O}_Q(1) = \mathcal{N}^\vee_{Q/\widetilde{X}} = j^*\mathcal{O}_{\widetilde{X}}(-Q)$, since Q is the exceptional Cartier divisor of a blow-up.

2.2.2 Semiorthogonal decompositions

We recall the definitions of admissible subcategories and exceptional collections, which are the main source of semiorthogonal decompositions. Denote by $\mathscr{T}$ a triangulated category.

Definition 2.2.3. *Let $\mathscr{A}_1, \dots, \mathscr{A}_m$ be a sequence of admissible subcategories of $\mathscr{T}$ (cf. Definition 1.2.1). Then we say that $\mathscr{A}_1, \dots, \mathscr{A}_m$ is a* semiorthogonal collection *if* $\operatorname{Hom}(\mathscr{A}_i, \mathscr{A}_j) = 0$ *for all $i > j$. If in addition this collection generates $\mathscr{T}$, we say that it forms a* semiorthogonal decomposition *of $\mathscr{T}$, which we denote by*

$$\mathscr{T} = \langle \mathscr{A}_1, \dots, \mathscr{A}_m \rangle.$$

Any admissible subcategory $\mathscr{A}$ induces a semiorthogonal decomposition: Set

$$\mathscr{A}^\perp = \{\mathcal{F} \in \mathscr{T} \mid \operatorname{Hom}(\mathscr{A}, \mathcal{F}) = 0\},$$
$${}^\perp\mathscr{A} = \{\mathcal{F} \in \mathscr{T} \mid \operatorname{Hom}(\mathcal{F}, \mathscr{A}) = 0\},$$

then there are two semiorthogonal decompositions

$$\mathscr{T} = \langle \mathscr{A}^\perp, \mathscr{A} \rangle, \qquad \mathscr{T} = \langle \mathscr{A}, {}^\perp\mathscr{A} \rangle.$$

We define the left mutation functor $\mathbb{L}_{\mathscr{A}}$ and the right mutation functor $\mathbb{R}_{\mathscr{A}}$ to fit into the following exact triangles, respectively,

$$\alpha\alpha^! \to \mathrm{id} \to \mathbb{L}_{\mathscr{A}}, \qquad \mathbb{R}_{\mathscr{A}} \to \mathrm{id} \to \alpha\alpha^*,$$

where $\alpha\colon \mathscr{A} \to \mathscr{T}$ is the embedding functor and $\alpha^!$ and α^* are its right and left adjoints, respectively. Note that the semiorthogonality ensures that the cones in the triangles above are functorial. Indeed, use that the decomposition of an object of $\mathscr{T}$ into semiorthogonal components is functorial to deduce that $\mathrm{im}(\mathbb{R}_{\mathscr{A}}) \subset {}^{\perp}\mathscr{A}$, and then consider the long exact sequences arising from applying, for example, $\mathrm{Hom}(\mathbb{R}_{\mathscr{A}}(\mathcal{F}), -)$ to the triangles above. The following lemmata describe the interaction between mutation functors and semiorthogonal decompositions.

Lemma 2.2.4 ([Kuz10, Corollary 2.9])**.** *Assume that $\mathscr{T} = \langle \mathscr{A}_1, \dots, \mathscr{A}_m \rangle$ is a semiorthogonal decomposition. Then for each $1 \leq i \leq m-1$ there is a semiorthogonal decomposition*

$$\mathscr{T} = \langle \mathscr{A}_1, \dots, \mathscr{A}_{i-1}, \mathbb{L}_{\mathscr{A}_i}(\mathscr{A}_{i+1}), \mathscr{A}_i, \mathscr{A}_{i+2}, \dots, \mathscr{A}_m \rangle$$

and for each $2 \leq i \leq m$ there is a semiorthogonal decomposition

$$\mathscr{T} = \langle \mathscr{A}_1, \dots, \mathscr{A}_{i-2}, \mathscr{A}_i, \mathbb{R}_{\mathscr{A}_i}(\mathscr{A}_{i-1}), \mathscr{A}_{i+1}, \dots, \mathscr{A}_m \rangle.$$

Lemma 2.2.5 ([Kuz19, Lemma 2.2])**.** *Let $\mathscr{A}$ be an admissible subcategory of $\mathscr{T}$. Assume that $\mathscr{A}$ admits a semiorthogonal decomposition $\mathscr{A} = \langle \mathscr{A}_1, \dots, \mathscr{A}_m \rangle$. Then*

$$\mathbb{L}_{\mathscr{A}} = \mathbb{L}_{\mathscr{A}_1} \circ \cdots \circ \mathbb{L}_{\mathscr{A}_m} \quad \textit{and} \quad \mathbb{R}_{\mathscr{A}} = \mathbb{R}_{\mathscr{A}_m} \circ \cdots \circ \mathbb{R}_{\mathscr{A}_1}.$$

Examples of admissible subcategories are given by exceptional objects.

Definition 2.2.6. *An object $\mathcal{E} \in \mathscr{T}$ is* exceptional *if* $\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{E}) = \mathbb{C}$.[2]

Definition 2.2.7. *A set of objects $\{\mathcal{E}_1, \dots, \mathcal{E}_m\}$ in $\mathscr{T}$ is an* exceptional collection *if each $\mathcal{E}_i$ is exceptional, and* $\mathrm{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = 0$ *when $i > j$.*

If $\mathcal{E}$ is an exceptional object in a triangulated category $\mathscr{T}$, then the full triangulated subcategory $\mathscr{A} = \langle \mathcal{E} \rangle$ generated by $\mathcal{E}$ is admissible, cf. [BK89]; the mutations of an object $\mathcal{F} \in \mathscr{T}$ can be described explicitly as

$$\mathbb{L}_{\mathcal{E}}(\mathcal{F}) = \mathrm{Cone}(\mathrm{Hom}^\bullet(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}), \qquad \mathbb{R}_{\mathcal{E}}(\mathcal{F}) = \mathrm{Cone}(\mathcal{F} \to \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{E})^\vee \otimes \mathcal{E})[-1].$$

Similarly, an exceptional collection gives rise to a semiorthogonal collection.

In this paper, we consider a special kind of semiorthogonal decomposition.

---

[2] If the category $\mathscr{T}$ is not proper, one also requires that the functors $\mathrm{Hom}^\bullet(\mathcal{E}, -)$ and $\mathrm{Hom}^\bullet(-, \mathcal{E})$ on $\mathscr{T}$ take values in the category of finite-dimensional graded vector spaces.

Definition 2.2.8 ([Kuz08b, Definition 2.16]). *Let X be a variety with a (not necessarily ample) line bundle $\mathcal{O}(1)$. A* Lefschetz decomposition *of $\mathbf{D}^{\mathrm{b}}(X)$ is a semiorthogonal decomposition of the form*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathscr{B}_0, \mathscr{B}_1(1), \dots, \mathscr{B}_{m-1}(m-1)\rangle \quad \textit{where} \quad 0 \subset \mathscr{B}_{m-1} \subset \cdots \subset \mathscr{B}_1 \subset \mathscr{B}_0 \subset \mathbf{D}^{\mathrm{b}}(X).$$

A Lefschetz decomposition is rectangular *if $\mathscr{B}_0 = \mathscr{B}_1 = \cdots = \mathscr{B}_{m-1}$. Similarly, a* dual Lefschetz decomposition *is a semiorthogonal decomposition of the form*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathscr{B}_{m-1}(1-m), \dots, \mathscr{B}_1(-1), \mathscr{B}_0\rangle \quad \textit{where} \quad 0 \subset \mathscr{B}_{m-1} \subset \cdots \subset \mathscr{B}_1 \subset \mathscr{B}_0 \subset \mathbf{D}^{\mathrm{b}}(X).$$

2.2.3 Spherical objects and Serre functors

Let $\mathscr{T}$ be a triangulated category. We recall the definition of spherical objects, which play an important role in this paper.

Definition 2.2.9 ([ST01, Definition 2.14, Lemma 2.15]). *Let $k \in \mathbf{N}$ be a natural number. An object $\mathcal{T} \in \mathscr{T}$ is called k-*spherical *if*

1. *the functors $\mathrm{Hom}^\bullet(\mathcal{T}, -)$ and $\mathrm{Hom}^\bullet(-, \mathcal{T})$ on $\mathscr{T}$ take values in the category of finite-dimensional graded vector spaces;*
2. *$\mathrm{Hom}^\bullet(\mathcal{T}, \mathcal{T}) = \mathbb{C} \oplus \mathbb{C}[-k]$;*
3. *for any $\mathcal{F} \in \mathscr{T}$ there is an isomorphism $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{T}[k])^\vee$, which is functorial in $\mathcal{F}$.*

Condition (c) in Definition 2.2.9 can be simplified in some situations, for instance when $\mathscr{T}$ has a Serre functor.

Definition 2.2.10. *Let $\mathscr{T}$ be a triangulated category. An equivalence $\mathbb{S}\colon \mathscr{T} \to \mathscr{T}$ is called* Serre functor *if for any two objects $\mathcal{F}, \mathcal{G} \in \mathscr{T}$ there is a bifunctorial isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{G}, \mathbb{S}(\mathcal{F}))^\vee.$$

For instance, by Grothendieck–Verdier duality [Huy06, Theorem 3.34] the Serre functor of the derived category $\mathbf{D}^{\mathrm{b}}(X)$ of a smooth projective variety X of dimension n is given by $\mathbb{T}_{\omega_X} \circ [n]$, where ω_X is the canonical bundle of X. The Serre functor is unique up to isomorphisms of exact functors. The following lemma describes the relation between Serre functors and semiorthogonal decompositions with two components.

Lemma 2.2.11 ([Kuz10, Lemma 2.11],[Kuz19, Lemma 2.6]). *Let $\mathscr{T} = \langle \mathscr{A}, \mathscr{B}\rangle$ be a semiorthogonal decomposition of a triangulated category. Assume that $\mathscr{T}$ has Serre functor $\mathbb{S}_{\mathscr{T}}$. Then*

1. *there are semiorthogonal decompositions $\mathscr{T} = \langle \mathbb{S}_{\mathscr{T}}(\mathscr{B}), \mathscr{A}\rangle = \langle \mathscr{B}, \mathbb{S}_{\mathscr{T}}^{-1}(\mathscr{A})\rangle$, and*

2. *$\mathscr{A}$ and $\mathscr{B}$ have Serre functors $\mathbb{S}_{\mathscr{A}}$ and $\mathbb{S}_{\mathscr{B}}$, respectively, satisfying the relations*

$$\mathbb{S}_{\mathscr{B}} = \mathbb{R}_{\mathscr{A}} \circ \mathbb{S}_{\mathscr{T}}, \qquad \mathbb{S}_{\mathscr{A}}^{-1} = \mathbb{L}_{\mathscr{B}} \circ \mathbb{S}_{\mathscr{T}}^{-1}.$$

Remark 2.2.12. Assume that $\mathscr{T}$ admits a Serre functor $\mathbb{S}$. By the Yoneda lemma, in the Definition 2.2.9 of a k-spherical object $\mathcal{T} \in \mathscr{T}$ we can replace condition (c) with $\mathbb{S}(\mathscr{T}) = \mathcal{T}[k]$.

This chapter is about varieties with an isolated nodal singularity. By the local nature of such singularities, it seems unnatural to focus just on projective varieties; we prefer instead to work with quasiprojective varieties. The smooth varieties arising from resolution of singularities will again be quasiprojective; in particular, their derived category will not have a Serre functor, but they will admit the following weaker version.

Definition 2.2.13 (Serre functor for a pair $(\mathscr{R}, \mathscr{T})$, [Ber04, Section 6.4])**.** *Let $\mathscr{T}$ be a triangulated category. Let $\mathscr{R} \subset \mathscr{T}$ be a full triangulated subcategory such that for any $\mathcal{F} \in \mathscr{R}$ the functors $\operatorname{Hom}^\bullet(\mathcal{F}, -)$ and $\operatorname{Hom}^\bullet(-, \mathcal{F})$ on $\mathscr{T}$ take values in the category of finite-dimensional graded vector spaces. An equivalence $\mathbb{S} \colon \mathscr{T} \to \mathscr{T}$ is called* Serre functor for the pair $(\mathscr{R}, \mathscr{T})$ *if*

1. *$\mathbb{S}$ leaves $\mathscr{R}$ stable and*

2. *for any two objects $\mathcal{F} \in \mathscr{R}, \mathcal{G} \in \mathscr{T}$ there is a bifunctorial isomorphism*

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(\mathcal{G}, \mathbb{S}(\mathcal{F}))^\vee.$$

In particular, the restriction of $\mathbb{S}$ to $\mathscr{R}$ is a Serre functor for $\mathscr{R}$.

Example 2.2.14. Let X be a smooth quasiprojective variety of dimension n. Let $j \colon E \to X$ be the embedding of a smooth projective divisor; denote by $\omega_j := \omega_E \otimes j^*\omega_X^\vee$ its relative dualizing bundle. Define the category $\mathbf{D}^\mathrm{b}{}_E(X)$ as the full subcategory of $\mathbf{D}^\mathrm{b}(X)$ consisting of complexes topologically supported on E. As a triangulated category, $\mathbf{D}^\mathrm{b}{}_E(X)$ is generated by $j_*\,\mathbf{D}^\mathrm{b}(E)$, a remark that is very useful in practice. For any $\mathcal{F} \in \mathbf{D}^\mathrm{b}{}_E(X)$, the functors $\operatorname{Hom}^\bullet(\mathcal{F}, -)$ and $\operatorname{Hom}^\bullet(-, \mathcal{F})$ take values in the category of finite-dimensional graded vector spaces: indeed, this holds true for an object of the form $j_*\mathcal{F}, \mathcal{F} \in \mathbf{D}^\mathrm{b}(E)$, because $\operatorname{Hom}^\bullet(-, j_*\mathcal{F}) = \operatorname{Hom}^\bullet(j^*(-), \mathcal{F})$ and $\operatorname{Hom}^\bullet(j_*\mathcal{F}, -) = \operatorname{Hom}^\bullet(\mathcal{F}, j^*(-) \otimes \omega_j[-1])$. We claim that $\mathbb{T}_{\omega_X} \circ [n]$ is a Serre functor for the pair $(\mathbf{D}^\mathrm{b}{}_E(X), \mathbf{D}^\mathrm{b}(X))$. Condition (a) in Definition 2.2.13 is clearly satisfied; as for condition (b), for any $\mathcal{F} \in \mathbf{D}^\mathrm{b}(E)$ and $\mathcal{G} \in \mathbf{D}^\mathrm{b}(X)$, by Grothendieck-Verdier duality we have

$$\begin{aligned}
\operatorname{Hom}_X(j_*\mathcal{F}, \mathcal{G}) &= \operatorname{Hom}_E(\mathcal{F}, j^*\mathcal{G} \otimes \omega_j[-1]) \\
&= \operatorname{Hom}_E(\mathcal{F}, j^*\mathcal{G} \otimes \omega_E \otimes j^*\omega_X^\vee[-1]) \\
&= \operatorname{Hom}_E(j^*(\mathcal{G} \otimes \omega_X^\vee)[-n], \mathcal{F})^\vee \\
&= \operatorname{Hom}_X(\mathcal{G}, j_*\mathcal{F} \otimes \omega_X[n])^\vee.
\end{aligned}$$

The following result is analogous to Lemma 2.2.11, and can be proven in the same way.

Lemma 2.2.15. *Let $\mathscr{T}$ be a triangulated category, and $\mathscr{R} \subset \mathscr{T}$ a full triangulated subcategory. Suppose that we have a full triangulated subcategory $\mathscr{A}$ of $\mathscr{R}$ that is admissible in both $\mathscr{R}$ and $\mathscr{T}$; in particular, we have semiorthogonal decompositions $\mathscr{R} = \langle \mathscr{A}, \mathscr{B} \rangle$ and $\mathscr{T} = \langle \mathscr{A}, \mathcal{C} \rangle$. Assume that the pair $(\mathscr{R}, \mathscr{T})$ has a Serre functor $\mathbb{S}_{\mathscr{R},\mathscr{T}}$. Then the pair $(\mathscr{B}, \mathcal{C})$ has a Serre functor, which is given by*

$$\mathbb{S}_{\mathscr{B},\mathcal{C}} = \mathbb{R}_{\mathscr{A}} \circ \mathbb{S}_{\mathscr{R},\mathscr{T}} .$$

Remark 2.2.16. Assume that an object $\mathcal{T} \in \mathscr{T}$ belongs to a full triangulated subcategory $\mathscr{R} \subset \mathscr{T}$ such that the pair $(\mathscr{R}, \mathscr{T})$ has a Serre functor $\mathbb{S}$. To check that $\mathcal{T}$ is k-spherical, condition (c) in Definition 2.2.9 can be replaced (again by the Yoneda lemma) with $\mathbb{S}(\mathcal{T}) = \mathcal{T}[k]$.

2.2.4 Categorical resolutions

We recall the material from [Kuz08b, §3].

Definition 2.2.17 (Geometric category)**.** *A triangulated category $\mathcal{D}$ is* geometric *if it is equivalent to an admissible subcategory of $\mathbf{D}^{\mathrm{b}}(X)$, where X is a smooth variety.*

Definition 2.2.18 ([Orl06, Definition 1.6])**.** *Let $\mathcal{D}$ be a triangulated category. An object $\mathcal{F} \in \mathcal{D}$ is* homologically finite *if for any $\mathcal{G} \in \mathcal{D}$ there exists only a finite number of $n \in \mathbb{Z}$ such that* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \mathcal{G}[n]) \neq 0$*. The category $\mathcal{D}^{\mathrm{perf}}$ is defined as the full subcategory of $\mathcal{D}$ whose objects are the homologically finite objects.*

Remark 2.2.19. The notation $\mathcal{D}^{\mathrm{perf}}$ is justified since the homologically finite objects in the bounded derived category of coherent sheaves on a quasiprojective variety X are nothing else than the perfect complexes, i.e. $\mathbf{D}^{\mathrm{b}}(X)^{\mathrm{perf}} = \mathbf{D}^{\mathrm{perf}}(X)$, cf. [Orl06, Proposition 1.11].

Definition 2.2.20 (Categorical resolution)**.** *A* categorical resolution *of a triangulated category $\mathcal{D}$ is a geometric triangulated category $\widetilde{\mathcal{D}}$ and a pair of functors*

$$\sigma_* \colon \widetilde{\mathcal{D}} \to \mathcal{D}, \qquad \sigma^* \colon \mathcal{D}^{\mathrm{perf}} \to \widetilde{\mathcal{D}},$$

such that σ^* is left adjoint to σ_* on $\mathcal{D}^{\mathrm{perf}}$, i.e.

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}}(\sigma^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \sigma_*\mathcal{G}) \qquad \textit{for any} \quad \mathcal{F} \in \mathcal{D}^{\mathrm{perf}},\ \mathcal{G} \in \widetilde{\mathcal{D}},$$

and the natural morphism of functors $\mathrm{id}_{\mathcal{D}^{\mathrm{perf}}} \to \sigma_*\sigma^*$ *is an isomorphism.*

A categorical resolution $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$ is weakly crepant *if σ^* is also right adjoint to σ_* on $\mathcal{D}^{\mathrm{perf}}$, i.e.*

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}}(\mathcal{G}, \sigma^*\mathcal{F}) = \mathrm{Hom}_{\mathcal{D}}(\sigma_*\mathcal{G}, \mathcal{F}) \qquad \textit{for any} \quad \mathcal{F} \in \mathcal{D}^{\mathrm{perf}},\ \mathcal{G} \in \widetilde{\mathcal{D}}.$$

We now focus on a particular construction of a (weakly crepant) categorical resolution starting from a classical resolution of singularities. Consider a resolution of rational singularities $\sigma\colon \widetilde{Y} \to Y$ whose exceptional locus E is an irreducible divisor. Let $Z = \sigma(E)$ and $\rho\colon E \to Z$ be the restriction of σ to E. Denote by $j\colon E \to \widetilde{Y}$ the inclusion morphism. Let

$$\mathbf{D}^{\mathrm{b}}(E) = \langle \mathscr{B}_{m-1}(1-m), \dots, \mathscr{B}_1(-1), \mathscr{B}_0 \rangle \tag{2.1}$$

be a dual Lefschetz decomposition with respect to $\mathcal{O}_E(1) \coloneqq \mathcal{N}^{\vee}_{E/\widetilde{Y}}$. Define $\widetilde{\mathcal{D}}$ as the subcategory

$$\widetilde{\mathcal{D}} \coloneqq \{\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\widetilde{Y}) \mid j^*\mathcal{F} \in \mathscr{B}_0\}.$$

Proposition 2.2.21 ([Kuz08b, Proposition 4.1]). *Consider the notation fixed in* (2.1). *The pushforward functor j_* is fully faithful on $\mathscr{B}_i(-i)$ for $1 \le i \le m-1$ and we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) = \langle j_*\mathscr{B}_{m-1}(1-m), \dots, j_*\mathscr{B}_1(-1), \widetilde{\mathcal{D}} \rangle.$$

Theorem 2.2.22 ([Kuz08b, Theorem 4.4, Proposition 4.5]). *Consider the notation fixed in* (2.1). *Suppose that $\mathscr{B}_0 \subset \mathbf{D}^{\mathrm{b}}(E)$ contains $\rho^*(\mathbf{D}^{\mathrm{perf}}(Z))$. Then the functor σ^* factors through $\widetilde{\mathcal{D}}$ and $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$ is a categorical resolution of $\mathbf{D}^{\mathrm{b}}(Y)$ where*

$$\sigma_*\colon \widetilde{\mathcal{D}} \to \mathbf{D}^{\mathrm{b}}(Y), \qquad \sigma^*\colon \mathbf{D}^{\mathrm{perf}}(Y) \to \widetilde{\mathcal{D}}.$$

If in addition Y is Gorenstein, and $\omega_{\widetilde{Y}} = \sigma^*\omega_Y \otimes \mathcal{O}((m-1)E)$, and $\rho^*(\mathbf{D}^{\mathrm{perf}}(Z)) \subset \mathscr{B}_{m-1}$, then the categorical resolution $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$ is weakly crepant.

2.2.5 Localization functors and their kernels

In this section we review results from [KL15; Efi20] which will allow us to compute the kernels of certain categorical resolutions.

Definition 2.2.23. *Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories.*

1. *An exact functor $F\colon \mathscr{T} \to \mathscr{T}'$ is a* localization *if the induced functor $\overline{F}\colon \mathscr{T}/\operatorname{Ker}(F) \to \mathscr{T}'$ is an equivalence.*
2. *An exact functor $F\colon \mathscr{T} \to \mathscr{T}'$ is a* localization up to direct summands *if $F\colon \mathscr{T} \to \operatorname{im}(F)$ is a localization onto a dense subcategory of $\mathscr{T}'$, in symbols $\operatorname{im}(F)^{\oplus} = \mathscr{T}'$.*[3]

---

[3] The terminology "categorical contraction" is preferred for this notion in [KS23a, Definition 1.10].

Definition 2.2.24 (Nonrational locus, [KL15, Definition 6.1])**.** *Let $\sigma\colon X \to Y$ be a proper birational morphism. A closed subscheme $Z \subset Y$ is called a* nonrational locus *of Y with respect to σ if the natural morphism*

$$\mathcal{I}_Z \to \sigma_*\mathcal{I}_{\sigma^{-1}(Z)}$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}(Y)$. Here $\mathcal{I}_Z \subset \mathcal{O}_Y$ denotes the ideal sheaf of $Z \subset Y$, and $\sigma^{-1}(Z)$ is the scheme-theoretic pre-image of Z, so that $\mathcal{I}_{\sigma^{-1}(Z)} = \sigma^{-1}\mathcal{I}_Z \cdot \mathcal{O}_X$.

Theorem 2.2.25 ([Efi20, Theorem 8.22])**.** *Let $\sigma\colon X \to Y$ be a proper morphism such that $\sigma_*\mathcal{O}_X = \mathcal{O}_Y$. Assume that there is a subscheme $Z \subset Y$, such that all its infinitesimal neighborhoods Z_k, for $k \geq 1$, are nonrational loci of Y with respect to σ. Consider the cartesian diagram*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\rho"] & X \arrow[d, "\sigma"'] \\ Z \arrow[r] & Y. \end{tikzcd}$$

Assume that the functor $\rho_*\colon \mathbf{D}^{\mathrm{b}}(E) \to \mathbf{D}^{\mathrm{b}}(Z)$ is a localization up to direct summands. If σ is an isomorphism outside Z, then $\sigma_*\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ is a localization up to direct summands with kernel classically generated by $j_*(\mathrm{Ker}(\rho_*))$.

We verify the hypotheses of Theorem 2.2.25 for blow-ups of certain affine cones. The following corollary is remarked in passing after [Efi20, Theorem 1.10]; we provide a proof for the sake of completeness.

Corollary 2.2.26. *Let $Y \subset \mathbb{A}^{n+1}$ be the cone over a projectively normal smooth Fano variety $W \subset \mathbb{P}^n$. Let $Z = \{0\}$ be the singular point of Y. Let $\sigma\colon \widetilde{Y} \to Y$ be the blow-up at the singular point Z and $E = W$ its exceptional divisor. Then, $\sigma_*\colon \mathbf{D}^{\mathrm{b}}(\widetilde{Y}) \to \mathbf{D}^{\mathrm{b}}(Y)$ is a localization up to direct summands with kernel classically generated by $j_*(\langle \mathcal{O}_E \rangle^{\perp})$, where the orthogonal $\langle \mathcal{O}_E \rangle^{\perp}$ is taken in $\mathbf{D}^{\mathrm{b}}(E)$.*

Proof. We verify that the hypotheses of Theorem 2.2.25 hold. First note that $\sigma\colon \widetilde{Y} \to Y$ is a resolution of singularities for Y; in particular, it is an isomorphism outside Z. Moreover, the exceptional locus E is isomorphic to the Fano variety W. As Y is an affine cone over W, its coordinate ring is isomorphic to the homogeneous coordinate ring of W, which is integrally closed as W is projectively normal, hence Y is normal.

Recall that a cone over a Fano variety has rational singularities by [Kol13, Corollary 3.4], hence, $\sigma_*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$. Let $\rho\colon E \to Z$ be the restriction of σ to E. As E is a Fano variety, we have that $\mathcal{O}_E$ is exceptional by Kodaira's vanishing theorem. As a consequence, we have $\rho_*\mathcal{O}_E = \mathrm{H}^\bullet(E, \mathcal{O}_E) = \mathbb{C} = \mathcal{O}_Z$. We now prove that ρ_* is a localization. Since the functor ρ_* has a left adjoint ρ^*, by [Efi20, Remark 3.3] it is a localization if and only if ρ^* is fully faithful. This is indeed the case by the projection formula applied to ρ_* using $\rho_*\mathcal{O}_E = \mathcal{O}_Z$

(see [Kuz08b, Lemma 2.4] for details). Finally, considering the decomposition induced on $\mathbf{D}^{\mathrm{b}}(E) = \langle \mathrm{Ker}(\rho_*), \rho^* \mathbf{D}^{\mathrm{b}}(Z)\rangle$, cf. [Kuz16, Lemma 2.3], we have

$$\mathrm{Ker}(\rho_*) = (\rho^* \mathbf{D}^{\mathrm{b}}(Z))^{\perp} = \langle \mathcal{O}_E\rangle^{\perp}.$$

The last thing to check in order to apply Theorem 2.2.25 is that the canonical map

$$\mathcal{I}_Z^k \to \sigma_*(\sigma^{-1}(\mathcal{I}_Z^k)\cdot \mathcal{O}_{\widetilde{Y}}) = \sigma_*\mathcal{I}_{\sigma^{-1}(Z_k)}$$

is an isomorphism for $k \geq 1$, where Z_k is the k-th formal neighbourhood of Z. By the construction of blow-ups, the variety $\widetilde{Y}$ is defined as $\mathrm{Proj}(\bigoplus_{i=0}^{\infty} \mathcal{I}_Z^i)$. On the other hand, the graded sheaf of modules corresponding to $\sigma^{-1}(\mathcal{I}_Z^k)\cdot\mathcal{O}_{\widetilde{Y}}$ is $\bigoplus_{i=0}^{\infty}\mathcal{I}_Z^{k+i}$, which is equal to $\mathcal{O}_{\widetilde{Y}/Y}(k)$, where $\mathcal{O}_{\widetilde{Y}/Y}(1)$ is the twisting sheaf on the blow-up $\widetilde{Y}$. We recall that $\mathcal{O}_{\widetilde{Y}/Y}(1) = \mathcal{O}_{\widetilde{Y}}(-E)$ and $\mathcal{O}_E(1) = \mathcal{O}_E(-E)$. Consider for $k \geq 0$ the short exact sequence of sheaves on Y

$$0 \to \mathcal{I}_Z^{k+1} \to \mathcal{I}_Z^k \to \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \to 0, \tag{2.2}$$

and the short exact sequence of sheaves on $\widetilde{Y}$

$$0 \to \mathcal{O}_{\widetilde{Y}}(-(k+1)E) \to \mathcal{O}_{\widetilde{Y}}(-kE) \to \mathcal{O}_E(-kE) \to 0, \tag{2.3}$$

as well as the morphism of exact triangles

$$\begin{tikzcd}
\mathcal{I}_Z^{k+1} \arrow[r] \arrow[d] & \mathcal{I}_Z^k \arrow[r] \arrow[d] & \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \arrow[d, dotted] \\
\sigma_*\mathcal{O}_{\widetilde{Y}}(-(k+1)E) \arrow[r] & \sigma_*\mathcal{O}_{\widetilde{Y}}(-kE) \arrow[r] & \sigma_*\mathcal{O}_E(-kE),
\end{tikzcd} \tag{2.4}$$

where the upper row is the triangle (2.2) and the lower row comes from the application of σ_* to (2.3). We claim that the induced map $\mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \to \sigma_*\mathcal{O}_E(-kE)$ is an isomorphism for $k \geq 0$. As Z is a point, it is enough to study the stalk of the morphism at Z. Let $\mathrm{R}(E)$ be the homogeneous coordinate ring of $E = W \subset \mathbb{P}^n$. By definition, the affine coordinate ring of Y, namely $\mathrm{K}[Y]$, is just $\mathrm{R}(E)$ without its grading. Identifying $\mathcal{I}_Z \subset \mathrm{K}[Y]$ with $(x_0, \ldots, x_n)$, we obtain that $\mathcal{I}_Z^k/\mathcal{I}_Z^{k+1}$ corresponds to the space of homogeneous polynomials of degree k in $\mathrm{K}[Y]$. On the other hand, by Kodaira vanishing, we have that $\mathrm{H}^i(E, \mathcal{O}_E(k)) = 0$ for any $i > 0$, so we obtain $\sigma_*\mathcal{O}_E(-kE) = \mathrm{H}^0(E, \mathcal{O}_E(k))$, which is isomorphic to the space of homogeneous polynomials of degree k in $\mathrm{R}(E)$. By projective normality, we have that the composition $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \to \mathrm{H}^0(E, \mathcal{O}_E(k))$ is surjective, cf. [Har77, Exercise II.5.14(d)], hence the map $\mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \to \mathrm{H}^0(E, \mathcal{O}_E(k))$ is surjective. As both source and target of the latter are vector spaces of the same dimension, the map is an isomorphism.

To conclude the proof, we prove inductively that the canonical maps $\mathcal{I}_Z^k \to \sigma_*\mathcal{O}_{\widetilde{Y}}(-kE)$ are isomorphisms. The base case $k = 0$ of the induction is given by the isomorphism $\sigma_*\mathcal{O}_{\widetilde{Y}} =$

$\mathcal{O}_Y$. Then by the induction hypothesis the map $\mathcal{I}_Z^k \to \sigma_* \mathcal{O}_{\widetilde{Y}}(-kE)$ is an isomorphism, hence the canonical morphism on the left in (2.4) is an isomorphism, concluding the inductive step. As we showed that Z_k is a nonrational locus of Y for $k \geq 1$, we can apply Theorem 2.2.25 and obtain the statement. □

Remark 2.2.27. Note that Corollary 2.2.26 remains valid for varieties Y with an isolated singular point y which look, upon restriction to a formal neighborhood of y in Y, like the cone singularity in the corollary. Indeed, the crucial part of the proof is the check that the infinitesimal neighborhoods of the singularity are nonrational loci. Now use that $\mathrm{Spec}(\widehat{\mathcal{O}_{Y,y}}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is faithfully-flat, cf. [Stacks, Tag 00MC], and $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$ is flat, so the nonrational locus condition can be checked after base-change to $\mathrm{Spec}(\widehat{\mathcal{O}_{Y,y}})$.

2.2.6 Derived base-change

The last ingredient we need in the derived categories setting is the following base-change result.

Proposition 2.2.28. *Consider a cartesian square of varieties*

$$\begin{tikzcd}
X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"'] \\
X \arrow[r, "f"] & S.
\end{tikzcd}$$

Suppose that g is a closed immersion and local complete intersection morphism, X is Cohen–Macaulay, and $\mathsf{codim}_X(X \times_S Y) = \mathsf{codim}_S(Y)$. Then

$$q_* p^* = g^* f_*.$$

Proof. The proposition is a corollary of Tor-independent base-change, cf. [Stacks, Tag 08IB]. In slightly more detail: Since local complete intersection immersions are Koszul-regular immersions, cf. [Stacks, Tag 09CC], one can use the Koszul complex to compute higher Tor groups. The regular sequences that define $Y \subset S$ locally stay regular on X because of the codimension assumption and the unmixedness theorem, cf. [Stacks, Tag 02JN]. So the Koszul complex stays exact after tensoring with $\mathcal{O}_X$, and we see that higher Tor groups vanish, as required to apply Tor-independent base-change.

A proof can also be found in [Kuz06, Corollary 2.27]. □

2.2.7 Spinor bundles on quadric hypersurfaces

In this subsection we summarize some properties of spinor bundles on quadric hypersurfaces. Let $Q \subset \mathbb{P}^{n+1}$ be the (unique up to isomorphism) smooth quadric hypersurface of dimension n. The definition of spinor bundles on Q, given in [Ott88], depends on the parity of n.

Assume first that $n = 2m+1$ is odd; in this case, the maximal dimension of a (projective) linear subspace contained in Q is m. The parameter space for the m-planes contained in Q is an irreducible smooth projective variety S. Let $\mathcal{O}_S(1)$ be the ample generator of $\operatorname{Pic}(S) \simeq \mathbb{Z}$; it can be shown that $\dim \mathrm{H}^0(S, \mathcal{O}_S(1)) = 2^{m+1}$. Now, for any $x \in Q$, consider the embedding

$$i_x \colon S_x := \{\mathbb{P}^m \subset Q \mid x \in \mathbb{P}^m\} \to S = \{\mathbb{P}^m \subset Q\}.$$

The induced restriction map $\mathrm{H}^0(S, \mathcal{O}_S(1)) \to \mathrm{H}^0(S_x, i_x^*\mathcal{O}_S(1))$ turns out to be surjective, so its dual yields an inclusion

$$\mathrm{H}^0(S_x, i_x^*\mathcal{O}_S(1))^\vee \to \mathrm{H}^0(S, \mathcal{O}_S(1))^\vee.$$

Since $\dim \mathrm{H}^0(S_x, i_x^*\mathcal{O}_S(1)) = 2^m$ for any $x \in Q$, we obtain a morphism

$$s \colon Q \to \mathrm{Gr}(2^m, 2^{m+1}).$$

The spinor bundle $\mathcal{S}$ on Q is defined as the pullback by s of the tautological subbundle on $\mathrm{Gr}(2^m, 2^{m+1})$.

Let us move on to the case of a quadric of even dimension $n = 2m$. The maximal dimension of a linear subspace contained in Q is m. The parameter space for the m-planes contained in Q has two connected components S' and S''. Both S' and S'' are smooth irreducible projective varieties. Let $\mathcal{O}_{S'}(1)$ and $\mathcal{O}_{S''}(1)$ be the ample generators of $\operatorname{Pic}(S') \simeq \mathbb{Z}$ and $\operatorname{Pic}(S'') \simeq \mathbb{Z}$, respectively; it can be shown that $\dim \mathrm{H}^0(S', \mathcal{O}_{S'}(1)) = \dim \mathrm{H}^0(S'', \mathcal{O}_{S''}(1)) = 2^m$. Now, for any $x \in Q$, consider the embeddings

$$i'_x \colon S'_x = \{\mathbb{P}^m \in S' \mid x \in \mathbb{P}^m\} \to S' \quad \text{and} \quad i''_x \colon S''_x = \{\mathbb{P}^m \in S'' \mid x \in \mathbb{P}^m\} \to S''.$$

The induced restriction maps $\mathrm{H}^0(S', \mathcal{O}_{S'}(1)) \to \mathrm{H}^0(S'_x, (i'_x)^*\mathcal{O}_{S'}(1))$ and $\mathrm{H}^0(S'', \mathcal{O}_{S''}(1)) \to \mathrm{H}^0(S''_x, (i''_x)^*\mathcal{O}_{S''}(1))$ turn out to be surjective. By passing to the duals we obtain the inclusions

$$\mathrm{H}^0(S'_x, (i'_x)^*\mathcal{O}_{S'}(1))^\vee \to \mathrm{H}^0(S', \mathcal{O}_{S'}(1))^\vee \quad \text{and} \quad \mathrm{H}^0(S''_x, (i''_x)^*\mathcal{O}_{S''}(1))^\vee \to \mathrm{H}^0(S'', \mathcal{O}_{S''}(1))^\vee.$$

Since $\dim \mathrm{H}^0(S'_x, (i'_x)^*\mathcal{O}_{S'}(1)) = \dim \mathrm{H}^0(S''_x, (i''_x)^*\mathcal{O}_{S''}(1)) = 2^{m-1}$ for any $x \in Q$, we obtain two morphisms

$$s' \colon Q \to \mathrm{Gr}(2^{m-1}, 2^m) \quad \text{and} \quad s'' \colon Q \to \mathrm{Gr}(2^{m-1}, 2^m).$$

The spinor bundle $\mathcal{S}'$ (resp. $\mathcal{S}''$) on Q is defined as the pullback by s' (resp. s'') of the tautological subbundle on $\mathrm{Gr}(2^{m-1}, 2^m)$. We write $\mathcal{S}$, respectively $\mathcal{S}'$, $\mathcal{S}''$, for the spinor bundle(s) on the odd, respectively even, dimensional quadric Q. These bundles enjoy the following properties.

Theorem 2.2.29.

1. *The spinor bundles are stable, cf. [Ott88, Theorem 2.1].*

2. *Suppose Q is an even dimensional quadric and let $\mathcal{S}'$, $\mathcal{S}''$ be the two spinor bundles. Let $i\colon Q' \to Q$ be the closed immersion of a smooth hyperplane section, with spinor bundle $\mathcal{S}$. Then $i^*\mathcal{S}' = i^*\mathcal{S}'' = \mathcal{S}$, cf. [Ott88, Theorem 1.4(i)].*

3. *If $\mathcal{S}$ is either the spinor bundle on the odd dimensional quadric or any of the two spinor bundles on the even dimensional quadric, then $\mathrm{H}^i(Q, \mathcal{S}(k)) = 0$ for $0 < i < n$ and arbitrary $k \in \mathbb{Z}$. Furthermore $\mathrm{H}^0(Q, \mathcal{S}(k)) = 0$ for $k \leq 0$, and $\dim \mathrm{H}^0(Q, \mathcal{S}(1)) = 2^{[(n+1)/2]}$, where n is the dimension of Q, cf. [Ott88, Theorem 2.3].*

4. *Suppose the quadric Q has odd dimension $n = 2m+1$. We have a short exact sequence*
$$0 \to \mathcal{S} \to \mathcal{O}_Q^{\oplus 2^{m+1}} \to \mathcal{S}(1) \to 0, \tag{2.5}$$
and $\mathcal{S}^\vee = \mathcal{S}(1)$, cf. [Ott88, Theorem 2.8(i)].

5. *Suppose the quadric Q has even dimension $n = 2m$. We have short exact sequences*
$$\begin{aligned} &0 \to \mathcal{S}' \to \mathcal{O}_Q^{\oplus 2^m} \to \mathcal{S}''(1) \to 0, \\ &0 \to \mathcal{S}'' \to \mathcal{O}_Q^{\oplus 2^m} \to \mathcal{S}'(1) \to 0. \end{aligned} \tag{2.6}$$
Furthermore, if $n \equiv 0 \pmod 4$, then $\mathcal{S}'^\vee = \mathcal{S}'(1)$ and $\mathcal{S}''^\vee = \mathcal{S}''(1)$, and if $n \equiv 2 \pmod 4$, then $\mathcal{S}'^\vee = \mathcal{S}''(1)$ and $\mathcal{S}''^\vee = \mathcal{S}'(1)$, cf. [Ott88, Theorem 2.8(ii)].

6. *Spinor bundles are exceptional. If Q is even dimensional, $\mathcal{S}'$ and $\mathcal{S}''$ are orthogonal to each other, cf. [Kap88].*

We summarize here some cohomology computations.

Lemma 2.2.30. *Let $Q \subset \mathbb{P}^{n+1}$ be the smooth quadric of dimension n. Then*
$$\omega_Q = \mathcal{O}_Q(-n) \tag{2.7}$$
and the following cohomology groups vanish:
$$\mathrm{H}^\bullet(Q, \mathcal{O}_Q(-k)) = 0 \quad \textit{for} \quad k = 1, \dots, n-1. \tag{2.8}$$

Proof. As Q is a smooth hypersurface of degree 2 in $\mathbb{P}^{n+1}$, by the adjunction formula we have $\omega_Q = \mathcal{O}_Q(-n-2+2) = \mathcal{O}_Q(-n)$. The vanishing statement (2.8) follows from Kodaira's vanishing theorem. □

Remark 2.2.31. Let $\mathcal{S}$ be any spinor bundle on the smooth quadric Q of dimension n. Using Serre duality and Theorem 2.2.29(c)-(e), we have $\mathrm{H}^n(Q, \mathcal{S}(k)) = 0$ for $k \geq 1-n$. In particular, $\mathrm{H}^\bullet(Q, \mathcal{S}(k)) = 0$ for $1-n \leq k \leq 0$.

Lemma 2.2.32. *Let $Q \subset \mathbb{P}^{n+1}$ be the smooth quadric of odd dimension $n = 2m+1$. We have*

$$\operatorname{Hom}^\bullet(\mathcal{S}(k), \mathcal{S}) = \begin{cases} \mathbb{C} & \text{if } k = 0 \\ \mathbb{C}[-1] & \text{if } k = 1. \end{cases} \tag{2.9}$$

Proof. The isomorphism for $k = 0$ follows from the exceptionality of $\mathcal{S}$, see Theorem 2.2.29(f). For the proof of the second isomorphism, we use sequence (2.5). Consider the long exact sequence induced by applying $\operatorname{Hom}^\bullet(-, \mathcal{S})$. This provides the exact triangle

$$\operatorname{Hom}^\bullet(\mathcal{S}, \mathcal{S}) \leftarrow \operatorname{Hom}^\bullet(\mathcal{O}_Q^{\oplus 2^{m+1}}, \mathcal{S}) \leftarrow \operatorname{Hom}^\bullet(\mathcal{S}(1), \mathcal{S}).$$

As the central term vanishes by Remark 2.2.31, we obtain

$$\operatorname{Hom}^\bullet(\mathcal{S}(1), \mathcal{S}) = \operatorname{Hom}^\bullet(\mathcal{S}, \mathcal{S})[-1] = \mathbb{C}[-1]. \qquad \square$$

Lemma 2.2.33. *Let $Q \subset \mathbb{P}^{n+1}$ be the smooth quadric of even dimension $n = 2m \geq 2$. Let $\mathcal{S}'$ and $\mathcal{S}''$ be its spinor bundles. We have*

$$\operatorname{Hom}^\bullet(\mathcal{S}'(k), \mathcal{S}') = \operatorname{Hom}^\bullet(\mathcal{S}''(k), \mathcal{S}'') = \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{if } k = 1, \end{cases} \tag{2.10}$$

and

$$\operatorname{Hom}^\bullet(\mathcal{S}''(k), \mathcal{S}') = \operatorname{Hom}^\bullet(\mathcal{S}'(k), \mathcal{S}'') = \begin{cases} 0 & \text{if } k = 0 \\ \mathbb{C}[-1] & \text{if } k = 1. \end{cases} \tag{2.11}$$

Proof. If $k = 0$, the isomorphism (2.10) holds because $\mathcal{S}'$ is exceptional by Theorem 2.2.29(f). To prove the vanishing of $\operatorname{Hom}^\bullet(\mathcal{S}'(1), \mathcal{S}')$, consider the defining sequence of a smooth hyperplane section $i\colon Q' \to Q$ tensored with $\mathcal{S}'$

$$0 \to \mathcal{S}'(-1) \to \mathcal{S}' \to i_* i^* \mathcal{S}' \to 0.$$

Applying $\operatorname{Hom}^\bullet(\mathcal{S}', -)$ and using adjunction we get

$$\operatorname{Hom}^\bullet(\mathcal{S}', \mathcal{S}'(-1)) \to \operatorname{Hom}^\bullet(\mathcal{S}', \mathcal{S}') \to \operatorname{Hom}^\bullet(\mathcal{S}', i_* i^* \mathcal{S}') = \operatorname{Hom}^\bullet(i^* \mathcal{S}', i^* \mathcal{S}').$$

Recall that by Theorem 2.2.29(b) we have $i^*\mathcal{S}' = \mathcal{S}$, where $\mathcal{S}$ is the spinor bundle on Q'. As spinor bundles are exceptional, we have $\operatorname{Hom}^\bullet(\mathcal{S}', \mathcal{S}') = \mathbb{C} = \operatorname{Hom}^\bullet(\mathcal{S}, \mathcal{S})$. Moreover, the map $\operatorname{Hom}^0(\mathcal{S}', \mathcal{S}') \to \operatorname{Hom}^0(\mathcal{S}', i_* i^* \mathcal{S}')$ is injective, hence an isomorphism. We conclude that $\operatorname{Hom}^\bullet(\mathcal{S}'(1), \mathcal{S}') = \operatorname{Hom}^\bullet(\mathcal{S}', \mathcal{S}'(-1)) = 0$.

We proceed with the proof of (2.11). The vanishing for $k = 0$ holds by Theorem 2.2.29(f). We calculate $\operatorname{Hom}^\bullet(\mathcal{S}''(1), \mathcal{S}')$. Applying $\operatorname{Hom}^\bullet(-, \mathcal{S}')$ to the sequence (2.6)

$$0 \to \mathcal{S}' \to \mathcal{O}_Q^{\oplus 2^m} \to \mathcal{S}''(1) \to 0,$$

we get

$$\mathrm{Hom}^\bullet(\mathcal{S}',\mathcal{S}') \leftarrow \mathrm{Hom}^\bullet(\mathcal{O}_Q^{\oplus 2^m},\mathcal{S}') \leftarrow \mathrm{Hom}^\bullet(\mathcal{S}''(1),\mathcal{S}').$$

As the central term vanishes by Remark 2.2.31, we obtain

$$\mathrm{Hom}^\bullet(\mathcal{S}''(1),\mathcal{S}') = \mathrm{Hom}^\bullet(\mathcal{S}',\mathcal{S}')[-1] = \mathbb{C}[-1]. \qquad \square$$

We end this section by recalling Kapranov's Lefschetz decomposition for quadrics.

Theorem 2.2.34 ([KP21b, Lemma 2.4])**.** *Let $Q \subset \mathbb{P}^{n+1}$ be the smooth quadric of dimension n. Then we have the dual Lefschetz decomposition*

$$\mathbf{D}^{\mathrm{b}}(Q) = \langle \mathscr{B}_{n-1}(1-n), \ldots, \mathscr{B}_1(-1), \mathscr{B}_0 \rangle. \tag{2.12}$$

Here, if n is odd, we have

$$\mathscr{B}_0 = \langle \mathcal{S}, \mathcal{O}_Q \rangle \quad \text{and} \quad \mathscr{B}_1 = \cdots = \mathscr{B}_{n-1} = \langle \mathcal{O}_Q \rangle,$$

where the bundle $\mathcal{S}$ is the unique spinor bundle on Q. If n is even, we have

$$\mathscr{B}_0 = \mathscr{B}_1 = \langle \mathcal{S}', \mathcal{O}_Q \rangle \quad \text{and} \quad \mathscr{B}_2 = \cdots = \mathscr{B}_{n-1} = \langle \mathcal{O}_Q \rangle,$$

where $\mathcal{S}'$ is any of the two spinor bundles on Q.

Proof. For an odd dimensional quadric we have by [Kap88] the semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(Q) = \langle \mathcal{S}, \mathcal{O}_Q, \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(n-1) \rangle.$$

It suffices to suitably group its components and apply Lemma 2.2.11(a) to get the desired dual Lefschetz decomposition.

For a quadric of dimension $n = 2m$ we have by [Kap88] the semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(Q) = \langle \mathcal{S}', \mathcal{S}'', \mathcal{O}_Q, \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(n-1) \rangle.$$

We claim that $\mathbb{R}_{\mathcal{O}_Q}\mathcal{S}'' = \mathcal{S}'(1)[-1]$. First, we have

$$\mathrm{Hom}^\bullet(\mathcal{S}'', \mathcal{O}_Q) = \mathrm{Hom}^\bullet(\mathcal{O}_Q, \mathcal{S}''^\vee) = \mathrm{H}^\bullet(Q, \mathcal{S}'''(1))$$

where $\mathcal{S}'''$ is one of the spinor bundles depending on the parity of m, see Theorem 2.2.29(e). By Theorem 2.2.29(c), we have the isomorphism

$$\mathrm{H}^\bullet(Q, \mathcal{S}'''(1)) = \mathbb{C}^{\oplus 2^m}.$$

Then, using the exact sequence (2.6), we obtain that $\mathrm{Cone}(\mathcal{S}'' \to \mathbb{C}^{\oplus 2^m} \otimes \mathcal{O}_Q) = \mathcal{S}'(1)$, which shows that $\mathbb{R}_{\mathcal{O}_Q}\mathcal{S}'' = \mathcal{S}'(1)[-1]$. By Lemma 2.2.4 we deduce the semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(Q) = \langle \mathcal{S}', \mathcal{O}_Q, \mathcal{S}'(1), \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(n-1) \rangle.$$

Tensoring by $\mathcal{O}_Q(-1)$ and applying Lemma 2.2.11(a) as before, we get the desired dual Lefschetz decomposition. $\square$

2.3 Categorical resolutions of nodal varieties

In this section we prove Theorem 2.1.1, which is obtained from Proposition 2.3.6, Proposition 2.3.7 and Proposition 2.3.8. Let Y be a quasiprojective variety with an isolated nodal singularity y. Let $\sigma\colon \widetilde{Y} \to Y$ be the resolution of singularities provided by the blow-up at the singular point. Recall that the exceptional divisor $j\colon Q \to \widetilde{Y}$ is isomorphic to the smooth quadric of dimension $\dim(Y) - 1$. Let $\mathcal{S}$ be the spinor bundle on Q if $\dim(Y)$ is even, and denote by $\mathcal{S}', \mathcal{S}''$ the spinor bundles if $\dim(Y)$ is odd. Recall from § 2.2.1 that $\mathcal{O}_Q(1) = j^*\mathcal{O}_{\widetilde{Y}}(-Q)$.

For the sake of simplicity, let us assume first that Y is projective: we shall explain how to adjust the proofs when Y is quasiprojective in Remark 2.3.9. We start with some observations on the properties of certain objects in $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$.

Lemma 2.3.1. *If* $\dim(Y) \geq 3$*, then* $j_*\mathcal{O}_Q(k)$ *is exceptional. Moreover, if* $\dim(Y)$ *is odd, then* $j_*\mathcal{S}'$ *and* $j_*\mathcal{S}''$ *are exceptional as well, and we have*

$$\mathrm{Hom}^\bullet(j_*\mathcal{S}', j_*\mathcal{S}'') = \mathbb{C}[-2]. \tag{2.13}$$

If $\dim(Y)$ *is even, then we have that*

$$\mathrm{Hom}^\bullet(j_*\mathcal{S}, j_*\mathcal{S}) = \mathbb{C} \oplus \mathbb{C}[-2]. \tag{2.14}$$

Proof. By Proposition 2.2.21, the functor j_* is fully faithful on the subcategory generated by the exceptional object $\mathcal{O}_Q$. It follows that $j_*\mathcal{O}_Q(k) = j_*\mathcal{O}_Q \otimes \mathcal{O}_{\widetilde{Y}}(-kQ)$ is exceptional too.

Now let us assume that $\dim(Y)$ is odd. Note that the role of the spinor bundles $\mathcal{S}'$ and $\mathcal{S}''$ is interchangeable. Applying Proposition 2.2.21 to the Lefschetz decomposition (2.12), we get that $j_*\mathcal{S}'$ and $j_*\mathcal{S}''$ are exceptional. Next, we compute $\mathrm{Hom}^\bullet(j_*\mathcal{S}', j_*\mathcal{S}'')$, which is isomorphic to $\mathrm{Hom}^\bullet(j^*j_*\mathcal{S}', \mathcal{S}'')$ by adjunction. Consider the exact triangle on Q

$$j^*j_*\mathcal{S}' \to \mathcal{S}' \to \mathcal{S}'(-Q)[2] = \mathcal{S}'(1)[2], \tag{2.15}$$

and the associated long exact sequence obtained by applying $\mathrm{Hom}^\bullet(-, \mathcal{S}'')$. By Theorem 2.2.29(f) and Lemma 2.2.33 we know

$$\mathrm{Hom}^\bullet(\mathcal{S}', \mathcal{S}'') = 0, \qquad \mathrm{Hom}^\bullet(\mathcal{S}'(1), \mathcal{S}'') = \mathbb{C}[-1].$$

Substituting these equalities, we obtain

$$\mathrm{Hom}^\bullet(j_*\mathcal{S}', j_*\mathcal{S}'') = \mathrm{Hom}^\bullet(\mathcal{S}'(1), \mathcal{S}'')[-1] = \mathbb{C}[-2],$$

proving the equality (2.13).

Following the same strategy, we compute $\mathrm{Hom}^\bullet(j_*\mathcal{S}, j_*\mathcal{S})$ when $\dim(Y)$ is even. By applying $\mathrm{Hom}^\bullet(-, \mathcal{S})$ to the exact triangle (2.15), we obtain

$$\mathrm{Hom}^\bullet(j^*j_*\mathcal{S}, \mathcal{S}) \leftarrow \mathrm{Hom}^\bullet(\mathcal{S}, \mathcal{S}) \leftarrow \mathrm{Hom}^\bullet(\mathcal{S}(1)[2], \mathcal{S}).$$

Recalling that

$$\mathrm{Hom}^\bullet(\mathcal{S},\mathcal{S}) = \mathbb{C}, \quad \mathrm{Hom}^\bullet(\mathcal{S}(1),\mathcal{S}) = \mathbb{C}[-1],$$

by Lemma 2.2.32, we obtain $\mathrm{Hom}^i(j^*j_*\mathcal{S},\mathcal{S}) = 0$ except for $i = 0, 2$, for which it is equal to $\mathbb{C}$. ☐

Remark 2.3.2. Note that by Lemma 2.3.1 the objects $j_*\mathcal{O}_Q(k)$ are exceptional, so the mutation functor $\mathbb{R}_{j_*\mathcal{O}_Q(k)}$ is well defined. The same remark holds for $j_*\mathcal{S}''$ when $\dim(Y)$ is odd.

Lemma 2.3.3. *If* $\dim(Y)$ *is even, we have the isomorphisms*

$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}) = j_*\mathcal{S}$$

for $2 - \dim(Y) \le k \le -1$*. Moreover, for all* $k \in \mathbb{Z}$*, we have*

$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}(k)) = j_*\mathcal{S}(k+1)[-1].$$

Proof. Again, we use the exact triangle on Q

$$j^*j_*\mathcal{S} \to \mathcal{S} \to \mathcal{S}(1)[2]. \tag{2.16}$$

We prove the first isomorphism. By adjunction, $\mathrm{Hom}^\bullet(j_*\mathcal{S}, j_*\mathcal{O}_Q(k)) = \mathrm{Hom}^\bullet(j^*j_*\mathcal{S}, \mathcal{O}_Q(k))$. Applying $\mathrm{Hom}^\bullet(-,\mathcal{O}_Q(k))$ to (2.16) we get the exact triangle

$$\mathrm{Hom}^\bullet(j^*j_*\mathcal{S},\mathcal{O}_Q(k)) \leftarrow \mathrm{Hom}^\bullet(\mathcal{S},\mathcal{O}_Q(k)) \leftarrow \mathrm{Hom}^\bullet(\mathcal{S}(1)[2],\mathcal{O}_Q(k)).$$

Now we have by Theorem 2.2.29(d) that

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathcal{S},\mathcal{O}_Q(k)) &= \mathrm{H}^\bullet(Q,\mathcal{S}^\vee(k)) = \mathrm{H}^\bullet(Q,\mathcal{S}(k+1)) \quad \text{and} \\ \mathrm{Hom}^\bullet(\mathcal{S}(1),\mathcal{O}_Q(k)) &= \mathrm{H}^\bullet(Q,\mathcal{S}^\vee(k-1)) = \mathrm{H}^\bullet(Q,\mathcal{S}(k)). \end{aligned}$$

Since $2 - \dim(Y) \le k \le -1$, both these terms vanish by Remark 2.2.31, thus

$$\mathrm{Hom}^\bullet(j^*j_*\mathcal{S},\mathcal{O}_Q(k)) = 0 \quad \text{for} \quad 2 - \dim(Y) \le k \le -1. \tag{2.17}$$

We conclude that

$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}) = j_*\mathcal{S}.$$

We prove now the second isomorphism. Let $2m+1 = \dim(Y) - 1$. Twisting the exact sequence (2.5) by $\mathcal{O}_Q(k)$ and taking the pushforward along j, we get the exact triangle

$$j_*\mathcal{S}(k+1)[-1] \to j_*\mathcal{S}(k) \to j_*\mathcal{O}_Q(k)^{\oplus 2^{m+1}}. \tag{2.18}$$

Since $\mathrm{Hom}^\bullet(j_*\mathcal{S}(k+1), j_*\mathcal{O}_Q(k)) = \mathrm{Hom}^\bullet(j^*j_*\mathcal{S}(1),\mathcal{O}_Q) = 0$ by (2.17), this is a mutation triangle. This immediately implies the statement. ☐

Lemma 2.3.4. *If* $\dim(Y)$ *is odd, we have the isomorphisms*

$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}') = j_*\mathcal{S}'$$

for $2 - \dim(Y) \leq k \leq -1$*. Moreover, for all* $k \in \mathbb{Z}$*, we have*

$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}'(k)) = j_*\mathcal{S}''(k+1)[-1],$$
$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}''(k)) = j_*\mathcal{S}'(k+1)[-1].$$

Finally, we have that

$$\mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}') = \operatorname{Cone}(j_*\mathcal{S}' \to j_*\mathcal{S}''[2])[-1].$$

Proof. The first three isomorphisms are proven in the same way as the previous lemma. The last one follows immediately from (2.13). □

We can now come to the study of the categorical resolution of $\mathbf{D}^{\mathrm{b}}(Y)$, where Y is the nodal variety from the beginning of this section.

Proposition 2.3.5. *With the notation introduced at the beginning of this section and in Theorem 2.2.34, set*

$$\widetilde{\mathcal{D}} \coloneqq \{\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\widetilde{Y}) \mid j^*\mathcal{F} \in \mathscr{B}_0\}.$$

Let $\sigma_* \colon \widetilde{\mathcal{D}} \to \mathbf{D}^{\mathrm{b}}(Y)$ *denote the restriction of the pushforward functor. Then the pullback functor* $\sigma^* \colon \mathbf{D}^{\mathrm{perf}}(Y) \to \mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ *factors as* $\sigma^* \colon \mathbf{D}^{\mathrm{perf}}(Y) \to \widetilde{\mathcal{D}}$*, and* $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$ *is a weakly crepant categorical resolution of* $\mathbf{D}^{\mathrm{b}}(Y)$*.*

Proof. Set $n \coloneqq \dim(Y) - 1$. Recall the dual Lefschetz decomposition of $\mathbf{D}^{\mathrm{b}}(Q)$ introduced in Theorem 2.2.34

$$\mathbf{D}^{\mathrm{b}}(Q) = \langle \mathscr{B}_{n-1}(1-n), \ldots, \mathscr{B}_1(-1), \mathscr{B}_0 \rangle,$$

where

1. $\mathscr{B}_0 = \langle \mathcal{S}, \mathcal{O}_Q \rangle$ and $\mathscr{B}_i = \langle \mathcal{O}_Q \rangle$ for $1 \leq i \leq n-1$, if Y is even dimensional,
2. $\mathscr{B}_0 = \mathscr{B}_1 = \langle \mathcal{S}', \mathcal{O}_Q \rangle$ and $\mathscr{B}_i = \langle \mathcal{O}_Q \rangle$ for $2 \leq i \leq n-1$, if Y is odd dimensional.

Denote by ρ the restriction of σ to Q; the image of ρ consists of the singular point y of Y. Since $\mathbf{D}^{\mathrm{perf}}(y) = \langle \mathcal{O}_y \rangle$, we have $\rho^* \mathbf{D}^{\mathrm{perf}}(y) = \langle \rho^*\mathcal{O}_y \rangle = \langle \mathcal{O}_Q \rangle \subset \mathscr{B}_0$. In fact, $\rho^* \mathbf{D}^{\mathrm{perf}}(y) = \langle \mathcal{O}_Q \rangle \subset \mathscr{B}_i$ for all i. Recall that a variety with nodal singularities is Gorenstein, as discussed in § 2.2.1. We now compute the discrepancy of the exceptional divisor Q. As σ is an isomorphism outside of Q, we have $\omega_{\widetilde{Y}} = \sigma^*\omega_Y \otimes \mathcal{O}(kQ)$ for some $k \in \mathbb{Z}$. By the adjunction formula and Lemma 2.2.30 we have that

$$\mathcal{O}_Q(-n) = \omega_Q = (\omega_{\widetilde{Y}} \otimes \mathcal{O}(Q))|_Q = (\sigma^*\omega_Y \otimes \mathcal{O}((k+1)Q))|_Q = \mathcal{O}_Q(-k-1).$$

As $\operatorname{Pic}(Q)$ is torsion free, cf. [Har77, Exercise II.6.5c], this implies $k = n-1$. Then the triple $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$ defined in the proposition is a weakly crepant categorical resolution by Theorem 2.2.22. □

Next, we compute the kernel of the categorical resolution from Proposition 2.3.5. For the rest of this section, we will use σ_* to denote the pushforward functor $\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) \to \mathbf{D}^{\mathrm{b}}(Y)$, and not its restriction to $\widetilde{\mathcal{D}}$.

Proposition 2.3.6. *The kernel* $\operatorname{Ker}(\sigma_*) \cap \widetilde{\mathcal{D}}$ *of the weakly crepant categorical resolution* $\widetilde{\mathcal{D}}$ *of* $\mathbf{D}^{\mathrm{b}}(Y)$ *is classically generated by a single object* $\mathcal{T}$*, where* $\mathcal{T} = j_*\mathcal{S}$ *if* $\dim(Y)$ *is even, and* $\mathcal{T} = \mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}'[1]) = \operatorname{Cone}(j_*\mathcal{S}' \to j_*\mathcal{S}''[2])$ *if* $\dim(Y)$ *is odd.*

Proof. Set $n := \dim(Y) - 1$. Note that the conditions of Theorem 2.2.25 are satisfied in our situation as explained in Corollary 2.2.26 and Remark 2.2.27. This gives that $\sigma_* : \mathbf{D}^{\mathrm{b}}(\widetilde{Y}) \to \mathbf{D}^{\mathrm{b}}(Y)$ is a localization functor up to direct summands, and its kernel is classically generated by $\mathscr{K} := \langle j_*(\langle \mathcal{O}_Q \rangle^{\perp}) \rangle$, that is, $\operatorname{Ker}(\sigma_*) = \mathscr{K}^{\oplus}$ is the idempotent completion of $\mathscr{K}$. We now determine $\operatorname{Ker}(\sigma_*) \cap \widetilde{\mathcal{D}}$.

On the one hand, $\langle \mathcal{O}_Q \rangle^{\perp}$ admits by Theorem 2.2.34 a semiorthogonal decomposition of the form

1. $\langle \mathcal{O}_Q \rangle^{\perp} = \langle \mathcal{O}_Q(1-n), \ldots, \mathcal{O}_Q(-1), \mathcal{S} \rangle$ if Y is even dimensional,
2. $\langle \mathcal{O}_Q \rangle^{\perp} = \langle \mathcal{O}_Q(1-n), \ldots, \mathcal{S}'(-1), \mathcal{O}_Q(-1), \mathcal{S}' \rangle$ if Y is odd dimensional,

thus the pushforwards of the components along j are a set of generators of $\mathscr{K}$.

On the other hand, the semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) = \langle j_*\mathscr{B}_{n-1}(1-n), \ldots, j_*\mathscr{B}_1(-1), \widetilde{\mathcal{D}} \rangle \tag{2.19}$$

induced by (2.12) and the fully faithfulness of j_* on $\mathscr{B}_i(-i)$ for $1 \leq i \leq n-1$ show that

1. $\{j_*\mathcal{O}_Q(1-n), \ldots, j_*\mathcal{O}_Q(-1)\}$ if Y is even dimensional,
2. $\{j_*\mathcal{O}_Q(1-n), \ldots, j_*\mathcal{S}'(-1), j_*\mathcal{O}_Q(-1)\}$ if Y is odd dimensional

are full exceptional collections of $\widetilde{\mathcal{D}}^{\perp}$.

Now, looking at the generators of $\widetilde{\mathcal{D}}^{\perp}$ and $\mathscr{K}$, we obtain $\widetilde{\mathcal{D}}^{\perp} \subset \mathscr{K} \subset \operatorname{Ker}(\sigma_*)$, which implies that

$$\operatorname{Ker}(\sigma_*) \cap \widetilde{\mathcal{D}} = \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}} \operatorname{Ker}(\sigma_*).$$

We first assume that Y is even dimensional. Notice that all the generators of $\mathscr{K}$ belong to $\widetilde{\mathcal{D}}^{\perp}$ except the pushforward of the spinor bundle, so that $\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}\mathscr{K} = \langle \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}) \rangle$. Since $\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(\mathscr{K}^{\oplus}) \subset (\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}\mathscr{K})^{\oplus}$, we have the inclusions

$$\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}) \subset \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(\operatorname{Ker}(\sigma_*)) \subset \langle \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}) \rangle^{\oplus}.$$

Now, as both $\operatorname{Ker}(\sigma_*)$ and $\widetilde{\mathcal{D}}$ are idempotent complete, so is their intersection $\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(\operatorname{Ker}(\sigma_*))$. Thus $\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}} \operatorname{Ker}(\sigma_*) = \langle \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}) \rangle^{\oplus}$. The same argument shows that $\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}} \operatorname{Ker}(\sigma_*) = \langle \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}') \rangle^{\oplus}$ when Y is odd dimensional.

To conclude, it suffices to compute the mutations of the spinor bundles through $\widetilde{\mathcal{D}}^{\perp}$. When Y is even dimensional, we have by Lemma 2.2.5 and Lemma 2.3.3

$$\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}) = (\mathbb{R}_{j_*\mathcal{O}_Q(-1)} \circ \cdots \circ \mathbb{R}_{j_*\mathcal{O}_Q(1-n)})(j_*\mathcal{S}) = j_*\mathcal{S}.$$

When Y is odd dimensional, we consider the exceptional collection of $\widetilde{\mathcal{D}}^{\perp}$ obtained by mutating $j_*\mathcal{S}'(-1)$ through $j_*\mathcal{O}_Q(-1)$. Since $\mathbb{R}_{j_*\mathcal{O}_Q(-1)}(j_*\mathcal{S}'(-1)) = j_*\mathcal{S}''[-1]$ by Lemma 2.3.4, we have

$$\widetilde{\mathcal{D}}^{\perp} = \langle j_*\mathcal{O}_Q(1-n), \dots, j_*\mathcal{O}_Q(-1), j_*\mathcal{S}''\rangle. \tag{2.20}$$

Using Lemma 2.2.5 and Lemma 2.3.4 we obtain

$$\mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}') = (\mathbb{R}_{j_*\mathcal{S}''} \circ \mathbb{R}_{j_*\mathcal{O}_Q(-1)} \circ \cdots \circ \mathbb{R}_{j_*\mathcal{O}_Q(1-n)})(j_*\mathcal{S}') = \mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}'),$$

and also $\mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}') = \mathrm{Cone}(j_*\mathcal{S}' \to j_*\mathcal{S}''[2])[-1]$.

These computations yield the desired classical generator of $\mathrm{Ker}(\sigma_*) \cap \widetilde{\mathcal{D}}$ in the even and odd dimensional case. □

Now let $\mathcal{T} = j_*\mathcal{S}$ or $\mathcal{T} = \mathrm{Cone}(j_*\mathcal{S}' \to j_*\mathcal{S}''[2])$, depending on the parity of the dimension of Y.

Proposition 2.3.7. *If Y is even dimensional, then $\mathcal{T}$ is a 2-spherical object in $\widetilde{\mathcal{D}}$.*

Proof. Set $n \coloneqq \dim(Y) - 1$. Let us prove that $\mathcal{T} = j_*\mathcal{S}$ satisfies the three conditions of Definition 2.2.9. Condition (a) is automatic, since $\widetilde{Y}$ is projective and $\widetilde{\mathcal{D}}$ is a semiorthogonal component of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ in the decomposition (2.19). By Lemma 2.3.1, we have that

$$\mathrm{Hom}^{\bullet}(\mathcal{T}, \mathcal{T}) = \mathbb{C} \oplus \mathbb{C}[-2],$$

so condition (b) holds true. It remains to check condition (c). Recall that $\widetilde{\mathcal{D}}$ has a Serre functor, given by Lemma 2.2.11(b); thus, by Remark 2.2.12, it is enough to show that $\mathbb{S}_{\widetilde{\mathcal{D}}}(\mathcal{T}) = \mathcal{T}[2]$. We have that

$$\mathbb{S}_{\widetilde{\mathcal{D}}}(j_*\mathcal{S}) = \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(\mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})}(j_*\mathcal{S})) = (\mathbb{R}_{j_*\mathcal{O}_Q(-1)} \circ \cdots \circ \mathbb{R}_{j_*\mathcal{O}_Q(1-n)} \circ \mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})})(j_*\mathcal{S}),$$

where $\mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})} = \mathbb{T}_{\omega_{\widetilde{Y}}} \circ [n+1]$. Since, by the adjunction formula, we have the equality

$$j^*\omega_{\widetilde{Y}} = \omega_Q \otimes j^*\mathcal{O}_{\widetilde{Y}}(-Q) = \mathcal{O}_Q(-\dim(Q)+1) = \mathcal{O}_Q(1-n),$$

we obtain

$$\mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})}(j_*\mathcal{S}) = j_*(\mathcal{S} \otimes j^*\omega_{\widetilde{Y}})[n+1] = j_*\mathcal{S}(1-n)[n+1].$$

Now, using Lemma 2.3.3, we have

$$\mathbb{R}_{j_*\mathcal{O}_Q(k)}(j_*\mathcal{S}(k)[2-k]) = j_*\mathcal{S}(k+1)[2-k-1].$$

Proceeding inductively, we obtain

$$\mathbb{S}_{\widetilde{\mathcal{D}}}(j_*\mathcal{S}) = j_*\mathcal{S}[2],$$

proving the statement. □

Proposition 2.3.8. *If Y is odd dimensional, then $\mathcal{T}$ is a 3-spherical object in $\widetilde{\mathcal{D}}$.*

Proof. Again, since the category $\widetilde{\mathcal{D}}$ is proper, condition (a) in Definition 2.2.9 is automatically satisfied. To check condition (b), recall that $\mathcal{T}$ sits in the exact triangle

$$j_*\mathcal{S}' \to j_*\mathcal{S}''[2] \to \mathcal{T}. \tag{2.21}$$

By definition we have $\mathrm{Hom}^\bullet(\mathcal{T}, j_*\mathcal{S}'') = 0$. Hence by applying $\mathrm{Hom}^\bullet(\mathcal{T}, -)$ to (2.21) we obtain that

$$\mathrm{Hom}^\bullet(\mathcal{T}, \mathcal{T}) = \mathrm{Hom}^\bullet(\mathcal{T}, j_*\mathcal{S}'[1]),$$

and by applying $\mathrm{Hom}^\bullet(-, j_*\mathcal{S}')$ to (2.21) we obtain

$$\mathrm{Hom}^\bullet(j_*\mathcal{S}', j_*\mathcal{S}') \leftarrow \mathrm{Hom}^\bullet(j_*\mathcal{S}''[2], j_*\mathcal{S}') \leftarrow \mathrm{Hom}^\bullet(\mathcal{T}, j_*\mathcal{S}').$$

We have by previous computations (cf. Lemma 2.3.1 and (2.13)) that

$$\mathrm{Hom}^\bullet(j_*\mathcal{S}', j_*\mathcal{S}') = \mathbb{C}, \quad \text{and} \quad \mathrm{Hom}^\bullet(j_*\mathcal{S}'', j_*\mathcal{S}') = \mathbb{C}[-2].$$

We get from the long exact sequence that

$$\mathrm{Hom}^\bullet(\mathcal{T}, \mathcal{T}) = \mathbb{C} \oplus \mathbb{C}[-3].$$

To complete the proof we need to show that $\mathbb{S}_{\widetilde{\mathcal{D}}}(\mathcal{T}) = \mathcal{T}[3]$. Using Lemma 2.2.11(b) with respect to the decomposition in (2.20), we have the factorization

$$\mathbb{S}_{\widetilde{\mathcal{D}}}(\mathcal{T}) = \mathbb{R}_{\widetilde{\mathcal{D}}^\perp}(\mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})}(\mathcal{T})) = (\mathbb{R}_{j_*\mathcal{S}''} \circ \mathbb{R}_{j_*\mathcal{O}_Q(-1)} \circ \cdots \circ \mathbb{R}_{j_*\mathcal{O}_Q(2-\dim(Y))} \circ \mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})})(\mathcal{T}).$$

For the sake of keeping a lighter presentation, we write, by abuse of notation, $\mathcal{T}(k)$ in place of $\mathcal{T} \otimes \mathcal{O}_{\widetilde{Y}}(-kQ)$, even though the object $\mathcal{T}$ does not belong to $\mathbf{D}^{\mathrm{b}}(Q)$. As in Proposition 2.3.7, we have

$$\mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y})}(\mathcal{T}) = \mathcal{T}(2 - \dim(Y))[\dim(Y)].$$

As $\mathbb{R}_{j_*\mathcal{O}_Q(k)}$ is an exact functor, by Lemma 2.3.4 we have

$$\begin{aligned} \mathbb{R}_{j_*\mathcal{O}_Q}(\mathcal{T}) &= \mathrm{Cone}(\mathbb{R}_{j_*\mathcal{O}_Q}(j_*\mathcal{S}') \to \mathbb{R}_{j_*\mathcal{O}_Q}(j_*\mathcal{S}''[2])) \\ &= \mathrm{Cone}(j_*\mathcal{S}''(1)[-1] \to j_*\mathcal{S}'(1)[1]) \\ &= \mathcal{T}'(1)[-1], \end{aligned}$$

where $\mathcal{T}' = \mathrm{Cone}(j_*\mathcal{S}'' \to j_*\mathcal{S}'[2])$. The arrow $j_*\mathcal{S}''(1)[-1] \to j_*\mathcal{S}'(1)[1]$ is nonzero (also similar for the arrows below), otherwise the object $\mathbb{S}_{\widetilde{\mathcal{D}}}(\mathcal{T})$ would become a direct sum of two objects, but this would contradict $\mathrm{Hom}^0(\mathcal{T}, \mathcal{T}) = \mathbb{C}$ as the Serre functor is an equivalence. Analogously, we obtain

$$(\mathbb{R}_{j_*\mathcal{O}_Q(1)} \circ \mathbb{R}_{j_*\mathcal{O}_Q})(\mathcal{T}) = \mathcal{T}(2)[-2],$$

and more generally

$$(\mathbb{R}_{j_*\mathcal{O}_Q(k+1)} \circ \mathbb{R}_{j_*\mathcal{O}_Q(k)})(\mathcal{T}(k)) = \mathcal{T}(k+2)[-2].$$

It follows that

$$(\mathbb{R}_{j_*\mathcal{O}_Q(-2)} \circ \cdots \circ \mathbb{R}_{j_*\mathcal{O}_Q(2-\dim(Y))})(\mathcal{T}(2-\dim(Y))[\dim(Y)]) = \mathcal{T}(-1)[3]. \tag{2.22}$$

Finally, we compute

$$\mathbb{R}_{j_*\mathcal{O}_Q(-1)}(\mathcal{T}(-1)[3]) = \mathcal{T}'[2],$$

and the last mutation

$$\begin{aligned}\mathbb{R}_{j_*\mathcal{S}''}(\mathcal{T}'[2]) &= \mathbb{R}_{j_*\mathcal{S}''}(\mathrm{Cone}(j_*\mathcal{S}'' \to j_*\mathcal{S}'[2])[2]) \\ &= \mathrm{Cone}(\mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}'') \to \mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}'[2]))[2] \\ &= \mathrm{Cone}(0 \to \mathbb{R}_{j_*\mathcal{S}''}(j_*\mathcal{S}'[2]))[2] \\ &= \mathrm{Cone}(0 \to \mathcal{T}[1])[2] \\ &= \mathcal{T}[3]. \qquad \square\end{aligned}$$

Remark 2.3.9. This concludes the proof of Theorem 2.1.1 in the case of a *projective* variety Y with an isolated nodal singularity y. We point out how to adjust the proofs when Y is only supposed *quasiprojective*. Let Y' be a projective compactification of Y; by resolution of singularities, we can assume that Y' is smooth outside y. We continue to denote by $\sigma\colon \widetilde{Y} \to Y$ the blow-up at the singular point, by $j\colon Q \to \widetilde{Y}$ the embedding of the exceptional divisor and by n the dimension of Q. The variety $\widetilde{Y}$ is quasiprojective, and can be regarded as an open subset of the blow-up $\widetilde{Y}'$ of Y' at y; we denote by $i\colon \widetilde{Y} \to \widetilde{Y}'$ the corresponding open immersion.

Let us focus on the categorical aspects. We will denote as $\mathbf{D}^{\mathrm{b}}{}_Q(\widetilde{Y})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ consisting of complexes topologically supported on Q; the functor i_* embeds it as a full subcategory of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y}')$. Now, Lemma 2.3.1 holds true even if $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ is not proper, because the functor j_* has both left and right adjoints. From Lemma 2.3.3 to Proposition 2.3.6, all results hold without any change. In fact, by going through the proofs, from the Lefschetz decomposition

$$\mathbf{D}^{\mathrm{b}}(Q) = \langle \mathscr{B}_{n-1}(1-n), \dots, \mathscr{B}_1(-1), \mathscr{B}_0 \rangle$$

of Theorem 2.2.34 we deduce semiorthogonal decompositions not only for $\mathbf{D}^{\mathrm{b}}(\widetilde{Y}')$ and $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$, but also for $\mathbf{D}^{\mathrm{b}}{}_Q(\widetilde{Y})$: explicitly, we have

$$\begin{aligned}\mathbf{D}^{\mathrm{b}}(\widetilde{Y}') &= \langle (i\circ j)_*\mathscr{B}_{n-1}(1-n), \dots, (i\circ j)_*\mathscr{B}_1(-1), \widetilde{\mathcal{D}}' \rangle \\ \mathbf{D}^{\mathrm{b}}(\widetilde{Y}) &= \langle j_*\mathscr{B}_{n-1}(1-n), \dots, j_*\mathscr{B}_1(-1), \widetilde{\mathcal{D}} \rangle \\ \mathbf{D}^{\mathrm{b}}{}_Q(\widetilde{Y}) &= \langle j_*\mathscr{B}_{n-1}(1-n), \dots, j_*\mathscr{B}_1(-1), \widetilde{\mathcal{D}}_Q \rangle,\end{aligned}$$

where $\widetilde{\mathcal{D}}'$ is defined as the left orthogonal of $\langle (i \circ j)_* \mathscr{B}_{n-1}(1-n), \dots, (i \circ j)_* \mathscr{B}_1(-1) \rangle$ in $\mathbf{D}^{\mathrm{b}}(\widetilde{Y}')$, and $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}_Q$ as the left orthogonal of $\langle j_* \mathscr{B}_{n-1}(1-n), \dots, j_* \mathscr{B}_1(-1) \rangle$ in $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ and ${\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y})$, respectively. The categories $\widetilde{\mathcal{D}}'$ and $\widetilde{\mathcal{D}}$ provide categorical resolutions of Y' and Y, respectively. Clearly $\widetilde{\mathcal{D}}_Q = \widetilde{\mathcal{D}} \cap {\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y})$, and we can easily verify that $\mathbb{R}_{\widetilde{\mathcal{D}'}^\perp} \circ i_* = i_* \circ \mathbb{R}_{\widetilde{\mathcal{D}_Q}^\perp}$ on ${\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y})$, so that $i_* \widetilde{\mathcal{D}}_Q = \widetilde{\mathcal{D}'} \cap i_* {\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y})$.

Consider now the classical generator $\mathcal{T}$ of $\mathrm{Ker}(\sigma_*)$ given by Proposition 2.3.6. We need to prove that it is spherical in the category $\widetilde{\mathcal{D}}$. Since $\mathcal{T}$ belongs to $\widetilde{\mathcal{D}}_Q$, the functors $\mathrm{Hom}^\bullet(\mathcal{T}, -)$ and $\mathrm{Hom}^\bullet(-, \mathcal{T})$ on $\widetilde{\mathcal{D}}$ take values in the category of finite-dimensional graded vector spaces, because so do they on $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ (see example 2.2.14); this shows that $\mathcal{T}$ satisfies condition (a) in Definition 2.2.9. For the other two conditions, we can reason as follows. By example 2.2.14 and Lemma 2.2.15, the pair $(\widetilde{\mathcal{D}}_Q, \widetilde{\mathcal{D}})$ has a Serre functor $\mathbb{S}$; moreover, we have

$$\begin{aligned} i_* \mathbb{S}(\mathcal{T}) &= i_* \mathbb{R}_{\widetilde{\mathcal{D}_Q}^\perp} (\mathcal{T} \otimes \omega_{\widetilde{Y}}[\dim(Y)]) = \mathbb{R}_{\widetilde{\mathcal{D}'}^\perp} i_* (\mathcal{T} \otimes i^* \omega_{\widetilde{Y}'}[\dim(Y)]) \\ &= \mathbb{R}_{\widetilde{\mathcal{D}'}^\perp} (i_* \mathcal{T} \otimes \omega_{\widetilde{Y}'}[\dim(Y)]) = \mathbb{R}_{\widetilde{\mathcal{D}'}^\perp} \circ \mathbb{S}_{\mathbf{D}^{\mathrm{b}}(\widetilde{Y}')}(i_* \mathcal{T}) = \mathbb{S}_{\widetilde{\mathcal{D}'}}(i_* \mathcal{T}). \end{aligned}$$

From the isomorphism $i_* \mathbb{S}(\mathcal{T}) = \mathbb{S}_{\widetilde{\mathcal{D}'}}(i_* \mathcal{T})$ and the full faithfulness of i_* on $\widetilde{\mathcal{D}}_Q$ we deduce that conditions (c) and (b) in Definition 2.2.9 are satisfied by $\mathcal{T}$ in $\widetilde{\mathcal{D}}$ if and only if they are satisfied by $i_* \mathcal{T}$ in $\widetilde{\mathcal{D}}'$. Hence, the k-sphericalness of $\mathcal{T}$ in $\widetilde{\mathcal{D}}$ is equivalent to the k-sphericalness of $i_* \mathcal{T}$ in $\widetilde{\mathcal{D}'}$, which was proven in Proposition 2.3.7 and Proposition 2.3.8.

This concludes the proof of Theorem 2.1.1. In [Kuz08b, Definition 3.5] another notion of crepancy was introduced in the categorical setting. A categorical resolution $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is *strongly crepant* if the relative Serre functor $\mathbb{S}_{\widetilde{\mathcal{D}}/\mathcal{D}}$ is isomorphic to the identity functor. We refer to [Kuz08b, Section 3] for the definition of relative Serre functor. We only recall this notion in the case we consider, namely $\mathcal{D} = \mathbf{D}^{\mathrm{b}}(Y)$ with a categorical resolution $\widetilde{\mathcal{D}} \subset \mathbf{D}^{\mathrm{b}}(\widetilde{Y})$, where $\pi \colon \widetilde{Y} \to Y$ is a geometrical resolution of singularities: a functor $\mathbb{S}_{\widetilde{\mathcal{D}}/Y} \colon \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ is a relative Serre functor if for every $\mathcal{F}, \mathcal{G} \in \widetilde{\mathcal{D}}$ there is a bifunctorial isomorphism

$$\mathrm{R}\mathcal{H}\mathrm{om}(\pi_* \mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}), \mathcal{O}_Y) \cong \pi_* \mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G}, \mathbb{S}_{\widetilde{\mathcal{D}}/Y}(\mathcal{F})).$$

In the next proposition, we show that the weakly crepant categorical resolution $\widetilde{\mathcal{D}}$ provided by Proposition 2.3.5 is not strongly crepant when the quasiprojective variety Y with isolated nodal singularity has dimension at least 4. We stick with the notations of Remark 2.3.9: $\sigma \colon \widetilde{Y} \to Y$ denotes the blow-up at the singular point, Q its exceptional divisor, ${\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y})$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ consisting of complexes topologically supported on Q, and $\widetilde{\mathcal{D}}_Q = \widetilde{\mathcal{D}} \cap {\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y})$. Recall that $\mathbb{T}_{\omega_{\widetilde{Y}}} \circ [\dim(Y)]$ is a Serre functor for the pair $({\mathbf{D}^{\mathrm{b}}}_Q(\widetilde{Y}), \mathbf{D}^{\mathrm{b}}(\widetilde{Y}))$, and induces a Serre functor $\mathbb{S}$ for the pair $(\widetilde{\mathcal{D}}_Q, \widetilde{\mathcal{D}})$.

Proposition 2.3.10. *The categorical resolution $\widetilde{\mathcal{D}}$ admits a relative Serre functor $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}$, given by $\mathbb{S}_{\widetilde{\mathcal{D}}/Y} = \mathbb{R}_{\widetilde{\mathcal{D}}^\perp} \circ \mathbb{T}_{\mathcal{O}((n-1)Q)}$.*

1. *For any* $\mathcal{F} \in \widetilde{\mathcal{D}}$ *such that* $j^*\mathcal{F} \in \langle \mathcal{O}_Q \rangle$ *we have* $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}(\mathcal{F}) = \mathcal{F}$.

2. *For any* $\mathcal{F} \in \widetilde{\mathcal{D}}_Q$ *we have* $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}(\mathcal{F}) = \mathbb{S}(\mathcal{F})[-\dim(Y)]$.

Therefore, if $\mathcal{T}$ *is the classical generator of the kernel computed in Proposition 2.3.6, we have* $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}(\mathcal{T}) = \mathcal{T}[2-\dim(Y)]$ *if* $\dim(Y)$ *is even and* $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}(\mathcal{T}) = \mathcal{T}[3-\dim(Y)]$ *if* $\dim(Y)$ *is odd. In particular, the categorical resolution* $(\widetilde{\mathcal{D}}, \sigma_*, \sigma^*)$ *is not strongly crepant if* $\dim(Y) > 3$.

Proof. The relative canonical bundle of σ is given by $\omega_{\widetilde{Y}/Y} = \mathcal{O}_{\widetilde{Y}}((n-1)Q)$. By [Kuz08b, Proposition 4.7], the relative Serre functor $\mathbb{S}_{\widetilde{Y}/Y} = \mathbb{T}_{\omega_{\widetilde{Y}/Y}}$ of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ induces a relative Serre functor $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}$ on $\widetilde{\mathcal{D}}$; its explicit expression, as well as and part (a), can be found in loc. cit. We prove part (b). Since $j^*\omega_{\widetilde{Y}/Y} = j^*\omega_{\widetilde{Y}}$, for any $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(Q)$ we have

$$\mathbb{S}_{\widetilde{Y}/Y}(j_*\mathcal{G}) = j_*\mathcal{G} \otimes \omega_{\widetilde{Y}/Y} = j_*(\mathcal{G} \otimes j^*\omega_{\widetilde{Y}/Y}) = j_*(\mathcal{G} \otimes j^*\omega_{\widetilde{Y}}) = j_*\mathcal{G} \otimes \omega_{\widetilde{Y}} = \mathbb{T}_{\omega_{\widetilde{Y}}}(j_*\mathcal{G}).$$

Hence, for any $\mathcal{F} \in \widetilde{\mathcal{D}}_Q$, we have

$$\mathbb{S}_{\widetilde{\mathcal{D}}/Y}(\mathcal{F}) = (\mathbb{R}_{\widetilde{\mathcal{D}}^\perp} \circ \mathbb{S}_{\widetilde{Y}/Y})(\mathcal{F}) = (\mathbb{R}_{\widetilde{\mathcal{D}}^\perp} \circ \mathbb{T}_{\omega_{\widetilde{Y}}})(\mathcal{F}) = \mathbb{S}(\mathcal{F})[-\dim(Y)].$$

Therefore, as soon as $\dim(Y) > 3$, the relative Serre functor $\mathbb{S}_{\widetilde{\mathcal{D}}/Y}$ is not the identity on the spherical object $\mathcal{T} \in \widetilde{\mathcal{D}}_Q$, so the categorical resolution $\widetilde{\mathcal{D}}$ is not strongly crepant. □

We now deduce Theorem 2.1.5 from Theorem 2.1.1.

Theorem 2.3.11. *If* $\mathscr{T}$ *is a geometric nodal category, then* $\mathscr{T}$ *is an abstract nodal category, i.e. there exists a categorical resolution* $\sigma_* \colon \widetilde{\mathcal{D}} \to \mathscr{T}$ *which is weakly crepant and whose kernel is classically generated by a single spherical object. Furthermore,* $\sigma_* \colon \widetilde{\mathcal{D}} \to \mathscr{T}$ *is a localization up to direct summands.*

Proof. By hypothesis, there exists a quasiprojective variety Y which has only an isolated nodal singularity, and a semiorthogonal decomposition $\mathbf{D}^{\mathrm{b}}(Y) = \langle \mathscr{T}, \mathscr{T}' \rangle$ with $\mathscr{T}^{\mathrm{perf}}$ not smooth. We claim that this forces $\mathscr{T}'^{\mathrm{perf}}$ to be smooth. For this, we look at the categories of singularities $\mathbf{D}^{\mathrm{sg}}(Y) := \mathbf{D}^{\mathrm{b}}(Y)/\mathbf{D}^{\mathrm{perf}}(Y)$ and $\mathscr{T}^{\mathrm{sg}} := \mathscr{T}/\mathscr{T}^{\mathrm{perf}}$. By [Orl04, §2, §3.3] and [Orl11, Thm. 2.10] we know that $\mathbf{D}^{\mathrm{sg}}(Y)^{\oplus} \simeq \mathbf{D}^{\mathrm{sg}}(\mathbb{C}[z]/(z^2))$ if $\dim(Y)$ is even, and $\mathbf{D}^{\mathrm{sg}}(Y)^{\oplus} \simeq \mathbf{D}^{\mathrm{sg}}(\mathbb{C}[x,y]/(xy))$ if $\dim(Y)$ is odd. In the even dimensional case, following [Orl04, §3.3], one sees that there exist non-zero morphisms between any pair of non-zero objects in $\mathbf{D}^{\mathrm{sg}}(\mathbb{C}[z]/(z^2))$. Hence the full subcategory $\mathbf{D}^{\mathrm{sg}}(Y) \subset \mathbf{D}^{\mathrm{sg}}(\mathbb{C}[z]/(z^2))$ admits no non-trivial semiorthogonal decomposition. But we have the semiorthogonal decomposition $\mathbf{D}^{\mathrm{sg}}(Y) = \langle \mathscr{T}^{\mathrm{sg}}, \mathscr{T}'^{\mathrm{sg}} \rangle$ by [Orl06, Prop. 1.10], so either $\mathscr{T}^{\mathrm{sg}} = 0$ or $\mathscr{T}'^{\mathrm{sg}} = 0$, as desired. In the odd dimensional case, the category $\mathbf{D}^{\mathrm{sg}}(\mathbb{C}[x,y]/(xy))$ is equivalent to the category of $\mathbb{Z}/2\mathbb{Z}$-graded finite-dimensional vector spaces, where the shift functor swaps the graded pieces cf. [KPS21, Ex. 2.18], so we can conclude as before.

Let us assume that the dimension $\dim(Y) \geq 2$ is even, the proof of the odd dimensional case is similar. Then, by Theorem 2.1.1, we know that there is a weakly crepant categorical resolution $\sigma_*\colon \widetilde{\mathcal{D}} \to \mathbf{D}^{\mathrm{b}}(Y)$ whose kernel $\mathrm{Ker}(\sigma_*)$ is classically generated by a 2-spherical object $\mathcal{S}$.

Let us denote by $\imath\colon \mathscr{T}' \to \mathbf{D}^{\mathrm{b}}(Y)$ the embedding functor. As $\mathscr{T}'$ is admissible, it has a left adjoint functor $\imath^*$ and a right adjoint functor $\imath^!$. We know that $\imath(\mathscr{T}') \subset \mathbf{D}^{\mathrm{perf}}(Y)$ since $\mathscr{T}'^{\mathrm{perf}} = \mathscr{T}'$ by hypothesis. Then we see that the functor $\sigma^* \circ \imath\colon \mathscr{T}' \to \widetilde{\mathcal{D}}$ is fully-faithful. Moreover, this functor has the right adjoint $\imath^! \circ \sigma_*$, thus making $\mathscr{T}'$ an admissible subcategory of $\widetilde{\mathcal{D}}$. So we can consider the semiorthogonal decomposition $\widetilde{\mathcal{D}} = \langle \widetilde{\mathscr{T}}, \mathscr{T}' \rangle$, where $\widetilde{\mathscr{T}} := \mathscr{T}'^{\perp}$.

Now we claim that the restriction of σ_* to $\widetilde{\mathscr{T}}$ provides a categorical resolution that satisfies the conditions in Definition 2.1.3. First, if $\mathcal{F} \in \mathscr{T} \cap \mathbf{D}^{\mathrm{perf}}(Y)$ and $\mathcal{G} \in \mathscr{T}' \subset \widetilde{\mathcal{D}}$, then $\mathrm{Hom}^\bullet(\mathcal{G}, \sigma^*\mathcal{F}) = \mathrm{Hom}^\bullet(\sigma_*\mathcal{G}, \mathcal{F}) = 0$, which implies that σ^* maps $\mathscr{T}^{\mathrm{perf}}$ to $\widetilde{\mathscr{T}}$. Second, if $\mathcal{F} \in \widetilde{\mathscr{T}}$ and $\mathcal{G} \in \mathscr{T}'$, then $\mathrm{Hom}^\bullet(\mathcal{G}, \sigma_*\mathcal{F}) = \mathrm{Hom}^\bullet(\sigma^*\mathcal{G}, \mathcal{F}) = 0$, which implies that σ_* maps $\widetilde{\mathscr{T}}$ to $\mathscr{T}$. Regarding adjointness and weak crepancy, let $\mathcal{F} \in \mathscr{T}^{\mathrm{perf}}$ and $\mathcal{G} \in \widetilde{\mathscr{T}}$, and considering them as objects of $\mathbf{D}^{\mathrm{b}}(Y)$ and $\widetilde{\mathcal{D}}$, respectively, we see that $\mathrm{Hom}(\sigma^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \sigma_*\mathcal{G})$ and $\mathrm{Hom}(\mathcal{G}, \sigma^*\mathcal{F}) = \mathrm{Hom}(\sigma_*\mathcal{G}, \mathcal{F})$. In the same vein we have that $\mathrm{id}_{\mathscr{T}^{\mathrm{perf}}} \to \sigma_*\sigma^*$ is an isomorphism.

By Theorem 2.1.1 we know that

$$\widetilde{\mathcal{D}}/\langle \mathcal{S} \rangle^{\oplus} \to \mathbf{D}^{\mathrm{b}}(Y)$$

is an equivalence onto its dense image. Since $\sigma_*(\mathcal{S}) = 0$ and for $\mathcal{G} \in \mathscr{T}'$ we have $\mathrm{Hom}(\sigma^*\mathcal{G}, \mathcal{S}) = \mathrm{Hom}(\mathcal{G}, \sigma_*\mathcal{S}) = 0$, we see that $\mathcal{S} \in \widetilde{\mathscr{T}}$. So, by the universal property of Verdier quotients, we can factor $\sigma_*|_{\widetilde{\mathscr{T}}}\colon \widetilde{\mathscr{T}} \to \mathscr{T}$ via

$$\overline{\sigma_*}\colon \widetilde{\mathscr{T}}/\langle \mathcal{S} \rangle^{\oplus} \to \mathscr{T}.$$

Furthermore, by [Orl06, Lemma 1.1] we have that the embedding $\widetilde{\mathscr{T}} \subset \widetilde{\mathcal{D}}$ descends to a fully-faithful functor $\widetilde{\mathscr{T}}/\langle \mathcal{S} \rangle^{\oplus} \to \widetilde{\mathcal{D}}/\langle \mathcal{S} \rangle^{\oplus}$, which implies that $\overline{\sigma_*}$ is fully-faithful.

Using that $\sigma_*(\mathscr{T}') \subset \mathscr{T}'$, we see that $\mathrm{im}(\overline{\sigma_*}) = \mathscr{T} \cap \mathrm{im}(\sigma_*)$. We need to check that the idempotent completion of the latter is $\mathscr{T}$. Since $\sigma_*\sigma^* = \mathrm{id}_{\mathbf{D}^{\mathrm{perf}}(Y)}$, we see that ${}^{\perp}\mathscr{T} = \mathscr{T}' \subset \mathrm{im}(\sigma_*)$, which implies that

$$\mathrm{im}(\sigma_*) \cap \mathscr{T} = \mathbb{L}_{\mathscr{T}'}(\mathrm{im}(\sigma_*)).$$

Since we know that $\mathrm{im}(\sigma_*)^{\oplus} = \mathbf{D}^{\mathrm{b}}(Y)$, we get

$$\begin{aligned} \mathcal{T} = \mathcal{T}'^{\perp} = \mathbb{L}_{\mathcal{T}'}(\mathbf{D}^{\mathrm{b}}(Y)) &= \mathbb{L}_{\mathcal{T}'}(\mathrm{im}(\sigma_*)^{\oplus}) \\ &\subset (\mathbb{L}_{\mathcal{T}'}(\mathrm{im}(\sigma_*)))^{\oplus} = (\mathrm{im}(\sigma_*) \cap \mathcal{T})^{\oplus}. \end{aligned}$$

Hence, since $\mathcal{T}$ is idempotent complete, we conclude $\mathcal{T} = (\mathrm{im}(\sigma_*) \cap \mathcal{T})^{\oplus}$, as desired. □

2.4 Categorical resolutions of nodal cubic fourfolds

In this section we focus on the special case of a cubic fourfold $Y \subset \mathbb{P}^5$ with a single isolated nodal singularity $P \in Y$. Our goal is to prove Theorem 2.1.8 by applying Theorem 2.1.1.

2.4.1 Geometric setting

We first recall the geometric setting following [Kuz10, Section 5], which can be summarized in the diagram

$$\begin{tikzcd}
 & Q \arrow[rr, hook, "j"] \arrow[dl] & & \widetilde{Y} \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & & D \arrow[ll, hook', "i"'] \arrow[dr, "p"] & \\
P \arrow[rr] & & Y & & \mathbb{P}^4 & & S. \arrow[ll, hook']
\end{tikzcd} \tag{2.23}$$

On the left hand side, the point P is the nodal singular point and the morphism $\sigma\colon \widetilde{Y} \to Y$ is the blow-up of Y at P; this yields the resolution of singularities $\widetilde{Y}$, whose exceptional divisor Q is a smooth quadric of dimension 3. On the right hand side, the linear projection from P induces a regular map $\pi\colon \widetilde{Y} \to \mathbf{P}^4$, which can be shown to be the blow-up of $\mathbf{P}^4$ along a smooth K3 surface S that is a $(2,3)$-complete intersection, cf. [Kuz10, Lemma 5.1]. We denote by D the exceptional divisor of the map π, and write $j\colon Q \hookrightarrow \widetilde{Y}$ and $i\colon D \hookrightarrow \widetilde{Y}$ for the inclusions, as well as p for the restriction $\pi|_D$.

Moreover, the restriction $\pi|_Q$ identifies the divisor Q with the defining quadric of S in $\mathbf{P}^4$. Also, the description of the map π shows that the surface S parametrizes the lines contained in Y passing through P. Such lines are contracted by the linear projection from P, and the divisor D is the union of their strict transforms in $\widetilde{Y}$.

The following result clarifies the relation between Q, D, and S.

Lemma 2.4.1. *The restriction $p|_Q = \pi|_{Q\cap D}$ of the projection map π identifies $Q \cap D \subset \widetilde{Y}$ with the K3 surface S. In other words, S is a retract of D and the diagram*

$$\begin{tikzcd}
Q \arrow[d, hook, "j"] & S \arrow[l, hook', "t"'] \arrow[d, hook, "s"'] \\
\widetilde{Y} & D \arrow[l, hook', "i"'] \arrow[u, bend right, "p"']
\end{tikzcd}$$

is cartesian, where t denotes the inclusion of S into Q, and $s\colon S \xrightarrow{\sim} Q \cap D \hookrightarrow D$ denotes the inclusion into D.

Proof. Recall that $\pi|_Q$ is an isomorphism between Q and the defining quadric of S in $\mathbf{P}^4$, and that $\pi(D) = S$. Therefore the intersection $Q \cap D$ is a closed subscheme of the pre-image $(\pi|_Q)^{-1}(S)$ in Q, which is a smooth K3 surface. Note that $Q \cap D$ is non-empty since each line in Y passing through P provides a point contained in $Q \cap D$. Then, by Krull's principal ideal theorem, the dimension of $Q \cap D$ is at least 2 everywhere. We conclude that $Q \cap D$

coincides with the surface $(\pi|_Q)^{-1}(S)$. In other words, $\pi|_{Q\cap D}$ identifies $Q \cap D$ with the K3 surface S. □

2.4.2 Computation of the kernel

We work in the geometric situation summarized in diagram (2.23). Let h be the class of a hyperplane in $\mathbb{P}^4$, and H be the class of a hyperplane section of $Y \subset \mathbb{P}^5$. By abuse of notation, we use the same notation for their pullbacks to $\widetilde{Y}$.

Recall that $\langle \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)\rangle$ is an exceptional sequence, so we have the semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(Y) = \langle \mathscr{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)\rangle, \tag{2.24}$$

where $\mathscr{A}_Y := \langle \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H)\rangle^{\perp}$. Now we consider two semiorthogonal decompositions of $\mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ arising from the two geometric interpretations of the variety $\widetilde{Y}$, cf. [Kuz10, (16), (17)]. First, we apply Proposition 2.2.21 using the Lefschetz decomposition of $\mathbf{D}^{\mathrm{b}}(Q)$ from Theorem 2.2.34 to write

$$\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) = \langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h), \widetilde{\mathcal{D}}\rangle,$$

where $\widetilde{\mathcal{D}} := {}^{\perp}\langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h)\rangle$ is a weakly crepant resolution of $\mathbf{D}^{\mathrm{b}}(Y)$. Then we consider the decomposition of $\widetilde{\mathcal{D}}$ induced by that of $\mathbf{D}^{\mathrm{b}}(Y)$ in (2.24). As Y has rational singularities, $\sigma^* : \mathbf{D}^{\mathrm{b}}(Y) \to \widetilde{\mathcal{D}}$ is fully faithful, cf. [Kuz08b, Lemma 2.4], so the pullbacks of $\mathcal{O}_Y$, $\mathcal{O}_Y(H)$ and $\mathcal{O}_Y(2H)$ along σ are an exceptional sequence in $\widetilde{\mathcal{D}}$, and we obtain

$$\widetilde{\mathcal{D}} = \langle \widetilde{\mathscr{A}_Y}, \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(H), \mathcal{O}_{\widetilde{Y}}(2H)\rangle.$$

Substituting this in the decomposition above, we get

$$\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) = \langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h), \widetilde{\mathscr{A}_Y}, \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(H), \mathcal{O}_{\widetilde{Y}}(2H)\rangle. \tag{2.25}$$

Note that the residual category $\widetilde{\mathscr{A}_Y}$ is a weakly crepant resolution of $\mathscr{A}_Y$ by [Kuz10, Lemma 5.8]. On the other hand, since $\widetilde{Y}$ is the blow-up of $\mathbf{P}^4$ along the K3 surface S, we have by Orlov's blow-up formula [Orl92] that

$$\mathbf{D}^{\mathrm{b}}(\widetilde{Y}) = \langle \Phi(\mathbf{D}^{\mathrm{b}}(S)), \mathcal{O}_{\widetilde{Y}}(-3h), \mathcal{O}_{\widetilde{Y}}(-2h), \mathcal{O}_{\widetilde{Y}}(-h), \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(h)\rangle, \tag{2.26}$$

where $\Phi\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(\widetilde{Y})$ is given by $\Phi = \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(D)} \circ i_* \circ p^*$. Recall that $\mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(D)}$ denotes the functor which twists by $\mathcal{O}_{\widetilde{Y}}(D)$.

Using a series of mutations, one may relate these two decompositions and show that there is an equivalence $\Phi''\colon \mathbf{D}^{\mathrm{b}}(S) \xrightarrow{\sim} \widetilde{\mathscr{A}_Y}$, cf. [Kuz10, Corollary 5.7], so $\mathbf{D}^{\mathrm{b}}(S)$ is also a weakly crepant categorical resolution of $\mathscr{A}_Y$. The equivalence is explicitly given by

$$\Phi'' = \mathbb{R}_{\mathcal{O}_{\widetilde{Y}}(-h)} \circ \mathbb{R}_{\mathcal{O}_{\widetilde{Y}}(-2h)} \circ \mathbb{T}_{\mathcal{O}_{\widetilde{Y}}(D-2h)} \circ i_* \circ p^*.$$

Now, applying Proposition 2.3.6, the weakly crepant categorical resolution of $\mathbf{D}^{\mathrm{b}}(Y)$ given by $\widetilde{\mathcal{D}}$ together with the restrictions of σ_* and σ^* has kernel classically generated by $j_*\mathcal{S}$. We show that the latter is also a classical generator of the kernel of $\widetilde{\mathscr{A}_Y} \to \mathscr{A}_Y$. Since $\widetilde{\mathscr{A}_Y}$ is an admissible subcategory, it is in particular thick, so it suffices to prove the following lemma.

Lemma 2.4.2. *The object $j_*\mathcal{S}$ lies in $\widetilde{\mathscr{A}_Y}$.*

Proof. By Proposition 2.3.6, we have that $j_*\mathcal{S}$ lies in $\widetilde{\mathcal{D}} = {}^{\perp}\langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h)\rangle$. It now suffices to verify that $j_*\mathcal{S} \in \big\langle \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(H), \mathcal{O}_{\widetilde{Y}}(2H)\big\rangle^{\perp}$.

Note that for all $k \in \mathbf{Z}$ we have

$$\operatorname{Hom}^{\bullet}_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}}(kH), j_*\mathcal{S}) = \operatorname{Hom}^{\bullet}_{Q}(\mathcal{O}_Q, \mathcal{S}) = 0,$$

since Q is the exceptional divisor of σ and the line bundle $\mathcal{O}_{\widetilde{Y}}(kH)$ pulls back to the trivial line bundle on Q. $\square$

Next we describe $j_*\mathcal{S}$ as an object in $\mathbf{D}^{\mathrm{b}}(S)$ using the left adjoint of Φ''. The latter has been computed in [Kuz10, Remark 5.9], but beware of a misprint in loc. cit., so we provide here its correct expression.

Proposition 2.4.3. *The left adjoint of Φ'' is*

$$\Psi = p_* \circ i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(-3h+D)[1]} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)}.$$

Proof. Recall that if $\mathcal{E}$ is an exceptional object in $\mathbf{D}^{\mathrm{b}}(X)$, where X is a smooth projective variety, then the functor $\mathbb{R}_{\mathbb{S}_X(\mathcal{E})}$ is right adjoint to $\mathbb{L}_{\mathcal{E}}$, where $\mathbb{S}_X$ is the Serre functor of $\mathbf{D}^{\mathrm{b}}(X)$. Using this fact and that the canonical class of $\widetilde{Y}$ is $-5h + D$, we obtain

$$\begin{aligned}\operatorname{Hom}_{\widetilde{Y}}(\mathscr{A}, \Phi''(\mathscr{B})) &= \operatorname{Hom}_{\widetilde{Y}}(\mathscr{A}, (\mathbb{R}_{\mathcal{O}_{\widetilde{Y}}(-h)} \circ \mathbb{R}_{\mathcal{O}_{\widetilde{Y}}(-2h)} \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(D-2h)} \circ i_* \circ p^*)(\mathscr{B}))\\ &= \operatorname{Hom}_D((i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(2h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)})(\mathscr{A}), p^*\mathscr{B}).\end{aligned}$$

Now we need to compute the left adjoint of p^*. The canonical bundle of D is by adjunction

$$\omega_D = (\omega_{\widetilde{Y}} \otimes \mathcal{O}(D))|_D = \big(\mathcal{O}(-5h + D) \otimes \mathcal{O}(D)\big)|_D.$$

So we have $\omega_D = i^*\mathcal{O}_{\widetilde{Y}}(-5h + 2D)$. We compute the left adjoint of p^* using Grothendieck–Verdier duality

$$\begin{aligned}\operatorname{Hom}_D(\mathcal{F}, p^*\mathscr{B}) &= \operatorname{Hom}_D(p^*\mathscr{B}, \mathcal{F} \otimes \omega_D[3])^{\vee}\\ &= \operatorname{Hom}_D(p^*\mathscr{B}, \mathcal{F}(-5h + 2D)[3])^{\vee}\\ &= \operatorname{Hom}_S(\mathscr{B}, p_*\mathcal{F}(-5h + 2D)[3])^{\vee}\\ &= \operatorname{Hom}_S(p_*\mathcal{F}(-5h + 2D)[3], \mathscr{B} \otimes \omega_S[2])\\ &= \operatorname{Hom}_S(p_*\mathcal{F}(-5h + 2D)[1], \mathscr{B}).\end{aligned}$$

This shows that the left adjoint of p^* is $p_* \circ \mathbf{T}_{i^*\mathcal{O}_{\widetilde{Y}}(-5h+2D)[1]}$. Putting everything together, we obtain

$$\begin{aligned}
&\operatorname{Hom}_D((i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(2h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)})(\mathscr{A}), p^*\mathscr{B}) \\
=&\operatorname{Hom}_S((p_* \circ \mathbf{T}_{i^*\mathcal{O}_{\widetilde{Y}}(-5h+2D)[1]} \circ i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(2h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)})(\mathscr{A}), \mathscr{B}) \\
=&\operatorname{Hom}_S((p_* \circ i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(-5h+2D)[1]} \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(2h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)})(\mathscr{A}), \mathscr{B}) \\
=&\operatorname{Hom}_S((p_* \circ i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(-3h+D)[1]} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)})(\mathscr{A}), \mathscr{B}),
\end{aligned}$$

and thus

$$\Psi = p_* \circ i^* \circ \mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(-3h+D)[1]} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(3h-D)} \circ \mathbb{L}_{\mathcal{O}_{\widetilde{Y}}(4h-D)},$$

proving the statement. □

We now identify $\Psi(j_*\mathcal{S})$ as the restriction of the spinor bundle on Q to S.

Proposition 2.4.4. *We have that* $\Psi(j_*\mathcal{S}) = t^*\mathcal{S}$, *where* $t\colon S \to Q$ *is the inclusion of* S *into the quadric* Q *which is embedded in* $\mathbb{P}^4$ *via* $\pi \circ j$.

Proof. Note that the first two mutations in the formula of Ψ have no effect, since we have, using the relation $D = 3h - H$,

$$\begin{aligned}
\operatorname{Hom}^\bullet_{\widetilde{Y}}(\mathcal{O}_{\widetilde{Y}}(4h-D), j_*\mathcal{S}) &= \operatorname{Hom}^\bullet_Q(j^*\mathcal{O}_{\widetilde{Y}}(4h-D), \mathcal{S}) \\
&= \operatorname{Hom}^\bullet_Q(j^*\mathcal{O}_{\widetilde{Y}}(h+H), \mathcal{S}) \\
&= \operatorname{Hom}^\bullet_Q(j^*\mathcal{O}_{\widetilde{Y}}(h), \mathcal{S}) \\
&= \operatorname{Hom}^\bullet_Q(\mathcal{O}_Q(h), \mathcal{S}) = 0,
\end{aligned}$$

and similarly for the second mutation. Applying $\mathbf{T}_{\mathcal{O}_{\widetilde{Y}}(-3h+D)}$, we get

$$j_*\mathcal{S} \otimes \mathcal{O}_{\widetilde{Y}}(D-3h) = j_*\mathcal{S} \otimes \mathcal{O}_{\widetilde{Y}}(-H) = j_*(\mathcal{S} \otimes j^*\mathcal{O}_{\widetilde{Y}}(-H)) = j_*(\mathcal{S} \otimes \mathcal{O}_Q) = j_*\mathcal{S}.$$

The last step is to calculate $p_*i^*j_*\mathcal{S}$. We consider the diagram

$$\begin{tikzcd}
Q \arrow[d, "j"] & S \arrow[l, "t"'] \arrow[d, "s"'] \arrow[dd, bend left, "\mathrm{id}"] \\
\widetilde{Y} \arrow[d, "\pi"] & D \arrow[l, "i"'] \arrow[d, "p"'] \\
\mathbb{P}^4 & S \arrow[l]
\end{tikzcd} \tag{2.27}$$

where the upper square is cartesian by Lemma 2.4.1. We prove $i^*j_* = s_*t^*$ by checking the conditions of the base-change result Proposition 2.2.28. Indeed, as Q and $\widetilde{Y}$ are smooth, they are Cohen–Macaulay. The closed immersion i of the exceptional divisor of the blow-up is by its very nature a local complete intersection in $\widetilde{Y}$. Finally, we have $\mathsf{codim}_{\widetilde{Y}}(D) = \mathsf{codim}_D(S) = 1$. Thus we obtain

$$p_* i^* j_* \mathcal{S} = p_* s_* t^* \mathcal{S} = t^* \mathcal{S}. \qquad \square$$

We conclude this section with the proof of Theorem 2.1.8.

Proof of Theorem 2.1.8. By Theorem 2.1.1 and Lemma 2.4.2 we have that the kernel of $\mathbf{D}^{\mathrm{b}}(S) \to \mathscr{A}_Y$ is classically generated by $j_* \mathcal{S}$. Then the statement follows from Proposition 2.4.4. $\square$

Appendix A

Grothendieck group of horospherical varieties of Picard rank one

Recall the notation introduced in Theorem 1.1.3. The following Proposition A.0.3 shows that the Grothendieck group $K_0(X)$ of a horospherical variety X is determined by the Grothendieck groups of its homogeneous pieces Y, Z. In particular, this addresses the case of odd isotropic Grassmannians. A similar results has been proved previously in singular cohomology (cf. [Gon+22, Fact 1.8]), with similar ideas, but the details of the proof were left to the reader. We provide a complete proof here.

For additional details on the following construction, we refer to [Pas09] and to [Gon+22, § 1.5]. We remark that there is an asymmetry in the choice of Z and Y, as one of the closed orbits is stable under the action of the non-reductive group $\mathrm{Aut}(X)$, while the other is not. This difference does not impact the proof and we can fix Y and Z in both ways.

Consider the following diagrams induced by the blowup of Z:

$$\begin{tikzcd}
E \arrow[r, "\epsilon"] \arrow[d, "\rho"] & \mathfrak{X} \arrow[r, "\pi"] \arrow[d, "\sigma"] & Y \\
Z \arrow[r, "\iota"] & X \arrow[ur, dotted] &
\end{tikzcd} \tag{A.1}$$

where $\mathfrak{X} \cong \mathrm{Bl}_Z X$ and the exceptional divisor E is isomorphic to the partial flag variety $\mathsf{G}/(\mathsf{P}_Y \cap \mathsf{P}_Z)$. The horizontal arrow $\pi : \mathfrak{X} \to Y$ is a $\mathbb{P}^c$-bundle, which restricts to an $\mathbb{A}^c$-bundle $\pi : \mathfrak{X} \setminus E \to Y$, where $c = \mathsf{dim}\, X - \mathsf{dim}\, Y$. Notice that the diagram (A.1) is G-equivariant with respect to the natural actions.

To prove Proposition A.0.3, we need to recall some facts.

Definition A.0.1 ([EH16, § 1.3.5])**.** *A smooth variety X admits a* finite affine stratification *if the following holds:*

1. *$X = \bigsqcup_i U_i$, where $\{U_i\}_i$ is a finite collection of irreducible locally closed subschemes,*

2. *$\overline{U}_i = \bigsqcup_{U_j \subseteq \overline{U}_i} U_j$, where $\overline{U}_i$ denotes the closure of U_i,*
3. *$U_i \cong \mathbb{A}^{n_i}$.*

We refer to U_i as the open strata.

Similarly to the analogous statement about Chow rings (cf. [EH16, Theorem 1.1.8]), we recall the following well known fact, which is a consequence of Quillen's localisation sequence.

Theorem A.0.2. *Let X be a smooth variety admitting a finite affine stratification $\{U_i\}_i$. Let $\{F_i\}_i$ be a finite collection of objects in $\mathbf{D}^{\mathrm{b}}(X)$ such that:*

- $\mathsf{supp}\, F_i \subseteq \overline{U}_i$,
- $\mathsf{rank}\,(F_i|_{U_i}) = 1$.

Then, $K_0(X)$ is a free abelian group and $\{[F_i]\}_i$ is a basis of $K_0(X)$.

We are ready for the main result of this appendix.

Proposition A.0.3. *Let X be a G-horospherical variety of Picard rank one, with closed G-invariant subvarieties Y, $Z \subset X$, then the canonical map induced by* (A.1)*:*

$$K_0(Z) \oplus K_0(Y) \xrightarrow{(\imath_*, \sigma_* \pi^*)} K_0(X). \tag{A.2}$$

is an isomorphism. In particular, $K_0(X)$ is a free abelian group and we have:

$$\mathsf{rank}\, K_0(X) = \mathsf{rank}\, K_0(Y) + \mathsf{rank}\, K_0(Z).$$

Proof. We prove that the morphism (A.2) is an isomorphism. Notice that the morphisms in (A.1) are equivariant with respect to the action of G. Since E, Y and Z are G-homogeneous varieties, fixing a Borel subgroup $\mathsf{B} \subset \mathsf{G}$, we obtain compatible finite affine stratifications by Schubert cells on E, Y and Z. Denote their open strata respectively as $\{E_i\}_i$, $\{Y_j\}_j$ and $\{Z_k\}_k$. Notably, the Schubert cells are exactly the orbits of B. As $E \cong \mathsf{G}/(\mathsf{P}_Y \cap \mathsf{P}_Z)$, the Schubert cells are compatible with the morphisms in (A.1), i.e. for every i, there exist j, k such that:

$$(\pi \circ \epsilon)(E_i) = Y_j \quad \text{and} \quad \rho(E_i) = Z_k. \tag{A.3}$$

We now define an affine stratification of $\mathfrak{X}$. Consider the following collection of locally closed subsets:

$$U_j^1 = \{\pi^{-1}(Y_j) \setminus E\}_j \quad \text{and} \quad U_i^2 = \{\epsilon(E_i)\}_i.$$

We claim that the collection defined by $\{U_j^1\}_j \sqcup \{U_i^2\}_i$ is a finite affine stratification of $\mathfrak{X}$. Indeed, it is immediate to see that the proposed collection is a partition of $\mathfrak{X}$, so that Definition A.0.1.(1) holds. The verification of the conditions Definition A.0.1.(2) and Definition A.0.1.(3) is straightforward for $\{U_i^2\}_i$, hence we point out the less intuitive steps for the subcollection $\{U_j^1\}_j$. Condition Definition A.0.1.(3) holds because $U_j^1 \cong \mathbb{A}^{n_j}$, as $\pi : \mathfrak{X} \setminus E \to Y$ is an $\mathbb{A}^c$-bundle, which is trivial on every Schubert cell of Y. We focus on Definition A.0.1.(2). From (A.3), we have:

$$(\pi \circ \epsilon)^{-1}(Y_j) = \bigsqcup_{(\pi\circ\epsilon)(E_s)\subseteq Y_j} E_s$$

As a consequence, we obtain:

$$\begin{aligned} \overline{U}_j^1 &= \pi^{-1}(\overline{Y}_j) = \\ &= \bigsqcup_{Y_t \subseteq \overline{Y}_j} \Big((\pi^{-1}(Y_t) \setminus E) \sqcup (\pi \circ \epsilon)^{-1}(Y_t) \Big) = \\ &= \bigsqcup_{Y_t \subseteq \overline{Y}_j} \Big((\pi^{-1}(Y_t) \setminus E) \sqcup \Big(\bigsqcup_{(\pi\circ\epsilon)(E_s)\subseteq Y_t} \epsilon(E_s) \Big) \Big) \end{aligned} \tag{A.4}$$

proving Definition A.0.1.(2).

We now show that the collection given by $\{\sigma(U_j^1)\}_j \sqcup \{\imath(Z_k)\}_k$ is a finite affine stratification of X. Again, the verification of the conditions in Definition A.0.1(2, 3) is straightforward for $\{\imath(Z_k)\}_k$. We point out the less intuitive steps for the strata $\{\sigma(U_j^1)\}_j$. Notice that σ is an isomorphism outside of the exceptional locus, hence Definition A.0.1.(3) holds by definition of U_j^1. To show Definition A.0.1.(2), we have:

$$\begin{aligned} \overline{\sigma(U_j^1)} &= \sigma(\overline{U}_j^1) = \\ &= \bigsqcup_{Y_t \subseteq \overline{Y}_j} \Big(\sigma(\pi^{-1}(Y_t) \setminus E) \sqcup \Big(\bigsqcup_{(\pi\circ\epsilon)(E_s)\subseteq Y_t} (\sigma \circ \epsilon)(E_s) \Big) \Big), \end{aligned}$$

where the first equality holds by properness of σ and the second is a consequence of (A.4). As $\sigma \circ \epsilon = \imath \circ \rho$, we conclude by (A.3) that $\{\sigma(U_j^1)\}_j \sqcup \{\imath(Z_k)\}_k$ is a finite affine stratification of X.

The collection $\{\sigma_* \pi^* \mathcal{O}_{\overline{Y}_j}\}_j \sqcup \{\imath_* \mathcal{O}_{\overline{Z}_k}\}_k$ induces a basis of $K_0(X)$ by Theorem A.0.2. Indeed, the support property is immediate, moreover, we have

$$(\sigma_* \pi^* \mathcal{O}_{\overline{Y}_j})|_{\sigma(U_j^1)} = \mathcal{O}_{\sigma(U_j^1)}, \qquad (\imath_* \mathcal{O}_{\overline{Z}_k})|_{Z_k} = \mathcal{O}_{Z_k}.$$

Finally, this shows that the morphism defined in (A.2) carries a basis of $K_0(Z) \oplus K_0(Y)$ to a basis of $K_0(X)$, proving that $(\imath_*, \sigma_* \pi^*)$ is an isomorphism. $\square$

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