

On the 6d origin of non-invertible symmetries in 4d

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ABSTRACT: It is well-known that six-dimensional superconformal field theories can be exploited to unravel interesting features of lower-dimensional theories obtained via compactifications. In this short note we discuss a new application of 6d (2,0) theories in constructing 4d theories with Kramers-Wannier-like non-invertible symmetries. Our methods allow to recover previously known results, as well as to exhibit infinitely many new examples of four dimensional theories with “M-ality” defects (arising from operations of order M generalizing dualities). In particular, we obtain examples of order $M = p^k$, where $p > 1$ is a prime number and k is a positive integer.

KEYWORDS: Anomalies in Field and String Theories, Discrete Symmetries, Field Theories in Higher Dimensions, Supersymmetry and Duality

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1 Introduction

Our understanding of generalizations of global symmetries, which capture the quantum numbers of extended operators in quantum fields [1–5], is undergoing a rapid evolution [6–90]. Symmetries of quantum fields are understood in terms of subsectors of extended quasi-topological defect operators whose fusion rules generalize the notion of groups, but can be characterized, exploiting higher categories, thus producing so-called *global categorical symmetries*. In particular, such generalized symmetry defects are not necessarily invertible [12, 35].

While such non-invertible symmetries in 1 + 1-dimensional QFTs are well known (see e.g. the discussion in [91]), it is a more recent result that 3+1 dimensional QFTs admit non-invertible symmetry defects as well [61, 62] (see also [74, 79, 81]). One can formulate two (pretty much related) strategies to produce such examples. The first builds on a generalization of the Kramers-Wannier duality defect, and involves finding models with a self-dual point in their moduli spaces, so that the duality defects become symmetries

for these systems, but with fusion rules happening to be non-invertible [62, 79, 81]. For the second, given a theory \mathcal{T} with a $\mathbb{Z}_M^{(0)} \times \mathbb{Z}_N^{(1)}$ symmetry with a mixed anomaly, one constructs a new theory by gauging the $\mathbb{Z}_N^{(1)}$ symmetry: the resulting theory has non-invertible codimension-one symmetry defects, descending from the original symmetry $\mathbb{Z}_M^{(0)}$. The corresponding fusion rules can be explicitly computed as a result of this construction [61]. This short note is the first in a series, whose purpose is to describe simple applications of 6d (2,0) superconformal field theories (SCFTs) in producing infinitely many examples of 3+1 dimensional theories with non-invertible global categorical symmetries of these kinds. This is yet another of the many applications of higher-dimensional SCFTs in unravelling interesting features of lower-dimensional dynamics.

More precisely, our technique can be explained as follows. 6d (2,0) SCFTs compactified on a Riemann surface Σ_g (possibly with punctures, suitably chosen and decorated) give rise to 4d $\mathcal{N} = 2$ SCFTs, whose conformal manifolds are identified with the moduli spaces of complex structures of Σ_g ; these theories are known in the literature as theories of class \mathcal{S} [92, 93]. In particular, the S-duality group of these models is identified with the mapping class group of Σ_g [93]. Since the action of the mapping class group is not free, there are points which admit a non-trivial stabilizer: the corresponding class \mathcal{S} theories have an enhanced 0-form symmetry provided their global structure is compatible with such an action — the stabilizer subgroup of the S-duality group becomes an ordinary 0-form symmetry for the theory corresponding to that point in moduli space. Since S-duality transformations typically rotate the global structure of the theory of interest, the resulting 0-form symmetry is expected to have a mixed anomaly with the corresponding 1-form symmetries. One can explicitly detect these mixed anomalies, exploiting the relative nature of 6d (2,0) SCFTs and its interplay with the global structure of class \mathcal{S} theories [94]. This gives a geometric origin for a large class of non-invertible symmetries in 4d theories with a mixed anomaly origin.

Whenever the stabilizer subgroup of the S-duality does not respect the global structure, we can compensate its action via gauging of a subgroup of the one-form symmetry, thus giving rise to intrinsic Kramers-Wannier duality defects, which do not have a mixed anomaly origin. In this paper we focus on cases that exhibit a mixed anomaly. In a follow up of this work we will study the more general case of intrinsic duality defects, building on uplifting the formalism introduced in [76] to our setup.

The power of the 6d approach is clear from the outset: our results can be obtained in few lines, starting from the simplest possible examples of class \mathcal{S} theories, arising from the compactification of 6d (2,0) SCFTs of type A_{n-1} on Riemann surfaces Σ_g of genus g without punctures. As an example of the power of this method, we showcase an infinity of theories with non-invertible symmetries of orders $M = p^k$, where $p > 1$ is a prime number and k is a positive integer.

The structure of this note is as follows. In section 2 we review the argument of [61] for obtaining Kramers-Wannier-like non-invertible symmetries in four-dimensional theories. In section 3, to establish some notations and conventions, we quickly review the Tachikawa's method for reading off the global structures of class \mathcal{S} theories from 6d [94]. In section 4 we discuss our general strategy in more detail, formulating a sufficient criterion for the existence of a mixed anomaly. In section 5 we discuss several applications of our method. In

particular, in §. 5.1 we rederive the original $\mathcal{N} = 4$ example in [61] from the 6d perspective, in §. 5.2 we give further 4d $\mathcal{N} = 4$ examples, and in §. 5.3 we construct infinitely many examples of models with non-invertible symmetry defects of orders $M = p^k$, where $p > 1$ is a prime number and k is a positive integer. In section 6 we present our conclusions and outlook. An alternative strategy to produce other kinds of non-invertible symmetry defects is briefly sketched in the appendix, building on the mechanism illustrated in [74].

2 “M-ality” from $\mathbb{Z}_M^{(0)} \times \mathbb{Z}_N^{(1)}$ mixed anomalies

In this section we review the construction of [61] for generating non-invertible M -ality defects, starting from a 4d theory \mathcal{T} with a $\mathbb{Z}_M^{(0)} \times \mathbb{Z}_N^{(1)}$ mixed anomaly.¹

Consider coupling \mathcal{T} to background gauge fields for the $\mathbb{Z}_M^{(0)} \times \mathbb{Z}_N^{(1)}$ symmetry, which we denote by $A^{(1)}$ and $B^{(2)}$. Assume \mathcal{T} has a mixed anomaly

$$Z_{\mathcal{T}}[A^{(1)} + d\lambda^{(0)}, B^{(2)}] = e^{i\frac{2\pi}{2N}p \int_X \lambda^{(0)} \mathcal{P}(B^{(2)})} Z_{\mathcal{T}}[A^{(1)}, B^{(2)}], \quad (2.1)$$

where p is an integer and $\mathcal{P}(B^{(2)})$ represents the Pontryagin square operation.² Let us denote the codimension one topological defects associated to the $\mathbb{Z}_M^{(0)}$ symmetry $D_3(M_3, B^{(2)})$, where we are emphasizing their explicit dependence on $B^{(2)}$. Because of the anomaly, the defect $D_3(M_3, B^{(2)})$ is not invariant with respect to background gauge transformations of $B^{(2)}$: only the following combination

$$D_3(M_3, B^{(2)}) e^{i\frac{2\pi}{2N}p \int_{M_4} \mathcal{P}(B^{(2)})}, \quad \partial M_4 = M_3, \quad (2.2)$$

is. As emphasized by Kaidi, Ohmori and Zheng [61], since this defect only depends on M_4 via the background, it is still a genuine defect of the theory. We are interested in gauging $\mathbb{Z}_N^{(1)}$, which leads to a new theory $\tilde{\mathcal{T}}$ with a dual \mathbb{Z}_N one form symmetry which we will label as $\tilde{\mathbb{Z}}_N^{(1)}$. Upon such gauging we are promoting $B^{(2)}$ to a dynamical gauge field $b^{(2)}$. The resulting defect is no longer a genuine operator of $\tilde{\mathcal{T}}$. To obtain a well-defined genuine topological defect of $\tilde{\mathcal{T}}$, one needs to cancel the dependence of $D_3(M_3, b^{(2)})$ on M_4 . When $\text{gcd}(N, p) = 1$, this can be done with a straightforward generalization of the argument in [61]: one can simply absorb the anomaly by stacking a copy of the minimal 3d TFT $\mathcal{A}_{N, -p}$ of [11] along $D_3(M_3, b^{(2)})$ to obtain a new genuine defect in the $\tilde{\mathcal{T}}$ theory:

$$\mathcal{N}(M_3) = D_3(M_3, b^{(2)}) \otimes \mathcal{A}_{N, -p}(M_3, b^{(2)}). \quad (2.3)$$

Then the theory $\tilde{\mathcal{T}}$ has a non-invertible M -ality defect $\mathcal{N}(M_3)$, with fusion rules determined from the properties of the 3d TFT $\mathcal{A}_{N, -p}$ and by the condensate of $\tilde{\mathbb{Z}}_N^{(1)}$ on M_3 .³

If instead $\text{gcd}(N, p) = k$, one can show that in this case the \mathbb{Z}_k subgroup of one-form symmetry is anomaly free and consequently anomaly only involves the quotient

¹The main emphasis in [61] is on duality defects: the M -ality case is somewhat implicit in the appendix of that paper. We are thankful to Justin Kaidi for sharing his insight on this more general argument with us.

²We refer our readers to the appendix titled *Pontryagin Square* on page 13 of the nice reference [2] for a summary about the definition of this operation, in the same set of conventions we adopt in this paper.

³We refer our readers that are not familiar with the notion of condensate to the beautiful papers [77, 79, 95].

$\mathbb{Z}_N/\mathbb{Z}_k \cong \mathbb{Z}_{N/k}$, assuming that X_4 is spin (and hence $\int_X \mathcal{P}(B^2)$ is divisible by two).⁴ Let us proceed by demonstrating this explicitly. Since $\gcd(N, p) = k$, we have an exact sequence

$$0 \rightarrow \mathbb{Z}_k \rightarrow \mathbb{Z}_N \rightarrow \mathbb{Z}_{N/k} \rightarrow 0, \tag{2.4}$$

and we can decompose

$$B^{(2)} = \frac{N}{k} B_k^{(2)} + B_{N/k}^{(2)}, \tag{2.5}$$

where $B_k^{(2)}$ is a background for $\mathbb{Z}_k^{(1)}$ and $B_{N/k}^{(2)}$ for $\mathbb{Z}_{N/k}^{(1)}$. By massaging the Pontrjagin square, one can show that.⁵

$$\frac{2\pi p}{2N} \int_X \mathcal{P}(B^{(2)}) = \frac{2\pi p}{2N} \frac{N}{k} \int_X \mathcal{P}(B_k^{(2)}) + \frac{2\pi p}{2N} \int_X \mathcal{P}(B_{N/k}^{(2)}) + 2\pi \times (\text{integers}). \tag{2.6}$$

The first term simplifies to

$$\frac{2\pi p}{2N} \frac{N}{k} \int_X \mathcal{P}(B_k^{(2)}) = \pi \ell \int_X \mathcal{P}(B_k^{(2)}), \quad \text{where } p = k\ell. \tag{2.7}$$

Since $\int_X \mathcal{P}(B_k^{(2)})$ is even on spin manifolds, this term does not contribute to the anomaly. Thus,

$$\frac{2\pi p}{2N} \int_X \mathcal{P}(B^{(2)}) = \frac{2\pi p}{2N} \int_X \mathcal{P}(B_{N/k}^{(2)}) + 2\pi \times (\text{integers}), \tag{2.8}$$

and since

$$\frac{2\pi p}{2N} = \frac{2\pi \ell}{2N/k}, \tag{2.9}$$

we obtain that a non-trivial $B_{N/k}^{(2)}$ flux causes the anomaly

$$\frac{2\pi p/k}{2N/k} \int_X \mathcal{P}(B_{N/k}^{(2)}). \tag{2.10}$$

Gauging the $\mathbb{Z}_k^{(1)}$ anomaly-free subgroup of the one-form symmetry, one obtains a theory with a symmetry group $\mathbb{Z}_{N/k}^{(1)}$ and the anomaly above. Gauging such $\mathbb{Z}_{N/k}^{(1)}$, we obtain yet another new theory $\tilde{\mathcal{T}}$, which has an M -ality defect obtained by

$$\mathcal{N}_3(M_3) = D_3(M_3, b_{N/k}^{(2)}) \otimes \mathcal{A}_{N/k, -p/k}(M_3, b_{N/k}^{(2)}). \tag{2.11}$$

Notice that this defect is also genuine: the anomaly now can be absorbed by stacking a 3d TFT $\mathcal{A}_{N/k, -p/k}$ since $\gcd(N/k, p/k) = 1$. This implies that also in this case we obtain an M -ality defect, whose fusion is proportional to the $\mathbb{Z}_{N/k}^{(1)}$ condensate along M_3 .

⁴Since we are working with supersymmetric theories in this paper, this is always the case.

⁵The relevant identities can be found in appendix A.

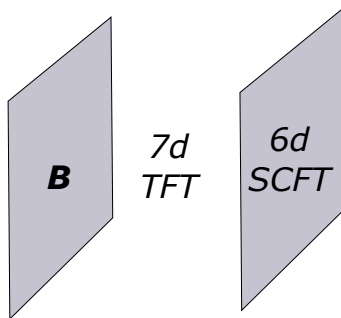


Figure 1. Schematic picture of 6d SCFTs as relative field theories to a 7d TFT.

3 Global structures from 6d — a quick review

3.1 The 6d partition vector

In this section we establish our notations and conventions by quickly summarising some of the features of 6d theories that will be useful below. We closely follow the presentation in [94]. For the sake of brevity and clarity, in this paper we focus on the 6d (2,0) theories of type A_{n-1} . It is well-known that the A_{n-1} 6d (2,0) theories are relative field theories [96, 97]: given a compact closed torsionless six-manifold Y , the 6d (2,0) A_{n-1} theory does not assign to it a complex number, a partition function, but rather a collection of partition functions, organized in a partition vector $|\mathcal{Z}(Y)\rangle$, which is an element of a Hilbert space. It is believed that the Hilbert space in question can be obtained from a non-invertible 7d TFT — see figure 1: the 6d SCFT are understood as non-topological boundary conditions for such a theory on a 7d spacetime with Y as a boundary. The Hilbert space can be characterised as a representation of a Heisenberg algebra of non-commuting discrete 3-form fluxes valued in \mathbb{Z}_n , the defect group of the 6d theory [6]. More precisely, the Heisenberg algebra in question is

$$\Phi(a)\Phi(b) = e^{i\langle a,b \rangle} \Phi(b)\Phi(a), \quad a, b \in H^3(Y, \mathbb{Z}_n), \quad (3.1)$$

where

$$\langle a, b \rangle = \frac{2\pi}{n} \int_Y a \cup b. \quad (3.2)$$

The Heisenberg algebra is also equipped with a canonical normal ordering prescription, which gives a group homomorphism from $H^3(Y, \mathbb{Z}_n)$ to the quantum torus in (3.1)⁶

$$\Phi(a+b) = \Phi(a)\Phi(b)e^{i\langle a,b \rangle/2} \quad (3.3)$$

By the Stone-Neumann-Mackay theorem, for each choice of a maximally isotropic sublattice \mathcal{L} of $H^3(Y, \mathbb{Z}_n)$ there is a unique ray in the Hilbert space $\mathcal{H}(Y)$, such that

$$\Phi(\ell)|\mathcal{L}, 0\rangle = |\mathcal{L}, 0\rangle \quad \forall \ell \in \mathcal{L}. \quad (3.4)$$

⁶Indeed: $\Phi(b+a) = \Phi(b)\Phi(a)e^{i\langle b,a \rangle/2} = \Phi(a)\Phi(b)e^{i\langle a,b \rangle/2} = \Phi(a+b)$ — we refer our readers that have caught the ambiguity in this equation to appendix B where we further clarify it.

The rest of the basis elements of $\mathcal{H}(Y)$ are obtained from the elements $v \in \mathcal{L}^\perp \equiv H^3(Y, \mathbb{Z}_n)/\mathcal{L}$:⁷

$$|\mathcal{L}, v\rangle = \Phi(v)|\mathcal{L}, 0\rangle \quad \forall v \in \mathcal{L}^\perp. \quad (3.5)$$

A crucial remark for us is that these are eigenvectors for the $\Phi(\ell)$ with $\ell \in \mathcal{L}$, with eigenvalues

$$\Phi(\ell)|\mathcal{L}, v\rangle = e^{i\langle \ell, v \rangle} |\mathcal{L}, v\rangle. \quad (3.6)$$

Another consequence of the Stone-Neumann-Mackay theorem is that, given two maximal isotropic subgroups of $H^3(Y, \mathbb{Z}_n)$, say \mathcal{L} and \mathcal{L}' , the representations constructed in this way are isomorphic, meaning that there is an invertible linear transformation such that

$$|\mathcal{L}', v'\rangle = \sum_{v \in \mathcal{L}^\perp} R_{v'} v |\mathcal{L}, v\rangle \quad \forall v' \in \mathcal{L}'^\perp. \quad (3.7)$$

For fixed \mathcal{L} , we can then write

$$|\mathcal{Z}(Y)\rangle = \sum_{v \in \mathcal{L}^\perp} \mathcal{Z}_v(Y) |\mathcal{L}, v\rangle. \quad (3.8)$$

The coefficients $\mathcal{Z}_v(Y)$ are the so-called 6d conformal blocks [97]. Clearly, choosing a different maximally isotropic lattice \mathcal{L}' , one has another set of 6d conformal blocks $\mathcal{Z}_{v'}(Y)$, but the two must be related. Indeed,

$$|\mathcal{Z}(Y)\rangle = \sum_{v' \in \mathcal{L}'^\perp} \mathcal{Z}_{v'}(Y) |\mathcal{L}', v'\rangle = \sum_{v' \in \mathcal{L}'^\perp} \mathcal{Z}_{v'}(Y) \sum_{v \in \mathcal{L}^\perp} R_{v'} v |\mathcal{L}, v\rangle, \quad (3.9)$$

which implies that

$$\mathcal{Z}_v(Y) = \sum_{v' \in \mathcal{L}'^\perp} \mathcal{Z}_{v'}(Y) R_{v'} v. \quad (3.10)$$

In order to extract values out of the partition vector, one can consider placing the 7d TFT on a finite interval times Y . On one side of the interval we have the relative 6d SCFT, on the other we insert a topological boundary condition, which in the figure we schematically denote \mathbf{B} . For example, we could be setting Dirichlet boundary conditions for all the 3-form fields, corresponding to the 3-form fluxes associated to the lattice \mathcal{L} . In this way we obtain a vector dual to $|\mathcal{L}, 0\rangle$. Hence,

$$Z_{\mathcal{L}}(Y) = \langle \mathcal{L}, 0 | \mathcal{Z}(Y) \rangle = \mathcal{Z}_0(Y). \quad (3.11)$$

Choosing different boundary conditions, corresponding to a different lattice, one obtains

$$Z_{\mathcal{L}'}(Y) = \mathcal{Z}_{0'}(Y) = \sum_{v \in \mathcal{L}^\perp} \mathcal{Z}_v(Y) (R^{-1})_v^{0'} \quad (3.12)$$

instead. This is the mechanism, which gives the 6d origin of the different partition functions, associated to possible global forms in lower dimensional field theories.

⁷Here we are implicitly choosing a representative of $v \in \mathcal{L}^\perp$ inside $H^3(Y)$ — of course the state $|\mathcal{L}, v\rangle$ depends on this choice only up to a phase which can always be absorbed in a local counterterm.

3.2 6d origin of global structure of class \mathcal{S} theories

Class \mathcal{S} theories are obtained by compactifying the 6d (2,0) theories on Riemann surfaces. In this work, to keep technicalities at a minimum, we focus on theories arising from Riemann surfaces Σ_g of genus g without punctures. For $g = 1$, i.e. when the Riemann surface is a torus, the corresponding class \mathcal{S} theories one obtains from this construction are the various $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{su}(n)$. For $g > 1$, one obtains models with an S-duality frame, where they can be interpreted as conformal gaugings of $2g - 2$ trinion T_n theories [93] (see also [98] for a thorough review), coupled to $3g - 3$ gauge groups with $\mathfrak{su}(n)$ gauge algebras.

The various global structures are captured from 6d exploiting the conformal block expansion. Consider the 6d (2,0) theory on a background $Y = \Sigma_g \times X$, where X is the 4d spacetime. Assuming X is compact, torsion-free, and that $H^1(X, \mathbb{Z})$ is trivial, from the Künneth formula and the universal coefficient theorem we obtain

$$H^3(Y, \mathbb{Z}_n) \simeq H^1(\Sigma_g, \mathbb{Z}_n) \otimes H^2(X, \mathbb{Z}_n). \tag{3.13}$$

From our discussion above, for each fixed $n \geq 2$, the additional discrete data needed to fully specify the theory is a maximal isotropic lattice L of $H^1(\Sigma_g, \mathbb{Z}_n)$ with respect to the canonical pairing induced by the intersection pairing on Σ_g . Then

$$\mathcal{L} = L \otimes H^2(X, \mathbb{Z}_n) \tag{3.14}$$

gives a maximal isotropic lattice of $H^3(Y, \mathbb{Z}_n)$ automatically. Different global forms for the 4d class \mathcal{S} theories, corresponding to the surface Σ_g , are parametrized by such choices of maximally isotropic sublattices. Fixing one such sublattice, the fluxes in \mathcal{L}^\perp parametrize the possible partition functions for the 4d theory with inequivalent 1-form symmetry backgrounds along X

$$\begin{aligned} Z_{\Sigma_g, L}(X, \xi) &= \langle \mathcal{L}, \xi | \mathcal{Z}(\Sigma_g \times X) \rangle \quad \xi \in \mathcal{L}^\perp \\ &= \langle \mathcal{L}, 0 | \Phi(\xi) | \mathcal{Z}(\Sigma_g \times X) \rangle, \end{aligned} \tag{3.15}$$

where we have used that $\langle \mathcal{L}, \xi | = \langle \mathcal{L}, 0 | \Phi(\xi)$ by definition.⁸

4 Non-invertible symmetries from 6d

4.1 0-form symmetry from mapping class group fixed points

Four-dimensional $\mathcal{N} = 2$ SCFTs have $\mathcal{N} = 2$ conformal manifolds which can be described as the space of exactly marginal deformations modulo S-duality. For the theories of class \mathcal{S} of interest for this paper the S-duality group has a beautiful geometrical description: the exactly marginal couplings of the $\mathcal{N} = 2$ theory are identified with the complex structure coordinates of the Teichmüller space of Σ_g , while the S-duality group is identified with the

⁸Morally this should be a Φ^\dagger , but since we always act on the left and never on the right, we won't pay attention to that.

mapping class group. The conformal manifold then is given by \mathcal{M}_g , the moduli space of complex structures modulo the mapping class group.⁹

Interestingly, the action of the mapping class group is not free: there can be points on the Teichmüller space that are fixed under the action of some subgroup of the mapping class group, and for that reason the conformal manifold is better characterized as a Deligne-Mumford stack. Whenever we have fixed points, naively one would claim that the theory corresponding to such a point Σ_g^* , which is stabilized by a subgroup \mathbb{G} of the mapping class group, has an enhanced $\mathbb{G}^{(0)}$ symmetry. This statement is slightly too naive: it is sufficient to consider the simplest case, i.e. genus one, to understand the problem. The mapping class group in that case is $SL(2, \mathbb{Z})$ and it acts on the complexified coupling τ , giving rise to the Montonen-Olive duality of $\mathcal{N} = 4$ SYM. If that statement above were to be true, all $\mathcal{N} = 4$ SYM theories would have an enhanced $\mathbb{Z}_2^{(0)}$ symmetry corresponding to the S transformation

$$\tau \rightarrow -1/\tau \tag{4.1}$$

at the self dual $\tau = i$. This statement is clearly false: $SU(n)$ is mapped to $PSU(n) = SU(n)/\mathbb{Z}_n$ by S-duality. This indicates there is a further requirement that needs to be imposed for \mathbb{G} to give rise to an enhanced $\mathbb{G}^{(0)}$ symmetry: the resulting symmetry must respect the global structure of the theory. This requirement is transparent from the 6d perspective. An element ϕ of the mapping class group acts on the surface Σ_g to give us a new surface $\phi(\Sigma_g)$. It also induces an isomorphism between the cohomology of Σ_g and that of $\phi(\Sigma_g)$ by pull-back. As a result, it generically changes the lattice $\mathcal{L} \simeq L \otimes H^2(X, \mathbb{Z}_n)$ which defines the 1-form symmetry of the theory (and hence its global structure), as follows:

$$\phi(\mathcal{L}) \equiv \phi_*(L) \otimes H^2(X, \mathbb{Z}_n) \tag{4.2}$$

In order for ϕ to truly generate a zero-form symmetry, it must fix not only Σ_g , but also the lattice L :

$$\phi(\Sigma_g^*) = \Sigma_g^* \quad \phi_*(L) = L \Rightarrow \phi(\mathcal{L}) = \mathcal{L}. \tag{4.3}$$

If these conditions are met, then

$$Z_{\Sigma_g^*, L}(X, 0) = Z_{\phi(\Sigma_g^*), \phi(L)}(X, 0) \tag{4.4}$$

and ϕ is indeed a symmetry of the theory.¹⁰

For genus $g > 1$ the mapping class group has a rather complicated structure,¹¹ but in this paper we are only interested in subgroups of the mapping class group that stabilize some surface Σ_g^* : describing these is much simpler because the stabilizer in the mapping

⁹This picture is slightly too naive because it does not keep track of the action of S-duality on the global structure of the theory. Including the global structure one obtains an extended conformal manifold. As we will see below, this remark has an important effect on our construction.

¹⁰When ϕ is not a symmetry it can give rise to more interesting effects, that give rise to intrinsic K -ality defects in the language of [76]. There one obtains a theory with a non-invertible symmetry starting from this setup as well, by coupling the 4d theory to an appropriate SPT to compensate to the given transformation [62, 79]. We will describe these in a separate work.

¹¹We refer to [99] for an instructive review.

class group of a surface Σ_g^* of genus $g > 1$ is isomorphic to the group of isometries of Σ_g^* .¹² For $g > 1$, the group of isometries is always finite, and like the finite symmetries of the torus, these isometries are a non-generic feature. The size of the group of isometries is bounded from above by $84(g-1)$, and the maximal order of a cyclic subgroup is $4g+2$.

Examples of surfaces with a \mathbb{Z}_{4g+2} group of isometries can be constructed in a very analogous manner to the tori with discrete symmetry: we start with a regular $4g+2$ -gon but, this time in hyperbolic space, and identify the diagonally opposite sides. This gives us a surface of genus g , which has rotations by an integer multiple of $\frac{2\pi}{4g+2}$ as its isometries (see e.g. figure 2 as an example). Compactification on these surfaces can be exploited to obtain class \mathcal{S} theories with a $\mathbb{Z}_{4g+2}^{(0)}$ enhanced symmetries.

4.2 Reading off the mixed anomaly from 6d

We now turn to the question of how we can use this technology as a diagnostic of mixed zero-form and one-form anomalies. In order to detect the 't Hooft anomaly, we act with our symmetry generator in presence of a non-trivial 1-form symmetry background. The simplest such background has a 6d avatar of the form $\beta \otimes v \in \mathcal{L}^\perp$ then

$$\begin{aligned} Z_{\phi(\Sigma_g^*), \phi(L)}(X, \phi(\beta \otimes v)) &= Z_{\Sigma_g^*, L}(X, \phi(\beta \otimes v)) \\ &= \langle \mathcal{L}, 0 | \Phi(\phi(\beta \otimes v)) | \mathcal{Z}(\Sigma_g^* \times X) \rangle \end{aligned} \quad (4.5)$$

by construction. Now the key point is that $\phi(\beta \otimes v)$ is not necessarily an element of \mathcal{L}^\perp . For instance, consider the case

$$\phi(\beta \otimes v) = \beta \otimes v + \alpha \otimes v \quad \text{where } \alpha \otimes v \in \mathcal{L} \quad (4.6)$$

If that is the case, via the normal ordering prescription in equation (3.3)

$$\Phi((\beta + \alpha) \otimes v) \equiv \Phi(\beta \otimes v) \Phi(\alpha \otimes v) e^{i\frac{1}{2}\langle \beta \otimes v, \alpha \otimes v \rangle} \quad (4.7)$$

and (4.5) equals

$$\begin{aligned} &\langle \mathcal{L}, 0 | \Phi(\beta \otimes v) \Phi(\alpha \otimes v) e^{i\frac{1}{2}\langle \beta \otimes v, \alpha \otimes v \rangle} | \mathcal{Z}(\Sigma_g^* \times X) \rangle \\ &= \langle \mathcal{L}, 0 | \Phi(\alpha \otimes v) \Phi(\beta \otimes v) e^{i\frac{1}{2}\langle \alpha \otimes v, \beta \otimes v \rangle} | \mathcal{Z}(\Sigma_g^* \times X) \rangle \\ &= e^{i\frac{1}{2}\langle \alpha \otimes v, \beta \otimes v \rangle} \langle \mathcal{L}, 0 | \Phi(\beta \otimes v) | \mathcal{Z}(\Sigma_g^* \times X) \rangle \\ &= e^{i\frac{1}{2}\langle \alpha \otimes v, \beta \otimes v \rangle} Z_{\Sigma_g, L}(X, \beta \otimes v) \end{aligned} \quad (4.8)$$

We obtain a mixed anomaly provided

$$e^{i\frac{1}{2}\langle \alpha \otimes v, \beta \otimes v \rangle} \neq 1. \quad (4.9)$$

Then *mutatis mutandis* the argument of [61] we can obtain a theory with a non-invertible symmetry by gauging the one-form symmetry associated to the choice of background $\beta \otimes v \in \mathcal{L}^\perp$. In the next section we illustrate this mechanism in practice.

¹²For $g = 1$ the isometry group always contains the group of translations and in addition there can be a discrete subgroup of isometries. This discrete subgroup is generically trivial, and is non-trivial only for the tori obtained by identifying the diagonally opposite sides of a square or a regular hexagon. In these cases it is isomorphic to the symmetries of the square or a regular hexagon, respectively.

5 Examples of applications

The technology developed in 4 suggests the following process for finding mixed zero-form one-form anomalies and consequently non-invertible duality defects:

1. Construct a class \mathcal{S} theory with a discrete global symmetry using a surface Σ_g with discrete isometries. For $g = 1$ this corresponds to the point $\tau = i$ which is self-dual under S duality. For $g > 1$ we obtain theories with \mathbb{Z}_{4g+2} 0-form symmetry using the surface obtained by identifying the diagonally opposite edges of a regular hyperbolic polygon.
2. Find a maximal isotropic sublattice \mathcal{L} of $H_1(\Sigma_g, \mathbb{Z}_n)$, which is invariant under the action of the discrete isometries constructed in the first step.
3. Find a background which under the action of the isometry changes by an element of the self dual lattice \mathcal{L} constructed in the second step. If (4.9) holds, we have found a mixed anomaly, and consequently a non-invertible duality defect in the theory obtained by gauging the anomalous one-form symmetry.

Below we present some examples where this strategy can be successfully carried out.

5.1 The $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(1)}$ mixed anomaly in $\text{SO}(3)_-$ at $\tau = i$ from 6d

As a warm-up and consistency check we now reproduce the mixed anomaly in $\text{SO}(3)_-$ using our formalism. The relevant surface in this case is a torus with complex structure τ . The \mathbb{Z}_2 valued cohomology of the torus is generated by A and B cycles with $A \cdot B = 1$ and has four elements

$$0, A, B, A + B. \quad (5.1)$$

The global form $\text{SO}(3)_-$ is obtained from the lattice $\mathcal{L} \simeq (A + B) \otimes H^2(X, \mathbb{Z}_n)$. In this case S duality acts by $\phi : \tau \mapsto -\frac{1}{\tau}$ and $A \mapsto B$, while $B \mapsto -A = A$ since $A + A = 0$. Since \mathcal{L} is invariant under the exchange of A and B , we obtain a zero-form symmetry at the self dual point $\tau = i$. For $\text{SO}(3)_-$, the S -duality then implies

$$Z_{\text{SO}(3)_-}(-\frac{1}{\tau}, X) = \langle \phi(\mathcal{L}), 0 | \mathcal{Z} \rangle = \langle \mathcal{L}, 0 | \mathcal{Z} \rangle = Z_{\text{SO}(3)_-}(\tau, X) \quad (5.2)$$

Now to detect the mixed anomaly, we wish to turn on a non-trivial background through a two cycle v in the spacetime, hence we consider

$$\begin{aligned} Z_{\text{SO}(3)_-}(-\frac{1}{\tau}, X, A \otimes v) &= \langle \phi(\mathcal{L}), \phi(A \otimes v) | \mathcal{Z} \rangle \\ &= \langle \mathcal{L}, B \otimes v | \mathcal{Z} \rangle \\ &= \langle \mathcal{L}, (A + (A + B)) \otimes v | \mathcal{Z} \rangle \\ &= e^{i\frac{1}{2} \langle A \otimes v, (A+B) \otimes v \rangle} \langle \mathcal{L}, A \otimes v | \mathcal{Z} \rangle \\ &= e^{i\frac{1}{2} \langle A \otimes v, (A+B) \otimes v \rangle} Z_{\text{SO}(3)_-}(\tau, X, A \otimes v) \\ &= e^{i\frac{1}{2} \langle A \otimes v, B \otimes v \rangle} Z_{\text{SO}(3)_-}(\tau, X, A \otimes v) \\ &= e^{i\frac{\pi}{2} \mathcal{P}(v)} Z_{\text{SO}(3)_-}(\tau, X, A \otimes v) \end{aligned} \quad (5.3)$$

which gives exactly the mixed anomaly at $\tau = i$ needed for generating a non-invertible duality symmetry along the lines discussed in [61].

5.2 More $\mathcal{N} = 4$ SYM examples from class \mathcal{S}

As a first example let us consider $\mathcal{N} = 4$ theory with gauge algebra $\mathfrak{su}(4n^2)$ at $\tau = i$. The surface in this case is a square torus and S duality acts as rotation by $\frac{\pi}{2}$. The cohomology $H^1(T^2, \mathbb{Z}_{4n^2})$ is generated by the usual A and B cycles with $A \cdot B = 1$ and relations $A^{4n^2} = B^{4n^2} = 1$ and has a total of $(2n)^4$ elements. We choose as the discrete data the lattice $\mathcal{L} = L \otimes H^2(X, \mathbb{Z}_n)$ where L is the sublattice of $H^1(T^2, \mathbb{Z}_{4n^2})$ generated by $2nA$ and $2nB$. The lattice $L \otimes H^2(X, \mathbb{Z}_n)$ is isotropic, since the intersection number of any two of its elements involves a factor of $4n^2$, which is zero in \mathbb{Z}_{4n^2} . Since it contains $(2n)^2$ elements, it is also maximal. This choice corresponds to the gauge group $SU(4n^2)/\mathbb{Z}_{2n}$. Moreover, this gauging is done without any discrete theta angle. The one-form symmetry group of this theory is $\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}$.

The lattice \mathcal{L} is manifestly self-dual under the S -duality which sends $A \rightarrow B$ and $B \rightarrow -A$. So the remaining task is to find a subgroup of the one form symmetry which has a mixed anomaly with S -duality. For this we consider a background given by $(nA + nB) \otimes v \in \mathcal{L}^\perp$. Since $2(nA + nB) \otimes v \in \mathcal{L}$, this background is for a \mathbb{Z}_2 subgroup of the one-form symmetry. Now, under S -duality

$$\phi((nA + nB) \otimes v) = (nB - nA) \otimes v = (nA + nB) \otimes v - 2nA \otimes v \quad (5.4)$$

and the phase in equation (4.9) is

$$\frac{1}{2} \langle (nA + nB) \otimes v, -2nA \otimes v \rangle = \frac{1}{2} \frac{2\pi}{4n^2} (-2n^2) \mathcal{P}(v) = -\frac{\pi}{2} \mathcal{P}(v) \quad (5.5)$$

which exhibits a mixed anomaly between the symmetry $\mathbb{Z}_2^{(0)}$ at $\tau = i$ and the $\mathbb{Z}_2^{(1)}$ subgroup of $\mathbb{Z}_{2n}^{(1)} \times \mathbb{Z}_{2n}^{(1)}$ corresponding to the element $(nA + nB) \otimes v \in \mathcal{L}^\perp$.

This example illustrates how it is possible to have further examples of $\mathfrak{su}(n)$ $\mathcal{N} = 4$ SYM theories which exhibit a $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_m^{(1)}$ mixed anomaly for a subgroup $\mathbb{Z}_m^{(1)}$ of the corresponding 1-form symmetry. Gauging such a subgroup one obtains a theory with a non-invertible duality defect.

5.3 Higher genus examples with non-invertible symmetries

In the discussion so far, we have only generated examples of non-invertible duality defects. In this section we use the class \mathcal{S} construction for higher genus Riemann surfaces to exhibit infinitely many example which give rise to M -ality non-invertible zero-form symmetries.

5.3.1 Geometry of Riemann surfaces Σ_g^* with \mathbb{Z}_{4g+2} isometry

Let us begin by describing some general features of the genus g surfaces Σ_g^* which exhibit an isometry group \mathbb{Z}_{4g+2} . These can be obtained by identifying the opposite edges of a regular hyperbolic $4g + 2$ -gon. There is a (redundant) collection of 1-cycles $C_1, C_2, \dots, C_{2g+1}$ which

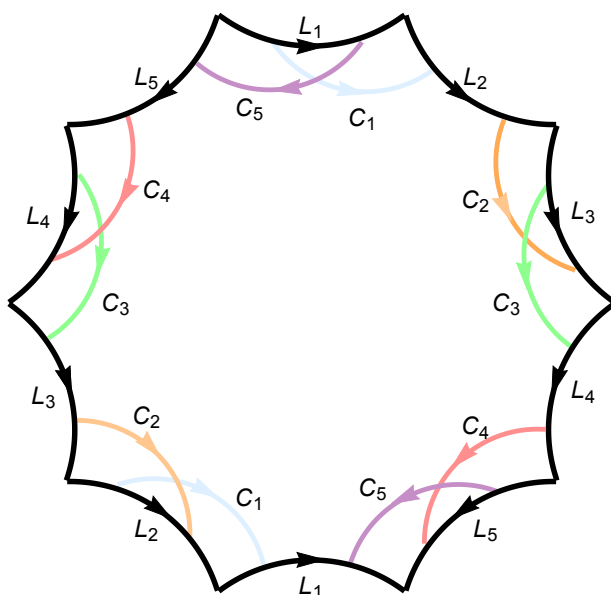


Figure 2. Decagon in the hyperbolic space, giving rise to a $g = 2$ surface upon identification of the opposite sides. Coloured lines indicate homology cycles on the surface, among which any four are linearly independent.

realise the \mathbb{Z}_{4g+2} action as $C_i \mapsto C_{i+1}$ where the index is identified modulo $2g + 1$ — see figure 2. The non-trivial intersection numbers are

$$C_i \cdot C_{i+1} = 1 \quad 1 \leq i \leq 2g \quad \text{and} \quad C_{2g+1} \cdot C_1 = -1 \quad (5.6)$$

and there is one relation between them:

$$C_{2g+1} = \sum_{i=1}^{2g} (-1)^i C_i. \quad (5.7)$$

A basis for $H_1(\Sigma_g, \mathbb{Z}_n)$ is given by C_i with $1 \leq i \leq 2g$. The isometry ϕ rotating the polygon by $\frac{2\pi}{4g+2}$ acts on the independent cycles as,

$$\phi(C_i) = C_{i+1}, \quad 1 \leq i \leq 2g. \quad (5.8)$$

A generic class \mathcal{S} theory obtained by compactifying 6d (2,0) theory with gauge algebra $\mathfrak{su}(n)$ on a genus g surface Σ_g has a one form symmetry group of the same order as the order of a maximal isotropic sublattice of $H_1(\Sigma_g, \mathbb{Z}_n)$ i.e. n^g . We would like to find subgroups of this one form symmetry, which have mixed anomaly with $\mathbb{Z}_{4g+2}^{(0)}$ zero-form symmetry generated by ϕ .

5.3.2 Infinitely many examples of p^k -ality defects

Armed with the geometry described in the previous section we can now describe a generalisation of the examples found above whenever the genus satisfies

$$2g + 1 = p^k \quad (5.9)$$

where p is a prime. In this case we choose to work with $6d$ $(2,0)$ A_{n-1} theories that are such that

$$n = p \tag{5.10}$$

so that we can view the cohomology group $H^1(\Sigma_g, \mathbb{Z}_p)$ as a $2g$ -dimensional vector space over a finite field \mathbb{F}_p . This has the advantage that we can use the Jordan decomposition to find subspaces invariant under ϕ .

We proceed by computing the characteristic polynomial of ϕ , i.e. $\det(\phi - x)$. We do this using the following trick: since for a prime p ,

$$\binom{p^k}{m} = 0 \pmod p \tag{5.11}$$

for $m \neq 0, p^k$, we obtain

$$(\phi - x)^{2g+1} = \phi^{2g+1} - x^{2g+1} \pmod p. \tag{5.12}$$

Since $\phi^{2g+1} = -1$,

$$\det((\phi - x)^{2g+1}) = (\det(\phi - x))^{2g+1} = (-1)^{2g+1}(x^{2g+1} + 1)^{2g+1}. \tag{5.13}$$

Next we use the fact that all the elements of the field \mathbb{F}_p except 0 form a multiplicative group of order $p - 1$ so $x^{p-1} = 1$. Hence,

$$\det(\phi - x) = (-1)^{2g+1}(x + 1)^{2g+1}. \tag{5.14}$$

One can easily check that there is only one eigenvector of eigenvalue -1 , by explicitly writing down the recurrence equation for the components of that eigenvector, therefore for each $\ell \leq 2g$ there is a unique subspace invariant under the action of ϕ given by the kernel of $(\phi + 1)^\ell$. Explicitly, there is a basis D_i such that,

$$\begin{aligned} \phi(D_1) &= -D_1, \\ \phi(D_i) &= -D_i + D_{i-1} \quad 1 < i \leq 2g. \end{aligned} \tag{5.15}$$

In terms of C_i , these new basis elements are given by,

$$D_i = \sum_{j=0}^{2g-i} \binom{2g-i}{j} C_{j+1}. \tag{5.16}$$

The unique invariant subspace of dimension d under the action of ϕ is given by the span of D_1, \dots, D_d . Specializing to $d = g$ we obtain the unique invariant subspace that can determine a global structure. We just need to check if it is isotropic. To do that we note that $D_i = (\phi + 1)^{2g-i} D_{2g}$. We then need to evaluate

$$D_i \cdot D_j = D_{2g}^T \left((\phi + 1)^{2g-i} \right)^T \Omega (\phi + 1)^{2g-j} D_{2g}, \tag{5.17}$$

where we denoted Ω the pairing induced by the intersection form on Σ_g . The middle part of this expression can be reorganised as¹³

$$\begin{aligned} ((\phi + 1)^{2g-i})^T \Omega(\phi + 1)^{2g-j} &= \Omega(\phi^{-1} + 1)^{2g-i}(\phi + 1)^{2g-j} \\ &= \Omega\phi^{-2g+i}(\phi + 1)^{4g-i-j}. \end{aligned} \tag{5.18}$$

Recalling that $(1 + \phi)^{2g}D_{2g} = 0$, we see that

$$D_i \cdot D_j = 0 \quad \forall i + j < 2g + 1. \tag{5.19}$$

Hence, the g -dimensional subspace spanned by D_1, \dots, D_g is indeed isotropic, we denote it

$$V = \text{span}(D_1, \dots, D_g) \subset H^1(\Sigma_g, \mathbb{Z}_p). \tag{5.20}$$

The theory $\mathcal{T}_{g,p}$ with global form corresponding to the maximal coisotropic lattice

$$\mathcal{L}_V = V \otimes H^2(X, \mathbb{Z}_n) \tag{5.21}$$

that we just identified has a $\mathbb{Z}_{4g+2}^{(0)}$ form enhancement of its symmetry at the self-dual point Σ_g^* of \mathcal{M}_g with \mathbb{Z}_{4g+2} isometry.

To exhibit an anomaly we can consider a background of the form $v \otimes D_{g+1}$, we have

$$\phi(D_{g+1}) = -D_{g+1} + D_g \quad \text{and} \quad \phi^2(D_{g+1}) = D_{g+1} - 2D_g + D_{g-1} \tag{5.22}$$

while the action of ϕ does not have the form $\phi(\beta \otimes v) = \beta \otimes v + \alpha \otimes v$, the action of ϕ^2 indeed does, and we have

$$\phi^2(\beta \otimes v) = \beta \otimes v + \alpha \otimes v \quad \text{with} \quad \begin{cases} \beta = D_{g+1} \\ \alpha = -2D_g + D_{g-1} \end{cases} \tag{5.23}$$

therefore we see that the $\mathbb{Z}_{2g+1}^{(0)}$ subgroup of $\mathbb{Z}_{4g+2}^{(1)}$ generated by ϕ^2 can have a mixed anomaly with a subgroup of the $(\mathbb{Z}_n^g)^{(1)}$ one-form symmetry for all these models. It is easy to see that there is an anomaly, since the phase in equation (4.9) is¹⁴

$$\frac{1}{2} \langle \alpha \otimes v, \beta \otimes v \rangle = \frac{\pi}{n} (D_{g+1} \cdot D_g) \int_X \mathcal{P}(v). \tag{5.24}$$

Then we produce a mixed anomaly of the form discussed in section 2, between the $\mathbb{Z}_{2g+1}^{(0)}$ subgroup of the duality symmetry and the $\mathbb{Z}_p^{(1)}$ subgroup of the one-form symmetry $((\mathbb{Z}_p)^g)^{(1)}$ which corresponds to the subspace of \mathcal{L}^\perp associated to $D_{g+1} \otimes H^2(X, \mathbb{Z}_n)$. It is the condensate of this subgroup which features in the resulting fusion algebra.

These anomalies are interesting in their own right are the smallest examples of mixed anomalies at a given genus when $2g + 1 = p^k$ for some prime p . When $2g + 1$ has more than one prime factors, we again need to look at the corresponding prime fields for the smallest examples. We expect these mixed anomalies, obtained when the homology of the

¹³Here we are using that $\phi^T \Omega \phi = \Omega$ since ϕ is an isometry, and therefore $\phi^T \Omega = \Omega \phi^{-1}$.

¹⁴Here we are using that $D_{g+1} \cdot D_{g-1} = 0$ by (5.19) and we also must have that $D_{g+1} \cdot D_g$ must be non-zero: if it was zero, this would violate the maximality of the isotropic sublattice generated by D_1, \dots, D_g .

Riemann surface is a vector space over a finite field, to be building blocks for more general mixed anomalies. We plan to return to a more systematic study of these phenomena in a future work.

6 Conclusions

In this note we have initiated the study of non-invertible defects in 4d $\mathcal{N} = 2$ theories of class \mathcal{S} . We have exploited the observation that such defects can be constructed, whenever a mixed anomaly between a zero-form symmetry and a one-form symmetry is present. Starting from this point, and using the insights coming from the 6d perspective, we were able to recover some known cases of 4d $\mathcal{N} = 4$ theories, possessing non-invertible defects, as well as to provide a bunch of new examples, coming from the higher-genus surfaces. In particular, we provide an infinite family of examples with increasing M -ality and $M = p^k$.

Our results presented here are just the tip of the iceberg, and several very natural directions for further explorations can be identified. First, a more systematic exploration of the M -ality defects we predict is interesting. We are currently studying the resulting fusion algebras, and we will present them in a follow-up of this short note.¹⁵ Second, so far we have considered only theories of A_n type. At the same time, it was observed [74] that for certain choices of the global structure also gauge theories with the gauge algebra of D_n type may have non-invertible defects in the operator spectrum.¹⁶ This is a good motivation to extend our analysis to the class \mathcal{S} theories of D_n and E_n types. Third, we were concentrating exclusively on the theories obtained from compactifications of 6d (2,0) theories on surfaces without punctures. It would certainly be interesting to extend the scope of examples by considering surfaces with punctures, regular or irregular ones. This exercise is interesting because in these examples we expect to be able to exhibit generalizations of symmetry defects corresponding to non-abelian finite zero-form symmetry groups arising from fixed points of the mapping class group.

It should also be mentioned that, while we have mostly been focused on the duality defects coming from the aforementioned mixed anomalies, there is another tool based on the self-duality a theory might obey and the corresponding Kramers-Wannier duality defects [62]. Exploiting our techniques it is easy to exhibit a broad scope of examples where the first method does not apply, while the second one is quite fruitful.

Finally, a potentially interesting class of examples to which our methods also apply is provided by compactifications of 6d (1,0) theories down to 4d (see e.g. [102–121] for a (partial) list of references on the subject). In this context the role of the Heisenberg algebra we discuss in this paper is played by the corresponding Heisenberg algebra arising from the 6d defect group of the 6d (1,0) SCFT [6, 13].

¹⁵Since the initial appearance of this note, this program has indeed been carried out by the authors of the current work in collaboration with Justin Kaidi [100] as well as independently in reference [101]. Further details about the defects presented here (including their fusion rules) can be found in these works to which we refer our readers for details.

¹⁶Fusion algebras arising from gauging an outer automorphisms acting non-trivially on the one-form symmetries [74] can be easily realized geometrically. We present some examples with these features in appendix C, as an appetizer.

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A On the proof of equation (2.6)

In order to prove (2.6) one needs to start from the following property of the Pontryagin square operation, which follows from the fact that it is a quadratic form

$$\int_X \mathcal{P}(A + B) = \int_X \mathcal{P}(A) + \int_X \mathcal{P}(B) + 2 \int_X A \cup B, \tag{A.1}$$

and applying it recursively one can show that

$$\int_X \mathcal{P}(\ell A) = \ell \int_X \mathcal{P}(A) + (2 \times \text{integer}) \int_X A \cup A \tag{A.2}$$

Indeed

$$\begin{aligned} \int_X \mathcal{P}(\ell A) &= \int_X \mathcal{P}(A) + \int_X \mathcal{P}((\ell - 1)A) + 2(\ell - 1) \int_X A \cup A \\ &= 2 \int_X \mathcal{P}(A) + \int_X \mathcal{P}((\ell - 2)A) + 2(\ell - 2) \int_X A \cup A + 2(\ell - 1) \int_X A \cup A \\ &= \dots \\ &= \ell \int_X \mathcal{P}(A) + (2 \times \text{integer}) \int_X A \cup A \end{aligned} \tag{A.3}$$

B Further details about the quantum torus algebra

The Heisenberg algebra given by (3.1) is well defined for $a, b \in H^3(Y, \mathbb{Z}_n)$. However, the computation of anomalies requires the quantum torus algebra (3.3), which involves expressions of the form $\frac{1}{2} \langle a, b \rangle$. The factor of $\frac{1}{2}$ in these expressions seem troubling at first glance. For the case when n is odd the issue can be dealt with easily: in that case 2 is invertible in \mathbb{Z}_n and hence (3.3) is well-defined. However, when n is even 2 is not invertible in \mathbb{Z}_n and so we need to explain what do we mean by the factor of $\frac{1}{2}$. One way out of this difficulty is to promote $\int_Y a \cup b$ to an element of \mathbb{Z}_{2n} . Let us look more closely at

how this can be done for the special case of interest in this paper, namely $Y = \Sigma_g \times X$ with X simply connected and Spin. In this case, let's consider for simplicity cycles of the form $a = \alpha \otimes v$ and $b = \beta \otimes v$ with $\alpha, \beta \in H^1(\Sigma_g, \mathbb{Z}_n)$ and $v \in H^2(X, \mathbb{Z}_n)$. Since Σ_g has no torsion in homology and X is simply connected (so that in particular has no torsional 1-cycles), the integral lifts for α, β, v , always exist. Let's choose some which we call $\tilde{\alpha}, \tilde{\beta}$ and \tilde{v} , respectively. Then the required promotion to \mathbb{Z}_{2n} is given by

$$\int_Y a \cup b \mapsto \left(\int_{\Sigma_g} \tilde{\alpha} \cup \tilde{\beta} \right) \left(\int_X \tilde{v} \cup \tilde{v} \right) \pmod{2n}. \tag{B.1}$$

Since any two integral lifts of v differ by nc with $c \in H^2(X, \mathbb{Z})$, $\int_X \tilde{v} \cup \tilde{v} \pmod{2n}$ is independent of the lift chosen.¹⁷ The cocycle $\tilde{v} \cup \tilde{v}$ reduces modulo $2n$ to the Pontryagin square $\mathcal{P}(v)$, in the same set of conventions of reference [2]. The map $v \rightarrow \int_X \mathcal{P}(v)$ provides a quadratic refinement of the intersection form of X and is applicable even in the more general setting when X has torsion in first homology. However, to ensure that the right hand side of (B.1) is also independent of the chosen lifts $\tilde{\alpha}$ and $\tilde{\beta}$, we also need $\int_X \tilde{v} \cup \tilde{v}$ to be even, i.e. the intersection form of X must be even. This is always the case for Spin manifolds.¹⁸

We note that $\mathcal{P}(v)$ is commonly used in literature to furnish a quadratic refinement for a single $\mathbb{Z}_n^{(1)}$ symmetry. However, for class \mathcal{S} theories equation (B.1) naturally gives the generalization to a quadratic refinement for an arbitrary number of one form symmetries. The ingredient required for this generalization is the intersection form of the Riemann surface Σ_g , which physically is interpreted in terms of the Dirac pairing between electric and magnetic charges of Wilson and 't Hooft loops.¹⁹

C Categorical symmetries from outer automorphisms

Another fruitful strategy to produce non-invertible fusion rules, that goes beyond condensation and higher gauging [77, 79, 95], arises when we are gauging an outer automorphism of the theory which is acting non-trivially on the one-form symmetry [74].

The class of models we are considering in this paper exhibits a natural outer automorphism: consider a degeneration of Σ_g such that it becomes a sphere with g handles located symmetrically. This arrangement gives rise to an action of the group \mathbb{Z}_g by cyclically permuting the various handles of Σ_g with one another. This is not a duality defect, rather an outer automorphism of the SCFT, but we can consider gauging such subgroup. Since the latter acts on the various factors of the 1-form symmetry of the system associated to the various handles, upon gauging we obtain non-invertible 2-form symmetries that have a twisted-sector like fusion.

¹⁷Indeed, $(\tilde{v} + nc) \cup (\tilde{v} + nc) = \tilde{v} \cup \tilde{v} + 2nc \cup \tilde{v} + n^2c \cup c = \tilde{v} \cup \tilde{v} \pmod{2n}$ where we have used that n is even to write $n^2 = (n/2)2n$, an integer multiple of $2n$.

¹⁸One can also easily verify that the quantum torus algebra defined with this quadratic refinement satisfies the Heisenberg algebra as expected.

¹⁹We thank the anonymous referee 1 from our JHEP submission for drawing our attention to the issues addressed here, which helped us clarifying the exposition of our results.

Labeling D_i the i -th codimension two topological surface defect corresponding to the i -th factor of the one-form symmetry $((\mathbb{Z}_n)^g)^{(1)}$, the gauged theory will have non-invertible codimension 2 defects of the form

$$\mathcal{N}_{\oplus_j (i_1^{k_1}, \dots, i_g^{k_g})_j} = \left[\bigoplus_j D_{i_1, j}^{k_1, j} \otimes D_{i_2, j}^{k_2, j} \otimes \dots \otimes D_{i_g, j}^{k_g, j} \right] \quad (\text{C.1})$$

which are labeled by gauge invariant orbits with respect to the $\mathbb{Z}_g^{(0)}$ action.

Consider for simplicity the case $g = 2$, then we have

$$\mathcal{N}(M_2) = [(D_1(M_2) \otimes 1) \oplus (1 \otimes D_2(M_2))] \quad (\text{C.2})$$

with a fusion algebra of the form

$$\mathcal{N}(M_2) \times \mathcal{N}(M_2) = \mathcal{N}_{(1^2, 0) \oplus (0, 1^2)} \oplus \mathcal{N}_{(1, 2)} \quad (\text{C.3})$$

where

$$\mathcal{N}_{(1^2, 0) \oplus (0, 1^2)} = [(D_1^2(M_2) \otimes 1) \oplus (1 \otimes D_2^2(M_2))] \quad (\text{C.4})$$

and

$$\mathcal{N}_{(1, 2)} = [D_1(M_2) \otimes D_2(M_2)]. \quad (\text{C.5})$$

The study of these structures can be carried out along the lines discussed in [74], and we plan to attack them using the methods discussed above in a subsequent work. We report of their existence in this appendix because these give a slightly different example of a 6d origin for 4d non-invertible symmetry defects thus complementing the results presented in this first exposition.

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